

Amenability: A (Somewhat) Brief Introduction

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Outline

- ① Definitions
- ② Paradoxical Decompositions
- ③ From Paradoxical Decompositions to Amenability
- ④ Equivalent Definitions and Other Criteria
 - A Taste of Functional Analysis
 - Introducing Approximations
 - Approximations with Representations and Operators
 - Review
- ⑤ Remarks and Acknowledgments

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then we call the pair (A, \star) a *group*.

We (usually) abbreviate $a \star b$ as ab . If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups

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- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

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- The group $\text{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group $E(3)$ consists of all translations, rotations, and flips in \mathbb{R}^3 , and is also known as the *isometry group* of \mathbb{R}^3 .

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Every group is equipped with a family of canonical actions, $\sigma_a: G \rightarrow G$ for each $a \in G$, given by $x \mapsto ax$, known as *left-multiplication*.

σ -Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

- ① $\emptyset, X \in \mathcal{A}$;
- ② for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
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If, for any countable collection, $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say \mathcal{A} is a σ -*algebra* of subsets.

σ -Algebras and Measures, Cont'd

If X is a set and \mathcal{A} is a σ -algebra, then a map $\mu: \mathcal{A} \rightarrow [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

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If $\{A_n\}_{n \geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

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- Are these questions even related?

Free Groups

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- We define $F(a, b)$ to be the set of all “words” in the alphabet $\{a, b, a^{-1}, b^{-1}\}$, subject to the condition that, for $w, w' \in F(a, b)$,

$$\begin{aligned} waa^{-1}w' &\sim wa^{-1}aw' \sim ww' \\ wbb^{-1}w' &\sim wb^{-1}bw' \sim ww'. \end{aligned}$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a, b)$.

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Thus, all we need to do is add back $W(b^{-1})$ to get $F(a, b)$ back.

$$F(a, b) = W(b^{-1}) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a , giving a decomposition of $F(a, b)$ in two separate ways:

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We're able to take part of the group $F(a, b)$, take some translations, and, miraculously, obtain the entire group back.

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$;

such that

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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Decompositions of Sets

If G acts on a set X , then a subset $A \subseteq X$ is *G-paradoxical* if there exist

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq A$; and
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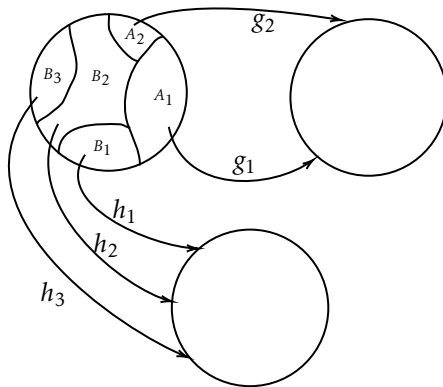
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A paradoxical group is a paradoxical set under the action of left-multiplication.

Depiction



Some Paradoxical Groups

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- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of $SO(3)$

The following two matrices (and their inverses) generate a subgroup of $SO(3)$ that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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Thus, $SO(3)$ is paradoxical — can we use it to find a paradoxical decomposition?

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of \mathbb{R}^3 with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B .

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- In other words, not all subsets of \mathbb{R}^3 have a definite “volume” invariant under isometry.

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Effectively, A and B are “equal” to each other up to the group action.

If A is G -paradoxical, then so too is B .

The Banach–Tarski Paradox: Proof Outline I

① We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a, b)$.

The Banach–Tarski Paradox: Proof Outline II

- ② We use the decomposition

$$\begin{aligned} F(a, b) &= a^{-1}W(a) \cup W(a^{-1}) \\ &= b^{-1}W(b) \cup W(b^{-1}) \end{aligned}$$

to duplicate the unit sphere in \mathbb{R}^3 , S^2 , except for a countable subset D . (The *Hausdorff Paradox*.)

