This is a set of notes I am taking for my Differential Topology class. The references occasionally used include

- *Geometry of Differential Forms* by Morita;
- *Topology from the Differentiable Viewpoint* by Milnor;
- *Differential Topology* by Hirsch
- *Introduction to Smooth Manifolds* by Lee;
- *A Short Course in Differential Topology* by Dundas.

Contents

Basic Properties

Definition: A topological space M is called a *manifold* if it satisfies the following:

- M is Hausdorff (points can be separated by open sets);
- M is second countable (the basis for the topology of M is countable);
- M is locally Euclidean (every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$ for some n).

In particular, the third condition says that for every $p \in M$, there is $U \in \mathcal{O}_p$ and a homeomorphism $\varphi\colon U \to \mathbb{R}^n$. The value of n is called the *dimension* of the manifold M.

Definition: Let M be an n-manifold. A *chart* on M is a pair (U, φ) such that $U \subseteq M$ is open, $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism.

A family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ is known as an *atlas* if

$$M = \bigcup_{i\in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point $p \in M$ and two charts (U, φ_U) and (V, φ_V) such that $p \in U$ and $p \in V$. The functions $\varphi_U\colon U \to \mathbb{R}^n$ and $\varphi_V\colon V \to \mathbb{R}^n$ are homeomorphism, meaning that $\varphi_V \circ \varphi_U^{-1}\colon \varphi_U(U\cap V)^n \to \mathbb{R}^n$ defined on the (nonempty) $U \cap V$ is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs $\varphi_V \circ \varphi_U^{-1}$ are *diffeomorphisms*. To do this, we need to first develop the derivative in $\mathbb{R}^n$.

Definition: Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say f is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear map $L \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\frac{|f(p+h) - f(p) - Lh|}{|h|} \to 0$$

as $h \to 0$.

The *derivative* of f is the association $f \mapsto L$ for each $p \in \mathbb{R}^n$. We write $D_p f$ to denote this map. Note that we consider elements of $\mathrm{Mat}_n(\mathbb{R})$ as points in $\mathbb{R}^{n^2}$ with the standard topology on $\mathbb{R}^{n^2}$.

A function f is called a *diffeomorphism* if it is (sufficiently) continuously differentiable and has a (similarly sufficiently) continuously differentiable inverse.

Definition: If (U, φ_U) and (V, φ_V) are charts such that $U \cap V \neq \emptyset$, the function $\varphi_V \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is known as the *transition map* between φ_U and φ_V.

A smooth structure for M is an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that for all i, j, the transition maps $\varphi_j \circ \varphi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are diffeomorphisms where defined.

If $\{(U_i, \varphi_i)\}_{i\in I}$ is a *maximal* smooth atlas — i.e., any other smooth atlas that contains $\{(U_i, \varphi_i)\}_{i\in I}$ is equal to $\{(U_i, \varphi_i)\}_{i\in I}$ — then we call $\{(U_i, \varphi_i)\}_{i\in I}$ a *smooth structure* for M.

Note: From now on, we use "manifold" to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

Definition: A *submanifold* of a manifold M with dimension $n + m$ is a subset $N \subseteq M$ such that for each $p \in N$, there is a chart $\varphi\colon U \supseteq M \to \mathbb{R}^{n+m}$ with $p \in U$ and $\varphi(U \cap N) = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+m}$.

Definition: A map $f\colon M \to N$ between manifolds is called *smooth* if for any chart (U, φ_U) in M and corresponding chart (V, φ_V) in N, the map $\varphi_V \circ f \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^k$ is (sufficiently) continuously differentiable.

The function f is a *diffeomorphism* if f is a smooth bijection with smooth inverse, and we say the manifolds M and N are diffeomorphic if they admit a diffeomorphism.

Remark: If $f\colon M \to N$ is smooth, then any representation of f is smooth. To see this, if (U, φ_1) and (U, φ_2) are charts in M, with corresponding charts (V, ψ_1) and (V, ψ_2), then

$$\psi_1 \circ f \circ \varphi_1^{-1} = \left(\psi_1 \circ \psi_2^{-1}\right) \circ \left(\psi_2 \circ f \circ \varphi_2^{-1}\right) \circ \left(\varphi_2 \circ \varphi_1^{-1}\right),$$

where the transition maps $\psi_1 \circ \psi_2^{-2}$ and $\varphi_2 \circ \varphi_1^{-1}$ are smooth.

More on Smooth Maps

Generally speaking, we will refer to charts on a dimension n smooth manifold by $(U, \varphi) = (U; x_1, \dots, x_n)$, where $x_i \colon U \to \mathbb{R}$ are the coordinates of U. Additionally, if $(\mathbb{R}^n; e_1, \dots, e_n)$ are the identity chart on $\mathbb{R}^n$, and the e_i are standard coordinates on $\mathbb{R}^n$, then the coordinate maps satisfy

$$x_i = e_i \circ \varphi.$$

Definition: Let $(U; x_1, \dots, x_n)$ be a chart on a manifold M of dimension n. If $f \colon M \to \mathbb{R}$ is a C^∞ function, we define the *partial derivative* of f with respect to x_i at p to be

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial \partial e_i}(\varphi(p)).$$

In particular,

$$\frac{\partial f}{\partial x_i} \circ \varphi^{-1} = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial e_i}$$

as functions on $\phi(U)$.

Proposition: The coordinate functions $x_1, \dots, x_n$ satisfy $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Proof. For any $p \in U$, we calculate

$$\begin{aligned}
\frac{\partial x_i}{\partial x_j}(p) &= \frac{\partial\left(x_i \circ \varphi^{-1}\right)}{\partial e_j}(\varphi(p)) \\
&= \frac{\partial((e_i \circ \varphi))}{\partial e_j}(\varphi(p)) \\
&= \frac{\partial e_i}{\partial e_j}(\varphi(p)) \\
&= \delta_{ij}.
\end{aligned}$$

$\square$

Examples

There are a couple special examples of (smooth) manifolds.

(i) Open subsets of $\mathbb{R}^n$ are always manifolds.

(ii) The general linear group, $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices, viewed as a subset of $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, is a manifold. Furthermore, it is an open subset of $\mathbb{R}^{n^2}$, as considering the map $\det\colon \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ given by $A \mapsto \det(A)$, we see that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$.

(iii) The special linear group, $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$, consisting of $n \times n$ matrices with determinant 1, is also a smooth manifold. Furthermore, this manifold is a closed subset of $\mathbb{R}^{n^2}$, as it is equal to $\det^{-1}(\{1\})$.

(iv) The n-sphere, S^n, given by

$$S^n = \left\{ (x_0, \dots, x_n) \,\middle|\, \sum_{i=0}^{n} x_i^2 = 1 \right\}$$

is a manifold in $\mathbb{R}^n$. That it is a smooth manifold is quite a bit less obvious.

Now, in low dimensions, we know that $S^2 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and that the continuously differentiable transformation $z \mapsto \frac{1}{z}$ takes the neighborhood basis of ∞ to deleted neighborhoods of 0, and takes the neighborhood basis of 0 to the neighborhood basis of ∞. This is our desired smooth structure.

In the case of the general S^n, we use two stereographic projections to construct our smooth structure. The first stereographic projection is via the north pole, N_p, and maps points on $S^n \setminus \{N_p\}$ bijectively to $\mathbb{R}^n$; this is a chart that is defined everywhere on S^n except N_p. Similarly, we may use a stereographic projection originating from the south pole, S_p, so as to create another chart defined everywhere except S_p. These two stereographic projections are our desired smooth structure, as these two charts are all that is necessary to cover S^n.

(v) The real projective plane, consisting of lines through the origin in $\mathbb{R}^{n+1}$, can be expressed as

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right)/\mathbb{R}^\times.$$

We will show that this is a manifold by constructing a family of charts mapping to $\mathbb{R}^n$.

Consider a point $(r_0, \dots, r_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. If $r_0 \neq 0$, then by dividing, we may associate this point's equivalence class in $\mathbb{RP}^n$ to

$$(1, r_1/r_0, \dots, r_n/r_0) \in \{1\} \times \mathbb{R}^n,$$

so we may associate all points of the form $[(r_0, \dots, r_n)]$ with $r_0 \neq 0$ with a chart (U_0, φ_0) that maps $\mathbb{RP}^n$ to $\mathbb{R}^n$.

Similarly, we may define U_k via

$$U_k = \{[(r_0, \dots, r_n)] \mid r_k \neq 0\}$$

with corresponding chart

$$\begin{aligned} \varphi_k \colon U_k &\to \mathbb{R}^n \\ [(r_0, \dots, r_n)] &\mapsto \frac{1}{r_k}(r_0, \dots, \widehat{r_k}, \dots, r_n), \end{aligned}$$

where $\widehat{r_k}$ denotes the exclusion of the r_k coordinate. Varying k from 0 to n, we see that

$$\mathbb{RP}^n = \bigcup_{k=0}^{n} U_k,$$

the chart functions $\varphi_k \colon U_k \to \mathbb{R}^n$ are homeomorphisms (as they are just division and projections). Furthermore, the transition maps $\varphi_j \circ \varphi_i^{-1}$ are coordinate-wise rational functions defined by

$$(u_1, \dots, u_n) \mapsto \left(\frac{u_1}{u_i}, \dots, \frac{1}{u_i}, \dots, \frac{u_n}{u_i}\right),$$

where the $\frac{1}{u_i}$ is at position j.

(vi) We now turn to a very important example from algebraic geometry: the Grassmannian, $\mathrm{Gr}(k, n)$, consisting of all the k-dimensional subspaces of $\mathbb{R}^n$.

This is a $k(n-k)$-dimensional manifold; we need to understand what the smooth structure is. To do this, we let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$, and for any $V \in \mathrm{Gr}(k, n)$, we consider maps in $\mathrm{Hom}(V, V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.

Now, we see that if $W \in \mathrm{Gr}(k, n)$ is any other k-dimensional subspace, the orthogonal $P_V \colon \mathbb{R}^n \to V$ restricted to W is a linear isomorphism if and only if $W \not\subseteq V^\perp$, or that $W \cap V^\perp = \{0\}$.

We see that if W is such that $P_V|_W \colon W \to V$ is a linear isomorphism, the inverse $(P_V|_W)^{-1} \colon V \to W$ is well-defined; so, we may make a correspondence between $\operatorname{Hom}(V, V^\perp)$ and $\operatorname{Gr}(k, n)$ by noting that any such $T \in \operatorname{Hom}(V, V^\perp)$ has a corresponding graph $(v, T(v))$, so we take $v \mapsto P_V|_W^{-1}(v)$, then project onto $V^\perp$ by taking $T\left(P_{V^\perp}\left(P_V|_W^{-1}(v)\right)\right) = T(v)$. We depict it as a diagram below.

$$\begin{tikzcd}
 & & V \\
V \arrow[r, "P_V|_W^{-1}"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "T"'] & W \arrow[r, hook, "\iota"] & \mathbb{R}^n \arrow[u, "P_V"'] \arrow[d, "P_{V^\perp}"] \\
 & & V^\perp
\end{tikzcd}$$

Defining $U_V = \{W \in \operatorname{Gr}(k, n) \mid W \cap V^\perp = \{0\}\}$, we may define the chart from U_V onto $\operatorname{Hom}(V, V^\perp)$ by $\varphi_V = P_{V^\perp} \circ P_V|_W^{-1}$. The family $\{(U_V, \varphi_V) \mid V \in \operatorname{Gr}(k, n)\}$ is our smooth atlas.

Inverse and Implicit Function Theorems

In order to replace manifolds with linear maps, we need to understand smooth maps on $\mathbb{R}^n$. The most important theorems in this regard are the inverse function theorem and the implicit function theorem.

Theorem (Inverse Function Theorem)**:** Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. If $D_p f$ is invertible as a linear map, then f has a local, continuously differentiable inverse $f^{-1} \colon V \to W$, where $p \in W \subseteq U$ and $f(p) \in V \subseteq \mathbb{R}^n$.

Additionally, $D_p\left(f^{-1}\right)$ is given by the inverse of the derivative's corresponding linear map evaluated at p, $D_p\left(f^{-1}\right) = \left(D_p f\left(f^{-1}(p)\right)\right)^{-1}$.

The proof uses the contraction mapping theorem. Recall that if X is a complete metric space, and $f \colon X \to X$ is a strict uniform contraction — that is, there exists $0 \leqslant \lambda < 1$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for all $x, y \in X$ — then f has a unique fixed point.

We begin with a technical lemma.

Lemma: If $U(0, r) \subseteq V$ for some $r > 0$ where V is a normed vector space, $g \colon V \to V$ is a uniform contraction, and $f = \mathrm{id} + g$, then the following hold:

- $(1 - \lambda)\|x - y\| \leqslant \|f(x) - f(y)\|$ (in particular, f is injective);
- if $g(0) = 0$, then
$$U(0, (1 - \lambda)r) \subseteq f(U(0, r)) \subseteq U(0, (1 + \lambda)r).$$

Proof of Lemma. To see the first item, we notice that by the triangle inequality,

$$\begin{aligned}
\|x - y\| - \|f(x) - f(y)\| &\leqslant \|x - y\| - \|x - y\| + \|g(x) - g(y)\| \\
&\leqslant \lambda\|x - y\|,
\end{aligned}$$

so $(1 - \lambda)\|x - y\| \leqslant \|f(x) - f(y)\|$, and f is injective. Furthermore, we see that if $g(0) = 0$, then

$$\begin{aligned}
f(U(0, r)) &= U(0, r) + g(U(0, r)) \\
&\subseteq U(0, r) + \lambda U(0, r) \\
&= U(0, (1 + \lambda)r).
\end{aligned}$$

Finally, if $y \in U(0, (1 - \lambda)r)$, then we want to find x such that $y = f(x) = x + g(x)$; equivalently, we see that we want x such that $x = y - g(x)$. Since the function $F(x) = y - g(x)$ is a translation of a uniform contraction, $F(x)$ is a contraction, so there is a fixed point, meaning $y \in f(U(0, r))$. $\square$

Note: We will use $|\cdot|$ to denote the norm on $\mathbb{R}^n$.

Proof of the Inverse Function Theorem. By using a series of affine maps — first by translating p to 0, then translating $f(p)$ to 0, then inverting $D_0 f$ as per our assumption, we may safely assume that $p = f(p) = 0$ and $D_0 f = \mathrm{Id}$.

Set $g = f - \mathrm{Id}$. We will show that g is a contraction in a sufficiently small ball. Fixing $x, y \in \mathbb{R}^n$, consider the map $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto g(x + t(y - x))$. Notice that by the Fundamental Theorem of Calculus,

$$|g(y) - g(x)| \leqslant |y - x| \sup_{0 \leqslant t \leqslant 1} |g'(x + t(y - x))|.$$

Furthermore, since $g'(0) = \mathbf{0}$ by the fact that $D_0 f = \mathrm{Id}$ and $(\mathrm{Id})' = \mathrm{Id}$, and since f is continuously differentiable, there is $r > 0$ such that

$$|g(y) - g(x)| \leqslant \frac{1}{2}|y - x|$$

for all $x, y \in U(0, r)$. Thus, g is a strict contraction on $U(0, r)$. By the previous lemma, we see that

$$U(0, r/2) \subseteq f(U(0, r));$$

by setting $U = U(0, r) \cap f^{-1}(U(0, r))$, we see that the map $f|_U \colon U \to V := U(0, r/2)$ is a bijection. The inverse function $f^{-1} \colon V \to U$ thus exists.

Now, we let $h = f^{-1}$, $x \in U$, $y \in V$ such that $h(x) = y$, and $A = D_x f$. We will show that $A^{-1} = D_y h$, which is enough to show that h is continuously differentiable, as we assume the map $x \mapsto D_x f$ is continuous, and inversion is continuous in $GL_n(\mathbb{R})$.

For sufficiently small vectors s and k, since f and h are bijections, we have

$$h(y + k) = x + s,$$

so

$$f(x + s) = y + k.$$

Furthermore, by unraveling the definitions of $f = g + \mathrm{Id}$, s, and k, and the fact that g is a uniform contraction on U, we get

$$\begin{aligned} |s - k| &= |(f(x + s) - f(x)) - s| \\ &= |(x + s + g(x + s)) - (x + g(x)) - s| \\ &= |g(x + s) - g(x)| \\ &\leqslant \frac{|s|}{2}. \end{aligned}$$

In particular, since

$$\begin{aligned} |s| &\leqslant |s - k| + |k| \\ &\leqslant |k| + \frac{|s|}{2}, \end{aligned}$$

we see that $|s|/2 \leqslant |k|$. We calculate

$$\begin{aligned} \left|h(y + k) - h(y) - A^{-1}k\right| &= \left|x + s - x - A^{-1}(f(x + s) - f(x))\right| \\ &= \left|s - A^{-1}(f(x + s) - f(x))\right| \\ &\leqslant \left\|A^{-1}\right\|_{op} |As - f(x + s) - f(x)|. \end{aligned}$$

Thus, since $|s|/2 \leqslant |k|$,

$$\frac{\left|h(y+k)-h(y)-A^{-1}k\right|}{|k|} \leqslant \frac{2\left\|A^{-1}\right\|_{op}|As-f(x+s)-f(x)|}{|s|} \to 0,$$

so $D_y h = A^{-1}$. □

One of the primary uses of the inverse function theorem is to prove the implicit function theorem.

