Banach Limits

Theorem (Hahn–Banach–Minkowski)**:** Let X be a real vector space, and let $p\colon X \to \mathbb{R}$ be such that $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ and $p(tx) = tp(x)$ for all $t \geq 0$. If $f\colon Y \to \mathbb{R}$ is a linear functional defined on a subspace Y such that $f(x) \leq p(x)$ for all $x \in Y$, then there is an extension $F\colon X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for all $x \in X$ and $F|_Y = f$.

Corollary: If X is a complex normed vector space with subspace $E \subseteq X$ and $\varphi \in E^*$, then there is $\Phi\colon X \to \mathbb{C}$ such that $\Phi|_E = \varphi$ and $\|\Phi\| = \|\varphi\|$.

Additionally, if there is $x_0 \in X \setminus E$, then there is $f \in X^*$ such that $f(x_0) = \operatorname{dist}_E(x_0)$ and $f|_E = 0$.

One of the most important vector spaces is the space ℓ_∞ of bounded sequences $x\colon \mathbb{N} \to \mathbb{C}$, which admits a subspace of convergent subspaces, often denoted c.

Proposition: There exists a linear functional $L\colon \ell_\infty \to \mathbb{C}$ with

(i) $\|L\| = 1$;

(ii) for any $x \in c$, $L(x) = \lim_{n\to\infty} x_n$;

(iii) for any $x \in \ell_\infty$ with $x_n \geq 0$ for each n, we have $L(x) \geq 0$;

(iv) for any $x \in \ell_\infty$, with $(S(x))_n \coloneqq x_{n+1}$, we have $L(S(x)) = L(x)$.

We will construct this linear functional using the Hahn–Banach theorem(s) by following the construction in Conway's book. We consider the real vector space $\operatorname{Re}(\ell_\infty)$, which we will write as ℓ_∞ for now.

To start we consider the subspace M of ℓ_∞ given by

$$M = \{x - S(x) \mid x \in \ell_\infty\}.$$

If $\mathbb{1}$ denotes the sequence of 1s in ℓ_∞, then we see that $\operatorname{dist}_M(\mathbb{1}) = 1$. First, $0 \in M$, so that $\operatorname{dist}_M(\mathbb{1}) \leq 1$. If there is n such that $(x - S(x))_n \leq 0$, then we see that

$$\begin{aligned} \|\mathbb{1} - (x - S(x))\| &\geq |\mathbb{1} - (x_n - (S(x))_n)| \\ &\geq 1. \end{aligned}$$

Else, if for all n, $0 \leq (x - S(x))_n = x_n - x_{n+1}$, then $x_{n+1} \leq x_n$ for all n. Since $x \in \ell_\infty$, there is $\alpha = \lim_{n\to\infty} x_n$. Therefore, $\lim_{n\to\infty}(x_n - x_{n+1}) = 0$, so $\|\mathbb{1} - (x - S(x))\| \geq 1$.

Thus, there is some linear functional $L\colon \ell_\infty \to \mathbb{R}$ such that $\|L\| = 1$ and $L(x) = L(S(x))$. This satisfies (i) and (iv) in the proposition.

Next, we show that $c_0 \subseteq \ker(L)$. Since (in our current focus) $c = c_0 + \mathbb{R}\mathbb{1}$, we would then obtain (ii). To see this, let $x \in c_0$. Observe that $S^n(x) - x$ is contained in M, meaning that $L(x) = L(S^n(x))$ for each n. If $\varepsilon > 0$, there is some N such that $|x_m| < \varepsilon$ for all $m > N$. Therefore,

$$|L(x)| = |L(S^n(x))|$$

$$\begin{aligned} &\leq \|S^n(x)\| \\ &< \varepsilon. \end{aligned}$$

Since ε is arbitrary, we thus have $x \in \ker(L)$.

Finally, to show (c), we let $x \in \ell_\infty$ be such that $x_n \geq 0$ for all n, and assume toward contradiction that $L(x) < 0$. Dividing out by $\|x\|$, we have that $L(x) < 0$ and $0 \leq x_n \leq 1$ for all n. Yet, this would imply that $\|\mathbb{1} - x\| \leq 1$ and $L(\mathbb{1} - x) = 1 - L(x) > 1$, contradicting (a).

