

Amenability: A (Somewhat) Brief Introduction

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Outline

- ① Definitions
- ② Paradoxical Decompositions
- ③ From Paradoxical Decompositions to Amenability
- ④ Equivalent Definitions

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Groups

If A is a set, and $\star: A \times A \rightarrow A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

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We abbreviate $a \star b$ as ab .

Subgroups, Quotient Groups

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_{\sim}$ known as the *quotient group* G/N .

Some Groups

- The integers \mathbb{Z} are a group under addition.
- The group of invertible $n \times n$ matrices over \mathbb{C} , $\mathrm{GL}_n(\mathbb{C})$, is a group under matrix multiplication.
- The subgroup $\mathrm{SO}(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under multiplication.

Group Actions

Let G be a group, and X a set. Let $\rho: G \times X \rightarrow X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X . We write $\rho(g, x) = g \cdot x$.

σ -Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

- ① $\emptyset, X \in \mathcal{A}$;
- ② for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- ③ for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say \mathcal{A} is a σ -*algebra* of subsets.

σ -Algebras and Measures, Cont'd

If X is a set and \mathcal{A} is a σ -algebra, then a map $\mu: \mathcal{A} \rightarrow [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

σ -Algebras and Measures, Cont'd

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then we say μ is a *finitely additive* measure. If $\{A_n\}_{n \geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n)$,

we say μ is a measure.

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Questions?

- If G is a group, is it possible to reconstruct G by using some subset of G ?
- When may we find a finitely additive probability measure $\mu: P(G) \rightarrow [0, 1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We begin by considering a special group, known as $F(a, b)$ or the *free group on two generators*.
- We define $F(a, b)$ to be the set of all “words” in the alphabet $\{a, b, a^{-1}, b^{-1}\}$, subject to the condition that, for $w, w' \in F(a, b)$,

$$\begin{aligned} waa^{-1}w' &\sim wa^{-1}aw' \sim ww' \\ wbb^{-1}w' &\sim wb^{-1}bw' \sim ww'. \end{aligned}$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a, b)$.

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