# Contents



# Introduction

This is going to be a part of my Honors thesis independent study, focused on amenability and $C^*$-algebras. This section of notes will be a deeper dive into group amenability. These notes will be taken from the notes my professor has prepared on group amenability, with supplementation from Volker Runde's *Lectures on Amenability* and Pierre de la Harpe's *Topics in Geometric Group Theory*.

I do not claim any of this work to be original.

# Basics of Amenable Groups and Subgroups

Let $G$ be a group, with $P(G)$ denoting the power set.

**Definition.** An invariant mean on $G$ is a set function $m : P(G) \to [0,1]$, which satisfies, for all $t \in G$ and $E, F \subseteq G$,

(1) $m(G) = 1$;

(2) $m(E \sqcup F) = M(E) + m(F)$;

(3) $m(tE) = m(E)$.

We say $G$ is amenable if it admits a mean.

We can also say that $m$ is a translation-invariant probability measure on $(G, P(G))$.

**Proposition** (Amenability of Subgroups and Quotient Groups)**:** Let $G$ be amenable, with $H \leqslant G$.

(1) $H$ is amenable;

(2) for $H \trianglelefteq G$, $G/H$ is amenable.

*Proof.*

(1) Let $R$ be a right transversal for $H$ (i.e., selecting one element of each right coset of $H$ to make up $R$).

If $m$ is a mean for $G$, we set

$$\lambda : \mathcal{P}(H) \to [0,1]$$

by $\lambda(E) = m(ER)$. We have

$$\lambda(H) = m(HR)$$

$$\begin{aligned} &= m(G)\\ &= 1. \end{aligned}$$

We claim that if $E \cap F = \emptyset$, then $ER \cap FR = \emptyset$, since if we suppose toward contradiction that $ER \cap FR \neq \emptyset$, then $xr_1 = yr_2$ for some $x \in E, y \in F$ and $r_1, r_2 \in R$. Then, we must have $r_2 r_1^{-1} = y^{-1}x \in H$, meaning $r_1 = r_2$ and $x = y$, which means $E \cap F \neq \emptyset$.

Thus, we have

$$\begin{aligned} \lambda(E \sqcup F) &= m((E \sqcup F)R)\\ &= m(ER \sqcup FR)\\ &= m(ER) + m(FR)\\ &= \lambda(E) + \lambda(F), \end{aligned}$$

and

$$\begin{aligned} \lambda(sE) &= m(sER)\\ &= m(ER)\\ &= \lambda(E). \end{aligned}$$

(2) For the canonical projection map $\pi : G \to G/H$ defined by $\pi(t) = tH$, we define

$$\lambda : P(G/H) \to [0, 1]$$

by $\lambda(E) = m\left(\pi^{-1}(E)\right)$. We have

$$\begin{aligned} \lambda(G/H) &= m\left(\pi^{-1}(G/H)\right)\\ &= m(G)\\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \lambda(E \sqcup F) &= m\left(\pi^{-1}(E \sqcup F)\right)\\ &= m\left(\pi^{-1}(E) \sqcup \pi^{-1}(F)\right)\\ &= m\left(\pi^{-1}(E)\right) + m\left(\pi^{-1}(F)\right)\\ &= \lambda(E) + \lambda(F). \end{aligned}$$

To show translation-invariance, we let $sH = \pi(s) \in G/H$, and $E \subseteq G/H$. Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for $r \in s\pi^{-1}(E)$, we have $r = st$ for $\pi(t) \in E$, so $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$.

Additionally, if $r \in \pi^{-1}(\pi(s)E)$, then $\pi(r) \in \pi(s)E$, so $\pi\left(s^{-1}r\right) \in E$, and $s^{-1}r \in \pi^{-1}(E)$. Thus, we have

$$\begin{aligned} \lambda(\pi(s)E) &= m\left(\pi^{-1}(\pi(s)E)\right)\\ &= m\left(s\pi^{-1}(E)\right)\\ &= m\left(\pi^{-1}(E)\right)\\ &= \lambda(E). \end{aligned}$$

$\square$

# Understanding Free Groups

In the Tarski's Theorem notes, we discussed a little bit the ramifications of the free group on two generators being paradoxical. In order to better understand free groups, we will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory* and Clara Löh's *Geometric Group Theory: An Introduction*.

## Groups specified by Generating Sets

**Definition.** Let $G$ be a group and $S \subseteq G$ be a subset. The subgroup generated by $S$ is the intersection of all subgroups of $G$ that contain $S$. We write $\langle S \rangle_G$. We say $S$ generates $G$ if $\langle S \rangle_G = G$.

A group is called finitely generated if it contains a finite subset that contains the group in question.

**Definition** (Characterization of a Generated Subgroup)**.** We can characterize a generated subgroup by $S$ as follows:

$$\langle S \rangle_G = \left\{ s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n} \;\middle|\; n \in \mathbb{N},\ s_1, \dots, s_n \in S,\ \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\} \right\}.$$

**Example** (Generating Sets)**.**

- If $G$ is a group, then $G$ is a generating set of $G$.
- The trivial group is generated by the empty set.
- The set $\{1\}$ generates $\mathbb{Z}$, as does $\{2, 3\}$. However, $\{2\}$ and $\{3\}$ alone do not generate $\mathbb{Z}$.
- Let $X$ be a set. The symmetric group $S_X$ is finitely generated if and only if $X$ is finite.

