# Contents



# Cardinality and Countability

## Section 1.1: Countable Sets

**Definition** (Denumerable Set)**.** A set $S$ is denumerable if there exists a function $f : S \to \mathbb{N}$ with $f$ a bijection. We also say $S$ is countably infinite.

**Definition** (Countable Set)**.** We say $S$ is countable if $S$ is either finite or denumerable.

**Theorem** (Countability of Unions)**:** If $A$ and $B$ are countable sets, then $A \cup B$ is countable.

**Theorem** (Countability of Subsets)**:** If $A \subseteq B$, then if $B$ is countable, then $A$ is countable.

**Theorem** (Union of Finite Sets)**:** If $A$ and $B$ are finite, then $A \cup B$ is finite.

*Proof.* If $A$ is finite and $B$ has one element, then we show that $A \cup B$ is finite (with two cases).

Afterward, for $|B| > 1$, we use induction on $|B|$. □

**Definition** (Finite Set)**.** A set $A$ is finite if there exists a bijection $f : S \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N} = \{0, 1, \dots\}$.

We write $|A| = n$.

**Theorem** (Disjoint Union of Countable Sets)**:** If $A$ is denumerable, $B$ is finite, and $A \cap B = \varnothing$, then $A \cup B$ is denumerable.

*Proof.* There exists a bijection $f : A \to \mathbb{N}$ (since $A$ is denumerable), and a bijection $g : B \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ (since $B$ is finite).

We create a new bijection $h : A \cup B \to \mathbb{N}$ by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since $A \cap B = \varnothing$, we know that $h$ is well-defined.

Now, we must show that $h$ is a bijection.

Suppose $h(x) = h(y)$.

**Case 1:** If $x, y \in B$, then $h(x) = g(x) - 1$, and $h(y) = g(y) - 1$, meaning $g(x) - 1 = g(y) - 1$, meaning $g(x) = g(y)$. Since $g$ is a bijection, $x = y$.

**Case 2:** If $x, y \in A$, a similar argument yields that $x = y$

**Case 3:** Without loss of generality, let $x \in A$ and $y \in B$. If $x \in A$, then $h(x) = f(x) + n$ and $h(y) = g(y) - 1$. Thus, $f(x) + n = g(y) - 1$. However, since $f(x) + n \geqslant n$ and $0 \leqslant g(y) - 1 \leqslant n - 1$. Thus, we get that $0 \leqslant n \leqslant n - 1$, which is a contradiction.

Thus, we have shown that $h$ is injective. □

**Theorem** (Cartesian Product of Natural Numbers)**:** $\mathbb{N} \times \mathbb{N}$ is denumerable.

*Proof.* We consider $\mathbb{N} \times \mathbb{N}$ as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \cdots,$$

$$\begin{array}{llllll}
\mathbb{N} \times \{0\} : & (0,0) & (1,0) & (2,0) & (3,0) & \cdots \\
\mathbb{N} \times \{1\} : & (0,1) & (1,1) & (2,1) & (3,1) & \cdots \\
\mathbb{N} \times \{2\} : & (0,2) & (1,2) & (2,2) & (3,2) & \cdots \\
\mathbb{N} \times \{3\} : & (0,3) & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{llllll}
\mathbb{N} \times \{0\} : & \cancel{(0,0)}^{0} & \cancel{(1,0)}^{2} & \cancel{(2,0)}^{5} & \cancel{(3,0)}^{9} & \cdots \\
\mathbb{N} \times \{1\} : & \cancel{(0,1)}^{1} & \cancel{(1,1)}^{4} & \cancel{(2,1)}^{8} & (3,1) & \cdots \\
\mathbb{N} \times \{2\} : & \cancel{(0,2)}^{3} & \cancel{(1,2)}^{7} & (2,2) & (3,2) & \cdots \\
\mathbb{N} \times \{3\} : & \cancel{(0,3)}^{6} & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can also find a bijection $P : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, with

$$P(x, y) = \frac{(x+y)(x+y+1)}{2} + x$$

A fun challenge is to prove that $P$ is a bijection. $\square$

**Theorem** (Countability of the Rationals)**:** $\mathbb{Q}$ is denumerable.

**Theorem** (Countability of the Integers)**:** The set $\mathbb{Z}$ is denumerable.

*Proof.* Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 2x & x \geqslant 0 \\ -2x-1 & x < 0 \end{cases}$$

$\square$

**Definition** (Cardinality)**.** We say two sets, $A$ and $B$, have the same cardinality if there exists a bijection $f : A \to B$.

**Theorem** (Finite Subset Cardinality)**:** If $m, n \in \mathbb{N}$ and $m \neq n$, then $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$ do not have the same cardinality.

**Theorem** (Infinitude of the Natural Numbers)**:** $\mathbb{N}$ is not finite.

**Example.** If $A \subsetneq B$ and $|A| = |B|$, then both $A$ and $B$ are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.

## Section 1.2: Uncountable Sets

**Definition** (Uncountable Set)**.** A set is uncountable if it is not countable.

**Theorem** (Uncountability of $\mathbb{R}$)**:** $\mathbb{R}$ is uncountable.

*Proof.* For all $x \in \mathbb{R}$, and for all $j \in \mathbb{N}$, we define $[x]_j$ to denote the $j+1$-th digit after the decimal point in the decimal expansion of $x$.

For example, $[\pi]_0 = 1$, $[\pi]_1 = 4$, etc.

Let $f : \mathbb{N} \to \mathbb{R}$. We will show that $f$ is not surjective.

Let $y \in [0, 1) \subseteq \mathbb{R}$ defined by $\forall j \in \mathbb{N}$,

$$[y]_j = \begin{cases} 0 & [f(j)]_j = 1 \\ 1 & [f(j)]_j \neq 1 \end{cases}.$$

We claim that $y \notin f(\mathbb{N})$. We will show that $\forall j \in \mathbb{N}$, $f(j) \neq y$.

We can see that if $[f(j)]_j = 1$, then $[y]_j = 0$. Similarly, if $[f(j)]_j \neq 1$, then $[y]_j = 1$. Either way, $[f(j)]_j \neq [y]_j$ for all $j \in \mathbb{N}$. $\square$

**Remark:** The above proof is an example of a diagonalization proof. It can be imagined as

$$\begin{array}{c|c} f(0) & *.\cancel{a_1}\overset{\neq}{a_2}a_3\dots \\ f(1) & *.b_1\cancel{b_2}\overset{\neq}{b_3}\dots \\ f(2) & *.c_1c_2\cancel{c_3}\overset{\neq}{\dots} \\ \vdots & \vdots \end{array}$$

**Note:** A substantial problem that we might need to deal with is that a real number does not necessarily have a unique decimal representation. For instance, $3.999\cdots = 4.000\dots$.

In order to resolve this issue, we can default to the option with trailing 0 over trailing 9.

**Definition** (Power Set)**.** The power set of a set $S$ is

$$P(S) = \{A \mid A \subseteq S\}.$$

**Theorem** (Power Set Surjection)**:** Let $f : S \to P(S)$. Then, $f$ is not surjective.

*Proof.* Let $T = \{x \in S \mid x \notin f(x)\}$. Then, $T \notin f(S)$.

Let $y \in S$. We want to show that $f(y) \neq T$. Suppose toward contradiction that $f(y) = T$. Then, if $y \in T$, then $y \in f(y)$, which implies that $y \notin T$.

If $y \notin T$, then $y \notin f(y)$, which implies that $y \in T$.

