Contents

Introduction

Finally, the last part of my notes on C*-algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of C*-algebras, discussing ideas such as amenability and nuclearity in C*-algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of $C^*_\lambda(G)$ and $C^*(G)$.

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's *C*-Algebras and Finite-Dimensional Approximations*.

Review: Representations, the Reduced Group C*-Algebra, and the Universal Group C*-Algebra

Left-Regular Representation

Let Γ be a group. Consider the space $\ell_2(\Gamma)$. For every $s \in \Gamma$, we define the operator

$$\lambda_s(\xi)(t) = \xi\left(s^{-1}t\right).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned}
\|\lambda_s(\xi)\|^2 &= \sum_{t\in\Gamma} |\lambda_s(\xi)(t)|^2\\
&= \sum_{t\in\Gamma} \left|\xi\left(s^{-1}t\right)\right|^2\\
&= \sum_{r\in\Gamma} |\xi(r)|^2\\
&= \|\xi\|^2.
\end{aligned}$$

Additionally, each λ_s admits an inverse, $\lambda_{s^{-1}} = \lambda_s^*$. Applying to the orthonormal basis $\{\delta_t\}_{t\in\Gamma}$, we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus, $\lambda_s \circ \lambda_r = \lambda_{sr}$, and we have the unitary representation of Γ, $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, where $\lambda(s) = \lambda_s$, for $s \in \Gamma$. This is the left-regular representation of Γ.

Note that the left regular representation is a faithful representation, hence injective.

Because the λ operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}\lambda_f(\xi)(t) &= \left(\sum_{s\in\Gamma} f(t)\lambda_s(\xi)\right)(t)\\ &= \sum_{s\in\Gamma} f(s)\xi\left(s^{-1}t\right)\end{aligned}$$

Note that the space of finitely supported functions on Γ, $\mathbb{C}[\Gamma]$,[1] is a $*$-algebra, where multiplication is given by convolution:

$$\begin{aligned}f * g(t) &= \sum_{s\in\Gamma} f(s)g\left(s^{-1}t\right)\\ &= \sum_{r\in\Gamma} f\left(tr^{-1}\right)g(r).\end{aligned}$$

Note that we are using $*$ both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on $\mathbb{C}[\Gamma]$ is given by

$$f^*(t) = \overline{f\left(t^{-1}\right)}.$$

A Bit on Representations and C^*-(Semi)norms

A C*-seminorm on a $*$-algebra is a seminorm such that defined by

- $\|ab\| \leqslant \|a\|\|b\|$;
- $\|a^*\| = \|a\|$;
- $\|a^*a\| = \|a\|^2$.

If A_0 is a $*$-algebra, then a representation of A_0 is a pair $(\pi_0, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi\colon A_0 \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism.

Additionally, if A_0 is a $*$-algebra with representation π_0, then we have C*-seminorm

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{op}.$$

If π_0 is injective, then $\|\cdot\|_{\pi_0}$ is a C*-norm. If π_0 is a C*-norm, then the completion of A_0 with respect to $\|\cdot\|_{\pi_0}$ is a C*-algebra.

The universal norm on A_0 is defined as

$$\|a\|_u = \sup_{p\in\mathcal{P}} p(a),$$

where $\mathcal{P}$ is the collection of all C*-seminorms on A_0. If $\|a\|_u < \infty$ for all $a \in A_0$, then $\|\cdot\|_u$ is a C*-seminorm on A_0. Note that if one of $p \in \mathcal{P}$ is a norm, then $\|\cdot\|_u$ defines a C*-norm on A_0.

[1] Also known as the free vector space over $\mathbb{C}$ with basis Γ.

If we have the unitary representation $u\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$, then

$$\pi_u(a) = \sum_{s\in\Gamma} u_s$$

is a representation of $\mathbb{C}[\Gamma]$. If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ is the left-regular representation, then the left-regular group C^*-algebra is the group $*$-algebra with C^*-norm defined by $\|a\| = \|\pi_\lambda(a)\|$.

The universal group C^*-algebra is defined as the norm completion of

$$\|a\|_{\max} = \sup\left\{\|\pi(a)\|_{op} \;\middle|\; \pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H}_\pi) \text{ is a representation}\right\}.$$

Note that

$$\begin{aligned}
\|\pi(a)\| &= \left\|\pi\left(\sum_{s\in\Gamma} a_s\delta_s\right)\right\| \\
&= \left\|\sum_{s\in\Gamma} a_s\pi(\delta_s)\right\| \\
&\leqslant \sum_{s\in\Gamma}\|a_s\pi(\delta_s)\| \\
&= \sum_{s\in\Gamma}|a_s|.
\end{aligned}$$

Note that since $\|\cdot\|_\lambda$ is a norm, we must have $a = 0$ if and only if $\|a\|_{\max} = 0$. The full group C^*-algebra admits a universal property.

Proposition: Let Γ be a discrete group. If $u\colon \Gamma \to \mathbb{B}(\mathcal{H})$, then there is a contractive $*$-homomorphism $\pi_u\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ that satisfies $\pi_u(\delta_s) = u(s)$.

Using the Left-Regular Representation to Establish Amenability

If $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\mathcal{H}$, then a vector $\xi \in \mathcal{H}$ is called invariant for π if $\pi(g)(\xi) = \xi$ for all $g \in \Gamma$.

Proposition: The left-regular representation for Γ admits an invariant vector if and only if Γ is finite.

Proof. Let Γ be finite. Since Γ is finite, all functions $a\colon \Gamma \to \mathbb{C}$ are square-summable. Thus, $\xi = \mathbb{1}_\Gamma$ is square-summable, and since $s\Gamma = \Gamma$ for all $s \in \Gamma$, we have $\mathbb{1}_\Gamma$ is invariant for λ.

Now, let $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ be the left-regular representation, and suppose there is $\xi \in \ell_2(\Gamma)$ such that for all $s \in \Gamma$, we have

$$\lambda_s(\xi) = \xi.$$

In particular, this means that for any $t \in \Gamma$, we have

$$\begin{aligned}
\lambda_s(\xi)(t) &= \xi\left(s^{-1}t\right) \\
&= \xi(t).
\end{aligned}$$

Since this holds for all $s \in \Gamma$, we have that $\xi = c\mathbb{1}_\Gamma$ for some $c \in \mathbb{C}$. However, since $\xi \in \ell_2(\Gamma)$, we must have that $\sum_{t\in\Gamma}|c|^2 < \infty$, which only holds if Γ is finite. □

An almost-invariant vector for a representation $\pi\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, as the name suggests,[II] a sequence (or net) of unit vectors $(\xi_i)_{i\in I}$ such that

$$\lim_{i\in I}\|\pi(g)(\xi_i) - \xi_i\| = 0.$$

Theorem: A group Γ is amenable if and only if the left-regular representation has an almost-invariant vector.

Proof. Let Γ be amenable, and let F_i be a Følner sequence, where $\frac{|sF_i \triangle F_i|}{|F_i|} \to 0$ for all $s \in \Gamma$.

Define $\xi_i = \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}$. Then,

$$\begin{aligned}
\|\lambda_s(\xi_i) - \xi_i\|^2 &= \sum_{t\in\Gamma}|\lambda_s(\xi_i)(t) - \xi_i(t)|^2\\
&= \sum_{t\in\Gamma}\left|\lambda_s\left(\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right)(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right|^2\\
&= \sum_{t\in\Gamma}\left|\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t)\right|^2\\
&= \frac{|sF_i \triangle F_i|}{|F_i|}.
\end{aligned}$$

Thus, λ has an almost-invariant vector.

Suppose there exists an almost-invariant vector $(\xi_i)_i \in \ell_2(\Gamma)$. It is sufficient to construct an approximate mean. Since $\xi_i \in \ell_2(\Gamma)$, we have that $\xi_i^2 \in \ell_1(\Gamma)$. Setting $\mu_i = \xi_i^2$, we plug this into the expression for an approximate mean, and obtain

$$\begin{aligned}
\|\lambda_s(u_i) - u_i\|_{\ell_1} &= \sum_{t\in\Gamma}\left|\lambda_s\left(\xi_i^2\right)(t) - \xi_i^2(t)\right|\\
&= \sum_{t\in\Gamma}|(\lambda_s(\xi_i)(t) - \xi_i(t))(\lambda_s(\xi_i)(t) + \xi_i(t))|\\
&= \|(\lambda_s(\xi_i) - \xi_i)(\lambda_s(\xi_i) + \xi_i)\|_{\ell_1}\\
&\leqslant \|\lambda_s(\xi_i) - \xi_i\|_{\ell_2}\|\lambda_s(\xi_i) + \xi_i\|\\
&\leqslant 2\|\lambda_s(\xi_i) - \xi_i\|\\
&\to 0.
\end{aligned}$$

Thus, μ_i is an approximate mean. □

Using the criterion of almost invariant vectors, we may show that a group is amenable if and only if the trivial representation — defined by $1_\Gamma\colon \Gamma \to \mathbb{C}$, $1_\Gamma(g) = 1$ is what is known as weakly contained in the left-regular representation.

A representation $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is weakly contained in another representation $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$, denoted $\pi \prec \rho$, if for every $\xi \in \mathcal{H}$, finite $E \subseteq \Gamma$, and $\varepsilon > 0$, then there are $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that

$$\left|\langle \pi(g)(\xi), \xi\rangle - \sum_{i=1}^{n}\langle \rho(g)(\eta_i), \eta_i\rangle\right| < \varepsilon.$$

Theorem: A discrete group Γ is amenable if and only if $1_\Gamma \prec \lambda$, where λ is the left-regular representation.

[II] I'm only mostly being facetious here.

Proof. We show that $1_\Gamma \prec \lambda$ is equivalent to the existence of an almost invariant vector for λ. We assume λ admits an almost-invariant vector. It is sufficient to show that for every $\varepsilon > 0$ and every finite set $E \subseteq \Gamma$, there are $\eta_1, \dots, \eta_n \in \ell_2(\Gamma)$ such that

$$\left|1 - \sum_{i=1}^{n} \langle \lambda_t(\eta_i), \eta_i \rangle\right| < \varepsilon$$

for every $t \in E$. If we take $n = 1$ and $\eta_1 = \xi$, where ξ is almost-invariant for all $g \in E$ — i.e., $\left\|\lambda_g(\xi) - \xi\right\|_{\ell_2} < \varepsilon$ for all $g \in E$. Note that we have

$$\begin{aligned}\left\|\lambda_g(\xi) - \xi\right\|^2 &= \left\langle \lambda_g(\xi) - \xi, \lambda_g(\xi) - \xi \right\rangle \\ &= \left\langle \lambda_g(\xi), \lambda_g(\xi) \right\rangle + \langle \xi, \xi \rangle - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi \right\rangle\right) \\ &= 2 - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi \right\rangle\right) \\ &= 2\operatorname{Re}\left(1 - \left\langle \lambda_g(\xi), \xi \right\rangle\right) \\ &\leqslant 2\left|1 - \left\langle \lambda_g(\xi), \xi \right\rangle\right|.\end{aligned}$$

Additionally,

$$\begin{aligned}\left|1 - \left\langle \lambda_g(\xi), \xi \right\rangle\right|^2 &= \left(1 - \left\langle \lambda_g(\xi), \xi \right\rangle\right)\left(1 - \overline{\left\langle \lambda_g(\xi), \xi \right\rangle}\right) \\ &= 1 - \overline{\left\langle \lambda_g(\xi), \xi \right\rangle} - \left\langle \lambda_g(\xi), \xi \right\rangle + \left|\left\langle \lambda_g(\xi), \xi \right\rangle\right|^2 \\ &\leqslant 2 - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi \right\rangle\right) \\ &= \left\|\lambda_g(\xi) - \xi\right\|^2.\end{aligned}$$

Thus, we have that

$$\begin{aligned}\left|1 - \left\langle \lambda_g(\xi), \xi \right\rangle\right| &\leqslant \left\|\lambda_g(\xi) - \xi\right\| \\ &< \varepsilon.\end{aligned}$$

We start by showing that $1_\Gamma \prec \lambda$ if and only if for every finite $S \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\|\lambda_s(\xi) - \xi\|_{\ell_2} < \varepsilon.$$

In the forward direction, we see that there exists a unit vector ξ such that $|1 - \langle \lambda_s(\xi), \xi \rangle| < \varepsilon^2/2$, meaning $\|\lambda_s(\xi) - \xi\| < \varepsilon$ by above. Similarly, if $\|\lambda_s(\xi) - \xi\| < \varepsilon$, then $1_\Gamma \prec \lambda$.

Now, we assume $1_\Gamma \prec \lambda$. Thus, for a finite $E \subseteq \Gamma$ and $\varepsilon > 0$, then there exists $f \in \ell_2(\Gamma)$ with $\|f\|_{\ell_2} = 1$ such that $\|\lambda_s(f) - f\| < \varepsilon$ for all $s \in E$.

