Here, we overview and discuss some of the most important results related to projections in von Neumann algebras.

Comparison of Projections: An Introduction

Recall that if H is a Hilbert space, an element $w \in B(H)$ is called a partial isometry if, for any $h \in \ker(w)^{\perp}$, we have $\|Wh\| = \|h\|$. We call $\ker(w)^{\perp}$ the initial space of W and $\operatorname{im}(w)$ the final space of W.

There are a variety of equivalent definitions for partial isometries.

Proposition: If $w \in B(H)$, then the following are equivalent:

(i) w is a partial isometry;

(ii) w^* is a partial isometry;

(iii) w^*w is a projection onto the initial space of w;

(iv) ww^* is a projection onto the final space of w;

(v) $ww^*w = w$;

(vi) $w^*ww^* = w^*$.

If $M \subseteq B(H)$ is a von Neumann algebra, then we say two projections $p, q \in P(M)$, where $P(M)$ denotes the space of projections of M, are (Murray–von Neumann) *equivalent* in M if there is a partial isometry $v \in P(M)$ such that $v^*v = p$ and $vv^* = q$. We will write $p \sim q$.

Note that projections have an ordering by saying that $p \leq q$ if $pq = qp = p$, or $\operatorname{im}(p) \subseteq \operatorname{im}(q)$. This allows us to say that p is *sub-equivalent* to q (in M), written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. In this scenario we will say that q majorizes p; the choice for this vocabulary will be seen below.

The sub-equivalence relation in fact forms a partial order, and equivalence as projections forms an equivalence relation. We will first show that it is a preorder.

Proposition: In a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\preceq$ is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry follows from the fact that if v is a partial isometry, then so is v^*.

Now, we will show transitivity for $\preceq$, from which we will see that $\sim$ is transitive. Letting $p, q, r \in P(M)$ be such that $p \preceq q$ and $q \preceq r$, we have partial isometries $u, v \in M$ with $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. Then, we have

$$\begin{aligned} qu &= quu^*u \\ &= (quu^*)u \\ &= uu^*u \\ &= u, \end{aligned}$$

so that

$$\begin{aligned}(vu)^*(vu) &= u^*v^*vu \\ &= u^*qu \\ &= u^*u \\ &= p \\ (vu)(vu)^* &= vuu^*v^* \\ &\leq vqv^* \\ &= vv^*vv^* \\ &= vv^* \\ &\leq r.\end{aligned}$$

Therefore, $p \preceq r$, so $\preceq$ is a transitive relation. □

To see that $\preceq$ is a partial order, we need an analogue of the Cantor–Schröder–Bernstein theorem for projections. In fact, it is proven in a similar manner.

Theorem: If $e \preceq f$ and $f \preceq e$, then $e \sim f$.

The projections in a von Neumann algebra form a complete lattice, as the collection of closed subspaces of H form a complete lattice under the operations

$$\begin{aligned}\bigvee_{i\in I} X_i &\coloneqq \overline{\sum_{i\in I} X_i} \\ \bigwedge_{i\in I} X_i &\coloneqq \bigcap_{i\in I} X_i.\end{aligned}$$

References

[Con00] John B. Conway. *A course in operator theory*. Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: 10.1090/gsm/021. URL: https://doi.org/10.1090/gsm/021.

[Dav25] Kenneth R. Davidson. *Functional analysis and operator algebras*. Vol. 13. CMS/CAIMS Books in Mathematics. Springer, Cham, [2025] ©2025, pp. xiv+797. ISBN: 978-3-031-63664-6; 978-3-031-63665-3. DOI: 10.1007/978-3-031-63665-3. URL: https://doi.org/10.1007/978-3-031-63665-3.