

These are assorted exercises and solutions from Conway's *A Course in Functional Analysis*.

## Spectral Theory

**Problem** ([Con90, Exercise IX.1.2]): Show that the unit ball of  $B(H)$  is WOT-compact.

**Solution:** Consider the set

$$K = \prod_{x,y \in B_H} \overline{\mathbb{D}},$$

where  $\mathbb{D}$  represents the complex unit disk and  $B_H$  denotes the closed unit ball of  $H$ . The space  $K$  is compact by Tychonoff's theorem. Let  $\phi: B_{B(H)} \rightarrow K$  be defined by

$$\phi(T) = (\langle Tx, y \rangle)_{x,y}.$$

Observe that by Cauchy-Schwarz, we have that

$$\begin{aligned} |\langle Tx, y \rangle| &\leq \|T\|_{\text{op}} \|x\| \|y\| \\ &\leq 1, \end{aligned}$$

so  $\phi$  is indeed well-defined. Furthermore,  $\phi$  is injective since for any two operators  $T$  and  $S$ , we have that  $T = S$  if and only if  $\langle Tx, y \rangle = \langle Sx, y \rangle$  for all  $x, y \in B_H$ , and  $\phi$  is continuous by the definition of the weak operator topology. Therefore, we only need to show that  $\phi$  has a closed range.

Let  $(T_i)_i$  be a net of operators in  $B_{B(H)}$  such that

$$\lim_{i \in I} (\langle T_i x, y \rangle)_{x,y} = (z_{x,y})_{x,y}.$$

Then, from the Cauchy-Schwarz inequality, it follows that  $(z_{x,y})_{x,y} \in K$ , and by the definition of convergence in the product topology, we have, for each  $x, y$ ,

$$\lim_{i \in I} \langle T_i x, y \rangle = z_{x,y}.$$

Therefore, we may define a semidefinite sesquilinear form  $F: H \times H \rightarrow \mathbb{C}$  given by

$$F(x, y) = \lim_{i \in I} \langle T_i x, y \rangle$$

for each  $x, y \in H$ . From the structure of sesquilinear forms, it then follows that there is some  $T \in B_{B(H)}$  such that  $F(x, y) = \langle Tx, y \rangle$ , and thus  $(T_i)_{i \in I} \rightarrow T$  in WOT.

**Problem** ([Con90, Exercise IX.1.13]): A representation  $\rho: A \rightarrow B(H)$  is *irreducible* if the only projections in  $B(H)$  that commute with every  $\rho(a)$  for  $a \in A$  are 0 and 1. Prove that if  $A$  is abelian and  $\rho$  is an irreducible representation, then  $\dim(H) = 1$ . Find the corresponding spectral measure.

**Solution:** Since  $A$  is abelian, so too is  $\rho(A)$ , meaning that  $\rho(A) \subseteq \rho(A)'$ . Since  $\rho(A)' = \mathbb{C}1$  by the assumption of irreducibility, it follows then that  $\rho(A) = \mathbb{C}1$ , whence  $H = [\rho(A)v] = \mathbb{C}v$ .

Without loss of generality, we may assume that  $A = C_0(X)$  for some locally compact Hausdorff space  $X$ , and  $\rho: C_0(X) \rightarrow \mathbb{C}$  is a character. The characters of  $C_0(X)$  are given by evaluation at  $x_0 \in X$ , meaning that their corresponding spectral measure is the Dirac mass  $\delta_{x_0}$ .

**Problem** ([Con90, Exercise IX.2.1]): Show that  $\lambda \in \sigma_p(N)$  if and only if  $E(\{\lambda\}) \neq 0$ . Moreover, if  $\lambda \in \sigma_p(N)$ , then  $E(\{\lambda\})$  is the orthogonal projection onto  $\ker(N - \lambda I)$ .

