

Problem (Problem 1): Describe the topology of the Grassmanian $\text{Gr}(k, n)$ in a uniform way, so that \mathbb{RP}^n becomes the special case of $\text{Gr}(1, n)$.

Solution: We let elements of $\text{Gr}(k, n)$ be defined as equivalence classes of linearly independent k -tuples of vectors in \mathbb{R}^n , where $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if $\text{span}\{v_1, \dots, v_k\} = \text{span}\{w_1, \dots, w_k\}$.

By extending (v_1, \dots, v_k) and (w_1, \dots, w_k) to ordered bases $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$, we see that these k -tuples are equivalent if and only if there is a change of basis transformation Q with matrix representation

$$Q = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n - k) \times (n - k)$ matrix. The subgroup of all such $Q \subseteq \text{GL}_n(\mathbb{R})$, which we call P , is the stabilizer of $\text{Gr}(k, n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $\text{GL}_n(\mathbb{R})$ acts transitively on all ordered bases of \mathbb{R}^n), we obtain $\text{Gr}(k, n) \cong \text{GL}_n(\mathbb{R})/P$, where the latter coset space is given the quotient topology.

Note that this definition comports with the definition of \mathbb{RP}^n as the space of one-dimensional subspaces, as the invertible 1×1 matrices are precisely the nonzero scalars.

Problem (Problem 2): Fix an inner product on \mathbb{R}^n . Show that the map $V \mapsto V^\perp$ induces a C^∞ diffeomorphism $\text{Gr}(k, n) \rightarrow \text{Gr}(n - k, n)$.

Solution: We know that, since there is an inner product, we may express the smooth atlas of $\text{Gr}(n, k)$ by $\{(\mathcal{U}_V, \varphi_V)\}$, where

$$\mathcal{U}_V = \{W \in \text{Gr}(k, n) \mid W \cap V^\perp = 0\},$$

and $\varphi = P_{V^\perp} P_V|_{\mathcal{U}_V}^{-1}$ is the sequence of projections. By pre-composing with the map $V \mapsto V^\perp$, we get the atlas $\{(\mathcal{U}_{V^\perp}, \varphi_{V^\perp})\}$ for $\text{Gr}(n - k, n)$ consisting of charts of the form

$$\begin{aligned} \mathcal{U}_{V^\perp} &= \{W \in \text{Gr}(n - k, n) \mid W \cap V = 0\} \\ \varphi_{V^\perp} &= P_V P_{V^\perp}|_{\mathcal{U}_{V^\perp}}^{-1} \end{aligned}$$

Since the maps $\varphi_V \circ (V \mapsto V^\perp) \circ \varphi_{V^\perp}^{-1}$ are a composition of smooth bijections with smooth inverses, we see that this is a C^∞ diffeomorphism between $\text{Gr}(k, n) \cong \text{Gr}(n - k, n)$.

Problem (Problem 3): Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Problem (Problem 4): Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either \mathbb{R} or S^1 , depending on whether it is compact or not.

Solution: Let M be a connected, paracompact manifold with dimension 1, and let $\{(\mathcal{U}_i, \varphi_i)\}_{i \in I}$ be an atlas for M , where φ_i are homeomorphisms between \mathcal{U}_i and \mathbb{R} .

Let $\{V_j\}_{j \in J}$ be a locally finite refinement of $\{\mathcal{U}_i\}_{i \in I}$, where the restrictions $\psi_j := \varphi_i|_{V_j}$ are homeomorphisms to $O_j \subseteq \mathbb{R}$. We see that for any $p \in M$, since the family of V_j with $p \in V_j$, which we call $\mathcal{V}_p = \{V_j \mid p \in V_j\}$, is finite, the intersection $\bigcap \mathcal{V}_p$ is open; similarly, the intersection $\bigcap \mathcal{O}_p \subseteq \mathbb{R}$ is open, where $\mathcal{O}_p = \{\varphi|_{V_j}(V_j) \subseteq \mathbb{R} \mid V_j \in \mathcal{V}_p\}$.

