Introduction

Introduction to Game Theory

Game Theory analyzes the *interaction* among a *group* of *rational* agents who *behave strategically*.

- A group consists of at least two individuals who are free to make decisions.
- An interaction means that the decisions of at least one member of the group must affect at least one other member of the group.
- In strategic behavior, members of the group account for the interaction in their decision making process.
- Rational agents act in their best decisions based on their knowledge.

Keynes's Beauty Contest: Choose the face that is the most chosen in a newspaper contest.

In many games, we are not asked to pick *our* favorite, we are asked to pick *everyone else's* favorite.

Applications of Game Theory

- Labor Economics (compensation interactions, promotions)
- Industrial Organization (pricing, entry, exit, etc.)
- Public Finance (public goods games)
- Political Economy (strategic voting)
- Trade (tariff wars)
- Biology (hunting and mating)
- Linguistics

It's important to remember that game theory is a subfield of *mathematics*, not economics.

Simultaneous Games of Complete Information

Static Games of Complete Information

We will begin by covering *static games of complete information*.

- Static: Play happens at once and payoffs are realized. Decisions are not necessarily made at the same time.
- Complete information: the following four are all common knowledge in the game
  - (i) all possible actions of the players
  - (ii) all possible outcomes
  - (iii) how each combination of actions of all players affects which outcome will materialize
  - (iv) the preferences of each and every player over outcomes
- An event, E, is common knowledge if everyone knows E, everyone knows everyone knows E, *ad infinitum*.

The Prisoner's Dilemma

- Two suspects are interrogated in separate rooms.
- There is enough evidence to convict each of them for a minor offense, but not enough to convict either of a major crime unless one finks (F).
- If they each stay quiet (Q), they only get 1 year in prison each.
- If only one finks, they are free, and the other gets 4 years in prison.
- If they both fink, they each will spend 3 years in prison.

We will try to write The Prisoner's Dilemma as a game. First, we can see this in a payoff matrix.

		Player Y: Q	Player Y: F
Player X	Q	$(2,2)$	$(0,3)$
	F	$(3,0)$	$(1,1)$

Normal-Form Game

The constituents of a *normal-form game* G consist of the following:

- A finite set of players: $N = \{1, 2, \dots, n\}$.
- For each player i, a set S_i denotes the *strategy space* of player i. We will let $S = S_1 \times S_2 \times \cdots S_n$ denote the strategy space of the entire game (i.e., the entire set of strategies possible).
  - Every element $s \in S$ is a *strategy profile*, where $s = (s_1, s_2, \dots, s_n)$.
  - We denote the strategy choices of all players except player i as $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.
- A payoff function: $v_i : S \to \mathbb{R}$. The payoff function depends on the strategies of *all players*.

Example

Let the following payoff matrix represent a game. Write the normal form.

	X	Y
A	$(5,1)$	$(2,6)$
B	$(0,9)$	$(3,2)$
C	$(4,4)$	$(4,7)$

- $n = 2$
- $S_1 = \{A, B, C\}$
  $S_2 = \{X, Y\}$

Strategic Dominance

Recall the prisoner's dilemma.

		Player Y	
		Q	F
Player X	Q	$(2,2)$	$(0,3)$
	F	$(3,0)$	$(1,1)$

Suppose you were player 1. If player 2 stays quiet, it is more optimal for you to fink than to stay quiet. Similarly, if player 2 finks, then it is more optimal for you to fink than to stay quiet.

In a similar vein, for player 2, it is more optimal to fink in both cases. Therefore, the proper strategy is (F, F).

Dominated Strategy

A strategy s_i' if *strictly dominated* for i if there is one other strategy $s_i \in S_i$ such that $v_i(s_i, s_{-i}) > v_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$.

Essentially, a strategy is strictly dominated if there is another strategy that yields a strictly greater payoff regardless of the other strategies.

A rational player will *never* play a strictly dominated strategy.

In the prisoner's dilemma, Q is strictly dominated by F in both cases. Oddly, this yields the worst outcome from a social perspective (i.e., it has the lowest aggregate welfare).

	L	M	R
T	2,2	1,1	4,0
B	1,2	4,1	3,5

Through *iterated elimination of strictly dominated strategies* (IESDS), we start by removing M from the strategy profile of player 2 as playing L is strictly better. Then, Player 1 realizes that player 2 is rational, and thus does not play B (as B is strictly dominated by T once M is removed from the strategy space of player 2). Finally, Player 2 does not play R, as R is strictly dominated by L given that player 1 will play T. Thus, we get our answer of **T**, **L**.

A game is *dominance solvable* if it can be solved via iterated elimination of strictly dominated strategies. However, only a small number of games are not dominance solvable.

