

**Problem** (Problem 1): Let  $R$  be a Euclidean domain,  $n \geq 2$  an integer.

- Use the proof of the Smith Normal Form to show that every matrix  $A \in \text{GL}_n(R)$  can be written as a product of elementary matrices  $E_{ij}(\lambda)$ , flip matrices  $F_{ij}$ , and a diagonal matrix  $D$ .
- Now show that the flip matrices can be eliminated from the product in (a), and one can assume that  $D = \text{diag}(d, 1, \dots, 1)$ . That is, all diagonal entries of  $D$  except possibly the  $(1, 1)$  entry are equal to 1.
- Deduce from (b) that  $\text{SL}_n(R)$  is generated by the elementary matrices  $E_{ij}(\lambda)$ .

**Solution:**

- Observe that a square matrix is in Smith normal form if and only if it is a diagonal matrix of the form  $D = \text{diag}(d_1, \dots, d_m, 0, \dots, 0)$  where  $d_1 | d_2 | \dots | d_m$ . By the proof of the Smith normal form, we have that the matrix  $UAV$  in Smith normal form is the product of three invertible matrices, so it is invertible, meaning that it is necessarily diagonal with  $d_1, \dots, d_n \in R^\times$ . Since the inverse of any  $E_{ij}(\lambda)$  is another matrix of the form  $E_{ij}(\lambda)$ , and the inverse of  $F_{ij}$  is itself, it follows that we may write any  $A \in \text{GL}_n(R)$  as

$$A = U^{-1}DV^{-1},$$

where  $U^{-1}$  and  $V^{-1}$  are collections of flips and  $E_{ij}(\lambda)$  and  $D$  is a diagonal matrix with  $d_1, \dots, d_n \in R^\times$ .

- We observe that the following relation holds between the matrices  $E_{ij}(\lambda)$  and  $F_{ij}$ :

$$\begin{aligned} E_{ij}(\lambda)F_{jk} &= F_{jk}E_{ik}(\lambda) \\ E_{ij}(\lambda)F_{ik} &= F_{ik}E_{kj}(\lambda) \\ E_{ij}(\lambda)F_{k\ell} &= F_{k\ell}E_{ij}(\lambda) \end{aligned}$$

where in the last case, we have both  $k, \ell$  not equal to either  $i$  or  $j$ . Since the form of these flip matrices is preserved, upon performing this reduction we may collect all the flip matrices at the front of the expression  $A = U^{-1}DV^{-1}$ .

**Problem** (Problem 2): Let  $R$  be a Euclidean domain, let  $k, n \in \mathbb{N}$ , and let  $i \leq \min(k, n)$ . Given a matrix  $A \in \text{Mat}_{k,n}(R)$ , define  $d_i(A)$  to be the greatest common divisor of all  $i \times i$  minors of  $A$ . Prove that  $d_i(PAQ) = d_i(A)$  for all  $P \in \text{GL}_k(R)$  and  $Q \in \text{GL}_n(R)$ .

**Problem** (Problem 3): Let  $R$  be a commutative ring with 1.

- Let  $C$  be an  $R$ -algebra, and  $A, B \subseteq C$   $R$ -subalgebras that commute with each other; that is,  $ab = ba$  for any  $a \in A$  and  $b \in B$ . Prove that there is an  $R$ -algebra homomorphism  $\varphi: A \otimes B \rightarrow C$  such that  $\varphi(a \otimes b) = ab$  for each  $a \in A$  and  $b \in B$ .
- Prove that  $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$  as rings.
- Now assume that  $R$  is a field, and let  $A$  be a finite-dimensional  $R$ -algebra. Prove that  $A \otimes A$  cannot be a field unless  $\dim(A) = 1$ .

**Solution:**

- Let  $\phi: A \times B \rightarrow C$  be defined by  $(a, b) \mapsto ab$ . Then,  $\phi$  is an  $R$ -bilinear map, so it induces a unique linear map on the tensor product  $\varphi: A \otimes B \rightarrow C$ . We claim that this map is compatible with the  $R$ -algebra structure of  $A \otimes B$ .

To see this, observe that if  $a_1, a_2 \in A$  and  $b_1, b_2 \in B$ , then

$$\begin{aligned} \varphi((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \varphi(a_1 a_2 \otimes b_1 b_2) \\ &= a_1 a_2 b_1 b_2 \\ &= a_1 b_1 a_2 b_2 \\ &= \varphi(a_1 \otimes b_1) \varphi(a_2 \otimes b_2). \end{aligned}$$

This gives our desired  $R$ -algebra homomorphism.

- (b) We observe that both  $\mathbb{R}$  and  $\mathbb{Z}[i]$  are  $\mathbb{Z}$ -subalgebras of  $\mathbb{C}$ . Therefore, from above, we have a  $\mathbb{Z}$ -algebra homomorphism

$$\begin{aligned}\varphi: \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] &\rightarrow \mathbb{C} \\ t \otimes (a + bi) &\mapsto ta + tbi.\end{aligned}$$

To see that this map is injective, observe that  $ta + tbi = 0$  if and only if  $ta = 0$  and  $tb = 0$ , meaning either that  $t = 0$  or  $a, b = 0$ ; in either case, the corresponding element of the tensor product is the zero tensor. As for surjectivity, if we have  $x + yi \in \mathbb{C}$ , then we may find the element  $x \otimes 1 + y \otimes i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$  that maps to  $x + yi$ . Since this is a bijective  $\mathbb{Z}$ -algebra homomorphism, it follows that  $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$  as  $\mathbb{Z}$ -algebras, hence as rings.

