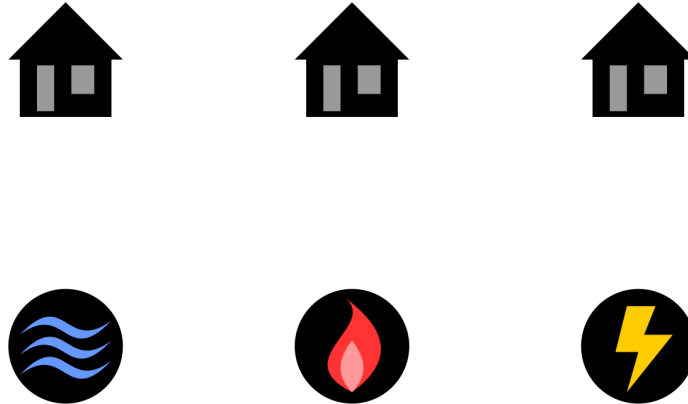
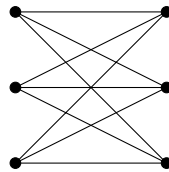


### Graphs and the Three Utilities Problem

We can imagine trying to connect three houses below with three utilities without the utility lines crossing.



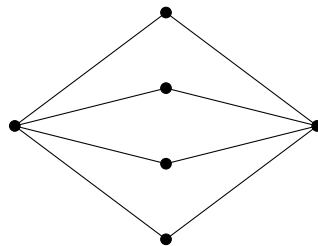
This problem is akin to the graph  $K_{3,3}$  (the complete bipartite graph with three vertices in each partite set).



A *graph* is an ordered pair of sets  $(V, E)$ , where  $E \subseteq V \times V$ .

For example, if  $V = \{a, b, c\}$  and  $E = \{(a, b), (a, c)\}$ , then  $(V, E)$  is a graph. The goal of the three utilities puzzle is to draw  $K_{3,3}$  in  $\mathbb{R}^2$  without any edges crossing. A graph that can be drawn as such is *planar*.

- $K_{3,3}$  is not planar.
- $K_{2,4}$  is planar.



### Euler's Theorem

Let  $G \subseteq \mathbb{R}^2$  be a planar graph (i.e., drawn in  $\mathbb{R}^2$  without edge crossings). Each disjoint subset of  $\mathbb{R}^2 - G$  is a *face* of  $G$ .

For every graph  $G$  embedded in  $\mathbb{R}^2$  (i.e., drawn without edge crossings) with  $V$  vertices,  $E$  edges, and  $F$  faces, the following is true:

$$V - E + F = 2$$

We will use this theorem to show that you cannot connect the three houses to the three utilities as follows:

### Outline Proof (of $K_{3,3}$ 's non-planarity)

Suppose toward contradiction that  $K_{3,3}$  is planar. Then, by Euler's Theorem, we know that  $V - E + F = 2$ .

We know that  $K_{3,3}$  has six vertices and nine edges, so we know that  $6 - 9 + F = 2$ . Therefore, we know that there must be 5 faces. In order to enclose a face, there must be at least four edges in  $K_{3,3}$  (as there is no edge between two members of a partite set). Additionally, each edge encloses two faces. Therefore,  $E \geq 2F$ . However, since  $E = 9$ , and we assume that  $F \geq 5$ , we have reached a contradiction (as  $9 < 10$ ). Thus,  $K_{3,3}$  is not planar.

### Four-Color Theorem

Every planar graph can be colored (adjacent vertices do not have the same color) with four colors. The planar graph can be colored by fewer colors.

### Polynomial Example

Let  $p(a, b, c, d) = ab + ac + ad + bc + bd + cd$ . When we factor, we get  $p(a, b, c, d) = a(b + c + d) + b(c + d) + cd$ . In the first equation, we had to carry out 6 multiplications, while in the second equation we only had to carry out 3 multiplications. We could factor differently:

$$\begin{aligned} p(a, b, c, d) &= ab + ac + ad + bc + bd + cd \\ &= a(b + c + d) + b(c + d) + cd \\ &= (a + b)(c + d) + ab + cd \end{aligned}$$

We have a lower bound of three multiplications to carry out.

