Problem (Problem 1)**:** Let $T: V \to W$ be a linear transformation between $\mathbb{F}$-vector spaces. Show that T is injective if and only if T maps $\mathbb{F}$-linearly independent subsets of V to $\mathbb{F}$-linearly independent subsets of W.

Solution: Let T be injective. We claim that if $\{v_1, \dots, v_n\}$ is linearly independent in V, then $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W. We see that if

$$\sum_{j=1}^{n} a_j T v_j = 0_W,$$

then

$$T\left(\sum_{j=1}^{n} a_j v_j\right) = 0_W,$$

meaning that

$$\sum_{j=1}^{n} a_j v_j \in \ker(T).$$

Now, since T is injective, $\ker(T) = \{0_V\}$, meaning that $\sum_{j=1}^{n} a_j v_j = 0_V$. Yet, since $\{v_1, \dots, v_n\}$ is linearly independent, this means $a_j = 0$ for each j, so $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W.

Now, let T map linearly independent subsets of V to linearly independent subsets of W. If $\mathcal{B}_V = \{v_i\}_{i \in I}$ is a basis for V, then since $\mathcal{B}_V$ is linearly independent, $\mathcal{C} = \{Tv_i\}_{i \in I}$ is a linearly independent subset of W, which can be extended to a basis $\mathcal{B}_W$. Since $\mathcal{C} \subseteq \mathcal{B}_W$, we see that any linear combination in $\mathcal{B}_W$ yields 0 if and only if every coefficient is zero, meaning that $\ker(T) = \{0_V\}$, so T is injective.

Problem (Problem 2)**:** Let $P_{n+1}(\mathbb{R})$ be the space of polynomials with real coefficients of degree $\leqslant n+1$. Prove that for any n points $a_1, \dots, a_n \in \mathbb{R}$, there exists a nonzero polynomial $f \in P_{n+1}(\mathbb{R})$ such that $f(a_j) = 0$ for each j, and $\sum_{j=1}^{n} f'(a_j) = 0$.

Solution: Based on the first condition, we see that the product $\prod_{j=1}^{n}(x - a_j)$ must divide the polynomial f, and since f has degree at most $n+1$, we must have $f(x) = (x - L)\prod_{j=1}^{n}(x - a_j)$ for some $a, b \in \mathbb{R}$. Writing $f'(x)$, we see that

$$f'(x) = \prod_{j=1}^{n}(x - a_j) + (x - L)\sum_{i=1}^{n}\prod_{j \neq i}(x - a_j),$$

implying that

$$\sum_{i=1}^{n} f'(a_i) = \sum_{i=1}^{n}(a_i - L)\prod_{j \neq i}(a_i - a_j).$$

By setting

$$0 = \sum_{i=1}^{n}(a_i - L)\prod_{j \neq i}(a_i - a_j),$$

we get

$$L = \frac{1}{\sum_{i=1}^{n}\prod_{j \neq i}(a_i - a_j)}\sum_{i=1}^{n} a_i \prod_{j \neq i}(a_i - a_j),$$

which is well-defined whenever the a_i are distinct.

Problem (Problem 3)**:** Let $T\colon V \to W$ be a linear map of finite-dimensional vector spaces, and let $W_0 \subseteq W$ be a subspace.

(a) Show that $T^{-1}(W_0) = \{v \in V \mid Tv \in W_0\}$ is a subspace of V.

(b) Assuming T is surjective, express $\dim\left(T^{-1}(W_0)\right)$ in terms of $\dim(W_0)$ and $\dim(\ker(T))$.

Solution:

(a) We see that if $v_1, v_2 \in T^{-1}(W_0)$ and $\alpha \in \mathbb{R}$, then since $Tv_1, \alpha Tv_2 \in W_0$, we have $Tv_1 + \alpha Tv_2 \in W_0$, so by linearity, $T(v_1 + \alpha v_2) \in W_0$, meaning $v_1 + \alpha v_2 \in T^{-1}(W_0)$, so $T^{-1}(W_0)$ is a subspace of V.

