2.1

Problem: Recall that an ordered pair (a, b) can be defined as the set $\{\{a\}, \{a, b\}\}$. Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Solution. Let $L = \{\{a\}, \{a, b\}\}$ and $R = \{c, \{c, d\}\}$. Suppose $L = R$. Since $\{a\} \in L$, we have $\{a\} \in R$. Thus, $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$.

Case 1: If $\{a\} = \{c\}$, then $a \in \{c\}$, meaning $a = c$.

Case 2: If $\{a\} = \{c, d\}$, then $c \in \{a\}$, meaning $c = a$.

Since $\{a, b\} \in L$, we have $\{a, b\} \in R$, meaning $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$.

Case 3: If $\{a, b\} = \{c\}$, then it must be the case that $\{a\} = \{c, d\}$, meaning $a = b = c = d$, so $b = d$.

Case 4: If $\{a, b\} = \{c, d\}$, then it must be the case that $\{a\} = \{c\}$, meaning $a = c$, and thus $b = d$.

2.2

Problem: Define the ordered triple (a, b, c) to be the ordered pair $((a, b), c)$, where the ordered pair is defined as usual. Show that

$$(a_1, b_1, c_1) = (a_2, b_2, c_2)$$

if and only if $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$.

Solution. Since

$$(a_1, b_1, c_1) = (a_2, b_2, c_2)$$

implies

$$((a_1, b_1), c_1) = ((a_2, b_2), c_2),$$

this is true if and only if $(a_1, b_1) = (a_2, b_2)$ and $c_1 = c_2$, which is true if and only if $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$.

2.3

Problem: Show that the replacement schema implies the comprehension schema.

Solution. Let $\psi(u, v) = \phi(v) \wedge u = v$. Then, the replacement schema becomes

$$\begin{aligned}
\forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow \exists u\, (u \in a \wedge \psi(u, v))) \\
\forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow \exists u\, (u \in a \wedge \forall u\, (\phi(v) \wedge u = v))) \\
\forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow v \in a \wedge \phi(v))
\end{aligned}$$

2.4

Problem: In this question, we show how the pairing axiom follows from the replacement schema. Let sets a and b be given.

(a) We originally used the pairing axiom to construct the set $\{\varnothing, \{\varnothing\}\}$. Instead, us the power set axiom.

(b) Let $\psi(u, v)$ be the formula

$$(u = \varnothing \wedge v = a) \vee (u \neq \varnothing \wedge v = b).$$

Show that this is a function-like formula.

(c) Use the replacement schema on the set $\{\varnothing, \{\varnothing\}\}$ and the function-like formula $\psi(u, v)$ to show the existence of the set with elements a and b.

Solution.

(a) Consider $\{\varnothing\}$. By the power set axiom, there exists a set c such that c consists of all subsets of $\{\varnothing\}$. Thus, $c = \{\varnothing, \{\varnothing\}\}$.

(b) Let $\psi(u, v) = (u = \varnothing \wedge v = a) \vee (u \neq \varnothing \wedge v = b)$. Then, if $\psi(u, v) = \psi(u, w) = \text{true}$,

$$(u = \varnothing \wedge v = a) \vee (u \neq \varnothing \wedge v = b)$$

and

$$(u = \varnothing \wedge w = a) \vee (u \neq \varnothing \wedge w = b)$$

If $v = b$, then $u \neq \varnothing$, implying $w = b$, and similarly, if $v = a$, then $w = a$. Thus, $u = w$.

(c) Using the replacement schema on $\{\varnothing, \{\varnothing\}\}$, we see there is a set b such that for $\varnothing \in \{\varnothing, \{\varnothing\}\}$, $\psi(u, v)$ maps $\varnothing$ to a, and for $\{\varnothing\} \in \{\varnothing, \{\varnothing\}\}$, $\psi(u, v)$ maps $\{\varnothing\}$ to b.

2.5

Problem:

(a) Define a relation on the set of ordered pairs of natural numbers as follows: $(a, b) \sim (c, d)$ if $a + d = b + c$. Show that this is an equivalence relation.

