The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of x).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

Some Functions and Sets

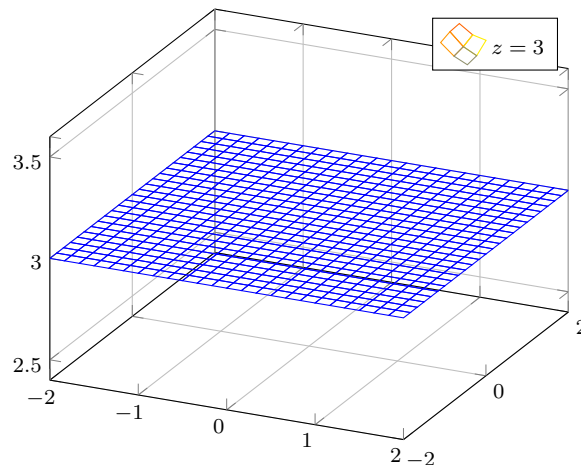$$f(x, y) = x^2 - y^2$$

DOMAIN: $\{(x, y) \mid \exists f(x, y)\}$

RANGE: $\{f(x, y) \mid (x, y) \in \mathrm{Dom}(f)\} = \mathbb{R}$

GRAPH: $\mathrm{Graph}(f) = \{x, y.f(x, y) \mid x, y \in \mathrm{Dom}(f)\}$. For example, $(1, 3, 4) \notin \mathrm{Graph}(f)$ since $1^2 - 3^2 \neq 4$.
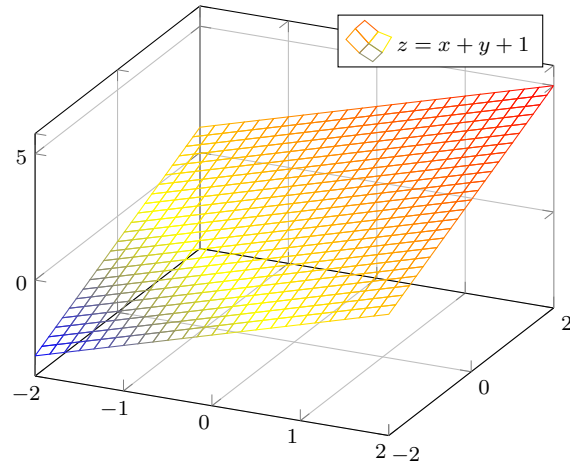
Examples

In $\mathbb{R}^3$, in x, y, z coordinates, $z = 3$ is a plane defined as follows:

- Parallel to the xy plane.

- Passes through the point $(0.0, 3)$.

Meanwhile, $y = 0$ would be a "wall" that passes through the origin that contains the line $y = 0$ in the xy plane.

