What follows will require a solid command of the continuous functional calculus.

The Positive Cone of a C^*-Algebra

Definition: Let A be a C^*-algebra. We say $a \in A$ is *positive* if $a \in A_{\text{s.a.}}$ and $\sigma(a) \subseteq \mathbb{R}_+$. We write $a \geq 0$, and say $a \in A_+$.

For example, an element $f \in C(X)$ is positive if and only if $f(x) \geq 0$ for all $x \in X$, and $\phi \in L_\infty(\mu)$ is positive if and only if $\phi(x) \geq 0$ for μ-a.e. x.

Proposition: If $a \in A_+$, then there are unique positive elements u and v in A such that $a = u - v$, $uv = vu = 0$.

Proof. Letting $f(t) = \max(0, t)$ and $g(t) = -\min(0, t)$, we have $f, g \in C(\mathbb{R})_+$ with $f(t) - g(t) = t$. Using the continuous functional calculus, let $u = f(a)$ and $v = g(a)$. Spectral mapping gives that $u, v \in A_+$ with $a = u - v$ and $uv = vu = 0$. $\square$

Lemma: If $a \in A$ is a C^*-algebra, $t \in \mathbb{R}$ with $\|a\| \leq t$, then $a \in A_+$ if and only if $\|a - t1\| \leq t$.

Proof. Since $\sigma(a) \subseteq [-t, t]$, and $a, 1$ are self-adjoint, we have

$$\|a - t1\| = r(a - t1)$$
$$= \sup_{s \in \sigma(a)} |s - t|$$
$$= \sup_{s \in \sigma(a)} (t - s),$$

so we have $\|a - t1\| \leq t$ if and only if $\sigma(a) \subseteq [0, \infty)$. $\square$

Theorem: Suppose A is a C^*-algebra.

Proposition: Let A be a C^*-algebra, $a \in A$. The following are equivalent:

(i) $a \in A_+$

(ii) there is $v \in A_+$ with $a = v^2$

(iii) there is $b \in A$ with $a = b^*b$.

Proof. To see that (i) implies (ii), we use the function $f(t) = t^{1/2}$ and apply the continuous functional calculus, which is well-defined since $\sigma(a) \subseteq [0, \infty)$. Similarly, since v is self-adjoint, it follows that (ii) implies (iii).

Now, if $a = b^*b$ for $b \in A$, we have the decomposition $a = a_+ - a_-$ as above. If we let $C = ba_-$, then

$$c^*c = a_- b^* b a_-$$
$$= a_-(a_+ - a_-)a_-$$
$$= -(a_-)^3.$$

Since $a_- \in A_+$, and $\sigma(a_-) = \{t^3 \mid t \in \sigma(a_-)\}$, it follows that $-c^*c = (a_-)^3$ is in A_+. Yet, this means that $c = 0$, so that $a = a_+ \in A_+$ $\square$

The element v such that $v^2 = a$ is known as the square root of a. We define the absolute value of an element a in a C^*-algebra by $|a| = (a^*a)^{1/2}$.

Corollary: If A is a C^*-algebra, $a \in A_+$, and $b \in A$, then $b^*ab \in A_+$.

Proof. We have that $b^*ab = (a^{1/2}b)^* a^{1/2}b$. $\square$

Proposition: The set A_+ is a norm-closed cone in A.

Proof. Let $A_+ \supseteq (a_n)_n \to a \in A$. Then, $a \in A_{\text{s.a.}}$, and we have from above that, since $\sigma(a_n) \subseteq [0, \|a_n\|]$,

$$\|a_n - \|a_n\|1\| \leq \|a_n\|,$$

so that

$$\|a - \|a\|1\| \leq \|a\|,$$

meaning that $a \geq 0$. It follows from spectral mapping that if $a \geq 0$ and $t \geq 0$, then $ta \geq 0$.

Finally, we must show that if $a, b \geq 0$, then so is $a + b$. It suffices to show that $\|a\|$ and $\|b\|$ are both less than or equal to 1. Yet, we then have

$$\left\|1 - \frac{1}{2}(a + b)\right\| = \frac{1}{2}\|(1 - a) + (1 - b)\|$$
$$\leq 1,$$

so we have $\frac{1}{2}(a + b) \geq 0$.

