Problem (Problem 1)**:** Use de Rham cohomology to prove that if B^n is a closed ball in $\mathbb{R}^n$, and $f\colon B^n \to B^n$ is smooth, then f has a fixed point.

Solution: Suppose $f\colon B^n \to B^n$ is a fixed-point free self-map of the ball. It follows then that by drawing a line between v and $f(v)$, we may define a smooth retraction of the ball to the sphere S^{n-1}. Call this retraction r.

We observe then that r induces a map in cohomology $r^*\colon H^*_{DR}(S^{n-1}) \to H^*_{DR}(B^n)$. In particular, since r is a retraction to S^{n-1}, it follows that r is homotopic to the identity map when restricted to S^{n-1}, meaning r^* is an isomorphism in de Rham cohomology of $H^*_{DR}(S^{n-1})$ and $H^*_{DR}(B^n)$.

Yet, we recognize that $H^{n-1}_{DR}(S^{n-1}) \cong \mathbb{R}$, while $H^{n-1}_{DR}(B^n) \cong 0$, the latter emerging from the fact that B^n is contractible via the straight-line homotopy and the Poincaré lemma. Thus, no such r exists, whence f cannot have a fixed point.

Problem (Problem 2)**:** Suppose M is a compact smooth manifold with a smooth triangulation, and let $f\colon M \to M$ be a smooth map preserving the triangulation. Write f^*_k for the induced map on $H^k_{DR}(M)$. Prove that if

$$\begin{aligned} L(f) &= \sum_{k=0}^{n} (-1)^k \operatorname{tr}\left(f^*_k\right) \\ &\neq 0, \end{aligned}$$

then f has a fixed point.

Solution: By abuse of notation, we treat $f^*\colon H^*(M;\mathbb{R}) \to H^*(M;\mathbb{R})$ to be the corresponding map on the simplicial cohomology rather than the de Rham cohomology, which follows from de Rham's theorem and the isomorphism between singular and simplicial cohomology.

Suppose f has no fixed points. Let $\Delta \subseteq M$ be a simplex. Then, by the definition of f, we observe that $f(\Delta) \subseteq M$ is also a simplex, which we call Λ. Suppose toward contradiction that $\Lambda = \Delta$. Then, restricting the map f to Δ, we observe that $f\colon \Delta \to \Delta$ is a smooth self-map of the k-simplex Δ. Yet, since $\Delta \cong B^n$ are diffeomorphic (when considering a small neighborhood of Δ), this implies that we have a smooth self-map on Δ, whence f has a fixed point by the result of Problem (1).

From the de Rham isomorphism and the fact that M is triangulated, an arbitrary cochain on M, I_ω, can be defined by

$$I_\omega(\Delta) = \int_\Delta \omega,$$

which induces the isomorphism $H^*_{DR}(M) \cong H^*(M;\mathbb{R})$. We observe that f^* yields a map on cochains by taking

$$\begin{aligned} f^*(I_\omega)(\sigma) &= \int_\sigma f^*\omega \\ &= I_{f^*\omega}(\sigma) \end{aligned}$$

for a k-simplex σ.

We seek to show that for an arbitrary cochain I_ω, that

$$\sum_{k=0}^{n} (-1)^k \operatorname{tr}\left(f^*_k I_\omega\right) = 0$$

Problem (Problem 3)**:** Compute the de Rham cohomology of $\mathbb{RP}^n$.

Solution: To start, we observe that $\mathbb{RP}^1 \cong S^1$, meaning that the de Rham cohomology of $\mathbb{RP}^1$ is

$$H^*_{DR}(\mathbb{RP}^1) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = 1 \\ 0 & \text{else} \end{cases}.$$

In higher dimensions, we consider the family of charts defined by

$$U_k = \{[x_0 : \cdots : x_k : \cdots : x_n] \mid x_{i\neq k} \in \mathbb{R}, x_k \neq 0\}.$$

We seek to understand the picture of

$$\begin{aligned} U_{k\neq 0} &= \bigcup_{k=1}^{n} U_k \\ &= \bigcup_{k=1}^{n} \{[x_0 : \cdots : n] \mid x_k \neq 0\}. \end{aligned}$$

In particular, the only elements of U_0 that are not in $U_{k\neq 0}$ are the ones of the form $[1 : 0 : \cdots : 0]$, whence $U_{k\neq 0} \cong \mathbb{R}^n \setminus \{0\}$.