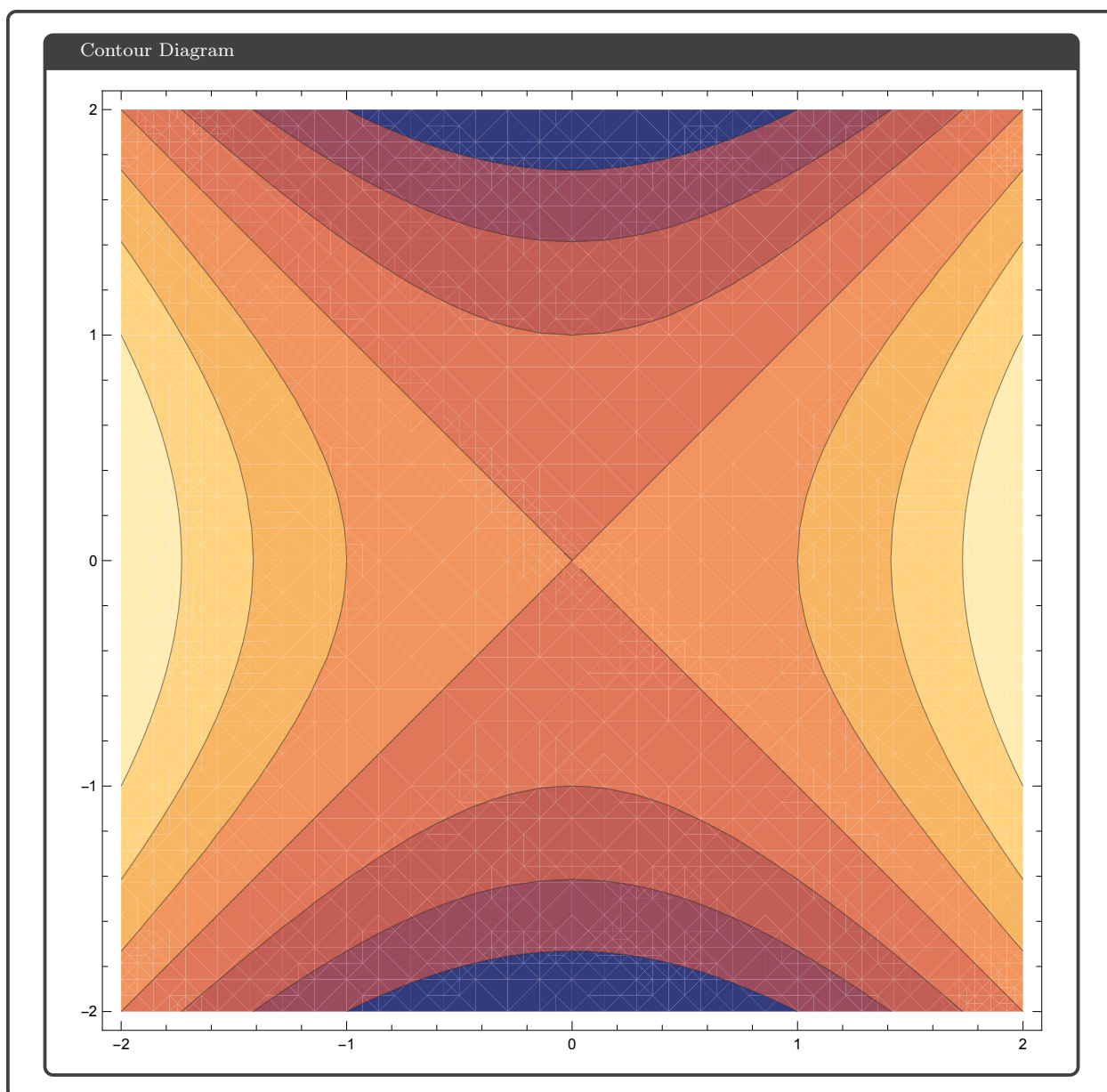Finally, $z = x + y + 1$ is a plane, as we can see below.

Visualizing a function of multiple variables

Consider the function $f(x, y) = x^2 - y^2$. We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$

- $f(0, y) = -y^2$

- $f(2, y) = 4 - y^2$

- $f(x, -2) = x^2 + 4$

- $f(x, 0) = x^2$

- $f(x, 2) = x^2 + 4$

Alternatively, we can visualize via contour diagrams (i.e., everywhere that z is a certain value), as seen in mathematica as follows:

Contour Diagram

Contour Example

Consider the function $f(x, y) = y - 3x^2$. We want to find the contours.