In the arbitrary case, we have the following. We want to find the lowest number of multiplications.

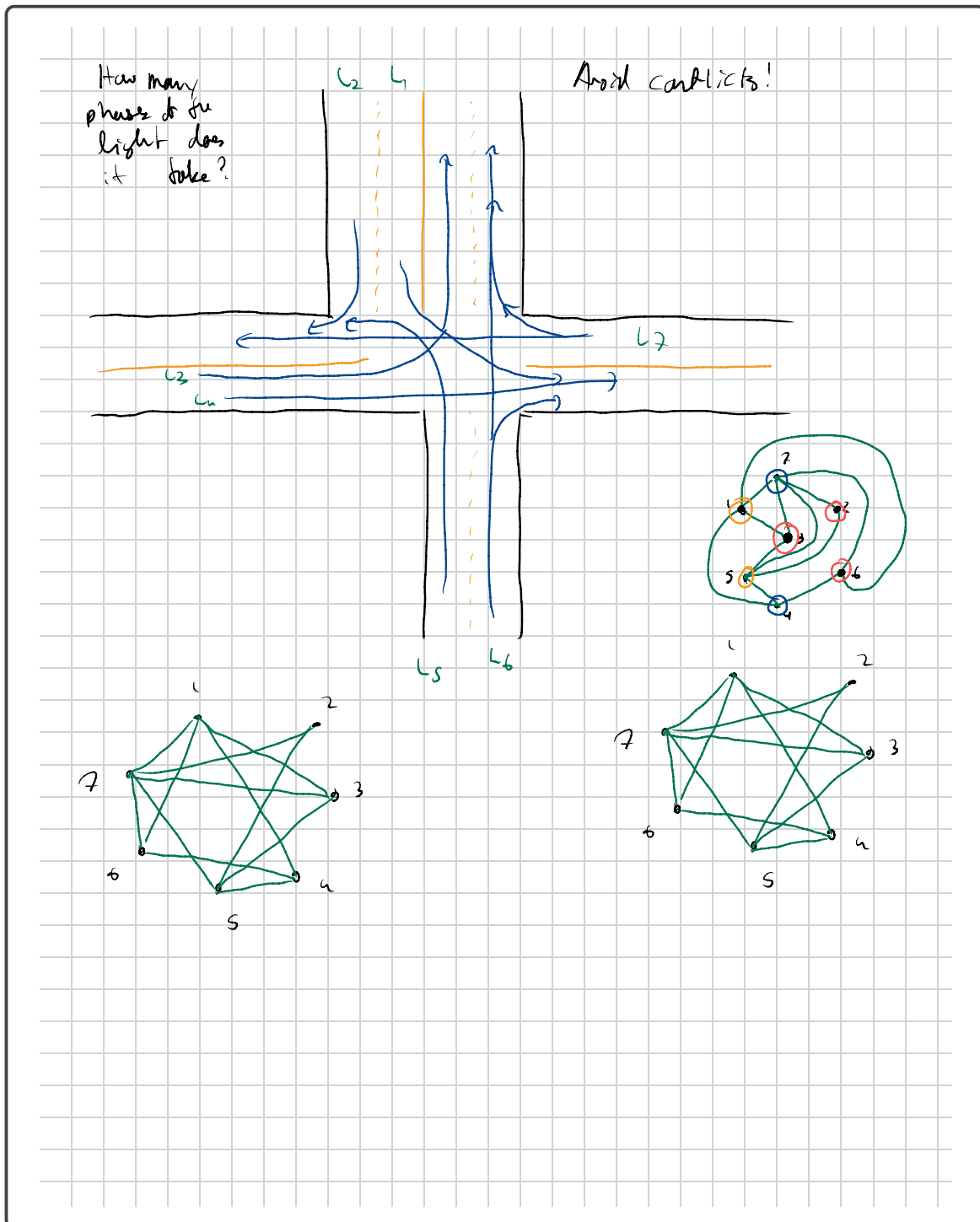
$$p(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n x_i x_j$$

