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Introduction

This is going to be a part of my Honors thesis independent study, focused on amenability and C^*-algebras. This section of notes will be a deeper dive into group amenability. These notes will be taken from the notes my professor has prepared on group amenability, with supplementation from Volker Runde's *Lectures on Amenability* and Pierre de la Harpe's *Topics in Geometric Group Theory*.

I do not claim any of this work to be original.

Basics of Amenable Groups and Subgroups

Let G be a group, with $P(G)$ denoting the power set.

Definition. An invariant mean on G is a set function $m \colon P(G) \to [0,1]$, which satisfies, for all $t \in G$ and $E, F \subseteq G$,

(1) $m(G) = 1$;

(2) $m(E \sqcup F) = M(E) + m(F)$;

(3) $m(tE) = m(E)$.

We say G is amenable if it admits a mean.

We can also say that m is a translation-invariant probability measure on $(G, P(G))$.

Proposition (Amenability of Subgroups and Quotient Groups)**:** Let G be amenable, with $H \leqslant G$.

(1) H is amenable;

(2) for $H \trianglelefteq G$, G/H is amenable.

Proof.

(1) Let R be a right transversal for H (i.e., selecting one element of each right coset of H to make up R).

If m is a mean for G, we set

$$\lambda\colon \mathcal{P}(H) \to [0,1]$$

by $\lambda(E) = m(ER)$. We have

$$\begin{aligned}\lambda(H) &= m(HR)\\ &= m(G)\\ &= 1.\end{aligned}$$

We claim that if $E \cap F = \emptyset$, then $ER \cap FR = \emptyset$, since if we suppose toward contradiction that $ER \cap FR \neq \emptyset$, then $xr_1 = yr_2$ for some $x \in E, y \in F$ and $r_1, r_2 \in R$. Then, we must have $r_2r_1^{-1} = y^{-1}x \in H$, meaning $r_1 = r_2$ and $x = y$, which means $E \cap F \neq \emptyset$.

Thus, we have

$$\begin{aligned}\lambda(E \sqcup F) &= m((E \sqcup F)R)\\ &= m(ER \sqcup FR)\\ &= m(ER) + m(FR)\\ &= \lambda(E) + \lambda(F),\end{aligned}$$

and

$$\begin{aligned}\lambda(sE) &= m(sER)\\ &= m(ER)\\ &= \lambda(E).\end{aligned}$$

(2) For the canonical projection map $\pi\colon G \to G/H$ defined by $\pi(t) = tH$, we define

$$\lambda\colon P(G/H) \to [0,1]$$

by $\lambda(E) = m\left(\pi^{-1}(E)\right)$. We have

$$\begin{aligned}\lambda(G/H) &= m\left(\pi^{-1}(G/H)\right)\\ &= m(G)\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(E \sqcup F) &= m\left(\pi^{-1}(E \sqcup F)\right)\\ &= m\left(\pi^{-1}(E) \sqcup \pi^{-1}(F)\right)\\ &= m\left(\pi^{-1}(E)\right) + m\left(\pi^{-1}(F)\right)\\ &= \lambda(E) + \lambda(F).\end{aligned}$$

To show translation-invariance, we let $sH = \pi(s) \in G/H$, and $E \subseteq G/H$. Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for $r \in s\pi^{-1}(E)$, we have $r = st$ for $\pi(t) \in E$, so $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$.

Additionally, if $r \in \pi^{-1}(\pi(s)E)$, then $\pi(r) \in \pi(s)E$, so $\pi\left(s^{-1}r\right) \in E$, and $s^{-1}r \in \pi^{-1}(E)$. Thus, we have

$$\begin{aligned}\lambda(\pi(s)E) &= m\left(\pi^{-1}(\pi(s)E)\right)\\ &= m\left(s\pi^{-1}(E)\right)\\ &= m\left(\pi^{-1}(E)\right)\\ &= \lambda(E).\end{aligned}$$

□

Understanding Free Groups

In the Tarski's Theorem notes, we discussed a little bit the ramifications of the free group on two generators being paradoxical. In order to better understand free groups, we will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory* and Clara Löh's *Geometric Group Theory: An Introduction*.

Groups specified by Generating Sets

Definition. Let G be a group and $S \subseteq G$ be a subset. The subgroup generated by S is the intersection of all subgroups of G that contain S. We write $\langle S\rangle_G$. We say S generates G if $\langle S\rangle_G = G$.

A group is called finitely generated if it contains a finite subset that contains the group in question.

Definition (Characterization of a Generated Subgroup)**.** We can characterize a generated subgroup by S as follows:

$$\langle S\rangle_G = \left\{s_1^{\varepsilon_1}s_2^{\varepsilon_2}\cdots s_n^{\varepsilon_n} \;\middle|\; n \in \mathbb{N},\ s_1,\dots,s_n \in S,\ \varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}\right\}.$$

Example (Generating Sets)**.**

- If G is a group, then G is a generating set of G.
- The trivial group is generated by the empty set.
- The set $\{1\}$ generates $\mathbb{Z}$, as does $\{2,3\}$. However, $\{2\}$ and $\{3\}$ alone do not generate $\mathbb{Z}$.
- Let X be a set. The symmetric group S_X is finitely generated if and only if X is finite.

Free Groups

Definition. Let S be a set. A group F containing S is said to be freely generated if, for every group G and every map $\varphi\colon S \to G$, there is a unique group homomorphism $\overline{\varphi} : F \to G$ extending φ. The following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\iota"'] & G \\ F \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

A group is free if it contains a free generating set.

Example.

- The additive group $\mathbb{Z}$ is freely generated by $\{1\}$. The additive group $\mathbb{Z}$ is *not* freely generated by $\{2,3\}$, or $\{2\}$, or $\{3\}$. In particular, not every generating set of a group contains a free generating set.

- The trivial group is freely generated by the empty set.
- Not every group is free — the additive groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$ are not free.

We will use the universal property of free groups to show their uniqueness up to isomorphism.

Proposition: Let S be a set. Then, there is at most one group freely generated by S up to isomorphism.

Proof. Let F and F' be two groups freely generated by S, with inclusions of φ and φ' respectively. Because F is freely generated by S, there is a group homomorphism $\overline{\varphi}'\colon F \to F'$ that extends φ — i.e., that $\overline{\varphi}' \circ \varphi = \varphi'$.

Similarly, there is a group homomorphism $\overline{\varphi}\colon F' \to F$ with $\overline{\varphi} \circ \varphi' = \varphi$.

$$\begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[d, hook, "\varphi"'] & F' \\ F \arrow[ur, "\overline{\varphi}'"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi'"'] & F \\ F' \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

We will show that $\overline{\varphi} \circ \overline{\varphi}' = \mathrm{id}_F$, and $\overline{\varphi}' \circ \overline{\varphi} = \mathrm{id}_{F'}$. The composition $\overline{\varphi} \circ \overline{\varphi}'$ is a group homomorphism that makes the following diagram commute.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi"'] & F \\ F \arrow[ur, "\overline{\varphi}\circ\overline{\varphi}'"'] & \end{tikzcd}$$

Since id_F is a group homomorphism contained in this diagram, and F is freely generated by S, we must have $\overline{\varphi} \circ \overline{\varphi}' = \mathrm{id}_F$. Similarly, we must have $\overline{\varphi}' \circ \overline{\varphi} = \mathrm{id}_{F'}$. □

Theorem (Existence of Free Groups)**:** Let S be a set. There exists a group freely generated by S. This group is unique up to isomorphism.

Proof. We want to construct a group consisting of "words" made up of the elements of S and their "inverses," then modding out by the natural cancellation rules.

We consider the alphabet

$$A = S \cup \hat{S}.$$

Here, $\hat{S} = \{\hat{s} \mid s \in S\}$ is a disjoint copy of S, such that $\hat{s}$ will serve as the inverse of s in the group we will construct.

We define A^* to be the set of all finite sequences over the alphabet A, including the empty word ϵ. We define the operation $A^* \times A^* \to A^*$ by concatenation. This operation is associative with neutral element ϵ.

We define

$$F(S) = A^*/\sim,$$

where $\sim$ is the equivalence relation generated by, for all $x, y \in A^*$ and $s \in S$, $xs\hat{s}y \sim xy$ and $x\hat{s}sy \sim xy$.

We denote the equivalence classes with respect to $\sim$ by $[\cdot]$.

Concatenation induces a well-defined operation $F(S) \times F(S) \to F(S)$ by

$$[x][y] = [xy]$$

for $x, y \in A^*$.

