Problem 1

Let $X = \{0,1\}^n$. Show that the Hamming distance:

$$d_H : X \times X \to [0,\infty)$$
$$d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) = \left|\{j \mid x_j \neq y_j\}\right|$$

defines a metric on X.

Proof:

- Symmetry:

$$\begin{aligned} d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) &= \left|\{j \mid x_j \neq y_j\}\right| \\ &= \left|\{j \mid y_j \neq x_j\}\right| \\ &= d_H\left((y_j)_{j=1}^n, (x_j)_{j=1}^n\right) \end{aligned}$$

- Definiteness: it is only the case that $d_H(x_j, y_j) = 0$ if $x_j = y_j$ for all j, by the definition of the distance.
- Similarly, since $x_j = x_j$ for all j, $d_H(x_j, x_j) = 0$.
- Let $(x_j)_j$, $(y_j)_j$, and $(z_j)_j$ be sequences of bits. The set $\{j \mid x_j \neq z_j\}$ is formed by taking all the values $\{j \mid x_j \neq y_j\}$ along with $\{j \mid y_j \neq z_j\}$, net of particular indices where $x_j = z_j$, but $x_j \neq y_j$. Therefore,

$$d(x,z) \leq d(x,y) + d(y,z).$$

Problem 2

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V, show that the induced metrics d and d' are equivalent.

Proof: Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms. Then, $\exists c_1, c_2 \in \mathbb{R}$ such that $\|v-w\|' \leq c_1\|v-w\|$ and $\|v-w\| \leq c_2\|v-w\|'$. However, this is the exact same statement as $d(v,w) \leq c_1 d'(v,w)$ and $d'(v,w) \leq c_2 d'(v,w)$. Thus, d and d' are equivalent metrics.

Problem 3

Let $\{X_k, d_k\}$ be a sequence of metric spaces with uniformly bounded metrics. Let

$$X := \prod_{k\geq 1} X_k$$

denote the product.

(a) Show that

$$D : X \times X \to [0,\infty)$$
$$D(x,y) := \sum_{k\geq 1} 2^{-k} d_k(x_k, y_k)$$

defines a metric on X.

(b) Consider the case where $\{X_k\} = \{0,2\}$ and $d_k(a,b) = |a-b|$ for every $k \geq 1$. We get the abstract Cantor set

$$\Delta := \prod_{k\geq 1}\{0,2\};$$
$$D(x,y) := \sum_{k=1}^{\infty} 3^{-k}|x_k - y_k|.$$

Prove that $D(x,z) = D(y,z)$ implies $x = y$.

Problem 4

Let $(V, \|\cdot\|)$ be a normed space, and suppose $E \subseteq V$. Show that the following are equivalent:

(1) E is bounded — $\operatorname{diam}(E) < \infty$;

(2) $\sup_{v \in E} \|v\| < \infty$;

(3) there is an $r > 0$ such that $E \subseteq B(0, r)$.

Proof: We will start by showing (i) implies (ii). Let E be a bounded subset of V. Thus, for all $v, w \in E$, $\|v - w\| \leq c$ for some $c \in \mathbb{R}^+$.

Problem 5

Let (X, d) be a metric space and suppose $A \subseteq X$. Show:

(i) $\overline{A^c} = (A^\circ)^c$

(ii) $(\overline{A})^c = (A^c)^\circ$

Proof:

(i) We have previously established that $\overline{A^c} \subseteq (A^\circ)^c$. Let $x \in (A^\circ)^c$. Then, $x \notin A^\circ$, meaning $\forall \delta > 0$, $U(x, \delta) \cap A^c \neq \emptyset$. Thus, $x \in \overline{A^c}$.

(ii) Let $x \in \overline{A}^c$. Then, $x \notin \overline{A}$, meaning $\exists \delta > 0$ such that $U(x, \delta) \cap A = \emptyset$. Thus, $U(x, \delta) \subseteq A^c$, meaning $x \in (A^c)^\circ$.

Let $x \in (A^c)^\circ$. Then, $\exists \delta > 0$ such that $U(x, \delta) \subseteq A^c$. Therefore, $U(x, \delta) \cap A = \emptyset$, meaning $x \notin \overline{A}$, so $x \in \overline{A}^c$.

Problem 6

In any metric space, show that open balls are open, closed balls are closed, and spheres are closed. Moreover, in a normed space, show that $\partial U(v, r) = \partial B(v, r) = S(v, r)$.

Problem 9

Show that c_0 with $\|\cdot\|_u$ is separable.

Proof: Let $z \in c_0$. Set $\varepsilon_1 > 0$, then finding N_1 large such that for all $n > N_1$, $z_n < \varepsilon_1$. Set $z' \in c_{00}$ to be equal to z on $1, \dots, N_1$ and equal to 0 for all $n > N_1$.

Recall that for

$$E_n = \left\{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\},$$
$$E = \bigcup E_n,$$

E is dense in c_{00}, meaning that there exists some $w \in c_{00}$ such that $\|z' - w\| < \varepsilon$ for any $\varepsilon > 0$. However, since $z' = z$ for all n from $1, \dots, N_1$, and the index of $\|z\|_u$ is contained in $1, \dots, N_1$, this means $\|z - w\| < \varepsilon$, meaning E is dense in c_0.

Since E is countable, this means c_0 is countable.

Problem 10

Let $\mathcal{C}$ denote the Cantor set. Show that $\mathcal{C}$ is nowhere dense.

Proof: We know that $\mathcal{C}$ is closed, meaning all we need show is that $\mathcal{C}^0 = \emptyset$.

Suppose toward contradiction that $\mathcal{C}^0$ is not empty. Then, $\exists x \in \mathcal{C}$ and $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}$.

Find m so large such that $3^{-m} < \varepsilon$. Then, $(x - \varepsilon, x + \varepsilon)$ must be contained in a subinterval with length $\frac{1}{3^m}$. However, $2\varepsilon > \frac{1}{3^m}$, and every subinterval in the element $\mathcal{C}_m$ has length $\frac{1}{3^m}$.