Solution (38.5)**:** Copying the template equation, we have

$$\frac{dv}{dt} = -\frac{c}{m}v^2 + g,$$

where c is some constant. We see that the terminal velocity is

$$v_t = \sqrt{\frac{mg}{c}}.$$

Separating variables, we have

$$\begin{aligned}\frac{dv}{-\frac{c}{m}v^2 + g} &= dt\\ \frac{1}{g}\left(\frac{dv}{1 - \frac{c}{mg}v^2}\right) &= dt\\ \frac{1}{g}\left(\frac{dv}{1 - (v/v_t)^2}\right) &= dt.\end{aligned}$$

Using the substitution $u \coloneqq v/v_t$, we have $du = \frac{1}{v_t}dv$, meaning that

$$v_t \int \frac{1}{1-u^2}\,du = \int g\,dt.$$

The integral of $\frac{1}{1-u^2}$ is $\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = \operatorname{arctanh}(u)$. Therefore, we have

$$\begin{aligned}\frac{v}{v_t} &= \tanh\left(\frac{g}{v_t}t\right) + K\\ v &= v_t\tanh\left(\frac{g}{v_t}t\right) + v_0\\ &= \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{c}{mg}}t\right) + v_0.\end{aligned}$$

Solution (38.6)**:**

(a) Using the chain rule and letting $\frac{dm}{dt} = km^{2/3}$, we have

$$\begin{aligned}\frac{dv}{dt} &= km^{2/3}\frac{dv}{dm}\\ \frac{dv}{dm} + \frac{v}{m} &= -\frac{b}{km}v + \frac{g}{km^{2/3}}.\end{aligned}$$

With integrating factor $m^{1+\frac{b}{k}}$, we have

$$\begin{aligned}m^{1+\frac{b}{k}}v &= \frac{g}{k}\frac{m^{\frac{4}{3}+\frac{b}{k}}}{\frac{4}{3}+\frac{b}{k}} + C\\ v &= \frac{g}{k\left(\frac{4}{3}+\frac{b}{k}\right)}m^{\frac{1}{3}+\frac{b}{k}} + Cm^{-1-\frac{b}{k}}.\end{aligned}$$

We let $v(m_0) = 0$, so that

$$C = -\frac{g}{k\left(\frac{4}{3}+\frac{b}{k}\right)}m_0^{\frac{4}{3}+\frac{b}{k}},$$

so

$$v = \frac{g}{\frac{4}{3}k + b}m^{\frac{1}{3}}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{4}{3}+\frac{b}{k}}\right).$$

Thus,

$$\begin{aligned}\frac{dv}{dt} &= g - \frac{1}{m}\frac{dm}{dt}v\\ &= g - \frac{1}{m}\left(km^{2/3}\right)\left(\frac{g}{\frac{4}{3}k+b}m^{\frac{1}{3}}\left(1-\left(\frac{m_0}{m}\right)^{\frac{4}{3}+\frac{b}{k}}\right)\right).\end{aligned}$$

(b) Using $\frac{dm}{dt} = km^{2/3}v$, and $\frac{dv}{dt} = km^{2/3}v\frac{dv}{dm}$, we obtain

$$\begin{aligned}m\frac{dv}{dt} + v\frac{dm}{dt} &= -bm^{2/3}v^2 + mg\\ v\,dv + \left(\frac{v^2}{m}\left(1+\frac{b}{k}\right) - \frac{g}{km[2/3]}\right)dm &= 0.\end{aligned}$$

This gives $\alpha = v$ and $\beta = \frac{v^2}{m}\left(1+\frac{b}{k}\right) - \frac{g}{km^{2/3}}$. Solving for $p(m)$, we get

$$\begin{aligned}p(m) &= \frac{1}{v}\left(\frac{2v}{m}\left(1+\frac{b}{k}\right)\right)\\ &= \frac{2}{m}\left(1+\frac{b}{k}\right).\end{aligned}$$

Therefore, our integrating factor is

$$w(x) = m^{2+\frac{2b}{k}}.$$

This gives

$$\begin{aligned}\frac{\partial\Phi}{\partial v} &= \alpha\\ \Phi &= \frac{1}{2}m^{2+\frac{2b}{k}}v^2 + c_1(m)\\ \frac{\partial\Phi}{\partial m} &= \beta\\ \Phi &= \frac{1}{2}m^{2+\frac{2b}{k}}v^2 - \frac{g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}} + c_2(v).\end{aligned}$$

Thus, $c_2(v) = 0$, and

$$\frac{1}{2}m^{2+\frac{2b}{k}}v^2 - \frac{g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}} = C.$$

Using $v(m_0) = 0$, we obtain the solution of

$$\frac{1}{2}m^{2+\frac{2b}{k}}v^2 = \frac{g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right).$$

Simplifying, this gives

$$v^2 = \frac{2g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{1}{3}}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right).$$

Therefore,

$$2v\frac{dv}{dm} = \frac{2g}{3k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{-2/3}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right) + \frac{2g}{km}\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}},$$

and

$$\begin{aligned}\frac{dv}{dt} &= \frac{k}{2}m^{2/3}\left(2v\frac{dv}{dm}\right)\\ &= \frac{g}{3\left(\frac{7}{3}+\frac{2b}{k}\right)}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right) + \frac{g}{m^{\frac{1}{3}}}\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}.\end{aligned}$$

Solution (38.7)**:**

Solution (39.5)**:**

Solution (39.7)**:**

Solution (39.8)**:**

Solution (39.13)**:**

Solution (39.17)**:**

Solution (39.18)**:**

Solution (39.21)**:**

Solution (39.22 (b))**:**

Solution (39.28)**:**