Problem 1

Let X be a metric space and consider a subset $Y \subseteq X$ viewed as a metric space. Show that $C \subseteq Y$ is connected in Y if and only if it is connected as a subset of X.

Proof: $C \subseteq Y$ is connected if and only if any splitting $C \subseteq (Y \cap U) \sqcup (Y \cap V)$ in Y is trivial, for $U, V \subseteq X$ open. Thus, $C \subseteq Y \cap (U \sqcup V)$ is a trivial splitting, if and only if $C \subseteq U \sqcup V$ is trivial.

Problem 2

If X is a metric space, and $Y \subseteq X$ is a connected subset of X, show that for every splitting $X = X_1 \sqcup X_2$, $X_i \subseteq X$ open, we must have $Y \subseteq X_1$ or $Y \subseteq X_2$.

Proof: Let $Y \subseteq X$ be connected. Then, for any splitting $Y \subseteq X_1 \cup X_2$, with $X_1, X_2 \subseteq X$ open, it is the case that $Y \cap X_1 \cap X_2 = \emptyset$.

Since the splitting is trivial, it is the case that either $Y \cap X_1 = \emptyset$ or $Y \cap X_2 = \emptyset$.

We also have that $Y \cap (X_1 \cup X_2) = (Y \cap X_1) \cup (Y \cap X_2) = Y$. Therefore, it must be the case that $Y \cap X_1 = Y$ or $Y \cap X_2 = Y$, so $Y \subseteq X_1$ or $Y \subseteq X_2$.

Problem 3

For $n = 0, 1, 2, 3 \dots$, let $X_n := [0, 1] \times \{2^{-n}\}$, and consider the space

$$X = \{(0, 0), (1, 0)\} \cup \left(\bigcup_{n=1}^{\infty} X_n \right).$$

(i) List all the connected components of X.

(ii) If $X = U \sqcup V$ is a nontrivial splitting of X, show that there is a finite subset $F \subseteq \mathbb{N}$ with

$$U = \bigcup_{n \in F} X_n, \quad V = X \setminus U.$$

Proof:

(i) Each of the X_n are connected components, as each X_n is connected, closed, and open in X (by selecting an open subsets of $\mathbb{R}^2$ that splits two X_n segments). Therefore, we must have $\{(0, 0)\}, \{(1, 0)\}, \{X_n\}_{n \geq 1}$ are the connected components of X.

Problem 4

Show that the n-sphere, $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\|_2 = 1\}$ is path-connected.

Proof: Let $x, y \in S^{n-1}$. Then, $\|x\|_2 = \|y\|_2 = 1$. Let $\gamma : [0, 1] \to S^{n-1}$ be defined by $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma(t) = \frac{(1-t)x+ty}{\|(1-t)x+ty\|}$ (for $(1 - t)x + ty \neq 0$). Since convex combinations and norms are continuous, $\gamma(t)$ is continuous and $\|\gamma(t)\| = 1$ for all t, meaning every element of $\gamma(t)$ is an element of S^{n-1}, so $\gamma(t)$ is a path.

If x and y are antipodes, then there is some x^* in a ε-neighborhood of x, and a path from x^* to y found by the previous method, so by appending paths, we have a path from x to y.

Problem 5

Let X be a metric space. We define a relation on X, $x \sim y$ if and only if there exists a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Show that this defines an equivalence relation on X. Equivalence classes are called path-connected components.

Proof: The relation is clearly reflexive.

For symmetry, if γ is a path from x to y, we define γ' as $\gamma(1 - t)$, which is a path from y to x.

If γ_1 is a path from x to y, and γ_2 is a path from y to z, we define $\gamma : [0, 1] \to X$ as

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

This is a path from x to z, and thus the relation is transitive.

Problem 6

Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Proof: Let $f : \mathbb{R} \to \mathbb{R}^2$ be a homeomorphism, meaning f is continuous.