(b) Let S be the set of ordered pairs of integers with a nonzero second component. Define a relation on S as follows: $(a, b) \sim (c, d)$ if $ad = bc$. Show that this is an equivalence relation.

Solution.

(a) Reflexivity follows from $a = a$ and $b = b$, while symmetry follows from the commutativity of addition. Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then,

$$\begin{aligned} a + d &= b + c && (*) \\ c + f &= d + e. && (**) \end{aligned}$$

Adding f to both sides of $(*)$, we have

$$\begin{aligned} a + d + f &= b + c + f \\ a + d + f &= b + d + e \\ a + f &= b + e, \end{aligned}$$

meaning $(a, b) \sim (e, f)$.

(b) Reflexivity follows from $a = a$ and $b = b$, while symmetry follows from the commutativity of multiplication.

Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then, $ad = bc$ and $cf = de$. Multiplying f on both sides of the first relation, we get

$$\begin{aligned} adf &= bcf \\ adf &= bde. \end{aligned}$$

Since $d \neq 0$, we have

$$af = be,$$

meaning $(a, b) \sim (e, f)$.

Extra Problem 1

Problem:

(a) Explain what would go wrong if we defined $(a, b) = \{a, \{b\}\}$.

(b) Can you figure out why the book defines $(a, b) = \{\{a\}, \{a, b\}\}$ instead of $\{a, \{a, b\}\}$.

Solution.

(a) I don't know how to do this one.

(b) If we consider $(a, b) = (a, b)$, we must then have $\{a, \{a, b\}\} = \{a, \{a, b\}\}$, meaning our cases would yield $a \in \{a, \{a, b\}\}$, or $a = \{a, b\}$, implying $a \in a$ or $a \in b$. In particular, for $a \in a$, we get a descending membership chain, which ends up requiring the regularity axiom.

Extra Problem 2

Problem: Let s be a set. Use mathematical symbols exclusively to express t, the set of all singleton subsets of s.

Solution.

$$\forall s\, \exists t\, \forall x\, (x \in t \Leftrightarrow x \in s \wedge \forall a\, \forall b\, (a \in x \wedge b \in x \Rightarrow a = b))$$

Extra Problem 3

Problem: Using the ZF Axioms, show that $A \times B$ exists for any sets A and B.

Solution. We know that for all $a \in A$, the pairing axiom allows for the existence of the set $\{a\}$. Similarly, for $a \in A$ and $b \in B$, the pairing axiom allows for creation of the set $\{a, b\}$. In particular, we let $\{a\}$ be shorthand for the pairing axiom applied to $a \in A$, and $\{a, b\}$ be shorthand for the pairing axiom applied to $a \in A$ and $b \in B$.

We can create the element $w \in A \times B$ by applying the pairing axiom to $\{a\}$ and $\{a, b\}$. We let $\{\{a\}, \{a, b\}\}$ be shorthand for the pairing axiom applied to $\{a\}$ and $\{a, b\}$.

This gives us

$$\forall A \forall B \exists C\ \ (w \in C \Leftrightarrow \forall a \forall b\, (a \in A \wedge b \in B \Rightarrow w = \{\{a\}, \{a, b\}\}))\,.$$

Extra Problem 4

Problem: Show that if A and B are nonempty sets, then $\bigcap (A \cup B) = \bigcap A \cup \bigcap B$.

Solution.

$$\begin{aligned}
\bigcap (A \cup B) &= \forall A \forall B \exists C\, \forall x\, (x \in C \wedge (x \in A \vee x \in B)) \\
&= \forall A \forall B \exists C\, \forall x\, ((x \in C \wedge x \in A) \vee (x \in C \wedge x \in B)) \\
&= \bigcap A \cup \bigcap B.
\end{aligned}$$

Extra Problem 5

Problem: Show there exists a set s such that $x \in s$ if and only if x is a natural number.

Solution.

$$\exists s\, \forall x \left(\underbrace{(x \in s \land x \cup \{x\} \in s)}_{\text{Axiom of Infinity}} \land \forall y \, (y \in s \Rightarrow \exists z \, (y = z \cup \{z\})) \right).$$