We claim that $F(S)$ with the defined concatenation is a group. We can see that $[\epsilon]$ is a neutral element for the operation, and associativity of the operation is inherited from the associativity of the operation on A^*.

To find inverses, we define $I\colon A^* \to A^*$ by $I(\epsilon) = \epsilon$, and

$$\begin{aligned} I(sx) &= I(x)\hat{s} \\ I(\hat{s}x) &= I(x)s \end{aligned}$$

for all $x \in A^*$ and $s \in S$. Induction shows that $I(I(x)) = x$, and

$$\begin{aligned} [I(x)][x] &= [I(x)x] \\ &= [\epsilon] \end{aligned}$$

for all $x \in A^*$. Thus, we must also have

$$\begin{aligned} [x][I(x)] &= [I(I(x))][I(x)] \\ &= [\epsilon]. \end{aligned}$$

Thus, we see that there are inverses in $F(S)$.

To see that $F(S)$ is freely generated by S, we let $\iota\colon S \to F(S)$ be the map given by sending a letter in $S \subseteq A^*$ to its equivalence class in $F(S)$. By construction, $F(S)$ is generated by the subset $\iota(S) \subseteq F(S)$.

We do not know yet that ι is injective, so we take a bit of a detour. We show that for every group G and every map $\varphi\colon S \to G$, there is a unique group homomorphism $\overline{\varphi}\colon F(S) \to G$ such that $\overline{\varphi} \circ \iota = \varphi$.

We construct a map $\varphi^*\colon A^* \to G$ inductively by

$$\begin{aligned} \epsilon &\mapsto e \\ sx &\mapsto \varphi(s)\varphi^*(x) \\ \hat{s}x &\mapsto (\varphi(s))^{-1}\varphi^*(x) \end{aligned}$$

for all $s \in S$ and $x \in A^*$. We can see that, since the definition of φ^* is compatible with the generating set of $\sim$, it is compatible with the equivalence relation of $\sim$ on A^*. Additionally, we can see that $\varphi^*(xy) = \varphi^*(x)\varphi^*(y)$ for all $x, y \in A^*$. Thus,

$$\begin{aligned} \overline{\varphi}\colon F(S) &\to G \\ [x] &\mapsto [\varphi^*(x)], \end{aligned}$$

is, as constructed, a group homomorphism, with $\overline{\varphi} \circ \iota = \varphi$. Since $\iota(S)$ generates $F(S)$, this group homomorphism is unique.

We must now show that ι is injective.

Let $s_1, s_2 \in S$. Consider the map $\varphi\colon S \to \mathbb{Z}$ given by $\varphi(s_1) = 1$ and $\varphi(s_2) = -1$. The corresponding homomorphism $\overline{\varphi}\colon F(S) \to G$ satisfies

$$\begin{aligned} \overline{\varphi}(\iota(s_1)) &= \varphi(s_1) \\ &= 1 \\ &\neq -1 \\ &= \varphi(s_2) \\ &= \overline{\varphi}(\iota(s_2)), \end{aligned}$$

meaning $\iota(s_1) \neq \iota(s_2)$. Thus, ι is injective. $\square$

Free Groups, Free Products, and the Ping Pong Lemma

We can now consider free groups in a more categorical context — specifically, as a special type of free object. Whereas the previous section drew information from Clara Löh's *Geometric Group Theory: An Introduction*, this section will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory*. Specifically, we are focused on chapter 2, which discusses free products, free groups, and the ping pong lemma (which will provide a more general proof of the paradoxicality of $SO(3)$).

Definition (Free Monoid)**.** A monoid is a set with multiplication that is associative and includes a neutral element.

Given a set A, the free monoid on A is the set $W(A)$ of finite sequences of elements of A (also known as words). We write an element of $W(A)$ as $w = a_1a_2\cdots a_n$, where each $a_j \in A$. We identify A with the subset of $W(A)$ of words with length 1.

Definition (Free Product)**.** Let $(\Gamma_i)_{i\in I}$ be a family of groups. Set

$$\begin{aligned} A &= \coprod_{i\in I}\Gamma_i\\ &= \{(g_i, i) \mid g_i \in \Gamma_i, i \in I\} \end{aligned}$$

to be the coproduct of this family.

Let $\sim$ be the equivalence relation generated by

$$\begin{aligned} we_iw' &\sim ww' && \text{where } e_i \in \Gamma_i \text{ is the neutral element}\\ wabw' &\sim wcw' && \text{where } a, b, c \in \Gamma_i,\ c = ab \text{ for some } i \in I \end{aligned}$$

for all $w, w' \in W(A)$. The quotient $W(A)/\sim$ with the operation of concatenation is a group, which is known as the free product of the groups $\{\Gamma_i\}_{i\in I}$. We write it as

$$\bigstar_{i\in I}\Gamma_i.$$

The inverse of the equivalence class for $w = a_1a_2\dots a_n$ is $w^{-1} = a_n^{-1}a_{n-1}^{-1}\cdots a_1^{-1}$. The neutral element is ϵ, which is the empty word.

A word $w = a_1a_2\cdots a_n \in W(A)$ with $a_j \in \Gamma_{i_j}$ is said to be reduced if $i_{j+1} \neq i_j$ and a_j is not the neutral element of Γ_{i_j}.

Proposition (Existence of the Free Product)**:** Let $\{\Gamma_i\}_{i\in I}$ be a family of groups, $A = \coprod_{i\in I}\Gamma_i$, and $\bigstar_{i\in I}\Gamma_i = W(A)/\sim$ be as above.

Then, any element in the free product $\bigstar_{i\in I}\Gamma_i$ is represented by a unique reduced word in $W(A)$.

Proof.

Existence: Consider an integer $n \geqslant 0$ and a reduced word $w = a_1a_2\cdots a_n$ in $W(A)$, an element $a \in A$, and the word $aw \in W(A)$. We set

$$\mathcal{R}(aw) = \begin{cases} w & a = e_i\\ aa_1a_2\cdots a_n & a \in \Gamma_i, a \neq e_i, i \neq k\\ ba_2\cdots a_n & a \in \Gamma_k, aa_1 = b \neq e_k\\ a_2\cdots a_n & a \in \Gamma_k, a_1 = a^{-1} \in \Gamma_k \end{cases},$$

where k is the index for which $a_1 \in \Gamma_k$.

Then, $\mathcal{R}(aw)$ is yet another reduced word, and $\mathcal{R}(aw) \sim aw$, meaning that any word $w \in W(A)$ is equivalent to some reduced word by inducting on the length of w.

Uniqueness: For each $a \in A$, Let $T(a) = \mathcal{R}(aw)$ be a self-map on the set of reduced words.

For each $w = b_1 b_2 \cdots b_n$, we set $T(w) = T(b_1)T(b_2)\cdots T(b_n)$. For $a, b, c \in \Gamma_i$ with $ab = c$, we have $T(a)T(b) = T(c)$, and $T(e_i) = \epsilon$ (the empty word) for all $i \in I$.

For each reduced word, notice that $T(w)\epsilon = w$.

Let w be some word in $W(A)$ with w_1, w_2 reduced words equivalent to w. Since $w_1 \sim w_2$, we have $T(w_1) = T(w_2)$, and

$$\begin{aligned} w_1 &= T(w_1)\epsilon \\ &= T(w_2)\epsilon \\ &= w_2. \end{aligned}$$

□

Corollary: Let $\{\Gamma_i\}_{i \in I}$ and $\Gamma = \bigstar_{i\in I}\Gamma_i$ as above. For each $i_0 \in I$, the canonical inclusion

$$\iota \colon \Gamma_{i_0} \to \Gamma$$

is injective.

Proof. For any $a \in \Gamma_{i_0} \setminus \{e_{i_0}\}$, $\iota(a)$ is represented by a unique one-letter reduced word not equivalent to the empty word. □

Now that we have an understanding of free products, we can conceptualize the free group as a particular type of free product.

Definition (Free Groups)**.** Let X be a set. The free group over X is the free product of a family of copies of $\mathbb{Z}$ indexed by X, denoted $F(X)$.

Equivalently, the free group over X is

$$F(X) = \bigstar_{a \in X}\langle a \rangle,$$

where $\langle a \rangle$ denotes the cyclic group generated by the element a.

We can also identify $F(X)$ with the set of reduced words in $X \sqcup X^{-1}$ (as was done in the previous subsection).

The cardinality of X is called the rank of $F(X)$.

If Γ is a group, then a free subset of Γ is a subset $X \subseteq \Gamma$ such that the inclusion $X \hookrightarrow F(X)$ extends to an isomorphism of $\langle X \rangle_\Gamma$ onto $F(X)$.