Strategic Dominance and Normal-Form Activity

Activity: Strategic Games and Dominance
Econ 305

Brandon Lehr

1 Strategic Games

For each of the games described below, determine the normal form of the game: number of players n, strategy space for each player S_i, and payoffs (as a matrix or function).

a. Matching Pennies (a zero-sum game). Two players simultaneously place a penny on a table. If the pennies match (e.g., both placed heads up), player 2 pays player 1 a dollar. If the pennies do not match, player 1 pays player 2 a dollar.

Players: $N = \{1, 2\}$

Strategy Space: $S_i = \{H, T\}$

Payoff Functions: $v_1 = \begin{cases} 1, & s_1 = s_2 \\ -1, & s_1 \neq s_2 \end{cases}$

$v_2 = \begin{cases} 1, & s_1 \neq s_2 \\ -1, & s_1 = s_2 \end{cases}$

		Player 2 H	Player 2 T
Player 1	H	1,-1	-1,1
	T	-1,1	1,-1

b. Bach or Stravinsky / Battle of the Sexes (a coordination game with some conflict). A couple wants to be together on their date night rather than alone, but they have different preferences over which type of concert they attend. They simultaneously — and without communication — choose to go to either the Bach or Stravinsky concert. Conditional on being together, player 1 prefers Bach and player 2 prefers Stravinsky.

Players: $N = \{1, 2\}$

Strategy Sets: $S_i = \{B, S\}$

Payoffs:

		Player 2 B	Player 2 S
Player 1	B	5,3	2,2
	S	0,0	3,5

1

c. Hawk vs. Dove / Chicken (an anti-coordination game). Two teenagers ride their bikes at high speed towards each other along a narrow ride. Neither of them wants to "chicken out" and lose their pride, but even worse is getting hurt by crashing into the oncoming biker.

Players: $n = 2$

Strategy Space: $\{D, H\}$

Payoffs:

		Player 2 D	Player 2 H
Player 1	D	1,1	1,2
	H	2,1	0,0

d. Cournot Competition (an industrial organization game). Two firms compete by simultaneously choosing how much to produce of a homogenous good (e.g., oil, soybeans) for a market.

Players: $n = 2$

Strategy Space: $S_i = [0, \infty)$

$P = d^{-1}(q_1 + q_2)$

Payoff: $\pi_i = d^{-1}(q_1 + q_2)q_i - c_i(q_i)$

2

2 Strict Dominance

Are the following games dominance solvable? Justify your answers.

a. A 4×4 game:

Yes, (B, X) is the result from IESDS on this game

	~~W~~	X	~~Y~~	~~Z~~
~~A~~	5,2	2,6	1,4	0,4
B	0,0	3,2	2,1	1,1
C	7,0	2,2	1,5	5,1
~~D~~	9,5	1,3	0,2	4,8

b. The beauty contest game, i.e., to win, come closest to guessing two-thirds the average of numbers between 0 and 100 selected by players.

Yes, every strategy is strictly dominated by 0 the game is dominance solvable

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Nash Equilibrium: Definition

A strategy profile s^* is a *pure strategy Nash equilibrium* if and only if the following holds.

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i, s_{-i}^*)$$

for all players i and all strategies $s_i \in S_i$.

Given what all other players are doing, no single player has an incentive to deviate to another action. This does not inform us about how to get to the Nash equilibrium, it just tells us that it is one.

For example, the prisoner's dilemma has a pure strategy Nash equilibrium: (F, F)

- For any other strategy profile, there is a profitable deviation.
- Similarly, result corresponds to the outcome of IESDS.