- -

For any c, we have that $c = y - 3x^3$, or $y = 3x^3 + c$. Therefore, every contour "looks like" $3x^3 + c$ for values of c. For example, in the following, we have $c = \{-2, -1, 0, 1, 2\}$
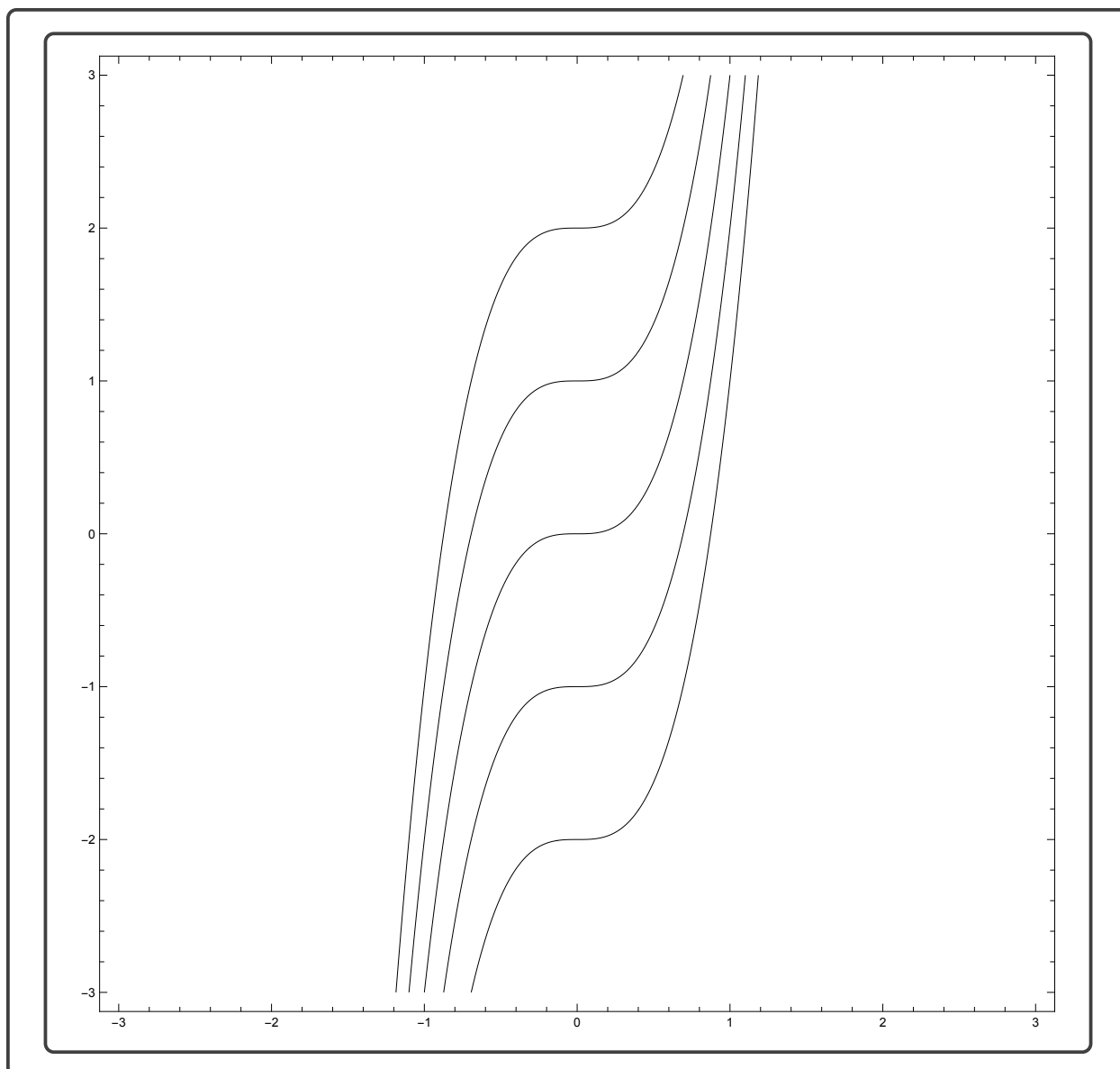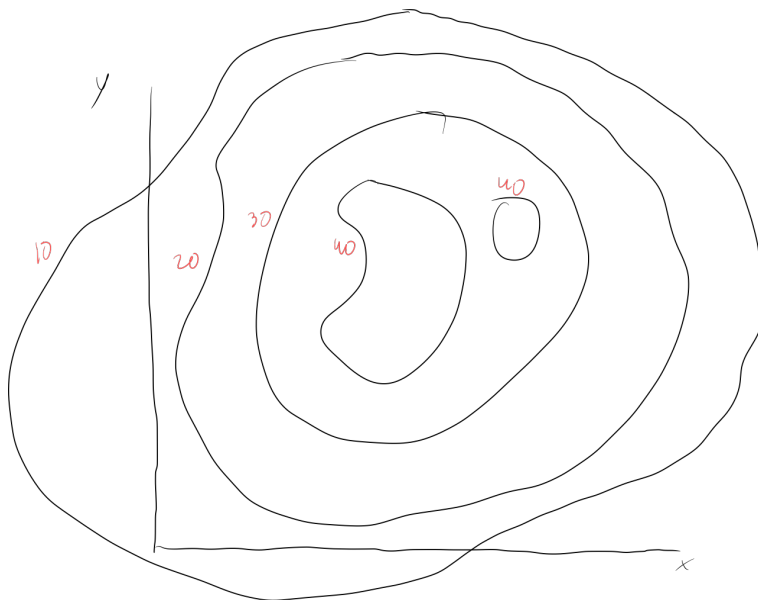
Distance

