Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: We know that $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$, meaning we need show $C_0(\mathbb{R})$ is closed under the uniform norm.

Let $(f_n)_n \to f$, with $(f_n)_n \in C_0(\mathbb{R})$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then,

$$\begin{aligned}
|f(x)| &= |f(x) - f_n(x) + f_n(x)| \\
&\leq |f_n(x) - f(x)| + |f_n(x)| \\
&\leq \|f_n - f\|_u + |f_n(x)|
\end{aligned}$$

By the definition of uniform convergence, for all $n \geq N_\varepsilon$, $\|f_n - f\| < \varepsilon/2$ and by the definition of vanishing at $\pm\infty$, for all $|x| > M$, $|f_n(x)| < \varepsilon/2$. Thus,

$$< \varepsilon,$$

meaning $f(x) \in C_0(\mathbb{R})$, so $C_0(\mathbb{R})$ is closed, so it is complete.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $(x_n)_n$ be a Cauchy sequence in ℓ_2. Let $x_n(k)$ denote the index k of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$\begin{aligned}
|x_n(k) - x_m(k)| &\leq c\,\|x_m(k) - x_n(k)\|_2 \\
&\to 0 && \text{since } (x_n)_n \text{ is Cauchy in } \ell_2.
\end{aligned}$$

Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, $x_n(k) \to x(k)$ as $k \to \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \to x(k)$ for each n.

We must show that $\|x_n - x\|_2 \to 0$ as $n \to \infty$. This is equivalent to

$$\begin{aligned}
\lim_{N\to\infty} \sum_{k=1}^{N} \lim_{m\to\infty} |x_n(k) - x_m(k)|^2 &\leq \lim_{m\to\infty} \sup_{m \geq M} \|x_n - x_m\|^2 \\
&\leq \varepsilon^2 && \text{since } (x_n)_n \text{ is Cauchy.}
\end{aligned}$$

Thus, $\|x_n - x_m\| \to 0$ as $m \to \infty$ and $n \to \infty$, so $\|x_n - x\| \to 0$ as $n \to \infty$.

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0,1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq d(x_2, x_1)\left(\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}\right) \\
&= d(x_2, x_1)\theta^{n-1}\left(1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}\right) \\
&\leq d(x_2, x_1)\frac{\theta^{n-1}}{1-\theta}.
\end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.

Problem 4

Let (X,d) be a complete metric space, and suppose $f : X \to X$ is a contractive map — i.e., there is a $\theta \in (0,1)$ with

$$d(f(x), f(y)) \leq \theta d(x,y).$$

Prove that f has a unique fixed point.

Proof: Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all n, we have

$$d(x_n, x_{n-1}) \leq \theta^n d(x_1, x_0).$$

Therefore, for x_m, x_n arbitrary in X with $m > n$, we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n)\\
&\leq \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0)\\
&= d(x_1, x_0)\theta^{n+1}\left(1 + \theta + \cdots + \theta^{m-n-1}\right)\\
&\leq d(x_1, x_0)\frac{\theta^{n+1}}{1-\theta}.
\end{aligned}$$

Since $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$ as $n \to \infty$, it must be the case that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Since X is complete, this means $(x_n)_n \to x$ for some $x \in X$, meaning $f(x) = x$.

Suppose it were the case that there existed s, t distinct with $f(s) = s$ and $f(t) = t$. Then, $d(f(s), f(t)) = d(s,t) \leq \theta d(s,t)$, but $\theta < 1$, which is a contradiction. Thus, x is unique.

Problem 5

If $(f_k)_k$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, show that the map

$$\begin{aligned}
\varphi &: \ell_2 \to \mathcal{H}\\
\varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k
\end{aligned}$$

is a well-defined isometry.

Proof: Let $\xi, \eta \in \ell_2$. Then,

$$\begin{aligned}
d(\xi, \eta) &= \|\xi - \eta\|_2\\
\varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k\\
\varphi(\eta) &= \sum_{k=1}^{\infty} \eta(k) f_k\\
d(\varphi(\xi), \varphi(\eta)) &= \left(\sum_{k=1}^{\infty} \langle \xi(k) - \eta(k), f_k\rangle\right)^{1/2}\\
&= \|\xi - \eta\|_2 && \text{Parseval's Identity.}
\end{aligned}$$

Problem 6

Let $T : V \to W$ be a continuous linear map between normed spaces which is bounded below; that is, there is a $C > 0$ such that $\|T(v)\| \geq C\|v\|$ for all $v \in V$. If V is complete, show that $\mathrm{ran}(T) \subseteq W$ is a closed subspace, and that $V \cong \mathrm{ran}(T)$ are uniformly isomorphic.