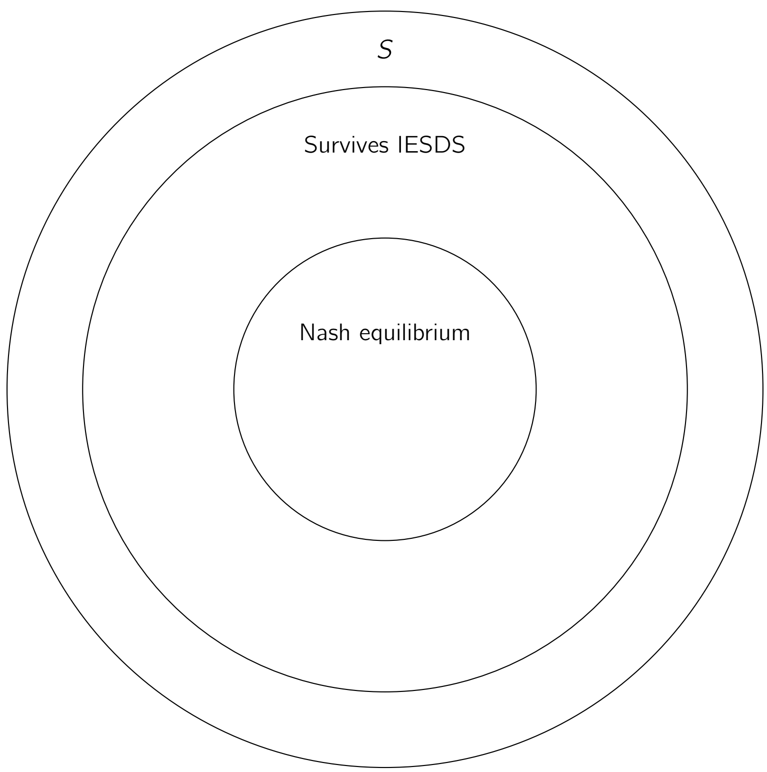
	L	M	R
T	2,2	1,1	4,0
B	1,2	4,1	3,5

In the above game, the pure strategy Nash equilibrium is (T, L). We can easily check that it is a Nash equilibrium, but in order to check that it is unique, we would need to look for deviations from the other strategy profiles. Or do we?

IESDS and Nash Equilibrium

- If s^* is a pure strategy Nash equilibrium of G, s^* survives IESDS.
- An action that is played in a Nash equilibrium is never eliminated in IESDS.
- If G is dominance solvable, then G has a unique Nash equilibrium:
  - The previous proposition tells us that G is dominance solvable $\Rightarrow$ there is at most one Nash equilibrium.

The relationship between the strategy set, S, the set of strategies that survive IESDS, and the Nash equilibrium can be seen below:



Voter Participation and Nash Equilibrium Activity

Activity: Voter Participation

Econ 305

Brandon Lehr

Two candidates, Joe and Donald, compete in a national Presidential election. Of the 200 million registered voters in the U.S., 100 million support Joe and 100 million support Donald. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives a payoff of 2 if the candidate she supports wins, 1 if this candidate ties, and 0 if this candidate loses. A citizen who votes receives the payoffs $2-c$, $1-c$, and $-c$ in these three cases, where $0 < c < 1$. Find the (pure strategy) Nash equilibria.

We can do this by considering different types of strategy profiles. For each case, we need to check whether or not any single citizen has an incentive to deviate to another strategy, given the strategies of all other citizens. If not, we have a Nash equilibrium.

Case 1: All Citizens Vote

Follow: $1-c$

Deviate (abstain): 0

all vote

Case 2: Not All Citizens Vote; the Candidates Tie

Vote: $2-c$

Abstain: 1

WV → vote
WA → vote
LV → vote
LA → vote

Case 3: A Candidate Wins by One Vote

Loser	Winner
vote: $1-c$	vote: $2-c$
abstain: 0	abstain: 2

WV → abstain
WA → abstain
LV → vote
LA → vote

Case 4: A Candidate Wins by at Least Two Votes ← stable solution

Loser Vote	Winner
vote: $-c$	vote: $2-c$
no vote: 0	no vote: 2

WV → abstain
WA → abstain
LV → abstain
LA → abstain

1

Bonus: Suppose that Joe has more supporters than Donald. What are the (pure strategy) Nash equilibria of this game?

Hint: Let n_J be the number of people who vote for Joe and n_D be the number of people who vote for Donald. Also, denote the number of Donald's supporters by $k < 100$ million. Proceed in cases as before.

There is no pure strategy nash equilibrium

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Bertrand Competition

Assumptions We have the following:

- **Players** Homogenous good produced by $n > 1$ firms (i.e., oil, soybeans)
- **Cost** The cost of producing q_i units to $c_i(q_i)$.
- **Demand** Total Market Demand is given by $D(p)$
- **Strategy Set** $S_i = \mathbb{R}^+$, where $p_i \in S_i$ denotes the price.

Normal-Form Game We have the following:

- **Players** $n = 2$
- **Cost Function** $c_i(q_i) = cq_i$ for some $c \in \mathbb{R}^+$ and for $i = 1, 2$
- **Payoffs** $v_i(p_i, p_j) = \begin{cases} 0, & p_i > p_j \\ (p_i - c)(D(p_i)), & p_i < p_j \\ \frac{1}{2}(p_i - c)(D(p_i)), & p_i = p_j \end{cases}$

Bertrand Duopoly

Activity: Bertrand Duopoly
Econ 305

Brandon Lehr

We can write profits in the symmetric Bertrand duopoly game ($n = 2$) with marginal costs, c, and demand function, $D(p)$, as follows:

$$v_i(p_i, p_j) = \begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_j \\ (p_i - c)D(p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

where $p_1, p_2 \in \mathbb{R}^+$.

Claim: The unique NE of the Bertrand game is $(p_1^*, p_2^*) = (c, c)$.

Proof: There are two parts we have to prove. First, we must show that the proposed action profile is in fact a NE. Next, we have to show that there is no other NE.

