# Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

**Proof:** We know that $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$, meaning we need show $C_0(\mathbb{R})$ is closed under the uniform norm.

Let $(f_n)_n \to f$, with $(f_n)_n \in C_0(\mathbb{R})$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then,

$$|f(x)| = |f(x) - f_n(x) + f_n(x)|$$
$$\leq |f_n(x) - f(x)| + |f_n(x)|$$
$$\leq \|f_n - f\|_u + |f_n(x)|$$

By the definition of uniform convergence, for all $n \geq N_\varepsilon$, $\|f_n - f\| < \varepsilon/2$ and by the definition of vanishing at $\pm\infty$, for all $|x| > M$, $|f_n(x)| < \varepsilon/2$. Thus,

$$< \varepsilon,$$

meaning $f(x) \in C_0(\mathbb{R})$, so $C_0(\mathbb{R})$ is closed, so it is complete.

# Problem 2

Show that $\ell_2$ is a Hilbert space.

**Proof:** Let $(x_n)_n$ be a Cauchy sequence in $\ell_2$. Let $x_n(k)$ denote the index $k$ of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$|x_n(k) - x_m(k)| \leq c \, \|x_m(k) - x_n(k)\|_2$$
$$\to 0 \qquad\qquad \text{since } (x_n)_n \text{ is Cauchy in } \ell_2.$$

Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, $x_n(k) \to x(k)$ as $k \to \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \to x(k)$ for each $n$.

We must show that $\|x_n - x\|_2 \to 0$ as $n \to \infty$. This is equivalent to

$$\lim_{N\to\infty} \sum_{k=1}^{N} \lim_{m\to\infty} |x_n(k) - x_m(k)|^2 \leq \lim_{m\to\infty} \sup_{m\geq M} \|x_n - x_m\|^2$$
$$\leq \varepsilon^2 \qquad\qquad \text{since } (x_n)_n \text{ is Cauchy.}$$

Thus, $\|x_n - x_m\| \to 0$ as $m \to \infty$ and $n \to \infty$, so $\|x_n - x\| \to 0$ as $n \to \infty$.

# Problem 3

Suppose $(X, d)$ is a complete metric space and $(x_n)_n$ is a sequence in $X$ such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

**Proof:** We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let $m, n$ be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$
$$\leq d(x_2, x_1) \left( \theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1} \right)$$
$$= d(x_2, x_1)\theta^{n-1} \left( 1 + \theta + \theta^2 + \cdots + \theta^{p-q-1} \right)$$
$$\leq d(x_2, x_1) \frac{\theta^{n-1}}{1 - \theta}.$$

Notice that the sequence $\left( \frac{\theta^{n-1}}{1-\theta} \right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since $X$ is complete, $(x_n)_n$ is convergent.

## Problem 4

Let $(X, d)$ be a complete metric space, and suppose $f : X \to X$ is a contractive map — i.e., there is a $\theta \in (0, 1)$ with

$$d(f(x), f(y)) \le \theta d(x, y).$$

Prove that $f$ has a unique fixed point.

**Proof:** Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all $n$, we have

$$d(x_n, x_{n-1}) \le \theta^n d(x_1, x_0).$$

Therefore, for $x_m, x_n$ arbitrary in $X$ with $m > n$, we have

$$\begin{aligned}
d(x_m, x_n) &\le d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\
&\le \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0) \\
&= d(x_1, x_0) \theta^{n+1} \left( 1 + \theta + \cdots + \theta^{m-n-1} \right) \\
&\le d(x_1, x_0) \frac{\theta^{n+1}}{1 - \theta}.
\end{aligned}$$

Since $\left( \frac{\theta^{n+1}}{1-\theta} \right)_n \to 0$ in $\mathbb{R}$ as $n \to \infty$, it must be the case that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Since $X$ is complete, this means $(x_n)_n \to x$ for some $x \in X$, meaning $f(x) = x$.

Suppose it were the case that there existed $s, t$ distinct with $f(s) = s$ and $f(t) = t$. Then, $d(f(s), f(t)) = d(s, t) \le \theta d(s, t)$, but $\theta < 1$, which is a contradiction. Thus, $x$ is unique.

## Problem 5

If $(f_k)_k$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, show that the map

$$\varphi : \ell_2 \to \mathcal{H}$$
$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_k$$

is a well-defined isometry.

**Proof:** Let $\xi, \eta \in \ell_2$. Then,

$$d(\xi, \eta) = \|\xi - \eta\|_2$$
$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_k$$
$$\varphi(\eta) = \sum_{k=1}^{\infty} \eta(k) f_k$$
$$d(\varphi(\xi), \varphi(\eta)) = \left( \sum_{k=1}^{\infty} \langle \xi(k) - \eta(k), f_k \rangle \right)^{1/2}$$
$$= \|\xi - \eta\|_2 \qquad\qquad \text{Parseval's Identity.}$$

## Problem 6

Let $T : V \to W$ be a continuous linear map between normed spaces which is bounded below; that is, there is a $C > 0$ such that $\|T(v)\| \geq C \|v\|$ for all $v \in V$. If $V$ is complete, show that $\operatorname{ran}(T) \subseteq W$ is a closed subspace, and that $V \cong \operatorname{ran}(T)$ are uniformly isomorphic.

