Complex Analysis

Analyticity and Path-Independence in the Complex Plane

Baby's First Complex Function Theory

We are interested in functions of the form $f(z)$, where $z = x + iy$ is some complex number. Note that this is specifically different from a function $g\colon \mathbb{R}^2 \to \Omega$ for some domain Ω; in the latter case, we have independent variables x and y, while in the former case, we must express $z = x + iy$.

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z)\,dz &= \oint_C w(z)\,(dx + i dy)\\ &= \oint_C w(z)\,dx + i\oint_C w(z)\,dy.\end{aligned}$$

Taking $A_x = w(z)$ and $A_y = iw(z)$, we have

$$= \oint_C \mathbf{A}\cdot\, d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla \times \mathbf{A})\,d\mathbf{a},$$

and when this integral is zero. Note that $(\nabla \times \mathbf{A})\cdot \hat{n} = 0$, so $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$.

Note that we can take

$$w(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$.

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

Theorem (Cauchy–Riemann Equations)**:**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that w is analytic,[1] which leads to Cauchy's theorem.

Theorem (Cauchy's Theorem)**:** If C is a simple closed curve in a simply connected region, then w is analytic if and only if

$$\oint_C w(z)\,dz = 0. \tag{†}$$

Fact. The function $w(z)$ is analytic inside the simply connected region R if any of these hold:

- w satisfies the Cauchy–Riemann equations;

[1] Equal to its Taylor series, also holomorphic.

- $w'(z)$ is unique and exists;
- $\frac{\partial w}{\partial \overline{z}} = 0$.
- w can be expanded as $w(z) = \sum_{n\geqslant 0} c_n(z-a)^n$, convergent on some open neighborhood of a for each a on its domain;[II]
- $w(z)$ is path-independent everywhere in R: $\oint_C w(z)\,dz = 0$.

Example. Considering $w(z) = z$, we have $u = x$ and $v = y$, so it satisfies the Cauchy–Riemann equations. However, neither $\operatorname{Re}(z)$ nor $\operatorname{Im}(z)$ are analytic, and neither is $\overline{z} = x - iy$.

Remark: Whenever we say "analytic at p," we mean "analytic in a neighborhood of p."

Note that since $\mathbb{C}$ is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of $\mathbb{C}$, by adjoining a point $\{\infty\}$, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. This compactified $\mathbb{C}^*$ is often represented as a unit sphere with the north pole, determined by $(0, 0, 1)$, is the point at infinity. The correspondence between $\mathbb{C}^* \setminus \{\infty\}$ and $\mathbb{C}$ is evaluated via stereographic projection.

We define $\frac{z}{\infty} = 0$ and $\frac{z}{0} = \infty$ for any $z \neq 0, \infty$. The correspondence between $z = x + iy$ in the plane to Z on the Riemann sphere with $\mathbb{R}^3$ coordinates (ξ_1, ξ_2, ξ_3) is

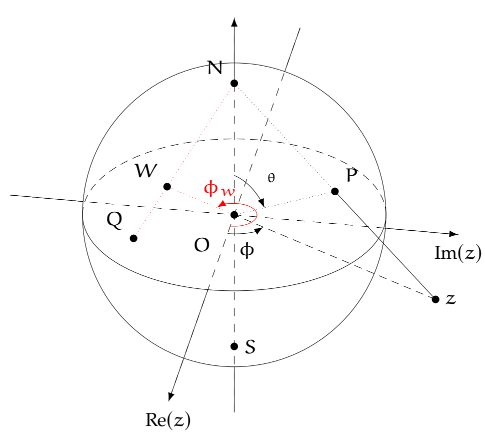
$$\xi_1 = \frac{2\operatorname{Re}(z)}{|z|^2 + 1}$$
$$\xi_2 = \frac{2\operatorname{Im}(z)}{|z|^2 + 1}$$
$$\xi_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Inverting, we may find

$$x = \frac{\xi_1}{1 - \xi_3}$$
$$y = \frac{\xi_2}{1 - \xi_3},$$

and with polar coordinates,

$$z = \cot(\theta/2)e^{i\phi}.$$



To determine analyticity at ∞, we set $\zeta = \frac{1}{z}$, and analyze the analyticity of $\tilde{w}(\zeta) = w(1/z)$ at 0.

[II]This is technically the real definition of analytic for the case when we're dealing with a function with domain $\mathbb{R}$.

Cauchy's Integral Formula

Consider the function $w(z) = c/z$, integrated around a circle of radius R. Then, writing $z = Re^{i\varphi}$, we get

$$\begin{aligned}\oint_\Gamma w(z)\,dz &= C\int_0^{2\pi} \frac{e^{-i\varphi}}{R} \underbrace{iRe^{i\varphi}\,d\varphi}_{dz}\\ &= ic\int_0^{2\pi} d\varphi\\ &= 2\pi ic.\end{aligned}$$

If our contour C runs around our singularity at $z = 0$ a total of n times, then we pick up a factor of n.

Now, when we consider

$$I = \oint_C \frac{dz}{z^n},$$

this integral actually yields 0 for any $n \neq 1$, despite the fact that 0 is a singularity for $f(z) = \frac{1}{z^n}$. This 0 is not a reflection of Cauchy's integral theorem, but of the fact that

$$z^{-n} = \frac{d}{dz}\left(\frac{z^{-n+1}}{n+1}\right),$$

meaning that z^{-n} is an exact differential, so integrating along a closed curve yields zero change. However, $\frac{1}{z} = \frac{d}{dz}(\ln z)$ may be an exact differential, but for complex z, $\ln z = \ln|z| + i\arg(z) = \ln r + i\varphi$. This yields

$$\begin{aligned}\oint_C \frac{c}{z}dz &= c\oint_C d(\ln z)\\ &= c(i(\varphi + 2\pi) - \varphi)\\ &= 2\pi ic.\end{aligned}$$

Ultimately, what this shows is that when we integrate any analytic function $f(\zeta)$ along a closed contour with a singularity at z, only the coefficient on $\frac{1}{\zeta - z}$ will remain. This coefficient is known as the residue at 0.

Theorem (Cauchy's Integral Formula)**:** If w is analytic in a simply connected region and C is a closed contour winding once around a point z in the region, then

$$w(z) = \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta. \tag{$**$}$$

Furthermore, this shows that any once-differentiable function is infinitely differentiable, as by differentiating under the integral sign, we get

$$\frac{d^n w}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Example (Deriving Liouville's Theorem)**.** Consider a circle C centered at radius r centered at at z, $\zeta - z = Re^{i\varphi}$. We take $d\zeta = iRe^{i\varphi}\,d\varphi$, and taking derivatives, we have

$$w'(z) = \frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi.$$

If w is bounded — i.e., $|w(z)| \leqslant M$ for all z in a given region — then

$$|w'(z)| = \left|\frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi\right|$$

$$
\begin{aligned}
&\leqslant \frac{1}{2\pi R}\int_{0}^{2\pi}\left|w\left(z+Re^{i\varphi}\right)\right|\,d\varphi\\
&\leqslant \frac{M}{R}
\end{aligned}
$$

for all R within the analytic region.

In the case where w is entire (i.e., analytic on $\mathbb{C}$), then this inequality holds for all $R \to \infty$. Thus, $|w'(z)| = 0$ for all z, meaning that w is constant.

This is known as Liouville's theorem — every bounded entire function is constant. This can be used to prove the fundamental theorem of algebra.

What Liouville's theorem tells us is that any nontrivial behavior will emerge from a function's singularities.

Singularities and Branches

To understand nontrivial behavior on the complex plane, we need to understand singularities. This will require us to develop understanding of Laurent series.

Taylor Series

We want to integrate $w(z)$ around some point a in an analytic region of $w(z)$. This yields the form

$$
\begin{aligned}
w(z) &= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta\\
&= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)-(z-a)}\,d\zeta\\
&= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)\left(1-\frac{z-a}{\zeta-a}\right)}\,d\zeta. \qquad (\ddagger)
\end{aligned}
$$

Since ζ is on the contour and z is in the contour, $\left|\frac{z-a}{\zeta-a}\right| < 1$, we may expand as a geometric series. Thus, we get

$$
= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)}\left(\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n\right)d\zeta.
$$

Since the series is uniformly convergent, we are allowed to exchange sum and integral, yielding

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right)}_{=c_n}(z-a)^n\\
&= \sum_{n=0}^{\infty} c_n (z-a)^n,
\end{aligned}
$$

where

$$
c_n = \frac{1}{n!}\left.\frac{d^n w}{dz^n}\right|_{z=a}.
$$

If our Taylor series reduces to a known series on the real axis, we find this very desirable. We say this is a type of analytic continuation from the real axis to the complex plane. For example,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is an analytic continuation of e^x.

However, more interestingly,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for all $s > 1$. However, we have also shown that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

converges for complex s for all real part greater than 1. Since values of this integral agree with the series representation of $\zeta(s)$ on real axis, we have that this is an analytic continuation of $\zeta(s)$ to the subset of $\mathbb{C}$ defined by $\text{Re}(s) > 1$.

Laurent Series

Now, what happens if, at (‡), we have $\left|\frac{z-a}{\zeta-a}\right| > 1$. The series as constructed would not converge, but what if we have a series that converges everywhere *outside* C? This would entail an expansion in reciprocal integer powers of $z - a$. This yields

$$\begin{aligned}
w(z) &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(z-a)\left(1 - \frac{\zeta-a}{z-a}\right)}\, d\zeta \\
&= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{z-a} \left(\sum_{n=0}^{\infty} \left(\frac{\zeta-a}{z-a} \right)^n \right) d\zeta \\
&= -\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C w(\zeta - a)^n\, d\zeta \right) \frac{1}{(z-a)^{n+1}} \\
&= \sum_{n=1}^{\infty} \underbrace{\left(-\frac{1}{2\pi i} \oint_C w(\zeta - a)^{n-1}\, d\zeta \right)}_{=c_{-n}} \frac{1}{(z-a)^n} \\
&= \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}
\end{aligned}$$

Note that this series has a singularity at $z = a$, but since our series is only defined outside a particular region, that doesn't matter. We call a series in reciprocal powers a Laurent series. More specifically, Laurent series may include expansions in negative powers as well as positive powers.