In $\mathbb{R}^5$, let $p = (3, 1, 4, 1, 5)$, and $q = (1, 0, -2, 0, 2)$. Using the Euclidean metric, we can find the distance between p and q is $d(p, q) = ((3 - 1)^2 + (1 - 0)^2 + (4 - (-2))^2 + (1 - 0)^2 + (5 - 2)^2)^{1/2} = (4 + 1 + 36 + 1 + 9)^{1/2} = \sqrt{51} = 7.14$. We can also call this the 2-norm.

$$d(p, q) = \left(\sum_{k=1}^{n} (p_k - q_k)^2 \right)^{1/2}$$

Derivatives

To denote a derivative, we can't talk about one value, we must use a *partial* derivative, $\frac{\partial f}{\partial x}$, or $\frac{\partial f}{\partial y}$. The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

x\y	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A "linear" approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and is an output in $z = f(x, y)$, and $m, n \in \mathbb{R}$.

For example, with the above table, we can see that the function is linear in x and y (i.e., the slope holding the other variable constant is constant).

Limits in Multivariable Functions

Consider the following:

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow $y = mx$

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 - y^2} = \lim_{(x,y) \to (0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2}$$
$$= \frac{1 + m^2}{1 - m^2}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of m

m	$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours "approach" a particular point, we can see that the limit is defined.

Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$. These vectors are equivalent because they are components of $\mathbb{R}^3$.

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

Properties of Vectors

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $c\vec{u} = \langle cu_1, cu_2, \ldots, cu_k \rangle$

Additionally, we define $\vec{u} \cdot \vec{v}$ as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

Partial Derivatives

Consider $f(x,y) = x^2 y + xe^y$.

$$f_x := \frac{\partial f}{\partial x}$$
$$f_x(a,b) = \frac{\partial f}{\partial x}\bigg|_{(a,b)}$$

We know that $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, meaning f is endlessly differentiable.

Functions and Approximations

Let $f(x,y) = x^2 - y^2$, $g(x,y) = 2xy$

- $f_{xx} + f_{yy} = 0$
- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For $f(x, y)$ at $(a, b, f(a, b))$, we have the following:

$$\ell(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(y - b)$$

$$q(x, y) = \ell(x, y) + \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right)$$

In order to get a sense of the "derivative," we can use the following:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

Directional Derivative and Gradient

Given $f(x, y)$ and (a, b), where $f \in C^2(\mathbb{R}^2)$. Then, the quadratic approximation is:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_x(a, b)(y - b)$$

$$+ \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b)\right)$$

$$df = f_x(a, b)dx + f)y(a, b)dy \qquad\qquad \text{a differential}$$

$$\Delta f = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

Evaluating $f(x, y) = xe^y$ at $(a, b) = (-1, 0)$

$$f_x = e^y$$

$$f_y = xe^y$$

$$f_x(-1, 0) = 1$$

$$f_y(-1, 0) = -1$$

$$\Delta f = \Delta x - \Delta u$$

On a given contour map, let $\vec{u} = \langle u_1, u_2\rangle$ denote a *unit* vector in a direction that we want to find the derivative of f in.

$$f_{\vec{u}}(x, y) = \nabla f(a, b) \cdot \vec{u}$$

Where

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

The directional derivative for all vectors $\vec{v}$ is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

Chain Rule

Let $f(x, y)$ be a function where $x - x(t)$ and $y = y(t)$. We want to find

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The chain rule works in higher dimensions too. Consider $k(x_1(t), x_2(t), \ldots, x_{152}(t))$. Then,

$$\frac{dk}{dt} = \sum_{i=1}^{152} \frac{\partial k}{\partial x_i}\frac{dx_i}{dt}$$

We can also view this as a vector. Let $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{152}(t) \end{pmatrix}$. Then, we can write $\frac{dk}{dt}$ more succinctly as follows:

$$\frac{dk}{dt} = \nabla k \cdot \frac{d\vec{x}}{dt}$$

For example, let $f(x, y, z) = 3x^2 y + zx + 2$, where $x = x(t), y = y(t), z = z(t)$

$$\frac{df}{dt} = \begin{pmatrix} 6xy + z \\ 3x^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

$$= (6xy + z) \, x'(t) + 3x^2 y'(t) + xz'(t)$$

So, if we let $x(t) = \sin(t)$, $y(t) = e^t$, and $z(t) = t^2 + 1$. Then, we have

$$\frac{df}{dt} = 6\sin(t)\cos(t)e^t + t^2 \cos(t) + \cos(t) + 3e^t \sin^2(t) + 2t\sin(t)$$

