Basic Properties

Definition: A topological space M is called a *manifold* if it satisfies the following:

- M is Hausdorff (points can be separated by open sets);
- M is second countable (the basis for the topology of M is countable);
- M is locally Euclidean (every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$ for some n).

In particular, the third condition says that for every $p \in M$, there is $U \in \mathcal{O}_p$ and a homeomorphism $\varphi\colon U \to \mathbb{R}^n$. The value of n is called the *dimension* of the manifold M.

Definition: Let M be an n-manifold. A *chart* on M is a pair (U, φ) such that $U \subseteq M$ is open, $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism.

A family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ is known as an *atlas* if

$$M = \bigcup_{i\in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point $p \in M$ and two charts (U, φ_U) and (V, φ_V) such that $p \in U$ and $p \in V$. The functions $\varphi_U\colon U \to \mathbb{R}^n$ and $\varphi_V\colon V \to \mathbb{R}^n$ are homeomorphism, meaning that $\varphi_V \circ \varphi_U^{-1}\colon \varphi_U(U \cap V)^n \to \mathbb{R}^n$ defined on the (nonempty) $U \cap V$ is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs $\varphi_V \circ \varphi_U^{-1}$ are *diffeomorphisms*. To do this, we need to first develop the derivative in $\mathbb{R}^n$.

Definition: Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say f is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear map $L \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\frac{\|f(p + h) - f(p) - Lh\|}{\|h\|} \to 0$$

as $h \to 0$.

The *derivative* of f is the association $f \mapsto L$ for each $p \in \mathbb{R}^n$. We write $D_p f$ to denote this map. Note that we consider elements of $\mathrm{Mat}_n(\mathbb{R})$ as points in $\mathbb{R}^{n^2}$ with the standard topology on $\mathbb{R}^{n^2}$.

A function f is called a *diffeomorphism* if it is (sufficiently) continuously differentiable and has a (similarly sufficiently) continuously differentiable inverse.

Definition: If (U, φ_U) and (V, φ_V) are charts such that $U \cap V \neq \emptyset$, the function $\varphi_V \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is known as the *transition map* between φ_U and φ_V.

A smooth structure for M is an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that for all i, j, the transition maps $\varphi_j \circ \varphi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are diffeomorphisms where defined.

If $\{(U_i, \varphi_i)\}_{i\in I}$ is a *maximal* smooth atlas — i.e., any other smooth atlas that contains $\{(U_i, \varphi_i)\}_{i\in I}$ is equal to $\{(U_i, \varphi_i)\}_{i\in I}$ — then we call $\{(U_i, \varphi_i)\}_{i\in I}$ a *smooth structure* for M.

Note: From now on, we use “manifold” to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

Definition: A map $f\colon M \to N$ between manifolds is called *smooth* if for any chart (U, φ_U) in M and corresponding chart (V, φ_V) in N, the map $\varphi_V \circ f \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^k$ is (sufficiently) continuously differentiable.

The function f is a *diffeomorphism* if f is a smooth bijection with smooth inverse, and we say the manifolds M and N are diffeomorphic if they admit a diffeomorphism.

Remark: If $f\colon M \to N$ is smooth, then any representation of f is smooth. To see this, if (U, φ_1) and (U, φ_2) are charts in M, with corresponding charts (V, ψ_1) and (V, ψ_2), then

$$\psi_1 \circ f \circ \varphi_1^{-1} = \left(\psi_1 \circ \psi_2^{-1}\right) \circ \left(\psi_2 \circ f \circ \varphi_2^{-1}\right) \circ \left(\varphi_2 \circ \varphi_1^{-1}\right),$$

where the transition maps $\psi_1 \circ \psi_2^{-2}$ and $\varphi_2 \circ \varphi_1^{-1}$ are smooth.

More on Smooth Maps

Generally speaking, we will refer to charts on a dimension n smooth manifold by $(U, \varphi) = (U; x_1, \dots, x_n)$, where $x_i\colon U \to \mathbb{R}$ are the coordinates of U. Additionally, if $(\mathbb{R}^n; e_1, \dots, e_n)$ are the identity chart on $\mathbb{R}^n$, and the e_i are standard coordinates on $\mathbb{R}^n$, then the coordinate maps satisfy

$$x_i = e_i \circ \varphi.$$

Definition: Let $(U; x_1, \dots, x_n)$ be a chart on a manifold M of dimension n. If $f\colon M \to \mathbb{R}$ is a C^∞ function, we define the *partial derivative* of f with respect to x_i at p to be

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial\partial e_i}(\varphi(p)).$$

In particular,

$$\frac{\partial f}{\partial x_i} \circ \varphi^{-1} = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial e_i}$$

as functions on $\phi(U)$.