Example (Annuli)**.** If we have a point a, we want to surround a by a special contour to apply Cauchy's integral formula.

In particular, for any z in the annulus, we get

$$w(z) = \frac{1}{2\pi i} \oint_{c_1 - c_2} \frac{w(\zeta)}{\zeta - z}\, d\zeta$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\oint_{c_1} \frac{w(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{c_2} \frac{w(\zeta)}{\zeta - z}\,d\zeta\\
&= \sum_{n=-\infty}^{\infty} c_n (z-a)^n\\
&= c_0 + \sum_{n=1}^{\infty}\left(c_{-n}(z-a)^n + c_n(z-a)^n\right).
\end{aligned}$$

Example. Consider the function

$$\begin{aligned}
w(z) &= \frac{1}{z^2+z-2}\\
&= \frac{1}{(z-1)(z+2)}\\
&= \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).
\end{aligned}$$

Now, we have three regions to expand w in.

- If $|z| < 1$, then our series is in both z^n and z^n.
- If $1 < |z| < 2$, then one of our series is going to in $\frac{1}{z^n}$ and one is in z^n.
- If $|z| > 2$, then both of our series are in the form of $\frac{1}{z^n}$ and $\frac{1}{z^n}$

Via tedious, heavily error-prone calculations, we find that

$$\begin{aligned}
w_1(z) &= -\frac{1}{3}\sum_{n=0}^{\infty}\left(1 + (-1)^n\left(\frac{1}{2}\right)^{n+1}\right)z^n\\
w_2(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}} + \left(-\frac{1}{2}\right)^{n+1} z^n\right)\\
w_3(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(1-(-2)^n\right)\frac{1}{z^{n+1}}.
\end{aligned}$$

Sewing all of w_1, w_2, w_3 together, then we get a full series representation of $w(z)$.

Definition. If $w(z)$ is a function that can be written as $w(z) = (z-a)^n g(a)$, where $g(a) \neq 0$, then we say w has an n-th order zero at $z = a$. If $n = 1$, then we say w has a simple zero at a.

Similarly, if we can write

$$w(z) = \frac{g(a)}{(z-a)^n}$$

with $g(a) \neq 0$, then we say w has a pole of order n at a. If $n = 1$, then we say w has a simple pole at a.

There are three types of isolated singularities (i.e., isolated points where $w(z)$ is not defined).

Definition. Let w be an analytic function with isolated singularity at a.

- If w remains bounded in any neighborhood of a, then it must be the case that $c_{-n} = 0$ for all $n > 1$, so the Laurent series is a pure Taylor expansion. We say $z = a$ is a removable singularity.

For instance, the function

$$\frac{\sin(z-a)}{z-a} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-a)^{2n}}{(2n+1)!}$$

has a removable singularity at $z = a$.

- If not all the c_{-n} are equal to zero, but there is a largest $n > 0$ such that c_{-n} is in the Laurent series expansion, then we say a is an n-th order pole. If $n = 1$, we say a is a simple pole.
- If there is no largest value of n such that c_{-n} is in the Laurent series — i.e., that $c_{-n} \neq 0$ for all n — then we say that a is an essential singularity.

  One of the most important facts about an essential singularity is that the behavior is path dependent. For instance,

  $$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!z^n}$$

  has an essential singularity at $z = 0$. We see that $e^{1/z}$ diverges as $z \to 0$ along the positive real axis, but if $z \to 0$ along the negative real axis, we get $e^{1/z} \to 0$.

Singularities can also occur at ∞, which occurs when $w(1/z)$ has a singularity at 0.

Multivalued Function

Consider the function

$$\begin{aligned} w(z) &= z^2 \\ &= \underbrace{\left(x^2 - y^2\right)}_{u(x,y)} + i\underbrace{(2xy)}_{v(x,y)} \\ &= r^2e^{2i\varphi}. \end{aligned}$$

Note that if we take a path around the origin going around by an angle of 2π, then the resulting path goes around twice. Note that this means the lines φ and $\varphi + \pi$ map to the same point in the w plane.

This isn't such a big deal in and of itself, but if we take $w(z) = z^{1/2}$, we get an issue. Instead of w being a two-to-one function, we now have w is a one-to-two function. This is an implicit problem in $\mathbb{R}$ with the function $w(x) = \sqrt{x}$, which we resolve by taking the "positive" square root. This is known as choosing a branch.

We have to do something similar in the complex plane. Note that if we go around by an angle of 2π in the z plane, then we only go around by an angle of π in the w-plane. As we keep going around the plane, we jump from branch to branch, which brings issues of continuity.

To resolve this, we create a "branch cut" that contours are not allowed to cross.

Example. The most common branch cut is to start from the branch point at $z = 0$, in the case of $w(z) = z^{1/2}$ or $w(z) = \ln(z)$, and extend along the real axis, meaning our branch cut is $(-\infty, 0]$.

This principal branch restricts *output* values of φ to $-\pi < \varphi \leqslant \pi$.

For instance, above the cut, we have $\varphi = \pi$, and below the branch cut, we have $\varphi = -\pi$, meaning we have

$$\sqrt{z} = \sqrt{r}e^{i\pi/2} \qquad \varphi \to \pi$$

$$\begin{aligned} &= i\sqrt{r} \\ \sqrt{z} &= \sqrt{r}e^{-i\pi/2} && \varphi \to -\pi \\ &= -i\sqrt{r}. \end{aligned}$$

This is why the branch cut "causes" a discontinuity across the branch, but in $\mathbb{C} \setminus (-\infty, 0]$.

Now, if we have

$$\begin{aligned} \sqrt{z_1}\sqrt{z_2} &= \left(r_1 e^{i\varphi_1}\right)^{1/2}\left(r_2 e^{i\varphi_2}\right)^{1/2} \\ &= \sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2}. \end{aligned}$$

However, if we want to calculate $\sqrt{z_1 z_2}$, and if $|\varphi_1 + \varphi_2| > \pi <$ then our product $z_1 z_2$ crosses the branch cut, and our discontinuity requires $\varphi_1 + \varphi_2$ to be converted to $\varphi_1 + \varphi_2 \pm 2\pi$ so as to bring the angle sum back into the principal branch. This means we have

$$\begin{aligned} \sqrt{z_1 z_2} &= \left(r_1 r_2 e^{i(\varphi_1+\varphi_2)/2}\right) \\ &= \begin{cases} \sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| \leqslant \pi \\ -\sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| > \pi \end{cases}. \end{aligned}$$

Example. Now, if we have $z_1 = 2e^{i(3\pi/4)}$ and $z_2 = e^{i(\pi/2)}$, then we have

$$\begin{aligned} \sqrt{z_1} &= \sqrt{2}e^{i3(\pi/8)} \\ \sqrt{z_2} &= e^{i(\pi/4)}. \end{aligned}$$

Note that if we take $\sqrt{z_1 z_2}$, then the argument of $z_1 z_2$ is $5\pi/4$, so we have to change our argument to $-3\pi/4$ to return to the principal branch before we may calculate the square root. This gives

$$\begin{aligned} \sqrt{z_1 z_2} &= \sqrt{2e^{-i(3\pi/4)}} \\ &= \sqrt{2}e^{-i\pi + i(5\pi/8)} \\ &= -\sqrt{2}e^{i(5\pi/8)} \\ &= -\sqrt{z_1}\sqrt{z_2}. \end{aligned}$$

Now, it is possible to have a branch point at ∞, by determining if $w\left(\frac{1}{z}\right)$ has a branch point at zero. For instance, if $w = z^{1/2}$, this gives

$$\begin{aligned} w\left(\frac{1}{z}\right) &= \frac{1}{\zeta^{1/2}} \\ &= \frac{1}{\sqrt{r}}e^{-i\varphi/2}, \end{aligned}$$

which has the multivalued behavior around the origin. Thus, $z = \infty$ is a branch point for z, and we consider the $(-\infty, 0]$ branch cut that connects the branch points at 0 and ∞.

Example. Consider

$$w(z) = \sqrt{(z-a)(z-b)}.$$

where $a, b \in \mathbb{R}$ with $a < b$. We expect the only finite branch points to be a and b. Introducing polar coordinates, we have

$$r_1 e^{i\varphi_1} = z - a$$

$$r_2 e^{i\varphi_2} = z - b,$$

giving

$$w(z) = \sqrt{r_1 r_2} e^{i\varphi_1} e^{i\varphi_2}.$$

Closed contours around *either* a or b are double-valued. However, if our closed contour goes around *both* a and b, then both φ_1 and φ_2 add up to 2π, meaning we don't have the multivalued behavior.

Now, to select our branch cut, we need to find out if the point at infinity is a branch point. We take $\zeta = \frac{1}{z}$, and we have

$$w(\zeta) = \frac{1}{\zeta}\sqrt{(1 - a\zeta)(1 - b\zeta)},$$

which blows up at ∞, but only takes a singular value.[III]

In general, $z^{1/m}$ for integral m will require m branch cuts.

Example. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{xe^{ikx}}{\sqrt{x^2 + a^2}}\, dx.$$

This is a hard integral to evaluate. To resolve this, we extend the integrand to the complex plane, and invoke Cauchy's theorem to deform the contour.