**Proof:** Since $T$ is bounded below, we know that $\|T\|_{\mathrm{op}} > 0$, meaning $T$ is not the zero transformation.

Let $(v_n)_n$ be a Cauchy sequence in $V$. Since $V$ is complete, $(v_n)_n \to v \in V$. Since $T$ is continuous, we have that $(T(v_n))_n \to T(v)$. Thus, $(T(v_n))_n$ is Cauchy in $W$, and thus $T$ is uniformly continuous.

It is also apparent that for any sequence $(v_n)_n \in V$, then since $(T(v_n))_n \to T(v)$ (as $T$ is continuous), any sequence in $T(V)$ is contained in $T(V)$, so $T(V) \subseteq W$ is closed.

Since $T' : V \to \operatorname{ran}(T)$ is surjective, it is bijective, so it must be the case that $V \cong \operatorname{ran}(T)$ are uniformly isomorphic.

## Problem 7

Let $X$ and $Y$ be metric spaces with completions $(\tilde{X}, \iota_X)$ and $(\tilde{Y}, \iota_Y)$ respectively. If $f : X \to Y$ is an isometry, show that there is a unique isometry $\tilde{f} : \tilde{X} \to \tilde{Y}$ that extends $f$. That is, the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ \tilde{f}\ } & \tilde{Y} \\
{\scriptstyle \iota_X}\big\uparrow & & \big\uparrow{\scriptstyle \iota_Y} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

**Proof:** We have that $\iota_X$ and $\iota_Y$ are isometries for $X$ and $Y$ into $\tilde{X}$ and $\tilde{Y}$ respectively. Consider the map $\ell : \iota_X(X) \to \tilde{Y}$, with $\iota_X(x) \mapsto \iota_Y(f(x))$. We have that since $\iota_Y$ and $f$ are isometries, $\iota_Y \circ f$ is an isometry, hence uniformly continuous.

Thus, we can extend $\ell$ to $\overline{\iota(X)} = \tilde{X}$, yielding $\tilde{f} : \tilde{X} \to \tilde{Y}$. Since $f$ is an isometry on $X$, and $X \subseteq \tilde{X}$ is dense, $\tilde{f}$ must also be an isometry.

## Problem 8

Let $V$ be a normed space, $W$ a Banach space, and $U \subseteq V$ a dense linear subspace. Let $T_0 : U \to W$ be a bounded linear map. Show that there is a unique bounded linear map $T : V \to W$ that extends $T_0$.

**Proof:** Since $T_0$ is bounded linear, $T_0$ is uniformly continuous. Thus, we have that $T_0$ is uniquely extendible to $V$, yielding $T : V \to W$ is continuous, and thus bounded linear.

## Problem 9

Let $X$ be a metric space. Show that the following are equivalent:

  (i) Every meager set has empty interior.

 (ii) The complement of a meager set is dense.

Moreover, show that these equivalent statements hold true if the metric space is complete.

**Proof:** Let $A = \bigcup_{i \geq 1} A_i$ be a meager subset of $X$. Suppose $A^\circ = \emptyset$. Then, $\overline{A^c} = (A^\circ)^c = X$, so $A^c$ is dense in $X$.

Suppose $\overline{A^c} = X$. Then, $(A^\circ)^c = X$, so $A^\circ = \emptyset$.

Let $A \subseteq X$ be meager in $X$ a complete metric space. Since $X$ is complete, it cannot be the case that $X \subseteq \bigcup A_i = A$ by Baire's theorem. Thus, for any $Y \subseteq X$ such that $Y^\circ \neq \emptyset$, it cannot be the case that $Y \subseteq \bigcup A_i$, so $A^\circ = \emptyset$.

# Problem 10

Let $V$ be an infinite-dimensional normed space with linear basis $B$.

(i) If $W \subset V$ is a proper subspace, show that $W^\circ = \emptyset$.

(ii) If $V$ is a Banach space, show that $B$ is uncountable. You may used the fact that finite-dimensional subspaces are always closed.

**Proof of (i):** Let $W \subset V$ be proper. Suppose $U(x, \varepsilon) \subseteq W$ for some $x \in V$ and $\varepsilon > 0$. Then, for $v \in V$, we have that $\frac{\varepsilon}{2} \frac{v}{\|v\|} + x \in U(x, \varepsilon)$, meaning $v \in W$, so $V \subseteq W$, which is a contradiction. Thus, $W^\circ = \emptyset$.

**Proof of (ii):** Let $\{e_n\}_{n \geq 1}$ be a countable, linearly independent set. Let $W_1 = \operatorname{span}\{e_1\}$, $W_2 = \operatorname{span}\{e_1, e_2\}$, and so on. We have that each $W_n \subseteq V$ is closed (by assumption), and $W_1 \subseteq W_2 \subseteq \cdots$. Since each $W_n$ has empty interior, it cannot be the case that $V = \bigcup W_n$ by Baire's Theorem.