Tensor Products, Bilinear Maps, and Linear Maps

First, we review some definitions of algebraic tensor products.

Definition of Tensor Products

From linear algebra, we know that if X and Y are vector spaces, then the *tensor product* of X and Y, denoted $X \otimes Y$, is the universal object such that if $t\colon X \times Y \to Z$ is a bilinear map, then there is a unique linear map $T\colon X \otimes Y \to Z$, alongside an injection $(x, y) \overset{\iota}{\mapsto} x \otimes y$ such that $T \circ \iota = t$.

$$\begin{tikzcd} X \times Y \arrow[r, "\iota"] \arrow[dr, "t"'] & X \otimes Y \arrow[d, "T"] \\ & Z \end{tikzcd}$$

Elements of the tensor product are linear combinations of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

Identifying Tensor Products as Spaces of Maps

We observe that if t is a bilinear *form* — i.e., a bilinear map $t\colon X \times Y \to \mathbb{F}$ for some field $\mathbb{F}$ — then there is a unique linear map $T\colon X \otimes Y \to \mathbb{F}$, meaning that the space of bilinear forms, $\hom(X \times Y, \mathbb{F})$, is in one-to-one correspondence with elements of the algebraic dual of the tensor product, $(X \otimes Y)'$.

We can in fact view tensors in and of themselves as bilinear forms. For any $x \in X$ and $y \in Y$, we may define a bilinear form $B_{x,y}\colon X' \times Y' \to \mathbb{F}$ by $B_{x,y}(\varphi, \psi) = \varphi(x)\psi(y)$. Thus, we have a unique linear map $X \otimes Y \to \hom(X' \times Y', \mathbb{F})$ taking $x \otimes y \mapsto B_{x,y}$.

The map is injective, since if

$$\sum_{i=1}^{n} B_{x_i, y_i} = 0,$$

then for any $\varphi \in X'$ and $\psi \in Y'$, we have

$$\sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = 0.$$

Since X' separates the points of X and similarly for Y' and the points of Y, we must have that

$$\sum_{i=1}^{n} x_i \otimes y_i = 0.$$

Thus, we have an embedding $X \otimes Y \hookrightarrow \hom(X' \times Y', \mathbb{F})$; if X and Y are dual spaces, then there is a corresponding embedding $X' \otimes Y' \hookrightarrow \hom(X \times Y, \mathbb{F})$ by identifying

$$\sum_{i=1}^{n} \varphi_i \otimes \psi_i \mapsto \left((x, y) \mapsto \sum_{i=1}^{n} \varphi_i(x)\psi_i(y)\right).$$

For any bilinear form B, there are two associated linear maps, $L_B\colon X \to Y'$ and $R_B\colon Y \to X'$, given by

$$\begin{aligned} B(x, y) &= \langle y, L_B(x) \rangle \\ &= \langle x, R_B(y) \rangle, \end{aligned}$$

where we let $\langle \cdot, \cdot \rangle \colon X \times X' \to \mathbb{F}$ denote the canonical duality given by

$$\langle x, \varphi \rangle = \varphi(x).$$

Thus, we see that every element of the tensor product,

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

gives us two linear maps

$$L_u(\varphi) = \sum_{i=1}^{n} \varphi(x_i) y_i$$
$$R_u(\psi) = \sum_{i=1}^{n} \psi(y_i) x_i,$$

giving two more identifications $X \otimes Y \hookrightarrow \hom(X', Y)$ and $X \otimes Y \hookrightarrow \hom(Y', X)$.

If one of X or Y is a dual space, we get $X' \otimes Y \hookrightarrow \hom(X, Y)$ and $X \otimes Y' \hookrightarrow \hom(Y, X)$.

Specifically, the elements of $\hom(X, Y)$ that correspond to elements of $X' \otimes Y$ are the finite-rank linear maps.

Examples

> **Example** (Matrices)**:** If we let $\mathbb{F}^n$ and $\mathbb{F}^m$ be endowed with the standard bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$, we may identify $\mathbb{F}^n \otimes \mathbb{F}^m \cong \mathbb{M}_{m,n}(\mathbb{F})$.
>
> We may identify $e_i \otimes f_j$ with the matrix unit e_{ij}.

