Problem (Problem 1 (a))**:** Fix topological spaces X and Y, and consider the set of all continuous maps $X \to Y$. Define a relation on this set by saying that f is related to g whenever f is homotopic to g. Prove that this relation is an equivalence relation.

Solution: For reflexivity, we may select the identity homotopy $F\colon X \times I \to Y$, given by $F(x,t) = f(x)$ for all $t \in I$ and all $x \in X$.

For symmetry, if $F\colon X \times I \to Y$ is a homotopy with $F(x,0) = f(x)$ and $F(x,1) = g(x)$, then we may define the homotopy $G\colon X \times I \to Y$ by taking $G(x,t) = F(x,1-t)$. This is a composition of continuous maps, so it is continuous, and has $G(x,0) = g(x)$ and $G(x,1) = f(x)$, so the relation is symmetric.

For transitivity, we consider two homotopies $F\colon X \times I \to Y$ and $G\colon X \times I \to Y$. Define a homotopy $H\colon X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

We claim that this map is continuous. To see this, we observe that if $U \subseteq Y$ is closed, then both $F^{-1}(U) \subseteq X \times I$ and $G^{-1}(U) \subseteq X \times I$ are closed; by taking intersections with the closed subsets $X \times [0,1/2]$ and $X \times [1/2,1]$, we see then that

$$H^{-1}(U) = \left(F^{-1}(U) \cap (X \times [0,1/2])\right) \cup \left(G^{-1}(U) \cap (X \times [1/2,1])\right),$$

which is a finite union of closed subsets, so it is closed. In particular, this means H is continuous.

Remark: The first two parts of this proof were sound, so I only edited my solution for the transitivity part of this problem.