Week 1

Problem (Exercise 1.17)**:** In this exercise, you will show that the moments of a standard Gaussian variable count pair partitions.

(i) Let X be a standard Gaussian variable. Prove that

$$\begin{aligned}\mathbb{E}\left(X^{2k}\right) &= (2k-1)!!\\ \mathbb{E}\left(X^{2k-1}\right) &= 0.\end{aligned}$$

(ii) Prove that $|P_2(2k)| = (2k-1)!!$ by putting $P_2(2k)$ in explicit bijection with a set of cardinality

$$(2k-1)|P_2(2k-2)|.$$

Remark: I had done the first part of this exercise earlier in a separate notes document, but had not written it up here for submission.

Solution:

(i) We see that

$$\begin{aligned}E[Z^m] &= \int_{-\infty}^{\infty} x^m e^{-x^2/2}\,dx\\ &= -x^m e^{-x^2/2}\Big|_{-\infty}^{\infty} + (m-1)\int_{-\infty}^{\infty} x^{m-2}e^{-x^2/2}\,dx\\ &= (m-1)E\left[Z^{m-2}\right].\end{aligned}$$

Therefore, we recover the recursion relation for $(2k-1)!!$ whenever $m = 2k$ and 0 otherwise.

(ii) Considering the set $[2k] = \{1, 2, \dots, 2k\}$, we see that there are $2k-1$ ways to pair 1 with any other element, and there are then $P_2(2k-2)$ pair partitions of the remaining $2k-2$ elements. This gives our desired bijection.

Week 2

Problem (Exercise 2.23):

(i) Find a recursion for $|NC_2(2k)|$.

(ii) Show that $\operatorname{Cat}(k) = \frac{1}{k+1}\binom{2k}{k}$ satisfies the same recursion relation shown in (i).

(iii) Let $d\mu = f(x)\,dx$, where

$$f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2} & -2 \leq x \leq 2 \\ 0 & \text{else} \end{cases}.$$

Show that

$$\int_{\mathbb{R}} x^{2k}\,d\mu = \operatorname{Cat}(k)$$
$$\int_{\mathbb{R}} x^{2k-1}\,d\mu = 0.$$

Solution:

(i) To find a recursion for $NC_2(2k)$, we start by counting the number of valid pairings of 1 with any other element of $[2k]$. For this, we observe that there must be an even number of elements between 1 and whatever it is paired with, meaning there are k valid pairings of 1.

For each of these pairings, there are two separate "sub-blocks" that we use to count the non-crossing partitions, the ones "between" $\{1, 2\ell\}$as ℓ ranges from 1 to k, and the ones "outside" the pairing $\{1, 2\ell\}$. Combined, this gives the recurrence relation

$$|NC_2(2k)| = \sum_{i=1}^{k} |NC_2(2i-2)||NC_2(2k-2i)|,$$

where $NC_2(0) = 1$ vacuously.

(ii) To evaluate the proposed expression for $C(z)$, we see that

$$\begin{aligned}
\frac{1}{2z}\left(1-\sqrt{1-4z}\right) &= \frac{1}{2z}\left(1-\sum_{k=0}^{\infty}\binom{1/2}{k}(-4z)^k\right)\\
&= \frac{1}{2z}\left(-\sum_{k=1}^{\infty}\left(\frac{1}{k!}\prod_{i=0}^{k}\left(\frac{1}{2}-i\right)\right)(-1)^k 2^{2k} z^k\right)\\
&= \frac{1}{2z}\left(-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-3)!!}{2^k k!}(-1)^k 2^{2k} z^k\right)\\
&= \frac{1}{2z}\left(\sum_{k=0}^{\infty}\frac{2^{k+1}(2k-1)!!}{(k+1)!}z^{k+1}\right)\\
&= \frac{1}{2z}\left(\sum_{k=0}^{\infty}\frac{2^{k+1}(2k)!}{2^k k!(k+1)!}z^{k}\right)\\
&= \sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}z^k\\
&= \sum_{k=0}^{\infty}\operatorname{Cat}(k)z^k.
\end{aligned}$$

By plugging the proposed functional equation into the closed-form expression into $C(z)$, we get

$$\begin{aligned}
\frac{1-\sqrt{1-4z}}{2z} &= 1 + z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^2\\
&= 1 + z\left(\frac{1+(1-4z)-2\sqrt{1-4z}}{4z^2}\right)\\
&= \frac{1-\sqrt{1-4z}}{2z}.
\end{aligned}$$

Yet, we also observe that

$$\begin{aligned}
1 + zC(z)^2 &= z\left(\sum_{k=0}^{\infty}\operatorname{Cat}(k)z^k\right)\left(\sum_{\ell=0}^{\infty}\operatorname{Cat}(\ell)z^\ell\right)\\
&= 1 + z\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\operatorname{Cat}(k)\operatorname{Cat}(k-\ell)z^k\\
&= 1 + \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\operatorname{Cat}(k)\operatorname{Cat}(k-\ell)z^{k+1}\\
&= 1 + \sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\operatorname{Cat}(k-1)\operatorname{Cat}(k-\ell)z^k\\
&= \sum_{k=0}^{\infty}\operatorname{Cat}(k)z^k,
\end{aligned}$$

so that $\operatorname{Cat}(k)$ satisfies the same recurrence relation as the one for the noncrossing pair partitions.