> **Example** (Vector-Valued Functions)**:** Let $\mathcal{F}(S)$ denote the vector space of functions from a set S into the field $\mathbb{F}$; if X is a vector space, then $\mathcal{F}(S, X)$ may denote the vector space of all functions from S to X with pointwise operations. Given $f \in \mathcal{F}(S)$ and any $x \in X$, we may define a function from S to X by taking $s \mapsto f(s)x$. We may write this as $f \cdot x$
>
> This defines a bilinear map $\mathcal{F}(S) \times X \to \mathcal{F}(S, X)$ taking $(f, x) \mapsto f \cdot x$.
>
> We may thus find a linear map
>
> $$\sum_{i=1}^{n} f_i \otimes x_i \mapsto \sum_{i=1}^{n} f_i \cdot x_i.$$
>
> We show that this map is injective. Suppose we have
>
> $$\sum_{i=1}^{n} f_i \cdot x_i = 0,$$
>
> so that
>
> $$\sum_{i=1}^{n} f_i(s) x_i = 0.$$
>
> Yet, since the evaluation functionals $f \mapsto f(s)$ are a separating subset of $\mathcal{F}(S)$, we have
>
> $$\sum_{i=1}^{n} f_i \otimes x_i = 0.$$
>
> This gives an embedding of $\mathcal{F}(S) \otimes X \hookrightarrow \mathcal{F}(S, X)$.

> **Example** (Vector-Valued Measures)**:** Let $\mathcal{A}$ be an algebra of subsets of S. The vector space $M_{\text{f.a.}}(S)$ denotes the space of all finitely additive scalar-valued measures on $\mathcal{A}$. An element of $M_{\text{f.a.}}(S)$ is a function from $\mathcal{A}$ to $\mathbb{F}$ such that $\mu(\emptyset) = 0$ and
>
> $$\mu\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$
>
> for every finite collection of pairwise disjoint sets in $\mathcal{A}$.
>
> For every vector space X, we can similarly define $M_{\text{f.a.}}(S, X)$ to be the space of finitely additive X-valued measures on on $\mathcal{A}$, and obtain an embedding of the tensor product $M_{\text{f.a.}}(S) \otimes X \hookrightarrow M_{\text{s.a.}}(S, X)$.

Summary

- If X and Y are vector spaces and $t\colon X \times Y \to Z$ is a bilinear map, then there is a unique linear map $T\colon X \otimes Y \to Z$ such that $T \circ \iota = t$.
- Letting $\hom(X \times Y, \mathbb{F})$ denote the space of bilinear forms on $X \times Y$ to an underlying field $\mathbb{F}$, there are identifications

$$\begin{aligned}(X \otimes Y)' &\leftrightarrow \hom(X \times Y, \mathbb{F})\\ X \otimes Y &\hookrightarrow \hom(X' \times Y', \mathbb{F})\\ X' \otimes Y &\hookrightarrow \hom(X, Y)\\ X \otimes Y' &\hookrightarrow \hom(Y, X).\end{aligned}$$

- We may identify vector-valued functions and measures as

$$\begin{aligned}\mathcal{F}(S) \otimes X &\hookrightarrow \mathcal{F}(S, X)\\ M_{\text{f.a.}}(S) \otimes X &\hookrightarrow M_{\text{f.a.}}(S, X).\end{aligned}$$

Projective Tensor Products

If X and Y are Banach spaces, there are a variety of ways we may seek to norm the tensor product $X \otimes Y$. The basic requirement we have is that we want

$$\|x \otimes y\| \leq \|x\|\|y\|,$$

and for any representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

we want

$$\|u\| \leq \sum_{i=1}^{n} \|x_i\|\|y_i\|.$$

Therefore, we must have

$$\|u\| \leq \inf\left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| \;\middle|\; u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

The latter value is thus the largest possible candidate for a norm on $X \otimes Y$ that has these desired qualities. We thus define

$$\|u\|_{\wedge} = \inf\left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| \;\middle|\; u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

to be the *projective norm* on $X \otimes Y$.

Proposition: Let X and Y be Banach spaces. Then, $\|\cdot\|_{\wedge}$ is a norm on $X \otimes Y$ with $\|x \otimes y\|_{\wedge} = \|x\|\|y\|$.

Proof. We start by showing homogeneity. Assume $\lambda \neq 0$. Then, if

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

we have

$$\lambda u = \sum_{i=1}^{n} (\lambda x_i) \otimes y_i,$$

we have

$$\begin{aligned} \|\lambda u\|_{\wedge} &\leq \sum_{i=1}^{n} \|\lambda x_i\|\|y_i\| \\ &= |\lambda| \sum_{i=1}^{n} \|x_i\|\|y_i\| \\ &= |\lambda|\|u\|_{\wedge}. \end{aligned}$$

Similarly,

$$\|u\|_{\wedge} \leq |\lambda|^{-1}\|\lambda u\|_{\wedge},$$

whence $\|\lambda u\|_{\wedge} = |\lambda|\|u\|_{\wedge}$.

