

## Complex Analysis

### Analyticity and Path-Independence in the Complex Plane

#### Baby's First Complex Function Theory

We are interested in functions of the form  $f(z)$ , where  $z = x + iy$  is some complex number. Note that this is specifically different from a function  $g: \mathbb{R}^2 \rightarrow \Omega$  for some domain  $\Omega$ ; in the latter case, we have independent variables  $x$  and  $y$ , while in the former case, we must express  $z = x + iy$ .

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z) dz &= \oint_C w(z) (dx + i dy) \\ &= \oint_C w(z) dx + i \oint_C w(z) dy.\end{aligned}$$

Taking  $A_x = w(z)$  and  $A_y = iw(z)$ , we have

$$= \oint_C \mathbf{A} \cdot d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla \times \mathbf{A}) \cdot \hat{n} da,$$

and when this integral is zero. Note that  $(\nabla \times \mathbf{A}) \cdot \hat{n} = 0$ , so  $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$ .

Note that we can take

$$w(z) = u(x, y) + iv(x, y),$$

where  $z = x + iy$ .

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

**Theorem** (Cauchy–Riemann Equations):

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that  $w$  is analytic,<sup>1</sup> which leads to Cauchy's theorem.

**Theorem** (Cauchy's Theorem): If  $C$  is a simple closed curve in a simply connected region, then  $w$  is analytic if and only if

$$\oint_C w(z) dz = 0. \tag{†}$$

**Fact.** The function  $w(z)$  is analytic inside the simply connected region  $R$  if any of these hold:

- $w$  satisfies the Cauchy–Riemann equations;

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<sup>1</sup>Equal to its Taylor series, also holomorphic.

- $w'(z)$  is unique and exists;
- $\frac{\partial w}{\partial \bar{z}} = 0$ .
- $w$  can be expanded as  $w(z) = \sum_{n \geq 0} c_n(z - a)^n$ , convergent on some open neighborhood of  $a$  for each  $a$  on its domain;<sup>II</sup>
- $w(z)$  is path-independent everywhere in  $\mathbb{R}$ :  $\oint_{\mathbb{C}} w(z) dz = 0$ .

**Example.** Considering  $w(z) = z$ , we have  $u = x$  and  $v = y$ , so it satisfies the Cauchy–Riemann equations. However, neither  $\text{Re}(z)$  nor  $\text{Im}(z)$  are analytic, and neither is  $\bar{z} = x - iy$ .

**Remark:** Whenever we say “analytic at  $p$ ,” we mean “analytic in a neighborhood of  $p$ .”

Note that since  $\mathbb{C}$  is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of  $\mathbb{C}$ , by adjoining a point  $\{\infty\}$ ,  $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ . This compactified  $\mathbb{C}^*$  is often represented as a unit sphere with the north pole, determined by  $(0, 0, 1)$ , is the point at infinity. The correspondence between  $\mathbb{C}^* \setminus \{\infty\}$  and  $\mathbb{C}$  is evaluated via stereographic projection.

We define  $\frac{z}{\infty} = 0$  and  $\frac{z}{0} = \infty$  for any  $z \neq 0, \infty$ . The correspondence between  $z = x + iy$  in the plane to  $Z$  on the Riemann sphere with  $\mathbb{R}^3$  coordinates  $(\xi_1, \xi_2, \xi_3)$  is

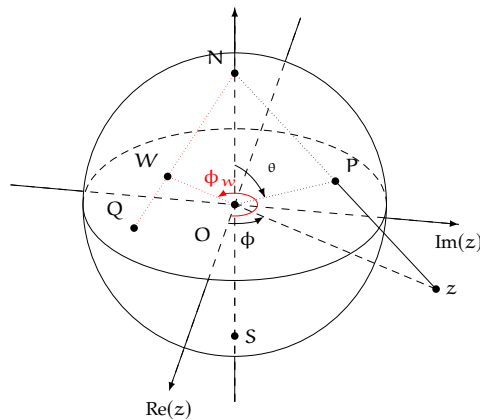
$$\begin{aligned}\xi_1 &= \frac{2 \text{Re}(z)}{|z|^2 + 1} \\ \xi_2 &= \frac{2 \text{Im}(z)}{|z|^2 + 1} \\ \xi_3 &= \frac{|z|^2 - 1}{|z|^2 + 1}.\end{aligned}$$

Inverting, we may find

$$\begin{aligned}x &= \frac{\xi_1}{1 - \xi_3} \\ y &= \frac{\xi_2}{1 - \xi_3},\end{aligned}$$

and with polar coordinates,

$$z = \cot(\theta/2)e^{i\phi}.$$



To determine analyticity at  $\infty$ , we set  $\zeta = \frac{1}{z}$ , and analyze the analyticity of  $\tilde{w}(\zeta) = w(1/z)$  at 0.

<sup>II</sup>This is technically the real definition of analytic for the case when we're dealing with a function with domain  $\mathbb{R}$ .

### Cauchy's Integral Formula

Consider the function  $w(z) = c/z$ , integrated around a circle of radius  $R$ . Then, writing  $z = Re^{i\varphi}$ , we get

$$\begin{aligned}\oint_{\Gamma} w(z) dz &= C \int_0^{2\pi} \frac{e^{-i\varphi}}{R} \underbrace{iRe^{i\varphi} d\varphi}_{dz} \\ &= ic \int_0^{2\pi} d\varphi \\ &= 2\pi ic.\end{aligned}$$

If our contour  $C$  runs around our singularity at  $z = 0$  a total of  $n$  times, then we pick up a factor of  $n$ .

Now, when we consider

$$I = \oint_C \frac{dz}{z^n},$$

this integral actually yields 0 for any  $n \neq 1$ , despite the fact that 0 is a singularity for  $f(z) = \frac{1}{z^n}$ . This 0 is not a reflection of Cauchy's integral theorem, but of the fact that

$$z^{-n} = \frac{d}{dz} \left( \frac{z^{-n+1}}{n+1} \right),$$

meaning that  $z^{-n}$  is an exact differential, so integrating along a closed curve yields zero change. However,  $\frac{1}{z} = \frac{d}{dz}(\ln z)$  may be an exact differential, but for complex  $z$ ,  $\ln z = \ln|z| + i \arg(z) = \ln r + i\varphi$ . This yields

$$\begin{aligned}\oint_C \frac{c}{z} dz &= c \oint_C d(\ln z) \\ &= c(i(\varphi + 2\pi) - \varphi) \\ &= 2\pi ic.\end{aligned}$$

Ultimately, what this shows is that when we integrate any analytic function  $f(\zeta)$  along a closed contour with a singularity at  $z$ , only the coefficient on  $\frac{1}{\zeta - z}$  will remain. This coefficient is known as the residue at 0.

**Theorem** (Cauchy's Integral Formula): If  $w$  is analytic in a simply connected region and  $C$  is a closed contour winding once around a point  $z$  in the region, then

$$w(z) = \frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{\zeta - z} d\zeta. \quad (**)$$

Furthermore, this shows that any once-differentiable function is infinitely differentiable, as by differentiating under the integral sign, we get

$$\frac{d^n w}{dz^n} = \frac{n!}{2\pi i} \oint_C \frac{w(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

**Example** (Deriving Liouville's Theorem). Consider a circle  $C$  centered at radius  $r$  centered at  $z$ ,  $\zeta - z = Re^{i\varphi}$ . We take  $d\zeta = iRe^{i\varphi} d\varphi$ , and taking derivatives, we have

$$w'(z) = \frac{1}{2\pi R} \int_0^{2\pi} w(z + Re^{i\varphi}) e^{-i\varphi} d\varphi.$$

If  $w$  is bounded — i.e.,  $|w(z)| \leq M$  for all  $z$  in a given region — then

$$|w'(z)| = \left| \frac{1}{2\pi R} \int_0^{2\pi} w(z + Re^{i\varphi}) e^{-i\varphi} d\varphi \right|$$

$$\begin{aligned} &\leq \frac{1}{2\pi R} \int_0^{2\pi} \left| w\left(z + Re^{i\varphi}\right) \right| d\varphi \\ &\leq \frac{M}{R} \end{aligned}$$

for all  $R$  within the analytic region.

In the case where  $w$  is entire (i.e., analytic on  $\mathbb{C}$ ), then this inequality holds for all  $R \rightarrow \infty$ . Thus,  $|w'(z)| = 0$  for all  $z$ , meaning that  $w$  is constant.

This is known as Liouville's theorem — every bounded entire function is constant. This can be used to prove the fundamental theorem of algebra.

What Liouville's theorem tells us is that any nontrivial behavior will emerge from a function's singularities.

## Singularities and Branches

To understand nontrivial behavior on the complex plane, we need to understand singularities. This will require us to develop understanding of Laurent series.

### Taylor Series

We want to integrate  $w(z)$  around some point  $a$  in an analytic region of  $w(z)$ . This yields the form

$$\begin{aligned} w(z) &= \frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(\zeta - a) - (z - a)} d\zeta \\ &= \frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(\zeta - a) \left(1 - \frac{z-a}{\zeta-a}\right)} d\zeta. \end{aligned} \tag{*}$$

Since  $\zeta$  is on the contour and  $z$  is in the contour,  $\left|\frac{z-a}{\zeta-a}\right| < 1$ , we may expand as a geometric series. Thus, we get

$$= \frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(\zeta - a)} \left( \sum_{n=0}^{\infty} \left( \frac{z-a}{\zeta-a} \right)^n \right) d\zeta.$$

Since the series is uniformly convergent, we are allowed to exchange sum and integral, yielding

$$\begin{aligned} &= \sum_{n=0}^{\infty} \underbrace{\left( \frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(\zeta - a)^{n+1}} d\zeta \right)}_{=c_n} (z - a)^n \\ &= \sum_{n=0}^{\infty} c_n (z - a)^n, \end{aligned}$$

where

$$c_n = \frac{1}{n!} \left. \frac{d^n w}{dz^n} \right|_{z=a}.$$

If our Taylor series reduces to a known series on the real axis, we find this very desirable. We say this is a type of analytic continuation from the real axis to the complex plane. For example,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is an analytic continuation of  $e^x$ .

However, more interestingly,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for all  $s > 1$ . However, we have also shown that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

converges for complex  $s$  for all real part greater than 1. Since values of this integral agree with the series representation of  $\zeta(s)$  on real axis, we have that this is an analytic continuation of  $\zeta(s)$  to the subset of  $\mathbb{C}$  defined by  $\text{Re}(s) > 1$ .

