

**Problem (Problem 1):**

- (a) Let  $G$  be a finite group. Show that for any subgroup  $H \leq G$ , we have  $n_p(H) \leq n_p(G)$ .
- (b) Let  $f: G \rightarrow G'$  be a surjective homomorphism of finite groups, and let  $p$  be a prime. Show that every  $p$ -Sylow subgroup  $P'$  of  $G'$  is the image of some  $p$ -Sylow subgroup  $P$  of  $G$ .

**Solution:**

- (a) Suppose  $|G| = p^r m$  and  $|H| = p^s \ell$ , with  $p \nmid m, \ell$ .

First, we observe that if  $s = r$ , then any  $p$ -Sylow subgroup of  $H$  is a  $p$ -Sylow subgroup of  $G$  that is contained in  $H$ , whence  $n_p(H) \leq n_p(G)$ .

Now, let  $s < r$ . We observe that if  $P \leq H \leq G$  is a  $p$ -Sylow subgroup of  $H$ , then by the second Sylow theorem,  $P$  is contained in some  $p$ -Sylow subgroup,  $P' \leq G$ . We claim that any two distinct  $p$ -Sylow subgroups of  $H$  must be contained in distinct  $p$ -Sylow subgroups of  $G$ . This follows from the fact that, if  $P_1, P_2 \leq H$  are two distinct  $p$ -Sylow subgroups, and  $P_1, P_2 \leq P'$ , then the subgroup  $\langle P_1, P_2 \rangle$  generated in  $H$  is contained in both  $H$  and  $P'$ , but has strictly larger order than either  $P_1$  or  $P_2$ , which contradicts the maximality of the orders of  $P_1$  and  $P_2$  respectively. Thus, any  $p$ -Sylow subgroup of  $H$  is of the form  $P' \cap H$  for some  $p$ -Sylow subgroup of  $G$ , whence  $n_p(H) \leq n_p(G)$ .

- (b) Let  $N = \ker(f)$ , and let  $P_0$  be a  $p$ -Sylow subgroup of  $N$ .

**Problem (Problem 2):** Let  $F = \mathbb{Z}/p\mathbb{Z}$  be the field with  $p$  elements. Show that the group of upper unitriangular matrices,

$$U = \left\{ (a_{ij})_{i,j} \in GL_n(F) \mid a_{ij} = 0 \text{ for } i > j \text{ and } a_{ii} = 1 \right\}$$

is a  $p$ -Sylow subgroup of  $G = GL_n(F)$ .

**Solution:** To start, we observe that the group of upper unitriangular matrices has an order of  $p^{n(n-1)/2}$ , as follows from the fact that all elements in the strict upper triangle of any given matrix can be pulled from  $\mathbb{Z}/p\mathbb{Z}$ .

The order of  $GL_n(F)$  can be seen to be

$$|GL_n(F)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}).$$

Taking out powers of  $p$  from each of the factors that divides  $p$ , we find that we get

$$\begin{aligned} |GL_n(F)| &= (p^n - 1)(p)(p^{n-1} - 1) \cdots (p^{n-1})(p - 1) \\ &= p \cdot p^2 \cdots p^{n-1} (p^n - 1)(p^{n-1} - 1) \cdots (p - 1) \\ &= p^{n(n-1)/2} (p^n - 1)(p^{n-1} - 1) \cdots (p - 1). \end{aligned}$$

Now, we observe that each of the factors of the form  $p^k - 1$  are coprime to  $p^k$ , meaning that  $p$  necessarily cannot divide  $p^k - 1$  for each  $k$ , so that any  $p$ -Sylow subgroup of  $GL_n(F)$  has the order  $p^{n(n-1)/2}$ .

**Problem (Problem 4):** Let  $G$  be a finite group of order  $p^n$  with  $n \geq 1$ . Show that for every  $m = 0, 1, \dots, n$ , the group  $G$  has a subgroup of order  $p^m$ .

**Solution:** We prove using induction. If  $n = 1$ , then  $G$  is a group of the form  $\mathbb{Z}/p\mathbb{Z}$ , meaning that  $G$  has a subgroup of order 1, which is  $\{e\}$ , and a group of order  $p$ , which is the group itself.

Now, let  $|G| = p^n$ , and suppose that for any group with order  $p^k$  with  $k < n$ , we have that said group contains subgroups of all prime orders from 0 to  $k$ . Letting  $G$  act on itself via conjugation, we obtain

from the class equation that

$$|G| = |Z(G)| + \sum_{a \in A} [G : Z_G(a)].$$

Taking residues modulo  $p$ , we observe that  $|Z(G)| \equiv |G|$  modulo  $p$ , and since  $|Z(G)| \geq 1$  as  $\{e\} \in Z(G)$ , it follows that  $|Z(G)| \geq p$ .

Now, if  $Z(G) = G$ , then  $G$  is abelian, so by the structure of finite abelian groups, we have

$$G \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k}\mathbb{Z},$$

where we may use powers of  $p$  as the sole factors by virtue of the fact that  $G$  is a  $p$  group. Taking residue classes modulo  $\mathbb{Z}/p\mathbb{Z} \times \{0\} \times \cdots \times \{0\}$ , we observe then that the quotient group  $G/(\mathbb{Z}/p\mathbb{Z})$  has subgroups of orders up to  $n - 1$  all of which contain  $\mathbb{Z}/p\mathbb{Z}$  by the fourth isomorphism theorem, whence  $G$  has subgroups of all orders up to  $n$ .

Meanwhile, if  $Z(G) \neq G$ , then  $Z(G)$  is a  $p$ -group of order less than  $p^n$ , and the quotient group  $G/Z(G)$  has order strictly less than  $p^n$  as well, meaning that the former contains  $p$ -subgroups up to the order of  $Z(G)$ , while the latter contains  $p$ -subgroups up to the order of  $G/Z(G)$ , each of which contains  $Z(G)$ , so that  $G$  contains  $p$ -subgroups of all orders up to  $p^n$ .

**Problem** (Problem 5): Show that a group of order 351 always has a normal  $p$ -Sylow subgroup for some prime  $p$  dividing the order.

**Solution:** The prime factorization of 351 yields  $3^3 \cdot 13$ . We observe that the number of 13-Sylow subgroups is congruent to 1 modulo 13 and divides 27; in particular, we have the cases of 1 and 27. If there is one 13-Sylow subgroup, this subgroup is normal, and we are done. Else, if there are 27 13-Sylow subgroups, these subgroups intersect at the identity (as each is isomorphic to  $\mathbb{Z}/13\mathbb{Z}$ ), and thus there are 324 elements of order 13, giving 27 elements with order not equal to 13. Since, by the first Sylow theorem, there is at least one 3-Sylow subgroup, this is the 3-Sylow subgroup, which is necessarily normal.

**Problem** (Problem 8): Let  $G$  be a group of order  $3 \cdot 5^2 \cdot 17$ .

- (a) Show that  $n_{17}(G) = 1$ . That is, a 17-Sylow subgroup  $H$  is normal.
- (b) The conjugation action of  $G$  on  $H$  defines a group homomorphism  $G \rightarrow \text{aut}(H)$ . Show that this homomorphism is trivial, and conclude that  $H \subseteq Z(G)$ .

**Solution:**

- (a) By the third Sylow theorem, we know that  $n_{17}(G) | 75$  and  $n_{17}(G) \equiv 1$  modulo 17. Writing out the possibilities for  $n_{17}$  under the second condition explicitly gives

$$n_{17}(G) = 1, 18, 35, 52, 69, 86, \dots$$

of which only 1 divides 75. Thus, there is only one 17-Sylow subgroup.