Problem (Problem 1)**:** Let R be a ring and M a left R-module.

(a) Prove that for every $m \in M$, the map $r \mapsto r \cdot m$ from R to M is a homomorphism of R-modules.

(b) Assume that R is commutative and M an R-module. Prove that there is an isomorphism $\hom_R(R, M) \cong M$ as left R-modules.

Solution:

(a) Let $m \in M$ be fixed, and define $\varphi_m \colon R \to M$ by

$$\varphi_m(r) = r \cdot m.$$

It follows from the axioms of left R-modules that

$$\begin{aligned}\varphi_m(r + s) &= (r + s) \cdot m\\ &= r \cdot m + s \cdot m\\ &= \varphi_m(r) + \varphi_m(s),\end{aligned}$$

and

$$\begin{aligned}\varphi_m(rs) &= (rs) \cdot m\\ &= r \cdot (s \cdot m)\\ &= r \cdot (\varphi_m(s)),\end{aligned}$$

so that φ_m is a homomorphism of left R-modules.

(b) If $\varphi_m \colon R \to M$ is the homomorphism as defined in part (a), we define a map $\varphi \colon M \to \hom_R(R, M)$ by

$$\varphi(m)(r) = \varphi_m(r).$$

First, we verify that φ is a homomorphism. If $r \in R$ is arbitrary, then

$$\begin{aligned}\varphi(m + n)(r) &= \varphi_{m+n}(r)\\ &= r \cdot (m + n)\\ &= r \cdot m + r \cdot n\\ &= \varphi_m(r) + \varphi_n(r)\\ &= (\varphi(m) + \varphi(n))(r).\end{aligned}$$

To see that φ is injective, we see that $\ker(\varphi)$ consists of all elements $m \in M$ such that $\varphi(m) = \varphi_0$, where $\varphi_0 \colon R \to M$ takes $r \mapsto 0$ for all $r \in R$. In particular, since $1 \in R$, it follows that $1 \cdot m = m = 0$, meaning that $\ker(\varphi) = \{0\}$.

To see that φ is surjective, we observe that for any $\psi \in \hom_R(R, M)$, ψ is fully determined by where it maps 1, as

$$\psi(r) = r \cdot \psi(1).$$

Therefore, if $\psi \in \hom_R(R, M)$, then we may find $m \in M$ corresponding to ψ by taking

$$m := \psi(1).$$

Thus, $M \cong \hom_R(R, M)$.

Problem (Problem 3)**:** Let R be a ring, and M a left R-module.

(a) Let N be a subset of M. The *annihilator* of N is defined to be the set

$$\operatorname{ann}_R(N) = \{r \in R \mid r \cdot n = 0 \text{ for all } n \in N\}.$$

Prove that $\operatorname{ann}_R(N)$ is a left-ideal of R.

(b) Show that if N is an R-submodule of M, then $\operatorname{ann}_R(N)$ is a two-sided ideal of R.

(c) For a subset I of R, the *annihilator* of I in M is defined to be the set

$$\operatorname{ann}_M(I) = \{m \in M \mid x \cdot m = 0 \text{ for all } x \in I\}.$$

Find a natural condition on I that guarantees $\operatorname{ann}_M(I)$ is a submodule of M.

(d) Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator.

Solution:

(a) First, we observe that $\operatorname{ann}_R(N)$ is nonempty, as $0 \in \operatorname{ann}_R(N)$. Additionally, if $s, t \in \operatorname{ann}_R(N)$, then for all $n \in N$,

$$\begin{aligned}(s-t)\cdot n &= s \cdot n - t \cdot n\\ &= 0,\end{aligned}$$

so that N is closed under subtraction. Finally, if $r \in R$ and $s \in \operatorname{ann}_R(N)$, then for all $n \in N$,

$$\begin{aligned}(rs)\cdot n &= r \cdot (s \cdot n)\\ &= r \cdot 0\\ &= 0,\end{aligned}$$

meaning that $rs \in \operatorname{ann}_R(N)$, or that $\operatorname{ann}_R(N)$ is a left-ideal of R.

(b) Let N be an R-submodule of M, and let $s \in \operatorname{ann}_R(N)$. If $r \in R$, then for all $n \in N$, $r \cdot n \in N$, so that $(sr)\cdot n = s \cdot (r \cdot n) = 0$, meaning that $sr \in \operatorname{ann}_R(N)$. Thus, $\operatorname{ann}_R(N)$ is a right-ideal, hence a two-sided ideal for R.

(c) We observe to start that $\operatorname{ann}_M(I)$ contains 0 and is additively closed, since if $m, n \in \operatorname{ann}_M(I)$ and $x \in I$ are arbitrary, then

$$\begin{aligned}x \cdot (m+n) &= x \cdot m + x \cdot n\\ &= 0.\end{aligned}$$

Therefore, if we desire for $\operatorname{ann}_M(I)$ to be a submodule of M, we would need $r \cdot m \in \operatorname{ann}_M(I)$ for all $m \in \operatorname{ann}_M(I)$, which would mean $r \cdot m$ would have to satisfy the condition

$$\begin{aligned}0 &= x \cdot (r \cdot m)\\ &= (xr)\cdot m,\end{aligned}$$

meaning that we would require $xr \in \operatorname{ann}_M(I)$. In other words, this means that $\operatorname{ann}_M(I)$ would have to be a right-ideal for R.