Problem 1

Let $G = (X, E, Y)$ be a bipartite graph, and let $d \in \mathbb{N}$ be a fixed constant. Show that if $|N(S)| \geq |S| - d$ for every $S \subseteq X$, then G contains a matching of cardinality $|X| - d$.

We add d vertices to the Y partition such that $|N(S)| + d \geq |S|$ for all $S \subseteq X$. Then, we will create an edge between every vertex $x \subseteq X$ and every auxiliary vertex. Let $G' = (X, E', Y')$ denote this new graph.

Let $S' \subseteq X$ be a set that contains all vertices of X — then, $N(S') \subseteq Y'$ must be of cardinality at least $|S'|$. So, for all $S' \subseteq X$, it follows that $|N(S')| \geq |S'|$, so G' has an X-perfect matching by Hall's Theorem.

Since there is a matching in G' that saturates every vertex in X, this matching maps every $x \in X$ to every $y' \in Y'$. We remove d edges from the matching, corresponding to the d auxiliary vertices in Y'. Thus, G has a matching of $|X| - d$ edges.

3.1.19

Let $\mathbf{A} = (A_1, \ldots, A_m)$ be a collection of subsets of Y. A **system of distinct representatives**, or SDR, for $\mathbf{A}$ is a set of *distinct* elements $a_1, \ldots, a_m \in Y$ where $a_i \in A_i$. Prove that $\mathbf{A}$ has a SDR if and only if $|\bigcup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$.

Let $G = (\{1, \ldots, m\}, E, \mathbf{A})$ where edges are defined from every element in $\{1, \ldots, m\}$ to every element in A_i for $i \in \{1, \ldots, m\}$. The definition of $N(S)$ that follows from this definition of G is $\bigcup_{i \in S} A_i$.

($\Rightarrow$) If $\mathbf{A}$ has a SDR, this is equivalent to a perfect matching in G from every $i \in \{1, \ldots, m\}$ to every $a_i \in A_i$ — so, by Hall's theorem, we know that $|N(S)| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$. So, $\mathbf{A} \Rightarrow |\bigcup_{i \in S} A_i| \geq |S|$.

($\Leftarrow$) Let $|\bigcup_{i \in S} A_i| \geq |S|$ for all $S \subseteq \{1, \ldots, m\}$. Then, by the definition of $N(S)$ in G, we know that $|N(S)| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$, meaning G has a perfect matching. This means that every $i \subseteq \{1, \ldots, m\}$ maps to a unique element a_i in $\mathbf{A}$, as G has a perfect matching. This is, thus, equivalent to $\mathbf{A}$ having a SDR.

3.1.25

A doubly stochastic matrix Q is a nonnegative real matrix in which every row and column sums to one. Prove that a doubly stochastic matrix Q can be expressed as a convex combination of permutation matrices — i.e., there exist permutation matrices $P_1, \ldots, P_m$ and *nonnegative* real coefficients $c_1, \ldots, c_m$ such that $Q = c_1 P_1 + c_2 P_2 + \cdots + c_m P_m$, where $\sum_{i=1}^{m} c_i = 1$.

We will prove via induction as follows:

Base Case If Q is a permutation matrix itself, then it is a doubly stochastic matrix and satisfies the base case.

Inductive Step Suppose that Q is a $m \times m$ matrix with at least $m+1$ nonzero entries. Let G represent a bipartite graph, where I represents the set of m rows, and J represents the set of m columns. Each nonzero entry in (i, j) represents an edge between the ith vertex in I and the jth vertex in J.

Let $S \subseteq I$ and $|S| = d$ for some $d \in \mathbb{N}$. Then, $N(S) \subseteq J$ represents the columns of at least one nonzero entry for each of the rows in S. The sum of each of these rows is 1, meaning the sum of the rows in S is d.

Each column sums to maximum 1, meaning $|S| \leq |N(S)|$, satisfying the condition for Hall's Theorem. Since $|I| = |J|$, the graph has a perfect matching, meaning we can find a permutation matrix P_1 and a positive number c_1. By the inductive hypothesis, $Q - c_1P_1 = c_2P_2 + \cdots + c_mP_m$, so $Q = c_1P_1 + c_2P_2 + \cdots + c_mP_m$.

3.1.29

Use the König-Egerváry theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ has a matching of size at least k.

Let G be bipartite. Then, from the König-Egerváry theorem, we know that $\alpha'(G) = \beta(G)$.

Let Q represent the minimum vertex cover, meaning $|Q| = \beta(G)$. Every edge is incident on some vertex $v \in Q$, and the upper bound on $d(v)$ is $\Delta(G)$. This means that $|Q|\Delta(G) \geq e(G)$ (assuming that there would be double counting in $|Q|\Delta(G)$). So, $\beta(G)\Delta(G) \geq n(G)$. Therefore, $\beta(G) \geq \frac{n(G)}{\Delta(G)}$. So, $\alpha'(G) \geq \frac{n(G)}{\Delta(G)}$, as $\alpha'(G) = \beta(G)$.