# Problem 1

(a)

$$\begin{aligned}\int_{C_1}\left(x^2+y^2\right)\,d\ell &= \int_0^1 x^2\,dx + \int_0^1 y^2+1\,dy\\ &= \frac{5}{3}.\end{aligned}$$

(b)

$$\begin{aligned}\int_{C_2}\left(x^2+y^2\right)\,d\ell &= \int_0^1 2x^2\,dx\\ &= \frac{2}{3}.\end{aligned}$$

(c)

$$\begin{aligned}\int_{C_3}\left(x^2+y^2\right)\,d\ell &= \int_0^1 x^2+x^4\,dx\\ &= \frac{8}{15}.\end{aligned}$$

# Problem 2

(a) Since $\oint_C\,d\ell$ "adds up" the infinitesimal lengths along C, this integral gives the length of C.

(b) Since $\oint_C\,d\vec{\ell}$ is a vector-valued integral along C, and since C is closed, this integral gives 0.

# Problem 3

(a) We have $d\ell = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, and $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}}$, so

$$\begin{aligned}\int_C\,d\ell &= \int\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx\\ &= \int\frac{1}{\sqrt{a^2-x^2}}\,dx\\ &= a\arcsin\left(\frac{x}{a}\right).\end{aligned}$$

Evaluated from $x=-a$ to $x=a$, we get that $\int_C\,d\ell = \pi a$.

(b) We have $d\ell = \sqrt{dr^2+r^2d\theta^2}$, so

$$\begin{aligned}\int\,d\ell &= \int_0^\pi a\,d\theta\\ &= \pi a\end{aligned}$$

# Problem 7

(a) $\oint_S\,dA$ gives the area of the sphere, as we do not have to integrate with respect to a direction.

(b) $\oint_S\,d\mathbf{A}$ yields zero, as $\hat{n}$ is symmetrical with respect to S.

# Problem 11

$$\begin{aligned}\int_S \mathbf{r}\cdot d\mathbf{A} &= \int (R\hat{r})\cdot\hat{r}\left(R^2 d\Omega\right)\\ &= R^3\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\ d\phi d\theta\\ &= 2\pi R^3\end{aligned}$$

# Problem 18

(a) Since $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$ are both zero, this field is both surface independent and path independent.

(b) Since both the curl and divergence of $\hat{r}$ is zero, this field is both surface independent and path independent.

(c)

$$\begin{aligned}\nabla\cdot\left(-\sin x\cosh y\hat{\imath}+\cos x\sinh y\hat{\jmath}\right) &= -\cos x\cosh y+\cos x\cosh y\\ &= 0\\ \nabla\times\left(-\sin x\cosh y\hat{\imath}+\cos x\sinh y\hat{\jmath}\right) &= \left(-\sin x\sinh y+\sin x\sinh y\right)\hat{k}\\ &= 0.\end{aligned}$$

Thus, the field is both surface independent and path independent.

(d)

$$\begin{aligned}\nabla\cdot\left(xy^2\hat{\imath}-x^2y\hat{\jmath}\right) &= 0\\ \nabla\times\left(xy^2\hat{\imath}-x^2y\hat{\jmath}\right) &= -4xy\hat{k}.\end{aligned}$$

Thus, the field is surface independent but not path independent.

(e)

$$\begin{aligned}\nabla\cdot\left(\rho z\hat{\phi}\right) &= 0\\ \nabla\times\left(\rho z\hat{\phi}\right) &\neq 0.\end{aligned}$$

Thus, the field is surface independent but not path independent.

# Problem 19

The integral along the path $dba$ is $-5$ and the integral along the path $ecdb$ is 3.

