

8.1

(a)

$$\begin{aligned}
 \int_0^1 2^x dx &= \int_0^1 e^{x(\ln 2)} dx \\
 &= \frac{1}{\ln 2} \left(e^{x(\ln 2)} \Big|_0^1 \right) && u = x(\ln 2) \\
 &= \frac{1}{\ln 2} \left(2^x \Big|_0^1 \right) \\
 &= \frac{1}{\ln 2} (2 - 1) \\
 &= \frac{1}{\ln 2}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^x dx &= \int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2} + x - \frac{1}{2}\right) + \frac{1}{2}} dx && \text{Completing the square.} \\
 &= e^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx \\
 &= \sqrt{2\pi} e && \text{Gaussian Integral}
 \end{aligned}$$

(c)

(d)

$$\int_{-a}^a \sin x e^{-\alpha x^2} dx = 0 \quad \text{Even/odd.}$$

(e)

$$\begin{aligned}
 \int_0^1 e^{\sqrt{x}} dx &= x e^{\sqrt{x}} \Big|_0^1 - \frac{1}{2} \int_0^1 x e^{\sqrt{x}} dx && \text{Integration by Parts} \\
 &= e - \int_0^1 u^3 e^u du && u = \sqrt{x} \\
 &= e - \left(u^3 e^u \Big|_0^1 - 3u^2 e^u \Big|_0^1 + 6u e^u \Big|_0^1 - 6e^u \Big|_0^1 \right) && \text{Repeated integration by parts.} \\
 &= 3e - 6.
 \end{aligned}$$

To evaluate $\int_0^1 u^3 e^u du$, we used tabular integration as follows:

Sign	Differentiate	Integrate
+	u^3	e^u
-	$3u^2$	e^u
+	$6u$	e^u
-	6	e^u
+	0	e^u

Taking the boundary integrals, we obtain

$$u^3 e^u \Big|_0^1 - 3u^2 e^u \Big|_0^1 + 6u e^u \Big|_0^1 - 6e^u \Big|_0^1 = 6 - 2e$$

(f)

$$\begin{aligned}
 \int \frac{1}{\sqrt{1+x^2}} dx &= \int \frac{1}{\cosh(u)} \cosh(u) du & x &= \sinh(u) \\
 &= u + C \\
 &= \sinh^{-1}(x) + C.
 \end{aligned}$$

(g)

$$\begin{aligned}
 \int \tanh x \, dx &= \int \frac{\sinh x}{\cosh x} \, dx \\
 &= \int \frac{1}{u} \, du & u &= \cosh x \\
 &= \ln |u| + C \\
 &= \ln |\cosh x| + C.
 \end{aligned}$$

(h)

$$\begin{aligned}
 \int \tan^{-1} x \, dx &= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx && \text{integration by parts} \\
 &= x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + C. && u\text{-substitution implicit}
 \end{aligned}$$

(i)

$$\begin{aligned}
 \int_S z^2 \, d\mathbf{a} &= \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\phi \, d\theta \\
 &= \frac{\pi}{2} \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \\
 &= -\frac{\pi}{2} \int_0^{-1} t^2 \, dt && t = \cos \theta \\
 &= \frac{\pi}{2} \left(\frac{t^3}{3} \Big|_{-1}^0 \right) \\
 &= \frac{\pi}{6}
 \end{aligned}$$

8.8

(a)

$$\begin{aligned}
 \int_0^\infty \frac{x}{e^x - 1} \, dx &= \int_0^\infty \frac{x e^{-x}}{1 - e^{-x}} \, dx \\
 &= \int_0^\infty x e^{-x} \left(\sum_{k=0}^\infty e^{-kx} \right) \, dx \\
 &= \sum_{k=0}^\infty \int_0^\infty x e^{-(k+1)x} \, dx \\
 &= \sum_{k=0}^\infty \frac{1}{(k+1)^2} \int_0^\infty u e^{-u} \, du && u = (k+1)x \\
 &= \frac{\pi^2}{6}. && \text{Basel Problem}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \int_0^{\infty} \frac{x}{e^x + 1} dx &= \int_0^{\infty} \frac{x e^{-x}}{1 + e^{-x}} dx \\
 &= \int_0^{\infty} x e^{-x} \sum_{k=0}^{\infty} (-1)^k e^{-kx} dx \\
 &= \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} x e^{-(k+1)x} dx \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} \int_0^{\infty} u e^{-u} dx \quad u = (k+1)x \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}.
 \end{aligned}$$

To resolve

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$$

we take

$$= \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) - \underbrace{\frac{1}{4} \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right)}_{\frac{\pi^2}{6}},$$

meaning

$$\int_0^{\infty} \frac{x}{e^x + 1} dx = \frac{\pi^2}{12}.$$

8.14

$$\begin{aligned}
 I_0(a) &= \int_0^{\infty} x^0 e^{-ax^2} dx \\
 &= \frac{1}{2} \sqrt{\frac{\pi}{a}}
 \end{aligned}$$

$$\begin{aligned}
 I_1(a) &= \int_0^{\infty} x e^{-ax^2} dx \\
 &= \frac{1}{2} \left(\frac{1}{-a} e^{-ax^2} \Big|_0^{\infty} \right) \\
 &= \frac{1}{2a}
 \end{aligned}$$

$$\begin{aligned}
 I_2(a) &= \int_0^{\infty} x^2 e^{-ax^2} dx \\
 &= -\frac{1}{2a} x e^{-ax^2} \Big|_0^{\infty} + \frac{1}{2a} \int_0^{\infty} x e^{-ax^2} dx
 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2a} I_1 \\ &= \frac{1}{4a^2} \end{aligned}$$

$$\begin{aligned} I_3(a) &= \int_0^\infty x^3 e^{-ax^2} dx \\ &= -\frac{1}{2a} x^2 e^{-ax^2} \Big|_0^\infty + \frac{1}{a} \int_0^\infty x e^{-ax^2} dx \\ &= \frac{1}{a} I_1 \\ &= \frac{1}{2a^2} \end{aligned}$$

$$\begin{aligned} I_4(a) &= \int_0^\infty x^4 e^{-ax^2} dx \\ &= -\frac{1}{2a} x^3 e^{-ax^2} \Big|_0^\infty + \frac{3}{2a} \int_0^\infty x^2 e^{-ax^2} dx \\ &= \frac{3}{2a} I_2 \\ &= \frac{3}{8a^3}. \end{aligned}$$