(b) First, since T is surjective, $T\left(T^{-1}(W_0)\right) = W_0$. Therefore, by restricting the map T, we get the surjective map $T'\colon T^{-1}(W_0) \to W_0$, and since $\ker(T) \subseteq T^{-1}(W_0)$, the First Isomorphism Theorem gives $T^{-1}(W_0)/\ker(T) \cong W_0$, so by rank-nullity (as each of W_0 and $T^{-1}(W_0)$ are finite-dimensional), $\dim\left(T^{-1}(W_0)\right) = \dim(\ker(T)) + \dim(W_0)$.

Problem (Problem 4)**:**

(a) Do there exist invertible matrices $A, B \in \mathrm{Mat}_2(\mathbb{R})$ such that

$$ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

(b) Do there exist matrices $A, B \in \mathrm{Mat}_2(\mathbb{R})$ such that

$$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

Solution:

(a) There do not. This follows from the fact that $\det\left(ABA^{-1}B^{-1}\right) = 1$, while the determinant of the latter matrix is 2.

Problem (Problem 5)**:**

(a) Find the inverse matrix A^{-1} for the matrix

$$A = \begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}.$$

(b) Prove that

$$\begin{vmatrix} a + x_1 & a & \cdots & a \\ a & a + x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a + x_n \end{vmatrix} = x_1 x_2 \cdots x_n \left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_n}\right).$$

Solution:

(a) We may find A^{-1} by trying to find the sequence of elementary matrices $E_1, \dots, E_n$ such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I.$$

First, we do row reduction on A, yielding

$$\begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ a & a & a+1 \end{pmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2a & a+1 \end{pmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3a+1 \end{pmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the product $E_n E_{n-1} \cdots E_2 E_1$ is our desired inverse, which we find by applying the elementary row operations to the identity matrix I, yielding

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & a & 1 \end{pmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & -a & 2a+1 \end{pmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}$$
$$\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ -a/(3a+1) & 1-(a/(3a+1)) & -1+((2a+1)/(3a+1)) \\ -(a/(3a+1)) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}$$
$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1-a/(3a+1) & -a/(3a+1) & -1+(2a+1)/(3a+1) \\ -a/(3a+1) & 1-(a/(3a+1)) & -1+((2a+1)/(3a+1)) \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix},$$

which is our desired inverse.

(b) We show the case for $n = 2$, then use induction from then on. By raw calculation, we see that

$$\begin{aligned} \begin{vmatrix} a + x_1 & a \\ a & a + x_2 \end{vmatrix} &= (a + x_1)(a + x_2) - a^2 \\ &= x_1 x_2 + a x_1 + a x_2 \end{aligned}$$

$$= x_1x_2\left(1 + \frac{a}{x_1} + \frac{a}{x_2}\right).$$

Now, for the general n case, we see that since determinants are multilinear,

$$\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x_n \end{vmatrix} = \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$
$$= a\begin{vmatrix} a+x_1 & a & \cdots & 1 \\ a & a+x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$

and since determinants are alternating,

$$= a\begin{vmatrix} x_1 & 0 & \cdots & 1 \\ 0 & x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$

and by the cofactor expansion,

$$= a(x_1x_2\cdots x_{n-1}) + x_n\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_{n-1} \end{vmatrix}$$

and by the induction hypothesis,

$$= a(x_1x_2\cdots x_{n-1}) + x_n(x_1x_2\cdots x_{n-1})\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}}\right)$$
$$= x_1x_2\cdots x_n\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}} + \frac{a}{x_n}\right),$$

we obtain our desired result.

Problem (Problem 6)**:** Let $A \in \text{Mat}_n(\mathbb{R})$, and $(a_{ij})_{ij}$ such that $|a_{ij}| < \frac{1}{n}$ for each i, j. Show that $\det(I_n - A) \neq 0$.

