

Abstract

We discuss and prove the three big theorems of real analysis — the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

Integration: An Introduction

In order to discuss integration, we need to start with the building blocks of all functions — simple functions.

Definition. Let X be a measure space, and let $\phi: X \rightarrow [0, \infty]$ be a function. We say ϕ is a *simple function* if it has finite range (and does not take the value $+\infty$).

The *standard form* of a simple function ϕ is

$$\phi = \sum_{k=1}^n c_k \mathbb{1}_{E_k},$$

where $\{c_1, \dots, c_n\} = \text{Ran}(\phi)$, and $E_k = \phi^{-1}(\{c_k\})$.

Recall that a function $f: X \rightarrow \mathbb{R}$, where (X, \mathcal{M}, μ) is a measure space, is called Borel-measurable (or just measurable) if, for every $E \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(E) \in \mathcal{M}$.

Definition. If $\phi: X \rightarrow [0, \infty]$ is a simple, measurable function defined on a measure space (X, \mathcal{M}, μ) , then the *integral* of ϕ is defined to be

$$\int_X \phi \, d\mu = \sum_{k=1}^n c_k \mu(E_k). \quad (\dagger)$$

Proposition: Let $\phi, \psi: X \rightarrow [0, \infty]$ be simple functions with standard forms

$$\begin{aligned} \phi &= \sum_{j=1}^n a_j \mathbb{1}_{E_j} \\ \psi &= \sum_{k=1}^m b_k \mathbb{1}_{F_k}. \end{aligned}$$

Then, the following hold

- (a) for all $c > 0$, $\int_X c\phi \, d\mu = c \int_X \phi \, d\mu$;
- (b) $\int_X \phi + \psi \, d\mu = \int_X \phi \, d\mu + \int_X \psi \, d\mu$;
- (c) if $\phi \leq \psi$ pointwise, then $\int_X \phi \, d\mu \leq \int_X \psi \, d\mu$.

Proof.

(a) We see that

$$\begin{aligned}\int_X c\phi \, d\mu &= \sum_{j=1}^n (c)(a_j)\mu(E_k) \\ &= c \sum_{k=1}^n a_j\mu(E_k) \\ &= c \int_X \phi \, d\mu.\end{aligned}$$

(b) Note that since

$$\begin{aligned}X &= \bigsqcup_{j=1}^n E_j \\ &= \bigsqcup_{k=1}^m F_k,\end{aligned}$$

we must have

$$\begin{aligned}E_j &= \bigsqcup_{k=1}^m E_j \cap F_k \\ F_k &= \bigsqcup_{j=1}^n F_k \cap E_j\end{aligned}$$

as a disjoint union. Therefore,

$$\begin{aligned}\int_X \phi \, d\mu + \int_X \psi \, d\mu &= \sum_{j=1}^n \sum_{k=1}^m (a_j + b_k)\mu(E_j \cap F_k) \\ &= \int_X \phi + \psi \, d\mu.\end{aligned}$$

(c) If $\phi \leq \psi$, $a_j \leq b_k$ whenever $E_j \cap F_k \neq \emptyset$. Therefore,

$$\begin{aligned}\int_X \phi \, d\mu &= \sum_{k=1}^m \sum_{j=1}^n a_j\mu(E_j \cap F_k) \\ &\leq \sum_{k=1}^m \sum_{j=1}^n b_k\mu(E_j \cap F_k) \\ &= \int_X \psi \, d\mu.\end{aligned}$$

□

Having established integrals for simple functions, we need to establish a convergence property for simple functions for all measurable functions.

Theorem: Let (X, \mathcal{M}, μ) be a measure space, and let $f: X \rightarrow [0, \infty]$ be a measurable function. Then, there is an increasing sequence $(\phi_n)_n$ of simple functions that converges pointwise to f . This sequence converges uniformly to f on any bounded sets.

