Problem (Problem 2)**:** Let R be a PID. For an R-module M, denote by $d(M)$ the minimal number of generators of M.

(a) Prove that if M is a finitely generated R-module, and N is a submodule of M, then $d(N) \leq d(M)$.

(b) Let $a \in R$ be a nonzero non-unit. Find (with proof) the number of submodules of R/aR in terms of the prime decomposition of a.

Solution:

(a) Let $\{v_1, \dots, v_n\}$ be a minimal generating set for M. Via the surjection $R^n \to M$ taking $(r_1, \dots, r_n) \mapsto \sum_{i=1}^{n} r_i v_i$, we observe that $M \cong R^n/G$ for some submodule G of R^n. Since N is a submodule of M, it follows from the fourth isomorphism theorem that N corresponds to a submodule of R^n containing G, which we will call N'; since N' is a submodule of a free module, it is free with rank $m \leq n$, and N' surjects onto N so that $d(N) \leq d(N') \leq n = d(M)$.

(b) Without loss of generality, let $a = p_1^{d_1} \cdots p_t^{d_t}$ be the prime decomposition for a, where $d_i \in \mathbb{N}$. From the Chinese Remainder Theorem, we have

$$R/(a) \cong R/\left(p_1^{d_1}\right) \oplus \cdots \oplus R/\left(p_t^{d_t}\right).$$

Observe that any submodule of $R/(a)$ is in correspondence with an ideal containing a (by the fourth isomorphism theorem). These ideals are precisely the ideals of products of prime powers that are less than or equal to a. Since, given $p_i^{d_i}$, there are $0, \dots, d_i$ potential options for the power of p_i, so that there are $(d_1 + 1) \cdots (d_t + 1)$ submodules in $R/(a)$.

Problem (Problem 3)**:**

(a) Let R be a PID, M a finitely generated R-module, and

$$M = \left(\bigoplus_{i=1}^{\ell} R/(a_i)\right) \oplus R^s$$

its invariant factor decomposition. Prove that $d(M) = m + s$.

(b) Again let R be a PID. Let F be a free R-module of rank n with basis $e_1, \dots, e_n$, N the submodule of F generated by some elements $v_1, \dots, v_n \in F$, and let $A \in \mathrm{Mat}_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Find a simple condition on the entries of A which holds if and only if $d(F/N) = n$.

Solution:

(a) Let p be a prime dividing a_1. Now, we observe that

$$\begin{aligned} M/pM &\cong \frac{\left(\bigoplus_{i=1}^{m} R/(a_i)\right) \oplus R^s}{\left(\bigoplus_{i=1}^{m} p(R/(a_i))\right) \oplus (pR)^s} \\ &\cong (R/(p))^{m+s}. \end{aligned}$$

Now, since R is a PID, any prime ideal is maximal, meaning that $d(M') = m + s$ is the dimension of $M' = (R/(p))^{m+s}$. Since M' is a quotient of M, it follows that $d(M) \geq m + s$.

Yet, since the set $\{e_i \mid 1 \leq i \leq m + s\}$ generates M as an R-module, it follows that $d(M) \leq m + s$, so that $d(M) = m + s$.

Problem (Problem 6)**:**

(i) Let V and W be R-modules over a commutative ring. Show that there is a natural homomorphism $\varphi\colon V^* \otimes W \to \operatorname{Hom}_R(V, W)$ such that $(\varphi(f \otimes w))(v) = f(v)w$.

(ii) Assume that W is a finitely generated free R-module. Prove that φ is an isomorphism.

(iii) Give examples showing that φ need not be surjective if W is either not free or W is free but not finitely generated.

Solution:

(i) We consider the map $\phi\colon V^* \times W \to \operatorname{Hom}_R(V, W)$ given by $\phi(f, w)v = f(v)w$. We observe that ϕ is bilinear, as

$$\begin{aligned}
\phi(f_1 + f_2, w)v &= (f_1 + f_2)(v)w\\
&= (f_1(v) + f_2(v))w\\
&= f_1(v)w + f_2(v)w\\
&= (\phi(f_1, w) + \phi(f_2, w))v\\
\phi(f, w_1 + w_2)v &= f(v)(w_1 + w_2)\\
&= (\phi(f, w_1) + \phi(f, w_2))v.
\end{aligned}$$

Therefore, ϕ induces a linear map $\varphi\colon V^* \otimes_R W \to \operatorname{Hom}_R(V, W)$.

(ii) Let W be a finitely-generated free R-module. Then, we have that

$$\begin{aligned}
V^* \otimes_R W &\cong V^* \otimes_R \left(\bigoplus_{i=1}^{n} R\right)\\
&\cong \bigoplus_{i=1}^{n} V^* \otimes_R R.
\end{aligned}$$

Now, we come to the question of what exactly $V^* \otimes_R R$ is.