Now, let $\varepsilon > 0$ and let $u, v \in X \otimes Y$ have representations

$$\begin{aligned} u &= \sum_{i=1}^{n} x_i \otimes y_i \\ v &= \sum_{j=1}^{m} w_j \otimes z_j \end{aligned}$$

such that

$$\begin{aligned} \sum_{i=1}^{n} \|x_i\|\|y_i\| &\leq \|u\|_{\wedge} + \varepsilon/2 \\ \sum_{j=1}^{m} \|w_j\|\|z_j\| &\leq \|v\|_{\wedge} + \varepsilon/2. \end{aligned}$$

Then, we have a representation

$$u + v = \sum_{i=1}^{n} x_i \otimes y_i + \sum_{j=1}^{m} w_j \otimes z_j,$$

so that

$$\begin{aligned} \|u + v\|_{\wedge} &\leq \sum_{i=1}^{n} \|x_i\|\|y_i\| + \sum_{j=1}^{m} \|w_j\|\|z_j\| \\ &\leq \|u\|_{\wedge} + \|v\|_{\wedge} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we obtain the triangle inequality.

Now, we show that the norm is definite. Let $\|u\|_{\wedge} = 0$. Then, for any $\varepsilon > 0$, there is a representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

such that

$$\sum_{i=1}^{n} \|x_i\|\|y_i\| \leq \varepsilon.$$

In particular, for any $\varphi \in X^*$ and $\psi \in Y^*$, we have

$$\left|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)\right| \leq \varepsilon\|\varphi\|\|\psi\|.$$

Since the quantity $\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)$ is independent of the representation of u, it follows that this sum equals zero. Yet, since X^* and Y^* separate the points of X and Y, it follows that $u = 0$.

Finally, we know that $\|x \otimes y\|_{\wedge} \leq \|x\|\|y\|$, so we let $\varphi \in B_{X^*}$ and $\psi \in B_{Y^*}$ such that $\varphi(x) = \|x\|$ and $\psi(y) = \|y\|$. We let $b\colon X \times Y \to \mathbb{F}$ be given by $B(w, z) = \varphi(w)\psi(z)$. The linearization B is a linear functional on $X \otimes Y$ with

$$\begin{aligned} \left|B\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right| &\leq \sum_{i=1}^{n} |B(x_i \otimes y_i)| \\ &= \sum_{i=1}^{n} |\varphi(x_i)\psi(y_i)| \\ &\leq \sum_{i=1}^{n} \|x_i\|\|y_i\|, \end{aligned}$$

so that $|B(u)| \leq \|u\|_{\wedge}$ for every $u \in X \otimes Y$. In particular, this means that B is a bounded linear functional on the normed space $(X \otimes Y, \|\cdot\|_{\wedge})$ with norm at most 1, whence $\|x\|\|y\| = B(x \otimes y) \leq \|x \otimes y\|_{\wedge}$. □

We may thus complete $(X \otimes Y, \|\cdot\|_{\wedge})$ with respect to the projective norm to obtain the *projective tensor product* of the Banach spaces X and Y, which we denote $X\hat{\otimes}Y$.

If $A \subseteq X$ and $B \subseteq Y$ are subsets, then we will let

$$A \otimes B := \{x \otimes y \mid x \in A, y \in B\}.$$

Proposition: The closed unit ball of $X\hat{\otimes}Y$ is the closed convex hull of $B_X \otimes B_Y$.

Proof. Since the closed unit ball is the closure of the unit ball in $X \otimes Y$, it suffices to prove the proposition for the space $(X \otimes Y, \|\cdot\|_{\wedge})$. Let u be an element of the open unit ball of $(X \otimes Y, \|\cdot\|_{\wedge})$.

By the definition of the projective norm, there is a representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

such that

$$\sum_{i=1}^{n} \|x_i\|\|y_i\| < 1.$$

Let

$$w_i = \|x_i\|^{-1} x_i$$

$$
\begin{aligned}
z_i &= \|y_i\|^{-1} y_i \\
\lambda_i &= \|x_i\|\|y_i\|.
\end{aligned}
$$

Then, we have

$$
u = \sum_{i=1}^{n} \lambda_i w_i \otimes z_i
$$

with $w_i \in B_X$, $z_i \in B_Y$, $\lambda_i \geq 0$, and $\sum_{i=1}^{n} \lambda_i < 1$. Thus, $u \in \operatorname{conv}(B_X \otimes B_Y)$, meaning that the closed unit ball of $X \otimes Y$ is contained in $\overline{\operatorname{conv}}(B_X \otimes B_Y)$. Yet, we must also have $B_X \otimes B_Y$ contained in the closed unit ball of $(X \otimes Y, \|\cdot\|_{\wedge})$, so it holds for $\overline{\operatorname{conv}}(B_X \otimes B_Y)$. □

Tensor Products of Linear Operators on Banach Spaces

In general, if we have two linear maps $S\colon X \to E$ and $T\colon Y \to F$, we have a bilinear map $X \times Y \to E \otimes F$ given by $(x, y) \mapsto (Sx) \otimes (Ty)$. From the universal property, we get a linear map $S \otimes T\colon X \otimes Y \to E \otimes F$ such that $(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$.

