Abstract

We detail the construction and some of the properties of the Lebesgue measure.

Premeasures, Outer Measures, and Measures

Consider a set-function $\lambda\colon P(\mathbb{R}) \to [0,\infty]$ that satisfies

- $\lambda(\varnothing) = 0$;
- for any finite or infinite sequence of disjoint sets, $\{E_j\}_{j=1}^{\infty}$, we have
$$\lambda\left(\bigsqcup_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty}\lambda(E_j);$$
- $\lambda(I) = b - a$, where I is an interval (either open, closed, or a half-interval);
- $\lambda(s + E) = \lambda(E)$.

Unfortunately, such a set-function doesn't exist.

In order to construct a set function on a restricted domain $\lambda\colon \mathcal{L} \to [0,\infty]$, we need to define a particular class of measurable subsets of $\mathbb{R}$. This is where the concept of an *outer measure* comes in.

Definition. Let X be a set, and let $\mu^*\colon P(X) \to [0,\infty]$ be a set function. We say μ^* is an *outer measure* if

- $\mu^*(\varnothing) = 0$;
- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$;
- $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty}\mu^*(A_j)$.

We will obtain an outer measure on the entirety of $P(X)$ by defining a notion of measure on some "satisfactory" subfamily $\mathcal{E} \subseteq P(X)$, then by approximating other subsets using this family.

Proposition: Let $\mathcal{E} \subseteq P(X)$ be a family of subsets such that $\varnothing \in \mathcal{E}$ and $X \in \mathcal{E}$, and let $\rho\colon \mathcal{E} \to [0,\infty]$ be a set function such that $\rho(\varnothing) = 0$. For any $A \subseteq X$, define

$$\mu^*(E) = \inf\left\{\sum_{j\geq 1}\rho(E_j) \;\middle|\; E_j \in \mathcal{E}, A \subseteq \bigcup_{j\geq 1} E_j\right\}.$$

Then, μ^* is an outer measure.

Proof. We start by showing well-definedness, which stems from the fact that we may select $E_j = X$ for all j.

Since we may take $E_j = \varnothing$ for all j, we must have $\mu^*(\varnothing) = 0$. Furthermore, if $A \subseteq B$, since the set over which the infimum is taken for the definition of $\mu^*(A)$ includes the corresponding set for B, we must have $\mu^*(A) \leq \mu^*(B)$.

Finally, let $\{A_j\}_{j\geq 1} \subseteq P(X)$, and let $\varepsilon > 0$. For each j, there exists $\{E_{j,k}\}_{k\geq 1} \subseteq \mathcal{E}$ such that $A_j \subseteq \bigcup_{k\geq 1} E_{j,k}$ and $\sum_{k\geq 1}\rho(E_{j,k}) \leq \mu^*(A_j) + \varepsilon 2^{-j}$.

Then, if $A = \bigcup_{j\geq 1} A_j$, we have $A \subseteq \bigcup_{j,k\geq 1} E_{j,k}$, and $\sum_{j,k\geq 1}\rho(E_{j,k}) \leq \sum_{j\geq 1}\mu^*(A_j) + \varepsilon$, so that $\mu^*(A) \leq \sum_{j\geq 1}\mu^*(A_j) + \varepsilon$. Since ε is arbitrary, we are done. $\square$

Definition. A subset $A \subseteq X$ is said to be μ^*-*measurable* if for any $E \subseteq X$, A serves as a good "cookie cutter" for E, in that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Equivalently, due to subadditivity, we have A is measurable if and only if for all $E \subseteq X$,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition. Let $\mathcal{A}$ be an algebra of subsets of X. We call a set function $\mu_0 \colon \mathcal{A} \to [0, \infty]$ a *premeasure* if

- $\mu_0(\varnothing) = 0$;
- for a collection of disjoint elements of $\mathcal{A}$, $\{A_j\}_{j=1}^{\infty}$ where $\bigcup_{j\geq 1} A_j \in \mathcal{A}$, we have

$$\mu_0\left(\bigsqcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} \mu_0(A_j).$$

Every premeasure gives rise to an outer measure by taking

$$\mu^*(E) = \inf\left\{ \sum_{j\geq 1} \mu_0(A_j) \;\middle|\; A_j \in \mathcal{A}, E \subseteq \bigcup_{j\geq 1} A_j \right\}. \tag{$*$}$$

A remarkable result by Caratheodory allows us to extend premeasures from algebras to measures on σ-algebras. To start, there is a little bit of build-up.