Thus, it cannot be the case that $f(y) = T$. □

**Definition** (Cardinality Comparison)**.** Let $A$ and $B$ be sets. Then, we write $\text{card}(A) \leqslant \text{card}(B)$ if there exists an injective map $f : A \hookrightarrow B$.

We write $\text{card}(A) < \text{card}(B)$ if there exists an injection $f : A \hookrightarrow B$ but no bijection.

**Example** (Cardinality of the Power Set)**.** For every set,

$$\text{card}(S) < \text{card}(P(S)).$$

(1) We know that $\text{card}(S) \leqslant \text{card}(P(S))$, defining $f : S \hookrightarrow P(S)$, $f(a) = \{a\}$, since if $f(x) = f(y)$, then $\{x\} = \{y\}$, meaning $x \in \{y\}$, so $x = y$.

In the case of $f : \emptyset \to \{\emptyset\}$, we define $\emptyset = f \subseteq \emptyset \times \{\emptyset\}$.

(2) Since there exists no bijection $f : S \to P(S)$, it is the case that $\text{card}(S) \neq \text{card}(P(S))$.

**Example** (Decimal Expansion)**.** We know that for some decimal expansion

$$\begin{aligned} 3.14159\ldots &= 3 + \frac{1}{10} + \frac{4}{100} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{n_i}{10^i}, \end{aligned}$$

with $0 \leqslant n_i \leqslant 9$ for $i \geqslant 1$.

However, we can also write any real number as

$$\sum_{i=0}^{\infty} \frac{n_i}{3^i}$$

with $0 \leqslant n_i \leqslant 2$ for all $i \geqslant 1$.

**Example** (Finite Strings)**.** Let $S$ be the set of all finite strings of 0 and 1. $S$ is countable.

**Proof 1:** We define $f : S \to \mathbb{N}$ by, for a string $x \in S$, $x$ starts with $n_1$ zeroes, then has $n_2$ ones, then $n_3$ zeroes, etc. We define $f(x) := 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times 7^{n_4} \times 11^{n_5} \cdots$, or

$$f(x) = \prod_{i}^{\infty} p_i^{n_i},$$

where $p_i$ denotes the $i$th prime number. We can see that $f$ is an injection.

Since $S$ is infinite (proof omitted), we can see that $f(S)$ is also infinite.[1] Since $f(S)$ is an infinite subset of $\mathbb{N}$, $f(S)$ is denumerable, meaning there exists a bijection $q : f(S) \to \mathbb{N}$. Therefore, we have $q \circ f : S \to \mathbb{N}$ is a bijection, meaning $S$ is denumerable.

**Proof 2:** List the elements of $S$ by length and lexicographic order: short strings come before long strings, and 0s come before 1s.

| Rank | String |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 00 |
| 3 | 01 |
| 4 | 10 |
| 5 | 11 |
| $\vdots$ | $\vdots$ |

This pattern yields a systematic way to map $S$ to the natural numbers.

**Proof 3:** We can see that

$$S = \bigcup_{i=1}^{\infty} S_i,$$

where $S_i$ is the set of all strings of length $i$, each of which contains $2^i$ elements.

Since each $S_i$ is finite, and $S_i \cap S_j = \varnothing$ (by definition). Thus, $S$ is a countable union of pairwise disjoint countable sets, so $S$ is countable.

**Example** (All Possible Writings)**.** Let $W$ be the set of all possible writings in English. We let $W_n$ denote the writing with $n$ characters. Then,

$$W = \bigcup_{n=1}^{\infty} W_n,$$

which is a countable union of disjoint finite sets, which is countable.

Similarly, we can list all the writings by length and lexicographic order.

This result implies that “almost all” real numbers, in a sense, are unable to be described.

## Section 1.3: Cantor–Schröder–Bernstein Theorem

**Example.** If we have $|A| \leqslant |B|$ and $|B| \leqslant |A|$, it does not necessarily imply $|A| = |B|$.

This is because the $\leqslant$ in the cardinality comparison implies there exist injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, not that the cardinalities are necessarily “less than or equal to” each other.

However, at the same time, this fact is true — this is what is known as the Cantor–Schröder–Bernstein Theorem.

**Theorem** (Cantor–Schröder–Bernstein)**:** Let $f : C \hookrightarrow D$ and $g : D \hookrightarrow C$ be injective maps. Then, $|C| = |D|$.

[1] If $f(S)$ is finite, then there exists a bijection $g : f(S) \to \{1, \dots, n\}$. Composing $g$ and $f$, we find $S$ is finite as $g \circ f|_S$ is a bijection.

*An Informal Proof Sketch.* Consider $C$ to be a set of cats and $D$ to be a set of dogs. Every cat chases a dog, and every dog chases a cat, with different cats chasing different dogs and vice versa.

There are four potential arrangements:

(1) A set of cats and dogs are chasing each other in a circle.

(2) A chain of dogs chasing cats that starts with a dog.

(3) A chain of cats chasing dogs that starts with a cat.

(4) An endless chain of cats chasing dogs with no discernible start or end point.

These four cases create a bijection from $C$ to $D$:

(1) Pair each cat with the dog that it is chasing.

(2) Pair each cat with the dog that it is chasing.

(3) Pair each cat with the dog that *is chasing it*.

(4) Pair each cat with the dog that it is chasing.

□

*A More Formal Proof Sketch.* For $C = \{c_i\}_{i\in I}$ and $D = \{d_i\}_i$, we have four types of sequences.

(i) Circular sequence: for some $m \in \mathbb{N}$, there exist $c_1, \dots, c_m$ and $d_1, \dots, d_m$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$, where $c_{m+1} = c_1$.

(ii) Cat sequence: there is $c_1, c_2, \dots$ and $d_1, d_2, \dots$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

(iii) Dog sequence: there is $c_1, c_2, \dots$ and $d_1, d_2, \dots$ such that $f(c_i) = d_{i+1}$ and $g(d_i) = c_i$.

(iv) Bi-infinite sequence: $\{c_i\}_{i\in\mathbb{Z}}$ and $\{d_i\}_{i\in\mathbb{Z}}$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

**Claim 1:** For every $c \in C$, $c$ is in exactly one sequence that is either a circular sequence, a cat sequence, a dog sequence, or a bi-infinite sequence.

We define our bijection $h : C \to D$ by

$$h(c) = \begin{cases} g^{-1}(c) & c \text{ in a dog sequence} \\ f(c) & \text{else} \end{cases}.$$

**Claim 2:** $h$ is well-defined.

**Claim 3:** $h$ is a bijection.

□

**Theorem:** For every set $A, B$, either $|A| \leqslant |B|$ or $|B| \leqslant |A|$.

In order to prove this, we need the axiom of choice.

**Example** (Cardinality of the Reals)**.** Recall that $|\mathbb{N}| < |P(\mathbb{N})|$ and $|\mathbb{N}| < |\mathbb{R}|$. According to the previous theorem, it is the case that either $|P(\mathbb{N})| \leqslant |\mathbb{R}|$ or $|\mathbb{R}| \leqslant |P(\mathbb{N})|$.

In particular, $|P(\mathbb{N})| = |\mathbb{R}|$.

*An Informal Proof.* Let $S$ be the set of all functions $f : \mathbb{N} \to \{0,1\}$. We will show that $|S| = |P(\mathbb{N})|$ and $|S| = |\mathbb{R}|$. This will show that $|P(\mathbb{N})| = |\mathbb{R}|$ (by composing bijections).

To show that $|S| = |P(\mathbb{N})|$, define a subset of $\mathbb{N}$ by the support[II] of some element of $S$. This is a bijection between $P(\mathbb{N})$ and $S$.

