

Week 1

Problem (Exercise 1.17): In this exercise, you will show that the moments of a standard Gaussian variable count pair partitions.

- (i) Let X be a standard Gaussian variable. Prove that

$$\begin{aligned}\mathbb{E}(X^{2k}) &= (2k - 1)!! \\ \mathbb{E}(X^{2k-1}) &= 0.\end{aligned}$$

- (ii) Prove that $|P_2(2k)| = (2k - 1)!!$ by putting $P_2(2k)$ in explicit bijection with a set of cardinality $(2k - 1)|P_2(2k - 2)|$.

Remark: I had done the first part of this exercise earlier in a separate notes document, but had not written it up here for submission.

Solution:

- (i) We see that

$$\begin{aligned}E[Z^m] &= \int_{-\infty}^{\infty} x^m e^{-x^2/2} dx \\ &= -x^m e^{-x^2/2} \Big|_{-\infty}^{\infty} + (m-1) \int_{-\infty}^{\infty} x^{m-2} e^{-x^2/2} dx \\ &= (m-1)E[Z^{m-2}].\end{aligned}$$

Therefore, we recover the recursion relation for $(2k - 1)!!$ whenever $m = 2k$ and 0 otherwise.

- (ii) Considering the set $[2k] = \{1, 2, \dots, 2k\}$, we see that there are $2k - 1$ ways to pair 1 with any other element, and there are then $P_2(2k - 2)$ pair partitions of the remaining $2k - 2$ elements. This gives our desired bijection.

Week 2

Problem (Exercise 2.23):

- (i) Find a recursion for $|NC_2(2k)|$.
- (ii) Show that $\text{Cat}(k) = \frac{1}{k+1} \binom{2k}{k}$ satisfies the same recursion relation shown in (i).
- (iii) Let $d\mu = f(x) dx$, where

$$f(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2} & -2 \leq x \leq 2 \\ 0 & \text{else} \end{cases}.$$

Show that

$$\int_{\mathbb{R}} x^{2k} d\mu = \text{Cat}(k)$$

$$\int_{\mathbb{R}} x^{2k-1} d\mu = 0.$$

Solution:

- (i) To find a recursion for $|NC_2(2k)|$, we start by counting the number of valid pairings of 1 with any other element of $[2k]$. For this, we observe that there must be an even number of elements between 1 and whatever it is paired with, meaning there are k valid pairings of 1.

For each of these pairings, there are two separate “sub-blocks” that we use to count the non-crossing partitions, the ones “between” $\{1, 2\ell\}$ as ℓ ranges from 1 to k , and the ones “outside” the pairing $\{1, 2\ell\}$. Combined, this gives the recurrence relation

$$|NC_2(2k)| = \sum_{i=1}^k |NC_2(2i-2)| |NC_2(2k-2i)|,$$

where $NC_2(0) = 1$ vacuously.

- (ii) To evaluate the proposed expression for $C(z)$, we see that

$$\begin{aligned} \frac{1}{2z} (1 - \sqrt{1 - 4z}) &= \frac{1}{2z} \left(1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-4z)^k \right) \\ &= \frac{1}{2z} \left(- \sum_{k=1}^{\infty} \left(\frac{1}{k!} \prod_{i=0}^{k-1} \left(\frac{1}{2} - i \right) \right) (-1)^k 2^{2k} z^k \right) \\ &= \frac{1}{2z} \left(- \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k-3)!!}{2^k k!} (-1)^k 2^{2k} z^k \right) \\ &= \frac{1}{2z} \left(\sum_{k=0}^{\infty} \frac{2^{k+1} (2k-1)!!}{(k+1)!} z^{k+1} \right) \\ &= \frac{1}{2z} \left(\sum_{k=0}^{\infty} \frac{2^{k+1} (2k)!}{2^k k! (k+1)!} z^k \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^k \\ &= \sum_{k=0}^{\infty} \text{Cat}(k) z^k. \end{aligned}$$

By plugging the proposed functional equation into the closed-form expression into $C(z)$, we get

$$\begin{aligned}\frac{1 - \sqrt{1 - 4z}}{2z} &= 1 + z \left(\frac{1 - \sqrt{1 - 4z}}{2z} \right)^2 \\ &= 1 + z \left(\frac{1 + (1 - 4z) - 2\sqrt{1 - 4z}}{4z^2} \right) \\ &= \frac{1 - \sqrt{1 - 4z}}{2z}.\end{aligned}$$

