Basic Properties

Definition: A topological space M is called a *manifold* if it satisfies the following:

- M is Hausdorff (points can be separated by open sets);
- M is second countable (the basis for the topology of M is countable);
- M is locally Euclidean (every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$ for some n).

In particular, the third condition says that for every $p \in M$, there is $U \in \mathcal{O}_p$ and a homeomorphism $\varphi\colon U \to \mathbb{R}^n$. The value of n is called the *dimension* of the manifold M.

Definition: Let M be an n-manifold. A *chart* on M is a pair (U, φ) such that $U \subseteq M$ is open, $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism.

A family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ is known as an *atlas* if

$$M = \bigcup_{i\in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point $p \in M$ and two charts (U, φ_U) and (V, φ_V) such that $p \in U$ and $p \in V$. The functions $\varphi_U\colon U \to \mathbb{R}^n$ and $\varphi_V\colon V \to \mathbb{R}^n$ are homeomorphism, meaning that $\varphi_V \circ \varphi_U^{-1}\colon \varphi_U(U \cap V)^n \to \mathbb{R}^n$ defined on the (nonempty) $U \cap V$ is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs $\varphi_V \circ \varphi_U^{-1}$ are *diffeomorphisms*. To do this, we need to first develop the derivative in $\mathbb{R}^n$.

Definition: Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say f is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear map $L \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\frac{\|f(p + h) - f(p) - Lh\|}{\|h\|} \to 0$$

as $h \to 0$.

The *derivative* of f is the association $f \mapsto L$ for each $p \in \mathbb{R}^n$. We write $D_p f$ to denote this map. Note that we consider elements of $\mathrm{Mat}_n(\mathbb{R})$ as points in $\mathbb{R}^{n^2}$ with the standard topology on $\mathbb{R}^{n^2}$.

A function f is called a *diffeomorphism* if it is continuously differentiable and has a continuously differentiable inverse.

Definition: If (U, φ_U) and (V, φ_V) are charts such that $U \cap V \neq \emptyset$, the function $\varphi_V \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is known as the *transition map* between φ_U and φ_V.

A smooth structure for M is an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that for all i, j, the transition maps $\varphi_j \circ \varphi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are diffeomorphisms where defined.

If $\{(U_i, \varphi_i)\}_{i\in I}$ is a *maximal* smooth atlas — i.e., any other smooth atlas that contains $\{(U_i, \varphi_i)\}_{i\in I}$ is equal to $\{(U_i, \varphi_i)\}_{i\in I}$ — then we call $\{(U_i, \varphi_i)\}_{i\in I}$ a *smooth structure* for M.

Note: From now on, we use "manifold" to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

Definition: A map $f\colon M \to N$ between manifolds is called *smooth* if for any chart (U, φ_U) in M and corresponding chart (V, φ_V) in N, the map $\varphi_V \circ f \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^k$ is continuously differentiable.

The function f is a *diffeomorphism* if f is a smooth bijection with smooth inverse, and we say the manifolds M and N are diffeomorphic if they admit a diffeomorphism.

Examples

There are a couple special examples of (smooth) manifolds.

(i) Open subsets of $\mathbb{R}^n$ are always manifolds.

(ii) The general linear group, $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices, viewed as a subset of $\text{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, is a manifold. Furthermore, it is an open subset of $\mathbb{R}^{n^2}$, as considering the map $\det\colon \text{Mat}_n(\mathbb{R}) \to \mathbb{R}$ given by $A \mapsto \det(A)$, we see that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$.

(iii) The special linear group, $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$, consisting of $n \times n$ matrices with determinant 1, is also a smooth manifold. Furthermore, this manifold is a closed subset of $\mathbb{R}^{n^2}$, as it is equal to $\det^{-1}(\{1\})$.

(iv) The n-sphere, S^n, given by

$$S^n = \left\{ (x_0, \dots, x_n) \;\middle|\; \sum_{i=0}^{n} x_i^2 = 1 \right\}$$

is a manifold in $\mathbb{R}^n$. That it is a smooth manifold is quite a bit less obvious.

Now, in low dimensions, we know that $S^2 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and that the continuously differentiable transformation $z \mapsto \frac{1}{z}$ takes the neighborhood basis of ∞ to deleted neighborhoods of 0, and takes the neighborhood basis of 0 to the neighborhood basis of ∞. This is our desired smooth structure.