Theorem (Implicit Function Theorem)**:** Let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable, and let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Assume

- $f(a,b) = 0$;
- the map $y \mapsto f(a,y)$ defined on $\mathbb{R}^m \to \mathbb{R}^m$ is invertible in a neighborhood of b — i.e., $D_b(y \mapsto f(a,y))$ as a linear map has rank m.

Then, there exists a continuously differentiable function $g\colon U \to V$, where $U \in \mathcal{O}_a$ and $V \in \mathcal{O}_b$ such that $f(x,g(x)) = 0$ on U.

Essentially, the theorem says that we can solve $f(x,y) = 0$ on a neighborhood of (a,b) by a function only depending on x. This means that about (a,b) in the graph $\Gamma(f)$, there is a coordinate representation as an m-manifold given by g.

Proof of the Implicit Function Theorem. Define a function $F\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$F(x,y) = (x, f(x,y)).$$

Since f is continuously differentiable, this function F is also continuously differentiable, so we may define $U \in \mathcal{O}_a$, $V \in \mathcal{O}_b$, and $W \in \mathcal{O}_{F(a,b)}$ such that

$$F\colon U \times V \to W$$

is continuously differentiable with continuously differentiable inverse (owing to the Inverse Function Theorem), so that $G = F^{-1} = (G_1, G_2)$ is defined on W. We see that

$$(x,y) = F(G_1(x,y), f(G_1(x,y), G_2(x,y))),$$

meaning that $G_1(x,y) = x$, and $y = f(x, G_2(x,y))$. Since at b, $f(a,b) = 0$, we have that $g(x) = G_2(x,0)$ is the desired function. □

Constructing C^∞ Maps on Manifolds

Definition: A function $f\colon U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open, is called C^∞ if the partial derivatives of all orders,

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

are continuous. Here, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a *multi-index*, where the α_i are positive integers for each i, and $|\alpha|$ is defined by $|\alpha| = \sum_{i=1}^n \alpha_i$.

We are concerned now with constructing C^∞ functions on C^∞-manifolds.[I] In order to do this, we introduce the bump functions.

[I] A C^∞ manifold is one where all the transition functions $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are C^∞ functions.

Definition: The *bump function* that is equal to 1 on $B(0,1)$ and is zero outside $U(0,2)$ is given by

$$h(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$
$$b(x) = \frac{h\left(4 - |x|^2\right)}{h\left(4 - |x|^2\right) + h\left(|x|^2 - 1\right)}. \tag{$*$}$$

Lemma: Let M be a C^∞ manifold. Let $U \in \mathcal{O}_p$, and let $f\colon U \to \mathbb{R}$ be an arbitrary C^∞ function defined on U.

Then, there exists $V \in \mathcal{O}_p$ with $\overline{V} \subseteq U$, and a C^∞ function $\widetilde{f}$ defined on M such that

$$\widetilde{f}(q) = \begin{cases} f(q) & q \in V \\ 0 & q \notin U. \end{cases}$$

Proof. Let (W, φ) be a chart centered at p with $\varphi(p) = 0$ and $U(0,3) \subseteq \varphi(W)$. Let $\overline{b} = b \circ \varphi$, where b is the bump function defined in $(*)$. Then, $\overline{b}$ is a C^∞ function on W, and is 0 outside $\varphi^{-1}(U(0,2)) \subseteq W$.

We define $\overline{b}$ to be equal to zero on W^c. Thus, if we define $V = \varphi^{-1}(U(0,1))$, then $V \in \mathcal{O}_p$, $\overline{V} \subseteq U$, and $\overline{b}$ is equal to 1 on V. Letting

$$\widetilde{f}(q) = \begin{cases} \overline{b}(q)f(q) & q \in W \\ 0 & q \notin W, \end{cases}$$

we see that $\widetilde{f}$ satisfies the required property. □

Given an atlas $\{(U_i, \varphi_i)\}$, we want to be able to "glue" functions together by using these charts. A fundamental construction for this purpose is known as a partition of unity.

Definition: Let X be a topological space.

- An open cover $\{U_i\}_{i\in I}$ is called *locally finite* if, for every $x \in X$, there is some $V \in \mathcal{O}_x$ such that $V \cap U_i = \emptyset$ for all but finitely many i.
- Another open cover $\left\{V_j\right\}_{j\in J}$ is called a refinement of another open cover $\{U_i\}_{i\in I}$ if for all $j \in J$, there exists some $i \in I$ such that $V_j \subseteq U_i$.
- We say X is *paracompact* if, for any open cover of X, there is a locally finite refinement.

Proposition: Let M be a topological manifold. Then, for any open cover $\{U_i\}_{i\in I}$ of M, there is a countable, locally finite refinement $\{V_k\}_{k=1}^{\infty}$ with the $\overline{V_k}$ compact. In particular, M is paracompact.

Additionally, we may select the coordinate maps $\psi_k\colon V_k \to \mathbb{R}^n$ such that $\psi_k(V_k) = U(0,3)$, and $\left\{\psi_k^{-1}(U(0,1))\right\}_{k=1}^{\infty}$ is an open cover of M.

Proof. Since M is a locally Euclidean and second countable, there is a countable basis of pre-compact open sets $\{O_\ell\}_{\ell=1}^{\infty}$. In particular, we may select an exhaustion of M by pre-compact sets by defining

$$E_1 = O_1$$
$$E_k = O_1 \cup O_2 \cup \cdots \cup O_{\ell_k},$$

where ℓ_k is some sufficiently large index as follows. Since $\overline{E_k}$ is compact, there is a sufficiently large ℓ such that $\overline{E_k} \subseteq O_1 \cup \cdots \cup O_\ell$. Defining ℓ_{k+1} to be the smallest index greater than ℓ_k that satisfy this property, we define

$$E_{k+1} = O_1 \cup \cdots \cup O_{\ell_{k+1}}.$$

For arbitrary k, each $\overline{E_k}$ is compact, and $\overline{E_k} \subsetneq E_{k+1}$, and $\bigcup_{k=1}^{\infty} E_k = M$. Note that if M is compact, this process terminates in a finite number of steps.

Now, let $\{U_i\}_{i\in I}$ be an arbitrary open cover of M, and fix $k \geqslant 1$. For each $p \in \overline{E_k} \setminus E_{k-1}$, select i_p such that $p \in U_{i_p}$, and select a chart (V_p, ψ_p) about p that satisfies $\psi_p(p) = 0$, $\psi_p(V_p) = U(0,3)$, and $V_p \subseteq U_{i_p} \cap \left(E_{k+1} \setminus \overline{E_{k-2}}\right)$, where we set $E_{-1} = E_0 = \emptyset$. Finally, set $W_p = \psi_p^{-1}(U(0,1))$.

Since $\overline{E_k} \setminus E_{k-1}$ is compact, we may select a finite number of such p such that the open sets W_p cover $\overline{E_k} \setminus E_{k-1}$. Applying this process to all k, and lining up the charts (V_p, ψ_p) corresponding to the finite number of points p chosen at each stage, we have the locally finite refinement of $\{U_i\}_{i\in I}$ with each $\overline{V_k}$ compact, $\psi_k(V_k) = U(0,3)$, and $\left\{\psi_k^{-1}(U(0,1))\right\}$ an open cover of M. □

Definition: Let M be a C^∞ manifold. A family $\{f_k\}_{k=1}^{\infty}$ of at most countably many C^∞ functions on M is called a *partition of unity* on M if it satisfies:

- for each k, $f_k(p) \geqslant 0$ for all $p \in M$, and the family $\left\{\operatorname{supp}(f_k)\right\}_{k=1}^{\infty}$ is locally finite;
- at all points p on M, $\sum_{k=1}^{\infty} f_k(p) = 1$.

If $\left\{\operatorname{supp}(f_k)\right\}_{k=1}^{\infty}$ is a refinement of an open cover $\{U_i\}_{i\in I}$, then we say the partition of unity is *subordinate* to the open cover.

Theorem: Let M be a C^∞ manifold, and let $\{U_i\}_{i\in I}$ be an open cover of M. Then, there exists a partition of unity $\{f_k\}_{k=1}^{\infty}$ that is subordinate to $\{U_i\}_{i\in I}$.

Proof. Let $\{V_k\}_{k=1}^{\infty}$ be a locally finite refinement of $\{U_i\}_{i\in I}$ such that the charts (V_k, ψ_k) have $\psi_k(V_k) = U(0,3)$.

For each k, using the bump function $(*)$, define

$$\widetilde{b_k}(q) = \begin{cases} b \circ \psi_k(q) & q \in V_k \\ 0 & q \notin V_k. \end{cases}$$

Then, $\widetilde{b_k}$ is a C^∞ function defined on M, and since $\operatorname{supp}\left(\widetilde{b_k}\right) \subseteq V_k$, we may set

$$f = \sum_{k=1}^{\infty} \widetilde{b_k}.$$

The function f is a C^∞ function defined on the whole of M. If we let $W_k = \psi_k^{-1}(U(0,1))$, then since $\{W_k\}_{k\geqslant 1}$ is an open cover of M, for any $q \in M$, there exists j such that $\widetilde{b_j}(q) = 1$. Thus, f never equals 0, so we if we set

$$f_k = \frac{\widetilde{b_k}}{f},$$

the family $\{f_k\}_{k\geqslant 1}$ is a partition of unity subordinate to $\{U_i\}_{i\in I}$. □

Tangent Spaces and Vector Fields

Smooth manifolds are able to be embedded into some Euclidean space,[II] so we start by considering them as such.

Definition: If $f\colon M \to N$ is a smooth map between an n-dimensional manifold M and a k-dimensional manifold N that are embedded into some Euclidean space $\mathbb{R}^\ell$, the *derivative* of f at p, defined for charts (U, φ) and (V, ψ), where $f(U) \subseteq V$, is defined by

$$D_p f = D_p\left(\psi^{-1} \circ F \circ \varphi\right),$$

where $F\colon \mathbb{R}^n \to \mathbb{R}^k$ is defined to be a map such that $f = \psi^{-1} \circ F \circ \varphi$.

Remark: This definition is independent of the chart representation. To see this, notice that as we have embedded both M and N into Euclidean space, the maps $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^k$ are diffeomorphisms, hence their derivatives are invertible linear maps.

$$\begin{tikzcd}
U \arrow[rr, "f"] \arrow[d, bend right, "\varphi_1"'] \arrow[d, bend left, "\varphi_2"] & & V \arrow[d, bend right, "\psi_2"'] \arrow[d, bend left, "\psi_1"] \\
\mathbb{R}^n \arrow[rr, bend left, "F_2"] \arrow[rr, bend right, "F_1"'] & & \mathbb{R}^k
\end{tikzcd}$$

Using some coordinate changes, we see that

$$F_1 = \left(\psi_1 \circ \psi_2^{-1}\right) \circ F_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)$$

so by the chain rule,

$$\begin{aligned}
Df &= D\left(\psi_1^{-1} \circ F_1 \circ \varphi_1\right)\\
&= D(\psi_1)^{-1} \circ D(F_1) \circ D(\varphi_1)\\
&= D\left(\psi_1^{-1}\right) \circ D\left(\left(\psi_1 \circ \psi_2^{-1}\right) \circ F_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)\right)\\
&= D(\psi_1)^{-1} \circ D(\psi_1) \circ D\left(\psi_2^{-1} \circ F_2 \circ \varphi_2\right) \circ D(\varphi_1)^{-1} \circ D(\varphi_1)\\
&= D\left(\psi_2^{-1} \circ F_2 \circ \varphi_2\right).
\end{aligned}$$

Note here that the chain rule is being used in $\mathbb{R}^\ell$, which Dundas calls the "flat chain rule,"[III] rather than the general case on a manifold.

One of the issues with this strategy, though, is that embeddings may carry different properties (though at high enough dimensions, any two embeddings are diffeomorphic to each other). For instance, embeddings $S^1 \hookrightarrow \mathbb{R}^3$ form the field of knot theory, which is a very intricate field.

As a result, we want to be able to define tangent spaces, derivatives, and the like without having to refer to coordinates. In order to do this, we need to discuss germs of functions.

Definition: Let $g\colon M \to N$ map $p \mapsto g(p)$. We define an equivalence relation on the space of functions $f\colon M \to N$ with $f(p) = g(p)$ by taking $f_1 \sim f_2$ whenever $f_1 = f_2$ on some open neighborhood $A \in \mathcal{O}_p$. The equivalence class $[g]_p$ is known as the *germ* of g at p.

[II] This is actually a very deep theorem.
[III] Flatness is always relative.

We denote by $\mathcal{C}_p$ the space of germs of C^∞ functions $f\colon M \to \mathbb{R}$ at p.

Remark: Often, books will use $\mathcal{O}_p$ to refer to the space of germs at p. We will use $\mathcal{C}_p$ for this purpose though, as we have defined $\mathcal{O}_p$ to refer to the system of open neighborhoods at p.

Proposition: The space $\mathcal{C}_p$ with the operations

- $[g] + [h] = [g + h]$;
- $\alpha[g] = [\alpha g]$;
- $[g] \cdot [h] = [g \cdot h]$

forms an algebra over $\mathbb{R}$.

Definition: Let W_p be the space of germs of smooth maps $\gamma\colon \mathbb{R} \to M$ that send $0 \mapsto p$. The *tangent space* T_pM is defined by $W_p/\sim$, where we define the equivalence relation $[g_1] \sim [g_2]$ for two germs at p $g_1\colon M \to \mathbb{R}$ and $g_2\colon M \to \mathbb{R}$ by

$$(\varphi \circ g_1)'(0) = (\varphi \circ g_2)(0)$$

for all $\varphi \in \mathcal{C}_p$.

Definition: If $f\colon M \to N$ is a smooth map, we define the map $T_pf\colon T_pM \to T_{f(p)}N$ to act via

$$T_pf([\gamma]) = [f \circ \gamma]$$

for all $\gamma \in W_p$.

Proposition (Chain Rule)**:** If $f\colon M \to N$ and $g\colon N \to L$ are smooth maps, then

$$T_{f(p)}g \circ T_pf = T_p(g \circ f).$$

Proof. If $\gamma \in W_p$, then

$$\begin{aligned} T_{f(p)}g \circ T_pf([\gamma]) &= T_{f(p)}g([f \circ \gamma]) \\ &= [g \circ f \circ \gamma] \\ &= T_p(g \circ f)([\gamma]). \end{aligned}$$

□

A terribly kept secret is that this function T_pf is actually the differential D_pf. This requires us to prove that this definition comports with the definition for the case of M as an embedded manifold. We require a few basic propositions for this purpose whose proofs follow from the inverse function theorem and various definitions.

Proposition: If $f\colon M \to N$ is a locally invertible smooth map about $p \in M$, then $T_pf\colon T_pM \to T_{f(p)}N$ is an isomorphism of vector spaces.

Proposition: If $0 \in \mathbb{R}^k$, then $T_0\mathbb{R}^k$ is represented by linear maps $t \mapsto \lambda t$ for some vector $\lambda \in \mathbb{R}^k$. Therefore, if M is k-dimensional, $T_pM \cong \mathbb{R}^k \cong \operatorname{Hom}(\mathbb{R}, \mathbb{R}^k)$.

Proposition: If $\varphi\colon U \to \mathbb{R}^k$ is a local diffeomorphism about p such that $\varphi(p) = 0$, then if $f \in C^\infty(\mathbb{R}^k)$, $f \circ \varphi \in C^\infty(U)$, which induces an algebra homomorphism

$$\begin{aligned} \varphi^*\colon \mathcal{C}_{0,\mathbb{R}^k} &\to \mathcal{C}_{p,M} \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Proposition: If M and N are embedded submanifolds of $\mathbb{R}^n$ with dimensions m and k respectively, and $f\colon M \to N$ is a smooth map, then

$$D_pf \equiv T_pf.$$

Proof. Let (U, φ) and (V, ψ) be charts for M and N respectively with $p \in U$ and $f(p) \in V$. Then, we may consider the coordinate maps $\varphi\colon U \to \mathbb{R}^m$ and $\psi\colon V \to \mathbb{R}^k$ to be such that $p \mapsto 0$ and $f(p) \mapsto 0$ respectively.

Now, we see that $T_p f$ and $D_p f$ can be written as

$$\begin{aligned} T_p f &= T_{f(p)}\psi^{-1} \circ T_0 F \circ T_{f(p)}\varphi \\ D_p f &= D_p\left(\psi^{-1} \circ F \circ \varphi\right). \end{aligned}$$

Yet, since $T_0 F = D_0 F$, $T_{f(p)}\psi^{-1} = D_{f(p)}\psi^{-1}$, and $T_p\varphi = D_p\varphi$, the chain rule gives

$$\begin{aligned} T_p f &= T_p\left(\psi^{-1} \circ F \circ \varphi\right) \\ &= D_p\left(\psi^{-1} \circ F \circ \varphi\right) \\ &= D_p f, \end{aligned}$$

implying that $T_p f = D_p f$. □

Now that we have established that we can consider manifolds as either standalone entities or as submanifolds of $\mathbb{R}^n$, we now shift our focus to understanding what information the derivative map $D_p f\colon T_p M \to T_{f(p)} M$ gives us about the underlying topology of M and N.

Definition: Let $f\colon M \to N$ be a smooth map, and let $p \in M$. We say p is a *critical point* for f if $D_p f$ does not have maximal rank.

If $D_p f$ has maximal rank, then we say p is a *regular point* of f.

