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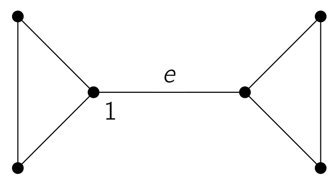
By Theorem 5.3, every Eulerian walk in a connected graph G must traverse each bridge of G twice. Suppose that G is a connected graph containing exactly one bridge e. Is it possible that G has an Eulerian walk that traverses e twice and all other edges of G once?

It is possible. The Eulerian walk in the following graph starting from vertex 1, satisfies the property.



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What is the length of an Eulerian walk in a tree of order $n \geq 2$?

A closed Eulerian walk in a tree must traverse every edge of the tree once, meaning that the length of the walk is $2(n-1)$.

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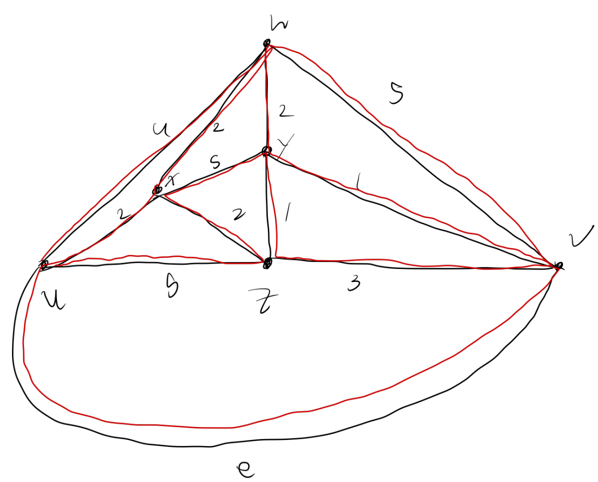
Determine the length of an Eulerian walk in the weighted graph of Figure 33 and describe an Eulerian walk for this graph.

We can see in the following image that an Eulerian circuit with u, v can be traced upon adding e. We take the shortest path from u to v as u, x, z, y, v to replace e.



Therefore, the Eulerian walk is $u, w, x, u, x, z, y, v, w, y, v, z, y, x, z, u$, with weight of $W(G) + 6$, or 38.

Ch. 8, 2

Show that if S_n is a Steiner triple system, then S_{2n+1} is also a Steiner triple system.

If S_n is a Steiner triple system, then S_n is a K_3-decomposition of $G = K_n$ where $n = 6q+1$ or $6q+3$. Then, S_{2n+1} must occur on $2(6q+1)+1 = 12q+3 = 6q'+3$, or $2(6q+3)+1 = 12(q+1)+1 = 6q''+1$. Therefore S_{2n+1} must be a Steiner Triple System.

Theorem 8.1

If S_n is a Steiner triple system, then either $n \geq 3$ and $n = 6q + 1$, or $n = 6q + 3$.

If S_n is a Steiner triple system, then K_n contains a number of edges divisible by 3. Therefore, we must have that

$$\frac{n(n-1)}{2} = 3k$$
$$n(n-1) = 6k,$$

implying that n divides 2 and $n - 1$ divides 3, or vice versa. Therefore, we must have that $n = 6q + 1$ or $n = 6q + 3$.

Theorem 8.4

Every Eulerian graph has a cycle decomposition

Let G be Eulerian. Then, since there is an Eulerian circuit, there must be a cycle contained within the Eulerian circuit (as every walk contains within it a path). Call this cycle C.

Upon deletion of $E(C)$, since every vertex in C has degree 2, it must be the case that every vertex of $G-C$ is even. Every component of $G-C$ is thus Eulerian, and contains within each component a cycle.

By repeating the process of finding cycles contained within Eulerian circuits of components, deleting these cycles, and finding ones in remaining components, we can find a cycle decomposition of G.