Complex Analysis

Analyticity and Path-Independence in the Complex Plane

Baby's First Complex Function Theory

We are interested in functions of the form $f(z)$, where $z = x + iy$ is some complex number. Note that this is specifically different from a function $g\colon \mathbb{R}^2 \to \Omega$ for some domain Ω; in the latter case, we have independent variables x and y, while in the former case, we must express $z = x + iy$.

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z)\,dz &= \oint_C w(z)\,(dx + i\,dy)\\ &= \oint_C w(z)\,dx + i\oint_C w(z)\,dy.\end{aligned}$$

Taking $A_x = w(z)$ and $A_y = iw(z)$, we have

$$= \oint_C \mathbf{A}\cdot\,d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla\times\mathbf{A})\,d\mathbf{a},$$

and when this integral is zero. Note that $(\nabla\times\mathbf{A})\cdot\hat{n} = 0$, so $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$.

Note that we can take

$$w(z) = u(x,y) + iv(x,y),$$

where $z = x + iy$.

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

Theorem (Cauchy–Riemann Equations)**:**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that w is analytic,[1] which leads to Cauchy's theorem.

Theorem (Cauchy's Theorem)**:** If C is a simple closed curve in a simply connected region, then w is analytic if and only if

$$\oint_C w(z)\,dz = 0. \tag{†}$$

Fact. The function $w(z)$ is analytic inside the simply connected region R if any of these hold:

- w satisfies the Cauchy–Riemann equations;

[1] Equal to its Taylor series, also holomorphic.

- $w'(z)$ is unique and exists;
- $\frac{\partial w}{\partial \overline{z}} = 0$.
- w can be expanded as $w(z) = \sum_{n\geqslant 0} c_n (z-a)^n$, convergent on some open neighborhood of a for each a on its domain;[II]
- $w(z)$ is path-independent everywhere in R: $\oint_C w(z)\,dz = 0$.

Example. Considering $w(z) = z$, we have $u = x$ and $v = y$, so it satisfies the Cauchy–Riemann equations. However, neither $\operatorname{Re}(z)$ nor $\operatorname{Im}(z)$ are analytic, and neither is $\overline{z} = x - iy$.

Remark: Whenever we say "analytic at p," we mean "analytic in a neighborhood of p."

Note that since $\mathbb{C}$ is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of $\mathbb{C}$, by adjoining a point $\{\infty\}$, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. This compactified $\mathbb{C}^*$ is often represented as a unit sphere with the north pole, determined by $(0, 0, 1)$, is the point at infinity. The correspondence between $\mathbb{C}^* \setminus \{\infty\}$ and $\mathbb{C}$ is evaluated via stereographic projection.

We define $\frac{z}{\infty} = 0$ and $\frac{z}{0} = \infty$ for any $z \neq 0, \infty$. The correspondence between $z = x + iy$ in the plane to Z on the Riemann sphere with $\mathbb{R}^3$ coordinates (ξ_1, ξ_2, ξ_3) is

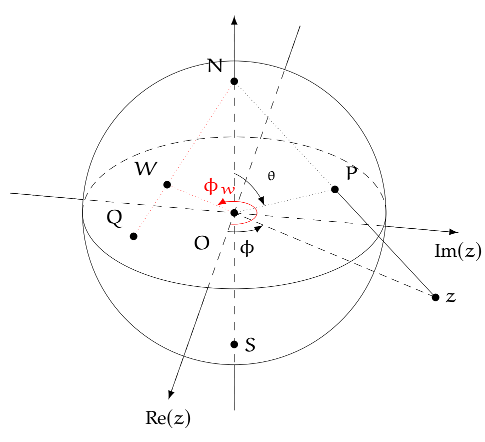
$$\xi_1 = \frac{2\operatorname{Re}(z)}{|z|^2 + 1}$$
$$\xi_2 = \frac{2\operatorname{Im}(z)}{|z|^2 + 1}$$
$$\xi_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Inverting, we may find

$$x = \frac{\xi_1}{1 - \xi_3}$$
$$y = \frac{\xi_2}{1 - \xi_3},$$

and with polar coordinates,

$$z = \cot(\theta/2) e^{i\phi}.$$



To determine analyticity at ∞, we set $\zeta = \frac{1}{z}$, and analyze the analyticity of $\tilde{w}(\zeta) = w(1/z)$ at 0.

[II]This is the real definition of analytic.

