Homotopies and the Fundamental Group

The fundamental goal of algebraic topology (and really topology in general) is to determine when two topological spaces X and Y are homeomorphic to each other. We will define a homeomorphism in a more uniform way for what we will discuss later; two topological spaces X and Y are homeomorphic if there are maps

$$f\colon X \to Y$$
$$g\colon Y \to X$$

such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

There is one issue though: this is a very hard problem. Most of the time, we restrict our view in some fashion or the other. For instance, differential topology deals with smooth manifolds and tries to understand equivalence through diffeomorphisms.

We unfortunately cannot do this for the general case, so we will instead try to relax the conditions on the functions f and g as follows.

Definition: A homotopy is a continuous map

$$F\colon X \times [0,1] \to Y$$

with $F(\cdot, t)\colon X \to Y$ a continuous map. We say two maps f_0 and f_1 are *homotopic* if there is a homotopy between them, and we write $f_0 \simeq f_1$.

We will write $[0,1]$ and I interchangeably.

Example: Consider the maps $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$, taking $x \mapsto x$, and $c\colon \mathbb{R}^n \to \mathbb{R}^n$, taking $x \mapsto 0$.

Then, id and c are homotopic via the homotopy

$$F(x,t) = (1-t)x.$$

Homotopies that fix subspaces are often quite useful.

Definition: Let $F\colon X \times [0,1] \to Y$ be a homotopy. We say that $f_0 = F(\cdot, 0)$ and $f_1 = F(\cdot, 1)$ are homotopic relative to $A \subseteq X$ if $F|_{A \times I}$ is constant. We write $f_0 \simeq f_1$ rel A.

Example: If $f_0 = \mathrm{id}$ on $\mathbb{R}^n$ and f_1 is given by

$$f_1(x) = \begin{cases} x & x \in D^n \\ \frac{x}{\|x\|} & x \in \mathbb{R}^n \setminus D^n \end{cases},$$

where D^n denotes the closed unit ball in $\mathbb{R}^n$, then we have that $f_0 \simeq f_1$ rel D^n.

Definition: We say two spaces X and Y are *homotopy equivalent* if there are continuous maps

$$f\colon X \to Y$$
$$g\colon Y \to X$$

such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In this case, we say X and Y have the same homotopy type, and write $X \simeq Y$.

The function g is called a homotopy inverse to f.

Example: If we let $f\colon S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$, then $g\colon \mathbb{R}^2 \setminus \{0\} \to S^1$ given by $x \mapsto \frac{x}{\|x\|}$ is a homotopy inverse to f. Though $\mathbb{R}^2 \setminus \{0\}$ and S^1 are of different dimensions (as manifolds), they are still homotopy equivalent, in particular meaning that homotopy equivalence is strictly weaker than homeomorphism.

Definition: A *deformation retraction* of a space X onto $A \subseteq X$ is a homotopy rel A from id_X to some map $r\colon X \to X$ with $r(X) = A$.

The map r, where $r|_A = \mathrm{id}_A$ and $r(X) = A$, is called a retraction of X onto A, and we call A a retract of X.

There are retractions that are not deformation retractions. For instance, if we let $r\colon S^1 \to S^1$ be defined by $(x, y) \mapsto (1, 0)$, then we have that $r(S^1) = A = \{(1, 0)\}$, but there is no homotopy from r to id_{S^1}.

Definition: A space with the homotopy type of a point is called contractible.

In other words, there are maps $f\colon X \to *$ and $g\colon * \to X$ such that $f \circ g = \mathrm{id}_*$ and $g \circ f = \mathrm{id}_X$, where $*$ denotes the one point space.

We have shown that $\mathbb{R}^n$ is contractible via straight line homotopy, as well as D^n, but neither are homeomorphic for $n \geq 0$. However, S^n is not contractible.

The Structure of Cell Complexes

Now, we can refine our original goal; we now seek to classify spaces up to homotopy equivalence. However, this is still a very hard question, which is where we introduce algebra. The question now is whether we can create an algebraic object (group, ring, module, etc.) assigned to a space such that, if two spaces are homotopy equivalent, the associated algebraic objects are isomorphic.

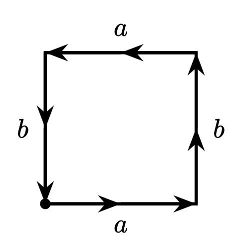
We call these objects *algebraic invariants*, and there are many of these objects. One of these we learned in differential topology was the de Rham cohomology for a differentiable manifold. However, we must note that these invariants lose information.[I] They are not *complete* invariants, but they are invariants nonetheless.

The two invariants we will discuss in this class are:

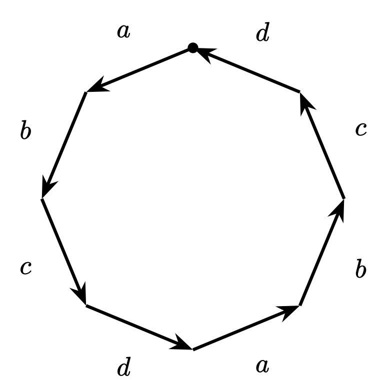
- the fundamental group, $\pi_1(X)$;
- and the homology groups, $H_n(X)$.

We must first introduce the primary setting we will examine these groups: cell complexes.

Consider the one-holed and two-holed tori. They can be created from quotients of a square and an octagon as follows respectively.



One-holed Torus



Two-holed Torus

[I]For instance, the Poincaré Lemma from differential topology shows that S^n and $\mathbb{R}^d \times S^n$ have the same cohomology groups, and so are homotopy equivalent, but they are most certainly not homeomorphic or diffeomorphic to each other.