When X and Y have norms, and we are concerned with continuity, we observe that if $u \in X \otimes Y$ has representation

$$
u = \sum_{i=1}^{n} x_i \otimes y_i,
$$

we have

$$
\begin{aligned}
\|(S \otimes T)(u)\|_{\wedge} &= \left\| \sum_{i=1}^{n} (Sx_i) \otimes (Ty_i) \right\| \\
&\leq \|S\|_{\text{op}} \|Y\|_{\text{op}} \sum_{i=1}^{n} \|x_i\|\|y_i\| <>
\end{aligned}
$$

whence

$$
\|(S \otimes T)(u)\|_{\wedge} \leq \|S\|_{\text{op}} \|T\|_{\text{op}} \|u\|_{\wedge}.
$$

This gives that $\|S \otimes T\|_{\text{op}} \leq \|S\|_{\text{op}} \|T\|_{\text{op}}$. Meanwhile, since $\|x \otimes y\|_{\wedge} = \|x\|\|y\|$, we have $\|S \otimes T\|_{\text{op}} \geq \|S\|_{\text{op}} \|T\|_{\text{op}}$, so that $\|S \otimes T\|_{\text{op}} = \|S\|_{\text{op}} \|T\|_{\text{op}}$.

Finally, we may extend $S \otimes T$ to the completions $X \hat{\otimes} Y$ and $E \hat{\otimes} F$. This gives the following proposition.

Proposition: Let $S\colon X \to E$ and $T\colon Y \to F$ be bounded linear operators. Then, there is a unique operator

$$
\begin{aligned}
S \hat{\otimes} T\colon X \hat{\otimes} Y &\to W \hat{\otimes} Z \\
(x \otimes y) &\mapsto (Sx) \otimes (Ty).
\end{aligned}
$$

Furthermore, $\left\|S \hat{\otimes} T\right\|_{\text{op}} = \|S\|_{\text{op}} \|T\|_{\text{op}}$.

Inheritance of the Projective Norm

In general, the projective tensor product does not respect subspaces, in the sense that if $W \leq X$ is a subspace, so that $W \otimes Y \leq X \otimes Y$ is an algebraic subspace, the norm on $W \otimes Y$ induced by $(X \otimes Y, \|\cdot\|_{\wedge})$ is not necessarily the same as the projective norm on $W \otimes Y$.

This follows from the fact that the definition of the norm $(W \otimes Y, \|\cdot\|_{\wedge})$ is restricted to all representations in $W \otimes Y$, and since there are more representations for u in $X \otimes Y$, it follows that the norm of u in $(X \otimes Y, \|\cdot\|_{\wedge})$ is lesser than or equal to the norm of u in $(W \otimes Y, \|\cdot\|_{\wedge})$.

We start by discussing the special case of complemented subspace. Recall that a closed subspace $E \leq X$ is called *complemented* if there is another closed subspace W such that $X = E \oplus W$. An equivalent characterization of a complemented subspace is that there is a continuous projection $P_E \colon X \to E$ such that $X = E \oplus \ker(P_E)$.

Proposition: Let E and F be complemented subspaces of X and Y respectively. Then, $E \otimes F$ is complemented in $X \otimes Y$, and the norm on $E \otimes F$ induced by the projective norm on $X \otimes Y$ is equivalent to the projective norm (in the sense of norm equivalence) on $E \otimes F$.

If E and F are complemented by projections of norm 1, then $E \otimes F$ is a subspace of $X \otimes Y$ that is also complemented by a projection of norm 1.

Proof. Let P and Q be projections from X, Y onto E, F respectively. Then, $P \otimes Q$ is a projection of $X \otimes Y$ onto $E \otimes F$.

