Math 395

Homework 7

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Problem 1

We say a field K/F is normal if K is the splitting field of a collection of polynomials. Equivalently, every polynomial in F[x] that has a root in K splits into linear factors over K. Let $\alpha \in \mathbb{R}$ such that $\alpha^4 = 5$. We will show that $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$, but $\mathbb{Q}(\alpha + i\alpha)$ is not normal over \mathbb{Q} .

Note that $(\alpha + i\alpha)^2 = 2i\alpha^2$. Thus, $\mathbb{Q}(\alpha + i\alpha) = \mathrm{Spl}_{\mathbb{Q}(i\alpha^2)}(x^2 - 2i\alpha^2)$, so $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$.

Suppose toward contradiction that $\mathbb{Q}(\alpha + i\alpha)$ is normal over \mathbb{Q} . Notice that $(\alpha + i\alpha)^4 = -20$, as is $(\alpha - i\alpha)^4$. Thus, $\alpha + i\alpha$ and $\alpha - i\alpha$ are roots of $x^4 + 20$. Since $\alpha, i, i\alpha \in \mathbb{Q}(\alpha + i\alpha)$, it is the case that $\mathbb{Q}(\alpha, i) \subseteq \mathbb{Q}(\alpha + i\alpha)$. However, we have

$$[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = [\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}]$$
$$= (2)(4)$$
$$= 8.$$

and $[\mathbb{Q}(\alpha + i\alpha) : \mathbb{Q}] = 4$, as $m_{\alpha + i\alpha, \mathbb{Q}}(x) = x^4 + 20$. \bot

Problem 2

The roots of $f(x)=(x^5-2)(x^2-2)$ are $\pm\sqrt{2},\zeta_5^k\sqrt[5]{2}$ for k=0,1,2,3,4. We can see that $\mathbb{Q}(\zeta_5,\sqrt{2},\sqrt[5]{2})$ contains the roots of $(x^5-2)(x^2-2)$, so $\mathrm{Spl}_{\mathbb{Q}}(f(x))\subseteq \mathbb{Q}(\zeta_5,\sqrt{2},\sqrt[5]{2})$. Additionally, we see that $\sqrt[5]{2}\in\mathrm{Spl}_{\mathbb{Q}}(f(x))$, $\zeta_5=\frac{\zeta_5\sqrt[5]{2}}{\sqrt[5]{2}}\in\mathrm{Spl}_{\mathbb{Q}}(f(x))$, and $\sqrt{2}\in\mathrm{Spl}_{\mathbb{Q}}(f(x))$. Thus, $\mathbb{Q}(\zeta_5,\sqrt[5]{2},\sqrt{2})=\mathrm{Spl}_{\mathbb{Q}}(f(x))$.

For x^6+x^3+1 , we have that $x^6+x^3+1=\frac{x^9-1}{x^3-1}$. Therefore, the roots of x^6+x^3+1 are ζ_9^d , where $\gcd(d,9)=1$ (since $9=3^2$, every $n\neq 0,3,6$ is a root of x^6+x^3+1). Therefore, we can see that $x^6+x^3+1=\Phi_9(x)$, meaning $\operatorname{Spl}_{\mathbb{Q}}(x^6+x^3+1)=\mathbb{Q}(\zeta_9)$.

Problem 3

For any prime p and any nonzero $a \in \mathbb{F}_p$, we will prove that $f(x) = x^p - x + a$ is irreducible and separable over \mathbb{F}_p .

First, we have that $D_x(f(x)) = px^{p-1} - 1 = -1$, meaning that $gcd(f(x), D_x(f(x))) = 1$, so f is separable.

Let α be a root of f. Then, we have that $\alpha^p - \alpha + a = 0$. Notice that for $j \in \mathbb{F}_p$, $(\alpha + j)^p = \alpha^p + j^p = \alpha^p + j$, meaning that $(\alpha + j)^p - (\alpha + j) + a = 0$, so $\alpha + j$ is a root of f.

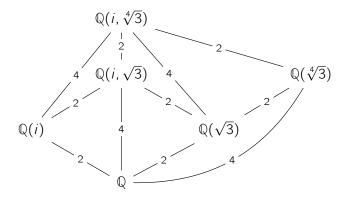
Suppose toward contradiction that f is reducible over \mathbb{F}_p . Then, for some $\alpha \in \mathbb{F}_p$, we must have

$$x^{p} - x + a = (x - \alpha)(x - (\alpha + 1))(x - (\alpha + 2)) \cdots (x - (\alpha + p - 1)),$$

However, by definition, this means that there is some $k \in \mathbb{F}_p$ such that $\alpha + k = 0$, meaning $a = \prod_{i=0}^{p-1} (\alpha + i) = 0$. \bot

Problem 6

To find the subfields of $\mathbb{Q}(i, \sqrt[4]{3})$, we see that the basis of $\mathbb{Q}(i, \sqrt[4]{3})$ over \mathbb{Q} is $\{1, \sqrt[4]{3}, \sqrt{3}, \sqrt[4]{27}, i, i\sqrt[4]{3}, i\sqrt{3}, i\sqrt{27}\}$, meaning $[\mathbb{Q}(i, \sqrt[4]{3}) : \mathbb{Q}] = 8$. Finding subspaces of $\mathbb{Q}(i, \sqrt[4]{3})$, we arrive at the following diagram.



For any subfield $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(i, \sqrt[4]{3})$, it must be the case that $[F : \mathbb{Q}] = 2^k$ for some k = 0, 1, 2, 3. Therefore, it must be the case that all subfields are of degree 1, 2, 4, 8.

Suppose there is any subfield $\mathbb{Q} \subseteq E \subseteq \mathbb{Q}(i)$. Then, it must be the case that $[E : \mathbb{Q}] = 1$ or $[E : \mathbb{Q}] = 2$, meaning $E = \mathbb{Q}$ or $E = \mathbb{Q}(i)$. The same argument applies for all degree 2 extensions in the above diagram.