Proposition: The coordinate functions $x_1, \dots, x_n$ satisfy $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Proof. For any $p \in U$, we calculate

$$\begin{aligned}
\frac{\partial x_i}{\partial x_j}(p) &= \frac{\partial\left(x_i \circ \varphi^{-1}\right)}{\partial e_j}(\varphi(p)) \\
&= \frac{\partial((e_i \circ \varphi))}{\partial e_j}(\varphi(p)) \\
&= \frac{\partial e_i}{\partial e_j}(\varphi(p)) \\
&= \delta_{ij}.
\end{aligned}$$

□

Examples

There are a couple special examples of (smooth) manifolds.

(i) Open subsets of $\mathbb{R}^n$ are always manifolds.

(ii) The general linear group, $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices, viewed as a subset of $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, is a manifold. Furthermore, it is an open subset of $\mathbb{R}^{n^2}$, as considering the map $\det\colon \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ given by $A \mapsto \det(A)$, we see that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$.

(iii) The special linear group, $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$, consisting of $n \times n$ matrices with determinant 1, is also a smooth manifold. Furthermore, this manifold is a closed subset of $\mathbb{R}^{n^2}$, as it is equal to $\det^{-1}(\{1\})$.

(iv) The n-sphere, S^n, given by

$$S^n = \left\{ (x_0, \dots, x_n) \;\middle|\; \sum_{i=0}^{n} x_i^2 = 1 \right\}$$

is a manifold in $\mathbb{R}^n$. That it is a smooth manifold is quite a bit less obvious.

Now, in low dimensions, we know that $S^2 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and that the continuously differentiable transformation $z \mapsto \frac{1}{z}$ takes the neighborhood basis of ∞ to deleted neighborhoods of 0, and takes the neighborhood basis of 0 to the neighborhood basis of ∞. This is our desired smooth structure.

In the case of the general S^n, we use two stereographic projections to construct our smooth structure. The first stereographic projection is via the north pole, N_p, and maps points on $S^n \setminus \{N_p\}$ bijectively to $\mathbb{R}^n$; this is a chart that is defined everywhere on S^n except N_p. Similarly, we may use a stereographic projection originating from the south pole, S_p, so as to create another chart defined everywhere except S_p. These two stereographic projections are our desired smooth structure, as these two charts are all that is necessary to cover S^n.

(v) The real projective plane, consisting of lines through the origin in $\mathbb{R}^{n+1}$, can be expressed as

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right)/\mathbb{R}^{\times}.$$

We will show that this is a manifold by constructing a family of charts mapping to $\mathbb{R}^n$.

Consider a point $(r_0, \dots, r_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. If $r_0 \neq 0$, then by dividing, we may associate this point's equivalence class in $\mathbb{RP}^n$ to

$$(1, r_1/r_0, \dots, r_n/r_0) \in \{1\} \times \mathbb{R}^n,$$

so we may associate all points of the form $[(r_0, \dots, r_n)]$ with $r_0 \neq 0$ with a chart (U_0, φ_0) that maps $\mathbb{RP}^n$ to $\mathbb{R}^n$.

Similarly, we may define U_k via

$$U_k = \{[(r_0, \dots, r_n)] \mid r_k \neq 0\}$$

with corresponding chart

$$\begin{aligned} \varphi_k \colon U_k &\to \mathbb{R}^n \\ [(r_0, \dots, r_n)] &\mapsto \frac{1}{r_k}(r_0, \dots, \widehat{r_k}, \dots, r_n), \end{aligned}$$

where $\widehat{r_k}$ denotes the exclusion of the r_k coordinate. Varying k from 0 to n, we see that

$$\mathbb{RP}^n = \bigcup_{k=0}^{n} U_k,$$

the chart functions $\varphi_k \colon U_k \to \mathbb{R}^n$ are homeomorphisms (as they are just division and projections). Furthermore, the transition maps $\varphi_j \circ \varphi_i^{-1}$ are coordinate-wise rational functions defined by

$$(u_1, \dots, u_n) \mapsto \left(\frac{u_1}{u_i}, \dots, \frac{1}{u_i}, \dots, \frac{u_n}{u_i}\right),$$

where the $\frac{1}{u_i}$ is at position j.

(vi) We now turn to a very important example from algebraic geometry: the Grassmannian, $\mathrm{Gr}(k, n)$, consisting of all the k-dimensional subspaces of $\mathbb{R}^n$.

This is a $k(n-k)$-dimensional manifold; we need to understand what the smooth structure is. To do this, we let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$, and for any $V \in \mathrm{Gr}(k, n)$, we consider maps in $\mathrm{Hom}(V, V^{\perp})$, where $V^{\perp}$ denotes the orthogonal complement of V.

Now, we see that if $W \in \mathrm{Gr}(k, n)$ is any other k-dimensional subspace, the orthogonal $P_V \colon \mathbb{R}^n \to V$ restricted to W is a linear isomorphism if and only if $W \not\subseteq V^{\perp}$, or that $W \cap V^{\perp} = \{0\}$.

