The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of x).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

Some Functions and Sets

$$f(x,y) = x^2 - y^2$$

Domain: $\{(x,y) \mid \exists f(x,y)\}$

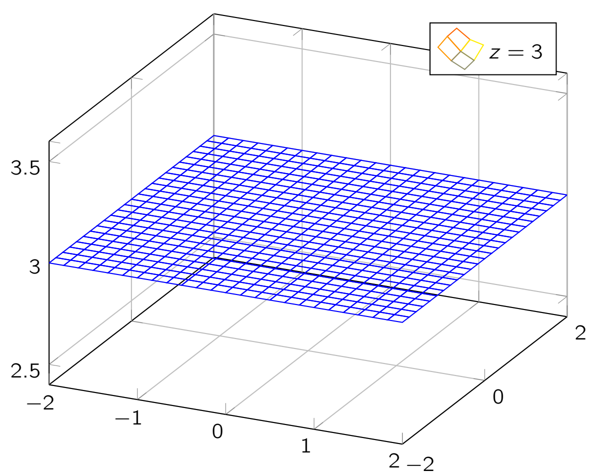
Range: $\{f(x,y) \mid (x,y) \in \mathrm{Dom}(f)\} = \mathbb{R}$

Graph: $\mathrm{Graph}(f) = \{x, y.f(x,y) \mid x, y \in \mathrm{Dom}(f)\}$. For example, $(1,3,4) \notin \mathrm{Graph}(f)$ since $1^2 - 3^2 \neq 4$.

Examples

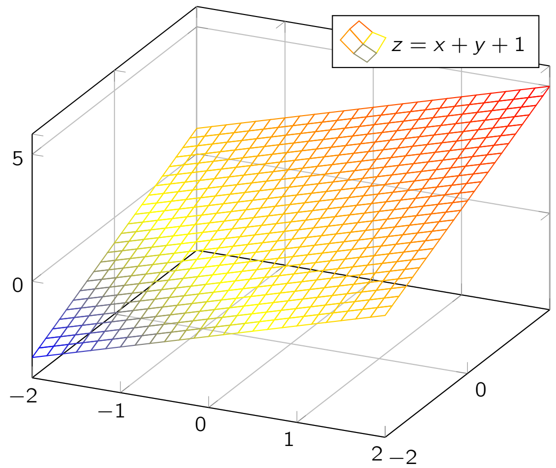
In $\mathbb{R}^3$, in x, y, z coordinates, $z = 3$ is a plane defined as follows:

- Parallel to the xy plane.
- Passes through the point $(0.0, 3)$.



Meanwhile, $y = 0$ would be a "wall" that passes through the origin that contains the line $y = 0$ in the xy plane.

Finally, $z = x + y + 1$ is a plane, as we can see below.



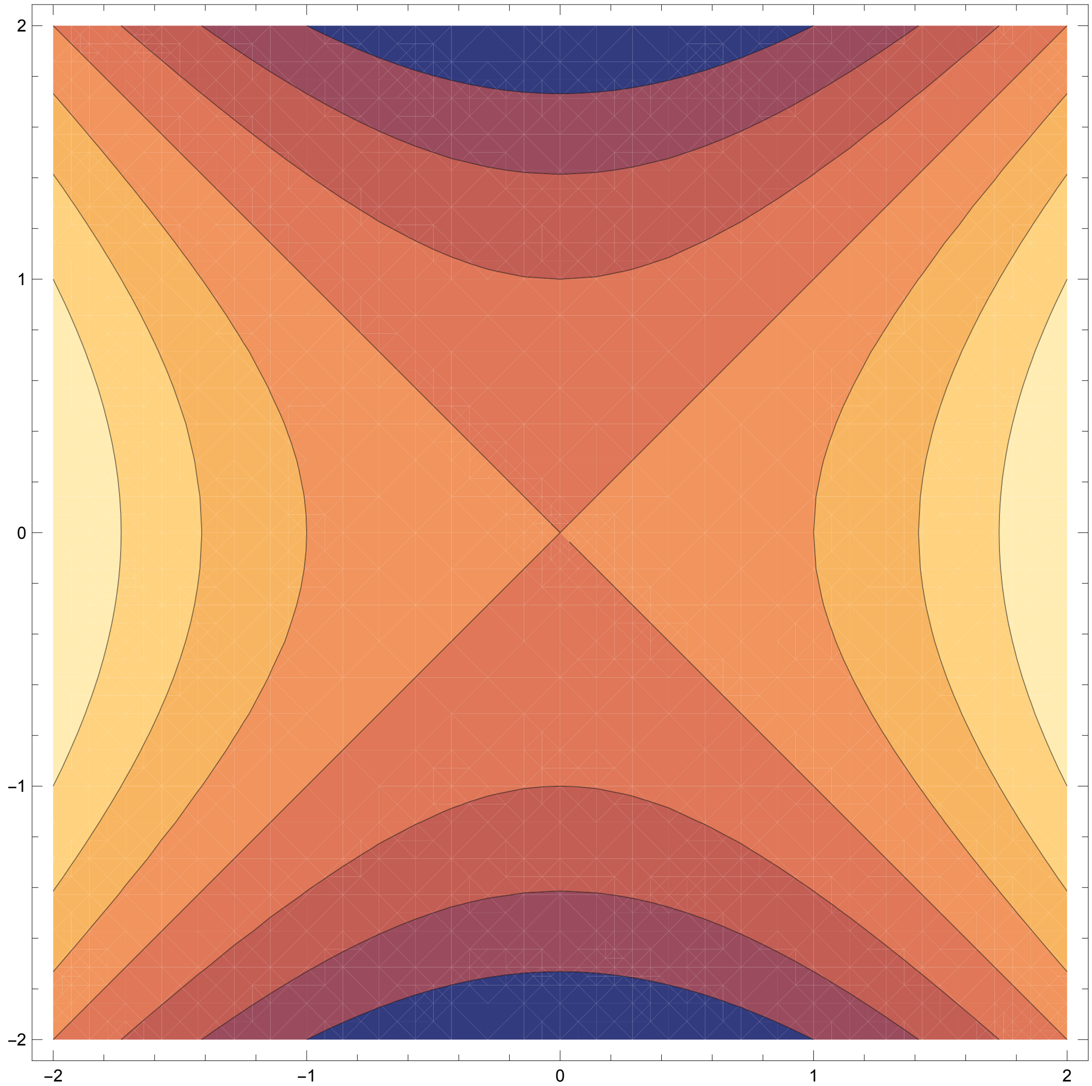
Visualizing a function of multiple variables

Consider the function $f(x, y) = x^2 - y^2$. We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$
- $f(0, y) = -y^2$
- $f(2, y) = 4 - y^2$
- $f(x, -2) = x^2 + 4$
- $f(x, 0) = x^2$
- $f(x, 2) = x^2 + 4$