1. *Profile is NE*

   - charge $\uparrow \to v = 0 = NE$
   - charge $\downarrow \to v < 0 < NE$

   no better than current situation

2. *Uniqueness*

   Consider all other possible action profiles:

   (a) If $p_i < c$ for either $i = 1$ or $i = 2$:

   $\exists i$ s.t. $v(p_i, p_j) < 0 \to$ profitable to charge marginal cost for $v(p_i, p_j) = 0$

   (b) If $p_i = c$ and $p_j > c$:

   $p_i \uparrow$ by $\varepsilon < (p_j - c) \to v(p_i + \varepsilon, p_j) > 0$

   (c) If $p_i > c$ and $p_j > c$ (without loss of generality let $p_i \geq p_j$):

   $v(p_i, p_j) = \begin{cases} 0, & p_i > p_j \\ \frac{1}{2}(p_i - c)D(p_i), & p_i = p_j \end{cases}$ profitable to go just below p_j, payoff: $0 \to > 0$

Bonus: Suppose we modify the game so that firms can only charge discrete prices measured to the precision of a cent (as opposed to all non-negative real numbers). Argue that $(c+1, c+1)$ is also a Nash equilibrium (where c is given in cents). Assume that $D(c+1) > 0$.

hold: $v_i = \frac{1}{2} D(c+1)$

Down by 1: $v_i = 0$

Down by > 1: $v_i < 0$

Up: $v_i = 0$

all worse than staying at $c+1$

Best Response Correspondence and finding Nash Equilibria

- To find a Nash Equilibrium, it is helpful to determine which actions are best for a player given the actions of their opponents.
- A *best response correspondence* is:

$$BR_i(s_{-i}) = \{s_i \in S \mid v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \ \forall s_i' \in S_i\}$$

- Any strategy in $BR_i(s_{-i})$ is at least as good for player i as every other strategy available to player i when the other players' strategies are given by s_{-i}.
- We call it a correspondence, as oppposed to a function, since $BR_i(s_i)$ can be set-valued

A strategy profile s^* is a Nash Equilibrium of G if and only if every player's strategy is a best response to the other players' strategies:

$$s^* \in BR_i(s_i^*) \ \forall i$$

Essentially, every player is playing their best response given that everyone else is also playing their best response.

There are two primary ways to find a Pure Strategy Nash Equilibrium:

- Refining educated guesses (e.g., Bertrand competition), necessary when best responses are set-valued
- Intersection of best responses (e.g., Payoff Matrix, Cournot competition)

Calculus and Best Response

In many cases, $BR_i(s_{-i})$ is a solution to the following problem:

$$\max_{s_i} v_i(s_i, s_{-i})$$

Essentially, we have to solve an optimization problem: find the value of s_i such that $v_i(s_i, s_{-i})$ is maximized.

Consider the following payoff function:

$$v_1(x_1, x_2) = 3x_1x_2 - x_1^2$$

We must use partial derivatives (as we're holding the strategy of x_2 constant in order to find the best response):

$$\begin{aligned} 0 &= \frac{\partial v_1}{\partial x_1} \\ &= 3x_2 - 2x_1 \\ x_1 &= \frac{3x_2}{2} \end{aligned}$$

Therefore, $BR_1(x_2) = x_1 = \frac{3x_2}{2}$.

In order to find the Nash Equilibrium, we have to find the best response function for every player and take the intersections of all the best response functions.

Cournot Competition

- n firms.
- Selling a homogenous identical good.
- *Quantity* Competition: $S_i = \mathbb{R}^+$
- Payoff function:

$$v_i(q_i, q_{-i}) = q_i P\left(\sum_{j=1}^{n} q_j\right) - c(q_i)$$

  What makes Cournot competition a game is that the payoff depends on the strategies of other players.

Cournot Duopoly: Example

- $n = 2$, $c(q_i) = 10q_i\ \forall i$
- $P(Q) = \begin{cases} 100 - Q, & Q \leq 100 \\ 0, & Q > 100 \end{cases}$

Three steps to find Nash Equilibrium:

(i) $BR_1(q_2)$

(ii) $BR_2(q_1)$

(iii) (q_1^*, q_2^*) is a Nash Equilibrium where $q_1^* = BR_1(BR_2(q_1^*))$ (this follows from the definition of a Nash Equilibrium).

$$\begin{aligned}
\max_{q_1} q_1(100 - q_1 - q_2) - 10q_1 &= \max_{q_1} q_1(90 - q_1 - q_2) \\
0 &= \frac{\partial v_1}{\partial q_1} \\
&= 90 - q_2 - 2q_1 \\
q_1 &= 45 - \frac{1}{2}q_2
\end{aligned}$$