**Solution:** Suppose  $E(\{\lambda\}) \neq 0$ , meaning that

$$E(\{\lambda\}) = \int_{\sigma(N)} \mathbb{1}_{\{\lambda\}} dE$$

$$\neq 0.$$

Since, for all  $x \in H$ , we have

$$\begin{aligned} (N - \lambda I)E(\{\lambda\})x &= \left( \int_{\sigma(N)} (z - \lambda) \mathbb{1}_{\{\lambda\}} dE \right) x \\ &= E(\{\lambda\})(N - \lambda I)x \\ &= 0, \end{aligned}$$

it follows that  $E(\{\lambda\})x \in \ker(N - \lambda I)$ , so that  $E(\{\lambda\}) \leq P_\lambda$ , where  $P_\lambda$  is the projection onto  $\ker(N - \lambda I)$ . Since  $E(\{\lambda\}) > 0$ , it follows that  $P_\lambda > 0$ , so  $P_\lambda$  is nontrivial, meaning  $\ker(N - \lambda I)$  is nontrivial, so  $\lambda \in \sigma_p(N)$ .

Now, let  $\lambda \in \sigma_p(N)$ . We start by supposing that  $\lambda \neq 0$ . If  $x \in \ker(T - \lambda I)$  is nonzero, then we have

$$\begin{aligned} Tx &= \lambda x \\ &= \left( \int_{\sigma(N)} \lambda dE \right) x \\ &= \left( \int_{\sigma(N)} \mathbb{1}_{\{\lambda\}} z dE \right) x \\ &= E(\{\lambda\})Tx \\ &= \lambda E(\{\lambda\})x, \end{aligned}$$

so  $E(\{\lambda\})x = x$ , meaning that  $E(\{\lambda\}) \geq P_\lambda$ . In particular, from what we have established above, this means that  $E(\{\lambda\}) = P_\lambda$ , so  $E(\{\lambda\}) \neq 0$ . If  $\lambda = 0$ , then we shift  $N$  by subtracting a factor of  $I$ , perform the same process, and shift back.

**Problem** ([Con90, Exercise IX.2.10]): Let  $A$  be a hermitian operator with spectral measure  $E$  on a separable space. For each real number  $t$ , define a projection  $P(t) = E(-\infty, t)$ . Show:

- (a)  $P(s) \leq P(t)$  for  $s \leq t$ ;
- (b) if  $t_n \leq t_{n+1}$  and  $(t_n)_n \rightarrow t$ , then  $P(t_n) \rightarrow P(t)$  in SOT;
- (c) for all but a countable number of points  $t$ ,  $P(t_n) \rightarrow P(t)$  in SOT if  $(t_n)_n \rightarrow t$ ;
- (d) for any  $f \in C(\sigma(A))$ , we have

$$f(A) = \int_{-\infty}^{\infty} f(t) dP(t),$$

where the integral is defined in the Riemann–Stieltjes sense.

**Solution:** Let  $x \in H$ . Then, we observe that

$$\begin{aligned} \langle P(s)x, x \rangle &= \int_{\sigma(A)} \mathbb{1}_{(-\infty, s)} dE_{x,x} \\ &= \int_{\sigma(A) \cap (-\infty, s)} dE_{x,x} \\ &\leq \int_{\sigma(A) \cap (-\infty, t)} dE_{x,x} \\ &= \langle P(t)x, x \rangle. \end{aligned}$$

Since  $x$  is arbitrary, and the measure  $E_{x,x}$  is real by the assumption that  $A$  is hermitian, it follows that  $P(s) \leq P(t)$  whenever  $s \leq t$ . This shows (a).

To show (b), we recall that a sequence of projections  $(P_n)_n \rightarrow P$  in SOT if and only if it converges in WOT,

as

$$\begin{aligned}\langle (P_n - P)x, x \rangle &= \langle (P_n - P)^2 x, x \rangle \\ &= \langle (P_n - P)x, (P_n - P)x \rangle \\ &= \|(P_n - P)x\|^2.\end{aligned}$$

We thus see that if  $(t_n)_n \nearrow t$ , we have for any  $x \in H$ ,

$$\begin{aligned}|\langle (P(t_n) - P(t))x, x \rangle| &= \left| \int_{\sigma(A)} \mathbb{1}_{(-\infty, t_n)} - \mathbb{1}_{(-\infty, t)} dE_{x,x} \right| \\ &= |E_{x,x}(\sigma(A) \cap (-\infty, t_n)) - E_{x,x}(\sigma(A) \cap (-\infty, t))|,\end{aligned}$$

which by continuity from below for measures converges to zero. Thus,  $(P(t_n))_n \rightarrow P(t)$  in WOT, so it converges in SOT.