Alternatively, we can visualize via contour diagrams (i.e., everywhere that z is a certain value), as seen in mathematica as follows:

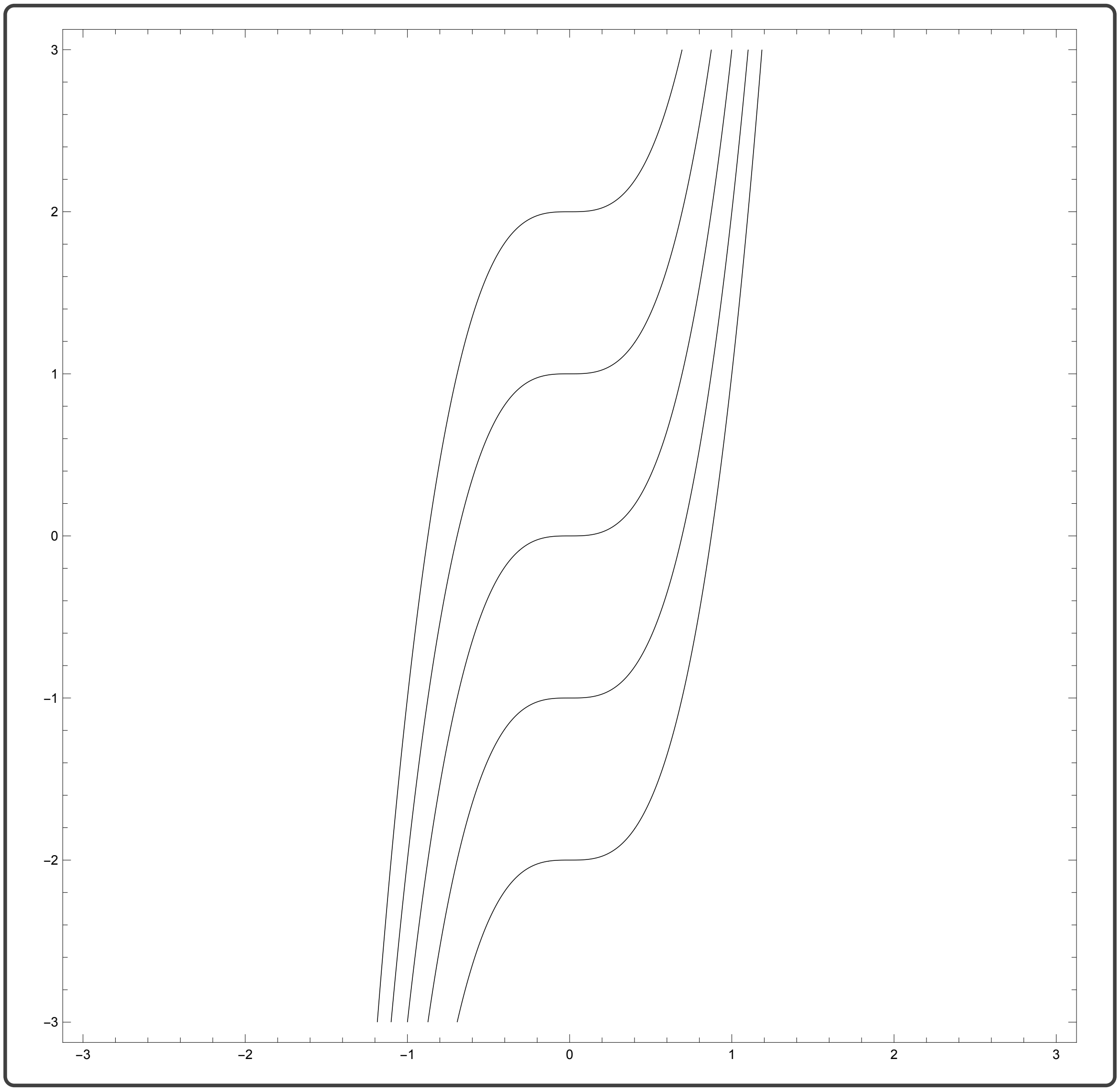
Contour Diagram



Contour Example

Consider the function $f(x, y) = y - 3x^2$. We want to find the contours.

For any c, we have that $c = y - 3x^3$, or $y = 3x^3 + c$. Therefore, every contour "looks like" $3x^3 + c$ for values of c. For example, in the following, we have $c = \{-2, -1, 0, 1, 2\}$

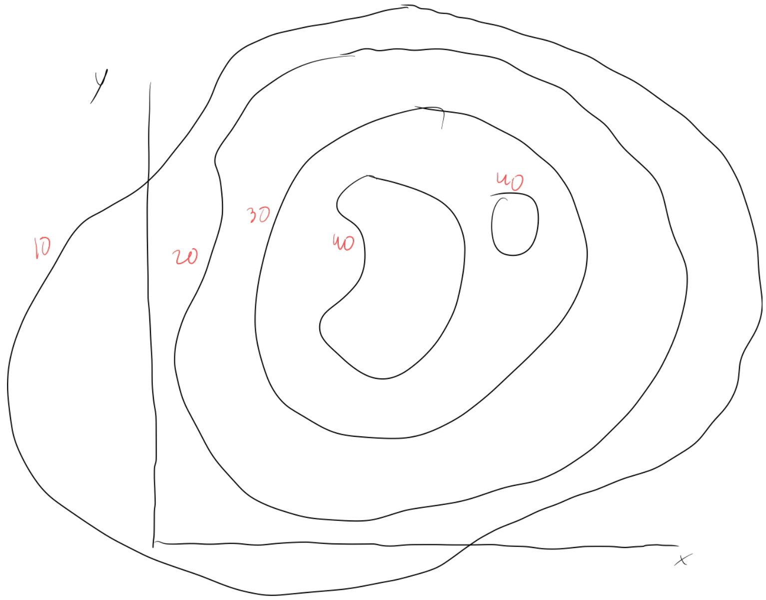


Distance

In $\mathbb{R}^5$, let $p = (3,1,4,1,5)$, and $q = (1,0,-2,0,2)$. Using the Euclidean metric, we can find the distance between p and q is $d(p,q) = ((3-1)^2 + (1-0)^2 + (4-(-2))^2 + (1-0)^2 + (5-2)^2)^{1/2} = (4+1+36+1+9)^{1/2} = \sqrt{51} = 7.14$. We can also call this the 2-norm.

$$d(p,q) = \left(\sum_{k=1}^{n} (p_k - q_k)^2\right)^{1/2}$$

Derivatives



To denote a derivative, we can't talk about one value, we must use a *partial* derivative, $\frac{\partial f}{\partial x}$, or $\frac{\partial f}{\partial y}$. The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

x\y	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A "linear" approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and is an output in $z = f(x, y)$, and $m, n \in \mathbb{R}$.

