Problem (Problem 1)**:** In this exercise, we prove another fundamental result in differential topology, called the tubular neighborhood theorem. Let M be a compact smooth manifold with orientable boundary N. For simplicity, assume that N is connected. The tubular neighborhood theorem asserts that N admits a neighborhood in M which is diffeomorphic to $N \times [0, 1)$.

(a) Choose a Riemannian metric on M, and show that N admits a nonvanishing vector field that is everywhere orthogonal to the tangent space of N. That is, a vector field X such that for all $p \in N$, $g(X_p, T_pN) = 0$.

(b) Use the flow generated by X to find the desired neighborhood.

Solution:

(a) If $p \in N$, then we observe that $T_pN < T_pM$ is a proper subspace with codimension 1. Letting $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for T_pN, then we may extend to a basis for T_pM by taking a representative for a basis for T_pM/T_pN, and observing that such a vector necessarily has

$$g_p(e_n, e_k) = 0$$

for all $k = 1, \dots, n-1$. By smoothly varying over all points $p \in N$, we get our desired everywhere nonvanishing vector field normal to T_pN.

(b) Let φ_t be the one-parameter diffeomorphism group generated by X, where $\varphi_t \colon M \to M$ is such that $\varphi_0(p) = p$ for all $p \in N$. Then, $\varphi \colon (-\varepsilon, \varepsilon) \to \operatorname{diff}(M)$ restricted to $[0, \varepsilon)$ gives our desired neighborhood in M diffeomorphic to $N \times [0, 1)$.

Problem (Problem Set 7, Problem 5)**:** Suppose G is a finite group acting freely on a manifold M by diffeomorphisms.

(a) Show that M/G is a manifold.

(b) Show that the de Rham cohomology of M/G is isomorphic to the G-invariant cohomology of M.

Solution:

(a) Let $p \in M$, and let U be a neighborhood of p. By shrinking U if necessary, the fact that G acts freely on M implies that for all $1 \neq g$, we have $g \cdot U \cap U = \emptyset$. In particular, this gives that the projection map $q \colon M \to M/G$ is a covering map. Thus, to find a chart about $[p] \in M/G$, we consider the image $\overline{U} := q(U) \subseteq M/G$. If φ is the coordinate map for U, define $\overline{\varphi} \colon \overline{U} \to \mathbb{R}^n$ by taking $\overline{\varphi}(q(U)) = \varphi \circ q^{-1}\left(\overline{U}\right)$.

Let $[w] \in \overline{U} \cap \overline{V}$. We have some $w_1 \in U$ and $w_2 \in V$ such that $q(w_1) = q(w_2) = [w]$; in particular, there is $g \in G$ such that $g \cdot w_1 = w_2$. We may define $U' = U \cap g^{-1} \cdot V$ and $V' = V \cap (g \cdot U)$, where $\overline{U'} \cap \overline{V'} \subseteq \overline{U} \cap \overline{V}$ and $[w] \in \overline{U'} \cap \overline{V'}$. Furthermore, we see that $q(U' \cap V') = \overline{U'} \cap \overline{V'}$, as any element in the latter is given by $[x]$, where $x_1 \in U'$ and $x_2 \in V'$ have $q(x_i) = [x]$, meaning that the element $k \cdot x_1 = x_2$ is uniquely determined as the action is free.

We observe now that the transition map $\overline{\psi} \circ \overline{\varphi}^{-1} \colon \overline{\varphi}\left(\overline{U'} \cap \overline{V'}\right) \to \overline{\psi}\left(\overline{U'} \cap \overline{V'}\right)$ is then given, by the definition of these maps, by the transition map between $\varphi\left(U \cap \left(g^{-1} \cdot V\right)\right)$ to $\psi((g \cdot U) \cap V)$. Therefore, M/G is a manifold.

(b) Consider a closed form $\omega \in \mathcal{A}^*(M/G)$. The pullback $q^*\omega \in \mathcal{A}^*(M)$ is necessarily G-invariant. This descends to a map in cohomology

$$q^* \colon H^*_{DR}(M/G) \to H^*_{DR}(M)^G.$$

Our goal is to show that this map is injective and surjective. First, let $[\omega] \in \ker(q^*)$. Then, ω is exact, meaning that $q^*\omega$ is exact, so there is some η such that $q^*\omega = d\eta$. If we let

$$\xi = \frac{1}{|G|}\sum_{g\in|G|} g^*\eta,$$

then we see that ξ is G-invariant and has $d\xi = \eta$. By the commutativity of pullback and d, there is then some $\overline{\xi} \in H^*_{DR}(M/G)$ such that $q^*\overline{\xi} = \xi$, meaning that $d\xi$ and ω are in the same cohomology class. Thus, $\ker(q^*) = \{0\}$.

