<div style="text-align: center;">Conquering an Island</div>

Two opposed armies are ready to conquer an island. Each army's general can choose to either attack (A) not attack (N). In addition, each army is either strong or weak, and there is common knowledge that these two events are equally likely and independent. The island is worth M if captured and it is captured either by attacking when its opponent is not or by attacking when its rival does if its own army is strong and the rival is weak. If two armies of equal strength both attack, neither captures the island. An army has a cost of fighting, s if strong and $w > s$ if weak ($M > 2w$). There is no cost of attacking if the rival does not. Find all of the pure strategy Bayesian Nash equilibria.

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Without loss of generality, we will analyze Player 1's payoffs.

Strong Army Strategies:

Not Attack:

$$E\left(v_1(N, s_2^*(\theta_2); S, \theta_2)\right) = 0$$

Attack, Opponent Attacks:

$$E\left(v_1(A, A; S, \theta_2)\right) = P(\theta_2 = S)v_1(A, A, ; S, \theta_2) + P(\theta_2 = W)v_1(A, A; S, \theta_2)$$
$$= \frac{1}{2}(-s) + \frac{1}{2}(M - s)$$
$$= \frac{1}{2}M - s$$
$$> 0$$

Attack, Opponent Does Not Attack:

$$E\left(v_1(A, N; S, \theta_2)\right) = M$$
$$> 0$$

Therefore, a strong army's dominant strategy is to Attack.

Weak Army Strategies: Not Attack:

$$E\left(v_1(N, s_2^*(\theta_2); W, \theta_2)\right) = 0$$

Attack, Opponent Attacks:

$$E\left(v_1(A, A; W, \theta_2)\right) = -w$$
$$< 0$$

meaning

$$BR_i(A; W, \theta_2) = N$$

Attack, Opponent Does Not Attack: If opponent does not attack, this means opponent must be of type W; a player of type S always attacks.

$$E\left(v_1(A, N; W, W)\right) = M$$
$$> 0$$

meaning

$$BR_i(N; W, W) = A$$

Since the best response functions are symmetric with respect to each other, this means the pure strategy Bayesian Nash equilibria are:

- $((A, A), (A, N))$

- $((A, N), (A, A))$

<div style="text-align: center;">Matching Pennies, Revisited</div>

Consider the modified version of Matching Pennies shown below. The players derive extra utility from playing Heads regardless of whether they win or lose. Player 1 gets extra utility a from playing Heads and player 2 gets extra utility b from playing heads. Assume that a and b are private information independently drawn from the uniform distribution on

[0, 1].

	H	T
H	$1 + a, -1 + b$	$-1 + a, 1$
T	$-1, 1 + b$	$1, -1$

(a)

Find a Bayesian Nash equilibrium of this game.

Cutoff Strategies:

$$s_1^*(a) = \begin{cases} H & a \geq a^* \\ T & a < a^* \end{cases}$$

$$s_2^*(b) = \begin{cases} H & b \geq b^* \\ T & b < b^* \end{cases}$$

Indifference Condition for Player 1:

$$Ev_1(H, s_2^*(\theta_2); \theta_1^*, \theta_2) = Ev_1(T, s_2^*(\theta_2); \theta_1^*, \theta_2)$$
$$(1 + a^*)(1 - b^*) + (-1 + a^*)(b^*) = (-1)(1 - b^*) + (1)(b^*)$$

Indifference Condition for Player 2:

$$Ev_2(s_1^*(\theta_1), H; \theta_1^*, \theta_2) = Ev_1(s_1^*(\theta_1), T; \theta_1^*, \theta_2)$$
$$(-1 + b^*)(1 - a^*) + (1 + b^*)(a^*) = (1)(1 - a^*) + (-1)(a^*)$$