### Laurent Series

Now, what happens if, at  $(\dagger)$ , we have  $\left| \frac{z-a}{\zeta-a} \right| > 1$ . The series as constructed would not converge, but what if we have a series that converges everywhere *outside*  $C$ ? This would entail an expansion in reciprocal integer powers of  $z - a$ . This yields

$$\begin{aligned} w(z) &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(z-a)\left(1 - \frac{\zeta-a}{z-a}\right)} d\zeta \\ &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{z-a} \left( \sum_{n=0}^{\infty} \left( \frac{\zeta-a}{z-a} \right)^n \right) d\zeta \\ &= -\sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \oint_C w(\zeta-a)^n d\zeta \right) \frac{1}{(z-a)^{n+1}} \\ &= \sum_{n=1}^{\infty} \underbrace{\left( -\frac{1}{2\pi i} \oint_C w(\zeta-a)^{n-1} d\zeta \right)}_{=c_{-n}} \frac{1}{(z-a)^n} \\ &= \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} \end{aligned}$$

Note that this series has a singularity at  $z = a$ , but since our series is only defined outside a particular region, that doesn't matter. We call a series in reciprocal powers a Laurent series. More specifically, Laurent series may include expansions in negative powers as well as positive powers.

**Example (Annuli).** If we have a point  $a$ , we want to surround  $a$  by a special contour to apply Cauchy's integral formula.

In particular, for any  $z$  in the annulus, we get

$$w(z) = \frac{1}{2\pi i} \oint_{c_1-c_2} \frac{w(\zeta)}{\zeta-z} d\zeta$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \oint_{c_1} \frac{w(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{c_2} \frac{w(\zeta)}{\zeta - z} d\zeta \\
&= \sum_{n=-\infty}^{\infty} c_n (z - a)^n \\
&= c_0 + \sum_{n=1}^{\infty} (c_{-n} (z - a)^n + c_n (z - a)^n).
\end{aligned}$$

**Example.** Consider the function

$$\begin{aligned}
w(z) &= \frac{1}{z^2 + z - 2} \\
&= \frac{1}{(z - 1)(z + 2)} \\
&= \frac{1}{3} \left( \frac{1}{z - 1} - \frac{1}{z + 2} \right).
\end{aligned}$$

Now, we have three regions to expand  $w$  in.

- If  $|z| < 1$ , then our series is in both  $z^n$  and  $z^n$ .
- If  $1 < |z| < 2$ , then one of our series is going to be in  $\frac{1}{z^n}$  and one is in  $z^n$ .
- If  $|z| > 2$ , then both of our series are in the form of  $\frac{1}{z^n}$  and  $\frac{1}{z^n}$ .

Via tedious, heavily error-prone calculations, we find that

$$\begin{aligned}
w_1(z) &= -\frac{1}{3} \sum_{n=0}^{\infty} \left( 1 + (-1)^n \left( \frac{1}{2} \right)^{n+1} \right) z^n \\
w_2(z) &= \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{1}{z^{n+1}} + \left( -\frac{1}{2} \right)^{n+1} z^n \right) \\
w_3(z) &= \frac{1}{3} \sum_{n=0}^{\infty} (1 - (-2)^n) \frac{1}{z^{n+1}}.
\end{aligned}$$

Sewing all of  $w_1, w_2, w_3$  together, then we get a full series representation of  $w(z)$ .

**Definition.** If  $w(z)$  is a function that can be written as  $w(z) = (z - a)^n g(a)$ , where  $g(a) \neq 0$ , then we say  $w$  has an  $n$ -th order zero at  $z = a$ . If  $n = 1$ , then we say  $w$  has a simple zero at  $a$ .

Similarly, if we can write

$$w(z) = \frac{g(a)}{(z - a)^n}$$

with  $g(a) \neq 0$ , then we say  $w$  has a pole of order  $n$  at  $a$ . If  $n = 1$ , then we say  $w$  has a simple pole at  $a$ .

There are three types of isolated singularities (i.e., isolated points where  $w(z)$  is not defined).

**Definition.** Let  $w$  be an analytic function with isolated singularity at  $a$ .

- If  $w$  remains bounded in any neighborhood of  $a$ , then it must be the case that  $c_{-n} = 0$  for all  $n > 1$ , so the Laurent series is a pure Taylor expansion. We say  $z = a$  is a removable singularity.

For instance, the function

$$\frac{\sin(z - a)}{z - a} = \sum_{n=0}^{\infty} (-1)^n \frac{(z - a)^{2n}}{(2n + 1)!}$$

has a removable singularity at  $z = a$ .

- If not all the  $c_{-n}$  are equal to zero, but there is a largest  $n > 0$  such that  $c_{-n}$  is in the Laurent series expansion, then we say  $a$  is an  $n$ -th order pole. If  $n = 1$ , we say  $a$  is a simple pole.
- If there is no largest value of  $n$  such that  $c_{-n}$  is in the Laurent series — i.e., that  $c_{-n} \neq 0$  for all  $n$  — then we say that  $a$  is an essential singularity.

One of the most important facts about an essential singularity is that the behavior is path dependent. For instance,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$$

has an essential singularity at  $z = 0$ . We see that  $e^{1/z}$  diverges as  $z \rightarrow 0$  along the positive real axis, but if  $z \rightarrow 0$  along the negative real axis, we get  $e^{1/z} \rightarrow 0$ .

Singularities can also occur at  $\infty$ , which occurs when  $w(1/z)$  has a singularity at 0.

## Multivalued Function

Consider the function

$$\begin{aligned} w(z) &= z^2 \\ &= \underbrace{(x^2 - y^2)}_{u(x,y)} + i \underbrace{(2xy)}_{v(x,y)} \\ &= r^2 e^{2i\varphi}. \end{aligned}$$

Note that if we take a path around the origin going around by an angle of  $2\pi$ , then the resulting path goes around twice. Note that this means the lines  $\varphi$  and  $\varphi + \pi$  map to the same point in the  $w$  plane.

This isn't such a big deal in and of itself, but if we take  $w(z) = z^{1/2}$ , we get an issue. Instead of  $w$  being a two-to-one function, we now have  $w$  is a one-to-two function. This is an implicit problem in  $\mathbb{R}$  with the function  $w(x) = \sqrt{x}$ , which we resolve by taking the "positive" square root. This is known as choosing a branch.

We have to do something similar in the complex plane. Note that if we go around by an angle of  $2\pi$  in the  $z$  plane, then we only go around by an angle of  $\pi$  in the  $w$ -plane. As we keep going around the plane, we jump from branch to branch, which brings issues of continuity.

To resolve this, we create a "branch cut" that contours are not allowed to cross.

**Example.** The most common branch cut is to start from the branch point at  $z = 0$ , in the case of  $w(z) = z^{1/2}$  or  $w(z) = \ln(z)$ , and extend along the real axis, meaning our branch cut is  $(-\infty, 0]$ .

This principal branch restricts *output* values of  $\varphi$  to  $-\pi < \varphi \leq \pi$ .

For instance, above the cut, we have  $\varphi = \pi$ , and below the branch cut, we have  $\varphi = -\pi$ , meaning we have

$$\sqrt{z} = \sqrt{r} e^{i\pi/2} \quad \varphi \rightarrow \pi$$

$$\begin{aligned}
&= i\sqrt{r} \\
\sqrt{z} &= \sqrt{r}e^{-i\pi/2} \\
&= -i\sqrt{r}.
\end{aligned}
\qquad \varphi \rightarrow -\pi$$

This is why the branch cut “causes” a discontinuity across the branch, but in  $\mathbb{C} \setminus (-\infty, 0]$ .

Now, if we have

$$\begin{aligned}
\sqrt{z_1}\sqrt{z_2} &= \left(r_1 e^{i\varphi_1}\right)^{1/2} \left(r_2 e^{i\varphi_2}\right)^{1/2} \\
&= \sqrt{r_1 r_2} e^{i(\varphi_1 + \varphi_2)/2}.
\end{aligned}$$

However, if we want to calculate  $\sqrt{z_1 z_2}$ , and if  $|\varphi_1 + \varphi_2| > \pi$  then our product  $z_1 z_2$  crosses the branch cut, and our discontinuity requires  $\varphi_1 + \varphi_2$  to be converted to  $\varphi_1 + \varphi_2 \pm 2\pi$  so as to bring the angle sum back into the principal branch. This means we have

$$\begin{aligned}
\sqrt{z_1 z_2} &= \left(r_1 r_2 e^{i(\varphi_1 + \varphi_2)/2}\right)^{1/2} \\
&= \begin{cases} \sqrt{r_1 r_2} e^{i(\varphi_1 + \varphi_2)/2} & |\varphi_1 + \varphi_2| \leq \pi \\ -\sqrt{r_1 r_2} e^{i(\varphi_1 + \varphi_2)/2} & |\varphi_1 + \varphi_2| > \pi \end{cases}.
\end{aligned}$$

**Example.** Now, if we have  $z_1 = 2e^{i(3\pi/4)}$  and  $z_2 = e^{i(\pi/2)}$ , then we have

$$\begin{aligned}
\sqrt{z_1} &= \sqrt{2}e^{i3(\pi/8)} \\
\sqrt{z_2} &= e^{i(\pi/4)}.
\end{aligned}$$

Note that if we take  $\sqrt{z_1 z_2}$ , then the argument of  $z_1 z_2$  is  $5\pi/4$ , so we have to change our argument to  $-3\pi/4$  to return to the principal branch before we may calculate the square root. This gives

$$\begin{aligned}
\sqrt{z_1 z_2} &= \sqrt{2e^{-i(3\pi/4)}} \\
&= \sqrt{2}e^{-i\pi + i(5\pi/8)} \\
&= -\sqrt{2}e^{i(5\pi/8)} \\
&= -\sqrt{z_1}\sqrt{z_2}.
\end{aligned}$$

Now, it is possible to have a branch point at  $\infty$ , by determining if  $w(\frac{1}{z})$  has a branch point at zero. For instance, if  $w = z^{1/2}$ , this gives

$$\begin{aligned}
w\left(\frac{1}{z}\right) &= \frac{1}{z^{1/2}} \\
&= \frac{1}{\sqrt{r}} e^{-i\varphi/2},
\end{aligned}$$

which has the multivalued behavior around the origin. Thus,  $z = \infty$  is a branch point for  $z$ , and we consider the  $(-\infty, 0]$  branch cut that connects the branch points at 0 and  $\infty$ .

**Example.** Consider

$$w(z) = \sqrt{(z-a)(z-b)}.$$

where  $a, b \in \mathbb{R}$  with  $a < b$ . We expect the only finite branch points to be  $a$  and  $b$ . Introducing polar coordinates, we have

$$r_1 e^{i\varphi_1} = z - a$$



$$r_2 e^{i\varphi_2} = z - b,$$

giving

$$w(z) = \sqrt{r_1 r_2} e^{i\varphi_1} e^{i\varphi_2}.$$

Closed contours around *either*  $a$  or  $b$  are double-valued. However, if our closed contour goes around *both*  $a$  and  $b$ , then both  $\varphi_1$  and  $\varphi_2$  add up to  $2\pi$ , meaning we don't have the multivalued behavior.

Now, to select our branch cut, we need to find out if the point at infinity is a branch point. We take  $\zeta = \frac{1}{z}$ , and we have

$$w(\zeta) = \frac{1}{\zeta} \sqrt{(1 - a\zeta)(1 - b\zeta)},$$

which blows up at  $\infty$ , but only takes a singular value.<sup>III</sup>

In general,  $z^{1/m}$  for integral  $m$  will require  $m$  branch cuts.