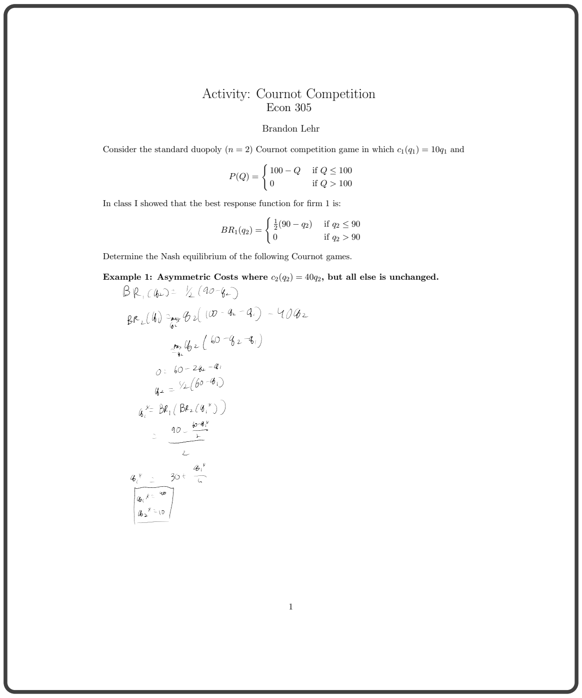
$$\begin{aligned}
\max_{q_2} q_2(100 - q_2 - q_1) - 10q_2 &= \max_{q_2} q_2(90 - q_1 - q_2) \\
0 &= \frac{\partial v_2}{q_2} \\
&= 90 - q_1 - 2q_2 \\
q_2 &= \frac{90 - q_1}{2}
\end{aligned}$$

Note In a symmetric game, every player's best response is identical with respect to every other player. You can only use this shortcut *after* finding the best response.

$$\begin{aligned}
q_1^* &= BR_1(BR_2(q_1^*)) \\
&= \frac{90 - \frac{90 - q_1^*}{2}}{2} \\
&= \frac{45}{2} + \frac{q_1^*}{4} \\
q_1^* &= 30 \\
q_2^* &= BR(30) \\
&= 30
\end{aligned}$$

Note In a symmetric game, $q_1^* = q_2^*$, and similar for the n player case.

Cournot Duopoly Variations



Activity: Cournot Competition
Econ 305

Brandon Lehr

Consider the standard duopoly ($n = 2$) Cournot competition game in which $c_1(q_1) = 10q_1$ and

$$P(Q) = \begin{cases} 100 - Q & \text{if } Q \leq 100 \\ 0 & \text{if } Q > 100 \end{cases}$$

In class I showed that the best response function for firm 1 is:

$$BR_1(q_2) = \begin{cases} \frac{1}{2}(90 - q_2) & \text{if } q_2 \leq 90 \\ 0 & \text{if } q_2 > 90 \end{cases}$$

Determine the Nash equilibrium of the following Cournot games.

Example 1: Asymmetric Costs where $c_2(q_2) = 40q_2$, but all else is unchanged.

$BR_1(q_2) = \frac{1}{2}(90 - q_2)$

$BR_2(q_1) = \max_{q_2} q_2(100 - q_2 - q_1) - 40q_2$

$= \max_{q_2} q_2(60 - q_2 - q_1)$

$0 = 60 - 2q_2 - q_1$

$q_2 = \frac{1}{2}(60 - q_1)$

$q_1^* = BR_1(BR_2(q_1^*))$

$= \dfrac{90 - \frac{60 - q_1^*}{2}}{2}$

$q_1^* = 30 + \frac{q_1^*}{4}$

$\boxed{\begin{array}{l} q_1^* = 40 \\ q_2^* = 10 \end{array}}$

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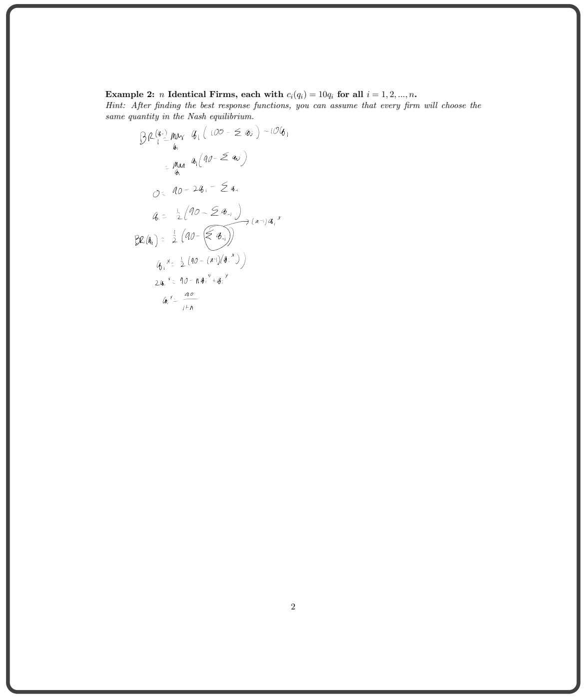


Example 2: n Identical Firms, each with $c_i(q_i) = 10q_i$ for all $i = 1, 2, ..., n$.
Hint: After finding the best response functions, you can assume that every firm will choose the same quantity in the Nash equilibrium.