For example, with the above table, we can see that the function is linear in x and y (i.e., the slope holding the other variable constant is constant).

Limits in Multivariable Functions

Consider the following:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow $y = mx$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2} &= \lim_{(x,y)\to(0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2} \\ &= \frac{1 + m^2}{1 - m^2}\end{aligned}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of m

m	$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x^2-y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours "approach" a particular point, we can see that the limit is defined.

Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$. These vectors are equivalent because they are components of $\mathbb{R}^3$.

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

Properties of Vectors

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $c\vec{u} = \langle cu_1, cu_2, \dots, cu_k \rangle$

Additionally, we define $\vec{u} \cdot \vec{v}$ as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

Partial Derivatives

Consider $f(x, y) = x^2 y + xe^y$.

$$f_x := \frac{\partial f}{\partial x}$$
$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

We know that $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, meaning f is endlessly differentiable.

Functions and Approximations

Let $f(x, y) = x^2 - y^2$, $g(x, y) = 2xy$

- $f_{xx} + f_{yy} = 0$
- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For $f(x, y)$ at $(a, b, f(a, b))$, we have the following:

$$\begin{aligned}
\ell(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(y - b) \\
q(x, y) &= \ell(x, y) + \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right)
\end{aligned}$$

In order to get a sense of the "derivative," we can use the following:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

Directional Derivative and Gradient

Given $f(x, y)$ and (a, b), where $f \in C^2(\mathbb{R}^2)$. Then, the quadratic approximation is:

$$\begin{aligned}
f(x, y) &\approx f(a, b) + f_x(a, b)(x - a) + f_x(a, b)(y - b) \\
&\quad + \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b)\right) \\
df &= f_x(a, b)dx + f)y(a, b)dy && \text{a differential} \\
\Delta f &= f_x(a, b)\Delta x + f_y(a, b)\Delta y
\end{aligned}$$

Evaluating $f(x, y) = xe^y$ at $(a, b) = (-1, 0)$

$$\begin{aligned}
f_x &= e^y \\
f_y &= xe^y \\
f_x(-1, 0) &= 1 \\
f_y(-1, 0) &= -1 \\
\Delta f &= \Delta x - \Delta u
\end{aligned}$$

On a given contour map, let $\vec{u} = \langle u_1, u_2\rangle$ denote a *unit* vector in a direction that we want to find the derivative of f in.

$$f_{\vec{u}}(x, y) = \nabla f(a, b) \cdot \vec{u}$$

Where

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

The directional derivative for all vectors $\vec{v}$ is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

Chain Rule

Let $f(x, y)$ be a function where $x - x(t)$ and $y = y(t)$. We want to find

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The chain rule works in higher dimensions too. Consider $k(x_1(t), x_2(t), \dots, x_{152}(t))$. Then,

$$\frac{dk}{dt} = \sum_{i=1}^{152} \frac{\partial k}{\partial x_i}\frac{dx_i}{dt}$$

We can also view this as a vector. Let $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{152}(t) \end{pmatrix}$. Then, we can write $\frac{dk}{dt}$ more succinctly as follows:

$$\frac{dk}{dt} = \nabla k \cdot \frac{d\vec{x}}{dt}$$

For example, let $f(x, y, z) = 3x^2y + zx + 2$, where $x = x(t), y = y(t), z = z(t)$

$$\begin{aligned} \frac{df}{dt} &= \begin{pmatrix} 6xy + z \\ 3x^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} \\ &= (6xy + z)\, x'(t) + 3x^2y'(t) + xz'(t) \end{aligned}$$

So, if we let $x(t) = \sin(t)$, $y(t) = e^t$, and $z(t) = t^2 + 1$. Then, we have

$$\frac{df}{dt} = 6\sin(t)\cos(t)e^t + t^2\cos(t) + \cos(t) + 3e^t\sin^2(t) + 2t\sin(t)$$

Alternatively, consider $f(x, y, z) = x^2 + yz + e^y$, where $x(s, t) = st, y = y(s, t) = t + s^2, z = z(s, t) = e^t$. Let

$$\vec{x} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix}$$

Then, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial t} \\ \frac{\partial f}{\partial s} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial s} \end{aligned}$$

Evaluating the first expression, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix} \cdot \begin{pmatrix} s \\ 1 \\ e^t \end{pmatrix} \\ &= 2s^2t + 3^t + e^{t+s^2} + (t + s^2)e^t \end{aligned}$$

Consider $f(x, y(x))$. Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

This is the technique we use to find implicit differentiation.

We know as a result that $\nabla f(a, b)$ is orthogonal to the contour curve at (a, b)

Recap

In $\mathbb{R}^3$, find the plane that contains $P = (P_1, P_2, P_3)$, Q, and R. We can find it by the following:

$$\begin{aligned} 0 &= \vec{n} \cdot \begin{pmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{pmatrix} \\ 0 &= n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3) \end{aligned}$$

where

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

Differentiability

A function $f(x)$ of one variable is differentiable at $x = a$ if

$$f(a) = \lim_{h \to 0} f(a + h)$$

and

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

exists and is bounded

We can also linearize the function. f is differentiable if

$$f(x) = f(a) + f'(a)(x-a) + E(x)$$

where $\lim_{h\to 0} \frac{E(a+h)}{h} = 0$.

In the multiple dimensions example, we have $f(x, y)$ is differentiable if

$$f(x, y) = \ell(x, y) + E(x, y)$$

where $\lim_{h\to 0, k\to 0} \frac{E(a+h, b+k)}{\sqrt{h^2+k^2}} = 0$

Local Maxima

Let $f(x, y) = x^2 + 2y^2$. We want to find (a, b) which are local maxima,minima, or other.

(a, b) is a local maximum if $f(a, b) \geq f(x, y)\ \forall (x, y) \in V_\varepsilon(a, b)$, where $\varepsilon > 0$.

(1) Find Critical Points for $f(x, y) : f_x(x, y), f_y(x, y) = 0,\ f_x(x, y), f_y(x, y)$ are undefined.

$$\begin{aligned} f_x(x, y) &= 2x \\ f_y(x, y) &= 4y \\ f_x(0, 0) &= 0 \\ f_y(0, 0) &= 0 \\ f(0, 0) &= 0 \\ f(x, y) &> 0 && \forall (x, y) \neq (0, 0) \end{aligned}$$

For all x, y, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 0$. Finally,

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) \cdot f_{yy}(x, y) + f_{xy}(x, y)^2 \\ &= 8 \\ &> 0 \end{aligned}$$

Since $D(x, y) > 0$, we look at the sign of f_{xx}. Since it is positive, $f(0, 0)$ has a local minimum.