**Example.** Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{x e^{ikx}}{\sqrt{x^2 + a^2}} dx.$$

This is a hard integral to evaluate. To resolve this, we extend the integrand to the complex plane, and invoke Cauchy's theorem to deform the contour.

Note that  $\sqrt{x^2 + a^2}$  is multivalued, with branch points at  $x = \pm ia$ . We choose the branch cut such that our integration contour does not cross the branch cut — i.e., from  $-ia$  to  $\infty$  to  $ia$ .

Now, we may deform the contour so as to closely wrap around the branch cut from  $ia$  to  $\infty$ . Remembering the sign discontinuity over the branch cut, this gives the integral

$$\begin{aligned} \int_{i\infty}^{i\infty} \frac{z e^{ikz}}{\sqrt{z^2 + a^2}} dz &= \int_{i\infty}^{ia} \frac{z e^{ikz}}{-i\sqrt{z^2 + a^2}} dz + \int_{-a}^{\infty} \frac{z e^{ikz}}{i\sqrt{z^2 + a^2}} dz \\ &= 2 \int_{ia}^{i\infty} \frac{z e^{ikz}}{i\sqrt{z^2 + a^2}} dz \\ &= 2 \int_a^{\infty} \frac{y e^{-ky}}{\sqrt{y^2 - a^2}} dy \quad z = iy \\ &= 2aK_1(ka) \\ &\sim e^{-ka} \end{aligned}$$

Here,  $K_1$  refers to the modified Bessel function.

## Logarithms

In the complex plane, we say

$$\begin{aligned} \ln z &= \ln(re^{i\varphi}) \\ &= \ln r + i\varphi \\ &= \ln|z| + i \arg(z). \end{aligned}$$

<sup>III</sup>Alternatively, we may see that a positively-oriented contour that surrounds both  $a$  and  $b$  is a negatively-oriented contour around  $\infty$ . Since such a contour is valid,  $\infty$  is not a branch point.

Unfortunately, this  $\ln z$  is a multivalued function — a very multivalued one indeed. This yields many branch points, including 0 and  $\infty$ :

$$\ln(1/\zeta) = -\ln(\zeta).$$

However, we choose the principal branch,  $\pi < \varphi \leq \pi$ , giving

$$\text{Ln } z = \text{Ln}|z| + i \text{Arg}(z).$$

**Example.** Consider  $\ln(z_1 z_2)$  and  $\text{Ln}(z_1 z_2)$ . If we have

$$z_1 = 1 + i$$

$$z_2 = i,$$

then

$$\arg(z_1) = \pi/4$$

$$\arg(z_2) = \pi/2,$$

so

$$\arg(z_1 z_2) = 3\pi/4$$

$$= \arg(z_1) + \arg(z_2)$$

$$= \text{Arg}(z_1 z_2).$$

However, if  $z_1 = z_2 = -1$ , then

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$= 2\pi$$

$$\text{Arg}(z_1 z_2) = \text{Arg}(1)$$

$$= 0.$$

Thus, we get that  $\text{Ln}(z_1 z_2) \neq \text{Ln}(z_1) + \text{Ln}(z_2)$ .

**Example** (Logarithms vs Inverse Trig). Here, we will derive  $\arctan(z)$  in terms of the complex logarithm.

Recall that

$$\cos(z) = \frac{1}{2} \left( e^{iz} + e^{-iz} \right)$$

$$\sin(z) = \frac{1}{2i} \left( e^{iz} - e^{-iz} \right),$$

so we have

$$z = \tan(w)$$

$$= -i \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}},$$

which after much tedious, error-prone symbolic manipulation, gives

$$e^{2iw} = \frac{i - z}{i + z}.$$

Thus, we have

$$w = \arctan(z)$$

$$= \frac{1}{2i} \ln \left( \frac{i-z}{i+z} \right).$$

Note that since  $\ln$  has branch points at 0 and  $\infty$ ,  $\ln \left( \frac{i-z}{i+z} \right)$  has branch points when  $z = \pm i$ .

Now, we must choose a branch cut. Specifically, we want our branch cut to continue the real  $\arctan(x)$ . We dub this  $\text{Arctan}(x)$ . Along the real axis, we have

$$\begin{aligned} \text{Arctan}(x) &= \frac{1}{2i} \text{Ln} \left( \frac{i-x}{i+x} \right) \\ &= \frac{1}{2i} \left( \text{Ln} \left| \frac{i-x}{i+x} \right| + i \text{Arg} \left( \frac{i-x}{i+x} \right) \right) \\ &= \frac{1}{2} \text{Arg} \left( \frac{i-x}{i+x} \right). \end{aligned}$$

The principal values are from  $-\pi$  to  $\pi$ , so the output of  $\text{Arctan}(x)$  ranges from  $-\pi/2$  to  $\pi/2$ .

## Conformal Maps

A conformal map is a special type of map  $w: \mathbb{C} \rightarrow \mathbb{C}$  that “preserves angles.” If, in  $z$ , we map curves whose intersections are at some angle  $\varphi$ , then the image of those curves also intersect at the angle  $\varphi$ .

**Example** (Our First Conformal Map). Consider the map

$$\begin{aligned} w(z) &= z^2 \\ &= (x^2 - y^2) + i(2xy) \\ &= u(x, y) + iv(x, y). \end{aligned}$$

Examining the line elements in the  $z$  and  $w$  planes, we have

$$\begin{aligned} ds^2 &= du^2 + dv^2 \\ &= \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right)^2 + \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right)^2 \\ &= \left( \frac{\partial u}{\partial x} dx - \frac{\partial v}{\partial x} dy \right)^2 + \left( \frac{\partial v}{\partial x} dx + \frac{\partial u}{\partial x} dy \right)^2 \\ &= \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right) (dx^2 + dy^2) \\ &= \left( \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) (dx^2 + dy^2) \\ &= 4(x^2 + y^2) (dx^2 + dy^2) \end{aligned}$$

Note that  $dx^2$  and  $dy^2$  have identical scale factors. Since angles are determined by the ratio of  $dx$  and  $dy$ , it is the case that *all* angles are preserved. This is what is meant by a conformal map.

**Example** (Analyticity and Conformality). Consider an analytic function  $w(z)$ , with its Taylor expansion about  $z_0$ .

$$w(z) = w(z_0) + w'(z_0)(z - z_0) + \cdots.$$

For a very small  $\xi = z - z_0$ , we may truncate it into first order, and place into polar form

$$w(z) - w(z_0) = w'(z_0)\xi,$$

$$= |w'(z_0)|e^{i\alpha_0}\xi.$$

Moving from  $z$  to  $w$ , we get a magnification (or shrinkage) by  $|w'(z_0)|$  and a rotation by  $\alpha_0$ .

Since, close to  $z_0$ ,  $\xi_1 = z_1 - z_0$  and  $\xi_2 = z_2 - z_0$  are magnified by (effectively) the same amount, and rotated by (effectively) the same amount, conformality is established.

**Definition.** A conformal map is an analytic function  $w(z)$  defined on a domain  $\Omega$  such that  $w'(z_0) \neq 0$  for all  $z_0 \in \Omega$ .

**Example (Möbius Transformations).** A Möbius transformation is a fractional linear transformation of the form

$$w(z) = \frac{az + b}{cz + d},$$

where  $ad - bc \neq 0$ . We can calculate  $w'(z)$  to be

$$w'(z) = \frac{ad - bc}{(cz + d)^2}.$$

Since  $w(z)$  is conformal, it is invertible, so

$$\begin{aligned} w^{-1}(z) &= z(w) \\ &= \frac{dw - b}{-cw + a}. \end{aligned}$$

The Möbius transformations include  $\infty$ , as we have  $w(\infty) = \frac{a}{c}$ , meaning that it is an automorphism of the Riemann sphere. Note that because of the constraint, we only need three numbers to specify a Möbius transformation.

Consider the Möbius transformation

$$w(z) = \frac{z - i}{z + i}.$$

We let  $z_1 = -1$ ,  $z_2 = 1$ , and  $z_3 = \infty$ . Then, we have

$$\begin{aligned} w(z_2) &= \frac{-1 - i}{-1 + i} \\ &= \frac{2i}{2} \\ &= i. \end{aligned}$$

Similarly, this gives  $w(z_3) = 1$ . After a bit more playing, we can find that this is a map of the (closed) upper half-plane to the (closed) unit disk,  $\mathbb{D}$ .

Now, if we look at the “ribbon” between the real axis and the line  $\text{Im}(z) = i$ , we see that it maps to the region

$$S = \mathbb{D} \setminus \left\{ z \mid \left| z - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$

**Example.** Consider the map  $w(z) = e^z$ . This gives

$$\begin{aligned} w(z) &= e^x e^{iy} \\ &= \rho e^{i\beta}. \end{aligned}$$

This sends curves of constant  $y$  to curves of constant argument, and maps curves of constant  $x$  to circles of constant radius.

## Complex Potentials

Consider the analytic function

$$\Omega(z) = \Phi(x, y) + i\Psi(x, y).$$

We know that

$$\begin{aligned}\frac{\partial \Phi}{\partial x} &= \frac{\partial \Psi}{\partial y} \\ \frac{\partial \Phi}{\partial y} &= -\frac{\partial \Psi}{\partial x}.\end{aligned}$$

Thus, we separate to get

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial \Psi}{\partial y} \\ &= \frac{\partial}{\partial y} \frac{\partial \Psi}{\partial x} \\ &= -\frac{\partial^2 \Phi}{\partial y^2},\end{aligned}$$

so

$$\begin{aligned}\nabla^2 \Phi &= 0 \\ \nabla^2 \Psi &= 0.\end{aligned}$$

The converse is also true — if there is some real harmonic function  $\Phi(x, y)$ , there is a conjugate harmonic function  $\Psi(x, y)$  such that  $\Omega(z) = \Phi(x, y) + i\Psi(x, y)$  is analytic.

If  $\Omega$  is analytic, then  $\Phi$  and  $\Psi$  must satisfy the Cauchy–Riemann equations, meaning that

$$\begin{aligned}\Psi(x, y) &= \int \frac{\partial \Psi}{\partial y} dy + \frac{\partial \Psi}{\partial x} dx \\ &= \int \frac{\partial \Phi}{\partial x} dy - \frac{\partial \Phi}{\partial y} dx.\end{aligned}$$

For  $\Psi$  to be a proper single-valued real function, the integral must be path-independent. Using Green's theorem, we may close the path in a simply connected region, and consider it as a surface integral. This gives

$$\begin{aligned}\oint_C \frac{\partial \Phi}{\partial x} dy - \frac{\partial \Phi}{\partial y} dx &= \int_S \left( \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y} \left( -\frac{\partial \Phi}{\partial y} \right) \right) dx dy \\ &= \int_S \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) dx dy \\ &= 0.\end{aligned}$$

We call  $\Omega(z) = \Phi(x, y) + i\Psi(x, y)$  the complex potential.