Solution: Let $\|x\| = \max_{i=1}^{n}|x_i|$. Let x_j be the component of x such that $|x_j| = \|x\|$. Then, we see that

$$\begin{aligned} \left|(Ax)_j\right| &= \left|\sum_{i=1}^{n} a_{ij}x_i\right| \\ &\leqslant \sum_{i=1}^{n}|a_{ij}||x_i| \\ &< \sum_{i=1}^{n}\frac{1}{n}|x_i| \\ &\leqslant \sum_{i=1}^{n}\frac{1}{n}\|x\| \end{aligned}$$

$$= |x_j|,$$

which means that $Ax \neq x$ at the component x_j, meaning $(I_n - A)x \neq 0$.

Problem (Problem 7)**:**

(a) Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that $A^2 = I_n$. Show that A is diagonalizable.

(b) Give an example of of $A \in \text{Mat}_2(\mathbb{C})$ satisfying $A^2 = \mathbf{0}_2$ (the zero matrix) which is not diagonalizable.

Solution:

(a) Since $A^2 - I_n = \mathbf{0}_n$, we see that the minimal polynomial of A is $m_A(t) = t^2 - 1$, which splits over $\mathbb{C}$ to yield $m_A(t) = (t-1)(t+1)$. In particular, since the minimal polynomial splits into a product of distinct linear factors, A is diagonalizable.

(b) The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfies $A^2 = \mathbf{0}_2$, but since $A \neq \mathbf{0}_2$, we see that $m_A(t) = t^2$. Since $m_A(t)$ does not split into distinct linear factors over $\mathbb{C}$, we see that A is necessarily not diagonalizable.

Problem (Problem 8)**:** Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that A^2 has n distinct eigenvalues. Show that A is diagonalizable.

Solution: Let $m_{A^2}(t)$ be the minimal polynomial for A^2, which since A^2 has n distinct eigenvalues, splits as

$$m_{A^2}(t) = (t - \lambda_1)\cdots(t - \lambda_n).$$

Observe that, if we set $p = m_{A^2}(t^2)$, that p then annihilates A. We may factor p as

$$p(t) = \left(t - \sqrt{\lambda_1}\right)\left(t + \sqrt{\lambda_1}\right)\cdots\left(t - \sqrt{\lambda_n}\right)\left(t + \sqrt{\lambda_n}\right).$$

Each of these factors are distinct, meaning that $m_A(t)$ consists entirely of distinct linear factors, so that A is diagonalizable.

Problem: Let $A \in \text{Mat}_n(\mathbb{R})$ satisfy $AA^T = I_n$, where A^T is the transpose of A and I_n is the identity matrix. Let $f(x) = \det(xI_n - A)$ be the characteristic polynomial of A.

(a) Show that if λ is an eigenvalue of A, then λ^{-1} is also an eigenvalue of A.

(b) Show that if $\det(A) = 1$ and n is odd, then $\lambda = 1$ is an eigenvalue of A.

Solution:

(a) Let λ be an eigenvalue for A with corresponding eigenvector v. Then,

$$Av = \lambda v.$$

Observe now that $\left(AA^T\right)^T = A^TA = I_n$, meaning that $A^T = A^{-1}$. Thus, we see that

$$\begin{aligned} v &= A^T(Av) \\ &= A^T\lambda v, \end{aligned}$$

so

$$A^Tv = \lambda^{-1}v.$$

Since A and A^T have the same eigenvalues, we thus get that λ^{-1} is an eigenvalue for A.

(b) We observe that $\det(A) = \lambda_1 \cdots \lambda_n$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues for A (counted with algebraic multiplicity). Since $\det(A) = 1$, and n is odd, it follows that, by pairing up eigenvalues with their inverses, there is at least one such λ_i with $\lambda_i = 1$. Thus, $\lambda = 1$ is an eigenvalue for A.