Setting $g = |f|^2$, we have $g \in \ell_1(\Gamma)$. From Hölder's inequality, we have

$$\begin{aligned}\|\lambda_s(g) - g\|_{\ell_1} &\leqslant \left\|\lambda_s\left(\overline{f}\right) + \overline{f}\right\|_{\ell_2} \|\lambda_s(f) - f\| \\ &\leqslant 2\|\lambda_s(f) - f\|_{\ell_2} \\ &< 2\varepsilon.\end{aligned}$$

Thus, Γ admits an approximate mean, hence is amenable. □

Having obtained some more resources on Kesten's criterion, we can now prove that.

Definition. Let $\lambda\colon \Gamma \to \mathbb{B}(\ell_2(\Gamma))$ be the left-regular representation. Then, for a finite set $E \subseteq \Gamma$, we define the Markov operator $M(E)$ by

$$M(E) = \sum_{t \in E} \lambda_t.$$

Note that since λ_t is an isometry for each t, we have

$$\begin{aligned}\|M(E)\|_{op} &= \left\|\frac{1}{|E|}\sum_{t\in E}\lambda_t\right\|_{op}\\ &= \frac{1}{|E|}\left\|\sum_{t\in E}\lambda_t\right\|_{op}\\ &\leqslant \frac{1}{|E|}\sum_{t\in E}\|\lambda_t\|_{op}\\ &= 1,\end{aligned}$$

so the Markov operator is a bounded operator (indeed, a contraction).

Theorem (Kesten's Criterion)**:** Let Γ contain a finite symmetric generating set S. Then, Γ is amenable if and only if

$$\|M(S)\|_{op} = 1.$$

Proof. Let Γ be amenable. Then, λ admits an almost-invariant vector, $(\xi_n)_n \subseteq S_{\ell_2(\Gamma)}$, such that

$$\|\lambda_s(\xi_n) - \xi_n\|_{\ell_2} \to 0$$

for all $s \in \Gamma$. In particular, we have

$$\begin{aligned}\left|\left(\left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n)\right\|_{\ell_2}\right) - \|\xi_n\|_{\ell_2}\right| &\leqslant \left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n) - \xi_n\right\|_{\ell_2}\\ &= \frac{1}{|S|}\left\|\left(\sum_{t\in S}\lambda_t\right)(\xi_n) - |S|\xi_n\right\|_{\ell_2}\\ &\leqslant \frac{1}{|S|}\sum_{t\in S}\|\lambda_t(\xi_n) - \xi_n\|_{\ell_2}\\ &\to 0,\end{aligned}$$

meaning that

$$\sup_{\xi\in S_{\ell_2(\Gamma)}}\left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi)\right\| = \|\xi\|,$$

and so the norm of the Markov operator is 1.

Suppose

$$\left\|\frac{1}{|S|}\sum_{t\in S}\lambda_t\right\|_{op} = 1,$$

or

$$\left\|\sum_{t\in S}\lambda_t\right\|_{op} = |S|.$$

Proposition: If $T \in \mathbb{B}(\mathcal{H})$ is a self-adjoint operator, then

$$\|T\|_{op} = \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|.$$

Proof. We have that

$$\begin{aligned}
|\langle T(x), x\rangle| &\leqslant \|T(x)\|\|x\| \\
&\leqslant \|T\|_{op}\|x\|^2 \\
&= \|T\|_{op}.
\end{aligned}$$

Now, we seek to establish the opposite direction. Note that since T is self-adjoint, we know that $\langle T(x), x\rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$, so by the polarization identity, we have that

$$\langle T(x), y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle).$$

Note that we know that

$$\|T\|_{op} = \sup_{x,y\in S_{\mathcal{H}}} |\langle T(x), y\rangle|.$$

Now, we set $\alpha = \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|$. Note that for any nonzero $x \in \mathcal{H}$, we have

$$\begin{aligned}
\left|\left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle\right| &\leqslant \alpha \\
|\langle T(x), x\rangle| &\leqslant \alpha\|x\|^2.
\end{aligned}$$

Now, for any $x, y \in \mathcal{H}$, we may assume that $\langle T(x), y\rangle \in \mathbb{R}$, as we may multiply $\langle T(x), y\rangle$ by $\operatorname{sgn}(\langle T(x), y\rangle)$. Thus, by the polarization identity and the fact that T is self-adjoint, we have

$$\begin{aligned}
\langle T(x), y\rangle &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle) \\
|\langle T(x), y\rangle| &= \left|\frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)\right| \\
&\leqslant \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|) \\
&\leqslant \frac{\alpha}{4}\left(\|x+y\|^2 + \|x-y\|^2\right) \\
&= \frac{\alpha}{4}\left(2\|x\|^2 + 2\|y\|^2\right) \\
&= \alpha.
\end{aligned}$$

Thus, we have $\|T\|_{op} \leqslant \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|$. □

Now, since S is symmetric, we have that $M(S)$ is self-adjoint. Therefore, we know that there is some $\xi_n \in S_{\mathcal{H}}$ such that

$$\begin{aligned}
1 - \frac{1}{n} &< \left\langle \left(\frac{1}{|S|}\sum_{t\in S} \lambda_t\right)(\xi_n), \xi_n \right\rangle \\
&\leqslant \left\langle \left(\frac{1}{|S|}\sum_{t\in S} \lambda_t\right)(|\xi_n|), |\xi_n| \right\rangle.
\end{aligned}$$

Thus, rearranging, we have

$$1 - \left\langle \left(\frac{1}{|S|}\sum_{t\in S} \lambda_t\right)(|\xi_n|), |\xi_n| \right\rangle < \frac{1}{n}.$$

Since $M(S)$ is a self-adjoint operator, we have that $\operatorname{Re}\left(\left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n),\xi_n\right\rangle\right) = \left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n),\xi_n\right\rangle$. This gives

$$\begin{aligned}\left\|\left(\frac{1}{S}\sum_{t\in S}\lambda_t\right)(\xi)-\xi\right\| &\leqslant \frac{1}{|S|}\sum_{t\in S}\|\lambda_t(\xi)-\xi\|\\ &\leqslant \frac{1}{|S|}\sum_{t\in S}\sqrt{2}|1-\langle\lambda_t(\xi),\xi\rangle|\\ &= \sqrt{2}\left|1-\frac{1}{|S|}\sum_{t\in S}\langle\lambda_t(\xi),\xi\rangle\right|\\ &\to 0.\end{aligned}$$

Thus, λ admits an almost-invariant vector. □

Next, we turn to Hulanicki's Criterion.

Definition. Let $f \in \ell_1(\Gamma)$. Then, we define the bounded operator

$$\lambda_{f(t)} = \sum_{t\in\Gamma} f(t)\lambda_t.$$

Theorem: If Γ is a discrete group, then Γ is amenable if and only if for every positive finitely-supported $f\colon \Gamma \to \mathbb{C}$, we have

$$\sum f(t) \leqslant \left\|\lambda_{f(t)}\right\|_{op}.$$

Proof. Suppose Γ is amenable. Let $f \geqslant 0$ be a finitely supported function, and let $(F_n)_n$ be a Følner sequence such that for every $g \in \operatorname{supp}(f)$, we have

$$\frac{|gF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}.$$

Let $\xi_n = \frac{1}{\sqrt{|F_n|}}\mathbb{1}_{F_n}$. Note that $\|\xi_n\|_{\ell_2} = 1$.

We will use the fact that

$$\sup_{x\in S_{\mathcal{H}}} |\langle T(x),x\rangle| \leqslant \|T\|_{op}.$$

We see that

$$\begin{aligned}\left|\left\langle\left(\sum_{t\in\Gamma} f(t)\lambda_t\right)(\xi_n),\xi_n\right\rangle\right| &= \left|\sum_{t\in\Gamma} f(t)\langle\lambda_t(\xi_n),\xi_n\rangle\right|\\ &= \left|\sum_{t,s\in\Gamma} f(t)\xi_n\left(t^{-1}s\right)\xi_n(s)\right|\\ &\leqslant \left\|\lambda_{f(t)}\right\|,\end{aligned}$$

meaning

$$\lim_{n\to\infty}\left|\left\langle\left(\sum_{t\in\Gamma} f(t)\lambda_t\right)(\xi_n),\xi_n\right\rangle\right| \leqslant \left\|\lambda_{f(t)}\right\|.$$

Notice that ξ_n is an almost-invariant vector for λ, meaning that $\xi_n\left(t^{-1}s\right) \to \xi_n(s)$. Therefore, this means

$$\begin{aligned}\lim_{n\to\infty}\left|\sum_{t,s\in\Gamma} f(t)\xi_n\left(t^{-1}s\right)\xi_n(s)\right| &= \lim_{n\to\infty}\left|\sum_{t,s\in\Gamma} f(t)|\xi_n(s)|^2\right|\\ &= \sum_{t\in\Gamma} f(t)\left|\sum_{s\in\Gamma}|\xi_n(s)|^2\right|\\ &= \sum_{t\in\Gamma} f(t)\\ &\leqslant \left\|\lambda_{f(t)}\right\|_{op}.\end{aligned}$$

This establishes that there is some $C > 0$ such that

$$\sum_{t\in\Gamma} f(t) \leqslant C\left\|\lambda_{f(t)}\right\|_{op}.$$

To show that $C = 1$, we note that, by the definition of convolution, we must have

$$\left(\sum_{t\in\Gamma} f(t)\right)^n = \sum_{t\in\Gamma}(f*\cdots*f)(t),$$

and

$$\begin{aligned}\left(\lambda_{f(t)}\right)^n &= \left(\sum_{t\in\Gamma} f(t)\lambda)_t\right)^n\\ &= \sum_{t\in\Gamma}(f*\cdots*f)(t)\lambda_t\\ &= \lambda_{(f*\cdots*f)(t)}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left(\sum_{t\in\Gamma} f(t)\right)^n &= \sum_{t\in\Gamma}(f*\cdots*f)(t)\\ &\leqslant C\left\|\lambda_{(f*\cdots*f)(t)}\right\|\\ &= C\left(\left\|\lambda_{f(t)}\right\|_{op}\right)^n.\end{aligned}$$

This means we have

$$\sum_{t\in\Gamma} f(t) \leqslant C^{1/n}\left\|\lambda_{f(t)}\right\|_{op}.$$

Since n is arbitrary, this means $C = 1$.

Now, if for all finitely supported f, we have

$$\sum_{t\in\Gamma} f(t) \leqslant \left\|\lambda_{f(t)}\right\|_{op}.$$

If $f = \mathbb{1}_E$ for some finite $E \subseteq \Gamma$, we see that

$$\left\|\sum_{t\in E}\lambda_t\right\|_{op} = |E|,$$

so by Kesten's criterion, we have that Γ is amenable. $\square$

Completely [Property] Maps

We begin this section with an overview of positive maps, completely positive maps, and extensions. These will be useful for understanding the theorem that a group is amenable if and only if the reduced group C*-algebra is nuclear. The ultimate goal here is to prove Arveson's extension theorem (i.e., that $\mathbb{B}(\mathcal{H})$ is injective with respect to completely positive maps). The primary text for this purpose will be Vern Paulsen's *Completely Bounded Maps and Operator Algebras*.

The idea behind completely positive maps is that they are positive when subjected to a certain amplification process related to the matrix algebras.

Definition. An element of a C*-algebra is positive if and only if it is self-adjoint and its spectrum is contained in the nonnegative reals. Alternatively, $b \in A$ is of the form $b = a^*a$ for some $a \in A$.

To introduce a norm such that $\operatorname{Mat}_n(A)$ becomes a C*-algebra, we begin with the most basic C*-algebra, $\mathbb{B}(\mathcal{H})$, and consider the n-fold amplification of $\mathcal{H}$, $\mathcal{H}^{(n)}$. This is a Hilbert space equipped with inner product

$$\left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} \langle h_j, k_j \rangle.$$

Meanwhile, we may consider $\operatorname{Mat}_n(\mathbb{B}(\mathcal{H}))$ as a linear map on $\mathcal{H}^{(n)}$, by taking

$$(T_{ij})_{ij} = \begin{pmatrix} \sum_{j=1}^{n} T_{1j}(h_j) \\ \vdots \\ \sum_{j=1}^{n} T_{nj}(h_j) \end{pmatrix}.$$

This yields a $*$-isomorphism between $\operatorname{Mat}_n(\mathbb{B}(\mathcal{H}))$ and $\mathbb{B}\left(\mathcal{H}^{(n)}\right)$.

Given any C*-algebra A, we may theorize $\operatorname{Mat}_n(A)$ by first isometrically representing $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, letting A be a C*-subalgebra of $\mathbb{B}(\mathcal{H})$, and then identifying $\operatorname{Mat}_n(A)$ as a $*$-subalgebra of $\operatorname{Mat}_n(\mathbb{B}(\mathcal{H}))$.

Using a faithful $*$-representation of A, we now have a way to turn $\operatorname{Mat}_n(A)$ into a C*-algebra. However, since the norm is unique on a C*-algebra, the norm on $\operatorname{Mat}_n(A)$ defined in this fashion is independent of the representation of A that we choose. Furthermore, since $*$-isomorphisms are positive maps, the positive elements of $\operatorname{Mat}_n(A)$ are uniquely determined. This means that every C*-algebra carries with it a set of canonically defined norms and orders on each $\operatorname{Mat}_n(A)$.