This gives

$$\begin{aligned}\frac{d\Omega}{dz} &= \frac{\partial \Phi}{\partial x} + i \frac{\partial \Psi}{\partial x} \\ &= \frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y}\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial \Psi}{\partial y} + i \frac{\partial \Psi}{\partial x} \\
&= \bar{\mathcal{E}},
\end{aligned}$$

where  $\mathcal{E}$  is the complex representation of the electric field,  $\mathbf{E}$ . We have

$$\begin{aligned}
\mathcal{E} &= \overline{\frac{\partial \Omega}{\partial z}} \\
&= \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y},
\end{aligned}$$

with

$$\mathbf{E} = \left| \frac{d\Omega}{dz} \right|.$$

The physics of electric fields is then determined entirely by the complex potential.

What makes harmonic functions useful is that, if there are complicated boundary conditions, we may apply a conformal map and the functions remain harmonic.

**Example (Cylindrical Capacitor).** Consider a cylindrical capacitor with nonconcentric plates meeting at insulated point  $u = 1$  and  $v = 0$ . The larger cylinder with radius 1 is grounded, and the smaller cylinder with radius  $1/2$  is held at voltage  $V_0$ . We want to find the electric field.

We want to find  $\tilde{\Phi}(w)$  such that

$$\nabla^2 \tilde{\Phi}(u, v) = 0.$$

This domain is kind of difficult, so we will solve the problem on a simpler domain and use a conformal map. Note that from Figure 20.4 in the book, we may use the Möbius transformation

$$w(z) = \frac{z - i}{z + i}$$

to transform *to* our cylindrical capacitor *from* a two-plate infinite capacitor with one plate at  $\text{Im}(z) = 1$  and one plate at  $\text{Im}(z) = 0$ . From physics, we know that  $\Phi(x, y) = \frac{V_0 y}{d}$ , where  $d = 1$ . Thus, the harmonic conjugate,  $\Psi = -V_0 x$ , gives us a complex potential of  $\Phi = -iV_0 z$ .

Solving

$$\frac{z - i}{z + i} = u(x, y) + iv(x, y),$$

we find

$$\begin{aligned}
x(u, v) &= -\frac{2v}{(1 - u)^2 + v^2} \\
y(u, v) &= \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}.
\end{aligned}$$

Now, this gives

$$\begin{aligned}
\tilde{\Phi}(u, v) &= \Phi(x(u, v), y(u, v)) \\
&= V_0 \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}.
\end{aligned}$$

**Example (Fluid Flow).** Consider fluid flow around a rock with disk of radius  $a$ ; far away from the rock, we have uniform flow speed of  $\alpha$ .

Symmetry allows us to focus only on the upper half-plane. Now, there is a conformal map in Table 20.1 of the textbook, which is the map  $w(z) = z + \frac{a^2}{z} = u(x, y) + iv(x, y)$  that maps the upper half-plane to the upper half-plane. Furthermore, this map sends the boundary hugging the rock into the  $u$ -axis.

After applying the conformal map, we get the stream lines  $\tilde{\Psi}(u, v) = \beta v$ , as they are streamlines of uniform horizontal flow.

Building the complex potential, we have

$$\begin{aligned}\tilde{\Omega}(w) &= \Phi(u, v) + i\Psi(u, v) \\ \tilde{\Omega}(w) &= \beta w,\end{aligned}$$

as we must have  $\frac{d\Phi}{du} = \frac{d\Psi}{dv} = \beta$ .

Mapping back into the  $z$ -plane, we have

$$\Omega(z) = \beta \left( z + \frac{a^2}{z} \right).$$

Note that as  $z$  becomes very big, the term  $\frac{a^2}{z}$  goes to 0, so we must have  $\beta = \alpha$ .

Now, we may find the streamlines and potentials. Note that we have

$$\begin{aligned}\Phi &= \text{Re}(\Omega) \\ \Psi &= \text{Im}(\Omega).\end{aligned}$$

Now, we have

$$\begin{aligned}\Omega(z) &= \alpha r \left( e^{i\varphi} + \frac{a^2}{r^2} e^{-i\varphi} \right) \\ &= \alpha r \left( \cos(\varphi) + i \sin(\varphi) + \frac{a^2}{r^2} (\cos(\varphi) - i \sin(\varphi)) \right).\end{aligned}$$

Taking real and imaginary parts, we have

$$\begin{aligned}\Phi &= \alpha r \left( 1 + \frac{a^2}{r^2} \right) \cos(\varphi) \\ \Psi &= \alpha r \left( 1 - \frac{a^2}{r^2} \right) \sin(\varphi).\end{aligned}$$

**Example.** Considering our conformal map

$$w(z) = z + \frac{a^2}{z}$$

again, we see that if  $|z| = a$ , then  $|u| \leq 2a$ . Meanwhile, if  $r > a$ , then

$$\begin{aligned}w(z) &= z + \frac{a^2}{z} \\ &= r e^{i\varphi} + \frac{a^2}{r} e^{-i\varphi}\end{aligned}$$

$$\begin{aligned}
&= \left(r + \frac{a^2}{r}\right) \cos(\varphi) + i \left(r - \frac{a^2}{r}\right) \sin(\varphi) \\
&= u + iv.
\end{aligned}$$

This gives

$$\frac{u^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{v^2}{\left(r - \frac{a^2}{r}\right)^2} = 1.$$

Note that  $w$  fails to be conformal when  $\frac{dw}{dz} = 0$ , meaning that it fails to be conformal at  $z = \pm a$ .

This is occasionally used in the real world<sup>IV</sup> to design airfoils.

## Residues

Consider a function  $f(z)$  with an  $n$ th order pole. Then,  $f$  can be written as

$$f(z) = \frac{g(z)}{(z - a)^n},$$

where  $g(z)$  is analytic and  $g(a) \neq 0$ . Recalling Cauchy's integral formula, we see that this expression for  $f$  is tantalizingly close to our desired state.

We may expand  $g$  in a Taylor series:

$$g(z) = \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!} (z - a)^m.$$

Letting  $C$  be a positively oriented contour in the analytic domain of  $f$  that encircles the singularity, we get

$$\oint_C f(z) dz = \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!} \oint_C (z - a)^{m-n} dz.$$

Note that if  $m - n \neq -1$ , then the integral on the right vanishes, so we only obtain a nonzero contribution at  $m = n - 1$ . Thus, we get

$$\oint_C f(z) dz = 2\pi i \frac{g^{(n-1)}(a)}{(n-1)!}.$$

**Definition.** Let  $f(z)$  be an analytic function with a pole at  $z = a$  with order  $n$ . We define the residue of  $f$  at  $a$  as

$$\text{Res}[f(z), a] := \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} ((z - a)^n f(z)).$$

This gives an alternative statement of Cauchy's integral formula, giving

$$\oint_C f(z) dz = 2\pi i \text{Res}[f(z), a].$$

However, when we have lots of poles for  $f$ , and  $C$  is a contour that surrounds all the poles, we may deform  $C$  such that it surrounds each pole. This gives the residue theorem.

**Theorem (Residue Theorem):**

$$\oint_C f(z) dz = 2\pi i \sum_{a \in C} \text{Res}[f(z), a] \quad (++)$$



Type	Method
n-th order pole	$\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} ((z-a)^n f(z))$
simple pole	$\lim_{z \rightarrow a} (z-a)f(z)$
$f = \frac{p}{q}, q(a)$ simple zero	$\frac{p(a)}{q'(a)}$
pole at infinity	$\lim_{z \rightarrow 0} \left( -\frac{1}{z^2} f\left(\frac{1}{z}\right) \right)$
pole at infinity, $\lim_{ z  \rightarrow \infty} f(z) = 0$	$-\lim_{ z  \rightarrow \infty} (zf(z))$

Table 1: Finding  $\text{Res}[f(z), a]$ 

We can find the residue in a variety of ways.

**Example.** We will find the residue for  $\cot(z)$  for each of the residues.

$$\begin{aligned}
 \text{Res}[\cot(z), n\pi] &= \lim_{z \rightarrow n\pi} (z - n\pi) \frac{\cos(z)}{\sin(z)} \\
 &= (-1)^n \lim_{z \rightarrow n\pi} \frac{z - n\pi}{\sin(z)} \\
 &= (-1)^n \lim_{z \rightarrow n\pi} \frac{z - n\pi}{(-1)^n \sin(z - n\pi)} \\
 &= 1.
 \end{aligned}$$

**Example.** We may find

$$\begin{aligned}
 \text{Res}\left[\frac{z}{\sinh(z)}, i\pi\right] &= \left. \frac{z}{\frac{d}{dz}(\sinh(z))} \right|_{z=i\pi} \\
 &= \frac{i\pi}{\cosh(i\pi)} \\
 &= (-1)^n i\pi
 \end{aligned}$$

**Example.** Let's evaluate

$$\oint_C \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

Finding the residue at each pole, we get

$$\begin{aligned}
 \text{Res}[f(z), 0] &= \frac{2}{3} \\
 \text{Res}[f(z), -1] &= -\frac{3}{2} \\
 \text{Res}[f(z), 3] &= -\frac{1}{6}.
 \end{aligned}$$

These are evaluated using the [cover-up method](#).

Now, we may find the integral by taking

$$\oint_{|z|=2} f(z) dz = -i \frac{5\pi}{3}.$$

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<sup>VI</sup>I guess people do things over there.

**Example.** Let

$$\begin{aligned} f(z) &= \frac{1}{z^2 \sinh(z)} \\ &= \frac{1}{-iz^2 \sin(iz)}. \end{aligned}$$

The simple zeros of  $\sinh(z)$  are at  $i\pi$ , so we have an order 3 pole at  $z = 0$

$$\begin{aligned} \text{Res}[f(z), 0] &= \frac{1}{(n-1)!} \frac{d^2}{dz^2} [z^3 f(z)] \Big|_{z=0} \\ &= \frac{1}{2} \frac{d^2}{dz^2} \left( \frac{z}{\sinh(z)} \right) \Big|_{z=0} \\ &= -\frac{1}{6}. \end{aligned}$$

Thus, integrating about the unit circle, we get

$$\oint_{|z|=1} = -\frac{i\pi}{3}.$$

If we were to evaluate via the Laurent series, we would have

$$\begin{aligned} \frac{1}{z^2 \sinh(z)} &= \frac{1}{z^2} \left( \frac{1}{z + z^2/3 + z^5/5! + \dots} \right) \\ &= \frac{1}{z^3} \left( \frac{1}{1 + z^2/3! + z^4/5! + \dots} \right) \\ &\approx \frac{1}{z^3} \left( 1 - \frac{z^2}{3!} + \dots \right) \\ &= \frac{1}{z^3} - \frac{1}{6z} + \dots, \end{aligned}$$

giving a residue of  $-\frac{1}{6}$ .