Note that $\sqrt{x^2 + a^2}$ is multivalued, with branch points at $x = \pm ia$. We choose the branch cut such that our integration contour does not cross the branch cut — i.e., from $-ia$ to ∞ to ia.

Now, we may deform the contour so as to closely wrap around the branch cut from ia to ∞. Remembering the sign discontinuity over the branch cut, this gives the integral

$$\begin{aligned}
\int_{i\infty}^{i\infty} \frac{ze^{ikz}}{\sqrt{x^2 + a^2}}\, dz &= \int_{i\infty}^{ia} \frac{ze^{ikz}}{-i\sqrt{x^2 + a^2}}\, dz + \int_{-a}^{\infty} \frac{ze^{ikz}}{i\sqrt{z^2 + a^2}}\, dz \\
&= 2\int_{ia}^{i\infty} \frac{ze^{ikz}}{i\sqrt{z^2 + a^2}}\, dz \\
&= 2\int_{a}^{\infty} \frac{ye^{-ky}}{\sqrt{y^2 - a^2}}\, dy && z = iy \\
&= 2aK_1(ka) \\
&\sim e^{-ka}
\end{aligned}$$

Here, K_1 refers to the modified Bessel function.

Logarithms

In the complex plane, we say

$$\begin{aligned}
\ln z &= \ln\left(re^{i\varphi}\right) \\
&= \ln r + i\varphi \\
&= \ln|z| + i\arg(z).
\end{aligned}$$

[III] Alternatively, we may see that a positively-oriented contour that surrounds both a and b is a negatively-oriented contour around ∞. Since such a contour is valid, ∞ is not a branch point.

Unfortunately, this $\ln z$ is a multivalued function — a very multivalued one indeed. This yields many branch points, including 0 and ∞:

$$\ln(1/\zeta) = -\ln(\zeta).$$

However, we choose the principal branch, $\pi < \varphi \leqslant \pi$, giving

$$\operatorname{Ln} z = \operatorname{Ln}|z| + i \operatorname{Arg}(z).$$

Example. Consider $\ln(z_1 z_1)$ and $\operatorname{Ln}(z_1 z_2)$. If we have

$$\begin{aligned} z_1 &= 1 + i \\ z_2 &= i, \end{aligned}$$

then

$$\begin{aligned} \arg(z_1) &= \pi/4 \\ \arg(z_2) &= \pi/2, \end{aligned}$$

so

$$\begin{aligned} \arg(z_1 z_2) &= 3\pi/4 \\ &= \arg(z_1) + \arg(z_2) \\ &= \operatorname{Arg}(z_1 z_2). \end{aligned}$$

However, if $z_1 = z_2 = -1$, then

$$\begin{aligned} \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\ &= 2\pi \\ \operatorname{Arg}(z_1 z_2) &= \operatorname{Arg}(1) \\ &= 0. \end{aligned}$$

Thus, we get that $\operatorname{Ln}(z_1 z_2) \neq \operatorname{Ln}(z_1) + \operatorname{Ln}(z_2)$.

Example (Logarithms vs Inverse Trig)**.** Here, we will derive $\arctan(z)$ in terms of the complex logarithm.

Recall that

$$\begin{aligned} \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \end{aligned}$$

so we have

$$\begin{aligned} z &= \tan(w) \\ &= -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}, \end{aligned}$$

which after much tedious, error-prone symbolic manipulation, gives

$$e^{2iw} = \frac{i - z}{i + z}.$$

Thus, we have

$$w = \arctan(z)$$

$$= \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right).$$

Note that since ln has branch points at 0 and ∞, $\ln\left(\frac{i-z}{i+z}\right)$ has branch points when $z = \pm i$.

Now, we must choose a branch cut. Specifically, we want our branch cut to continue the real $\arctan(x)$. We dub this $\operatorname{Arctan}(x)$. Along the real axis, we have

$$\begin{aligned}\operatorname{Arctan}(x) &= \frac{1}{2i}\operatorname{Ln}\left(\frac{i-x}{i+x}\right)\\ &= \frac{1}{2i}\left(\operatorname{Ln}\left|\frac{i-x}{i+x}\right| + i\operatorname{Arg}\left(\frac{i-x}{i+x}\right)\right)\\ &= \frac{1}{2}\operatorname{Arg}\left(\frac{i-x}{i+x}\right).\end{aligned}$$

The principal values are from $-\pi$ to π, so the output of $\operatorname{Arctan}(x)$ ranges from $-\pi/2$ to $\pi/2$.

Conformal Maps

A conformal map is a special type of map $w\colon \mathbb{C} \to \mathbb{C}$ that "preserves angles." If, in z, we map curves whose intersections are at some angle φ, then the image of those curves also intersect at the angle φ.

Example (Our First Conformal Map)**.** Consider the map

$$\begin{aligned}w(z) &= z^2\\ &= \left(x^2 - y^2\right) + i(2xy)\\ &= u(x,y) + iv(x,y).\end{aligned}$$

Examining the line elements in the z and w planes, we have

$$\begin{aligned}ds^2 &= du^2 + dv^2\\ &= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2\\ &= \left(\frac{\partial u}{\partial x}dx - \frac{\partial v}{\partial x}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy\right)^2\\ &= \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)\left(dx^2 + dy^2\right)\\ &= \left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)\left(dx^2 + dy^2\right)\\ &= 4\left(x^2 + y^2\right)\left(dx^2 + dy^2\right)\end{aligned}$$

Note that dx^2 and dy^2 have identical scale factors. Since angles are determined by the ratio of dx and dy, it is the case that *all* angles are preserved. This is what is meant by a conformal map.

Example (Analyticity and Conformality)**.** Consider an analytic function $w(z)$, with its Taylor expansion about z_0.

$$w(z) = w(z_0) + w'(z_0)(z - z_0) + \cdots.$$

For a very small $\xi = z - z_0$, we may truncate it into first order, and place into polar form

$$w(z) - w(z_0) = w'(z_0)\xi$$

$$= |w'(z_0)|e^{i\alpha_0}\xi.$$

Moving from z to w, we get a magnification (or shrinkage) by $|w'(z_0)|$ and a rotation by α_0.

Since, close to z_0, $\xi_1 = z_1 - z_0$ and $\xi_2 = z_2 - z_0$ are magnified by (effectively) the same amount, and rotated by (effectively) the same amount, conformality is established.

Definition. A conformal map is an analytic function $w(z)$ defined on a domain Ω such that $w'(z_0) \neq 0$ for all $z_0 \in \Omega$.

Example (Möbius Transformations)**.** A Möbius transformation is a fractional linear transformation of the form

$$w(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. We can calculate $w'(z)$ to be

$$w'(z) = \frac{ad-bc}{(cz+d)^2}.$$

Since $w(z)$ is conformal, it is invertible, so

$$\begin{aligned} w^{-1}(z) &= z(w) \\ &= \frac{dw-b}{-cw+a}. \end{aligned}$$

The Möbius transformations include ∞, as we have $w(\infty) = \frac{a}{c}$, meaning that it is an automorphism of the Riemann sphere. Note that because of the constraint, we only need three numbers to specify a Möbius transformation.

Consider the Möbius transformation

$$w(z) = \frac{z-i}{z+i}.$$

We let $z_1 = -1$, $z_2 = 1$, and $z_3 = \infty$. Then, we have

$$\begin{aligned} w(z_2) &= \frac{-1-i}{-1+i} \\ &= \frac{2i}{2} \\ &= i. \end{aligned}$$

Similarly, this gives $w(z_3) = 1$. After a bit more playing, we can find that this is a map of the (closed) upper half-plane to the (closed) unit disk, $\mathbb{D}$.

Now, if we look at the "ribbon" between the real axis and the line $\operatorname{Im}(z) = i$, we see that it maps to the region

$$S = \mathbb{D} \setminus \left\{ z \;\middle|\; \left| z - \frac{1}{2} \right| \leqslant \frac{1}{2} \right\}.$$

Example. Consider the map $w(z) = e^z$. This gives

$$\begin{aligned} w(z) &= e^x e^{iy} \\ &= \rho e^{i\beta}. \end{aligned}$$

This sends curves of constant y to curves of constant argument, and maps curves of constant x to circles of constant radius.

Complex Potentials

Consider the analytic function

$$\Omega(z) = \Phi(x,y) + i\Psi(x,y).$$

We know that

$$\begin{aligned}\frac{\partial \Phi}{\partial x} &= \frac{\partial \Psi}{\partial y}\\ \frac{\partial \Phi}{\partial y} &= -\frac{\partial \Psi}{\partial x}.\end{aligned}$$

Thus, we separate to get

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial y}\\ &= \frac{\partial}{\partial y}\frac{\partial \Psi}{\partial x}\\ &= -\frac{\partial^2 \Phi}{\partial y^2},\end{aligned}$$

so

$$\begin{aligned}\nabla^2\Phi &= 0\\ \nabla^2\Psi &= 0.\end{aligned}$$

The converse is also true — if there is some real harmonic function $\Phi(x,y)$, there is a conjugate harmonic function $\Psi(x,y)$ such that $\Omega(z) = \Phi(x,y) + i\Psi(x,y)$ is analytic.

If Ω is analytic, then Φ and Ψ must satisfy the Cauchy–Riemann equations, meaning that

$$\begin{aligned}\Psi(x,y) &= \int \frac{\partial \Psi}{\partial y}\,dy + \frac{\partial \Psi}{\partial x}\,dx\\ &= \int \frac{\partial \Phi}{\partial x}\,dy - \frac{\partial \Phi}{\partial x}\,dx.\end{aligned}$$

For Ψ to be a proper single-valued real function, the integral must be path-independent. Using Green's theorem, we may close the path in a simply connected region, and consider it as a surface integral. This gives

$$\begin{aligned}\oint_C \frac{\partial \Phi}{\partial x}\,dy - \frac{\partial \Phi}{\partial y}\,dx &= \int_S \left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \Phi}{\partial y}\right)\right)dx\,dy\\ &= \int_x \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)dx\,dy\\ &= 0.\end{aligned}$$

We call $\Omega(z) = \Phi(x,y) + i\Psi(x,y)$ the complex potential.

