Problem 1

Let $X = \{0,1\}^n$. Show that the Hamming distance:

$$d_H : X \times X \to [0,\infty)$$
$$d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) = \left|\{j \mid x_j \neq y_j\}\right|$$

defines a metric on X.

Proof:

- Symmetry:
$$\begin{aligned} d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) &= \left|\{j \mid x_j \neq y_j\}\right| \\ &= \left|\{j \mid y_j \neq x_j\}\right| \\ &= d_H\left((y_j)_{j=1}^n, (x_j)_{j=1}^n\right) \end{aligned}$$
- Definiteness: it is only the case that $d_H(x_j, y_j) = 0$ if $x_j = y_j$ for all j, by the definition of the distance.
- Similarly, since $x_j = x_j$ for all j, $d_H(x_j, x_j) = 0$.
- Let $(x_j)_j$, $(y_j)_j$, and $(z_j)_j$ be sequences of bits. The set $\{j \mid x_j \neq z_j\}$ is formed by taking all the values $\{j \mid x_j \neq y_j\}$ along with $\{j \mid y_j \neq z_j\}$, net of particular indices where $x_j = z_j$, but $x_j \neq y_j$. Therefore,
$$d(x,z) \leq d(x,y) + d(y,z).$$

Problem 2

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V, show that the induced metrics d and d' are equivalent.

Proof: Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms. Then, $\exists c_1, c_2 \in \mathbb{R}$ such that $\|v-w\|' \leq c_1 \|v-w\|$ and $\|v-w\| \leq c_2 \|v-w\|'$. However, this is the exact same statement as $d(v,w) \leq c_1 d'(v,w)$ and $d'(v,w) \leq c_2 d'(v,w)$. Thus, d and d' are equivalent metrics.

Problem 3

Let $\{X_k, d_k\}$ be a sequence of metric spaces with uniformly bounded metrics. Let

$$X := \prod_{k\geq 1} X_k$$

denote the product.

(a) Show that

$$D : X \times X \to [0,\infty)$$
$$D(x,y) := \sum_{k\geq 1} 2^{-k} d_k(x_k, y_k)$$

defines a metric on X.

(b) Consider the case where $\{X_k\} = \{0,2\}$ and $d_k(a,b) = |a-b|$ for every $k \geq 1$. We get the abstract Cantor set

$$\Delta := \prod_{k\geq 1} \{0,2\};$$
$$D(x,y) := \sum_{k=1}^{\infty} 3^{-k} |x_k - y_k|.$$

Prove that $D(x,z) = D(y,z)$ implies $x = y$.

Problem 10

Let $\mathcal{C}$ denote the Cantor set. Show that $\mathcal{C}$ is nowhere dense.

Proof: We know that $\mathcal{C}$ is closed, meaning all we need show is that $\mathcal{C}^0 = \emptyset$.

Suppose toward contradiction that $\mathcal{C}^0$ is not empty. Then, $\exists x \in \mathcal{C}$ and $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{C}$.

Find m so large such that $3^{-m} < \varepsilon$. Then, $(x-\varepsilon, x+\varepsilon)$ must be contained in a subinterval with length $\frac{1}{3^m}$. However, $2\varepsilon > \frac{1}{3^m}$, and every subinterval in the element $\mathcal{C}_m$ has length $\frac{1}{3^m}$.