Math 395: Homework 7
Due: November 14, 2024
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Problem 16

Problem: Use the definition to compute the determinant of a 3×3 matrix over a field F. Check that your result agrees with the familiar definition of the determinant of a matrix.

Solution: Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be the standard basis. Let $T \in \operatorname{Hom}_F\left(F^3, F^3\right)$ be defined by the following set of maps

$$\begin{aligned} e_1 &\mapsto ae_1 + de_2 + ge_3 \\ e_2 &\mapsto be_1 + ee_2 + he_3 \\ e_3 &\mapsto ce_1 + fe_2 + ie_3. \end{aligned}$$

The matrix for this linear transformation is

$$[T]_{\mathcal{E}_3} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We apply the definition of the determinant to find

$$\begin{aligned} \Lambda^3(T)(e_1 \wedge e_2 \wedge e_3) &= T(e_1) \wedge T(e_2) \wedge T(e_3) \\ &= (ae_1 + de_2 + ge_3) \wedge (be_1 + ee_2 + he_3) \wedge (ce_1 + fe_2 + ie_3) \\ &= ae_1 \wedge ((ee_2 + he_3) \wedge (fe_2 + ie_3)) + de_2 \wedge (((be_1 + he_3)) \wedge (ce_1 + ie_3)) \\ &+ ge_3 \wedge ((be_1 + ee_2) \wedge (ce_1 + fe_2)) \\ \\ &= ae_1 \wedge ((ei - hf)(e_2 \wedge e_3)) + de_2 \wedge (bi - ch)(e_1 \wedge e_3) + ge_3 \wedge (bf - ce)(e_1 \wedge e_2) \\ &= \underbrace{(a(ei - hf) - d(bi - ch) + g(bf - ce))}_{\det(T)}(e_1 \wedge e_2 \wedge e_3). \end{aligned}$$

Taking the cofactor expansion of $[T]_{\mathcal{E}}$ along the first column, we get

$$\begin{aligned} \operatorname{Det}([T]_{\mathcal{E}}) &= a \operatorname{Det}\begin{pmatrix} e & f \\ h & i \end{pmatrix} - d \operatorname{Det}\begin{pmatrix} b & c \\ h & i \end{pmatrix} + g \operatorname{Det}\begin{pmatrix} b & c \\ e & f \end{pmatrix} \\ &= a(ei - hf) - d(bi - ch) + g(bf - ce). \end{aligned}$$

Thus, the cofactor expansion and the definition of the determinant are equal to each other.

Problem 17

Problem: Let $v_1, \dots, v_k \in V$. Prove that $v_1 \wedge \cdots \wedge v_k = 0_{\Lambda^k(V)}$ if $v_1, \dots, v_k$ are linearly dependent.

Solution: Without loss of generality, let $v_1 = \sum_{i=2}^{k} a_i v_i$ for some $a_i \in F$. Then,

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &= \left(\sum_{i=2}^{k} a_i v_i\right) \wedge v_2 \wedge \cdots \wedge v_k \\ &= \sum_{i=2}^{k} a_i \left(v_i \wedge v_2 \wedge \cdots \wedge v_k\right) \\ &= 0_{\Lambda^k(V)} \qquad (*) \end{aligned}$$

To recover $(*)$, we used the fact that $v_i \wedge v_i = 0$ for any v_i.

Problem 20

Problem: Use the definition from this chapter to prove that if $A \in GL_n(F)$, then $\det\left(A^{-1}\right) = \det(A)^{-1}$, without using the fact that $\det(AB) = \det(A)\det(B)$.

Solution: Let T_A be the transformation corresponding to $A \in GL_n(F)$. Let $\mathcal{E}_n = \{e_1, \ldots, e_n\}$ be the standard basis for F^n, and let $\mathcal{C}_n = \{v_1, \ldots, v_n\}$ be a basis for F^n defined by $v_i = T_A(e_i)$. It is the case that $\mathcal{C}_n$ exists, as T_A is a bijective linear transformation.

We can thus see that

$$\begin{aligned} \Lambda^n\left(T_A^{-1}\right)(e_1 \wedge \cdots \wedge e_n) &= \left(\frac{1}{\det(T_A)}\right)(\det(T_A))\,\Lambda^n\left(T_A^{-1}\right)(e_1 \wedge \cdots \wedge e_n) \\ &= \frac{1}{\det(T_A)}\Lambda^n\left(T_A^{-1}\right)\left(\det(T_A)(e_1 \wedge \cdots \wedge e_n)\right) \\ &= \frac{1}{\det(T_A)}\Lambda^n\left(T_A^{-1}\right) \circ \Lambda^n(T_A)(e_1 \wedge \cdots \wedge e_n) \\ &= \frac{1}{\det(T_A)}\Lambda^n\left(T_A^{-1}\right)\left(T_A(e_1) \wedge \cdots \wedge T_A(e_n)\right) \\ &= \frac{1}{\det(T_A)}\Lambda^n\left(T_A^{-1}\right)(v_1 \wedge \cdots \wedge v_n) \\ &= \frac{1}{\det(T_A)}\left(T_A^{-1}(v_1) \wedge \cdots \wedge T_A^{-1}(v_n)\right) \\ &= \frac{1}{\det(T_A)}(e_1 \wedge \cdots \wedge e_n). \end{aligned}$$

Thus, it is the case that $\det\left(T_A^{-1}\right) = (\det(T_A))^{-1}$, so $\det\left(A^{-1}\right) = (\det(A))^{-1}$.

Exercise

Problem: Let $B \in \mathrm{Mat}_n(F)$. Define φ on $V = F^n$ by taking

$$\varphi(v, w) = (Bv) \cdot w.$$

Show $\varphi \in \mathrm{Hom}_F(F^n, F^n; F)$. What is the relationship between φ_B and φ.

Solution: To see that $\varphi \in \operatorname{Hom}_F(F^n, F^n; F)$, we let $v, v_1, v_2, w, w_1, w_2 \in F^n$, and let $\alpha \in F$. Then,

$$\begin{aligned}
\varphi(v, w_1 + \alpha w_2) &= (Bv) \cdot (w_1 + \alpha w_2) \\
&= (Bv) \cdot w_1 + (Bv) \cdot (\alpha w_2) \\
&= (Bv) \cdot w_1 + \alpha (Bv) \cdot w_2 \\
&= \varphi(v, w_1) + \alpha \varphi(v, w_2)
\end{aligned}$$

$$\begin{aligned}
\varphi(v_1 + \alpha v_2, w) &= (B(v_1 + \alpha v_2)) \cdot w \\
&= (Bv_1 + B(\alpha v_2)) \cdot w \\
&= (Bv_1 + \alpha B v_2) \cdot w \\
&= (Bv_1) \cdot w + \alpha (Bv_2) \cdot w \\
&= \varphi(v_1, w) + \alpha \varphi(v_2, w).
\end{aligned}$$

Thus, we can see that φ is bilinear.

To see how φ relates to

$$\varphi_B(v, w) = v^T B w,$$

we observe that for $v, w \in F^n$,

$$v \cdot w = w^T v.$$

Thus, we see that

$$\begin{aligned}
\varphi(v, w) &= (Bv) \cdot w \\
&= w^T B v \\
&= \varphi_B(w, v).
\end{aligned}$$