Let $u \in E \otimes F$. We have that $\|u\|_{\wedge, X\otimes Y} \leq \|u\|_{\wedge, E\otimes F}$, so we let

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

be a representation of u in $X \otimes Y$. Then,

$$\begin{aligned} u &= P \otimes Q(u) \\ &= \sum_{i=1}^{n} (Px_i) \otimes (Qy_i) \end{aligned}$$

is a representation of u in $E \otimes F$, whence

$$\begin{aligned} \|u\|_{\wedge, E\otimes F} &\leq \sum_{i=1}^{n} \|Px_i\| \|Qy_i\| \\ &\leq \|P\|_{\mathrm{op}} \|Q\|_{\mathrm{op}} \sum_{i=1}^{n} \|x_i\| \|y_i\|, \end{aligned}$$

so it follows that for every representation of u in $X \otimes Y$, we have

$$\begin{aligned} \|u\|_{\wedge, X\otimes Y} &\leq \|u\|_{\wedge, E\otimes F} \\ &\leq \|P\|_{\mathrm{op}} \|Q\|_{\mathrm{op}} \|u\|_{\wedge, X\otimes Y}. \end{aligned}$$

If E and F are complemented by projections of norm 1, we have $\|u\|_{\wedge, X\otimes Y} = \|u\|_{\wedge, E\otimes F}$ for every $u \in E \otimes F$, as $\|P \otimes Q\|_{\mathrm{op}} = \|P\|_{\mathrm{op}} \|Q\|_{\mathrm{op}}$. □

Now, we may explain why exactly the norm is known as the projective norm. Recall that a linear map $Q \colon X \to Y$ is known as a 1-quotient map if Q is surjective and $Q(B_X) = B_Y$, meaning that Y is isometrically isomorphic to $X/\ker(Q)$. An equivalent condition to this is

$$\|y\| = \inf\{\|x\| \mid x \in X, Qx = y\}$$

for every $y \in Y$.

Proposition: Let $Q \colon W \to X$ and $R \colon Z \to Y$ be 1-quotient maps. Then, $Q \otimes R$ is a quotient operator mapping $W \hat{\otimes} Z \to X \hat{\otimes} Y$.

Proof. It suffices to show that

$$Q \otimes R \colon W \otimes Z \to X \otimes Y$$

is a 1-quotient map. To see that $Q \otimes R$ is surjective, let $\sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. There exist $w_i \in W$ and

$z_i \in Z$ with $Qw_i = x_i$ and $Rz_i = y_i$, so that

$$Q \otimes R\left(\sum_{i=1}^{n} w_i \otimes z_i\right) = \sum_{i=1}^{n} x_i \otimes y_i.$$

Thus, $Q \otimes R$ is surjective.

Now, let $u \in X \otimes Y$. Letting $(Q \otimes R)v = u$, so that

$$\begin{aligned}\|u\|_\wedge &\leq \|Q\|_{\mathrm{op}}\|R\|_{\mathrm{op}}\|v\|_\wedge \\ &= \|v\|_\wedge.\end{aligned}$$

Given $\varepsilon > 0$, pick a representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

such that

$$\sum_{i=1}^{n} \|x_i\|\|y_i\| \leq \|u\|_\wedge + \varepsilon.$$

For each i, select $w_i \in W$ and $z_i \in Z$ with $Qw_i = x_i$, $Rz_i = y_i$, and

$$\begin{aligned}\|w_i\| &\leq \left(1 + 2^{-n}\varepsilon\right)\|x_i\| \\ \|y_i\| &\leq \left(1 + 2^{-n}\varepsilon\right)\|y_i\|.\end{aligned}$$

Then,

$$(Q \otimes R)\left(\sum_{i=1}^{n} w_i \otimes z_i\right) = u,$$

and by using the estimate

$$\prod_{i=1}^{n}(1 + a_i) \leq e^{\sum_{i=1}^{n} a_i},$$

we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} w_i \otimes z_i\right\|_\wedge &\leq e^{4\varepsilon}\left(\sum_{i=1}^{n}\|x_i\|\|y_i\|\right) \\ &\leq e^{4\varepsilon}(\|u\|_\wedge + \varepsilon).\end{aligned}$$

Since this holds for every $\varepsilon > 0$, it follows that

$$\|u\|_\wedge = \inf\{\|v\|_\wedge \mid v \in W \otimes Z, (Q \otimes R)v = u\}.$$

□

The Dual Space of $X\hat{\otimes}Y$

Recall that a bilinear map $B\colon X \times Y \to Z$ is called bounded if there exists a constant C such that

$$\|B(x, y)\| \leq C\|x\|\|y\|$$

for every $x \in X$ and $y \in Y$.

