

# Observations on Excess Area Identities and Operator Symbols in Bergman Spaces

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# Definition and Notations

- $\Omega$  : a region in  $\mathbb{C}$  e.g.  $\mathbb{D}$ ,  $D(0, r)$ ,  $\mathbb{A}(0, r, 1)$ ,  $\mathbb{C}$
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$ : weight function

## Definition ( $\lambda$ -weighted Square-Integrable Functions)

$$L^2(\Omega, \lambda) = \left\{ f : \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f(z)|^2 \lambda(z) dA(z) < \infty \right\}$$

- $L^2(\Omega, \lambda)$  forms a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega, \lambda)} = \int_{\Omega} f(z) \overline{g(z)} \lambda(z) dA(z)$$

inducing the norm

$$\|f\|_{L^2(\Omega, \lambda)}^2 = \int_{\Omega} |f(z)|^2 \lambda(z) dA(z)$$

# Definitions and Notations

## Definition (Holomorphic Function on $\Omega$ )

We say  $h$  is holomorphic on  $\Omega$ , or  $h \in \mathcal{O}(\Omega)$ , if, for all  $z \in \Omega$

$$\begin{aligned}\frac{\partial h(z)}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial h(z)}{\partial x} + i \frac{\partial h(z)}{\partial y} \right) \\ &= \frac{1}{2} \left( \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \\ &= 0.\end{aligned}$$

## Definition ( $\lambda$ -weighted Bergman Space)

$$A^2(\Omega, \lambda) := \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

# Definitions and Notations

## Definition ( $A^{1,2}(\Omega, \lambda)$ )

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \mid \frac{\partial h}{\partial \bar{z}} \in A^2(\Omega, \lambda) \right\}$$

## Definition (Weighted Image-Area)

Let  $h \in A^{1,2}(\Omega, \lambda)$ .

$$\begin{aligned} A_{\Omega, \lambda}(h) &= \int_{\Omega} \left| \frac{\partial h}{\partial \bar{z}} \right|^2 \lambda(z) dA(z) \\ &= \left\| \frac{\partial h}{\partial \bar{z}} \right\|_{L^2(\Omega, \lambda)}^2 \end{aligned}$$

# Definitions and Notations

- $A^2(\Omega, \lambda)$  has a reproducing kernel i.e  $\exists! K_{\Omega}^{\lambda}(\cdot, z) \in A^2(\Omega, \lambda) :$

$$h(z) = \langle h(\cdot), K_{\Omega}^{\lambda}(\cdot, z) \rangle_{L^2(\Omega, \lambda)}$$

- $A^2(\Omega, \lambda)$  is a closed subspace of  $L^2(\Omega, \lambda)$ .

## Definition (Bergman Projection)

Let  $P^{\Omega, \lambda} : L^2(\Omega, \lambda) \rightarrow A^2(\Omega, \lambda)$

$$\begin{aligned} (P^{\Omega, \lambda} h)(z) &:= \langle h(\cdot), K_{\Omega}^{\lambda}(\cdot, z) \rangle_{L^2(\Omega, \lambda)} \\ &= \int_{\Omega} h(w) \overline{K_{\Omega}^{\lambda}(w, z)} \lambda(w) dA(w) \end{aligned}$$

## Definition (Multiplication Operator)

Let  $M_\varphi : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda)$  where  $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

## Definition (Toeplitz Operator)

$T_\varphi^{\Omega, \lambda} : A^2(\Omega, \lambda) \rightarrow A^2(\Omega, \lambda)$ , where  $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega, \lambda} := P^{\Omega, \lambda} M_\varphi$$



# Definitions and Notations

## Definition (Commutator)

Let  $[P^{\Omega,\lambda}, M_\varphi] : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda)$

$$[P^{\Omega,\lambda}, M_\varphi] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

## Definition (Hankel Operator)

Let  $H_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \rightarrow (A^2(\Omega, \lambda))^\perp$

$$\begin{aligned} H_\varphi^{\Omega,\lambda} &:= - [P^{\Omega,\lambda}, M_\varphi] \Big|_{A^2(\Omega,\lambda)} \\ &= (I - P^{\Omega,\lambda}) M_\varphi \\ &= M_\varphi - P^{\Omega,\lambda} M_\varphi \\ &= M_\varphi - T_\varphi^{\Omega,\lambda} \end{aligned}$$

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# Motivations

- $\{z^n\}_{n=0}^{\infty}$  form a *complete orthogonal basis* for  $A^2(\mathbb{D})$
- If  $h$  is holomorphic, then  $h$  is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

and

$$S_N := \sum_{n=0}^N h_n z^n$$

converges uniformly on compact subsets.

