

Here, we overview and discuss some of the most important results related to projections in von Neumann algebras.

Comparison of Projections

Recall that if H is a Hilbert space, an element $w \in B(H)$ is called a partial isometry if, for any $h \in \ker(w)^\perp$, we have $\|Wh\| = \|h\|$. We call $\ker(w)^\perp$ the initial space of W and $\text{im}(w)$ the final space of W .

There are a variety of equivalent definitions for partial isometries.

Proposition: If $w \in B(H)$, then the following are equivalent:

- (i) w is a partial isometry;
- (ii) w^* is a partial isometry;
- (iii) w^*w is a projection onto the initial space of w ;
- (iv) ww^* is a projection onto the final space of w ;
- (v) $ww^*w = w$;
- (vi) $w^*ww^* = w^*$.

Theorem (Polar Decomposition): Let $a \in B(H)$. Then, there is a partial isometry $w \in B(H)$ with initial space $\ker(a)^\perp$ and final space $\text{im}(a)$ such that $a = w|a|$.

If $a \in M \subseteq B(H)$, where M is a von Neumann algebra, then both $|a|$ and w are in M .

Equivalence of Projections

If $M \subseteq B(H)$ is a von Neumann algebra, then we say two projections $p, q \in P(M)$, where $P(M)$ denotes the space of projections of M , are (Murray–von Neumann) *equivalent* in M if there is a partial isometry $v \in P(M)$ such that $v^*v = p$ and $vv^* = q$. We will write $p \sim q$.

Note that projections have an ordering by saying that $p \leq q$ if $pq = qp = p$, or $\text{im}(p) \subseteq \text{im}(q)$. This allows us to say that p is *sub-equivalent* to q (in M), written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$.¹

The sub-equivalence relation in fact forms a partial order, and equivalence as projections forms an equivalence relation. We will first show that it is a preorder.

Proposition: In a von Neumann algebra, the relation \sim is an equivalence relation on $P(M)$, and the relation \preceq is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry follows from the fact that if v is a partial isometry, then so is v^* .

Now, we will show transitivity for \preceq , from which we will see that \sim is transitive. Letting $p, q, r \in P(M)$ be such that $p \preceq q$ and $q \preceq r$, we have partial isometries $u, v \in M$ with

¹We will say that the projection q majorizes p if $p \preceq q$, and we will say that q dominates p if $p \leq q$.

$u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. Then, we have

$$\begin{aligned} qu &= quu^*u \\ &= (quu^*)u \\ &= uu^*u \\ &= u, \end{aligned}$$

so that

$$\begin{aligned} (vu)^*(vu) &= u^*v^*vu \\ &= u^*qu \\ &= u^*u \\ &= p \\ (vu)(vu)^* &= vu u^*v^* \\ &\leq vqv^* \\ &= vv^*vv^* \\ &= vv^* \\ &\leq r. \end{aligned}$$

Therefore, $p \preceq r$, so \preceq is a transitive relation. \square

To see that \preceq is a partial order, we need an analogue of the Cantor–Schröder–Bernstein theorem for projections. In fact, it can be proven in a similar manner. First, we discuss a simple lemma.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are pairwise orthogonal families of projections with $p_i \preceq q_i$, then $\sum_{i \in I} p_i \preceq \sum_{i \in I} q_i$.

Proof. Let u_i be the partial isometries with $u_i^*u_i = p_i$ and $r_i := u_iu_i^* \leq q_i$. Then, the r_i are pairwise orthogonal since the q_i are pairwise orthogonal, and for any $i \neq j$,

$$\begin{aligned} u_i^*u_j &= u_i^*u_iu_i^*u_ju_j^*u_j \\ &= u_i^*r_i r_j u_j \\ &= 0 \\ u_iu_j^* &= u_iu_i^*u_iu_j^*u_ju_j^* \\ &= u_i p_i p_j u_j^* \\ &= 0. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \left(\sum_{i \in I} u_i^* \right) \left(\sum_{j \in I} u_j \right) &= \sum_{i \in I} u_i^*u_i \\ &= \sum_{i \in I} p_i \end{aligned}$$

$$\begin{aligned} \left(\sum_{i \in I} u_i \right) \left(\sum_{j \in I} u_j^* \right) &= \sum_{i \in I} u_i u_i^* \\ &\leq \sum_{i \in I} q_i. \end{aligned}$$

This gives $\sum_{i \in I} p_i \preceq \sum_{i \in I} q_i$. □

Theorem: If $e \preceq f$ and $f \preceq e$, then $e \sim f$.

Proof. We will let $e_0 := e$ and $f_0 := f$. Let v and w be partial isometries with $v^*v = e$, $vv^* = f_1 \leq f$, $w^*w = f$, $ww^* = e_1 \leq e$. Inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of some projection dominated by f_1 , it follows that we may write $f_2 := ve_1(ve_1)^*$ with $f_2 \leq f_1$. Since w maps the range of f_1 onto the range of some projection dominated by e_1 , it follows that we may write $wf_1(wf_1)^* =: e_2$. Observe also that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

Inductively, we obtain decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1} , and w maps the range of f_n isometrically onto that of e_{n+1} . Defining $e_\infty := \inf_n e_n$ and $f_\infty = \inf_n f_n$, we have that v maps the range of e_∞ onto that of f_∞ , and w that of f_∞ onto the range of e_∞ . Note that we have $e_\infty \sim f_\infty$.

As discussed earlier, we have that $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$, so since sums of pairwise orthogonal families of projections respects equivalence, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (e_{2n} - e_{2n+1}) &\sim \sum_{n=0}^{\infty} (f_{2n+1} - f_{2n+2}) \\ \sum_{n=0}^{\infty} (e_{2n+1} - e_{2n+2}) &\sim \sum_{n=0}^{\infty} (f_{2n} - f_{2n+1}). \end{aligned}$$

Therefore, we get

$$\begin{aligned} e &= e_\infty + \sum_{n=0}^{\infty} (e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty} (e_{2n+1} - e_{2n+2}) \\ &\sim f_\infty + \sum_{n=0}^{\infty} (f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty} (f_{2n} - f_{2n+1}) \\ &= f. \end{aligned}$$

□

Central Projections and the Comparison Theorem

The projections in a von Neumann algebra form a complete lattice, as the collection of closed subspaces of H form a complete lattice under the operations

$$\bigvee_{i \in I} X_i := \overline{\sum_{i \in I} X_i}$$

$$\bigwedge_{i \in I} X_i := \bigcap_{i \in I} X_i.$$

If $S \subseteq H$ is any subset, then we will define the range projection of S by

$$[S] := P_{\overline{\text{span}(S)}}.$$

Proposition: If $M \subseteq B(H)$ is a von Neumann algebra, and $x \in M$, then $[xH]$ and $[x^*H]$ are in M , with $[xH] \sim [x^*H]$ in M .

