Chapter 5 Theorems

Definitions: A **walk** in a graph is a list of vertices such that any two consecutive vertices are adjacent to each other.

A **trail** is a walk that does not repeat edges, but can repeat vertices.

A **path** is a trail that does not repeat vertices.

A **closed walk** is a walk that ends at the same vertex that it started at. A walk is open if it is not closed. A **circuit** is a closed trail, and a **cycle** is a closed path.

An **Eulerian circuit** is a circuit that traverses all the edges of a graph. A graph is Eulerian if it contains an Eulerian circuit. An **Eulerian trail** traverses all the edges of a graph, and does not return to the same vertex it started from.

Theorem 5.1: A connected graph G is Eulerian if and only if every vertex of G has even degree.

Corollary 5.2: A connected graph G contains an (open) Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, every Eulerian trail of G begins at one of these odd vertices and ends at the other.

Chapter 6 Theorems

Definitions: A **Hamiltonian cycle** is a cycle that contains all the vertices of a graph. A graph is Hamiltonian if it contains a Hamiltonian cycle.

Theorem 6.2 (Dirac's Theorem): If G is a graph of order $n \geq 3$ such that $d(v) \geq n/2$ for all vertices of G, then G is Hamiltonian.

Theorem 6.3 (Ore's Theorem): If G is a graph of order $n \geq 3$ such that $d(u) + d(v) \geq n$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

Theorem 6.5: For any graph G, if there is a positive integer k such that deleting k vertices results in a graph with more than k components, then G is not Hamiltonian.

Chapter 7 Theorems

Definitions: A **matching** is a set of pairwise disjoint edges. A **perfect matching** is a matching that is incident on every vertex.

The subgraph whose edges are a perfect matching is a **1-factor**.

If G contains 1-factors $F_1, F_2, \ldots, F_k$ such that $E(G)$ is partitioned by $E(F_1), E(F_2), \ldots, E(F_k)$, then $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is a **1-factorization** of G, and G is **1-factorable**.

A **bridge** is an edge upon whose deletion the number of components in a connected graph increases.

If a graph can be decomposed into edge-disjoint Hamiltonian cycles, then the graph is **Hamiltonian-factorable**.

Theorem 7.1 (Hall's Theorem): A sequence $(C_1, \ldots, C_n)$ of n nonempty finite sets has a system of distinct representatives $(s_1, \ldots, s_i)$ where $s_i \in C_i$ if and only if for each subsequence Y, the union of the sets in Y has at least as many elements as Y.

Alternatively, if G is a bipartite graph on vertices $C \sqcup S$, where $C = \{c_1, \ldots, c_n\}$ and $S = \{s_1, \ldots, s_m\}$, then G has a C-perfect matching (a matching that contains every vertex in C) if and only if $\forall r$ where $1 \leq r \leq n$, any r vertices in C are adjacent to at least r vertices in S.

Theorem 7.7 (Petersen's Theorem): Every bridgeless 3-regular graph contains a perfect matching.

Theorem 7.10: For every even integer $n \geq 2$, K_n is 1-factorable.

Theorem 7.13: For every odd integer $n \geq 3$, K_n is Hamiltonian-factorable.

Chapter 8 Theorems

Definitions: For an integer $n \geq 3$, a system S_n of n symbols is a **Steiner triple system** if for every pair of symbols, such a pair only exists in one triple. This is comparable to decomposing K_n into copies of K_3.

Theorem 8.1: If S_n is a Steiner triple system for $n \geq 3$, then $n = 6q + 1$ or $n = 6q + 3$ for some $q \in \mathbb{Z}$.

Theorem 8.4: Every Eulerian graph has a cycle decomposition.

Nate Hall's Theorem: All four-regular graphs of order 7 are nonplanar.

Chapter 9 Theorems

Definition: A graph orientation is **strong** if for each distinct $u, v \in V(G)$, there is a path from u to v and a path from v to u.

Robbins's Theorem: A graph G has a strong orientation if and only if G is bridgeless and connected.

Theorem 9.2: Every strongly oriented complete graph contains a directed Hamiltonian path.

Chapter 10 Theorems

Definitions: A graph embedding is **planar** if it can be drawn in the plan without any edge crossings. A graph **subdivision** occurs when one "inserts" a vertex within an edge. A graph **minor** is constructed by deleting edges, vertices, or by contracting edges.

A set of graphs S is **minor-closed** if for any H such that H is a minor of $G \in S$, $H \in S$. A graph G is **minor-minimal** in S if no minor of G is an element of S. If S is minor closed, there is a finite set of **forbidden minors** that define S.

Euler Identity: If G is planar and embedded in the plane, then $V - E + F = 2$.

Theorem 10.2: If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n - 6$.

Kuratowski's Theorem: A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

Wagner's Theorem: A graph G is planar if and only if K_5 and $K_{3,3}$ are not minors of the graph.

Graph Minor Theorem: For any infinite set of graphs, there exists a graph that is a minor of another graph.

Corollary to the Graph Minor Theorem: For any minor-closed set of graphs S, there is a finite set M of forbidden minors for S.