We say $q \in N$ is a *critical value* if $f^{-1}(\{q\})$ contains a critical point for f. Else, we say q is a *regular value*.

Remark: To show that $D_p f\colon T_p M \to T_{f(p)} N$ is surjective, we take an element of $T_{f(p)} N$ (i.e., a smooth map $\gamma\colon \mathbb{R} \to N$) and show that there is a smooth map $\delta\colon \mathbb{R} \to M$ such that $f \circ \delta = \gamma$ on a neighborhood of $f(p)$.

The study of critical points is actually very vital in understanding the underlying manifold's global topology. This is the field known as Morse theory, and we will discuss it later in the course.

Definition: Let $f\colon M \to \mathbb{R}$ be a smooth function, with M a manifold. We say f is *Morse* if all the critical points of f are isolated, and the critical points are nondegenerate, in the sense that the Hessian matrix, given by

$$H_p f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j=1}^{n}$$

has nonzero determinant, where $(x_1, \dots, x_n)$ is a coordinate system about the critical point p.

Morse functions allow us to calculate a quantity known as the index of the manifold at any given value of $\mathbb{R}$, thereby allowing us to reconstruct the manifold from the information that the functions give us.

There are two important theorems related to critical points/values and regular points/values.

Theorem (Sard's Theorem)**:** Let $f\colon M \to N$ be a smooth map of manifolds. The set of critical values in N is of measure zero (where measure is defined by the measure of the image under a coordinate map).

We will not prove Sard's Theorem now, but we will prove a very useful result that is often used in conjunction with Sard's Theorem.

Theorem (Regular Value Theorem)**:** Let $f\colon M \to N$ be a smooth map of manifolds with dimensions $m \geqslant n$ respectively. If $q \subseteq N$ is a regular value, then $f^{-1}(\{q\}) \subseteq M$ is a submanifold of dimension $m - n$.

Proof. Let $p \in f^{-1}(\{q\})$, and let (U, φ) be a chart of p where $\varphi\colon U \to \mathbb{R}^m \cong T_p M$ are identified as such. Since $D_p f$ is of full rank, we know that $K = \ker\left(D_p f\right)$ is of dimension $m - n$, so that $K \cong \mathbb{R}^{m-n}$.

Let $L: \mathbb{R}^m \to \mathbb{R}^{m-n} \cong K$ be a projection, and define

$$\begin{aligned} F: \mathbb{R}^m \supseteq U &\to N \times \mathbb{R}^{m-n} \\ x &\mapsto (f(x), L(x)). \end{aligned}$$

Notice then that $D_p F = \left(D_p f, L\right)$, where the latter comes about from the fact that L is a linear map. Then, we have that $D_p f$ is of rank n, and L is of rank $m - n$, meaning that $D_p F: \mathbb{R}^m \to \mathbb{R}^m$ is of full rank, hence is invertible on a neighborhood $V \times W \subseteq N \times \mathbb{R}^{m-n}$, where W is a neighborhood of $0 \in \mathbb{R}^{m-n}$, so we may identify $U \cong V \times W$.

By composing with the projection $\pi: N \times \mathbb{R}^{m-n} \to N$ given by $(q, W) \to q$, we have that $f = \pi \circ F$, so that $f^{-1}(\{q\}) = F^{-1} \circ \pi^{-1}(\{q\})$, meaning that $f^{-1}(\{q\}) \cong \mathbb{R}^{m-n}$. □

One of the central uses of the regular value theorem is the fact that it allows us to prove a smooth version of Brouwer's Fixed Point Theorem for the general case of manifolds, known as the No Retraction Theorem.

Theorem (No Retraction Theorem)**:** Let M be a compact smooth n-dimensional manifold with boundary, and let $N = \partial M$ be the boundary. There does not exist any smooth surjective function $r: M \to N$ such that $r(x) = x$ for every $x \in N$.

Proof. Suppose toward contradiction that there were such a retraction, which we call r. Let X be the set of critical points for r in M; then, by Sard's Theorem, $r(X) \subseteq N$ has measure zero, so there exists a regular value $y \in N$.

By the Regular Value Theorem, $r^{-1}(\{y\})$ is a smooth 1-dimensional manifold, as N is a $n - 1$ dimensional manifold, meaning that $r^{-1}(\{y\})$ is either S^1 or an open interval. If $r^{-1}(\{y\})$ is a circle, then $r^{-1}(\{y\})$ necessarily must be contained in the interior of M, which would contradict the fact that $y \in \partial M$. Therefore, $r^{-1}(\{y\})$ is an interval, and specifically is one that has both of its endpoints on N, as on the interior of M, such an interval must be identified to a 1-dimensional subspace of M, so there is some $z \neq y \in N$ such that $z \in r^{-1}(\{y\})$. Yet, that means that $r(z) = y \neq z$, which is a contradiction. □

The Tangent Bundle

Recall that we defined the differential $D_p f$ via the action on the manifold about the point p. Unfortunately, the issue with this formulation is that it is purely local — the main reason we study manifolds is that we want to be able to use local information about the function to obtain insights about the global topology of the manifold. We need a construction that allows us to collect information about all the differentials at points of M together. This is the tangent bundle.

Definition: Let M be a manifold. The *tangent bundle* TM is the disjoint union

$$TM = \coprod_{p \in M} T_p M.$$

Now, if M is a manifold of dimension m, then TM is a manifold of dimension $2m$. To see this, observe that if $p \in \mathbb{R}^m$, then $T_p\mathbb{R}^m \cong \mathbb{R}^m$.

Therefore, if at each point in $\mathbb{R}^m$, we assign a copy of the tangent space, we have that

$$T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m.$$

If $f: \mathbb{R}^m \to \mathbb{R}^n$ is smooth, we get the map $Tf: T\mathbb{R}^m \to T\mathbb{R}^n$ given by

$$(x, v) \mapsto (f(x), D_x f(v)).$$

Now, given a coordinate map $\varphi: M \supseteq U \to \mathbb{R}^m$, we may define

$$T\varphi(U) = \varphi(U) \times \mathbb{R}^m.$$

Thus, if $\{(U_i, \varphi_i)\}_{i \in I}$ is an atlas for M, we have transition maps

$$\begin{aligned} T\psi(U \cap V) &\to T\varphi(U \cap V) \\ (x, v) &\mapsto \left(\varphi \circ \psi^{-1}(x), D_x\left(\varphi \circ \psi^{-1}\right)(v)\right). \end{aligned}$$

Thus, if $f\colon M \to N$ is a smooth map, it induces a differential map on the tangent bundles $Df\colon TM \to TN$.

> **Remark:** If M and N only have C^1 structures, it turns out that there is a compatible C^∞ structure, meaning that we may safely assume that any C^1 manifold is C^∞.

Vector Fields

> **Definition:** If M is a manifold, then a *vector field* on M is a smooth right-inverse of the projection map
>
> $$\begin{aligned} \pi\colon TM &\to M \\ (x, v) &\mapsto x. \end{aligned}$$

When we consider vector fields on manifolds, some basic questions crop up. The most basic of them all is the following: does there exist a nowhere-vanishing vector field on M?

- In any Euclidean space, we may take a constant nonzero vector as our assignment, so the answer is yes.
- For S^1, we can embed it into $\mathbb{R}^2$, then take the map $(x, y) \mapsto ((x, y), (-y, x))$, which is the family of tangent vectors to the unit circle in $\mathbb{R}^2$.
- For S^2, the answer is no. This is the much-celebrated "hairy ball theorem."
- For S^3, the answer is yes. In fact, for $S^{(2n-1)}$ where n is a natural number, the answer is yes, while for S^{2n}, the answer is no.

Now, if M is a manifold, with (U, φ) a chart on M with $p \in U$, identification $\varphi\colon U \to \mathbb{R}^n$, and coordinate representation $(x_1, \dots, x_n)$. Since $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, we may write the local coordinates for T_pM formally as $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. In other words, we consider the $\frac{\partial}{\partial x_i}$ as a basis for the vector space T_pM.

If X is a vector field on M, we may express X locally about p formally as

$$X = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},$$

where each $a_i\colon U \to \mathbb{R}$ is a C^∞ function.

The first question we have is whether this is well-defined. To do this, we consider another chart, (V, ψ) with $p \in V$, identification $\psi\colon V \to \mathbb{R}^n$, and coordinate representation $(y_1, \dots, y_n)$. Given the coordinate change $\psi \circ \varphi^{-1}$, we want to consider a corresponding coordinate change $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \mapsto \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right)$. Via the chain rule on $\mathbb{R}^n$, we find that this corresponding coordinate change is

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j},$$

which emerges from applying the differential to $\psi \circ \varphi^{-1}$.

The space of vector fields on M has algebraic structure.

- If X and Y are vector fields, then so is $\alpha X + \beta Y$ for all $\alpha, \beta \in \mathbb{R}$.

- If $f \in C^\infty(M)$, then $f \cdot X$ is also a vector field on X, which is locally represented by

$$f \cdot X = \sum_{i=1}^{n} (f a_i) \frac{\partial}{\partial x_i}.$$

Furthermore, X acts on the space $C^\infty(M)$ via differentiation. If $p \in U$ with local coordinates $(x_1, \dots, x_n)$, we have $X(f)$ locally defined by

$$X(f)(p) := \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p).$$

This action has two properties:

- Linearity: $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$;
- Leibniz Rule: $X(fg) = f \cdot X(g) + g \cdot X(f)$.

More generally, we say that an $\mathbb{R}$-linear map with these two properties is a *derivation*. It turns out that the span of the set of derivations on M is actually equal to the space of vector fields of M.

Finally, vector fields on M admit an intrinsic multiplication.

Definition: If X and Y are vector fields on M, the *Lie Bracket* of X and Y is defined for any $f \in C^\infty(M)$ by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Proposition: Let $p \in M$ have local chart (U, φ), where $\varphi = (x_1, \dots, x_n)$ is the coordinate map, and let X and Y be vector fields on M with local representation on (U, φ) given by

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$
$$Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}.$$

Then, $[X, Y]$ has local representation given by

$$[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_i \frac{\partial b_i}{\partial x_j} - b_i \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Proposition: The Lie Bracket $[X, Y]$ of vector fields on M satisfies three properties:

- bilinearity: $[\alpha X_1 + \beta X_2, Y] = \alpha[X_1, Y] + \beta[X_2, Y]$ and $[X, \alpha Y_1 + \beta Y_2] = \alpha[X, Y_1] + \beta[X, Y_2]$ for all $\alpha, \beta \in \mathbb{R}$;
- alternating: $[X, Y] = -[Y, X]$;
- Jacobi identity: $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$;
- extended bilinearity for $C^\infty(M)$: $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

Submanifolds of Dimension 1

Definition: Let X be a vector field on M, with $p \in M$. An *integral curve* for X through p is a C^∞ map $c\colon \mathbb{R} \to M$ such that $0 \mapsto p$ and $Dc\colon T\mathbb{R} \to TM$ maps $\frac{\partial}{\partial t} \mapsto X$, where $\left(t, \frac{\partial}{\partial t}\right)$ are the global coordinates for the tangent bundle $T\mathbb{R}$.

In local coordinates, we may express c and Dc via

$$\begin{aligned} c(t) &= (x_1(t), \dots, x_n(t)) \\ Dc\left(\frac{\partial}{\partial t}\right) &= \sum_{i=1}^{n} \frac{dx_i}{dt}\frac{\partial}{\partial x_i} \\ &= \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}\right) \end{aligned}$$

when $p \in U \subseteq M$ is a chart with local coordinates $(x_1, \dots, x_n)$. We then say that c is an integral curve if, for a local representation

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

we have

$$Dc\left(\frac{\partial}{\partial t}\right) = \sum_{i=1}^{n} \frac{dx_i}{dt}\frac{\partial}{\partial x_i},$$

or that

$$\frac{dx_i}{dt} = a_i(x_1(t), \dots, x_n(t))$$

for all i, and $(x_1(0), \dots, x_n(0)) = p$.

One can imagine an integral curve as a "flow" following a vector field traced out by a particle.[IV]

Theorem: Given a vector field X on M, there is a unique integral curve passing through every $p \in M$.

In order to prove this theorem, we need to recall the Picard–Lindelöf theorem from ordinary differential equations that gives us a sufficient condition for existence and uniqueness of solutions to initial value problems.

Theorem (Picard–Lindelöf)**:** Let $x\colon \mathbb{R} \to \mathbb{R}^n$ and $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\begin{aligned} \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}\right) &= \dot{x} \\ &= f(t, x), \end{aligned}$$

with $x(0) = x_0$. If f is Lipschitz, then the initial value problem has a unique solution defined on $(-\varepsilon, \varepsilon)$ for a sufficiently small ε.

Recall that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if there is $L < \infty$ such that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leqslant L.$$

Notice that if f is continuously differentiable and defined on a compact domain, this follows immediately from the extreme value theorem.

The proof of the Picard–Lindelöf theorem follows from a technique known as Picard iteration. Essentially, we rewrite the differential equation in integral form

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds,$$

[IV] This is where the connection between differential topology and partial differential equations begins to appear.

and start by setting

$$x_0(t) = x_0.$$

Then, we inductively define

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))\,ds.$$

Defining the integral operator

$$K(y(t)) = x_0 + \int_0^t f(s, y(s))\,ds,$$

we essentially desire to show that K has a fixed point for all continuous y on a sufficiently small neighborhood of 0. This will follow from showing that K is a contraction in the supremum metric on this sufficiently small neighborhood of 0 whenever f is Lipschitz, which will allow us to use the contraction mapping theorem, as continuous functions on a compact set are complete under the supremum metric.

Corollary: If M is a manifold, and X a vector field on M with $p \in M$, then there exists an integral curve $c\colon (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$.

Definition: We call the vector field X *complete* if every integral curve along every point of p can be extended to all of $\mathbb{R}$.

There are many vector fields on manifolds that aren't complete. For instance, if $M = \mathbb{R}^2 \setminus \{0\}$, and $X = \frac{\partial}{\partial x_1}$, then an integral curve through $(1, 0)$ cannot be extended to all of $\mathbb{R}$, as it would hit the missing point at the origin.

Flows and Diffeomorphism Groups

Complete vector fields on manifolds enable us to create diffeomorphisms. Furthermore, it can be shown that if M has dimension greater than or equal to 2, then $\operatorname{diff}(M)$ is k-transitive, in that any k-tuple of distinct elements can be mapped to any other k-tuple of distinct elements.

Definition: A *flow* on M is a one-parameter group of diffeomorphisms of M, defined by

$$\begin{aligned} \varphi\colon \mathbb{R} &\to \operatorname{diff}(M) \\ t &\mapsto \varphi_t, \end{aligned}$$

where $\varphi_t(p) := c_p(t)$ when c_p is the integral curve through p.

In particular, $\operatorname{im}(\varphi)$ is the flow.

Proposition: If M is a connected manifold with $\dim(M) \geqslant 2$, then $\operatorname{diff}(M)$ is k-transitive.

Remark: The reason this does not work if M is a 1-dimensional manifold is that $\mathbb{R}$ is linearly ordered and S^1 is cyclically ordered.

Proof. We start with the case of $k = 1$.

Let p and q be in the same chart, (U, φ), where $\varphi = (x_1, \dots, x_n)$ is the coordinate map. By composing with a series of affine transformations of $\mathbb{R}^n$, we may assume that $\varphi(p)$ is the origin and q is on the coordinate axis x_1. Furthermore, let N be a compact subset of $\mathbb{R}^n$ such that $p, q \in N$.

Define f to be a smooth bump function on M such that f is 1 on N and 0 outside a neighborhood of N. Then, if $X = f\frac{\partial}{\partial x_1}$ on U and zero outside, we observe that the integral curve through p passes through q, and is a flow with $\varphi(t) = \psi_t(p) = q$ for some $t \in \mathbb{R}$.

Thus $\operatorname{diff}(M)$ acts transitively on points in the same chart. Meanwhile, if p and q are not in some chart, we find a finite length "chain" of intersecting charts that move from the chart at p to the chart at q, then by composing a finite collection of diffeomorphisms, we find our desired diffeomorphism. $\square$

Differential Forms

Now that we have a (reasonable) understanding of the tangent spaces of a manifold, we now concern ourselves with the dual to the tangent space, which are known as the *cotangent spaces*. Similar to how vector fields emerge from the tangent bundle and its projection onto M, the "dual" to vector fields, known as differential forms, emerge from the cotangent bundle..

Note that if T_pM is a tangent space that is isomorphic to $\mathbb{R}^n$, then the corresponding cotangent space, denoted T_p^*M is isomorphic to $(\mathbb{R}^n)^* := \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$.

Recall that the basis for T_pM is given by $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ for local coordinates $(x_1, \dots, x_n)$. The corresponding dual basis for T_p^*M is established from the dual basis for a vector space V. Recall that if $(v_1, \dots, v_n)$ is a basis for V, then $\left(v_1^*, \dots, v_n^*\right)$ is a basis for V^*, where

$$\begin{aligned} v_i^*(v_j) &= \delta_i^j \\ &:= \begin{cases} 1 & i = j \\ 0 & \text{else.} \end{cases} \end{aligned}$$

The corresponding (formal) basis for T_p^*M is given by $(dx_1, \dots, dx_n)$.