Proof. Let $x = v|x|$ be the polar decomposition. Note that $v \in M$. Now, vv^* is the projection onto \overline{xH} and v^*v is the projection onto $\ker(x)^\perp = \overline{x^*H}$. Thus, these projections are equivalent in M . \square

Definition: Let $x \in M$. We define the *central support* to be the projection

$$z(x) = \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say p and q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $M \subseteq B(H)$ is a von Neumann algebra, then the central support of any $p \in P(M)$ is given by

$$z(p) = [MpH].$$

Let $w = [MpH]$. Since M is unital, it follows that $p \leq w$, and since \overline{MpH} is a reducing subspace for both M and M' , we have $w \in M \cap M'$, so $z(p) \leq w$.

Conversely, if $x \in M$, then

$$\begin{aligned} xpH &= xz(p)pH \\ &= z(p)xpH, \end{aligned}$$

meaning that $[xpH] \leq z(p)$, so $w \leq z(p)$ as x was arbitrary.

Proposition: Let M be a von Neumann algebra, and let $p, q \in P(M)$ be projections. The following are equivalent:

- (i) p and q are centrally orthogonal;
- (ii) $pMq = \{0\}$;
- (iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ with $p_0 \sim q_0$.

Proof. Let p and q be centrally orthogonal. Then, for any $x \in M$, we have

$$pxq = pz(p)xz(q)q$$

$$\begin{aligned}
&= pxz(p)z(q)q \\
&= 0.
\end{aligned}$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, so $p \leq 1 - z(q)$. Since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, meaning that $z(p)z(q) = 0$. Therefore, (i) and (ii) are equivalent.

Suppose (ii) is not the case. Let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$. Defining

$$\begin{aligned}
p_0 &= [pxqH] \\
q_0 &= [qx^*pH],
\end{aligned}$$

we have that $p_0 \leq p$, $q_0 \leq q$, and since $(pxq)^* = qx^*p$, we have $p_0 \sim q_0$.

Now, if there are nonzero projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim q_0$, then if v is a partial isometry with $v^*v = p_0$, $vv^* = q_0$, then $v^* = p_0v^*q_0$, meaning

$$\begin{aligned}
pv^*q &= pp_0v^*q_0q \\
&= p_0v^*q_0 \\
&= v^* \\
&\neq 0,
\end{aligned}$$

meaning $pMq \neq \{0\}$. □

Theorem (Comparison Theorem): Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there is a central projection $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1 - z) \preceq p(1 - z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ of pairwise orthogonal projections with $p_i \sim q_i$ and, setting

$$\begin{aligned}
p_0 &= \sum_{i \in I} p_i \\
q_0 &= \sum_{i \in I} q_i,
\end{aligned}$$

we have $p_0 \preceq q_0$. From above, we have that $p_0 \sim q_0$.

Let $w := z(q - q_0)$. Since $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are maximal, it follows that $z(q - q_0)$ and $z(p - p_0)$ are centrally orthogonal, yielding $(p - p_0)w = 0$, meaning $pw = p_0w$.

If we let v be a partial isometry implementing the equivalence $p_0 \sim q_0$, then we have that vw is a partial isometry implementing the equivalence $p_0w \sim q_0w$. Therefore, we have

$$\begin{aligned}
pw &= p_0w \\
&\sim q_0w \\
&\leq q.
\end{aligned}$$

Similarly, $p_0(1 - w) \sim q_0(1 - w)$, so since $q - q_0 \leq w$, we have $q(1 - w) \preceq p(1 - w)$. □

Recall that a factor is a von Neumann algebra M such that $Z(M) = \mathbb{C}1$.

| **Corollary:** If M is a factor, then any two projections in M can be compared.

The Type Decomposition

| **Definition:** Let M be a von Neumann algebra, and $p \in B(H)$ a projection not necessarily in M . The algebra pMp is known as a corner (or compression) of M .

| **Theorem:** Let $M \subseteq B(H)$ be a von Neumann algebra, and let $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, and $(pMp)' = M'p$, $(M'p)' = pMp$.

| **Corollary:** If M is a factor and $p \in P(M)$, then pMp and $M'p$ are both factors.

| **Definition:** Let $M \subseteq B(H)$ be a von Neumann algebra. We say a projection $p \in P(M)$ is

- minimal if $p \neq 0$ and the only subprojections of p are 0 and p ;
- abelian if pMp is abelian;
- finite if $q \leq p$ and $q \sim p$ implies $q = p$;
- semifinite if there are pairwise orthogonal finite projections $p_i \in P(M)$ such that $p = \sum_{i \in I} p_i$;
- purely infinite if $p \neq 0$ and there is no nonzero finite projection $q \leq p$;
- properly infinite if $p \neq 0$ and zp is not finite for any nonzero central projection $z \in Z(P(M))$.

We say that the von Neumann algebra M is finite/semifinite/purely infinite/properly infinite if the projection 1 satisfies its respective condition. Additionally, if M has no minimal projections, we say it is diffuse.

What we will be working towards is known as the type decomposition, which forms the basis for the dimension theory of von Neumann algebras. Eventually, we will show that every von Neumann algebra M can be decomposed as

$$M = M_{\text{sf}} \oplus M_{\text{III}},$$

where M_{sf} is a semifinite von Neumann algebra and M_{III} is a type III von Neumann algebra, a definition we will discuss shortly. First, we must expand on some of the ways that comparison of projections interacts with these properties of those projections.

| **Lemma:** Let $\{p_i\}_{i \in I}$ be a family of centrally orthogonal projections in a von Neumann algebra $M \subseteq B(H)$. If each p_i is abelian (finite), then the sum $\sum_{i \in I} p_i$ is also abelian (finite).

| *Proof.* If each p_i is abelian, then since they are centrally orthogonal for any $i \neq j$, then for any $x, y \in M$, we have $p_i x p_j y p_j = 0$. Therefore, we have

$$\begin{aligned} (p x p)(p y p) &= \sum_{i \in I} p_i x p_i y p_i \\ &= (p y p)(p x p), \end{aligned}$$

so p is abelian.

Now, if each p_i is finite, and $u \in M$ is such that $uu^* \leq u^*u = p$, then for all i we have $z(p_i)u^*uz(p_i) = p$, and $uz(p_i)u^* = z(p_i)uu^* \leq p_i$, meaning $uz(p_i)u^* = p_i$, and

$$\begin{aligned} uu^* &= uz(p)u^* \\ &= \sum_{i \in I} uz(p_i)u^* \\ &= p. \end{aligned}$$

□

Proposition: Let p, q be nonzero projections in a von Neumann algebra with $p \preceq q$. If q is finite (purely infinite), then p is also finite (purely infinite).

Proof. Suppose q is finite, and $p \sim q$, with $v \in M$ implementing the equivalence. Let $u \in M$ be such that $u^*u = p$ and $uu^* \leq p$. Then,

$$\begin{aligned} (vuv^*)^*(vuv^*) &= q \\ (vuv^*)(vuv^*)^* &\leq q, \end{aligned}$$

so since q is finite, $(vuv^*)(vuv^*)^* = q$, so $uu^* = p$.

Now, if $p \leq q$, then if $u^*u = p$ with $uu^* \leq p$, then by setting $w = u + (q - p)$, we have that $w^*w = q$ and $ww^* \leq q$, so $u^*u + (q - p) = ww^* = q$, meaning $uu^* = p$.

