8.1

(a)

$$\begin{aligned}
\int_0^1 2^x \, dx &= \int_0^1 e^{x(\ln 2)} \, dx \\
&= \frac{1}{\ln 2}\left(e^{x(\ln 2)}\Big|_0^1\right) && u = x(\ln 2) \\
&= \frac{1}{\ln 2}\left(2^x\Big|_0^1\right) \\
&= \frac{1}{\ln 2}(2-1) \\
&= \frac{1}{\ln 2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^x \, dx &= \int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2}+x-\frac{1}{2}\right)+\frac{1}{2}} \, dx && \text{Completing the square.} \\
&= e^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2} \, dx \\
&= \sqrt{2\pi e} && \text{Gaussian Integral}
\end{aligned}$$

(c)

(d)

$$\int_{-a}^{a} \sin x e^{-\alpha x^2} \, dx = 0 \qquad \text{Even/odd.}$$

(e)

$$\begin{aligned}
\int_0^1 e^{\sqrt{x}} \, dx &= x e^{\sqrt{x}}\Big|_0^1 - \frac{1}{2}\int_0^1 x e^{\sqrt{x}} \, dx && \text{Integration by Parts} \\
&= e - \int_0^1 u^3 e^u \, du && u = \sqrt{x} \\
&= e - \left(u^3 e^u\Big|_0^1 - 3u^2 e^u\Big|_0^1 + 6u\Big|_0^1 - 6e^u\Big|_0^1\right) && \text{Repeated integration by parts.} \\
&= 3e - 6.
\end{aligned}$$

To evaluate $\int_0^1 u^3 e^u \, du$, we used tabular integration as follows:

Sign	Differentiate	Integrate
+	u^3	e^u
-	$3u^2$	e^u
+	$6u$	e^u
-	6	e^u
+	0	e^u

Taking the boundary integrals, we obtain

$$u^3 e^u\Big|_0^1 - 3u^2 e^u\Big|_0^1 + 6ue^u\Big|_0^1 - 6e^u\Big|_0^1 = 6 - 2e$$

(f)

$$\begin{aligned}
\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\cosh(u)}\cosh(u)\,du && x = \sinh(u)\\
&= u + C\\
&= \sinh^{-1}(x) + C.
\end{aligned}$$

(g)

$$\begin{aligned}
\int \tanh x\,dx &= \int \frac{\sinh x}{\cosh x}\,dx\\
&= \int \frac{1}{u}\,du && u = \cosh x\\
&= \ln|u| + C\\
&= \ln|\cosh x| + C.
\end{aligned}$$

(h)

$$\begin{aligned}
\int \tan^{-1} x\,dx &= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx && \text{integration by parts}\\
&= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C. && \text{u-substitution implicit}
\end{aligned}$$

(i)

$$\begin{aligned}
\int_S z^2\,d\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2} \cos^2\theta\sin\theta\,d\phi d\theta\\
&= \frac{\pi}{2}\int_0^{\pi/2} \cos^2\theta\sin\theta\,d\theta\\
&= -\frac{\pi}{2}\int_0^{-1} t^2\,dt && t = \cos\theta\\
&= \frac{\pi}{2}\left(\frac{t^3}{3}\Big|_{-1}^{0}\right)\\
&= \frac{\pi}{6}
\end{aligned}$$

8.8

(a)

$$\begin{aligned}
\int_0^\infty \frac{x}{e^x - 1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx\\
&= \int_0^\infty xe^{-x}\left(\sum_{k=0}^{\infty} e^{-kx}\right)dx\\
&= \sum_{k=0}^{\infty}\int_0^\infty xe^{-(k+1)x}\,dx\\
&= \sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\int_0^\infty ue^{-u}\,du && u = (k+1)x\\
&= \frac{\pi^2}{6}. && \text{Basel Problem}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty \frac{x}{e^x+1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx\\
&= \int_0^\infty xe^{-x}\sum_{k=0}^{\infty}(-1)^k e^{-kx}\,dx\\
&= \sum_{k=0}^{\infty}(-1)^k\int_0^\infty xe^{-(k+1)x}\,dx\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\int_0^\infty ue^{-u}\,dx && u=(k+1)\,x\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}.
\end{aligned}$$