Now, the tangent bundle for M is given by

$$TM = \bigcup_{p\in M} T_pM,$$

and similarly, the cotangent bundle for M is given by

$$T^*M = \bigcup_{p\in M} T_p^*M.$$

Next, we concern ourselves with the manifold structure of T^*M. To start, if $U \subseteq M$ is a chart, then $T^*U := U \times (\mathbb{R}^n)^*$. Now we concern ourselves with the transition maps.

Recall from linear algebra that if $A\colon V \to W$ is a linear map, then $A^*\colon W^* \to V^*$ is the dual map (or transpose) given by $A^*\varphi = \varphi \circ A$. Similarly, recall that for two charts (U, φ) and (V, ψ), the tangent bundle admits the map

$$\begin{aligned} T(\varphi(U \cap V)) &\to T(\psi(U \cap V)), \\ (x, v) &\mapsto \left(\psi \circ \varphi^{-1}(x), D_x\left(\psi \circ \varphi^{-1}\right)\right). \end{aligned}$$

Thus, we may consider the transition map for the cotangent bundle $T^*(\psi(U \cap V)) \to T^*(\varphi(U \cap V))$ to be the dualization of the transition map for the tangent bundle.

Definition: A differential (1-)form on M is a smooth section of the projection map $\pi^*\colon T^*M \to M$.

Locally, if $U \subseteq M$ has coordinates $(x_1, \dots, x_n)$, then $(dx_1, \dots, dx_n)$ are coordinates for the cotangent space over U, with forms denoted

$$\omega = \sum_{i=1}^{n} f_i \, dx_i,$$

where $f_i \in C^\infty(M)$ for all i. The differentials d_{x_i} satisfy

$$d_{x_i}\left(\frac{\partial}{\partial x_j}\right) = \delta_i^j.$$

Some Exterior Algebra

One of the most important use cases for differential forms is that they enable us to perform integration on manifolds. This requires a construction known as the exterior algebra.

If $\mathbb{R}^n$ admits a basis $(v_1, \dots, v_n)$, we define the exterior algebra

$$\Lambda(\mathbb{R}^n) = \bigoplus_{i=1}^{n} \Lambda^i(\mathbb{R}^n),$$

where

$$\begin{aligned} \Lambda^0(\mathbb{R}^n) &\cong \mathbb{R} \\ \Lambda^1(\mathbb{R}^n) &\cong \mathbb{R}^n \\ \Lambda^i(\mathbb{R}^n) &= \operatorname{span}\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k, v_i \wedge v_i = 0\}. \end{aligned}$$

In other words, the family of $\Lambda^k(\mathbb{R}^n)$ are alternating and multilinear. In fact, the exterior algebra is the "universal" algebra for alternating multilinear maps.

Theorem: If $t\colon V \times \cdots \times V \to W$ is an alternating k-multilinear map, then if we define

$$\begin{aligned} \iota\colon V \times \cdots \times V &\hookrightarrow \Lambda^k(V) \\ (v_1, \dots, v_k) &\mapsto v_1 \wedge \cdots \wedge v_k, \end{aligned}$$

then there is a unique linear map $T\colon \Lambda^k(V) \to W$ such that $T \circ \iota = t$.

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\iota"] \arrow[dr, "t"'] & \Lambda^k(V) \arrow[d, "T"] \\ & W \end{tikzcd}$$

The multiplication in $\Lambda(\mathbb{R}^n)$ is given by the exterior product,

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_\ell) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_\ell.$$

Additionally, we see that for $\sigma \in S_k$,

$$v_{i_{\sigma(1)}} \wedge \cdots \wedge v_{i_{\sigma(k)}} = \operatorname{sgn}(\sigma) v_{i_1} \wedge \cdots \wedge v_{i_k}.$$

Definition: A k-form on a finite-dimensional real vector space V is an alternating k-multilinear map

$$\omega\colon V^k \to \mathbb{R}.$$

We denote the set of all k-forms on V by $\mathcal{A}^k(V)$.

Proposition:

$$\mathcal{A}^k(V) \cong \Lambda^k(V^*).$$

Proof. Apply the universal property. □

A Coordinate-Free Definition of the Cotangent Space

Technically, the "cotangent space" that we defined above with the formal coordinates $(dx_1, \dots, dx_n)$ is not the real, coordinate-free definition of the cotangent space. In order to define the cotangent space in a coordinate-free fashion, we need to go back to the definition of the tangent space using germs of functions $\mathcal{C}_p$.

Recall that the tangent space of M is defined by an equivalence relation on W_p, which is the space of germs of smooth maps $\gamma\colon \mathbb{R} \to M$ where $\gamma(0) = p$. Specifically, the equivalence relation $[g_1] \sim [g_2]$ for two germs $g_1, g_2\colon \mathbb{R} \to M$ with $0 \mapsto p$ is given by $(\varphi \circ g_1)'(0) = (\varphi \circ g_2)'(0)$ for all $\overline{\varphi} \in \mathcal{C}_p$.

Inside $\mathcal{C}_p$, we can look at the ideal J_p defined by

$$J_p = \left\{\overline{\varphi} \in \mathcal{C}_p \;\middle|\; \varphi(p) = 0\right\}.$$

Notice that J_p is a maximal ideal in $\mathcal{C}_p$, as every germ in $\mathcal{C}_p$ differs from an element of J_p by a real number. We then define J_p^2 to be functions of the form $\varphi\psi$, where $\varphi, \psi \in J_p$. In essence, elements of J_p^2 vanish to (at least) order 2 at p.

Definition: The *cotangent space* of p at M is given by

$$T_p^* M = J_p / J_p^2.$$

Definition: If $\varphi \in \mathcal{C}_p$, then the *exterior derivative* of φ is defined by

$$d\varphi = (\varphi - \varphi(p)) + J_p^2.$$

In order to see that this is actually the derivative (i.e., the best linear approximation), we note that by Taylor's Theorem, if $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function, we have

$$f(t) \approx f(0) + c_1 t + c_2 t^2 + O\left(t^3\right)$$

so

$$\begin{aligned} f(t) - f(0) &\approx c_1 t + \underbrace{c_2 t^2 + O\left(t^3\right)}_{\in J_p^2} \\ f(t) - f(0) + J_p^2 &= c_1 t + J_p^2. \end{aligned}$$

Proposition: The exterior derivative satisfies $d(\varphi\psi) = d(\varphi)\psi + \varphi d(\psi)$.

Proof. We must show that

$$d(\varphi)\psi + \varphi d(\psi) - d(\varphi\psi) \in J_p^2.$$

Expanding definitions, we have

$$\begin{aligned} d(\varphi)\psi + \varphi d(\psi) - d(\varphi\psi) &= \psi(\varphi - \varphi(p)) + \varphi(\psi - \psi(p)) - (\varphi\psi - \varphi(p)\psi(p)) \\ &= (\varphi - \varphi(p))(\psi - \psi(p)) \\ &\in J_p^2. \end{aligned}$$

□

We now encounter a very important question: does this definition of the cotangent space comport with the definition above as the dual of the tangent space? The answer is yes.

Proposition: Let $d\varphi \in T_p^*(M)$. Define the map

$$\begin{aligned} \alpha\colon T_p^*(M) &\to \left(T_p M\right)^* \\ d\varphi &\mapsto \left\{[\gamma] \mapsto (\varphi \circ \gamma)'(0)\right\}. \end{aligned}$$

Furthermore, the map α satisfies the following.

(a) α is linear and well-defined.

(b) α is natural, in the sense that if $f\colon (M, p) \to (N, q)$ is a smooth map, then the following diagram commutes:

$$\begin{tikzcd} T_q^*N \arrow[r, "T_f^*"] \arrow[d, "\alpha_{N,q}"'] & T_p^*M \arrow[d, "\alpha_{M,p}"] \\ (T_qN)^* \arrow[r, "(T_f)^*"'] & (T_pM)^* \end{tikzcd}$$

Here, $T_f^*(d\varphi) = d(\varphi \circ f)$.

(c) α is an isomorphism.

(d) In local coordinates $(x_1, \dots, x_n)$ about p, we have

$$d\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} dx_i.$$

Proof.

(a) This follows from the definition of the tangent space (and specifically, the equivalence relation between germs $\gamma\colon \mathbb{R} \to M$ that defined the tangent space).

(b) Set $\omega_\varphi = \{[\gamma] \mapsto (\varphi \circ \gamma)'(0)\}$. Then, we observe that, if $d\varphi \in T_q^*N$ and $[\gamma] \in T_pM$, that

$$\begin{aligned} (Tf)^* \circ \alpha_{N,q}(d\varphi)[\gamma] &= (Tf)^*\omega_\varphi([\gamma]) \\ &= \omega_\varphi(Tf([\gamma])) \\ &= \omega_\varphi([f \circ \gamma]) \\ &= (\varphi \circ f \circ \gamma)'(0). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \alpha_{M,p} \circ T^*f(d\varphi)([\gamma]) &= \alpha_{M,p}(d(\varphi \circ f))([\gamma]) \\ &= \omega_{\varphi \circ f}([\gamma]) \\ &= (\varphi \circ f \circ \gamma)'(0). \end{aligned}$$

□

Derivations and the Dual of the Cotangent Space

We review the different spaces that we have discussed thus far.

(i) The tangent space, T_pM, is defined to be equivalence classes of germs of smooth curves $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = p$, where two curves γ_1 and γ_2 are set to be equal if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for all $\varphi \in C^\infty(M)$.

(ii) The cotangent space T_p^*M is defined to be J_p/J_p^2, where J_p is the ideal in $\mathcal{C}_{p,M}$[V] consisting of all functions φ such that $\varphi(p) = 0$. If $\varphi \in \mathcal{C}_{p,M}$, we define the exterior derivative $d\varphi$ by $(\varphi - \varphi(p)) + J_p^2$ — i.e., taking the residue class modulo J_p^2 of $\varphi - \varphi(p)$.

[V] Recall that this is the set of germs of smooth functions $\varphi\colon M \to \mathbb{R}$.

(iii) The relationship between the cotangent space and the dual of the tangent space, $\left(T_pM\right)^*$, is given by the map $\alpha_{p,M}(d\varphi)([\gamma]) = (\varphi \circ \gamma)'(0)$. This $\alpha_{p,M}$ is a well-defined natural isomorphism of vector spaces.

Now, we consider a new space, $\mathcal{D}_pM$, consisting of all derivations at p.

> **Definition:** Let $p \in M$. A *derivation* of p at M is a linear map $X_p \colon \mathcal{C}_{p,M} \to \mathbb{R}$ satisfying the Leibniz rule:
>
> $$X_p(f \cdot g) = X_p(f)g(p) + f(p)X_p(g),$$
>
> The space of derivations at p is denoted $\mathcal{D}_pM$.

In order to understand $\mathcal{D}_pM$ for a general manifold, we start by examining the case of $\mathbb{R}^n$; specifically, $D_0\mathbb{R}^n$. The linear maps $\mathcal{C}_{0,\mathbb{R}^n} \to \mathbb{R}$ are spanned by partial derivatives — i.e., the maps $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$, where we use the derivation

$$\left.\frac{\partial}{\partial x_i}\right|_0 (f) = \frac{\partial f}{\partial x_i}(0).$$

The partial derivatives are linearly independent and satisfy the relations

$$\frac{\partial}{\partial x_i}\left(x_j\right) = \delta_i^j.$$

Our hope is that a general derivation on a manifold is similar — i.e., spanned by partial derivatives in local coordinates. Toward this end, we start by understanding two facts about derivations at 0.

- These derivations annihilate constant functions, which follows from the Leibniz rule. If X is such a derivation, then

$$\begin{aligned} X_0(1) &= X_0(1 \cdot 1) \\ &= 2X_0(1), \end{aligned}$$

  so that $X(1) = 0$.

- Derivations annihilate functions of higher order than the constants — if X is a derivation, and we have coordinates $x_1, \dots, x_n$, then

$$\begin{aligned} X_0\left(x_ix_j\right) &= X_0(x_i)x_j(0) + x_i(0)X_0\left(x_j\right) \\ &= 0. \end{aligned}$$

Thus, if we take a Taylor expansion of $f \colon \mathbb{R}^n \to \mathbb{R}$ about 0, which we write as

$$f(x) \approx c + \sum_{i=1}^{n} a_ix_i + O\left(x^2\right),$$

then

$$\begin{aligned} D_0(f) &= \sum_{i=1}^{n} a_iD_0(x_i) \\ &=: \sum_{i=1}^{n} a_ic_i, \end{aligned}$$

so that

$$D = \sum_{i=1}^{n} c_i\frac{\partial}{\partial x_i}.$$

Definition: Let $f\colon M \to N$ be a smooth map, such that $f(p) = q$. Then, there is an induced map $f_*\colon \mathcal{D}_p M \to \mathcal{D}_q N$ given by

$$f_* X(\varphi) = X(\varphi \circ f)$$

for all $\varphi \in \mathcal{C}_{q,N}$ and all $X \in \mathcal{D}_p M$. In particular, if we let X_p denote the evaluation of the derivation at p, then we have

$$(f_* X)_q(\varphi) = X_p(\varphi \circ f).$$

This is known as the *pushforward* of the derivation.

We know from linear algebra that if we have a vector space V, there is a canonical map from V to V^{**} given by $x \mapsto \hat{x}$, where $\hat{x}(\varphi) = \varphi(x)$. If V is finite-dimensional, then this is an isomorphism. This isomorphism is known as *canonical evaluation*, and is often denoted ev.

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend right, "x \mapsto \hat{x}"'] & V^* \arrow[r] & V^{**} \end{tikzcd}$$

Just as we showed that $T_p^* M \cong \left(T_p M\right)^*$ are canonically isomorphic, we would ideally like a similar canonical isomorphism between $\left(T_p^* M\right)^*$ and $\mathcal{D}_p M$, so that we may use a variety of compositions to establish that $\mathcal{D}_p M$ is canonically isomorphic to $T_p M$, and let derivations be considered as equivalent to vector fields.

Towards this end, we observe that the exterior derivative, $d\colon \mathcal{C}_{p,M} \to T_p^* M$, is a derivation. Define now the map $\beta\colon \left(T_p M^*\right)^* \to \mathcal{D}_p$ by

$$g \mapsto \left\{\mathcal{C}_{p,M} \overset{d}{\mapsto} T_p^* M \overset{g}{\mapsto} \mathbb{R}\right\}.$$

Proposition: The map β is a natural isomorphism, in the sense that if $f\colon M \to N$ is a smooth map, then the following diagram commutes.

$$\begin{tikzcd} \left(T_p^* M\right)^* \arrow[r, "\beta_{p,M}"] \arrow[d, "(D^* f)^*"'] & \mathcal{D}_p M \arrow[d, "f_*"] \\ \left(T_q^* N\right)^* \arrow[r, "\beta_{q,N}"] & \mathcal{D}_q N \end{tikzcd}$$

Here, recall that $T^* f(d\varphi) \equiv D^* f(d\varphi) = d(\varphi \circ f)$, as we defined for the map $\alpha_{p,M}$, while $(D^* f)^*$ is the dualization of $D^* f$.

The proposition then gives us the family of canonical isomorphisms given by the following.

$$\begin{tikzcd} T_p M \arrow[r, "\mathrm{ev}"] \arrow[rrr, bend right, "{[\gamma] \mapsto X_\gamma}"'] & \left(T_p M\right)^{**} \arrow[r, "\alpha^*"] & \left(T_p^* M\right)^* \arrow[r, "\beta"] & \mathcal{D}_p M \end{tikzcd}$$

Here, X_γ is the derivation given by $X_\gamma(\varphi) = (\varphi \circ \gamma)'(0)$ for any $\varphi \in C^\infty(M)$. Note that α^* is the dualization of the linear map $\alpha\colon \left(T_p M\right)^* \to T_p^* M$.

Operations on Differential Forms

Recall that $\mathcal{A}^1(M)$ is the set of differential 1-forms on M, which is equivalent to the space of smooth sections of T^*M. These forms have local expansions of the form

$$\omega = \sum_{i=1}^{n} f_i \, dx_i,$$

where the $f_i \in C^\infty(M)$. The exterior derivative of $f \in \mathcal{C}_{p,M}$ is then given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

We may then view $C^\infty(M) = \mathcal{A}^0(M)$. This follows from the fact that $\lambda^0(T_p^*M) = \mathbb{R}$, implying that $M \times \mathbb{R} = T^*M$, so that smooth sections are smooth functions $f\colon M \to \mathbb{R}$.

The exterior derivative is then a linear map $d\colon \mathcal{A}^0(M) \to \mathcal{A}^1(M)$. Similarly, if X is a vector field on M, and $\omega \in \mathcal{A}^1(M)$, then we may consider $\omega(X) \in C^\infty(M)$ to be defined as follows. In local coordinates, if we write

$$\begin{aligned} X &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \\ \omega &= \sum_{i=1}^{n} f_i \, dx_i, \end{aligned}$$

then

$$\begin{aligned} \omega(X)(p) &= \sum_{i=1}^{n} f_i(p) dx_i \left(\sum_{j=1}^{n} a_i(p) \frac{\partial}{\partial x_j}\right) \\ &= \left(\sum_{i=1}^{n} f_i a_i\right)(p). \end{aligned}$$

We may then expand to differential k-forms, denoted $\mathcal{A}^k(M)$. These are smooth sections of $\Lambda^k(T^*M)$, which are the k-multilinear (as C^∞ modules) alternating maps $(\mathfrak{X}(M))^k \to \mathbb{R}$, where $\mathfrak{X}(M)$ denotes the vector fields on M.

