

Solution (29.5):

(a) We have

$$\begin{aligned} \left(\vec{w} \cdot \overset{\leftrightarrow}{T} \right)_k &= \sum_{i,j} w_i T_{jk} \delta_{ij} \\ &= \sum_{i,k} w_i T_{ik}, \end{aligned}$$

which is a first-rank tensor.

(b) Since $\vec{w} \cdot \overset{\leftrightarrow}{T}$ is a first-rank tensor, and we are taking the dot product of two first rank tensors the expression $\vec{w} \cdot \overset{\leftrightarrow}{T} \cdot \vec{v}$ is a scalar (or rank zero tensor).

(c) We have

$$\begin{aligned} \overset{\leftrightarrow}{T} \cdot \vec{u} &= \left(\sum_{i,j} T_{ij} e_i \otimes e_j \right) \cdot \left(\sum_{k,\ell} u_{k\ell} e_k \otimes e_\ell \right) \\ &= \sum_{i,j,k,\ell} T_{ij} u_{k\ell} (e_k \cdot e_i) (e_j \cdot e_\ell), \end{aligned}$$

which is a scalar.

(d) The expression $\overset{\leftrightarrow}{T} \vec{v}$ expresses the operation of the linear map

$$\overset{\leftrightarrow}{T} = \sum_{i,j} T_{ij} e_i \otimes e_j$$

on

$$\vec{v} = \sum_i v_i e_i,$$

meaning that $\overset{\leftrightarrow}{T} \vec{v}$ is a vector.

(e) The expression $\overset{\leftrightarrow}{T} \vec{u}$ is a composition of two linear maps on $V \otimes V$, so it is a rank 2 tensor (or another linear map on $V \otimes V^*$).

Solution (29.7): We have 2^4 or 16 components in Λ_{ijkl} .

Solution (29.10): We have

$$T_{ij} = \sum_{k,\ell} R_{ik} R_{j\ell} T_{k\ell},$$

so that

$$\begin{aligned} \sum_{ij} \epsilon_{ij} T_{ij} &= \sum_{i,j,k,\ell} \epsilon_{ij} R_{ik} R_{j\ell} T_{k\ell} \\ &= \sum_{i,j,k,\ell} R_{ik} R_{j\ell} \epsilon_{k\ell} T_{k\ell} \\ &= \sum_{k,\ell} \epsilon_{k\ell} T_{k\ell}. \end{aligned}$$

Solution (29.11):

(a) We may write T_{ij} as $T = \frac{1}{2}(T + T^T) + \frac{1}{2}(T - T^T)$, which are the symmetric and antisymmetric components.

(b) Taking

$$S_{ij} = \sum_{k,\ell} R_{ik} R_{j\ell} S_{k\ell},$$

we have

$$\begin{aligned} S_{ji} &= \sum_{k,\ell} R_{jk} R_{i\ell} S_{k\ell} \\ &= \sum_{k,\ell} R_{j\ell} R_{ik} S_{\ell k} \\ &= \sum_{k,\ell} R_{ik} R_{j\ell} S_{\ell k} \\ &= \sum_{k,\ell} R_{ik} R_{j\ell} S_{k\ell} \\ &= S_{ij}. \end{aligned}$$

Similarly,

$$\begin{aligned} A_{ij} &= \sum_{k,\ell} R_{ik} R_{j\ell} A_{k\ell} \\ A_{ji} &= \sum_{k,\ell} R_{jk} R_{i\ell} A_{k\ell} \\ &= - \sum_{k,\ell} R_{ik} R_{j\ell} A_{k\ell} \\ &= -A_{ij}. \end{aligned}$$

In matrix form, we have

$$\begin{aligned} S_{ji} &= S_{ij}^T \\ &= \left(R S_{k\ell} R^T \right)^T \\ &= R S_{k\ell} R^T. \end{aligned}$$

and similarly,

$$\begin{aligned} -A_{ji} &= \left(A_{ij} \right)^T \\ &= \left(R A_{k\ell} R^T \right)^T \\ &= R A_{k\ell} R^T. \end{aligned}$$

Solution (29.12):

(a) Let $\mathbf{v} = R\mathbf{v}'$, where R is a rotation matrix. Then, we must have

$$\begin{aligned} \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y &= \nabla' \cdot \mathbf{v} \\ &= \nabla \cdot R\mathbf{v}' \\ &= \nabla \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \end{pmatrix} \\ &= \nabla \cdot \begin{pmatrix} v'_x \cos(\theta) - v'_y \sin(\theta) \\ v'_x \sin(\theta) + v'_y \cos(\theta) \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial x} \left(v'_x \cos(\theta) - v'_y \sin(\theta) \right) + \frac{\partial}{\partial y} \left(v'_x \sin(\theta) + v'_y \cos(\theta) \right) \\
&= \left(\cos(\theta) \frac{\partial}{\partial x} + \sin(\theta) \frac{\partial}{\partial y} \right) v'_x + \left(\cos(\theta) - \sin(\theta) \frac{\partial}{\partial y} \right) v'_y \\
&= \mathbf{R}^T \nabla \cdot \mathbf{v}'
\end{aligned}$$

(b) Let $\mathbf{v} = \mathbf{R}\mathbf{v}'$, where \mathbf{R} is a rotation matrix. Then, we must have

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \sum_i \frac{\partial}{\partial x_i} v_i \\
&= \sum_{i,j} \frac{\partial}{\partial x_i} R_{ij} v'_j \\
&= \sum_{i,j} R_{ji} \frac{\partial}{\partial x_j} v'_j \\
&= \nabla' \cdot \mathbf{v}',
\end{aligned}$$

where we were able to switch the indices on \mathbf{R} by the fact that it is not a function of x_i .

Solution (29.14): We have

$$\begin{aligned}
\sigma'_{ij} &= \sum_{k,\ell} R_{ik} R_{j\ell} \sigma_{k\ell} \\
&= \sum_{k,\ell} R_{ik} R_{j\ell} (Y_{k\ell mn} \epsilon_{mn}) \\
&= \sum_{k,\ell} R_{ik} R_{j\ell} \left(\sum_{o,p} R_{mo} R'_{np} \epsilon'_{op} \right) \\
&= \sum_{k,\ell,o,p} R_{ik} R_{j\ell} R_{mo} R_{np} Y'_{k\ell op} \epsilon'_{op},
\end{aligned}$$

so that

$$Y_{ijmn} = \sum_{k,\ell,o,p} R_{ik} R_{j\ell} R_{mo} R_{np} Y'_{k\ell op},$$

meaning the elastic modulus is a rank 4 tensor.

Solution (29.23): We have

$$\begin{aligned}
T_{ij} &= \sum_{k,\ell} R_{ik} R_{j\ell} \left(\frac{1}{3} \text{tr}(\mathbf{T}) \delta_{k\ell} + \frac{1}{2} (T_{k\ell} - T_{\ell k}) + \frac{1}{2} (T_{k\ell} + T_{\ell k} - \frac{2}{3} \text{tr}(\mathbf{T}) \delta_{ij}) \right) \\
&= \frac{1}{3} \delta_{ij} + \frac{1}{2} (T_{ij} - T_{ji}) + \frac{1}{2} (T_{ij} + T_{ji} - \frac{2}{3} \delta_{ij}).
\end{aligned}$$

Solution (29.24):

Solution (29.25):