We see that if W is such that $P_V|_W \colon W \to V$ is a linear isomorphism, the inverse $(P_V|_W)^{-1} \colon V \to W$ is well-defined; so, we may make a correspondence between $\mathrm{Hom}(V, V^{\perp})$ and $\mathrm{Gr}(k, n)$ by noting that any such $T \in \mathrm{Hom}(V, V^{\perp})$ has a corresponding graph $(v, T(v))$, so we take $v \mapsto P_V|_W^{-1}(v)$, then project onto $V^{\perp}$ by taking $T\left(P_{V^{\perp}}\left(P_V|_W^{-1}(v)\right)\right) = T(v)$. We depict it as a diagram below.

$$\begin{tikzcd}
 & & V \\
V \arrow[r, "P_V|_W^{-1}"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "T"'] & W \arrow[r, hook, "\iota"] & \mathbb{R}^n \arrow[u, "P_V"'] \arrow[d, "P_{V^{\perp}}"] \\
 & & V^{\perp}
\end{tikzcd}$$

Defining $U_V = \{W \in \mathrm{Gr}(k, n) \mid W \cap V^{\perp} = \{0\}\}$, we may define the chart from U_V onto $\mathrm{Hom}(V, V^{\perp})$ by $\varphi_V = P_{V^{\perp}} \circ P_V|_W^{-1}$. The family $\{(U_V, \varphi_V) \mid V \in \mathrm{Gr}(k, n)\}$ is our smooth atlas.

Inverse and Implicit Function Theorems

In order to replace manifolds with linear maps, we need to understand smooth maps on $\mathbb{R}^n$. The most important theorems in this regard are the inverse function theorem and the implicit function theorem.

Theorem (Inverse Function Theorem)**:** Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. If $D_p f$ is invertible as a linear map, then f has a local, continuously differentiable inverse $f^{-1} \colon V \to W$, where $p \in W \subseteq U$ and $f(p) \in V \subseteq \mathbb{R}^n$.

Additionally, $D_p\left(f^{-1}\right)$ is given by the inverse of the derivative's corresponding linear map evaluated at p, $D_p\left(f^{-1}\right) = \left(D_p f\left(f^{-1}(p)\right)\right)^{-1}$.

The proof uses the contraction mapping theorem. Recall that if X is a complete metric space, and $f \colon X \to X$ is a strict uniform contraction — that is, there exists $0 \leqslant \lambda < 1$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for all $x, y \in X$ — then f has a unique fixed point.

We begin with a technical lemma.

Lemma: If $U(0, r) \subseteq V$ for some $r > 0$ where V is a normed vector space, $g \colon V \to V$ is a uniform contraction, and $f = \mathrm{id} + g$, then the following hold:

- $(1-\lambda)\|x - y\| \leqslant \|f(x) - f(y)\|$ (in particular, f is injective);
- if $g(0) = 0$, then

$$U(0, (1-\lambda)r) \subseteq f(U(0, r)) \subseteq U(0, (1+\lambda)r).$$

Proof of Lemma. To see the first item, we notice that by the triangle inequality,

$$\begin{aligned}
\|x - y\| - \|f(x) - f(y)\| &\leqslant \|x - y\| - \|x - y\| + \|g(x) - g(y)\| \\
&\leqslant \lambda \|x - y\|,
\end{aligned}$$

so $(1-\lambda)\|x-y\| \leqslant \|f(x)-f(y)\|$, and f is injective. Furthermore, we see that if $g(0)=0$, then

$$\begin{aligned} f(U(0,r)) &= U(0,r) + g(U(0,r)) \\ &\subseteq U(0,r) + \lambda U(0,r) \\ &= U(0,(1+\lambda)r). \end{aligned}$$

Finally, if $y \in U(0,(1-\lambda)r)$, then we want to find x such that $y = f(x) = x + g(x)$; equivalently, we see that we want x such that $x = y - g(x)$. Since the function $F(x) = y - g(x)$ is a translation of a uniform contraction, $F(x)$ is a contraction, so there is a fixed point, meaning $y \in f(U(0,r))$. □

Note: We will use $|\cdot|$ to denote the norm on $\mathbb{R}^n$.

Proof of the Inverse Function Theorem. By using a series of affine maps — first by translating p to 0, then translating $f(p)$ to 0, then inverting $D_0 f$ as per our assumption, we may safely assume that $p = f(p) = 0$ and $D_0 f = \mathrm{Id}$.

Set $g = f - \mathrm{Id}$. We will show that g is a contraction in a sufficiently small ball. Fixing $x, y \in \mathbb{R}^n$, consider the map $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto g(x + t(y-x))$. Notice that by the Fundamental Theorem of Calculus,

$$|g(y) - g(x)| \leqslant |y - x| \sup_{0 \leqslant t \leqslant 1} |g'(x + t(y-x))|.$$

Furthermore, since $g'(0) = \mathbf{0}$ by the fact that $D_0 f = \mathrm{Id}$ and $(\mathrm{Id})' = \mathrm{Id}$, and since f is continuously differentiable, there is $r > 0$ such that

$$|g(y) - g(x)| \leqslant \frac{1}{2}|y - x|$$

for all $x, y \in U(0,r)$. Thus, g is a strict contraction on $U(0,r)$. By the previous lemma, we see that

$$U(0, r/2) \subseteq f(U(0,r));$$

by setting $U = U(0,r) \cap f^{-1}(U(0,r))$, we see that the map $f|_U \colon U \to V := U(0,r/2)$ is a bijection. The inverse function $f^{-1} \colon V \to U$ thus exists.

