Problem (Problem 1)**:** For two ideals $I, J \subseteq R$, prove the following.

(a) The intersection $I \cap J$ is an ideal of R.

(b) The product $IJ \subseteq I \cap J$.

(c) Let $f\colon R \to R/(IJ)$ be the canonical homomorphism. Then, for any $x \in I \cap J$, the image $f(x)$ is nilpotent.

(d) If $I + J = R$, then $IJ = I \cap J$.

Solution:

(a) If $x, y \in I \cap J$, then $x - y \in I \cap J$ since $x - y \in I$ and $x - y \in J$. Furthermore, if $r \in R$, then $rx \in I$ and $rx \in J$, so $rx \in I \cap J$, so $I \cap J$ is an ideal.

(b) We observe that for any $q \in IJ$, we may express

$$q = \sum_{k=1}^{n} x_k y_k,$$

where $x_k \in I$ and $y_k \in J$. In particular, each $x_k y_k \in I \cap J$, so $q \in I \cap J$, meaning $IJ \subseteq I \cap J$.

(c) Let $x \in I \cap J$. Then, following from the well-definedness of operations in the quotient ring, we see that $(x + IJ)^n = x^n + IJ$. In particular, if $n = 2$, then x^2 is a linear combination of an element of I multiplied by an element of J, so $x^2 \in IJ$, meaning that $(x + IJ)^2 = x^2 + IJ = IJ = 0 + IJ$, meaning that x is nilpotent.

(d) We will show that if $q \in I \cap J$, then q can be written as a linear combination of elements of I multiplied by elements of J. In particular, we start by letting $i \in I$ and $j \in J$ be such that $i + j = 1$. Then, $q(i + j) = q$, meaning that $qi + qj = q$, and since $q \in I \cap J$, we have expressed q as a linear combination of elements of I multiplied by elements of J. Thus, $I \cap J \subseteq IJ$, meaning $IJ = I \cap J$.

Problem (Problem 3)**:** Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers.

(a) Show that every nonzero ideal $I \subseteq R$ contains a nonzero integer.

(b) Identify the quotient R/I where $I = (2 + i)$ is the principal ideal generated by $2 + i$.

Solution:

(a) Let $I \subseteq R$ be a nonzero ideal, and let $a + ib \in I$ with $a, b \in \mathbb{Z} \setminus \{0\}$. Since multiplication by any element of R yields another element in I, we see that

$$\begin{aligned}(a + ib)(a - ib) &= a^2 + b^2 \\ &\in R,\end{aligned}$$

and since $a, b \neq 0$, so too is $a^2 + b^2$, so any nonzero ideal of R contains a nonzero integer.

(b) Consider the map $\varphi\colon \mathbb{Z} \to R/I$ given by $z \mapsto z + I$. Since this is a composition of the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ and the projection map $\pi\colon \mathbb{Z}[i] \to \mathbb{Z}[i]/(2 + i)$, this is a ring homomorphism. We will show that this ring homomorphism is surjective.

Let $(a + bi) + I \in R/I$. We will show that there is some $k \in \mathbb{Z}$ such that $k - (a + bi) \in (2 + i)$. For this purpose, let

$$(x + yi)(2 + i) = (a - k) + bi,$$

so that

$$2x - y = (a - k)$$

$$2y + x = b.$$

We thus get that

$$\begin{aligned} 5x &= 2a + b - 2k \\ 5y &= 2b - a + k. \end{aligned}$$

Reducing modulo 5, we thus have that

$$\begin{aligned} 0 &\equiv 2a + b - 2k \\ &\equiv 2b - a + k, \end{aligned}$$

meaning that $k = 3b + a$ (modulo 5). We thus have that

$$\begin{aligned} (3b + a) - (a + bi) &= 3b - bi \\ &= b(3 - i) \\ &= b(1 - i)(2 + i), \end{aligned}$$

so $z \mapsto z + I$ is surjective. We observe furthermore that $5\mathbb{Z} \subseteq \ker(\varphi)$, and since 5 is prime, it is a subset of no other ideal, and since the homomorphism φ is nontrivial, we thus have that $\ker(\varphi) = 5\mathbb{Z}$, so by the first isomorphism theorem, $\mathbb{Z}[i]/(2 + i) \cong \mathbb{Z}/5\mathbb{Z}$.

Problem (Problem 4)**:** Let R_1 and R_2 be rings. We consider the Cartesian product $R = R_1 \times R_2$ and introduce the operations

$$\begin{aligned} (a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2) \\ (a_1, a_2)(b_1, b_2) &= (a_1 a_2, b_1 b_2). \end{aligned}$$

Show that R is a ring with these operations.

Solution: If $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$, then since $a_1 - a_2 \in R_1$ and $b_1 - b_2 \in R_2$, it is clear that $R_1 \times R_2$ endowed with the $+$ operation is an abelian group with additive identity $(0, 0)$ as we have endowed $R_1 \times R_2$ with coordinate-wise operations inherited from R_1 and R_2.

Similarly, if $c_1 \in R_1$ and $c_2 \in R_2$, then since multiplication is associative in R_1 and R_2, we have

$$\begin{aligned} (a_1, a_2) \cdot ((b_1, b_2) \cdot (c_1, c_2)) &= (a_1 \cdot (b_1 \cdot c_1), a_2 \cdot (b_2 \cdot c_2)) \\ &= ((a_1 \cdot b_1) \cdot c_1, (a_2 \cdot b_2) \cdot c_2) \\ &= ((a_1, a_2) \cdot (b_1, b_2)) \cdot (c_1, c_2), \end{aligned}$$

so multiplication is associative in $R_1 \times R_2$. Finally, since multiplication distributes over addition in R_1 and in R_2, we have that

$$\begin{aligned} (a_1, a_2) \cdot ((b_1, b_2) + (c_1, c_2)) &= (a_1 \cdot (b_1 + c_1), a_2 \cdot (b_2 + c_2)) \\ &= (a_1 \cdot b_1 + a_1 \cdot c_1, a_2 \cdot b_2 + a_2 \cdot c_2) \\ &= (a_1 \cdot b_1, a_2 \cdot b_2) + (a_1 \cdot c_1, a_2 \cdot c_2) \\ &= (a_1, a_2) \cdot (b_1, b_2) + (a_1, a_2) \cdot (c_1, c_2), \end{aligned}$$

meaning that multiplication distributes over addition in $R_1 \times R_2$.

Problem (Problem 7)**:** Let I, J be ideals such that $I + J = R$ and $IJ = 0$. Show that the map

$$f \colon R \to \mathbb{R}/I \times \mathbb{R}/J$$

given by $x \mapsto (x + I, x + J)$ is a ring isomorphism.

Solution: By the result from Problem 1, we know that since $I + J = R$ and $IJ = 0$, then $I \cap J = \{0\}$. Therefore, since $r \in \ker(f)$ if and only if $r + I = 0 + I$ and $r + J = 0 + J$, or that $r \in I$ and $r \in J$, we have that $\ker(f) = 0$.

Furthermore, for any $(r + I, s + J) \in R/I \times R/J$, we use the fact that $r + I \neq 0 + I$ if and only if $r \in J$, and $s + J \neq 0 + J$ if and only if $s \in I$, meaning that $x = r + s$ satisfies $x + I = r + I$ and $x + J = r + J$. Moreover, if $r + I = 0 + I$ and $s + J \neq 0 + J$, then $x = s$ satisfies the desired result, while if $r + I \neq 0$ and $s + J = 0$, then $x = r$ satisfies the desired result. Thus, f is surjective, hence an isomorphism.