Problem (Problem 1)**:** Let M be a Riemannian manifold with metric g, f a C^∞ function on M, and let X be a vector field on M. Find an expression in local coordinates for the gradient of f and the divergence of X.

Solution: Define

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}.$$

Let $p \in M$ and $U \subseteq M$ a chart for p with coordinates $(x_1, \dots, x_n)$. Then, we observe that the 1-form ω_X defined via the Riemannian metric, $\omega_X(Y) = g(X, Y)$, is given locally by

$$\begin{aligned} g(X, Y) &= \sum_{i=1}^{n} f_i \sum_{j=1}^{n} a_{ij} g_j \\ &= \sum_{i=1}^{n} f_i \sum_{j=1}^{n} a_{ij}\, dx_j \left(\sum_{k=1}^{n} g_k \frac{\partial}{\partial x_k}\right) \end{aligned}$$

whence

$$\omega_X = \sum_{i=1}^{n} f_i \left(\sum_{j=1}^{n} a_{ij}\, dx_j\right).$$

Computing the divergence, which is given by $*d*(\omega_X)$, we find that

$$\begin{aligned} *\omega_X &= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{j-1} f_i a_{ij}\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ d(*\omega_X) &= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{j-1} d\left(f_i a_{ij}\right) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial\left(f_i a_{ij}\right)}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_n \\ *(d(*\omega_X)) &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial f_i}{\partial x_j} + f_i \frac{\partial a_{ij}}{\partial x_j}. \end{aligned}$$