- (c) Suppose  $A$  is an  $R$ -algebra such that  $A \otimes_R A$  is a field.

**Problem (Problem 5):** Let  $V$  and  $W$  be finite-dimensional vector spaces over  $F$ , with  $\{v_1, \dots, v_n\}$  a basis for  $V$  and  $\{w_1, \dots, w_m\}$  a basis for  $W$ . Let  $\varphi: V \otimes W \rightarrow \text{Mat}_{n,m}(F)$  be given by  $\varphi(v_i \otimes w_j) = e_{ij}$ , where  $e_{ij}$  is the matrix unit whose  $(i, j)$  entry is 1 and all other entries are 0.

- (a) Prove that for a matrix  $A \in \text{Mat}_{n,m}(F)$ , the following are equivalent:

- (i)  $A = \varphi(v \otimes w)$  for some elements  $v \in V$  and  $w \in W$ ;
- (ii)  $\text{rk}(A) \leq 1$ .

- (b) Let  $A \in \text{Mat}_{n,m}(F)$ . Prove that  $\text{rk}(A)$  is the smallest  $d$  such that  $\varphi^{-1}(A)$  can be written as a sum of  $d$  simple tensors.

**Solution:**

- (a) Suppose that  $A = \varphi(v \otimes w)$  for some  $v \in V$  and  $w \in W$ . We may write

$$\begin{aligned}v \otimes w &= \sum_{i=1}^n \sum_{j=1}^m s_i t_j e_i \otimes e_j \\ A &= \sum_{i=1}^n \sum_{j=1}^m s_i t_j e_{ij}.\end{aligned}$$

Then, using the identity

$$e_{ij}(e_k) = \delta_{jk} e_i,$$

where  $\delta_{jk}$  denotes the Kronecker delta, we get that for an arbitrary vector

$$x = \sum_{k=1}^m r_k e_k$$

in  $F^m$ , we have

$$\begin{aligned}Ax &= \left( \sum_{i=1}^n \sum_{j=1}^m s_i t_j e_{ij} \right) \left( \sum_{k=1}^m r_k e_k \right) \\ &= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m s_i t_j r_k e_{ij}(e_k) \\ &= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m s_i t_j r_k \delta_{jk} e_i \\ &= \sum_{i=1}^n \sum_{j=1}^m t_j r_j s_i e_i\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^m t_j r_j \left( \sum_{i=1}^n s_i e_i \right) \\
&\in \text{span} \left\{ \sum_{i=1}^n s_i e_i \right\}.
\end{aligned}$$

Therefore,  $\text{rk}(A) \leq 1$ .

If  $\text{rk}(A) = 0$ , then  $v \otimes w$  is the zero tensor since  $\varphi$  is an isomorphism. Else, we assume  $\text{rk}(A) = 1$ . Then, there are some coefficients  $s_1, \dots, s_n$  such that

$$\text{im}(A) = \text{span} \left\{ \sum_{i=1}^n s_i e_i \right\}.$$

Now, let

$$x = \sum_{k=1}^m r_k e_k.$$

We may then define

$$w = \sum_{j=1}^m t_j e_j$$

to be such that

$$\sum_{j=1}^m t_j r_j = c,$$

so that

$$\varphi \left( \left( \sum_{i=1}^n s_i e_i \right) \otimes \left( \sum_{j=1}^m t_j e_j \right) \right) \left( \sum_{k=1}^m r_k e_k \right) = c \sum_{i=1}^n s_i e_i.$$

Thus, we find  $v \otimes w$  such that  $A = \varphi(v \otimes w)$ .

(b)

**Problem (Problem 6):** Let  $R$  be a ring with 1, and let  $M$  be a left  $R$ -module,  $N$  a submodule. Prove that  $M$  is Noetherian if and only if  $N$  and  $M/N$  are both Noetherian.

**Solution:** Suppose  $M$  is a Noetherian module. Then, any submodule of  $M$  is finitely generated, so since any submodule of  $N$  is a submodule of  $M$ ,  $N$  is Noetherian. Similarly, since any submodule of  $M/N$  corresponds to a submodule of  $M$  that contains  $N$  by the Fourth Isomorphism Theorem, it follows that  $M/N$  is also Noetherian.

Now, suppose  $M$  is a module such that  $M/N$  and  $N$  are Noetherian. Let  $P_1 \leq P_2 \leq \dots$  be an ascending chain of submodules for  $M$ . Then,  $P_1 \cap N \leq P_2 \cap N \leq \dots$  is an ascending chain of submodules of  $N$ , so there is some index  $k_1$  such that  $P_{k_1+i} \cap N = P_{k_1} \cap N$  for all  $i \in \mathbb{N}$ . Similarly, the set of submodules  $P_1 + N \leq P_2 + N \leq \dots$  is an ascending chain of submodules that contains  $N$ , so the submodules  $(P_1 + N)/N \leq (P_2 + N)/N \leq \dots$  forms an ascending chain of submodules in  $M/N$ , so there is some index  $k_2$  such that  $P_{k_2+i} + N = P_{k_2} + N$  for all  $i \in \mathbb{N}$ . In particular, this means that for all  $i \in \mathbb{N}$ ,  $P_{k+i} = P_k$ , where  $k = \max(k_1, k_2)$ , so  $M$  is Noetherian.