Problem 1

Find $\sup(A)$ and $\inf(A)$ where

(a) $A := \left\{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$

(b) $A := \left\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbb{N}\right\}$

(c) $A := \left\{\frac{m}{n} \mid m, n \in \mathbb{N},\ m + n \leq 10\right\}$

(a)

$\sup(A) = 2$: For any $t \in A,\ t < 2$, we can find a_t as follows:

$$a_t := \begin{cases} 1,\ t < 1 \\ \frac{4}{3},\ 1 \leq t < \frac{4}{3} \\ 2,\ t = \frac{4}{3} \end{cases}$$

$\inf(A) = \frac{1}{2}$: For any $t \in A$, $t > \frac{1}{2}$, we can find a_t as follows:

$$a_t := \begin{cases} 1,\ t > 1 \\ \frac{3}{4},\ \frac{3}{4} < t \leq 1 \\ \frac{1}{2},\ t < \frac{3}{4} \end{cases}$$

(b)

$\sup(A) = 1$: For any $t \in A$, $t < 1$, we can find $a_t > t$ as follows:

(1) Take $|t| \geq t$.

(2) If $|t| < \frac{1}{2}$, find m such that $\frac{1}{m} < |t|$ (which exists by the Archimedean Property corollary). Set $a_t = 1 - \frac{1}{m}$.

(3) If $|t| > \frac{1}{2}$, then find m such that $\frac{1}{m} < 1 - |t|$, and set $a_t = 1 - \frac{1}{m}$.

In all three cases, $a_t > t$, meaning $\sup(A) = 1$

$\inf(A) = -1$

(c)

Since A is finite, $\sup(A) = \max(A) = 9$ and $\inf(A) = \min(A) = \frac{1}{9}$

Problem 2

Suppose $u = \sup(A)$ such that $u \notin A$. Show that there is a strictly increasing sequence

$$t_1 < t_2 < t_3 < \dots$$

With $t_n \in A$ and $t_n + \frac{1}{n} > u$ for all $n \geq 1$

Let $t_n = u - \frac{1}{2n}$. t_n must be a strictly increasing sequence because $t_{n+1} = u - \frac{1}{2n+2} > u - \frac{1}{2n} = t_n$.

Additionally, $t_n + \frac{1}{n} = u - \frac{1}{n} < u$, meaning $t_n \in A$.

Problem 3

If m is a lower bound for $A \subseteq \mathbb{R}$, show that the following are equivalent:

(i) $m = \inf(A)$

(ii) $\forall t > m,\ \exists a_t \in A \ni a_t < t$

(iii) $\forall \varepsilon > 0,\ \exists a_\varepsilon \ni m + \varepsilon > a_\varepsilon$

Problem 4

Let $A, B \in \mathbb{R}$ be bounded subsets.

(a) Show that

$$\sup(A + B) = \sup(A) + \sup(B)$$
$$\inf(A + B) = \inf(A) + \inf(B)$$

(b) If $t > 0$, show that

$$\sup(tA) = t\sup(A)$$
$$\inf(tA) = t\inf(A)$$