

Problem (Problem 1): Let G be a Lie group, which is a topological group that is also a smooth manifold and where all group operations are smooth. For convenience, we will always assume that G is path-connected. Prove that the tangent bundle TG of G is trivial — i.e., TG composes as a direct product.

Solution: From Cayley's Theorem, we know that G acts on itself transitively by left-multiplication. That is, for any $g \in G$, there is a map $L_g: G \rightarrow G$ that takes $h \mapsto gh$. This is a diffeomorphism of smooth manifolds since L_g is smooth and admits the smooth inverse $L_{g^{-1}}$. In particular, this means that

$$D_e(L_g): T_e G \rightarrow T_g G$$

is invertible as a linear map. Letting $T_e G \cong \mathbb{R}^n$ have a local basis $\mathcal{B}_e = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$, we then observe that $D_e(L_g)$ then maps this basis to a basis for $T_g G$ since $D_e(L_g)$ is an isomorphism, meaning that

$$\begin{aligned} TG &= \bigsqcup_{g \in G} T_g G \\ &= \bigsqcup_{g \in G} D_e(L_g)(T_e G) \\ &\cong \bigsqcup_{g \in G} \mathbb{R}^n \\ &\cong G \times \mathbb{R}^n. \end{aligned}$$

Thus, TG is trivial.

Problem (Problem 2): Note that a Lie group can act on itself by left or right multiplication. A vector field on G is called *left-invariant* if it is invariant under the differential of left multiplication L_g for every $g \in G$. Prove that $T_e G$ can be identified with left invariant vector fields on G .