

Problem (Problem 1):

- (a) Let G be a finite group. Show that for any subgroup $H \leq G$, we have $n_p(H) \leq n_p(G)$.
- (b) Let $f: G \rightarrow G'$ be a surjective homomorphism of finite groups, and let p be a prime. Show that every p -Sylow subgroup P' of G' is the image of some p -Sylow subgroup P of G .

Solution:

- (a) Suppose $|G| = p^r m$ and $|H| = p^s \ell$, with $p \nmid m, \ell$.

First, we observe that if $s = r$, then any p -Sylow subgroup of H is a p -Sylow subgroup of G that is contained in H , whence $n_p(H) \leq n_p(G)$.

Now, let $s < r$. We observe that if $P \leq H \leq G$ is a p -Sylow subgroup of H , then by the second Sylow theorem, P is contained in some p -Sylow subgroup, $P' \leq G$. We claim that any two distinct p -Sylow subgroups of H must be contained in distinct p -Sylow subgroups of G . This follows from the fact that, if $P_1, P_2 \leq H$ are two distinct p -Sylow subgroups, and $P_1, P_2 \leq P'$, then the subgroup $\langle P_1, P_2 \rangle$ generated in H is contained in both H and P' , but has strictly larger order than either P_1 or P_2 , which contradicts the maximality of the orders of P_1 and P_2 respectively. Thus, any p -Sylow subgroup of H is of the form $P' \cap H$ for some p -Sylow subgroup of G , whence $n_p(H) \leq n_p(G)$.

Problem (Problem 2): Let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Show that the group of upper unitriangular matrices,

$$U = \left\{ (a_{ij})_{i,j} \in GL_n(F) \mid a_{ij} = 0 \text{ for } i > j \text{ and } a_{ii} = 1 \right\}$$

is a p -Sylow subgroup of $G = GL_n(F)$.

Solution: To start, we observe that the group of upper unitriangular matrices has an order of $p^{n(n-1)/2}$, as follows from the fact that all elements in the strict upper triangle of any given matrix can be pulled from $\mathbb{Z}/p\mathbb{Z}$.

The order of $GL_n(F)$ can be seen to be

$$|GL_n(F)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}).$$

Taking out powers of p from each of the factors that divides p , we find that we get

$$\begin{aligned} |GL_n(F)| &= (p^n - 1)(p)(p^{n-1} - 1) \cdots (p^{n-1})(p - 1) \\ &= p \cdot p^2 \cdots p^{n-1} (p^n - 1)(p^{n-1} - 1) \cdots (p - 1) \\ &= p^{n(n-1)/2} (p^n - 1)(p^{n-1} - 1) \cdots (p - 1). \end{aligned}$$

Now, we observe that each of the factors of the form $p^k - 1$ are coprime to p^k , meaning that p necessarily cannot divide $p^k - 1$ for each k , so that any p -Sylow subgroup of $GL_n(F)$ has the order $p^{n(n-1)/2}$.

Problem (Problem 4): Let G be a finite group of order p^n with $n \geq 1$. Show that for every $m = 0, 1, \dots, n$, the group G has a subgroup of order p^m .

Solution: We prove using induction. If $n = 1$, then G is a group of the form $\mathbb{Z}/p\mathbb{Z}$, meaning that G has a subgroup of order 1, which is $\{e\}$, and a group of order p , which is the group itself.

Now, let $|G| = p^n$, and suppose that for any group with order p^k with $k < n$, we have that said group contains subgroups of all prime orders from 0 to k . Letting G act on itself via conjugation, we obtain

from the class equation that

$$|G| = |Z(G)| + \sum_{a \in A} [G : Z_G(a)].$$

Taking residues modulo p , we observe that $|Z(G)| \equiv |G|$ modulo p , and since $|Z(G)| \geq 1$ as $\{e\} \in Z(G)$, it follows that $|Z(G)| \geq p$.

Now, if $Z(G) = G$, then G is abelian, so by the structure of finite abelian groups, we have

$$G \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k}\mathbb{Z},$$

where we may use powers of p as the sole factors by virtue of the fact that G is a p group. Taking residue classes modulo $\mathbb{Z}/p\mathbb{Z} \times \{0\} \times \cdots \times \{0\}$, we observe then that the quotient group $G/(\mathbb{Z}/p\mathbb{Z})$ has subgroups of orders up to $n - 1$ all of which contain $\mathbb{Z}/p\mathbb{Z}$ by the fourth isomorphism theorem, whence G has subgroups of all orders up to n .

Meanwhile, if $Z(G) \neq G$, then $Z(G)$ is a p -group of order less than p^n , and the quotient group $G/Z(G)$ has order strictly less than p^n as well, meaning that the former contains p -subgroups up to the order of $Z(G)$, while the latter contains p -subgroups up to the order of $G/Z(G)$, each of which contains $Z(G)$, so that G contains p -subgroups of all orders up to p^n .

Problem (Problem 5): Show that a group of order 351 always has a normal p -Sylow subgroup for some prime p dividing the order.

Solution: Taking a prime factorization of 351 gives $351 = 3^2 \cdot 29$. Therefore, this is a group of the form $p^2 \cdot q$, where $p < q$. In particular, the number of 3-Sylow subgroups must divide 29 and be equivalent to 1 modulo 3, so that there is exactly one 3-Sylow subgroup, which is necessarily normal.

Problem (Problem 8): Let G be a group of order $3 \cdot 5^2 \cdot 17$.

- (a) Show that $n_{17}(G) = 1$. That is, a 17-Sylow subgroup H is normal.
- (b) The conjugation action of G on H defines a group homomorphism $G \rightarrow \text{aut}(H)$. Show that this homomorphism is trivial, and conclude that $H \subseteq Z(G)$.

Solution:

- (a) By the third Sylow theorem, we know that $n_{17}(G) | 75$ and $n_{17}(G) \equiv 1$ modulo 17. Writing out the possibilities for n_{17} under the second condition explicitly gives

$$n_{17}(G) = 1, 18, 35, 52, 69, 86, \dots$$

of which only 1 divides 75. Thus, there is only one 17-Sylow subgroup.