A k-form evaluates on a k-tuple of vector fields and outputs a C^∞ function. In local coordinates, k-forms appear as

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Definition: We define the *exterior product*

$$\begin{aligned} \mathcal{A}^k(M) \times \mathcal{A}^\ell(M) &\to \mathcal{A}^{k+\ell}(M) \\ (\omega, \eta) &\mapsto \omega \wedge \eta, \end{aligned}$$

where in local coordinates, if we have

$$\begin{aligned} \omega &= \sum_I f_I \, dx_I \\ \eta &= \sum_J g_J \, dx_J, \end{aligned}$$

then

$$\omega \wedge \eta = \sum_{I,J} f_I g_J \, dx_I \wedge dx_J.$$

We observe that the exterior product has the property that, for arbitrary vector fields $X_1, \dots, X_{k+\ell} \in \mathfrak{X}(M)$,

$$\omega \wedge \eta(X_1, \dots, X_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) \omega\left(X_{\sigma(1)}, \dots, X_{\sigma(k)}\right) \eta\left(X_{\sigma(k+1)}, \dots, X_{\sigma(\ell)}\right).$$

Definition: We define the *exterior derivative* to be a linear map

$$d\colon \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

given by, if

$$\omega = \sum_I f_I \, dx_I,$$

the expression

$$d\omega = \sum_{j=1}^{n} \sum_I \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I.$$

Proposition: The exterior derivative acts on the exterior product via a graded Leibniz rule. That is, if $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^\ell(M)$, then

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta).$$

The $(-1)^k$ can be viewed as a correction term for the exterior power of T^*M that ω is defined over.

Proposition: For any k and for any $\omega \in \mathcal{A}^k(M)$, we have

$$d(d\omega) = 0.$$

Proof. Observe that for the case of $\mathcal{A}^0(M) = C^\infty(M)$, Clairaut's theorem and the alternating property of the exterior product provides

$$\begin{aligned} d^2 f &= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \\ &= 0. \end{aligned}$$

The argument in higher orders follows similarly. □

Thus, we get a family of vector vector spaces with corresponding linear maps as below, which is known as a *chain complex*.

$$0 \longrightarrow \mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^k(M) \xrightarrow{d} \cdots$$

Since the maps d are linear, we may consider their kernels and images, which yields a construction known as the k-th *de Rham cohomology group*, given by

$$H^k_{DR} = \frac{\ker\left(d|_{\mathcal{A}^k(M)}\right)}{\operatorname{im}\left(d|_{\mathcal{A}^{k-1}(M)}\right)}.$$

The de Rham cohomology groups essentially quantify the failure of this chain complex to be exact (in the sense that the image of one map is the kernel of the next map). This has significant implications for understanding the geometry of the underlying manifold, and we will work with it soon in this course.

Definition: Let $f\colon M \to N$ be a C^∞ map of smooth manifolds. The *pullback* of $\omega \in \mathcal{A}(N)$ by f is the map

$$f^*\colon \mathcal{A}(N) \to \mathcal{A}(M)$$

defined in the natural way discussed below.

Note that the existence of f follows from

- f induces the bundle map $Df\colon TM \to TN$;
- which induces a dual map $D^*f\colon T^*N \to T^*M$;
- which extends to exterior powers $\Lambda D^*f\colon \Lambda T^*N \to \Lambda T^*M$;
- from which we take sections to obtain $f^*\colon \mathcal{A}(N) \to \mathcal{A}(M)$.

To understand the importance of the pullback, we consider the case of $\varphi\colon \mathbb{R}^n \supseteq U \to U' \subseteq \mathbb{R}^n$, which maps coordinates $(y_1, \dots, y_n) \mapsto (x_1, \dots, x_n)$. Recall that a smooth atlas essentially allows us to say that coordinate maps in one local chart "work nicely with" coordinate maps in another local chart given by diffeomorphisms on their overlaps. We want to extend these beyond coordinate charts themselves to differential forms more generally; to do this, we start by trying to understand the pullback map on $\mathbb{R}^n$.

The pullback of φ on x_i, written $\varphi^*(x_i)$, is defined by $x_i(\varphi)$. One can consider the pullback as taking the x_i coordinate of the image of $(y_1, \dots, y_n)$. Similarly, we may examine $\varphi^*\colon \mathcal{A}(U') \to \mathcal{A}(U)$ given by

$$\begin{aligned}\varphi(dx_i) &= d(\varphi^* x_i)\\ &= d(x_i \circ \varphi).\end{aligned}$$

Therefore, if $x_i = x_i(y_1, \dots, y_n)$, then by chain rule, we know that

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial y_j}\, dy_j.$$

For a k-form, the computation $\varphi^*(dx_1 \wedge \cdots \wedge dx_k)$ is given by

$$\begin{aligned}\varphi^*(dx_1 \wedge \cdots \wedge dx_k) &= d(x_1 \circ \varphi) \wedge \cdots \wedge d(x_k \circ \varphi)\\ &= \sum_{j_1 < \cdots < j_k} \frac{\partial(x_1, \dots, x_k)}{\partial\left(y_{j_1}, \dots, y_{j_k}\right)}\, dy_{j_i},\end{aligned}$$

where

$$\frac{\partial(x_1, \dots, x_k)}{\partial\left(y_{j_1}, \dots, y_{j_k}\right)} = \det\left(\left(\frac{\partial x_i}{\partial y_{j_\ell}}\right)_{i,\ell}\right)$$

denotes the Jacobian determinant.

Essentially, the pullback allows us to "amplify" smooth transition maps that define our manifold structure to differential forms on the manifold.

As discussed earlier, we know that k-forms are k-multilinear alternating maps acting on k-tuples of vector fields to yield C^∞ functions. Their particular action can be described relatively succinctly below.

Definition: If $\omega \in \mathcal{A}^k(M)$, then if we write $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$, we have

$$\omega(X_1, \dots, X_k) = \frac{1}{k!} \det\left(\left(\alpha_i(X_j)\right)_{i,j}\right).$$

It then follows that, if we desire to harmonize this action on vector fields with the exterior derivative, that we are forced to use the following characterization.

Proposition: If $\omega \in \mathcal{A}^k(M)$, then if $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$ are vector fields, we have

$$d\omega(X_1, \dots, X_{k+1}) = \frac{1}{k+1}\sum_{i=1}^{k+1}(-1)^{i+1}X_i\left(\omega\left(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\right)\right) + \frac{1}{k+1}\sum_{j=2}^{k+1}\sum_{i=1}^{j}(-1)^{i+j}\omega\left(\left[X_i, X_j\right], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\right).$$

Definition: Let M be a smooth manifold, and $X \in \mathfrak{X}(M)$. We define the *interior product*

$$\iota_X \colon \mathcal{A}^k(M) \to \mathcal{A}^k(M)$$

by

$$(\iota_X\omega)(X_1, \dots, X_{k-1}) = k\omega(X, X_1, \dots, X_{k-1}).$$

If $k = 0$, then $\iota_X = 0$.

Note that ι_X is an anti-derivation of degree -1. That is,

$$\iota_X(\omega \wedge \eta) = \iota_X\omega \wedge \eta + (-1)^k\omega \wedge \iota_X\eta$$

whenever $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^\ell(M)$.

Definition: If $X \in \mathfrak{X}(M)$ is a fixed vector field, then the *Lie Derivative* $L_X \colon \mathcal{A}^k(M) \to \mathcal{A}^k(M)$ is given by

$$(L_X\omega)(X_1, \dots, X_k) = X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^{k}\omega(X_1, \dots, [X, X_i], \dots, X_k).$$

Proposition (Cartan's Magic Formula)**:** Let $X, Y \in \mathfrak{X}(M)$ be vector fields. Then, the following hold:

(i) $L_X\iota_Y - \iota_Y L_X = \iota_{[X,Y]}$;

(ii) $L_X = \iota_X d + d\iota_X$.

Proof.

(i) Let ω be a k-form with $k > 0$. Then, for any $X_1, \dots, X_{k-1} \in \mathfrak{X}(M)$,

$$\begin{aligned} L_X\iota_Y\omega(X_1, \dots, X_{k-1}) &= X(\iota_Y\omega(X_1, \dots, X_{k-1})) - \sum_{i=1}^{k-1}(\iota_Y\omega)(X_1, \dots, [X, X_i], \wedge, X_{k-1}) \\ &= kX(\omega(Y, X_1, \dots, X_{k-1})) - k\sum_{i=1}^{k-1}\omega(Y, X_1, \dots, [X, X_i], \dots, X_{k-1}). \end{aligned}$$

Meanwhile,

$$\begin{aligned} \iota_Y L_X\omega(X_1, \dots, X_{k-1}) &= kL_X(Y, X_1, \dots, X_{k-1}) \\ &= kX(\omega(Y, X_1, \dots, X_{k-1})) - k\omega([X, Y], X_1, \dots, X_{k-1}) \\ &\quad - k\sum_{i=1}^{k-1}\omega(Y, X_1, \dots, [X, X_i], \dots, X_{k-1}). \end{aligned}$$

Subtracting, we get

$$L_X\iota_Y\omega - \iota_Y L_X\omega = \iota_{[X,Y]}\omega.$$

(ii) When $k = 0$, we have

$$L_X f = X(f)$$

$$\begin{aligned}\iota_X f &= 0\\ \iota_X df &= df(X)\\ &= X(f),\end{aligned}$$

so the equivalence holds when $k = 0$.

Now, let $k > 0$, and let ω be a k-form, with $X_1, \dots, X_k$ vector fields. We have

$$\begin{aligned}\iota_X d\omega(X_1, \dots, X_k) &= (k+1)d\omega(X, X_1, \dots, X_k)\\ &= X(\omega(X_1, \dots, X_k)) + \sum_{i=1}^{k} (-1)^i X_i\left(\omega\left(X, X_1, \dots, \widehat{X_i}, \dots, X_k\right)\right)\\ &+ \sum_{j=1}^{k} (-1)^j \omega\left(\left[X, X_j\right], X_1, \dots, \widehat{X}_j, \dots, X_k\right)\\ &+ \sum_{j=1}^{k}\sum_{i=1}^{j-1} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X, \dots, \widehat{X_i}, \dots, \widehat{X}_j, \dots, X_k\right),\end{aligned}$$

and

$$\begin{aligned}d\iota_X\omega(X_1, \dots, X_k) &= \sum_{i=1}^{k} (-1)^{i+1} X_i\left(\omega\left(X, X_1, \dots, \widehat{X_i}, \dots, X_k\right)\right)\\ &+ \sum_{j=1}^{k}\sum_{i=1}^{j-1} (-1)^{i+j} \omega\left(X, \left[X_i, X_j\right], X_1, \dots, \widehat{X_i}, \dots, \widehat{X}_j, \dots, X_k\right).\end{aligned}$$

Adding, we get

$$\begin{aligned}(\iota_X d + d\iota_X)\omega(X_1, \dots, X_k) &= X(\omega(X_1, \dots, X_k)) + \sum_{j=1}^{k} (-1)^j \omega\left(\left[X, X_j\right], X_1, \dots, \widehat{X}_j, \dots, X_k\right)\\ &= L_X\omega(X_1, \dots, X_k).\end{aligned}$$

□

Cartan's formulae allow us to prove some basic facts about the Lie derivative.

Proposition: The following formulae hold:

(i) $L_X(\omega \wedge \eta) = L_X\omega \wedge \eta + \omega \wedge L_X\eta$;

(ii) $(L_X d)\omega = d(L_X\omega)$;

(iii) $L_X L_Y - L_Y L_X = L_{[X,Y]}$.

The Lie derivative is fundamentally related to the one-parameter diffeomorphism group generated by a vector field. To see this, we observe that, given a vector field $X \in \mathfrak{X}(M)$, the "directional derivative" of $f \in C^\infty(M)$ in the direction of X is given by $X(f)$.

There is a different way to consider a directional derivative — that being via the one-parameter diffeomorphism group. We observe that

$$\begin{aligned}X_p(f) &= \lim_{t\to 0} \frac{\left(\varphi_t^* f\right)(p) - f(p)}{t}\\ &= \lim_{t\to 0} \frac{f(\varphi_t(p)) - f(p)}{t},\end{aligned}$$

since $\varphi_t(p) = c_p(t)$, and $\dot{c}_p(0) = X_p$.

Note that the Lie derivative of a vector field is given by $[X, Y]$. Since $((\varphi_{-t})_*Y)(f) = \varphi_t^* Y(f \circ \varphi_{-t})$, we get

$$\begin{aligned}
\lim_{t\to 0} \frac{(\varphi_{-t})_*Y - Y}{t}(f) &= \lim_{t\to 0} \frac{\varphi_t^*(Y(f\circ\varphi_{-t})) - \varphi_t^*Y(f) + \varphi_t^*Y(f) - Y(f)}{t}\\
&= \lim_{t\to 0} \varphi_t^*\left(Y\left(\frac{f\circ\varphi_{-t} - f}{t}\right)\right) + \lim_{t\to 0}\frac{\varphi_t^*Y(f) - Y(f)}{t}\\
&= \lim_{t\to 0} \varphi_t^* Y\left(\frac{\varphi_{-t}^* f - f}{t}\right) + \lim_{t\to 0}\frac{\varphi_t^*Y(f) - Y(f)}{t}\\
&= Y(-X(f)) + X(Y(f))\\
&= [X, Y](f).
\end{aligned}$$

Therefore, as a rule, we write

$$[X, Y] = \lim_{t\to 0}\frac{(\varphi_{-t})_*Y - Y}{t}.$$

Recall that $((\varphi_{-t})_*Y)(f) = \varphi_t^*(Y(f\circ\varphi_{-t}))$.

The Lie derivative of a differential form is thus given as follows.

Proposition: Let X be a vector field with corresponding one-parameter diffeomorphism group $\varphi\colon \mathbb{R} \to \operatorname{diff}(M)$. Then, for a k-form $\omega \in \mathcal{A}^k(M)$, we have

$$L_X\omega = \lim_{t\to 0}\frac{\varphi_t^*\omega - \omega}{t}.$$

Proof. We show that if $\varphi\colon M \to M$ is an arbitrary diffeomorphism, we have

$$(\varphi^*\omega)(X_1, \dots, X_k) = \varphi^*(\omega(\varphi_* X_1, \dots, \varphi_* X_k)).$$

This follows from the fact that, by the definition of the pullback,

$$(\varphi^*\omega)_p(X_1, \dots, X_k) = \omega_{\varphi(p)}(\varphi_* X_1, \dots, \varphi_* X_k).$$

Computing the right-hand side of our desired formula for the Lie bracket, we obtain

$$\begin{aligned}
\lim_{t\to 0}\frac{\varphi_t^*\omega(X_1,\dots,X_k) - \omega(X_1,\dots,X_k)}{t} &= \lim_{t\to 0}\frac{\varphi_t^*(\omega((\varphi_t)_*X_1, \dots, (\varphi_t)_*X_k)) - \omega(X_1,\dots,X_k)}{t}\\
&= \lim_{t\to 0}\frac{\varphi_t^*(\omega((\varphi_t)_*X_1, \dots, (\varphi_t)_*X_k)) - \varphi_t^*(\omega(X_1,\dots,X_k))}{t}\\
&\quad + \lim_{t\to 0}\frac{\varphi_t^*(\omega(X_1,\dots,X_k)) - \omega(X_1,\dots,X_k)}{t}.
\end{aligned}$$

We can see that the latter term evaluates to $X(\omega(X_1, \dots, X_k))$. We may then write the former term by taking successive addition/subtraction replacing a $(\varphi_t)_*X_j$ in the input to ω with simply X_j. This gives

$$\lim_{t\to 0}\frac{\varphi_t^*(\omega((\varphi_t)_*X_1, \dots, (\varphi_t)_*X_k)) - \varphi_t^*(\omega(X_1,\dots,X_k))}{t} = \sum_{i=1}^{k}\omega(X_1, \dots, [-X, X_i], \dots, X_k),$$

so that we get the Lie derivative upon addition. □

The Frobenius Theorem

If X is a vector field on M, then an integral curve through each $p \in M$ is determined via X. If the integral curves through all $p \in M$ can be extended to M, then the integral curves do not intersect each other and completely cover M.

Now, the question arises: what if we look at two vector fields? Three? When can we extend vector fields to partition M into a family of submanifolds with a particular dimension?

Definition: An r-dimensional *distribution* $\mathcal{D}$ on M is an assignment $\mathcal{D}_p$ of r-dimensional subspaces of T_pM for each $p \in M$ such that the family $\mathcal{D}$ varies smoothly with respect to p.

By "varies smoothly," we mean that for each $p \in M$, there are r locally-defined vector fields $X_1, \dots, X_r$ such that $\{X_1, \dots, X_r\}$ form a basis for all q in a neighborhood of p.

Definition: Given a distribution $\mathcal{D}$ on M, we say a submanifold N is an *integral submanifold* for $\mathcal{D}$ if $T_pN = \mathcal{D}_p$ for all $p \in N$. If an integral manifold exists through each point of M (equivalently, if M can be partitioned into a family of integral submanifolds), then we say $\mathcal{D}$ is *completely integrable*.

Proposition: Let $\mathcal{D}$ be a distribution on a manifold M. If $\mathcal{D}$ is completely integrable, then for any $X, Y \in \mathcal{D}$, we have $[X, Y] \in \mathcal{D}$.

