Déjà Vu

Consider the game in which the following stage game is repeated twice.

	L	C	R
T	$8, 8$	$0, 0$	$16, 0$
M	$0, 0$	$4, 4$	$16, -1$
B	$0, 16$	$-1, 16$	$12, 12$

For each of the action profiles below, construct a SPE in which the action profile is played in the first stage of the game, or show that no such SPE exists.

(a) (B, L)

(b) (B, C)

In order to deduce the potential action profiles, we start by finding the Nash equilibria of the game.

- R is strictly dominated by $\frac{1}{2}L + \frac{1}{2}C$.

	L	C	~~R~~
T	$8, 8$	$0, 0$	~~$16, 0$~~
M	$0, 0$	$4, 4$	~~$16, -1$~~
B	$0, 16$	$-1, 16$	~~$12, 12$~~

- Now, B is strictly dominated by $\frac{1}{2}T + \frac{1}{2}M$.

	L	C	~~R~~
T	$8, 8$	$0, 0$	~~$16, 0$~~
M	$0, 0$	$4, 4$	~~$16, -1$~~
~~B~~	~~$0, 16$~~	~~$-1, 16$~~	~~$12, 12$~~

We can see that there are three Nash equilibria:

- (T, L), with payoffs $(8, 8)$
- (M, C), with payoffs $(4, 4)$
- $\left(\frac{1}{3}T + \frac{2}{3}M, \frac{1}{3}L + \frac{2}{3}C\right)$, with payoffs $\left(\frac{16}{3}, \frac{16}{3}\right)$

Thus, we would expect that conceivable SPE play as follows:

- In stage 1, if both players cooperate, the second stage plays the (T, L) Nash equilibrium.
- In stage 1, if a player defects, the second stage plays the (M, C) Nash equilibrium.

(a) If we were to construct a SPE with (B, L) in the first stage, we have the following:

  - The incentive for Player 1 to deviate to playing T yields a net payoff increase of 8
  - The maximum possible punishment is a net loss of 4 — therefore, Player 1 has an incentive to deviate.

(b) If we were to construct a SPE with (B, C) in the first stage, we have the following:

  - The incentive for Player 1 to deviate to playing M yields a net payoff increase of 5.
  - The maximum possible punishment is a net loss of 4 — therefore, Player 1 has an incentive to deviate.

Therefore, both of the proposed strategies cannot be constructed into SPE.

Two Seagulls

Seagulls love shellfish. In order to break the shell, they need to fly high up and drop the shellfish. However, the other seagulls will steal the shellfish from the seagull that dropped it. Consider the case of two seagulls, Nina (player 1) and Irina (player 2). The seagulls have two options: Up or Down. There is a cost of 10 to going up and the value of eating the shellfish is 20. Nina and Irina repeat this game infinitely many times, with a common discount factor of δ. In particular, the stage game is given by the following payoff matrix:

	Up	Down
Up	$-10, -10$	$-10, 20$
Down	$20, -10$	$0, 0$

For each strategy profile below, find the range of discount factors (if any) such that the strategy profile is a subgame perfect equilibrium.

(a) (No Punishment) In the first stage Nina plays Up and Irina plays Down. They alternate Up and Down each day thereafter, irrespective of the history. A deviation by some player in some period does not change the prescription of play.

(b) (Grim Trigger) In the first stage Nina plays Up and Irina plays Down. They alternate Up and Down each day thereafter. If some player ever fails to follow this scheme, then both birds switch to playing Down forever.

(a)

Current period, Player i plays down:

Follow:

$$\begin{aligned} v_i &= (1-\delta)\left(-10 - 10\delta^2 - 10\delta^4 - \cdots + 20\delta + 20\delta^3 + \cdots\right) \\ &= (1-\delta)\left(\frac{20\delta - 10}{1-\delta^2}\right) \\ &= \frac{20\delta - 10}{1+\delta} \end{aligned}$$

Deviate in current period:

$$\begin{aligned} v_i &= (1-\delta)\left(0 - 10\delta^2 - 10\delta^4 - \cdots + 20\delta + 20\delta^3 + \cdots\right) \\ &= (1-\delta)\left(\frac{20\delta - 10\delta^2}{1-\delta^2}\right) \\ &= \frac{20\delta - 10\delta^2}{1+\delta} \end{aligned}$$

SPE Condition:

$$\begin{aligned} \frac{20\delta - 10}{1+\delta} &\geq \frac{20\delta - 10\delta^2}{1+\delta} \\ 20\delta - 10 &\geq 20\delta - 10\delta^2 \\ \delta^2 &\geq 1 \\ \delta &\geq 1 \end{aligned}$$

However, since $\delta < 1$ necessarily by the One Stage Deviation Property, the given strategy is not a SPE.

(b)

Current period, Player i plays down:

Follow:

$$\begin{aligned} v_i &= (1-\delta)\left(-10 - 10\delta^2 - 10\delta^4 - \cdots + 20\delta + 20\delta^3 + \cdots\right) \\ &= (1-\delta)\left(\frac{20\delta - 10}{1-\delta^2}\right) \\ &= \frac{20\delta - 10}{1+\delta} \end{aligned}$$

Deviate in current period:

$$\begin{aligned} v_i &= (1-\delta)(0) \\ &= 0 \end{aligned}$$

SPE Condition:

$$\begin{aligned} \frac{20\delta - 10}{1+\delta} &\geq 0 \\ \delta &\geq \frac{1}{2} \end{aligned}$$