Now, we let $h = f^{-1}$, $x \in U$, $y \in V$ such that $h(x) = y$, and $A = D_x f$. We will show that $A^{-1} = D_y h$, which is enough to show that h is continuously differentiable, as we assume the map $x \mapsto D_x f$ is continuous, and inversion is continuous in $GL_n(\mathbb{R})$.

For sufficiently small vectors s and k, since f and h are bijections, we have

$$h(y + k) = x + s,$$

so

$$f(x + s) = y + k.$$

Furthermore, by unraveling the definitions of $f = g + \mathrm{Id}$, s, and k, and the fact that g is a uniform contraction on U, we get

$$\begin{aligned} |s - k| &= |(f(x+s) - f(x)) - s| \\ &= |(x + s + g(x+s)) - (x + g(x)) - s| \\ &= |g(x+s) - g(x)| \\ &\leqslant \frac{|s|}{2}. \end{aligned}$$

In particular, since

$$|s| \leqslant |s - k| + |k|$$

$$\leqslant |k| + \frac{|s|}{2},$$

we see that $|s|/2 \leqslant |k|$. We calculate

$$\begin{aligned}\left|h(y+k) - h(y) - A^{-1}k\right| &= \left|x + s - x - A^{-1}(f(x+s) - f(x))\right| \\ &= \left|s - A^{-1}(f(x+s) - f(x))\right| \\ &\leqslant \left\|A^{-1}\right\|_{op} |As - f(x+s) - f(x)|.\end{aligned}$$

Thus, since $|s|/2 \leqslant |k|$,

$$\begin{aligned}\frac{\left|h(y+k) - h(y) - A^{-1}k\right|}{|k|} &\leqslant \frac{2\left\|A^{-1}\right\|_{op} |As - f(x+s) - f(x)|}{|s|} \\ &\to 0,\end{aligned}$$

so $D_y h = A^{-1}$. □

One of the primary uses of the inverse function theorem is to prove the implicit function theorem.

Theorem (Implicit Function Theorem)**:** Let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable, and let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Assume

- $f(a, b) = 0$;
- the map $y \mapsto f(a, y)$ defined on $\mathbb{R}^m \to \mathbb{R}^m$ is invertible in a neighborhood of b — i.e., $D_b(y \mapsto f(a, y))$ as a linear map has rank m.

Then, there exists a continuously differentiable function $g\colon U \to V$, where $U \in \mathcal{O}_a$ and $V \in \mathcal{O}_b$ such that $f(x, g(x)) = 0$ on U.

Essentially, the theorem says that we can solve $f(x, y) = 0$ on a neighborhood of (a, b) by a function only depending on x. This means that about (a, b) in the graph $\Gamma(f)$, there is a coordinate representation as an m-manifold given by g.

Proof of the Implicit Function Theorem. Define a function $F\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$F(x, y) = (x, f(x, y)).$$

Since f is continuously differentiable, this function F is also continuously differentiable, so we may define $U \in \mathcal{O}_a$, $V \in \mathcal{O}_b$, and $W \in \mathcal{O}_{F(a,b)}$ such that

$$F\colon U \times V \to W$$

is continuously differentiable with continuously differentiable inverse (owing to the Inverse Function Theorem), so that $G = F^{-1} = (G_1, G_2)$ is defined on W. We see that

$$(x, y) = F(G_1(x, y), f(G_1(x, y), G_2(x, y))),$$

meaning that $G_1(x, y) = x$, and $y = f(x, G_2(x, y))$. Since at b, $f(a, b) = 0$, we have that $g(x) = G_2(x, 0)$ is the desired function. □

Constructing C^∞ Maps on Manifolds

Definition: A function $f\colon U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open, is called C^∞ if the partial derivatives of all orders,

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

are continuous. Here, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a *multi-index*, where the α_i are positive integers for each i, and $|\alpha|$ is defined by $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

We are concerned now with constructing C^∞ functions on C^∞-manifolds.[I] In order to do this, we introduce the bump functions.

Definition: The *bump function* that is equal to 1 on $B(0,1)$ and is zero outside $U(0,2)$ is given by

$$h(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$
$$b(x) = \frac{h\left(4 - |x|^2\right)}{h\left(4 - |x|^2\right) + h\left(|x|^2 - 1\right)}. \tag{$*$}$$

Lemma: Let M be a C^∞ manifold. Let $U \in \mathcal{O}_p$, and let $f \colon U \to \mathbb{R}$ be an arbitrary C^∞ function defined on U.