To show $|S| = |\mathbb{R}|$, we place a decimal point in front of the string, and consider it as a real number in base 2, which yields a bijection between $S$ and $[0,1]$.

Next, we show that $|[0,1]| = |(0,1)|$.

Finally, we show that $|(0,1)| = \mathbb{R}$. Take $f : (0,1) \to \mathbb{R}$ to be $\cot(\pi x)$ — or $\tan(\pi x - \pi/2)$. These are bijections from $(0,1)$ to $\mathbb{R}$. □

**Definition** (Continuum Hypothesis)**.** We are aware that

$$|\mathbb{N}| < |\mathbb{R}| = |P(\mathbb{N})|.$$

The continuum hypothesis states that there exists no set $S$ such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

The continuum hypothesis is independent of the ZFC axioms.[III]

**Exercise** (Challenge Problem)**:** Let $T = \left\{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{N};\ \text{finitely many nonzero } a_i\right\}$. Is $T$ countable? We also write

$$T = \bigoplus_{i=0}^{\infty} \mathbb{N}.$$

# Axiomatic Set Theory

**Question:** Is there a set $A$ such that $A \in A$?

**Answer: Yes.**

There is the set $\{\cdots\{\}\cdots\}$, which contains infinitely many sets in itself. Additionally, there is the set $A = \{x \mid x \text{ is a set}\}$.

**Example** (Russell's Paradox)**.** Consider the set

$$R = \{x \mid x \notin x\}.$$

The question is if $R \in R$. However, this cannot be true, because if $R \in R$, then $R \notin R$ and vice versa.

## Axioms of Set Theory

We cannot just say

$$S = \{x \mid x \text{ is blah}\},$$

as evidenced by Russell's paradox. We need to carefully construct rules to create a rigorous description of formal set theory.

**Axiom** (Existence)**:** The existence axiom states that there exists a set:

$$\exists a\,(a = a).$$

[II] The elements that $f$ does not map to 0 for some $f \in S$.

[III] Zermelo–Fraenkel Axioms with the Axiom of Choice.

**Axiom** (Empty Set)**:** The empty set axiom states that there exists a set with no elements:

$$\exists a\, \forall x\, (x \notin a)\,.$$

**Axiom** (Pairing)**:** The pairing axiom states that, given any sets $a$ and $b$, there is a set $c$ such that the only elements of $c$ are $a$ and $b$:

$$\forall a\, \forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = a \vee x = b)$$

**Axiom** (Extensionality)**:** The axiom of extensionality states that if two sets have the same elements, they are the same sets:

$$\forall a\, \forall b\, (\forall x\, (x \in a \Leftrightarrow x \in b) \Rightarrow a = b)$$

**Question:** What is a set?

**Answer:** The unsatisfying answer is that "set" and "element" have no meaning *per se*. The main reason we define these axioms is to define relationships between objects (rather than objects themselves).

**Example.** We want to prove that for every set $b$, there exists a set $\{b\}$.

Symbolically, we want to show

$$\forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = b)\,.$$

In particular, we can see that, in the pairing axiom, there is no requirement that $a$ and $b$ be distinct. Therefore, we can use the pairing axiom of $a = b$ and $b = b$. Therefore, the pairing axiom becomes

$$\forall b\, \forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = b \vee x = b)\,,$$

which reduces to

$$\forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = b)\,.$$

In particular, if $b = \{\}$ in the previous example, then the pairing axiom implies the uniqueness of the empty set. We will denote $\{\} = \emptyset$. We can create a tower

$$\{\}\,, \{\{\}\}\,, \{\{\}\,, \{\{\}\}\}\,, \dots,$$

entirely consisting of the empty set.

**Axiom** (Union)**:** The axiom of union states that for any set $a$, there exists a set consisting of all the elements of $a$

$$\forall a \exists u \forall x \forall y\, ((x \in y \wedge y \in a) \Rightarrow x \in u)$$

**Definition.** The string $a \subseteq b$ is shorthand for

$$\forall x\, (x \in a \Rightarrow x \in b)\,.$$

**Axiom** (Power Set)**:** The power set axiom states that for all $a$, there is a set $b$ such that all elements of $b$ are subsets of $a$ and all subsets of $a$ are contained in $b$:

$$\forall a\, \exists b\, \forall y\, (y \in b \Leftrightarrow y \subseteq a)\,.$$

**Definition.** We let $(a, b)$ be shorthand for the set

$$\{a, \{a, b\}\}\,.$$

**Exercise:** If $\{a, \{a, b\}\} = \{c, \{c, d\}\}$, it is the case that $a = c$ and $b = d$.

Recall that

$$c = \{x \mid x \text{ is blah}\}$$

is a problematic definition of a set. However, if $a$ is a set, we can define

$$c = \{x \mid x \in a \wedge x \text{ is blah}\},$$

which does not cause any contradictions. The following axiom schema formalizes this fact.

**Axiom** (Comprehension schema)**:** The comprehension schema says that, given any formula $\varphi(x)$, in which $x$ is a free variable, there exists a set $c$ whose elements are those in $a$ that satisfy $\varphi$:

$$\forall a \, \exists c \, \forall x \, (x \in c \Leftrightarrow x \in a \wedge \varphi(x)).$$

**Remark:** There are infinitely many axioms in the comprehension schema, one for each formula $\varphi$. This is why it is known as a schema rather than an axiom.

**Remark:** Since we can specify a formula $\varphi(x) : x \neq x$, the comprehension schema obviates the empty set axiom.

**Example** (Some Logic)**.** An example of a formula is $\forall p \, \exists q (p \Rightarrow q)$.

In the formula $\exists q \ (p \Rightarrow q)$, we say $p$ is a free variable.

The main symbols in logic are $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, ()$ (the symbols that make up propositional logic), as well as $\forall, \exists$ (which form the basis of first-order logic).

In propositional logic, the only two symbols that are needed are $\wedge$ and $\neg$ (or $\vee$ and $\neg$).[IV]

When we get to set theory, the last symbol we need is $\in$.

We can build larger formulae by substituting formulae into other formulae.

**Example** (Using the Comprehension Schema)**.** Let $\phi(x) : \exists y \, (y \in X)$. This is an axiom:

$$\forall a \, \exists b \, \forall x \, (x \in b \Leftrightarrow x \in a \wedge \exists y \, (y \in x))$$

In particular, this axiom is equivalent to saying

$$\forall a \, \exists b \text{ s.t. } b = \{x \in a \mid x \neq \varnothing\}.$$

**Axiom** (Union)**:** The union axiom states that for a collection of sets $T$, there is a union of the sets, $a = \bigcup T$.

$$\forall t \, \exists a \, \forall x \, (x \in a \Leftrightarrow \exists y \, (y \in t \wedge x \in y)).$$

Alternatively, we can say

$$\forall t \;\; a = \{x \mid x \in \text{some element of } t\}$$

is a set.

**Axiom** (Infinity)**:** There exists an infinite set.

$$\exists a \, (\varnothing \in a \wedge \forall x \, (x \in a \Rightarrow x \cup \{x\} \in a))$$

**Remark:** To see that this set, $a$ has an element, $\varnothing$. Thus,

$$a = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \dots\}$$

We define $0 = \varnothing$, $1 = \{\varnothing, \{\varnothing\}\}$, etc. Thus, the axiom of infinity defines the natural numbers.

[IV] In computers, the only gate that is necessary is the NAND gate.

