Quasi-Review: Locally Compact Groups and the Banach ∗-algebra $L_1(G)$

Basic Properties of Topological Groups

A topological group is a group G equipped with a topology such that the operations

$$(x, y) \mapsto xy$$
$$x \mapsto x^{-1}$$

are continuous. In general, we will let 1 denote the identity of G.

We call G a locally compact group if the topology of G is locally compact. Equivalently, the topology of G is locally compact if there there is a neighborhood system about 1 consisting of pre-compact open sets.

We will refer to the following subset operations in G regularly:

$$Ax = \{\, ax \mid a \in A \,\}$$
$$xA = \{\, xa \mid a \in A \,\}$$
$$A^{-1} = \{\, a^{-1} \mid a \in A \,\}$$
$$AB = \{\, ab \mid a \in A, b \in B \}.$$

A subset V is called *symmetric* if $V = V^{-1}$.

These are some useful propositions.

Proposition: Let G be a topological group.

(i) The topology of G is invariant under translations and inversion.

(ii) For every neighborhood U of 1, there is a symmetric neighborhood V of 1 such that $VV \subseteq U$.

(iii) If H is a subgroup of G, then so is $\overline{H}$.

(iv) Every open subgroup of G is closed.

(v) If A and B are compact subsets of G, then so is AB.

Proposition: Suppose H is a subgroup of the topological group G.

(i) If H is closed, then G/H is Hausdorff.

(ii) If G is locally compact, then so is G/H.

(iii) If H is normal, then G/H is a topological group.

We will assume all the time that G is Hausdorff, via the following proposition.

Corollary: If G is a T1 topological group, then G is Hausdorff. If G is not T1, then $\overline{\{1\}}$ is a closed normal subgroup with $G/\overline{\{1\}}$ is a Hausdorff topological group.

Proposition: Every locally compact group G has a subgroup G_0 that is open, closed, and σ-compact.

Considering various functions $f\colon G \to \mathbb{C}$, we define the left and right translates of f as

$$L_y f(x) = f\big(y^{-1}x\big)$$
$$R_y f(x) = f(xy),$$

and say that f is left (right) uniformly continuous if $\|L_y f - f\|_u \to 0$ ($\|R_y f - f\|_u \to 0$) as $y \to 1$.

Proposition: If $f \in C_c(G)$, then f is left and right uniformly continuous.

A left *Haar measure* is a nonzero Radon measure μ on G such that $\mu(xE) = \mu(E)$ for every Borel subset $E \subseteq G$.

Proposition: Every locally compact group G admits a left Haar measure λ. This Haar measure is unique up to a constant multiple.

If we have a left Haar measure λ, then if we define

$$\lambda_x(E) = \lambda(Ex),$$

we have that λ_x is again a left Haar measure, so there is some number $\Delta(x)$ such that $\lambda_x = \Delta(x)\lambda$, where $\Delta(x)$ is independent of the original choice of λ.

The function $\Delta\colon G \to (0, \infty)$ defined as such is known as the *modular function* of G.

Proposition: The function Δ is a continuous homomorphism from G to $\mathbb{R}_{>0}$, and for any $f \in L_1(\lambda)$, we have

$$\int R_y f \, d\lambda = \Delta\big(y^{-1}\big) \int f \, d\lambda.$$

We call G *unimodular* if $\Delta \equiv 1$.

Proposition: If $G/[G, G]$ is compact, then G is unimodular.

Convolutions and $L_1(G)$

If G is a locally compact group, we let $M(G)$ denote the space of complex-valued Radon measures on G. The convolution of two measures $\mu, \nu \in M(G)$ is given as follows. If we let

$$I(\phi) = \iint \phi(xy) \, d\mu(x)d\nu(y),$$

then we observe that $I(\phi)$ is a linear functional on $C_0(G)$ that satisfies

$$|I(\phi)| \leq \|\phi\|_u \|\mu\| \|\nu\|,$$

meaning that it is given by a measure $\mu * \nu \in M(G)$ with $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$. We call $\mu * \nu$ the convolution of μ and ν.

Observe that if $\delta_x \in M(G)$ is the point mass at $x \in G$, then

$$\int \phi \, d(\delta_x * \delta_y) = \iint \phi(uv) \, d\delta_x(u)\delta_y(v)$$

$$= \phi(xy)$$
$$= \int \phi \, d\delta_{xy},$$

meaning that $\delta_x * \delta_y = \delta_{xy}$.

