Math 395: Homework 5
Due: October 22, 2024
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Problem 16

Problem: Let

$$A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

(a) Find the characteristic polynomial of A.

(b) Compute E_λ^j for all eigenvalues λ of A and all j.

(c) Give the Jordan canonical form of A and the Jordan basis $\mathcal{B}$ of $\mathbb{F}^4$.

Solution:

(a) Using computational assistance, we find

$$\begin{aligned} c_A(x) &= \det(A - xI_4) \\ &= x^4 - 12x^3 + 52x^2 - 96x + 64 \\ &= (x-4)^2 (x-2)^2. \end{aligned}$$

(b) The eigenvalues of A are 2 and 4. Thus, we calculate

$$A - 2I_4 = \begin{pmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{pmatrix}.$$

In reduced row echelon form (with computational assistance), we get

$$\simeq \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $A - 2I_4$ is of rank 2, so $\dim(\ker(A - 2I_4)) = 2$. Thus, the vectors $v_1 = 2e_1 + e_2 + e_3 + 3e_4$ and $v_2 = 2e_1 + 3e_2 + 3e_3 + 5e_4$ form a basis for the kernel. Additionally, $E_2^\infty = E_2^1$, since the degree on $(x-2)$ is 2.

Now, we turn our attention to $A - 4I_4$. We have

$$A - 4I_4 = \begin{pmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{pmatrix},$$

which row-reduces to

$$\simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $A - 4I_4$ is of rank 3, so $\dim(\ker(A - 4I_4)) = 1$. The vector $v_3 = 2e_1 + 3e_2 + 3e_3 + 3e_4$ is a basis for E_4^1. Then, we have

$$(A - 4I_4)^2 = \begin{pmatrix} 4 & 8 & -4 & -4 \\ 4 & 4 & 0 & -4 \\ 4 & 0 & 4 & -4 \\ 4 & 8 & -4 & -4 \end{pmatrix},$$

which row-reduces to

$$\simeq \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $(A - 4I_4)^2$ is of rank 2, meaning $\dim\left(\ker\left((A - 4I_4)^2\right)\right) = 2$, The vector $v_4 = 4e_1 + 4e_3 - 4e_4$, along with v_3, forms a basis for E_4^2.

(c) Thus, via finding the generalized eigenspaces E_2^1 and E_4^2, we get the Jordan basis of

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \\ -4 \end{pmatrix} \right\},$$

with the Jordan canonical form of

$$[T_A]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$