

Basics

Definition: Let A be a C^* -algebra. A *representation* of A is a $*$ -homomorphism $\pi: A \rightarrow B(\mathcal{H})$ for some Hilbert space \mathcal{H} .

Definition: Two representations $\pi: A \rightarrow B(\mathcal{H}_\pi)$ and $\rho: A \rightarrow B(\mathcal{H}_\rho)$ are called *unitarily equivalent* if there is a unitary map $U: \mathcal{H}_\rho \rightarrow \mathcal{H}_\pi$ such that

$$\pi(a) = U\rho(a)U^*$$

for all $a \in A$.

Definition: If $\pi: A \rightarrow B(\mathcal{H}_\pi)$ and $\rho: A \rightarrow B(\mathcal{H}_\rho)$ be representations. Then, the formula

$$\pi \oplus \rho(a)(h, k) := (\pi(a)h, \rho(a)k)$$

defines the *direct sum* of π and ρ . If π is unitarily equivalent to a direct sum $\rho_1 \oplus \rho_2$, then we consider $\rho_1 \oplus \rho_2$ to be a decomposition of π in terms of the “smaller” representations.

Definition: A closed subspace \mathcal{K} of \mathcal{H}_π is *invariant* under π if $\pi(a)k \in \mathcal{K}$ for all $a \in A$ and $k \in \mathcal{K}$.

Observe that if \mathcal{K} is an invariant subspace, then the orthogonal complement \mathcal{K}^\perp is also invariant. This follows from the fact that if $y \in \mathcal{K}^\perp$, then

$$\begin{aligned} \langle k, \pi(a)y \rangle &= \langle \pi(a)^*k, y \rangle \\ &= \langle \pi(a^*)k, y \rangle \\ &= 0 \end{aligned}$$

for all $k \in \mathcal{K}$.

Conversely, if \mathcal{K} is invariant, then we can recover $\pi = \pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp}$, via the canonical unitary isomorphism $U: \mathcal{K} \oplus \mathcal{K}^\perp \rightarrow \mathcal{H}_\pi$ given by $(k, y) \mapsto k + y$.

Definition: A representation π is *irreducible* if there are no closed invariant subspaces apart from $\{0\}$ and \mathcal{H}_π .

Lemma: A representation π of a C^* -algebra A is irreducible if and only if $\pi(A)' = \mathbb{C}I_{\mathcal{H}}$, where $\pi(A)'$ denotes the commutant of $\pi(A)$.

Proof. Suppose \mathcal{V} is a nontrivial invariant subspace for π . Then, the orthogonal projection $P_{\mathcal{V}}$ commutes with every $\pi(A)$ and is not a scalar multiple of $I_{\mathcal{H}}$.

Now, suppose there is a non-scalar operator T commuting with $\pi(A)$. Then, either the real or imaginary part of T is a self-adjoint operator S that commutes with $\pi(A)$. From the continuous functional calculus, since $\sigma(S)$ is not one point, there are some nonzero continuous $f, g \in C(\sigma(S))$ such that $fg = 0$. Then, since $f(S), g(S) \in C^*(S)$, and $f(S), g(S)$ commute with $\pi(A)$, it follows that $\overline{f(S)\mathcal{H}}$ and $\overline{g(S)\mathcal{H}}$ are nonzero mutually orthogonal invariant subspaces, so π is reducible. \square

Definition: If π is a representation of the C^* -algebra A , then we call the subspace

$$[\pi(A)\mathcal{H}_\pi] = \overline{\text{span}}\{\pi(a)h \mid h \in \mathcal{H}_\pi, a \in A\}$$

the *essential subspace* of \mathcal{H}_π . The representation π is called *nondegenerate* if the essential subspace \mathcal{K} is equal to \mathcal{H}_π .

Note that the representation π being nondegenerate is equivalent to $\pi(1) = I_{\mathcal{H}_\pi}$ if A has an identity, or $\pi(e_i) \rightarrow I_{\mathcal{H}_\pi}$ strongly for any approximate identity $(e_i)_{i \in I}$.

The essential subspace is always invariant, and π is equivalent to $\pi|_{\mathcal{K}} \oplus 0$. Generally, if I is an ideal in A , then the subspace

$$\mathcal{K} = \overline{\text{span}}\{\pi(a)h \mid h \in \mathcal{H}_\pi, a \in I\}$$

is invariant, but π is not zero on \mathcal{K}^\perp unless I is an essential ideal.¹ Any nondegenerate representation of an ideal I extends canonically to a nondegenerate representation π of A on the same space.