Local Maxima and Minima Approach

Given $f(x, y)$, we want

(1) Find critical points:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned}$$

(2) Compute $f_{xx}, f_{yy}, f_{xy}, D = f_x x f_y y - (f_x y)^2$

(3)

f_{xx}	D	Critical Point
+	+	Local Minimum
-	+	Local Maximum
$\pm$	-	Saddle Point
$\pm$	0	Nothing

Consider the function

$$\begin{aligned} f(x,y) &= \ln(x^2+y^2+1) \\ f(0,0) &= 0 \\ f(x,y) &> 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2x}{x^2+y^2+1} \\ \frac{\partial f}{\partial y} &= \frac{2y}{x^2+y^2+1} \end{aligned}$$

Critical Points: $(0,0)$

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} &= \frac{2(x^2+y^2+1)-4x^2}{\left(x^2+y^2+1\right)^2} \\ &= 2 \\ \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} &= \frac{2(x^2+y^2+1)-4y^2}{\left(x^2+y^2+1\right)^2} \\ &= 2 \\ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} &= \frac{-4xy}{\left(x^2+y^2+1\right)^2} \\ &= 0 \end{aligned}$$

Now, consider the function

$$\begin{aligned} f(x,y) &= x^2-2xy+y^2 \\ \frac{\partial f}{\partial x} &= 2x-2y \\ \frac{\partial f}{\partial y} &= -2x+2y \\ \frac{\partial^2 f}{\partial x^2} &= 2 \\ \frac{\partial^2 f}{\partial y^2} &= 2 \\ \frac{\partial^2 f}{\partial x \partial y} &= -2 \\ D &= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= 0 \end{aligned}$$

Therefore, the critical points of this function are indeterminate with the given approach. However, we know that $f(x,y) = (x-y)^2 = 0$ when $x = y$, so the line $y = x$ is a local minimum trough in 3-space.

Now, consider the function

$$\begin{aligned} f(x,y) &= (x-1)^2(y+2) \\ \frac{\partial f}{\partial x} &= 2(x-1)(y+2) \\ \frac{\partial f}{\partial y} &= (x-1)^2 \end{aligned}$$

Critical points: $\{(1,y) \mid y \in \mathbb{R}\}$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2(y+2) \\ \frac{\partial^2 f}{\partial y^2} &= 0 \\ \frac{\partial^2 f}{\partial x \partial y} &= 2(x-1) \\ D &= 0 - \left(2(x-1)\right)^2 \\ &= 0 && \text{Evaluating } D \text{ at critical points} \end{aligned}$$

Finding Critical Points

Let $f(x,y) = (y^2+2)\sin(x)$. on $[-2,2]\times[-2,2]$

$$\begin{aligned}\frac{\partial f}{\partial x} &= (y^2+2)\cos(x)\\ &= 0\\ \frac{\partial f}{\partial y} &= 2y\sin(x)\\ &= 0\\ (x,y) &= \left(\frac{(2n+1)\pi}{2}, 0\right)\\ &= \{(\pi/2,0),(-\pi/2,0)\}\\ \frac{\partial^2 f}{\partial x^2} &= -(y^2+2)\sin(x)\\ \frac{\partial^2 f}{\partial y^2} &= 2\sin(x)\\ \frac{\partial^2 f}{\partial x\partial y} &= 2y\cos(x)\\ D(x,y) &= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\\ &= -2(y^2+2)\sin^2(x) - 4y^2\cos^2(x)\\ &< 0\end{aligned}$$

Therefore, the critical points are saddle points. If there is no domain restriction, we have a series of saddle points all along $y=0$.

Why Finding Critical Points Works

We create the Taylor series of $f(x,y)$ at (x_0,y_0):

$$\begin{aligned}f(x,y) &\approx \ell(x_0,y_0) + \frac{1}{2}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2\right)\\ &= \underbrace{f(x_0,y_0) + \nabla f(x,y)\cdot\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix}}_{=0 \text{ at critical points}} + \frac{1}{2}\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix}^T \underbrace{\begin{vmatrix}f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0)\\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0)\end{vmatrix}}_{\text{Hessian}}\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix}\end{aligned}$$

If the Hessian is positive definite, then $\lambda_1,\lambda_2>0$ and the critical point is a local min. If the Hessian is negative definite, then $\lambda_1,\lambda_2<0$ and the critical point is a local max.

In any given 2×2 matrix, the eigenvalues λ_1,λ_2 are such that $\lambda_1+\lambda_2 = \mathrm{Tr}(A)$ and $\lambda_1\lambda_2 = \mathrm{Det}(A)$.

Optimization

Let $f(x,y) = 2x-y$. We want to optimize f with respect to $g(x,y) = x^2-y^2-4=0$.

Define $L(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c)$. Given $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}$, then $L:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$.

Then, we take

$$\begin{aligned}\nabla L &= \nabla f = \lambda\nabla g\\ &= 0 && \text{critical points of } L\end{aligned}$$

We find x, y, λ for each critical point.

$$\begin{aligned}
\nabla f &= \begin{pmatrix} 2 \\ -3 \end{pmatrix}\\
\nabla g &= \begin{pmatrix} 2x \\ -2y \end{pmatrix}\\
\nabla f &= \lambda \nabla g\\
2 &= 2\lambda x\\
-3 &= -2\lambda y\\
x^2 - y^2 &= 4\\
\\
\lambda &= \frac{1}{x}\\
\lambda &= \frac{3}{2y}\\
x &= \frac{2y}{3}\\
\frac{4y^2}{9} - y^2 &= 4\\
-\frac{5}{9}y^2 &= 4
\end{aligned}$$

No Solution

However, if $g(x, y) = x^2 + y^2 - 4 = 0$, we have

$$\begin{aligned}
\nabla f &= \begin{pmatrix} 2 \\ -3 \end{pmatrix}\\
\nabla g &= \begin{pmatrix} 2x \\ 2y \end{pmatrix}\\
\nabla f &= \lambda \nabla g\\
2 &= 2\lambda x\\
-3 &= 2\lambda y\\
x^2 + y^2 &= 4\\
\\
\lambda &= \frac{1}{x}\\
\lambda &= \frac{-3}{2y}\\
x &= \frac{-2y}{3}\\
\frac{4y^2}{9} + y^2 &= 4\\
-\frac{13}{9}y^2 &= 4\\
y &= \pm\frac{6}{\sqrt{13}}\\
x &= \mp\frac{4}{\sqrt{13}}\\
f_{\max} &= 2\sqrt{13}\\
f_{\min} &= -2\sqrt{13}
\end{aligned}$$

This system of Lagrange multipliers applies in the n dimensional case.