To extend to $\mathbb{C}$, and rewriting the functional on $\mathbb{R}$ as L_1, we may write any element $x \in \ell_\infty$ as $x = x_1 + ix_2$. Observe then that

$$L(x) = L_1(x_1) + iL_1(x_2)$$

is a linear functional on ℓ_∞. Now, observe that $\|L\| \geq 1$ almost by design. Since L is a nonzero linear functional, we let x be such that $L(x) \neq 0$, and set

$$\alpha = \frac{|L(x)|}{L(x)}.$$

We have that $|\alpha| = 1$ and $\alpha L(x) = |L(x)|$. We may then compute

$$\begin{aligned} |L(x)| &= L(\alpha x) \\ &= \operatorname{Re}(L(\alpha x)) \\ &= L_1(\alpha x) \\ &\leq \|L_1\|\|\alpha x\| \\ &= \|x\|. \end{aligned}$$

In particular, this means $\|L\| \leq 1$, so $\|L\| = 1$.

We call such a shift-invariant extension of the limit to all of ℓ_∞ a *Banach limit*. A quick observation gives that this limit functional cannot in fact be an algebra homomorphism. Considering the case of $a_n = (-1)^n$, we have that $a_{n+1} = -a_n$, so that

$$\begin{aligned} L(S(a)) &= L(a) \\ &= L(-a) \\ &= -L(a), \end{aligned}$$

or that the Banach limit must be equal to 0. However, if L were instead an algebra homomorphism (where the multiplication operation on ℓ_∞ is given pointwise), we would have

$$\begin{aligned} L\left(a^2\right) &= L(\mathbb{1}) \\ &= 1 \\ &= (L(a))^2, \end{aligned}$$

meaning that we would have $L(a) = \pm 1$, This means that such a limit that is an algebra homomorphism cannot be shift-invariant in the general case.

Regarding the case of $(-1)^n$, we observe that from our work above that 0 is the only Banach limit for this sequence. The sequences where every Banach limit assigns the same value for them are known as the *almost convergent* sequences. Lorentz (1948) showed that an equivalent criterion for almost convergence is that, for all $\varepsilon > 0$, there exists p_0 such that for all $p > p_0$ and all $n \in \mathbb{N}$, we have

$$\left|\frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} - L\right| < \varepsilon.$$

Every convergent sequence is almost convergent by definition.

Generalized Limits beyond Banach Limits

Constructing Banach Limits from Generalized Limits

We have obtained one limit. However, there are lots of other extensions of the limit, each of which has norm 1. In fact, if we consider the restriction of ℓ_∞ to the reals, we know from undergrad real analysis that

$$p(x) = \limsup_{n\to\infty}((x_n)_n)$$

is in fact a sublinear functional. Furthermore, since

$$\liminf((x_n)_n) = -\limsup((-x_n)_n),$$

we can specify an extension to the limit functional to $L\colon \ell_\infty \to \mathbb{R}$ with $\|L\| = 1$ such that for any $(x_n)_n \in \ell_\infty$, we have

$$\liminf_{n\to\infty}((x_n)_n) \leq L(x) \leq \limsup_{n\to\infty}((x_n)_n)$$

However, these generalized limits may not align to the same requirement of shift-invariance as the aforementioned Banach limit. We can in fact construct a Banach limit from any generalized limit by using Césaro-type summation: letting $x = (x_n)_n \in \ell_\infty$, we may define

$$c_n = \frac{1}{n}\sum_{k=1}^{n} a_k,$$

and then, for a generalized limit L, defining L' by

$$L'(a) = L(c).$$

From undergrad real analysis, we know that if $(a_n)_n \to a$ is convergent, then we have

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)_n \to a$$

as well. Thus, we see that

$$\begin{aligned} L'(S(a)) - L'(a) &= L'((a_{n+1})_n) - L((a_n)_n) \\ &= L\left(\left(\frac{1}{n}\sum_{k=1}^{n} a_{k+1} - \frac{1}{n}\sum_{k=1}^{n} a_k\right)_n\right) \\ &= L\left(\left(\frac{1}{n}(a_{n+1} - a_1)\right)_n\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{n}(a_{n+1} - a_1)\right) \\ &= 0, \end{aligned}$$

with the second-to-last equality emerging from the fact that $a \in \ell_\infty$, so the sequence $(a_{n+1} - a_1)_n$ is necessarily bounded.