Alternatively, consider $f(x, y, z) = x^2 + yz + e^y$, where $x(s, t) = st, y = y(s, t) = t + s^2, z = z(s, t) = e^t$. Let

$$\vec{x} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix}$$

Then, we have

$$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t}$$
$$\frac{\partial f}{\partial s} = \nabla f \cdot \frac{\partial \vec{x}}{\partial s}$$

Evaluating the first expression, we have

$$\frac{\partial f}{\partial t} = \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix} \cdot \begin{pmatrix} s \\ 1 \\ e^t \end{pmatrix}$$

$$= 2s^2 t + 3^t + e^{t+s^2} + (t + s^2)e^t$$

Consider $f(x, y(x))$. Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

This is the technique we use to find implicit differentiation.

We know as a result that $\nabla f(a, b)$ is orthogonal to the contour curve at (a, b)

Recap

In $\mathbb{R}^3$, find the plane that contains $P = (P_1, P_2, P_3)$, Q, and R. We can find it by the following:

$$0 = \vec{n} \cdot \begin{pmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{pmatrix}$$

$$0 = n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3)$$

where

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

Differentiability

A function $f(x)$ of one variable is differentiable at $x = a$ if

$$f(a) = \lim_{h \to 0} f(a+h)$$

and

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists and is bounded

We can also linearize the function. f is differentiable if

$$f(x) = f(a) + f'(a)(x - a) + E(x)$$

where $\lim_{h \to 0} \frac{E(a+h)}{h} = 0$.

In the multiple dimensions example, we have $f(x, y)$ is differentiable if

$$f(x, y) = \ell(x, y) + E(x, y)$$

where $\lim_{h \to 0, k \to 0} \frac{E(a+h, b+k)}{\sqrt{h^2 + k^2}} = 0$

Local Maxima

Let $f(x, y) = x^2 + 2y^2$. We want to find (a, b) which are local maxima, minima, or other.

(a, b) is a local maximum if $f(a, b) \geq f(x, y) \ \forall (x, y) \in V_\varepsilon(a, b)$, where $\varepsilon > 0$.

(1) Find Critical Points for $f(x, y) : f_x(x, y), f_y(x, y) = 0, \ f_x(x, y), f_y(x, y)$ are undefined.

$$f_x(x, y) = 2x$$
$$f_y(x, y) = 4y$$
$$f_x(0, 0) = 0$$
$$f_y(0, 0) = 0$$
$$f(0, 0) = 0$$
$$f(x, y) > 0 \qquad\qquad \forall (x, y) \neq (0, 0)$$

For all x, y, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 0$. Finally,

$$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) + f_{xy}(x, y)^2$$
$$= 8$$
$$> 0$$

Since $D(x, y) > 0$, we look at the sign of f_{xx}. Since it is positive, $f(0, 0)$ has a local minimum.

Local Maxima and Minima Approach

Given $f(x, y)$, we want

(1) Find critical points:

$$\frac{\partial f}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} = 0$$

(2) Compute $f_{xx}, f_{yy}, f_{xy}, D = f_{xx}f_{yy} - (f_{xy})^2$

(3)

f_{xx}	D	Critical Point
+	+	Local Minimum
-	+	Local Maximum
±	-	Saddle Point
±	0	Nothing

Consider the function

$$f(x, y) = \ln(x^2 + y^2 + 1)$$
$$f(0, 0) = 0$$
$$f(x, y) > 0$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$
$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

Critical Points: $(0, 0)$

$$\frac{\partial^2 f}{\partial x^2}\bigg|_{(0,0)} = \frac{2(x^2 + y^2 + 1) - 4x^2}{(x^2 + y^2 + 1)^2}$$
$$= 2$$
$$\frac{\partial^2 f}{\partial x^2}\bigg|_{(0,0)} = \frac{2(x^2 + y^2 + 1) - 4y^2}{(x^2 + y^2 + 1)^2}$$
$$= 2$$
$$\frac{\partial^2 f}{\partial x \partial y}\bigg|_{(0,0)} = \frac{-4xy}{(x^2 + y^2 + 1)^2}$$
$$= 0$$

Now, consider the function

$$f(x, y) = x^2 - 2xy + y^2$$
$$\frac{\partial f}{\partial x} = 2x - 2y$$
$$\frac{\partial f}{\partial y} = -2x + 2y$$
$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial y^2} = 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2$$
$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$= 0$$

Therefore, the critical points of this function are indeterminate with the given approach. However, we know that $f(x, y) = (x - y)^2 = 0$ when $x = y$, so the line $y = x$ is a local minimum trough in 3-space.