## Free Groups

**Definition.** Let $S$ be a set. A group $F$ containing $S$ is said to be freely generated if, for every group $G$ and every map $\varphi : S \to G$, there is a unique group homomorphism $\overline{\varphi} : F \to G$ extending $\varphi$. The following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\iota"'] & G \\ F \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

A group is free if it contains a free generating set.

**Example.**

- The additive group $\mathbb{Z}$ is freely generated by $\{1\}$. The additive group $\mathbb{Z}$ is *not* freely generated by $\{2, 3\}$, or $\{2\}$, or $\{3\}$. In particular, not every generating set of a group contains a free generating set.
- The trivial group is freely generated by the empty set.
- Not every group is free — the additive groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are not free.

We will use the universal property of free groups to show their uniqueness up to isomorphism.

**Proposition:** Let $S$ be a set. Then, there is at most one group freely generated by $S$ up to isomorphism.

*Proof.* Let $F$ and $F'$ be two groups freely generated by $S$, with inclusions of $\varphi$ and $\varphi'$ respectively. Because $F$ is freely generated by $S$, there is a group homomorphism $\overline{\varphi}' : F \to F'$ that extends $\varphi$ — i.e., that $\overline{\varphi}' \circ \varphi = \varphi'$.

Similarly, there is a group homomorphism $\overline{\varphi} : F' \to F$ with $\overline{\varphi} \circ \varphi' = \varphi$.

$$\begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[d, hook, "\varphi"'] & F' \\ F \arrow[ur, "\overline{\varphi}'"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi'"'] & F \\ F' \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

We will show that $\overline{\varphi} \circ \overline{\varphi}' = \mathrm{id}_F$, and $\overline{\varphi}' \circ \overline{\varphi} = \mathrm{id}_{F'}$. The composition $\overline{\varphi} \circ \overline{\varphi}'$ is a group homomorphism that makes the following diagram commute.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi"'] & F \\ F \arrow[ur, "\overline{\varphi}\circ\overline{\varphi}'"'] & \end{tikzcd}$$

Since $\mathrm{id}_F$ is a group homomorphism contained in this diagram, and $F$ is freely generated by $S$, we must have $\overline{\varphi} \circ \overline{\varphi}' = \mathrm{id}_F$. Similarly, we must have $\overline{\varphi}' \circ \overline{\varphi} = \mathrm{id}_{F'}$. $\square$

**Theorem** (Existence of Free Groups)**:** Let $S$ be a set. There exists a group freely generated by $S$. This group is unique up to isomorphism.

*Proof.* We want to construct a group consisting of "words" made up of the elements of $S$ and their "inverses," then modding out by the natural cancellation rules.

We consider the alphabet

$$A = S \cup \hat{S}.$$

Here, $\hat{S} = \{\hat{s} \mid s \in S\}$ is a disjoint copy of $S$, such that $\hat{s}$ will serve as the inverse of $s$ in the group we will construct.

We define $A^*$ to be the set of all finite sequences over the alphabet $A$, including the empty word $\epsilon$. We define the operation $A^* \times A^* \to A^*$ by concatenation. This operation is associative with neutral element $\epsilon$.

We define

$$F(S) = A^*/\sim,$$

where $\sim$ is the equivalence relation generated by, for all $x, y \in A^*$ and $s \in S$, $xs\hat{s}y \sim xy$ and $x\hat{s}sy \sim xy$.

We denote the equivalence classes with respect to $\sim$ by $[\cdot]$.

Concatenation induces a well-defined operation $F(S) \times F(S) \to F(S)$ by

$$[x][y] = [xy]$$

for $x, y \in A^*$.

We claim that $F(S)$ with the defined concatenation is a group. We can see that $[\epsilon]$ is a neutral element for the operation, and associativity of the operation is inherited from the associativity of the operation on $A^*$.

To find inverses, we define $I : A^* \to A^*$ by $I(\epsilon) = \epsilon$, and

$$I(sx) = I(x)\hat{s}$$
$$I(\hat{s}x) = I(x)s$$

for all $x \in A^*$ and $s \in S$. Induction shows that $I(I(x)) = x$, and

$$[I(x)][x] = [I(x)x]$$

$$= [\epsilon]$$

for all $x \in A^*$. Thus, we must also have

$$\begin{aligned}
[x]\,[I(x)] &= [I\,(I\,(x))]\,[I\,(x)]\\
&= [\epsilon]\,.
\end{aligned}$$

Thus, we see that there are inverses in $F(S)$.

To see that $F(S)$ is freely generated by $S$, we let $\iota : S \to F(S)$ be the map given by sending a letter in $S \subseteq A^*$ to its equivalence class in $F(S)$. By construction, $F(S)$ is generated by the subset $\iota(S) \subseteq F(S)$.

We do not know yet that $\iota$ is injective, so we take a bit of a detour. We show that for every group $G$ and every map $\varphi : S \to G$, there is a unique group homomorphism $\overline{\varphi} : F(S) \to G$ such that $\overline{\varphi} \circ \iota = \varphi$.