This gives

$$\begin{aligned}\frac{d\Omega}{dz} &= \frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial x}\\ &= \frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y}\end{aligned}$$

$$\begin{aligned}&= \frac{\partial \Psi}{\partial y} + i\frac{\partial \Psi}{\partial x}.\\ &= \overline{\mathcal{E}},\end{aligned}$$

where $\mathcal{E}$ is the complex representation of the electric field, $\mathbf{E}$. We have

$$\begin{aligned}\mathcal{E} &= \overline{\frac{\partial \Omega}{\partial z}}\\ &= \frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y},\end{aligned}$$

with

$$E = \left|\frac{d\Omega}{dz}\right|.$$

The physics of electric fields is then determined entirely by the complex potential.

What makes harmonic functions useful is that, if there are complicated boundary conditions, we may apply a conformal map and the functions remain harmonic.

Example (Cylindrical Capacitor)**.** Consider a cylindrical capacitor with nonconcentric plates meeting at insulated point $u = 1$ and $v = 0$. The larger cylinder with radius 1 is grounded, and the smaller cylinder with radius $1/2$ is held at voltage V_0. We want to find the electric field.

We want to find $\widetilde{\Phi}(w)$ such that

$$\nabla^2\widetilde{\Phi}(u, v) = 0.$$

This domain is kind of difficult, so we will solve the problem on a simpler domain and use a conformal map. Note that from Figure 20.4 in the book, we may use the Möbius transformation

$$w(z) = \frac{z-i}{z+i}$$

to transform *to* our cylindrical capacitor *from* a two-plate infinite capacitor with one plate at $\operatorname{Im}(z) = 1$ and one plate at $\operatorname{Im}(z) = 0$. From physics, we know that $\Phi(x, y) = \frac{V_0 y}{d}$, where $d = 1$. Thus, the harmonic conjugate, $\Psi = -V_0 x$, gives us a complex potential of $\Phi = -iV_0 z$.

Solving

$$\frac{z-i}{z+i} = u(x, y) + iv(x, y),$$

we find

$$\begin{aligned}x(u, v) &= -\frac{2v}{(1-u)^2 + v^2}\\ y(u, v) &= \frac{1-u^2-v^2}{(1-u)^2+v^2}.\end{aligned}$$

Now, this gives

$$\begin{aligned}\widetilde{\Phi}(u, v) &= \Phi(x(u, v), y(u, v))\\ &= V_0\frac{1-u^2-v^2}{(1-u)^2+v^2}.\end{aligned}$$

Example (Fluid Flow)**.** Consider fluid flow around a rock with disk of radius a; far away from the rock, we have uniform flow speed of α.

Symmetry allows us to focus only on the upper half-plane. Now, there is a conformal map in Table 20.1 of the textbook, which is the map $w(z) = z + \frac{a^2}{z} = u(x,y) + iv(x,y)$ that maps the upper half-plane to the upper half-plane. Furthermore, this map sends the boundary hugging the rock into the u-axis.

After applying the conformal map, we get the stream lines $\widetilde{\Psi}(u,v) = \beta v$, as they are streamlines of uniform horizontal flow.

Building the complex potential, we have

$$\begin{aligned}\widetilde{\Omega}(w) &= \Phi(u,v) + i\Psi(u,v)\\ \widetilde{\Omega}(w) &= \beta w,\end{aligned}$$

as we must have $\frac{d\Phi}{du} = \frac{d\Psi}{dv} = \beta$.

Mapping back into the z-plane, we have

$$\Omega(z) = \beta\left(z + \frac{a^2}{z}\right).$$

Note that as z becomes very big, the term $\frac{a^2}{z}$ goes to 0, so we must have $\beta = \alpha$.

Now, we may find the streamlines and potentials. Note that we have

$$\begin{aligned}\Phi &= \operatorname{Re}(\Omega)\\ \Psi &= \operatorname{Im}(\Omega).\end{aligned}$$

Now, we have

$$\begin{aligned}\Omega(z) &= \alpha r\left(e^{i\varphi} + \frac{a^2}{r^2}e^{-i\varphi}\right)\\ &= \alpha r\left(\cos(\varphi) + i\sin(\varphi) + \frac{a^2}{r^2}(\cos(\varphi) - i\sin(\varphi))\right).\end{aligned}$$

Taking real and imaginary parts, we have

$$\begin{aligned}\Phi &= \alpha r\left(1 + \frac{a^2}{r^2}\right)\cos(\varphi)\\ \Psi &= \alpha r\left(1 - \frac{a^2}{r^2}\right)\sin(\varphi).\end{aligned}$$

Example. Considering our conformal map

$$w(z) = z + \frac{a^2}{z}$$

again, we see that if $|z| = a$, then $|u| \leqslant 2a$. Meanwhile, if $r > a$, then

$$\begin{aligned}w(z) &= z + \frac{a^2}{z}\\ &= re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\end{aligned}$$

$$
\begin{aligned}
&= \left(r + \frac{a^2}{r}\right)\cos(\varphi) + i\left(r - \frac{a^2}{r}\right)\sin(\varphi)\\
&= u + iv.
\end{aligned}
$$

This gives

$$\frac{u^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{v^2}{\left(r - \frac{a^2}{r}\right)} = 1.$$

Note that w fails to be conformal when $\frac{dw}{dz} = 0$, meaning that it fails to be conformal at $z = \pm a$.

This is occasionally used in the real world[IV] to design airfoils.

Residues

Consider a function $f(z)$ with an nth order pole. Then, f can be written as

$$f(z) = \frac{g(z)}{(z-a)^n},$$

where $g(z)$ is analytic and $g(a) \neq 0$. Recalling Cauchy's integral formula, we see that this expression for f is tantalizingly close to our desired state.

We may expand g in a Taylor series:

$$g(z) = \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!}(z-a)^m.$$

Letting C be a positively oriented contour in the analytic domain of f that encircles the singularity, we get

$$\oint_C f(z)\,dz = \sum_{m=0}^{\infty} \frac{g^{(m)(a)}}{m!}\oint_C (z-a)^{m-n}\,dz.$$

Note that if $m - n \neq -1$, then the integral on the right vanishes, so we only obtain a nonzero contribution at $m = n-1$. Thus, we get

$$\oint_C f(z)\,dz = 2\pi i \frac{g^{(n-1)(a)}}{(n-1)!}.$$

Definition. Let $f(z)$ be an analytic function with a pole at $z = a$ with order n. We define the residue of f at a as

$$\operatorname{Res}[f(z), a] := \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right).$$

This gives an alternative statement of Cauchy's integral formula, giving

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f(z), a].$$

However, when we have lots of poles for f, and C is a contour that surrounds all the poles, we may deform C such that it surrounds each pole. This gives the residue theorem.

Theorem (Residue Theorem)**:**

$$\oint_C f(z)\,dz = 2\pi i \sum_{a\in C}\operatorname{Res}[f(z), a] \tag{††}$$

Type	Method
n-th order pole	$\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right)$
simple pole	$\lim\limits_{z\to a}(z-a)f(z)$
$f=\frac{p}{q}, q(a)$ simple zero	$\dfrac{p(a)}{q'(a)}$
pole at infinity	$\lim\limits_{z\to 0}\left(-\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right)\right)$
pole at infinity, $\lim_{\|z\|\to\infty} f(z)=0$	$-\lim\limits_{\|z\|\to\infty}(zf(z))$

Table 1: Finding $\text{Res}[f(z), a]$

We can find the residue in a variety of ways.

Example. We will find the residue for $\cot(z)$ for each of the residues.

$$\begin{aligned}
\text{Res}[\cot(z), n\pi] &= \lim_{z\to n\pi}(z-n\pi)\frac{\cos(z)}{\sin(z)}\\
&= (-1)^n \lim_{z\to n\pi}\frac{z-n\pi}{\sin(z)}\\
&= (-1)^n \lim_{z\to n\pi}\frac{z-n\pi}{(-1)^n\sin(z-n\pi)}\\
&= 1.
\end{aligned}$$

Example. We may find

$$\begin{aligned}
\text{Res}\left[\frac{z}{\sinh(z)}, in\pi\right] &= \left.\frac{z}{\frac{d}{dz}(\sinh(z))}\right|_{z=in\pi}\\
&= \frac{in\pi}{\cosh(in\pi)}\\
&= (-1)^n in\pi
\end{aligned}$$

Example. Let's evaluate

$$\oint_C \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

Finding the residue at each pole, we get

$$\begin{aligned}
\text{Res}[f(z), 0] &= \frac{2}{3}\\
\text{Res}[f(z), -1] &= -\frac{3}{2}\\
\text{Res}[f(z), 3] &= -\frac{1}{6}.
\end{aligned}$$

These are evaluated using the cover-up method.

Now, we may find the integral by taking

$$\oint_{|z|=2} f(z)\,dz = -i\frac{5\pi}{3}.$$

[iv] I guess people do things over there.