In the general case, we have some $q_0 \leq q$ such that $p \sim q_0 \leq q$.

Since projections are purely infinite when they have no nonzero finite subprojections, the purely infinite case follows from the finite case. □

Proposition: A projection $p \in P(M)$ is semifinite if and only if p is the supremum of finite projections. In particular, a supremum of semifinite projections is also semifinite.

Proof. If p is semifinite, then p is the sum (and hence the supremum) of a family of pairwise orthogonal finite projections.

Now, suppose $p = \bigvee_{\alpha} p_{\alpha}$, where each p_{α} is finite. Let $\{q_{\beta}\}_{\beta}$ be a maximal family of pairwise orthogonal finite subprojections of p . If we set

$$q_0 = p - \sum_{\beta} q_{\beta},$$

and suppose that $q_0 \neq 0$, then there exists some p_{α} such that p_{α} and q_0 are not orthogonal (else it would contradict maximality), so they are not centrally orthogonal. Therefore, we have a nonzero subprojection $q_1 \leq q_0$ such that $q_1 \preceq p_{\alpha}$, so it is finite by what we showed previously; this contradicts the maximality of the set $\{q_{\beta}\}_{\beta}$. □

Corollary: Let p be a projection in a von Neumann algebra M . If p is semifinite (purely infinite), then the central support $z(p)$ is also semifinite (purely infinite).

Proof. The central support is the supremum over all equivalent projections, so since the supremum of semifinite projections is again semifinite, it follows from the previous proposition.

Furthermore, a nonzero projection is purely infinite if and only if it is centrally orthogonal to every semifinite projection, so we obtain the corollary in this case. \square

Corollary: Let p, q be nonzero projections in a von Neumann algebra such that $p \preceq q$. If q is semifinite, then so is p .

Proof. It is enough to consider the case when q is central, in which case we may take $p \leq q$. Let p_0 be the maximal semifinite subprojection of p . Since q is semifinite, it is the supremum of its finite subprojections. Since $z(p - p_0) \leq q = z(q)$, it follows that if $p - p_0 \neq 0$, then there would exist a nonzero finite subprojection that would be equivalent to a subprojection of $p - p_0$, which contradicts the definition of p_0 . Thus, p is the supremum of its finite subprojections, so it is semifinite. \square

Lemma: Let M be a properly infinite von Neumann algebra. Then, there exists a projection $p \in P(M)$ such that $p \sim 1 - p \sim 1$.

Proof. Since M is properly infinite, there exists $u \in M$ with $uu^* \leq u^*u = 1$. Set $p_0 = 1 - uu^*$. Then, $p_n = u^n p_0 (u^n)^*$ is a pairwise orthogonal family of equivalent projections. Let $\{q_i\}_{i \in I}$ be a maximal family of pairwise orthogonal equivalent projections in M extending $\{p_n\}_{n \in \mathbb{N}}$, and set $q_0 = 1 - \sum_{i \in I} q_i$.

By the comparison theorem, there is $z \in Z(P(M))$ such that $q_0 z \leq q_{i_0} z$ and $q_{i_0} (1 - z) \leq q_0 (1 - z)$. If it were the case that $z = 0$, then we would have $q_{i_0} \leq q_0$, contradicting maximality of $\{q_i\}_{i \in I}$, so $z \neq 0$, and we have

$$\begin{aligned} z &= q_0 z + \sum_{i \in I} q_i z \\ &\preceq q_{i_0} z + \sum_{i \neq i_0} q_i z \\ &= \sum_{i \in I} q_i z \leq z, \end{aligned}$$

so $z \sim \sum_{i \in I} q_i z$ by Cantor–Schröder–Bernstein for projections. Decomposing $\{q_i\}_{i \in I}$ into infinite subsets, we may construct two projections p and $z - p$ such that $p \sim z - p \sim z$.

Now, let $\{r_j\}_{j \in J}$ be a maximal family of centrally orthogonal projections with $r_j \sim z(r_j) - r_j \sim z(r_j)$. Then, the argument above shows that $\sum_{j \in J} z(r_j) = 1$, so by setting $p = \sum_{j \in J} r_j$, we obtain our desired result. \square

Proposition: Let p and q be finite projections in a von Neumann algebra M . Then, $p \vee q$ is finite.

Proof. We use Kaplansky's formula for this, which gives $p \vee q - p \sim q - p \wedge q$. This follows from observing that $x = (1 - p)q$ has $\ker(x) = \ker(q) \oplus (qH \cap pH)$, meaning that $[x^*H] = 1 - ((1 - q) + q \wedge p) = q - q \wedge p$. Symmetrically, this gives

$$\begin{aligned} [xH] &= (1 - p) - (1 - p) \wedge (1 - q) \\ &= p \vee q - p. \end{aligned}$$

In particular, since $q - p \wedge q \leq q$, we may assume that p and q are orthogonal, replacing q with $p \vee q - p$. We may also assume that $p + q = 1$ by passing to $(p + q)M(p + q)$.

Let z_0 be the supremum of all finite central projections. It follows that z_0 is finite. If $z_0 = 1$, then we are done. Else, we may use $(1 - z_0)p$ and $(1 - z_0)q$, wherein $z_0 = 0$ and thus we assume that M is properly infinite.

Therefore, we have a projection $r \in P(M)$ such that $r \sim 1 - r \sim 1$. By comparison, there is $z \in P(Z(M))$ such that

$$\begin{aligned} z(p \wedge r) &\preceq z(q \wedge (1 - r)) \\ (1 - z)(q \wedge (1 - r)) &\preceq (1 - z)(p \wedge r). \end{aligned}$$

Additionally, $zr \sim z(1 - r) \sim z$, and

$$\begin{aligned} z(p \wedge r) &= zp \wedge zr \\ &\preceq z(1 - r) \wedge zq, \end{aligned}$$

so by using Kaplansky's formula, we have

$$\begin{aligned} zr &= z(r - r \wedge p) + z(r \wedge p) \\ &\preceq z(r \vee p - p) + z(q \wedge (1 - r)) \\ &= zq, \end{aligned}$$

Therefore, $zr = 0$ since zq is finite and M is properly infinite, so $z \sim 0$. In particular, this gives $q \wedge (1 - r) \preceq p \wedge r$. Replacing p with q and r with $1 - r$, we get

$$\begin{aligned} 1 - r &= (1 - r - (1 - r) \wedge q) + ((1 - r) \wedge q) \\ &\preceq ((1 - r) \vee q - q) + (p \wedge r) \\ &= p, \end{aligned}$$

which gives a contradiction since p is finite. □

Proposition: Let p and q be finite projections with $p \sim q$. Then, $1 - p$ and $1 - q$ are equivalent.