Instead of using the contour on the unit circle, if we want to use a circle of radius 4, we get the residues at  $z = \pm i\pi$ . To evaluate this, we take

$$\begin{aligned} \text{Res}[f(z), i\pi] &= \frac{1}{-\pi^2(-1)} \\ &= \frac{1}{\pi^2} \\ \text{Res}[f(z), -i\pi] &= \frac{1}{\pi^2}. \end{aligned}$$

Evaluating the integral, we would get

$$\begin{aligned} \oint_C f(z) dz &= 2\pi i \left( -\frac{1}{6} + \frac{2}{\pi^2} \right) \\ &= -\frac{i\pi}{3} + \frac{4i}{\pi}. \end{aligned}$$

**Example.** We will now use the residue theorem to evaluate a real-valued integral. Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx.$$

Since this integral goes to zero, we will evaluate

$$I' = \oint_C \frac{1}{z^2 + 1} dz,$$

where  $C$  is a semicircle with radius  $r$  along the real axis from  $-r$  to  $r$  “pointing upward,” so to speak.

This gives

$$\oint_C \frac{1}{z^2 + 1} dz = \int_{C_r} f(z) dz + \int_{-r}^r f(x) dx,$$

which, sending  $r$  to infinity, is equal to

$$I = \int_{-\infty}^{\infty} f(x) dx.$$

However, since our expression  $\frac{1}{z^2+1}$  has poles at  $i$  and  $-i$ , our semicircle gives

$$\begin{aligned} \oint_C \frac{1}{z^2 + 1} &= 2\pi i \operatorname{Res}[f(z), i] \\ &= 2\pi i \lim_{z \rightarrow i} \frac{1}{z + i} \\ &= 2\pi i \frac{1}{2i} \\ &= \pi. \end{aligned}$$

If we have a finite number of isolated singularities, we are always able to draw a contour that encloses all of them, which allows us to use the residue theorem.

Now, we know that we can have poles at infinity — and that any positively-oriented contour in the plane is a negatively-oriented contour around  $\infty$ . Thus, if we have a contour surrounding all our finite singularities, we get

$$\begin{aligned} \sum_i \operatorname{Res}[f(z), a_i] &= -\operatorname{Res}[f(z), \infty] \\ \operatorname{Res}[f(z), \infty] + \sum_i \operatorname{Res}[f(z), a_i] &= 0, \end{aligned}$$

as we’re doing the same integral, but in negative orientation about  $\infty$  and positive orientation about our singularities.

We have

$$\operatorname{Res}[f(z), \infty] = \operatorname{Res}\left[-\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right].$$

**Example.** Now, recalling

$$f(z) = \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

The residues are

$$\begin{aligned} \operatorname{Res}[f(z), 0] &= 2/3 \\ \operatorname{Res}[f(z), -1] &= -3/2 \end{aligned}$$

$$\text{Res}[f(z), 3] = -1/6.$$

Now, calculating the residue at infinity, we have

$$\begin{aligned}\text{Res}[f(z), \infty] &= \text{Res}\left[-\frac{1}{z^2} \frac{(1/z - 1)(1/z - 2)}{1/z(1/z + 1)(3 - 1/z)}, 0\right] \\ &= -\text{Res}\left[\frac{(z - 1)(2z - 1)}{z(z + 1)(3z - 1)}\right] \\ &= 1.\end{aligned}$$

Now, if  $\lim_{|z| \rightarrow \infty} f(z) = 0$ , then  $f$  is pure Laurent series. In that case, if there is a residue, then we find the residue by evaluating

$$\text{Res}[f(z), \infty] = - \lim_{|z| \rightarrow \infty} zf(z)$$

**Example.** Consider functions of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where  $q$  is a higher-order polynomial than  $p$ .

If  $q$  has first-order zeros  $a$  and second-order zeros at  $b$ , then

$$f(z) = \sum_{k=1}^n \frac{A_k}{z - a_k} + \frac{B_k}{z - b_k} + \frac{C_k}{(z - b_k)^2}.$$

Note that the coefficients are actually residues. This gives

$$\begin{aligned}A_k &= \text{Res}[f(z), a_k] \\ B_k &= \text{Res}[f(z), b_k] \\ C_k &= \text{Res}[(z - b_k)f(z), b_k].\end{aligned}$$

For instance,

$$\frac{(z - 1)(z - 2)}{z(z + 1)(3 - z)} = \frac{2}{3} \frac{1}{z} - \frac{1}{6} \frac{1}{z - 3} - \frac{3}{2} \frac{1}{z + 1}.$$

Now, we may also have

$$\frac{(z - 1)(z - 2)}{z(z + 1)^2(3 - z)} = \frac{2}{3} \frac{1}{z} - \frac{1}{24} \frac{1}{z - 3} - \frac{5}{8} \frac{1}{z + 1} - \frac{3}{2} \frac{1}{(z + 1)^2}.$$

### Integrating around a Circle

We want to evaluate angular integrals of the form

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi)) d\varphi.$$

Now, while this is a real integral over a domain, we may reformulate it about the unit circle by using the substitutions

$$\begin{aligned}z &= e^{i\varphi} \\ d\varphi &= \frac{dz}{iz},\end{aligned}$$

which yields

$$\begin{aligned}\sin(n\varphi) &= \frac{1}{2i} \left( z^n - \frac{1}{z^n} \right) \\ \cos(m\varphi) &= \frac{1}{2} \left( z^m + \frac{1}{z^m} \right).\end{aligned}$$

Thus, our integral becomes

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi)) d\varphi = \oint_{|z|=1} f\left(\frac{1}{2i} \left( z^n - \frac{1}{z^n} \right), \frac{1}{2} \left( z^m + \frac{1}{z^m} \right)\right) \frac{dz}{iz}.$$

**Example.** Consider

$$\begin{aligned}\int_0^{2\pi} \sin^2(\varphi) d\varphi &= -\frac{1}{4} \oint_{|z|=1} \left( z - \frac{1}{z} \right)^2 \frac{dz}{iz} \\ &= \frac{i}{4} \oint_{|z|=1} \frac{1}{z^3} (z^4 - 2z^1 + 1) dz \\ &= -\frac{1}{2} \pi \operatorname{Res} \left[ \frac{1}{z^3} (z^4 - 2z^1 + 1), 0 \right].\end{aligned}$$

The residue at  $z = 0$  is  $-2$  — this can be found by dividing out by  $z^3$ .

Thus, we get the answer of

$$\int_0^{2\pi} \sin^2(\varphi) d\varphi = \pi.$$

**Example.** Using residues, we can evaluate a lot of integrals that are quite tricky on their face.

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\varphi)}{5 - 4\sin(\varphi)} d\varphi &= \oint_{|z|=1} \frac{\frac{1}{2} \left( z^2 + \frac{1}{z^2} \right)}{5 - \frac{4}{2i} \left( z - \frac{1}{z} \right)} \frac{dz}{iz} \\ &= -\oint_{|z|=1} \frac{z^4 + 1}{2z^2(2z - i)(z - 2i)} dz.\end{aligned}$$

Now, we have a simple pole at  $i/2$ , a simple pole at  $2i$ , and a pole of order 2 at 0. We only evaluate the residues at 0 and  $i/2$ . We get

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= -\frac{d}{dz} \left( \frac{z^4 + 1}{2z^2(2z - i)(z - 2i)} \right) \Big|_{z=0} \\ &= -\frac{5i}{8} \\ \operatorname{Res}[f(z), i/2] &= \frac{17i}{24}.\end{aligned}$$

Thus, we get the result of

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\varphi)}{5 - 4\sin(\varphi)} d\varphi &= 2\pi i \left( -\frac{5i}{8} + \frac{17i}{24} \right) \\ &= -\frac{\pi}{6}.\end{aligned}$$

### Integrating along the Real Axis

If we want to evaluate integrals along the real axis, such as

$$I = \int_{-\infty}^{\infty} f(x) dx,$$

we may be curious as to how we may evaluate this.

To do this, we recall that we created a contour in the upper half-plane of large enough radius  $r$ , and evaluated the residues inside the contour. We consider the contour to be equal to  $C = C_r + l_r$ , where  $l_r$  is along the real axis and  $C_r$  closes our contour. Thus, we get

$$\oint_C f(z) dz = \lim_{r \rightarrow \infty} \left( \int_{-r}^r f(x) dx + \int_{C_r} f(z) dz \right).$$

Note that the polar coordinate Jacobian gives us the requirement that  $\lim_{|z| \rightarrow \infty} |zf(z)| = 0$ .

When  $f(z) = \frac{p(z)}{q(z)}$ , this is satisfied when  $q$  is of degree at least two more than that of  $p$ .

**Example.** Consider

$$\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4} dx = \oint_C \frac{2z+1}{z^4+5z^2+4} dz.$$

Factoring, we get

$$\oint_C \frac{2z+1}{z^4+5z^2+4} dz = \oint_C \frac{2z+1}{(z-2i)(z+2i)(z-i)(z+i)} dz.$$

We only care about the residues in the upper half-plane. We have residues of

$$\begin{aligned} \text{Res}[f(z), 2i] &= -\frac{1}{3} + \frac{i}{12} \\ \text{Res}[f(z), i] &= \frac{1}{3} - \frac{i}{6}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4} dx &= 2\pi i \left( \frac{1}{3} - \frac{i}{6} - \frac{1}{3} + \frac{i}{12} \right) \\ &= \frac{\pi}{6}. \end{aligned}$$

Note that if we chose our contour to be in the lower half-plane, then we would have a *negatively* oriented contour, and evaluate at the residues in the lower half-plane.

**Example.** Consider

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^3-i} dx &= \oint_C \frac{1}{z^3-i} dz \\ &= \oint_C \frac{1}{(z+i)(z-e^{i\pi/6})(z-e^{5i\pi/6})}. \end{aligned}$$

Closing  $C$  in the lower half-plane, we only need the residue at  $-i$ . This gives

$$\begin{aligned} \oint_C \frac{1}{z^3-i} &= -2\pi i \left( -\frac{1}{3} \right) \\ &= \frac{2\pi i}{3}. \end{aligned}$$

Consider integrals of the form

$$\int_{-\infty}^{\infty} g(x)e^{ikx} dx,$$

where  $k$  is real.

Now, we want to know when exactly we are allowed to “close up” the semicircle contour.

We start by assuming  $k$  is positive. Closing in the upper half-plane so as to ensure exponential decay, we have

$$\begin{aligned} \left| \int_{C_r} g(z)e^{ikz} dz \right| &\leq \int_{C_r} |g(z)e^{ikz}| dz \\ &= \int_0^\pi |g(re^{i\varphi})| re^{-kr \sin(\varphi)} d\varphi. \end{aligned}$$

Since  $\sin(\varphi) \geq 0$  on the range of integration, the integral vanishes as  $r \rightarrow \infty$ . Therefore, we are allowed to close up the contour whenever  $|g(z)| \rightarrow 0$  as  $|z| \rightarrow \infty$ .