Proof. Select a local coordinate system about p, $(U; x_1, \dots, x_n)$ such that p is the origin, and the submanifold N is given by the set of all $(x_1, \dots, x_n)$ such that $x_{r+1} = \cdots = x_n = 0$.

Then, for any $q \in N$, $\mathcal{D}_q$ is the span of $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$. Letting

$$
\begin{aligned}
X &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \\
Y &= \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},
\end{aligned}
$$

we have that

$$
\begin{aligned}
a_i(x_1, \dots, x_r, 0, \dots, 0) &= 0 \\
b_i(x_1, \dots, x_r, 0, \dots, 0) &= 0
\end{aligned}
$$

for any $i > r$, by the definition of $\mathcal{D}$. It follows that for any $i \leqslant r$ and $j > r$, we have

$$
\begin{aligned}
\frac{\partial a_j}{\partial x_i}(0) &= 0 \\
\frac{\partial b_j}{\partial x_i}(0) &= 0.
\end{aligned}
$$

Thus,

$$
[X, Y] = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n}\left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\right)}_{=: c_j} \frac{\partial}{\partial x_j},
$$

whence the coefficient c_j evaluates to 0 for any $j > r$. Thus, $[X, Y]_p \in \mathcal{D}_p$. □

Definition: A distribution $\mathcal{D}$ on a manifold is said to be *involutive* if the Lie bracket of any two vector fields $X, Y \in \mathcal{D}$ also belongs to $\mathcal{D}$.

Theorem (Frobenius Theorem)**:** A distribution $\mathcal{D}$ is completely integrable if and only if $\mathcal{D}$ is involutive.

In order to understand and prove the Frobenius theorem, we need to discuss commutative vector fields.

Definition: We say two vector fields X and Y are commutative if $[X, Y] = 0$.

Commutative vector fields have very nice geometric properties; specifically, there's a way to characterize commutative vector fields via their corresponding flows.

Proposition: Let X and Y be vector fields with corresponding flows φ_t and ψ_t. The following are equivalent:

(i) X and Y commute;

(ii) Y is invariant under φ_t — that is, $(\varphi_t)_* Y = Y$ whenever defined;

(iii) φ_t and ψ_t commute, in that for any $t, s \in \mathbb{R}$, $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ where defined.

Proof. We observe that

$$\begin{aligned}\frac{d}{dt}((\varphi_t)_* Y)\bigg|_{t=t_0} &= \lim_{t\to 0}\frac{\left(\varphi_{t_0+t}\right)_* Y - \left(\varphi_{t_0}\right)_* Y}{t}\\ &= \lim_{t\to 0}\left(\varphi_{t_0}\right)_*\frac{(\varphi_t)_* Y - Y}{t}\\ &= \left(\varphi_{t_0}\right)_*[-X, Y]\\ &= 0.\end{aligned}$$

Thus, $(\varphi_t)_* Y$ does not depend on t, so $(\varphi_t)_* Y = (\varphi_0)_* Y = Y$.

Given a diffeomorphism φ of M, the vector field $\varphi_* Y$ is defined, and the integral curve of $\varphi_* Y$ through $p \in M$ is given by $\varphi \circ c$, where c is the integral curve of y through $\varphi^{-1}(p)$. Thus, the one-parameter diffeomorphism group generated by $\varphi_* Y$ is given by $\varphi \circ \psi_t \circ \varphi^{-1}$.

Now, if we apply this to $(\varphi_t)_* Y$ for each t, we get the one-parameter diffeomorphism group for $(\varphi_t)_* Y$ is given by $\varphi_t \circ \psi_s \circ \varphi_t^{-1}$ with s as the parameter, so since Y is invariant under φ_t, we have $\varphi_t \circ \psi_s \circ \varphi_t^{-1} = \psi_s$, so that $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$.

Finally, we observe that for any $p \in M$, since $\psi_s(p)$ is the integral curve of Y through p, we have

$$\frac{d}{ds}\psi_s(p)\bigg|_{s=0} = Y_p.$$

Now, since $\varphi_t \circ \psi_s \circ \varphi_t^{-1}(p)$ is the integral curve of $(\varphi_t)_* Y$ through p, we have

$$\frac{d}{ds}\varphi_t \circ \psi_s \circ \varphi_t^{-1}(p)\bigg|_{s=0} = ((\varphi_t)_* Y)_p,$$

so that $(\varphi_t)_* Y = Y$, meaning

$$\begin{aligned}[X, Y] &= \lim_{t\to 0} -\frac{(\varphi_{-t})_* Y - Y}{t}\\ &= 0.\end{aligned}$$

$\square$

Integration and de Rham Cohomology

So far, we have only been treating differential forms as these abstract things completely disconnected from the fundamental structure of the manifold. Yet, they are very useful, precisely because we can integrate over differential forms.

Of course, this integration is absolutely not the same as the traditional Riemann or Lebesgue integral, because the measure of a set (that has nonzero measure) depends on the choice of charts.

Consider a differential form $\omega = h(x_1, x_2)dx_1 \wedge dx_2$ on $\mathbb{R}^2$, where

$$\begin{aligned} x_1 &= f(y_1, y_2) \\ x_2 &= g(y_1, y_2). \end{aligned}$$

Then, we showed earlier that, via a change of coordinates, we may write

$$dx_1 \wedge dx_2 = \frac{\partial(f, g)}{\partial(y_1, y_2)} dy_1 \wedge dy_2,$$

where the expression in partials denotes the Jacobian determinant as a part of change of coordinates. In particular, this suggests that

$$\omega = h(f(y_1, y_2), g(y_1, y_2)) \frac{\partial(f, g)}{\partial(y_1, y_2)} dy_1 \wedge dy_2,$$

meaning that the integral over $\mathbb{R}^2$ does not depend on the choice of coordinates.

Now, if M is a connected manifold, we venture to ask exactly what

$$\int_M \omega$$

means.

First, if ω vanishes outside a compact subset of a Euclidean chart, then we may write

$$\omega = f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n,$$

for some $f \in C_c^\infty(\mathbb{R}^n)$. Then, we may define

$$\int_M \omega = \int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dx_1 dx_2 \cdots dx_n,$$

where we use the traditional Riemann integral. This is well-defined as we discussed earlier due to the change of variables formula.

Next, we concern ourselves with compactly supported differential n-forms on M. For now, we assume that all the differential forms we concern ourselves with are n-forms, where n is the dimension of M. We will discuss the case of general k-forms later, as we go deeper into de Rham cohomology.

> **Definition:** We say a differential form ω is *compactly supported* on M if ω vanishes outside a compact subset of M.

In particular, this means that if we have an open cover $\{U_i\}_{i\in\mathbb{N}}$ of M, then we may find $U_1, \dots, U_n$ such that

$$\operatorname{supp}(\omega) \subseteq \bigcup_{i=1}^{n} U_i,$$

by compactness. If $\{f_i\}_{i\in\mathbb{N}}$ is a partition of unity subordinate to $\{U_i\}_{i\in\mathbb{N}}$, then we may define the integral of ω as follows.

> **Definition:** Let ω be a compactly supported differential form on M, and let $\{f_i\}_{i\in\mathbb{N}}$ be a partition of unity subordinate to the open cover $\{U_i\}_{i\in\mathbb{N}}$. Define
>
> $$\omega_i = f_i \omega,$$

so that

$$\omega = \sum_{i=1}^{n} f_i \omega.$$

Then, we define the integral of ω over M to be

$$\int_M \omega = \sum_{i=1}^{n} \int_M \omega_i.$$

Observe that this definition works since each of the ω_i vanishes outside a compact subset of a Euclidean chart.

The question then arises: we know that we can determine the sign of this integral upon the choice of a chart, but what happens if we select a different chart? How do we know that the sign is well-defined? This has to do with the orientation of the underlying manifold.

Orientation

Definition: Let M be a smooth manifold. An *orientation* on M is a coherent choice of ordered basis on T_pM for all $p \in M$. In particular, it is an everywhere non-vanishing section of $\Lambda^n TM$.

Remark: We may consider $\Lambda^n TM$ to be the "determinant bundle" of the manifold. This follows from the fact that linear maps on V extended to $\Lambda^n V$ using the universal property of the exterior product are simply scaling by the determinant of the linear map.

Since smooth sections of $\Lambda^n TM$ are continuous, it follows that if we are able to move along any path in a manner that preserves this "determinant bundle," then we never encounter linear dependence with our tangent vectors.

Not every manifold is oriented; a famous example is that of the Möbius strip.

Another way to conceive of the orientation of a manifold is via a selection of orientation on charts such that they agree on overlaps. In other words, if we have the ordered basis $(x_1, \dots, x_n)$ for a chart (U, φ), and the ordered basis $(y_1, \dots, y_n)$ for a different chart (V, ψ), then the linear map $D(\psi \circ \varphi^{-1})$ is a local diffeomorphism (hence has nonzero determinant). If this determinant is positive, then these two charts have the same orientation.

For now, we may only define the integral $\int_M \omega$ whenever M is oriented.

Remark: If M is not oriented, then there is an oriented "double cover" created by essentially gluing both orientations of M together.

In the next section, we will concern ourselves with manifolds that have boundary, so we discuss a little bit about orientation on the boundary.

Recall that the boundary of M consists of those points whose charts can be identified with the half-space

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \mid x_n \geqslant 0\}.$$

Letting (U, φ) be a chart about $p \in \partial M$, we note that for any $q \in U^\circ$, the local basis for T_qM is given by $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, while for $T_p\partial M$, it is given by $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)$. The induced orientation on $T_p\partial M$ is given by an "outward pointing" tangent vector.

Definition: We say the ordered basis

$$\mathcal{B} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)$$

is positively oriented for $T_p\partial M$ if

$$\mathcal{B}' = \left(-\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)$$

is positively oriented for T_pM.

Stokes's Theorem

The most powerful theorem when it comes to integration on manifold is known as Stokes's Theorem.[VI]

Theorem: Let M be an oriented manifold (possibly with boundary). Then, if $\omega \in \mathcal{A}^{n-1}(M)$ is compactly supported, we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. It suffices to prove the theorem for ω compactly supported on one Euclidean chart. For a chart U on M, we have $U \cong \mathbb{R}^n$ or $U \cong \mathbb{R}^n_+$ Since $\omega \in \mathcal{A}^{n-1}(M)$, we may write

$$\omega = \sum_{i=1}^{n} a_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\widehat{dx_i}$ denotes the exclusion of the index, and a_i are compactly supported C^∞ functions.

Now, if $U \cong \mathbb{R}^n$, we have $\partial\mathbb{R}^n = \emptyset$, so

$$\int_{\partial\mathbb{R}^n} \omega = 0.$$

Ideally, we would find

$$\int_U d\omega = 0.$$

Towards this end, we write

$$d\omega = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial a_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n.$$

Therefore, integrating, we find

$$\int_{\mathbb{R}^n} d\omega = \sum_{i=1}^{n} (-1)^{i+1} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} \, dx_i\right) dx_1 \cdots \widehat{dx_i} \cdots dx_n.$$

By the fundamental theorem of calculus, it follows that the integral in the parentheses is equal to the evaluation of a_i (as an improper integral) at the bounds $-\infty$ and ∞; since a_i is compactly supported, each of these integrals is zero, so we get 0.

Meanwhile, if $U \cong \mathbb{R}^n_+$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n_+} d\omega &= (-1)^n \int_{\partial\mathbb{R}^n_+} \left(\int_0^{\infty} \frac{\partial a_n}{\partial x_n} \, dx_n\right) dx_1 \cdots dx_{n-1} \\
&= \int_{\partial\mathbb{R}^n_+} a_n(x_1, \dots, x_{n-1}, 0) \, dx_1 \cdots dx_{n-1}
\end{aligned}$$

[VI] In my view, it is one of the most beautiful theorems, right up there with the Hahn–Banach Theorems and the Baire Category Theorem.

$$= \int_{\partial \mathbb{R}^n_+} \omega,$$

where the equivalence between the first and second lines follows from the Fundamental Theorem of Calculus for $\frac{\partial a_n}{\partial x_n}$. Note that the sign of the integral depends on the induced orientation of the boundary. □

Observe that Stokes's Theorem is basically a supercharged version of the fundamental theorem of calculus.

Real Simplicial/Singular Homology

To be able to integrate general k-forms on manifolds, we need a structure known as k-chains. Eventually, we will use these differential forms to obtain a graded $\mathbb{R}$-algebra written

$$H^*_{DR}(M) = \bigoplus_{i=0}^{n} H^i_{DR}(M),$$

where the H^i_{DR} are real vector spaces based on the spaces differential forms. Here, "grading" means, if $x \in H^i$, and $y \in H^j$, then $x \cdot y \in H^{i+j}$. Even though the H^i_{DR} are generated by differential forms, we will eventually see that H^*_{DR} is a topological invariant (under homotopy equivalence).

The realm of de Rham cohomology is based on the following fundamental question: if ω is a k-form on M, is $\omega = d\eta$ for some $\eta \in \mathcal{A}^{k-1}(M)$.

The way to motivate de Rham cohomology is to start with simplicial homology, which is concerned with structures known as simplices.

Definition: The k-simplex, $\Delta^k \subseteq \mathbb{R}^k$, is defined as follows:

$$\Delta^k = \left\{ (x_1, \dots, x_k) \;\middle|\; x_i \geqslant 0, \sum_{i=1}^{k} x_i = 1 \right\}.$$

Essentially, Δ^k is the archetypal convex set.

Definition: A *face* of Δ^k consists of

- the images of Δ^k under coordinate projection;
- the opposite face of 0 — i.e., the set $\{(x_1, \dots, x_k) \mid x_1 + \cdots + x_k = 1\}$.

Definition: A topological space X is called a *simplicial complex* if it can be written as a union of simplices where

- if σ_1, σ_2 are simplices in X, then $\sigma_1 \cap \sigma_2$ is either empty or a common face;
- every neighborhood of a point in X intersects finitely many simplices.

Definition: If $v_1, \dots, v_k$ are the vertices of a k-simplex σ, an *orientation* of σ is a choice of order for $v_1, \dots, v_k$. Orientations are *equivalent* if they differ by an even permutation.

Definition: Given an orientation on a set of vertices $(v_0, \dots, v_k)$ identified with the k-simplex, we define the *boundary* to be a weighted sum of the faces of Δ^k, defined as follows:

$$\partial \Delta^k = \sum_{i=0}^{k} (-1)^i (v_0, \dots, \widehat{v_i}, \dots, v_k),$$

where the hat denotes exclusion.

If X is a simplicial complex, we define k-*chains* to be formal $\mathbb{R}$-linear combinations of k-simplices inside X. That is, the free vector space over $\mathbb{R}$ generated by the k-simplices of X.

The boundary map,

$$\partial\colon C_k(X,\mathbb{R}) \to C_{k-1}(X,\mathbb{R}),$$

is defined on the k-simplices of X as before, then extended by linearity.

Proposition (Boundary of the Boundary)**:**

$$\partial \circ \partial \equiv 0.$$

What this means in practice is that $\operatorname{im}(\partial|_{C_{k+1}}) \subseteq \ker(\partial|_{C_k})$. We define

$$\begin{aligned} Z_k(X,\mathbb{R}) &= \ker(\partial\colon C_k(X,\mathbb{R}) \to C_{k-1}(X,\mathbb{R})) \\ B_k(X,\mathbb{R}) &= \operatorname{im}(\partial\colon C_{k+1}(X,\mathbb{R}) \to C_k(X,\mathbb{R})) \end{aligned}$$

to be the *cycles* and *boundaries* of the simplicial complex X respectively.

This gives a *chain complex* as follows.

$$\cdots \longrightarrow C_{k+1}(X,\mathbb{R}) \xrightarrow{\partial} C_k(X,\mathbb{R}) \xrightarrow{\partial} C_{k-1}(X,\mathbb{R}) \longrightarrow \cdots$$

Definition: The k-th *real simplicial homology* of X is defined as

$$H_k(X,\mathbb{R}) = Z_k(X,\mathbb{R})/B_k(X,\mathbb{R}).$$

The primary issue with simplicial homology is that, while it is easier to compute, it is not immediately apparent that it is a topological invariant. The proof that it is, in fact, is extremely difficult. This requires us to create a more canonical homology that is more clearly a topological invariant. This is known as the singular homology.

Definition: If X is a topological space, a *singular* k-*chain* is a continuous map

$$\sigma\colon \Delta^k \to X.$$

The space of singular k-chains is the free $\mathbb{R}$-vector space generated by these continuous maps, and is denoted C_k^{sing}.

If X is a smooth manifold, then we consider the C^∞ maps from Δ^k to X. Specifically, the C^∞ maps on Δ^k are ones that extend to a smooth function on an open neighborhood U of Δ^k.

In order to specify the boundary maps, we recall that Δ^k has a family of $(k-1)$-dimensional faces, with natural inclusions given by

$$\begin{aligned} \varepsilon_0\colon \Delta^{k-1} &\to \Delta^k \\ (x_1,\dots,x_{k-1}) &\mapsto \left(1 - \sum_{i=1}^{k-1} x_i, x_1,\dots,x_{k-1}\right), \end{aligned}$$

and

$$\begin{aligned} \varepsilon_j\colon \Delta^{k-1} &\to \Delta^k \\ (x_1,\dots,x_{k-1}) &\mapsto (x_1,\dots,0,x_j,\dots,x_{k-1}), \end{aligned}$$

where the 0 is at position j.