Example. Let

$$\begin{aligned} f(z) &= \frac{1}{z^2\sinh(z)} \\ &= \frac{1}{-iz^2\sin(iz)}. \end{aligned}$$

The simple zeros of $\sinh(z)$ are at $in\pi$, so we have an order 3 pole at $z = 0$

$$\begin{aligned} \operatorname{Res}[f(z), 0] &= \frac{1}{(n-1)!}\frac{d^2}{dz^2}\left[z^3 f(z)\right]\Big|_{z=0} \\ &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z}{\sinh(z)}\right)\Big|_{z=0} \\ &= -\frac{1}{6}. \end{aligned}$$

Thus, integrating about the unit circle, we get

$$\oint_{|z|=1} = -\frac{i\pi}{3}.$$

If we were to evaluate via the Laurent series, we would have

$$\begin{aligned} \frac{1}{z^2\sinh(z)} &= \frac{1}{z^2}\left(\frac{1}{z + z^2/3 + z^5/5! + \cdots}\right) \\ &= \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right) \\ &\approx \frac{1}{z^3}\left(1 - \frac{z^2}{3!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{6z} + \cdots, \end{aligned}$$

giving a residue of $-\frac{1}{6}$.

Instead of using the contour on the unit circle, if we want to use a circle of radius 4, we get the residues at $z = \pm i\pi$. To evaluate this, we take

$$\begin{aligned} \operatorname{Res}[f(z), i\pi] &= \frac{1}{-\pi^2(-1)} \\ &= \frac{1}{\pi^2} \\ \operatorname{Res}[f(z), -i\pi] &= \frac{1}{\pi^2}. \end{aligned}$$

Evaluating the integral, we would get

$$\begin{aligned} \oint_C f(z)\,dz &= 2\pi i\left(-\frac{1}{6} + \frac{2}{\pi^2}\right) \\ &= -\frac{i\pi}{3} + \frac{4i}{\pi}. \end{aligned}$$

Example. We will now use the residue theorem to evaluate a real-valued integral. Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx.$$

Since this integral goes to zero, we will evaluate

$$I' = \oint_C \frac{1}{z^2+1}\,dz,$$

where C is a semicircle with radius r along the real axis from $-r$ to r "pointing upward," so to speak.

This gives

$$\oint_C \frac{1}{z^2+1}\,dz = \int_{C_r} f(z)\,dz + \int_{-r}^{r} f(x)\,dx,$$

which, sending r to infinity, is equal to

$$I = \int_{-\infty}^{\infty} f(x)\,dx.$$

However, since our expression $\frac{1}{z^2+1}$ has poles at i and $-i$, our semicircle gives

$$\begin{aligned}\oint_C \frac{1}{z^2+1} &= 2\pi i \operatorname{Res}[f(z), i]\\ &= 2\pi i \lim_{z\to i} \frac{1}{z+i}\\ &= 2\pi i \frac{1}{2i}\\ &= \pi.\end{aligned}$$

If we have a finite number of isolated singularities, we are always able to draw a contour that encloses all of them, which allows us to use the residue theorem.

Now, we know that we can have poles at infinity — and that any positively-oriented contour in the plane is a negatively-oriented contour around ∞. Thus, if we have a contour surrounding all our finite singularities, we get

$$\begin{aligned}\sum_i \operatorname{Res}[f(z), a_i] &= -\operatorname{Res}[f(z), \infty]\\ \operatorname{Res}[f(z), \infty] + \sum_i [f(z), a_i] &= 0,\end{aligned}$$

as we're doing the same integral, but in negative orientation about ∞ and positive orientation about our singularities.

We have

$$\operatorname{Res}[f(z), \infty] = \operatorname{Res}\left[-\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right].$$

Example. Now, recalling

$$f(z) = \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

The residues are

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= 2/3\\ \operatorname{Res}[f(z), -1] &= -3/2\end{aligned}$$

$$\text{Res}[f(z), 3] = -1/6.$$

Now, calculating the residue at infinity, we have

$$\begin{aligned}\text{Res}[f(z), \infty] &= \text{Res}\left[-\frac{1}{z^2}\frac{(1/z-1)(1/z-2)}{1/z(1/z+1)(3-1/z)}, 0\right]\\ &= -\text{Res}\left[\frac{(z-1)(2z-1)}{z(z+1)(3z-1)}\right]\\ &= 1.\end{aligned}$$

Now, if $\lim_{|z|\to\infty} f(z) = 0$, then f is pure Laurent series. In that case, if there is a residue, then we find the residue by evaluating

$$\text{Res}[f(z), \infty] = -\lim_{|z|\to\infty} zf(z)$$

Example. Consider functions of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where q is a higher-order polynomial than p.

If q has first-order zeros a and second-order zeros at b, then

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z-a_k} + \frac{B_k}{z-b_k} + +\frac{C_k}{(z-b_k)^2}.$$

Note that the coefficients are actually residues. This gives

$$\begin{aligned}A_k &= \text{Res}[f(z), a_k]\\ B_k &= \text{Res}[f(z), b_k]\\ C_k &= \text{Res}[(z-b_k)f(z), b_k].\end{aligned}$$

For instance,

$$\frac{(z-1)(z-2)}{z(z+1)(3-z)} = \frac{2}{3}\frac{1}{z} - \frac{1}{6}\frac{1}{z-3} - \frac{3}{2}\frac{1}{z+1}.$$

Now, we may also have

$$\frac{(z-1)(z-2)}{z(z+1)^2(3-z)} = \frac{2}{3}\frac{1}{z} - \frac{1}{24}\frac{1}{z-3} - \frac{5}{8}\frac{1}{z+1} - \frac{3}{2}\frac{1}{(z+1)^2}.$$

Integrating around a Circle

We want to evaluate angular integrals of the form

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi))\, d\varphi.$$

Now, while this is a real integral over a domain, we may reformulate it about the unit circle by using the substitutions

$$\begin{aligned}z &= e^{i\varphi}\\ d\varphi &= \frac{dz}{iz},\end{aligned}$$

which yields

$$\begin{aligned}\sin(n\varphi) &= \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)\\ \cos(m\varphi) &= \frac{1}{2}\left(z^m + \frac{1}{z^m}\right).\end{aligned}$$

Thus, our integral becomes

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi))\, d\varphi = \oint_{|z|=1} f\left(\frac{1}{2i}\left(z^n - \frac{1}{z^n}\right), \frac{1}{2}\left(z^m + \frac{1}{z^m}\right)\right)\frac{dz}{iz}.$$

Example. Consider

$$\begin{aligned}\int_0^{2\pi} \sin^2(\varphi)\, d\varphi &= -\frac{1}{4}\oint_{|z|=1}\left(z - \frac{1}{z}\right)^2\frac{dz}{iz}\\ &= \frac{i}{4}\oint_{|z|=1}\frac{1}{z^3}\left(z^4 - 2z^1 + 1\right)\, dz\\ &= -\frac{1}{2}\pi\operatorname{Res}\left[\frac{1}{z^3}\left(z^4 - 2z^1 + 1\right), 0\right].\end{aligned}$$

The residue at $z = 0$ is -2 — this can be found by dividing out by z^3.

Thus, we get the answer of

$$\int_0^{2\pi}\sin^2(\varphi)\, d\varphi = \pi.$$

Example. Using residues, we can evaluate a lot of integrals that are quite tricky on their face.

$$\begin{aligned}\int_0^{2\pi}\frac{\cos(2\varphi)}{5 - 4\sin(\varphi)}\, d\varphi &= \oint_{|z|=1}\frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{5 - \frac{4}{2i}\left(z - \frac{1}{z}\right)}\frac{dz}{iz}\\ &= -\oint_{|z|=1}\frac{z^4 + 1}{2z^2(2z - i)(z - 2i)}\, dz.\end{aligned}$$

Now, we have a simple pole at $i/2$, a simple pole at $2i$, and a pole of order 2 at 0. We only evaluate the residues at 0 and $i/2$. We get

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= -\frac{d}{dz}\left(\frac{z^4 + 1}{2z^2(2z - i)(z - 2i)}\right)\Bigg|_{z=0}\\ &= -\frac{5i}{8}\\ \operatorname{Res}[f(z), i/2] &= \frac{17i}{24}.\end{aligned}$$

Thus, we get the result of

$$\begin{aligned}\int_0^{2\pi}\frac{\cos(2\varphi)}{5 - 4\sin(\varphi)}\, d\varphi &= 2\pi i\left(-\frac{5i}{8} + \frac{17i}{24}\right)\\ &= -\frac{\pi}{6}.\end{aligned}$$

Integrating along the Real Axis

If we want to evaluate integrals along the real axis, such as

$$I = \int_{-\infty}^{\infty} f(x)\,dx,$$

we may be curious as to how we may evaluate this.

To do this, we recall that we created a contour in the upper half-plane of large enough radius r, and evaluated the residues inside the contour. We consider the contour to be equal to $C = C_r + l_r$, where l_r is along the real axis and C_r closes our contour. Thus, we get

$$\oint_C f(z)\,dz = \lim_{r\to\infty}\left(\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz\right).$$

Note that the polar coordinate Jacobian gives us the requirement that $\lim_{|z|\to\infty}|zf(z)| = 0$.

When $f(z) = \frac{p(z)}{q(z)}$, this is satisfied when q is of degree at least two more than that of p.

Example. Consider

$$\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4}\,dx = \oint_C \frac{2z+1}{z^4+5z^2+4}\,dz.$$

Factoring, we get

$$\oint_C \frac{2z+1}{z^4+5z^2+4}\,dz = \oint_C \frac{2z+1}{(z-2i)(z+2i)(z-i)(z+i)}\,dz.$$

We only care about the residues in the upper half-plane. We have residues of

$$\begin{aligned}\operatorname{Res}[f(z), 2i] &= -\frac{1}{3} + \frac{i}{12}\\ \operatorname{Res}[f(z), i] &= \frac{1}{3} - \frac{i}{6}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4}\,dx &= 2\pi i\left(\frac{1}{3} - \frac{i}{6} - \frac{1}{3} + \frac{i}{12}\right)\\ &= \frac{\pi}{6}.\end{aligned}$$

Note that if we chose our contour to be in the lower half-plane, then we would have a *negatively* oriented contour, and evaluate at the residues in the lower half-plane.

