

## Problem 1

If  $F$  is a finite set and  $k : F \rightarrow F$  is a self-map, prove that  $k$  is injective if and only if  $k$  is surjective.

## Problem 2

Prove that a set  $A$  is infinite if and only if there is a non-surjective injection  $f : A \rightarrow A$ .

## Problem 3

Let  $A$ ,  $B$ , and  $C$  be sets and suppose  $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$ . Prove that  $\text{card}(A) < \text{card}(C)$ .

## Problem 4

If  $A \subseteq B$  is an inclusion of sets with  $A$  countable and  $B$  uncountable, show that  $B \setminus A$  is uncountable.

## Problem 5

Is the set  $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$  countable?

## Problem 6

Consider the set  $\mathcal{F}(\mathbb{N})$  of all finite subsets of  $\mathbb{N}$ . Is  $\mathcal{F}(\mathbb{N})$  countable?

## Problem 7

Let  $k \in \mathbb{N}$ .

(i) Prove that  $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$  is countable.

(ii) Show that the set  $\mathbb{N}^\infty := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}\}$  consisting of all sequences of natural numbers is uncountable.

(iii) Prove that the set of **finitely-supported** natural sequences  $c_c(\mathbb{N}) := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}, n_k = 0 \text{ for all but finitely many } k\}$  is countable.

## Problem 8

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range  $\text{ran}(f)$  cannot contain any interval.

## Problem 9

Prove that the set

$$\mathcal{P} := \left\{ \sum_{k=0}^n a_k x^k \mid n \in \mathbb{N}_0, a_k \in \mathbb{Q} \right\}$$

consisting of all polynomials with rational coefficients, is countable.

## Problem 10

A real number  $t$  is called **algebraic** if there is a nonzero polynomial  $p$  with rational coefficients such that  $p(t) = 0$ . If  $t \in \mathbb{R}$  is not algebraic, then it is called **transcendental**. For example,  $\sqrt{2}$  is algebraic, but  $\pi$  is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.