Cauchy's Integral Formula

Consider the function $w(z) = c/z$, integrated around a circle of radius R. Then, writing $z = Re^{i\varphi}$, we get

$$\begin{aligned}\oint_\Gamma w(z)\,dz &= C\int_0^{2\pi}\frac{e^{-i\varphi}}{R}\underbrace{iRe^{i\varphi}\,d\varphi}_{dz}\\ &= ic\int_0^{2\pi}\,d\varphi\\ &= 2\pi ic.\end{aligned}$$

If our contour C runs around our singularity at $z = 0$ a total of n times, then we pick up a factor of n.

Now, when we consider

$$I = \oint_C \frac{dz}{z^n},$$

this integral actually yields 0 for any $n \neq 1$, despite the fact that 0 is a singularity for $f(z) = \frac{1}{z^n}$. This 0 is not a reflection of (†), but of the fact that

$$z^{-n} = \frac{d}{dz}\left(\frac{z^{-n+1}}{n+1}\right),$$

meaning that z^{-n} is an exact differential, so integrating along a closed curve yields zero change. However, $\frac{1}{z} = \frac{d}{dz}(\ln z)$ may be an exact differential, but for complex z, $\ln z = \ln|z| + i\arg(z) = \ln r + i\varphi$. This yields

$$\begin{aligned}\oint_C \frac{c}{z}dz &= c\oint_C d(\ln z)\\ &= c(i(\varphi + 2\pi) - \varphi)\\ &= 2\pi ic.\end{aligned}$$

Ultimately, what this shows is that when we integrate any analytic function $f(\zeta)$ along a closed contour with a singularity at z, only the coefficient on $\frac{1}{\zeta - z}$ will remain. This coefficient is known as the residue at 0.

Theorem (Cauchy's Integral Formula)**:** If w is analytic in a simply connected region and C is a closed contour winding once around a point z in the region, then

$$w(z) = \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta.$$

Furthermore, this shows that any once-differentiable function is infinitely differentiable, as by differentiating under the integral sign, we get

$$\frac{d^n w}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Example (Deriving Liouville's Theorem)**.** Consider a circle C centered at radius r centered at at z, $\zeta - z = Re^{i\varphi}$. We take $d\zeta = iRe^{i\varphi}\,d\varphi$, and taking derivatives, we have

$$w'(z) = \frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi.$$

If w is bounded — i.e., $|w(z)| \leqslant M$ for all z in a given region — then

$$|w'(z)| = \left|\frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi\right|$$

$$
\begin{aligned}
&\leqslant \frac{1}{2\pi R}\int_{0}^{2\pi}\left|w\left(z+Re^{i\varphi}\right)\right|\,d\varphi\\
&\leqslant \frac{M}{R}
\end{aligned}
$$

for all R within the analytic region.

In the case where w is entire (i.e., analytic on $\mathbb{C}$), then this inequality holds for all $R \to \infty$. Thus, $|w'(z)| = 0$ for all z, meaning that w is constant.

This is known as Liouville's theorem — every bounded entire function is constant. This can be used to prove the fundamental theorem of algebra.

What Liouville's theorem tells us is that any nontrivial behavior will emerge from a function's singularities.

Singularities and Branches

To understand nontrivial behavior on the complex plane, we need to understand singularities. This will require us to develop understanding of Laurent series.

Taylor Series

We want to integrate $w(z)$ around some point a in an analytic region of $w(z)$. This yields the form

$$
\begin{aligned}
w(z) &= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{\zeta - z}\,d\zeta\\
&= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)-(z-a)}\,d\zeta\\
&= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)\left(1-\frac{z-a}{\zeta-a}\right)}\,d\zeta. \qquad (\ddagger)
\end{aligned}
$$

Since ζ is on the contour and z is in the contour, $\left|\frac{z-a}{\zeta-a}\right| < 1$, we may expand as a geometric series. Thus, we get

$$
= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)}\left(\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^{n}\right)d\zeta.
$$

Since the series is uniformly convergent, we are allowed to exchange sum and integral, yielding

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right)}_{=c_n}(z-a)^{n}\\
&= \sum_{n=0}^{\infty}c_n(z-a)^{n},
\end{aligned}
$$

where

$$
c_n = \frac{1}{n!}\left.\frac{d^n w}{dz^n}\right|_{z=a}.
$$

If our Taylor series reduces to a known series on the real axis, we find this very desirable. We say this is a type of analytic continuation from the real axis to the complex plane. For example,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is an analytic continuation of e^x.

However, more interestingly,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for all $s > 1$. However, we have also shown that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

converges for complex s for all real part greater than 1. Since values of this integral agree with the series representation of $\zeta(s)$ on real axis, we have that this is an analytic continuation of $\zeta(s)$ to the subset of $\mathbb{C}$ defined by $\text{Re}(s) > 1$.