We construct a map $\varphi^* : A^* \to G$ inductively by

$$\begin{aligned}
\epsilon &\mapsto e\\
sx &\mapsto \varphi(s)\varphi^*(x)\\
\hat{s}x &\mapsto (\varphi(s))^{-1}\varphi^*(x)
\end{aligned}$$

for all $s \in S$ and $x \in A^*$. We can see that, since the definition of $\varphi^*$ is compatible with the generating set of $\sim$, it is compatible with the equivalence relation of $\sim$ on $A^*$. Additionally, we can see that $\varphi^*(xy) = \varphi^*(x)\,\varphi^*(y)$ for all $x, y \in A^*$. Thus,

$$\begin{aligned}
\overline{\varphi} : F(S) &\to G\\
[x] &\mapsto [\varphi^*(x)]\,,
\end{aligned}$$

is, as constructed, a group homomorphism, with $\overline{\varphi} \circ \iota = \varphi$. Since $\iota(S)$ generates $F(S)$, this group homomorphism is unique.

We must now show that $\iota$ is injective.

Let $s_1, s_2 \in S$. Consider the map $\varphi : S \to \mathbb{Z}$ given by $\varphi(s_1) = 1$ and $\varphi(s_2) = -1$. The corresponding homomorphism $\overline{\varphi} : F(S) \to G$ satisfies

$$\begin{aligned}
\overline{\varphi}\,(\iota\,(s_1)) &= \varphi\,(s_1)\\
&= 1\\
&\neq -1\\
&= \varphi\,(s_2)\\
&= \overline{\varphi}\,(\iota\,(s_2))\,,
\end{aligned}$$

meaning $\iota(s_1) \neq \iota(s_2)$. Thus, $\iota$ is injective. □

## Free Groups, Free Products, and the Ping Pong Lemma

We can now consider free groups in a more categorical context — specifically, as a special type of free object. Whereas the previous section drew information from Clara Löh's *Geometric Group Theory: An Introduction*, this section will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory*. Specifically, we are focused on chapter 2, which discusses free products, free groups, and the ping pong lemma (which will provide a more general proof of the paradoxicality of SO(3)).

**Definition** (Free Monoid). A monoid is a set with multiplication that is associative and includes a neutral element.

Given a set $A$, the free monoid on $A$ is the set $W(A)$ of finite sequences of elements of $A$ (also known as words). We write an element of $W(A)$ as $w = a_1 a_2 \cdots a_n$, where each $a_j \in A$. We identify $A$ with the subset of $W(A)$ of words with length 1.

**Definition** (Free Product)**.** Let $(\Gamma_i)_{i\in I}$ be a family of groups. Set

$$\begin{aligned} A &= \coprod_{i\in I} \Gamma_i \\ &= \{(g_i, i) \mid g_i \in \Gamma_i, i \in I\} \end{aligned}$$

to be the coproduct of this family.

Let $\sim$ be the equivalence relation generated by

$$\begin{aligned} w e_i w' &\sim w w' && \text{where } e_i \in \Gamma_i \text{ is the neutral element} \\ w a b w' &\sim w c w' && \text{where } a, b, c \in \Gamma_i,\ c = ab \text{ for some } i \in I \end{aligned}$$

for all $w, w' \in W(A)$. The quotient $W(A)/\sim$ with the operation of concatenation is a group, which is known as the free product of the groups $\{\Gamma_i\}_{i\in I}$. We write it as

$$\bigstar_{i\in I}\Gamma_i.$$

The inverse of the equivalence class for $w = a_1 a_2 \dots a_n$ is $w^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$. The neutral element is $\epsilon$, which is the empty word.

A word $w = a_1 a_2 \cdots a_n \in W(A)$ with $a_j \in \Gamma_{i_j}$ is said to be reduced if $i_{j+1} \neq i_j$ and $a_j$ is not the neutral element of $\Gamma_{i_j}$.

**Proposition** (Existence of the Free Product)**:** Let $\{\Gamma_i\}_{i\in I}$ be a family of groups, $A = \coprod_{i\in I} \Gamma_i$, and $\bigstar_{i\in I}\Gamma_i = W(A)/\sim$ be as above.

Then, any element in the free product $\bigstar_{i\in I}\Gamma_i$ is represented by a unique reduced word in $W(A)$.

*Proof.*

Existence: Consider an integer $n \geqslant 0$ and a reduced word $w = a_1 a_2 \cdots a_n$ in $W(A)$, an element $a \in A$, and the word $aw \in W(A)$. We set

$$\mathcal{R}(aw) = \begin{cases} w & a = e_i \\ a a_1 a_2 \cdots a_n & a \in \Gamma_i, a \neq e_i, i \neq k \\ b a_2 \cdots a_n & a \in \Gamma_k, a a_1 = b \neq e_k \\ a_2 \cdots a_n & a \in \Gamma_k, a_1 = a^{-1} \in \Gamma_k \end{cases},$$

where $k$ is the index for which $a_1 \in \Gamma_k$.

Then, $\mathcal{R}(aw)$ is yet another reduced word, and $\mathcal{R}(aw) \sim aw$, meaning that any word $w \in W(A)$ is equivalent to some reduced word by inducting on the length of $w$.

Uniqueness: For each $a \in A$, Let $T(a) = \mathcal{R}(aw)$ be a self-map on the set of reduced words.

For each $w = b_1 b_2 \cdots b_n$, we set $T(w) = T(b_1) T(b_2) \cdots T(b_n)$. For $a, b, c \in \Gamma_i$ with $ab = c$, we have $T(a) T(b) = T(c)$, and $T(e_i) = \epsilon$ (the empty word) for all $i \in I$.

For each reduced word, notice that $T(w)\epsilon = w$.