In particular, this means that there are plenty of Banach limits, since we can take *any* generalized extension of the limit to ℓ_∞, then use this Césaro-type summation to yield a Banach limit.

Generalized Limits that are Algebra Homomorphisms

We know from undergraduate real analysis that the limit satisfies the various properties described above — positive, unital, shift-invariant — and used some Hahn–Banach magic to construct a similar type of limit on all of ℓ_∞. Yet, there is one more property from undergraduate real analysis that classical limits have and we did not make use of; namely, that such limits are algebra homomorphisms. We will now discuss how to construct such types of limits.

To start, we observe that ℓ_∞ and its subspace c_0 are C^*-algebras. In particular, if we endow $\mathbb{N}$ with the discrete topology, then

$$\begin{aligned} \ell_\infty &= C_b(\mathbb{N}) \\ c_0 &= C_0(\mathbb{N}). \end{aligned}$$

Since ℓ_∞ is unital,the Gelfand–Naimark theorem says that we may view ℓ_∞ as the space of continuous functions over the space of characters on the space. That is, we let $\Omega(\ell_\infty) \subseteq \ell_\infty^*$ be the space of all algebra homomorphisms $\phi\colon \ell_\infty \to \mathbb{C}$ satisfying $\phi(\mathbb{1}) = 1$, and endow $\Omega(\ell_\infty)$ with the weak* topology. Such characters are automatically positive and unital, so we may view them, in a sense, as limits.

Characters are in one-to-one correspondence with maximal ideals (one direction is obvious, and the other direction follows from the Gelfand–Mazur theorem characterizing Banach algebras that are also division algebras). In particular, we may define a corresponding space of maximal ideals, $\beta\mathbb{N}$, endowed with the hull–kernel topology, where if $X \subseteq \beta\mathbb{N}$ is some set, then

$$\overline{X} = \left\{ p \in \beta\mathbb{N} \;\middle|\; p \supseteq \bigcap_{q\in X} q \right\}.$$

It can be shown that $\beta\mathbb{N}$ satisfies the universal property of the Stone–Cech compactification. From some more hard work (see the book *Algebra in the Stone–Cech Compactification*), it can be shown that the set $\beta\mathbb{N}$ is the space of ultrafilters on $\mathbb{N}$. Furthermore, any character $\phi\colon \ell_\infty \to \mathbb{C}$ is in one-to-one correspondence with an ultrafilter by defining the set $A \subseteq \mathbb{N}$ to be an element of $\mathcal{U}_\phi$ if $\phi(\chi_A) = 1$.

Yet, this idea of viewing limits as every such character is not really accurate. For instance, the map $\phi\colon \ell_\infty \to \mathbb{C}$ taking $\phi(x) = x_1$ is a character, but it doesn't really evaluate a sequence at a limit if it has one. For this purpose, we desire that any c_0 sequence evaluates to 0; that is, we want the set of all elements of $\phi \in \Omega(\ell_\infty)$ such that $c_0 \subseteq \ker(\phi)$. This induces a character, $\varphi\colon \ell_\infty/c_0 \to \mathbb{C}$, following from the first isomorphism theorem (for C^*-algebras). Finally, we may view these as generalized limits, by setting

$$\lim_{\varphi}((a_n)_n) = \phi((a_n)_n + c_0).$$

To find the ultrafilters that correspond to these particular characters, observe that if $A \subseteq \mathbb{N}$ is finite, then χ_A is a c_0 sequence; in particular, this means that it evaluates to zero. Therefore, if $\phi\colon \ell_\infty \to \mathbb{C}$ is a character with $c_0 \subseteq \ker(\phi)$, then the ultrafilter $\mathcal{U}_\phi$ must not have any finite subsets — i.e., it is a free ultrafilter (rather than a principal ultrafilter).