**Axiom** (Regularity)**:** There is no infinite chain of the form

$$\cdots \in d \in c \in b \in a.$$

$$\forall s \exists x \left(s = \varnothing \vee s \neq \varnothing \Rightarrow (x \in s \wedge x \cap s = \varnothing)\right)$$

**Remark:** The existence of this axiom is meant to obviate the case where we imagined a set $a$ with $a \in a$.

**Definition** (Function-like Formula)**.** Let $\psi(x, y)$ be a formula with $x, y$ free variables such that $\forall x, y, z$, $\psi(x, y) \wedge \psi(x, z) \Rightarrow y = z$.

**Axiom** (Replacement Schema)**:**

$$\forall a\, \exists b\, \forall x \left(x \in b \Leftrightarrow \exists y \left(y \in a \wedge \psi(x, y)\right)\right)$$

**Remark:** It is possible to prove the comprehension schema from the replacement schema.

The axioms that we have discussed so far are known as the Zermelo–Fraenkel axioms.

**Question:** If $A$ and $B$ are nonempty, is it the case that $A \times B \neq \varnothing$

**Answer: Yes.**

There exists $a \in A$ and $b \in B$ such that $(a, b) \in A \times B$. This can be proven using the ZF axioms.

**Question:** If $A_1, A_2, \ldots, \neq \varnothing$, then is $A_1 \times A_2 \times \cdots \neq \varnothing$?

**Answer:** This requires the axiom of choice.

**Axiom** (Choice)**:** If $T$ is a collection of sets, $\exists b$ such that $\forall a \in T$, $a \cap b \neq \varnothing$.

$$\forall t\, \exists b \left(\forall a \left(a \in t \Rightarrow \exists x \left(x \in a \wedge x \in b\right)\right)\right).$$

**Remark:** We define $x \in (a \cap b)$ as shorthand for $x \in a \wedge x \in b$.

**Remark:** The axiom of choice is controversial.

**Remark:** The axiom of choice entails certain counterintuitive results, such as the Banach–Tarski paradox[v] and the existence of non-measurable sets.

The Banach–Tarski paradox states that for any two bounded subsets of $\mathbb{R}^3$ with nonempty interior, one of the sets can be partitioned into finitely many subsets, with certain isometries applied to said partition, and reconstituted into the second set.

**Recall:**

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

**Definition.** For any sets $A$ and $B$, each subset of $A \times B$ is a relation from $A$ to $B$.

**Definition.** A relation $R \subseteq A \times B$ is a function if

$$\forall x \forall y \forall z \left((x, y) \in R \wedge (x, z) \in R \Rightarrow y = z\right).$$

**Definition.** A function $F \subseteq A \times B$ is injective if

$$\forall x \forall x' \forall y \left((x, y) \in F \wedge (x', y) \in F \Rightarrow x = x'\right)$$

**Notation:** For some statement $\varphi$,

$$\forall x \in A\, (\varphi)$$

is shorthand for

$$\forall x \left(x \in A \Rightarrow \varphi\right)$$

**Notation:** If $F \subseteq A \times B$ and $\forall x \in A, (x, y) \in F$, then we write $F : A \to B$.

Also, $\forall (x, y) \in F$, we write $F(x) = y$.

[v] Hey, one of the topics for my Honors thesis is on this.

**Definition.** A function $F$ is onto $B$ if

$$\forall y \in B\, \exists x\, (x, y) \in F.$$

**Remark:** Do not say "onto" without mentioning $B$. It is okay to say $F : A \to B$ is onto (or surjective).

**Example.** We wish to show that if $f : A \xrightarrow{\text{onto}} B$, then there exists a function $g : B \to A$ such that $g$ is an injection.

Since $f$ is onto $B$, for every $b \in B$, there exists $a \in A$ such that $f(a) = b$. We define $g(b)$ to be a particular choice function on the set of all $a$ such that $f(a) = b$.

**Remark:** The above statement (that every surjective function has a right-inverse, which is necessarily injective) is an equivalent statement to the axiom of choice.

**Example** (Natural Numbers)**.** Since the empty set exists, we can define $\varnothing = \{\} = 0$. We set $1 = \{0\}$, $2 = \{0, 1\}$, etc. We have $n = \{0, \dots, n-1\}$.

If we take $n \cup \{n\}$, we have

$$\begin{aligned} \{0, \dots, n-1\} \cup \{n\} &= \{0, \dots, n\} \\ &= n + 1. \end{aligned}$$

In other words, we define addition by taking $n \cup \{n\}$.

**Question:** Is $n \in n + 1$? Is $n \subseteq n + 1$?

**Answer:** **Yes.** and **yes**.

**Definition.** We say $m < n$ if $m \in n$, or $m \subseteq n$.

**Example.** We will use the ZF axioms to show that there exists a set whose elements are all the natural numbers.

Defining using the axiom of infinity, we get

$$\exists s\ (\varnothing \in s \wedge \forall x\, (x \in s \Rightarrow x \cup \{x\} \in s) \wedge \forall y\, (y \in s \Rightarrow y = \varnothing \vee \exists x\, (x \cup \{x\} = y)))$$

## Ordinal Numbers and Well-Orderings

**Recall:** Recall that we define $\varnothing = 0$, $1 = 0 \cup \{0\}$, and $n + 1 = n \cup \{n\}$.

Notice that $n \in n + 1$, meaning $0 \in 1 \in 2 \in \cdots$, and $n \subseteq n + 1$, meaning $0 \subseteq 1 \subseteq 2 \subseteq \cdots$.

**Notation:** For any set $x$, $x^+ = x \cup \{x\}$. We call $x^+$ the successor of $x$.

**Recall:** The infinity axiom states that

$$\exists A\, (\varnothing \in A \wedge \forall x\, (x \in A \Rightarrow x \cup \{x\} \in A)).$$

One of our previous homework problems showed that there exists a set that contains all natural numbers and only natural numbers.

$$\exists \omega \forall x\, (x \in \omega \Leftrightarrow x \in A \wedge (x = \varnothing \vee \exists y\, (y \in \omega \wedge x = y^+)))$$

**Definition** (Natural Numbers)**.** For $\omega$ defined by

$$\exists \omega \forall x\, \left(x \in \omega \Leftrightarrow x \in A \wedge \left(x = \varnothing \vee \exists y\, \left(y \in \omega \wedge x = y^+\right)\right)\right),$$

we say $\omega$ is the set of all natural numbers.

**Remark:** Given a relation $R$, we write $(x, y) \in R$ if $x R y$.

**Definition** (Total/Linear Order)**.** Given a set $A$, a (strict) total/linear order is a relation $R$ such that $\forall x, y \in A$, then exactly one of the following holds:

$$xRy \vee yRx \vee x = y.$$

Additionally, $\forall x, y, z \in A$, $xRy \wedge yRz \Rightarrow xRz$, meaning $R$ is transitive.

**Remark:** This is a strict inequality.

**Notation:** For a total ordering $R$, we use the symbol $<$. This does not imply that a given ordering is a "less than" type of ordering.

**Example.** The relation $x < y$ is a total ordering on $\mathbb{Q}$ (or $\mathbb{R}$).

**Definition** (Well-Ordering)**.** A well-ordering on $A$ is a total ordering $R$ on $A$ such that every nonempty subset of $A$ has a least element.

$$\forall S \left(S \subseteq A \wedge S \neq \emptyset \Rightarrow \exists x \in S \forall y \in S \left(x < y \vee x = y\right)\right)$$

**Question:** Is $\mathbb{Q}$ well-ordered by $<$?

**Answer:** No.

Consider the set $\left\{q \mid q > \sqrt{2}\right\}$. Since $\sqrt{2} \notin \mathbb{Q}$,[VI], this set has no least element, meaning $\mathbb{Q}$ is not well-ordered.