Yet, we also observe that

$$\begin{aligned}1 + zC(z)^2 &= z \left(\sum_{k=0}^{\infty} \text{Cat}(k)z^k \right) \left(\sum_{\ell=0}^{\infty} \text{Cat}(\ell)z^{\ell} \right) \\ &= 1 + z \sum_{k=0}^{\infty} \sum_{\ell=0}^k \text{Cat}(k) \text{Cat}(k-\ell)z^k \\ &= 1 + \sum_{k=0}^{\infty} \sum_{\ell=0}^k \text{Cat}(k) \text{Cat}(k-\ell)z^{k+1} \\ &= 1 + \sum_{k=1}^{\infty} \sum_{\ell=1}^k \text{Cat}(k-1) \text{Cat}(k-\ell)z^k \\ &= \sum_{k=0}^{\infty} \text{Cat}(k)z^k,\end{aligned}$$

so that $\text{Cat}(k)$ satisfies the same recurrence relation as the one for the noncrossing pair partitions.

(iii) The fact that

$$\begin{aligned}\int_{\mathbb{R}} x^{2k-1} d\mu &= \frac{1}{2\pi} \int_{-2}^2 x^{2k-1} \sqrt{1 - 4x^2} dx \\ &= 0,\end{aligned}$$

follows from the fact that this is an odd integrand over a symmetric interval. Else, we find

$$\begin{aligned}\frac{1}{2\pi} \int_{-2}^2 x^{2k} \sqrt{4 - x^2} dx &= \frac{1}{2\pi} \int_0^\pi 2^{2k+2} \sin^2(\theta) \cos^{2k}(\theta) d\theta \\ &= 2^{2k+1} \left(\frac{(2k-1)!!}{(2k)!!} - \frac{(2k+1)!!}{(2k+2)!!} \right) \\ &= 2^{2k+1} \left(1 - \frac{2k+1}{2k+2} \right) \frac{(2k-1)!!}{(2k)!!} \\ &= 2^{2k+1} \frac{1}{2k+2} \frac{(2k-1)!!}{(2k)!!} \\ &= \frac{(2k)!}{(k+1)!k!} \\ &= \frac{1}{k+1} \binom{2k}{k} \\ &= \text{Cat}(k).\end{aligned}$$

Week 3

Problem (Exercise 4.12): A C^* -probability space is a $*$ -probability space (\mathcal{A}, φ) where \mathcal{A} is a C^* -algebra and φ is a state.

- (i) Let (\mathcal{A}, φ) be a C^* -probability space, and suppose $a \in \mathcal{A}$ is normal. Prove that there is a unique compactly supported measure μ such that

$$\varphi(a^p(a^*)^q) = \int_{\mathbb{C}} z^p \bar{z}^q d\mu.$$

- (ii) Let (\mathcal{A}, φ) be a C^* -probability space, and let $\{\mathcal{A}_i\}_{i \in I}$ be a family of freely independent $*$ -subalgebras of \mathcal{A} . For each i , let \mathcal{B}_i be the norm closure of \mathcal{A}_i . Show that $\{\mathcal{B}_i\}_{i \in I}$ are freely independent.

Solution:

- (i) Let (π_φ, H_φ) be the GNS representation of \mathcal{A} with respect to φ , admitting a cyclic vector ξ_φ . Let $T = \pi_\varphi(a)$ admit spectral measure E such that

$$T = \int_{\sigma(T)} z dE.$$

Following from the functional calculus, we observe that for any bounded Borel function ψ on $\sigma(T)$, we have

$$\psi(T) = \int_{\sigma(T)} \psi dE.$$

In particular, this yields

$$T^p(T^*)^q = \int_{\sigma(T)} z^p \bar{z}^q dE.$$

Now, define the measure μ to be the unique measure (emerging from the Riesz Representation Theorem) such that

$$\left\langle \left(\int_{\sigma(T)} \psi dE \right) \xi_\varphi, \xi_\varphi \right\rangle = \int_{\sigma(T)} \psi d\mu.$$

Then, from the definition of the cyclic vector in the GNS construction, we get

$$\begin{aligned} \varphi(a^p(a^*)^q) &= \langle \pi_\varphi(a^p(a^*)^q) \xi_\varphi, \xi_\varphi \rangle \\ &= \langle T^p(T^*)^q \xi_\varphi, \xi_\varphi \rangle \\ &= \int_{\sigma(T)} z^p \bar{z}^q d\mu. \end{aligned}$$

Since the spectrum is compact, μ is our desired compactly supported measure.