Let $w$ be some word in $W(A)$ with $w_1, w_2$ reduced words equivalent to $w$. Since $w_1 \sim w_2$, we have $T(w_1) = T(w_2)$, and

$$\begin{aligned} w_1 &= T(w_1)\,\epsilon \\ &= T(w_2)\,\epsilon \\ &= w_2. \end{aligned}$$

□

**Corollary:** Let $\{\Gamma_i\}_{i\in I}$ and $\Gamma = \bigstar_{i\in I}\Gamma_i$ as above. For each $i_0 \in I$, the canonical inclusion

$$\iota : \Gamma_{i_0} \to \Gamma$$

is injective.

*Proof.* For any $a \in \Gamma_{i_0} \setminus \{e_{i_0}\}$, $\iota(a)$ is represented by a unique one-letter reduced word not equivalent to the empty word. □

Now that we have an understanding of free products, we can conceptualize the free group as a particular type of free product.

**Definition** (Free Groups)**.** Let $X$ be a set. The free group over $X$ is the free product of a family of copies of $\mathbb{Z}$ indexed by $X$, denoted $F(X)$.

Equivalently, the free group over $X$ is

$$F(X) = \bigstar_{a\in X} \langle a\rangle\,,$$

where $\langle a\rangle$ denotes the cyclic group generated by the element $a$.

We can also identify $F(X)$ with the set of reduced words in $X \sqcup X^{-1}$ (as was done in the previous subsection).

The cardinality of $X$ is called the rank of $F(X)$.

If $\Gamma$ is a group, then a free subset of $\Gamma$ is a subset $X \subseteq \Gamma$ such that the inclusion $X \hookrightarrow F(X)$ extends to an isomorphism of $\langle X\rangle_\Gamma$ onto $F(X)$.

We can now state and prove a universal property for free products (which naturally simplifies in the case of a free group.)

**Theorem** (Universal Property for Free Products)**:** Let $\Gamma$ be a group, and $\{\Gamma_i\}_{i\in I}$ be a family of groups. Let $\{h_i : \Gamma_i \to \Gamma\}_{i\in I}$ be a family of homomorphisms.

Then, there exists a unique homomorphism $h : \bigstar_{i\in I}\Gamma_i \to \Gamma$ such that the following diagram commutes for each $i_0 \in I$.

$$\begin{tikzcd} \Gamma_{i_0} \arrow[r, "h_{i_0}"] \arrow[d, hook, "\iota"'] & \Gamma \\ \bigstar_{i\in I}\Gamma_i \arrow[ur, "h"'] & \end{tikzcd}$$

In particular, if $\Gamma$ is a group, $X$ is a set, and $\phi : X \to \Gamma$ is a set map, there exists a unique homomorphism $\Phi : F(X) \to \Gamma$ such that $\Phi(x) = \phi(x)$ for each $x \in X$.

*Proof.* For a reduced word $w = a_1a_2\cdots a_n \in \bigstar_{i\in I}\Gamma_i$ with $a_j \in \Gamma_{i_j} \setminus \{e_{i_j}\}$, and $i_{j+1} \neq i_j$ for each $j \in \{1,\dots,n-1\}$, we set

$$h(w) = h_{i_1}(a_1)\,h_{i_2}(a_2)\cdots h_{i_n}(a_n),$$

which defines $h$ uniquely in terms of $h_i$. □

Note that for any two sets $X, Y$, the universal property provides that any map $X \to Y$ extends canonically to a group homomorphism, $F(X) \to F(Y)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

We can now prove an important lemma that will be useful in understanding paradoxical groups.

**Theorem** (Ping Pong Lemma)**:** Let $G$ be a group acting on a set $X$, and let $\Gamma_1, \Gamma_2$ be subgroups of $G$. Let $\Gamma = \langle \Gamma_1, \Gamma_2\rangle$. Assume $\Gamma_1$ contains at least 3 elements and $\Gamma_2$ contains at least two elements.

Suppose there exist nonempty subsets $X_1, X_2 \subseteq X$ with $X_1 \triangle X_2 \neq \emptyset$, such that for all $\gamma_1 \in \Gamma_1$ with $\gamma_1 \neq e_G$, and for all $\gamma_2 \in \Gamma_2$ with $\gamma_2 \neq e_G$,

$$\begin{aligned} \gamma(X_2) &\subseteq X_1 \\ \gamma(X_1) &\subseteq X_2. \end{aligned}$$

Then, $\Gamma$ is isomorphic to the free product $\Gamma_1 \star \Gamma_2$.

*Proof.* Let $w$ be a nonempty reduced word spelled with letters from the disjoint union of $\Gamma_1 \setminus \{e_G\}$ and $\Gamma_2 \setminus \{e_G\}$. We must show that the element of $\Gamma$ defined by $w$ is not the identity.

If $w = a_1b_1a_2b_2\cdots a_k$ with $a_1,\dots,a_k \in \Gamma_1 \setminus \{e_G\}$ and $b_1,\dots,b_{k-1} \in \Gamma_2 \setminus \{e_G\}$. Then,

$$\begin{aligned} w(X_2) &= a_1b_1\cdots a_{k-1}b_{k-1}a_k(X_2) \\ &\subseteq a_1b_1\cdots a_{k-1}b_{k-1}(X_1) \\ &\subseteq a_1b_1\cdots a_{k-1}(X_2) \\ &\vdots \\ &\subseteq a_1(X_2) \\ &\subseteq X_1. \end{aligned}$$

Since $X_2 \not\subseteq X_1$, this implies $w \neq e_G$.