Example. Consider

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^3 - i}\,dx &= \oint_C \frac{1}{z^3 - i}\,dz\\ &= \oint_C \frac{1}{(z+i)\left(z - e^{i\pi/6}\right)\left(z - e^{5i\pi/6}\right)}.\end{aligned}$$

Closing C in the lower half-plane, we only need the residue at $-i$. This gives

$$\begin{aligned}\oint_C \frac{1}{z^3 - i} &= -2\pi i\left(-\frac{1}{3}\right)\\ &= \frac{2\pi i}{3}.\end{aligned}$$

Consider integrals of the form

$$\int_{-\infty}^{\infty} g(x)e^{ikx}\,dx,$$

where k is real.

Now, we want to know when exactly we are allowed to "close up" the semicircle contour.

We start by assuming k is positive. Closing in the upper half-plane so as to ensure exponential decay, we have

$$\begin{aligned}\left|\int_{C_r} g(z)e^{ikz}\,dz\right| &\leqslant \int_{C_r}\left|g(z)e^{ikz}\right|dz\\ &= \int_0^{\pi}\left|g\left(re^{i\varphi}\right)\right|re^{-kr\sin(\varphi)}\,d\varphi.\end{aligned}$$

Since $\sin(\varphi) \geqslant 0$ on the range of integration, the integral vanishes as $r \to \infty$. Therefore, we are allowed to close up the contour whenever $|g(z)| \to 0$ as $|z| \to \infty$.

Example. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+4}\,dx.$$

This gives

$$\begin{aligned}I &= \operatorname{Re}\left(\int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+4}\,dx\right)\\ &= \operatorname{Re}\left(\oint_C \frac{e^{ikz}}{z^2+4}\,dz\right)\\ &= \oint_C \frac{e^{ikz}}{(z-2i)(z+2i)}\,dz.\end{aligned}$$

We assume $k > 0$. Then, evaluating at $2i$, we have

$$I = \frac{\pi}{2}e^{-2k}.$$

Now, if $k < 0$, we close our contour in the lower half-plane, we get

$$I = \frac{\pi}{2}e^{2k}.$$

Thus, our integral is always

$$I = \frac{\pi}{2}e^{-2|k|}.$$

Non-Circular Contours

Sometimes, semicircles don't work.

Example. Consider

$$\int_{-\infty}^{\infty}\frac{e^{bx}}{e^x+1}\,dx,$$

where $0 < b < 1$. Writing our integral, we have

$$I = \int \frac{e^{bz}}{e^z + 1}\, dz$$

This gives poles at $z = (2n+1)i\pi$, which means we cannot close this contour with a semicircular arc at ∞.

What may work in this case is by drawing a rectangular contour from $-a$ to a such that it encloses exactly one of the poles of our integrand. The vertical segments of this contour go to zero as we send $a \to \infty$. We call the segment of the contour along the line $a + 2\pi i$ to $a - 2\pi i$ as I'.

This gives

$$I + I' = \oint_C \frac{e^{bz}}{e^z + 1}\, dz.$$

Now, we constructed I' such that

$$\begin{aligned} I' &= \int_{\infty}^{-\infty} \frac{e^{b(x+2\pi i)}}{e^{x+2\pi i} + 1}\, dx \\ &= -e^{2\pi i b} \int_{-\infty}^{\infty} \frac{e^{bx}}{e^x + 1}\, dx \\ &= -e^{2\pi i b} I. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \oint_C \frac{e^{bz}}{e^z + 1}\, dz &= I\left(1 - e^{2\pi i b}\right) \\ &= 2\pi i \operatorname{Res}\left[\frac{e^{bz}}{e^z + 1}, i\pi\right], \end{aligned}$$

giving

$$I = \frac{\pi}{\sin(\pi b)}.$$

Example. We want to evaluate

$$\int_0^\infty \cos\left(x^2\right)\, dx$$
$$\int_0^\infty \sin\left(x^2\right)\, dx.$$

To evaluate this, we draw a slice-shaped contour going along the real axis and returning to 0 along $z = re^{i\pi/4}$. Therefore, we evaluate

$$\begin{aligned} \oint_C e^{iz^2}\, dz &= \int_0^\infty e^{ix^2}\, dx + 0 + \int_\infty^0 e^{i\left(re^{i\pi/4}\right)^2} e^{i\pi/4}\, dr \\ &= \int_0^\infty e^{ix^2}\, dx + 0 + \int_\infty^0 e^{-r^2} e^{i\pi/4}\, dr. \end{aligned}$$

Thus, we get

$$\begin{aligned} \int_0^\infty \cos\left(x^2\right)\, dx &= \int_0^\infty \sin\left(x^2\right)\, dx \\ &= \frac{1}{2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Integrating with Branch Cuts

When we're integrating with residues, branch cuts are a feature rather than a bug.

Example. Consider the integral

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^3}\,dx.$$

We need a branch cut to avoid the multivalued behavior. Our poles are at $e^{i\pi/3}, -1, e^{-i\pi/3}$. Since our integral is along the real axis, we take our branch cut along the domain $[0,\infty]$.

We draw our contour of radius R by hugging the branch without crossing it, with a small circle of radius ε just outside 0. This gives the integral

$$\oint \frac{\sqrt{z}}{1+z^3}\,dz = \int_0^\infty \frac{\sqrt{z}}{1+z^3}\,dx + \int_{C_R} \frac{\sqrt{z}}{1+z^3}\,dz + \int_\infty^0 \frac{\sqrt{z}}{1+z^3}\,dz + \int_{C_\varepsilon} \frac{\sqrt{z}}{1+z^3}\,dz.$$

Note that since $\lim_{|z|\to\infty}|zf(z)| = 0$, and $\lim_{|z|\to 0}|zf(z)| = 0$, our integrals along C_R and C_ε go to zero, giving the integral

$$\begin{aligned} I' &= \int_\infty^0 \frac{\sqrt{z}}{1+z^3}\,dz \\ &= \int_\infty^0 \frac{\sqrt{e^{2i\pi}x}}{1+(e^{2i\pi}x)^3}\,dx \\ &= \int_0^\infty \frac{\sqrt{x}}{1+x^3}\,dx \\ &= I. \end{aligned}$$

Thus,

$$\oint \frac{\sqrt{z}}{1+z^3}\,dz = 2I.$$

Evaluating the residues, we have

$$\begin{aligned} \operatorname{Res}\left[f(z), e^{i\pi/3}\right] &= \lim_{z\to e^{i\pi/3}} \frac{\sqrt{z}}{3z^2} \\ &= -\frac{i}{3} \\ \operatorname{Res}[f(z), -1] &= \lim_{z\to -1} \frac{\sqrt{z}}{3z^2} \\ &= \frac{i}{3} \\ \operatorname{Res}\left[f(z), e^{5\pi i/3}\right] &= -\frac{i}{3}, \end{aligned}$$

giving the solution of

$$\begin{aligned} I &= \frac{1}{2}2\pi i\left(-\frac{i}{3}\right) \\ &= \frac{\pi}{3}. \end{aligned}$$

Example. To evaluate

$$I = \int_0^\infty \frac{1}{1+x^3}\,dx,$$

we start by evaluating

$$\int_0^\infty \frac{\ln(x)}{1+x^3}\,dx$$

with the branch cut along the real axis. Using the keyhole contour in the previous example, we have that C_R and C_ε contribute nothing, and ln picks up a phase of $2\pi i$, so that

$$\begin{aligned}\oint_C \frac{\ln(z)}{1+z^3} &= \int_0^\infty \frac{\ln(x)}{1+x^3}\,dx + \int_\infty^0 \frac{\ln(x)+2\pi i}{1+x^3}\,dx\\ &= -2\pi i I.\end{aligned}$$

Therefore,

$$I = -\sum \operatorname{Res}\left[\frac{\ln(x)}{1+x^3}\right].$$

Thus, we get the solution of

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

Example. Consider the integral

$$I = \int_0^1 \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx.$$

The poles are around $\pm ia$.

Our problem is that we have multivalued behavior at ± 1. We may take the cut from -1 to 1 along the real axis, and our contour gives a sign flip across the cut.

We draw a dog-bone style contour hugging the cut in negative orientation to give us $2I$. Thus, we get

$$\begin{aligned}\oint_C \frac{\sqrt{1-z^2}}{z^2+a^2}\,dz &= \int_{-1}^1 \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx - \int_1^{-1} \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx\\ &= 4I,\end{aligned}$$

where the sign flip in the second integral comes from crossing the branch cut.