- Relationship between  $L^2$  norm of  $h$  to the  $\ell^2$  norm of  $\{h_k\}_{k=0}^{\infty}$ :

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $[T_{\bar{z}}^{\mathbb{D}} M_z, D M_z](z^m) = 0$

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

# Literature Review on Previous Results I

- D'Angelo's Excess Area identity [D'A19]

Let  $h \in A^{1,2}(\mathbb{D})$ . Then,

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial(zh)}{\partial z} \right\|_{L^2(\mathbb{D})}^2 - \left\| \frac{\partial h}{\partial z} \right\|_{L^2(\mathbb{D})}^2 \\ &= \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \\ &= \pi \|Sh\|_{L^2(b\mathbb{D})}^2 \end{aligned}$$

where  $Sh$  is the restriction of  $h$  to the unit circle.

# Literature Review on Previous Results II

- Excess Area identity with Blaschke product multiplier
- 'Excess Area' identity for harmonic functions [BÇGH22]
- Generating symbols for Toeplitz operators for a given initial  $p$  and target polynomial  $q$  on unit disc and polydisc,  $T_{\varphi}^{\mathbb{D}}(p) = q$  and  $T_{\varphi}^{\mathbb{D}^n}(p) = q$  [ÇDTR<sup>+</sup>24]
- Substituted derivatives for Toeplitz operators in Excess Area identity [ÇDTR<sup>+</sup>24]

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# Summary of Results

1. Results and Observations influenced by the Excess Area identity:
  - i. On  $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2})$ ,  $A^2(\mathbb{D}, \lambda)$ ,  $A^2(D(0, r))$
  - ii. On convergence of identities on certain weighted discs.
- 2 Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
  - i. On unweighted and weighted Toeplitz operators relation
  - ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on  $A^2(\mathbb{D})$



# Methods Used

- Relation between  $L^2$  norms of functions and  $\ell^2$  norms of Taylor series:

$$\|h\|_{L^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- Integration by parts via Stokes's theorem on forms:

$$\oint_{\partial\Omega} f \, dz = \int_{\Omega} \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz$$
$$\oint_{\partial\Omega} f \, d\bar{z} = \int_{\Omega} \frac{\partial f}{\partial z} \, dz \wedge d\bar{z}.$$

- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Beta, Gamma, and Hypergeometric functions

# Excess Area on Fock Spaces

D'Angelo's Excess Area identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \|Sh\|_{L^2(b\mathbb{D})}^2$$

## Excess Area on Fock Space

Given  $h \in \mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2})$  with  $\frac{\partial h}{\partial \bar{z}} \in \mathcal{F}^2$ ,

$$\begin{aligned} A_{\mathcal{F}^2}(zh) - A_{\mathcal{F}^2}(h) \\ = \pi \left\| M_z T_{\bar{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 + \pi \|M_z h\|_{\mathcal{F}^2}^2 \end{aligned}$$

Here, the restriction of  $h$  to the unit circle in D'Angelo's Excess Area identity is replaced with the Bergman projection on  $\mathbb{C}$ .

## Excess Area on $A^2(\mathbb{D}, \lambda)$

D'Angelo's Excess Area identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \|Sh\|_{L^2(b\mathbb{D})}^2$$

### Excess Area on $A^2(\mathbb{D}, \lambda)$

Let  $h \in A^{1,2}(\mathbb{D}, \lambda)$ ,  $\lambda(z) = 1 - |z|^2$ . Then,

$$A_{\mathbb{D},\lambda}(z^{m+1}h) - A_{\mathbb{D},\lambda}(z^m h) = \pi \|z^m h\|_{L^2(\mathbb{D},\lambda)}^2.$$

Here, the restriction of  $h$  to the unit circle is replaced with the function itself.

