Problem 1

Let $\mathbb{F}$ be a field. Show that the following hold:

(i) $-1(a) = -a$

(ii) $-(-a) = a$

(iii) $-(a+b) = (-a) + (-b)$

(iv) $(-a)^{-1} = -(a^{-1})$

(v) $(ab)^{-1} = a^{-1}b^{-1}$

(i)

$$\begin{aligned}
0 &= (1 + (-1)) \\
0(a) &= (1 + (-1))a \\
0 &= 1(a) + (-1)(a) \\
0 &= a + (-1)(a) \\
-a &= (-1)(a)
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &= -(-a) + (-a) \\
a &= -(-a) + ((-a) + a) \\
a &= -(-a)
\end{aligned}$$

(iii)

$$\begin{aligned}
0 &= -(a+b) + (a+b) \\
-b &= -(a+b) + a + (b-b) \\
-a + (-b) &= -(a+b) + (a-a) \\
(-a) + (-b) &= -(a+b)
\end{aligned}$$

(iv)

$$\begin{aligned}
1 &= (-a)^{-1}(-a) \\
-1 &= (-a)^{-1}(a) \\
-1(a^{-1}) &= (-a)^{-1} \\
-(a^{-1}) &= (-a)^{-1}
\end{aligned}$$

(v)

$$\begin{aligned} 1 &= (ab)^{-1}(ab) \\ b^{-1} &= (ab)^{-1}(a) \\ a^{-1}b^{-1} &= (ab)^{-1} \end{aligned}$$

Problem 2

Consider the set

$$K := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

Show that:

(i) $x, y \in K \Rightarrow x + y \in K \hat{x} y \in K$

(ii) $x \neq 0 \Rightarrow x^{-1} \in K$

(i)

Let $x, y \in K$. Then, $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$, where $a, b, c, d \in \mathbb{Q}$.

$x + y = (a + c) + (b + d)\sqrt{2} \in K$, as $\mathbb{Q}$ is closed under addition.

$xy = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}$, as $\mathbb{Q}$ is closed under multiplication.

(ii)

Let $x = a + b\sqrt{2} \neq 0 \in K$. Thus, at least one of $a, b \neq 0$.

$$\begin{aligned} x^{-1} &= \frac{1}{a + b\sqrt{2}} \\ &= \frac{a - b\sqrt{2}}{a^2 - 2b^2} \\ &= \frac{a}{a^2 - 2b^2} + \frac{-b\sqrt{2}}{a^2 - 2b^2} \end{aligned}$$

Since $a/(a^2 - 2b^2)$ and $(-b)/(a^2 - 2b^2)$ are both in $\mathbb{Q}$, $x^{-1} \in K$.