The Gelfand–Naimark–Segal Construction

Definition: An element a of a C^* -algebra A is called *positive* if there is $b \in A$ with $a = b^*b$. Equivalently, a is positive if and only if $\sigma(a) \subseteq [0, \infty)$.

There are a few useful identities for positive elements. Specifically, the following hold:

$$\begin{aligned}\|a\|^2 1_A &\geq a^*a \\ \|a\|^2 b^*b - b^*a^*ab &\geq 0.\end{aligned}$$

Definition: A linear functional $\rho: A \rightarrow \mathbb{C}$ is called *positive* if $\rho(a) \geq 0$ whenever $a \geq 0$. A positive linear functional of norm 1 is called a *state*.

Lemma: Let f be a positive linear functional on a C^* -algebra A . Then, for all $a, b \in A$, we have

$$f(b^*a) = \overline{f(a^*b)}$$

and

$$|f(b^*a)|^2 \leq f(b^*b)f(a^*a).$$

Proof. To see the first identity, we let $\lambda \in \mathbb{C}$, and observe that

$$\begin{aligned}0 &\leq f((\lambda a + b)^*(\lambda a + b)) \\ &= |\lambda|^2 f(a^*a) + \bar{\lambda} f(a^*b) + \lambda f(b^*a) + f(b^*b).\end{aligned}$$

Now, since $|\lambda|^2 f(a^*a) + f(b^*b)$ is always real, we must have

$$\operatorname{Im}(\bar{\lambda} f(a^*b) + \lambda f(b^*a)) = 0$$

for all λ . By taking $\lambda = 1$ and $\lambda = i$, we get equality of imaginary and real parts of $f(a^*b)$ and $\overline{f(b^*a)}$.

As for the Cauchy–Schwarz inequality, we observe that if $\lambda = \overline{x f(b^*a)}$ for some $x \in \mathbb{R}$, we have

$$\begin{aligned}0 &\leq x^2 |f(b^*a)|^2 f(a^*a) + x |f(a^*b)|^2 + x |f(b^*a)|^2 + f(b^*b) \\ &= x^2 |f(b^*a)|^2 f(a^*a) + 2x |f(b^*a)|^2 + f(b^*b).\end{aligned}$$

The right-hand side is a quadratic in x that is always greater than or equal to 0, so

$$4|f(b^*a)|^4 - 4|f(b^*a)|^2 f(a^*a)f(b^*b) \leq 0.$$

□

To understand the GNS construction, we start by taking a state τ on a C^* -algebra A . Then, defining

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\},$$

we observe that $\tau(b^*a) = 0$ if either a or b are in N_τ . In particular, we get the inner product on A/N_τ given by

$$\langle a + N_\tau, b + N_\tau \rangle = \tau(b^*a).$$

Define \mathcal{H}_τ to be the Hilbert space completion of A/N_τ . Since $\|a\|^2 b^*b - b^*a^*ab$ is of the form c^*c , we have

$$\|a(b + N_\tau)\|^2 = \tau(b^*a^*ab)$$

¹An essential ideal is one that has nonzero intersection with any other closed ideal of A .

$$\begin{aligned}
&= \|a\|^2 \tau(b^*b) - \tau(c^*c) \\
&\leq \|a\|^2 \tau(b^*b) \\
&= \|a\|^2 \|b + N_\tau\|^2.
\end{aligned}$$

In particular, this means that the elements of A act as bounded operators on A/N_τ , which we extend to operators $\pi_\tau(a)$ in the completion. This gives a nondegenerate representation π_τ of A on the Hilbert space \mathcal{H}_τ .

Lemma: Suppose A is a non-unital C^* -algebra, and $\rho \in S(A)$. Then, if $(e_i)_{i \in I}$ is an approximate identity for A , $\rho(e_i) \rightarrow 1$. Furthermore, the formula $\tau(a + \lambda 1) = \rho(a) + \lambda$ defines a state τ on the unitization \tilde{A} .