We can now state and prove a universal property for free products (which naturally simplifies in the case of a free group.)

Theorem (Universal Property for Free Products)**:** Let Γ be a group, and $\{\Gamma_i\}_{i\in I}$ be a family of groups. Let $\{h_i \colon \Gamma_i \to \Gamma\}_{i \in I}$ be a family of homomorphisms.

Then, there exists a unique homomorphism $h \colon \bigstar_{i\in I}\Gamma_i \to \Gamma$ such that the following diagram commutes for each $i_0 \in I$.

$$\begin{tikzcd} \Gamma_{i_0} \arrow[r, "h_{i_0}"] \arrow[d, hook, "\iota"'] & \Gamma \\ \bigstar_{i\in I}\Gamma_i \arrow[ur, "h"'] & \end{tikzcd}$$

In particular, if Γ is a group, X is a set, and $\phi\colon X \to \Gamma$ is a set map, there exists a unique homomorphism $\Phi\colon F(X) \to \Gamma$ such that $\Phi(x) = \phi(x)$ for each $x \in X$.

Proof. For a reduced word $w = a_1a_2\cdots a_n \in \bigstar_{i\in I}\Gamma_i$ with $a_j \in \Gamma_{i_j} \setminus \{e_{i_j}\}$, and $i_{j+1} \neq i_j$ for each $j \in \{1,\dots,n-1\}$, we set

$$h(w) = h_{i_1}(a_1)h_{i_2}(a_2)\cdots h_{i_n}(a_n),$$

which defines h uniquely in terms of h_i. □

Note that for any two sets X, Y, the universal property provides that any map $X \to Y$ extends canonically to a group homomorphism, $F(X) \to F(Y)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

We can now prove an important lemma that will be useful in understanding paradoxical groups.

Theorem (Ping Pong Lemma)**:** Let G be a group acting on a set X, and let Γ_1, Γ_2 be subgroups of G. Let $\Gamma = \langle \Gamma_1, \Gamma_2\rangle$. Assume Γ_1 contains at least 3 elements and Γ_2 contains at least two elements.

Suppose there exist nonempty subsets $X_1, X_2 \subseteq X$ with $X_1 \triangle X_2 \neq \emptyset$, such that for all $\gamma_1 \in \Gamma_1$ with $\gamma_1 \neq e_G$, and for all $\gamma_2 \in \Gamma_2$ with $\gamma_2 \neq e_G$,

$$\begin{aligned} \gamma(X_2) &\subseteq X_1 \\ \gamma(X_1) &\subseteq X_2. \end{aligned}$$

Then, Γ is isomorphic to the free product $\Gamma_1 \star \Gamma_2$.

Proof. Let w be a nonempty reduced word spelled with letters from the disjoint union of $\Gamma_1 \setminus \{e_G\}$ and $\Gamma_2 \setminus \{e_G\}$. We must show that the element of Γ defined by w is not the identity.

If $w = a_1b_1a_2b_2\cdots a_k$ with $a_1,\dots,a_k \in \Gamma_1 \setminus \{e_G\}$ and $b_1,\dots,b_{k-1} \in \Gamma_2 \setminus \{e_G\}$. Then,

$$\begin{aligned} w(X_2) &= a_1b_1\cdots a_{k-1}b_{k-1}a_k(X_2) \\ &\subseteq a_1b_1\cdots a_{k-1}b_{k-1}(X_1) \\ &\subseteq a_1b_1\cdots a_{k-1}(X_2) \\ &\vdots \\ &\subseteq a_1(X_2) \\ &\subseteq X_1. \end{aligned}$$

Since $X_2 \nsubseteq X_1$, this implies $w \neq e_G$.

If $w = b_1a_2b_2a_2\cdots b_k$, we select $a \in \Gamma_1 \setminus \{e_G\}$, and apply the previous argument to awa^{-1}. Since $awa^{-1} \neq e_G$, neither is w.

Similarly, if $w = a_1b_1\cdots a_kb_k$, we select $a \in \Gamma_1 \setminus \{e_G, a_1^{-1}\}$, and apply the argument to awa^{-1}, and if $w = b_1a_2b_2\cdots a_k$, we select $a \in \Gamma_1 \setminus \{e_G, a_k\}$, and apply the argument to awa^{-1}. □

Example. We can use the Ping Pong Lemma to see that

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a subgroup of $SL(2, \mathbb{Z})$ which is free of rank 2.

Corollary: The special orthogonal group $SO(3)$ contains a subgroup isomorphic to the free group on two generators.

To prove this, we state a different version of the Ping Pong Lemma that we will apply to a particular space.

Theorem (Ping Pong Lemma for Cyclic Groups)**:** Let G act on a set X, and suppose there exist disjoint subsets $A_+, A_-, B_+, B_- \subseteq X$ whose union is not all of X. If there exist elements a and b in G such that

$$\begin{aligned} a \cdot (X \setminus A_-) &\subseteq A_+ \\ a^{-1} \cdot (X \setminus A_+) &\subseteq A_- \\ b \cdot (X \setminus B_-) &\subseteq B_+ \\ b \cdot (X \setminus B_+) &\subseteq B_-, \end{aligned}$$

then it is the case that the group generated by a and b is free of rank 2.

Proof of Corollary. We let

$$\begin{aligned} a &= \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a^{-1} &= \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ b &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \\ b^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}. \end{aligned}$$

We specify

$$X = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} A_+ &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, x \equiv 3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ A_- &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, x \equiv -3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \end{aligned}$$

$$B_{+} = \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, z \equiv 3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}$$
$$B_{-} = \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, z \equiv -3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}.$$

To verify that the conditions of the Ping Pong Lemma hold, we calculate

$$\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 3x+4y \\ -4x+3y \\ 5z \end{pmatrix} \tag{1}$$
$$\begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 3x-4y \\ 4x+3y \\ 5z \end{pmatrix} \tag{2}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 5x \\ 3y-4z \\ 4y+3z \end{pmatrix} \tag{3}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 5x \\ 3y+4z \\ -4y+3z \end{pmatrix}. \tag{4}$$

We verify that the conditions for the Ping Pong Lemma hold for each of these four conditions.

(1) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_{-},$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x + 4y \equiv 3(-4x+3y)$ modulo 5, and that $z' = 5z \equiv 0$ modulo 5.

(2) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_{+},$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x - 4y \equiv -3(4x+3y)$ modulo 5, and $z' = 5z \equiv 0$ modulo 5.

(3) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_{-},$$

we see that $k+1 \in \mathbb{Z}$, $z' = 4y + 3z \equiv 3(3y-4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

(4) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_{+},$$

we see that $k+1 \in \mathbb{Z}$, $z' = -4y + 3z \equiv -3(3y+4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

Since we have verified that the conditions for the Ping Pong Lemma hold for each of the conditions, we have that $\{a, b\} \subseteq SO(3)$ generate a group isomorphic to the free group on two generators. $\square$

States and Means on $\ell_\infty(G)$

Definition. Let G be a group.

(1) The space $\mathcal{F}(G,\mathbb{R})$ is defined by
$$\mathcal{F}(G,\mathbb{R}) = \{f \mid f\colon G \to \mathbb{R} \text{ is a function}\}.$$

(2) A function $f \in \mathcal{F}(G,\mathbb{R})$ is positive if $f(x) \geqslant 0$ for all $x \in G$.

(3) A function $f \in \mathcal{F}(G,\mathbb{R})$ is simple if $\operatorname{Ran}(f)$ is finite. We say
$$\Sigma = \left\{f\colon \mathcal{F}(G,\mathbb{R}) \;\middle|\; f \text{ is simple}\right\}.$$

Fact. $\Sigma \subseteq \mathcal{F}(G,\mathbb{R})$ is a subspace. To see this, if f, g are such that $\operatorname{Ran}(f), \operatorname{Ran}(g)$ are finite, and $\alpha \in \mathbb{R}$, then
$$\operatorname{Ran}(f + \alpha g) \leqslant \operatorname{Ran}(f) + \operatorname{Ran}(g),$$
so $f + \alpha g$ has finite range.

Definition. For $E \subseteq G$, set
$$\mathbb{1}_E \colon G \to \mathbb{R}$$
defined by
$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$
This is the characteristic function of E.

Fact.
$$\operatorname{span}\{\mathbb{1}_E \mid E \subseteq G\} = \Sigma.$$

Proof. We see that $\mathbb{1}_E \in \Sigma$ for any $E \subseteq G$, and Σ is a subspace.