Let $f(x,y,z) = x + 2y + z^2$ subject to the constraint $g(x,y,z) = x^2 + y^2 + z^2 = 1$.

$$\begin{aligned}
\nabla f &= \lambda \nabla g\\
\begin{pmatrix} 1\\ 2\\ 2z \end{pmatrix} &= \lambda \begin{pmatrix} 2x\\ 2y\\ 2z \end{pmatrix}\\
2\lambda x &= 1\\
2\lambda y &= 2\\
2\lambda z &= 2z \tag{*}\\
x^2 + y^2 + z^2 &= 1
\end{aligned}$$

Consider (*):

$$\begin{aligned}
\lambda &= 1\\
x &= 1/2\\
y &= 1\\
\frac{1}{4} + 1 + z^2 &= 1
\end{aligned}$$

no solution

$$\begin{aligned}
z &= 0\\
x^2 + y^2 &= 1\\
\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} &= 1\\
\frac{5}{4\lambda^2} &= 1\\
\lambda &= \pm\frac{\sqrt{5}}{2}
\end{aligned}$$

Case 1:

$$\begin{aligned}
\lambda &= \frac{\sqrt{5}}{2}\\
x &= \frac{1}{\sqrt{5}}\\
y &= \frac{2}{\sqrt{5}}
\end{aligned}$$

Case 2:

$$\begin{aligned}
\lambda &= -\frac{\sqrt{5}}{2}\\
x &= -\frac{1}{\sqrt{5}}\\
y &= -\frac{2}{\sqrt{5}}
\end{aligned}$$

Evaluating f:

x	y	z	λ	$f(x,y,z)$
$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	$\frac{\sqrt{5}}{2}$	$\sqrt{5}$
$-\frac{1}{\sqrt{5}}$	$-\frac{2}{\sqrt{5}}$	0	$-\frac{\sqrt{5}}{2}$	$-\sqrt{5}$

If we want to optimize f with respect to multiple constraint functions $g_1, g_2, g_3, \dots, g_k$, we would do:

$$\nabla f = \sum_{i=1}^{k} \lambda_i \nabla g_i$$

Integration

Consider $f(x,y)$. We want to integrate along the rectangle $D = [0,3] \times [0,2]$. We can find this as follows:

$$\begin{aligned}\int_D f(x,y) &= \int_0^3 \int_0^2 f(x,y)\,dy\,dx\\ &= \int_0^3 dx \int_0^2 dy f(x,y)\end{aligned}$$

For any two regions D_1 and D_2, we have:

$$\begin{aligned}\int_{D_1} f(x,y) + \int_{D_2} f(x,y) &= \int_{D_1 \ominus D_2} f(x,y)\\ &= \int_{D_1 \cup D_2 \setminus D_1 \cap D_2} f(x,y)\end{aligned}$$

Multidimensional Integral Approximation

We want to find

$$\int_0^2 dy \int_0^3 dx f(x,y)$$

where $f(x,y)$ is expressed as below.

$y \backslash x$	0	1	2	3
0	1	2	5	4
1	2	1	2	0
2	1	-1	1	-2

Just as we can use the left/right endpoint method for evaluating integrals in one dimension, we can use left/right and top/bottom endpoints to approximate the integral.

Evaluating a Multidimensional Integral

$$\underbrace{\int_0^1}_{dy}\underbrace{\int_0^1}_{dx} xe^y dx dy = \left(\int_0^1 e^y dy\right)\left(\int_0^1 x dx\right)$$
$$= \left(e^y\Big|_0^1\right)\left(\frac{x^2}{2}\Big|_0^1\right)$$
$$= (e-1)\left(\frac{1}{2}-0\right)$$
$$= \frac{e-1}{2}$$

This can scale up into multiple dimensions:

$$\int_0^1\int_2^4\int_{-1}^2 x+y+z^2\ dx\ dy\ dz = \int_0^1\int_2^4\left(\int_{-1}^2 x+y+z^2 dx\right) dy\ dz$$
$$= \int_0^1\int_2^4\left(\frac{x^2}{2}+yx+xz^2\Big|_{x=-1}^{x=2}\right) dy\ dz$$
$$= \int_0^1\int_2^4\left(\left(2-\frac{1}{2}\right)+(2y-(-y))+(2z^2-(-z^2))\right) dy\ dz$$
$$= \int_0^1\int_2^4 3z^2+3y+\frac{3}{2} dy\ dz$$
$$= \int_0^1\left(\frac{3}{2}y^2+\frac{3}{2}y+3yz^2\Big|_{y=2}^{y=4}\right) dz$$
$$= \int_0^1 6z^2+21dz$$
$$= 2z^3+21z\Big|_{z=0}^{z=1}$$
$$= 23$$

Consider the integral below:

$$\int\int_D xe^y\ dx\ dy = \int_0^1\int_y^1 xe^y\ dx\ dy$$
$$= \int_0^1 e^y\ dy\left(\frac{x^2}{2}\Big|_{x=y}^{x=1}\right)$$
$$= \int_0^1 e^y\left(\frac{1}{2}-\frac{y^2}{2}\right) dy$$
$$= \frac{1}{2}\left(\int_0^1 e^y\ dy - \int_0^1 y^2e^y\ dy\right)$$
$$= \frac{1}{2}\left((e-1)-(e-2)\right)$$
$$= \frac{1}{2}$$

Example Integrals

Consider the domain $D : \{(x, y) \mid 1 \leq x \leq 2,\ 0 \leq y \leq \ln 2,\ y = \ln x\}$. We are going to evaluate $f(x, y) = 1$.

$$\begin{aligned}
\int_D f(x,y)dD &= \int\limits_0^{\ln 2}\int\limits_{e^y}^{2} 1\ dx\ dy \\
&= \int\limits_0^{\ln 2}\left(x\Big|_{x=e^y}^{x=2}\right)dy \\
&= \int\limits_0^{\ln 2}(2-e^y)dy \\
&= 2\ln 2 - 1
\end{aligned}$$

Consider $A = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2 \leq (x, y) \leq \sqrt{x}\}$ and evaluating $f(x, y) = x + 2y$

$$\begin{aligned}
\int_A x + 2y dA &= \int\limits_0^{1}\int\limits_{y^2}^{\sqrt{y}} x+2y\ dx\ dy \\
&= \int_0^1 \left(\frac{x^2}{2} + 2xy\Big|_{x=y^2}^{x=\sqrt{y}}\right)dy \\
&= \int_0^1 \frac{y}{2} + 2y^{3/2} - \left(\frac{y^4}{2} + 2y^3\right)dy \\
&= \frac{y^2}{4} + \frac{4}{5}y^{5/2} - \frac{y^5}{10} - \frac{y^4}{2}\Big|_{y=0}^{y=1} \\
&= \frac{9}{20}
\end{aligned}$$