Proof. We have that $p \vee q$ is finite, so we may assume that M is finite. By the comparison theorem, there are projections p_1 and q_1 and a central projection $z \in P(Z(M))$ such that

$$\begin{aligned} (1 - p)z &\sim q_1 \leq (1 - q)z \\ (1 - q)(1 - z) &\sim p_1 \leq (1 - p)(1 - z). \end{aligned}$$

Then, we have

$$\begin{aligned} z &= (1 - p)z + pz \\ &\sim q_1 + qz \\ &\leq (1 - q)z + qz \end{aligned}$$

$$= z$$

and

$$\begin{aligned} (1 - z) &= (1 - q)(1 - z) + q(1 - z) \\ &\sim p_1 + p(1 - z) \\ &\leq (1 - z), \end{aligned}$$

so since both z and $(1 - z)$ are finite, we have $q_1 = (1 - q)z$ and $p_1 = (1 - p)(1 - z)$. Thus, $1 - q \sim 1 - p$. \square

Definition: Let M be a von Neumann algebra.

- We say M is type I if every nonzero central projection in M majorizes a nonzero abelian projection in M .
- We say M is type II_1 if it is finite, has no nonzero abelian projections, and every nonzero central projection in M majorizes a nonzero finite projection.
- We say M is type II_∞ if every nonzero central projection majorizes a nonzero finite projection, and has no nonzero finite central projections.
- We say M is type III if it is purely infinite.

Theorem: Every von Neumann algebra M uniquely decomposes into a direct sum of those of type I, II_1 , II_∞ , and III. Moreover, every projection e in M can be uniquely written as the sum of centrally orthogonal projections e_1 and e_2 in M such that e_1 is finite and e_2 is properly infinite.

Proof. Let $\{e_i\}_{i \in I}$ be a maximal family of centrally orthogonal abelian projections in M , and let $e = \sum_{i \in I} e_i$. Then, e is abelian; define $z_I = z(e)$.

If z is a nonzero central projection majorized by z_I , then ze is a nonzero abelian projection, so Mz_I is of type I.

By construction, there is then no nonzero abelian projection in $M(1 - z_I)$, so it has no non-trivial direct summand of type I.

Let $\{f_j\}_{j \in J}$ be a maximal family of centrally orthogonal finite projections in $M(1 - z_I)$, and let $f = \sum_{j \in J} f_j$. Then, f is finite. Set $z_{\text{II}} = z(f)$. By construction, we have that Mz_{II} has no nonzero abelian projections, and every nonzero projection z in Mz_{II} majorizes a nonzero finite projection zf . Thus, Mz_{II} is of type II.

By the maximality of $\{f_j\}_{j \in J}$, it follows that $1 - z_{\text{II}} = z_{\text{III}}$ does not majorize any finite projection, so Mz_{III} is of type III. We have that $z_I + z_{\text{II}} + z_{\text{III}} = 1$.

Now, let $\{z_k\}_{k \in K}$ be a maximal orthogonal family of finite central projections in Mz_{II} , and set $z_{\text{II}_1} = \sum_{k \in K} z_k$, and $z_{\text{II}_\infty} = z_{\text{II}} - z_{\text{II}_1}$. It follows that Mz_{II_1} is of type II_1 and Mz_{II_∞} is of type II_∞ . Therefore, we get the direct sum decomposition

$$M = Mz_I \oplus Mz_{\text{II}_1} \oplus Mz_{\text{II}_\infty} \oplus Mz_{\text{III}}.$$

As for uniqueness, we suppose that there is another orthogonal decomposition $1 = w_I +$

$w_{\text{II}_1} + w_{\text{II}_\infty} + w_{\text{III}}$. Then, we must have that $w_I(1 - z_I) = 0$ since $1 - z_I$ does not majorize any nonzero abelian projection, while $w_I(1 - z_I)$ is a central projection in Mw_I . Therefore, $w_I \leq z_I$. Similarly, $z_I(1 - w_I) = 0$ for the same reason, meaning $z_I \leq w_I$, so they are equal. By similar arguments, all the other summands are equal to each other.

Finally, we let e be a nonzero projection in M . By considering $M_e := eMe$, we may assume that $e = 1$. We let e_1 be the sum of a maximal orthogonal family of finite central projections. Then, e_1 is finite and central, with $1 - e_1$ properly infinite. The uniqueness of this decomposition has the same flavor as the arguments for $\{z_I, \dots, z_{\text{III}}\}$. \square

Definition: We say a factor is atomic if it contains a minimal projection, and otherwise we say it is diffuse.

Structure of Type I and II von Neumann Algebras

Now, we will discuss some structural results related to type I and II von Neumann algebras. Before we can do this, we must discuss tensor products.

Tensor Products of Hilbert Spaces and Operators

Definition: Let H and K be Hilbert spaces. There is an inner product on the algebraic tensor product $H \odot K$ given by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle,$$

whenever $\xi_1, \eta_1 \in H$ and $\xi_2, \eta_2 \in K$. The Hilbert space tensor product of H and K , denoted $H \otimes K$, is the completion of $H \odot K$ with respect to the norm induced by this inner product.

Proposition: Let H and K be Hilbert spaces with orthonormal bases $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$. Then,

- (i) $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal basis for $H \otimes K$;
- (ii) if $|J| = \alpha$ for some cardinal α , then $H \otimes K \cong H^{(\alpha)} \cong \bigoplus_{j \in J} H$;
- (iii) if $H = L_2(X, \mu)$ for some σ -finite regular Borel measure space (X, μ) , and K is separable, then $H \otimes K \cong L_2(X, \mu, K)$, where the latter denotes the space of square-integrable Borel functions with respect to the norm on K .

Proof.

- (i) We observe that the set $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal set. The spans of these elementary tensors are all the vectors of the form $x \otimes y$, provided that x is a finite linear combination of the e_i and y is a finite linear combination of the f_j . Therefore, the completion is equal to the completion of $H \otimes K$, so the set is an orthonormal basis.
- (ii) We find that

$$H \otimes K \cong \bigoplus_{j \in J} H \otimes \mathbb{C}f_j$$

is an ℓ_2 direct sum of $|J|$ copies of H .

(iii) Define maps $Y_j: L_2(X, \mu, K) \rightarrow L_2(X, \mu)$ by taking

$$(Y_j f)(x) = \langle f(x), f_j \rangle,$$

a representative of an equivalence class modulo μ . Then, we have that

$$f \mapsto \sum_{j \in J} Y_j f$$

defines a map from $L_2(X, \mu, K) \rightarrow \bigoplus_{j \in J} H \otimes \mathbb{C} f_j$. By Tonelli's theorem and Parseval's identity, we get

$$\begin{aligned} \|Yf\|^2 &= \sum_{j \in J} \|Y_j f\|^2 \\ &= \sum_{j \in J} \int |\langle f(x), f_j \rangle|^2 d\mu \\ &= \int \sum_{j \in J} |\langle f(x), f_j \rangle|^2 d\mu \\ &= \int \|f(x)\|^2 d\mu \\ &= \|f\|^2. \end{aligned}$$

Therefore, Y is an isometry. The range is dense since, if $h \in L_2(\mu)$, then $f := hf_j$ is mapped to the vector with h in position j and 0 elsewhere. Thus, Y is a unitary map from $L_2(X, \mu, K)$ onto $H \otimes K$. □

Proposition: Let $A \in B(H)$ and $B \in B(K)$. Then, there is a unique $A \otimes B \in B(H \otimes K)$ such that $(A \otimes B)(\xi \otimes \eta) = A\xi \otimes B\eta$. Furthermore, $\|A \otimes B\| = \|A\| \|B\|$.