**Example.** Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + 4} dx.$$

This gives

$$\begin{aligned} I &= \operatorname{Re} \left( \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + 4} dx \right) \\ &= \operatorname{Re} \left( \oint_C \frac{e^{ikz}}{z^2 + 4} dz \right) \\ &= \oint_C \frac{e^{ikz}}{(z - 2i)(z + 2i)} dz. \end{aligned}$$

We assume  $k > 0$ . Then, evaluating at  $2i$ , we have

$$I = \frac{\pi}{2} e^{-2k}.$$

Now, if  $k < 0$ , we close our contour in the lower half-plane, we get

$$I = \frac{\pi}{2} e^{2k}.$$

Thus, our integral is always

$$I = \frac{\pi}{2} e^{-2|k|}.$$

### Non-Circular Contours

Sometimes, semicircles don't work.

**Example.** Consider

$$\int_{-\infty}^{\infty} \frac{e^{bx}}{e^x + 1} dx,$$

where  $0 < b < 1$ . Writing our integral, we have

$$I = \int \frac{e^{bz}}{e^z + 1} dz$$

This gives poles at  $z = (2n + 1)i\pi$ , which means we cannot close this contour with a semicircular arc at  $\infty$ .

What may work in this case is by drawing a rectangular contour from  $-a$  to  $a$  such that it encloses exactly one of the poles of our integrand. The vertical segments of this contour go to zero as we send  $a \rightarrow \infty$ . We call the segment of the contour along the line  $a + 2\pi i$  to  $a - 2\pi i$  as  $I'$ .

This gives

$$I + I' = \oint_C \frac{e^{bz}}{e^z + 1} dz.$$

Now, we constructed  $I'$  such that

$$\begin{aligned} I' &= \int_{\infty}^{-\infty} \frac{e^{b(x+2\pi i)}}{e^{x+2\pi i} + 1} dx \\ &= -e^{2\pi i b} \int_{-\infty}^{\infty} \frac{e^{bx}}{e^x + 1} dx \\ &= -e^{2\pi i b} I. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \oint_C \frac{e^{bz}}{e^z + 1} dz &= I(1 - e^{2\pi i b}) \\ &= 2\pi i \operatorname{Res} \left[ \frac{e^{bz}}{e^z + 1}, i\pi \right], \end{aligned}$$

giving

$$I = \frac{\pi}{\sin(\pi b)}.$$

**Example.** We want to evaluate

$$\begin{aligned} \int_0^{\infty} \cos(x^2) dx \\ \int_0^{\infty} \sin(x^2) dx. \end{aligned}$$

To evaluate this, we draw a slice-shaped contour going along the real axis and returning to 0 along  $z = re^{i\pi/4}$ . Therefore, we evaluate

$$\begin{aligned} \oint_C e^{iz^2} dz &= \int_0^{\infty} e^{ix^2} dx + 0 + \int_{\infty}^0 e^{i(re^{i\pi/4})^2} e^{i\pi/4} dr \\ &= \int_0^{\infty} e^{ix^2} dx + 0 + \int_{\infty}^0 e^{-r^2} e^{i\pi/4} dr. \end{aligned}$$

Thus, we get

$$\begin{aligned} \int_0^{\infty} \cos(x^2) dx &= \int_0^{\infty} \sin(x^2) dx \\ &= \frac{1}{2} \sqrt{\frac{\pi}{2}}. \end{aligned}$$



### Integrating with Branch Cuts

When we're integrating with residues, branch cuts are a feature rather than a bug.

**Example.** Consider the integral

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^3} dx.$$

We need a branch cut to avoid the multivalued behavior. Our poles are at  $e^{i\pi/3}, -1, e^{-i\pi/3}$ . Since our integral is along the real axis, we take our branch cut along the domain  $[0, \infty]$ .

We draw our contour of radius  $R$  by hugging the branch without crossing it, with a small circle of radius  $\epsilon$  just outside 0. This gives the integral

$$\oint \frac{\sqrt{z}}{1+z^3} dz = \int_0^\infty \frac{\sqrt{z}}{1+z^3} dx + \int_{C_R} \frac{\sqrt{z}}{1+z^3} dz + \int_\infty^0 \frac{\sqrt{z}}{1+z^3} dz + \int_{C_\epsilon} \frac{\sqrt{z}}{1+z^3} dz.$$

Note that since  $\lim_{|z| \rightarrow \infty} |zf(z)| = 0$ , and  $\lim_{|z| \rightarrow 0} |zf(z)| = 0$ , our integrals along  $C_R$  and  $C_\epsilon$  go to zero, giving the integral

$$\begin{aligned} I' &= \int_\infty^0 \frac{\sqrt{z}}{1+z^3} dz \\ &= \int_\infty^0 \frac{\sqrt{e^{2i\pi}x}}{1+(e^{2i\pi}x)^3} dx \\ &= \int_0^\infty \frac{\sqrt{x}}{1+x^3} dx \\ &= I. \end{aligned}$$

Thus,

$$\oint \frac{\sqrt{z}}{1+z^3} dz = 2I.$$

Evaluating the residues, we have

$$\begin{aligned} \text{Res}\left[f(z), e^{i\pi/3}\right] &= \lim_{z \rightarrow e^{i\pi/3}} \frac{\sqrt{z}}{3z^2} \\ &= -\frac{i}{3} \\ \text{Res}[f(z), -1] &= \lim_{z \rightarrow -1} \frac{\sqrt{z}}{3z^2} \\ &= \frac{i}{3} \\ \text{Res}\left[f(z), e^{5\pi i/3}\right] &= -\frac{i}{3}, \end{aligned}$$

giving the solution of

$$\begin{aligned} I &= \frac{1}{2} 2\pi i \left(-\frac{i}{3}\right) \\ &= \frac{\pi}{3}. \end{aligned}$$

**Example.** To evaluate

$$I = \int_0^{\infty} \frac{1}{1+x^3} dx,$$

we start by evaluating

$$\int_0^{\infty} \frac{\ln(x)}{1+x^3} dx$$

with the branch cut along the real axis. Using the keyhole contour in the previous example, we have that  $C_R$  and  $C_\epsilon$  contribute nothing, and  $\ln$  picks up a phase of  $2\pi i$ , so that

$$\oint_C \frac{\ln(z)}{1+z^3} dz = \int_0^{\infty} \frac{\ln(x)}{1+x^3} dx + \int_{\infty}^0 \frac{\ln(x) + 2\pi i}{1+x^3} dx = -2\pi i I.$$

Therefore,

$$I = -\sum \text{Res} \left[ \frac{\ln(x)}{1+x^3} \right].$$

Thus, we get the solution of

$$\int_0^{\infty} \frac{1}{1+x^3} dx = \frac{2\pi}{3\sqrt{3}}.$$

**Example.** Consider the integral

$$I = \int_0^1 \frac{\sqrt{1-x^2}}{x^2+a^2} dx.$$

The poles are around  $\pm ia$ .

Our problem is that we have multivalued behavior at  $\pm 1$ . We may take the cut from  $-1$  to  $1$  along the real axis, and our contour gives a sign flip across the cut.

We draw a dog-bone style contour hugging the cut in negative orientation to give us  $2I$ . Thus, we get

$$\oint_C \frac{\sqrt{1-z^2}}{z^2+a^2} dz = \int_{-1}^1 \frac{\sqrt{1-x^2}}{x^2+a^2} dx - \int_1^{-1} \frac{\sqrt{1-x^2}}{x^2+a^2} dx = 4I,$$

where the sign flip in the second integral comes from crossing the branch cut.

Now, to evaluate the sum of the residues, we need to evaluate at three poles —  $ia$ ,  $-ia$ , and the pole at  $\infty$ . Thus, we get

$$\begin{aligned} \text{Res}[f(z), \pm ia] &= \frac{\sqrt{a^2+1}}{2ia} \\ -\text{Res}[f(z), \infty] &= \lim_{|z| \rightarrow \infty} zf(z) \\ &= i. \end{aligned}$$

Therefore,

$$\begin{aligned} 4I &= 2\pi i \left( \frac{\sqrt{a^2+1}}{ia} - i \right) \\ &= \frac{\pi}{2a} (\sqrt{a^2+1} - a). \end{aligned}$$

### Poles on the axis

If we want to evaluate integrals with the pole on the contour, we need to use principal values.

$$\text{PV} \int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \left( \int_a^{x_0 - \varepsilon} f(x) dx + \int_{x_0 + \varepsilon}^b f(x) dx \right).$$

Similarly, we want to apply this for the calculus of residues. To do this, we take

$$\oint_C f(z) dz = \text{PV} \int_{-\infty}^{\infty} f(x) dx + \lim_{\varepsilon \rightarrow 0} \int_{c_{\pm}} f(z) dz,$$

where  $c_{\pm}$  are small semicircular contour additions of radius  $\varepsilon$  to  $C$  that hug our pole on the real axis, with  $c_-$  excluding the pole and  $c_+$  including the pole. Thus, we have

$$\oint_C f(z) dz = \text{PV} \int_{-\infty}^{\infty} f(x) dx + \lim_{\varepsilon \rightarrow 0} \int_{c_{\pm}} \frac{(z - x_0)f(z)}{z - x_0} dz.$$

Introducing  $z - x_0 = \varepsilon e^{i\varphi}$ , we have  $dz = i\varepsilon e^{i\varphi} d\varphi$ , giving

$$\oint_C f(z) dz = \text{PV} \int_{-\infty}^{\infty} f(x) dx + \text{Res}[f(z), x_0] \int_{c_{\pm}} i d\varphi.$$

Thus, we have

$$\begin{aligned} \oint_C f(z) dz &= \text{PV} \int_{-\infty}^{\infty} f(x) dx \pm i\pi \text{Res}[f(z), x_0]. \\ &= 2\pi i \sum_{z_i} \text{Res}[f(z) - z_i]. \end{aligned}$$

Thus, we have

$$\text{PV} \int_{-\infty}^{\infty} f(x) dx = \sum_{z_i \text{ in } C} \text{Res}[f(z), z_i] \mp i\pi \text{Res}[f(z), x_0].$$

Note that we only have *half* the residue when the pole is on the contour. Therefore, we have the result of

$$\text{PV} \int_{-\infty}^{\infty} f(x) dx = 2\pi i \left( \sum_{z_i \text{ in } C} \text{Res}[f(z), z_i] + \frac{1}{2} \sum_{z_i \text{ on } C} \text{Res}[f(z), z_i] \right).$$

**Example.** Consider

$$\text{PV} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x - a} dx.$$

We have a simple pole at  $x = a$ .

