**Abstract**

We discuss the much celebrated Regular Value Theorem and Sard's Theorem, and discuss some of the consequences and applications of these results.

A smooth map between manifolds $f\colon M \to N$ includes a certain family of local information; for instance, the derivative $D_p f\colon T_p M \to T_{f(p)}N$, which is a linear map between tangent spaces at $p$ and $q$, is defined on a coordinate chart $U \subseteq M$ for $p$ and a corresponding coordinate chart $V \subseteq N$ for $f(p)$. Yet, the properties of this linear map can give us information about the underlying map $f$.

To understand this, we need to dive into the world of regular and critical values.

Much of this document is based on the book *Topology from the Differentiable Viewpoint* and assorted notes from my Differential Topology class.

# Sard's Theorem

**Definition:** Let $f\colon M \to N$ be a smooth map, and let $p \in M$. We say $p$ is a *critical point* for $f$ if $D_p f$ does not have the same rank as the dimension of $T_{f(p)}N$. If $D_p f$ has the same rank as the dimension of $T_{f(p)}N$, then we say that $p$ is a *regular point* of $f$.

We say $q \in N$ is a *critical value* for $f$ if $f^{-1}(\{q\})$ contains a critical point for $f$. Else,we say that $q$ is a *regular value*.

We start with the case of Sard's Theorem on $\mathbb{R}^n$. Then, we will expand this to the case of any arbitrary manifold by means of a technical lemma.

**Theorem** (Sard's Theorem)**:** Let $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be a smooth map. Then, if $C$ is the set of critical points for $f$, we have $f(C) \subseteq \mathbb{R}^m$ has measure zero.

*Proof.* We use induction on $n$. The statement only makes sense for $n \geq 0$ and $p \geq 1$. Clearly, the theorem is true for $n = 0$.

Let $C_1 \subseteq C$ be the set of all $x \in U$ such that $D_x f$ is zero, and similarly, let $C_i$ be the set of all $x$ such that $(D_x)^j f$ is zero for all $j \leq i$. We obtain a descending sequence of closed sets $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$.

We start by showing that $f(C \setminus C_1)$ has measure zero. For each $x \in C\setminus C_1$, we find an open neighborhood $V \subseteq \mathbb{R}^n$ such that $f(V \cap C)$ has measure zero. Since $\mathbb{R}^n$ is second countable, $C \setminus C_1$ is covered by countably many such open neighborhoods, it follows that $f(C \setminus C_1)$ has measure zero.

Since $x \notin C_1$, there is some partial derivative, which we use change of coordinates to write as $\frac{\partial f}{\partial x_1}$, that is not zero at $x$. Let

$$h(x) = (f_1(x), x_2, \dots, x_n).$$

Then, since $D_x h$ is nonsingular, by the inverse function theorem, $h$ maps some neighborhood $V$ of $x$ diffeomorphically onto an open set $V' \subseteq \mathbb{R}^n$. The composition $f \circ h^{-1}$ then maps $V'$ to $\mathbb{R}^m$ then maps $V'$ to $\mathbb{R}^m$.

Observe that the set of critical points of $g$ is precisely $h(V \cap C)$, so the set $g(C')$ is equal to $f(V \cap C)$.

For each hyperplane $(t, x_2, \dots, x_n) \in V'$, we observe that $g(t, x_2, \dots, x_n)$ is contained in $t \times \mathbb{R}^{m-1} \subseteq \mathbb{R}^m$, meaning that $g$ maps hyperplanes to hyperplanes. Let

$$g^t\colon \left(t \times \mathbb{R}^{n-1}\right) \cap V' \to t \times \mathbb{R}^{m-1}$$

be the restriction of $g$. A point in $t \times \mathbb{R}^{n-1}$ is a critical value for $g^t$ if and only if it is critical for $g$, since the matrix of first derivatives for $g$ is of the form

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} 1 & 0 \\ * & \left(\frac{\partial g_i^t}{\partial x_j}\right) \end{pmatrix}.$$

From the induction hypothesis, it follows that the critical values of $g^t$ has measure zero in $t \times \mathbb{R}^{m-1}$. In particular, the critical values of $g$ intersects each hyperplane in $t \times \mathbb{R}^{m-1}$ in a set of measure zero, meaning that by Fubini's theorem, $g(C') = f(V \cap C)$ has measure zero.

Now, for each $x_0 \in C_k \setminus C_{k+1}$, there is some $(k+1)$-th derivative that is not zero, which we write

$$\frac{\partial^{k+1} f_r}{\partial x_{s_1} \cdots \partial x_{s_{k+1}}}.$$

Then, writing

$$w(x) = \frac{\partial^k f_r}{\partial x s_2 \cdots \partial x_{s_{k+1}}},$$

we observe that $w(x)$ vanishes at $x_0$, but $\frac{\partial w}{\partial x_{s_1}}$ does not. For definiteness, we let $s_1 = 1$. The map $h \colon U \to \mathbb{R}^n$, defined by

$$h(x) = (w(x), x_2, \dots, x_n)$$

then carries a neighborhood $V$ of $x_0$ diffeomorphically onto an open set $V'$. Then, $h$ carries $C_k \cap V$ into the hyperplane $\{0\} \times \mathbb{R}^{n-1}$. Again, we consider the map $g = f \circ h^{-1}$, and define

$$g_0 \colon \left(\{0\} \times \mathbb{R}^{n-1}\right) \cap V' \to \mathbb{R}^m$$

to be the restriction of $g$. Inductively, the critical values of $g_0$ has measure zero in $\mathbb{R}^m$. Yet, each point in $h(C_k \cap V)$ is a critical point of $g_0$ as all derivatives of order $\leq k$ vanish, meaning that

$$g_0 \circ h(C_k \cap V) = f(C_k \cap V)$$

has measure zero. □