**Definition.** Let $R_1$ be a relation on $A_1$, and $R_2$ a relation on $A_2$.

We say $(A_1, R_1)$ is order-isomorphic to $(A_2, R_2)$ if

$$\exists f : A_1 \xrightarrow{\text{bijection}} A_2$$

and $\forall x, y \in A_1$, $xR_1y \Leftrightarrow f(x)R_2f(y)$.

**Remark:** If $R_1$ and $R_2$ are understood, we say $A_1$ is order-isomorphic to $A_2$, and we write $A_1 \cong A_2$.

**Example.** If $\omega = \{1, 2, \dots\}$, $R_1 = R_2 = <$, then if $A = \{0, 2, 4, \dots\}$, $\omega \cong A$.

**Question:** Is $\in$ a total order on $\omega^+ = \omega \cup \{\omega\}$?

**Answer: Yes.**

Notice that

$$\begin{aligned} \omega^+ &= \{0, 1, 2, \dots, \omega\} \\ &= \{0, 1, 2, \dots, \{0, 1, 2, \dots\}\}. \end{aligned}$$

This is also a well-ordering.

**Example.** Consider, now

$$\begin{aligned} Y &= \left(\omega^+\right)^+ \\ &= \omega^+ \cup \left\{\omega^+\right\} \\ &= \left\{0, 1, \dots, \omega, \omega^+\right\}. \end{aligned}$$

**Question:** Is $\in$ a total ordering on $Y$?

**Answer: Yes.**

**Question:** Is $\in$ a well-ordering on $Y$?

**Answer: Yes.**

**Question:** Is $(\omega, \in) \cong (\omega^+ \in)$.

[VI] I am not proving this here.

**Answer:** If there exists $f : \omega \to \omega^+$, then $f(n) = \omega$ for some $n$. Since $f(n+1) \in \omega^+$, and $f(n) \in f(n+1)$, it is the case that $\omega \in f(n+1)$.

However, $f(n+1) \in \omega^+ \setminus \{\omega\}$, meaning $f(n+1) \in \omega = \omega$.

Thus, we have $\omega \in f(n+1) \in \omega$, which violates the axiom of regularity.

**Question:** Suppose $A, B, C$ are well-ordered by $R_A, R_B, R_C$.
**True/False:** $A \cong A$.
**True/False:** If $A \cong B$, then $B \cong A$.
**True/False:** If $A \cong B$ and $B \cong C$, then $A \cong C$.

**Answer: True** for all three.

Therefore, we can talk about $\cong$ as an equivalence relation on the ~~set~~ class of well-ordered sets.

**Example.** The following are representatives of separate equivalence classes in the class of well-ordered sets with respect to order-isomorphism.

$$\begin{aligned} \omega &= \{0, 1, 2, \dots\} \\ \underbrace{\omega^+}_{\omega+1} &= \{0, 1, 2, \dots, \omega\} \\ \omega + 2 &= \{0, 1, 2, \dots, \omega, \omega+1\}, \qquad \vdots \end{aligned}$$

Notice that these sets are all denumerable, but they are not order-isomorphic.

**Theorem:** Every such equivalence class has exactly one element that is well-ordered by $\in$ and is $\in$-transitive.

This element is called an ordinal.

**Definition.** A set $A$ is $\in$-transitive if $a \in b$ and $b \in A$ implies $a \in A$. Alternatively, every element of $a$ is a subset of $A$.

**Example.** We can see that $\omega$ is $\in$-transitive, since for any $a \in b$ and $b \in \omega$, then $a \in \omega$ (by definition of $\omega$).

**Question:** Is 3 $\in$-transitive?

**Answer: Yes.**

**Theorem:** For any two ordinals $\alpha, \beta$, either $\alpha \in \beta$, $\beta \in \alpha$, or $\beta = \alpha$.

**Recall:** An ordinal is a set that is $\in$-transitive and well-ordered by $\in$.

A set $t$ is $\in$-transitive if $a \in b$ and $b \in t$ implies $a \in t$. Equivalently, $b \in t \Rightarrow b \subseteq t$.

**Example.** The set

$$\{a < b < c\} \cong 3 = \{0, 1, 2\},$$

since $0 < 1 < 2$.

The set

$$\{a_0 < a_1 < \cdots\} \cong \omega,$$

while

$$\{a_0 < a_1 < \cdots < b_0\} \cong \omega^+ := \omega + 1 = \omega \cup \{\omega\}.$$

We can also see that

$$\begin{aligned} \{a_0 < a_1 < a_2 < \cdots < b_0 < b_1 < b_3 < \cdots\} &= \omega + \omega \\ &= \omega 2 \end{aligned}$$

**Example.** Let $S = \{p^n \mid p \text{ prime}, n \in \omega\}$.

We place the ordering

$$2^0 \prec 2^1 \prec \cdots 3^1 \prec 3^2 \prec \cdots \prec 5^1 \prec 5^2 \prec \cdots$$

In other words,

$$\begin{aligned} p_k^m &\prec p_{k+1}^n \\ p_k^m &\prec p_k^{m+1}. \end{aligned}$$

We can see that this ordering must be isomorphic to $\omega\omega$, since it must be greater than $\omega k$ for all $k \in \omega$.

**Example.** We define

$$\begin{aligned} 1 + \omega &\cong \{b_0 < a_0 < a_1 < a_2 < \cdots\} \\ &\cong \omega. \end{aligned}$$

This means $1 + \omega = \omega$, while $\omega + 1 \neq \omega$.

This is because $\omega + 1$ has a greatest element, while $\omega$ does not.

**Definition** (Addition)**.** For any ordinals $\alpha$ and $\beta$, $\alpha + \beta$ is the ordinal that is order isomorphic to the following well-ordered set.

$$S = \{0\} \times \alpha \cup \{1\} \times \beta.$$

The ordering for this set is the lexicographical ordering. We declare

$$(x, y) \prec (x', y')$$

$x \in x'$ or $x = x'$ and $y \in y'$.

**Example.**

$$\begin{aligned} 2 + 3 &= \{0, 1\} + \{0, 1, 2\} \\ S &= \{0\} \times \{0, 1\} \cup \{1\} \times \{0, 1, 2\} \\ &= \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2)\} \\ &= \{(0, 0) \prec (0, 1) \prec (1, 0) \prec (1, 1) \prec (1, 2)\} \\ &\cong \{0, 1, 2, 3, 4\} \\ &= 5 \end{aligned}$$

**Definition** (Multiplication)**.** For any ordinals $\alpha$ and $\beta$, $\alpha\beta$ is the ordinal that is order-isomorphic to the following well-ordered set

$$S = \alpha \times \beta,$$

ordered by

$$(a, b) \prec (a', b')$$

if $a \in a'$ or $a = a'$ and $b \in b'$

**Remark:** For general ordinals, addition and multiplication are *not* commutative.

For instance, $1 + \omega \neq \omega + 1$, since $1 + \omega = \omega$. However, addition and multiplication of ordinals is associative.

**Theorem:**

$$\begin{aligned}(\alpha + \beta) + \gamma &= \alpha + (\beta + \gamma)\\ (\alpha\beta)\gamma &= \alpha(\beta\gamma).\end{aligned}$$

**Remark:** We define

$$\begin{aligned}\omega^2 &:= \omega\omega,\\ \omega^3 &:= \omega\omega\omega.\end{aligned}$$

However, we may ask how to define

$$\omega^\omega.$$

**Definition** (Exponentiation)**.** For any ordinals $\alpha$ and $\beta$, we define

$$\alpha^\beta = \begin{cases} 1 & \text{if } \beta = 0 \\ \alpha^\gamma\alpha & \text{if } \beta = \gamma^+ \text{ for some } \gamma \\ \bigcup_{\gamma<\beta} \alpha^\gamma & \text{else} \end{cases}$$

**Remark:** If an ordinal $\alpha \neq 0$ and $\alpha$ has no predecessor, then $\alpha$ is known as a limit ordinal. For instance, $\omega$ is a limit ordinal.