To establish (c), we fix  $x \in H$ . Then, the function

$$f(t) = \langle P(t)x, x \rangle$$

is a monotone increasing right-continuous function on  $\mathbb{R}$ . In particular, this means that  $f(t)$  has at most a countable number of discontinuities, meaning that for any sequence  $(t_n)_n \rightarrow t$ , it follows that  $P(t_n) \rightarrow P(t)$  in WOT, hence in SOT, everywhere outside these countable number of discontinuities.

**Problem** ([Con90, Exercise IX.2.11], with some modifications): Suppose  $H$  is a separable Hilbert space; show that  $\sigma_P(N)$  is such that  $E(\sigma(N) \setminus \sigma_p(N)) = 0$  if and only if  $N$  is diagonalizable. That is, there is a basis of  $H$  consisting of eigenvectors for  $N$ .

**Solution:** Consider the internal direct sum space

$$X = \bigoplus_{\lambda \in \sigma_p(N)} \ker(N - \lambda I).$$

Since  $H$  is separable, it follows that  $\sigma_p(N)$  is countable, so this is a countable direct sum. Consider the spectral projection given by

$$P = \sum_{\lambda \in \sigma_p(N)} E(\{\lambda\}).$$

Observe then that

$$\begin{aligned}P &= \int_{\sigma_p(N)} dE \\ &= \int_{\sigma_p(N)} dE + 0 \\ &= \int_{\sigma_p(N)} dE + \int_{\sigma(N) \setminus \sigma_p(N)} dE \\ &= \int_{\sigma(N)} dE,\end{aligned}$$

so  $P$  is the identity.

**Problem** ([Con90, Exercise IX.2.14]): Prove that if  $A$  is hermitian,  $\exp(iA)$  is unitary. Is the converse true?

**Solution:** Since  $\sigma(A) \subseteq \mathbb{R}$ , we have

$$\exp(iA) = \int_{\sigma(A)} e^{ix} dE$$

$$\begin{aligned}\exp(iA)^* &= \int_{\sigma(A)} e^{-ix} dE \\ \exp(iA) \exp(iA)^* &= \int_{\sigma(A)} dE \\ &= \exp(iA)^* \exp(iA),\end{aligned}$$

so that  $\exp(iA)$  is unitary.

Similarly, there is a continuous bijection  $f: [0, 2\pi) \rightarrow S^1$  given by  $t \mapsto e^{it}$ , with Borel-measurable inverse  $g$ , so that if  $V$  is any unitary operator, we may define

$$A = g(V),$$

which has  $\exp(iA) = V$ . Since  $g$  is real-valued, it follows that  $A$  is hermitian.

**Problem** ([Con90, Exercise IX.2.22]): Prove that if  $U$  is any unitary operator on  $H$ , then there is a continuous function  $u: [0, 1] \rightarrow B(H)$  such that  $u(0) = U$ ,  $u(1) = I$ , and  $u(t)$  is unitary for each  $t$ .

**Solution:** Since  $U$  is unitary, there is a hermitian operator  $A$  such that  $U = \exp(iA)$ . We may then define the continuous map

$$\begin{aligned}u: [0, 1] &\rightarrow B(H) \\ t &\mapsto \exp(i(1-t)A).\end{aligned}$$

Since  $(1-t)A$  is also a hermitian operator, and dominated convergence gives that this is a continuous map, this is thus our desired map.

**Problem** ([Con90, Exercise IX.2.23]): If  $N$  is normal, show that there is a sequence of invertible normal operators that converges to  $N$ .

**Solution:** We observe that the sequence of functions

$$f_n(z) = z \mathbb{1}_{\sigma(N) \setminus \{0\}} + \frac{1}{n} \mathbb{1}_{\{0\}}$$

converges pointwise to  $z$ , is nonzero everywhere on  $\sigma(N)$ , and is bounded above by the necessarily integrable function

$$h(z) = \left( \sup_{z \in \sigma(N)} |z| + 1 \right) \mathbb{1}_{\sigma(N)}.$$

So, the operator

$$f_n(N) = \int_{\sigma(N)} f_n(z) dE$$

is invertible for each  $N$ , with  $f_n(N) \rightarrow N$  by dominated convergence.

## References

[Con90] John B. Conway. *A Course in Functional Analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.