The minimum number of multiplications we can do is  $n - 1$ . We can find this via a graph with  $n$  vertices  $\{x_1, \dots, x_n\}$ , and for  $x_i x_j$  in  $p$ , we have an edge from  $x_i$  to  $x_j$ . This is the complete graph on  $n$  vertices,  $K_n$ . Each complete bipartite subgraph represents a multiplication — so our question can be restated as follows:

Given a complete graph on  $n$  vertices,  $K_n$ , partition its edges into as few complete graphs as possible.

The answer for this is  $n - 1$ , with a proof in linear algebra. However, there is no graph theory-specific proof for this question.

## Light Cycles



Diestel book: Overview

A **graph** is an ordered pair  $G = (V, E)$  of sets such that  $\forall e \in E, e = \{v, w\}$  for some  $v, w \in V$ .

### Paths and Cycles

A graph  $H$  is a **subgraph** of a graph  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .

A **path** is a subgraph  $P$  of  $G$  such that  $V(P) = \{v_0, \dots, v_k\}$  and  $E(P) = \{v_0v_1, \dots, v_{k-1}v_k\}$ . We say the **length** of  $P$  is equal to  $|E(P)|$ .

If  $v_kv_0 \in E(G)$ , then  $C = P + v_kv_0$  is a **cycle**.  $V(C) = V(P)$  and  $E(C) = E(P) \cup \{v_kv_0\}$ .

**Abbreviations:**  $P = v_0 \dots v_k$ , and  $C = v_0 \dots v_kv_0$

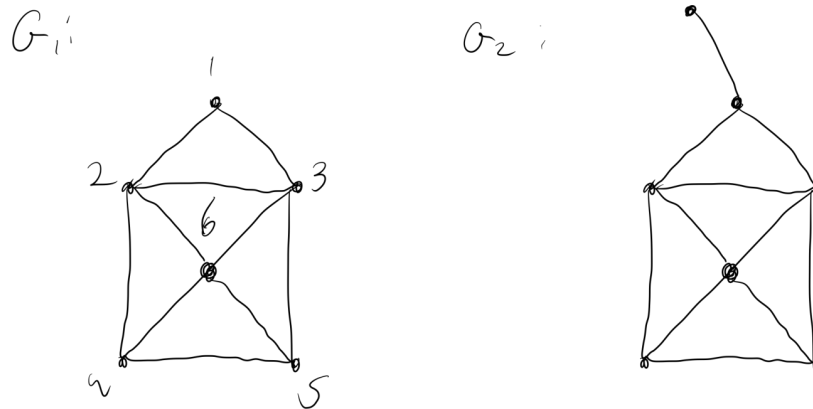
### Degree, Order, and Size

Given  $v \in V(G)$ , the **degree** of  $v$   $d(v) = |\{vw \mid v \in E(G)\}|$ . The edge  $vw$  is **incident** to  $v$ .

The **order** of  $G$  is  $|V(G)|$ , or  $|G|$ , and the **size** of  $G$  is  $|E(G)|$ , or  $\|G\|$ .

### Hamiltonian Cycles

A cycle  $C \subseteq G$  is **Hamiltonian** if  $V(C) = V(G)$ . A graph is Hamiltonian if it contains a Hamiltonian cycle.



For example,  $G_1$  has a Hamiltonian cycle  $\{1, 2, 4, 5, 6, 3, 1\}$ , while  $G_2$  does not have one as the stray vertex cannot be reached without going over an edge.

For example, the Knight's Tour (where you visit every square on a chess board) involves finding a particular kind of Hamiltonian cycle.

## Dirac's Theorem

If  $G$  is a graph of order  $\geq 3$  such that every vertex has degree  $\geq \left\lceil \frac{|G|}{2} \right\rceil$ , then  $G$  is Hamiltonian.