If $w = b_1a_2b_2a_2\cdots b_k$, we select $a \in \Gamma_1 \setminus \{e_G\}$, and apply the previous argument to $awa^{-1}$. Since $awa^{-1} \neq e_G$, neither is $w$.

Similarly, if $w = a_1b_1\cdots a_kb_k$, we select $a \in \Gamma_1 \setminus \{e_G, a_1^{-1}\}$, and apply the argument to $awa^{-1}$, and if $w = b_1a_2b_2\cdots a_k$, we select $a \in \Gamma_1 \setminus \{e_G, a_k\}$, and apply the argument to $awa^{-1}$. □

**Example.** We can use the Ping Pong Lemma to see that

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

generate a subgroup of $SL(2,\mathbb{Z})$ which is free of rank 2.

**Corollary:** The special orthogonal group $SO(3)$ contains a subgroup isomorphic to the free group on two generators.

To prove this, we state a different version of the Ping Pong Lemma that we will apply to a particular space.

**Theorem** (Ping Pong Lemma for Cyclic Groups)**:** Let $G$ act on a set $X$, and suppose there exist disjoint subsets $A_+, A_-, B_+, B_- \subseteq X$ whose union is not all of $X$. If there exist elements $a$ and $b$ in $G$ such that

$$\begin{aligned} a \cdot (X \setminus A_-) &\subseteq A_+ \\ a^{-1} \cdot (X \setminus A_+) &\subseteq A_- \\ b \cdot (X \setminus B_-) &\subseteq B_+ \\ b \cdot (X \setminus B_+) &\subseteq B_-, \end{aligned}$$

then it is the case that the group generated by $a$ and $b$ is free of rank 2.

*Proof of Corollary.* We let

$$\begin{aligned} a &= \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a^{-1} &= \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ b &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \\ b^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}. \end{aligned}$$

We specify

$$X = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} A_+ &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| k \in \mathbb{Z}, x \equiv 3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ A_- &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| k \in \mathbb{Z}, x \equiv -3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ B_+ &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| k \in \mathbb{Z}, z \equiv 3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\} \\ B_- &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| k \in \mathbb{Z}, z \equiv -3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}. \end{aligned}$$

To verify that the conditions of the Ping Pong Lemma hold, we calculate

$$\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left( \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 3x + 4y \\ -4x + 3y \\ 5z \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left( \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 3x - 4y \\ 4x + 3y \\ 5z \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \left( \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 5x \\ 3y - 4z \\ 4y + 3z \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix} \left( \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 5x \\ 3y + 4z \\ -4y + 3z \end{pmatrix}. \tag{4}$$

We verify that the conditions for the Ping Pong Lemma hold for each of these four conditions.

(1) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_-,$$

we see that $k + 1 \in \mathbb{Z}$, $x' = 3x + 4y \equiv 3(-4x + 3y)$ modulo 5, and that $z' = 5z \equiv 0$ modulo 5.

(2) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_+,$$

we see that $k + 1 \in \mathbb{Z}$, $x' = 3x - 4y \equiv -3(4x + 3y)$ modulo 5, and $z' = 5z \equiv 0$ modulo 5.

(3) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_-,$$

we see that $k + 1 \in \mathbb{Z}$, $z' = 4y + 3z \equiv 3(3y - 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

(4) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_+,$$

we see that $k + 1 \in \mathbb{Z}$, $z' = -4y + 3z \equiv -3(3y + 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

Since we have verified that the conditions for the Ping Pong Lemma hold for each of the conditions, we have that $\{a, b\} \subseteq SO(3)$ generate a group isomorphic to the free group on two generators. □

## States and Means on $\ell_\infty(G)$

**Definition.** Let G be a group.

(1) The space $\mathcal{F}(G, \mathbb{R})$ is defined by

$$\mathcal{F}(G, \mathbb{R}) = \{ f \mid f : G \to \mathbb{R} \text{ is a function} \}.$$

(2) A function $f \in \mathcal{F}(G, \mathbb{R})$ is positive if $f(x) \geqslant 0$ for all $x \in G$.

(3) A function $f \in \mathcal{F}(G, \mathbb{R})$ is simple if $\operatorname{Ran}(f)$ is finite. We say

$$\Sigma = \left\{ f : \mathcal{F}(G, \mathbb{R}) \;\middle|\; f \text{ is simple} \right\}.$$

**Fact.** $\Sigma \subseteq \mathcal{F}(G, \mathbb{R})$ is a subspace. To see this, if $f, g$ are such that $\operatorname{Ran}(f), \operatorname{Ran}(g)$ are finite, and $\alpha \in \mathbb{R}$, then

$$\operatorname{Ran}(f + \alpha g) \leqslant \operatorname{Ran}(f) + \operatorname{Ran}(g),$$

so $f + \alpha g$ has finite range.

**Definition.** For $E \subseteq G$, set

$$\mathbb{1}_E : G \to \mathbb{R}$$

defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

This is the characteristic function of $E$.

**Fact.**

$$\operatorname{span}\{\mathbb{1}_E \mid E \subseteq G\} = \Sigma.$$

*Proof.* We see that $\mathbb{1}_E \in \Sigma$ for any $E \subseteq G$, and $\Sigma$ is a subspace.

If $\phi \in \Sigma$, with $\operatorname{Ran}(\phi) = \{t_1, \dots, t_n\}$ with $t_i$ distinct, we set

$$E_i = \phi^{-1}(\{t_i\}),$$

meaning

$$\phi = \sum_{i=1}^{n} t_i \mathbb{1}_{E_i}.$$

□

**Definition.**

(1) A function $f \in \mathcal{F}(G, \mathbb{R})$ is bounded if there exists $M > 0$ such that $\operatorname{Ran}(f) \subseteq [-M, M]$.