Proof. Let $A \otimes I_K$ be the amplification to $H \otimes K$. Fixing an orthonormal basis $\{\eta_j\}_{j \in J}$ for K , then any vector $\omega \in H \otimes K$ (from part (ii) above) can be expressed as

$$\omega = (\zeta_j \otimes \eta_j)_j$$

for a collection of $\zeta_j \in H$, and we may define

$$\begin{aligned} \|\omega\|^2 &= \sum_{j \in J} \|\zeta_j\|^2 \\ &< \infty \end{aligned}$$

Therefore, we may rewrite

$$(A \otimes I)\omega = (A\zeta_j \otimes \eta_j)_j,$$

giving

$$\begin{aligned}
\|(A \otimes I)\omega\|^2 &= \left\| (A\zeta_j \otimes \eta_j)_j \right\|^2 \\
&= \sum_{j \in J} \|A\zeta_j\|^2 \\
&\leq \|A\|^2 + \sum_{j \in J} \|\zeta_j\|^2 \\
&= (\|A\| \|\omega\|)^2.
\end{aligned}$$

It follows that $\|A \otimes I\| = \|A\|$. Similarly, we may define $I \otimes B$, and set $A \otimes B = (A \otimes I)(I \otimes B)$. Thus, we have

$$\begin{aligned}
\|A \otimes B\|(\xi \otimes \eta) &= (A \otimes I)(I \otimes B)(\xi \otimes \eta) \\
&= (A \otimes I)(\xi \otimes B\eta) \\
&= A\xi \otimes B\eta.
\end{aligned}$$

We have that $\|A \otimes B\| \leq \|A \otimes I\| \|I \otimes B\| = \|A\| \|B\|$. Since $\|(A \otimes B)(\xi \otimes \eta)\| = \|A\xi\| \|B\eta\|$, we may choose unit vectors ξ and η appropriately to approximate $\|A\| \|B\|$. Finally, since $H \odot K$ is dense in $H \otimes K$, we have that $A \otimes B$ is well-defined. \square

Definition: Let M and N be von Neumann algebras. Then, the von Neumann algebra tensor product $M \bar{\otimes} N$ is the WOT-closure of $M \odot N$ in $B(H \otimes K)$. If $N = \mathbb{C}1$, then $M \bar{\otimes} \mathbb{C}1 = M^{(\alpha)}$ is an amplification of M .

Definition: A system $\{w_{ij} \mid i, j \in I\}$ of elements in a von Neumann algebra is called a system of *matrix units* if

- (i) $w_{ij}^* = w_{ji}$;
- (ii) $w_{ij}w_{kl} = \delta_{jk}w_{il}$;
- (iii) $\sum_{i \in I} w_{ii} = 1$.

Proposition: Let $M \subseteq B(H)$ be a von Neumann algebra. Let $\{w_{ij} \mid i, j \in I\}$ be a system of matrix units. Fix $i_0 \in I$, and let $e = w_{i_0 i_0}$. Then,

$$M \cong (eMe) \otimes B(\ell_2(I)).$$

Proof. Let $\{\varepsilon_i\}_{i \in I}$ be an orthonormal basis in $\ell_2(I)$, and let $v_i = w_{i, i_0}$ for each $i \in I$. Then, $w_{ij} = v_i v_j^*$ and $v_i^* v_i = e$. Define a map

$$U \left(\sum_{i \in I} \xi_i \otimes \varepsilon_i \right) = \sum_{i \in I} v_i \xi_i$$

for each $\sum_{i \in I} \xi_i \otimes \varepsilon_i \in K \otimes \ell_2(I)$. Then, we have

$$\left\langle U \left(\sum_{i \in I} \xi_i \otimes \varepsilon_i \right), U \left(\sum_{j \in I} \eta_j \otimes \varepsilon_j \right) \right\rangle = \left\langle \sum_{i \in I} v_i \xi_i, \sum_{j \in I} v_j \eta_j \right\rangle$$

$$\begin{aligned}
&= \sum_{i,j \in I} \langle v_j^* v_i \xi_i, \eta_j \rangle \\
&= \sum_{i \in I} \langle \xi_i, \eta_i \rangle \\
&= \left\langle \sum_{i \in I} \xi_i \otimes \varepsilon_i, \sum_{j \in I} \eta_j \otimes \varepsilon_j \right\rangle,
\end{aligned}$$

so that U is a well-defined isometry of $K \otimes \ell_2(I)$ onto H . Since the range of U contains all $v_i K = w_{ii} H$, it follows that U is surjective as $\sum_{i \in I} w_{ii} = 1$.

Now, letting $x = (x_{ij}) \in eMe \otimes B(\ell_2(I))$, we have

$$\begin{aligned}
UxU^*\xi &= Ux \left(\sum_{j \in I} v_j^* \xi \otimes \varepsilon_j \right) \\
&= U \left(\sum_{i,j \in I} x_{ij} v_j \xi \otimes \varepsilon_i \right) \\
&= \sum_{i,j \in I} v_i x_{ij} v_j^* \xi,
\end{aligned}$$

meaning that $UxU^* = \sum_{i,j \in I} v_i x_{ij} v_j^*$ is an element of M . Meanwhile, if $x \in M$, then

$$\begin{aligned}
U^*xu \left(\sum_{j \in I} \xi_j \otimes \varepsilon_j \right) &= U^*x \sum_{j \in I} v_j \xi_j \\
&= U^* \sum_{j \in I} x v_j \xi_j \\
&= \sum_{i,j \in I} (v_i^* x v_j \xi_j \otimes \varepsilon_i).
\end{aligned}$$

Therefore, $(U^*xU)_{i,j} = v_i^* x v_j$, so that $U^*xU \in eMe \bar{\otimes} B(\ell_2(I))$. □

Type I von Neumann Algebras

We start by considering the structure and classification of type I von Neumann algebras. For starters, we know that every projection dominates a rank one projection (which is minimal), so since any minimal projection is abelian, it follows that $B(H)$ is a factor of type I. Furthermore, we can show that any atomic von Neumann algebra is type I.

Proposition:

- (i) If A is an abelian von Neumann algebra, then $A \bar{\otimes} B(H)$ is type I for any Hilbert space H .
- (ii) If A is an abelian von Neumann algebra, then the matrix algebra $M_n(A)$ is a finite von

Neumann algebra of type I, and the A -valued map

$$\Phi(x) = \frac{1}{n} \sum_{i=1}^n x_{ii}$$

is

- center-valued: $\Phi((\delta_{ij}a)_{i,j}) = a$ whenever $a \in A$;
- A -modular: $\Phi(axb) = a\Phi(x)b$ whenever $a, b \in A$;
- positive definiteness: $\Phi(x^*x) \geq 0$, with equality only when $x = 0$;
- traciality: $\Phi(x^*x) = \Phi(xx^*)$.

Proof.