Now, consider the function

$$f(x, y) = (x - 1)^2(y + 2)$$
$$\frac{\partial f}{\partial x} = 2(x - 1)(y + 2)$$
$$\frac{\partial f}{\partial y} = (x - 1)^2$$

Critical points: $\{(1, y) \mid y \in \mathbb{R}\}$

$$\frac{\partial^2 f}{\partial x^2} = 2(y + 2)$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2(x - 1)$$

$$D = 0 - (2(x - 1))^2$$

$$= 0 \qquad\qquad \text{Evaluating } D \text{ at critical points}$$

Finding Critical Points

Let $f(x, y) = (y^2 + 2)\sin(x)$. on $[-2, 2] \times [-2, 2]$

$$\frac{\partial f}{\partial x} = (y^2 + 2)\cos(x)$$

$$= 0$$

$$\frac{\partial f}{\partial y} = 2y\sin(x)$$

$$= 0$$

$$(x, y) = \left(\frac{(2n + 1)\pi}{2}, 0\right)$$

$$= \{(\pi/2, 0), (-\pi/2, 0)\}$$

$$\frac{\partial^2 f}{\partial x^2} = -(y^2 + 2)\sin(x)$$

$$\frac{\partial^2 f}{\partial y^2} = 2\sin(x)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y\cos(x)$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= -2(y^2 + 2)\sin^2(x) - 4y^2\cos^2(x)$$

$$< 0$$

Therefore, the critical points are saddle points. If there is no domain restriction, we have a series of saddle points all along $y = 0$.

Why Finding Critical Points Works

We create the Taylor series of $f(x, y)$ at (x_0, y_0):

$$f(x, y) \approx \ell(x_0, y_0) + \frac{1}{2}\left(f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2\right)$$

$$= f(x_0, y_0) + \underbrace{\nabla f(x, y) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}_{=0 \text{ at critical points}} + \frac{1}{2}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^T \underbrace{\begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}}_{\text{Hessian}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

If the Hessian is positive definite, then $\lambda_1, \lambda_2 > 0$ and the critical point is a local min. If the Hessian is negative definite, then $\lambda_1, \lambda_2 < 0$ and the critical point is a local max.

In any given 2×2 matrix, the eigenvalues λ_1, λ_2 are such that $\lambda_1 + \lambda_2 = \text{Tr}(A)$ and $\lambda_1\lambda_2 = \text{Det}(A)$.

Optimization

Let $f(x, y) = 2x - y$. We want to optimize f with respect to $g(x, y) = x^2 - y^2 - 4 = 0$.

Define $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$. Given $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, then $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Then, we take

$$\nabla L = \nabla f = \lambda \nabla g$$
$$= 0 \qquad\qquad\qquad \text{critical points of } L$$

We find x, y, λ for each critical point.

$$\nabla f = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g$$
$$2 = 2\lambda x$$
$$-3 = -2\lambda y$$
$$x^2 - y^2 = 4$$

$$\lambda = \frac{1}{x}$$
$$\lambda = \frac{3}{2y}$$
$$x = \frac{2y}{3}$$
$$\frac{4y^2}{9} - y^2 = 4$$
$$-\frac{5}{9}y^2 = 4$$

No Solution

However, if $g(x, y) = x^2 + y^2 - 4 = 0$, we have

$$\nabla f = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$2 = 2\lambda x$$

$$-3 = 2\lambda y$$

$$x^2 + y^2 = 4$$

$$\lambda = \frac{1}{x}$$

$$\lambda = \frac{-3}{2y}$$

$$x = \frac{-2y}{3}$$

$$\frac{4y^2}{9} + y^2 = 4$$

$$-\frac{13}{9}y^2 = 4$$

$$y = \pm\frac{6}{\sqrt{13}}$$

$$x = \mp\frac{4}{\sqrt{13}}$$

$$f_{\max} = 2\sqrt{13}$$

$$f_{\min} = -2\sqrt{13}$$