Complex Numbers

A complex number is an ordered pair of real numbers, $(a, b) = a + bi$. A vector in $\mathbb{R}^2$ is also an ordered pair, (a, b) of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with $\mathbb{R}^2$. However, unlike vectors in $\mathbb{R}^2$, there is also an operation $\cdot$. We desire for $(0, 1) \cdot (0, 1) = (-1, 0)$; essentially, $i^2 = -1$. We say that i is a square foot of -1; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di)\\ &:= a(c) + adi + bci + bd(i^2)\\ &:= (ac - bd) + (ad + bc)i\\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus, $\mathbb{R}^2$ with the operations $+$ and the above defined complex multiplication is known as $\mathbb{C}$. We write as $a + bi$ instead of (a, b).

Given $z = (a + bi) \in \mathbb{C}$, we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$. If $\text{Im}(z) = 0$, then $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$. However, many people say that $\mathbb{R} \subseteq \mathbb{C}$, even if $\mathbb{C}$ isn't defined as such.

Reciprocals of Complex Numbers

Let $z \in \mathbb{C}$, where $z \neq 0$. Then, $\exists w \in \mathbb{C}$ such that $zw = 1$.

Let $w = c + di$. We want to show that $zw = 1$.

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1\\ ad + bc &= 0.\end{aligned}$$

Thus, let $w = c + di$, with $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2}\\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every $z \neq 0$, with $z = a + bi$, the *reciprocal* of z is defined as $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Then, for $w \in \mathbb{C}$, we define

$$\frac{w}{z} := w\left(\frac{1}{z}\right).$$

Properties of Complex Numbers

Let $z = a + bi \in C$. Then, the (Euclidean) norm (or absolute value) of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

(i) $z\overline{z} = |z|^2$

(ii) $\overline{(\overline{z})} = z$

(iii) $\overline{(z+w)} = \overline{z} + \overline{w}$

(iv) $\overline{zw} = \overline{z} \cdot \overline{w}$

(v) $z + \overline{z} = 2\text{Re}(z)$, so $\text{Re}(z) = \frac{z+\overline{z}}{2}$

(vi) $z - \overline{z} = 2\text{Im}(z)i$, so $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

Polar Representation

Let $z = a+bi$ (or $z = (a, b)$). Then, $|z| = \sqrt{a^2+b^2}$ is the *radius*, and the *argument* is found by $\theta = \arctan(b/a)$ for $a \neq 0$. Therefore, the full polar representation is as follows:

$$z = |z|\left(\cos\theta + i\sin\theta\right). \qquad \theta \in [0, 2\pi)$$

If $z = 0$, then $|z| = 0$, and $\arg z$ is undefined.

For example, we can find $\arg i$ in $[\pi, 3\pi)$ as $\frac{5\pi}{2}$.

For z_1 and z_2 in polar form, we have:

$$\begin{align} |z_1 z_2| &= |z_1||z_2| \tag{1}\\ \arg(z_1 z_2) &= \arg z_1 + \arg z_2 \mod 2\pi \tag{2} \end{align}$$

Proof of (1):

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)\overline{(z_1 z_2)} \\ &= z_1 z_2 \overline{z_1}\,\overline{z_2} \\ &= z_1 \overline{z_1} z_2 \overline{z_2} \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

Since $|z| \geq 0$, we get $|z_1 z_2| = |z_1||z_2|$.

Let $z = 2(\cos\pi/6 + i\sin\pi/6)$, and let $f : \mathbb{C} \to \mathbb{C}$ defined as $f(w) = zw$. Then, f rotates w by $\pi/6$ and scales w by 2.

Theorem: For $n \in \mathbb{N}$, if $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Proof: Induct on n. For the base case, we know that $n = 1$ satisfies this property. For $n > 1$, we have:

$$\begin{aligned} z^{n+1} &= (z^n)(z) \\ &= \left(r^n(\cos(n\theta) + i\sin(n\theta))\right) r(\cos\theta + i\sin\theta) \\ &= (r^n)(r)\left(\cos(n\theta+\theta) + i\sin(n\theta+\theta)\right) && \text{Polar Representation Definition} \\ &= r^{n+1}(\cos\left((n+1)\theta\right) + i\sin((n+1)\theta)) \end{aligned}$$

We can use this technique to find the "roots of unity." For example, to find all z such that $z^3 = 1$, we use our technique:

$$\begin{aligned}
z^3 &= 1\\
|z| &= 1\\
\arg z^3 &= 0\\
3\arg z &= 0 \mod 2\pi\\
\arg z &= \frac{k2\pi}{3}\\
&= 0, \frac{2\pi}{3}, \frac{4\pi}{3}\\
z_1 &= 1\\
z_2 &= (\cos 2\pi/3 + i\sin 2\pi/3)\\
z_3 &= (\cos 4\pi/3 + i\sin 4\pi/3)
\end{aligned}$$

We can see that $z_2^2 = z_3$.

For the n case, we find $z_2 = \cos(2\pi/n) + i\sin(2\pi/n)$, and $z_k = z_2^{k-1}$.