We close our contour with a semicircle on the upper half-plane. Since we have no poles inside the contour, we have

$$\begin{aligned} \text{PV} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x - a} dx &= \pi i \text{Res} \left[ \frac{e^{ikx}}{x - a}, a \right] \\ &= \pi i e^{ika}. \end{aligned}$$

Notice that if  $k < 0$ , we must close the contour in the lower half-plane, giving

$$\text{PV} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x - a} dx = \text{sgn}(k) \pi i e^{ika}.$$

Taking real and imaginary components, we get

$$\begin{aligned} \text{PV} \int_{-\infty}^{\infty} \frac{\cos(kx)}{x-a} dx &= -\pi \sin(ka) \\ \text{PV} \int_{-\infty}^{\infty} \frac{\sin(kx)}{x} dx &= \pi \cos(ka). \end{aligned}$$

**Example.** We will evaluate

$$I = \text{PV} \int_0^{\infty} \frac{\ln(x)}{x^2 + a^2} dx.$$

We have a troublesome portion at  $x = 0$ , so we draw our contour to exclude 0.

We may close the contour with a large semicircle  $C_R$ . Since  $\lim_{|z| \rightarrow \infty} |zf(z)| = 0$  and  $\lim_{|z| \rightarrow 0} |zf(z)| = 0$ , we may take these limits to give

$$\begin{aligned} \oint_C \frac{\ln(z)}{z^2 + a^2} &= \int_{-\infty}^0 \frac{\ln(e^{i\pi}x)}{x^2 + a^2} dx + \int_0^{\infty} \frac{\ln(e^{i0}x)}{x^2 + a^2} dx \\ &= \text{PV} \int_{-\infty}^{\infty} \frac{\ln(x)}{x^2 + a^2} dx + i\pi \int_0^{\infty} \frac{1}{x^2 + a^2} dx \\ &= 2\text{PV} \int_0^{\infty} \frac{\ln(x)}{x^2 + a^2} dx + \frac{i\pi^2}{2a} \\ &= 2\pi i \text{Res}[f(z), ia]. \end{aligned}$$

Thus, we get  $I = \frac{\pi}{2a} \ln(a)$ .

**Example.** Instead of moving our contour up or down by  $\varepsilon$  to include (or exclude) a pole, we may move the pole up or down by  $\varepsilon$ . We consider

$$\int_{-\infty}^{\infty} \frac{g(x)}{x - x_0} dx = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(x)}{x - (x_0 \pm i\varepsilon)} dx.$$

Breaking into real or imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{g(x)}{x - (x_0 \pm i\varepsilon)^2} dx = \int_{-\infty}^{\infty} g(x) \frac{x - x_0}{(x - x_0)^2 + \varepsilon^2} dx \pm i\varepsilon \int_{-\infty}^{\infty} \frac{g(x)}{(x - x_0)^2 + \varepsilon^2} dx.$$

Now, notice that

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}.$$

Now, we may take

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} dx = \varepsilon \oint_C \frac{dz}{z + \varepsilon^2},$$

where  $z = (x - x_0)^2$ . This gives

$$\begin{aligned} \varepsilon \oint_C \frac{dz}{z + \varepsilon^2} &= \varepsilon (2\pi i) \left( \frac{1}{2i\varepsilon} \right) \\ &= \pi. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} dx = \pi \delta(x - x_0).$$

Therefore, we recover

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^{\infty} \frac{g(x)}{(x - x_0)^2 + \varepsilon^2} dx = \pi g(x_0).$$

This gives the identity under the integral of

$$\frac{1}{z \mp i\varepsilon} = \text{PV} \frac{1}{x} \pm i\pi \delta(x).$$

**Example.** Consider the integral

$$\begin{aligned} \oint_{C_r} \frac{\cos(z)}{z} dz &= \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{\cos(x)}{x \pm i\varepsilon} dx \\ &= \text{PV} \int_{-\infty}^{\infty} \frac{\cos(x)}{x} dx \mp i\pi \int_{-\infty}^{\infty} \cos(x) \delta(x) dx \\ &\quad \underbrace{\hspace{10em}}_{=0} \\ &= \mp i\pi \end{aligned}$$

### Sommerfeld–Watson Transform and Series Summation

Thus far, we've been replacing integrals with sums. Now, we're interested in going the other way around.

Consider the sum

$$S = \sum_{n=-\infty}^{\infty} f(n),$$

given the condition that  $f(z)$  is analytic for  $z \in \mathbb{R}$  and  $\lim_{|z| \rightarrow \infty} |z^2 f(z)| = 0$ .

We will introduce the auxiliary function

$$\begin{aligned} g(z) &= \pi \cot(\pi z) \\ &= \pi \frac{\cos(\pi z)}{\sin(\pi z)}. \end{aligned}$$

Note that  $g(z)$  has an infinite number of poles at  $z = n$  for each  $n \in \mathbb{Z}$ .

Now, what we will do here is integrate the product  $f(z)g(z)$  around a long enough symmetric contour hugging the real axis. This gives

$$\begin{aligned} \frac{1}{2\pi i} \oint_C f(z)g(z) dz &= \sum_{n=-\infty}^{\infty} \text{Res}[\pi \cot(\pi z)f(z), n] \\ &= \sum_{n=-\infty}^{\infty} f(z) \frac{\pi \cos(\pi z)}{\frac{d}{dz}(\sin(\pi z))} \Big|_{z=n} \\ &= \sum_{n=-\infty}^{\infty} f(n). \end{aligned}$$

Now, this doesn't *seem* that helpful, until we remember that our contour  $C$  surrounds all the other poles of  $f$  in negative orientation.

$$\frac{1}{2\pi} \oint_C f(z)g(z) dz = - \sum_i \text{Res}[\pi \cot(\pi z)f(z), z_i].$$

Thus, we have converted our infinite sum into a finite sum.

Similarly, if we have an alternating sign series

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty} (-1)^n f(n) \\ &= - \sum_i \text{Res}[\pi \csc(\pi z)f(z), z_i] \end{aligned}$$

**Example.** Consider

$$S = \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2}.$$

Our analogous function is

$$f(z) = \frac{1}{z^2 + a^2}.$$

Then,

$$\begin{aligned} S' &= - \text{Res} \left[ \frac{\pi \csc(\pi z)}{z^2 + a^2}, \pm ia \right] \\ &= - \frac{\pi}{2a} \coth(\pi a). \end{aligned}$$

Therefore, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a).$$

Now, we write

$$S = \frac{1}{2a^2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}.$$

Thus, we get the sum

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \frac{1}{2a^2} (1 + \pi a \coth(\pi a)).$$

**Example.** Now, we may consider

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} \\ &= - \text{Res} \left[ \frac{\pi \csc(\pi z)}{z^2 + a^2}, \pm ia \right] \\ &= - \frac{\pi}{2a} \frac{1}{\sinh(\pi a)}, \end{aligned}$$

giving

$$S = \frac{1}{2a^2} \left( 1 + \frac{\pi a}{\sinh(\pi a)} \right).$$

## Oscillators and Forcing

Consider a damped harmonic oscillator with position  $u(t)$ . Then,  $u$  obeys Newton's second law,

$$\ddot{u} + 2\beta\dot{u} + \omega_0^2 u = 0.$$

Here,  $\beta$  is the damping factor, and  $\omega_0$  denotes the natural frequency.

The solutions of this equation are

$$\begin{aligned} u(t) &= e^{-\beta t} (a e^{i\Omega t} + b e^{-i\Omega t}) \\ &= e^{-\beta t} (a \cos(\Omega t) + b \sin(\Omega t)), \end{aligned}$$

where  $\Omega^2 = \omega_0^2 - \beta^2$ . This is known as a transient solution.

There are three types of motion.

- An underdamped system occurs when  $\omega_0 > \beta$ , so  $\Omega$  is real, meaning we get oscillation that is damped out.
- An overdamped system occurs when  $\beta > \omega_0$ , so  $\Omega$  is imaginary, and the damping slows down the return of the wave.
- When  $\omega = \beta$ , then  $\Omega = 0$ , and the solution is of the form  $u(t) = e^{-\beta t}(at + b)$ , and the system returns to equilibrium as quickly as possible. This is known as critical damping.

A forced system occurs when we have the differential equation

$$\ddot{u} + \beta\dot{u} + \omega_0^2 u = f(t). \quad (\dagger)$$

We may consider a forcing function of the form  $f(t) = f_0 \cos(\omega t)$ . We may also write

$$f(t) = f_0 \operatorname{Re}(e^{i\omega t}).$$

We expect to have a complex steady-state solution of the form

$$U_\omega = C(\omega) e^{i\omega t}.$$

We solve for  $U$  by sticking it into the differential equation of  $\dagger$ . This will give the equation

$$U_\omega = \frac{f_0 e^{i\omega t}}{(\omega_0^2 - \omega^2) + 2i\beta\omega}.$$

Note that the real solution is  $u = \operatorname{Re}(U_\omega)$ , or

$$\begin{aligned} u(t) &= \frac{1}{2}(U_\omega + U_{-\omega}) \\ &= \frac{1}{2}(U_\omega + \overline{U_\omega}). \end{aligned}$$

Now, when we consider a generalized forcing function  $f(t)$ , where  $f$  is a continuum sum of forcing frequencies where the amplitudes are functions of  $\omega$ ,  $\hat{f}(\omega)$ , we get an integral:

$$u(t) = \int \frac{F(\omega) e^{i\omega t}}{(\omega_0^2 - \omega^2) + 2i\beta\omega} d\omega.$$

Plugging this solution into the differential equation, we get

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega,$$

which is a Fourier transform (see [Math Methods 1](#)).

## Impulse Forcing

Consider a hammer blow forcing function, known as an impulse forcing function.

The impulse forcing is of the form

$$\begin{aligned} f(t) &= f_0 \delta(\omega(t - t_0)) \\ &= \frac{\hat{f}_0}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-t_0)} d\omega, \end{aligned}$$

where

$$\hat{f}_0 = \frac{f_0}{\omega_0}.$$

We want to find the impulse solution,

$$\begin{aligned} G(t) &:= u_\delta(t) \\ &= \frac{\hat{f}_0}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-t_0)}}{(\omega_0 - \omega^2) + 2i\beta\omega} d\omega. \end{aligned}$$

To do this integral, we will make use of residues. Writing our denominator as

$$(\omega_0^2 - \omega^2) + 2i\beta\omega = (\omega - \omega_+)(\omega - \omega_-),$$

where

$$\begin{aligned} \omega_{\pm} &= i\beta \pm \sqrt{\omega_0^2 - \beta^2} \\ &= i\beta \pm \Omega. \end{aligned}$$

We close our contour in the upper half-plane so that get a decaying exponential. Evaluating the residues, we get

$$\begin{aligned} G(t) &= \hat{f}_0 \frac{e^{-\beta(t-t_0)}}{2i\Omega} \left( e^{i\Omega(t-t_0)} - e^{-i\Omega(t-t_0)} \right) \\ &= \frac{f_0}{\Omega} \sin(\Omega(t - t_0)) e^{-\beta(t-t_0)}, \end{aligned}$$

where  $t > t_0$ .