$BR_1(q_{-i}) = \max_{q_1} q_1(100 - \sum q_i) - 10q_1$

$= \max_{q_1} q_1(90 - \sum q_i)$

$0 = 90 - 2q_1 - \sum q_{-i}$

$q_i = \frac{1}{2}(90 - \sum q_{-i})$

$BR_i(q_i) = \frac{1}{2}(90 - \sum q_{-i})$ $\to (n-1)q_i^*$

$q_i^* = \frac{1}{2}(90 - (n-1)(q_i^*))$

$2q_i^* = 90 - nq_i^* + q_i^*$

$q_i^* = \frac{90}{1+n}$

2

Mixed Strategies

		IRS	
		Audit (A)	Don't (D)
Taxpayer	Honest (H)	$10, 9$	$10, 10$
	Cheat (C)	$-35, 10$	$15, 5$

In a game like Rochambeau or the Tax Audit game (above), there is no pure strategy Nash equilibrium (i.e., there is always some incentive to deviate regardless of the outcome). Many other games, especially in the realms of sports and the military, feature a lack of pure strategy Nash equilibria.

Players must randomize over the possible strategy spaces to maximize their payoff, rather than depending on a pure strategy.

- A *mixed strategy*, σ_i, of player i is a probability distribution over S_i.
- There are two primary properties of a probability distribution:
  - $\sigma_i(s_i) \geq 0$ for all $s_i \in S_i$ (something cannot have negative probability).
  - $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$ (everything must happen).
- There are multiple ways to write the mixed strategy:
  - Functions: $\sigma_1(H) = 1/3$, $\sigma_1(T) = 2/3$
  - Vector: $\sigma_1 = \langle 1/3, 2/3 \rangle$
  - Type 2 Vector: $\sigma_1 = \frac{1}{3}H + \frac{2}{3}T$

Mixed Strategy Profiles and Expected Payoffs

A mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a set of mixed strategies in the game.

The expected payoff to player i with the mixed strategy profile (σ_i, σ_{-i}) is as follows:

$$v_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})v_i(s_i, s_{-i})$$

Note: This is equivalent to the weighted average of the payoffs of the pure strategy profiles.

For example, in the tax game, let $\sigma_1 = \frac{3}{4}H + \frac{1}{4}C$ and $\sigma_2 = \frac{1}{3}A + \frac{2}{3}D$. Then, we have the total payoff as follows:

$$\begin{aligned} v_1(\sigma_1, \sigma_2) &= \frac{3}{4}\frac{1}{3}(10) + \frac{3}{4}\frac{2}{3}(10) + \frac{1}{4}\frac{1}{3}(-35) + \frac{1}{4}\frac{2}{3}(15) \\ &= \frac{85}{12} \end{aligned}$$

Therefore, $\frac{85}{12}$ is the *expected payoff*.

Mixed Strategy Nash Equilibrium

Let σ^* be a mixed strategy profile. Then, if

$$v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(\sigma_i, \sigma_{-i}^*)$$

for all players i and all mixed strategies σ_i, σ_i^* is the *mixed strategy Nash equilibrium*.

Instead of checking all mixed strategies, we only need check if switching to any *pure* strategy makes a player better off. If it is not, the mixed strategy profile is a Nash equilibrium.

$$v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(s_i, \sigma_{-i}^*)\ \forall s_i \in S_i$$

Many mixed strategy Nash equilibria are not strict Nash equilibria (i.e., there are other strategies with similar payoffs).

Finding Mixed Strategy Nash Equilibria

In a finite game, the *support* of a mixed strategy σ_i, $\text{supp}(\sigma_i)$ is the set of strategies to which σ_i has strictly positive probability:

$$\text{supp}(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$$

If σ^* is a mixed strategy Nash equilibrium, and $s_i', s_i'' \in \text{supp}(\sigma_i)$, then

$$v_i(s_i', \sigma_i) = v_i(s_i'', \sigma_i)$$

Essentially, if a player is playing two strategies with positive probability, they have to be indifferent between the two strategies (or else they would play the one with a higher payoff).