Definition: If $\sigma\colon \Delta^k \to X$ is a singular k-simplex, then the *boundary* of σ is the formal weighted sum

$$\partial(\sigma) = \sum_{i=0}^{k-1} (-1)^i \sigma \circ \varepsilon_i.$$

The *real singular homology* is given by

$$H_k^{\text{sing}}(X, \mathbb{R}) = \frac{Z_k^{\text{sing}}(X, \mathbb{R})}{B_k^{\text{sing}}(X, \mathbb{R})},$$

where the singular k-cycles and singular k-boundaries are defined analogously to the case of the simiplicial homology.

Theorem: For any simplicial complex X and for all k,

$$H_k^{\text{sing}}(X, \mathbb{R}) \cong H_k(X, \mathbb{R}).$$

These isomorphisms extend to the smooth case if X is a smooth manifold, as follows from a fundamental result from Cairns and Whitehead.

Theorem (Cairns–Whitehead Theorem)**:** If X is a smooth manifold, then X admits a smooth simplicial structure.

Cohomology and Fundamental Classes

An issue with relating real homology to the exterior derivatives on differential forms is that, with the case of forms, the arrows point in the "wrong" direction, relative to the simplicial and singular homology. To fix this issue, we dualize.

Definition: If X is a topological space, then the k-*cochains* are defined by

$$C^k(X, \mathbb{R}) := \hom(C_k(X, \mathbb{R}), \mathbb{R}).$$

The *coboundary map*,

$$\delta\colon C^k(X, \mathbb{R}) \to C^{k+1}(X, \mathbb{R}),$$

is given by the dualization of the boundary map. The k-cocycles and k-coboundaries are defined analogously, and the k-th cohomology group is defined by

$$H^k(X, \mathbb{R}) = \frac{Z^k(X, \mathbb{R})}{B^k(X, \mathbb{R})}.$$

Remark: By this construction, we automatically get two short exact sequences of vector spaces. Here, i denotes the inclusion of the cocycles into C^k, while ι denotes the inclusion of coboundaries into cocycles.

$$\begin{tikzcd} 0 \arrow[r] & Z^k \arrow[r, hook, "i"] & C^k \arrow[r, "\delta_k"] & B^{k+1} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & B^k \arrow[r, hook, "\iota"] & Z^k \arrow[r, two heads, "\pi"] & H^k \arrow[r] & 0 \end{tikzcd}$$

Theorem (Poincaré Duality)**:** If M is a closed (i.e., boundaryless) and orientable manifold, then

$$H^k(M; \mathbb{R}) \cong H_{n-k}(M; \mathbb{R}),$$

where n is the dimension of M.

Now, a simple question emerges: what does all this homology and cohomology have to do with differential forms?

Definition: If M is a smooth manifold, then we define

$$H_{DR}^{k} = \frac{\ker\left(d\colon \mathcal{A}^{k} \to \mathcal{A}^{k+1}\right)}{\operatorname{im}(d\colon \mathcal{A}^{k-1} \to \mathcal{A}^{k})}$$

to be the kth *de Rham cohomology* of M.

Theorem (de Rham's Theorem)**:**

$$H_{DR}^{k}(M) \cong H^{k}(M; \mathbb{R})$$

for all k and all smooth manifolds M.

Now, if M is a connected, closed, smooth manifold, then we know that $M \cong K$ for some simplicial complex K that we are able to extend to an open neighborhood of all simplices. We then define

$$H_{*}(M; \mathbb{R}) := H_{*}(K; \mathbb{R}),$$

and observe that since M is connected, $H_0(M; \mathbb{R}) \cong \mathbb{R}$. If M has dimension n and M is orientable, then $H_n(M; \mathbb{R}) \cong \mathbb{R}$ as well.

Definition: A generator of $H_n(M; \mathbb{R})$ is called a *fundamental class*.

Before we start discussing fundamental classes for simplicial homology over $\mathbb{R}$, we start by discussing fundamental classes for simplicial homology over $\mathbb{F}_2$, which allows us to ignore orientation.

If Δ^{k} is a k-simplex over $\mathbb{F}_2$, then $\partial\Delta^{k}$ is a sum over faces. Then, if M is a closed manifold with simplicial structure K, we have

$$[M] = \sum \left\{\sigma \,\middle|\, \sigma \text{ is a max-dimensional simplex}\right\}.$$

Now, we observe that if we have a point p on the face of one of these maximum-dimensional simplices, then we must have a second simplex that whose face touches p so as to ensure a local homeomorphism to $\mathbb{R}^{n}$. Therefore, if we compute $\partial[M]$, we observe that every face occurs twice in the computation, whence $[M] \neq 0$ in $H_n(M, \mathbb{F}_2)$.

Other candidates for elements of $H_n(M; \mathbb{F}_2)$ are sub-sums; these are either equal to zero or an element not in the kernel of ∂.

Moving to the case of homology over $\mathbb{R}$, we start by letting Δ^{n} be an n-simplex in a closed, orientable manifold. Each such n-simplex inherits an orientation from M that defines a sign. Then, we may define the fundamental class for M as

$$[M] = \sum \left\{\sigma \,\middle|\, \sigma \text{ is a max-dimensional simplex with inherited orientation}\right\}.$$

In particular, to get $\partial[M] = 0$, which will allow $H_n(M; \mathbb{R}) = \langle [M] \rangle$, we need the codimension-1 faces to cancel out by opposite orientations. This means that

$$\partial[M] = \sum \left\{\text{oriented faces of max-dimensional simplices}\right\},$$

where adjacent max-dimensional simplices have opposite signs on the common face.

Now, the question emerges: how do we know that this is the *only* generator for $H_n(M; \mathbb{R})$? This follows from the fact that $H_n(M; \mathbb{R}) \cong Z_n(M; \mathbb{R})$, as we see that $B_n(M; \mathbb{R}) = 0$. We see that for any simplices σ_1 and σ_2 with common face τ and weights of a and b respectively, that

$$\partial(a\sigma_1 + b\sigma_2) = a\partial(\sigma_1) + b\partial(\sigma_2) \pm (a - b)\tau.$$

Observe that if these weights are not constant on $C \in C_n(M; \mathbb{R})$, then $C \notin Z_n(M; \mathbb{R})$.

Definition: If M and N are closed, orientable, connected manifolds with $f\colon M \to N$ a smooth map, then f induces a map $f_*\colon H_n(M;\mathbb{R}) \to H_n(N;\mathbb{R})$, which is a linear map from $\mathbb{R}$ to $\mathbb{R}$. The multiplier is equal to the *degree* of f.

Generalized Stokes's Theorem on k-Forms

Let $\sigma\colon \Delta^k \to M$ be a smooth, singular k-simplex. We observe that the pullback of $\omega \in \mathcal{A}^k(M)$ with respect to σ, $\sigma^*\omega$, is a k-form on Δ^k. Then, the boundary map ∂ can have its definition extended to a map from singular k-chains to singular $(k-1)$-chains.

Theorem (Stokes's Theorem)**:** If $\omega \in \mathcal{A}^{k-1}(M)$, and σ is a singular k-simplex, then

$$\int_{\partial\sigma} \omega = \int_{\sigma} d\omega,$$

where

$$\int_{\sigma} d\omega := \int_{\Delta^k} \sigma^*(d\omega).$$

Proof. We assume that C is a sole k-simplex, $\sigma\colon \Delta^k \to M$. Observe then that

$$\int_{\sigma} \eta = \int_{\Delta^k} \sigma^*\eta$$

where $\sigma^*\eta$ is the pullback to $\mathbb{R}^n$, and may be written as

$$\sigma^*\eta = \sum_{i=1}^{n} a_i(x)\, dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_k,$$

where the hat denotes exclusion. Therefore, we may assume that $\sigma^*\omega$ is of the form

$$\sigma^*\omega = a(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k,$$

meaning

$$\begin{aligned}\sigma^*(d\omega) &= \frac{\partial a}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k\\ &= (-1)^{j-1}\frac{\partial a}{\partial x_j} dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_k.\end{aligned}$$

Letting $\varepsilon_i\colon \Delta^{k-1} \hookrightarrow \Delta^k$, $0 \leqslant i \leqslant k-1$ be the family of inclusion maps defined by

$$\begin{aligned}\varepsilon_0(x_1,\dots,x_{k-1}) &= \left(1 - \sum_{i=1}^{k-1} x_i, x_1, \dots, x_{k-1}\right)\\ \varepsilon_i(x_1,\dots,x_{k-1}) &= (x_1,\dots,0,\dots,x_{k-1}),\end{aligned}$$

where the 0 is at position i when $i > 0$, we observe that

$$d\sigma = \sum_{i=0}^{k} (-1)^i \sigma \circ \varepsilon_i.$$

Therefore, it suffices to show that

$$(-1)^{j-1}\int_{\Delta^k} \frac{\partial a}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_k = \sum_{i=0}^{k} \int_{\Delta^{k-1}} \varepsilon_i^*\Big(a(x)\, dx_1 \wedge \cdots \widehat{dx_j} \wedge \cdots \wedge dx_k\Big) \tag{$*$}$$

Evaluating $\varepsilon_i^*(dx_\ell)$, we see that

$$\begin{aligned}
\varepsilon_0(dx_1) &= -\sum_{i=1}^{k-1} dx_i \\
\varepsilon_0(dx_j) &= dx_{j-1} && j \neq 1 \\
\varepsilon_i(dx_j) &= dx_j && j < i \\
\varepsilon_i(dx_j) &= 0 && j = i \\
\varepsilon_i(dx_j) &= dx_{j-1}. && j > i
\end{aligned}$$

The right-hand-side of (*) becomes

$$\begin{aligned}
&(-1)^{j-1}\int_{\Delta^k} a\left(1-\sum_{i=1}^{k-1} x_i, x_1, \dots, x_{k-1}\right) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&+ (-1)^j \int_{\Delta^{k-1}} a\left(x_1, \dots, x_{j-1}, 0, x_j, \dots, x_{k-1}\right) dx_1 \wedge \cdots \wedge dx_{k-1}.
\end{aligned}$$

By taking a change of coordinates on Δ^{k-1}mapping

$$(x_1, \dots, x_{k-1}) \mapsto \left(x_2, \dots, x_{j-1}, \sum_{i=1}^{k-1} x_i, x_j, \dots, x_{k-1}\right),$$

the second term disappears and the first collapses to

$$(-1)^{j-1}\int a\left(x_1, \dots, x_{j-1}, 1-\sum_{i=1}^{n} x_i, x_{j+1}, \dots, x_{k-1}\right) dx_1 \wedge \cdots \wedge dx_{k-1}.$$

Meanwhile, evaluating the left-hand-side, we see that by the fundamental theorem of calculus,

$$\int_{\Delta^k} \frac{\partial a}{\partial x_j}\, dx_1 \wedge \cdots \wedge x_k = \int_{\Delta^{k-1}} \int_1^{1-\sum_{i\neq j} x_i} \frac{\partial a}{\partial x_j}\, dx_j\, dx_1 \wedge \cdots \widehat{dx_j} \wedge \cdots \wedge dx_k,$$

where the Δ^{k-1} refers to the face where the j coordinate is equal to zero. This yields

$$\int_{\Delta^{k-1}} a\left(x_1, \dots, x_{j-1}, 1-\sum_{i\neq j} x_i, x_{j+1}, \dots, x_k\right) - a\left(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_k\right) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k.$$

□

Closed and Exact Forms

Definition: We call $\omega \in \mathcal{A}^k(M)$ *closed* if $d\omega = 0$, and *exact* if there is $\eta \in \mathcal{A}^{k-1}(M)$ such that $\omega = d\eta$.

We define

$$H_{DR}^k = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

If ω is a closed form, then the *cohomology class* of ω is the coset of ω summed with exact forms.

We know already that if $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^\ell(M)$, then $\omega \wedge \eta \in \mathcal{A}^{k+\ell}(M)$. We would hope that this comports with the de Rham cohomology, in the sense that H_{DR}^* would then have a grading structure.

Toward this end, if $\omega \in H^k_{DR}$ and $\eta \in H^\ell_{DR}$, then

$$\begin{aligned} d(\omega \wedge \eta) &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \\ &= 0. \end{aligned}$$

Yet, for this to be well-defined, we need to show that it is independent of coset representative. Let $\omega' = \omega + d\xi$, and $\eta' = \eta + d\tau$. We would require $\omega' \wedge \eta' - \omega \wedge \eta$. Observe that

$$\begin{aligned} (\omega + d\xi) \wedge (\eta + d\tau) &= \omega \wedge \eta + \omega \wedge d\tau + d\xi \wedge \eta + d\xi \wedge d\tau \\ &= d\Big((-1)^k \omega \wedge \tau + \xi \wedge \eta + \xi \wedge d\tau\Big). \end{aligned}$$

Thus, integration defines a linear map from $\mathcal{A}^k(M) \to S^k_\infty(M)$, where $S^k_\infty(M)$ denotes the C^∞ singular k-cochains on M, given by

$$I(\omega)(\sigma) = \int_{\Delta^k} \sigma^* \omega,$$

and extended by linearity. By Stokes, it follows that

$$\begin{aligned} I(d\omega)(c) &= \int_c d\omega \\ &= \int_{\partial c} \omega \\ &= I(\omega)(\partial c) \\ &= \delta(I(\omega))(c), \end{aligned}$$

whence $I \circ d = \delta \circ I$. Thus, the following diagram commutes.

$$\begin{tikzcd} \mathcal{A}^k(M) \arrow[r, "d"] \arrow[d, "I"'] & \mathcal{A}^{k+1}(M) \arrow[d, "I"] \\ S^k_\infty(M) \arrow[r, "\delta"'] & S^{k+1}_\infty(M) \end{tikzcd}$$

Theorem (de Rham's Theorem)**:** The map I induces an isomorphism

$$H^*_{DR}(M) \to H^*(M; \mathbb{R}),$$

where the wedge product in de Rham cohomology maps to the *cup product* in the real singular cohomology.

In order to prove de Rham's Theorem, we need to venture into the realm of Čech cohomology; here, the Poincaré lemma will be of much help.

Theorem (Poincaré Lemma)**:** Let M be an arbitrary smooth manifold. Define two maps

$$\begin{aligned} \pi\colon M \times \mathbb{R} &\to M \\ (p, t) &\mapsto p \\ \iota\colon M &\to M \times \mathbb{R} \\ p &\mapsto (p, 0). \end{aligned}$$

Then, the map $\pi^*\colon H^*_{DR}(M) \to H^*_{DR}(M \times \mathbb{R})$ is an isomorphism in de Rham cohomology with inverse ι^*.

Proof. We observe that $\pi \circ \iota = \mathrm{id}_M$, whence $\iota^* \circ \pi^* = \mathrm{id}_{H^*_{DR}(M)}$. Therefore, it suffices to show $\pi^* \circ \iota^* = \mathrm{id}_{H^*_{DR}(M \times \mathbb{R})}$.

Toward this end, we try to find a map $\Phi\colon \mathcal{A}^k(M \times \mathbb{R}) \to \mathcal{A}^{k-1}(M)$ such that

$$\mathrm{id} - \pi^* \circ \iota^* = d\Phi + \Phi d.$$

We observe that the map on the right-hand-side maps closed forms to exact forms, meaning that it induces the zero map in de Rham cohomology. We call such a Φ a *chain homotopy*.

In local coordinates, define Φ acting on

$$\omega = \sum_I a_i\, dx_I + \sum_J b_J\, dt \wedge dx_J,$$

where I consists of k indices and J consists of $k-1$ indices, by taking

$$\Phi(\omega) = \sum_J \left(\int_0^t b_J\, d\tau\right) dx_J.$$

We claim that Φ is our desired chain homotopy. Toward this end, we consider two cases as follows.

)i If $\omega = a(x,t)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then

$$\begin{aligned} \Phi(\omega) &= 0 \\ \Phi(d\omega) &= \left(\int_0^t \frac{\partial a}{\partial \tau}\, d\tau\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (a(x,t) - a(x,0))\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \omega - \pi^* \circ \iota^*(\omega), \end{aligned}$$

where $\iota^*(\omega) \in \mathcal{A}^k(M)$ involves forgetting the t coordinate, while $\pi^*(\iota^*(\omega)) \in \mathcal{A}^k(M \times \mathbb{R})$ uses inclusion with t-coordinate 0.

)ii If $\omega = b(x,t)\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$, then we observe that

$$\iota^*(\omega) = 0,$$

as follows from the fact that $\iota^*(dt) = 0$, so that

$$(\mathrm{id} - \pi^* \circ \iota^*)(\omega) = \omega,$$

while

$$\begin{aligned} d(\Phi(\omega)) &= d\left(\left(\int_0^t b(x,\tau)\, d\tau\right) dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}\right) \\ &= \omega + \sum_{m=1}^{n} \left(\int_0^t b(x,\tau)\, d\tau\right) dx_m \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}} \\ \Phi(d\omega) &= \Phi\left(-\sum_{m=1}^{n} \frac{\partial b}{\partial t}\, dt \wedge dx_m \wedge \cdots \wedge dx_{j_{k-1}}\right) \\ &= -\sum_{m=1}^{n} \left(\int_0^t \frac{\partial b}{\partial \tau}\, d\tau\right) dx_m \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}. \end{aligned}$$

□

Corollary: For all k and all n, we have

$$H^k_{DR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{else} \end{cases}.$$

Corollary: If $f\colon M \to N$ and $g\colon M \to N$ are smoothly homotopic smooth maps, then f^* and g^* induce the same map on H^*_{DR}.