Theorem: Let $b\colon X \to Y$ be a bounded bilinear map. Then, there exists a unique linear map $B\colon X\hat{\otimes}Y \to Z$ such that $B(x \otimes y) = b(x, y)$ for every x, y. The correspondence $b \leftrightarrow B$ is an isometric isomorphism between $B(X \times Y, Z)$ and $B\bigl(X\hat{\otimes}Y, Z\bigr)$.

Proof. The existence of the linear map follows from the universal property of tensor products. We start by showing that B is bounded on the projective norm of $X \otimes Y$. For $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, we have

$$\begin{aligned} \|Bu\| &= \left\|\sum_{i=1}^{n} b(x_i, y_i)\right\| \\ &\leq \|b\|_{\text{op}} \sum_{i=1}^{n} \|x_i\|\|y_i\|. \end{aligned}$$

This holds for every representation of u, meaning that $\|Bu\| \leq \|b\|_{\text{op}}\|u\|_{\wedge}$, so B is bounded and satisfies $\|B\|_{\text{op}} \leq \|b\|_{\text{op}}$. Yet, since

$$\begin{aligned} \|b(x,y)\| &= \|B(x \otimes y)\| \\ &\leq \|B\|_{\text{op}}\|x\|\|y\|, \end{aligned}$$

so that $\|b\|_{\text{op}} = \|B\|_{\text{op}}$. The operator B has a unique extension to $B\colon X\hat{\otimes}Y \to Z$ with the same norm. We only need to show now that this is surjective.

If $T \in B\bigl(X\hat{\otimes}Y, Z\bigr)$, then the bounded bilinear map $b\colon X \times Y \to Z$ may be defined by $b(x, y) = T(x \otimes y)$. □

This gives a canonical identification

$$B(X \times Y, Z) \cong B\bigl(X\hat{\otimes}Y, Z\bigr).$$

In particular, if $Z = \mathbb{F}$, then

$$\bigl(X\hat{\otimes}Y\bigr)^* \cong B(X \times Y, \mathbb{F}),$$

with the action given by

$$\left\langle \sum_{i=1}^{n} x_i \otimes y_i, b \right\rangle = \sum_{i=1}^{n} b(x_i, y_i),$$

where $\langle \cdot, \cdot \rangle$ is the canonical duality for X.[I] In particular, from the Hahn–Banach Theorems, we get the expression for the projective norm

$$\|u\|_{\wedge} = \sup\Bigl\{ |u, b| \Bigm| b \in B(X \times Y, \mathbb{F}), \|b\|_{\text{op}} \leq 1 \Bigr\}.$$

This can be expanded further by using the identification

$$B(X \times Y, \mathbb{F}) \cong B(X, Y^*),$$

whence

$$\bigl(X\hat{\otimes}Y\bigr)^* \cong B(X, Y^*),$$

with the action of $S \in B(X, Y^*)$ given by

$$\left\langle \sum_{i=1}^{n} x_i \otimes y_i, S \right\rangle = \sum_{i=1}^{n} \langle y_i, Sx_i \rangle.$$

This gives two more variations on the duality for a projective norm, given by

$$\begin{aligned} \|u\|_{\wedge} &= \sup\Bigl\{ |\langle u, S \rangle| \Bigm| S \in B(X, Y^*), \|S\|_{\text{op}} \leq 1 \Bigr\} \\ &= \sup\Bigl\{ |\langle u, T \rangle| \Bigm| T \in B(Y, X^*), \|T\|_{\text{op}} \leq 1 \Bigr\}. \end{aligned}$$

[I] I.e., if $\varphi \in X^*$ and $x \in X$, then $\langle x, \varphi \rangle = \hat{x}(\varphi) = \varphi(x)$.

Corollary: Let W be a subspace of X. Then, $W\hat{\otimes}Y$ is a subspace of $X\hat{\otimes}Y$ if and only if every operator $S\colon W \to Y^*$ extends to an operator with the same norm from X into Y^*.

Recall that a Banach space Z is called *injective* if it has the property that, for every Banach space X and every subspace W of X, every operator $S\colon W \to Z$ extends to an operator $\widetilde{S}\colon X \to Z$ with the same norm.

Equivalently, Z is injective if and only if Z is complemented by a norm-one projection in any Banach space that contains Z as a subspace.

Theorem: Every bounded bilinear form on $X \times Y$ admits an extension to a bounded bilinear form on $X^{**} \times Y^{**}$ with the same norm.

Proof. Let A be a bounded bilinear form on $X \times Y$, and let S be the operator from X into Y^* given such that $A(x,y) = \langle y, Sx\rangle$ for every $x \in X$ and $y \in Y$.