Laurent Series

Now, what happens if, at (‡), we have $\left|\frac{z-a}{\zeta-a}\right| > 1$. The series as constructed would not converge, but what if we have a series that converges everywhere *outside* C? This would entail an expansion in reciprocal integer powers of $z - a$. This yields

$$\begin{aligned}
w(z) &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(z-a)\left(1 - \frac{\zeta-a}{z-a}\right)}\, d\zeta \\
&= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{z-a}\left(\sum_{n=0}^{\infty}\left(\frac{\zeta-a}{z-a}\right)^n\right) d\zeta \\
&= -\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_C w(\zeta-a)^n\, d\zeta\right)\frac{1}{(z-a)^{n+1}} \\
&= \sum_{n=1}^{\infty}\underbrace{\left(-\frac{1}{2\pi i}\oint_C w(\zeta-a)^{n-1}\, d\zeta\right)}_{=c_{-n}}\frac{1}{(z-a)^n} \\
&= \sum_{n=1}^{\infty}\frac{c_{-n}}{(z-a)^n}
\end{aligned}$$

Note that this series has a singularity at $z = a$, but since our series is only defined outside a particular region, that doesn't matter. We call a series in reciprocal powers a Laurent series. More specifically, Laurent series may include expansions in negative powers as well as positive powers.

Example (Annuli)**.** If we have a point a, we want to surround a by a special contour to apply Cauchy's integral formula.

In particular, for any z in the annulus, we get

$$w(z) = \frac{1}{2\pi i}\oint_{c_1 - c_2} \frac{w(\zeta)}{\zeta - z}\, d\zeta$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\oint_{c_1} \frac{w(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{c_2} \frac{w(\zeta)}{\zeta - z}\,d\zeta \\
&= \sum_{n=-\infty}^{\infty} c_n (z-a)^n \\
&= c_0 + \sum_{n=1}^{\infty} \left(c_{-n}(z-a)^n + c_n(z-a)^n\right).
\end{aligned}
$$

Example. Consider the function

$$
\begin{aligned}
w(z) &= \frac{1}{z^2 + z - 2} \\
&= \frac{1}{(z-1)(z+2)} \\
&= \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).
\end{aligned}
$$

Now, we have three regions to expand w in.

- If $|z| < 1$, then our series is in both z^n and z^n.
- If $1 < |z| < 2$, then one of our series is going to in $\frac{1}{z^n}$ and one is in z^n.
- If $|z| > 2$, then both of our series are in the form of $\frac{1}{z^n}$ and $\frac{1}{z^n}$

Via tedious, heavily error-prone calculations, we find that

$$
\begin{aligned}
w_1(z) &= -\frac{1}{3}\sum_{n=0}^{\infty}\left(1 + (-1)^n \left(\frac{1}{2}\right)^{n+1}\right) z^n \\
w_2(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}} + \left(-\frac{1}{2}\right)^{n+1} z^n\right) \\
w_3(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(1 - (-2)^n\right)\frac{1}{z^{n+1}}.
\end{aligned}
$$

Sewing all of w_1, w_2, w_3 together, then we get a full series representation of $w(z)$.

Definition. If $w(z)$ is a function that can be written as $w(z) = (z-a)^n g(a)$, where $g(a) \neq 0$, then we say w has an n-th order zero at $z = a$. If $n = 1$, then we say w has a simple zero at a.

Similarly, if we can write

$$
w(z) = \frac{g(a)}{(z-a)^n}
$$

with $g(a) \neq 0$, then we say w has a pole of order n at a. If $n = 1$, then we say w has a simple pole at a.

There are three types of isolated singularities (i.e., isolated points where $w(z)$ is not defined).

Definition. Let w be an analytic function with isolated singularity at a.

- If w remains bounded in any neighborhood of a, then it must be the case that $c_{-n} = 0$ for all $n > 1$, so the Laurent series is a pure Taylor expansion. We say $z = a$ is a removable singularity.

For instance, the function

$$\frac{\sin(z-a)}{z-a} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-a)^{2n}}{(2n+1)!}$$

has a removable singularity at $z = a$.

- If not all the c_{-n} are equal to zero, but there is a largest $n > 0$ such that c_{-n} is in the Laurent series expansion, then we say a is an n-th order pole. If $n = 1$, we say a is a simple pole.
- If there is no largest value of n such that c_{-n} is in the Laurent series — i.e., that $c_{-n} \neq 0$ for all n — then we say that a is an essential singularity.

