Problem 1

Let $(x_k)_k$ be a sequence of strictly positive numbers such that

$$(kx_k)_k \to L > 0.$$

Show that $\sum_k x_k$ diverges.

Since $(kx_k)_k \to L$, every subsequence of $(kx_k)_k$ converges to L. Let $n_k = 2^k$. Then,

$$(2^k x_{2^k})_k \to L > 0,$$

implying that

$$\sum_k 2^k x_{2^k} = \infty.$$

By the Cauchy Condensation test, this implies that $\sum_k x_k$ diverges.

Problem 2

Let $(x_k)_k$ be a sequence of strictly positive numbers. Show the following:

(i) If $\limsup_{k\to\infty} \frac{x_{k+1}}{x_k} < 1$, then $\sum_k x_k$ converges.

(ii) If $\liminf_{k\to\infty} \frac{x_{k+1}}{x_k} > 1$, then $\sum_k x_k$ diverges.

(a)

Let $\varepsilon > 0$.

$$\begin{aligned}\limsup_{k\to\infty} \frac{x_{k+1}}{x_k} &:= u < 1\\ &= \inf_{n\geq 1}\left(\sup_{k\geq n}\frac{x_{k+1}}{x_k}\right)\end{aligned}$$

By the definition of inf, we have that $\exists N \in N$ large such that

$$\sup_{k\geq N}\frac{x_{k+1}}{x_k} < u + \varepsilon.$$

By the definition of sup, we have that $\forall k \geq N$,

$$\begin{aligned}\frac{x_{k+1}}{x_k} &< u+\varepsilon\\ x_{k+1} &< (u+\varepsilon)x_k.\end{aligned}$$

Inductively on x_k, we have that

$$x_{k+m} < (L+\varepsilon)^m x_k,$$

and series-wise, we have

$$\sum_{k=N}^{\infty} x_k < x_N \sum_{m=1}^{\infty}(u+\varepsilon)^m.$$

For sufficiently small ε, the sum on the right-hand side converges, implying that the sum on the left-hand side must converge. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{N-1} x_k + \sum_{k=N}^{\infty} x_k\\ &< \sum_{k=1}^{N-1} x_k + x_n\sum_{m=1}^{\infty}(u+\varepsilon)^m,\end{aligned}$$

meaning that $\sum_k x_k$ is bounded above by a convergent series, so it is convergent.

(b)

Let $\varepsilon > 0$.

$$\begin{aligned}\liminf_{k\to\infty} \frac{x_{k+1}}{x_k} &:= \ell > 1 \\ &= \sup_{n\geq 1}\left(\inf_{k\geq n} \frac{x_{k+1}}{x_k}\right)\end{aligned}$$

By the definition of sup, we have that for large $N \in \mathbb{N}$, and for $k \geq N$,

$$\inf_{k\geq n} \frac{x_{k+1}}{x_k} > \ell - \varepsilon.$$

By the definition of inf, we also have that

$$\begin{aligned}\frac{x_{k+1}}{x_k} &> \ell - \varepsilon \\ x_{k+1} &> (\ell - \varepsilon)x_k\end{aligned}$$

Inductively, we have that

$$x_{k+m} > (\ell - \varepsilon)^m x_k,$$

and via series, we have

$$\sum_{k=N}^{\infty} x_k > x_N \sum_{m=1}^{\infty} (\ell - \varepsilon)^m.$$

For sufficiently small ε, the sum on the right-hand side diverges. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{N-1} x_k + \sum_{k=N}^{\infty} x_k \\ &> x_N \sum_{k=1}^{\infty} (\ell - \varepsilon)^m + \sum_{k=1}^{N-1} x_k,\end{aligned}$$

and since $\sum_k x_k$ is bounded below by a divergent series, the sum diverges.

Problem 3

Consider the sequence of functions

$$\begin{aligned}f_n &: \mathbb{R} \to \mathbb{R}; \\ f_n(x) &= \arctan(nx)\end{aligned}$$

(i) Show that $(f_n)_n \to \frac{\pi}{2}\text{sgn}$ point-wise.

(ii) Show that the convergence in (i) is nonuniform on $(0, \infty)$.

(iii) Show that the convergence in (i) is uniform on $[a, \infty)$ for a fixed $a > 0$.

(i)

Let $\varepsilon > 0$. We know that, $\exists N \in N$ such that $\forall n \geq N$, $|\arctan(n) - \pi/2| < \varepsilon$.

Case 1: Let $x = 0$. Then,

$$\arctan(nx) = 0 \qquad \forall n \geq 1$$

Case 2: Let $x > 0$. Then, set $N' = \lceil N/x \rceil$. So, for $n' \geq N'$, we have

$$\begin{aligned} |\arctan(nx) - \pi/2| &= |\arctan(n') - \pi/2| \\ &< \varepsilon \end{aligned}$$

implying that $\arctan(nx) \to \pi/2$ when $x > 0$.

Case 3: Let $x < 0$. Then, set $x^* = -x$, and we have the same result as in Case 2, where $\arctan(nx^*) \to \pi/2$.

Since $\arctan(nx^*) = \arctan(n(-x)) = -\arctan(nx)$, we have that $\arctan(nx) \to -\pi/2$.

(ii)

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Set $\varepsilon_0 = \frac{\pi}{4}$. Then, we have that

$$\begin{aligned} |\arctan(n_k x_k) - \pi/2| &= \left|\arctan\left(k\frac{1}{k}\right) - \frac{\pi}{2}\right| \\ &= \left|\arctan(1) - \frac{\pi}{2}\right| \\ &= \left|\frac{\pi}{4} - \frac{\pi}{2}\right| \\ &= \frac{\pi}{4} \\ &\geq \varepsilon. \end{aligned}$$

(iii)