**Example.** From this definition,

$$\omega^\omega = \bigcup_{n\in\omega} \omega^n.$$

**Remark:** Notice that $\omega^\omega$ is countable, since it is the countable union of countable sets.

**Definition.**

$$\begin{aligned}\omega^{\omega^\omega} &:= \omega^{(\omega^\omega)}\\ \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} &:= \bigcup_{n\in\omega} \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\\ &= \epsilon_0.\end{aligned}$$

**Definition.** We define

$$\omega_1 := \{\alpha \mid \alpha \text{ is an ordinal and } \alpha \text{ is countable}\}.$$

**Remark:** It can be proven that $\omega_1$ is indeed an ordinal.

Every subset of $\omega_1$ is well-ordered (or else we would violate the Axiom of Regularity).

**Theorem:** It is not the case that $\omega_1$ is countable.

## Induction and Recursion

**Definition** (Principle of Mathematical Induction)**.** Let $\phi$ be a formula such that

$$\phi(0) \wedge \forall n \in \omega\,(\phi(n) \Rightarrow \phi(n+1))$$

Then, $\forall n \in \omega,\ \phi(n)$.

Equivalently, let S be a set such that

$$0 \in S \wedge \forall n \in \omega\,(n \in S \Rightarrow n+1 \in S).$$

Then, $\omega \subseteq S$.

**Definition** (Strong Principle of Mathematical Induction)**.** Let $S$ be a set such that

$$0 \in S \wedge \forall n \in \omega \left(n \subseteq S \Rightarrow n \in S\right).$$

Then, $\omega \subseteq S$.

**Remark:** Strong induction implies weak induction, since the antecedent in strong induction is more restrictive than the antecedent in weak induction.

*Proof.* Suppose toward contradiction that $\omega \nsubseteq S$. Then, since $\omega \setminus S \subseteq \omega$ must be nonempty, and $\omega$ is well-ordered, there exists $n_0$ such that $n_0 \in \omega \setminus S$. Thus, for every $m < n_0$, $m \in S$.

Thus, $\forall m \in n_0$, $m \in S$, meaning $n_0 \subseteq S$. Thus, $n_0 \in S$, meaning $n_0 \in S \wedge n_0 \notin S$. $\perp$ □

**Remark:** The above proof shows that everything you can prove by induction, you can prove by contradiction (since induction follows from contradiction).

**Example.** Suppose $\prec$ is a well-ordering on $\mathbb{R}$.[VII] Define $x \in \mathbb{R}$ to be "good" if a certain condition is satisfied. We wish to show that $x \in \mathbb{R}$ — in particular, we cannot use either weak or strong induction.

*Proof Idea.* Suppose there exists some real number $x$ that fails the condition. Let $x_0$ the least element that fails the condition. Then, $\forall y \prec x_0$, $y$ is good. Then, we need to use some inductive step to show that such a condition implies that $x_0$ is good. □

**Example.** Suppose that for all $m, n \in \mathbb{N}$, Then, $G_{m,n}$ is some graph, group, etc.

We want to show that every $G_{m,n}$ satisfies some condition.

Suppose there is a bad $G_{a,b}$. Take the smallest such $G_{a,b}$ (via the lexicographical order), and we can use strong induction to show that such a $G_{a,b}$ also satisfies the condition.

**Example** (Transfinite Induction)**.** Suppose we want to show that for all $\alpha \in \omega 2$, $\phi(\alpha)$.

**Question:** Is the following enough?

$$\phi(0) \wedge \forall \alpha \in \omega 2 \left(\phi(\alpha) \Rightarrow \varphi\left(\alpha \cup \{\alpha\}\right)\right).$$

**Answer: No**.

The reason why the above cannot work (as a statement of induction) is because $\omega$ is a limit ordinal (i.e., $\omega$ is not a successor to any particular ordinal).

We can use contradiction.

*Proof by Contradiction.* Suppose toward contradiction that $\phi(\alpha)$ is not true for all $\alpha \in \omega 2$. Let $\alpha_0$ be the smallest ordinal in $\omega 2$ such that $\phi\left(\alpha_0\right)$ is false.

Then, for every $\alpha \in \alpha_0$, $\phi(\alpha)$. Then, we would have to conclude $\phi\left(\alpha_0\right)$, implying a contradiction. □

The above is an example of transfinite induction.

**Example** (Recursion)**.** Recall the Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

We define the Fibonacci numbers recursively:

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n+2) &= F(n+1) + F(n). \end{aligned}$$

[VII] All nonempty sets contain a well-ordering, which is another statement of the Axiom of Choice

**Question:** Which of the following are valid recursive definitions?

(a) $f : \mathbb{N} \to \mathbb{N}$, with

$$f(n) = \begin{cases} n^2 & n \text{ odd} \\ f(n/2) & n \text{ even, and} n > 0 \\ 1 & n = 0 \end{cases}.$$

(b) Let $f : [0, \infty) \to [0, \infty)$ defined by $f(0) = 1$, $f(x) = 2f(x/2)$.

(c) Let $f : \mathbb{N} \to \mathbb{N}$, $f(0) = 1$, $f(1) = 1$, and $f(n) = 2f(n-2)$ for all $n \geqslant 2$.

(d) Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(0) = 1$, and

$$f(n) = \begin{cases} 2f(n-1) & n > 0 \\ 3f(n+1) & n < 0 \end{cases}.$$

(e) Let $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by

$$A(m, n) = \begin{cases} n + 1 & m = 0 \\ A(m-1, 1) & m > 0 \\ A(m-1, A(m, n-1)) & m > 0 \;\&\; n > 0 \end{cases}$$

We can also write $A(m, n)$ as $A_m(n)$, with $A_0(n) = n + 1$, $A_{m+1}(n) = \underbrace{A_m \circ \cdots \circ A_m}_{n+1 \text{ times}}(1)$

(f) Let

$$C(n) = \begin{cases} n/2 & n \text{ even} \\ 3n + 1 & n \text{ odd}, n \neq 1 \\ 1 & n = 1 \end{cases}.$$

We define $f : \mathbb{N} \to \mathbb{N}$ by $f(0) = f(1) = 0$, and

$$f(n) = \begin{cases} f(n/2) & n \text{ even} \\ f(3n+1)+ & n \text{ odd} \end{cases}.$$

**Answer:**

(a) Since $f$ is defined for either odd elements or some smaller element, and there is a base case of $n = 0$, this should be a valid definition.

(b) This isn't a valid definition, since a recursive definition needs to reach some "stopping point."

(c) This is a valid definition, since we ultimately reach some stopping point with $n = 0$ or $n = 1$.

(d) This is a valid definition.

(e) This is a valid definition — notice that the function is always defined in terms of some value "less than" the input, and it always has a minimum value. If we know $A(a, b)$ for all $(a, b) < (m, n)$,[VIII] then we can find $(m, n)$. The function $A(m, n)$ is known as the Ackermann function.

(f) If you prove the Collatz conjecture, then this is a valid definition.

**Example** (Using Induction to show Validity of Recursion Formula)**.** Show there exists a unique $F : \mathbb{N} \to \mathbb{N}$ such that $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$.