Next, we observe that

$$\begin{aligned} U_0 \cap U_{k\neq 0} &= \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\} \cap \bigcup_{k=1}^{n} \{[x_0 : \cdots : x_n] \mid x_k \neq 0\} \\ &= \{[x_0 : \cdots : x_n] \mid x_0 \neq 0, x_k \neq 0 \text{ for at least one } 1 \leqslant k \leqslant n\} \\ &= U_0 \setminus \{[1 : 0 : \cdots : 0]\} \\ &\cong \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

Thus, by Mayer–Vietoris, we obtain the following short exact sequence.

$$0 \longrightarrow H^*(\mathbb{RP}^n) \longrightarrow H^*(\mathbb{R}^n) \oplus H^*(\mathbb{R}^n \setminus \{0\}) \longrightarrow H^*(\mathbb{R}^n \setminus \{0\}) \longrightarrow 0$$

Focusing on the case of H^0, this yields the following exact sequence, whence $H^0(\mathbb{RP}^n) \cong \mathbb{R}$.

$$0 \longrightarrow H^0(\mathbb{RP}^n) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \cdots$$

Since the $H^k(\mathbb{R}^n)$ are zero for all $k \geqslant 1$, it follows that we have $H^k(\mathbb{RP}^n) \cong 0$ for $1 \leqslant k < n$.

Finally, concerning ourselves with $H^n(\mathbb{RP}^n)$, we concern ourselves with orientability; specifically, $H^n(\mathbb{RP}^n) \cong \mathbb{R}$ if n is odd and $H^n(\mathbb{RP}^n) \cong 0$ if n is even, as $\mathbb{RP}^n$ is orientable if and only if n is odd.

Problem (Problem 4)**:** Prove the Five Lemma. Namely, consider the following commutative diagram of vector spaces, where the horizontal sequences are exact. Show that if f_1, f_2, f_4, f_5 are isomorphisms, that f_3 is also an isomorphism.

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5 \end{tikzcd}$$

Solution: We start by showing that f_3 is injective. Let $x \in \ker(f_3)$.

- By commutativity, we have

$$\begin{aligned} 0 &= \beta_3 \circ f_3(x) \\ &= f_4 \circ \alpha_3(x), \end{aligned}$$

so it follows that $\alpha_3(x) = 0$ as f_4 is injective, so $x \in \ker(\alpha_3)$. By exactness, we let $a_2 \in A_2$ be such that $\alpha_2(a_2) = x$, and define $f_2(a_2) = b_2$.

- By commutativity,

$$\begin{aligned} \beta_2(b_2) &= \beta_2(f_2(a_2)) \\ &= f_3(\alpha_2(a_2)) \\ &= f_3(x) \\ &= 0, \end{aligned}$$

so $b_2 \in \ker(\beta_2)$, meaning that by exactness, there is $b_1 \in B_1$ such that $\beta_1(B_1) = b_2$. Since f_1 is surjective, we let $a_1 \in A_1$ be such that $f_1(a_1) = b_1$.

- Finally, by commutativity, we have

$$\begin{aligned} f_2(\alpha_1(a_1)) &= \beta_2(f_1(a_1)) \\ &= \beta_1(b_1) \\ &= b_2 \\ &= f_2(a_2), \end{aligned}$$

and since f_2 is injective, we have $a_2 = \alpha_1(a_1)$.

- Thus, since $x = \alpha_2(a_2)$, we have

$$\begin{aligned} x &= \alpha_2(\alpha_1(a_1)) \\ &= (\alpha_2 \circ \alpha_1)(a_1) \\ &= 0, \end{aligned}$$

so f is injective.

Now, we show that f is surjective.