(2) The space $\ell_\infty(G)$ is defined by

$$\ell_\infty(G) = \{f \in \mathcal{F}(G, \mathbb{R}) \mid f \text{ is bounded}\}.$$

(3) The norm on $\ell_\infty(G)$ is defined by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

**Proposition:** The space $\ell_\infty(G)$ is complete, Additionally, $\overline{\Sigma} = \ell_\infty(G)$.

*Proof.* Let $(f_n)_n$ be Cauchy. For $x \in G$, it is the case that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n - f_m)(x)| \\ &\leqslant \|f_n - f_m\|, \end{aligned}$$

meaning $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We must show that $f \in \ell_\infty(G)$ and $\|f_n - f\| \to 0$.

$$\begin{aligned} |f(x)| &= \left|\lim_{n\to\infty} f_n(x)\right| \\ &= \lim_{n\to\infty} |f_n(x)| \\ &\leqslant \limsup_{n\to\infty} \|f_n\| \\ &\leqslant C, \end{aligned}$$

as Cauchy sequences are always bounded. Thus, $\sup_{x\in G} |f(x)| \leqslant C$.

Given $\varepsilon > 0$, we find $N$ such that for all $m, n \geqslant N$, $\|f_n - f_m\| \leqslant \varepsilon$. Thus, for $x \in G$, we have

$$\begin{aligned} |f_n(x) - f)m(x)| &\leqslant \|f_n - f_m\| \\ &\leqslant \varepsilon. \end{aligned}$$

Taking $m \to \infty$, we get $|f_n(x) - f(x)| \leqslant \varepsilon$ for all $n \geqslant N$, meaning $\|f_n - f\| \leqslant \varepsilon$ for all $n \geqslant N$.

Now, for $f \in \ell_\infty(G)$, let $\operatorname{Ran}(f) \subseteq [-M, M]$ for some $M > 0$. Let $\varepsilon > 0$. Since $[-M, M]$ is compact, it is totally bounded, so we can find intervals $I_1, \dots, I_n$ with $[-M, M] = \bigsqcup_{k=1}^{n} I_k$, with the length of each $I_k$ less than $\varepsilon$.

Set $E_k = f^{-1}(I_k)$. Pick $t_k \in I_k$. Then, we set

$$\phi = \sum_{i=1}^{n} t_k \mathbb{1}_{E_k}.$$

We see that $\|\phi - f\| < \varepsilon$. □

**Corollary:** For any $f \in \ell_\infty(G)$, there is a sequence $(\phi_n)_n$ in $\Sigma$ with $\|\phi_n - f\| \to 0$. If $f \geqslant 0$, then it is possible to select $\phi_n \geqslant 0$.

**Proposition:** Let $G$ be a group. There is an action

$$G \xrightarrow{\lambda_s} \operatorname{Isom}(\ell_\infty(G))$$

defined by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right).$$

*Proof.* We have

$$\begin{aligned} \lambda_s(f + \alpha g)(t) &= (f + \alpha g)\left(s^{-1}t\right) \\ &= f\left(s^{-1}t\right) + \alpha g\left(s^{-1}t\right) \\ &= \lambda_s(f)(t) + \alpha\lambda_s(g)(t) \\ &= (\lambda_s(f) + \alpha\lambda_s(g))(t). \end{aligned}$$

Thus, $\lambda_s$ is a linear operator.

We have

$$\|\lambda_s(f)\| = \sup_{t\in G} |\lambda_s(f)(t)|$$

$$= \sup_{t \in G} \left| f\left(s^{-1}t\right) \right|$$
$$= \|f\|,$$

hence

$$\begin{aligned} \|\lambda_s(f) - \lambda_s(f)\| &= \|\lambda_s(f-g)\| \\ &= \|f-g\|. \end{aligned}$$

Thus, $\lambda_s$ is an isometry.

We have

$$\begin{aligned} \lambda_s \circ \lambda_r(f)(t) &= \lambda_r(f)\left(s^{-1}t\right) \\ &= f\left(r^{-1}s^{-1}t\right) \\ &= f\left((sr)^{-1}t\right) \\ &= \lambda_{sr}(f)(t), \end{aligned}$$

meaning $\lambda_s \circ \lambda_r = \lambda_{sr}$. □

**Remark:** By a similar process, we find that $\lambda_s(\mathbb{1}_E) = \mathbb{1}_{sE}$ for any subset $E \subseteq G$ and $s \in G$.

**Definition.** A state on $\ell_\infty(G)$ is a continuous linear functional $\mu \in (\ell_\infty(G))^*$ that satisfies the following.

(1) $\mu$ is positive;

(2) $\mu(\mathbb{1}_G) = 1$.

A state is called left-invariant if

$$\mu(\lambda_s(f)) = \mu(f).$$

**Example.** Let $G$ be a group.

- If $x \in G$, then $\delta_x : \ell_\infty(G) \to \mathbb{F}$ defined by

  $$\delta_x(f) = f(x)$$

  is a state. However, note that it is not necessarily invariant.

  $$\begin{aligned} \delta_x(\lambda_s(f)) &= \lambda_s(f)(x) \\ &= f\left(s^{-1}x\right) \\ &\neq f(x). \end{aligned}$$

- If $G$ is finite, then

  $$\mu = \frac{1}{|G|} \sum_{x \in G} \delta_x$$

  is an invariant state.