Consider $f(\mathbb{R}\backslash\{0\})$. We have that $\mathbb{R}\backslash\{0\} = f^{-1}\left(f(\mathbb{R} \setminus \{0\})\right)$ is disconnected. However, $f(\mathbb{R}\backslash\{0\}) = f(\mathbb{R})\backslash f(\{0\}) = \mathbb{R}^2\backslash f(0)$, but $\mathbb{R}^2 \setminus f(0)$ is connected. $\perp$

Problem 7

Let V be a normed space and suppose $Y \subseteq V$ is an open and connected subset. Fix a vector $y_0 \in Y$, and set

$$W := \{w \in Y \mid \text{there is a path from } y_0 \text{ to } w\}.$$

(i) Show that W is open in Y.

(ii) Show that W is closed in Y.

(iii) Conclude that Y is path-connected.

Problem 8

A group is a nonempty set G with a binary operation $G \times G \to G$, $(s, t) \mapsto st$ satisfying

- $(st)r = s(tr)$;
- there is a unique neutral element $e \in G$ with $te = et$ for all $t \in G$;
- for every $t \in G$ there is a unique inverse $t^{-1} \in G$ with $t^{-1}t = tt^{-1} = e$.

A subgroup of G is a nonempty subset $H \subseteq G$ such that $s, t \in H \Rightarrow st, t^{-1} \in H$. The subgroup H is normal if $t \in G, s \in H$ implies $tst^{-1} \in H$.

Consider a group G equipped with a metric so that the operations $G \times G \to G$, $(s, t) \mapsto st$ and $G \to G$, $t \mapsto t^{-1}$ are both continuous. Show that the connected component containing the neutral element e, G_0, is a closed and normal subgroup of G.

Proof: Let $s, t \in G_0$. We have some connected set C such that $s, e \in C$, and some connected set D such that $t, e \in D$. Notice that CD is connected, since $C \times D$ is continuous; since $st \in CD$ and $e \in CD$, CD is a connected set containing st and e, so $st \in G_0$.

By a similar argument, we see that if $s, e \in D$, then $D \to D^{-1}$ is a connected set containing s^{-1} and e (as $e^{-1} = e$), meaning s^{-1} is in a connected set with e. Thus, G_0 is a subgroup.

Problem 9

Show that the Cantor set is totally disconnected.

Proof: Let $a, b \in \mathcal{C}$. We will show that the only components in $\mathcal{C}$ are singletons.

Suppose $a \neq b$. Then, since $\mathbb{R}$ is Hausdorff, there exists ε so small such that $(a - \varepsilon, a + \varepsilon) \cap (b - \varepsilon, b + \varepsilon) = \emptyset$. We find n large such that $\frac{1}{3^n} < \varepsilon$, Then, it is the case that $[a - 3^{-n}, a + 3^{-n}] \cap [b - 3^{-n}, b + 3^{-n}] = \emptyset$.

Since $a, b \in \mathcal{C}_n$ for all n, we have that for all $m \geq n$, $[a - 3^{-m}, a + 3^{-m}] \cap [b - 3^{-m}, b + 3^{-m}] = \emptyset$.

Therefore, $(-\infty, a + 3^{-m}) \cup (b - 3^{-m}, \infty) \cap \mathcal{C}$ is a non-trivial splitting.

Since $\{a\}$ and $\{b\}$ are connected sets, it is the case that the only connected sets in $\mathcal{C}$ are singletons.

Problem 10

A metric space X is called zero-dimensional if for any $x, y \in X$ with $x \neq y$, there are open subsets $U, V \subseteq X$ with $x \in U, y \in V$ and $X = U \sqcup V$.

(i) Show that every zero-dimensional metric space is totally disconnected.

(ii) If $Y \subseteq \mathbb{R}$ is totally disconnected, show that Y is zero-dimensional.

(iii) Conclude that $\mathbb{Q}$ and the Cantor set are zero-dimensional.

Bonus

Let X be a compact metric space. Show that X is zero-dimensional if and only if X admits a basis of compact-open subsets.