General Multivariable Integration

In any given area of integration A, we have the following general form:

$$\begin{aligned}
\int_A f(x,y)\ dA &= \underbrace{\int_{c_1}^{c_2}}_{\text{always constants}} \int_{g(y)}^{h(y)} !dx!dy \\
&= \int_{d_1}^{d_2}\int_{p(x)}^{q(x)} f(x,y)\ dy\ dx
\end{aligned}$$

In the three-dimensional case, we have

$$\int_V f(x,y,z)\ dV = \int_{c_1}^{c_2}\int_{g(z)}^{h(z)}\int_{p(y,z)}^{q(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Consider an integral with domain as follows:

$$D = \left\{x, y, z \mid -1 \leq x \leq 1,\ 0 \leq y \leq 10,\ 0 \leq z \leq 1 - x^2\right\}$$

For volume, we have that $f(x, y, z) = 1$

$$\begin{aligned}
V &= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{10} 1\ dy\ dz\ dx \\
&= \int_{-1}^{1}\int_0^{1-x^2} 10\ dz\ dx \\
&= 10\int_{-1}^{1} 1\ dx - 10\int_{-1}^{1} x^2\ dx \\
&= \frac{40}{3}
\end{aligned}$$

For another example, consider the pyramid defined by the points $(0,0,0),(0,2,0),(1,0,0),(0,0,3)$. As established, we do $f(x,y,z)=1$ integrated over the domain.

$$V=\int_0^3\int_0^{g(y)}\int_0^{h(y,z)} f(x,y,z)\,dx\,dy\,dz$$

We find the cross product for the plane

$$\begin{aligned}
\vec{n} &= \begin{pmatrix}-1\\2\\0\end{pmatrix}\times\begin{pmatrix}-1\\0\\3\end{pmatrix}\\
&= \begin{pmatrix}6\\3\\2\end{pmatrix}\\
6 &= 6x+3y+2z\\
x &= 1-\frac{1}{2}y-\frac{1}{3}z\\
y &= 2-\frac{2}{3}z\\
V &= \int_0^3\int_0^{2-\frac{2z}{3}}\int_0^{1-\frac{1}{2}y-\frac{1}{3}z} 1\,dx\,dy\,dz\\
&= \int_0^3\int_0^{2-\frac{2z}{3}}\left(1-\frac{1}{2}y-\frac{1}{3}z\right)dy\,dz\\
&= \int_0^3 2-\frac{2z}{3}-\frac{\left(2-\frac{2z}{3}\right)^2}{4}-\frac{\left(2-\frac{2z}{3}\right)z}{3}\,dz
\end{aligned}$$

u substitution:

$$\begin{aligned}
u &= 2-\frac{2z}{3}\\
z &= \frac{3(u-2)}{-2}\\
dz &= -\frac{3}{2}du\\
&= \int_2^0\left(u-\frac{u^2}{4}-\frac{3u(u-2))}{-6}\right)\left(-\frac{3}{2}du\right)\\
&= \int_2^0\frac{u^2}{4}\\
&= -\frac{3}{2}\int_2^0\frac{u^2}{4}\\
&= -\frac{3}{2}\frac{u^3}{12}\Big|_2^0\\
&= 1
\end{aligned}$$

The Jacobian

Let $f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)$ be differentiable.

$$J=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix}$$

If we seek to enact a change of variables, we compute $\det J$.

For example, let $x = r\cos\theta = x(r,\theta)$, $y = r\sin\theta = y(r,\theta)$.

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ \det J &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r \end{aligned}$$

Consider a circle of radius 5. We will integrate $f(x,y) = x^2 + y$ over this circle.

(1): Cartesian

$$\begin{aligned} \int_A f(x,y)dA &= \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \left(x^2+y\right)\ dy\ dx \\ &= \int_{-5}^{5} \left(x^2y + \frac{y^2}{2}\bigg|_{y=\sqrt{25-x^2}}^{y=\sqrt{25-x^2}} \right)\ dx \\ &= \int_{-5}^{5} 2x^2\sqrt{25-x^2}\ dx \\ &\vdots \end{aligned}$$

(2): Polar

$$\begin{aligned} \int_A f(r,\theta)\ dr\ d\theta &= \int_A f(r,\theta)\left(\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}\right)\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^5 \left(r^2\cos^2\theta + r\sin\theta\right)(r)\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^5 r^3\cos^2\theta\ dr\ d\theta + \int_0^{2\pi}\int_0^5 r^2\sin\theta\ dr\ d\theta \\ &= \left(\int_0^{2\pi}\cos^2\theta\ d\theta\right)\left(\int_0^5 r^3\ dr\right) + \left(\int_0^{2\pi}\sin\theta\ d\theta\right)\left(\int_0^5 r^2\ dr\right) \\ &= \frac{625}{4}\left(\int_0^{2\pi}\cos^2\theta\ d\theta\right) \\ &= \frac{625\pi}{4} \end{aligned}$$

Cylindrical Coordinates

We can define a new coordinate base as follows:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Thus, any integral over the cylinder has the following form:

$$\begin{aligned} \int_V f(x,y,z)\ dV &= \int_V f(r,\theta,z)\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}\ dr\ d\theta\ dz \\ &= \int_V f(r,\theta,z) r\ dr\ d\theta\ dz \end{aligned}$$

Spherical Coordinates

If we have spherical symmetry, we will need to include a second angle φ, relative to the positive z axis:

$$\begin{aligned}
x &= \rho\cos\theta\sin\varphi \\
y &= \rho\sin\theta\sin\varphi \\
z &= \rho\cos\varphi \\
J &= \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} \end{vmatrix} \\
&= \rho^2\sin\varphi
\end{aligned}$$

Integrating with Spherical and Cylindrical Coordinates

Consider a cylinder defined by $z \in [0,4]$ and $r = 1$, and a half-sphere defined by $r = 1$ and $\varphi \in [0, \pi/2]$, located on "top" of the cylinder. We want to find

$$\int_V f(x,y,z)\ dV = \int_{V_1} f(x,y,z)\ dV_1 + \int_{V_2} f(x,y,z)\ dV_2$$

where $f(x,y,z=z)$. Then,

$$\begin{aligned}
\int_V z\ dV &= \int_{V_1} (z) r dr\ d\theta\ dz + \int_{V_2} (z)(\rho^2\sin\varphi) d\rho\ d\theta\ d\varphi \\
I_c &= \int_0^4 \int_0^{2\pi} \int_0^1 (z)(r) dr\ d\theta\ dz \\
&= \int_0^4 \int_0^{2\pi} \left(\frac{z}{2} r^2 \bigg|_0^1 \right)\ d\theta\ dz \\
&= \int_0^4 \int_0^{2\pi} \frac{z}{2}\ d\theta\ dz \\
&= \int_0^4 \pi z\ dz \\
&= 8\pi \\
I_s &= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (4 + \cos\varphi)(r^2\sin\varphi) d\rho\ d\theta\ d\varphi \\
&= 2\pi \int_0^{\pi/2} \int_0^1 (4+\cos\varphi)(r^2\sin\varphi) d\rho\ d\varphi \\
&= 2\pi \int_0^1 r^2\ dr \int_0^{\pi/2} 4\sin\varphi + \cos\varphi\sin\varphi\ d\varphi \\
&= \frac{2\pi}{3}\frac{9}{2} \\
&= 3\pi \\
I &= I_c + I_s \\
&= 11\pi
\end{aligned}$$