If $\phi \in \Sigma$, with $\operatorname{Ran}(\phi) = \{t_1, \dots, t_n\}$ with t_i distinct, we set
$$E_i = \phi^{-1}(\{t_i\}),$$
meaning
$$\phi = \sum_{i=1}^{n} t_i \mathbb{1}_{E_i}.$$
□

Definition.

(1) A function $f \in \mathcal{F}(G,\mathbb{R})$ is bounded if there exists $M > 0$ such that $\operatorname{Ran}(f) \subseteq [-M, M]$.

(2) The space $\ell_\infty(G)$ is defined by
$$\ell_\infty(G) = \{f \in \mathcal{F}(G,\mathbb{R}) \mid f \text{ is bounded}\}.$$

(3) The norm on $\ell_\infty(G)$ is defined by
$$\|f\| = \sup_{x \in G} |f(x)|.$$

Proposition: The space $\ell_\infty(G)$ is complete, Additionally, $\overline{\Sigma} = \ell_\infty(G)$.

Proof. Let $(f_n)_n$ be Cauchy. For $x \in G$, it is the case that

$$\begin{aligned}|f_n(x) - f_m(x)| &= |(f_n - f_m)(x)|\\ &\leqslant \|f_n - f_m\|,\end{aligned}$$

meaning $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We must show that $f \in \ell_\infty(G)$ and $\|f_n - f\| \to 0$.

$$\begin{aligned}|f(x)| &= \left|\lim_{n\to\infty} f_n(x)\right|\\ &= \lim_{n\to\infty}|f_n(x)|\\ &\leqslant \limsup_{n\to\infty}\|f_n\|\\ &\leqslant C,\end{aligned}$$

as Cauchy sequences are always bounded. Thus, $\sup_{x\in G}|f(x)| \leqslant C$.

Given $\varepsilon > 0$, we find N such that for all $m, n \geqslant N$, $\|f_n - f_m\| \leqslant \varepsilon$. Thus, for $x \in G$, we have

$$\begin{aligned}|f_n(x) - f)m(x)| &\leqslant \|f_n - f_m\|\\ &\leqslant \varepsilon.\end{aligned}$$

Taking $m \to \infty$, we get $|f_n(x) - f(x)| \leqslant \varepsilon$ for all $n \geqslant N$, meaning $\|f_n - f\| \leqslant \varepsilon$ for all $n \geqslant N$.

Now, for $f \in \ell_\infty(G)$, let $\operatorname{Ran}(f) \subseteq [-M, M]$ for some $M > 0$. Let $\varepsilon > 0$. Since $[-M, M]$ is compact, it is totally bounded, so we can find intervals $I_1, \dots, I_n$ with $[-M, M] = \bigsqcup_{k=1}^{n} I_k$, with the length of each I_k less than ε.

Set $E_k = f^{-1}(I_k)$. Pick $t_k \in I_k$. Then, we set

$$\phi = \sum_{i=1}^{n} t_k \mathbb{1}_{E_k}.$$

We see that $\|\phi - f\| < \varepsilon$. ◻

Corollary: For any $f \in \ell_\infty(G)$, there is a sequence $(\phi_n)_n$ in Σ with $\|\phi_n - f\| \to 0$. If $f \geqslant 0$, then it is possible to select $\phi_n \geqslant 0$.

Proposition: Let G be a group. There is an action

$$G \xrightarrow{\lambda_s} \operatorname{Isom}(\ell_\infty(G))$$

defined by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right).$$

Proof. We have

$$\begin{aligned}\lambda_s(f + \alpha g)(t) &= (f + \alpha g)\left(s^{-1}t\right)\\ &= f\left(s^{-1}t\right) + \alpha g\left(s^{-1}t\right)\\ &= \lambda_s(f)(t) + \alpha\lambda_s(g)(t)\\ &= (\lambda_s(f) + \alpha\lambda_s(g))(t).\end{aligned}$$

Thus, λ_s is a linear operator.

We have

$$\begin{aligned}\|\lambda_s(f)\| &= \sup_{t\in G}|\lambda_s(f)(t)| \\ &= \sup_{t\in G}\left|f\left(s^{-1}t\right)\right| \\ &= \|f\|,\end{aligned}$$

hence

$$\begin{aligned}\|\lambda_s(f) - \lambda_s(f)\| &= \|\lambda_s(f-g)\| \\ &= \|f-g\|.\end{aligned}$$

Thus, λ_s is an isometry.

We have

$$\begin{aligned}\lambda_s \circ \lambda_r(f)(t) &= \lambda_r(f)\left(s^{-1}t\right) \\ &= f\left(r^{-1}s^{-1}t\right) \\ &= f\left((sr)^{-1}t\right) \\ &= \lambda_{sr}(f)(t),\end{aligned}$$

meaning $\lambda_s \circ \lambda_r = \lambda_{sr}$. □

Remark: By a similar process, we find that $\lambda_s(\mathbb{1}_E) = \mathbb{1}_{sE}$ for any subset $E \subseteq G$ and $s \in G$.

Definition. A state on $\ell_\infty(G)$ is a continuous linear functional $\mu \in (\ell_\infty(G))^*$ that satisfies the following.

(1) μ is positive;

(2) $\mu(\mathbb{1}_G) = 1$.

A state is called left-invariant if

$$\mu(\lambda_s(f)) = \mu(f).$$

Example. Let G be a group.

- If $x \in G$, then $\delta_x : \ell_\infty(G) \to \mathbb{F}$ defined by

  $$\delta_x(f) = f(x)$$

  is a state. However, note that it is not necessarily invariant.

  $$\begin{aligned}\delta_x(\lambda_s(f)) &= \lambda_s(f)(x) \\ &= f\left(s^{-1}x\right) \\ &\neq f(x).\end{aligned}$$

- If G is finite, then

  $$\mu = \frac{1}{|G|}\sum_{x\in G}\delta_x$$

  is an invariant state.

Lemma (Characterization of States)**:**

(1) If μ is a state on $\ell_{\infty}(G)$, then

$$\|\mu\|_{op} = 1.$$

(2) If $\mu \in (\ell_{\infty}(G))^*$ is such that

$$\begin{aligned}\|\mu\| &= \mu(\mathbb{1}_G)\\ &= 1,\end{aligned}$$

then μ is positive and a state.

Proof.

(1) Given $f \in \ell_{\infty}(G)$, we have

$$\begin{aligned}\|f\|\mathbb{1}_G - f \geqslant 0\\ \|f\|\mathbb{1}_G + f \geqslant 0,\end{aligned}$$

so

$$\begin{aligned}0 &\leqslant \mu(\|f\|\mathbb{1}_G - f)\\ &= \|f\|\mu(\mathbb{1}_G) - \mu(f)\\ 0 &\leqslant \mu(\|f\|\mathbb{1}_G + f)\\ &= \|f\|\mu(\mathbb{1}_G) + \mu(f).\end{aligned}$$

Thus, we have $\pm\mu(f) \leqslant \|f\|\mu(\mathbb{1}_G) = \|f\|$, so $|\mu(f)| \leqslant \|f\|$, so $\|\mu\| \leqslant 1$. Additionally, since $\mu(\mathbb{1}_G) = 1$, we must have $\|\mu\| = 1$.

(2) Suppose $\|\mu\| = \mu(\mathbb{1}_G) = 1$. Let $f \geqslant 0$. Set $g = \frac{1}{\|f\|_u} f$.

Then, $\operatorname{Ran}(g) \subseteq [0, 1]$, and $\operatorname{Ran}(g - \mathbb{1}_G) \subseteq [-1, 1]$, so $\|g - \mathbb{1}_G\|_u \leqslant 1$.

Since $\|\mu\| = 1$, we must have

$$\begin{aligned}|\mu(g - \mathbb{1}_G)| &\leqslant 1\\ |\mu(g) - 1| &\leqslant 1,\end{aligned}$$

and since $\mu(\mathbb{1}_G) = 1$, we must have $\mu(g) \in [0, 2]$, so $\mu(f) = \|f\|\mu(g) \geqslant 0$.

□

Corollary: The set of states on $(\ell_{\infty}(G))^*$ forms a w^*-compact subset of $B_{(\ell_{\infty}(G))^*}$.

Proof. It has been proven in functional analysis that a convex subset of $(\ell_{\infty}(G))^*$ is w^*-compact if it is norm bounded and w^*-closed. Since the set of states is convex and norm-bounded, all we need to show is that $S(\ell_{\infty}(G))$ is w^*-closed.

