

Section 4.1

Solution (Problem 4): Evaluating with the initial conditions, we get

$$\begin{aligned}c_1 - c_2 &= 0 \\ -c_3 &= 2 \\ c_2 &= -1.\end{aligned}$$

We see that $c_1 = -1$, $c_2 = -1$, and $c_3 = -2$. This yields the particular solution of

$$y = -1 - \cos x - 2 \sin x.$$

Solution (Problem 10): The interval $(-\pi, \pi)$ contains a unique solution to the initial value problem.

Solution (Problem 14):

(a) We have

$$\begin{aligned}c_1 + c_2 + 3 &= 0 \\ c_1 + c_2 + 3 &= 4,\end{aligned}$$

which is not possible.

(b) We have

$$\begin{aligned}3 &= 0 \\ c_1 + c_2 + 3 &= 2,\end{aligned}$$

which is yet again not possible.

(c) We have

$$\begin{aligned}3 &= 3 \\ c_1 + c_2 + 3 &= 0,\end{aligned}$$

meaning that the solution set is all pairs (c_1, c_2) such that $c_1 + c_2 = -3$.

(d) We have

$$\begin{aligned}c_1 + c_2 + 3 &= 3 \\ 4c_1 + 16c_2 + 3 &= 15,\end{aligned}$$

or

$$\begin{aligned}c_1 + c_2 &= 0 \\ 4c_1 + 16c_2 &= 12\end{aligned}$$

meaning

$$\begin{aligned}c_1 &= -1 \\ c_2 &= 1.\end{aligned}$$

Solution (Problem 22): Since

$$\sinh(x) = \frac{1}{2}(e^x + e^{-x}),$$

the functions are not linearly independent anywhere on $(-\infty, \infty)$.

Solution (Problem 28): First, we verify that both solutions work.

$$\begin{aligned}x^2 \frac{d^2}{dx^2}(\cos(\ln(x))) + x \frac{d}{dx}(\cos(\ln(x))) + \cos(\ln(x)) &= x^2 \left(-\frac{\cos(\ln(x))}{x^2} + \frac{\sin(\ln(x))}{x^2} \right) + x \left(-\frac{\sin(\ln(x))}{x} \right) + \cos(\ln(x)) \\ &= 0\end{aligned}$$

$$x^2 \frac{d^2}{dx^2}(\sin(\ln(x))) + x \frac{d}{dx}(\sin(\ln(x))) + \sin(\ln(x)) = x^2 \left(-\frac{\cos(\ln(x))}{x^2} - \frac{\sin(\ln(x))}{x^2} \right) + x \left(\frac{\cos(\ln(x))}{x} \right) + \sin(\ln(x)) = 0.$$

Additionally, we find that

$$\det \begin{pmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{pmatrix} = \frac{1}{x} \neq 0,$$

so the solutions are linearly independent. Since the differential equation $x^2 y'' + xy' + y = 0$ is a second order equation, there are no other linearly independent solutions. Thus, we have the general solution of

$$y = \alpha \cos(\ln(x)) + \beta \sin(\ln(x)).$$

Solution (Problem 30): I'm not checking the Wronskian on this one, they're clearly linearly independent. However, I will be doing the derivatives.

$$\begin{aligned} \frac{d^4}{dx^4}(1) + \frac{d^2}{dx^2}(1) &= 0 \\ \frac{d^4}{dx^4}(x) + \frac{d^2}{dx^2}(x) &= 0 \\ \frac{d^4}{dx^4}(\cos(x)) + \frac{d^2}{dx^2}(\cos(x)) &= \cos(x) - \cos(x) = 0 \\ \frac{d^4}{dx^4}(\sin(x)) + \frac{d^2}{dx^2}(\sin(x)) &= \sin(x) - \sin(x) = 0. \end{aligned}$$

Thus, since the solutions are linearly independent and have dimension 4, they form a basis for the general solution of $y^{(4)} + y'' = 0$. The general solution is

$$y(x) = c_1 + c_2 x + c_3 \cos(x) + c_4 \sin(x).$$

Solution (Problem 36):

- (a) We have $y = 5$ is a particular solution to $y'' + 2y = 10$.
- (b) We have $y = -2x$ is a particular solution to $y'' + 2y = 10$.
- (c) Using linearity, we get that $y = -2x + 5$ is a particular solution to $y'' + 2y = -4x + 10$.
- (d) Using a similar process, we have a particular solution of $y = 4x + \frac{5}{2}$.
- (e) Neither of these linear combinations are general solutions of the differential equation, as the linearity principle only applies to solutions of the corresponding homogeneous equation.

Section 4.2

Solution (Problem 2): Using the power of inspection, we find that our other solution is $y_2 = e^{-2x}$.

Solution (Problem 8):

$$\begin{aligned} y_2(x) &= e^{x/3} \int \frac{e^{-x/6}}{e^{2x/3}} dx \\ &= -\frac{6}{5} e^{-x/2} \end{aligned}$$

Solution (Problem 16):

$$y_2(x) = \int e^{-\int \frac{2x}{1-x^2} dx} dx$$

$$= \frac{1}{3}x^2 - x.$$

Solution (Problem 20): Using the power of inspection, the other homogeneous solution is $y_2(x) = e^{3x}$. Letting $y_p(x) = v(x)e^x$, we get

$$\begin{aligned}(v'' - 2v')e^x &= x \\ v'' - 2v' &= xe^{-x}\end{aligned}$$

$$\frac{d}{dx}(v'(x)) - 2v'(x) = xe^{-x}.$$

Using the integrating factor e^{-2x} , we have

$$\begin{aligned}\frac{d}{dx}(e^{-2x}v'(x)) &= xe^{-x} \\ e^{-2x}v'(x) &= \int xe^{-x} dx \\ &= -xe^{-x} - e^{-x} \\ v(x) &= -xe^x.\end{aligned}$$

Thus, we have the general solution of

$$y(x) = c_1 e^x + c_2 e^{3x} + c_3 x e^{2x}.$$

Solution (Problem 22):

Section 4.3

Solution (Problem 4):

Solution (Problem 6):

Solution (Problem 12):

Solution (Problem 16):

Solution (Problem 22):

Solution (Problem 36):

Solution (Problem 38):

Solution (Problem 50):