Proof: Since T is bounded below, we know that $\|T\|_{\mathrm{op}} > 0$, meaning T is not the zero transformation.

Let $(v_n)_n$ be a Cauchy sequence in V. Since V is complete, $(v_n)_n \to v \in V$. Since T is continuous, we have that $(T(v_n))_n \to T(v)$. Thus, $(T(v_n))_n$ is Cauchy in W, and thus T is uniformly continuous.

It is also apparent that for any sequence $(v_n)_n \in V$, then since $(T(v_n))_n \to T(v)$, any sequence in $T(V)$ is contained in $T(V)$, so $T(V) \subseteq W$ is closed.

Since $T' : V \to \mathrm{ran}(T)$ is surjective, it is bijective, so it must be the case that $V \cong \mathrm{ran}(T)$ are uniformly isomorphic.

Problem 7

Let X and Y be metric spaces with completions $(\tilde{X}, \iota_X)$ and $(\tilde{Y}, \iota_Y)$ respectively. If $f : X \to Y$ is an isometry, show that there is a unique isometry $\tilde{f} : \tilde{X} \to \tilde{Y}$ that extends f. That is, the following diagram commutes:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] & \tilde{Y} \\
X \arrow[u, hook, "\iota_X"] \arrow[r, "f"] & Y \arrow[u, hook, "\iota_Y"']
\end{tikzcd}$$

Proof: We have that ι_X and ι_Y are isometries for X and Y into $\tilde{X}$ and $\tilde{Y}$ respectively. Consider the map $\ell : \iota_X(X) \to \tilde{Y}$, with $\iota_x(X) \mapsto \iota_Y(f(x))$. We have that since ι_Y and f are isometries, $\iota_Y \circ f$ is an isometry, hence uniformly continuous.

Thus, we can extend ℓ to $\overline{\iota(X)} = \tilde{X}$, yielding $\tilde{f} : \tilde{X} \to \tilde{Y}$. Since f is an isometry on X, and $X \subseteq \tilde{X}$ is dense, $\tilde{f}$ must also be an isometry.

Problem 8

Let V be a normed space, W a Banach space, and $U \subseteq V$ a dense linear subspace. Let $T_0 : U \to W$ be a bounded linear map. Show that there is a unique bounded linear map $T : V \to W$ that extends T_0.

Proof: Since T_0 is bounded linear, T_0 is continuous, and thus uniformly continuous. Since U is dense in V, we know that T_0 is uniquely extendible to V, so $T : V \to W$ is continuous, hence uniformly continuous.

Problem 9

Let X be a metric space. Show that the following are equivalent:

(i) Every meager set has empty interior.

(ii) The complement of a meager set is dense.

Moreover, show that these equivalent statements hold true if the metric space is complete.

Proof: Let $A = \bigcup_{i\geq 1} A_i$ be a meager subset of X. Suppose $A^\circ = \emptyset$. Then, $\overline{A^c} = \left(A^\circ\right)^c = X$, so A^c is dense in X.

Suppose $\overline{A^c} = X$. Then, $\left(A^\circ\right)^c = X$, so $A^\circ = \emptyset$.

Let $A \subseteq X$ be meager in X a complete metric space. Since X is complete, it cannot be the case that $X \subseteq \bigcup A_i = A$ by Baire's theorem. Thus, for any $Y \subseteq X$ such that $Y^\circ \neq \emptyset$, it cannot be the case that $Y \subseteq \bigcup A_i$, so $A^\circ = \emptyset$.

Problem 10

Let V be an infinite-dimensional normed space with linear basis B.

(i) If $W \subset V$ is a proper subspace, show that $W^\circ = \emptyset$.

(ii) If V is a Banach space, show that B is uncountable. You may used the fact that finite-dimensional subspaces are always closed.

Proof of (i): Let $W \subset V$ be proper. Suppose $U(x,\varepsilon) \subseteq W$ for some $x \in V$ and $\varepsilon > 0$. Then, for $v \in V$, we have that $\frac{\varepsilon}{2}\frac{v}{\|v\|} + x \in U(x,\varepsilon)$, meaning $v \in W$, so $V \subseteq W$, which is a contradiction. Thus, $W^\circ = \emptyset$.

Proof of (ii): Let $\{e_n\}_{n\geq 1}$ be a countable, linearly independent set. Let $W_1 = \text{span}\{e_1\}$, $W_2 = \text{span}\{e_1, e_2\}$, and so on. We have that each $W_n \subseteq V$ is closed (by assumption), and $W_1 \subseteq W_2 \subseteq \cdots$. Since each W_n has empty interior, it cannot be the case that $V = \bigcup W_n$ by Baire's Theorem.