For example, consider $\operatorname{Mat}_k(\mathbb{C})$, which can be identified with $\mathcal{L}\left(\mathcal{C}^k\right)$. We identify $\operatorname{Mat}_n(\operatorname{Mat}_k(\mathbb{C})) \cong \operatorname{Mat}_{nk}(\mathbb{C})$. With this identification, the usual multiplication and involution on $\operatorname{Mat}_n(\operatorname{Mat}_k(\mathbb{C}))$ become multiplication and involution on $\operatorname{Mat}_{nk}(\mathbb{C})$.

Now, let X be a compact Hausdorff space, and let $C(X)$ be the C*-algebra of continuous functions with $f^*(x) = \overline{f(x)}$, equipped with the norm $\|f\| = \sup_{x \in X} |f(x)|$. Then, an element $F = (f_{ij})_{ij}$ of $\operatorname{Mat}_n(C(X))$ can be considered as a continuous $\operatorname{Mat}_n(\mathbb{C})$-valued function. Addition, multiplication, and involution in $\operatorname{Mat}_n(C(X))$ are pointwise. Recalling that the norm on $\operatorname{Mat}_n(C(X))$ is unique, we may let $\|F\| = \sup_{x \in X} \|F(x)\|$, where the latter norm is the canonical matrix norm on $\operatorname{Mat}_n(C(X))$. The positive elements of $\operatorname{Mat}_n(C(X))$ are those F for which $F(x)$ is a positive matrix for all x.

Now, given two C*-algebras A and B and a map $\phi\colon A \to B$, there are maps $\phi_n\colon \operatorname{Mat}_n(A) \to \operatorname{Mat}_n(B)$, given by

$$\phi_n\left((a_{ij})_{ij}\right) = (\phi(a_{ij}))_{ij}.$$

In general, when we say that ϕ is completely [property], then we say that all the ϕ_n have that property. For instance, if ϕ is positive, in that it maps positive elements of A to positive elements of B, then we say ϕ is completely positive if ϕ_n is a positive map for each n, where the positive elements of $\operatorname{Mat}_n(A)$ and $\operatorname{Mat}_n(B)$ are defined canonically.

Unfortunately, it's not always the case that (e.g.) positive maps are completely positive, or even that $\|\phi_n\|_{op} = \|\phi\|_{op}$ for each n.

There is an isomorphism between $\operatorname{Mat}_n(A)$ and the tensor product $\operatorname{Mat}_n(\mathbb{C}) \otimes A$. We detail it here. The proof is from Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*.

Theorem: Let A be an algebra, and let $\operatorname{Mat}_n(A)$ denote the matrix algebra of A. Then, there is a $*$-isomorphism

$$\operatorname{Mat}_n(A) \cong \operatorname{Mat}_n(\mathbb{C}) \otimes A.$$

Proof. Define $\varphi\colon \operatorname{Mat}_n(A) \to \operatorname{Mat}_n(\mathbb{C}) \otimes A$ by

$$\varphi\left(\left(a_{ij}\right)_{ij}\right) = \sum_{i,j=1}^{n} e_{ij} \otimes x_{ij}.$$

Recall that if A and B are two algebras, multiplication in $A \otimes B$ is defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

and if A and B are $*$-algebras, then the involution is defined by

$$(a \otimes b)^* = a^* \otimes b^*.$$

We start by showing that $\operatorname{Mat}_n(A) \cong \operatorname{Mat}_n(\mathbb{C}) \otimes A$ as vector spaces. By the definition of the tensor product, the map φ is linear.

Now, suppose

$$\begin{aligned}\varphi\left(\left(a_{ij}\right)_{ij}\right) &= \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij}\\ &= 0.\end{aligned}$$

Then, since $\left\{e_{ij}\right\}_{ij}$ is linearly independent, we know that $x_{ij} = 0$ for all i, j, so $\left(a_{ij}\right)_{ij} = 0$, so φ is injective.

Now, let $t \in \operatorname{Mat}_n(\mathbb{C}) \otimes A$ be given by

$$t = \sum_{k} m_k \otimes a_k,$$

where $m_k \in \operatorname{Mat}_n(\mathbb{C})$ and $a_k \in A$. Then, using the matrix units, we write each m_k as

$$m_k = \sum_{i,j=1}^{n} m_k(i,j) e_{ij}.$$

This gives

$$\begin{aligned}t &= \sum_{k}\left(\sum_{i,j=1}^{n} m_k(i,j) e_{ij}\right) \otimes a_k\\ &= \sum_{i,j=1}^{n} e_{ij} \otimes \left(\sum_{k} m_k(i,j) a_k\right).\end{aligned}$$

Defining $a_{ij} \coloneqq \sum_{k} m_k(i,j)a_k$, we get

$$t = \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij},$$

meaning that

$$\varphi\left(\left(x_{ij}\right)_{ij}\right) = t.$$

Thus, φ is surjective.

We will show now that φ is multiplicative and $*$-preserving. If $\left(a_{ij}\right)_{ij}$ and $\left(b_{ij}\right)_{ij}$ belong to $\operatorname{Mat}_n(A)$.

$$\begin{aligned}
\varphi((a_{ik})_{ik})\varphi\left(\left(b_{lj}\right)_{lj}\right) &= \left(\sum_{i,k=1}^{n} e_{ik} \otimes a_{ik}\right)\left(\sum_{l,j=1}^{n} e_{lj} \otimes b_{lj}\right)\\
&= \sum_{i,j,k,l=1}^{n} (e_{ik} \otimes a_{ik})\left(e_{lj} \otimes b_{lj}\right)\\
&= \sum_{i,j,k,l=1}^{n} e_{ik}e_{lj} \otimes a_{ik}b_{lj}\\
&= \sum_{i,j,k=1}^{n} e_{ik}e_{kj} \otimes a_{ik}b_{kj}\\
&= \sum_{ij,k=1}^{n} e_{ij} \otimes a_{ik}b_{kj}\\
&= \sum_{i,j=1}^{n} e_{ij} \otimes \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)\\
&= \varphi\left(\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{ij}\right)\\
&= \varphi\left(\left(a_{ij}\right)_{ij}\left(b_{ij}\right)_{ij}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi\left(\left(a_{ij}\right)_{ij}\right)^* &= \left(\sum_{i=1}^{n} e_{ij} \otimes a_{ij}\right)^*\\
&= \sum_{i,j=1}^{n} \left(e_{ij} \otimes a_{ij}\right)^*\\
&= \sum_{i,j=1}^{n} e_{ij}^* \otimes a_{ij}^*\\
&= \sum_{i,j=1}^{n} e_{ji} \otimes a_{ij}^*\\
&= \sum_{i,j=1}^{n} e_{ij} \otimes a_{ji}^*
\end{aligned}$$

$$\begin{aligned} &= \varphi\left(\left(a_{ji}^{*}\right)_{ij}\right)\\ &= \varphi\left(\left(a_{ij}\right)_{ij}^{*}\right). \end{aligned}$$

□

There are lots of useful results using amplification to the matrix algebras.

Example (Dilating an Isometry)**.** Let V be an isometry, and let $P = I_{\mathcal{H}} - VV^{*}$ be the projection onto $\operatorname{Ran}(V)^{\perp}$. Define U on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ by

$$U = \begin{pmatrix} V & P \\ 0 & V^{*} \end{pmatrix}.$$

We find that

$$\begin{aligned} U^{*} &= \begin{pmatrix} V^{*} & 0 \\ P & V \end{pmatrix}\\ UU^{*} &= \begin{pmatrix} V & P \\ 0 & V^{*} \end{pmatrix}\begin{pmatrix} V^{*} & 0 \\ P & V \end{pmatrix}\\ &= \begin{pmatrix} VV^{*} + P & PV \\ V^{*}P & V^{*}V \end{pmatrix}\\ &= \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{H}} \end{pmatrix}\\ &= I_{\mathcal{K}}\\ U^{*}U &= \begin{pmatrix} V^{*} & 0 \\ P & V \end{pmatrix}\begin{pmatrix} V & P \\ 0 & V^{*} \end{pmatrix}\\ &= I_{\mathcal{K}}. \end{aligned}$$

Thus, U is a unitary on $\mathcal{K}$. We may identify $\mathcal{H} \cong \mathcal{H} \oplus 0$, and take

$$V^{n} = P_{\mathcal{H}} U^{n}|_{\mathcal{H}}$$

for all $n \geqslant 0$. Thus, we are able to realize any isometry V as the restriction of some unitary to a subspace that respects powers.

Example (Dilating a Contraction)**.** Similarly, we may define the isometric dilation of a contraction. Let T be an operator on $\mathcal{H}$ with $\|T\| \leqslant 1$, and define $D_T = (I - T^{*}T)^{1/2}$. We see that

$$\begin{aligned} \|T(h)\|^2 + \|D_T(h)\|^2 &= \langle T^{*}T(h), h\rangle + \left\langle D_T^2(h), h\right\rangle\\ &= \|h\|^2. \end{aligned}$$

We consider now the sequence space

$$\ell_2(\mathcal{H}) = \left\{ (h_n)_{n\in\mathbb{N}} \;\middle|\; h_n \in \mathcal{H}, \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \right\}.$$

We have the norm

$$\|(h_n)_n\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2$$

and the inner product

$$\langle (h_n)_n, (k_n)_n \rangle = \sum_{n=1}^{\infty} \langle h_n, k_n \rangle.$$

We define the operator $V\colon \ell_2(\mathcal{H}) \to \ell_2(\mathcal{H})$ by

$$V((h_n)_n) = (T(h_1), D_T(h_1), h_2, \dots).$$

It then follows that V is an isometry on $\ell_2(\mathcal{H})$, and that if we identify $\mathcal{H} \cong \mathcal{H} \oplus 0 \oplus \cdots$, then $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$.

Theorem (Sz.-Nagy's Dilation Theorem)**:** Let T be a contraction operator on $\mathcal{H}$. There is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and a unitary operator U on $\mathcal{K}$ such that $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$.

Proof. Take $\mathcal{K} = \ell_2(\mathcal{H}) \oplus \ell_2(\mathcal{H})$, and identify $\mathcal{H}$ as $(\mathcal{H} \oplus 0 \oplus \cdots) \oplus 0$. Let V be the isometric dilation of T on $\ell_2(\mathcal{H})$, and let U be the unitary dilation of V on $\ell_2(\mathcal{H}) \oplus \ell_2(\mathcal{H})$. Then, since $\mathcal{H} \subseteq \ell_2(\mathcal{H}) \oplus 0$, we have that $P_{\mathcal{H}} U^n|_{\mathcal{H}} = P_{\mathcal{H}} V^n|_{\mathcal{H}} = T^n$ for all $n \geqslant 0$. □

Whenever Y is an operator on $\mathcal{K}$, $\mathcal{H}$ a (closed) subspace of $\mathcal{K}$, and $X = P_{\mathcal{H}} Y|_{\mathcal{H}}$, then we say X is a compression of Y.

Corollary (Von Neumann's Inequality)**:** Let T be a contraction on a Hilbert space. Then, for any polynomial p,

$$\|p(T)\| \leqslant \sup_{|z| \leqslant 1} |p(z)|.$$

Proof. Let U be a unitary dilation of T. Since $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$, linearity means we have $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$. Since U is defined on a larger space than T, then $\|p(T)\| \leqslant \|p(U)\|$. Furthermore, since unitaries are normal, we have

$$\|p(U)\| = \sup_{\lambda \in \sigma(U)} |p(\lambda)|,$$

where $\sigma(U)$ is the spectrum of U. Since U is unitary, $\sigma(U) \subseteq \mathbb{T}$, so von Neumann's inequality follows. □

Positive and Completely Positive Maps

Positive Maps

There are certain results on positive maps that are useful in the study of completely positive maps. We introduce them here.

Definition. If S is a subset of a C*-algebra A, we say S is an operator system if A is unital and S is a self-adjoint sub*space* of A with $1_A \in S$.

Note that if S is an operator system and $h \in S$ is self-adjoint, then though the values h_+ and h_-, defined by the continuous functional calculus with

$$\begin{aligned} f^+(x) &= \max\{0, x\} \\ f^-(x) &= \min_{0, -x} \end{aligned}$$

may not belong to S, we can write h as the difference of two positive elements in s by

$$h = \frac{1}{2}(\|h\| 1_A + h) - \frac{1}{2}(\|h\| 1_A - h).$$

Definition. If S is an operator system, B is a C*-algebra, and $\phi\colon S \to B$ is a linear map, then ϕ is called positive if it maps positive elements of S to positive elements of B.

Theorem: If ϕ is a positive linear functional on an operator system S, then $\|\phi\| = \phi(1_A)$.

When the range of ϕ is not $\mathbb{C}$, but rather a C*-algebra, then the situation is a bit different.

