Problem (Problem 1)**:** Let $(a_n)_n$ be a sequence for which $\sum_{n=0}^{\infty}|a_n|^2$ is finite. For each positive N, define $f_N(z) = \sum_{n=0}^{\infty} a_n z^n$, and define $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(a) Show that f is holomorphic on $\mathbb{D}$.

(b) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f_N\left(re^{i\theta}\right)\right|^2 \, d\theta.$$

(c) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \, d\theta.$$

(d) Determine in terms of $(a_n)_n$ the limit

$$\lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \, d\theta.$$

Solution:

(a) Let $0 < r < 1$. Since each f_N is analytic, we can use the Cauchy Integral Formula to compute a_N explicitly, yielding

$$\begin{aligned} |a_N| &= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f_N(\xi)}{\xi^{N+1}}\, d\xi\right| \\ &\leqslant \frac{1}{r^N}\sup_{|z|=r}|f_N(z)|. \end{aligned}$$

Therefore, if we are able to show that the value

$$\sup_{|z|=r}|f_N(z)|$$

is uniformly bounded by a constant for all N, we will be able to use the Cauchy–Hadamard theorem to show that $\limsup_{N\to\infty}|a_N|^{1/N} \leqslant 1$. Toward this end, we use the Cauchy–Schwarz inequality, which yields

$$\begin{aligned} \sup_{|z|=r}|f_N(z)| &= \sup_{|z|=r}\left|\sum_{n=0}^{N} a_n z^n\right| \\ &\leqslant \sup_{|z|=r}\left(\sum_{n=0}^{N}|a_n|^2\right)^{1/2}\left(\sup_{m=0}^{N}|z|^{2m}\right)^{1/2} \\ &\leqslant \sup_{|z|=r}\underbrace{\left(\sum_{n=0}^{\infty}|a_n|^2\right)^{1/2}}_{=:K}\left(\sum_{m=0}^{\infty}|z|^{2m}\right)^{1/2} \\ &= \frac{K}{(1-|r|^2)^{1/2}}. \end{aligned}$$

Since we have established this uniform bound, we thus find that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence at least 1, so f is analytic on $\mathbb{D}$, hence holomorphic on $\mathbb{D}$.

(b) We write out the integral to yield

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|f_N\left(re^{i\theta}\right)\right|^2\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{N}a_nr^ne^{in\theta}\right)\overline{\left(\sum_{m=0}^{N}a_mr^me^{im\theta}\right)}\,d\theta\\ &= \frac{1}{2\pi}\sum_{n=0}^{N}\sum_{m=0}^{N}a_n\overline{a_m}r^{m+n}\int_0^{2\pi}e^{i(n-m)\theta}\,d\theta\\ &= \sum_{n=0}^{N}|a_n|^2r^{2n}.\end{aligned}$$

(c) Since f is holomorphic with radius of convergence at least 1, the series expression on $S(0,r)$ converges uniformly, so that we may exchange sum and integral. This yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n\overline{a_m}r^{m+n}\int_0^{2\pi}e^{i(n-m)\theta}\,d\theta\\ &= \sum_{n=0}^{\infty}|a_n|^2r^{2n}.\end{aligned}$$

(d) Since the sequence $(a_n)_n$ is square-summable, the limit is well-defined, and we get

$$\begin{aligned}\lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta &= \lim_{r\nearrow 1}\sum_{n=0}^{\infty}|a_n|^2r^{2n}\\ &= \sum_{n=0}^{\infty}|a_n|^2.\end{aligned}$$

Problem (Problem 2)**:** Let $\varphi\colon[0,1]\to\mathbb{C}$ be continuous, and define $f\colon\mathbb{C}\setminus[0,1]\to\mathbb{C}$ by

$$f(z) = \int_0^1\frac{\varphi(t)}{t-z}\,dt.$$

Show that f is holomorphic and determine the derivative of f in terms of φ.