- (i) Let p be a minimal projection in $B(H)$, and let $e = 1 \otimes p$. Then,

$$\begin{aligned} e(A \bar{\otimes} B(H))e &= A \bar{\otimes} pB(H)p \\ &= A \otimes \mathbb{C}p, \end{aligned}$$

so e is an abelian projection. Since the center of $A \bar{\otimes} B(H)$ is $A \otimes \mathbb{C}$, it follows that the central support $z(e)$ of e is the identity. Thus, $A \bar{\otimes} B(H)$ is type I.

- (ii) The first two properties of Φ follow from the definition, while the latter two properties follow from taking

$$\begin{aligned} \Phi(x^*x) &= \frac{1}{n} \sum_{i,j=1}^n x_{ij}^* x_{ij} \\ &= \frac{1}{n} \sum_{i,j=1}^n x_{ij} x_{ij}^* \\ &= \Phi(xx^*). \end{aligned}$$

□

Definition: An A -valued map satisfying the properties in (ii), except for possibly the faithfulness condition, is known as an A -valued trace.

We will discuss traces later. First, we will discuss further characterizations and structures of type I von Neumann algebras.

Proposition: A von Neumann algebra is type I if and only if it has an abelian projection whose central support is 1. We call such projections faithful.

Proof. Let e be a faithful abelian projection in M , and let g be any nonzero central projection. Then, $eg \neq 0$. Yet, $(eg)M(eg) = (eMe)g$, which is abelian; since $eg \leq g$, it follows that M is type I.

Now, if M is type I, then by Zorn's lemma, there is a maximal family of centrally orthogonal abelian projections $\{e_i\}_{i \in I}$. Set $z = \sum_i z(e_i)$. Then, $(1 - z)M(1 - z)$ is type I, so if

$1 - z \neq 0$ we get a contradiction by the maximality of $\{e_i\}_{i \in I}$. Thus, $\sum_{i \in I} z(e_i) = 1$.

If we let $e = \sum_{i \in I} e_i$, then e is abelian and $z(e) = 1$. \square

Definition: Let n be a cardinal number. We say a type I von Neumann algebra is type I_n if 1 is a sum of n equivalent nonzero abelian projections. We say it is type I_∞ if it is type I_n for an infinite cardinal n .

Furthermore, if z is any projection that is the sum of n equivalent abelian projections with central support equal to z , then we say z is n -homogeneous.

Lemma: Let M be a von Neumann algebra, and let $p \in P(M)$ be a projection with central support $z(p)$. Then,

$$M'z(p) \rightarrow M'p$$

given by $yz(p) \mapsto yp$ is an isomorphism of $*$ -algebras. In particular, $Z(pMp) = Z(Mz(p))p$.

Proof. Recalling that $z(p) = [MpH]$, we have that if $yp = 0$, then for all $x \in M$ and $\xi \in H$,

$$\begin{aligned} yxp\xi &= xyp\xi \\ &= 0. \end{aligned}$$

Therefore, $yz(p) = 0$, meaning that $yz(p) \mapsto yp$ is injective. Since this map is surjective, it follows that we get this as an isomorphism.

In particular, we have that $Z(pMp) = Z(M'p)$, and since $yp \in Z(M'p)$ if and only if $yz(p) \in Z(M'z(p))$, we find our desired equality. \square

An important fact about abelian projections is that their central supports fully encode their order relation (in the following sense).

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $p, q \in P(M)$ are abelian projections with $z(p) = z(q)$, then $p \sim q$.

Note that we already have that if $p \sim q$, then $z(p) = z(q)$, but this provides a type of converse.

Proof. Suppose $p \preceq q$, and let $v \in M$ implement this subequivalence, where $vv^* = q_0 \leq q$. Since q is abelian, we have

$$\begin{aligned} qMq &= Z(qMq) \\ &= Z(Mz(q))q, \end{aligned}$$

so since $q_0 \in qMq$, we have $q_0 = zq$ for some central projection $z \in Z(Mz(q))$. Therefore, we get

$$\begin{aligned} z(p) &= z(q_0) \\ &= zz(q) \\ &= z(q) \\ &= z(p). \end{aligned}$$

In particular, this gives $zz(q) = z(q)$, so $z(q) \leq z$, and thus $z = z(q)$, $q_0 = q$, and $p \sim q$.

In the case that p is not subequivalent to q , then by the comparison theorem, there is $z \in Z(M)$ such that $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$. Now, we have $z(pz) = z(p)z = z(qz)$, so $pz \sim qz$, and $q(1-z) \sim p(1-z)$, so $p \sim q$. \square

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If n and M are cardinals such that M is type I_n and I_m , then $n = m$.

Proof. Let $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ be pairwise orthogonal equivalent abelian projections with $|I| = n$ and $|J| = m$ such that

$$\begin{aligned} 1 &= \sum_{i \in I} p_i \\ &= \sum_{j \in J} q_j. \end{aligned}$$

Note that $z(p_i)$ is constant across i , so that

$$\begin{aligned} 1 - z(p_i) &= (1 - z(p_i)) \sum_{k \in I} p_k \\ &= 0, \end{aligned}$$

so that $z(p_i) = 1$ for all $i \in I$. Similarly, $z(q_j) = 1$ for all $j \in J$, meaning $p_i \sim q_j$ for all $i \in I$ and $j \in J$.

Now, we start by assuming that $n < \infty$. For each $i \in I$ and each $j \in J$, let $v_i^* v_i = u_j^* u_j = p_{i_0}$, $v_i v_i^* = p_i$, and $u_j u_j^* = q_j$. Now, consider the map

$$\Phi(x) = \frac{1}{n} \sum_{i \in I} v_i^* x v_i.$$

Then,

$$\begin{aligned} \Phi(xy) &= \frac{1}{n} \sum_{i \in I} v_i^* x y v_i \\ &= \frac{1}{n} \sum_{i, k \in I} v_i^* x v_k v_k^* y v_i \\ &= \frac{1}{n} \sum_{i, k \in I} (p_i v_i^* x v_k p_{i_0}) (p_{i_0} v_k^* y v_i p_{i_0}) \\ &= \frac{1}{n} \sum_{i, k \in I} (p_{i_0} v_i^* y v_k p_{i_0}) (p_{i_0} v_k^* x v_i p_{i_0}) \\ &= \Phi(yx). \end{aligned}$$

Consequently, $\Phi(p_i) = \Phi(q_j) = \Phi(p_{i_0})$ for all $i \in I$ and all $j \in J$, with common image p_{i_0} . Therefore, we get

$$n p_{i_0} = \sum_{i \in I} \Phi(p_i)$$

$$\begin{aligned}
&= \Phi(1) \\
&= \sum_{j \in J} \Phi(q_j) \\
&= mp_{i_0},
\end{aligned}$$

so $n = m$.