Čech Cohomology and the Proof of the de Rham Theorem

Recall the statement of de Rham's theorem.

Theorem (de Rham's Theorem)**:**

$$H^*_{DR}(M) \cong H^*(M; \mathbb{R}).$$

This is a very hard theorem, not least of which is that the left-hand side consists of equivalence classes of closed forms modulo exact forms, while the right-hand side consists of cocycles modulo coboundaries.

To prove this theorem, we enter the realm of Čech cohomology, which is a type of cohomology related to an open cover, and is written $\check{H}^*(M; \mathcal{U})$. We will show that for a particularly nice class of open covers, we have

$$\check{H}^*(M; \mathcal{U}) \cong H^*_{DR}(M).$$

Definition: Let X be any topological space, and let $\mathcal{U}$ be an open cover. The *nerve* of $\mathcal{U}$, written $\mathcal{N}(\mathcal{U})$, is a simplicial complex defined as follows.

- The vertices are elements of $\mathcal{U}$.
- The face relations are defined by nonempty intersections of elements of $\mathcal{U}$.

If $X = M$ is a manifold, then the Čech cohomology of M with respect to a locally finite open cover $\mathcal{U}$ is

$$\check{H}^*(M; \mathcal{U}) := H^*(\mathcal{N}(\mathcal{U}); \mathbb{R}).$$

We observe that the choice of $\mathcal{U}$ influences $\check{H}^*(M; \mathcal{U})$, so we need to find a way to choose a $\mathcal{U}$ that "preserves" the fundamental structure of M in the process.

Definition: An open subset V of a manifold M is called *contractible* if for any $p \in V$, the identity map is homotopic to the constant map $x \mapsto p$.

An open cover of a manifold is called a *good cover* if for each $U_\alpha \in \mathcal{U}$, U_α is contractible, and every finite nonempty intersection of elements of $\mathcal{U}$ is also contractible.

Proposition: If M is a manifold, there is a locally finite good cover of M, $\mathcal{U}$, such that $H^*(M; \mathbb{R}) \cong \check{H}^*(M; \mathcal{U})$.

Proof. Let K be the simplicial complex for M. We define the *open star cover* of K to be the family of open sets defined by

$$\mathcal{O}_v := \bigcup_{\substack{v \in \sigma \\ \sigma \in K}} \sigma^\circ$$

where σ° denotes the open simplex of σ obtained by removing the boundary.

First, we observe that this is a good open cover. This follows from the fact that simplices themselves are contractible, and the fact that the open star $\mathcal{O}_v$ for a vertex v always contains v itself, meaning that any homotopy to any point of the simplex can be "routed" through v.

We see that $\mathcal{N}(\mathcal{U}) = K$ by observing an ℓ-simplex is contained in K if and only if

$$\mathcal{O}_{v_1} \cap \cdots \cap \mathcal{O}_{v_{\ell+1}} \neq \emptyset.$$

Therefore, we find that M admits a good open cover such that $H^*(M;\mathbb{R}) \cong \check{H}^*(M;\mathcal{U})$. $\square$

Theorem: Let $\mathcal{U}$ be a good open cover of a smooth manifold M. Then, there is a natural isomorphism

$$H^*_{DR}(M) \cong \check{H}^*(M;\mathcal{U}).$$

To prove this theorem, we must prepare the particular structure and double chain complex that we will do a diagram chase on. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$, and let $\mathcal{A}^{k,\ell}(\mathcal{U})$ be the set of assignments

$$\omega(\alpha_0,\dots,\alpha_k) \in \mathcal{A}^\ell\left(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}\right),$$

where the α_i are ordered such that their permutation is equal to the original multiplied by the sign of the permutation, and $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \neq \emptyset$.

Define two boundary operators

$$\begin{aligned}
\delta\colon \mathcal{A}^{k,\ell}(\mathcal{U}) &\to \mathcal{A}^{k+1,\ell}(\mathcal{U})\\
d\colon \mathcal{A}^{k,\ell}(\mathcal{U}) &\to \mathcal{A}^{k,\ell+1}(\mathcal{U}),
\end{aligned}$$

given by

$$\begin{aligned}
(\delta\omega)(\alpha_0,\dots,\alpha_{k+1}) &= \sum_{i=0}^{k+1}(-1)^i\omega(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_{k+1})\\
(d\omega)(\alpha_0,\dots,\alpha_k) &= d(\omega(\alpha_0,\dots,\alpha_k)).
\end{aligned}$$

We observe that the following relations hold:

$$\begin{aligned}
\delta\circ\delta &= 0\\
d\circ d &= 0\\
\delta\circ d &= d\circ\delta.
\end{aligned}$$

This enables us to create the following double complex, where i is the inclusion of the locally constant functions $C^k(M;\mathcal{U})$, and r is the restriction of $\omega \in \mathcal{A}^\ell(M)$ to each open set U_α.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \vdots & \\
0 \arrow[r] & \mathcal{A}^\ell(M) \arrow[r, "r"] \arrow[u, "d"] & \mathcal{A}^{0,\ell}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \mathcal{A}^{1,\ell}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \arrow[r, "\delta"] & \mathcal{A}^{k,\ell}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & & \vdots \arrow[u, "d"] & \\
0 \arrow[r] & \mathcal{A}^1(M) \arrow[r, "r"] \arrow[u, "d"] & \mathcal{A}^{0,1}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \mathcal{A}^{1,1}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \arrow[r, "\delta"] & \mathcal{A}^{k,1}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \mathcal{A}^0(M) \arrow[r, "r"] \arrow[u, "d"] & \mathcal{A}^{0,0}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \mathcal{A}^{1,0}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \arrow[r, "\delta"] & \mathcal{A}^{k,0}(\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & & C^0(M;\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "i"] & C^1(M;\mathcal{U}) \arrow[r, "\delta"] \arrow[u, "i"] & \cdots \arrow[r, "\delta"] & C^k(M;\mathcal{U}) \arrow[r, "\delta"] \arrow[u] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

In this diagram, the left-most column is the de Rham complex, and the bottom row is the Čech complex.

Proposition: Each row and column of the above double complex is exact. That is, for arbitrary k and ℓ, the sequences

$$0 \longrightarrow \mathcal{A}^{\ell}(M) \xrightarrow{r} \mathcal{A}^{0,\ell}(\mathcal{U}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{A}^{k,\ell}(\mathcal{U}) \xrightarrow{\delta} \cdots$$

$$0 \longrightarrow C^{k}(M;\mathcal{U}) \xrightarrow{i} \mathcal{A}^{k,0}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{k,\ell}(\mathcal{U}) \xrightarrow{d} \cdots$$

are exact.

Proof. We start by showing that the first sequence is exact. It is the case that $r\colon \mathcal{A}^{\ell}(M) \to \mathcal{A}^{0,\ell}(\mathcal{U})$ is injective, as the latter space contains forms that are able to not be consistently defined outside an element of $\mathcal{U}$.

Now, suppose that $\omega \in \mathcal{A}^{0,\ell}(\mathcal{U})$ satisfies $\delta\omega = 0$. Then, for any $\alpha, \beta \in A$, we have

$$\omega(\alpha)|_{U_\alpha \cap U_\beta} = \omega(\beta)|_{U_\alpha \cap U_\beta},$$

whence ω is a differential form defined on the whole of M, meaning $\omega \in \operatorname{im}(r)$.

For general $k > 0$, define

$$\Phi\colon \mathcal{A}^{k,\ell}(\mathcal{U}) \to \mathcal{A}^{k,\ell-1}(\mathcal{U})$$

by selecting a partition of unity $\{f_\alpha\}_{\alpha\in A}$ subordinate to $\mathcal{U}$, and for $\omega \in \mathcal{A}^{k,\ell}(\mathcal{U})$, define

$$(\phi(\omega))(\alpha_0, \dots, \alpha_{k-1}) = \sum_{\alpha} f_\alpha \omega(\alpha, \alpha_0, \dots, \alpha_{k-1}).$$

Upon tedious computations, we can show that $\delta(\phi(\omega)) + \phi(\delta(\omega)) = \omega$. Since $\delta(\omega) = 0$, this gives $\omega = \delta(\phi(\omega))$, where $\phi(\omega) \in \mathcal{A}^{k-1,\ell}(\mathcal{U})$. Thus, the first sequence is exact.

Now, we consider the column sequence. Since i is an inclusion of locally constant functions on $\mathcal{U}$ to $\mathcal{A}^{k,0}(\mathcal{U})$, the sequence is exact at i. Suppose $\omega \in \mathcal{A}^{k,0}(\mathcal{U})$ satisfies $d\omega = 0$. Observe that ω assigns a C^∞ function to each nonempty $(k+1)$-fold intersect

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \neq \emptyset.$$

Since the intersection is contractible, it is path-connected, whence ω is constant on each nonempty intersection. Thus, we observe that ω can be defined via elements of $C^k(M;\mathcal{U})$, so that ω is exact.

Next, if $\ell > 0$ and $\omega \in \mathcal{A}^{k,\ell}(\mathcal{U})$ satisfies $d\omega = 0$, we use the fact that the de Rham cohomology of contractible open subsets is trivial, whence there is some $\eta \in \mathcal{A}^{k,\ell-1}$ such that $d\eta = \omega$. □

Theorem (Isomorphism between de Rham Cohomology and Čech Cohomology)**:**

$$H^*_{DR}(M) \cong \check{H}^*(M;\mathcal{U}).$$

Proof. The method of proof we will use is a diagram chase. In particular, we will find a candidate isomorphism $\varphi\colon H^{\ell}_{DR}(M) \to \check{H}^{\ell}(M;\mathcal{U})$ that will take a de Rham cohomology class $x \in H^{\ell}_{DR}(M)$ and output a cocycle in $\check{H}^*(M;\mathcal{U})$.

Choose a closed form $\omega \in \mathcal{A}^{\ell}(M)$ representing x. Put $r(\omega) = \omega_0 \in \mathcal{A}^{0,\ell}(\mathcal{U})$. Since $d\omega_0 = d(r(\omega)) = r(d\omega) = 0$, it follows from exactness that there is $\eta_0 \in \mathcal{A}^{0,\ell-1}(\mathcal{U})$ with $d\eta_0 = \omega_0$. Put $\omega_1 = \delta\eta_0$.

$$\begin{tikzcd}
 & {} & \\
\cdots \arrow[r, "\delta"] & \mathcal{A}^{i,\ell-i}(\mathfrak{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \mathcal{A}^{i+1,\ell-i}(\mathfrak{U}) \\
 & \mathcal{A}^{i,\ell-i-1}(\mathfrak{U}) \arrow[r, "\delta"] \arrow[u, "d"] & \mathcal{A}^{i+1,\ell-i-1}(\mathfrak{U}) \arrow[u, "d"] \\
 & & {} \arrow[u, "d"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & {} & \\
\cdots \arrow[r, "\delta"] & \omega_i \arrow[r, "\delta"] \arrow[u, "d"] & 0 \\
 & \eta_i \arrow[r, "\delta"] \arrow[u, "d"] & \omega_{i+1} \arrow[u, "d"] \\
 & & {} \arrow[u, "d"]
\end{tikzcd}$$

Putting $\omega_1 = \delta\eta_0$, it follows that

$$\begin{aligned}
d\omega_1 &= d(\delta\eta_0)\\
&= \delta(d\eta_0)\\
&= \delta\omega_0\\
&= \delta(r(\omega))\\
&= 0,
\end{aligned}$$

whence there exists $\eta_1 \in \mathcal{A}^{1,\ell-2}(\mathfrak{U})$ constructed similarly. Inductively, we find ω_i such that $d\omega_i = 0$, whence $\omega_i = d\eta_i$, and $\omega_{i+1} = \delta\eta_{i+1}$. This yields $\omega_\ell \in \mathcal{A}^{\ell,0}(\mathfrak{U})$. Therefore, there exists $c \in C^\ell(\mathfrak{U})$ such that $\omega_\ell = i(c)$. Since $\delta c = 0$, we see that c is a cocycle.

Define $\varphi(x) = [c] \in \check{H}^\ell(M;\mathfrak{U})$.

Now, we prove that $\varphi(x)$ is well-defined. Assuming that $\omega' = \omega + d\gamma_0$ for some $\gamma_0 \in \mathcal{A}^{\ell-1}(M)$, we have $\omega_0' = \omega_0 + r(d\gamma_0) = \omega_0 + d(r(\gamma_0))$.

Inductively, we show that for arbitrary $i = 0, \dots, \ell$, there is $\gamma_i \in \mathcal{A}^{i-1,\ell-i-1}(\mathfrak{U})$ such that

$$\omega_i' = \omega_i + d(\delta\gamma_i).$$

We have shown it for the case of $i = 0$. Since $\omega_i = d\eta_i$, and $\omega_i' = d\eta_i'$ by definition, whence

$$\begin{aligned}
d\left(\eta_i' - \eta_i - \delta\gamma_i\right) &= \omega_i' - \omega_i - d(\delta\gamma_i)\\
&= 0,
\end{aligned}$$

so by exactness, there is $\gamma_{i+1} \in \mathcal{A}^{i,\ell-i-2}(\mathfrak{U})$ such that $d\gamma_{i+1} = \eta_i' - \eta_i - \delta\gamma_i$, whence

$$\begin{aligned}
\omega_i' &= \delta\eta_{i'}\\
&= \delta\eta_i + \delta(d\gamma_{i+1})\\
&= \omega_{i+1} + d(\delta\gamma_{i+1}).
\end{aligned}$$

Now, if we put $i = \ell$, we find that there is $\gamma_\ell \in C^{\ell-1}(\mathfrak{U})$ such that $\omega_\ell' = \omega_\ell + i(\delta\gamma_\ell)$. Yet, since $\omega_\ell' = i(c')$ and $\omega = i(c)$, we have $c' = c + \delta\gamma_\ell$, whence c and c' represent the same cohomology class. Thus, we have

> defined our homomorphism $\varphi\colon H^*_{DR}(M) \to \check{H}^*(M; \mathcal{U})$.
>
> In order to see that this is an isomorphism, we observe that there are inverses; in the diagram, we followed a "zig-zag" path down from $\mathcal{A}^\ell(M)$ to $C^\ell(\mathcal{U})$, but we can start at $C^\ell(\mathcal{U})$ and follow the same zig-zag path using exactness to obtain an element of $\mathcal{A}^\ell(M)$. $\square$

Now, we have established two separate cohomology isomorphisms; the one between de Rham and Čech cohomology, $\varphi\colon H^*_{DR}(M) \to \check{H}^*(M; \mathcal{U}) \cong H^*(M; \mathbb{R})$, and integration over forms, $I\colon H^*_{DR}(M) \to H^*(M; \mathbb{R})$, giving two maps for a representative of a cohomology class ω,

$$\begin{aligned} \varphi(\omega) &= c \\ I(\omega) &= c', \end{aligned}$$

two cochains in $H^*(M; \mathbb{R})$. As it turns out, these cochains are of the same class, up to a sign,

$$I_{H^\ell} = (-1)^{\frac{\ell(\ell+1)}{2}} \varphi_{H^\ell}.$$

Product Structure in Simplicial/Singular Cohomology

Thus, we have finally established the isomorphism

$$H^*_{DR}(M) \cong H^*(M; \mathbb{R}).$$

Yet, we know that there is a product structure in $H^*_{DR}(M)$, as for any $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^\ell(M)$, we have the wedge product $\omega \wedge \eta \in \mathcal{A}^{k+\ell}(M)$. We will now develop the corresponding product structure on $H^*(M; \mathbb{R})$.

Let K be a simplicial complex for M. Then, there is a cell structure (not necessarily a simplicial complex) on $K \times K$. For any simplices $\sigma, \tau \in K$, we need to subdivide the corresponding simplex $\sigma \times \tau \in K \times K$. Assuming we can do this, given cochains c, c' of degree k, ℓ respectively on K, we may build a cochain $c \times c' \in C^{k+\ell}(K \times K)$ by taking

$$c' \times c'(\sigma \times \tau) = c(\sigma)c'(\tau)$$

for $\sigma \in C^k(K)$ and $\tau \in C^\ell(K)$.

Notations

- A general normed space V will have its norm denoted by $\|\cdot\|$. If $V = \mathbb{R}^n$, then we denote the norm by $|\cdot|$.
- We denote topological spaces by (X, τ).
- $U(x, r) = \{y \in V \mid \|x - y\| < r\}$.
- $B(x, r) = \{y \in V \mid \|x - y\| \leqslant r\}$.
- $\mathcal{N}_p$: neighborhood system centered at $p \in X$.
- $\mathcal{O}_p$: system of *open* neighborhoods centered at $p \in X$.
- When we say a number n is positive, we mean that $n \geqslant 0$. Similarly, a sequence $(a_n)_n$ is decreasing (increasing) if $a_n \geqslant a_{n+1}$ ($a_n \leqslant a_{n+1}$).
- We define $\mathcal{C}_{p,M}$ to be the space of C^∞ functions $\varphi\colon (-\varepsilon, \varepsilon) \to M$ such that $\varphi(0) = p$.
- We write $\mathcal{X}(M)$ to be the space of vector fields on M.