Then, there exists $V \in \mathcal{O}_p$ with $\overline{V} \subseteq U$, and a C^∞ function $\widetilde{f}$ defined on M such that

$$\widetilde{f}(q) = \begin{cases} f(q) & q \in V \\ 0 & q \notin U. \end{cases}$$

Proof. Let (W, φ) be a chart centered at p with $\varphi(p) = 0$ and $U(0,3) \subseteq \varphi(W)$. Let $\overline{b} = b \circ \varphi$, where b is the bump function defined in $(*)$. Then, $\overline{b}$ is a C^∞ function on W, and is 0 outside $\varphi^{-1}(U(0,2)) \subseteq W$.

We define $\overline{b}$ to be equal to zero on W^c. Thus, if we define $V = \varphi^{-1}(U(0,1))$, then $V \in \mathcal{O}_p$, $\overline{V} \subseteq U$, and $\overline{b}$ is equal to 1 on V. Letting

$$\widetilde{f}(q) = \begin{cases} \overline{b}(q)f(q) & q \in W \\ 0 & q \notin W, \end{cases}$$

we see that $\widetilde{f}$ satisfies the required property. □

Given an atlas $\{(U_i, \varphi_i)\}$, we want to be able to "glue" functions together by using these charts. A fundamental construction for this purpose is known as a partition of unity.

Definition: Let X be a topological space.

- An open cover $\{U_i\}_{i \in I}$ is called *locally finite* if, for every $x \in X$, there is some $V \in \mathcal{O}_x$ such that $V \cap U_i = \emptyset$ for all but finitely many i.
- Another open cover $\left\{V_j\right\}_{j \in J}$ is called a refinement of another open cover $\{U_i\}_{i \in I}$ if for all $j \in J$, there exists some $i \in I$ such that $V_j \subseteq U_i$.
- We say X is *paracompact* if, for any open cover of X, there is a locally finite refinement.

Proposition: Let M be a topological manifold. Then, for any open cover $\{U_i\}_{i \in I}$ of M, there is a countable, locally finite refinement $\{V_k\}_{k=1}^{\infty}$ with the $\overline{V_k}$ compact. In particular, M is paracompact.

Additionally, we may select the coordinate maps $\psi_k \colon V_k \to \mathbb{R}^n$ such that $\psi_k(V_k) = U(0,3)$, and $\left\{\psi_k^{-1}(U(0,1))\right\}_{k=1}^{\infty}$ is an open cover of M.

Proof. Since M is a locally Euclidean and second countable, there is a countable basis of pre-compact open sets $\{O_\ell\}_{\ell=1}^{\infty}$. In particular, we may select an exhaustion of M by pre-compact sets by defining

$$E_1 = O_1$$

[I] A C^∞ manifold is one where all the transition functions $\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are C^∞ functions.

$$E_k = O_1 \cup O_2 \cup \cdots \cup O_{\ell_k},$$

where ℓ_k is some sufficiently large index as follows. Since $\overline{E_k}$ is compact, there is a sufficiently large ℓ such that $\overline{E_k} \subseteq O_1 \cup \cdots \cup O_\ell$. Defining ℓ_{k+1} to be the smallest index greater than ℓ_k that satisfy this property, we define

$$E_{k+1} = O_1 \cup \cdots \cup O_{\ell_{k+1}}.$$

For arbitrary k, each $\overline{E_k}$ is compact, and $\overline{E_k} \subsetneq E_{k+1}$, and $\bigcup_{k=1}^{\infty} E_k = M$. Note that if M is compact, this process terminates in a finite number of steps.

Now, let $\{U_i\}_{i\in I}$ be an arbitrary open cover of M, and fix $k \geqslant 1$. For each $p \in \overline{E_k} \setminus E_{k-1}$, select i_p such that $p \in U_{i_p}$, and select a chart (V_p, ψ_p) about p that satisfies $\psi_p(p) = 0$, $\psi_p(V_p) = U(0,3)$, and $V_p \subseteq U_{i_p} \cap \left(E_{k+1} \setminus \overline{E_{k-2}}\right)$, where we set $E_{-1} = E_0 = \emptyset$. Finally, set $W_p = \psi_p^{-1}(U(0,1))$.

Since $\overline{E_k} \setminus E_{k-1}$ is compact, we may select a finite number of such p such that the open sets W_p cover $\overline{E_k} \setminus E_{k-1}$. Applying this process to all k, and lining up the charts (V_p, ψ_p) corresponding to the finite number of points p chosen at each stage, we have the locally finite refinement of $\{U_i\}_{i\in I}$ with each $\overline{V_k}$ compact, $\psi_k(V_k) = U(0,3)$, and $\left\{\psi_k^{-1}(U(0,1))\right\}$ an open cover of M. □

Definition: Let M be a C^∞ manifold. A family $\{f_k\}_{k=1}^{\infty}$ of at most countably many C^∞ functions on M is called a *partition of unity* on M if it satisfies:

- for each k, $f_k(p) \geqslant 0$ for all $p \in M$, and the family $\left\{\operatorname{supp}(f_k)\right\}_{k=1}^{\infty}$ is locally finite;
- at all points p on M, $\sum_{k=1}^{\infty} f_k(p) = 1$.