Proposition: Let S be an operator system, and let B be a C*-algebra. If $\phi\colon S \to B$ is a positive map, then ϕ is bounded, with

$$\|\phi\| \leqslant 2\|\phi(1_A)\|.$$

Proof. Note that if p is positive, then $0 \leqslant p \leqslant \|p\|1_A$, so $0 \leqslant \phi(p) \leqslant \|p\|\phi(1_A)$ since positive functions are order-preserving. Thus, we get $\|\phi(p)\| \leqslant \|p\|\|\phi(1)\|$ when $p \geqslant 0$.

Note that when p_1 and p_2 are positive, then $\|p_1 - p_2\| \leqslant \max\{\|p_1\|, \|p_2\|\}$. If h is self-adjoint, then we have

$$\|\phi(h)\| = \frac{1}{2}\phi(\|h\|1_A + h) - \frac{1}{2}\phi(\|h\|1_A - h),$$

which is the difference of two positive elements in B. Thus, we have

$$\begin{aligned}\|\phi(h)\| &\leqslant \frac{1}{2}\max\{\|\phi(\|h\|1_A + h)\|, \|\phi(\|h\|1_A - h)\|\}\\ &\leqslant \|h\|\|\phi(1)\|.\end{aligned}$$

Finally, if a is arbitrary then write $a = h + ik$ via the Cartesian decomposition, where $\|h\|, \|k\| \leqslant \|a\|$, and h, k are self-adjoint. Thus, we have

$$\begin{aligned}\|\phi(a)\| &\leqslant \|\phi(h)\| + \|\phi(k)\|\\ &\leqslant 2\|a\|\|\phi(1_A)\|.\end{aligned}$$

□

As it turns out, 2 is the best constant.

Example. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and $C(\mathbb{T})$ be the continuous functions on z. Let z be the cordinate function, and let $S \subseteq C(\mathbb{T})$ be the subspace spanned by $1, z, \overline{z}$. Defining

$$\phi(a + bz + c\overline{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix},$$

An element of S is positive if and only if $c = \overline{b}$ and $a \geqslant 2|b|$, and an element of $\operatorname{Mat}_2(\mathbb{C})$ is positive if and only if its diagonal entries and determinant are nonnegative real numbers. Thus, it is the case that ϕ is a positive map, but also

$$\begin{aligned}2\|\phi(1)\| &= 2\\ &= \|\phi(z)\|\\ &\leqslant \|\phi\|,\end{aligned}$$

meaning $\|\phi\| = 2\|\phi(1)\|$.

We are interested in seeing when unital, positive maps are contractive.

Lemma: Let A be a C*-algebra, and let p_i be positive elements of A such that

$$\sum_{i=1}^{n} p_i \leqslant 1.$$

If λ_i are scalars with $|\lambda_i| \leqslant 1$, then

$$\left\|\sum_{i=1}^{n} \lambda_i p_i\right\| \leqslant 1.$$

Proof. Note that

$$\begin{pmatrix} \sum_{i=1}^{n}\lambda_i p_i & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix} = \begin{pmatrix} p_1^{1/2} & \cdots & p_n^{1/2}\\ 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{pmatrix}\operatorname{diag}(\lambda_1,\dots,\lambda_n)\begin{pmatrix} p_1^{1/2} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ p_n^{1/2} & 0 & \cdots & 0\end{pmatrix}.$$

The norm on the matrix on the left is $\left\|\sum_{i=1}^{n}\lambda_i p_i\right\|$, while the three matrices on the right have norm less than 1, using the fact that $\|a^*a\| = \|a\|^2$. □

Theorem: Let B be a C*-algebra, X a compact Hausdorff space, and $C(X)$ the continuous functions on X. Let $\phi\colon C(X) \to B$ be a positive map. Then, $\|\phi\| = \|\phi(1)\|$.

Proof. We may assume $\phi(1) \leqslant 1$. Let $f \in C(X)$ with $\|f\| \leqslant 1$, and let $\varepsilon > 0$. Now, we may choose a finite open cover $\{U_i\}_{i=1}^{n}$ of X such that $|f(x) - f(x_i)| < \varepsilon$ for all $x \in U_i$, and let $\{p_i\}_{i=1}^{n}$ be a partition of unity subordinate to the cover. That is, $\{p_i\}_{i=1}^{n}$ are nonnegative continuous functions satisfying $\sum_{i=1}^{n} p_i = 1$ and $p_i(x) = 0$ for $x \notin U_i$.

Set $\lambda_i = f(x_i)$, and note that if $p_i(x) \neq 0$ for some i, then $x \in U_i$ and $|f(x) - \lambda_i| < \varepsilon$. Hence, for any x, we have

$$\begin{aligned}\left|f(x) - \sum_{i=1}^{n}\lambda_i p_i(x)\right| &= \left|\sum_{i=1}^{n}(f(x) - \lambda_i)p_i(x)\right|\\ &\leqslant \sum_{i=1}^{n}|f(x) - \lambda_i|p_i(x)\\ &< \sum_{i=1}^{n}\varepsilon p_i(x)\\ &= \varepsilon.\end{aligned}$$

By above, we know that $\left\|\sum_{i=1}^{n}\lambda_i p_i\right\| \leqslant 1$, we have

$$\begin{aligned}\|\phi(f)\| &\leqslant \left\|\phi\left(f - \sum_{i=1}^{n}\lambda_i p_i\right)\right\| + \left\|\sum_{i=1}^{n}\phi(p_i)\right\|\\ &< 1 + \varepsilon\|\phi\|.\end{aligned}$$

Since ε was arbitrary, we have $\|\phi\| \leqslant 1$. □

Lemma (Riesz–Fejér Theorem)**:** Let $\tau\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$ be a strictly positive function on $\mathbb{T}$. Then, there is a polynomial $p(z) = \sum_{n=0}^{n} p_n z^n$ such that

$$\tau\left(e^{i\theta}\right) = \left|p\left(e^{i\theta}\right)\right|^2.$$

Proof. Note that τ is real-valued, so $a_{-n} = \overline{a_n}$, and a_0 is real. Assuming $a_{-N} \neq 0$, we take $g(z) = \sum_{n=-N}^{N} a_n z^{n+N}$, so that g is a polynomial of degree $2n$, $g(0) \neq 0$.

We have $g\left(e^{i\theta}\right) = \tau\left(e^{i\theta}\right)e^{iN\theta} \neq 0$, and that $\overline{g(1/\overline{z})} = z^{-2N}g(z)$.

We write the $2N$ zeros of g as $z_1, \dots, z_N, 1/\overline{z_1}, \dots, 1/\overline{z_N}$.

Set $q(z) = (z - z_1)\cdots(z - z_N)$ and $h(z) = (z - 1/\overline{z_1})\cdots(z - 1/\overline{z_N})$. We have that

$$g(z) = a_N q(z)h(z),$$

where

$$\overline{h(z)} = \frac{(-1)^N \overline{z}^N q(1/\overline{z})}{z_1 \cdots z_N}.$$

Thus, we have

$$\begin{aligned}\tau\left(e^{i\theta}\right) &= e^{-iN\theta} g\left(e^{i\theta}\right)\\ &= \left|g\left(e^{i\theta}\right)\right|\\ &= \left|a_N q\left(e^{i\theta}\right)\overline{h}\left(e^{i\theta}\right)\right|\\ &= \frac{a_N}{z_1 \cdots z_N}\left|q\left(e^{i\theta}\right)\right|^2.\end{aligned}$$

□

Theorem: Let T be an operator on $\mathcal{H}$ with $\|T\| \leqslant 1$, and let $S \subseteq C(\mathbb{T})$ be the operator system defined by

$$S = \left\{p\left(e^{i\theta}\right) + \overline{q(e^{i\theta})} \;\middle|\; p, q \text{ are polynomials}\right\}.$$

Then, $\phi\colon S \to \mathbb{B}(\mathcal{H})$, given by $\phi(p + \overline{q}) = p(T) + q(T)^*$ is positive.

Proof. It is enough to prove that $\phi(\tau)$ is positive for every *strictly* positive τ.

Let $\tau\left(e^{i\theta}\right)$ be strictly positive in S, meaning $\tau\left(e^{i\theta}\right) = \sum_{\ell,k=0}^{n} \alpha_\ell \overline{\alpha_k} e^{i(\ell-k)\theta}$. We must prove that

$$\phi(\tau) = \sum_{\ell,k=0}^{n} \alpha_\ell \overline{\alpha_k} T(\ell - k),$$

where

$$T(j) = \begin{cases} T^j & j \geqslant 0 \\ (T^*)^{-j} & j < 0. \end{cases}$$

Fix $x \in \mathcal{H}$. Note that

$$\langle \phi(\tau)(x), x\rangle = \left\langle \begin{pmatrix} I & T^* & \cdots & (T^*)^n \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^n & \cdots & T & I \end{pmatrix}\begin{pmatrix} \overline{\alpha_1} x \\ \overline{\alpha_2} x \\ \vdots \\ \overline{\alpha_n} x \end{pmatrix}, \begin{pmatrix} \overline{\alpha_1} x \\ \overline{\alpha_2} x \\ \vdots \\ \overline{\alpha_n} x \end{pmatrix}\right\rangle, \tag{$*$}$$

where our matrix operator acts on $\mathcal{H}^{(n)}$. Thus, we only need to show that this matrix operator is positive.

To that end, define the $n \times n$ matrix

$$R = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ T & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T & 0 \end{pmatrix},$$

and note that $R^{n+1} = 0$, with $\|R\|_{op} \leqslant 1$ (as T is a contraction).

We let I denote the identity operator on $\mathcal{H}^{(n)}$. The matrix operator $(*)$ can be written as

$$I + R + R^2 + \cdots + R^n + R^* + \cdots + (R^*)^n = (I - R)^{-1} + (I - R^*)^{-1} - I,$$

where we used the fact that $R^{n+1} = 0$ in the geometric series for $(I - R)^{-1}$ and $(I - R^*)^{-1}$. To see that this operator is positive, we let $h \in \mathcal{H}^{(n)}$, and let $h = (I - R)y$ for some $y \in \mathcal{H}^{(n)}$. Then,

$$\begin{aligned} \left\langle \left((I - R)^{-1} + (I - R^*)^{-1} - I \right)(h), h \right\rangle &= \langle y, (I - R)y \rangle + \langle (I - R)(y), y \rangle - \langle (I - R)(y), (I - R)(y) \rangle \\ &= \|y\|^2 - \|R(y)\|^2 \\ &\geqslant 0, \end{aligned}$$

since R is a contraction. □

Now, we may prove von Neumann's inequality in a different way.

Theorem (von Neumann's Inequality)**:** Let T be an operator on a Hilbert space with $\|T\|_{op} \leqslant 1$. Then, for any polynomial p, we have

$$\|p(T)\|_{op} \leqslant \|p\|,$$

where $\|p\| = \sup_\theta \left|p\left(e^{i\theta}\right)\right|$.

Proof. The operator system defined by

$$S = \left\{ p\left(e^{i\theta}\right) + \overline{q(e^{i\theta})} \;\middle|\; p, q \text{ polynomials} \right\}$$

is a $*$-algebra that separates points, so by the Stone–Weierstrass theorem, S is dense in $C(\mathbb{T})$. We know that ϕ is bounded, so it extends $C(\mathbb{T})$. The extension to $\overline{S} = C(\mathbb{T})$ also positive, so ϕ is contractive. □

Note that if $A(\mathbb{D})$ denotes the functions analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, we know that by the maximum modulus principle that the supremum of any function in $A(\mathbb{D})$ occurs on $\mathbb{T}$. We may thus consider $A(\mathbb{D})$ as a closed subalgebra of $C(\mathbb{T})$.

Furthermore, polynomials are dense in $A(\mathbb{D})$. Thus, the homomorphism $p \mapsto p(T)$ extends to a homomorphism $f \mapsto f(T)$ that satisfies $\|f(T)\|_{op} \leqslant \|f\|$ for all $f \in A(\mathbb{D})$.

Another consequence is that if a is an element of some unital C*-algebra A with $\|a\| \leqslant 1$, then there is a unital, positive map $\phi \colon C(\mathbb{T}) \to A$ such that $\phi(p) = p(a)$.

Corollary: Let B and C be unital C*-algebras. Let A be a unital subalgebra of B, and let $S = A + A^*$ be an operator space. If $\phi \colon S \to C$ is positive, then $\|\phi(a)\| \leqslant \|\phi(1)\|\|a\|$.

Proof. Let $a \in A$ with $\|a\| \leqslant 1$. We may extend ϕ to a positive map on $\overline{S}$. There is also a positive map $\psi \colon C(\mathbb{T}) \to B$ with $\psi(p) = p(a)$. Since A is an algebra, we must have $\operatorname{Ran}(\psi) \subseteq \overline{S}$.

The composition of positive maps is positive, so we have

$$\begin{aligned} \|\phi(a)\| &= \left\| \phi \circ \psi\left(e^{i\theta}\right) \right\| \\ &\leqslant \|\phi \circ \psi(1)\| \left\| e^{i\theta} \right\| \\ &= \|\phi(1)\|. \end{aligned}$$

□

If $\phi(1) = 1$, then ϕ is a contraction on A, though ϕ may not be a contraction on all of S.