In the case of the general S^n, we use two stereographic projections to construct our smooth structure. The first stereographic projection is via the north pole, N_p, and maps points on $S^n \setminus \{N_p\}$ bijectively to $\mathbb{R}^n$; this is a chart that is defined everywhere on S^n except N_p. Similarly, we may use a stereographic projection originating from the south pole, S_p, so as to create another chart defined everywhere except S_p. These two stereographic projections are our desired smooth structure, as these two charts are all that is necessary to cover S^n.

(v) The real projective plane, consisting of lines through the origin in $\mathbb{R}^{n+1}$, can be expressed as

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right)/\mathbb{R}^\times.$$

We will show that this is a manifold by constructing a family of charts mapping to $\mathbb{R}^n$.

Consider a point $(r_0, \dots, r_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. If $r_0 \neq 0$, then by dividing, we may associate this point's equivalence class in $\mathbb{RP}^n$ to

$$(1, r_1/r_0, \dots, r_n/r_0) \in \{1\} \times \mathbb{R}^n,$$

so we may associate all points of the form $[(r_0, \dots, r_n)]$ with $r_0 \neq 0$ with a chart (U_0, φ_0) that maps $\mathbb{RP}^n$ to $\mathbb{R}^n$.

Similarly, we may define U_k via

$$U_k = \{[(r_0, \dots, r_n)] \mid r_k \neq 0\}$$

with corresponding chart

$$\begin{aligned} \varphi_k\colon U_k &\to \mathbb{R}^n \\ [(r_0, \dots, r_n)] &\mapsto \frac{1}{r_k}(r_0, \dots, \widehat{r_k}, \dots, r_n), \end{aligned}$$

where $\widehat{r_k}$ denotes the exclusion of the r_k coordinate. Varying k from 0 to n, we see that

$$\mathbb{RP}^n = \bigcup_{k=0}^{n} U_k,$$

the chart functions $\varphi_k \colon U_k \to \mathbb{R}^n$ are homeomorphisms (as they are just division and projections). Furthermore, the transition maps $\varphi_j \circ \varphi_i^{-1}$ are coordinate-wise rational functions defined by

$$(u_1, \dots, u_n) \mapsto \left(\frac{u_1}{u_i}, \dots, \frac{1}{u_i}, \dots, \frac{u_n}{u_i}\right),$$

where the $\frac{1}{u_i}$ is at position j.

(vi) We now turn to a very important example from algebraic geometry: the Grassmannian, $\mathrm{Gr}(k, n)$, consisting of all the k-dimensional subspaces of $\mathbb{R}^n$.

This is a $k(n-k)$-dimensional manifold; we need to understand what the smooth structure is. To do this, we let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$, and for any $V \in \mathrm{Gr}(k, n)$, we consider maps in $\mathrm{Hom}(V, V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.

Now, we see that if $W \in \mathrm{Gr}(k, n)$ is any other k-dimensional subspace, the orthogonal $P_V \colon \mathbb{R}^n \to V$ restricted to W is a linear isomorphism if and only if $W \not\subseteq V^\perp$, or that $W \cap V^\perp = \{0\}$.

We see that if W is such that $P_V|_W \colon W \to V$ is a linear isomorphism, the inverse $(P_V|_W)^{-1} \colon V \to W$ is well-defined; so, we may make a correspondence between $\mathrm{Hom}(V, V^\perp)$ and $\mathrm{Gr}(k, n)$ by noting that any such $T \in \mathrm{Hom}(V, V^\perp)$ has a corresponding graph $(v, T(v))$, so we take $v \mapsto P_V|_W^{-1}(v)$, then project onto $V^\perp$ by taking $T\big(P_{V^\perp}\big(P_V|_W^{-1}(v)\big)\big) = T(v)$. We depict it as a diagram below.

$$\begin{tikzcd}
 & & V \\
V \arrow[r, "P_V|_W^{-1}"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "T"'] & W \arrow[r, hook, "\iota"] & \mathbb{R}^n \arrow[u, "P_V"'] \arrow[d, "P_{V^\perp}"] \\
 & & V^\perp
\end{tikzcd}$$

Defining $U_V = \{W \in \mathrm{Gr}(k, n) \mid W \cap V^\perp = \{0\}\}$, we may define the chart from U_V onto $\mathrm{Hom}(V, V^\perp)$ by $\varphi_V = P_{V^\perp} \circ P_V|_W^{-1}$. The family $\{(U_V, \varphi_V) \mid V \in \mathrm{Gr}(k, n)\}$ is our smooth atlas.