Let  $P$  be a path in  $G$  with maximum length (i.e., a longest path). **Outline:**

**Step 1** Show that  $|V(P)| > \frac{|G|}{2}$

**Step 2** Show  $\exists C \subseteq G$  such that  $V(C) = V(P)$ .

**Step 3** Show that  $C$  is a Hamiltonian cycle.

**Step 1** Let  $P = (v_1, v_2, \dots, v_k)$  be a path in  $G$  with maximum length. Suppose toward contradiction that  $|P| < n/2$ , meaning  $k < n/2$ . Then,  $\nexists v_i$  such that  $v_i$  is connected to any of  $v_1, \dots, v_k$ , or else we would be able to extend  $P$ . Thus,  $\forall v \in \{v_1, \dots, v_k\}$ ,  $v$  is only adjacent to other members in  $v_1, \dots, v_k$ . However, this means that the maximum value  $v$  can take is  $k-1$ , and since  $k < n/2$ , this means  $k-1 < n/2$ , or that  $v$  would not satisfy one of the conditions of  $G$ .  $\perp$

**Step 2** Let  $P = v_0 \dots v_k$ . It suffices to show that  $\exists j \in \{2, \dots, k\}$  such that  $v_1 \leftrightarrow v_j$  and  $v_{j-1} \leftrightarrow v_k$ . Since  $P$  has maximum length,  $v_1$  has no neighbor outside  $P$  (or else  $P$  could be extended). Similarly,  $v_k$  has no neighbor outside  $P$ . However, every vertex has degree at least 2, meaning  $v_1$  must have a neighbor in  $P$ . Suppose toward contradiction that  $\nexists j-1$  such that  $v_{j-1} \leftrightarrow v_k$ . Then,  $N = \{v_{2-1}, \dots, v_{k-1-1}\} \geq \frac{n}{2}$  are not neighbors of  $v_k$ . This means  $k \leq n$ , so  $v_k$  has  $k-1-N$  neighbors, implying  $d(v_k) < \frac{n}{2}$ , which is our contradiction.

**Step 3** Let  $P$  is a path of maximum length in  $G$ , and  $C$  be a cycle in  $G$  such that  $V(C) = V(P)$ . Suppose toward contradiction that  $|P| < n$ . Then,  $\exists v \in G$  such that  $v \notin P$ . Since  $d(v) \geq \frac{n}{2}$ ,  $v$  is adjacent to at least one vertex  $w \in P$  (as there are not enough vertices outside  $P$  for  $v$  to be adjacent to). Let  $C = (v_{i_1}, \dots, v_{i_k}, v_{i_1})$ . WLOG,  $v$  is adjacent to  $v_{i_1}$ . Then,  $P' = v, v_{i_1}, \dots, v_{i_k}$  is a path that is longer than  $P$ , which is a contradiction.

## Ore's Theorem

If  $|G| \geq 3$  and  $\forall v, w \in V(G)$  where  $v \nleftrightarrow w$  and  $d(v) + d(w) \geq n$ , then  $G$  is Hamiltonian.

We can use Ore's Theorem to prove Dirac's Theorem.

## Vertex Deletion

Let  $v \in G$ . Then,  $G - v$  is the subgraph of  $G$  with vertices  $V(G) \setminus \{v\}$ , and edges  $E(G) \setminus \{vw \mid vw \in E(G)\}$ .

## Theorem 6.4

Let  $v_1, \dots, v_k \in V(G)$ . Then,  $G - v_1 - v_2 - \dots - v_k$  has at most  $k$  components.

## Connectedness

A graph  $G$  is **connected** if  $\forall v, w \in V(G)$ ,  $\exists P : v \dots w$ .

## Distinct Representatives

Suppose we want to pick one student representative from every Oxy math class. No student should be chosen more than once. Say there are  $n$  classes:  $c_1, \dots, c_n$ , where  $c_i = \{s_1, \dots, s_{k_i}\}$ , where  $1 \leq i \leq n$ .