Let $G$ be the set of all $n \in \mathbb{N}$ such that there exists a unique $g : \{0, \dots, n\} \to \mathbb{N}$ defined by $g(0) = 0$, $g(1) = 1$, and $g(k) = g(k-1) + g(k-2)$ for all $2 \leqslant k \leqslant n$.

We will show that $G = \mathbb{N}$.

[VIII] Lexicographically, meaning $(a, b) < (c, d)$ if $a < b$ or if $a = c$ and $b < d$.

Let $n_0 = \min(\mathbb{N} \setminus G)$. It must be the case $n_0 \neq 0$ and $n_0 \neq 1$. Then, there exists a unique function $g' : \{0, \dots, n_0 - 1\} \to \mathbb{N}$ such that $g'(0) = 0$, $g'(1) = 1$, and $g'(k) = g'(k-1) + g'(k-2)$ for all $2 \leqslant k \leqslant n_0 - 1$. Define $g : \{0, \dots, n_0\} \to \mathbb{N}$ by $g(n_0) = g'(n_0 - 1) + g'(n_0 - 2)$ and $g(k) = g'(k)$ for $2 \leqslant k \leqslant n_0 - 1$.

Thus, we have shown existence. Suppose $\exists f : \{0, \dots, n_0\} \to \mathbb{N}$ such that $f(0) = 0$, $f(1) = 1$, and $f(k) = f(k-1) + f(k-2)$. However, $f|_{\{0,\dots,n_0-1\}} = g'$, by uniqueness meaning for all $k < n_0$, $f(k) = g'(k)$. Thus, $f(n_0) = f(n_0 - 1) + f(n_0 - 2) = g'(n_0 - 1) + g'(n_0 - 2) = g(n_0)$.

Thus, for each $n \in \mathbb{N}$, there exists a unique $g_n$ that satisfies the given conditions. Let $F = \bigcup_{n \in \mathbb{N}} g_n$.

## Cardinal Numbers

Define a relation $\sim$ on sets by $A \sim B \Leftrightarrow |A| = |B|$.

**Question:** Is this an equivalence relation?

**Answer: Yes.** Since bijections are invertible, the identity map is a bijection, and composing bijections yields another bijection, this is an equivalence relation.

**Example.**

$$\{3, 5\} \sim \{\varnothing, \omega\} \sim \{\{\omega\}, \mathbb{R}\} \sim 2 = \{0, 1\}.$$

From this, we intuitively select 2 to be the representative of this equivalence class.

**Example.**

$$\omega \sim \omega 2 \sim \omega 3 \sim \cdots \sim \omega^2 \sim \cdots \sim \omega^{\omega^{\omega}}$$

Similarly, we select $\omega$ to be the representative of $|\omega|$.

**Definition** (Cardinality of a Set)**.** Let $A$ be a set. The cardinality of $A$ is the least ordinal $\alpha$ such that there exists a bijection $f : A \to \alpha$. This ordinal $\alpha$ is denoted $|A|$.

**Remark:** Before today, $|A|$ had no definition. We did write $|A| = |B|$, but that was shorthand for $\exists f : A \xrightarrow{\text{bijection}} B$.

**Question:** What is $\left|\omega^2\right|$?

**Answer:** $\omega$

What is $|\omega|$?

**Answer:** $\omega$

What is $|3|$?

**Answer:** 3

What is $|\mathbb{R} \times \mathbb{R}|$ and its relation to $|\mathbb{R}|$ or $|P(\omega)|$.

**Answer:** $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}| = |P(\omega)| = \omega_1$ (assuming the continuum hypothesis)

**Definition** (Cardinal Number)**.** Let $\alpha$ be an ordinal. If $|\alpha| = \alpha$, we say $\alpha$ is a cardinal number.

Every natural number is an ordinal and a cardinal.

**Notation:** When dealing with cardinals, it is customary to write $\aleph_0$ to denote $\omega$.

We wrote $|A| = |B|$ to be shorthand for $\exists f : A \xrightarrow{\text{bijection}} B$. However, now there is a new meaning, since $|A|$ is actually a set. This means that when we write $|A| = |B|$, then the ordinals referring to $|A|$ and $|B|$ are equal to each other.

We need to derive the "old meaning."

**Theorem:** $|A| = |B|$ if and only if there exists a bijection $f : A \to B$.

*Proof.* Let $\alpha = |A|$. Then, $\alpha = |B|$. By definition, there exist bijections $f : A \to \alpha$ and $g : B \to \alpha$. Composing $f \circ g^{-1} : A \to B$, we get a bijection.

Suppose there exists a bijection $f : A \to B$. Let $\alpha = |A|$. Thus, there exists a bijection $g : A \to \alpha$. So, taking $g \circ f^{-1}$, we get a bijection from $B$ to $\alpha$. We have $\alpha$ is a cardinal as $\alpha = |A|$, meaning $\alpha = |B|$. Thus, $|A| = |B|$. □

**Question:** What does $|A| < |B|$ mean?

**Answer:** Before today, $|A| < |B|$ meant there exists $f : A \hookrightarrow B$ and no bijection $g : A \to B$.

However, now, we mean $|A| < |B|$ means $|A| \in |B|$

**Theorem:** $|A| \in |B| \Leftrightarrow \exists f : A \hookrightarrow B$ and there is no bijection $g : A \to B$

*Proof.* Homework problem. □

**Definition** (Cardinal Arithmetic)**.** Let $\kappa, \lambda$ be cardinals. Then,

$$\begin{aligned} \kappa +_{\text{card}} \lambda &:= |(\kappa \times \{0\}) \cup (\lambda \times \{1\})| \\ \kappa \cdot_{\text{card}} \lambda &:= |\kappa \times \lambda| \end{aligned}$$

**Question:** Is $\kappa \cdot_{\text{card}} \lambda = \kappa \cdot_{\text{ord}} \lambda$?

**Remark:** If we use $\kappa$ and $\lambda$, then we are referring to cardinal operations, while if we use $\alpha$ and $\beta$, we are referring to ordinal operations.

**Theorem:** Let $\kappa$, $\lambda$, and $\mu$ be cardinals.

(i) $\kappa + \lambda = \lambda + \kappa$ and $\kappa \cdot \lambda = \lambda \cdot \kappa$;

(ii) if $\kappa \leqslant \lambda$, then $\kappa + \mu \leqslant \lambda + \mu$ and $\kappa \cdot \mu \leqslant \lambda \times \mu$.

*Proof.* Homework problem. □

**Theorem:** If $\lambda$ is an infinite cardinal, then $\lambda \cdot \lambda = \lambda$.

**Example.** In particular $\left|\mathbb{R}^2\right| = |\mathbb{R}|$, since

$$\begin{aligned} \left|\mathbb{R}^2\right| &= |\mathbb{R} \times \mathbb{R}| \\ &= |\mathbb{R}| \cdot |\mathbb{R}| \\ &= |\mathbb{R}|. \end{aligned}$$

**Question:** Is $|\omega| + |\mathbb{R}| \geqslant |\mathbb{R}|$?

**Answer:** No.

**Corollary:** If $\lambda$ is an infinite cardinal, and $0 \neq \kappa \leqslant \lambda$, then $\kappa + \lambda = \lambda$, and $\kappa \cdot \lambda = \lambda$.