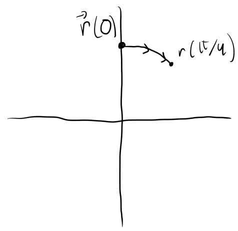
Vector Valued Functions

Let

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \qquad 0 \leq t \leq \frac{\pi}{4}$$

$$\vec{r}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{r}\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$



We can expand to $\mathbb{R}^3$:

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ 15 \end{pmatrix} \qquad \text{Circle at } z = 15$$

$$\vec{r}'(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ t\sin(t) \end{pmatrix} \qquad \text{Helix}$$

$$\vec{r}''(t) = \begin{pmatrix} t\sin(t) \\ t\cos(t) \\ t \end{pmatrix} \qquad \text{Conical Helix}$$

We can also add parameters:

$$\vec{r}(t) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \end{pmatrix} \qquad \text{Disc}$$

Differentiation in Vector Valued Functions

For

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

we have

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

For example, in uniform circular motion, we have

$$\begin{aligned}
\vec{x}(t) &= r\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}\\
\frac{d\vec{x}}{dt} &= \omega r\begin{pmatrix}-\sin\omega t\\ \cos\omega t\end{pmatrix}\\
\frac{d^2\vec{x}}{dt^2} &= -\omega^2 r\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}\\
&= -\omega^2 r\vec{x}(t)
\end{aligned}$$

Examining the Helix

Let

$$\begin{aligned}
\vec{r}(t) &= \begin{pmatrix}x(t)\\ y(t)\\ z(t)\end{pmatrix}\\
&= \begin{pmatrix}r\cos\omega t\\ r\sin\omega t\\ ct\end{pmatrix}\\
\vec{v}(t) &= \frac{d\vec{r}}{dt}\\
&= \begin{pmatrix}-\omega r\sin\omega t\\ \omega r\cos\omega t\\ c\end{pmatrix}\\
\vec{a}(t) &= \begin{pmatrix}\omega^2 r\cos\omega t\\ \omega^2 r\sin\omega t\\ 0\end{pmatrix}\\
\|\vec{r}(t)\| &= \sqrt{r^2+m^2t^2}\\
\|\vec{v}(t)\| &= \sqrt{\omega^2r^2+m^2}\\
\|\vec{a}(t)\| &= \omega^2 r
\end{aligned}$$

Path Length of Helix

Along $\vec{r}(t)$, the total length traveled along $t \in [a,b]$ is

$$\ell = \int_a^b \|r'(t)\|\ dt$$

So, for the helix, we have

$$\begin{aligned}
\ell &= \int_a^b \sqrt{\omega^2r^2+m^2}\ dt\\
&= \frac{2\pi\sqrt{\omega^2r^2+m^2}}{\omega}
\end{aligned}$$

Vector Fields

The following are vector fields:

$$\vec{F}(x,y) = \begin{pmatrix} x+y \\ y^2 - x \end{pmatrix}$$

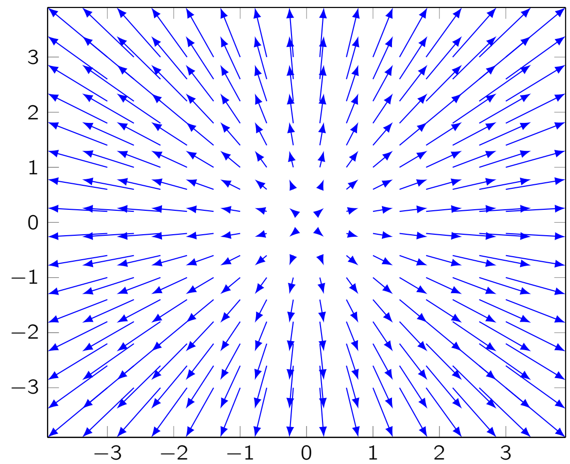
$$\vec{F}(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{F}(x,y,z,w) = \begin{pmatrix} xy \\ yz \\ 0 \\ w \end{pmatrix}$$

For the vector field

$$\vec{F}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix},$$

we select various points and draw their resulting vectors.



Curves in a Vector Field

Given a field $\vec{F}(x,y)$, a flow line $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is such that $\vec{F}(\vec{r}(t)) = \vec{r}(t)$. For example, let

$$\begin{aligned}
\vec{F}(x,y) &= \begin{pmatrix} x \\ y \end{pmatrix} \\
\vec{r}(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
x'(t) &= x(t) \\
y'(t) &= y(t) \\
x(0) &= x_0 \\
y(0) &= y_0 \\
x(t) &= x_0 e^t \\
y(t) &= y_0 e^t.
\end{aligned}$$

This time, let

$$\begin{aligned}
\vec{F}(x,y) &= \begin{pmatrix} x \\ 1 \end{pmatrix} \\
\vec{r}'(t) &= \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\
x'(t) &= x(t) \\
y'(t) &= 1 \\
x(t) &= x_0 e^t \\
y(t) &= t + y_0 \\
t &= \ln(x) - \ln(x_0) && x \neq 0 \\
y &= \ln(x) + (y_0 - \ln(x_0))
\end{aligned}$$

Flow Lines in Conservative Vector Fields

Let $f(x,y) = x^2 + y$. If

$$\begin{aligned}
\vec{F}(x,y) &= \nabla f(x,y) \\
&= \begin{pmatrix} 2x \\ 1 \end{pmatrix},
\end{aligned}$$

then $\vec{F}(x,y)$ is known as a **conservative** vector field — i.e., it is derived from a gradient.