Obviously, there must be at least  $n$  students in all classes combined: i.e.,

$$\left| \bigcup c_i \right| \geq n$$

However, this goes deeper:

$$\begin{aligned} |c_1 \cup c_2| &\geq 2 \\ |c_3 \cup c_5 \cup c_6| &\geq 3 \\ &\vdots \\ |c_{i_1} \cup \dots \cup c_{i_r}| &\geq r \quad \forall r \end{aligned} \quad (*)$$

Obviously, condition  $(*)$  is necessary.

We want  $c_i$  and  $c_j$  to be distinct, (even when they are equal as sets).

Let  $Z = (c_1, \dots, c_n)$  be a finite sequence. Then,  $(c_{i_1}, \dots, c_{i_k})$  is a subsequence of  $Z$  if  $i_1 < \dots, i_k$ .

## Hall's Theorem

Let  $Z = (c_1, \dots, c_n)$  be a sequence of sets  $c_i$ . Suppose that for every subsequence  $Y$  of  $Z$  with  $Y = (c_{i_1}, \dots, c_{i_k})$  such that  $|c_{i_1} \cup \dots \cup c_{i_k}| \geq k$ . Then,  $\exists$  pairwise distinct  $s_1, \dots, s_n$  with  $s_i \in c_i$ .

**Note**  $(*)$  is a sufficient condition

Informally, we can restate the premise as follows: Let  $G$  be a bipartite graph. One set of vertices  $c_1, \dots, c_n$ , is the classes, and the other set  $s_1, \dots, s_m$  is the set of all students. Each vertex  $c_i$  is connected by edges to its students.

## Hall's Theorem (In Graphs)

Let  $G$  be a bipartite graph on vertices  $C \sqcup S$ , where  $C = \{c_1, \dots, c_n\}$  and  $S = \{s_1, \dots, s_m\}$ . Then,  $G$  has a matching (i.e., a set of pairwise disjoint edges) if and only if  $\forall r$   $1 \leq r \leq n$ , any  $r$  vertices in  $C$  are connected to at least  $r$  vertices in  $S$ .

## Proof of Hall's Theorem

**Base Case:** The theorem holds for  $n = 1$ .  $S_1 \neq \emptyset$  by the theorem's hypothesis, as if  $Y := (S_1)$ , then  $|\bigcup_{S \in Y} S| \geq 1$ , so  $|S_1| \geq 1$ .

**Induction Hypothesis** Assume the theorem holds for  $n - 1$  and every  $m < n - 1$ . We will show the theorem holds for  $n$

**Proof**

CASE 1: Assume every proper subsequence  $Y$  of  $Z$  is loose. Let  $x_1 \in S_1$  ( $S_1 \neq \emptyset$  as proved in the base case). Let  $S'_i = S_i \setminus \{x_1\}$ , where  $2 \leq i \leq n$ . Let  $Z' = (S'_2, \dots, S'_n)$ .

Let  $Y'$  be a subsequence of  $Z'$ . We want to show that

$$\left| \bigcup S'_i \in Y' S'_i \right| \geq |Y'|$$

We know that  $Y$  consists of all  $S_i$  such that  $S'_i \in Y'$ . Since  $Y$  is loose (as  $S_1 \notin Y$ ), and  $\left| \bigcup_{S_i \in Y} S_i \right| \geq |Y|$ .

$$\begin{aligned} \left| \bigcup_{S'_i \in Y'} S'_i \right| &\geq \left| \bigcup_{S_i \in Y} S_i \right| - 1 \\ &> |Y| - 1 \\ &\geq |Y| \\ &= |Y'| \end{aligned}$$

CASE 2: Suppose  $\exists$  a tight proper subsequence of  $Z$ ,  $Y$ . Without loss of generality,  $Y = (S_1, \dots, S_m)$ , where  $1 \leq m < n$ . Since  $Y$  satisfies the theorem hypothesis, and  $m < n$ , so the induction hypothesis must hold.