## Contraction and Dilation from $A^2(D(0, r))$ to $A^2(\mathbb{D})$

### Contraction from $A^2(\mathbb{D})$ to $A^2(D(0, r))$

Given some  $h \in A^{1,2}(\mathbb{D})$  and taking  $h_r = h(rz)$  for some  $0 < r < 1$ ,

$$A_{\mathbb{D}}(zh_r) - A_{\mathbb{D}}(h_r) = \pi \|Sh_r\|_{L^2(b\mathbb{D})}^2 \quad (1)$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \|Sh\|_{L^2(bD(0,r))}^2. \quad (2)$$

### Dilation from $A^2(D(0, r))$ to $A^2(\mathbb{D})$

Given some  $h \in A^{1,2}(D(0, r))$  and taking  $h_{\frac{1}{r}} = h(\frac{z}{r})$  for some  $0 < r < 1$

$$A_{D(0,r)}(zh_{1/r}) - r^2 A_{D(0,r)}(h_{1/r}) = \pi r^2 \|Sh_{1/r}\|_{L^2(bD(0,r))}^2 \quad (3)$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \|Sh\|_{L^2(b\mathbb{D})}^2 \quad (4)$$

# Approximation for Sequences of Berezin Transform

## Weighted Area on $D(0, r)$

Let  $\lambda_r(z) = \chi_{D(0,r)} \left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$  where  $r > 0$ . Then,

$$A_{D(0,r),\lambda_r}(h) = \int_{D(0,r)} |h'(z)|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} dA(z)$$

We find that, as  $r \rightarrow \infty$ ,  $A_{D(0,r),\lambda_r}(h) \rightarrow A_{\mathcal{F}^2}(h)$ .

Separately,

$$A_{\mathcal{F}^2}(h) = \left\| T_{\bar{z}}^{\mathcal{F}^2} h \right\|_{\mathcal{F}^2}^2$$

## Berezin Transform Convergence, Cont'd

### Reproducing Kernel on $A^2(D(0, r), \lambda_r)$

$$\begin{aligned} K_{D(0,r)}^{\lambda_r}(w, z) &= \sum_{k=0}^{\infty} \frac{\bar{z}^k w^k}{\|w^k\|_{L^2(D(0,r), \lambda_r)}^2} \\ &= \frac{1}{\left(1 - \frac{\bar{z}w}{r^2}\right)^{r^2+2}} \end{aligned}$$

$K_{D(0,r)}^{\lambda_r}(w, z)$  uniformly converges on compact subsets of  $D(0, r)$ .

### Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w, z) = e^{\bar{z}w}$$

## Definition (Berezin Transform ([Zhu07]))

Let

$$k_z^{\Omega, \lambda}(w) := \frac{K_{\Omega}^{\lambda}(w, z)}{\sqrt{K_{\Omega}^{\lambda}(z, z)}}$$

Then, for some bounded operator  $T$  on  $L^2(\Omega, \lambda)$ , define  $\mathcal{B}^{\Omega, \lambda} : B(L^2(\Omega, \lambda)) \rightarrow L^2(\Omega, \lambda)$

$$(\mathcal{B}^{\Omega, \lambda} T)(z) := \langle Tk_z^{\Omega, \lambda}, k_z^{\Omega, \lambda} \rangle_{L^2(\Omega, \lambda)}$$

# Berezin Transform Convergence, Cont'd

Previous results:

- For  $\varphi \in L^\infty(\Omega, \lambda)$ ,  $\mathcal{B}^{\Omega, \lambda} T_\varphi = \mathcal{B}^{\Omega, \lambda} M_\varphi$ . (see [AZ98a]).
- $\varphi$  is harmonic if and only if  $\mathcal{B}^{\Omega, \lambda} M_\varphi = \varphi$  (proof in [Eng94]).

New results:

- For  $T_\varphi^{D(0,r), \lambda_r} = P^{D(0,r), \lambda_r} M_\varphi$ , the Berezin transform  $\mathcal{B}^{D(0,r), \lambda_r} T_\varphi^{D(0,r), \lambda_r}$  converges pointwise to  $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$  as  $r \rightarrow \infty$ .
- By Dini's Theorem, this convergence is uniform on compact subsets of  $\mathbb{C}$  (proof inspired by [GŞ20]).