Now, to see surjectivity, let $[\omega] \in H^*_{DR}(M)$ be G-invariant. By using the same averaging process, we have a representative of the cohomology class that can be found by pullback of a closed form in $H^*_{DR}(M/G)$, so that $H^*_{DR}(M/G) \cong H^*_{DR}(M)^G$.

Problem (Problem Set 8, Problem 3)**:** Compute the de Rham cohomology of $\mathbb{RP}^n$.

Solution: We observe that the antipodal map, $x \mapsto -x$, is a finite free action on the manifold S^n, and is such that the orbit space is $\mathbb{RP}^n$, given by $\mathbb{Z}/2\mathbb{Z}$.

We know that the cohomology for S^n is given by $\mathbb{R}$ at H^0, and $\mathbb{R}$ at H^n, with 0 everywhere else. The antipodal map is of degree 1 if and only if n is odd, which means that the antipodal map is thus a sign-preserving local diffeomorphism. In particular, this means that for an n-form $\omega \in \mathcal{A}^n(\mathbb{RP}^n)$, we have

$$\int_{S^n} q^*\omega = \int_{\mathbb{RP}^n}\omega,$$

so that the simplicial cohomology is identical. Thus, if n is odd, then $\mathbb{RP}^n$ and S^n have the same cohomology.

Now, if n is even, we know that the degree of the antipodal map is -1. Yet, this means that there are no invariant closed n-forms, meaning that the top-dimensional cohomology of $\mathbb{RP}^n$ is 0.

Problem (Problem Set 8, Problem 5)**:** Use the Mayer–Vietoris sequence to prove the Künneth Formula: if M and N are smooth manifolds, then $H^*_{DR}(M\times N)$ is the tensor product of $H^*_{DR}(M)$ and $H^*_{DR}(N)$. Specifically, in each degree ℓ, we have

$$H^\ell_{DR}(M\times N) = \bigoplus_{i+j=\ell} H^i_{DR}(M)\otimes H^j_{DR}(N).$$

Solution: For the sake of being able to solve this problem, we focus on the case where M and N are closed smooth manifolds.

Let $V = M \times N$ be the product manifold for M and N. If $\pi_1 : V \to M$ and $\pi_2 : V \to N$ are the projection maps on M and N respectively, we get the composed maps

$$\mathcal{A}^k(M)\times\mathcal{A}^\ell(N) \to \mathcal{A}^{k+\ell}(V)$$

given by $(\omega, \eta) \mapsto \pi_1^*\omega \wedge \pi_2^*\eta$. If ω and η are closed forms, then we observe that

$$\begin{aligned} d\left(\pi_1^*\omega\wedge\pi_2^*\eta\right) &= d\pi_1^*\omega\wedge\pi_2^*\eta + (-1)^k\pi_1^*\omega\wedge d\pi_2^*\eta\\ &= \pi_1^*(d\omega)\wedge\pi_2^*\eta + (-1)^k\pi_1^*\omega\wedge\pi_2^*(d\eta)\\ &= 0. \end{aligned}$$

Furthermore, if we let $\omega' = \omega + d\tau$ and $\eta' = \eta + d\rho$, then we know from earlier work that $\pi_1^*\omega'\wedge\pi_2^*\eta'$ can be expressed as $\pi_1^*\omega\wedge\pi_2^*\eta + d\sigma$ for some form σ by using the fact that d and the pullback commute. Thus, it follows that the map descends to a map in cohomology, given by

$$H^k_{DR}(M)\times H^\ell_{DR}(N) \to H^{k+\ell}(M\times N)$$

$$([\omega],[\eta]) \mapsto \left[\pi_1^*\omega \wedge \pi_2^*\eta\right],$$

whence via the universal property of tensor products and direct sums, we get the map

$$\psi\colon H^*_{DR}(M)\otimes H^*_{DR}(N) \to H^*(M\times N).$$

Our goal now is to show that ψ is indeed an isomorphism.

Toward this end, suppose we have two open sets in the good cover for M, given by U_1 and U_2. From the Mayer–Vietoris sequence, this yields the following exact sequence in cohomology for a fixed k, where D_k denote the connecting homomorphisms from $H^k(U_1\cap V_2)$ to $H^{k+1}(M)$.

$$\cdots \xrightarrow{D_{k-1}} H^k_{DR}(M) \xrightarrow{i} H^k_{DR}(U_1)\oplus H^k_{DR}(U_2) \xrightarrow{j} H^k_{DR}(U_1\cap U_2) \xrightarrow{D_k} \cdots$$

Since the tensor product preserves exact sequences, we observe that by taking the tensor product with $H^\ell_{DR}(N)$, giving the following.