Now, if  $t < t_0$ , then we must close our contour in the lower half-plane, and since there are no poles in the lower half-plane, we get  $G(t) = 0$  for  $t < t_0$ . Thus, we must have

$$G(t) = \frac{f_0}{\Omega} \sin(\Omega(t - t_0)) e^{-\beta(t-t_0)} \Theta(t - t_0),$$

where  $\Theta$  is the Heaviside step function.

- The imaginary and real parts of  $\omega_{\pm}$  give the damping,  $\beta$ , and parameter,  $\Omega$ , respectively. Now, we may interpret the different types of damping in this respect.
  - If  $\Omega$  is real (i.e., underdamped motion), then  $\omega_{\pm}$  have constant magnitude of  $\omega_0$ , meaning that varying the damping only moves the poles around in a circle.
  - If  $\beta = \omega_0$  (i.e., critically damped motion), then the poles converge at  $i\beta$  along the imaginary axis.
  - If  $\beta > \omega_0$  (i.e., overdamped motion), then the poles separate along the imaginary axis, giving non-oscillatory motion.
- The poles also encode resonance characteristics, where we have  $\omega_{\text{res}}^2 = \omega_0^2 - 2\beta^2$ .
- If  $\beta \ll \omega_0$ , then the damping is mathematically equivalent to the  $i\epsilon$  prescription moving the resonance pole at  $\omega_0$  off the real axis and into the upper half-plane.



### Waves on a String

Whereas an undamped oscillator is harmonic only in time, a wave is harmonic in both space and time.

A wave satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where  $c$  is the wave speed.

The general solution is of the form

$$u(x, t) = ce^{i(kx - \omega t)}.$$

A forced wave occurs via

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial x^2} = f(x, t).$$

Now, we start with the impulse solution,

$$\begin{aligned} f(x, t) &= f_0 \delta(x - x_0) \delta(t - t_0) \\ &= f_0 \int_{-\infty}^{\infty} \frac{e^{ik(x-x_0)}}{2\pi} dk \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{2\pi} d\omega. \end{aligned}$$

Now, we have

$$G(x, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2/c^2 - k^2} d\omega.$$

To evaluate this integral, we start with the integral in  $\omega$ , given by

$$I = \frac{c^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - c^2 k^2} d\omega.$$

Unfortunately here, our simple poles lie on the real axis at  $\pm ck$ .

To find the solution, we need boundary and initial conditions to know where we want to make our  $i\epsilon$  adjustment.

If there is no wave before our impulse hits, we need our integral to vanish whenever  $t < 0$  which occurs when we close our contour in the lower half plane, so we subtract  $i\epsilon$  from  $\pm ck$ . Factoring, we have

$$I = \frac{c^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - (ck - i\epsilon))(\omega + (ck + i\epsilon))} d\omega.$$

Thus, we have

$$I(t > 0) = -\frac{c}{k} \sin(ckt) e^{-\epsilon t},$$

and in the limit as  $\epsilon \rightarrow 0$ , we have

$$I(t) = -\frac{c}{k} \sin(ckt).$$

Now, sticking our value of  $I$  into the integral in  $k$ , we have

$$\begin{aligned} G(x, t > 0) &= -\frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ickt} - e^{-ickt}}{2ik} e^{ikx} dk \\ &= -\frac{c}{2} \Theta(ct - |x|) \Theta(t). \end{aligned}$$

There are some comments in order.

- The factor of  $\Theta(t)$  effectively states that nothing happens before  $t = 0$ .
- The factor of  $\Theta(ct - |x|)$  denotes causality. The term  $|x|$  denotes the physical symmetry, while we need  $ct - |x| > 0$  in order to feel an effect.

## Quantum Mechanics

The Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = 0.$$

Here,  $\psi$  denotes the wavefunction. The probability of a particle being found within  $dx$  of  $x$  is  $|\psi|^2 dx$ .

In quantum mechanics, we define operators for energy and momentum as

$$E := +i\hbar \frac{\partial}{\partial t}$$

$$P := -i\hbar \frac{\partial}{\partial x}.$$

The Schrödinger equation falls from the fact that  $E = \frac{p^2}{2m}$ . These operators are Hermitian, so their eigenvalues are real.

We want to figure out the Green's function for the quantum hammer blow at  $t = 0$ , which we call the propagator.

$$i\hbar \frac{\partial G}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 G}{\partial x^2} = i\hbar \delta(x) \delta(t).$$

When we try to evaluate it, we get

$$G(x, t > 0) = \Theta(t) \sqrt{\frac{m}{2\pi i\hbar t}} e^{imx^2/2\hbar t}.$$

This solution has a problem, though — there is no sense of causality. This says that, as long as  $t > 0$ , there is going to be a measurable reaction at all  $x$ .

The main reason is that the Schrödinger equation is not relativistic. Accounting for the relativistic relationship, we have

$$E^2 - P^2 c^2 = m^2 c^4.$$

This gives an equation known as the Klein–Gordon equation:

$$-\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = \frac{m^2 c^2}{\hbar^2} \phi.$$

Solving the Klein–Gordon equation gives infinitely many solutions that have both positive and negative energy, the latter of which is a major issue. However, it can be shown that the Klein–Gordon equation applies to particles with integral spin, with wave function

$$\phi(x, t) = C e^{i(kx - \omega t)}$$

so long as

$$\frac{\omega^2}{c^2} - k^2 = \frac{m^2 c^2}{\hbar^2}.$$

To see how causality fares for the Klein–Gordon wavefunction, we solve it for the unit impulse acting on  $x = 0$  at  $t = 0$ .

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right)G(x, t) = i\delta(x)\delta(t),$$

where we set  $\hbar = c = 1$ . We get the Green's function

$$G(x, t) = i \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{ipx} dp \int_{-\infty}^{\infty} \frac{e^{-iEt}}{2\pi(E^2 - p^2 - m^2)} dE.$$

We have two poles at  $\pm\sqrt{p^2 + m^2}$ , and when we close the contour we obtain different physical results.

When  $t < 0$ , we must close the contour in the upper-half plane, so we exclude the poles from the contour. Since there are no poles in the upper half-plane, we get a factor of  $\Theta(t)$ .

Thus, when  $t > 0$ , we close in the lower half-plane, giving

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{-iEt}}{(E + iE)^2 - p^2 - m^2} dE \\ &= -\frac{1}{2\pi} 2\pi i \left( \frac{e^{-i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} - \frac{e^{i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} \right). \end{aligned}$$

Defining  $E_p := \sqrt{p^2 + m^2}$ , we have

$$I = -\frac{i}{2E_p} (e^{-iE_p t} - e^{iE_p t}).$$

Thus, we have

$$\begin{aligned} G(x, t > 0) &= \Theta(t) \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{ipx}}{2E_p} (e^{-iE_p t} - e^{iE_p t}) dp \\ &= \frac{\Theta(t)}{4\pi} \int_{-\infty}^{\infty} \frac{1}{E_p} (e^{-i(E_p t - px)} - e^{i(E_p t - px)}) dp, \end{aligned}$$

where we were allowed to flip the sign of  $p$  because the integral is taken over all space.

## Tensors

### Cartesian Tensors

Recall from intro mechanics that when we have a block on a ramp, we may choose two different orientations of our axes. However, despite our choice of basis, we still get the same result after applying Newton's second law — in that sense, the vectors represent something physical separate from their components. Using a rotation matrix, we are able to convert the equations

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y, \end{aligned}$$

into

$$\begin{aligned} F'_x &= ma'_x \\ F'_y &= ma'_y. \end{aligned}$$

This means that an expression like  $\mathbf{F} = m\mathbf{a}$  is a *covariant* expression; it is the same equation regardless of reference frame.

## Stress and Strain

If we have a continuous body, the stress is defined as the internal force per unit area exerted at an imaginary surface within the body. In other words, stress times area equals force.

$$dF_i = \sum_j \sigma_{ij} da_j,$$

where  $\sigma_{ij}$  denotes the stress *tensor*. The reason why it is a two-indexed object rather than a one-indexed object is because there are two different types of stress: pressure and shear.

Pressure is the normal component to the surface, while shear is the tangent component, so  $\sigma_{ij}$  denotes the two components; then, the diagonal components denote the pressure and the off-diagonal components the shear.

Whereas stress describes the internal forces of an object, strain describes deformations due to external forces.

For instance, in a rod of length  $L$  and cross section  $A$ , if we strain it to a length of  $L + \Delta L$ , the relationship between induced stress  $\sigma = \frac{F}{A}$  normal to surface and strain  $\epsilon = \Delta L / L$  is given by Hooke's law:

$$F = k\Delta L$$

$$\sigma = Y\epsilon,$$

where  $Y = \frac{kL}{A}$  is the elastic modulus of the material. Now, since  $\Delta L$  can be negative, this equation describes both tensile stress and strain and compressive stress and strain.

Now, if stress is a tensor, then so is strain; moving beyond one dimension, we define a vector field  $\mathbf{u}(\mathbf{r})$  which gives the magnitude and direction of a body's displacement at position  $\mathbf{r}$ . To understand internal stress, we need to focus not on the rigid body motion, but only on the relative displacement of points within the body. This means we need to consider

$$u_i(\mathbf{r} + d\mathbf{r}) = u_i(\mathbf{r}) + \nabla u_i \cdot d\mathbf{r} + \dots,$$

where we allow  $|\nabla u_i| \ll 1$ , meaning we take the first order expansion. This gives

$$\mathbf{u}(\mathbf{r} + d\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \sum_j \delta_j \mathbf{u} dr_j,$$

where

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

This is where strain resides. Specifically, we may decompose  $\nabla \mathbf{u}$  into symmetric and antisymmetric components,

$$\nabla \mathbf{u} = \overset{\leftrightarrow}{\epsilon} + \overset{\leftrightarrow}{\phi},$$

where

$$\epsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

$$\phi_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i).$$

The symmetric/antisymmetric decomposition is covariant, so we may express

$$\mathbf{u}(\mathbf{r} + d\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \vec{\epsilon} \cdot d\mathbf{r} + \vec{\phi} \cdot d\mathbf{r}.$$

Turning our attention to  $\vec{\phi}$ , we may use the Levi-Civita symbol to write

$$\phi_{ij} = - \sum_k \epsilon_{ijk} \varphi_k,$$

and take

$$\begin{aligned} \sum_j \phi_{ij} dr_j &= - \sum_{j,k} \epsilon_{ijk} \varphi_k dr_j \\ &= \sum_{j,k} \epsilon_{ikj} \varphi_k dr_j, \end{aligned}$$

meaning

$$d\mathbf{u} = \vec{\phi} \times d\mathbf{r},$$

describing the rotation of  $d\mathbf{r}$  by  $\varphi$  around  $\hat{\phi}$ . This is rigid body motion.

Thus, deformations are described by the strain tensor,  $\vec{\epsilon}$ . Hooke's law then becomes

$$\sigma_{ij} = \sum_{k,l} Y_{ijkl} \epsilon_{kl},$$

where  $Y_{ijkl}$  is the elasticity tensor.