Problem (Problem 1): Let F be a field, $a(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in F[x]$ a nonconstant monic polynomial, and let $A = C_{a(x)}$ be its companion matrix. Prove by direct computation that $\mathrm{SNF}(xI - A) = \mathrm{diag}(1, \ldots, 1, a(x))$.

Solution: We observe that

$$
xI - A = \begin{pmatrix}
x & 0 & \cdots & 0 & a_0 \\
-1 & x & \cdots & 0 & a_1 \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & x + a_{n-1}
\end{pmatrix}.
$$

Focusing on the bottom 2 rows, we use the following reduction method

$$
\begin{pmatrix} x & a_{n-2} \\ -1 & x + a_{n-1} \end{pmatrix} \xrightarrow{R_{n-1} \leftarrow x R_n + R_{n-1}} \begin{pmatrix} 0 & x^2 + a_{n-1}x + a_{n-2} \\ -1 & x + a_{n-1} \end{pmatrix}
$$

$$
\xrightarrow{C_n \leftarrow (x + a_{n-1})C_{n-1} + C_n} \begin{pmatrix} 0 & x^2 + a_{n-1}x + a_{n-2} \\ -1 & 0 \end{pmatrix}.
$$

Inductively repeating this reduction method, we say at step i that we perform the following two operations consecutively

- $R_{n-i} \leftarrow x R_{n-i+1} + R_{n-i}$;
- $C_{n-i+1} \leftarrow \left(x^i + a_{n-1}x^{i-1} + \cdots + a_{n-i}\right)C_{n-i} + C_{n-i+1}$

Upon completion of this process at step n, we obtain a matrix consisting entirely of -1 along the subdiagonal and $a(x)$ in position $(1, n)$. Next, we perform the following procedure as i ranges from 1 to $n - 1$.

- $R_i \leftarrow (-1)R_{i+1} + R_i$;
- $R_{i+1} \leftarrow R_i + R_{i+1}$.

This gives a matrix with 1 along the diagonal and $a(x)$ along column n. Then, upon performing the operation

- $R_i \leftarrow (-1)R_n + R_i$

for each $1 \leq i \leq n - 1$, we obtain our desired diagonal matrix in Smith normal form, where we have $\mathrm{diag}(1, \ldots, 1, a(x))$.

Problem (Problem 2): Prove that the constant term in the characteristic polynomial of the $n \times n$ matrix A is $(-1)^n \det(A)$, and that the coefficient of x^{n-1} is the negative of the sum of the diagonal entries of A. Prove that $\det(A)$ is the product of the eigenvalues of A and that the trace of A is the sum of the eigenvalues of A.

Solution: We start by showing that this holds for a companion matrix, $A = C_{a(x)}$. Note that in our computation showing that $\mathrm{SNF}(xI - A) = \mathrm{diag}(1, 1, \ldots, a(x))$, we exclusively used row and column operations (and employed no flips); as a result, it follows that the characteristic polynomial of a companion matrix for $a(x)$ is exactly $a(x)$. Then, we observe that

$$
\begin{aligned}
a_0 &= \chi_A(0) \\
&= \det(-A) \\
&= \det((-I)A) \\
&= \det(-I)\det(A) \\
&= (-1)^n \det(A),
\end{aligned}
$$

and that the coefficient on the x^{n-1} is equal to a_{n-1}, or $-(-a_{n-1})$, which is the trace of the companion matrix.

In the general case, we observe that A is similar to a matrix in rational canonical form,

$$A \sim \operatorname{diag}(A_1, \ldots, A_r),$$

and has

$$\chi_A(x) = \chi_{A_1}(x) \cdots \chi_{A_r}(x),$$

where we use the fact that characteristic polynomials are invariant under similarity transformation, so that

$$
\begin{aligned}
\chi_A(0) &= \chi_{A_1}(0) \cdots \chi_{A_r}(0) \\
&= a_{0,1} \cdots a_{0,r} \\
&= (-1)^{n_1} \det(A_1) \cdots (-1)^{n_r} \det(A_r) \\
&= (-1)^n \det(A_1) \cdots \det(A_r) \\
&= (-1)^n \det(A),
\end{aligned}
$$

where we let n_i denote the dimension of the specific companion matrix A_i. Additionally, we observe that the coefficient on the $n-1$ degree term on $\chi_A(x)$ is given summing the coefficient of an $n_i - 1$ degree term with the n_j degree terms for all $j \neq i$. In particular, this means that we get

$$
\begin{aligned}
a_{n-1} &= \sum_{i=1}^{r} a_{n_i - 1} \\
&= \sum_{i=1}^{r} - \operatorname{Tr}(A_i) \\
&= - \operatorname{Tr}(A).
\end{aligned}
$$

From basic properties of polynomials, we know that the constant term of a polynomial of degree n is equal to $(-1)^n$ multiplied by the product of the roots, while the coefficient on the degree $n-1$ term is equal to -1 multiplied by the sum of the roots. In particular, applying this to the characteristic polynomial, we get that the trace is the sum of the eigenvalues of A and the determinant is the product of the eigenvalues.

Problem (Problem 3): Determine the number of possible RCFs of 8×8 matrices over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$.

Solution: Factoring over $\mathbb{Q}$, we have that

$$\chi_A(x) = x^4 (x^2 + 1)(x - 1)(x + 1).$$

In order to determine the possible rational canonical forms, we need to determine the possible invariant factors, $a_1(x) | a_2(x) | \cdots | a_d(x)$, subject to the constraint that $a_d(x) = \mu_A(x)$ has the same roots as $\chi_A(x)$. In particular, we must have that $\mu_A(x)$ can only be one of the following, where we observe that we cannot have $x^2 + 1$ anywhere in the invariant factor decomposition outside of the minimal polynomial since it has multiplicity 1:

- $p_1(x) = x(x^2 + 1)(x - 1)(x + 1)$;

- $p_1(x) = x^2(x^2 + 1)(x - 1)(x + 1)$;

- $p_2(x) = x^3(x^2 + 1)(x - 1)(x + 1)$;

- $p_4(x) = x^4(x^2 + 1)(x - 1)(x + 1)$.

We find that the possible decompositions are thus

$$
\begin{aligned}
A_1 &= [x, x, x, p_1(x)] \\
A_2 &= [1, x, x^2, p_1(x)] \\
A_3 &= [1, 1, x^3, p_1(x)]
\end{aligned}
$$

$$B_1 = [x, x, p_2(x)]$$
$$B_2 = [1, x^2, p_2(x)]$$
$$C = [1, x, p_3(x)]$$
$$D = [p_4(x)].$$

Problem (Problem 4): Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing this does not hold for 4×4 matrices.