For  $m+1 \leq k \leq n$ , let  $S'_k = S_k \setminus \{x_1, \dots, x_m\}$ . Let  $Z' = (S'_{m+1}, \dots, S'_n)$ . Let  $Y'$  be any subsequence of  $Z'$ . We want to show that  $\left| \bigcup_{S'_i \in Y'} S'_i \right| \geq |Y'|$ .

Let  $\bar{Y}$  be the subsequence of  $Z$  corresponding to  $Y'$ , i.e.,  $S_i \in \bar{Y} \Leftrightarrow S'_i \in Y'$ .

$$\begin{array}{c|c} Y & \bar{Y} \\ \hline S_1, \dots, S_m & S_{m+1}, \dots, S_n \end{array}$$

Let  $W = Y + \bar{Y}$ , where  $+$  denotes concatenation. Since  $W$  is a subsequence of  $Z$ , and  $Z$  satisfies the Hall hypothesis, we have

$$\left| \bigcup W \right| \geq |W|$$

since

$$\begin{aligned} \bigcup W &= \bigcup Y \cup \bigcup \bar{Y} \\ &= \bigcup Y \cup \bigcup Y' \end{aligned}$$

as everything in  $\bar{Y}$  is either in  $Y'$  or in  $Y$ , and due to double counting, we have

$$\left| \bigcup W \right| \leq \left| \bigcup Y \right| + \left| \bigcup Y' \right|$$

since

$$\begin{aligned} \left| \bigcup Y \right| &= |Y| \\ &= m \end{aligned}$$

as  $Y$  is tight, and

$$\begin{aligned} |W| &= |Y| + |Y'| \\ &= |Y| + |Y'| \end{aligned}$$

so, we have

$$\begin{aligned} |W| &\leq \left| \bigcup W \right| \\ m + |Y'| &\leq m + \left| \bigcup Y' \right| \\ |Y'| &\leq \left| \bigcup Y' \right| \end{aligned}$$



**$k$ -factorable Graphs**

Let  $H$  be a subgraph of  $G$ . Let  $k \in \mathbb{Z}^+$ .  $H$  is a  $k$ -factor of  $G$  if

- (i)  $H$  is  $k$ -regular (i.e., every vertex of  $H$  is of degree  $k$ )
- (ii)  $V(H) = V(G)$  ( $H$  is a spanning subgraph)

$k$ -factors are not necessarily connected subgraphs.

A graph  $G$  is  $k$ -factorable if its edges can be partitioned  $k$ -factors of  $G$ . If  $G$  has  $k$ -factors  $H_1, \dots, H_m$  such that  $\{E(H_1), \dots, E(H_m)\}$  is a partition of  $E(G)$ .

For example,  $K_4$  is 1-factorable.

**1-factorability of  $K_n$** 

$K_n$  is 1-factorable if and only if  $n$  is even.

( $\Rightarrow$ ) The proof is trivial.

( $\Leftarrow$ ) Number the vertices of  $K_n$ . Redraw the graph such that vertex 1 is in the center of a  $n - 1$ -gon. Connect vertex 1 to vertex 2, and draw all the edges that are perpendicular to this edge. Let this 1-factor be denoted  $H_1$ .

Connect vertex 1 to vertex 3, and draw the edges perpendicular to that edge. This 1-factor is denoted  $H_2$ .

Continue until we finish connecting vertex 1 to vertex  $n/2$ , and  $H_1, \dots, H_{n/2}$  must partition the edges of  $K_n$ .

**2-factorability of Graphs**

A graph  $G$  is 2-factorable if and only if  $G$   $k$ -regular for some even integer  $k$ .

An edge  $vw$  of  $G$  is a *bridge* if  $v$  and  $w$  are in different components of  $G - vw$ .

**Chinese Postman Problem**

- A **walk** is a sequence  $(v_1, \dots, v_k)$  of vertices such that  $\exists v_i v_{i+1} \in E(G)$ .
- A **trail** is a walk that does not repeat edges.
- A **path** is a trail that does not repeat vertices.
- A closed path is a **cycle**, and a closed trail is a **circuit**.