Solution: Let $z, z+h\in\mathbb{C}\setminus[0,1]$, so we may calculate

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{1}{h}\int_0^1\frac{\varphi(t)}{t-(z+h)}-\frac{\varphi(t)}{t-z}\,dt\\ &= \lim_{h\to 0}\frac{1}{h}\int_0^1\frac{(t-z)\varphi(t)-(t-(z+h))\varphi(t)}{(t-(z+h))(t-z)}\,dt\\ &= \lim_{h\to 0}\frac{1}{h}\int_0^1\frac{h\varphi(t)}{(t-(z+h))(t-z)}\,dt\\ &= \lim_{h\to 0}\int_0^1\frac{\varphi(t)}{(t-(z+h))(t-z)}\,dt\\ &= \int_0^1\frac{\varphi(t)}{(t-z)^2}\,dt.\end{aligned}$$

Let $(z_n)_n\subseteq\mathbb{C}\setminus[0,1]$ converge to $z\in\mathbb{C}\setminus[0,1]$. Define the sequence of functions given by

$$(h_n)_n\colon[0,1]\to\mathbb{C}$$

$$t \mapsto \frac{\varphi(t)}{(t-z_n)^2}.$$

We claim that the $(h_n)_n$ converge uniformly to

$$h(t) = \frac{\varphi(t)}{(t-z)^2}.$$

Observe that the pointwise convergence is clear, and $h(t)$ is well-defined for all t by definition. Now, we observe as well that the value $K = \operatorname{dist}_{[0,1]}(\{z_n \mid n \in \mathbb{N}\})$ is nonzero, as the closure of $[0,1]$ in $\mathbb{C}$ is $[0,1]$, and $\{z_n \mid n \in \mathbb{N}\}$ are explicitly contained in $\mathbb{C} \setminus [0,1]$. Similarly, since z is not contained in the closure of $[0,1]$, we find that $L = \operatorname{dist}_{[0,1]}(\{z\})$ is also nonzero.

Thus, we find that

$$\begin{aligned}
\left|\frac{\varphi(t)}{(t-z_n)^2} - \frac{\varphi(t)}{(t-z)^2}\right| &= \frac{|\varphi(t)|\left|2t(z_n - z) + \left(z^2 - z_n^2\right)\right|}{|t-z_n|^2|t-z|^2} \\
&\leqslant \frac{\|\varphi\|_u\left|2t(z_n - z) + \left(z^2 - z_n^2\right)\right|}{|t-z_n|^2|t-z|^2} \\
&\leqslant \|\varphi\|_u \frac{2|z_n - z| + \left|z^2 - z_n^2\right|}{K^2L^2} \\
&= \frac{2\|\varphi\|_u}{K^2L^2}\left(|z_n - z| + \left|z^2 - z_n^2\right|\right),
\end{aligned}$$

meaning that the supremum of the left-hand side is less than or equal to a constant multiplied by $|z_n - z| + \left|z^2 - z_n^2\right|$. Since $z \mapsto z^2$ is continuous, it follows that $(h_n)_n \to h$ uniformly. Thus, we may exchange limit and integral, so that

$$\begin{aligned}
\lim_{z_n \to z} f'(z_n) &= \lim_{z_n \to z} \int_0^1 \frac{\varphi(t)}{(t-z_n)^2}\,dt \\
&= \int_0^1 \lim_{z_n \to z} \frac{\varphi(t)}{(t-z_n)^2}\,dt \\
&= \int_0^1 \frac{\varphi(t)}{(t-z)^2}\,dt \\
&= f'(z),
\end{aligned}$$

meaning $f'(z)$ is continuous, so f is holomorphic.

Problem (Problem 3)**:** Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire.

(a) Suppose there exist $C, R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leqslant C|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree at most n.

(b) Suppose that $g\colon \mathbb{C} \to \mathbb{C}$ is also entire and $|f(z)| \leqslant |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists some $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$ such that $f(z) = \alpha g(z)$ for all $z \in \mathbb{C}$.

(c) Suppose that there exists some $\theta \in \mathbb{R}$ such that $f(\mathbb{C}) \cap \left\{re^{i\theta} \,\middle|\, r > 0\right\} = \emptyset$. Show that f is constant.

Solution:

(a) Let $r > R$. Then, by the Cauchy estimate, we get that

$$\left|f^{(n+1)}(0)\right| \leqslant \frac{(n+1)!}{r^{n+1}} \sup_{|z|=r}|f(z)|$$

$$\leqslant \frac{(n+1)!}{r^{n+1}} \sup_{|z|=r} \left(C|z|^n\right)$$
$$= \frac{C(n+1)!}{r},$$

so since r is arbitrary and f is entire, we find that $f^{(n+1)}(0) = 0$, so that the power series expansion of f about 0 terminates beyond $n+1$. Since f is entire, its power series expansion about any $z_0 \in \mathbb{C}$ is equal to $f(z)$ everywhere in $\mathbb{C}$, so in particular, this holds for f at 0, meaning f is a polynomial of degree at most n.

