Math 395

Homework 3
Due: 2/15/2024

Name: Avinash Iyer

Collaborators: Nate Hall, Antonio Cabello, Gianluca Crescenzo

Problem 1

Let $\varphi : R \to S$ be a ring homomorphism. Let $\mathfrak{p} \in \text{Spec}(S)$. We will prove that $\varphi^{-1}(\mathfrak{p}) \subset R$ is an element of $\text{Spec}(R)$.

Let $\mathfrak{p} \in \text{Spec}(S)$. Let $ab \in \varphi^{-1}(\mathfrak{p})$. Then, $\varphi(ab) \in \mathfrak{p}$. So, $\varphi(a)\varphi(b) \in \mathfrak{p}$, meaning either $\varphi(a) \in \mathfrak{p}$ or $\varphi(b) \in \mathfrak{p}$. Therefore, $a \in \varphi^{-1}(\mathfrak{p})$ or $b \in \varphi^{-1}(\mathfrak{p})$. Therefore, $\varphi^{-1}(\mathfrak{p})$.

Problem 4

Let I, J be ideals of R with $I \subseteq J$. We will show that J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.

We know that J/I is non-empty, as it contains 0_R, so we will show that J/I is closed under subtraction and multiplication by elements of R/I. Thus, by the rules of subrings, for $j_1, j_2, j \in J$, we have

$$\begin{aligned}(j_1 + I) - (j_2 + I) &= (j_1 - j_2) + I \\ &\in J/I,\end{aligned}$$

and, since $j \in R$, by the properties of the quotient ring,

$$(r + I)(j + I) = (rj) + I,$$

and since $rj \in J$ as J is an ideal,

$$(rj + I) \in J/I.$$

Similarly,

$$\begin{aligned}(j + I)(r + I) &= (jr) + I \\ &\in J/I.\end{aligned}$$

Therefore, since J/I is non-empty, closed under subtraction, and closed under multiplication by elements of R/I, it is the case that J/I is an ideal R/I.

Let $\varphi : R/I \to R/J$, $r + I \mapsto r + J$. We will show that φ is a well-defined homomorphism with $\ker(\varphi) = J/I$.

Let $r_1 \sim_{R/I} r_2$. Then, $r_1 = r_2 + i$ for some $i \in I$. Then,

$$\begin{aligned}\varphi(r_1) &= r_1 + J \\ &= (r_2 + i) + J,\end{aligned}$$

and since $I \subseteq J$,

$$= r_2 + J.$$

Thus, φ is well-defined. Additionally,

$$\begin{aligned}\varphi((r_1+I)+(r_2+I)) &= \varphi((r_1+r_2)+I)\\ &= (r_1+r_2)+J\\ &= (r_1+J)+(r_2+J)\\ &= \varphi(r_1+I)+\varphi(r_2+I),\end{aligned}$$

and

$$\begin{aligned}\varphi((r_1+I)(r_2+I)) &= \varphi(r_1r_2+I)\\ &= r_1r_2+J\\ &= (r_1+J)(r_2+J)\\ &= \varphi(r_1+I)\varphi(r_2+I).\end{aligned}$$

Therefore, φ is a homomorphism. The elements that φ maps to $0+J$ are precisely the elements of $j+I$ where $j \in J$, as

$$\begin{aligned}\varphi(j+I) &= j+J\\ &= 0+J.\end{aligned}$$

Thus, $\ker(\varphi) = J/I$.

By the first isomorphism theorem, $(R/I)/(J/I) \cong R/J$.

Problem 5

Define $\varphi : \mathbb{F}_p \to \mathbb{F}_p$, where $\varphi(x) = x^p$ for $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We will show that φ is an isomorphism.

We will start by showing that φ is a well-defined homomorphism. Let $[a]_p = [b]_p$. Then, $a = b + kp$ for some $k \in \mathbb{Z}$. By Fermat's Little Theorem, $\varphi(a) = a^p \equiv [a]_p$, and $\varphi(b+kp) = b^p + p(\ell)$ for some ℓ, so $\varphi(b) \equiv [b]_p$ as well. Thus, $\varphi([a]_p) = \varphi([b]_p)$.

Since φ is well-defined, we find that, for $a, b \in \mathbb{F}_p$,

$$\begin{aligned}\varphi(a+b) &= ([a+b]_p)^p\\ &\equiv [a+b]_p\\ &= [a]_p + [b]_p\\ &\equiv ([a]_p)^p + ([b]_p)^p\\ &= \varphi(a)+\varphi(b),\end{aligned}$$

and

$$\begin{aligned}\varphi(ab) &= ([ab]_p)^p\\ &\equiv [ab]_p\\ &= [a]_p[b]_p\\ &\equiv ([a]_p)^p\,([b]_p)^p\\ &= \varphi(a)\varphi(b),\end{aligned}$$

meaning φ is a homomorphism.

Since, for all $x \in \mathbb{F}_p$, $x \equiv x^p$, it is the case that φ is surjective. Finally, since $\varphi(0) = 0$, and for $x \neq 0$, $\varphi(x) \neq 0$, it is the case that $\ker(\varphi) = \{0\}$, meaning φ is injective.

Since φ is a bijective homomorphism, φ is an isomorphism.