Now, suppose n is infinite. By a symmetric argument, we have m is infinite. Let φ be a normal state with support $z \in Z(M)$; that is, $w \in \ker(\varphi)$ if and only if $zwz = 0$. For each $i \in I$, define $\sigma_i: Z(M) \rightarrow p_iMp_i$ by $\sigma_i(z) = zp_i$. This is an isomorphism from the earlier lemma, so we may define φ_i by

$$\varphi_i(x) = \varphi(\sigma_i^{-1}(p_i x p_i)).$$

We have that φ_i is a state with support zp_i . Since σ_i is σ -WOT continuous, we have

$$\begin{aligned}
1 &= \varphi_i(1) \\
&= \varphi_i\left(\sum_{j \in J} q_j\right) \\
&= \sum_{j \in J} \varphi_i(q_j).
\end{aligned}$$

Since $\varphi_i(q_j) \geq 0$ for each $j \in J$, the set of all j for which $\varphi_i(q_j) > 0$ is countable. Yet, if $j \notin J_i$, we have $\varphi_i(q_j) = 0$, meaning $(zp_i)q_j(zp_i) = 0$, and by centrality, we have $zp_iq_jp_iz = 0$, so that $q_jp_iz = 0$.

Yet, for any $j \in J$, we have

$$\sum_{i \in I} q_j p_i z = q_j z,$$

which is nonzero as $z(q_j) = 1$. In particular, this means $q_j p_i z \neq 0$ for some $i \in I$, meaning $j \in J_i$ for some $i \in I$. That is, $J = \bigcup_{i \in I} J_i$, so that $m \leq n\aleph_0 = n$, and by symmetry, $m = n$. \square

Theorem: Let M be a von Neumann algebra of type I. Then, there exists a unique family of orthogonal central projections $\{z_\alpha\}_\alpha$ indexed by cardinal numbers such that $\sum_\alpha z_\alpha = 1$, and $z_\alpha M z_\alpha$ is isomorphic to the tensor product of an abelian von Neumann algebra A_α and $B(H_\alpha)$, where $\dim(H_\alpha) = \alpha$. In particular, we have

$$M \cong \bigoplus_{\alpha} A_\alpha \bar{\otimes} B(H_\alpha).$$

Proof. We note that if $\{z_i\}_{i \in I}$ is a family of α -homogeneous projections, then their sum is also α -homogeneous. Therefore, upon taking suprema, we may find a greatest z_α that is α -homogeneous.

If $z_\alpha \neq 0$, then there exists an orthogonal family $\{e_i\}_{i \in I_\alpha}$ of abelian projections with $z(e_i) = z_\alpha$, and I_α has cardinality α . Since these projections are mutually equivalent, we have

$$z_\alpha M z_\alpha \cong e_1 M e_1 \bar{\otimes} B(\ell_2(I_\alpha)).$$

Since $e_1 M e_1$ is abelian, we only have to show that z_α and z_β are orthogonal for distinct α and β .

If α is infinite, then both I_α and $I_\alpha \setminus \{i_0\}$ have the same cardinality, meaning that $z_\alpha = \sum_{i \in I_\alpha} e_i \sim \sum_{i \in I_\alpha \setminus \{i_0\}} e_i < z_\alpha$, meaning z_α is not a finite projection. In particular, α must be finite if M is finite. \square

Proposition: A von Neumann algebra M is type I if and only if M admits a faithful normal representation $\pi: M \rightarrow B(H)$ such that $\pi(M)'$ is abelian.

Proof. Let $M \subseteq B(H)$ be a von Neumann algebra with abelian commutant M' . Then, M' coincides with $Z(M)$. For each $\xi \in H$, let e_ξ denote the projection of H onto $[Z(M)\xi]$. Since e_ξ commutes with $Z(M)$, it follows that e_ξ is contained in M . Since $e_\xi Z(M) e_\xi \subseteq B([Z(M)\xi]H)$ admits a cyclic vector, namely ξ , it follows that $e_\xi Z(M) e_\xi$ is abelian, so $(e_\xi Z(M) e_\xi)' = e_\xi M e_\xi$ is abelian. Thus, e_ξ is abelian for any $\xi \in H$. Furthermore, for any projection $p \in M$, we have that $e_\xi \leq p$ whenever $\xi \in pH$, so every nonzero projection majorizes an abelian projection, so M is type I.

Now, if M is type I, then we may represent M as

$$M \cong \bigoplus_{\alpha} A_{\alpha} \bar{\otimes} B(H_{\alpha})$$

and take A_{α} as a masa of $B(H_{\alpha})$, meaning that on the Hilbert space

$$H := \bigoplus_{\alpha} H_{\alpha} \otimes H_{\alpha},$$

we have

$$\begin{aligned} M' &= \left(\bigoplus_{\alpha} A'_{\alpha} \bar{\otimes} B(H_{\alpha}) \right)' \\ &= \bigoplus_{\alpha} (A_{\alpha} \otimes \mathbb{C}) \\ &= \bigoplus_{\alpha} A_{\alpha}, \end{aligned}$$

so M' is abelian. \square

Corollary: If M is a von Neumann algebra of type I, then so is M' .

Traces on Finite von Neumann Algebras

Lemma: Let M be a finite von Neumann algebra, and $p \in P(M)$ a nonzero projection. If we have $\{p_i\}_{i \in I}$ a family of pairwise orthogonal projections with $p_i \sim p$ for each i , then I is finite.

Proof. If this were not the case, then we would have a proper subset $J \subsetneq I$ with $\sum_{i \in I} p_i \sim \sum_{j \in J} p_j < \sum_{i \in I} p_i$, implying $\sum_{i \in I} p_i$ is not finite, but the sum of pairwise orthogonal finite projections is finite. \square

Lemma: Let $M \subseteq B(H)$ be a type II_1 von Neumann algebra. Then, there exists a projection $p_{1/2} \in P(M)$ such that $p_{1/2} \sim 1 - p_{1/2}$.

Furthermore, there is a family of projections $\{p_r\}_r$ indexed by dyadic rationals in $[0, 1]$ such that $p_r \leq p_s$ if $r \leq s$, $p_s - p_r \sim p_{s'} - p_{r'}$ whenever $0 \leq r \leq s \leq 1$, $0 \leq r' \leq s' \leq 1$, and $s - r = s' - r'$, and each of the p_r has central support equal to 1.

Proof. Let $\{p_i, q_i\}_{i \in I}$ be a maximal family of pairwise orthogonal projections with $p_i \sim q_i$ for all $i \in I$. Define

$$p_{1/2} = \sum_{i \in I} p_i$$

$$q = \sum_{i \in I} q_i.$$

We have that $p_{1/2} \sim q$, and we claim that $q \sim 1 - p_{1/2}$. If not, then $1 - (p_{1/2} + q) \neq 0$, so since M is type II, it follows that $1 - (p_{1/2} + q)$ is not abelian, so there exists some projection in the corner of M with respect to $1 - (p_{1/2} + q)$ that is strictly less than its central support in the corner. We denote this central support by z . Then, if we set $q_0 = z - p_0$, we have that p_0 and q_0 are not centrally orthogonal, so they have equivalent subprojections. Yet, this contradicts maximality of $\{p_i, q_i\}_{i \in I}$, so it must be the case that $q = 1 - p_{1/2}$.