Mixed Strategy Nash Equilibrium Example

IRS

		Audit (A)	Don't (D)
Taxpayer	Honest (H)	$10, 9$	$10, 10$
	Cheat (C)	$-35, 10$	$15, 5$

$$\sigma_1 = pH + (1-p)C$$
$$\sigma_1 = qA + (1-q)D$$

Player 1's Indifference:

$$\begin{aligned} v_1(H, \sigma_2) &= 10 \\ v_2(C, \sigma_2) &= -35q + 15(1-q) \\ 10 &= 15 - 50q && \text{when taxpayer will be indifferent} \\ q^* &= \frac{1}{10} \end{aligned}$$

Player 2's Indifference:

$$\begin{aligned} v_2(\sigma_1, A) &= 9p + 10(1-p) \\ v_2(\sigma_1, D) &= 10p + 5(1-p) \\ 10 - p &= 5 + 5p \\ p^* &= \frac{5}{6} \end{aligned}$$

The Nash equilibrium is where $p^* = 5/6$ and $q^* = 1/10$.

Mixed Strategy Nash Equilibrium: D-Day

Activity: D-Day

Econ 305

Brandon Lehr

Find *all* of the Nash equilibria of the following game:

	Defend Normandy (DN)	Defend Calais (DC)
Invade Normandy (IN)	-2,2	1,-1
Invade Calais (IC)	3,-3	-1,1

What is the probability of invading an undefended site (outcomes of (IN, DC) or (IC, DN))?
Let $p = \sigma_1(IN)$ and $q = \sigma_2(DN)$.

$\sigma_1 = p(IN) + (1-p)(IC)$

$\sigma_2 = q(DN) + (1-q)(DC)$

$u_1(IN, \sigma_2) = -2(q) + 1(1-q)$

$u_1(IC, \sigma_2) = 3(q) + (-1)(1-q)$

$-2q + 1 - q = 3q - 1 + q$

$2 = 7q$

$q^* = 2/7$

$u_2(\sigma_1, DN) = 2p + (-3)(1-p)$

$u_2(\sigma_1, DC) = -p + 1(1-p)$

$5p - 3 = 1 - 2p$

$p^* = 4/7$

$\boxed{(4/7, 2/7)}$

Finding Mixed Strategy Nash Equilibria in Larger Games

In the 2×2 games, we did not have to worry about which strategies were in the support of the strategies played in Nash equilibrium, but we do in higher dimensional games.

	L	C	R
U	$10, 9$	$10, 6$	$10, 10$
M	$-5, 9$	$15, 10$	$11, 12$
D	$-35, 10$	$10, 7$	$15, 5$

Reducing Size of Larger Games

A strategy s_i' is *strictly dominated* if there exists a mixed strategy σ_i such that

$$v_i(\sigma_i, s_{-i}) > v_i(s_i', s_{-i})$$

for every $s_{-i} \in S_{-i}$.

Essentially, if we can find a mixed strategy profile that is strictly better than the pure strategy, the pure strategy is strictly dominated.

In the previous game, we can use IESDS as follows:

- No strategies are strictly dominated by pure strategies.
- However, C is strictly dominated by $\frac{1}{2}L + \frac{1}{2}R$

	L	C	R
U	$10, 9$	$\cancel{10, 6}$	$10, 10$
M	$-5, 9$	$\cancel{15, 10}$	$11, 12$
D	$-35, 10$	$\cancel{10, 7}$	$15, 5$

- Additionally, M is strictly dominated by $\frac{3}{4}U + \frac{1}{4}D$

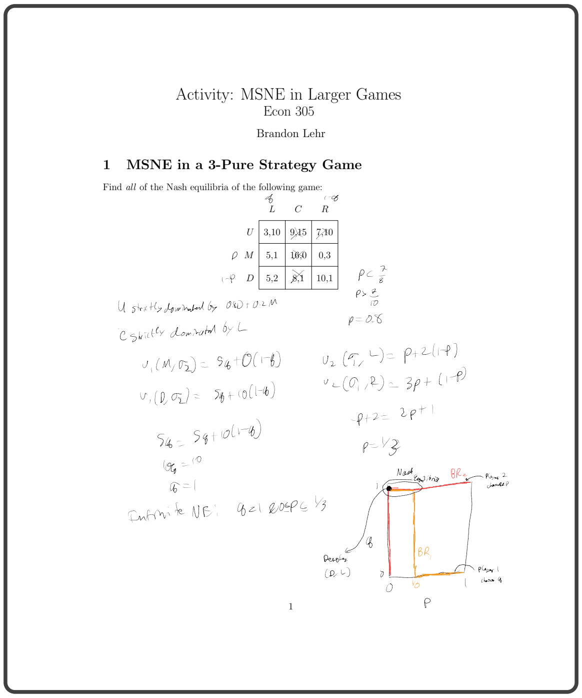
	L	C	R
U	$10, 9$	$\cancel{10, 6}$	$10, 10$
M	$\cancel{-5, 9}$	$\cancel{15, 10}$	$\cancel{11, 12}$
D	$-35, 10$	$\cancel{10, 7}$	$15, 5$

Since, from here, we cannot find any strictly dominated strategies, we have to use our method for finding mixed strategy Nash equilibria.