To this end, let $f \in \ell_{\infty}(G)$ be positive and $(\varphi_i)_i$ be a net in $S(\ell_{\infty}(G))$ with $(\varphi_i)_i \to \varphi$. We must show that φ is positive and satisfies $\varphi(\mathbb{1}_G) = 1$. To this end, we see that

$$\varphi_i(f) \geqslant 1$$

for all $i \in I$, so we must necessarily have $\varphi(f) \geqslant 0$, and similarly, since $\varphi_i(\mathbb{1}_G) = 1$ for each $i \in I$, we also have $\varphi(\mathbb{1}_G) = 1$. □

Proposition: If $\mu \in (\ell_\infty(G))^*$ is a state, then $m\colon P(G) \to [0,1]$ defined by $m(E) = \mu(\mathbb{1}_E)$ is a finitely additive probability measure on G. Moreover, if μ is invariant, then m is a mean.

Proof. We have

$$\begin{aligned}
m(G) &= \mu(\mathbb{1}_G)\\
&= 1\\
m(\varnothing) &= \mu(0)\\
&= 0\\
m(E \sqcup F) &= \mu(\mathbb{1}_{E\sqcup F})\\
&= \mu(\mathbb{1}_E + \mathbb{1}_F)\\
&= \mu(\mathbb{1}_E) + \mu(\mathbb{1}_F)\\
&= m(E) + m(F).
\end{aligned}$$

Additionally, since $0 \leqslant \mathbb{1}_E \leqslant \mathbb{1}_G$, we have $0 \leqslant \mu(\mathbb{1}_E) \leqslant 1$, so $0 \leqslant m(E) \leqslant m(G) = 1$.

If μ is invariant, then

$$\begin{aligned}
m(sE) &= \mu(\mathbb{1}_{sE})\\
&= \mu(\lambda_s(\mathbb{1}_E))\\
&= \mu(\mathbb{1}_E)\\
&= m(E).
\end{aligned}$$

□

Proposition: If G admits a mean, then $(\ell_\infty(G))^*$ admits an invariant state.

Proof. Let m be a finitely-additive probability measure. Define

$$\mu_0\colon \Sigma \to \mathbb{R}$$

by

$$\mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right) = \sum_{k=1}^{n} t_k m(E_k).$$

Since m is finitely additive, it is the case that μ_0 is well-defined, linear, and positive.

Note that $\mu_0(\mathbb{1}_G) = m(G) = 1$.

If m is a mean, then for $f = \sum_{k=1}^{n} t_k E_k$,

$$\begin{aligned}
\mu_0(\lambda_s(f)) &= \mu_0\left(\lambda_s\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right)\right)\\
&= \mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{sE_k}\right)\\
&= \sum_{k=1}^{n} t_k m(sE_k)\\
&= \sum_{k=1}^{n} t_k m(E_k)
\end{aligned}$$

$$= \mu_0(f).$$

Additionally, we see that

$$\begin{aligned}
|\mu_0(f)| &= \left|\sum_{k=1}^{n} t_k m(E_k)\right| \\
&\leqslant \sum_{k=1}^{n} |t_k| m(E_k) \\
&\leqslant \sum_{k=1}^{n} \|f\| m(E_k) \\
&= \|f\| \sum_{k=1}^{n} m(E_k) \\
&\leqslant \|f\|.
\end{aligned}$$

Thus, μ_0 is continuous, so μ_0 is uniformly continuous.

Since $\overline{\Sigma} = \ell_\infty(G)$, we see that μ_0 extends to a continuous linear functional $\mu\colon \ell_\infty(G) \to \mathbb{R}$, with $\mu(\mathbb{1}_G) = \mu_0(\mathbb{1}_G) = 1$.

If $f \geqslant 0$, we find a sequence $(\phi_n)_n$ in Σ with $\phi_n \geqslant 0$, $\|\phi_n - f\| \xrightarrow{n\to\infty} 0$, and we set

$$\begin{aligned}
\mu(f) &= \lim_{n\to\infty} \mu(\phi_n) \\
&= \lim_{n\to\infty} \mu_0(\phi_n) \\
&\geqslant 0,
\end{aligned}$$

meaning μ is a state.

If $f \in \ell_\infty(G)$, $s \in G$, and $(\phi_n)_n$ in Σ with $(\phi_n)_n \to f$, then

$$\begin{aligned}
\|\lambda_s(\phi_n) - \lambda_s(f)\| &= \|\lambda_s(\phi_n - f)\| \\
&= \|\phi_n - f\| \\
&\to 0
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mu(\lambda_s(\phi_n)) &= \mu_0(\lambda_s(\phi_n)) \\
&= \mu_0(\phi_n) \\
&= \mu(\phi_n) \\
&\to \mu(f),
\end{aligned}$$

so $\mu(f) = \mu(\lambda_s(f))$. Thus, $\mu \in (\ell_\infty(G))^*$ is an invariant state. □

Using Invariant States

Proposition: $\mathbb{Z}$ is amenable.

Proof. We know that $\lambda_1\colon \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$, defined by

$$\lambda_1(f)(k) = f(k-1)$$

is an isometry.

We set $Y = \operatorname{Ran}(\mathrm{id} - \lambda_1) \subseteq \ell_\infty(\mathbb{Z})$.

We claim that $\operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$.

Suppose toward contradiction that there is $y \in Y$ with $\|\mathbb{1}_{\mathbb{Z}} - y\|_u = \rho < 1$. Then, $y = f - \lambda_1(f)$ for some $f \in \ell_\infty(\mathbb{Z})$, meaning

$$\|\mathbb{1} - (f - \lambda_1(f))\| = \rho.$$

Thus, for all $k \in \mathbb{Z}$, we have

$$|1 - (f(k) - f(k-1))| \leqslant \rho,$$

meaning $|f(k) - f(k-1)| \geqslant 1 - \rho > 0$. However, such an f cannot be bounded.

Since $\operatorname{dist}_{\overline{Y}}(\mathbb{1}_Z) = \operatorname{dist}_Y(\mathbb{1}_Z) \geqslant 1$, the Hahn–Banach theorem provides $\mu \in (\ell_\infty(\mathbb{Z}))^*$ with $\|\mu\| = 1$, $\mu|_{\overline{Y}} = 0$, and $\mu(\mathbb{1}) = \operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$.

Since $\|\mu\| = 1$ and $\mu(\mathbb{1}) \geqslant 1$, we must have $\mu(\mathbb{1}) = 1$.

Since $\|\mu\| = \mu(\mathbb{1}_{\mathbb{Z}}) = 1$, it is the case that μ is a state on $\ell_\infty(\mathbb{Z})$. Since $\mu(y) = 0$ for all $y \in Y$, we have

$$\begin{aligned}\mu(f - \lambda_1(f)) = 0 \\ \mu(f) &= \mu(\lambda_1(f)),\end{aligned}$$

so inductively, we have $\mu(f) = \mu(\lambda_k(f))$ for all $k \in \mathbb{Z}$, meaning μ is an invariant state on $\ell_\infty(\mathbb{Z})$. Thus, $\mathbb{Z}$ is amenable. □

Proposition: If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.

Proof. Let $\rho \in (\ell_\infty(G/N))^*$ be an invariant state, and $p\colon P(N) \to [0,1]$. For $E \subseteq G$, we define

$$f_E\colon G/N \to \mathbb{R}$$

by $f_E(tN) = p\left(N \cap t^{-1}E\right)$.

We verify that this is well-defined — for $tN = sN$, we have $s^{-1}t \in N$, so

$$\begin{aligned}p\left(N \cap t^{-1}E\right) &= p\left(s^{-1}t\left(N \cap t^{-1}E\right)\right)\\ &= p\left(s^{-1}TN \cap s^{-1}E\right)\\ &= p\left(N \cap s^{-1}E\right).\end{aligned}$$

We also see that f_E is bounded, and

$$\begin{aligned}f_{E\sqcup F}(tN) &= p\left(N \cap t^{-1}(E \sqcup F)\right)\\ &= p\left(N \cap \left(t^{-1}E \sqcup t^{-1}F\right)\right)\\ &= p\left(\left(N \cap t^{-1}E\right) \sqcup \left(N \cap t^{-1}F\right)\right)\\ &= p(N \cap t^{1}E) + p\left(N \cap t^{-1}F\right)\\ &= f_E(tN) + f_F(tN)\end{aligned}$$

$$= (f_E + f_F)(tN).$$

Thus, $f_{E \sqcup F} = f_E + f_F$.

Additionally,

$$\begin{aligned} f_{sE}(tN) &= p\left(N \cap t^{-1}sE\right) \\ &= f_E\left(s^{-1}tN\right) \\ &= \lambda_{sN}(f_E)(tN), \end{aligned}$$

so $f_{sE} = \lambda_{sN}(f_E)$.