- ③ We show that S^2 and $S^2 \setminus D$ are $\text{SO}(3)$ -equidecomposable — there is thus a paradoxical decomposition of S^2 .
- ④ We show that the unit ball, $B(0, 1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group $E(3)$.

The Banach–Tarski Paradox: Proof Outline III

- ⑤ Define a relation $A \leq B$ if A is G -equidecomposable with a subset of B , and show that if $A \leq B$ and $B \leq A$, then A and B are G -equidecomposable.
- ⑥ Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

Ill-Behaved Groups

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- The way that our copy of $F(a, b)$ helped “create” the Banach–Tarski paradox suggests that $F(a, b)$ is a particularly ill-behaved group.
- Let $\nu: F(a, b) \rightarrow [0, 1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a, b), E \subseteq F(a, b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a, b) = W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}),$$

we have

$$1 = \nu(W(a)) + \nu(W(a^{-1})) + \nu(W(b)) + \nu(W(b^{-1}))$$

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$$\begin{aligned} 1 &= \nu(W(a)) + \nu(W(a^{-1})) + \nu(W(b)) + \nu(W(b^{-1})) \\ &= \nu(a^{-1}W(a)) + \nu(W(a^{-1})) + \nu(b^{-1}W(b)) + \nu(W(b^{-1})) \\ &= \nu(a^{-1}W(a) \sqcup W(a^{-1})) + \nu(b^{-1}W(b) \sqcup W(b^{-1})) \\ &= \nu(F(a, b)) + \nu(F(a, b)) \\ &= 2. \end{aligned}$$

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu: G \rightarrow [0, 1]$ such that

$$\nu(tE) = \nu(E)$$

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If G admits a mean, we say G is *amenable*.

- In other words, G is sufficiently “well-behaved.”

Inheritance Properties of Amenability

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- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

is a mean.

- Abelian groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

Paradoxical Groups and Amenability

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Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

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Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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Why Find Alternative Characterizations?

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Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

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- homogeneity: $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in \mathbb{C}$;
- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of \mathbb{R}^n or \mathbb{C}^n with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}$$

However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f: \Gamma \rightarrow \mathbb{C} \mid \sup_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f: \Gamma \rightarrow \mathbb{C} \mid \sum_{t \in \Gamma} |f(t)| < \infty \right\};$$

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They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t \in \Gamma} |f(t)|$;
- $\|f\|_{\ell_1} := \sum_{t \in \Gamma} |f(t)|$;
- $\|f\|_{\ell_2} := \left(\sum_{t \in \Gamma} |f(t)|^2 \right)^{1/2}$.

Linear Maps and Linear Functionals

A linear transformation $T: V \rightarrow W$ is called *bounded* if

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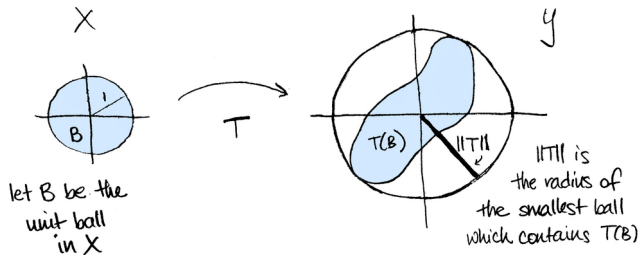
$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted $\|T\|_{\text{op}}$.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Operator Norm Pictorial Depiction

Courtesy of Tai-Danae Bradley.



Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi: \ell_\infty(\Gamma) \rightarrow \mathbb{C}$ is a linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbf{1}_\Gamma) = \|\varphi\|_{\text{op}}$.

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- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}}$. All positive linear functionals are automatically continuous.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s: \ell_\infty(\Gamma) \rightarrow \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f(s^{-1}t)$$

for all $t \in \Gamma$ and fixed $s \in \Gamma$.

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If $\varphi: \ell_\infty(\Gamma) \rightarrow \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

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If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

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Approximations and Amenability

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Remember when we decomposed

$$F(a, b) = W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was “significantly” “bigger” than $W(a^{-1})$; specifically, it gave us $F(a, b) \setminus W(a^{-1})$.