(iii) The fact that

$$\begin{aligned}
\int_{\mathbb{R}} x^{2k-1}\,d\mu &= \frac{1}{2\pi}\int_{-2}^{2} x^{2k-1}\sqrt{1-4x^2}\,dx\\
&= 0,
\end{aligned}$$

follows from the fact that this is an odd integrand over a symmetric interval. Else, we find

$$\begin{aligned}
\frac{1}{2\pi}\int_{-2}^{2} x^{2k}\sqrt{4-x^2}\,dx &= \frac{1}{2\pi}\int_{0}^{\pi} 2^{2k+2}\sin^2(\theta)\cos^{2k}(\theta)\,d\theta\\
&= 2^{2k+1}\left(\frac{(2k-1)!!}{(2k)!!} - \frac{(2k+1)!!}{(2k+2)!!}\right)\\
&= 2^{2k+1}\left(1-\frac{2k+1}{2k+2}\right)\frac{(2k-1)!!}{(2k)!!}\\
&= 2^{2k+1}\frac{1}{2k+2}\frac{(2k-1)!!}{(2k)!!}\\
&= \frac{(2k)!}{(k+1)!k!}\\
&= \frac{1}{k+1}\binom{2k}{k}\\
&= \operatorname{Cat}(k).
\end{aligned}$$

Week 3

Problem (Exercise 4.12)**:** A C^*-probability space is a $*$-probability space $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a C^*-algebra and φ is a state.

(i) Let $(\mathcal{A}, \varphi)$ be a C^*-probability space, and suppose $a \in \mathcal{A}$ is normal. Prove that there is a unique compactly supported measure μ such that

$$\varphi(a^p(a^*)^q) = \int_{\mathbb{C}} z^p \overline{z}^q \, d\mu.$$

(ii) Let $(\mathcal{A}, \varphi)$ be a C^*-probability space, and let $\{\mathcal{A}_i\}_{i\in I}$ be a family of freely independent $*$-subalgebras of $\mathcal{A}$. For each i, let $\mathcal{B}_i$ be the norm closure of $\mathcal{A}_i$. Show that $\{\mathcal{B}_i\}_{i\in I}$ are freely independent.

Solution:

(i) Let (π_φ, H_φ) be the GNS representation of $\mathcal{A}$ with respect to φ, admitting a cyclic vector ξ_φ. Let $T = \pi_\varphi(a)$ admit spectral measure E such that

$$T = \int_{\sigma(T)} z \, dE.$$

Following from the functional calculus, we observe that for any bounded Borel function ψ on $\sigma(T)$, we have

$$\psi(T) = \int_{\sigma(T)} \psi \, dE.$$

In particular, this yields

$$T^p(T^*)^1 = \int_{\sigma(T)} z^p \overline{z}^q \, dE.$$

Now, define the measure μ to be the unique measure (emerging from the Riesz Representation Theorem) such that

$$\left\langle \left(\int_{\sigma(T)} \psi \, dE \right) \xi_\varphi, \xi_\varphi \right\rangle = \int_{\sigma(T)} \phi \, d\mu.$$

Then, from the definition of the cyclic vector in the GNS construction, we get

$$\begin{aligned} \varphi(a^p(a^*)^q) &= \langle \pi_\varphi(a^p(a^*)^q)\xi_\varphi, \xi_\varphi \rangle \\ &= \langle T^p(T^*)^q \xi_\varphi, \xi_\varphi \rangle \\ &= \int_{\sigma(T)} z^p \overline{z}^q \, d\mu. \end{aligned}$$

Since the spectrum is compact, μ is our desired compactly supported measure.

(ii) Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of freely independent $*$-subalgebras of $\mathcal{A}$. We wish to show that for $a_{i_j} \in B_{i_j}$ with $i_j \neq i_{j+1}$, we have

$$\varphi(a_{i_1} \cdots a_{i_k}) = 0.$$

For this, we see that for each a_{i_j}, there is a sequence $\left(a_{i_j}(m)\right)_{m\in M} \subseteq A$ converging to a_{i_j}. Since φ is continuous, this gives

$$\begin{aligned} \varphi(a_{i_1} \cdots a_{i_k}) &= \varphi\left(\lim_{m\to\infty} a_{i_1}(m) \cdots a_{i_k}(m) \right) \\ &= \lim_{m\to\infty} \varphi(a_{i_1}(m) \cdots a_{i_k}(m)) \\ &= 0. \end{aligned}$$

Thus, the family $\{\mathcal{B}_i\}_{i\in I}$ is freely independent.