Chapter 33 Problems

Problem 2

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\left(1+4x^2-x^3\right)\,dx\\
&= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\left(1+4x^2\right)\,dx\\
&= \frac{2}{L}\int_{0}^{L} \left(\cos\left(\frac{n\pi x}{L}\right)+4x^2\cos\left(\frac{n\pi x}{L}\right)\right)\,dx\\
&= \frac{8}{L}\int_{0}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\,dx\\
&= (-1)^n\,\frac{16L^2}{n^2\pi^2}\\
a_0 &= \frac{1}{L}\int_{-L}^{L} \left(1+4x^2-x^3\right)\,dx\\
&= \frac{2}{L}\int_{0}^{L} 1+4x^2\,dx\\
&= \frac{2}{L}\left(L+\frac{4}{3}L^3\right)\\
&= 2\left(1+\frac{4}{3}L^2\right)\\
b_n &= \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\left(1+4x^2-x^3\right)\,dx\\
&= -\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)x^3\,dx.\\
&= (-1)^n\,\frac{2L^3}{\pi^3 n^3}\left(\pi^2n^2-6\right).
\end{aligned}$$

Problem 4

(a) We have

$$\begin{aligned}
f(x) &= \frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\\
&= c_0+\sum_{n=1}^{\infty} a_n\frac{e^{in\pi x/L}+e^{i(-n)\pi x/L}}{2}-ib_n\frac{e^{in\pi x/L}-e^{i(-n)\pi x/L}}{2}\\
&= c_0+\sum_{n=1}^{\infty}\frac{1}{2}\left(a_n-ib_n\right)e^{in\pi x/L}+\frac{1}{2}\left(a_n+ib_n\right)e^{i(-n)\pi x/L}\\
&= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}.
\end{aligned}$$

(b)

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}f(x)\,dx$$

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L}\left(\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right) f(x)\,dx\right)\\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right) f(x)\,dx + \frac{i}{L}\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right) f(x)\,dx\right)\\
&= \frac{1}{2}\left(a_0 + a_n - \operatorname{sgn}(n)\, i b_n\right).
\end{aligned}$$

Problem 5

We have that, for $-\pi < a < \pi$,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x-a)\,e^{-in\pi x}\,dx\\
&= \frac{1}{2\pi}e^{-in\pi a},
\end{aligned}$$

meaning

$$\delta(x-a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in\pi a}.$$

Problem 7

Since $\sin x\cos x$ is odd, we only have sines in our Fourier expansion, yielding

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\sin(x)\cos(x)\,\,dx\\
&= \frac{1}{2}\sin(2x).
\end{aligned}$$

This yields the familiar identity

$$\sin(2x) = 2\sin x\cos x.$$

Problem 11

(a) For $\sin(x+\pi/4)$ considered as a function with period 2π, we have

$$\sin(x+\pi/4) = \frac{\sqrt{2}}{2}\cos(x) + \frac{\sqrt{2}}{2}\sin(x).$$

(b) For $\sin(x+\pi/4)$ considered as a function with period π, we have

$$\begin{aligned}
a_0 &= \frac{2\sqrt{2}}{\pi}\\
a_n &= \frac{\sqrt{2}\left(1+(-1)^n\right)}{\pi(1-n^2)}\\
b_n &= \frac{\sqrt{2}n\left(1+(-1)^n\right)}{\pi(n^2-1)}.
\end{aligned}$$

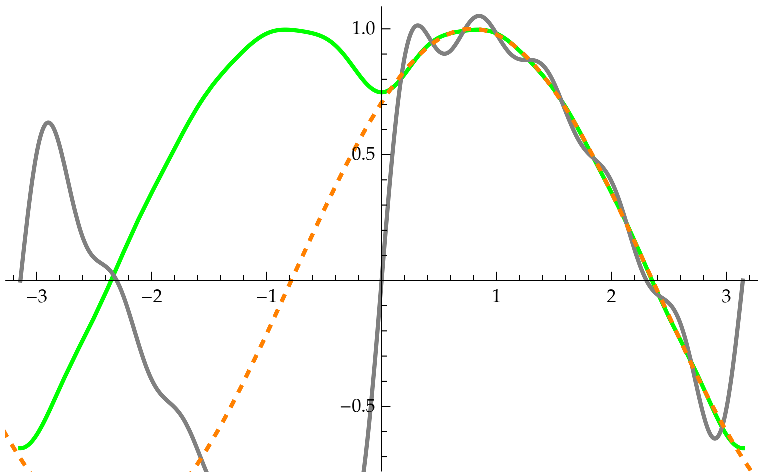
(c) For $\sin(x+\pi/4)$ considered as an even function with period 2π, we have

$$\begin{aligned}
a_0 &= \frac{2\sqrt{2}}{\pi}\\
a_n &= \frac{\sqrt{2}\left(1+(-1)^n\right)}{\pi(1-n^2)},
\end{aligned}$$

(d) For $\sin(x + \pi/4)$ considered as an odd function with period π, we have

$$b_n = \frac{\sqrt{2}n\left(1 + (-1)^n\right)}{\sqrt{2}\left(n^2 - 1\right)}.$$

Plotting, we get the following graph.



Problem 16

(a) We have

$$\begin{aligned}
a_0 &= \frac{1}{2}\pi \\
a_n &= \frac{1}{\pi n^2}\left(-1 + (-1)^n\right) \\
b_n &= \frac{\pi}{n}(-1)^{n+1} \\
f(x) &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\pi}{n}(-1)^{n+1}\sin(nx) + \frac{1}{\pi n^2}\left(1 + (-1)^n\right)\cos(nx).
\end{aligned}$$

(b) Evaluating at $x = 0$, we find the sine portion goes away, yielding

$$\begin{aligned}
0 &= \frac{\pi}{4}\sum_{n \text{ odd}} -\frac{2}{\pi n^2} \\
\frac{\pi^2}{8} &= \sum_{n \text{ odd}} \frac{1}{n^2}.
\end{aligned}$$

Evaluating at $x = \pi$, we get

$$\frac{3\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}.$$

The value at $x = 0$ is correct due to ringing at the jump discontinuity at $x = \pi$.

Problem 17

(a)

$$\zeta(2) = \frac{\pi^2}{6}$$

(b)

$$\frac{1}{16} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

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Problem 5

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx &= \langle f \mid f \rangle \\
&= \left\langle \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \,\middle|\, \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right\rangle \\
&= \left|\frac{a_0}{2}\right|^2 + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 \\
&= \left\langle \sum_{n=-\infty}^{\infty} c_n e^{inx} \,\middle|\, \sum_{n=-\infty}^{\infty} c_n e^{inx} \right\rangle \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2 .
\end{aligned}$$

Problem 8

We have

$$\begin{aligned}
c_0 &= \frac{1}{12} \\
c_n &= \frac{(-1)^n}{2n^2\pi^2} \\
\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 \, dx &= \frac{1}{80} \\
\frac{1}{80} &= \frac{1}{144} + \sum_{n=1}^{\infty} \frac{1}{4n^4\pi^4} \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}.
\end{aligned}$$