Math 395

Homework 7
Due: 4/18/2024

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Problem 1

We say a field K/F is normal if K is the splitting field of a collection of polynomials. Equivalently, every polynomial in $F[x]$ that has a root in K splits into linear factors over K. Let $\alpha \in \mathbb{R}$ such that $\alpha^4 = 5$. We will show that $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$, but $\mathbb{Q}(\alpha + i\alpha)$ is not normal over $\mathbb{Q}$.

Problem 3

For any prime p and any nonzero $a \in \mathbb{F}_p$, we will prove that $f(x) = x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$.

First, we have that $D_x(f(x)) = px^{p-1} - 1 = -1$, meaning that $\gcd(f(x), D_x(f(x))) = 1$, so f is separable.

Let α be a root of f. Then, we have that $\alpha^p - \alpha + a = 0$. Notice that for $j \in \mathbb{F}_p$, $(\alpha + j)^p = \alpha^p + j^p = \alpha^p + j$, meaning that $(\alpha + j)^p - (\alpha + j) + a = 0$, so $\alpha + j$ is a root of f.

Suppose toward contradiction that f is reducible over $\mathbb{F}_p$. Then, for some $\alpha \in \mathbb{F}_p$, we must have

$$x^p - x + a = (x - \alpha)(x - (\alpha + 1))(x - (\alpha + 2)) \cdots (x - (\alpha + p - 1)),$$

However, by definition, this means that there is some $k \in \mathbb{F}_p$ such that $\alpha + k = 0$, meaning $a = \prod_{i=0}^{p-1} (\alpha + i) = 0$. $\perp$

Problem 4

Let K be a finite extension of $\mathbb{Q}$. We will prove there are only a finite number of roots of unity in K.

Problem 6