Corollary: Let A and B be unital C*-algebras with $\phi\colon A \to B$ a positive map. Then, $\|\phi\|_{op} = \|\phi(1)\|$.

Lemma: Let A be a C*-algebra, $S \subseteq A$ an operator system, and $f\colon S \to \mathbb{C}$ a linear functional with $f(1) = 1 = \|f\|$. If a is a normal element of A, and $a \in S$, then $f(a) \in \overline{\operatorname{conv}}(\sigma(a))$.

Proof. Suppose not.

The convex hull of a compact set is the intersection of all closed disks containing the set. Then, there exists λ and $r > 0$ such that $|f(a) - \lambda| > r$, where

$$\sigma(a) \subseteq \{z \mid |z - \lambda| \leqslant r\}.$$

Then, $\sigma(a - \lambda 1) \subseteq \{z \mid |z| \leqslant r\}$. Since norm and spectral radius agree for normal elements, we have $\|a - \lambda 1\| \leqslant r$, while $|f(a - \lambda 1)| > r$. This contradicts the fact that $\|f\| \leqslant 1$. □

Proposition: Let S be an operator system, B a unital C*-algebra, and let $\phi\colon S \to B$ be a unital contraction. Then, ϕ is positive.

Proof. Since we can represent B on $\mathbb{B}(\mathcal{H})$, we assume $B = \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Fix $x \in \mathcal{H}$ with $\|x\| = 1$.

Setting $f(a) = \langle \phi(a)(x), x\rangle$, we have $f(1) = 1$ and $\|f\| \leqslant \|\phi\|$. If a is positive, then $f(a)$ is positive by the previous lemma, so since x was arbitrary, $\phi(a)$ is also positive. □

Proposition: Let A be a unital C*-algebra, and let M be a unital subspace of A. If B is a unital C*-algebra, and $\phi\colon M \to B$ is a unital contraction, then the map $\widetilde{\phi}\colon M + M^* \to B$, given by

$$\widetilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$$

is well-defined and the unique positive extension of ϕ to $M + M^*$.

Proof. To prove that $\widetilde{\phi}$ is well-defined, it is enough to prove that if a and a^* belong to M, then $\phi(a)^* = \phi(a^*)$. Set

$$S_1 = \{a \mid a \in M \text{ and } a^* \in M\}.$$

Then, S_1 is an operator system, and ϕ is a unital, contractive map on S_1, hence positive by the previous proposition. Since ϕ is positive, ϕ is self-adjoint, so $\phi(a^*) = \phi(a)^*$, meaning $\widetilde{\phi}$ is well-defined.

To see that $\widetilde{\phi}$ is positive, we may assume $B = \mathbb{B}(\mathcal{H})$. Fix $x \in S_{\mathcal{H}}$, and set $\widetilde{\rho}(a) = \left\langle \widetilde{\phi}(a)(x), x\right\rangle$. We will show that $\widetilde{\rho}$ is positive.

Let $\rho\colon M \to C$ be defined by $\rho(a) = \langle \phi(a)(x), x\rangle$. Then, $\|\rho\| = 1$, and so by the Hahn–Banach theorem, ρ extends to $\rho_1\colon M + M^* \to \mathbb{C}$ with $\|\rho_1\| = 1$. Since ρ_1 is positive, $\rho_1(a + b^*) = \rho(a) + \overline{\rho(b)} = \widetilde{\rho}(a + b^*)$. Thus $\widetilde{\rho}$ is positive. □

Completely Positive Maps

Definition. If A is a C* algebra and $M \subseteq A$ is a linear subspace, then we call M an operator space.

We may regard $\operatorname{Mat}_n(M)$ as a subspace of $\operatorname{Mat}_n(A)$, with the norm structure inherited from the unique norm structure on $\operatorname{Mat}_n(A)$. The primary distinguishing feature of an operator space is the fact that $\operatorname{Mat}_n(M)$ has a unique norm for all $n \geqslant 1$.

Similarly, if $S \subseteq A$ is an operator system, then we endow $\operatorname{Mat}_n(S)$ with the norm and order it inherits from $\operatorname{Mat}_n(A)$.

Definition. If a matrix $S \in \mathrm{Mat}_n(\mathbb{C})$ is positive definite and Hermitian, then S is positive.

Proof. If S is Hermitian, then we know that all the eigenvalues of S are real and that S is diagonalizable with orthonormal vectors $\{v_1, \dots, v_n\}$. Therefore, if

$$\langle S(x), x\rangle \geqslant 0$$

for all $x \in \mathbb{C}^n$, then so too does this hold for v_j and corresponding λ_j. Thus, $\lambda_j \geqslant 0$ for all j, so S is positive. □

Lemma (Ordering of $\mathrm{Mat}_n(\mathbb{B}(\mathcal{H}))$)**:** We have that $\left(T_{ij}\right)_{ij} \in \mathrm{Mat}_n(\mathbb{B}(\mathcal{H}))_+$ if and only if, for all $x_1, \dots, x_n \in \mathcal{H}$, we have $\left(\left\langle T_{ij}\left(x_j\right), x_i\right\rangle\right)_{ij} \in \mathrm{Mat}_n(\mathbb{C})_+$.

Definition. If B is a C*-algebra, and $\phi\colon S \to B$ is a linear map, then $\phi_n\colon \mathrm{Mat}_n(S) \to \mathrm{Mat}_n(B)$ is defined by $\phi_n\left(\left(a_{ij}\right)_{ij}\right) = \left(\phi\left(a_{ij}\right)\right)_{ij}$. We call ϕ n-positive if ϕ_n is positive, and ϕ is called completely positive if it is n-positive for all n.

We call ϕ completely bounded if $\sup_n \|\phi_n\|$ is finite. We set

$$\|\phi\|_{cb} = \sup_n \|\phi_n\|.$$

We say ϕ is completely isometric or completely contractive if each ϕ_n is isometric and that $\|\phi\|_{cb} \leqslant 1$ respectively.

We investigate some of the properties of classes of completely positive maps such that we may prove when they are automatically completely positive.

Lemma: Let A be a C*-algebra, and let $a, b \in A$. Then, the following hold.

(i) We have $\|a\| \leqslant 1$ if and only if

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$$

is positive in $\mathrm{Mat}_2(A)$.

(ii) We have

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$$

is positive in $\mathrm{Mat}_2(A)$ if and only if $a^*a \leqslant b$.

Proof. Let A be represented by $\pi\colon A \to \mathbb{B}(\mathcal{H})$, and set $T = \pi(a)$. If $\|T\| \leqslant 1$, then for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\left\langle \begin{pmatrix} I & T \\ T^* & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle &= \langle x, x\rangle + \langle T(y), x\rangle + \langle x, T(y)\rangle + \langle y, y\rangle \\ &\geqslant \|x\|^2 - 2\|T\|_{op}\|y\|\|x\| + \|y\|^2 \\ &\geqslant 0.\end{aligned}$$

Conversely, if $\|T\|_{op} > 1$, then there exist unit vectors x and y such that $\langle T(y), x\rangle < -1$, and the above inner product is negative.

□

Exercise (Exercise 3.2)**:** Let P, Q, A be operators on a Hilbert space $\mathcal{H}$, with P, Q positive.

(i) Show that
$$\begin{pmatrix}P & A\\ A^{*} & Q\end{pmatrix} \geqslant 0$$
if and only if
$$|\langle Ax,y\rangle|^2 \leqslant \langle Py,y\rangle\langle Qx,x\rangle.$$

(ii) Show that
$$\begin{pmatrix}1 & A\\ A^{*} & B\end{pmatrix} \geqslant 0$$
if and only if $B \geqslant A^{*}A$.

(iii) Show that if
$$\begin{pmatrix}P & A\\ A^{*} & Q\end{pmatrix} \geqslant 0,$$
then for any $x \in \mathcal{H}$, we have
$$\begin{aligned}0 &\leqslant \langle (P + A + A^{*} + Q)x,x\rangle\\ &\leqslant \left(\sqrt{\langle Px,x\rangle} + \sqrt{\langle Qx,x\rangle}\right)^2,\end{aligned}$$
hence
$$\|P + AA^{*} + Q\| \leqslant \left(\|P\|^{1/2} + \|Q\|^{1/2}\right)^2.$$

(iv) Show that if
$$\begin{pmatrix}P & A\\ A^{*} & P\end{pmatrix} \geqslant 0,$$
then $A^{*}A \leqslant \|P\|P$, implying $\|A\| \leqslant \|P\|$.

Solution:

(i) We see that
$$\begin{pmatrix}P & A\\ A^{*} & Q\end{pmatrix} \geqslant 0$$
if and only if, for any $x,y \in \mathcal{H}$, we have
$$\begin{pmatrix}\langle Px,x\rangle & \langle Ay,x\rangle\\ \langle A^{*}x,y\rangle & \langle Qy,y\rangle\end{pmatrix} \geqslant 0.$$
Thus, we have
$$\begin{aligned}\det\begin{pmatrix}\langle Px,x\rangle & \langle Ay,x\rangle\\ \langle A^{*}x,y\rangle & \langle Qy,y\rangle\end{pmatrix} &= \langle Px,x\rangle\langle Qy,y\rangle - |\langle Ay,x\rangle|^2\\ &\geqslant 0,\end{aligned}$$
so that
$$|\langle Ay,x\rangle|^2 \leqslant \langle Px,x\rangle\langle Qy,y\rangle.$$
Suppose that
$$|\langle Ay,x\rangle|^2 \leqslant \langle Px,x\rangle\langle Qy,y\rangle.$$

Now, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\left\langle \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle &= \langle Px, x\rangle + \langle Ay, x\rangle + \langle A^*x, y\rangle + \langle Qy, y\rangle \\ &= \langle Px, x\rangle + 2\operatorname{Re}(\langle Ay, x\rangle) + \langle Qy, y\rangle \\ &\geqslant \langle Px, x\rangle - 2|\langle Ay, x\rangle| + \langle Qy, y\rangle \\ &\geqslant \langle Px, x\rangle - 2\langle Px, x\rangle^{1/2}\langle Qy, y\rangle^{1/2} + \langle Qy, y\rangle \\ &= \left(\langle Px, x\rangle^{1/2} + \langle Qy, y\rangle^{1/2}\right)^2 \\ &\geqslant 0.\end{aligned}$$

(ii) We begin by assuming that $B \geqslant A^*A$. Since $B \geqslant A^*A$, we have

$$\langle (B - A^*A)(y), y\rangle \geqslant 0,$$

so that

$$\langle By, y\rangle \geqslant \|Ay\|^2.$$

Thus, in the 2×2 case, we have, for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}^{(2)}$,

$$\begin{aligned}\left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle &= \langle x, x\rangle + \langle Ay, x\rangle + \langle A^*x, y\rangle + \langle By, y\rangle \\ &\geqslant \langle x, x\rangle + \langle Ay, x\rangle + \langle A^*x, y\rangle + \langle Ay, Ay\rangle \\ &= \langle x, x\rangle + \langle Ay, Ay\rangle + 2\operatorname{Re}(\langle Ay, x\rangle) \\ &\geqslant \langle x, x\rangle + \langle Ay, Ay\rangle - 2\|Ay\|\|x\| \\ &= \|x\|^2 + \|Ay\|^2 - 2\|Ay\|\|x\| \\ &\geqslant 0.\end{aligned}$$

Thus, the matrix is positive.

For the converse direction, we suppose $B \ngeqslant A^*A$. Then, there is some $y \in \mathcal{H}$ such that $\langle (B - A^*A)(y), y\rangle < 0$. This gives $\langle By, y\rangle < \|Ay\|^2$. We may select y such that $\|Ay\|^2 = 1$. Setting $x = -Ay$, we have

$$\begin{aligned}\left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle &= \langle x, x\rangle + \langle Ay, x\rangle + \langle A^*x, y\rangle + \langle By, y\rangle \\ &= \langle x, x\rangle + \langle Ay, x\rangle + \langle x, Ay\rangle + \langle By, y\rangle \\ &= \langle -Ay, -Ay\rangle + \langle Ay, -Ay\rangle + \langle -Ay, Ay\rangle + \langle By, y\rangle \\ &= \|Ay\|^2 - 2\|Ay\|^2 + \langle By, y\rangle \\ &= -1 + \langle By, y\rangle \\ &< -1 + \|Ay\|^2 \\ &= 0.\end{aligned}$$

Thus, the matrix is negative.