# Unweighted and Weighted Toeplitz Operators Relation

Using an extension of [ÇDTR<sup>+</sup>24, Lemma 2.1]

For weight  $\lambda(z) = (1 - |z|^2)^\alpha$  ( $\alpha \geq 0$ ) on the unit disc,  
 $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ :

$$\frac{T_{\bar{z}^m}^{\mathbb{D}, \lambda_\alpha}(z^n)}{T_{\bar{z}^m}^{\mathbb{D}}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leq n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T_{\bar{z}^m}^{\mathbb{D}, \lambda_\alpha}(z^n) = s_{n,m,\alpha} T_{\bar{z}^m}^{\mathbb{D}}(z^n), \text{ and } \lim_{n \rightarrow \infty} s_{n,m,\alpha} = 1$$

# Unweighted and Weighted Commutator on $A^2(\mathbb{D})$

## Existence of Commutator Symbols

Given  $p$  and  $q$  are harmonic polynomials and  $\frac{\partial}{\partial \bar{z}}(p) \neq 0$ , there does not exist a polynomial symbol  $\phi$ , such that  $[P^{\mathbb{D}}, M_{\phi}](p) = q$  or  $[P^{\mathbb{D}, \lambda}, M_{\phi}](p) = q$ .

Compare to [ÇDTR<sup>+</sup>24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

# Unweighted and Weighted Hankel Operator on $A^2(\mathbb{D})$

## Existence of Hankel Operator Symbols

Given some holomorphic polynomials  $p, q$  where  $p$  is not constant, there does not exist a polynomial symbol  $\phi$  such that  $H_{\phi}^{\mathbb{D}}(p) = \bar{q}$  or

$$H_{\phi}^{\mathbb{D}, \lambda}(p) = \bar{q}$$

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## Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0, r, 1))$

For all integers  $m$  and  $n$ ,

$$T_{\bar{z}^m}^{\mathbb{A}(0, r, 1)}(z^n) = \begin{cases} \frac{2mr^{2m} \ln(r)}{(r^{2m}-1)} z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m}-1}{2m \ln(r)} z^{-1} & \text{if } n = m-1 . \\ \frac{(n-m+1)(1-r^{2n+2})}{(n+1)(1-r^{2n-2m+2})} z^{n-m} & \text{else} \end{cases}$$

We attempted to find an algorithm to generate  $\varphi \in L^\infty(\mathbb{A}(0, r, 1))$  such that  $T_\varphi^{\mathbb{A}(0, r, 1)}(p) = q$  for given holomorphic Laurent polynomials  $p$  and  $q$ , but could not prove lack of existence if  $p$  has roots inside  $\overline{\mathbb{A}(r, 0, 1)}$ .

# Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial  $p$  and target polynomial  $q$  on  $\mathbb{A}(0, r, 1)$ ,  $T_{\varphi}^{\mathbb{A}(0, r, 1)}(p) = q$
- Extension of 'Excess Area' identity to harmonic functions in  $L^2(\mathbb{C}, e^{-|z|^2})$ .
- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential,  
 $(1 - |z|^2)^A e^{\frac{-B}{(1-|z|^2)^\alpha}} (A \geq 0, B > 0, \alpha > 0)$ .

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# What is Research Like?

- The complex analysis group consisted of myself, Jennifer Yuan (NYU Abu Dhabi), and Sakia Akamah (Rose–Hulman Institute of Technology).
- Weeks were 9am to 5pm, mostly doing various calculations and updating our collected results document.
- Every weekday morning, our mentor met with us to go over the previous day's calculations and suggest new ideas/directions.
- We rotated project directions every day after the morning's meeting so we could bring fresh perspectives and ideas to the work the previous person had done yesterday.
- We did not fully understand what we were doing a lot of the time (especially in the beginning), but that's honestly very expected; our mentor was always there to help us along the way and answer any questions we had.



# Activities and Life Outside Research

- In addition to the research itself, we also received visitors from academia and industry to provide some guidance on our potential future paths.
- We also visited the University of North Texas in Denton to talk with PhD students and faculty who gave us important advice on how to succeed as a graduate student.
- We also had some fun — usually after work was over for the day we'd watch anime in the university housing common room, or play badminton at the gym, and occasionally played board games.
- There were four project cohorts at the REU, but ours was the coolest since it wasn't applied.

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