Solution to System:

$$1 + a^* - b^* - a^*b^* + a^*b^* - b^* = 2b^* - 1 \qquad \text{Player 1 Indifference Condition}$$
$$4b^* - 2 = a^*$$
$$a^* + a^*b^* - 1 + a^* - a^*b^* + b^* = 1 - 2a^* \qquad \text{Player 2 Indifference Condition}$$
$$b^* = 1 - 4a^*$$
$$1 - 4(4b^* - 2) = b^*$$
$$b^* = \frac{10}{17}$$
$$a^* = \frac{6}{17}$$

Therefore, the Bayesian Nash equilibrium is:

$$s_1^*(a) = \begin{cases} H & a \geq \frac{6}{17} \\ T & a < \frac{6}{17} \end{cases}$$

$$s_2^*(b) = \begin{cases} H & b \geq \frac{10}{17} \\ T & b < \frac{10}{17} \end{cases}$$

(b)

The probability that player 1 plays T in the BNE is $\frac{6}{17}$ and the probability that player 2 plays T in the BNE is $\frac{10}{17}$.

(c)

We would expect the probabilities to approach the mixed strategy Nash equilibrium of the regular matching pennies game, which are $\frac{1}{2}$ and $\frac{1}{2}$ respectively.

Final Times

Consider the following model of Prof. Brandon Lehr and Prof. Andy Jalil choosing dates for the 305 and 251 final exams, respectively. Each exam will either be on Tuesday or on Friday. The exam dates are chosen by Brandon and Andy simultaneously; they send messages of either Tuesday or Friday to the final exam office, which schedules each exam on the requested date.

Brandon and Andy, however, like to have the exam on different days because they share some of the same students. Specifically, they get 1 util if the exams are on different dates. However, both prefer to have their exam on Tuesday. Brandon gets extra utility θ_B from holding his exam on Tuesday and Andy gets extra utility θ_A from holding his exam on Tuesday. Suppose that θ_B and θ_A are private information, uniformly distributed on $[0, \bar{\theta}]$.

	Tuesday	Friday
Tuesday	θ_B, θ_A	$1 + \theta_B, 1$
Friday	$1, 1 + \theta_A$	$0, 0$

(a)

Find a symmetric pure strategy Bayesian Nash equilibrium of this game.

Cutoff Strategy:

$$s_B^*(\theta_B) = \begin{cases} T & \theta_B \geq \theta_B^* \\ F & \theta_B < \theta_B^* \end{cases}$$

$$s_A^*(\theta_A) = \begin{cases} T & \theta_A \geq \theta_A^* \\ F & \theta_A < \theta_A^* \end{cases}$$

Indifference Condition for Player B:

$$Ev_A(T, s_A^*(\theta_A); \theta_B^*, \theta_A) = Ev_A(F, s_A^*(\theta_A); \theta_B^*, \theta_A)$$

$$\theta_B^* \frac{1}{\overline{\theta}}(\overline{\theta} - \theta_A^*) + (1 + \theta_B^*)\frac{1}{\overline{\theta}}(\theta_A^*) = (1)\frac{1}{\overline{\theta}}(\overline{\theta} - \theta_A^*)$$

Indifference Condition for Player A:

$$Ev_A(s_B^*(\theta_B), T; \theta_B, \theta_A^*) = Ev_A(s_B^*(\theta_B), F; \theta_B, \theta_A^*)$$

$$\theta_A^* \frac{1}{\overline{\theta}}(\overline{\theta} - \theta_B^*) + (1 + \theta_A^*)\frac{1}{\overline{\theta}}(\theta_B^*) = (1)\frac{1}{\overline{\theta}}(\overline{\theta} - \theta_B^*)$$

