Quasi-Review: Locally Compact Groups and the Banach $*$-algebra $L_1(G)$

Basic Properties of Topological Groups

A topological group is a group G equipped with a topology such that the operations

$$\begin{aligned}(x,y) &\mapsto xy\\ x &\mapsto x^{-1}\end{aligned}$$

are continuous. In general, we will let 1 denote the identity of G.

We call G a locally compact group if the topology of G is locally compact. Equivalently, the topology of G is locally compact if there there is a neighborhood system about 1 consisting of pre-compact open sets.

We will refer to the following subset operations in G regularly:

$$\begin{aligned}Ax &= \{ax \mid a \in A\}\\ xA &= \{xa \mid a \in A\}\\ A^{-1} &= \left\{a^{-1} \mid a \in A\right\}\\ AB &= \{ab \mid a \in A, b \in B\}.\end{aligned}$$

A subset V is called *symmetric* if $V = V^{-1}$.

These are some useful propositions.

> **Proposition:** Let G be a topological group.
>
> (i) The topology of G is invariant under translations and inversion.
>
> (ii) For every neighborhood U of 1, there is a symmetric neighborhood V of 1 such that $VV \subseteq U$.
>
> (iii) If H is a subgroup of G, then so is $\overline{H}$.
>
> (iv) Every open subgroup of G is closed.
>
> (v) If A and B are compact subsets of G, then so is AB.

> **Proposition:** Suppose H is a subgroup of the topological group G.
>
> (i) If H is closed, then G/H is Hausdorff.
>
> (ii) If G is locally compact, then so is G/H.
>
> (iii) If H is normal, then G/H is a topological group.

We will assume all the time that G is Hausdorff, via the following proposition.

> **Corollary:** If G is a T1 topological group, then G is Hausdorff. If G is not T1, then $\overline{\{1\}}$ is a closed normal subgroup with $G/\overline{\{1\}}$ is a Hausdorff topological group.

> **Proposition:** Every locally compact group G has a subgroup G_0 that is open, closed, and σ-compact.

Considering various functions $f\colon G \to \mathbb{C}$, we define the left and right translates of f as

$$\begin{aligned}L_y f(x) &= f\left(y^{-1}x\right)\\ R_y f(x) &= f(xy),\end{aligned}$$

and say that f is left (right) uniformly continuous if $\|L_y f - f\|_u \to 0$ ($\|R_y f - f\|_u \to 0$) as $y \to 1$.

> **Proposition:** If $f \in C_c(G)$, then f is left and right uniformly continuous.

A left *Haar measure* is a nonzero Radon measure μ on G such that $\mu(xE) = \mu(E)$ for every Borel subset $E \subseteq G$.

> **Proposition:** Every locally compact group G admits a left Haar measure λ. This Haar measure is unique up to a constant multiple.

If we have a left Haar measure λ, then if we define

$$\lambda_x(E) = \lambda(Ex),$$

we have that λ_x is again a left Haar measure, so there is some number $\Delta(x)$ such that $\lambda_x = \Delta(x)\lambda$, where $\Delta(x)$ is independent of the original choice of λ.

The function $\Delta\colon G \to (0,\infty)$ defined as such is known as the *modular function* of G.

> **Proposition:** The function Δ is a continuous homomorphism from G to $\mathbb{R}_{>0}$, and for any $f \in L_1(\lambda)$, we have
>
> $$\int R_y f\, d\lambda = \Delta\left(y^{-1}\right) \int f\, d\lambda.$$

We call G *unimodular* if $\Delta \equiv 1$.

> **Proposition:** If $G/[G,G]$ is compact, then G is unimodular.

Convolutions and $L_1(G)$

If G is a locally compact group, we let $M(G)$ denote the space of complex-valued Radon measures on G. The convolution of two measures $\mu,\nu \in M(G)$ is given as follows. If we let

$$I(\phi) = \iint \phi(xy)\, d\mu(x)d\nu(y),$$

then we observe that $I(\phi)$ is a linear functional on $C_0(G)$ that satisfies

$$|I(\phi)| \leq \|\phi\|_u \|\mu\|\|\nu\|,$$

meaning that it is given by a measure $\mu * \nu \in M(G)$ with $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$. We call $\mu * \nu$ the convolution of μ and ν.

Observe that if $\delta_x \in M(G)$ is the point mass at $x \in G$, then

$$\begin{aligned}\int \phi\, d(\delta_x * \delta_y) &= \iint \phi(uv)\, d\delta_x(u)\delta_y(v)\\ &= \phi(xy)\\ &= \int \phi\, d\delta_{xy},\end{aligned}$$

meaning that $\delta_x * \delta_y = \delta_{xy}$.

The estimate $\|\mu * \nu\| \leq \|\mu\|\|\mu\|$ gives that convolution makes $M(G)$ a Banach algebra, which we call the *measure algebra* of G. Furthermore, $M(G)$ admits an involution

$$\mu^*(E) = \overline{\mu(E^{-1})},$$

so that

$$\int \phi\, d\mu^* = \int \phi\left(x^{-1}\right) d\overline{\mu(x)}.$$

We may identify the space $L_1(G)$ to be the subspace of $M(G)$ where a function f is identified with the measure $f(x)dx$. If $f,g \in L_1(G)$, then the convolution of f and g is the function

$$f * g(x) = \int f(y) g\left(y^{-1}x\right) dy.$$

With convolution and the involution given by

$$\begin{aligned} f^*(x)dx &= \overline{f(x^{-1})}d\left(x^{-1}\right) \\ f^*(x) &= \Delta\left(x^{-1}\right)\overline{f(x^{-1})}, \end{aligned}$$

we have that $L_1(G)$ is a Banach $*$-algebra known as the *group algebra* of G.

Representations of a Group and its Group $*$-Algebra

Functions of Positive Type

References

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