The estimate $\|\mu * \nu\| \leq \|\mu\| \|\mu\|$ gives that convolution makes $M(G)$ a Banach algebra, which we call the *measure algebra* of G. Furthermore, $M(G)$ admits an involution

$$\mu^*(E) = \overline{\mu(E^{-1})},$$

so that

$$\int \phi \, d\mu^* = \int \phi(x^{-1}) \, \overline{d\mu(x)}.$$

We may identify the space $L_1(G)$ to be the subspace of $M(G)$ where a function f is identified with the measure $f(x)dx$. If $f, g \in L_1(G)$, then the convolution of f and g is the function

$$f * g(x) = \int f(y)g(y^{-1}x) \, dy.$$

With convolution and the involution given by

$$f^*(x)dx = \overline{f(x^{-1})}d(x^{-1})$$
$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})},$$

we have that $L_1(G)$ is a Banach $*$-algebra known as the *group algebra* of G.

Now, we observe that if G is discrete, then if δ_e is the point mass at 1, we have that $f * \delta = \delta * f = f$ for any function f. If G is not discrete, we must use an *approximate identity* for G. In particular, we can select a family of mollifiers $\{\psi_U\}_{U \in \mathcal{U}}$ such that

$$\|\psi_U * f - f\| \to 0$$
$$\|f * \psi_U - f\| \to 0$$

if f is uniformly continuous and $U \to \{1\}$ in a neighborhood system $\mathcal{U}$ of 1.

Homogeneous Spaces

If G is a locally compact group, then G can act on a locally compact Hausdorff space by homeomorphisms. Recall from algebra that the group action is transitive if there is one orbit. We call S a G-space.

The standard example of a transitive G-space is the quotient space G/H for some closed subgroup H of G. These are, to an extent, the only G-spaces, as follows from the orbit-stabilizer theorem. If S is a G-space, then we may define a map $\phi \colon G \to S$ by $\phi(x) = x \cdot s_0$, and take the quotient by the stabilizer subgroup

$$H = \{x \in G \mid x \cdot s_0 = s_0\},$$

so that $\Phi \colon G/H \to S$ has $\Phi \circ q = \phi$ for the quotient map $q \colon G \to G/H$ is a continuous bijection.

Proposition: If G is σ-compact, then Φ is a homeomorphism.

Proof. It suffices to show that ϕ maps open sets in G to open sets in S. Suppose U is open in G, $x_0 \in U$. Pick a compact symmetric neighborhood V of 1 such that $x_0 VV \subseteq U$. Since G is σ-compact, there is a countable $\{y_n\}_{n \geq 1} \subseteq G$ such that $\{y_n V\}_{n \geq 1}$ covers G. Then, we have

$$S = \bigcup_{n=1}^{\infty} \phi(y_n V).$$

The sets $\phi(y_n V)$ are homeomorphic to $\phi(V)$ since the map $s \mapsto y_n \cdot s$ is a homeomorphism of S, and all the $y_n V$ are compact, hence closed.

By Baire Category Theorem for LCH spaces, it follows that $\phi(V)$ has an interior point, which we call $\phi(x_1)$ for $x_1 \in V$. Then, $\phi(x_0)$ is an interior point of $\phi(x_0 x_1^{-1} V)$, and $x_0 x_1^{-1} V \in x_0 VV \subseteq U$, so that $\phi(x_0)$ is an interior point of $\phi(U)$. Thus $\phi(U)$ is open. $\qquad\square$

If S is a transitive G-space that is isomorphic to a quotient space G/H, then will write $S \cong G/H$, and call S a *homogeneous space*. The identification is dependent on the choice of base point, but the identity $s_0' = x_0 \cdot x_0$ induces a map $H' = x_0 H x_0^{-1}$, inducing a G-equivariant homeomorphism $G/H \cong G/H'$.

We will address the question of whether there is a G-invariant Radon measure on G/H — that is, a radon measure μ such that $\mu(xE) = \mu(E)$ for every $x \in G$.

We assume that G is a locally compact group with left Haar measure dx, a H is a closed subgroup of G with left Haar measure $d\xi$, and $q: G \to G/H$ is the quotient map $q(x) = xH$, and Δ_G, Δ_H the corresponding modular functions.