$$\cdots \xrightarrow{D_{k-1}} H^k_{DR}(M)\otimes H^\ell_{DR}(N) \xrightarrow{i} H^k_{DR}(U_1)\otimes H^\ell_{DR}(N)\oplus H^k_{DR}(U_2)\otimes H^\ell_{DR}(N) \xrightarrow{j} H^k(U_1\cap U_2)\otimes H^\ell(N) \xrightarrow{D_k} \cdots$$

Taking direct sums with the same dimension, we obtain the following diagram.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\bigoplus_{k+\ell=i-1} H^k(U_1\cap U_2)\otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "D_{i-1}"'] & H^{i-1}((U_1\cap U_2)\times N) \arrow[d, "D_{i-1}"] \\
\bigoplus_{k+\ell=i} H^k(M)\otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "i"'] & H^i(M\times N) \arrow[d, "i"] \\
\bigoplus_{k+\ell=i}(H^k(U_1)\otimes H^\ell(N)\oplus H^k(U_2)\otimes H^\ell(N)) \arrow[r, "\psi"] \arrow[d, "j"'] & H^i(U_1\times N)\oplus H^i(U_2\times N) \arrow[d, "j"] \\
\bigoplus_{k+\ell=i} H^k(U_1\cap U_2)\otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "D_i"'] & H^i((U_1\cap U_2)\times N) \arrow[d, "D_i"] \\
\vdots & \vdots
\end{tikzcd}$$

Since U_1, U_2, and $U_1\cap U_2$ are contractible, under the good cover assumption, it follows from the Poincaré Lemma that the following subsection of the diagram is commutative, with ψ necessarily an isomorphism in each of the columns.

$$\begin{tikzcd}
\bigoplus_{k+\ell=i}(H^k(U_1)\otimes H^\ell(N)\oplus H^k(U_2)\otimes H^\ell(N)) \arrow[r, "\psi"] \arrow[d, "j"'] & H^i(U_1\times N)\oplus H^i(U_2\times N) \arrow[d, "j"] \\
\bigoplus_{k+\ell=i} H^k(U_1\cap U_2)\otimes H^\ell(N) \arrow[r, "\psi"] & H^i((U_1\cap U_2)\times N)
\end{tikzcd}$$

Similarly, we have that the following diagram is commutative, following from the Mayer–Vietoris sequence.

$$\begin{tikzcd}
\bigoplus_{k+\ell=i} H^k(M)\otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "i"'] & H^i(M\times N) \arrow[d, "j"] \\
\bigoplus_{k+\ell=i}(H^k(U_1)\otimes H^\ell(N)\oplus H^k(U_2)\otimes H^\ell(N)) \arrow[r, "\psi"] & H^i(U_1\times N)\oplus H^i(U_2\times N)
\end{tikzcd}$$

Therefore, we only need to verify commutativity for the following square.

$$\begin{tikzcd}
\bigoplus_{k+\ell=i-1} H^k(U_1 \cap U_2) \otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "D_{i-1}"'] & H^{i-1}((U_1 \cap U_2) \times N) \arrow[d, "D_{i-1}"] \\
\bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) \arrow[r, "\psi"] & H^i(M \times N)
\end{tikzcd}$$

First, from the Mayer–Vietoris sequence and the fact that the coboundary map in de Rham cohomology emerges from the exterior derivative, we have that the map D_i is given by

$$D_{i-1}([\omega]) = \begin{cases} [d(-f_U\omega)] \\ [d(f_V\omega)] \end{cases}$$

for any cohomology class representative ω. Now, we observe that

$$\begin{aligned} \psi(D_{i-1}([\omega],[\eta])) &= \left[\pi_1^*(D_{i-1}(\omega)) \wedge \pi_2^*\eta\right] \\ D_{i-1}(\psi([\omega],[\eta])) &= \left[D_{i-1}\left(\pi_1^*\omega \wedge \pi_2^*\eta\right)\right] \end{aligned}$$

In particular, since $\pi_1^* f_U$ and $\pi_1^* f_V$ form a partition of unity for $M \times F$, we have

$$\begin{aligned} \pi_1^*(D_{i-1}(\omega)) \wedge \pi_2^*\eta &= \pi_1^*(d(f_V\omega)) \wedge \pi_2^*\eta \\ &= d\left(\pi_1^*(f_V\omega)\right) \wedge \pi_2^*\eta \\ D_{i-1}\left(\pi_1^*\omega \wedge \pi_2^*\eta\right) &= d\left(\pi_1^* f_V \pi_1^*\omega \wedge \pi_2^*\eta\right) \\ &= d(\pi^*(f_V\omega)) \wedge \pi_2^*\eta. \end{aligned}$$

Since ψ at each of U, V, and $U \cap V$ is an isomorphism, and the diagram commutes, the Five Lemma gives that ψ at M is an isomorphism.

For any finite good cover with more than 2 elements, induction gives the desired result.