Consider the bounded bilinear form B on $X^{**} \times Y^{**}$ given by

$$B(x^{**}, y^{**}) = \langle S^* y^{**}, x^{**}\rangle,$$

where $S^*\colon Y^{**} \to X^*$ is the adjoint of S. For any $x \in \iota(X)$ and $y \in \iota(Y)$, where ι is the canonical injection into the double dual, we have

$$\begin{aligned} B(\hat{x}, \hat{y}) &= \langle S^*\hat{y}, \hat{x}\rangle \\ &= \langle y, Sx\rangle \\ &= A(x,y), \end{aligned}$$

so that B is an extension of A satisfying $\|B\|_{\mathrm{op}} = \|A\|_{\mathrm{op}}$. □

The Bochner Integral

We now focus on the special case of the tensor product $L_1(\mu)\hat{\otimes}X$, which is a Banach space of integrable vector-valued functions. To understand this space further, we need to develop an integration theory for functions with values in a Banach space.

Let (Ω, Σ, μ) be a measure space. Recall that a function $f\colon \Omega \to X$ is called simple there are a finite number of distinct values, where we may partition Ω into $E_1, \dots, E_n$ and distinct nonzero $x_1, \dots, x_n$ in X such that $f(\omega) = x_i$ whenever $\omega \in E_i$. We may write

$$f = \sum_{i=1}^{n} \chi_{E_i} x,$$

where χ_{E_i} denotes the characteristic function of the set E_i. If $E_i \in \Sigma$ for each i, then f is also a measurable function.

We will fire off a bunch of useful notions, most of which originate in measure theory.

- An arbitrary function $f\colon \Omega \to X$ is called measurable if there is a sequence $(f_n)_n$ of simple measurable functions converging almost everywhere to f.
- If f is measurable, then so too is $\|f\|$ (by the reverse triangle inequality).
- The function $f\colon \Omega \to X$ is called *weakly* measurable if the scalar-valued function $\varphi(f)$ is measurable for each $\varphi \in X^*$.
- If $f^{-1}(S)$ is measurable for every open subset $S \subseteq X$, then we say f is Borel measurable.
- The function $f\colon \Omega \to X$ is called essentially separably valued if there is $E \subseteq \Omega$ with $\mu(\Omega \setminus E) = 0$ and $f(E)$ is contained in a separable subspace Y of X.
- If μ is σ-finite, then f is μ-measurable if and only if $\chi_E f$ is μ-measurable for every $E \in \Sigma$ with finite measure.

There is a mild distinction between weakly measurable, measurable, and Borel measurable as we will show below.

Proposition (Pettis Measurability Theorem)**:** Let (Ω, Σ, μ) be a σ-finite measure space. If $f\colon \Omega \to X$ is a function, then the following are equivalent:

(i) f is measurable;

(ii) f is weakly measurable and essentially separably valued;

(iii) f is Borel measurable and essentially separably valued.

Proof. We start by showing that (i) implies (ii). If f is measurable, and $(f_n)_n \to f$ converges μ-a.e., then for any $\varphi \in X^*$, we have that $(\varphi(f_n))_n$ is a sequence of scalar-valued measurable functions that converge almost everywhere to $\varphi(f)$, so $\varphi(f)$ is measurable. Letting Y be the subspace generated by the ranges of the f_n, then Y is separable and $f(\Omega)$ is contained in Y.

We show that (ii) implies (iii). If f is weakly measurable, and $f(\Omega)$ is essentially contained in a separable subspace Y of X, then we may assume without loss of generality that $f(\Omega) \subseteq Y$. Since the embedding of Y into X is continuous, it suffices to prove that f is Borel measurable as a function from Ω into Y. Since $\varphi(f)$ is measurable for every $\varphi \in X^*$, then f is Borel measurable in the weak topology of Y; that is, $f^{-1}(U) \in \Sigma$ for every weakly open subset U of Y. Thus, this holds for weakly closed subsets of Y. Any norm-closed ball in Y is convex, hence weakly closed, and since Y is separable, every open $U \subseteq Y$ is the union of a countable number of closed balls, so $f^{-1}(U) \in \Sigma$, meaning f is Borel measurable.