Let $P: C_c(G) \to C_c(G/H)$ be defined by

$$Pf(xH) = \int_H f(x\xi) \, d\xi.$$

This is well-defined by left-invariance of $d\xi$. If $\phi \in C(G/H)$, then

$$P((\phi \circ q) \cdot f) = \phi \cdot Pf.$$

Lemma: If $E \subseteq G/H$ is compact, then there is a compact $K \subseteq G$ with $q(K) = E$.

Proof. Let V be an open neighborhood of 1 in G with compact closure. Since q is an open map, $q(xV)$ is an open cover of E, so there is a finite subcover $q(x_j V)$. Let $K = q^{-1}(E) \cap \bigcup_{j=1}^{n} x\overline{V}$. Then, since $q^{-1}(E)$ is closed, K is compact with $q(K) = E$. $\qquad\square$

Lemma: If $F \subseteq G/H$ is compact, then there is $f \geq 0$ in $C_c(G)$ with $Pf = 1$ on F.

Proof. Let E be a compact neighborhood of F in G/H. We find $K \subseteq G$ compact such that $q(K) = E$. Select positive $g \in C_c(G)$ with $g > 0$ on K, and $\phi \in C_c(G/H)$ supported in E with $\phi = 1$ on F. Set

$$f = \frac{\phi \circ q}{Pg \circ q} g,$$

with the fraction equal to zero whenever the numerator is zero. We have f is continuous, since

$Pg > 0$ on $\text{supp}(\phi)$, has support contained in $\text{supp}(g)$, and $Pf = \frac{\phi}{Pg}Pg = \phi$. $\square$

Proposition: If $\phi \in C_c(G/H)$, then there exists $f \in C_c(G)$ with $Pf = \phi$ and $q(\text{supp}(f)) = \text{supp}(\phi)$, and has $f \geq 0$ if $\phi \geq 0$.

Proof. If $\phi \in C_c(G)$, then by the previous lemma, then there exists $g \geq 0$ in $C_c(G)$ with $Pg = 1$ on $\text{supp}(\phi)$. Letting $f = (\phi \circ q)g$, then $Pf = \phi(Pg) = \phi$. $\square$

Theorem: Let G be a locally compact group, H a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, we have

$$\int_G f(x)\,dx = \int_{G/H} Pf\,d\mu$$
$$= \int_{G/H}\int_H f(x\xi)\,d\xi\,d\mu(xH)$$

for any $f \in C_c(G)$.

Proof. Suppose there is a G-invariant measure μ. Then, $f \mapsto \int Pf\,d\mu$ is a nonzero left-invariant positive linear functional on $C_c(G)$, so by the uniqueness of Haar measure, we have $\int Pf\,d\mu = c\int f(x)\,dx$ for some c.

This formula fully determines μ, meaning that μ is unique up to the arbitrary constant factor in Haar measure. We may assume that $c = 1$, so we have for any $\eta \in H$,

$$\Delta_G(\eta)\int_G f(x)\,dx = \int_G f\left(x\eta^{-1}\right)\,dx$$
$$= \int_{G/H}\int_H f\left(x\xi\eta^{-1}\right)\,d\xi\,d\mu(xH)$$
$$= \Delta_H(\eta)\int_{G/H}\int_H f(x\xi)\,d\xi\,d\mu(xH)$$
$$= \Delta_H(\eta)\int_G f(x)\,dx,$$

so that $\Delta_G(\eta) = \Delta_H(\eta)$. $\square$

Representations of a Group and its Group $*$-Algebra

If G is a locally compact group, then a *unitary representation* of G is a homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$, where $\mathcal{U}(\mathcal{H}_\pi)$ denotes the unitary group of a Hilbert space $\mathcal{H}_\pi$. We call $\mathcal{H}_\pi$ the *representation space* of π, and the dimension of $\mathcal{H}_\pi$ is called the dimension (or degree) of the representation.

We do not require π to be continuous in the norm topology of $\mathcal{B}(\mathcal{H}_\pi)$, but as it turns out, both weak and strong continuity are equivalent as the WOT and SOT coincide on $\mathcal{U}(\mathcal{H}_\pi)$. If $(T_\alpha)_\alpha \to T$ is a net of unitary operators converging in WOT, then for any $u \in \mathcal{H}_\pi$, we have

$$\|(T_\alpha - T)u\|^2 = \|T_\alpha u\|^2 - 2\,\text{Re}\langle T_\alpha u, Tu\rangle + \|Tu\|^2$$

$$= \|u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu \rangle,$$

and the latter term converges to $2\|Tu\|^2 = 2\|u\|^2$, so that $\|T_\alpha u - Tu\| \to 0$.