If $\left\{\operatorname{supp}(f_k)\right\}_{k=1}^{\infty}$ is a refinement of an open cover $\{U_i\}_{i\in I}$, then we say the partition of unity is *subordinate* to the open cover.

Theorem: Let M be a C^∞ manifold, and let $\{U_i\}_{i\in I}$ be an open cover of M. Then, there exists a partition of unity $\{f_k\}_{k=1}^{\infty}$ that is subordinate to $\{U_i\}_{i\in I}$.

Proof. Let $\{V_k\}_{k=1}^{\infty}$ be a locally finite refinement of $\{U_i\}_{i\in I}$ such that the charts (V_k, ψ_k) have $\psi_k(V_k) = U(0,3)$.

For each k, using the bump function (∗), define

$$\widetilde{b_k}(q) = \begin{cases} b \circ \psi_k(q) & q \in V_k \\ 0 & q \notin V_k. \end{cases}$$

Then, $\widetilde{b_k}$ is a C^∞ function defined on M, and since $\operatorname{supp}\left(\widetilde{b_k}\right) \subseteq V_k$, we may set

$$f = \sum_{k=1}^{\infty} \widetilde{b_k}.$$

The function f is a C^∞ function defined on the whole of M. If we let $W_k = \psi_k^{-1}(U(0,1))$, then since $\{W_k\}_{k\geqslant 1}$ is an open cover of M, for any $q \in M$, there exists j such that $\widetilde{b_j}(q) = 1$. Thus, f never equals 0, so we if we set

$$f_k = \frac{\widetilde{b_k}}{f},$$

the family $\{f_k\}_{k\geqslant 1}$ is a partition of unity subordinate to $\{U_i\}_{i\in I}$. □

Tangent Space, Vector Fields, and Cotangent Space

Smooth manifolds are able to be embedded into some Euclidean space,[II] so we start by considering them as such.

Definition: If $f\colon M \to N$ is a smooth map between an n-dimensional manifold M and a k-dimensional manifold N that are embedded into some Euclidean space $\mathbb{R}^\ell$, the *derivative* of f at p, defined for charts (U, φ) and (V, ψ), where $f(U) \subseteq V$, is defined by

$$D_p f = D_p\left(\psi^{-1} \circ F \circ \varphi\right),$$

where $F\colon \mathbb{R}^n \to \mathbb{R}^k$ is defined to be a map such that $f = \psi^{-1} \circ F \circ \varphi$.

Remark: This definition is independent of the chart representation. To see this, notice that as we have embedded both M and N into Euclidean space, the maps $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^k$ are diffeomorphisms, hence their derivatives are invertible linear maps.

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, bend right, "\varphi_1"'] \arrow[d, bend left, "\varphi_2"] & V \arrow[d, bend right, "\psi_2"'] \arrow[d, bend left, "\psi_1"] \\
\mathbb{R}^n \arrow[r, bend left, "F_2"] \arrow[r, bend right, "F_1"'] & \mathbb{R}^k
\end{tikzcd}$$

Using some coordinate changes, we see that

$$F_1 = \left(\psi_1 \circ \psi_2^{-1}\right) \circ F_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)$$

so by the chain rule,

$$\begin{aligned}
Df &= D\left(\psi_1^{-1} \circ F_1 \circ \varphi_1\right) \\
&= D(\psi_1)^{-1} \circ D(F_1) \circ D(\varphi_1) \\
&= D\left(\psi_1^{-1}\right) \circ D\left(\left(\psi_1 \circ \psi_2^{-1}\right) \circ F_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)\right) \\
&= D(\psi_1)^{-1} \circ D(\psi_1) \circ D\left(\psi_2^{-1} \circ F_2 \circ \varphi_2\right) \circ D(\varphi_1)^{-1} \circ D(\varphi_1) \\
&= D\left(\psi_2^{-1} \circ F_2 \circ \varphi_2\right).
\end{aligned}$$

Note here that the chain rule is being used in $\mathbb{R}^\ell$, which Dundas calls the "flat chain rule,"[III] rather than the general case on a manifold.

One of the issues with this strategy, though, is that embeddings may carry different properties (though at high enough dimensions, any two embeddings are diffeomorphic to each other). For instance, embeddings $S^1 \hookrightarrow \mathbb{R}^3$ form the field of knot theory, which is a very intricate field.

As a result, we want to be able to define tangent spaces, derivatives, and the like without having to refer to coordinates. In order to do this, we need to discuss germs of functions.

