**Abstract**

We discuss and prove some fundamental results about differentiation, after which prove the fundamental theorem of calculus for Lebesgue integrals.

# Preliminary

In our discussion of the Radon–Nikodym Theorem, we were able to define an abstract derivative of a ($\sigma$-finite) complex measure with respect to a different ($\sigma$-finite) measure. In Euclidean space, $\mathbb{R}^n$, we may consider trying to define a "pointwise" derivative by taking

$$F(x) = \lim_{r\to 0} \frac{\nu(U(x,r))}{m(U(x,r))},$$

where $m$ is the Lebesgue measure, and $\nu$ is our given complex measure. If we take the Lebesgue–Radon–Nikodym decomposition

$$d\nu = d\lambda + f\, dm,$$

we would hope that $F = f$ almost everywhere. Indeed, we will show this to be the case, after which we may prove a stronger version of the fundamental theorem of calculus, this time for Lebesgue integrals.

Note that from now on, every measure-theoretic term (i.e., integrable, almost everywhere, etc.) is taken with respect to the Lebesgue measure on $\mathbb{R}^n$.

We start with a fundamental lemma in measure theory for Euclidean spaces.

**Theorem** (Vitali Covering Lemma)**:** Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B\in\mathcal{C}} B$.

If $c < m(U)$, then there exist disjoint $B_1, \dots, B_k$ such that

$$3^{-n}c \leq \sum_{j=1}^{k} m(B_j).$$

*Proof.* By inner regularity, there is a compact $K \subseteq U$ such that $m(K) > c$; finitely many balls in $\mathcal{C}$, which we call $A_1, \dots, A_m$, cover $K$.

We proceed via exhaustion; select $B_1$ to be the largest of the $A_j$, $B_2$ to be the largest of the $A_j$ disjoint from $B_1$, $B_3$ the largest of the $A_j$ disjoint from $B_2$ and $B_1$, etc. According to this construction, if $A_i$ is not among the $B_j$, then there is $j$ such that $A_i \cap B_j \neq \emptyset$, and if $j$ is the smallest such index, then the radius of $A_i$ is at most that of $B_j$. Via some triangle inequality magic, we see that $A_i \subseteq B_j^*$, where $B_j^*$ is defined to the ball with the same center as $B_j$ and three times the radius.

Then, $K \subseteq \bigcup_{j=1}^{k} B_j^*$, so that

$$\begin{aligned} c &< m(K) \\ &\leq \sum_{j=1}^{k} m\left(B_j^*\right) \\ &= 3^n \sum_{j=1}^{k} m(B_j). \end{aligned}$$

$\square$

## The Lebesgue Differentiation Theorem

**Definition:** A function $f\colon \mathbb{R}^n \to \mathbb{C}$ is called *locally integrable* if $\int_K |f|\,dm < \infty$ for every bounded measurable $K \subseteq \mathbb{R}^n$.[I]

The space of locally integrable functions is denoted $L_{1,\text{loc}}$.

**Definition:** If $f \in L_{1,\text{loc}}$, and $x \in \mathbb{R}^n$, and $r > 0$, define

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy$$

to be the *average* of $f$ on $B(x,r)$.

**Lemma:** If $f \in L_{1,\text{loc}}$, then $A_r f$ is jointly continuous in $r$ and $x$.

*Proof.* We know that $m(B(x,r)) = cr^n$, where $c = m(B(0,1))$, and $m(S(x,r)) = 0$, where $S(x,r) = \{y \mid |y-x| = r\}$.

Moreover, as $r \to r_0$ and $x \to x_0$, $\mathbb{1}_{B(x,r)} \to \mathbb{1}_{B(x_0,r_0)}$ pointwise on $\mathbb{R}^n \setminus S(x_0,r_0)$, so the convergence is pointwise almost everywhere. Furthermore, note that $\left|\mathbb{1}_{B(x,r)}\right| \leq \mathbb{1}_{B(x_0,r_0+1)}$ for $r < r_0 + 1/2$ and $|x - x_0| < 1/2$. Thus, by dominated convergence, it follows that $\int_{B(x,r)} f(y)\,dy$ is continuous in $r$ and $x$, and so is $A_r f(x)$. $\square$

**Definition:** If $f \in L_{1,\text{loc}}$, we define the *Hardy–Littlewood Maximal Function*, $Hf$, by

$$\begin{aligned} Hf(x) &= \sup_{r>0} A_r|f|(x) \\ &= \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\,dy. \end{aligned}$$

**Theorem** (The Maximal Theorem)**:** There is a constant $C > 0$ such that for all $f \in L_1$ and all $\alpha > 0$,

$$m(\{x \mid Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)|\,dx.$$

*Proof.* Let $E_\alpha = \{x \mid Hf(x) > \alpha\}$. For each $x \in E_\alpha$, we may find $r_x > 0$ such that $A_{r_x}|f|(x) > \alpha$. The balls $U(x, r_x)$ cover $E_\alpha$, so by the Vitali Covering Lemma, if $c < m(E_\alpha)$, then there are $x_1, \dots, x_k$ such that $B_j = B\left(x_j, r_{x_j}\right)$ are disjoint and $\sum_{j=1}^{k} m(B_j) > 3^{-n}c$.

Then, we see that

$$\begin{aligned} c &< 3^n \sum_{j=1}^{k} m(B_j) \\ &\leq \frac{3^n}{\alpha} \sum_{j=1}^{k} \int_{B_j} |f(y)|\,dy \\ &\leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f(y)|\,dy. \end{aligned}$$

Thus, letting $c \to m(E_\alpha)$, we obtain our desired result. $\square$

## The Fundamental Theorem of Calculus for Lebesgue Integration

[I] Note that we still use the convention $0 \cdot \infty = 0$.