Finally, if $a \in A_+ \cap (-A_+)$, then $\sigma(a) = 0$, so $r(a) = 0$, but $r(a) = \|a\|$ since a is self-adjoint, and thus normal. $\qquad\square$

The fact that A_+ is a closed cone in A allows us to define an order on $A_{\text{s.a.}}$ given by $a \leq b$ if $b - a \in A_+$.

Note that an element $a \in A_+$ is invertible if and only if $a \geq t1$ for some $t > 0$. This follows from the fact that $a \geq t1$ if and only if $a - t1 \in A_+$; by spectral mapping, this means that $\sigma(a - t1) = \{\lambda - t \mid \lambda \in \sigma(a)\}$, which occurs only if $\sigma(a) \subseteq [t, \infty)$. In particular, this means that $0 \notin \sigma(a)$, so that a is invertible.

Proposition: Let $a, b \in A_{\text{s.a.}}$. The following hold:

(i) if $-b \leq a \leq b$, then $\|a\| \leq \|b\|$;

(ii) if $0 \leq a \leq b$, then $a^{1/2} \leq b^{1/2}$;

(iii) if $0 \leq a \leq b$ and a is invertible, then b is invertible and $b^{-1} \leq a^{-1}$.

Proof.

(i) Since $-\|b\|1 \leq -b \leq a \leq b \leq \|b\|1$, it follows that $\|a\| \leq \|b\|$.

(ii) From $(*)$ in the proof of (iii),

$$\left\|b^{-1/4}a^{1/2}b^{-1/4}\right\| = r\left(b^{-1/4}a^{1/2}b^{-1/4}\right)$$
$$= r\left(a^{1/2}b^{-1/4}b^{-1/4}\right)$$
$$\leq \left\|a^{1/2}b^{-1/2}\right\|$$
$$\leq 1,$$

meaning that $b^{-1/4}a^{1/2}b^{-1/4} \leq 1$, and thus $a^{1/2} \leq b^{1/2}$. This shows the case where a is invertible

If a is not invertible, then for any $\varepsilon > 0$, we have that $(a + \varepsilon 1)^{1/2} \leq (b + \varepsilon 1)^{1/2}$. For some $c \in A_{\text{s.a.}}$, define $f_\varepsilon \in C(\sigma(c))$ by $f_\varepsilon(t) = (t + \varepsilon)^{1/2}$, meaning that $f_\varepsilon(c) \in A_+$ and $(f_\varepsilon(c))^2 = c + \varepsilon 1$. In particular, this means that $(c + \varepsilon 1)^{1/2} = f_\varepsilon(c)$, and since $f_\varepsilon(t) \to t^{1/2}$ uniformly as $\varepsilon \to 0$, we have that $\left\|(c + \varepsilon 1)^{1/2} - c^{1/2}\right\| \to 0$, so we get $a^{1/2} \leq b^{1/2}$ in the general caseomte.

(iii) If $0 \leq a \leq b$ and a is invertible, then $a \geq t1$ for some $t > 0$, so $b \geq t1$ meaning b is invertible. Moreover, we see that

$$0 \leq b^{-1/2}ab^{-1/2}$$
$$\leq b^{-1/2}bb^{-1/2}$$
$$= 1,$$

so that $\left\|b^{-1/2}ab^{-1/2}\right\| \leq 1$. Therefore, we have

$$\left\|a^{1/2}b^{-1/2}\right\| = \left\|\left(a^{1/2}b^{-1/2}\right)^* a^{1/2}b^{-1/2}\right\|^{1/2} \qquad *$$

$$= \left\|b^{-1/2}ab^{-1/2}\right\|^{1/2}$$

$$\leq 1,$$

so we have

$$\left\|a^{1/2}b^{-1}a^{1/2}\right\| = \left\|a^{1/2}b^{-1/2}\left(a^{1/2}b^{-1/2}\right)^*\right\|$$

$$\leq 1,$$

so $a^{1/2}b^{-1}a^{1/2} \leq 1$, meaning $b^{-1} \leq a^{-1}$.