A courier wants to deliver the mail on every street in a neighborhood. The goal is to minimize the number of streets to repeat.

We can represent this by letting vertices be intersections and edges to be streets. The goal is to create a closed walk with as few edges repeated as possible.

- A closed walk that minimizes the number of repeated edges is an *Eulerian* walk.

Finding an Eulerian closed walk:

- (1) Let  $v_1, \dots, v_{2k}$  be all the odd vertices.
- (2) Let  $G' = G + v_1v_2 + v_3v_4 + \dots + v_{2k-1}v_{2k}$ .
- (3) Every vertex in  $G'$  has even degree, so  $G'$  contains within it an Eulerian circuit  $C$ .
- (4)  $C$  contains every edge  $v_1v_2, \dots, v_{2k-1}v_{2k}$ . Replace each edge  $v_i v_{i+1}$  in this set with a shortest path in  $G$  from  $v_i$  to  $v_{i+1}$ . This gives a closed walk in  $G$  that contains all edges.
- (5) Do steps (1)–(4) for all possible pairings of the odd vertices, and choose the shortest walk.

We can do the same problem on a weighted graph, where each edge is assigned a weight in a real number. In this case, an Eulerian closed walk is a closed walk that contains all edges and minimizes the total weight.

### Graph Decomposition

Let  $a_1, \dots, a_n$  be distinct items. A **Steiner Triple System** on  $a_1, \dots, a_n$  is a set  $S$  of triples  $\{a, b, c\} \subseteq \{a_1, \dots, a_n\}$ , such that every pair  $\{a_i, a_j\}$ ,  $i \neq j$  is a subset of exactly one element of  $S$ .

#### Example

$n = 4$ ,  $a_1, \dots, a_4$  are distinct.

Let

$$S = \{\{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}\}$$

Is  $S$  a Steiner Triple System on  $a_1, \dots, a_4$ ?

$S$  is not a Steiner Triple System.  $\{a_2, a_4\}$  is not a subset of any element of  $S$ , and  $\{a_1, a_3\}$  appear in both elements.

Represent each  $a_i$  as a vertex, each pair  $a_i a_j$  as an edge, then a STS corresponds to  $K_n$  with  $E(K_n)$  partitioned into 3-cycles.

#### Partitioning into 3-cycles

$K_n$  can be decomposed into 3-cycles if and only if  $n \equiv 1 \pmod{6}$  or  $n \equiv 3 \pmod{6}$ .

### Decomposing into Trails

Let  $G$  be a connected graph with exactly four odd vertices. Show that  $G$  decomposes into two trails,  $T_1$  and  $T_2$ . Furthermore,  $T_1$  and  $T_2$  contain exactly two of the odd vertices.

Let  $a, b, c, d$  be the odd vertices. Let  $G' = G + ab + cd$ . In  $G'$ , every vertex has even degree, so  $G'$  has an Eulerian circuit  $C$ .

$$C = a \underbrace{b, \dots, c}_{T_1} d, \underbrace{\dots, a}_{T_2}$$

### Trail Decomposition, Even Length

Find  $T'_1$  and  $T'_2$  such that  $T'_1$  and  $T'_2$  are of even length, given the same conditions as the previous problem.

We know that  $G$  has even size, and

$$|T_1| + |T_2| = |G|$$

If  $T_1$  and  $T_2$  are of even length, then  $T'_1 = T_1$  and  $T'_2 = T_2$ , and we are done.

Suppose  $T_1$  and  $T_2$  are both of odd length. If  $T_1$  and  $T_2$  do not share any vertices,  $V(T_1) \sqcup V(T_2) = V(G)$ , meaning  $G$  is disconnected.

Let  $v \in V(T_1) \cap V(T_2)$ .  $T_1$  is from  $b$  to  $v$  to  $c$ ,  $T_2$  is from  $d$  to  $v$  to  $a$ . Let  $T_i = R_i \cup S_i$ .