Exponential, Logarithm, and Trigonometric Functions in $\mathbb{C}$

Exponential

Let $z = a + bi$. We define e^{a+bi} as follows:

$$e^{a+bi} = e^a\left(\cos b + i\sin b\right)$$

Recall that for every nonzero complex number, $z = |z|\left(\cos\theta + i\sin\theta\right)$, where $\theta = \arg z$. Thus,

$$\begin{aligned}
z &= |z|e^{i\theta}\\
&= |z|e^{i\arg z}.
\end{aligned}$$

The function e^z has some properties similar to the function e^x in real numbers, and some properties varying with the real numbers.

$$\begin{aligned}
e^z e^w &= e^{z+w}\\
e^z &\neq 0
\end{aligned}$$

However, there are some differences:

$$\begin{aligned}
|e^{i\theta}| &= 1 && \forall\theta\\
e^{a+bi} &= e^a
\end{aligned}$$

From these properties, we find Euler's equation:

$$e^{i\pi} + 1 = 0$$

Additionally, e^z is periodic, while $f(x) = e^x$ is injective:

$$\begin{aligned}
e^{z+2n\pi} &= e^z\left(\cos(2n\pi) + i\sin 2n\pi\right)\\
&= e^z
\end{aligned}$$

When examining the function $f : \mathbb{C} \to \mathbb{C}\setminus\{0\}$, $z \mapsto e^z$, we find that the following happen:

- $f(\mathbb{R}) = (0, \infty)$ — we apply $f(x) = e^x$.
- $f(a+bi) = e^a e^{bi}$ — e^a is rotated by b.
- $f(\mathbb{R} + bi)$ is expressed as the line along b radians through the origin.
- Therefore, $f(A_0) = \mathbb{C} \setminus \{0\}$, where $A_0 = \{a + bi \mid a \in \mathbb{R}, b \in [0, 2\pi)\}$.

Logarithm

Recall that for a function $f : A \to B$, f^{-1} is a function if f is injective. However, for any f, it is the case that $f^{-1}(b)$ does exist, defined as follows:

$$f^{-1}(b) = \{a \mid f(a) = b\}.$$

For the function $f(z) = e^z$, f is not one to one, so for $w = e^z$, $f^{-1}(w) = \{z' \in \mathbb{C} \mid e^{z'} = w\}$. We can find this as $f^{-1}(w) = \{z + 2n\pi i \mid n \in \mathbb{Z}\}$.

We define $\log(w) := \{z \in \mathbb{C} \mid e^z = w\}$. For a fixed $\theta \in \mathbb{R}$, we define $\log_{A_\theta}(w) := \{z \mid e^z = w, z \in A_\theta\}$.

Let $z = 1 + \frac{5\pi}{2} i$. Then,

$$\log_{A_{-\pi}} e^z = 1 + \frac{\pi}{2} i$$

Let $w \in \mathbb{C} \setminus \{0\}$. To find $\log w$ (all values), then

$$\begin{aligned} z &\in \log w \\ e^z &= w \\ &= |w| e^{i \arg w} \\ e^{a+bi} &= |w| e^{i \arg w} \\ e^a e^{ib} &= |w| e^{i \arg w}. \end{aligned}$$

Therefore, $a = \ln|w|$ and $b = \arg w$. Additionally, the following hold, for $z_1, z_2 \in \mathbb{C}$:

$$\log_{A_\theta}(z_1 z_2) = \log_{A_\theta}(z_1) + \log_{A_\theta}(z_2) + 2n\pi i$$

Cosine and Sine

$$\begin{aligned} e^{ib} &= \cos b + i \sin b \\ e^{-ib} &= \cos b - i \sin b \\ \cos z &:= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &:= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Complex Powers

Recall that for $s, t \in \mathbb{R}$, $s^t = e^{t \ln s}$, where $s > 0$. For $z, w \in \mathbb{C}$, $z^w = e^{w \log z}$., where $z \neq 0$.

$$\begin{aligned} (-2)^i &= e^{i \log(-2)} \\ &= e^{i(\ln(2) + i\pi)} \\ &= e^{i \ln 2 - (\pi + 2\pi n)} \\ &= e^{-\pi + 2\pi n + i \ln 2} \end{aligned}$$

This has *infinitely* many values.

Let $\alpha = u + vi$. Then,

$$\begin{aligned} z^{\alpha} &= e^{\alpha \log z} \\ &= e^{(u+vi)(\ln|z| + i \arg z)} \\ &= e^{(u \ln|z| - v \arg z)} e^{i(v \ln|z| + u \arg z)} \end{aligned}$$

Since $\arg z = \theta + 2\pi n$ for some real $\theta \in [0, 2\pi)$,

$$= e^{u \ln z} e^{-v(\theta + 2\pi n)} e^{iv \ln|z|} e^{iu(\theta + 2\pi n)}$$

Therefore, complex exponentiation is single-valued if $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Z}$, then z^{α} has only one value; if $\alpha \in \mathbb{Q}$, where $\alpha = \frac{p}{q}$ and $\gcd(p, q) = 1$, then z^{α} takes q distinct values, which are the qth-roots.

Continuous Functions with Complex Domains

Let $z \in \mathbb{C}$, let $r > 0$.

- The set $D(z; r) := \{w \mid w \in \mathbb{C}, |z - w| < r\}$ is the r-neighborhood of z.
- A subset $A \subseteq \mathbb{C}$ is open if $(\forall z \in A)\,(\exists r > 0) \ni D(z; r) \subseteq A$.

For example, if $A = \{z \mid \text{Re}(z) > 0\}$, we can find r equal to half the magnitude of the real component of z for any $z \in A$, meaning A is open.

Meanwhile, if $A = \{z \mid \text{Re}(z) \geq 0\}$, this is not the case. If $z = 0$, then $\nexists r > 0$ such that $D(z; r) \subseteq A$, as any open ball of radius r will have some element in $\overline{A}$.

- A subset $B \subseteq \mathbb{C}$ is closed if $\overline{B} \subseteq \mathbb{C}$ is open.

For example, $A = \emptyset$ is open, by vacuous truth, so $\overline{A} = \mathbb{C}$ is closed. Similarly, since $\mathbb{C}$ is open, $\emptyset$ is closed.

Meanwhile, $A = \{x + iy \mid -1 \leq x < 1\}$ is neither open nor closed.