**Lemma** (Characterization of States)**:**

(1) If $\mu$ is a state on $\ell_\infty(G)$, then

$$\|\mu\|_{op} = 1.$$

(2) If $\mu \in (\ell_{\infty}(G))^{*}$ is such that

$$\begin{aligned} \|\mu\| &= \mu(\mathbb{1}_G) \\ &= 1, \end{aligned}$$

then $\mu$ is positive and a state.

*Proof.*

(1) Given $f \in \ell_{\infty}(G)$, we have

$$\begin{aligned} \|f\| \mathbb{1}_G - f &\geqslant 0 \\ \|f\| \mathbb{1}_G + f &\geqslant 0, \end{aligned}$$

so

$$\begin{aligned} 0 &\leqslant \mu(\|f\| \mathbb{1}_G - f) \\ &= \|f\| \mu(\mathbb{1}_G) - \mu(f) \\ 0 &\leqslant \mu(\|f\| \mathbb{1}_G + f) \\ &= \|f\| \mu(\mathbb{1}_G) + \mu(f). \end{aligned}$$

Thus, we have $\pm\mu(f) \leqslant \|f\| \mu(\mathbb{1}_G) = \|f\|$, so $|\mu(f)| \leqslant \|f\|$, so $\|\mu\| \leqslant 1$. Additionally, since $\mu(\mathbb{1}_G) = 1$, we must have $\|\mu\| = 1$.

(2) Suppose $\|\mu\| = \mu(\mathbb{1}_G) = 1$. Let $f \geqslant 0$. Set $g = \frac{1}{\|f\|_u} f$.

Then, $\operatorname{Ran}(g) \subseteq [0,1]$, and $\operatorname{Ran}(g - \mathbb{1}_G) \subseteq [-1,1]$, so $\|g - \mathbb{1}_G\|_u \leqslant 1$.

Since $\|\mu\| = 1$, we must have

$$\begin{aligned} |\mu(g - \mathbb{1}_G)| &\leqslant 1 \\ |\mu(g) - 1| &\leqslant 1, \end{aligned}$$

and since $\mu(\mathbb{1}_G) = 1$, we must have $\mu(g) \in [0,2]$, so $\mu(f) = \|f\| \mu(g) \geqslant 0$.

□

**Corollary:** The set of states on $(\ell_{\infty}(G))^{*}$ forms a $w^{*}$-compact subset of $B_{(\ell_{\infty}(G))^{*}}$.

*Proof.* It has been proven in functional analysis that a convex subset of $(\ell_{\infty}(G))^{*}$ is $w^{*}$-compact if it is norm bounded and $w^{*}$-closed. Since the set of states is convex and norm-bounded, all we need to show is that $S(\ell_{\infty}(G))$ is $w^{*}$-closed.

To this end, let $f \in \ell_{\infty}(G)$ be positive and $(\varphi_i)_i$ be a net in $S(\ell_{\infty}(G))$ with $(\varphi_i)_i \to \varphi$. We must show that $\varphi$ is positive and satisfies $\varphi(\mathbb{1}_G) = 1$. To this end, we see that

$$\varphi_i(f) \geqslant 1$$

for all $i \in I$, so we must necessarily have $\varphi(f) \geqslant 0$, and similarly, since $\varphi_i(\mathbb{1}_G) = 1$ for each $i \in I$, we also have $\varphi(\mathbb{1}_G) = 1$. □

**Proposition:** If $\mu \in (\ell_{\infty}(G))^{*}$ is a state, then $m : P(G) \to [0,1]$ defined by $m(E) = \mu(\mathbb{1}_E)$ is a finitely additive probability measure on G. Moreover, if $\mu$ is invariant, then $m$ is a mean.

*Proof.* We have

$$\begin{aligned} m(G) &= \mu\left(\mathbb{1}_G\right) \\ &= 1 \\ m\left(\varnothing\right) &= \mu\left(0\right) \\ &= 0 \\ m\left(E \sqcup F\right) &= \mu\left(\mathbb{1}_{E \sqcup F}\right) \\ &= \mu\left(\mathbb{1}_E + \mathbb{1}_F\right) \\ &= \mu\left(\mathbb{1}_E\right) + \mu\left(\mathbb{1}_F\right) \\ &= m\left(E\right) + m\left(F\right). \end{aligned}$$

Additionally, since $0 \leqslant \mathbb{1}_E \leqslant \mathbb{1}_G$, we have $0 \leqslant \mu\left(\mathbb{1}_E\right) \leqslant 1$, so $0 \leqslant m(E) \leqslant m(G) = 1$.

If $\mu$ is invariant, then

$$\begin{aligned} m\left(sE\right) &= \mu\left(\mathbb{1}_{sE}\right) \\ &= \mu\left(\lambda_s\left(\mathbb{1}_E\right)\right) \\ &= \mu\left(\mathbb{1}_E\right) \\ &= m\left(E\right). \end{aligned}$$

□

**Proposition:** If $G$ admits a mean, then $\left(\ell_\infty\left(G\right)\right)^*$ admits an invariant state.

*Proof.* Let $m$ be a finitely-additive probability measure. Define

$$\mu_0 : \Sigma \rightarrow \mathbb{R}$$

by

$$\mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right) = \sum_{k=1}^{n} t_k m\left(E_k\right).$$

Since $m$ is finitely additive, it is the case that $\mu_0$ is well-defined, linear, and positive.

