Solution (40.7)**:** We have

$$\begin{aligned}\langle \psi \mid \mathcal{L}\phi\rangle &= \int_a^b \overline{\psi(x)}\left(\alpha(x)\frac{d^2\phi}{dx^2} + \beta(x)\frac{d\phi}{dx} + \gamma(x)\phi(x)\right) dx\\ &= \int_a^b \overline{\psi(x)}\alpha(x)\frac{d^2\phi}{dx^2}\,dx + \int_a^b \overline{\psi(x)}\beta(x)\frac{d\phi}{dx}\,dx + \int_a^b \overline{\psi(x)}\gamma(x)\phi(x)\,dx.\end{aligned}$$

We evaluate these integrals separately. Assuming that α, β, γ are real-valued, we have

$$\begin{aligned}\int_a^b \overline{\psi(x)}\alpha(x)\frac{d^2\phi}{dx^2}\,dx &= \frac{d\phi}{dx}\overline{\psi(x)}\alpha(x)\Big|_a^b - \int_a^b \left(\frac{d\alpha}{dx}\overline{\psi(x)} + \overline{\frac{d\psi}{dx}}\alpha(x)\right)\frac{d\phi}{dx}\,dx\\ &= \underbrace{\left(\frac{d\phi}{dx}\alpha(x)\overline{\psi(x)} - \phi(x)\left(\frac{d\alpha}{dx}\overline{\psi(x)} + \overline{\frac{d\psi}{dx}}\alpha(x)\right)\right)\Big|_a^b}_{S_1}\\ &\quad + \int_a^b \overline{\left(\alpha(x)\frac{d^2}{dx^2} + 2\frac{d\alpha}{dx}\frac{d}{dx} + \frac{d^2\alpha}{dx^2}\right)\psi(x)}\phi(x)\,dx.\\ \int_a^b \overline{\psi(x)}\beta(x)\frac{d\phi}{dx}\,dx &= \underbrace{\left(\phi(x)\beta(x)\overline{\psi(x)}\right)\Big|_a^b}_{S_2} - \int_a^b \phi(x)\left(\frac{d\beta}{dx}\overline{\psi(x)} + \overline{\frac{d\psi}{dx}}\beta(x)\right) dx.\end{aligned}$$

Thus, we have

$$\int_a^b \overline{\psi(x)}\left(\alpha(x)\frac{d^2\phi}{dx^2}\right) dx = S_1 + S_2 + \int_a^b \overline{\left(\alpha(x)\frac{d^2}{dx^2} + \left(2\frac{d\alpha}{dx} - \beta(x)\right)\frac{d}{dx} + \left(\frac{d^2\alpha}{dx^2} - \frac{d\beta}{dx} + \gamma(x)\right)\right)\psi(x)}\phi(x)\,dx.$$

Solution (40.23)**:**

(a) We have $p(x) = 1$, and

$$\begin{aligned}\int_0^a \overline{\sin(n\pi x/a)}\sin(m\pi x/a)\,dx &= \frac{a}{m\pi - n\pi}\left(n\pi\cos(n\pi x/)\overline{\sin(m\pi x/a)} - m\pi\cos(m\pi x/a)\overline{\sin(n\pi x/a)}\right)\Big|_0^a\\ &= 0.\end{aligned}$$

(b) With the eigenfunctions $J_0(\alpha_i r/a)$, we have

$$\int_0^a r\overline{J_0\left(\frac{\alpha_m}{a}r\right)}J_0\left(\frac{\alpha_n}{a}r\right) dx = \frac{r\left(\frac{\alpha_n}{a}J_0'\left(\frac{\alpha_n}{a}r\right)\right)\Big|_0^a}{\frac{\alpha_m}{a} - \frac{\alpha_n}{a}}.$$

We use the identity that

$$J_0' = -J_1$$

to use $J_1(0) = 0$ and $J_0\left(\frac{\alpha_i}{a}(a)\right) = 0$, so we recover the orthogonality relation.

(c) We have

$$\begin{aligned}\int_0^\infty \operatorname{Ai}(\kappa x + \alpha_n)\operatorname{Ai}(\kappa x + \alpha_m)\,dx &= \frac{\kappa x(\operatorname{Ai}'(\kappa x + \alpha_n)\operatorname{Ai}(\kappa x + \alpha_m) - \operatorname{Ai}'(\kappa x + \alpha_m)\operatorname{Ai}'(\kappa x + \alpha_n))\big|_0^\infty}{\kappa^2(\alpha_n - \alpha_m)}\\ &= 0.\end{aligned}$$

Solution (40.27)**:**

Solution (41.8)**:**

Solution (41.13)**:**

Solution (41.14)**:**

Solution (41.16)**:**

Solution (41.25)**:**

Solution (41.28)**:**

Solution (42.1)**:**

Solution (42.2)**:**

Solution (42.11)**:**