(b) If g is 0, or f is 0, we are done. Else, assume that g and f are not identically zero. Observe that if g is everywhere non-vanishing, then the function $\frac{f(z)}{g(z)}$ is entire, and satisfies

$$\left|\frac{f(z)}{g(z)}\right| \leqslant 1,$$

hence $\frac{f(z)}{g(z)} = \alpha$ for some α with $|\alpha| \leqslant 1$.

If $k(z) = \frac{f(z)}{g(z)}$ is such that $g(z)$ admits zeros, then they must be isolated zeros, or else by the identity theorem, g would be identically zero everywhere. Let a be one of these zeros for g. If ε is small, we observe that for $0 < |z-a| < \varepsilon$, $k(z)$ is bounded. If we let M be this bound, then we observe that the value

$$\left|\frac{1}{2\pi i} \int_{|\zeta-a|=\varepsilon} \frac{k(\zeta)}{\zeta-a}\,d\zeta\right| \leqslant \frac{1}{2\pi} \int_{|\zeta-a|=\varepsilon} \frac{|k(\zeta)|}{|\zeta-a|}\,|d\zeta|$$
$$\leqslant \frac{1}{2\pi} \int_{|\zeta-a|=\varepsilon} \frac{M}{|\zeta-a|}\,|d\zeta|$$
$$= \frac{M}{\varepsilon},$$

so that the integral is well-defined. In particular, this means that we may define a holomorphic extension of $k(z)$ by

$$h(z) = \begin{cases} k(z) & g(z) \neq 0 \\ \frac{1}{2\pi i} \int_{|\zeta-z|=\varepsilon} \frac{k(\zeta)}{\zeta-z}\,d\zeta & g(z) = 0. \end{cases}$$

The function $h(z)$ is thus entire, and bounded by 1, so by Liouville's theorem, $h(z) = \alpha$ for some α with $|\alpha| \leqslant 1$. This means that whenever $g(z) \neq 0$, we have $f(z) = \alpha g(z)$, and clearly when $g(z) = 0$, we have $f(z) = \alpha g(z)$, so that $f(z) = \alpha g(z)$.

(c) Let f be such that $f(\mathbb{C}) \cap \left\{re^{i\theta} \mid r > 0\right\} = \emptyset$. By adding a sufficient multiple of $2\pi k$, we may assume $\theta > 0$.

Define a branch of the logarithm $\log_\theta(z)$ by taking

$$S_\theta = \{z \in \mathbb{C} \mid \theta < \operatorname{Im}(z) < \theta + 2\pi\}.$$

Then, we observe that $\sqrt{z} = e^{\frac{1}{2}\log_\theta(z)}$ maps S_θ to the set

$$\mathbb{H}_\theta = \left\{z \mid \theta/2 < \arg(z) < \theta/2 + \pi\right\} \cup \{0\}.$$

Observe that the map $z \mapsto e^{-i\theta/2}z$ is entire and maps $\mathbb{H}_\theta$ to the upper half plane plus $\{0\}$, and the Cayley transform, $\varphi(w) = \frac{w-i}{w+i}$, is holomorphic on $\mathbb{C} \setminus \{i\}$, maps $0 \mapsto -1$, and maps the open upper half-plane to the unit disk. Therefore, if we fix some $\varepsilon > 0$, we find that the composition

$$\varphi \circ \left(z \mapsto e^{-i\theta/2}z\right) \circ \sqrt{\cdot} \circ f \colon \mathbb{C} \to U(0, 1+\varepsilon)$$

is an entire function that is bounded in modulus by $1+\varepsilon$. In particular, since all of $\sqrt{\cdot}$, $z \mapsto e^{-i\theta/2}z$, and φ are nonconstant and holomorphic on the specified domains, this implies that f is constant.