# Problem 20

(a) Since $\nabla\times\mathbf{E}=0$, we find the integral by taking

$$\int_C \mathbf{E}\cdot d\vec{\ell} = \int_{C_1}\mathbf{E}\cdot d\vec{\ell}$$

$$\begin{aligned} &= \int_{C_2} \mathbf{E} \cdot d\vec{\ell} \\ &= \frac{1}{2} x^2 y^2 \Big|_{(0,a)}^{(a,0)} \\ &= 0. \end{aligned}$$

(b) We have

$$\begin{aligned} \int_{C_1} \mathbf{B} \cdot d\vec{\ell} &= \int_{C_1} B_x \, dx + B_y \, dy \\ &= \sqrt{2} \int_0^a x^2 \left(-x + a\right) + \sqrt{2} \int_0^a \left(-y + a\right) y^2 \, dy \\ &= 2\sqrt{2} \int_0^a t^2(-t + a) \, dt \\ &= 2\sqrt{2} \int_0^a at^2 - t^3 \, dt \\ &= 2\sqrt{2} \left(\frac{a^4}{3} - \frac{a^4}{4}\right) \\ &= \frac{a^4}{6} \sqrt{2}. \\ \int_{C_2} \mathbf{B} \cdot d\vec{\ell} &= \int_0^{\pi/2} \begin{pmatrix} a^3 \sin^2 t \cos t \\ -a^3 \sin t \cos^2 t \end{pmatrix} \cdot \begin{pmatrix} a \cos t \\ -a \sin t \end{pmatrix} \, dt \\ &= a^4 \int_0^{\pi/2} 2 \sin^2 t \cos^2 t \, dt \\ &= \frac{a^4}{4} \int_0^{\pi/2} \sin^2 (2t) \, dt \\ &= \frac{\pi}{4} a^4. \end{aligned}$$

# Problem 21

(a)

$$\begin{aligned} \int_C \mathbf{F} \cdot d\vec{\ell} &= \int_{C_1 + C_2 + C_3 + C_4} x \, dx - \int_{C_1 + C_2 + C_3 + C_4} y \, dy \\ &= 0. \end{aligned}$$

(b)

$$\begin{aligned} \int_C \mathbf{F} \cdot d\vec{\ell} &= \int_0^{2\pi} \left(-a^2 \cos t \sin t\right) + \left(-a^2 \sin^2 t\right) + \left(3a^3 \cos^2 t \sin t\right) \, dt \\ &= -2\pi a^2. \end{aligned}$$

(c)

$$\begin{aligned} \int_C \mathbf{F} \cdot d\vec{\ell} &= \int_0^{2\pi} R^2 \, d\phi \\ &= 2\pi R^2. \end{aligned}$$

(d)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\vec{\ell} &= \int_0^{2\pi} r^2\sin^2\theta\sin^2\phi\Big|_{r=R,\theta=\alpha}\,d\phi\\ &= \pi R^2\sin^2\alpha.\end{aligned}$$

As $\alpha$ approaches $\pi$, this value approaches 0, which would be expected as we are integrating about a point at $\alpha = \pi$.

# Problem 22

(a)

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{a} &= \int_{S_1+\cdots+S_6}\mathbf{F}\cdot d\mathbf{a}\\ &= 6a^5.\end{aligned}$$

(b)

$$\begin{aligned}\int_S\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)\cdot d\mathbf{a} &= \int r\hat{r}\cdot d\mathbf{a}\\ &= a^3\int d\Omega\\ &= 4\pi a^3.\end{aligned}$$

(c)

$$\begin{aligned}\int_S\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)\cdot d\mathbf{a} &= \int\left(r\hat{r}+z\hat{k}\right)\cdot\hat{r}\,dA\\ &= 4\pi a.\end{aligned}$$

(d)

$$\int_S r^2\hat{r}\cdot d\mathbf{a} = R^3.$$

(e)

$$\int_S\left(\hat{\imath}+\hat{\jmath}+z\left(x^2+y^2\right)\hat{k}\right)\cdot d\mathbf{a} = 0.$$

# Problem 26

Since $\nabla\cdot\mathbf{E}\neq 0$, $\mathbf{E}$ is not surface-independent.

# Problem 28

(a) Inside the sphere,

$$Q_0 = \int \rho\,dV$$

$$\begin{aligned}
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{s} \alpha r^3 \sin\theta \, dr d\phi d\theta \\
&= \pi\alpha a^4,
\end{aligned}$$

meaning that

$$q_{\text{encl}}\left(r < a\right) = \pi\alpha r^4,$$

so we have

$$\begin{aligned}
\mathbf{E}\left(r > a\right) &= \frac{Q_0}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} \\
\mathbf{E}\left(r < a\right) &= \frac{\alpha r^2}{4\epsilon_0}\hat{r}
\end{aligned}$$

(b) I don't know how to do this problem.