Due: 09/05/2024
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Problem 4

Problem. Let $T \in \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}, \mathbb{F})$. Prove there exists $\alpha \in \mathbb{F}$ such that $T(v) = \alpha v$ for all $v \in \mathbb{F}$.

Solution. Since $\dim_{\mathbb{F}}(\mathbb{F}) = 1$, we know that the basis of $\mathbb{F}$ is $\{\beta\}$ for some $\beta \in \mathbb{F}$. For $v \in \mathbb{F}$, it is then the case that v is a linear combination of the basis of $\mathbb{F}$ over $\mathbb{F}$, meaning $v = v_0\beta$ for some $v_0 \in \mathbb{F}$, implying $\beta = \left(v_0^{-1}\right) v$.

Considering a linear transformation $T(v)$, we have

$$T(v) = T(v_0\beta).$$

Substituting $\beta = v_0^{-1}v$, and using the commutativity and associativity of multiplication under $\mathbb{F}$, we have

$$T(v) = T\left(v\left(v_0^{-1}v\right)\right).$$

Using the fact that T is linear and $v \in \mathbb{F}$, we have

$$\begin{aligned} &= vT\left(v_0^{-1}v_0\right) \\ &= vT(1). \end{aligned}$$

Thus, $\alpha = T(1)$.

Problem 6

Problem. Let V be an $\mathbb{F}$-vector space. Prove that if $\{v_1, \dots, v_n\}$ is linearly independent, then so is the set $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$.

Solution. To prove that $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ is linearly independent, we consider the sum

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_n v_n,$$

and show that this sum equals zero if and only if $a_i = 0$ for each i. Rearranging the sum, we have

$$a_1v_1 + (a_2 - a_1)v_2 + \cdots + (a_{n-1} - a_{n-2})v_{n-1} + (a_n - a_{n-1})v_n.$$

Since the set $\{v_1, \dots, v_n\}$ are linearly independent, this linear combination equals 0_V if and only if $a_1 = (a_2 - a_1) = \cdots = a_n - a_{n-1} = 0$. In particular, since $a_1 = 0$, it must be the case that $a_2 = 0$, $a_3 = 0$, and so on.

Thus, $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ are linearly independent.

Problem 9

Problem. Let V be a finite-dimensional vector space and $T \in \operatorname{Hom}_{\mathbb{F}}(V, V)$ with $T^2 = T$.

(a) Prove that $\operatorname{im}(T) \cap \ker(T) = \{0\}$.

(b) Prove that $V = \operatorname{im}(T) \oplus \ker(T)$.

(c) Let $V = \mathbb{F}^n$. Prove that there is a basis of V such that the matrix of T with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

Problem 13

Problem. Let p be a prime and V a dimension n vector space over $\mathbb{F}_p$. Show there are

$$(p^n - 1)(p^n - p)\left(p^n - p^2\right)\cdots\left(p^n - p^{n-1}\right)$$

distinct bases of V.