We see that $M = \bigcup_{p \in M} \bigcap \mathcal{V}_p$. Note that for any distinguished point p_1 , the corresponding sets $\bigcap \mathcal{V}_{p_1}$ and $\bigcup_{p \neq p_1} \bigcap \mathcal{V}_p$ must have nonempty (open) intersection, by the assumption that M is connected. Thus, the corresponding union $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ is an open and connected subset of \mathbb{R} . We may similarly map $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ into S^1 by composing with the quotient map.

Now, if M is compact, then $\bigcup_{p \in M} \bigcap \mathcal{V}_p$ covers M , so there is a finite subcover $M = \bigcup_{i=1}^n \bigcap \mathcal{V}_{p_i}$, so that $\bigcup_{i=1}^n \bigcap \mathcal{O}_{p_i}$ fully covers the corresponding range, meaning that, composing with the quotient map $\bigcup_{i=1}^n \bigcap \mathcal{O}_{p_i}$, we have that $M \cong S^1$. Similarly, if M is non-compact, then $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ is an open and connected subset of \mathbb{R} that does not admit any finite subcover, hence it is homeomorphic to \mathbb{R} .

Problem (Problem 5): In this problem, we prove a weak version of the Whitney Embedding Theorem.

- Find a C^∞ function λ on \mathbb{R}^n with values in $[0, 1]$ such that λ takes the value 1 on the closed ball $B(0, 1)$, and vanishes outside the closed ball $B(0, 2)$.
- Suppose M is a compact C^k manifold of dimension n . Find a C^k atlas $\{U_i, \varphi_i\}_{i \in I}$ such that $\varphi_i(U_i)$ contains $B(0, 2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0, 1))^\circ$.
- Let λ_i be defined by $\lambda \circ \varphi_i$ on U_i , and 0 outside U_i . Let $f_i: M \rightarrow \mathbb{R}^n$ be defined by $\lambda_i \circ \varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Problem (Problem 6): Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Problem (Problem 7): Let X and Y be topological spaces, and let $C(X, Y)$ be the set of continuous maps from X to Y . Equip $C(X, Y)$ with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space and let X be compact. We note that a neighborhood basis in the topology of uniform convergence on $C(X, Y)$ consists of sets of the form

$$U_{f,\varepsilon} = \left\{ g \mid \sup_{x \in X} d(f(x), g(x)) < \varepsilon \right\}.$$

Similarly, a subbase for the compact open topology consists of sets of the form

$$U_{f,K,\varepsilon} = \left\{ g \mid \sup_{x \in K} d(f(x), g(x)) < \varepsilon \right\};$$

the fact that Y is a metric space allows us to take this refinement of the compact-open topology.

Thus, to prove that the compact-open topology and the topology of uniform convergence are equivalent, we show that any basis element of the topology of uniform convergence is contained in a basis element of the compact-open topology, and vice versa.

First, we see that almost by definition, if $K \subseteq X$ is any compact subset, then

$$U_{f,\varepsilon} \subseteq U_{f,K,\varepsilon},$$

as any function whose supremum distance is less than ε over X must have that supremum distance hold over $K \subseteq X$.

Now, in the reverse direction, we fix f and ε . We wish to show that there is a finite family of subsets U_{K_i, V_i} with $f \in U_{K_i, V_i}$ for each i , and their intersection lies in $U_{f,\varepsilon}$. We see that every point $x \in X$ has a pre-compact open neighborhood U_x such that $f(\overline{U_x}) \subseteq U(f(x), \varepsilon/3)$. The family $\{x \in X\} U_x$ is an open cover for X , so admits a finite subcover $\{U_{x_i}\}_{i=1}^n$. Since each $\overline{U_{x_i}}$ is compact, and $f \in U_{\overline{U_{x_i}}, U(f(x_i), \varepsilon/3)}$

for each i , we see that

$$V = \bigcap_{i=1}^n \overline{U_{U_{x_i}, U(f(x_i), \varepsilon/3)}}$$

is an open subset in the compact-open topology on $C(X, Y)$ that contains f and is contained in $U_{f, \varepsilon}$, so the topologies are thus equal.