We start with a discrete set of points, which are known as 0-cells and form the 0-skeleton, denoted X^0. Inductively, we form the n-skeleton, written X^n, from the space X^{n-1} by attaching n-cells (n-dimensional disks), denoted D^n_α, via attaching maps $\varphi_\alpha\colon S^{n-1}_\alpha \to X^{n-1}$, and forming via

$$X^n = X^{n-1} \sqcup \left(\bigsqcup_\alpha D^n_\alpha / x \sim \varphi_\alpha(x)\right).$$

This process may terminate at some n, or it continues to infinity, in which case we write $X = \bigcup X^n$. If $X = X^n$ for some n, the smallest such n is called the *dimension* of X. Sometimes we may denote the n-cells by e^n.

We also refer to these types of spaces as CW complexes. Note that all CW complexes are Hausdorff, so in particular, they cannot be the backbone of every topological space.

Example: The space S^n can be formed as an n-dimensional CW complex in two different ways. Either we may take one 0-cell and attach an n-cell via the constant map, or take two 0-cells, and inductively attach two n-cells to S^{n-1}.

Example: A more intricate example is that of $\mathbb{RP}^n$, real projective space in n-dimensions, which is given by

$$\mathbb{RP}^n = S^n / x \sim -x.$$

The cell decomposition of $\mathbb{RP}^n$ is given by a quotient of the second method we discussed for constructing S^n, by identifying "opposite cells" of dimension n. It then follows that

$$\mathbb{RP}^n \cong e^0 \cup e^1 \cup \cdots \cup e^n$$

with satisfactory attaching maps.

Example: Similarly, as was discussed at the end of differential topology, it can be shown that complex projective space, $\mathbb{CP}^n$, is given by

$$\mathbb{CP}^n \cong e^0 \cup e^2 \cup \cdots \cup e^{2n}.$$

Definition: If D^n_α is a cell in a cell complex X, then there is a *characteristic map*

$$\Phi_\alpha\colon D^n_\alpha \to X$$

such that $\Phi_\alpha|_{S^{n-1}_\alpha} = \varphi_\alpha$ and $\Phi^\alpha|_{(D^n_\alpha)^\circ}$ is a homeomorphism onto its range. That is, Φ_α is the composite given by

$$\begin{aligned} D^n_\alpha &\overset{\iota}{\hookrightarrow} X^{n-1} \sqcup \left(\bigsqcup_\alpha D^\alpha_n\right) \\ &\xrightarrow{/\sim} X^n \\ &\overset{\iota}{\hookrightarrow} X. \end{aligned}$$

Point-Set Constructions with Cell Complexes

Definition: A sub-complex of a cell complex X is a closed subspace $A \subseteq X$ that is a union of cells of X. The pair (X, A) is called a CW-pair.

Note that by "closed subspace" of X, we mean that all the attaching maps of A have their images contained in A.

Example:

(i) If we use the construction

$$S^n = 2e^n \cup 2e^{n-1} \cup \cdots \cup 2e^0,$$

then we have that $S^i \subseteq S^n$ for each $0 \leq i \leq n$ are subcomplexes. This is not the case with the $e^0 \cup e^n$ structure for S^n.

(ii) The only subcomplexes of $\mathbb{RP}^n$ are the $\mathbb{RP}^i$ with $0 \leq i \leq n$.

(iii) Similarly, the only subcomplexes of $\mathbb{CP}^n$ are the $\mathbb{CP}^i$ for $0 \leq i \leq n$.

Next, given two cell complexes X and Y, we can obtain a cell complex structure on the product given by the products of cells in X with cells in Y. Attaching maps are defined to be the restrictions of the products of the corresponding characteristic maps.

If we are given a CW-pair (X, A), the quotient X/A is a cell complex as follows:

- there is a 0-cell corresponding to everything in A, and all other cells are those of $X \setminus A$;
- given an n-cell $e_\alpha^n \in X \setminus A$, the attaching map $\varphi_\alpha \colon S_\alpha^{n-1} \to X^{n-1}$ gives rise to the attaching map

$$\overline{\varphi_\alpha} \colon S_\alpha^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}.$$

Definition: Given a topological space X, the suspension SX is defined as

$$SX = (X \times I/X \times \{1\})/X \times \{0\}.$$

The cone CX is defined as

$$CX = X \times I/X \times \{1\}.$$

Pointed Spaces

In order to introduce the fundamental group, we need to discuss a more detailed type of space, known as a pointed space. Pointed spaces are just what they say they are — topological spaces with a specified basepoint.

If X is a cell complex, we will strategically choose the base-point to be one of the 0-cells, so that the pair (X, x_0) is a CW-pair in addition to being a regular pointed space.

As it stands, many of our traditional constructions need to be modified for pointed spaces so as to preserve the basepoint.

Example: If (X, x_0) and (Y, y_0) are pointed spaces, we want to specify the basepoint of their disjoint union, $X \sqcup Y$, in such a fashion that preserves these basepoints. As it stands, there is no canonical choice, so we make an identification.

The *wedge product* of two pointed spaces (X, x_0) and (Y, y_0) is the space

$$X \vee Y = X \sqcup Y/x_0 \sim y_0.$$

Over any indexing set α, we may define

$$\bigvee_\alpha X_\alpha = \bigsqcup_\alpha X_\alpha/(x_\alpha \sim x_\beta)$$

for all α, β.

There is a similar process for products in order to ensure we have a canonical basepoint.

Definition: Let (X, x_0) and (Y, y_0) be pointed spaces. The *smash product* is the space

$$X \wedge Y = X \times Y/X \vee Y.$$

Note that $X \wedge Y$ need not be homeomorphic to $X \times Y$.