Problem 1

Let V be a vector space and suppose $\{W_i\}$ is a family of subspaces of V.

(i) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of V contained in every W_i.

Proof: We will show that (a) $\bigcap_{i\in I} W_i$ is a subspace of V, and (b) there is is no larger subspace of V contained within every W_i.

(a) Let $v_i, v_j \in \bigcap_{i\in I} W_i$, $\alpha, \beta \in \mathbb{F}$. We want to show that $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$. Since $v_i \in \bigcap_{i\in I} W_i$, $v_i \in W_i$ for some W_i, and $v_j \in W_j$ for some W_j. Additionally, WLOG, $v_j \in W_i$, as both v_i and v_j are contained within their intersection. Therefore, $\alpha v_i + \beta v_j \in W_i$, so $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$.

(b) Suppose there is a subspace U of V such that every W_i is contained in U, and $U \supset \bigcap_{i\in I} W_i$.

(ii) Show that

$$\sum_{i\in I} W_i := \left\{ \sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite} \right\}$$

is the smallest subspace containing each W_i.

Problem 2

Let V be a vector space and suppose $S \subseteq V$ is any subset. Show that

$$\operatorname{span}(S) = \bigcap \{W \mid S \subseteq W,\ W \subseteq V \text{ subspace}\}$$

Deduce that $\operatorname{span}(S)$ is the smallest subspace of V containing S.

Proof: Let W be a subspace containing S. Since W is a subspace, every linear combination of every element of S is inside W, as every element of S is an element of W. Therefore, for *every* subspace W such that $S \subseteq W$, any linear combination of every element in S is also in W — thus, $\operatorname{span}(S) = W$.

From this, we can see that $\operatorname{span}(S)$ can be no smaller than any subspace containing S, meaning $\operatorname{span}(S)$ is the smallest subspace of V containing S.

Problem 3

Let V be a vector space with subspaces $W_i \subseteq V$ for $i = 1, 2$. If $W_1 \cup W_2 \subseteq V$ is a subspace, show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Problem 4

Let V be a vector space over $\mathbb{F}$ and suppose $W \subseteq V$ is a subspace.

(i) Show that the quotient space $V/W = \{[v]_W \mid v \in V\}$ is a vector space with operations

$$[u]_W + [v]_W := [u+v]_W$$
$$\alpha[v]_W := [\alpha v]_W$$

(ii) Show that $\|\cdot\|$ is a norm on V. Show that

$$\|[v]_W\|_{V/} := \inf_{w\in W} \|v - w\|$$

is a seminorm on V/W.

Problem 5

Show that the quantity

$$\|f\|_1 := \int_0^1 |f(t)|dt$$

defines a norm on $C([0,1])$ with $\|f\|_1 \leq \|f\|_u$. Are $\|\cdot\|_1$ and $\|\cdot\|_u$ equivalent norms?

Non-Negativity: Since $|f(t)| \geq 0$ for $t \in [0,1]$ by the definition of absolute value, it is the case that $\int_0^1 |f(t)|dt \geq 0$.

Positive Definite: Clearly, $\|\mathbb{0}\|_1 = 0$. Additionally, since f is continuous, $|f|$ is continuous, and since $|f(t)| \geq 0$ for $t \in [0,1]$, it must be the case that $\int_0^1 |f(t)|dt = 0$ only when $f = \mathbb{0}$.

Absolute Homogeneity: Let $\alpha \in \mathbb{R}$

$$\begin{aligned}
\|\alpha f\|_1 &= \int_0^1 |\alpha f(t)|dt \\
&= \int_0^1 |\alpha||f(t)|dt \\
&= |\alpha| \int_0^1 |f(t)|dt \\
&= |\alpha|\|f\|_1
\end{aligned}$$

Triangle Inequality:

$$\begin{aligned}
\|f+g\|_1 &= \int_0^1 |f(t)+g(t)|dt \\
&\leq \int_0^1 (|f(t)| + |g(t)|)\,dt \\
&= \int_0^1 |f(t)|dt + \int_0^1 |g(t)|dt \\
&= \|f\|_1 + \|g\|_1
\end{aligned}$$

Problem 6

Show that all the p-norms, $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ on $\mathbb{F}^n$ are equivalent. Also, show that if $1 \leq p \leq q \leq \infty$, then $\ell_p \subseteq \ell_q$.

Problem 7

Let $\mathbb{M}_{m,n}(\mathbb{C})$ denote the linear space of all $m \times n$ matrices with coefficients from $\mathbb{C}$. For $a \in \mathbb{M}_{m,n}(\mathbb{C})$, set

$$\|a\|_{\text{op}} := \sup_{\xi \in B_{\ell_2^n}} \|a\xi\|_{\ell_2^m}.$$

Show that $\|\cdot\|_{\text{op}}$ is a norm on $\mathbb{M}_{m,n}(\mathbb{C})$. This is the operator norm.