Definition: Let $g\colon M \to N$ map $p \mapsto g(p)$. We define an equivalence relation on the space of functions $f\colon M \to N$ with $f(p) = g(p)$ by taking $f_1 \sim f_2$ whenever $f_1 = f_2$ on some open neighborhood $A \in \mathcal{O}_p$. The equivalence class $[g]_p$ is known as the *germ* of g at p.

[II] This is actually a very deep theorem.
[III] Flatness is always relative.

We denote by $\mathcal{C}_p$ the space of germs of C^∞ functions $f\colon M \to \mathbb{R}$ at p.

Remark: Often, books will use $\mathcal{O}_p$ to refer to the space of germs at p. We will use $\mathcal{C}_p$ for this purpose though, as we have defined $\mathcal{O}_p$ to refer to the system of open neighborhoods at p.

Proposition: The space $\mathcal{C}_p$ with the operations

- $[g] + [h] = [g + h]$;
- $\alpha[g] = [\alpha g]$;
- $[g] \cdot [h] = [g \cdot h]$

forms an algebra over $\mathbb{R}$.

Definition: Let W_p be the space of germs of smooth maps $\gamma\colon \mathbb{R} \to M$ that send $0 \mapsto p$. The *tangent space* T_pM is defined by $W_p/\sim$, where we define the equivalence relation $[g_1] \sim [g_2]$ for two germs at p $g_1\colon M \to \mathbb{R}$ and $g_2\colon M \to \mathbb{R}$ by

$$(\varphi \circ g_1)'(0) = (\varphi \circ g_2)(0)$$

for all $\varphi \in \mathcal{C}_p$.

Definition: If $f\colon M \to N$ is a smooth map, we define the map $T_pf\colon T_pM \to T_{f(p)}N$ to act via

$$T_pf([\gamma]) = [\gamma \circ f]$$

for all $\gamma \in W_p$.

Proposition (Chain Rule)**:** If $f\colon M \to N$ and $g\colon N \to L$ are smooth maps, then

$$T_{f(p)}g \circ T_pf = T_p(g \circ f).$$

Proof. If $\gamma \in W_p$, then

$$\begin{aligned} T_{f(p)}g \circ T_pf([\gamma]) &= T_{f(p)}g([f \circ \gamma]) \\ &= [g \circ f \circ \gamma] \\ &= T_p(g \circ f)([\gamma]). \end{aligned}$$

□

A terribly kept secret is that this function T_pf is actually the differential D_pf. This requires us to prove that this definition comports with the definition for the case of M as an embedded manifold. We require a few basic propositions for this purpose whose proofs follow from the inverse function theorem and various definitions.

Proposition: If $f\colon M \to N$ is a locally invertible smooth map about $p \in M$, then $T_pf\colon T_pM \to T_{f(p)}N$ is an isomorphism of vector spaces.

Proposition: If $0 \in \mathbb{R}^k$, then $T_0\mathbb{R}^k$ is represented by linear maps $t \mapsto \lambda t$ for some vector $\lambda \in \mathbb{R}^k$. Therefore, if M is k-dimensional, $T_pM \cong \mathbb{R}^k \cong \operatorname{Hom}(\mathbb{R}, \mathbb{R}^k)$.

Proposition: If $\varphi\colon U \to \mathbb{R}^k$ is a local diffeomorphism about p such that $\varphi(p) = 0$, then if $f \in C^\infty(\mathbb{R}^k)$, $f \circ \varphi \in C^\infty(U)$, which induces an algebra homomorphism

$$\begin{aligned} \varphi^*\colon \mathcal{C}_{0,\mathbb{R}^k} &\to \mathcal{C}_{p,M} \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Proposition: If M and N are embedded submanifolds of $\mathbb{R}^n$ with dimensions m and k respectively, and $f\colon M \to N$ is a smooth map, then

$$D_pf \equiv T_pf.$$

Proof. Let (U, φ) and (V, ψ) be charts for M and N respectively with $p \in U$ and $f(p) \in V$. Then, we may consider the coordinate maps $\varphi\colon U \to \mathbb{R}^m$ and $\psi\colon V \to \mathbb{R}^k$ to be such that $p \mapsto 0$ and $f(p) \mapsto 0$ respectively.

Now, we see that $T_p f$ and $D_p f$ can be written as

$$\begin{aligned} T_p f &= T_{f(p)}\psi^{-1} \circ T_0 F \circ T_{f(p)}\varphi \\ D_p f &= D_p\left(\psi^{-1} \circ F \circ \varphi\right). \end{aligned}$$

Yet, since $T_0 F = D_0 F$, $T_{f(p)}\psi^{-1} = D_{f(p)}\psi^{-1}$, and $T_p\varphi = D_p\varphi$, the chain rule gives

$$\begin{aligned} T_p f &= T_p\left(\psi^{-1} \circ F \circ \varphi\right) \\ &= D_p\left(\psi^{-1} \circ F \circ \varphi\right) \\ &= D_p f, \end{aligned}$$

implying that $T_p f = D_p f$. ◻

Now that we have established that we can consider manifolds as either standalone entities or as submanifolds of $\mathbb{R}^n$, we now shift our focus to understanding what information the derivative map $D_p f\colon T_p M \to T_{f(p)} M$ gives us about the underlying topology of M and N.