Problem (Problem 4)**:** Let $U = \{z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1\}$. Suppose $f\colon U \to \mathbb{C}$ is holomorphic, and there exists $C > 0$ and $\eta \in \mathbb{R}$ such that

$$|f(z)| \leqslant C(1+|z|)^{\eta}$$

for all $z \in U$. Show that for each $n \geqslant 0$, there exists a constant $C_{n,\eta} \geqslant 0$ dependent only on n and η such that

$$\left|f^{(n)}(x)\right| \leqslant C_{n,\eta}(1+|x|)^{\eta}$$

for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$, $0 < r < 1$, and to start, assume $\eta \geqslant 0$. Then, from Cauchy's estimate, a bunch of triangle inequalities, and the fact that $\eta \geqslant 0$ and $r < 1$, we find that

$$\begin{aligned}
\left|f^{(n)}(x)\right| &\leqslant \frac{n!}{r^n} \sup_{|w-x|=r} |f(w)| \\
&\leqslant \frac{n!}{r^n} \sup_{|w-x|=r} \left(C(1+|w|)^{\eta}\right) \\
&\leqslant \frac{Cn!}{r^n} \sup_{|w-x|=r} \left(1 + \left|w - \frac{3}{2}x\right| + \frac{3}{2}|x|\right)^{\eta} \\
&\leqslant \frac{Cn!}{r^n} \sup_{|w-x|=r} (1+|w-x|+2|x|)^{\eta} \\
&\leqslant \frac{Cn!}{r^n} \sup_{|w-x|=r} (1+r+2|x|)^{\eta} \\
&\leqslant \frac{Cn!}{r^n} (2+2|x|)^{\eta} \\
&\leqslant \frac{C2^{\eta}n!}{r^n}(1+|x|)^{\eta}.
\end{aligned}$$

In particular, since this inequality holds for every $0 < r < 1$, it holds for $r = 1/2$, so that $C_{n,\eta} = C2^{\eta+n}n!$.

Now, if $\eta < 0$, we see that

$$\begin{aligned}
\sup_{|w-x|=r} (1+|w|)^{\eta} &= \left(\inf_{|w-x|=r} (1+|w|)\right)^{\eta} \\
&= \begin{cases} (1+|x-r|)^{\eta} & x \geqslant 0 \\ (1+|x+r|)^{\eta} & x < 0 \end{cases}',
\end{aligned}$$

and by the triangle inequality,

$$\leqslant (1-r+|x|)^{\eta}.$$

Finally, we observe that, for $0 < r < 1$ and fixed $|x|$, since $(1-r)+|x| \geqslant (1-r)+(1-r)|x|$, the order reverses. Thus, by the Cauchy estimates, we have

$$\begin{aligned}
\left|f^{(n)}(x)\right| &\leqslant \frac{n!}{r^n} \sup_{|w-x|=r} |f(w)| \\
&\leqslant \frac{n!}{r^n} \sup_{|w-x|=r} \left(C(1+|w|)^{\eta}\right)
\end{aligned}$$

$$\leqslant \frac{C(1-r)^{n} n!}{r^{n}}(1+|x|)^{n}.$$

Since this holds for any r, it holds for $r = 1/2$, so that we get $C_{n,\eta} = C2^{n-\eta}n!$.

Problem (Problem 5)**:** Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geqslant 1$, where $a_0, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$.

(a) Show that there exist n complex numbers $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ not necessarily distinct such that $P(z) = a_n(z-\alpha_1)\cdots(z-\alpha_n)$.

(b) Suppose $|\alpha_0| > |\alpha_n|$. Show that there exists some $\alpha \in \mathbb{C}$ for which $|\alpha| > 1$ and $P(\alpha) = 0$.

Solution:

(a) Dividing out by a_n, we take

$$P(z) = a_n\left(z^n + \frac{a_{n-1}}{a_n}z^{n-1} + \cdots + \frac{a_1}{a_n}z + \frac{a_0}{a_n}\right).$$

By the fundamental theorem of algebra, we can find some α_1 such that $P(\alpha_1) = 0$. Therefore, by polynomial division, we have a monic polynomial $q(z)$ with degree $n-1$ such that

$$P(z) = a_n q(z)(z-\alpha_1).$$

If $q(z)$ is a constant polynomial, it is necessarily equal to 1 and we are done. Else, inductively, we may find $\alpha_2, \dots, \alpha_n \in \mathbb{C}$ such that $q(z) = (z-\alpha_2)\cdots(z-\alpha_n)$, meaning that

$$P(z) = a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

(b) If P is a polynomial, then we may factor

$$P(z) = a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

Observe that

$$a_0 = a_n \prod_{i=1}^{n} \alpha_i,$$

so that

$$\left|\frac{a_0}{a_n}\right| = \prod_{i=1}^{n}|\alpha_i|.$$

Since $|a_0| > |a_n|$, it follows that

$$\prod_{i=1}^{n}|\alpha_i| > 1.$$

By the pigeonhole principle, there must be at least one α_i such that $|\alpha_i| > 1$.