$\square$

We say a function like $f(t) = t^{1/2}$ defined on $S \subseteq \mathbb{R}$ is *operator-monotone* if $f(a) \leq f(b)$ whenever $a \leq b$ in $A_{\text{s.a.}}$ and $\sigma(a) \cup \sigma(b) \subseteq S$.

Ideals in C^*-Algebras

We seek to understand the structure of ideals in a C^*-algebra, which will allow us to establish a C^*-algebraic version of the first isomorphism theorem. We will also use some of these ideas when discussing approximate identities in C^*-algebras next section.

Proposition: If I is a closed left (or right) ideal in A, with $a \in I$ and $a = a^*$, then if $f \in C(\sigma(a))$ is such that $f(0) = 0$, then $f(a) \in I$.

Proof. We note that if I is proper, we must have $0 \in \sigma(a)$ since a cannot be invertible. Since $\sigma(a) \subseteq \mathbb{R}$, the Stone–Weierstrass theorem implies there is a sequence $(p_n(t))_n \to f(t)$ uniformly, so $p_n(0) \to f(0) = 0$. In particular, $q_n(t) = p_n(t) - p_n(0)$ converges to $f(t)$ uniformly on $\sigma(a)$ with $q_n(0) = 0$ for all n. Therefore, we see that $q_n(a) \in I$ for each n by the definition of an ideal, and $\|q_n(a) - f(a)\| \to 0$, so $f(a) \in I$. $\square$

Corollary: If I is a closed left or right ideal, and $a \in I$ with $a = a^*$, then $a_+, a_-, |a|, |a|^{1/2} \in I$.

Theorem: If I is a closed ideal in A, then I is self-adjoint.

Proof. Fix $a \in I$. Since I is an ideal, $a^*a \in I$. We will construct a sequence of continuous functions $(u_n)_n$ defined on $[0, \infty)$ such that $u_n(0) = 0$ and $u_n(t) \geq 0$ for all t, such that

$$\|au_n(a^*a) - a\| \to 0.$$

If such a sequence can be constructed, then $u_n(a^*a) \in I$, $u_n(a^*a) \in I$, and $u_n(a^*a)a^* \in I$ as I is an ideal, giving

$$\|u_n(a^*a)a^* - a^*\| = \|au_n(a^*a) - a\|$$

$$\to 0,$$

so that $a^* \in I$ whenever $a \in I$. We observe that

$$\|au_n(a^*a) - a\|^2 = \left\|(au_n(a^*a) - a)^*(au_n(a^*a) - a)\right\|$$

$$= \|u_n(a^*a)a^*au_n(a^*a) - a^*au_n(a^*a) - u_n(a^*a)a^*a + a^*a\|,$$

meaning that if $b = a^*a$, we have that $bu_n(b) = u_n(b)b$ implies that, if we set

$$f_n(t) = tu_n(t)^2 - 2tu_n(t) + t$$

$$= t(u_n(t) - 1)^2,$$

then we have $\|f_n(b)\| \leq \sup_{t \geq 0} |f_n(t)|$. If we set $u_n(t) = nt$ for $0 \leq t \leq \frac{1}{n}$ and $u(t) = 1$ for all $t \geq \frac{1}{n}$, then we have that

$$\sup_{t \geq 0} |f_n(t)| = \frac{4}{27n},$$

which gives our desired result. □

The function u_n actually gives us a sequence $(e_n)_n$ of positive elements corresponding to each a such that $e_1 \leq e_2 \leq \cdots$, $\|e_n\| \leq 1$, and $\|ae_n - a\| \to 0$. We will discuss a similar, more useful version in the following section.

Now, we discuss quotients of C^*-algebras.