Finally,

$$\begin{aligned} f_G(tN) &= p\left(N \cap t^{-1}G\right) \\ &= p(N) \\ &= 1, \end{aligned}$$

so $f_G = \mathbb{1}_{G/N}$.

We define $m\colon P(G) \to [0,1]$ by

$$m(E) = \rho(f_E).$$

Then, we have

$$\begin{aligned} m(E \sqcup F) &= m(E) + m(F) \\ m(G) &= 1 \\ m(sE) &= \rho(f_{sE}) \\ &= \rho(\lambda_{sN}(f_E)) \\ &= \rho(f_E) \\ &= m(E), \end{aligned}$$

meaning m is a mean. ◻

Corollary: The finite direct product of amenable groups is amenable.

Proof. If H and K are amenable, then we know that

$$K \cong \frac{H \times K}{H}$$

is amenable, and H is amenable, so $H \times K$ is amenable. ◻

Corollary: Finitely generated abelian groups are amenable.

Proof. All finitely generated abelian groups are isomorphic to $\mathbb{Z}^d \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$ by the Fundamental Theorem of Finitely Generated Abelian Groups. Since $\mathbb{Z}^d$ is a finite direct product of $\mathbb{Z}$ (which is amenable), and the torsion group $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \mathbb{Z}/n_k\mathbb{Z}$ is finite, we have that a finitely generated abelian group is amenable. ◻

Corollary: If $\{G_i\}_{i\in I}$ is a directed family of amenable groups — i.e., that for any two groups G_j and G_k, there is G_ℓ with $G_j \subseteq G_\ell$ and $G_k \subseteq G_\ell$ — then the direct union,

$$G = \bigcup_{i \in I} G_i,$$

is also amenable.

Proof. Let $\mu_i \in (\ell_\infty(G_i))^*$ be the respective invariant states.

Set

$$M_i = \{\mu \in S((\ell_\infty(G))^*) \mid \mu(\lambda_s(f)) = \mu(f) \text{ for all } s \in G_i\},$$

and set $\mu(f) = \mu_i(f|_{G_i})$. We see that M_i is w^*-closed in $B_{(\ell_\infty(G))^*}$, as we have established the state space as a w^*-closed subset of $B_{(\ell_\infty(G))^*}$.

For $i_1, \dots, i_n$, we find $G_j \supseteq G_{i_1}, \dots, G_{i_n}$, which necessarily exists as $\{G_i\}_{i\in I}$ is directed. Thus, $M_j \subseteq \bigcap_{k=1}^{n} M_{i_k}$, meaning $\{M_i\}_{i\in I}$ has the finite intersection property.

By compactness, there is $\mu \in \bigcap_{i\in I} M_i$, meaning μ is an invariant state. □

Corollary: All abelian groups are amenable.

Proof. Every abelian group is the direct union of its finitely generated subgroups. □

Corollary: All solvable groups are amenable.

Proof. Let $e_G = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ be such that $G_{j-1} \trianglelefteq G_j$ for $j = 1, \dots, n$, and G_j/G_{j-1} abelian.

Since G_0 is abelian, it is amenable. Similarly, G_1/G_0 is abelian, so it is amenable, so G_1 is amenable. Continuing in this fashion, we see that G is amenable. □

Følner's Condition and Invariant Approximate Means

Definition. A group G is said to satisfy the Følner condition if, for every $\varepsilon > 0$, and for all $E \subseteq G$ finite, there is a nonempty $F \subseteq G$ finite such that for all $t \in E$,

$$\frac{|tF \triangle F|}{|F|} \leqslant \varepsilon.$$

Equivalently, since

$$\frac{|tF \triangle F|}{|F|} = 2\left(1 - \frac{|tF \cap F|}{|F|}\right),$$

we have the equivalent formulation that

$$\frac{|tF \triangle F|}{|F|} \leqslant \varepsilon \text{ if and only if } 1 - \frac{|tF\cap|}{|f|} \leqslant \varepsilon/2.$$

Thus, G satisfies the Følner condition if and only if, for all $\varepsilon > 0$ and for all finite $E \subseteq G$, there exists a nonempty $F \subseteq G$ with

$$\frac{|tF \cap F|}{|f|} \geqslant 1 - \varepsilon.$$

Example. All finite groups satisfy Følner's condition by taking $F = G$ for each subset $E \subseteq G$.

Lemma: A countable group G satisfies the Følner condition if and only if G admits a Følner sequence, $(F_n)_n$ with $F_n \subseteq G$ finite, such that

$$\left(\frac{|tF_n \triangle F_n|}{|F_n|}\right)_n \xrightarrow{n\to\infty} 0,$$

or equivalently,

$$\left(\frac{|tF_n \cap F_n|}{|F_n|}\right)_n \xrightarrow{n\to\infty} 1,$$

for all t in G.

Proof. Let G admit a Følner sequence, $(F_n)_n$. Given $\varepsilon > 0$, and $E \subseteq G$ finite, find N such that for all $s \in E$ and $n \geqslant N$,

$$\frac{|sF_n \triangle F_n|}{|F_n|} \leqslant \varepsilon.$$

We take $F = F_N$.

Let G satisfy the Følner condition. We write $G = \bigcup_{n\geqslant 1} E_n$, with $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, and define F_n such that for all $t \in E_n$,

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}.$$

Then, given $t \in G$, it is the case that $t \in E_N$ for some N, so $t \in E_n$ for all $n \geqslant N$, so

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}$$

for all $n \geqslant N$, meaning that

$$\frac{|tF_n \triangle F_n|}{|F_n|} \xrightarrow{n\to\infty} 0.$$

□

Lemma: Let G be a finitely generated group with generating set S (where S may not be symmetric[1]). If $(F_n)_n$ is a sequence of finite subsets of G such that

$$\left(\frac{|sF_n \triangle F_n|}{|F_n|}\right)_n \to 0$$

for all $s \in S$, then $(F_n)_n$ is a Følner sequence for G.

Proof. We start by showing that we can assume S to be symmetric. The following are both true:

- $s(A\triangle B) = sA \triangle sB$;
- $A\triangle C \subseteq (A\triangle B) \cup (B\triangle C)$.

Thus, if we have s^{-1} rather than s, our assumption provides, for all $s \in S$,

$$\begin{aligned}\frac{\left|s^{-1}F_n \triangle F_n\right|}{|F_n|} &= \frac{s^{-1}(F_n \triangle sF_n)}{|F_n|}\\ &= \frac{|F_n \triangle sF_n|}{|F_n|}\\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

Thus, we may assume S is symmetric.

[1]Closed under inversion.

For $s, t \in F_n$, we have

$$\begin{aligned}\frac{|stF_n \triangle F_n|}{|F_n|} &\leqslant \frac{|StF_n \triangle sF_n|}{F_n} + \frac{|sF_n \triangle F_n|}{|F_n|} \\ &= \frac{|s(tF_n \triangle F_n)|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\ &= \frac{|tF_n \triangle F_n|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

We use induction to find the general case. □

Example. Considering $\mathbb{Z}$ again, we remember that $\{1\}$ is the generating set for $\mathbb{Z}$. If we let $F_n = \{-n, -n+1, \dots, -1, 0, 1, \dots$ we have

$$\begin{aligned}\frac{|(F_n + 1)\triangle F_n|}{|F_n|} &= \frac{2}{2n+1} \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

Thus, we have that $\mathbb{Z}$ satisfies the Følner condition.

We now turn our attention to approximate means, from which with Følner's condition, we will be able to construct a different, equivalent condition for group amenability.

Definition. If G is a group, we define

$$\operatorname{Prob}(G) := \left\{ f\colon G \to [0,\infty) \,\middle|\, \left|\operatorname{supp}(f)\right| < \infty,\ \sum_{t\in G} f(t) = 1 \right\}.$$

Note that $\operatorname{Prob}(G) \subseteq B_{\ell_1(G)}$. Given $f \in \operatorname{Prob}(G)$, we set

$$\varphi_f\colon \ell_\infty(G) \to \mathbb{C},$$

defined by

$$\varphi_f(g) = \sum_{t\in G} g(t)f(t).$$

We claim that φ_f is a state on $\ell_\infty(G)$.

Proof. If $g \geqslant 0$, then $\varphi_f(g) \geqslant 0$, and $\varphi_f(\mathbb{1}_G) = 1$. It is also clear that φ_f is linear.