Finally, we show that (iii) implies (i). Assume that f is Borel measurable, with $f(\Omega)$ contained in a separable subspace Y of X (without loss of generality). It suffices to assume that μ is finite. Let $\{y_k\}_{k\geq 1}$ be a countable dense subset of Y; then, for any n, the union of the open balls $U(y_k, 1/n)$ is Y. By the Borel measurability of f, $E_k^n := f^{-1}(U(y_k, 1/n))$ are measurable and cover Ω for each n. Convert this to a disjoint cover by setting $F_1^n = E_1^n$ and $F_k^n = E_k^n \setminus \left(E_1^n \cup \cdots \cup E_{k-1}^n\right)$ for each k. For each n, let

$$g_n = \sum_{k=1}^{\infty} \chi_{F_k^n} y_k.$$

Then, $\|f(\omega) - g(\omega)\| < 1/n$ for each $\omega \in \Omega$. Therefore, the sequence $(g_n)_n$ of countably valued functions converges uniformly to f. We will truncate these expressions to get a sequence of measurable simple functions that converge μ-a.e. to f. Choose m_n such that

$$\mu\left(\bigcup_{k=m_n+1}^{\infty} F_k^n\right) \leq 2^{-n},$$

and let

$$h_n = \bigcup_{k=1}^{m_n} \chi_{F_k^n} y_k,$$

so that $\|f(\omega) - h_n(\omega)\| < 1/n$ for each $\omega \in \Omega \setminus C_n$, where we let

$$C_n = \bigcup_{k=m_n+1}^{\infty} F_k^n.$$

Let $C = \limsup_{k\to\infty} C_k$; then by Borel–Cantelli, C is a null set, and for any $\omega \notin C$, there is n such that $\omega \notin C_k$ for any $k \geq n$, meaning that $\|f(\omega) - h_n(\omega)\| < 1/k \leq 1/n$ for every $k \geq n$. $\square$

We may now define the integral here. Let f be a measurable simple function with representation

$$f = \sum_{i=1}^{n} \chi_{E_i} x_i,$$

and suppose each A_i has finite measure. We define

$$\int_E f\, d\mu = \sum_{i=1}^{n} \mu(A_i \cap E) x_i.$$

A measurable function $f\colon \Omega \to X$ is called *Bochner integrable* if there exists a sequence of simple functions $(f_n)_n \to f$ μ-a.e. and

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\|\, d\mu = 0.$$

The Bochner integral of f over $E \in \Sigma$ is then given by

$$\int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu.$$

We can actually characterize Bochner integrability as follows.

Theorem (Bochner's Theorem)**:** If $f\colon \Omega \to X$ is a measurable function, then f is Bochner-integrable if and only if $\|f\|$ is integrable.

Proof. Suppose f is Bochner integrable. Let $(f_n)_n \to f$ be a sequence of simple functions converging μ-a.e. as in the definition of the Bochner integral. Then,

$$\begin{aligned}\int_\Omega \|f\|\, d\mu &= \int_\Omega \|(f - f_n) + f_n\|\, d\mu \\ &\leq \int_\Omega \|f_n - f\|\, d\mu + \int_\Omega \|f_n\|\, d\mu,\end{aligned}$$

for each n, so the integral of $\|f\|$ is finite. Conversely, if $\|f\|$ is integrable, and $(f_n)_n$ converges μ-a.e. to f, we fix $\delta > 0$, and define

$$g_n(\omega) := \begin{cases} f_n(\omega) & \|f_n(\omega)\| \leq (1+\delta)\|f(\omega)\| \\ 0 & \text{else} \end{cases}.$$

Then, $(g_n)_n$ is a sequence of measurable simple functions that converges μ-a.e. to f. The sequence of scalar functions $(\|f - g_n\|)_n$ is dominated by the integrable function $(2+\delta)\|f\|$, so by dominated convergence, we have

$$\int_\Omega \|f - g_n\|\, d\mu \to 0.$$

□

There is a version of the triangle inequality for the Bochner integral.

Corollary: If $f\colon \Omega \to X$ is Bochner-integrable, then

$$\left\| \int_\Omega f\, d\mu \right\| \leq \int_\Omega \|f\|\, d\mu.$$

Proof. Let $(f_n)_n \to f$ be a sequence of measurable simple functions converging μ-a.e. to f, and

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu.$$

Then, $(\|f_n\|)_n \to \|f\|$, so that

$$\begin{aligned}\left\| \int_\Omega f\, d\mu \right\| &= \left\| \lim_{n\to\infty} \int_\Omega f_n\, d\mu \right\| \\ &\leq \lim_{n\to\infty} \int_\Omega \|f_n\|\, d\mu\end{aligned}$$

$$= \int_{\Omega} \|f\| \, d\mu.$$

□

Additionally, operators respect the Bochner integral.

Proposition: Let $f \colon \Omega \to X$ be Bochner integrable. If $T \in B(X, Y)$, then $Tf \colon \Omega \to Y$ is Bochner integrable, with

$$\int_{\Omega} Tf \, d\mu = T\left(\int_{\Omega} f \, d\mu\right).$$

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