*Proof.*

$$\begin{aligned} \lambda &= 1 \cdot \lambda && \text{Needs proof.} \\ &\leqslant \kappa\lambda\lambda \\ &\leqslant \lambda \cdot \lambda \\ &= \lambda. \end{aligned}$$

Thus, all the inequalities are equalities, meaning $\lambda = \kappa \cdot \lambda$.

$$\lambda = 0 + \lambda$$

$$\begin{aligned}
&\leqslant \kappa + \lambda \\
&\leqslant \lambda + \lambda \\
&= |\lambda +_{\text{ord}} \lambda| \\
&= |\lambda \cdot_{\text{ord}} 2| \\
&= \lambda \cdot 2 \\
&= 2 \cdot \lambda \\
&\leqslant \lambda \cdot \lambda \\
&= \lambda.
\end{aligned}$$

□

**Example.** Let $S = \{f \mid f : 3 \to 2\}$, or $S = \{f \mid f : \{0, 1, 2\} \to \{0, 1\}\}$. Then, $S = 2 \times 2 \times 2 = 2^3$.

In general, if $A$ and $B$ are finite sets, we define $|\{f \mid f : A \to B\}| = |B|^{|A|}$.

**Definition.** Let $A$ and $B$ be arbitrary sets. Then,

$$|A|^{|B|} = |\{f \mid f : B \to A\}|$$

**Example.**

$$\begin{aligned}
2^{\aleph_0} &= |\{f \mid f : \omega \to \{0, 1\}\}| \\
&= |P(\omega)| \\
&= |\mathbb{R}| \\
&= \omega_1
\end{aligned}$$

**Theorem:**

$$\left(\kappa^{\lambda}\right)^{\mu} = \kappa^{\lambda \cdot \mu}$$

**Theorem:** If $\kappa$ is an infinite cardinal, then

$$\kappa^{\kappa} = 2^{\kappa}.$$

*Proof.*

$$\begin{aligned}
\kappa^{\kappa} &= (2^{\kappa})^{\kappa} \\
&= 2^{\kappa \cdot \kappa} \\
&= 2^{\kappa} \\
&\leqslant \kappa^{\kappa}.
\end{aligned}$$

□

## Equivalent Versions of the Axiom of Choice

**Theorem** (Traditional Statement of the Axiom of Choice)**:** If $S$ is a set, and $\forall x \in S$, $x \notin \emptyset$, then

$$\exists f : S \to \bigcup S$$

such that $\forall x \in S$, $f(x) \in x$.

We say $f$ is a choice function.

**Theorem** (Well-Ordering Theorem)**:** Every nonempty set admits a well-ordering.

**Theorem** (Zorn's Lemma)**:** In every partially ordered set $S$, if every chain has an upper bound in $S$, then $S$ contains a maximal element.

The common joke is that the axiom of choice is obviously true, the well-ordering theorem is obviously false, and Zorn's lemma is unclear.

**Definition** (Partially Ordered Set)**.** A relation $\leq$ is known as a partial order if

- $\forall x \in S\,(x \leq x)$;
- $\forall x, y \in S\,(x \leq y \wedge y \leq x \Rightarrow x = y)$;
- $\forall x, y, z \in S\,(x \leq y \wedge y \leq z \Rightarrow x \leq z)$.

A partial order may or may not be total. A total ordering includes a fourth condition:

- $\forall x, y \in S\,(x \leq y \vee y \leq x)$.

A set equipped with a partial ordering is known as a partially ordered set.

**Definition** (Chain)**.** A chain in $S$ is a subset of $S$ that is totally ordered by $\leq$.

**Definition** (Upper Bound)**.** An upper bound of a subset of $S$ is an element $u \in S$ such that $\forall x \in T\,(x \leq u)$.

**Definition** (Maximal Element)**.** An element $m \in S$ is maximal if $\forall x \in S\,(x \geq m \Rightarrow x = m)$.

**Example** (Using Zorn's Lemma)**.** We want to know if there exists an uncountable set $T$ such that

(1) $\forall A \in T$, $A \subseteq \mathbb{R}$ and $A$ is countable;

(2) $(T, \subseteq)$ is totally ordered.

The answer is yes.

*Proof of Zorn's Lemma.* Suppose $S$ does not have a maximal element. Then, every chain $C$ in $S$ has a strict upper bound; i.e., for any upper bound $b$ of $C$, $b \notin C$.

The Axiom of Choice implies that there exists $f : H = \{C \mid C \text{ is a chain in } S\} \to S$ such that $f(C)$ is a strict upper bound for $c$.

Let $\Gamma$ be an arbitrary ordinal, $\alpha \in \Gamma$. Define $g : \Gamma \to H$ recursively by

$$g(\alpha) = \begin{cases} \emptyset & \alpha = \emptyset \\ g(\beta) \cup \{f(g(3))\} & \alpha = \beta + 1 \\ \bigcup_{\beta \in \alpha} g(\beta) & \alpha \text{ is a limit ordinal} \end{cases}.$$

We must show that $g$ is injective.

If $g$ is injective, then we have $|\Gamma| \leqslant |H|$. However, since $\Gamma$ is arbitrary, we can find $\kappa$ that is a cardinal for $|H|$, but this implies that $|H| \geqslant \kappa$. □

**Theorem:** Every vector space has a basis.

*Proof.* Let $V$ be a vector space. Let $L = \left\{S \subseteq V \mid S \text{ is linearly independent}\right\}$. Then, $(L, \subseteq)$ is a partially ordered set.

Every chain $C$ in $L$ has an upper bound:

$$U = \bigcup_{A \in C} A.$$

Then, $C$ is necessarily linearly independent, as otherwise, we would have $a_1v_1 + \cdots a_nv_n = 0$ with $a_1, \dots, a_n \neq 0$, implying $v_1, \dots, v_n \in A$ for some $A \in C$, implying $A$ is linearly dependent.

Thus, by Zorn's lemma, $L$ has a maximal element, $S_{max}$. Then, $S_{max} \in L$, so $S_{max}$ is linearly independent.

Additionally, $S_{max}$ spans $V$, because if there were some $w \in V$ with $w \notin \operatorname{span}(S_{max})$, then we could take $S_{max} \cup \{w\}$, which would still be linearly independent, contradicting the maximality of $S$. $\square$

**Example.** Let $\Gamma = \{f : \mathbb{R} \to \mathbb{R}\}$, and let $\Gamma_C \left\{f : \mathbb{R} \xrightarrow{\text{continuous}} \mathbb{R}\right\}$. We want to prove that $|\Gamma_C| < |\Gamma|$.

**Lemma:** If $f, g \in \Gamma_C$ are continuous, and for every $x \in \mathbb{Q}$, $f(x) = g(x)$, then $f = g$.

*Proof.* Suppose toward contradiction that $\exists x$ with $f(x) \neq g(x)$. Then, $(f - g)(x) \neq 0$. Since $f - g$ is continuous, there is some $\delta$ such that on $(x - \delta, x + \delta)$, $f - g$ is never zero. However, since $\exists r \in \mathbb{Q}$ such that $r \in (x - \delta, x + \delta)$, this implies that $(f - g)(r) \neq 0$. $\square$

Let $\gamma_{\mathbb{Q}} = \left\{f|_{\mathbb{Q}} \mid f \in \Gamma_C\right\}$. Let $\varphi : \Gamma_C \to \Gamma_{\mathbb{Q}}$ defined by $\varphi(f) = f|_{\mathbb{Q}}$. Then, $\varphi$ is injective. Thus, $|\Gamma_C| \leqslant \left|\Gamma_{\mathbb{Q}}\right| \leqslant |\mathbb{R}|^{|\mathbb{Q}|} < |\mathbb{R}|^{|\mathbb{R}|}$ since $|\mathbb{Q}| < |\mathbb{R}|$, so $|\Gamma_C| < |\Gamma|$.