We only need show that $\|\varphi_f\| = 1$. We see

$$\begin{aligned}|\varphi_f(g)| &= \left|\sum_{t\in G} g(t)f(t)\right| \\ &\leqslant \sum_{t\in G} |g(t)||f(t)| \\ &\leqslant \|g\|_\infty \sum_t |f_t| \\ &= \|g\|_\infty.\end{aligned}$$

□

Proposition: There is an action $\lambda\colon G \xrightarrow{\text{Isom}} (\ell_1(G))$ such that $\operatorname{Prob}(G)$ is invariant.

Proof. We let $\lambda_s(f)(t) = f\left(s^{-1}t\right)$. Then,

$$\begin{aligned}
\|\lambda_s(f)\|_1 &= \sum_{t\in G} |\lambda_s(f)(t)|\\
&= \sum_{t\in G}\left|f\left(s^{-1}t\right)\right|\\
&= \sum_{r\in G}|f(r)|\\
&= \|f\|.
\end{aligned}$$

We also see that λ_s is linear.

Additionally,

$$\begin{aligned}
\lambda_r\circ\lambda_s(f)(t) &= \lambda_s(f)\left(r^{-1}t\right)\\
&= f\left(s^{-1}r^{-1}t\right)\\
&= f\left((rs)^{-1}t\right)\\
&= \lambda_{rs}(f)(t).
\end{aligned}$$

We see that if $f \in \text{Prob}(G)$, then for $f \geqslant 0$, we have $\lambda_s(f) \geqslant 0$, and

$$\operatorname{supp}(\lambda_s(f)) = s\left(\operatorname{supp}(f)\right),$$

which is also finite.

Thus,

$$\begin{aligned}
\sum_{t\in G}\lambda_s(f)(t) &= \sum_{t\in G} f\left(s^{-1}t\right)\\
&= \sum_{r\in G} f(r)\\
&= 1
\end{aligned}$$

for $f \in \text{Prob}(G)$. □

Definition. For a countable group G, a sequence $(f_k)_k$ in Prob(G) is an approximate invariant mean if, for all $s \in G$,

$$\|f_k - \lambda_s(f_k)\|_1 \xrightarrow{k\to\infty} 0.$$

Proposition: If G admits a Følner sequence $(F_k)_k$, then it admits an approximate mean.

Proof. Set $f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k} \in \text{Prob}(G)$. Then,

$$\begin{aligned}
\|f_k - \lambda_s(f_k)\|_1 &= \frac{1}{|F_k|}\left\|\mathbb{1}_{F_k} - \lambda_s(\mathbb{1}_{F_k})\right\|\\
&= \frac{1}{|F_k|}\left\|\mathbb{1}_{F_k} - \mathbb{1}_{sF_k}\right\|\\
&= \frac{|F_k\triangle sF_k|}{|F_k|},
\end{aligned}$$

which thus converges to 0 as $k \to \infty$. □

Proposition: If G has an approximate mean, then G is amenable.

Proof. Let $(f_k)_k$ be an approximate mean. We define $\varphi_k = (\varphi_{f_k})_k$ to be a sequence of states on $\ell_\infty(G)$.

Since the state space on $\ell_\infty(G)$ is w^*-compact, there is a state μ and a subnet $\left(\varphi_{k_j}\right)_j$ with $\left(\varphi_{k_j}\right)_j \xrightarrow{w^*} \mu$.

We only need to show that μ is invariant. Note that

$$|\mu(g) - \mu(\lambda_s(g))| \leqslant \left|\mu_g - \varphi_{k_j}(g)\right| + \left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| + \left|\varphi_{k_j}(\lambda_s(g)) - \mu(\lambda_s(g))\right|$$

holds for all $g \in \ell_\infty(G)$, $s \in G$, and for all j.

Given $\varepsilon > 0$, we find J such that for $j \geqslant J$, we have

$$\begin{aligned}\left|\mu(g) - \varphi_{k_j}(g)\right| &< \varepsilon/3 \\ \left|\mu(\lambda_s(g)) - \varphi_{k_j}(\lambda_s(g))\right| &< \varepsilon/3.\end{aligned}$$

We see that

$$\begin{aligned}\left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| &= \left|\sum_{t\in G} g(t) f_{k_j}(t) - \sum_{t\in G} g\left(s^{-1}t\right) f_{k_j}(t)\right| \\ &= \left|\sum_{t\in G} g(t) f_{k_j}(t) - \sum_{r\in G} g(r) f_{k_j}(sr)\right| && r = s^{-1}t \\ &= \left|\sum_{t\in G} g(t)\left(f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right)\right| \\ &\leqslant \|g\|_\infty \sum_{t\in G} \left|f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right| \\ &= \|g\|_\infty \left\|f_{k_j} - \lambda_{s^{-1}}\left(f_{k_j}\right)\right\| \\ &< \varepsilon/3\end{aligned}$$

for large j. Thus, we have

$$|\mu(g) - \mu(\lambda_s(g))| < \varepsilon,$$

for all $\varepsilon > 0$, so $\mu(g) = \mu(\lambda_s(g))$. ◻

Equivalence between Means and Approximate Means

We wish to show that if G is amenable, then G has an approximate mean.

Theorem: Let G be amenable. Then, G has an approximate mean.

Proof. Recall that a net $(f_i)_{i\in I}$ in $\mathrm{Prob}(G)$ has its domain as the set

$$I = \{(E, \varepsilon) \mid E \subseteq G \text{ finite}, \varepsilon > 0\},$$

directed by $(E, \varepsilon) \leqslant (E', \varepsilon')$ if $E \subseteq E'$ and $\varepsilon \geqslant \varepsilon'$.

Suppose there is no approximate mean. Then, there exists a finite subset $E_0 \subseteq G$ and $\varepsilon_0 > 0$ such that for all $s \in E_0$ and $f \in \mathrm{Prob}(G)$, we have

$$\|f - \lambda_s(f)\| \geqslant \varepsilon_0.$$

Let $X = \bigoplus_{|E_0|} \ell_1(G)$ be endowed with the 1-norm.

Consider the set

$$C = \left\{ (f - \lambda_s(f))_{s \in E_0} \;\middle|\; f \in \mathrm{Prob}(G) \right\}.$$

Since $\mathrm{Prob}(G)$ is convex, C is convex, and since $|E_0|$ is finite, C is necessarily bounded. Additionally, note that it is the case that $0 \notin \overline{C}$.

By the Hahn–Banach separation theorem, there is a real-valued $\varphi \in X^*$ such that $\varphi(C) \geqslant 1$.

Note also that

$$\begin{aligned} X^* &\cong \bigoplus_{|E_0|} \ell_1(G)^* \\ &\cong \bigoplus_{|E_0|} \ell_\infty(G) \end{aligned}$$

with the ∞ norm. We let $\varphi = \left(\varphi_{g_s}\right)_{s \in E_0}$ with $g_s \in \ell_\infty(G)$. From the duality between $\ell_1(G)$ and $\ell_\infty(G)$, for any $f \in \ell_1(G)$ and $s \in E_0$, we have

$$\varphi_{g_s}(f) = \sum_{t \in G} f(t)g(t).$$

Thus, for all $f \in \mathrm{Prob}(G)$, we have

$$\begin{aligned} 1 &\leqslant \varphi\left((f - \lambda_s(f))_{s \in E_0}\right) \\ &= \sum_{s \in E_0} \varphi_{g_s}(f - \lambda_s(f)) \\ &= \sum_{s \in E_0} \sum_{t \in G} (f - \lambda_s(f))(t) g_s(t) \\ &= \sum_{s \in E_0} \left(\sum_{t \in G} f(t) g_s(t) - \sum_{t \in G} f\left(s^{-1}t\right) g_s(t) \right) \\ &= \sum_{s \in E_0} \left(\sum_{t \in G} f(t) g_s(t) - \sum_{r \in G} f(r) g(sr) \right) && s^{-1}t = r \\ &= \sum_{s \in E_0} \left(\sum_{r \in T} f(r) g_s(r) - \sum_{r \in G} f(r) \lambda_{s^{-1}}(g)(r) \right) \\ &= \sum_{s \in E_0} \sum_{r \in G} f(r)(g_s - \lambda_{s^{-1}}(g))(r). \end{aligned}$$

Note that this holds for any $f \in \mathrm{Prob}(G)$. In particular, if $f = \delta_t$ for a given $t \in \mathrm{Prob}(G)$, then we must have

$$\begin{aligned} \sum_{s \in E_0} \sum_{r \in G} f(r)(g_s(r) - \lambda_{s^{-1}}(g_s)(r)) &= \sum_{s \in E_0} \sum_{r \in G} \delta_t(r)(g_s(r) - \lambda_{s^{-1}}(g_s)(r)) \\ &= \sum_{s \in E_0} (g_s - \lambda_{s^{-1}} g_s)(t) \\ &\geqslant 1. \end{aligned}$$

Since G is amenable, there is a mean $\mu : \ell_\infty(G) \to \mathbb{C}$ with $\mu(g_s) = \mu(\lambda_{s^{-1}}(g_s))$, meaning

$$\mu\left(\sum_{s \in E_0} (g_s - \lambda_{s^{-1}}(g_s))(t) \right) = 0$$

$$\geqslant 1,$$

which is a contradiction. ◻

Growth Rates and Amenability

Definition. Let G be a group, and let S be a finite symmetric generating set (i.e., $\langle S\rangle = G$ and $s \in S \Leftrightarrow s^{-1} \in S$) for G. We define

$$\ell_{G,S}(g) = \min\{n \mid g = s_1 \cdots s_n | s_i \in S\},$$

where $\ell_{G,S}(e_G) = 0$.