If G acts on a locally compact Hausdorff space S, then G acts on $C(S)$ by $(\pi(g)f)(s) = f(g^{-1} \cdot s)$. If S has a G-invariant Radon measure, then π defines a unitary representation on $L_2(\mu)$.

The most important representation is the representation on $L_2(G)$ induced by the action of G on itself by left-multiplication. This defines $(\lambda(g)f)(y) = f(x^{-1}y)$.

Any unitary representation π on $\mathcal{H}_\pi$ induces a representation $\overline{\pi}$ on the dual space $\overline{\mathcal{H}_\pi}$, determined by $\overline{\pi}(x) = \langle \cdot, \pi(x^{-1}) \rangle$. We call $\overline{\pi}$ the contragradient of π.

If π_1 and π_2 are unitary representations of G, then an intertwining operator for π_1 and π_2 is a bounded linear map $T \colon H_1 \to H_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. We say that π_1 and π_2 are *unitarily equivalent* if the set of intertwiners admits a unitary map.

If $\mathcal{M}$ is a closed subspace of $\mathcal{H}_\pi$, then we say $\mathcal{M}$ is invariant for π if $\pi(x)\mathcal{M} \subseteq \mathcal{M}$ for all $x \in G$. If $\mathcal{M}$ is a nontrivial invariant subspace, then $\pi|_{\mathcal{M}}$ defines a representation of G on $\mathcal{M}$, known as a subrepresentation of π. If π admits a nontrivial invariant subspace, then we say π is reducible; else, π is irreducible.

If $\{\pi_i\}_{i \in I}$ is a family of unitary representations, their direct sum $\bigoplus_{i \in I} \pi_i$ is the representation on $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ given by

$$\pi(x)((v_i)_i) = \sum_{i \in I} \pi_i(x)v_i.$$

We can see that subrepresentations always arise as direct summands.

Proposition: If $\mathcal{M}$ is invariant under π, then so is $\mathcal{M}^\perp$.

Proof. Let $u \in \mathcal{M}$ and $v \in \mathcal{M}^\perp$. Then,

$$\langle \pi(x)v, u \rangle = \langle v, \pi(x^{-1})u \rangle$$
$$= 0,$$

so $\pi(x)v \in \mathcal{M}^\perp$. $\qquad\qquad\square$

If π is a unitary representation, and $u \in \mathcal{H}_\pi$, then the closed linear span of $\{\pi(x)u \mid x \in G\}$ is called the *cyclic subspace* generated by u. Observe that $\mathcal{M}_u$ is invariant under π; if $\mathcal{M}_u = \mathcal{H}_\pi$, then u is called a cyclic vector for π, and π is called a cyclic representation if it has a cyclic vector.

Proposition: Every unitary representation is a direct sum of cyclic representations.

Proof. Let π be a representation on $\mathcal{H}_\pi$. By Zorn's Lemma, there is a maximal collection $\{\mathcal{M}_i\}_{i \in I}$ of mutually orthogonal cyclic subspaces of $\mathcal{H}_\pi$. If there were some $u \in \mathcal{H}_\pi$ orthogonal to all the $\mathcal{M}_i$, then the cyclic subspace generated by u would also be orthogonal to the $\mathcal{M}_i$, which would contradict maximality. Therefore,

$$\mathcal{H}_\pi = \bigoplus_{i \in I} \mathcal{M}_i$$

$$\pi = \bigoplus_{i \in I} \pi|_{\mathcal{M}_i}.$$

$\square$

Proposition: Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}_\pi$, and let P be the orthogonal projection onto $\mathcal{M}$. Then, $\mathcal{M}$ is invariant under π if and only if P is an intertwiner for π.

Proof. If $P\pi(x) = \pi(x)P$, then for any $v \in \mathcal{M}$, we have $\pi(x)v = \pi(x)Pv = P\pi(x)v \in \mathcal{M}$, so $\mathcal{M}$ is invariant.

Conversely, if $\mathcal{M}$ is invariant, then we have $\pi(x)Pv = \pi(x)v = P\pi(x)v$ whenever $v \in \mathcal{M}$ and $\pi(x)Pv = 0 = P\pi(x)v$ whenever $v \in \mathcal{M}^\perp$, so $\pi(x)P = P\pi(x)$. $\square$

Functions of Positive Type

References

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