Math 395: Homework 2
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Exercise 1

Problem:

(1) Let $\mathcal{A}$ be a basis of U, $\mathcal{B}$ be a basis of V, and $\mathcal{C}$ be a basis of W. Let $S \in \operatorname{Hom}_{\mathbb{F}}(U, V)$ and $T \in \operatorname{Hom}_{\mathbb{F}}(V, W)$. Show that

$$[T \circ S]_{\mathcal{A}}^{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [S]_{\mathcal{A}}^{\mathcal{B}}.$$

(2) We know that, given $A \in \operatorname{Mat}_{m,p}(\mathbb{F})$ and $B \in \operatorname{Mat}_{n,m}(\mathbb{F})$, we have corresponding T_A and T_B linear maps. Show that you recover the definition of matrix multiplication by using part (1) to define matrix multiplication.

Solution.

(1) Assuming that U, V, W are $\mathbb{F}$-vector spaces with dimensions of n, m, and p respectively, we can see that the following diagram commutes.

$$\begin{tikzcd}
U \arrow[r, "S"] \arrow[d, dashed, "T_{\mathcal{A}}"] & V \arrow[r, "T"] \arrow[d, dashed, "T_{\mathcal{B}}"] & W \arrow[d, dashed, "T_{\mathcal{C}}"] \\
\mathbb{F}^n \arrow[r, "{[S]_{\mathcal{A}}^{\mathcal{B}}}"'] & \mathbb{F}^m \arrow[r, "{[T]_{\mathcal{B}}^{\mathcal{C}}}"'] & \mathbb{F}^p
\end{tikzcd}$$

Therefore, it must be the case that $[T \circ S]_{\mathcal{A}}^{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [S]_{\mathcal{A}}^{\mathcal{B}}$.

(2) For $(a_{ij}) = A \in \operatorname{Mat}_{m,p}(\mathbb{F})$ and $(b_{ij}) = B \in \operatorname{Mat}_{n,m}(\mathbb{F})$, we have

$$\begin{aligned}
T_B(e_j) &= \sum_{k=1}^{m} b_{kj} f_k \\
T_A(f_k) &= \sum_{i=1}^{p} a_{ik} g_i.
\end{aligned}$$

In particular, since we know that

$$[T_A \circ T_B]_{\mathcal{A}}^{\mathcal{C}} = [T_A]_{\mathcal{B}}^{\mathcal{C}} [T_B]_{\mathcal{A}}^{\mathcal{B}},$$

we have

$$\begin{aligned}
[T_A \circ T_B]_{\mathcal{A}}^{\mathcal{C}}(e_j) &= \sum_{i=1}^{p} c_{ij} g_i \\
&= [T_A]_{\mathcal{B}}^{\mathcal{C}} [T_B]_{\mathcal{A}}^{\mathcal{B}}(e_j), \\
&= [T_A]_{\mathcal{B}}^{\mathcal{C}} \left(\sum_{k=1}^{m} b_{kj} f_k\right) \\
&= \sum_{i=1}^{p} \underbrace{\left(\sum_{k=1}^{m} a_{ik} b_{kj}\right)}_{c_{ij}} g_i.
\end{aligned}$$

Thus, we recover the definition of matrix multiplication.

Exercise 2

Problem: Let $A_1, A_2 \in \mathrm{Mat}_{m,n}(\mathbb{F})$, $c \in \mathbb{F}$. Use the definition of the transpose to show

$$\begin{aligned} (A_1 + A_2)^T &= A_1^T + A_2^T \\ (cA_1)^T &= cA_1^T. \end{aligned}$$

Solution. For bases $\mathcal{E}_n = \{e_1, \dots, e_n\}$ and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ for $\mathbb{F}^n$ and $\mathbb{F}^m$, and corresponding linear transformations T_{A_1} and T_{A_2}, we have

$$\begin{aligned} (A_1 + A_2)^T &= \left[(T_{A_1} + T_{A_2})'\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= \left[T'_{A_1} + T'_{A_2}\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= \left[T'_{A_1}\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} + \left[T'_{A_2}\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= A_1^T + A_2^T \end{aligned}$$

$$\begin{aligned} (cA_1)^T &+ \left[(T_{cA_1})'\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= \left[(cT_{A_1})'\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= \left[cT'_{A_1}\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= c\left[T'_{A_1}\right]_{\mathcal{F}'_m}^{\mathcal{E}'_n} \\ &= cA_1^T. \end{aligned}$$

Problem 1

Problem: Let $V = P_n(\mathbb{F})$. Let $\mathcal{B} = \{1, x, \dots, x^n\}$ be a basis of V. Let $\lambda \in \mathbb{F}$, and set $C = \left\{1, x - \lambda, \dots, (x - \lambda)^{n-1}, (x - \lambda)^n\right\}$. Define a linear transformation $T \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ by taking $T\left(x^j\right) = (x - \lambda)^j$. Determine the matrix of this linear transformation. Use this to conclude that C is also a basis of V.

Solution. Considering our basis $\mathcal{B} = \{1, x, \dots, x^n\}$, we evaluate $T\left(x^j\right)$ for each j. In particular, this yields

$$T\left(x^j\right) = \sum_{k=0}^{j} \binom{j}{k} (-\lambda)^{j-k} x^k,$$

meaning that our linear transformation is

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & -\lambda & (-\lambda)^2 & \cdots & (-\lambda)^n \\ 0 & 1 & 2(-\lambda) & \cdots & \binom{n}{1}(-\lambda)^{n-1} \\ 0 & 0 & 1 & \cdots & \binom{n}{2}(-\lambda)^{n-1} \\ & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We can see that $[T]_{\mathcal{B}}^{\mathcal{B}}$ is nonsingular (since it is an upper triangular matrix that is nonzero along the diagonal), meaning that T is injective (and thus, bijective), so it is an isomorphism.

Since T is an isomorphism, and $T\left(x^j\right) = (x - \lambda)^j$, this means C is a basis.

Problem 4

Problem: Let $V = P_5(\mathbb{Q})$ and let $\mathcal{B} = \{1, x, \ldots, x^5\}$. Prove that the following are elements of V'< and express them as linear combinations of the dual basis.

(a) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 t^2 p(t)\, dt$.

(b) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = p'(5)$, where $p'(x)$ denotes the derivative of $p(x)$.

Solution. We define $\mathcal{B} = \{1, x, \ldots, x^5\} = \{e_0, e_1, \ldots, e_5\}$.

In particular, we can see that for $p(x) = \sum_{i=0}^5 a_i x^i$, $a_i = e_i'(p)$.

(a) Let $p(x) = \sum_{i=0}^5 a_i x^i$. Then,

$$\begin{aligned}
\int_0^1 t^2 p(t)\, dt &= \int_0^1 t^2 \sum_{i=0}^5 a_i t^i\, dt \\
&= \int_0^1 \sum_{i=0}^5 a_i t^{i+2}\, dt \\
&= \sum_{i=0}^5 \frac{1}{i+3} a_i \\
&= \sum_{i=0}^5 \frac{1}{i+3} e_i'(p).
\end{aligned}$$

(b) Let $p(x) = \sum_{i=0}^5 a_i x^i$. Then,

$$\begin{aligned}
p'(x) &= \sum_{i=1}^5 a_i x^{i-1} \\
&= \sum_{i=0}^4 a_{i+1} x^i \\
p'(5) &= \sum_{i=0}^4 a_{i+1} \left(5^i\right) \\
&= \sum_{i=0}^4 \left(5^i\right) e_{i+1}(p).
\end{aligned}$$