Problem 1

Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function with $f(0) = f(1)$. Show that there is a $c \in [0, 1/2]$ with $f(c) = f(c+1/2)$. Conclude that there are always antipodal points on the earth's equator with the same temperature.

Consider $g(x) = f(x) - f(x + 1/2)$ on $[0, 1/2]$. Then, $g(0) = f(0) - f(1/2)$, and $g(1/2) = f(1/2) - f(1)$. Since $f(0) = f(1)$, it must be the case that $g(0) = -g(1/2)$.

Therefore, on $[0, 1/2]$, if $g(0) = k$ for $k \in \mathbb{R}$, then $g(1/2) = -k$, meaning that by the Intermediate Value Theorem, $\exists c \in [0, 1/2]$ with $g(c) = 0$. This is equivalent to $f(c) = f(c + 1/2)$ by the definition of g.

For any two antipodes on the earth's equator, let $t(x)$ be the temperature at point x. Then, moving from x to $-x$, where $-x$ denotes the opposite point on the earth's equator, it must be the case that the values of t at x and $-x$ flip. Therefore, there is a point where $t(c) = t(-c)$.

Problem 2

Suppose $f : [a, b] \to \mathbb{R}$ is injective and continuous. Show that f is strictly monotone.

Problem 3

Suppose $f : [0, 1] \to \mathbb{R}$ is a map that takes on each of its values exactly twice. Show that f cannot be continuous at every point.

Problem 4

Show that the function $f(x) = \frac{1}{x^2}$ is continuous on $[1, \infty)$ but not on $(0, \infty)$.

Problem 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p; that is,

$$f(x + p) = f(x) \qquad \forall x \in \mathbb{R}$$

If f is continuous, show that f is bounded and uniformly continuous on $\mathbb{R}$.

Problem 6

Show that $f(x) = x$ and $g(x) = \sin(x)$ are both uniformly continuous on $\mathbb{R}$, but the product

$$h(x) = x \sin(x)$$

is not uniformly continuous on $\mathbb{R}$.

Problem 7

If $f : D \to \mathbb{R}$ is uniformly continuous and $|f(x)| \geq k > 0$ for some k, show that $\frac{1}{f}$ is uniformly continuous on D.

Problem 8

If $D \subseteq \mathbb{R}$ is a bounded set and $f : D \to \mathbb{R}$ is uniformly continuous, show that f is bounded.

Problem 9

Suppose $f_n : D \to \mathbb{R}$ is a sequence of continuous functions such that $(f_n)_n \to f$ uniformly on D. Show that f is also continuous.

Problem 10

Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$$
$$f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}.$$