(iii) We apply the result in (i) to the vector $\begin{pmatrix} x \\ x \end{pmatrix}$. This gives

$$\begin{aligned}\left\langle \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} x \\ x \end{pmatrix}\right\rangle &= \langle Px, x\rangle + \langle Ax, x\rangle + \langle A^*x, x\rangle + \langle Qx, x\rangle \\ &= \langle Px, x\rangle + 2\operatorname{Re}(\langle Ax, x\rangle) + \langle Qx, x\rangle \\ &\leqslant \langle Px, x\rangle + 2|\langle Ax, x\rangle| + \langle Qx, x\rangle \\ &\leqslant \langle Px, x\rangle + 2\langle Px, x\rangle^{1/2}\langle Qx, x\rangle^{1/2} + \langle Qx, x\rangle \\ &= \left(\langle Px, x\rangle^{1/2} + \langle Qx, x\rangle^{1/2}\right)^2.\end{aligned}$$

(iv) Setting $Q = P$ in the result from (i), we have

$$|\langle Ay, x\rangle|^2 \leqslant \langle Px, x\rangle\langle Py, y\rangle,$$

which holds for all $x, y \in \mathcal{H}$. In particular, setting $x = Ay$, we have

$$\begin{aligned}|\langle Ay, Ay\rangle| &\leqslant \langle PAy, Ay\rangle\langle Py, y\rangle\\ &\leqslant \|PAy\|\|Ay\|\langle Py, y\rangle\\ &\leqslant \|P\|\|Ay\|^2\langle Py, y\rangle.\end{aligned}$$

This gives

$$\begin{aligned}\|Ay\|^4 &\leqslant \|P\|\|Ay\|^2\langle Py, y\rangle\\ \|Ay\|^2 &\leqslant \|P\|\langle Py, y\rangle\\ \langle A^*Ay, y\rangle &\leqslant \|P\|\langle Py, y\rangle,\end{aligned}$$

or that $A^*A \leqslant \|P\|P$.

Proposition: Let S be an operator system, B a unital C*-algebra, and $\phi\colon S \to B$ a unital 2-positive map. Then, ϕ is contractive.

Proof. Let $a \in S$ with $\|a\| \leqslant 1$. Then,

$$\phi_2\begin{pmatrix}1 & a\\ a^* & 1\end{pmatrix} = \begin{pmatrix}1 & \phi(a)\\ \phi(a)^* & 1\end{pmatrix}$$

is positive, hence $\|\phi(a)\| \leqslant 1$. □

Proposition (Cauchy–Schwarz for 2-positive Maps)**:** Let A, B be unital C*-algebras, and let $\phi\colon A \to B$ be a unital 2-positive map. Then,

$$\phi(a)^*\phi(a) \leqslant \phi(a^*a)$$

for all $a \in A$.

Proof. We have that

$$\begin{aligned}\begin{pmatrix}1 & a\\ 0 & 0\end{pmatrix}^*\begin{pmatrix}1 & a\\ 0 & 0\end{pmatrix} &= \begin{pmatrix}1 & \phi(a)\\ \phi(a)^* & \phi(a^*a)\end{pmatrix}\\ &\geqslant 0,\end{aligned}$$

meaning that $\phi(a)^*\phi(a) \leqslant \phi(a^*a)$ by above. □

Proposition: Let A and B be unital C*-algebras, and let M be a unital subspace of M, with $S = M + M^*$. If $\phi\colon M \to B$ is unital and 2-contractive, then $\widetilde{\phi}\colon S \to B$ given by $\widetilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ is 2-positive and contractive.

Proof. Since ϕ is contractive, we know from above that $\widetilde{\phi}$ is well-defined. Furthermore, note that

$$\mathrm{Mat}_2(S) = \mathrm{Mat}_2(M) + \mathrm{Mat}_2(M)^*,$$

and

$$\left(\widetilde{\phi}\right)_2 = \left(\widetilde{\phi_2}\right).$$

Now, since ϕ_2 is contractive, we have that $\widetilde{\phi_2}$ is positive, so $\widetilde{\phi}$ is contractive. □

Proposition: Let A and B be unital C*-algebras, let M be a unital subspace, and let $S = M + M^*$. If $\phi\colon M \to B$ is unital and completely contractive, then $\widetilde{\phi}\colon S \to B$ is completely positive and completely contractive.

Proof. Since ϕ_n is unital and contractive, $\widetilde{\phi}_n$ is positive. Additionally, since $\left(\widetilde{\phi}_n\right)_2$ is positive, $\widetilde{\phi}_n$ is contractive. ◻

Note that since $\operatorname{Mat}_2(\operatorname{Mat}_n(A)) \cong \operatorname{Mat}_{2n}(A)$ are $*$-isomorphic, the norm on $\operatorname{Mat}_2(\operatorname{Mat}_n(A))$ is equal to the norm on $\operatorname{Mat}_{2n}(A)$.

Now, we may see some examples that belong to these categories.

Example. If A and B are C*-algebras, and $\pi\colon A \to B$ is a $*$-homomorphism, then π is completely positive and completely contractive, since each $\pi_n\colon \operatorname{Mat}_n(A) \to \operatorname{Mat}_n(B)$ is a $*$-homomorphism, and $*$-homomorphisms are both positive and contractive.

Example. Fixing $x, y \in A$, we may define $\phi\colon A \to A$ by $\phi(a) = xay$. Note that if $(a_{ij})_{ij} \in \operatorname{Mat}_n(A)$, then

$$\begin{aligned}\left\|\phi_n\left((a_{ij})_{ij}\right)\right\| &= \left\|(xa_{ij}y)_{ij}\right\| \\ &= \left\|(xI_n)\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}(yI_n)\right\| \\ &\leqslant \|x\|\left\|(a_{ij})_{ij}\right\|\|y\|.\end{aligned}$$

This means ϕ is completely bounded with $\|\phi\|_{cb} \leqslant \|x\|\|y\|$. Similarly, if $x = y^*$, then ϕ_n is positive.

This gives us the archetype of a completely bounded map. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, and $v_i\colon \mathcal{H}_1 \to \mathcal{H}_2$ are bounded operators for $i = 1, 2$, then if $\pi\colon A \to \mathbb{B}(\mathcal{H}_2)$ is a $*$-homomorphism, we may define $\phi\colon A \to \mathbb{B}(\mathcal{H}_1)$ by $\phi(a) = v_2^*\pi(a)v_1$. This function ϕ is completely bounded with $\|\phi\|_{cb} \leqslant \|v_1\|\|v_2\|$.

In fact, we will show that *every* completely bounded map is of this form.

Proposition: Let $S \subseteq A$ be an operator system, B a C*-algebra, and $\phi\colon S \to B$ completely positive. Then, ϕ is completely bounded, and $\|\phi(1)\| = \|\phi\| = \|\phi\|_{cb}$.

Proof. We have $\|\phi(1)\| \leqslant \|\phi\| \leqslant \|\phi\|_{cb}$, so it is sufficient to show that $\|\phi\|_{cb} \leqslant \|\phi(1)\|$. Let $A = (a_{ij})_{ij}$ be in $\operatorname{Mat}_n(S)$ with $\|A\| \leqslant 1$, and let I_n be the unit of $\operatorname{Mat}_n(A)$. Then, since

$$T = \begin{pmatrix} I_n & A \\ A^* & I_n\end{pmatrix}$$

is positive, the map

$$\phi_{2n}\left(\begin{pmatrix} I_n & A \\ A^* & I_n\end{pmatrix}\right) = \left(\begin{pmatrix} \phi_n(I_n) & \phi_n(A) \\ \phi_n(A)^* & \phi_n(I_n)\end{pmatrix}\right)$$

is positive, so $\|\phi_n(A)\| \leqslant \|\phi_n(I_n)\| = \|\phi(1)\|$. ◻

Schur Products and Tensor Products

We will apply the previous results on positive and completely positive maps on the Schur product.

Definition. If $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$, then the Schur product is defined by

$$A * B = (a_{ij}b_{ij})_{ij}.$$

Note that for a fixed A, we get a linear map

$$S_A(B) = A * B.$$

To study the Schur product, we review some results on tensor products.

Let $A \in \operatorname{Mat}_n(\mathbb{C})$ and $B \in \operatorname{Mat}_m(\mathbb{C})$. Then, $A \otimes B$ is the linear transformation on $\mathbb{C}^n \otimes \mathbb{C}^m = \mathbb{C}^{nm}$, defined by $A \otimes B(x \otimes y) = Ax \otimes By$ with the unique linear extension provided by the tensor product.

Note that we have $\|A \otimes B\| = \|A\|\|B\|$, which is shown by writing $A \otimes B = (A \otimes I)(I \otimes B)$.

Now, letting $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ be our canonical orthonormal bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, we may order our basis as $e_1 \otimes f_i$, then $e_2 \otimes f_i$, etc., yielding the block matrices for $A \otimes B$ is

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

This matrix is known as the Kronecker product of A and B. Now, similarly, we may order our basis by $e_i \otimes f_1$, then $e_i \otimes f_2$, etc., yielding a different block matrix of the form

$$\begin{pmatrix} b_{11}A & \cdots & b_{1m}A \\ \vdots & \ddots & \vdots \\ b_{m1}A & \cdots & b_{mm}A \end{pmatrix},$$

which is the Kronecker product of B and A.

Now, since both of these matrices represent the same linear transformation, they are unitarily equivalent, given by the permutation matrix that reorders the basis vectors. One obtains the (k, ℓ) entry of the (i, j) block of $b_{ij}A$ by taking the (i, j) entry of the (k, ℓ) block $a_{k,\ell}B$. We will call this the *canonical shuffle*.

Now, we let A and B be elements of $\operatorname{Mat}_n(\mathbb{C})$, and define $V\colon \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ to be the isometry given by $V(e_i) = e_i \otimes e_i$. We will show that $V^*(A \otimes B)V = A * B$. Note that

$$\begin{aligned} \left\langle V^*(A \otimes B)Ve_j, e_i\right\rangle &= \left\langle (A \otimes B)\left(e_j \otimes e_j\right), e_i \otimes e_i\right\rangle \\ &= \left\langle Ae_j, e_i\right\rangle\left\langle Be_j, e_i\right\rangle \\ &= a_{ij}b_{ij} \\ &= \left\langle A * Be_j, e_i\right\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \|S_A(B)\| &\leqslant \|V^*(A \otimes B)V\| \\ &\leqslant \|A\|\|B\|, \end{aligned}$$

so that

$$\|S_A\| \leqslant \|A\|.$$

Now, if $\left(B_{ij}\right)_{ij} \in \operatorname{Mat}_k(\operatorname{Mat}_n(\mathbb{C}))$, then

$$\begin{aligned} (S_A)\left(\left(B_{ij}\right)_{ij}\right) &= \left(V^*\left(A \otimes B_{ij}\right)V\right)_{ij} \\ &= \begin{pmatrix} V^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V^* \end{pmatrix} A \otimes \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}\begin{pmatrix} V & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V \end{pmatrix}, \end{aligned}$$

so that $\|(S_A)_k\| \leqslant \|A\|$. Thus, $\|S_A\|_{cb} \leqslant \|A\|$.

However, this isn't a really good estimate. For instance, if A is the matrix consisting of all 1s, then the norm of A is n, while $\|S_A\| = 1$.

We will prove that if A is positive, then S_A is completely positive. Thus, for positive matrices, we are able to obtain $\|S_A\|_{cb}$ by finding

$$\begin{aligned}\|S_A\| &= \|S_A(I)\| \\ &= \|S_A\|_{cb} \\ &= \max\{a_{ii} \mid i = 1, \dots, n\}.\end{aligned}$$

Now, if A is not positive, then obtaining this norm is a bit more difficult. We can decompose $A = (P_1 - P_2) + i(P_3 - P_4)$, and get

$$\|S_A\|_{cb} \leqslant \|S_{P_1}\|_{cb} + \|S_{P_2}\|_{cb} + \|S_{P_3}\|_{cb} + \|S_{P_4}\|_{cb},$$

but unfortunately this estimate isn't really enough.

Now, we will characterize when the Schur product is completely positive.

Theorem: Let $A = \left(a_{ij}\right)_{ij} \in \mathrm{Mat}_n(\mathbb{C})$. The following are equivalent:

(i) A is positive;

(ii) $S_A\colon \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ is positive;

(iii) $S_A\colon \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ is completely positive.

Proof. We have that (iii) implies (ii), and (ii) implies (i) by choosing J to be the matrix consisting of 1, which is positive, meaning $S_A(J) = A$. Thus, we must prove that (i) implies (iii).

Note that if A and B are positive, then $A \otimes B$ is positive. This follows from the fact that $A \otimes B = \left(A^{1/2} \otimes B^{1/2}\right)^2$.

Now, if $B \in \mathrm{Mat}_n(\mathbb{C})$ is positive, then

$$\begin{aligned}S_A(B) &= V^*(A \otimes B)V \\ &= \left(\left(A^{1/2} \otimes B^{1/2}\right)V\right)^*\left(\left(A^{1/2} \otimes B^{1/2}\right)V\right)\end{aligned}$$

is positive, meaning (i) implies (ii).