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But what does “bigger” actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n \rightarrow \infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

Approximate Means

The Følner condition allows us to find an “approximate” version of a mean.

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Keeping $\lambda_s(f)(t) = f(s^{-1}t)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k \rightarrow \infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|} \mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a “sufficient” finitely supported function g ,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

then use a “layer cake” decomposition to find our Følner sets:

$$g = \sum_{i=1}^n c_i \mathbb{1}_{F_i},$$

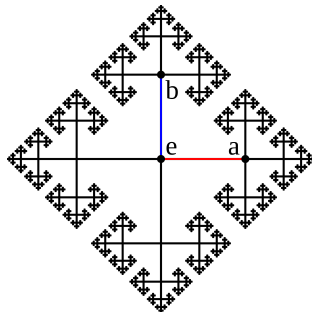
where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S , we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by “walking” along the generators.

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Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G , the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

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If $S \subseteq V(G)$ is a subset of vertices of a graph G , the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ , then Γ is amenable if and only if

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- This is proven with the Følner condition.

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Hilbert Spaces

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If \mathcal{H} is complete with respect to this norm, we call \mathcal{H} a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T: \mathcal{H} \rightarrow \mathcal{H}$, include a special structure called an adjoint that “plays nicely” with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

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then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$

$$\lambda(s^{-1}) = \lambda(s)^*$$

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All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma: \Gamma \rightarrow \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

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The map $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$ is a very special representation, known as the *left-regular representation*.

This is because it “encodes” the group’s left-multiplication action, in the sense that $\lambda_s(\delta_t) = \delta_{st}$, where δ_t is the point mass at $t \in \Gamma$.

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* for $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ if

$$\lim_{k \rightarrow \infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

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If $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

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Introduction to C^* -Algebras

The space of *all* bounded linear operators, $T: \mathcal{H} \rightarrow \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\text{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
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Furthermore, the operator norm “plays well” with operator composition and the adjoint, in the sense that:

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These make $\mathbb{B}(\mathcal{H})$ a C^* -algebra. However, there are other C^* -algebras.

A Group C^* -Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t) \delta_t,$$

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This becomes a $*$ -algebra when endowed with multiplication (by convolution) and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g(s^{-1}t)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^* -Algebra, cont'd

If we represent $\pi_\lambda: \mathbb{C}[\Gamma] \rightarrow \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\text{op}},$$

we get the *reduced group C^* -algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

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We can use these sufficient approximations to establish amenability.

Nuclearity

A C^* -algebra, A , is called *nuclear* if there exist two sequences of maps, $\varphi_n: A \rightarrow \text{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n: \text{Mat}_{k(n)}(\mathbb{C}) \rightarrow A$, such that

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- Essentially, any $a \in A$ is “close enough” to a certain family of finite-dimensional analogues.

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Specifically, by showing that the approximation of $\frac{|sF_n \cap F_n|}{|F_n|} \rightarrow 1$ corresponds to the existence of maps $\varphi_n: C_\lambda^*(\Gamma) \rightarrow \text{Mat}_{|F_n|}(\mathbb{C})$ and $\psi_n: \text{Mat}_{|F_n|}(\mathbb{C}) \rightarrow C_\lambda^*(\Gamma)$ that satisfy

$$\|x - \psi_n \circ \varphi_n(x)\| \xrightarrow{n \rightarrow \infty} 0.$$

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Final Remarks

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Nuclear C^* -algebras are classified, so active research areas primarily concern whether or not certain classes of C^* -algebras are nuclear (hence classifiable).

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Nuclear C^* -algebras are classified, so active research areas primarily concern whether or not certain classes of C^* -algebras are nuclear (hence classifiable).

There are also a lot of other directions that amenability can take the eager student, but I think this was a pretty nice overview of some of the ways that amenability touches all sorts of other fields of math.

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A large thank you goes to

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