Proof. For each n , partition the interval $[0, 2^n]$ into subintervals of length 2^{-n} . There are 2^{2n} subintervals, with

$$I_{n,0} = \left[0, \frac{1}{2^n}\right]$$

$$I_{n,k} = \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right],$$

where $0 \leq k \leq 2^{2n} - 1$. We define $J_n = (2^n, \infty]$. Define

$$E_{n,k} = f^{-1}(I_{n,k})$$

$$F_n = f^{-1}(J_n).$$

Then, we may take

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n}.$$

The family ϕ_n are simple, measurable, positive, and increasing.

Fix $x \in X$ such that $f(x) < \infty$, and find N such that $f(x) \leq 2^N$. Then, for a fixed $n \geq N$, there is $0 \leq k \leq 2^{2n} - 1$ such that $x \in E_{n,k}$. Thus,

$$|\phi_n(x) - f(x)| = \left| f(x) - \frac{k}{2^n} \right| \tag{*}$$

$$\leq \frac{1}{2^n}.$$

Thus, this family is pointwise convergent.

If $f(x) = +\infty$, then $\phi_n(x) = 2^n$ for all n , meaning $\phi_n(x)$ also converges to $f(x)$.

If $f(x)$ is bounded, then for a sufficiently large n , $F_n = \emptyset$, and the construction in (*) is valid for all $x \in X$, meaning $\|\phi_n - f\|_u \leq \frac{1}{2^n}$, and $\sup_n \|\phi_n\|_u \leq \|f\|_u$. \square

Remark: By decomposing any complex-valued function f using the Cartesian decomposition to yield $f = (f_+ - f_-) + i(g_+ - g_-)$, the above theorem can be extended to all complex-valued functions. There, the modulus of the simple functions, $(|\phi_n|)_n$ can be taken to be pointwise increasing and bounded above by $|f|$, with uniform convergence on sets where f is bounded in modulus.

The Monotone Convergence Theorem

Since any measurable function $f: X \rightarrow [0, \infty]$ is a pointwise limit of simple functions, we may define the integral of a function as follows.

Definition. Let (X, \mathcal{M}, μ) be a measure space, and let $f: X \rightarrow [0, \infty]$ be a measurable function. The *integral* of f is defined to be

$$\int_X f \, d\mu = \sup \left\{ \int_X \phi \, d\mu \mid \phi \text{ simple, } 0 \leq \phi \leq f \right\}.$$

This definition of the integral agrees with the definition in (\dagger) whenever f is simple. Furthermore, it follows that, for all $c \in [0, \infty)$,

$$\int_X cf \, d\mu = c \int_X f \, d\mu,$$

and whenever $f \leq g$,

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

Yet, the issue is that our family of simple functions is uncountable. In order to (more easily) establish this integral, we need to be able to extract a sequence.

Theorem (Monotone Convergence Theorem): Let $(f_n)_n$ be a family of $[0, \infty]$ -valued measurable functions on X such that $f_j \leq f_{j+1}$ for all j . Define

$$\begin{aligned} f &= \lim_{n \rightarrow \infty} f_n \\ &= \sup_{n \in \mathbb{N}} f_n. \end{aligned}$$

Then,

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Proof. The sequence $(\int_X f_n \, d\mu)$ is an increasing sequence of real numbers, so it has a limit (which may be equal to ∞). Furthermore, $\int_X f_n \, d\mu \leq \int_X f \, d\mu$ for all n , meaning $\sup(\int_X f_n \, d\mu) \leq \int_X f \, d\mu$.

To establish the reverse inequality, let $\alpha \in (0, 1)$, $0 \leq \phi \leq f$ a simple function, and let

$$E_n = \{x \mid f_n(x) \geq \alpha\phi(x)\}.$$

The family $\{E_n\}_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets whose union is X .^I We have

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu$$

^ITo see that their union is equal to X , recall that f is the pointwise limit of f_n .

$$\geq \alpha \int_{E_n} \phi \, d\mu.$$

Since

$$\lim_{n \rightarrow \infty} \int_{E_n} \phi \, d\mu = \int_X \phi \, d\mu,$$

we have

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \geq \alpha \int_X \phi \, d\mu.$$

We may take the supremum over all $\alpha \in (0, 1)$, meaning

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \geq \int_X \phi \, d\mu.$$

Taking the supremum over all simple $0 \leq \phi \leq f$, we obtain

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu.$$

□