Inverse and Implicit Function Theorems

In order to replace manifolds with linear maps, we need to understand smooth maps on $\mathbb{R}^n$. The most important theorems in this regard are the inverse function theorem and the implicit function theorem.

> **Theorem** (Inverse Function Theorem)**:** Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. If $D_p f$ is invertible as a linear map, then f has a local, continuously differentiable inverse $f^{-1} \colon V \to W$, where $p \in W \subseteq U$ and $f(p) \in V \subseteq \mathbb{R}^n$.

The proof uses the contraction mapping theorem. Recall that if X is a complete metric space, and $f \colon X \to X$ is a strict uniform contraction — that is, there exists $0 \leqslant \lambda < 1$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for all $x, y \in X$ — then f has a unique fixed point.

We begin with a technical lemma.

Lemma: If $U(0,r) \subseteq V$ for some $r > 0$ where V is a normed vector space, $g\colon V \to V$ is a uniform contraction, and $f = \mathrm{id} + g$, then the following hold:

- $(1-\lambda)\|x-y\| \leqslant \|f(x)-f(y)\|$ (in particular, f is injective);
- if $g(0) = 0$, then
$$U(0,(1-\lambda)r) \subseteq f(U(0,r)) \subseteq U(0,(1+\lambda)r).$$

Proof of Lemma. To see the first item, we notice that by the triangle inequality,

$$\begin{aligned}\|x-y\| - \|f(x)-f(y)\| &\leqslant \|x-y\| - \|x-y\| + \|g(x)-g(y)\| \\ &\leqslant \lambda\|x-y\|,\end{aligned}$$

so $(1-\lambda)\|x-y\| \leqslant \|f(x)-f(y)\|$, and f is injective. Furthermore, we see that if $g(0) = 0$, then

$$\begin{aligned}f(U(0,r)) &= U(0,r) + g(U(0,r)) \\ &\subseteq U(0,r) + \lambda U(0,r) \\ &= U(0,(1+\lambda)r).\end{aligned}$$

Finally, if $y \in U(0,(1-\lambda)r)$, then we want to find x such that $y = f(x) = x + g(x)$; equivalently, we see that we want x such that $x = y - g(x)$. Since the function $F(x) = y - g(x)$ is a translation of a uniform contraction, $F(x)$ is a contraction, so there is a fixed point, meaning $y \in f(U(0,r))$. □

Note: We will use $|\cdot|$ to denote the norm on $\mathbb{R}^n$.

Proof of the Inverse Function Theorem. By using a series of affine maps — first by translating p to 0, then translating $f(p)$ to 0, then inverting $D_0 f$ as per our assumption, we may safely assume that $p = f(p) = 0$ and $D_0 f = \mathrm{Id}$.

Set $g = f - \mathrm{Id}$. We will show that g is a contraction in a sufficiently small ball. Fixing $x, y \in \mathbb{R}^n$, consider the map $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto g(x + t(y-x))$. Notice that by the Fundamental Theorem of Calculus,

$$|g(y) - g(x)| \leqslant |y-x| \sup_{0\leqslant t\leqslant 1} |g'(x + t(y-x))|.$$

Furthermore, since $g'(0) = \mathbf{0}$ by the fact that $D_0 f = \mathrm{Id}$ and $(\mathrm{Id})' = \mathrm{Id}$, and since f is continuously differentiable, there is $r > 0$ such that

$$|g(y) - g(x)| \leqslant \frac{1}{2}|y-x|$$

for all $x, y \in U(0,r)$. Thus, g is a strict contraction on $U(0,r)$. By the previous lemma, we see that

$$U(0,r/2) \subseteq f(U(0,r));$$

by setting $U = U(0,r) \cap f^{-1}(U(0,r))$, we see that the map $f|_U\colon U \to V := U(0,r/2)$ is a bijection. The inverse function $f^{-1}\colon V \to U$ thus exists.

Now, we let $h = f^{-1}$, $x \in U$, $y \in V$ such that $h(x) = y$, and $A = D_x f$. We will show that $A^{-1} = D_y h$, which is enough to show that h is continuously differentiable, as we assume the map $x \mapsto D_x f$ is continuous, and inversion is continuous in $GL_n(\mathbb{R})$.

