Part 1

1.8, Problem 4

To solve

$$\frac{dy}{dt} = 2y + \sin 2t,$$

we start by solving the homogeneous equation

$$\frac{dy}{dt} = 2y,$$

which yields $y_h = ke^{2t}$.

For the particular solution, we guess that y is of the form

$$y_p(t) = A\cos 2t + B\sin 2t.$$

Plugging this into our equation, we get

$$\begin{aligned} -2A\sin 2t + 2B\cos 2t + 2\left(A\cos 2t + B\sin 2t\right) &= \sin 2t \\ (2B - 2A)\sin 2t + (2B + 2A)\cos 2t &= \sin 2t, \end{aligned}$$

meaning $A = -\frac{1}{4}$ and $B = \frac{1}{4}$. Thus, our general solution is

$$y(t) = -\frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t + ke^{2t}.$$

1.8, Problem 8

To solve

$$\frac{dy}{dt} - 2y = 3e^{-2t},$$

with the initial condition of $y(0) = 10$, we start by solving the homogeneous equation, which yields $y_h(t) = ke^{2t}$.

For the particular solution, we guess that y is of the form

$$y_p(t) = Ae^{-2t}.$$

Substituting into our equation, we have

$$\begin{aligned} -2Ae^{-2t} - 2\left(Ae^{-2t}\right) &= 3e^{-2t} \\ -4Ae^{-2t} &= 3e^{-2t}, \end{aligned}$$

which yields $A = -\frac{3}{4}$. Thus, our general solution is of the form

$$y(t) = -\frac{3}{4}e^{-2t} + ke^{2t}.$$

The initial condition yields $k = \frac{43}{4}$.

1.8, Problem 9

To solve

$$\frac{dy}{dt} + y = \cos 2t,$$

with the initial condition of $y(0) = 5$, we start by solving the homogeneous equation, which yields $y_h(t) = ke^{-t}$.

For the particular solution, we guess that y is of the form

$$y_p(t) = A\cos 2t + B\sin 2t.$$

Substituting into our equation, we get

$$\begin{aligned} -2A\sin 2t + 2B\cos 2t + (A\cos 2t + B\sin 2t) &= \cos 2t \\ (2B + A)\cos 2t + (B - 2A)\sin 2t &= \cos 2t, \end{aligned}$$

meaning $A = \frac{1}{3}$ and $B = \frac{2}{3}$. Thus, our general solution is

$$y(t) = \frac{1}{3}\cos 2t + \frac{2}{3}\sin 2t + ke^{-t}.$$

Solving the initial condition yields $k = \frac{14}{3}$.

1.8, Problem 17

(a)

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{1-t}\right) &= \frac{1}{(1-t)^2} \\ &= \left(\frac{1}{1-t}\right)^2. \end{aligned}$$

(b)

$$\begin{aligned} \frac{d}{dt}\left(\frac{2}{1-t}\right) &= \frac{2}{(1-t)^2} \\ &\neq \left(\frac{2}{1-t}\right)^2. \end{aligned}$$

(c) These two facts do not contradict the linearity principle since the equation $\frac{dy}{dt} = y^2$ is not a linear equation.

1.8, Problem 18

(a) The solution of $y(t) = 2$ is an equilibrium solution for this equation.

(b)

$$\begin{aligned} \frac{d}{dt}\left(2 - e^{-t}\right) &= e^{-t} \\ &= 2 - \left(2 - e^{-t}\right) \\ &= 2 - y. \end{aligned}$$

(c) The uniqueness of solutions implies that the only initial solution to an IVP with $y(0) = a$ for $a \neq 2$ is one that satisfies $y(t) = 2 - e^{-t}$, which is not able to be added or multiplied by a constant.

1.8, Problem 20

$$
\begin{aligned}
(2at + b) + 2\left(at^2 + bt + c\right) &= 3t^2 + 2t - 1\\
2at^2 + (2a + 2b)\,t + (b + c) &= 3t^2 + 2t - 1.
\end{aligned}
$$

Thus, $a = \frac{3}{2}$, $b = -\frac{1}{2}$, and $c = -\frac{1}{2}$.

1.8, Problem 31

To represent the first 30 years, we get

$$\frac{dy}{dt} = 5000 + 0.07y$$

Solving the initial value problem with $y(0) = 0$, we get

$$y(t) = \frac{5000}{0.07}e^{0.07t} - \frac{5000}{0.07}.$$

Then, $y(30) = 511869$.