  One of the most important facts about an essential singularity is that the behavior is path dependent. For instance,

  $$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

  has an essential singularity at $z = 0$. We see that $e^{1/z}$ diverges as $z \to 0$ along the positive real axis, but if $z \to 0$ along the negative real axis, we get $e^{1/z} \to 0$.

Singularities can also occur at ∞, which occurs when $w(1/z)$ has a singularity at 0.

Multivalued Function

Consider the function

$$\begin{aligned} w(z) &= z^2 \\ &= \underbrace{\left(x^2 - y^2\right)}_{u(x,y)} + i\underbrace{(2xy)}_{v(x,y)} \\ &= r^2 e^{2i\varphi}. \end{aligned}$$

Note that if we take a path around the origin going around by an angle of 2π, then the resulting path goes around twice. Note that this means the lines φ and $\varphi + \pi$ map to the same point in the w plane.

This isn't such a big deal in and of itself, but if we take $w(z) = z^{1/2}$, we get an issue. Instead of w being a two-to-one function, we now have w is a one-to-two function. This is an implicit problem in $\mathbb{R}$ with the function $w(x) = \sqrt{x}$, which we resolve by taking the "positive" square root. This is known as choosing a branch.

We have to do something similar in the complex plane. Note that if we go around by an angle of 2π in the z plane, then we only go around by an angle of π in the w-plane. As we keep going around the plane, we jump from branch to branch, which brings issues of continuity.

To resolve this, we create a "branch cut" that contours are not allowed to cross.

Example. The most common branch cut is to start from the branch point at $z = 0$, in the case of $w(z) = z^{1/2}$ or $w(z) = \ln(z)$, and extend along the real axis, meaning our branch cut is $(-\infty, 0]$.

This principal branch restricts *output* values of φ to $-\pi < \varphi \leqslant \pi$.

For instance, above the cut, we have $\varphi = \pi$, and below the branch cut, we have $\varphi = -\pi$, meaning we have

$$\sqrt{z} = \sqrt{r}e^{i\pi/2} \qquad \varphi \to \pi$$

$$
\begin{aligned}
&= i\sqrt{r}\\
\sqrt{z}&\sqrt{r}e^{-i\pi/2} && \varphi \to -\pi\\
&= -i\sqrt{r}.
\end{aligned}
$$

This is why the branch cut "causes" a discontinuity across the branch, but in $\mathbb{C} \setminus (-\infty, 0]$.

Now, if we have

$$
\begin{aligned}
\sqrt{z_1}\sqrt{z_2} &= \left(r_1 e^{i\varphi_1}\right)^{1/2}\left(r_2 e^{i\varphi_2}\right)^{1/2}\\
&= \sqrt{r_1 r_2} e^{i(\varphi_1+\varphi_2)/2}.
\end{aligned}
$$

However, if we want to calculate $\sqrt{z_1 z_2}$, and if $|\varphi_1 + \varphi_2| > \pi <$ then our product $z_1 z_2$ crosses the branch cut, and our discontinuity requires $\varphi_1 + \varphi_2$ to be converted to $\varphi_1 + \varphi_2 \pm 2\pi$ so as to bring the angle sum back into the principal branch. This means we have

$$
\begin{aligned}
\sqrt{z_1 z_2} &= \left(r_1 r_2 e^{i(\varphi_1+\varphi_2)/2}\right)\\
&= \begin{cases} \sqrt{r_1 r_2} e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| \leqslant \pi \\ -\sqrt{r_1 r_2} e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| > \pi \end{cases}.
\end{aligned}
$$

Example. Now, if we have $z_1 = 2e^{i(3\pi/4)}$ and $z_2 = e^{i(\pi/2)}$, then we have

$$
\begin{aligned}
\sqrt{z_1} &= \sqrt{2}e^{i3(\pi/8)}\\
\sqrt{z_2} &= e^{i(\pi/4)}.
\end{aligned}
$$

Note that if we take $\sqrt{z_1 z_2}$, then the argument of $z_1 z_2$ is $5\pi/4$, so we have to change our argument to $-3\pi/4$ to return to the principal branch before we may calculate the square root. This gives

$$
\begin{aligned}
\sqrt{z_1 z_2} &= \sqrt{2e^{-i(3\pi/4)}}\\
&= \sqrt{2}e^{-i\pi + i(5\pi/8)}\\
&= -\sqrt{2}e^{i(5\pi/8)}\\
&= -\sqrt{z_1}\sqrt{z_2}.
\end{aligned}
$$