For sufficiently small vectors s and k, since f and h are bijections, we have

$$h(y+k) = x + s,$$

so

$$f(x+s) = y + k.$$

Furthermore, by unraveling the definitions of $f = g + \mathrm{Id}$, s, and k, and the fact that g is a uniform contraction on U, we get

$$\begin{aligned} |s-k| &= |(f(x+s)-f(x))-s| \\ &= |(x+s+g(x+s))-(x+g(x))-s| \\ &= |g(x+s)-g(x)| \\ &\leqslant \frac{|s|}{2}. \end{aligned}$$

In particular, since

$$\begin{aligned} |s| &\leqslant |s-k|+|k| \\ &\leqslant |k|+\frac{|s|}{2}, \end{aligned}$$

we see that $|s|/2 \leqslant |k|$. We calculate

$$\begin{aligned} \left|h(y+k)-h(y)-A^{-1}k\right| &= \left|x+s-x-A^{-1}(f(x+s)-f(x))\right| \\ &= \left|s-A^{-1}(f(x+s)-f(x))\right| \\ &\leqslant \left\|A^{-1}\right\|_{op}|As-f(x+s)-f(x)|. \end{aligned}$$

Thus, since $|s|/2 \leqslant |k|$,

$$\begin{aligned} \frac{\left|h(y+k)-h(y)-A^{-1}k\right|}{|k|} &\leqslant \frac{2\left\|A^{-1}\right\|_{op}|As-f(x+s)-f(x)|}{|s|} \\ &\to 0, \end{aligned}$$

so $D_y h = A^{-1}$. □

Constructing C^∞ Maps on Manifolds

Definition: A function $f\colon U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open, is called C^∞ if the partial derivatives of all orders,

$$\frac{\partial^{|\alpha|} f}{\partial\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

are continuous. Here, $\alpha = (\alpha_1,\dots,\alpha_n)$ is a *multi-index*, where $\alpha_i \geqslant 0$ for each i, and $|\alpha| = \sum_{i=1}^{n}\alpha_i$.

We are concerned now with constructing C^∞ functions on C^∞-manifolds.[I] In order to do this, we introduce the bump functions.

Definition: The *bump function* that is equal to 1 on $B(0,1)$ and is zero outside $U(0,2)$ is given by

$$\begin{aligned} h(x) &= \begin{cases} e^{-1/x} & x>0 \\ 0 & x\leqslant 0 \end{cases} \\ b(x) &= \frac{h\left(4-|x|^2\right)}{h\left(4-|x|^2\right)+h\left(|x|^2-1\right)}. \end{aligned} \tag{$*$}$$

Lemma: Let M be a C^∞ manifold. Let $U \in \mathcal{O}_p$, and let $f\colon U \to \mathbb{R}$ be an arbitrary C^∞ function defined on U.

[I] A C^∞ manifold is one where all the transition functions $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i\cap U_j) \to \varphi_j(U_i \cap U_j)$ are C^∞ functions.

Then, there exists $V \in \mathcal{O}_p$ with $\overline{V} \subseteq U$, and a C^∞ function $\widetilde{f}$ defined on M such that

$$\widetilde{f}(q) = \begin{cases} f(q) & q \in V \\ 0 & q \notin U. \end{cases}$$

Proof. Let (W, φ) be a chart centered at p with $\varphi(p) = 0$ and $U(0, 3) \subseteq \varphi(W)$. Let $\overline{b} = b \circ \varphi$, where b is the bump function defined in $(*)$. Then, $\overline{b}$ is a C^∞ function on W, and is 0 outside $\varphi^{-1}(U(0, 2)) \subseteq W$.

We define $\overline{b}$ to be equal to zero on W^c. Thus, if we define $V = \varphi^{-1}(U(0, 1))$, then $V \in \mathcal{O}_p$, $\overline{V} \subseteq U$, and $\overline{b}$ is equal to 1 on V. Letting

$$\widetilde{f}(q) = \begin{cases} \overline{b}(q)f(q) & q \in W \\ 0 & q \notin W, \end{cases}$$

we see that $\widetilde{f}$ satisfies the required property. □

The Tangent Space

Notations

- A general normed space V will have its norm denoted by $\|\cdot\|$. If $V = \mathbb{R}^n$, then we denote the norm by $|\cdot|$.
- We denote topological spaces by (X, τ).
- $U(x, r) = \{y \in V \mid \|x - y\| < r\}$.
- $B(x, r) = \{y \in V \mid \|x - y\| \leqslant r\}$.
- $\mathcal{N}_p$: neighborhood system centered at $p \in X$.
- $\mathcal{O}_p$: system of *open* neighborhoods centered at $p \in X$.