Now, to evaluate the sum of the residues, we need to evaluate at three poles — ia, $-ia$, and the pole at ∞. Thus, we get

$$\begin{aligned}\operatorname{Res}[f(z), \pm ia] &= \frac{\sqrt{a^2+1}}{2ia}\\ -\operatorname{Res}[f(z), \infty] &= \lim_{|z|\to\infty} zf(z)\\ &= i.\end{aligned}$$

Therefore,

$$\begin{aligned}4I &= 2\pi i\left(\frac{\sqrt{a^2+1}}{ia} - i\right)\\ &= \frac{\pi}{2a}\left(\sqrt{a^2+1} - a\right).\end{aligned}$$

Poles on the axis

If we want to evaluate integrals with the pole on the contour, we need to use principal values.

$$\mathrm{PV}\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\left(\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx\right).$$

Similarly, we want to apply this for the calculus of residues. To do this, we take

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \lim_{\varepsilon\to 0} f(z)\,dz,$$

where $c_\pm$ are small semicircular contour additions of radius ε to C that hug our pole on the real axis, with c_- excluding the pole and c_+ including the pole. Thus, we have

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \lim_{\varepsilon\to 0}\int_{c_\pm}\frac{(z-x_0)f(z)}{z-x_0}\,dz.$$

Introducing $z - x_0 = \varepsilon e^{i\varphi}$, we have $dz = i\varepsilon e^{i\varphi}\,d\varphi$, giving

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \operatorname{Res}[f(z), x_0]\int_{c_\pm} i\,d\varphi.$$

Thus, we have

$$\begin{aligned}\oint_C f(z)\,dz &= \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx \pm i\pi\operatorname{Res}[f(z), x_0].\\ &= 2\pi i\sum_{z_i}\operatorname{Res}[f(z) - z_i].\end{aligned}$$

Thus, we have

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = \sum_{z_i \text{ in } C}\operatorname{Res}[f(z), z_i] \mp i\pi\operatorname{Res}[f(z), x_0].$$

Note that we only have *half* the residue when the pole is on the contour. Therefore, we have the result of

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\sum_{z_i \text{ in } C}\operatorname{Res}[f(z), z_i] + \frac{1}{2}\sum_{z_i \text{ on } C}\operatorname{Res}[f(z), z_i]\right).$$

Example. Consider

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx.$$

We have a simple pole at $x = a$.

We close our contour with a semicircle on the upper half-plane. Since we have no poles inside the contour, we have

$$\begin{aligned}\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx &= \pi i\operatorname{Res}\left[\frac{e^{ikx}}{x-a}, a\right]\\ &= \pi i e^{ika}.\end{aligned}$$

Notice that if $k < 0$, we must close the contour in the lower half-plane, giving

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx = \operatorname{sgn}(k)\pi i e^{ika}.$$

Taking real and imaginary components, we get

$$\begin{aligned}
\text{PV}\int_{-\infty}^{\infty}\frac{\cos(kx)}{x-a}\,dx &= -\pi\sin(ka)\\
\text{PV}\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx &= \pi\cos(ka).
\end{aligned}$$

Example. We will evaluate

$$I = \text{PV}\int_{0}^{\infty}\frac{\ln(x)}{x^2+a^2}\,dx.$$

We have a troublesome portion at $x = 0$, so we draw our contour to exclude 0.

We may close the contour with a large semicircle C_R. Since $\lim_{|z|\to\infty}|zf(z)| = 0$ and $\lim_{|z|\to 0}|zf(z)| = 0$, we may take these limits to give

$$\begin{aligned}
\oint_C \frac{\ln(z)}{z^2+a^2} &= \int_{-\infty}^{0}\frac{\ln\left(e^{i\pi}x\right)}{x^2+a^2}\,dx + \int_{0}^{\infty}\frac{\ln\left(e^{i0}x\right)}{x^2+a^2}\,dx\\
&= \text{PV}\int_{-\infty}^{\infty}\frac{\ln(x)}{x^2+a^2}\,dx + i\pi\int_{0}^{\infty}\frac{1}{x^2+a^2}\,dx\\
&= 2\text{PV}\int_{0}^{\infty}\frac{\ln(x)}{x^2+a^2}\,dx + \frac{i\pi^2}{2a}\\
&= 2\pi i\,\text{Res}[f(z), ia].
\end{aligned}$$

Thus, we get $I = \frac{\pi}{2a}\ln(a)$.

Example. Instead of moving our contour up or down by ε to include (or exclude) a pole, we may move the pole up or down by ε. We consider

$$\int_{-\infty}^{\infty}\frac{g(x)}{x-x_0}\,dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(x)}{x-(x_0\pm i\varepsilon)}\,dx.$$

Breaking into real or imaginary parts, we have

$$\int_{-\infty}^{\infty}\frac{g(x)}{x-(x_0\pm i\varepsilon)^2}\,dx = \int_{-\infty}^{\infty}g(x)\frac{x-x_0}{(x-x_0)^2+\varepsilon^2}\,dx \pm i\varepsilon\int_{-\infty}^{\infty}\frac{g(x)}{(x-x_0)^2+\varepsilon^2}\,dx.$$

Now, notice that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2} = \begin{cases}0 & x\neq x_0\\ \infty & x = x_0\end{cases}.$$

Now, we may take

$$\int_{-\infty}^{\infty}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}\,dx = \varepsilon\oint_C\frac{dz}{z+\varepsilon^2},$$

where $z = (x-x_0)^2$. This gives

$$\begin{aligned}
\varepsilon\oint_C\frac{dz}{z+\varepsilon^2} &= \varepsilon(2\pi i)\left(\frac{1}{2i\varepsilon}\right)\\
&= \pi.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}\,dx = \pi\delta(x-x_0).$$

Therefore, we recover

$$\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{\infty}\frac{g(x)}{(x-x_0)^2+\varepsilon^2}\,dx = \pi g(x_0).$$

This gives the identity under the integral of

$$\frac{1}{z\mp i\varepsilon} = \text{PV}\frac{1}{x}\pm i\pi\delta(x).$$

Example. Consider the integral

$$\begin{aligned}\oint_{C_r}\frac{\cos(z)}{z}\,dz &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{\cos(x)}{x\pm i\varepsilon}\,dxj\\ &= \underbrace{\text{PV}\int_{-\infty}^{\infty}\frac{\cos(x)}{x}\,dx}_{=0}\mp i\pi\int_{-\infty}^{\infty}\cos(x)\delta(x)\,dx\\ &= \mp i\pi\end{aligned}$$

Sommerfeld–Watson Transform and Series Summation

Thus far, we've been replacing integrals with sums. Now, we're interested in going the other way around.

Consider the sum

$$S = \sum_{-\infty}^{\infty} f(n),$$

given the condition that $f(z)$ is analytic for $z\in\mathbb{R}$ and $\lim_{|z|\to\infty}\left|z^2 f(z)\right| = 0$.

We will introduce the auxiliary function

$$\begin{aligned}g(z) &= \pi\cot(\pi z)\\ &= \pi\frac{\cos(\pi z)}{\sin(\pi z)}.\end{aligned}$$

Note that $g(z)$ has an infinite number of poles at $z = n$ for each $n\in\mathbb{Z}$.

Now, what we will do here is integrate the product $f(z)g(z)$ around a long enough symmetric contour hugging the real axis. This gives

$$\begin{aligned}\frac{1}{2\pi i}\oint_C f(z)g(z)\,dz &= \sum_{n=-\infty}^{\infty}\text{Res}[\pi\cot(\pi z)f(z), n]\\ &= \sum_{n=-\infty}^{\infty} f(z)\frac{\pi\cos(\pi z)}{\frac{d}{dz}(\sin(\pi z))}\bigg|_{z=n}.\\ &= \sum_{n=-\infty}^{\infty} f(n).\end{aligned}$$

Now, this doesn't *seem* that helpful, until we remember that our contour C surrounds all the other poles of f in negative orientation.

$$\frac{1}{2\pi}\oint_C f(z)g(z)\,dz = -\sum_i \operatorname{Res}[\pi\cot(\pi z)f(z), z_i].$$

Thus, we have converted our infinite sum into a finite sum.

Similarly, if we have an alternating sign series

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty} (-1)^n f(n) \\ &= -\sum_i \operatorname{Res}[\pi\csc(\pi z)f(z), z_i] \end{aligned}$$

Example. Consider

$$S = \sum_{n=0}^{\infty} \frac{1}{n^2+a^2}.$$

Our analogous function is

$$f(z) = \frac{1}{z^2+a^2}.$$

Then,

$$\begin{aligned} S' &= -\operatorname{Res}\left[\frac{\pi\csc(\pi z)}{z^2+a^2}, \pm ia\right] \\ &= -\frac{\pi}{2a}\coth(\pi a). \end{aligned}$$

Therefore, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a).$$

Now, we write

$$S = \frac{1}{2a^2} + \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2}.$$

Thus, we get the sum

$$\sum_{n=0}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2}(1+\pi a\coth(\pi a)).$$

Example. Now, we may consider

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} \\ &= -\operatorname{Res}\left[\frac{\pi\csc(\pi z)}{z^2+a^2}, \pm ia\right] \\ &= -\frac{\pi}{2a}\frac{1}{\sinh(\pi a)}, \end{aligned}$$

giving

$$S = \frac{1}{2a^2}\left(1+\frac{\pi a}{\sinh(\pi a)}\right).$$

Oscillators and Forcing

Consider a damped harmonic oscillator with position $u(t)$. Then, u obeys Newton's second law,

$$\ddot{u} + 2\beta\dot{u} + \omega_0^2 u = 0.$$

Here, β is the damping factor, and ω_0 denotes the natural frequency.

The solutions of this equation are

$$\begin{aligned} u(t) &= e^{-\beta t}\left(ae^{i\Omega t} + be^{-i\Omega t}\right) \\ &= e^{-\beta t}(a\cos(\Omega t) + b\sin(\Omega t)), \end{aligned}$$

where $\Omega^2 = \omega_0^2 - \beta^2$. This is known as a transient solution.

There are three types of motion.

- An underdamped system occurs when $\omega_0 > \beta$, so Ω is real, meaning we get oscillation that is damped out.
- An overdamped system occurs when $\beta > \omega_0$, so Ω is imaginary, and the damping slows down the return of the wave.
- When $\omega = \beta$, then $\Omega = 0$, and the solution is of the form $u(t) = e^{-\beta t}(at + b)$, and the system returns to equilibrium as quickly as possible. This is known as critical damping.

