Problem (Problem 1)**:** A subset $A \subseteq \mathbb{R}^n$ is said to have *measure zero* if, for all $\varepsilon > 0$, the set A can be covered by open balls of total volume at most ε. Prove that a countable subset of $\mathbb{R}^n$ has measure zero, and that the standard middle-thirds cantor set in $[0,1] \subseteq \mathbb{R}$ has measure zero.

Solution: Let A be countable, and let $\{a_k\}_{k\geqslant 1}$ be an enumeration of the points in A. Let $\varepsilon > 0$. Let c_n be the constant dependent on n such that the volume of $U(x,r) = c_n r^n$. For each k, define

$$r_k = \left(\frac{1}{2^k c_n}\varepsilon\right)^{1/n}.$$

Then, we see that the family $\{U(a_k, r_k)\}_{k=1}^{\infty}$ has total volume no more than ε, seeing as if all the open balls are disjoint, their union has total volume ε. Thus, countable subsets of $\mathbb{R}^n$ have measure zero.

If $\mathcal{C} \subseteq [0,1]$ is the traditional middle-thirds Cantor set, then we calculate the measure of its complement by taking

$$\begin{aligned}\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k &= \frac{1}{3}\frac{1}{1-\left(\frac{2}{3}\right)}\\ &= 1,\end{aligned}$$

meaning that the Cantor set has measure zero.

Problem (Problem 2)**:** Prove that if $A \subseteq U \subseteq \mathbb{R}^n$ has measure zero (with U open), and $f\colon U \to \mathbb{R}^n$ is smooth, show that $f(A)$ has measure zero.

Problem (Problem 5)**:** Prove that $SL_2(\mathbb{R})$, the 2×2 real matrices of determinant one, is diffeomorphic to $\mathbb{R}^2 \times S^1$.

Solution: We consider the action of $SL_2(\mathbb{R})$ on the upper half-plane of $\mathbb{C}$, $\mathbb{H} = \{z \mid \operatorname{Im}(z) > 0\}$, given by

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

In particular, if $z = x + iy$ with $y > 0$, then

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d\end{pmatrix} z &= \frac{(ax+b)+iay}{(cx+d)+icy}\\ &= \frac{1}{(cx+d)^2 + c^2y^2}\left(\left((ax+b)(cx+d)+acy^2\right) + i(acxy - acxy + ady - bcy)\right)\\ &= \frac{1}{(cx+d)^2 + c^2y^2}\left(\left((ax+b)(cx+d)+acy^2\right) + iy\right),\end{aligned}$$

In particular, this is a fractional linear transformation on $\mathbb{C}$ that is an automorphism of $\mathbb{H}$, so by composing these fractional linear transformations, we can see that $SL_2(\mathbb{R})$ acting on $\mathbb{H}$ via this map is a group action.

This action is transitive, since for any $x + iy \in \mathbb{H}$, we may map $i \mapsto x + iy$ by using the transformation

$$\frac{ai+b}{ci+d} = i$$

which via multiplication and matching parts gives

$$\begin{aligned}a &= cx + dy\\ b &= xd - yc\end{aligned}$$

so by multiplying and back-substituting, we get

$$c^2 + d^2 = \frac{1}{y}.$$

By guessing that $c = 0$, we get

$$\begin{aligned} d &= \frac{1}{\sqrt{y}} \\ a &= \sqrt{y} \\ b &= \frac{x}{\sqrt{y}}. \end{aligned}$$

Now, to understand the stabilizer of some $z \in \mathbb{H}$, we only need to understand the stabilizer of i. For this, we see that

$$\begin{aligned} \frac{ai+b}{ci+d} &= i \\ ai + b &= di - c \end{aligned}$$

so

$$\begin{aligned} a &= d \\ b &= -c, \end{aligned}$$

and by back-substituting into the determinant, we get

$$a^2 + c^2 = 1,$$

so the stabilizer of i is all matrices of the form

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Thus, by orbit-stabilizer, $\mathbb{H} \cong SL_2(\mathbb{R})/P$, where P is the group of rotation matrices and the action is left-multiplication. In particular, since every rotation matrix corresponds one-to-one with an element of $S^1 \subseteq \mathbb{C}$, given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto e^{i\theta},$$

we find that $\mathbb{H} \cong SL_2(\mathbb{R})/S^1$, or that $\mathbb{H} \times S^1 \cong SL_2(\mathbb{R})$.