Now, to see that (i) implies (iii), we let $B = \left(B_{ij}\right)_{ij} \in \mathrm{Mat}_k(\mathrm{Mat}_n(\mathbb{C}))$, and write $B = \left(X_{ij}\right)_{ij}^*\left(X_{ij}\right)_{ij}$. We see that

$$\begin{aligned}(S_A)_k(B) &= \left(V^*\left(A \otimes B_{ij}\right)V\right) \\ &= \left(\left(A^{1/2} \otimes X_{ij}\right)V\right)^*\left(\left(A^{1/2} \otimes X_{ij}\right)V\right),\end{aligned}$$

meaning $(S_A)_k$ is positive. □

There is an analogous theory of Schur products in the space $\mathbb{B}(\ell_2)$, where we consider the bounded operators as infinite matrices. If we mandate that $A \in \mathbb{B}(\ell_2)_+$, then we can use a similar line of argumentation to show that S_A is completely positive., but this requires a bit more care as the matrix consisting of all 1s regarded as an operator on ℓ_2 is not a bounded operator.

Now, we can show a pretty useful result, which is that bounded linear functionals are not only positive, but completely positive.

Proposition: Let S be an operator space, and let $f\colon S \to \mathbb{C}$ be a bounded linear functional. Then,

$$\|f\|_{cb} = \|f\|_{op},$$

and if S is an operator system with f positive, then f is completely positive.

Proof. Let $\left(a_{ij}\right)_{ij} \in \operatorname{Mat}_n(S)$, and let $x, y \in \mathbb{C}^n$ be unit vectors. Then,

$$\begin{aligned}
\left|\left\langle f\left(\left(a_{ij}\right)_{ij}\right)(x), y\right\rangle\right| &= \left|\sum_{i,j=1}^{n} f\left(a_{ij}\right)\left(x_j\right)\overline{y_i}\right| \\
&= \left|f\left(\sum_{i,j=1}^{n} a_{ij}x_j\overline{y_i}\right)\right| \\
&\leqslant \|f\|_{op}\left\|\sum_{i,j=1}^{n} a_{ij}x_j\overline{y_i}\right\|.
\end{aligned}$$

Now, all we need to show is that the latter element has norm less than $\left\|\left(a_{ij}\right)_{ij}\right\|$. Note that this sum is the entry on the first row and column of the matrix that represents the product

$$\begin{pmatrix} \overline{y_1}1 & \cdots & \overline{y_n}1 \\ 0 & \cdots & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n 1 & 0 & \cdots & 0 \end{pmatrix}.$$

The outer two factors have norm 1, since x and y are chosen to be unit vectors.

To show that f is completely positive, we only need to show that

$$\left\langle f_n\left(\left(a_{ij}\right)_{ij}\right)(x), x\right\rangle = f\left(\sum_{i,j=1}^{n} a_{ij}x_j\overline{x_i}\right)$$

is positive whenever $\left(a_{ij}\right)_{ij}$ is positive. However, using the above product, we see that the summation is equal to the first row and column entry of a positive matrix, hence positive. □

Now, we examine the positivity and boundedness of maps with codomain $C(X)$, where X is a compact Hausdorff space.

Note that every element $F = \left(f_{ij}\right)_{ij}$ of $\operatorname{Mat}_n(C(X))$ can be considered as a continuous matrix-valued function, with multiplication and $*$-operation as pointwise multiplication and involution of the matrix-valued functions.

To make $\operatorname{Mat}_n(C(X))$ into a C*-algebra is to set $\|f\| = \sup\{\|F(x)\| \mid x \in X\}$, and by uniqueness of C*-norms, this is the only way to create a C*-norm.

Theorem: Let S be an operator space, and let $\phi\colon S \to C(X)$ be a bounded linear map. Then, $\|\phi\|_{cb} = \|\phi\|_{op}$. Furthermore, if S is an operator system and ϕ is positive, then ϕ is completely positive.

Proof. Let $x \in X$, and define ϕ^x to be pointwise evaluation — i.e., $\phi^x(a) = \phi(a)(x)$. Then,

$$\begin{aligned}
\|\phi_n\| &= \sup\{\|\phi_n^x\| \mid x \in X\} \\
&= \sup\{\|\phi^x\| \mid x \in X\} \\
&= \|\phi\|_{op}.
\end{aligned}$$

Similarly, $\phi_n\left(\left(a_{ij}\right)_{ij}\right)$ is positive if and only if $\phi_n^x\left(\left(a_{ij}\right)_{ij}\right)$ is positive for all $x \in X$. □

Thus, when the codomain C*-algebra is commutative, boundedness and complete boundedness, as well as positivity and complete positivity, coincide. A commutative domain *is* enough to show that positive maps are completely positive, but unfortunately a commutative domain is not enough to guarantee that bounded maps are completely bounded.

Lemma: Let $\left(p_{ij}\right)_{ij}$ be a positive scalar matrix, and let q be a positive element of some C*-algebra B. Then, $\left(p_{ij}q\right)_{ij}$ is positive in $\operatorname{Mat}_n(B)$.

Proof. We write $\left(p_{ij}\right)_{ij}$ as $\left(s_{ij}\right)_{ij}^*\left(s_{ij}\right)_{ij}$, and write $q = n^*n$. This gives

$$\begin{aligned}
\left(p_{ij}q\right) &= \left(p_{ij}\right)_{ij}\operatorname{diag}(q,\dots,q)\\
&= \left(s_{ij}\right)_{ij}^*\left(s_{ij}\right)\operatorname{diag}(n^*n,\dots,n^*n)\\
&= \left(s_{ij}\right)_{ij}^*\left(s_{ij}\right)_{ij}\operatorname{diag}(n^*,\dots,n^*)\operatorname{diag}(n,\dots,n)\\
&= \left(s_{ij}\right)_{ij}^*\left(s_{ij}\right)_{ij}\operatorname{diag}(n,\dots,n)^*\operatorname{diag}(n,\dots,n)\\
&= \operatorname{diag}(n,\dots,n)^*\left(s_{ij}\right)_{ij}^*\left(s_{ij}\right)_{ij}\operatorname{diag}(n,\dots,n) && \operatorname{diag}(n,\dots,n)\in\operatorname{Mat}_n(B),\ \left(s_{ij}\right)_{ij}\in\operatorname{Mat}_n(\mathbb{C})\\
&= \left(\left(s_{ij}\right)_{ij}\operatorname{diag}(n,\dots,n)\right)^*\left(\left(s_{ij}\right)_{ij}\operatorname{diag}(n,\dots,n)\right).
\end{aligned}$$

Thus, $\left(p_{ij}q\right)_{ij}$ is positive in $\operatorname{Mat}_n(B)$. ◻

Theorem: Let B be a C*-algebra, and let $\phi\colon C(X)\to B$ be a positive map. Then, ϕ is completely positive.

Proof. Let $P(x)$ be positive in $\operatorname{Mat}_n(C(X))$. We prove that $\phi_n(P)$ is positive.

Given $\varepsilon > 0$, we may find a partition of unity $\{u_\ell(x)\}_{\ell=1}^m$ and positive matrices $P_\ell = \left(p_{ij}^\ell\right)_{ij}$ such that

$$\left|P - \sum_{\ell=1}^m u_\ell(x)P_\ell\right| < \varepsilon.$$

However, we know that

$$\begin{aligned}
\phi_n(u_\ell P_\ell) &= \phi_n\left(\left(u_\ell p_{ij}^\ell\right)_{ij}\right)\\
&= \left(\phi(u_\ell)p_{ij}^\ell\right)_{ij},
\end{aligned}$$

which is positive. Therefore, $\phi_n(P)$ is within $\varepsilon\|\phi_n\|\|P\|$ of a sum of positive elements. Since $\operatorname{Mat}_n(B)_+$ is a closed set, we have that $\phi_n(P)$ is positive. ◻

Corollary: Let T be a contractive operator on $\mathcal{H}$, and let $\left(p_{ij}\right)_{ij}$ be a $n\times n$ matrix of polynomials. Then,

$$\left\|\left(p_{ij}(T)\right)_{ij}\right\|_{op} \leqslant \sup\left\{\left\|\left(p_{ij}(z)\right)_{ij}\right\| \;\middle|\; |z| = 1\right\}.$$

Proof. The map given by $\phi(p+\overline{q}) = p(T) + q(T)^*$ extends to a positive map $\varphi\colon C(\mathbb{T})\to\mathbb{B}(\mathcal{H})$. This map is completely positive as $C(\mathbb{T})$ is a commutative C*-algebra. Thus, $\|\varphi\|_{cb} = \|\varphi(1)\| = 1$. Thus,

$$\begin{aligned}
\left\|\left(p_{ij}(T)\right)_{ij}\right\| &= \left\|\phi_n\left(\left(p_{ij}\right)_{ij}\right)\right\|\\
&\leqslant \left\|\left(p_{ij}(1)\right)_{ij}\right\|.
\end{aligned}$$

◻

Lemma: Let A be a C*-algebra. Then, every positive element of $\mathrm{Mat}_n(A)$ is a sum of n positive elements of the form $\left(a_i^* a_j\right)_{ij}$, where $\{a_1, \dots, a_n\} \subseteq A$.

Proof. Note that if R is the element of $\mathrm{Mat}_n(A)$ whose kth row is $a_1, \dots, a_n$ and 0 elsewhere, then $R^*R = \left(a_i^* a_j\right)_{ij}$, so such an element is positive.

Now, let P be positive, yielding $P = B^*B$. Write $B = R_1 + \cdots R_n$, where R_k is the kth row of B and 0 elsewhere.

Then, since $R_i^* R_j = 0$ whenever $i \neq j$, we have that $P = R_1^* R_1 + \cdots + R_n^* R_n$. □

Thus, it suffices to check that $\phi\colon A \to B$ is n-positive by verifying that $\left(\phi\left(a_i^* a_j\right)\right)_{ij}$ is positive for all $a_1, \dots, a_n \in A$.

Theorem: Let B be a C*-algebra, let $\phi\colon \mathrm{Mat}_n(\mathbb{C}) \to B$ be a linear map, and let $\left\{e_{ij}\right\}_{i,j=1}^n$ denote the standard matrix units for $\mathrm{Mat}_n(\mathbb{C})$. The following are equivalent:

(i) ϕ is completely positive;

(ii) ϕ is n-positive;

(iii) $\left(\phi\left(e_{ij}\right)\right)_{ij}$ is positive in $\mathrm{Mat}_n(B)$.

Proof. It suffices to show that (iii) implies (i), as (i) implies (ii) and $\left(e_{ij}\right)_{ij}$ is positive for each i, j, giving (ii) implies (iii).

It is sufficient to assume that $B = \mathbb{B}(\mathcal{H})$. Fix k, and let $x_1, \dots, x_k \in \mathcal{H}$, $B_1, \dots, B_k \in \mathrm{Mat}_n(\mathbb{C})$. It is sufficient to prove that

$$\sum_{i,j}^{k} \left\langle \phi\left(B_i^* B_j\right) x_j, x_j \right\rangle \geqslant 0.$$

Write $B_\ell = \sum_{r,s=1}^n b_{rs,\ell} e_{rs}$, such that

$$B_i^* B_j = \sum_{r,s,t=1}^{n} \overline{b_{rs,i}} b_{rt,j} e_{st}.$$

Set $y_{t,r} = \sum_{j=1}^k b_{rt,j} x_j$. Then,

$$\begin{aligned} \sum_{i,j=1}^{k} \left\langle \phi\left(B_i^* B_j\right) x_j, x_i \right\rangle &= \sum_{r=1}^{n} \sum_{s,t=1} \left\langle \phi(e_{st}) \left(\sum_{i,j=1}^{k} b_{rs,i} b_{rt,j} x_j \right), x_i \right\rangle \\ &= \sum_{r=1}^{n} \sum_{s,t} \langle \phi(e_{st}) y_{t,r}, y_{s,r} \rangle. \end{aligned}$$

However, this latter sum is positive, since $(\phi(e_{st}))st$ is positive, so we have expressed our original sum as the sum of n positive quantities. □

Now, we may obtain some fairly deep results in operator theory via the properties of positive maps.

Definition. If $T \in \mathbb{B}(\mathcal{H})$, we define the numerical radius of T by

$$w(T) = \sup_{x \in B_{\mathcal{H}}} |\langle Tx, x \rangle|.$$

Exercise: Let S_n be the cyclic forward shift on $\mathbb{C}^n$. That is, $S_n e_j = e_{j+1}$ mod n, where $e_0, \dots, e_{n-1}$ is the canonical basis for $\mathbb{C}^n$.

(i) Show that S_n is unitarily equivalent to a diagonal matrix whose entries are the nth roots of unity.

(ii) Let $T \in \mathbb{B}(\mathcal{H})$. Show that $w(T) = w(T \otimes S_n)$.

(iii) Let R_n be the $n \times n$ matrix of operators whose subdiagonals are T and 0 elsewhere. Show that $w(R_n) \leqslant w(T \otimes S_n)$.

(iv) Show that $\operatorname{Re}(\langle R_n y, y\rangle) \leqslant 1$ for all $\|y\| = 1$ if and only if $w(R_n) \leqslant 1$.