Let

$$\begin{aligned}
\vec{F}(x,y) &= \begin{pmatrix} 2x+y \\ y \end{pmatrix} \\
\vec{r}'(t) &= \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}
\end{aligned}$$

Solve the system:

$$x'(t) = 2x + y$$
$$y'(t) = y$$

Guess:

$$\begin{aligned}
x(t) &= Ae^t + Be^{2t} \\
x(0) &= x_0 \\
x_0 &= A + B \\
x'(t) &= Ae^t + 2Be^{2t} \\
&= 2\left(Ae^t + Be^{2t}\right) + y_0e^t \\
A &= 2A + y_0 \\
A &= -y_0 \\
B &= x_0 + y_0 \\
x(t) &= -y_0e^t + (x_0 + y_0)e^{2t} \\
y(t) &= y_0e^t \\
x(t) &= -y + (x_0 + y_0)\left(\frac{y(t)}{y_0}\right)^2
\end{aligned}$$

Let

$$\vec{F} = \begin{pmatrix} y \\ -x \\ 2 \end{pmatrix}$$

$$\begin{aligned}
x'(t) &= y(t) \\
y'(t) &= -x(t) \\
z'(t) &= z(t) \\
z(t) &= 2 \\
x''(t) &= -x(t) \\
y''(t) &= -y(t)
\end{aligned}$$

$$\begin{aligned}
x(t) &= A\cos(t) + B\sin(t) \\
y(t) &= P\cos(t) + Q\sin(t) \\
z(t) &= 2t + z_0
\end{aligned}$$

Conservative Vector Fields and Calculating Work

We can find work using the traditional formula from physics:

$$W = \int \vec{F} \cdot d\vec{r}$$

Let

$$\vec{F} = \begin{pmatrix} x \\ y^2 \end{pmatrix}$$
$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad 0 \leq t \leq 1$$
$$d\vec{r} = \vec{r}\,'(t)dt$$
$$W = \int \begin{pmatrix} x(t) \\ y(t)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$
$$= \int_0^1 t + 2t^5 \; dt$$
$$= \frac{5}{6}$$

Green's Theorem

Let

$$\vec{F}(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}$$

be differential. Let

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

where $a \leq t \leq b$ and $\vec{r}(a) = \vec{r}(b)$. Suppose C, the curve parametrized by $\vec{r}$, is simply connected. Then,

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y} \right) \; dx \; dy$$

We can show that Green's Theorem is true by taking the line integral along a square curve ($L \leq x \leq R$ and $B \leq y \leq T$)—by the properties of curve-tracing, we can show that this is equivalent to a double integral along an area.

$$\int_C \vec{F} \cdot d\vec{r} = \int_L^R F_1(x,B)x'(t)dt + \int_R^L F_1(x,T)x'(t)dt + \int_B^T F_2(R,y)y'(t)dt + \int_T^B F_2(L,y)y'(t)dt$$
$$= \int_L^R \left(F_1(x,B) - F_1(x,T)\right) x'(t)dt + \int_B^T \left(F_2(R,y) - F_2(L,y)\right) y'(t)dt$$

as $\Delta x, \Delta y \to 0$:

$$= \int_L^R \int_B^T \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \; dx \; dy$$

Green's Theorem under a Conservative Field

$$\oint_C \vec{F} \cdot d\vec{r} = \int \int_A \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \; dx \; dy$$

If and only if $\vec{F}$ is conservative is $\oint_C \vec{F} \cdot d\vec{r} = 0$.

The curl of $\vec{F}$ is

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial z} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}\end{aligned}$$

Let

$$\vec{F} = \begin{pmatrix} xe^y \\ ye^x \end{pmatrix}$$

and C is the square defined by $(0,0)$ and $(1,1)$, counterclockwise.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \int_0^1 (ye^x - xe^y)\ dx\ dy \\ &= 0\end{aligned}$$

Surface Integrals

Consider a vector field $\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}$ along the window with corner points $(0,0), (0,L), (H,0), (H,L)$.

$$\vec{A} = |A|\vec{n}$$

where $\vec{n}$ is the unit vector orthogonal to the area. We can find the *flux* of $\vec{F}$ through area $\vec{A}$.

$$\text{Flux} = \vec{F} \cdot \vec{A}$$

In the above case, we have $\text{Flux} = 100HL$. However, we can investigate deeper.

Suppose A is not a "nice" surface — we might want to refine via

$$\sum \vec{F} \cdot \Delta\vec{A} \to \int \vec{F} \cdot d\vec{A}$$

Developing Flux Integrals

Given $\vec{F}(x,y,z)$, where $d\vec{A}$ depends on geometry, and S a closed surface, we can find

$$\int_S \vec{F} \cdot d\vec{A} = \int_V \nabla \cdot \vec{F} d\vec{V}$$

Let

$$\vec{F}(x,y,z) = \begin{pmatrix} 1 \\ y \\ z \end{pmatrix}$$

Find

$$\int_S \vec{F} \cdot d\vec{A}$$

for each of the six sides of the positive unit cube.

(1) $x = 0$, $0 \leq y, z \leq 1$; $\vec{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$; $\vec{F}(x, y, z) = \begin{pmatrix} 1 \\ y \\ z \end{pmatrix}$; Flux = -1

(2) $x = 1$, $0 \leq y, z \leq 1$; $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\vec{F}(x, y, z) = \begin{pmatrix} 1 \\ y \\ z \end{pmatrix}$; Flux = 1

(3) $y = 0$, $0 \leq x, z \leq 1$; $\vec{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$; $\vec{F}(x, y, z) = \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$; Flux = 0

(4) $y = 1$, $0 \leq x, z \leq 1$; $\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\vec{F}(x, y, z) = \begin{pmatrix} 1 \\ 1 \\ z \end{pmatrix}$; Flux = 1

(5) $z = 0$, $0 \leq x, y \leq 1$; $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$; $\vec{F}(x, y, z) = \begin{pmatrix} 1 \\ y \\ 0 \end{pmatrix}$; Flux = 0

(6) $z = 1$, $0 \leq x, y \leq 1$; $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\vec{F}(x, y, z) = \begin{pmatrix} 1 \\ y \\ 1 \end{pmatrix}$; Flux = 1

Therefore, total flux is the sum of these respective fluxes, or 2.

∇ Operator

With $f : \mathbb{R}^n \to \mathbb{R}$ defined where $f(x_1, \dots, x_n)$ is differentiable,

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

With $\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \qquad \text{curl}$$

Surface Integral, Yet Another Example

Consider $\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ along the pyramid

(1) S_1: $x = 0$, $y, z \geq 0$; $\hat{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$; Flux = 0

(2) S_2: $y = 0$, $x, z \geq 0$; $\hat{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$; Flux = 0

(3) S_3: $z = 0$, $x, y \geq 0$; $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$; Flux = $\frac{1}{2}$

(4) S_4: $x + y + z = 1$; $\hat{n} = \frac{\sqrt{3}}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; Flux = $\sqrt{3}/3 \cdot \sqrt{3}/2 = 1/2$

Therefore, total flux is 0.