Now, using a different initial value problem, we solve

$$\frac{dy}{dt} = -36000 + 0.07y$$

under the initial value of $y(0) = 511869$. Thus, we get

$$y(t) = Ke^{0.07t} + \frac{36000}{0.07}.$$

Solving for K, we get $K = -2416.43$. Thus,

$$
\begin{aligned}
t &= \frac{1}{0.07}\left(\ln \frac{36000}{(0.07)(2416.43)}\right)\\
&= 76.
\end{aligned}
$$

1.9, Problem 4

$$
\begin{aligned}
\frac{dy}{dt} &= -2ty + 4e^{-t^2}\\
\frac{dy}{dt} + 2ty &= 4e^{-t^2}\\
e^{t^2}\frac{dy}{dt} + 2te^{t^2}y &= 4e^{-t^2}e^{t^2}\\
\frac{d}{dt}\left(e^{t^2}y\right) &= 4\\
e^{t^2}y &= 4t + C\\
y &= \frac{4t}{e^{t^2}} + Ce^{-t^2}.
\end{aligned}
$$

1.9, Problem 5

$$\begin{aligned}
\frac{dy}{dt} - \frac{2t}{1+t^2}y &= 3\\
\frac{1}{1+t^2}\frac{dy}{dt} - \frac{2t}{1+t^2}y &= \frac{3}{1+t^2}\\
\frac{d}{dt}\left(y\frac{1}{1+t^2}\right) &= \frac{3}{1+t^2}\\
y\frac{1}{1+t^2} &= 3\arctan(t) + C\\
y &= 3\arctan(t)\left(1+t^2\right) + C\left(1+t^2\right).
\end{aligned}$$

1.9, Problem 9

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{y}{t} + 2\\
\frac{dy}{dt} + \frac{y}{t} &= 2\\
t\frac{dy}{dt} + y &= 2t\\
\frac{d}{dt}(ty) &= 2t\\
ty &= t^2 + C\\
y &= t + \frac{C}{t}.\\
y(1) &= 1 + \frac{C}{1}\\
&= 3\\
C &= 2\\
y(t) &= t + \frac{2}{t}.
\end{aligned}$$

1.9, Problem 12

$$\begin{aligned}
\frac{dy}{dt} - \frac{3}{t}y &= 2t^3e^{2t}\\
\frac{1}{t^3}\frac{dy}{dt} - \frac{3}{t^4}y &= 2e^{2t}\\
\frac{d}{dt}\left(\frac{1}{t^3}y\right) &= 2e^{2t}\\
\frac{1}{t^3}y &= e^{2t} + C\\
y &= t^3e^{2t} + Ct^3.\\
y(1) &= 1 + C\\
&= 0\\
C &= -1\\
y(t) &= t^3e^{2t} - t^3.
\end{aligned}$$

1.9, Problem 19

The only value of a for which there is an explicit solution to the differential equation $\frac{dy}{dt} = aty + 4e^{-t^2}$ is with $a = t$.

1.9, Problem 21

(a)

$$\begin{aligned}
\frac{dv}{dt} + 0.4v &= 3\cos(2t)\\
e^{0.4t}\frac{dv}{dt} + 0.4e^{0.4t}v &= 3e^{0.4t}\cos(2t)\\
\frac{d}{dt}\left(e^{0.4t}v\right) &= 3e^{0.4t}\cos(2t)\\
e^{0.4t}v &= \frac{3}{\left(\frac{(0.4)^2}{4}+1\right)}e^{0.4t}\left(\frac{1}{2}\sin(2t) + \frac{1}{4}\cos(2t)\right) + C\\
v &= \frac{3}{\left(\frac{(0.4)^2}{4}+1\right)}\left(\frac{1}{2}\sin(2t) + \frac{1}{4}\cos(2t)\right) + \frac{C}{e^{0.4t}}.
\end{aligned}$$

(b) The homogeneous equation $\frac{dv}{dt} = -0.04v$ is solved by $v_h(t) = e^{-0.4t}$.

For the particular solution, we will use the guess that $v = A\cos 2t + B\sin 2t$. Then,

$$\begin{aligned}
-A\sin 2t + B\cos 2t + 0.4\left(A\cos 2t + B\sin 2t\right) &= 3\cos 2t\\
(0.4A + B)\cos 2t + (0.4B - A)\sin 2t &= 3\cos 2t,
\end{aligned}$$

meaning $A = 0.4B$, $1.4B = 3$, so $B = \frac{3}{1.4}$ and $A = 0.4\frac{3}{1.4}$. Thus, the general solution is

$$y(t) = \frac{3}{1.4}\sin 2t + \frac{(0.4)(3)}{1.4}\cos 2t + Ce^{-0.4}t.$$

The latter method was quite a bit easier than the former method.