Solution:

(i) By the definition of the cyclic forward shift, defining $A := S_n$, we have $A^n = I_n$, or $A^n - I = 0$. This means that the minimal polynomial for S_n is $m_{S_n}(x) = x^n - 1$, meaning that the nth roots of unity are eigenvalues for S. Since S_n is an operator acting on $\mathbb{C}^n$, there are at n eigenvalues (with multiplicity) for S_n, meaning that the nth roots of unity are in fact *the* eigenvalues of S_n. Thus, S_n is unitarily equivalent to a diagonal matrix with the nth roots of unity on the diagonal.

(ii) The operator $T \otimes S_n$ acts on $\mathcal{H} \otimes \mathbb{C}^n$ such that $(T \otimes S_n)(y \otimes v) = Ty \otimes S_n v$. Thus, we have

$$\begin{aligned} w(T \otimes S_n) &= \sup_{y \otimes v \in \mathbb{B}_{\mathcal{H} \otimes \mathbb{C}^n}} |\langle (T \otimes S_n)(y \otimes v), y \otimes v\rangle| \\ &= \sup_{y \otimes v \in B_{\mathcal{H} \otimes \mathbb{C}^n}} |\langle Ty \otimes S_n v, y \otimes v\rangle| \\ &= \sup_{y \in B_{\mathcal{H}}} \sup_{v \in B_{\mathbb{C}^n}} |\langle Ty, y\rangle||\langle S_n v, v\rangle| \\ &= w(T) w(S_n) \\ &= w(T). \end{aligned}$$

(iii) We consider the non-cyclic shift S_n', and note that $w(S_n') \leqslant w(S_n)$, as applying the non-cyclic shift will yield zero in the first entry of the vector v. We have that $R_n \cong T \otimes S_n'$, meaning that $w(R_n) \leqslant w(T)$.

(iv) If $w(R_n) \leqslant 1$ for all $\|y\| = 1$, then since $\operatorname{Re}(\langle R_n y, y\rangle) \leqslant |\langle R_n y, y\rangle| \leqslant 1$, it is clear that $\operatorname{Re}(\langle R_n y, y\rangle) \leqslant 1$ for all $\|y\| = 1$.

Now, suppose $\operatorname{Re}(\langle R_n y, y\rangle) \leqslant 1$ for all $\|y\| = 1$.

Theorem: Let $T \in \mathbb{B}(\mathcal{H})$, let $S \subseteq C(\mathbb{T})$ be the operator system defined by $S = \left\{p + \overline{q} \mid p, q \text{ polynomials}\right\}$. The following are equivalent:

(i) $w(T) \leqslant 1$;

(ii) the map $\phi \colon S \to \mathbb{B}(\mathcal{H})$, defined by

$$\phi(p + \overline{q}) = p(T) + q(T)^* + \left(p(0) + \overline{q(0)}\right) I$$

is positive.

Proof. We start by showing that (i) implies (ii).

Let R_n be the $n \times n$ operator matrix with subdiagonal entry T and remaining entries 0. Note that $w(R_n) \leqslant w(T)$.

Now, we see that ϕ is positive so long as the matrix

$$\begin{pmatrix} 2 & T^* & \cdots & (T^*)^n \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^n & \cdots & T & 2 \end{pmatrix} \tag{$*$}$$

is positive for all n.

Note that $R_n^{n+1} = 0$, so $(*)$ can be written as $(I - R_n)^{-1} + (I - R_n^*)^{-1}$.

Fix $x = (I - R_n)y$, and compute

$$\left\langle \left((I - R_n)^{-1} + (I - R_n^*)^{-1}\right)x, x\right\rangle = 2\|y\|^2 - 2\operatorname{Re}(\langle R_n y, y\rangle).$$

Thus, $(*)$ is positive if and only if $w(R_n) \leqslant 1$. Since $w(T) \leqslant 1$ implies $w(R_n) \leqslant 1$, we have $(*)$ is positive, meaning ϕ is positive.

Conversely, if ϕ is positive, since $\overline{S} = C(\mathbb{T})$, ϕ is completely positive by the fact that if $\phi\colon C(X) \to B$ is positive, then ϕ is completely positive.

Note that

$$\begin{pmatrix} 1 & \overline{z} & \cdots & \overline{z}^n \\ z & 1 & \cdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{z} \\ z^n & \cdots & z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & z & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^n \end{pmatrix}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \overline{z} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{z}^n \end{pmatrix}$$

is positive in $\operatorname{Mat}_n(C(\mathbb{T}))$, so its image under ϕ_n is also positive. However, since this image is equal to $(*)$, we have that $(*)$ is positive for all n, meaning $w(R_n) \leqslant 1$.

Let $x \in \mathcal{H}$, $\|x\| = 1$, and $y = \frac{1}{\sqrt{n}}(x \oplus \cdots \oplus x)$ be a unit vector $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Then, we have

$$\begin{aligned} 1 &\leqslant |\langle R_n y, y\rangle| \\ &= \frac{n-1}{n}|\langle Tx, x\rangle|, \end{aligned}$$

meaning $w(t) \leqslant \frac{n}{n-1}$ for all n, meaning $w(T) \leqslant 1$. □

If $w(T) \leqslant 1$, we may extend the functional calculus from the circle to the disk algebra, $A(\mathbb{D})$.

Corollary: Let $T \in \mathbb{B}(\mathcal{H})$ with $w(T) \leqslant 1$. Let $f \in A(\mathbb{D})$ with $f(0) = 0$. Then, $w(f(T)) \leqslant \|f\|$.

Proof. It is sufficient to assume that f is a polynomial, and $\|f\| \leqslant 1$. □

Dilations

We saw our first example of a dilation theorem earlier in our first proof of von Neumann's inequality when we showed that if T is contractive, there is some projection P from $\mathcal{K} \supseteq \mathcal{H}$ and some unitary $U \in \mathbb{B}(\mathcal{K})$ such that $T^n = PU^n|_{\mathcal{H}}$.

Now, we will show an incredibly powerful result that characterizes all the completely positive maps, known as Stinespring's dilation theorem.

Theorem (Stinespring's Dilation)**:** Let A be a unital C*-algebra, and let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be a completely positive map. Then, there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi\colon \mathbb{B}(\mathcal{K})$, and a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|_{\text{op}}^2$, such that

$$\phi(a) = V^*\pi(a)V$$

for all $a \in A$.

Proof. Consider the algebraic tensor product $A \otimes \mathcal{H}$, and define the symmetric bilinear map $\langle\cdot,\cdot\rangle$ on the space by setting

$$\langle a \otimes x, b \otimes y\rangle = \langle \phi(b^*a)x, y\rangle_{\mathcal{H}},$$

and extending linearly.

Since we have

$$\left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right\rangle = \left\langle \phi_n\left(\left(a_i^* a_j\right)_{ij}\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}}$$
$$\geqslant 0,$$

and ϕ is completely positive, we have that $\langle a \otimes x, b \otimes y\rangle$ is a positive semidefinite bilinear form.

Since positive semidefinite bilinear forms satisfy the Cauchy–Schwarz inequality, $|\langle u, v\rangle| \leqslant \langle u, u\rangle\langle v, v\rangle$, we may define the "null set"

$$N := \{u \in A \otimes \mathcal{H} \mid \langle u, u\rangle = 0\}$$

as a subspace of $A \otimes \mathcal{H}$. The induced inner product on $(A \otimes \mathcal{H})/N$ is

$$\langle u + N, v + N\rangle = \langle u, v\rangle.$$

Remark: The construction here is very similar to the GNS construction.

We will let $\mathcal{K}$ be the completion of $(A \otimes \mathcal{H})/N$.

Now, if $a \in A$, define $\pi(a)\colon A \otimes \mathcal{H} \to A \otimes \mathcal{H}$ by

$$\pi(a)\left(\sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i=1}^{n} (a a_i) \otimes x_i.$$

We begin by showing that

$$\left(a_i^* a^* a a_j\right)_{ij} \leqslant \|a^* a\| \left(a_i^* a_j\right)_{ij},$$

where the inequality is in $\operatorname{Mat}_n(A)_+$. This follows from the fact that

$$\begin{aligned}
\left(a_i^* a^* a a_j\right)_{ij} &= \left(a I_n \left(a_{ij}\right)_{ij}\right)^* \left(a I_n \left(a_{ij}\right)_{ij}\right)\\
&\leqslant \|(a I_n)^* a I_n\| \left(a_i^* a_j\right)_{ij}\\
&= \|a^* a\| \left(a_i^* a_j\right)_{ij},
\end{aligned}$$

where the last line follows from the fact that for any elements of a C*-algebra, a, b, we have $0 \leqslant b^* a^* a b \leqslant \|a^* a\| b^* b$.

Now, this gives

$$\begin{aligned}
\left\langle \pi(a)\left(\sum_{j=1}^{n} a_j \otimes x_j\right), \pi(a)\left(\sum_{i=1}^{n} a_i \otimes x_i\right) \right\rangle &= \sum_{i,j=1}^{n} \left\langle \pi\left(a_i^* a^* a a_j\right) x_j, x_i \right\rangle_{\mathcal{H}}\\
&\leqslant \|a^* a\| \sum_{i,j=1}^{n} \left\langle \phi\left(a_i^* a_j\right) x_j, x_i \right\rangle\\
&= \|a\|^2 \left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right\rangle.
\end{aligned}$$

Thus, $\pi(a)$ vanishes on N, meaning it induces a quotient map that we will write as $\overline{\pi}(a)$. The above inequality shows that $\overline{\pi}(a)$ is bounded, with $\|\overline{\pi}(a)\| \leqslant \|a\|$. Thus, $\overline{\pi}(a)$ extends to a bounded linear operator on $\mathcal{K}$, denoted $\widetilde{\pi}(a)$.

Now, the map $\widetilde{\pi}\colon A \to \mathbb{B}(\mathcal{K})$ is a unital $*$-homomorphism.

Define $V\colon \mathcal{H} \to \mathcal{K}$ by $V(x) = 1 \otimes x + N$. Then, since

$$\begin{aligned}\|Vx\|^2 &= \langle 1 \otimes x, 1 \otimes x\rangle\\ &= \langle \phi(1)x, x\rangle_{\mathcal{H}}\\ &\leqslant \|\phi(1)\|\|x\|^2,\end{aligned}$$

V is bounded. Furthermore,

$$\begin{aligned}\|V\|_{op}^2 &= \sup_{x \in B_{\mathcal{H}}} \langle \phi(1)x, x\rangle\\ &= \|\phi(1)\|.\end{aligned}$$

Finally, we see that

$$\begin{aligned}\langle V^*\widetilde{\pi}(a)Vx, y\rangle &= \langle (\pi(a)1) \otimes x, 1 \otimes y\rangle_{\mathcal{H}}\\ &= \langle \phi(a)x, y\rangle_{\mathcal{H}},\end{aligned}$$

so that $V^*\widetilde{\pi}(a)V = \phi(a)$. □

There are some remarks to be made. First, any map of the form $\phi(a) = V^*\pi(a)V$ is already completely positive, so Stinespring's dilation is a complete characterization of completely positive maps from any C*-algebra into any $\mathbb{B}(\mathcal{H})$. Furthermore, if ϕ is unital, then V is an isometry, and we may identify $\mathcal{H}$ with $V\mathcal{H} \subseteq \mathcal{K}$. This identification gives V^* as the projection of $\mathcal{K}$ onto $\mathcal{H}$, or $P_{\mathcal{H}}$. Thus,

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}.$$

If $T \in \mathbb{B}(\mathcal{K})$, then $P_{\mathcal{H}}T|_{\mathcal{H}}$ is called the compression of T to $\mathcal{H}$. We may decompose $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, and consider T as the 2×2 operator matrix whose compression is equal to the $(1,1)$ entry of the operator matrix. Thus, Stinespring's dilation shows that every completely positive map into $\mathbb{B}(\mathcal{H})$ is the compression to $\mathcal{H}$ of a $*$-homomorphism into a Hilbert space that contains $\mathcal{H}$.

Additionally, Stinespring's dilation is a generalization of the GNS construction, which was used to convert from states to representations of C*-algebras as subalgebras of $\mathbb{B}(\mathcal{H})$. In particular, if $\mathcal{H} = \mathbb{C}$, then the isometry $V\colon \mathbb{C} \to \mathcal{K}$ is determined by $V(1) = x$< and>

$$\begin{aligned}\phi(a) &= \phi(a)(1) \cdot 1\\ &= V^*\pi(a)V(1) \cdot 1\\ &= \langle \pi(a)V(1), V(1)\rangle_{\mathcal{K}}\\ &= \langle \pi(a)x, x\rangle.\end{aligned}$$

Furthermore, if we reread the proof with $\mathcal{H} = \mathbb{C}$ and $A \otimes \mathbb{C} = A$, we recover the proof of the GNS representation of states.

Finally, if $\mathcal{H}$ and A are separable, then so too is $\mathcal{K}$, and if $\mathcal{H}$ and A are finite-dimensional, then so too is $\mathcal{K}$.

Now, we turn our attention to the uniqueness of the Stinespring representations, $(\pi, V, \mathcal{K})$.