Note that $\mu_0\left(\mathbb{1}_G\right) = m(G) = 1$.

If $m$ is a mean, then for $f = \sum_{k=1}^{n} t_k E_k$,

$$\begin{aligned} \mu_0\left(\lambda_s\left(f\right)\right) &= \mu_0\left(\lambda_s\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right)\right) \\ &= \mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{sE_k}\right) \\ &= \sum_{k=1}^{n} t_k m\left(sE_k\right) \\ &= \sum_{k=1}^{n} t_k m\left(E_k\right) \\ &= \mu_0\left(f\right). \end{aligned}$$

Additionally, we see that

$$\begin{aligned}|\mu_0(f)| &= \left|\sum_{k=1}^{n} t_k m(E_k)\right|\\ &\leqslant \sum_{k=1}^{n} |t_k|\, m(E_k)\\ &\leqslant \sum_{k=1}^{n} \|f\|\, m(E_k)\\ &= \|f\| \sum_{k=1}^{n} m(E_k)\\ &\leqslant \|f\|.\end{aligned}$$

Thus, $\mu_0$ is continuous, so $\mu_0$ is uniformly continuous.

Since $\overline{\Sigma} = \ell_\infty(G)$, we see that $\mu_0$ extends to a continuous linear functional $\mu : \ell_\infty(G) \to \mathbb{R}$, with $\mu(\mathbb{1}_G) = \mu_0(\mathbb{1}_G) = 1$.

If $f \geqslant 0$, we find a sequence $(\phi_n)_n$ in $\Sigma$ with $\phi_n \geqslant 0$, $\|\phi_n - f\| \xrightarrow{n\to\infty} 0$, and we set

$$\begin{aligned}\mu(f) &= \lim_{n\to\infty} \mu(\phi_n)\\ &= \lim_{n\to\infty} \mu_0(\phi_n)\\ &\geqslant 0,\end{aligned}$$

meaning $\mu$ is a state.

If $f \in \ell_\infty(G)$, $s \in G$, and $(\phi_n)_n$ in $\Sigma$ with $(\phi_n)_n \to f$, then

$$\begin{aligned}\|\lambda_s(\phi_n) - \lambda_s(f)\| &= \|\lambda_s(\phi_n - f)\|\\ &= \|\phi_n - f\|\\ &\to 0\end{aligned}$$

Thus, we have

$$\begin{aligned}\mu(\lambda_s(\phi_n)) &= \mu_0(\lambda_s(\phi_n))\\ &= \mu_0(\phi_n)\\ &= \mu(\phi_n)\\ &\to \mu(f),\end{aligned}$$

so $\mu(f) = \mu(\lambda_s(f))$. Thus, $\mu \in (\ell_\infty(G))^*$ is an invariant state. □

## Using Invariant States

**Proposition:** $\mathbb{Z}$ is amenable.

*Proof.* We know that $\lambda_1 : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$, defined by

$$\lambda_1(f)(k) = f(k-1)$$

is an isometry.

We set $Y = \operatorname{Ran}(\operatorname{id} - \lambda_1) \subseteq \ell_\infty(\mathbb{Z})$.

We claim that $\operatorname{dist}_Y(\mathbb{1}_\mathbb{Z}) \geqslant 1$.

Suppose toward contradiction that there is $y \in Y$ with $\|\mathbb{1}_\mathbb{Z} - y\|_u = \rho < 1$. Then, $y = f - \lambda_1(f)$ for some $f \in \ell_\infty(\mathbb{Z})$, meaning

$$\|\mathbb{1} - (f - \lambda_1(f))\| = \rho.$$

Thus, for all $k \in \mathbb{Z}$, we have

$$|1 - (f(k) - f(k-1))| \leqslant \rho,$$

meaning $|f(k) - f(k-1)| \geqslant 1 - \rho > 0$. However, such an $f$ cannot be bounded.

Since $\operatorname{dist}_{\overline{Y}}(\mathbb{1}_Z) = \operatorname{dist}_Y(\mathbb{1}_Z) \geqslant 1$, the Hahn–Banach theorem provides $\mu \in (\ell_\infty(\mathbb{Z}))^*$ with $\|\mu\| = 1$, $\mu|_{\overline{Y}} = 0$, and $\mu(\mathbb{1}) = \operatorname{dist}_Y(\mathbb{1}_\mathbb{Z}) \geqslant 1$.

Since $\|\mu\| = 1$ and $\mu(\mathbb{1}) \geqslant 1$, we must have $\mu(\mathbb{1}) = 1$.

Since $\|\mu\| = \mu(\mathbb{1}_\mathbb{Z}) = 1$, it is the case that $\mu$ is a state on $\ell_\infty(\mathbb{Z})$. Since $\mu(y) = 0$ for all $y \in Y$, we have

$$\begin{aligned} \mu(f - \lambda_1(f)) = 0 \\ \mu(f) = \mu(\lambda_1(f)), \end{aligned}$$

so inductively, we have $\mu(f) = \mu(\lambda_k(f))$ for all $k \in \mathbb{Z}$, meaning $\mu$ is an invariant state on $\ell_\infty(\mathbb{Z})$. Thus, $\mathbb{Z}$ is amenable. $\square$

**Proposition:** If $N \trianglelefteq G$ and $G/N$ are amenable, then $G$ is amenable.