Solution to System:

$$\theta_B^* \frac{1}{\overline{\theta}}(\overline{\theta} - \theta_A^*) + (1 + \theta_B^*)\frac{1}{\overline{\theta}}(\theta_A^*) = (1)\frac{1}{\overline{\theta}}(\overline{\theta} - \theta_A^*) \qquad \text{Player A Indifference Condition}$$

$$\theta_A^* \frac{1}{\overline{\theta}}(\overline{\theta} - \theta_B^*) + (1 + \theta_A^*)\frac{1}{\overline{\theta}}(\theta_B^*) = (1)\frac{1}{\overline{\theta}}(\overline{\theta} - \theta_B^*) \qquad \text{Player B Indifference Condition}$$

$$\overline{\theta}\theta_B^* - \theta_B^*\theta_A^* + \theta_A^* + \theta_A^*\theta_B^* = \overline{\theta} - \theta_A^*$$

$$\theta_B^* = 1 - \frac{2\theta_A^*}{\overline{\theta}}$$

$$\theta_A^*\overline{\theta} - \theta_B^*\theta_A^* + \theta_B^* + \theta_A^*\theta_B^* = \overline{\theta} - \theta_B^*$$

$$\theta_A^* = 1 - \frac{2\theta_B^*}{\overline{\theta}}$$

$$\theta_B^* = 1 - \frac{2\left(1 - \frac{2\theta_B^*}{\overline{\theta}}\right)}{\overline{\theta}}$$

$$\overline{\theta}\theta_B^* = \overline{\theta} - 2 + \frac{2\theta_B^*}{\overline{\theta}}$$

$$\frac{\theta_B^*\left(\overline{\theta}^2 - 2\right)}{\overline{\theta}} = \overline{\theta} - 2$$

$$\theta_B^* = \frac{\overline{\theta}\left(\overline{\theta} - 2\right)}{\overline{\theta}^2 - 2}$$

$$\theta_A^* = \frac{\overline{\theta}^2 - 2\overline{\theta} + 4}{\overline{\theta}^2 - 2}$$

(b)

What does Harsanyi's theorem say about the equilibria of this game when $\overline{\theta}$ is small?

As $\overline{\theta}$ becomes small, the Bayesian Nash equilibrium of the game approaches the mixed strategy Nash equilibrium.

Two players, 1 and 2, each own a house. Each player i values their own house at v_i. The value of player i's house to the other player is $\frac{3}{2} v_i$. Each player knows v_i to themselves, but not the value of the other player's house. The values v_i are drawn independently from $[0, 1]$ with uniform distribution.

(a)

Suppose players announce simultaneously whether they want to exchange their houses. If both players agree to an exchange, the exchange takes place. Otherwise no exchange takes place. Find a Bayesian Nash equilibrium of this game in pure strategies in which each player i accepts an exchange if and only if the value v_i does not exceed some threshold θ_i.

The simultaneous exchange game has the structure below, where E denotes "exchange" and D denotes "don't exchange.'

	E	D
E	$\frac{3}{2} v_2 - v_1, \frac{3}{2} v_1 - v_2$	v_1, v_2
D	v_1, v_2	v_1, v_2

Let $\theta_i \in [0, \bar{\theta}]$

Cutoff Strategy:

$$s_i^*(v_i) = \begin{cases} D & v_i \geq \theta_i \\ E & v_i < \theta_i \end{cases}$$

Indifference Condition for Player 1:

$$Ev_1(D, s_2^*(v_2); \theta_1, v_2) = Ev_1(E, s_2^*(v_2); \theta_1, v_2)$$

$$\theta_1 = \frac{\theta_2}{\bar{\theta}} \left(\frac{3}{2} v_2 - \theta_1 \right) + \left(1 - \frac{\theta_2}{\bar{\theta}} \right) \theta_1$$

Indifference Condition for Player 2:

$$Ev_2(s_1^*(v_1), D; v_1, v_2) = Ev_2(s_1^*(v_1))$$

Symmetric Bayesian Nash Equilibrium: $s_i^*(\theta_i) = k\theta_i + c$.

(1) Everyone except player 1 is playing $s_i^*(\theta_i) = k\theta_i + c$

(2) Find $Ev_1(b_1, s_{-1}(\theta_{-1}); \theta_1, \theta_{-1})$

(3) Find b_1^* such that $\frac{\partial Ev_1}{\partial b_1} = 0$

(4) In BNE, $b_1^* = k\theta_1 + c$