Proposition: Let μ_0 be a premeasure on $\mathcal{A}$, with μ^* defined by $(*)$. Then,

(a) $\mu^*|_{\mathcal{A}} = \mu_0$;

(b) every set in $\mathcal{A}$ is μ^*-measurable.

Proof. Suppose $E \in \mathcal{A}$. If $E \subseteq \bigcup_{j\geq 1} A_j$ with $A_j \in \mathcal{A}$, we let $B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right)$. The B_n are disjoint members of $\mathcal{A}$ whose union is E, so

$$\begin{aligned}\mu_0(E) &= \sum_{j=1}^{\infty} \mu_0(B_j)\\ &\leq \sum_{j=1}^{\infty} \mu_0(A_j).\end{aligned}$$

It follows that $\mu_0(E) \leq \mu^*(E)$. The reverse inequality is clear from the fact that we may specify $A_1 = E$ and $A_{j>1} = \varnothing$.

Meanwhile, if $A \in \mathcal{A}$, $E \subseteq X$, and $\varepsilon > 0$, then there is a collection $\{B_j\}_{j\geq 1} \subseteq \mathcal{A}$ with $E \subseteq \bigcup_{j\geq 1} B_j$ and $\sum_{j\geq 1} \mu_0(B_j) \leq \mu^*(E) + \varepsilon$. By additivity on $\mathcal{A}$, we get

$$\begin{aligned}\mu^*(E) + \varepsilon &\geq \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \mu_0(B_j \cap A^c)\\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c),\end{aligned}$$

so A is measurable. $\square$

Theorem (Caratheodory's Theorem)**:** Let $\mathcal{A} \subseteq P(X)$ be an algebra, let μ_0 be a premeasure on $\mathcal{A}$, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 — namely, $\mu - \mu^*|_{\mathcal{M}}$, where μ^* is given by $(*)$.

If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$, with equality for all $\mu(E) < \infty$. Furthermore, if μ_0 is σ-finite, then μ is unique.

Proof. We start by showing that if μ^* is an outer measure, then if $\mathcal{M}^*$ is the collection of μ^*-measurable sets, $\mathcal{M}^*$ is a σ-algebra and $\mu^*|_{\mathcal{M}^*}$ is a complete measure.[I]

By definition, $\mathcal{M}^*$ is closed under complements, as the definition of μ^*-measurability is symmetric in A and A^c. To show finite additivity, if $A, B \in \mathcal{M}^*$ and $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c)\\ &\quad + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

We note that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, so subadditivity gives

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B)).$$

Therefore,

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Therefore, $A \cup B \in \mathcal{M}^*$, so $\mathcal{M}^*$ is an algebra. Moreover, if $A, B \in \mathcal{M}^*$ are disjoint, then

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c)\\ &= \mu^*(A) + \mu^*(B).\end{aligned}$$

To show that $\mathcal{M}^*$ is a σ-algebra, we show that $\mathcal{M}^*$ is closed under countable *disjoint* unions. Let $\{A_j\}_{j\geq 1}$ be a sequence of disjoint sets in $\mathcal{M}^*$, and let $B_n = \bigsqcup_{j=1}^{n} A_j$, with $B = \bigsqcup_{j\geq 1} A_j$. Then, for any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}),\end{aligned}$$

so by induction, we have

$$\mu^*(E \cap B_n) = \sum_{j=1}^{n} \mu^*(E \cap A_j).$$

This gives

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c)\\ &\geq \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B^c),\end{aligned}$$

and taking $n \to \infty$, we have

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c)$$

[I]This is Theorem 1.11 in Folland's *Real Analysis*.

$$
\begin{aligned}
&\geq \mu^{*}\left(\bigsqcup_{j\geq 1} E\cap A_j\right) + \mu^{*}(E\cap B^{c})\\
&= \mu^{*}(E\cap B) + \mu^{*}(E\cap B^{c})\\
&\geq \mu^{*}(E).
\end{aligned}
$$

$\square$