Chapter 36 Problems

Problem 2

$$\begin{aligned}\mathcal{F}\left[e^{-ax^2}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-ikx}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2-ikx}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left(e^{(ik)^2/4a}\sqrt{\frac{\pi}{a}}\right) \\ &= \frac{1}{\sqrt{2a}}e^{-k^2/4a}.\end{aligned}$$

Problem 3

$$\begin{aligned}\mathcal{F}[f] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{i}{k}\left(e^{-ikx}\Big|_{-a}^{a}\right)\right) \\ &= -\sqrt{\frac{2}{\pi}}\frac{\sin(ka)}{k}.\end{aligned}$$

As we take a to infinity, the range of x values in $f(x)$ increases to infinity while the range of k values approaches 0.

Problem 4

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-a|x|}\cos(bx)\,dx &= \left\langle e^{-a|x|}, \cos(bx)\right\rangle \\ &= \left\langle \mathcal{F}\left[e^{-a|x|}\right], \mathcal{F}[\cos(bx)]\right\rangle \\ &= \int_{-\infty}^{\infty}\frac{a}{k^2+a^2}\left(\delta(k+b)+\delta(k-b)\right)\,dk \\ &= \frac{2a}{a^2+b^2}.\end{aligned}$$

Problem 5

$$\begin{aligned}\int_{-\infty}^{\infty}\overline{f(x)}g(x)\,dx &= \langle f, g\rangle && \text{Inner Product.} \\ &= \langle \mathcal{F}[f], \mathcal{F}[g]\rangle && \text{Plancherel's Identity} \\ &= \int_{-\infty}^{\infty}\overline{\widehat{f}(k)}\widehat{g}(k)\,dk.\end{aligned}$$