Definition: Let $f\colon M \to N$ be a smooth map, and let $p \in M$. We say p is a *critical point* for f if $D_p f$ does not have maximal rank.

If $D_p f$ has maximal rank, then we say p is a *regular point* of f.

We say $q \in N$ is a *critical value* if $f^{-1}(\{q\})$ contains a critical point for f. Else, we say q is a *regular value*.

The study of critical points is actually very vital in understanding the underlying manifold's global topology. This is the field known as Morse theory, and we will discuss it later in the course.

Definition: Let $f\colon M \to \mathbb{R}$ be a smooth function, with M a manifold. We say f is *Morse* if all the critical points of f are isolated, and the critical points are nondegenerate, in the sense that the Hessian matrix, given by

$$H_p f = \left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p\right)_{i,j=1}^{n}$$

has nonzero determinant, where $(x_1, \dots, x_n)$ is a coordinate system about the critical point p.

Morse functions allow us to calculate a quantity known as the index of the manifold at any given value of $\mathbb{R}$, thereby allowing us to reconstruct the manifold from the information that the functions give us.

The Tangent Bundle

Recall that we defined the differential $D_p f$ via the action on the manifold about the point p. Unfortunately, the issue with this formulation is that it is purely local — the main reason we study manifolds is that we want to be able to use local information about the function to obtain insights about the global topology of the manifold. We need a construction that allows us to collect information about all the differentials at points of M together. This is the tangent bundle.

Definition: Let M be a manifold. The *tangent bundle* TM is the disjoint union

$$TM = \coprod_{p \in M} T_p M.$$

Now, if M is a manifold of dimension m, then TM is a manifold of dimension $2m$. To see this, observe that if $p \in \mathbb{R}^m$, then $T_p\mathbb{R}^m \cong \mathbb{R}^m$.

Therefore, if at each point in $\mathbb{R}^m$, we assign a copy of the tangent space, we have that

$$T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m.$$

If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is smooth, we get the map $Tf\colon T\mathbb{R}^m \to T\mathbb{R}^n$ given by

$$(x, v) \mapsto (f(x), D_x f(v)).$$

Now, given a coordinate map $\varphi\colon M \supseteq U \to \mathbb{R}^m$, we may define

$$T\varphi(U) = \varphi(U) \times \mathbb{R}^m.$$

Thus, if $\{(U_i, \varphi_i)\}_{i\in I}$ is an atlas for M, we have transition maps

$$\begin{aligned} T\psi(U \cap V) &\to T\varphi(U \cap V) \\ (x, v) &\mapsto \left(\varphi \circ \psi^{-1}(x), D_x\left(\varphi \circ \psi^{-1}\right)(v)\right). \end{aligned}$$

Thus, if $f\colon M \to N$ is a smooth map, it induces a differential map on the tangent bundles $Df\colon TM \to TN$.

> **Remark:** If M and N only have C^1 structures, it turns out that there is a compatible C^∞ structure, meaning that we may safely assume that any C^1 manifold is C^∞.

Vector Fields

> **Definition:** If M is a manifold, then a *vector field* on M is a smooth right-inverse of the projection map
>
> $$\begin{aligned} \pi\colon TM &\to M \\ (x, v) &\mapsto x. \end{aligned}$$

When we consider vector fields on manifolds, some basic questions crop up. The most basic of them all is the following: does there exist a nowhere-vanishing vector field on M?

- In any Euclidean space, we may take a constant nonzero vector as our assignment, so the answer is yes.
- For S^1, we can embed it into $\mathbb{R}^2$, then take the map $(x, y) \mapsto ((x, y), (-y, x))$, which is the family of tangent vectors to the unit circle in $\mathbb{R}^2$.
- For S^2, the answer is no. This is the much-celebrated "hairy ball theorem."
- For S^3, the answer is yes. In fact, for $S^{(2n-1)}$ where n is a natural number, the answer is yes, while for S^{2n}, the answer is no.

Notations

- A general normed space V will have its norm denoted by $\|\cdot\|$. If $V = \mathbb{R}^n$, then we denote the norm by $|\cdot|$.
- We denote topological spaces by (X, τ).
- $U(x, r) = \{y \in V \mid \|x - y\| < r\}$.
- $B(x, r) = \{y \in V \mid \|x - y\| \leqslant r\}$.
- $\mathcal{N}_p$: neighborhood system centered at $p \in X$.
- $\mathcal{O}_p$: system of *open* neighborhoods centered at $p \in X$.
- When we say a number n is positive, we mean that $n \geqslant 0$. Similarly, a sequence $(a_n)_n$ is decreasing (increasing) if $a_n \geqslant a_{n+1}$ ($a_n \leqslant a_{n+1}$).