Cournot Competition Collusion

Suppose that two firms are engaged in an infinitely repeated Cournot competition game with market demand in each period given by $P(Q) = 1 - Q$, where $Q = q_1 + q_2$. Assume that each firm has zero costs. Show that by using a grim trigger strategy of permanent reversion to the static Cournot equilibrium ($q_1 = q_2 = 1/3$), firms can sustain full collusion ($q_1 = q_2 = 1/4$) as a subgame perfect equilibrium for δ sufficiently high. Find the range of δ such that this is true.

$$v_i = q_i(1 - q_i - q_{-i})$$

Best response of firm i:

$$\begin{aligned} \frac{\partial v_i}{\partial q_i} &= 0 \\ 1 - 2q_i - q_{-i} &= 0 \\ q_i &= \frac{1 - q_{-i}}{2} \end{aligned}$$

Best response to attempted collusion:

$$\begin{aligned} q_i &= \frac{1 - \frac{1}{4}}{2} \\ &= \frac{3}{8} \end{aligned}$$

Payoff from best response to attempted collusion:

$$\begin{aligned} v_i &= \frac{3}{8}\left(1 - \frac{3}{8} - \frac{1}{4}\right) \\ &= \frac{9}{64} \end{aligned}$$

Payoff in Nash equilibrium of stage game:

$$\begin{aligned} v_i &= \frac{1}{3}\left(1 - \frac{1}{3} - \frac{1}{3}\right) \\ &= \frac{1}{9} \end{aligned}$$

Discounted average payoff from full collusion with grim trigger:

$$\begin{aligned} v_i^* &= (1-\delta)\left(\frac{1}{8} + \frac{1}{8}\delta + \frac{1}{8}\delta^2 + \cdots\right) \\ &= (1-\delta)\frac{1}{8(1-\delta)} \\ &= \frac{1}{8} \end{aligned}$$

Discounted average payoff from deviating in Period 1 with grim trigger:

$$\begin{aligned} v_i' &= (1-\delta)\left(\frac{9}{64} + \underbrace{\frac{1}{9}\delta + \frac{1}{9}\delta^2 + \cdots}_{\substack{\text{discounted future payoff}\\ \text{from grim trigger}}}\right) \\ &= \frac{9}{64} - \frac{9}{64}\delta + (1-\delta)\frac{\delta}{9(1-\delta)} \\ &= \frac{9}{64} - \frac{9}{64}\delta + \frac{\delta}{9} \end{aligned}$$

One Stage Deviation Condition:

$$\begin{aligned} \frac{1}{8} &\geq \frac{9}{64} - \frac{9}{64}\delta + \frac{\delta}{9} \\ \frac{17}{576}\delta &\geq \frac{1}{64} \\ \delta &\geq \frac{9}{17} \end{aligned}$$

Not-So-Grim Trigger

Consider the infinitely repeated Prisoner's Dilemma with discount factor $\delta < 1$ described by the following matrix:

	m	f
M	$4, 4$	$-1, 5$
F	$5, -1$	$1, 1$

Instead of using grim-trigger strategies to support a pair of actions (a_1, a_2) other than (F, f) as a subgame perfect equilibrium, assume that the players wish to choose a less draconian punishment called a "length-T punishment" strategy. If there is a deviation from (a_1, a_2) then the players will play (F, f) for T periods and then resume playing (a_1, a_2). Let δ_T be the critical discount factor so that if $\delta > \delta_T$ then the adequately defined strategies will implement the desired path of play with length-T punishment as the threat.

(a)

Let $T = 1$. What is the critical value δ_1 to support the pair of actions (M, m) played in every period?

Follow:

$$\begin{aligned} v_i &= (1-\delta_1)\left(4 + 4\delta_1 + 4\delta_1^2 + 4\delta_1^3 + \cdots\right) \\ &= 4 \end{aligned}$$

Deviate in period 1, punishment in period 2, then return:

$$\begin{aligned} v_i &= (1-\delta_1)\left(5 + \delta_1 + 4\delta_1^2 + 4\delta_1^3 + \cdots\right) \\ &= 5(1-\delta_1) + \delta_1(1-\delta_1) + 4\delta_1^2 \end{aligned}$$

Nash equilibrium condition:

$$\begin{aligned} 4 &= 5 - 5\delta_1 + \delta_1 - \delta_1^2 + 4\delta_1^2 \\ 0 &= 1 - 4\delta_1 + 3\delta_1^2 \\ 0 &= (3\delta_1 - 1)(\delta_1 - 1) \\ \delta_1 &= \frac{1}{3}. \end{aligned}$$

(b)

Let $T = 2$. What is the critical value δ_2 to support the pair of actions (M, m) played in every period?

Follow:

$$\begin{aligned} v_i &= (1-\delta_2)\left(4 + 4\delta_2 + 4\delta_2^2 + 4\delta_2^3 + \cdots\right) \\ &= 4 \end{aligned}$$

Deviate in period 1, punishment in period 2 and 3, then return:

$$\begin{aligned} v_i &= (1-\delta_2)\left(5 + \delta_2 + \delta_2^2 + 4\delta_2^3 + \cdots\right) \\ &= 5(1-\delta_2) + \frac{\delta_2(1-\delta_2^3)(1-\delta_2)}{1-\delta_2} + 4\delta_2^3 \\ &= 5 - 4\delta_2 + 4\delta_2^3 - \delta_2^4 \end{aligned}$$

Nash equilibrium condition:

$$\begin{aligned} 4 &= 5 - 4\delta_2 + 4\delta_2^3 - \delta_2^4 \\ 0 &= 1 - 4\delta_2 + 4\delta_2^3 - \delta_2^4 \\ \delta_2 &= 2 - \sqrt{3} && \text{found using Wolfram|Alpha} \\ &\approx 0.267. \end{aligned}$$

(c)

The critical values in (a) and (b) differ because the longer a punishment drags on, the less patient the players need to be for it to “bite” (i.e., affect their average payoff).