Now, we construct the family of projections indexed by the dyadic rationals inductively. Let $p_{1/2}$ be defined as above, $p_0 = 0$, and $p_1 = 1$. If $v \in M$ is such that $v^*v = p_{1/2}$ and $vv^* = 1 - p_{1/2}$. Then, $p_{1/2}Mp_{1/2}$ is type II, and specifically type II_1 , since $p_{1/2}$ is a finite projection. If we let $q \sim p_{1/2}$ with $q < p_{1/2}$, then $q + (1 - p_{1/2}) \sim 1$ with $q + (1 - p_{1/2}) < 1$, which contradicts the finitude of 1. Therefore, we may obtain $p_{1/4} \leq p_{1/2}$ with $p_{1/4} \sim p_{1/2} - p_{1/4}$, and set $p_{3/4} = p_{1/2} + vp_{1/2}v^*$, with $p_0 \leq p_{1/4} \leq p_{1/2} \leq p_{3/4} \leq p_1$, with $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for each $k = 0, 1, 2, 3$. Thus, inductively, we obtain a family indexed by dyadic rationals satisfying $p_s - p_r \sim p_{s'} - p_{r'}$ whenever $s - r = s' - r'$.

Now, we verify central supports. Fix a dyadic rational r , and set $z = 1 - z(p_r)$. Let $n \in \mathbb{N}$ be large enough that $s = \frac{1}{2^n} \leq r$. Then, we have that $zp_s \leq zp_r = 0$, and $zp_s \sim z(p_{ks} - p_{(k-1)s})$ for every $1 \leq k \leq 2^n$, meaning that $z(p_{ks} - p_{(k-1)s}) = 0$. Thus,

$$z = z \sum_{k=1}^n (p_{ks} - p_{(k-1)s})$$

$$= 0,$$

so $z(p_r) = 1$. \square

Lemma: Let M be a type II_1 von Neumann algebra, and let $\{p_r\}_r$ be the above collection of projections. If $p \in P(M)$, with $p \neq 0$, then there exists $z \in P(Z(M))$ such that $pz \neq 0$ and $p_r z \preceq pz$ for some positive dyadic rational r .

Proof. Passing to $Mz(p)$ allows us to assume that $z(p) = 1$. If this does not hold, then we would have $p \preceq p_r$ for every $r > 0$, meaning that p would be equivalent to a subprojection of $p_{2^{-k}} - p_{2^{-(k+1)}}$, which contradicts the fact that any family of pairwise orthogonal projections equivalent to a fixed projection is finite. \square

We say a projection $p \in P(M)$ is *monic* if there exists a finite collection of pairwise orthogonal projections $\{p_1, \dots, p_n\}$ such that $p_i \sim p$ and $\sum_{i=1}^n p_i \in Z(M)$. The projections $p_{1/2^k}$ defined above in the II_1 case are monic.

Proposition: If M is a finite von Neumann algebra, then every projection is the sum of pairwise orthogonal monic projections.

Proof. By passing to a maximal sub-family, it suffices to show that every nonzero projection has a nonzero monic subprojection. Furthermore, by passing to the type decomposition, it suffices to consider the cases where M is type II_1 or M is type I_n .

In the case of type II_1 , we get this from the above discussion, and in the case of type I_n , this follows from considering a nonzero abelian subprojection. \square

Definition: Let M be a von Neumann algebra. We say a linear map $\varphi: M \rightarrow Z(M)$ is a *center-valued state* if $\varphi(x^*x) \geq 0$, $\varphi|_{Z(M)} = \text{id}$, and $\varphi(zx) = z\varphi(x)$ whenever $z \in Z(M)$ and $x \in M$.

Lemma: If M is a von Neumann algebra and $\varphi: M \rightarrow Z(M)$ is a center-valued state, then φ is bounded with $\|\varphi\| = 1$.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. Then, M has a normal center-valued state.

Proof. Since $Z(M)$ is an abelian von Neumann algebra, $Z(M)'$ is type I , so it has an abelian projection q with full central support. Then, $qMq \subseteq qZ(M)'q = Z(M)q$, so $\theta(z) = zq$ defines a normal isomorphism from $Z(M)$ onto $Z(M)q$. Setting $\varphi(x) = \theta^{-1}(qxq)$, then φ is a normal center-valued state. \square

Lemma: Let M be a von Neumann algebra, and let $\tau: M \rightarrow Z(M)$ be a center-valued state. The following are equivalent:

- (i) $\tau(xy) = \tau(yx)$ for all $x, y \in M$;
- (ii) $\tau(xx^*) = \tau(x^*x)$ for all $x \in M$;
- (iii) $\tau(p) = \tau(q)$ for all equivalent projections $p, q \in P(M)$.

Proof. We only need to show (iii) implies (i). First, for all $p \in P(M)$ and all unitaries u , we have $\tau(upu^*) = \tau(p)$. Since τ is bounded, then by applying the functional calculus, we have that $\tau(uxu^*) = \tau(x)$ for all self-adjoint $x \in M$. Separating real and imaginary parts, and replacing x with xu , it follows that $\tau(ux) = \tau(xu)$ for all $x \in M$ and all unitaries u . Since every operator is a linear combination of four unitaries, this shows (i). \square

As discussed above, such a center-valued state is known as a center-valued trace.

Lemma: Let $\varphi: M \rightarrow Z(M)$ be a normal center-valued state. Then, for every $\varepsilon > 0$,

there exists $p \in P(M)$ with $\varphi(p) \neq 0$ and for all $x \in pMp$,

$$\varphi(xx^*) \leq (1 + \varepsilon)\varphi(x^*x).$$

Proof. Let $q_0 = 1 - \sum_{i \in I} q_i$, where $\{q_i\}_{i \in I}$ is a maximal family of pairwise orthogonal projections with $\varphi(q_0) = 0$. By normality, we must have that $\varphi(q_0) = 1$, and φ is faithful when restricted to $q_0 M q_0$.

Let $\{e_i, f_i\}_{i \in I}$ be a maximal family of projections with $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ pairwise orthogonal, $e_i \sim f_i$ for each i , and $\varphi(e_i) > \varphi(f_i)$ for each i . Setting $e = q_0 - \sum_{i \in I} e_i$ and $f = q_0 - \sum_{i \in I} f_i$, then unless φ is already a trace, we have $\varphi(f) > \varphi(e) \geq 0$. Hence, $f \neq 0$, so $e \sim f$, meaning $e \neq 0$.

Let μ be the smallest number such that $\varphi(\bar{e}) \leq \mu\varphi(\bar{f})$ whenever $\bar{e} \leq e$, $\bar{f} \leq f$, and $\bar{e} \sim \bar{f}$. Then, $\mu \neq 0$ since $\varphi(e) \neq 0$, and there exists $\bar{e} \leq e$, $\bar{f} \leq f$ such that $\bar{e} \sim \bar{f}$, $(1 + \varepsilon)\varphi(\bar{e}) \not\leq \mu\varphi(\bar{f})$. By cutting by a suitable central projection, we may assume that $(1 + \varepsilon)\varphi(\bar{e}) > \mu\varphi(\bar{f})$. \square

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