2.12

Problem: Let κ and λ be cardinals. Show that $\kappa \in \lambda$ if and only if there exists an injective function from κ to λ and there does not exist a bijective map between κ and λ.

Solution: Let $\kappa \in \lambda$. Then, $\kappa \subset \lambda$ since ordinals are transitive. Then, $\iota : \kappa \hookrightarrow \lambda$, the inclusion map, is injective.

Let $S = \{\alpha \mid \exists g : \alpha \to \kappa\}$ with g bijective, and similarly, let $T = \{\alpha \mid \exists h : \alpha \to \lambda\}$. Since $S = T$, then κ is the least element of S and λ is the least element of S as both κ and λ are cardinals, meaning $\kappa = \lambda$.

Suppose there exists $f : \kappa \hookrightarrow \lambda$ that is injective, and there does not exist $g : \kappa \to \lambda$ that is bijective.

By trichotomy, either $\kappa = \lambda$, $\kappa \in \lambda$, or $\lambda \in \kappa$. Since $\kappa \neq \lambda$ (as otherwise, $\mathrm{id} : \kappa \to \lambda$ would be a bijection). If $\lambda \in \kappa$, then there would exist an injection $h : \lambda \hookrightarrow \kappa$, then there would be a bijection by Cantor–Schröder–Bernstein, which would be a contradiction to the assumption that there does not exist a bijection.

2.13

Problem: Let A be a set. Given a subset B of A, define $f_B : A \to \{0,1\}$ by

$$f_B(x) = \begin{cases} 1 & x \in B, \\ 0 & x \notin B \end{cases}.$$

Let C be the set of all functions mapping from A from $\{0,1\}$, and define $\Phi : P(A) \to C$ by $\Phi(B) = f_B$. Show that Φ is bijective.

Solution: Let $\Phi(B) = \Phi(C)$. Then, we have $f_B = f_C$, meaning that $f_B(x) = f_C(x)$ for all $x \in A$. Thus, for $x \in B$, we have $f_B(x) = 1 = f_C(x)$, meaning $x \in C$, and for $x \notin B$, $f_B(x) = 0 = f_C(x)$, meaning $x \notin C$. Thus, $B = C$. This shows injectivity.

To show surjectivity, we let $f \in C$. Then, $\mathrm{Graph}(f)$ is some collection of the form $(a, 0)$ and $(a, 1)$ in $A \times \{0,1\}$. We find $B \subseteq A$ by taking $B = \left\{a \in A \mid (a,1) \in \mathrm{Graph}(f)\right\}$. Since f is a function, it must be the case that B is well-defined, and $B \subseteq A$. Thus, Φ is surjective.

Let $|A| = \kappa$. We can define a bijection $P(A)$ to $\{0,1\}^A$, meaning $|P(A)| = \left|\{0,1\}^A\right|$, and $\left|\{0,1\}^A\right| = 2^\kappa$, so $|P(A)| = 2^\kappa$.

Extra Problem 1

Problem: Show that for cardinals A and B, $A + B = B + A$ and $AB = BA$.

Solution: We have

$$\begin{aligned} A + B &= |A \times \{0\} \cup B \times \{1\}| = |S| \\ B + A &= |A \times \{1\} \cup B \times \{0\}| = |T|. \end{aligned}$$

We define a bijection $S \to T$ by $(a, 0) \mapsto (a, 1)$ for $a \in A$ and $(b, 1) \mapsto (b, 0)$ for $b \in B$. Thus, $|S| = |T|$, so $A + B = B + A$.

We also have

$$\begin{aligned} AB &= |A \times B| = |P| \\ BA &= |B \times A| = |Q|. \end{aligned}$$

We define a bijection $P \to Q$ by $(a, b) \mapsto (b, a)$. Thus, $|P| = |Q|$, meaning $AB = BA$.

Extra Problem 2

Problem: Use the "contradiction format" of induction to prove the Pigeonhole Principle.

Solution: Suppose toward contradiction that the Pigeonhole Principle fails. Let $n_0 \in \mathbb{N}$ be the smallest value such that the Pigeonhole principle fails. Then, there exists an injection from $\{0, \dots, n_0\} \to A$, where $A \subsetneq \{0, \dots, n_0\}$. In particular, the ordinal corresponding to $|\{0, \dots, n_0\}|$ is $n_0 + 1$, so $|A| \in n_0 + 1$. However, since $|A| \in n_0 + 1$, there is an injection from A to $n_0 + 1$, meaning there is a bijection from A to $\{0, \dots, n_0\}$. However, this implies that $n_0 + 1 = A \in n_0 + 1$, or $n_0 + 1 \in n_0 + 1$, which is a violation of the Axiom of Regularity.

Extra Problem 3

Problem: Prove that if $A \subsetneq B$ and $|A| = |B|$, then A and B are infinite.

Solution: Let $c_A : A \to \lambda$ and $c_B : B \to \lambda$ be bijections, where $\lambda = \min\left\{\alpha \mid c_A, c_B \text{ are bijections}\right\}$.

Since $A \subsetneq B$, $\iota : A \to B$ defined by $\iota(x) = x$ is an injection that is not a bijection. Thus, $c_B \circ \iota : A \to \lambda$ is an injection. However, since there does not exist $\kappa \in \lambda$ with $c_B \circ \iota : A \to \kappa$ as a bijection, it must be the case that λ is a limit ordinal (i.e., infinite).

Extra Problem 4

Problem: Prove that if γ is an infinite ordinal, then $\omega \subseteq \gamma$.

Solution: If γ is an infinite ordinal, then $\gamma = \omega$, in which case $\omega \subseteq \omega = \gamma$, $\omega \in \gamma$, in which case $\omega \subseteq \gamma$, or $\gamma \in \omega$, meaning γ is finite (contradicting the assumption that γ is infinite).

Extra Problem 5

Problem: Show that every infinite set contains a denumerable subset.

Solution: Let S be an infinite set, and let α denote the cardinality of S.

There exists some bijection $f : S \to \alpha$. Since S is infinite, α is infinite, since if α were to be finite, then $f^{-1} : \alpha \to S$ would be a bijection with a finite domain, meaning S would be finite.

By the previous problem, $\omega \subseteq \alpha$, meaning we can take $U \subseteq S$ to be $U = f^{-1}(\omega)$, which is a denumerable set as ω is denumerable.

Extra Problem 6

Problem: Show that every infinite subset S has a proper subset with the same cardinality as S.

Solution: I don't know how to do this problem.