Problem 9

Given any function $f : [0,1] \to \mathbb{C}$, we define

$$N(f) := \sup_{x \neq y, x,y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|}$$

and

$$\|f\|_{\Lambda} := |f(0)| + N(f).$$

Moreover, set

$$\Lambda[0,1] := \{f : [0,1] \to \mathbb{C} \mid \|f\|_{\Lambda} < \infty\}$$

(i) Show that $\Lambda[0,1]$ is precisely the set of Lipschitz continuous functions on $[0,1]$.

Proof: Let $f \in \Lambda[0,1]$. Then, $\|f\|_{\Lambda} = c$ for some finite c. Then, for $x, y \in [0,1]$

$$\begin{aligned} \frac{|f(x) - f(y)|}{|x - y|} &\leq N(f) \\ &\leq \|f\|_{\Lambda} \\ &= c. \end{aligned}$$

So,

$$|f(x) - f(y)| \leq c|x - y|,$$

which defines a Lipschitz continuous function.

(ii) Verify that $\Lambda[0,1]$ is a vector space with norm $\|f\|_{\Lambda}$, which is the Lipschitz norm.

Proof of Vector Space: Let $f, g \in \Lambda[0,1]$. Then, f and g are Lipschitz continuous. Let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned} |(\alpha f)(x) - (\alpha f)(y)| &= |\alpha||f(x) - f(y)| \\ &\leq |alpha|c|x - y| \\ &= h|x - y|, \end{aligned}$$

and

$$\begin{aligned} |(f + g)(x) - (f + g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq c|x - y| + d|x - y| \\ &= \ell|x - y|, \end{aligned}$$

meaning that $\Lambda[0,1]$ is closed under addition and scalar multiplication.

Proof of Norm:

Non-Negativity: Since, for any f, $|f(0)| \geq 0$, and $\|f\|_{\Lambda} \geq |f(0)|$, it must be the case that $\|f\|_{\Lambda} \geq 0$.

Positive Definiteness:

$$\begin{aligned} \|f\|_{\Lambda} &= 0 \\ |f(0)| + \sup_{x,y \in [0,1], x \neq y} \frac{|f(x) - f(y)|}{|x - y|} &= 0, \end{aligned}$$

meaning that for $x, y \in [0,1]$ and $x \neq y$

$$f(x) = f(y)$$

and

$$f(0) = 0$$

so $f = \mathbb{0}_f$. Additionally, if $f = \mathbb{0}_f$, then $\|f\|_\Lambda = 0$ since $|f(0)| = 0$ and $f(x) = f(y) = 0$ for all $x, y \in [0,1]$.

Absolute Homogeneity: Let $\alpha \in \mathbb{C}$.

$$\begin{aligned}
\|\alpha f\| &= |\alpha f(0)| + N(\alpha f)\\
&= |\alpha||f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|\alpha f(x) - \alpha f(y)|}{|x-y|}\\
&= |\alpha| \left(|f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|}\right)\\
&= |\alpha|\|f\|_\Lambda
\end{aligned}$$

Triangle Inequality: Let $f, g \in \Lambda[0,1]$. Then,

$$\begin{aligned}
\|f+g\| &= |f(0)+g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)+g(x) - (f(y)+g(y))}{|x-y|}\\
&\leq \left(|f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|}\right) + \left(|g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|g(x)-g(y)|}{|x-y|}\right)\\
&= \|f\|_\Lambda + \|g\|_\Lambda
\end{aligned}$$

Therefore, $\Lambda[0,1]$ is a normed vector space with $\|\cdot\|_\Lambda$ as the Lipschitz norm.

(iii) Show that $\|f\|_u \leq \|f\|_\Lambda$ for every $f : [0,1] \to \mathbb{R}$.

Problem 10

Let p be a seminorm on a vector space V.

(i) Show that $N_p := \{w \in V \mid p(w) = 0\}$ is a subspace of V.

Proof: Let $v, w \in N_p$. Then, $p(v) = 0$ and $p(w) = 0$. Since p is a seminorm, for $\alpha, \beta \in \mathbb{F}$, we have:

$$\begin{aligned}
p(\alpha v + \beta w) &\leq p(\alpha v) + p(\beta w)\\
&= |\alpha| p(v) + |\beta| p(w)\\
&= 0.
\end{aligned}$$

Since p is definitionally non-negative, $p(\alpha v + \beta w) = 0$. Therefore, N_p is a vector space.

(ii) We form the quotient vector space V/N_p. Show that

$$\|[v]_{N_p}\|_p := p(v)$$

defines a norm on V/N_p.

(iii) If $(E, \|\cdot\|)$ is a normed space and $T : V \to E$ is a linear map, show that $p(v) := \|T(v)\|$ is a seminorm on V. In this case, what is N_p.