To resolve

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2} = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots$$

we take

$$= \left(1+\frac{1}{9}+\frac{1}{25}+\cdots\right)-\frac{1}{4}\underbrace{\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots\right)}_{\frac{\pi^2}{6}},$$

meaning

$$\int_0^\infty \frac{x}{e^x+1}\,dx = \frac{\pi^2}{12}.$$

8.14

$$\begin{aligned}
I_0(a) &= \int_0^\infty x^0 e^{-ax^2}\,dx\\
&= \frac{1}{2}\sqrt{\frac{\pi}{a}}
\end{aligned}$$

$$\begin{aligned}
I_1(a) &= \int_0^\infty xe^{-ax^2}\,dx\\
&= \frac{1}{2}\left(\frac{1}{-a}e^{-ax^2}\Big|_0^\infty\right)\\
&= \frac{1}{2a}
\end{aligned}$$

$$\begin{aligned}
I_2(a) &= \int_0^\infty x^2 e^{-ax^2}\,dx\\
&= -\frac{1}{2a}\left(xe^{-ax^2}\Big|_0^\infty\right)+\frac{1}{2a}\int_0^\infty e^{-ax^2}\,dx
\end{aligned}$$

$$= \frac{1}{4a}\sqrt{\frac{\pi}{a}}.$$

$$\begin{aligned}
I_3(a) &= \int_0^\infty x^3 e^{-ax^2}\,dx\\
&= -\frac{1}{2a}\left(x^2 e^{-ax^2}\Big|_0^\infty\right) + \frac{1}{a}\int_0^\infty x e^{-ax^2}\,dx\\
&= \frac{1}{2a^2}
\end{aligned}$$

$$\begin{aligned}
I_4(a) &= \int_0^\infty x^4 e^{-ax^2}\,dx\\
&= -\frac{1}{2a}x^3 e^{-ax^2}\Big|_0^\infty + \frac{3}{2a}\int_0^\infty x^2 e^{-ax^2}\,dx\\
&= \frac{3}{2a}I_2\\
&= \frac{3}{8a^3}\sqrt{\frac{\pi}{a}}.
\end{aligned}$$

8.24

$$\begin{aligned}
J(a) &= \lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 e^{-a\frac{\sum_{i=1}^n x_i}{n}}\frac{n}{\sum_{i=1}^n x_i}\,dx_1 dx_2\cdots dx_n\\
J'(a) &= -\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 e^{-a}\,dx_1 dx_2\cdots dx_n\\
&= -e^{-a}
\end{aligned}$$

meaning

$$\begin{aligned}
J(a) &= e^{-a}\\
J(0) &= 1.
\end{aligned}$$

8.26

(a)

$$\begin{aligned}
\int_0^\infty e^{-ax}\sin x\,dx &= -\frac{1}{k}e^{-ax}\cos x\Big|_0^\infty - \frac{a}{k}e^{-ax}\sin x\Big|_0^\infty - \frac{a^2}{k^2}\int_0^\infty e^{-ax}\sin x\,dx\\
\left(1+a^2\right)\int_0^\infty e^{-ax}\sin x\,dx &= -e^{-ax}\cos x\Big|_0^\infty - ae^{-ax}\sin x\Big|_0^\infty\\
\left(1+a^2\right)\int_0^\infty e^{-ax}\sin x\,dx &= -\frac{1}{k}\\
\int_0^\infty e^{-ax}\sin x\,dx &= -\frac{1}{k\left(1+\frac{a^2}{k^2}\right)}
\end{aligned}$$