Part 1

3.6, Problem 3

We define $x_1 = y$ and $x_2 = \frac{dy}{dt}$. We get

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2\\
\frac{dx_2}{dt} &= -9x_1 - 6x_2\\
\frac{d\vec{X}}{dt} &= \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix}\vec{X}\\
\det\begin{pmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{pmatrix} &= \lambda(\lambda+6)+9\\
\lambda_1 &= -3\\
\vec{v}_1 &= \begin{pmatrix} 1 \\ -3 \end{pmatrix}\\
\vec{X}(t) &= e^{3t}\begin{pmatrix} s_0 \\ t_0 \end{pmatrix} + te^{3t}\begin{pmatrix} 3s_0 + t_0 \\ -9s_0 - t_0 \end{pmatrix}\\
y(t) &= s_0 e^{3t} + (3s_0 + t_0)\, te^{3t}.
\end{aligned}$$

3.6, Problem 4

Using the lucky guess, we let $y(t) = e^{st}$, yielding

$$\begin{aligned}
e^{st}\left(s^2 - 4s + 4\right) &= 0\\
s &= 2\\
y(t) &= k_1 e^{2t} + k_2 te^{2t}
\end{aligned}$$

3.6, Problem 11

Using the lucky guess method, we find the general solution of the form

$$y(t) = k_1 e^{4t} + k_2 te^{4t}.$$

Finding $y(0)$, we get $k_1 = 3$. Taking derivatives, we then get

$$11 = 4k_1 + k_2,$$

yielding $k_2 = -1$. The solution to the IVP is, then,

$$y(t) = 3e^{4t} - te^{4t}.$$

3.6, Problem 12

Using the lucky guess method, we find the general solution of the form

$$y(t) = k_1 e^{2t} + k_2 te^{2t}.$$

Finding $y(0)$, we get $k_1 = 1$, and

$$\begin{aligned}
1 &= 2k_1 + k_2\\
k_2 &= -1.
\end{aligned}$$

The solution to the IVP is, then,

$$y(t) = e^{2t} - te^{2t}.$$

3.6, Problem 31

We have

$$\begin{aligned}
\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy &= 0\\
\frac{d^2}{dt^2}\left(\operatorname{Re}(y(t)) + i\operatorname{Im}(y(t))\right) + p\frac{d}{dt}\left(\operatorname{Re}(y(t)) + i\operatorname{Im}(y(t))\right) + q\left(\operatorname{Re}(y(t)) + i\operatorname{Im}(y(t))\right) &= 0\\
\underbrace{\frac{d^2}{dt^2}\left(\operatorname{Re}(y(t))\right) + p\frac{d}{dt}\left(\operatorname{Re}(y(t))\right) + q\operatorname{Re}(y(t))}_{\text{Real}} + i\underbrace{\left(\frac{d^2}{dt^2}\left(\operatorname{Im}(y(t))\right) + p\frac{d}{dt}\left(\operatorname{Im}(y(t))\right) + q\operatorname{Im}(y(t))\right)}_{\text{Imaginary}} &= 0,
\end{aligned}$$

meaning that both the Real and Imaginary portions must be equal to zero.

Thus, both $\operatorname{Re}(y(t))$ and $\operatorname{Im}(y(t))$ are solutions.

6.1, Problem 3

$$\begin{aligned}
\mathcal{L}[h] &= \int_0^\infty \left(-5t^2\right)e^{-st}\,dt\\
&= -5\int_0^\infty t^2e^{-st}\,dt\\
&= -5\left(-\frac{t^2}{s}e^{-st}\Big|_0^\infty - \frac{2t}{s^2}e^{-st}\Big|_0^\infty - \frac{2}{s^3}e^{-st}\Big|_0^\infty\right)\\
&= -\frac{10}{s^3}
\end{aligned}$$

6.1, Problem 4

$$\begin{aligned}
\mathcal{L}[k] &= \int_0^\infty t^5e^{-st}\,dt\\
&= \left(-\frac{t^5}{s}e^{-st} - \frac{5t^4}{s^2}e^{-st} - \frac{20t^3}{s^3}e^{-st} - \frac{60t^2}{s^4}e^{-st} - \frac{120t}{s^5}e^{-st} - \frac{120}{s^6}e^{-st}\right)\Big|_0^\infty\\
&= \frac{120}{s^6}
\end{aligned}$$

6.1, Problem 6

$$\begin{aligned}
\mathcal{L}[p] &= \sum_{k=0}^{n} a_k\mathcal{L}\left[t^k\right]\\
&= \sum_{k=0}^{n}\frac{a_k k!}{s^{k+1}}.
\end{aligned}$$