A forced system occurs when we have the differential equation

$$\ddot{u} + \beta\dot{u} + \omega_0^2 u = f(t). \tag{$\dagger$}$$

We may consider a forcing function of the form $f(t) = f_0\cos(\omega t)$. We may also write

$$f(t) = f_0 \operatorname{Re}\left(e^{i\omega t}\right).$$

We expect to have a complex steady-state solution of the form

$$U_\omega = C(\omega)e^{i\omega t}.$$

We solve for U by sticking it into the differential equation of $\dagger$. This will give the equation

$$U_\omega = \frac{f_0 e^{i\omega t}}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}.$$

Note that the real solution is $u = \operatorname{Re}(U_\omega)$, or

$$\begin{aligned} u(t) &= \frac{1}{2}(U_\omega + U_{-\omega}) \\ &= \frac{1}{2}\left(U_\omega + \overline{U_\omega}\right). \end{aligned}$$

Now, when we consider a generalized forcing function $f(t)$, where f is a continuum sum of forcing frequencies where the amplitudes are functions of ω, $\hat{f}(\omega)$, we get an integral:

$$u(t) = \int \frac{F(\omega)e^{i\omega t}}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}\,d\omega.$$

Plugging this solution into the differential equation, we get

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega,$$

which is a Fourier transform (see Math Methods 1).

Impulse Forcing

Consider a hammer blow forcing function, known as an impulse forcing function.

The impulse forcing is of the form

$$\begin{aligned} f(t) &= f_0\delta(\omega(t-t_0)) \\ &= \frac{\hat{f}_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t_0)}\,d\omega, \end{aligned}$$

where

$$\hat{f}_0 = \frac{f_0}{\omega_0}.$$

We want to find the impulse solution,

$$\begin{aligned} G(t) &\coloneqq u_\delta(t) \\ &= \frac{\hat{f}_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-t_0)}}{(\omega_0-\omega^2)+2i\beta\omega}\,d\omega. \end{aligned}$$

To do this integral, we will make use of residues. Writing our denominator as

$$\left(\omega_0^2-\omega^2\right)+2i\beta\omega = (\omega-\omega_+)(\omega-\omega_-),$$

where

$$\begin{aligned} \omega_\pm &= i\beta \pm \sqrt{\omega_0^2-\beta^2} \\ &= i\beta \pm \Omega. \end{aligned}$$

We close our contour in the upper half-plane so that get a decaying exponential. Evaluating the residues, we get

$$\begin{aligned} G(t) &= \hat{f}_0\frac{e^{-\beta(t-t_0)}}{2i\Omega}\left(e^{i\Omega(t-t_0)}-e^{-i\Omega(t-t_0)}\right) \\ &= \frac{f_0}{\Omega}\sin(\Omega(t-t_0))e^{-\beta(t-t_0)}, \end{aligned}$$

where $t > t_0$.

Now, if $t < t_0$, then we must close our contour in the lower half-plane, and since there are no poles in the lower half-plane, we get $G(t) = 0$ for $t < t_0$. Thus, we must have

$$G(t) = \frac{f_0}{\Omega}\sin(\Omega(t-t_0))e^{-\beta(t-t_0)}\Theta(t-t_0),$$

where Θ is the Heaviside step function.

- The imaginary and real parts of $\omega_\pm$ give the damping, β, and parameter, Ω, respectively. Now, we may interpret the different types of damping in this respect.
  - If Ω is real (i.e., underdamped motion), then $\omega_\pm$ have constant magnitude of ω_0, meaning that varying the damping only moves the poles around in a circle.
  - If $\beta = \omega_0$ (i.e., critically damped motion), then the poles converge at $i\beta$ along the imaginary axis.
  - If $\beta > \omega_0$ (i.e., overdamped mption), then the poles separate along the imaginary axis, giving non-oscillatory motion.
- The poles also encode resonance characteristics, where we have $\omega_{\text{res}}^2 = \omega_0^2 - 2\beta^2$.
- If $\beta \ll \omega_0$, then the damping is mathematically equivalent to the $i\varepsilon$ prescription moving the resonance pole at ω_0 off the real axis and into the upper half-plane.

Waves on a String

Whereas an undamped oscillator is harmonic only in time, a wave is harmonic in both space and time.

A wave satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where c is the wave speed.

The general solution is of the form

$$U(x,t) = ce^{i(kx=\omega t)}.$$

A forced wave occurs via

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(x,t).$$

Now, we start with the impulse solution,

$$\begin{aligned} f(x,t) &= f_0\delta(x-x_0)\delta(t-t_0) \\ &= f_0 \int_{-\infty}^{\infty} \frac{e^{ik(x-x_0)}}{2\pi}\,dk \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{2\pi}\,d\omega. \end{aligned}$$

Now, we have

$$G(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{ikx}\,dk \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2/c^2 - k^2}\,d\omega.$$

To evaluate this integral, we start with the integral in ω, given by

$$I = \frac{c^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - c^2k^2}\,d\omega.$$

Unfortunately here, our simple poles lie on the real axis at $\pm ck$.

To find the solution, we need boundary and initial conditions to know where we want to make our $i\varepsilon$ adjustment.

If there is no wave before our impulse hits, we need our integral to vanish whenever $t < 0$ which occurs when we close our contour in the lower half plane, so we subtract $i\varepsilon$ from $\pm ck$. Factoring, we have

$$I = \frac{c^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - (ck - i\varepsilon))(\omega + (ck + i\varepsilon))}\,d\omega.$$

Thus, we have

$$I(t>0) = -\frac{c}{k}\sin(ckt)e^{-\varepsilon t},$$

and in the limit as $\varepsilon \to 0$, we have

$$I(t) = -\frac{c}{k}\sin(ckt).$$

Now, sticking our value of I into the integral in k, we have

$$\begin{aligned} G(x,t>0) &= -\frac{c}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ictk} - e^{-ictk}}{2ik} e^{ikx}\,dk \\ &= -\frac{c}{2}\Theta(ct - |x|)\Theta(t). \end{aligned}$$

There are some comments in order.

- The factor of $\Theta(t)$ effectively states that nothing happens before $t = 0$.
- The factor of $\Theta(ct - |x|)$ denotes causality. The term $|x|$ denotes the physical symmetry, while we need $ct - |x| > 0$ in order to feel an effect.

Quantum Mechanics

The Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} + \frac{\partial^2 \psi}{\partial x^2} = 0.$$

Here, ψ denotes the wavefunction. The probability of a particle being found within dx of x is $|\psi|^2 dx$.

In quantum mechanics, we define operators for energy and momentum as

$$\begin{aligned} E &:= +i\hbar \frac{\partial}{\partial t} \\ P &:= -i\hbar \frac{\partial}{\partial x}. \end{aligned}$$

The Schrödinger equation falls from the fact that $E = \frac{P^2}{2m}$. These operators are Hermitian, so their eigenvalues are real.

We want to figure out the Green's function for the quantum hammer blow at $t = 0$, which we call the propagator.

$$i\hbar \frac{\partial G}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2} = i\hbar\delta(x)\delta(t).$$

When we try to evaluate it, we get

$$G(x, t > 0) = \Theta(t)\sqrt{\frac{m}{2\pi i\hbar t}}e^{imx^2/2\hbar t}.$$

This solution has a problem, though — there is no sense of causality. This says that, as long as $t > 0$, there is going to be a measurable reaction at all x.

The main reason is that the Schrödinger equation is not relativistic. Accounting for the relativistic relationship, we have

$$E^2 - P^2c^2 = m^2c^4.$$

This gives an equation known as the Klein–Gordon equation:

$$-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = \frac{m^2c^2}{\hbar^2}\phi.$$

Solving the Klein–Gordon equation gives infinitely many solutions that have both positive and negative energy, the latter of which is a major issue. However, it can be shown that the Klein–Gordon equation applies to particles with integral spin, with wave function

$$\phi(x, t) = Ce^{i(kx - \omega t)}$$

so long as

$$\frac{\omega^2}{c^2} - k^2 = \frac{m^2c^2}{\hbar^2}.$$

To see how causality fares for the Klein–Gordon wavefunction, we solve it for the unit impulse acting on $x = 0$ at $t = 0$.

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right)G(x,t) = i\delta(x)\delta(t),$$

where we set $\hbar = c = 1$. We get the Green's function

$$G(x,t) = i\int_{-\infty}^{\infty} \frac{1}{2\pi}e^{ipx}\, dp \int_{-\infty}^{\infty} \frac{e^{-iEt}}{2\pi(E^2 - p^2 - m^2)}\, dE.$$

We have two poles at $\pm\sqrt{p^2 + m^2}$, and when we close the contour we obtain different physical results.

When $t < 0$, we must close the contour in the upper-half plane, so we exclude the poles from the contour. Since there are no poles in the upper half-plane, we get a factor of $\Theta(t)$.

Thus, when $t > 0$, we close in the lower half-plane, giving

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{-iEt}}{(E + iE)^2 - p^2 - m^2}\, dE \\ &= -\frac{1}{2\pi}2\pi i\left(\frac{e^{-i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} - \frac{e^{i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}}\right). \end{aligned}$$

Defining $E_p := \sqrt{p^2 + m^2}$, we have

$$I = -\frac{i}{2E_p}\left(e^{-iE_pt} - e^{iE_pt}\right).$$

Thus, we have

$$\begin{aligned} G(x, t > 0) &= \Theta(t)\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{e^{ipx}}{2E_p}\left(e^{-iE_pt} - e^{iE_pt}\right) dp \\ &= \frac{\Theta(t)}{4\pi}\int_{-\infty}^{\infty} \frac{1}{E_p}\left(e^{-i(E_pt - px)} - e^{i(E_pt - p_x)}\right) dp, \end{aligned}$$

where we were allowed to flip the sign of p because the integral is taken over all space.