Cardinality and Countability

Section 1.1: Countable Sets

Definition (Denumerable Set)**.** A set S is denumerable if there exists a function $f : S \to \mathbb{N}$ with f a bijection. We also say S is countably infinite.

Definition (Countable Set)**.** We say S is countable if S is either finite or denumerable.

Theorem (Countability of Unions)**.** *If* A *and* B *are countable sets, then* $A \cup B$ *is countable.*

Theorem (Countability of Subsets)**.** *If* $A \subseteq B$*, then if* B *is countable, then* A *is countable.*

Theorem (Union of Finite Sets)**.** *If* A *and* B *are finite, then* $A \cup B$ *is finite.*

Proof. If A is finite and $|B|$ has one element, then we show that $A \cup B$ is finite (with two cases).

Afterward, for $|B| > 1$, we use induction on $|B|$. □

Definition (Finite Set)**.** A set A is finite if there exists a bijection $f : S \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N} = \{0, 1, \dots\}$.

We write $|A| = n$.

Theorem (Disjoint Union of Countable Sets)**.** *If* A *is denumerable,* B *is finite, and* $A \cap B = \emptyset$*, then* $A \cup B$ *is denumerable.*

Proof. There exists a bijection $f : A \to \mathbb{N}$ (since A is denumerable), and a bijection $g : B \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ (since B is finite).

We create a new bijection $h : A \cup B \to \mathbb{N}$ by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since $A \cap B = \emptyset$, we know that h is well-defined.

Now, we must show that h is a bijection.

Suppose $h(x) = h(y)$.

Case 1: If $x, y \in B$, then $h(x) = g(x) - 1$, and $h(y) = g(y) - 1$, meaning $g(x) - 1 = g(y) - 1$, meaning $g(x) = g(y)$. Since g is a bijection, $x = y$.

Case 2: If $x, y \in A$, a similar argument yields that $x = y$

Case 3: Without loss of generality, let $x \in A$ and $y \in B$. If $x \in A$, then $h(x) = f(x) + n$ and $h(y) = g(y) - 1$. Thus, $f(x) + n = g(y) - 1$. However, since $f(x) + n \geqslant n$ and $0 \leqslant g(y) - 1 \leqslant n - 1$. Thus, we get that $0 \leqslant n \leqslant n - 1$, which is a contradiction.

Thus, we have shown that h is injective. □

Theorem (Cartesian Product of Natural Numbers)**.** $\mathbb{N} \times \mathbb{N}$ *is denumerable.*

Proof. We consider $\mathbb{N} \times \mathbb{N}$ as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \cdots,$$

$$\begin{array}{llllll}
\mathbb{N}\times\{0\}: & (0,0) & (1,0) & (2,0) & (3,0) & \cdots \\
\mathbb{N}\times\{1\}: & (0,1) & (1,1) & (2,1) & (3,1) & \cdots \\
\mathbb{N}\times\{2\}: & (0,2) & (1,2) & (2,2) & (3,2) & \cdots \\
\mathbb{N}\times\{3\}: & (0,3) & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{llllll}
\mathbb{N}\times\{0\}: & \cancel{(0,0)}^{0} & \cancel{(1,0)}^{2} & \cancel{(2,0)}^{5} & \cancel{(3,0)}^{9} & \cdots \\
\mathbb{N}\times\{1\}: & \cancel{(0,1)}^{1} & \cancel{(1,1)}^{4} & \cancel{(2,1)}^{8} & (3,1) & \cdots \\
\mathbb{N}\times\{2\}: & \cancel{(0,2)}^{3} & \cancel{(1,2)}^{7} & (2,2) & (3,2) & \cdots \\
\mathbb{N}\times\{3\}: & \cancel{(0,3)}^{6} & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can also find a bijection $P : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, with

$$P(x,y) = \frac{(x+y)(x+y+1)}{2} + x$$

A fun challenge is to prove that P is a bijection. $\square$

Theorem (Countability of the Rationals). $\mathbb{Q}$ *is denumerable.*

Theorem (Countability of the Integers). *The set* $\mathbb{Z}$ *is denumerable.*

Proof. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 2x & x \geqslant 0 \\ -2x-1 & x < 0 \end{cases}$$

$\square$

Definition (Cardinality). We say two sets, A and B, have the same cardinality if there exists a bijection $f : A \to B$.

Theorem (Finite Subset Cardinality). *If* $m, n \in \mathbb{N}$ *and* $m \neq n$, *then* $\{1,2,\dots,m\}$ *and* $\{1,2,\dots,n\}$ *do not have the same cardinality.*

Theorem (Infinitude of the Natural Numbers). $\mathbb{N}$ *is not finite.*

Example. If $A \subsetneq B$ and $|A| = |B|$, then both A and B are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.