Proof. Since approximate identities in C^* -algebras are increasing, ρ is positive, and $\rho(e_i) \leq 1$ for each e_i , it follows that $(\rho(e_i))_{i \in I}$ converges to L for some $L \leq 1$. Now, since

$$\begin{aligned}
e_i^2 &= (e_i^{1/2})^* e_i e_i^{1/2} \\
&\leq (e_i^{1/2})^* e_i^{1/2} \\
&= e_i.
\end{aligned}$$

Thus, if $a \in A$, Cauchy–Schwarz gives

$$\begin{aligned}
|\rho(e_i a)|^2 &\leq \rho(e_i^2) \rho(a^* a) \\
&\leq \rho(e_i) \|a\|^2 \\
&\leq L \|a\|^2.
\end{aligned}$$

This holds for all $a \in A$ if and only if $L \geq 1$ as $\|\rho\| = 1$, so since $L \leq 1$ it follows that $\rho(e_i) \rightarrow 1$.

By the earlier equation, we also have that $|\rho(a)|^2 \leq \rho(a^* a)$. Combining with the identity $\rho(a^*) = \overline{\rho(a)}$, we get

$$\begin{aligned}
\tau((\lambda 1 + a)^*(\lambda 1 + a)) &= \tau(|\lambda|^2 1 + \bar{\lambda} a + \lambda a^* + a^* a) \\
&= |\lambda| + 2 \operatorname{Re}(\bar{\lambda} \rho(a)) + \rho(a^* a) \\
&\geq |\lambda| - 2|\lambda| |\rho(a)| + \rho(a^* a) \\
&\geq |\lambda|^2 - 2|\lambda| |\rho(a)| + |\rho(a)|^2 \\
&\geq (|\lambda| - |\rho(a)|)^2 \\
&\geq 0.
\end{aligned}$$

Thus, τ is positive and has $\tau(1) = 1$. □

Definition: If $\pi: A \rightarrow B(\mathcal{H}_\pi)$ is a representation, then a vector $h \in \mathcal{H}_\pi$ is called cyclic for π if the set

$$[\pi(A)h] := \{\pi(a)h \mid a \in A\}$$

spans a dense subspace of \mathcal{H}_π .

Proposition: If ρ is a state on a C^* -algebra A , then there is a unit cyclic vector h_ρ in \mathcal{H}_ρ such that

$$\rho(a) = \langle \pi_\rho(a) h_\rho, h_\rho \rangle$$

for all $a \in A$.

Conversely, if h is a unit cyclic vector for a representation $\pi: A \rightarrow B(\mathcal{H}_\pi)$, then the map $\tau: A \rightarrow \mathbb{C}$ given by

$$\tau(a) = \langle \pi(a)h, h \rangle$$

is a state on A . The map $a \mapsto \pi(a)h$ induces a unitary isomorphism between \mathcal{H}_τ and \mathcal{H}_π such that $\pi(a) = U\pi_\tau(a)U^*$.

Proof. If A is unital, then we set $h_\rho = 1 + N_\rho$ as our cyclic vector. It then follows that

$$\begin{aligned} [\pi(A)h_\rho] &= \{\pi(a) + N_\rho \mid a \in A\} \\ &= A/N_\rho, \end{aligned}$$

which is norm-dense in \mathcal{H}_ρ .

If A is non-unital, we may use the lemma to extend ρ to $\tau: \tilde{A} \rightarrow \mathbb{C}$; the inclusion $A \hookrightarrow \tilde{A}$ induces an isometry V of \mathcal{H}_ρ into \mathcal{H}_τ mapping $a + N_\rho$ to $a + N_\tau$. This isometry is such that

$$V\pi_\rho(a) = \pi_\tau(a)V$$

for all $a \in A$. We may identify \mathcal{H}_ρ with the subspace $V\mathcal{H}_\rho$ of \mathcal{H}_τ , since \mathcal{H}_ρ is then essential subspace $[\pi_\tau(a)\mathcal{H}_\tau]$ of $\pi_\tau|_A$, with $\pi_\tau|_A = \pi_\rho \oplus 0$. The projection of $1 + N_\tau$ onto \mathcal{H}_ρ satisfies

$$\begin{aligned} \pi_\rho(a)h_\rho &= \pi_\tau(a)(1 + N_\tau) \\ &= a + N_\tau. \end{aligned}$$

Thus, h_ρ is cyclic for π_ρ , and has

$$\begin{aligned} \langle \pi_\rho(a)h_\rho, h_\rho \rangle &= \langle \pi_\tau(a)(1 + N_\tau), 1 + N_\tau \rangle \\ &= \tau(a) \\ &= \rho(a). \end{aligned}$$

Conversely, let τ be a positive functional of norm at most 1; then, τ has norm equal to 1 since $\pi(e_i)h \rightarrow h$ for any approximate identity $(e_i)_{i \in I}$ of A . \square