Some easy facts we can see from this definition:

- $\ell_{G,S}(g) = \ell_{G,S}\left(g^{-1}\right)$;
- $\ell_{G,S}(gh) \leqslant \ell_{G,S}(g) + \ell_{G,S}(h)$.

The following is a more substantive fact that we will use in our discussion of growth rates and amenability. Specifically, this will allow us to talk about "a" growth rate, as all generating sets are, in a sense, equivalent.

Fact. Let S, T be finite symmetric generating sets for G. Then, there exists some $K \in \mathbb{N}$ such that, for all $g \in G$,

$$\frac{1}{K}\ell_{G,S}(g) \leqslant \ell_{G,T}(g) \leqslant K\ell_{G,S}(g).$$

Proof. Let

$$\begin{aligned} M &= \max\{\ell_{G,T}(s) \mid s \in S\}\\ N &= \max\{\ell_{G,S}(t) \mid t \in T\}. \end{aligned}$$

Let $n = \ell_{G,S}(g)$, such that $g = s_1 \cdots s_n$ for $s_i \in S$. Then, we see that

$$\begin{aligned} \ell_{G,T}(g) &= \ell_{G,T}(s_1, \dots, s_n)\\ &\leqslant \ell_{G,T}(s_1) + \cdots + \ell_{G,T}(s_n)\\ &\leqslant nM\\ &= M\ell_{G,S}(g). \end{aligned}$$

Similarly, we find that $\ell_{G,S}(g) \leqslant N\ell_{G,T}(g)$. We set $K = \max(M, N)$, and find

$$\frac{1}{K}\ell_{G,S}(g) \leqslant \ell_{G,T}(g) \leqslant K\ell_{G,S}(g).$$

◻

Since $\ell_{G,S}(g)$ is, in a sense, a "norm" on G, we may also define a metric — the word metric — with respect to the generating set S.

Fact. Let S be a finite symmetric generating set for G.

$$d_S(g, h) = \ell_{G,S}\left(g^{-1}h\right),$$

we obtain a metric on G. If S and T are finite symmetric generating sets for G, then the metrics d_S and d_T are equivalent.

Proof.

$$\begin{aligned} d_S(g,h) &= \ell_{G,S}\left(g^{-1}h\right)\\ &= \ell_{G,S}\left(h^{-1}g\right)\\ &= d_S(h,g) \end{aligned}$$

$$\begin{aligned} d_S(g,h) &= \ell_{G,S}\left(g^{-1}h\right)\\ &= \ell_{G,S}\left(g^{-1}kk^{-1}h\right)\\ &\leqslant \ell_{G,S}\left(g^{-1}k\right) + \ell_{G,S}\left(k^{-1}h\right)\\ &= d_S(g,k) + d_S(k,h). \end{aligned}$$

$$\begin{aligned} d_S(g,g) &= \ell_{G,S}\left(g^{-1}g\right)\\ &= \ell_{G,S}(e)\\ &= 0\\ d_S(g,h) = 0 &\Leftrightarrow \ell_{G,S}\left(g^{-1}h\right) = 0\\ &\Leftrightarrow g^{-1}h = e\\ &\Leftrightarrow g = h. \end{aligned}$$

The equivalence of the induced metrics follows from the previous fact. □

Definition. Let G be a group with finite generating symmetric set S. Let $n \geqslant 0$. Let

$$\begin{aligned} B_{G,S}(n) &= \{g \in G \mid \ell_{G,S} \leqslant n\}\\ \gamma_{G,S}(n) &= |B_{G,S}(n)|. \end{aligned}$$

Fact. The following hold:

(1) $\gamma_{G,S}(n)$ is increasing;

(2) $\gamma_{G,S}(n+m) \leqslant \gamma_{G,S}(n)\gamma_{G,S}(m)$;

(3) $\lim_{n\to\infty}(\gamma_{G,S}(n))^{1/n} = \rho_{G,S}$ exists;

(4) if S, T are symmetric finite generating sets for G, then there exists $K \in \mathbb{N}$ such that $\gamma_{G,T}(n) \leqslant \gamma_{G,S}(Kn)$ for all $n \in \mathbb{N}$, and $\rho_{G,S} = \rho_{G,T}$.

Proof.

(1) We have $B_{G,S}(n) \subseteq B_{G,S}(n+1)$, so $\gamma_{G,S}(n)$ is increasing.

(2) We claim that $B_{G,S}(n)B_{G,S}(m) = B_{G,S}(n+m)$. Let $g \in B_{G,S}(n), h \in B_{G,S}(m)$. Then, we have $\ell_{G,S}(gh) \leqslant \ell_{G,S}(g) + \ell_{G,S}(h) \leqslant m+n$, so $B_{G,S}(n)B_{G,S}(m) \subseteq B_{G,S}(n+m)$. Meanwhile, if $g \in B_{G,S}(n+m)$, we may write $g = s_1 \cdots s_k$, where $k \leqslant n+m$ and $s_i \in S$. Then, we may factor

$$g = \underbrace{s_1 \cdots s_\ell}_{g_1}\underbrace{s_{\ell+1} \cdots s_k}_{g_2},$$

where $\ell \leqslant n$ and $k-\ell \leqslant m$, meaning $g_1 \in B_{G,S}(n)$ and $g_2 \in B_{G,S}(m)$. Thus, we have $B_{G,S}(n)B_{G,S}(m) = B_{G,S}(n+m)$.

We thus have

$$\begin{aligned}\gamma_{G,S}(n+m) &= |B_{G,S}(n+m)|\\ &= |B_{G,S}(n)B_{G,S}(m)|\\ &\leqslant |B_{G,S}(n)||B_{G,S}(m)|\\ &= \gamma_{G,S}(n)\gamma_{G,S}(m).\end{aligned}$$

(3) We see that $\gamma_{G,S}(n) \leqslant \gamma_{G,S}(1)^n$. Inductively, we have

$$\begin{aligned}\gamma_{G,S}(n+1) &\leqslant \gamma_{G,S}(n)\gamma_{G,S}(1)\\ &\leqslant \gamma_{G,S}(1)^n\gamma_{G,S}(1)\\ &= \gamma_{G,S}(1)^{n+1},\end{aligned}$$

hence

$$1 \leqslant \gamma_{G,S}(n)^{1/n} \leqslant \gamma_{G,S}(1).$$

(4) We know that there exists K such that

$$\frac{1}{K}\ell_{G,S} \leqslant \ell_{G,T} \leqslant K\ell_{G,S}.$$

Thus, if $g \in B_{G,T}(n)$, then $\ell_{G,T}(g) \leqslant n$, so $\ell_{G,S}(g) \leqslant Kn$, so $g \in B_{G,S}(Kn)$, and $B_{G,T}(n) \subseteq B_{G,S}(Kn)$. We also have $\gamma_{G,T}(n) \leqslant \gamma_{G,S}(Kn)$.

Similarly, if $g \in B_{G,S}(n)$, then $\ell_{G,S}(g) \leqslant n$, so $\ell_{G,T}(g) \leqslant Kn$, and $g \in B_{G,T}(Kn)$. Therefore, $B_{G,S}(n) \subseteq B_{G,T}(Kn)$, so $\gamma_{G,S}(n) \leqslant \gamma_{G,T}(Kn)$. Thus, we get

$$\begin{aligned}\gamma_{G,S}\left(\frac{n}{K}\right)^{1/n} &\leqslant \gamma_{G,T}(n)^{1/n}\\ &\leqslant \left(\gamma_{G,S}(Kn)^{1/Kn}\right)^{K}.\end{aligned}$$

Sending $n \to \infty$, we get

$$\rho_{G,S} \leqslant \rho_{G,T} \leqslant \rho_{G,S},$$

and $\rho_{G,S} = \rho_{G,T}$.

□