Lemma: If I is an ideal in A, and $a \in A$, then

$$\|a + I\| = \inf\{\|a - ax\| \mid x \in I, x \geq 0, \|x\| \leq 1\}.$$

Proof. Let

$$(B_I)_+ = \{x \in (B_X \cap I) \cap A_+\}.$$

Then, we have that

$$\|a + I\| \leq \inf_{x \in (B_I)_+} \|a - ax\|.$$

Now, if $(e_n)_n$ is a sequence in $(B_I)_+$ such that $\|y - ye_n\| \to 0$ as $n \to \infty$, we have that $0 \leq 1 - e_n \leq 1$, meaning

$$\|(a + y)(1 - e_n)\| \leq \|a + y\|.$$

Hence,

$$\begin{aligned}
\|a + y\| &\geq \liminf_{n \to \infty} \|(a + y)(1 - e_n)\| \\
&= \liminf_{n \to \infty} \|a - ae_n + y - ye_n\| \\
&= \liminf_{n \to \infty} \|a - ae_n\|,
\end{aligned}$$

meaning that

$$\begin{aligned}
\|a + y\| &\geq \inf_n \|a - ae_n\| \\
&\geq \inf_{x \in (B_I)_+} \|a - ax\|.
\end{aligned}$$

This gives our desired result. □

Theorem: Let A be a C^*-algebra, and let I be a closed ideal of A. Defining $(a + I)^* = a^* + I$, we have that A/I with the quotient norm is a C^*-algebra.

Proof. Since A is a C^*-algebra, it is a Banach algebra, so since I is a closed subspace of A, it follows that A/I is a Banach algebra. The main thing we must prove is the C^*-identity. Since I is self-adjoint, it follows that

$$\|a^* + I\| = \|a + I\|$$

for all $a \in A$, so since A/I is a Banach algebra, we have

$$\begin{aligned}
\|a^*a + I\| &= \|(a^* + I)(a + I)\| \\
&\leq \|a^* + I\| \|a + I\| \\
&= \|a + I\|^2.
\end{aligned}$$

Meanwhile, from the lemma, we have

$$\begin{aligned}
\|a + I\|^2 &= \inf_{x \in (B_I)_+} \|a - ax\|^2 \\
&= \inf_{x \in (B_I)_+} \|a(1 - x)\|^2 \\
&= \inf_{x \in (B_I)_+} \|(1 - x)a^*a(1 - x)\| \\
&\leq \inf_{x \in (B_I)_+} \|a^*a(1 - x)\| \\
&= \inf_{x \in (B_I)_+} \|a^*a - a^*ax\| \\
&= \|a^*\|.
\end{aligned}$$

This gives our desired result. $\qquad\square$

Theorem: If A and B are C^*-algebras, and $\rho \colon A \to B$ is a $*$-homomorphism, then ρ is a contraction, and $\operatorname{im}(\rho)$ is closed in B. If ρ is injective, then ρ is an isometry.

Proof. It suffices to assume that A and B are unital (else we simply use the unitization), so we know that $\sigma(\rho(x)) \subseteq \sigma(x)$ for each x, so $r(\rho(x)) \subseteq r(x)$. This gives the case for all normal elements, and by the C^*-identity, we get

$$\begin{aligned}
\|\rho(a)\|^2 &= \|\rho(a^*a)\| \\
&= r(\rho(a^*a)) \\
&\leq r(a^*a) \\
&= \|a\|^2.
\end{aligned}$$

Since ρ is a contraction, it follows that $I = \ker(\rho)$ is a closed ideal, and by the first isomorphism theorem, $\bar{\rho} \colon A/I \to B$ is injective, so it is an isometry, meaning $\operatorname{im}(\bar{\rho}) = \operatorname{im}(\rho)$ is closed. $\qquad\square$

Approximate Identities

Not all C^*-algebras have a unit (even though we can always pass to the unitization), but it is often useful to work with approximations of an identity without choosing to adjoin a unit.

Definition: If A is a C^*-algebra, a net $(e_i)_i$ is called an approximate identity for A if

(i) $0 \leq e_i \leq 1$ for each i;

(ii) $e_i \leq e_j$ whenever $i \leq j$;

(iii) $\lim_{i \in I} \|xe_i - x\| = 0$ for all $x \in A$.

If the net is a sequence, we call it a sequential approximate identity.

Lemma: If A is a C^*-algebra, $a, b \in A$ with $0 \leq a \leq b \leq 1$, then

$$\begin{aligned}
\|x - bx\|^2 &\leq \|x^*(1 - a)x\| \\
\|x - xb\|^2 &= \|x^*(1 - a)x\|
\end{aligned}$$

for all $x \in A$.

