

Revised Problems

Problem (Problem 1): Prove that our cell complex structure for T^2 coincides with a product cell complex structure on $S^1 \times S^1$.

Solution: Consider the cell complex structure on S^1 given by one 1-cell, e^1 , and one 0-cell, e^0 , where the endpoints of e^1 are identified via the constant map to e^0 ; the corresponding characteristic map is $\Phi: [0, 1] \rightarrow S^1$, identifying $0 \sim 1$.

Considering two copies of S^1 in this fashion, the product CW complex structure is then one consisting of

- one 0-cell, $e_1^0 \times e_2^0$
- two 1-cells, $e_1^1 \times e_2^0$ and $e_1^0 \times e_2^1$;
- one 2-cell, $e_1^1 \times e_2^1$.

There are then characteristic maps

$$\Phi_1: e_1^1 \times e_2^0 \rightarrow S^1 \times S^1$$

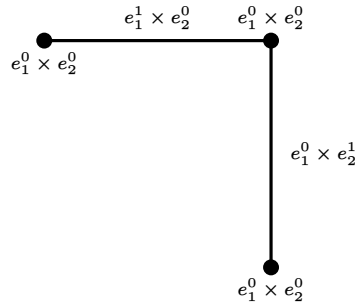
$$\Phi_2: e_1^0 \times e_2^1 \rightarrow S^1 \times S^1,$$

with

$$\Phi_1|_{\partial(e_1^1 \times e_2^0)}(e_1^1 \times e_2^0) = e_1^0 \times e_2^0$$

$$\Phi_2|_{\partial(e_1^0 \times e_2^1)}(e_1^0 \times e_2^1) = e_1^0 \times e_2^0.$$

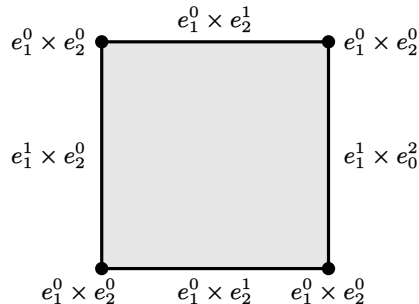
In particular, this means we may view the 1-skeleton as a wedge of two circles; since this is difficult to draw in TikZ, we instead simply label all the vertices and edges of the figure below assuming the necessary identifications.



The 2-skeleton is given by $e_1^1 \times e_2^1$, with a characteristic map Ψ given by the product of the characteristic maps of each of e_1^1 and e_2^1 (see Hatcher Theorem A.6). Observe that the attaching map is then given by the restriction of Ψ to the boundary, which is

$$\partial(e_1^1 \times e_2^1) = (e_1^1 \times e_2^0) \sqcup (e_1^0 \times e_2^1).$$

Since the 1-skeleton is precisely $(e_1^0 \times e_2^1) \sqcup (e_1^1 \times e_2^0)$, it follows that the attaching map for the 2-skeleton identifies the boundary of $e_1^1 \times e_2^1$ with the 1-skeleton, giving the following figure:



which is the cell complex structure of the torus.

Problem (Problem 2): Prove that if X is a cell complex, then so is the suspension SX .

Solution: We observe that the product $X \times [0, 1]$ is a cell complex, as it is a Cartesian product of a cell complex with one 1-cell (the interval itself) and two 0-cells (the endpoints of the interval). Since the characteristic maps on $X \times [0, 1]$ are the products of the characteristic maps on X and $[0, 1]$ (see Hatcher Theorem A.6), we see that the attaching maps for $X \times \{0\}$ and $X \times \{1\}$ are the products of the attaching maps for X and the constant maps representing $\{0\}$ and $\{1\}$.

In particular, this means that $X \times \{0\}$ and $X \times \{1\}$ are subcomplexes of $X \times [0, 1]$ (as they contain all their attaching maps). Since the quotient of a cell complex by a subcomplex is a cell complex, it follows that $SX = X \times [0, 1] / X \times \{0\} \cup X \times \{1\}$ is a cell complex.

Problem (Problem 3 (b)): Prove that S^∞ is contractible.

Solution: We view S^∞ as the subspace of \mathbb{R}^∞ , which is the space of finitely supported sequences. Specifically, the space S^∞ is the set of the finitely supported sequences (x_n) such that, if k is the index of the largest nonzero element, then (x_0, \dots, x_k) is an element of S^k — i.e.,

$$\sum_{i=0}^k x_i^2 = 1.$$

In general, we define the norm of the sequence $\|(x_n)\|$ to be the finite sum

$$\|(x_n)\| = \sum_{i=0}^{\infty} x_i^2.$$

Consider now the map $H: S^\infty \times [0, 1] \rightarrow S^\infty$

$$H((x_n), t) = \begin{cases} (1 - 2t)(x_n) + 2t(x_{n+1}) & 0 \leq t \leq 1/2 \\ (2 - 2t)(x_{n+1}) + (2t - 1)(1, 0, \dots) & 1/2 \leq t \leq 1. \end{cases}$$

where we insert 0 into index 0 in the first half of the homotopy. Then, H is continuous along each of $S^\infty \times [0, 1/2]$ and $S^\infty \times [1/2, 1]$ as it is continuous in each variable, and since the piecewise definitions are equal at $t = 1/2$, it follows that H is continuous along $[0, 1]$. Then, we see that $H(\cdot, t)/\|H(\cdot, t)\|$ is contained within S^∞ , and is a homotopy between the identity and a constant map, so the identity is null-homotopic, meaning S^∞ is contractible.