Proof. For any $x \in A$, we have

$$\begin{aligned}
\|x - bx\|^2 &= \|(1 - b)x\|^2 \\
&= \left\| x^*(1 - b)^2 x \right\| \\
&\leq \|x^*(1 - b)x\| \\
&\leq \|x^*(1 - a)x\|.
\end{aligned}$$

The other inequality follows similarly.   □

Theorem: Let I be a dense ideal of A. Letting

$$E = \{e \in I \mid 0 \leq e < 1\}$$

be equipped with the natural ordering, then E is directed and an approximate identity for A.

Proof. To establish that E is directed, let $a, b \in E$, and define

$$\psi(t) = 1 - (1+t)^{-1}$$
$$\phi(t) = t(1-t)^{-1},$$

where $\psi(t)$ is defined for all $t \geq 0$ and $\phi(t)$ is defined for $0 \leq t < 1$. Define $c = \phi(a) + \phi(b)$ and $d = \psi(c)$. For any $e \in E$, we then have

$$\phi(e) = e(1-e)^{-1}$$
$$\in I_+.$$

Therefore, $c \in I_+$ and $d = c(1+c)^2 \in I$. Since $0 \leq \psi(t) < 1$ for all $t \geq 0$, we have $d \in E$ by functional calculus. Now, if we let $x = \phi(a)$, then $x \leq c$, so $(1+x)^{-1} \geq (1+c)^{-1}$, and since $\psi(\phi(t)) = t$, we have

$$a = \psi(x)$$
$$= 1 - (1+x)^{-1}$$
$$\leq \psi(c)$$
$$= d,$$

where the inequality emerges from the fact that continuous functional calculus preserves order. Thus, d dominates both a and b, so E is directed.

For any $x \in I_+$, we define $a_n = \psi(nx)$, so $a_n \in E$, and take $\eta(t) = t^2(1 - \psi(nt)) = t^2(1+nt)^{-1} \leq t/n$. Therefore, $\eta(x) = x^2(1 - \psi(nx)) = x(1-a_n)x$, meaning

$$\|x(1-a_n)x\| \leq \|\eta\|_u$$
$$\leq \|x\|/n.$$

If $\varepsilon > 0$, then there is n large enough such that for any $e \in E$ with $e \geq a_n$,

$$\|x - xe\|^2 \leq \varepsilon,$$

so that

$$\lim_{e \in E} \|xe - x\| = 0$$

whenever $x \in I_+$.

For any $x \in A_+$, let $(y_n)_n \subseteq I_+$ such that $(y_n)_n \to x^{1/2}$. Then, $x_n = y_n^* y_n \to x$ with $x_n \in I_+$. Since E is bounded, the limit holds for any $x \in A_+$. Finally, since any element of A can be written as a linear combination of elements in A_+, this holds for all $x \in A$.   □

Corollary: A separable C^*-algebra has a sequential approximate identity.

Proof. Let $\{a_n\}_{n \geq 1}$ be a countable dense subset of A, $(e_i)_i$ an approximate identity. By inducting, we have a sequence $(i_n)_n$ such that $\|a_k e_i - a_k\| < 2^{-n}$ for $1 \leq k \leq n$ and $i \geq i_n$. If we let $x_n = e_{i_n}$, then for all $k \geq 1$, $\|a_k x_n - a_k\| \to 0$ as $n \to \infty$, and since $(x_n)_n$ is bounded, $\|ax_n - a\| \to 0$ for all $a \in A$.   □

References

[Con90]   John B. Conway. *A Course in Functional Analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.

[KR97]   Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*. Vol. 15. Graduate Studies in Mathematics. Elementary theory, Reprint of the 1983 original. American Mathematical Society, Providence, RI, 1997, pp. xvi+398. ISBN: 0-8218-0819-2. DOI: 10.1090/gsm/015. URL: https://doi.org/10.1090/gsm/015.

[Con00]   John B. Conway. *A course in operator theory*. Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: 10.1090/gsm/021. URL: https://doi.org/10.1090/gsm/021.