By using the method, we find that the solutions are $p = 5/6$ on U and $q = 1/10$ on L.

Note: After IESDS, the game is the same as the Tax Audit Game from earlier.

Mixed Strategy Nash Equilibria in Larger Game Example



Activity: MSNE in Larger Games
Econ 305

Brandon Lehr

1 MSNE in a 3-Pure Strategy Game

Find *all* of the Nash equilibria of the following game:

		q L	C	$1-q$ R
	U	3,10	~~9,15~~	~~7,10~~
p	M	5,1	~~10,0~~	0,3
$1-p$	D	5,2	~~8,1~~	10,1

U strictly dominated by $0.8D + 0.2M$

C strictly dominated by L

$p < \frac{7}{8}$, $p > \frac{7}{10}$, $p = 0.8$

$u_1(M, \sigma_2) = 5q + 0(1-q)$

$u_1(D, \sigma_2) = 5q + 10(1-q)$

$5q = 5q + 10(1-q)$

$10q = 10$

$q = 1$

$u_2(\sigma_1, L) = p + 2(1-p)$

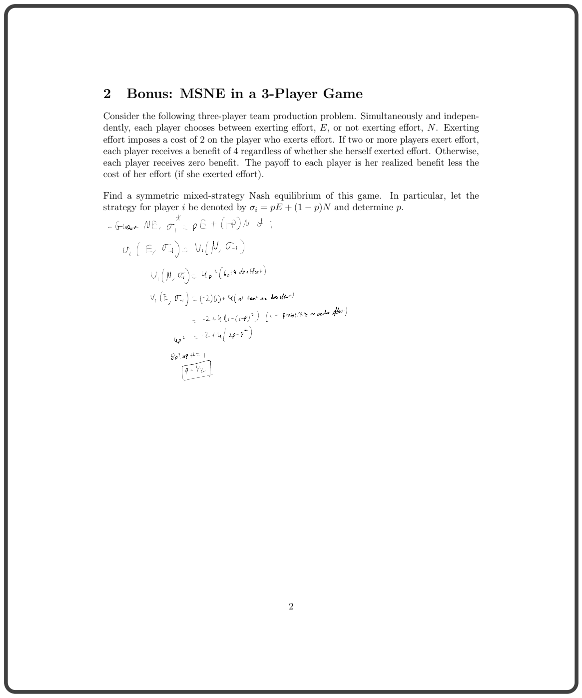
$u_2(\sigma_1, R) = 3p + (1-p)$

$-p + 2 = 2p + 1$

$p = 1/3$

Infinite NE: $q = 1$ & $0 \le p \le 1/3$

1



2 Bonus: MSNE in a 3-Player Game

Consider the following three-player team production problem. Simultaneously and independently, each player chooses between exerting effort, E, or not exerting effort, N. Exerting effort imposes a cost of 2 on the player who exerts effort. If two or more players exert effort, each player receives a benefit of 4 regardless of whether she herself exerted effort. Otherwise, each player receives zero benefit. The payoff to each player is her realized benefit less the cost of her effort (if she exerted effort).

Find a symmetric mixed-strategy Nash equilibrium of this game. In particular, let the strategy for player i be denoted by $\sigma_i = pE + (1-p)N$ and determine p.

- Guess NE, $\sigma_i^* = pE + (1-p)N \ \forall i$

$u_i(E, \sigma_{-i}) = u_i(N, \sigma_{-i})$

$u_i(N, \sigma_{-i}) = 4p^2$ (both do effort)

$u_i(E, \sigma_{-i}) = (-2)(1) + 4$(at least one does effort)

$= -2 + 4(1 - (1-p)^2)$ (1 − probability no one does effort)

$4p^2 = -2 + 4(2p - p^2)$

$\boxed{p = 1/2}$

2

Multiple Equilibria

Meet in New York: you and your friend have agreed to meet in New York at 12pm. Both of your phones have died. Where do you go?

There are multiple Nash equilibria to this game (essentially, basically anywhere).

Interpreting Nash Equilibria

IESDS is a constructive process to *predict* play in a game; it does not assume that players know the strategy choices of other players.

In a Nash equilibrium, players have self-fulfilling beliefs about the play of others. Essentially, Nash equilibria survive IESDS, and then players respond to *correct* beliefs.

We might ask where the beliefs come from:

Play Prescription: An outside party tells people how to play.

Pre-play Communication: Players communicate and agree on how to play the game.

Rational Introspection: A Nash equilibrium seems reasonable because my beliefs of what other players do are consistent with them being rational. This works in games with a unique Nash equilibrium.

Focal Point: Social norms lead players to prefer certain actions over others.

Learning: Players play many times, and learn what other people prioritize by doing so.

Evolution: Players are programmed to play a certain action; if they do not play a Nash equilibrium, they "mutate." If this deviation is profitable, the players will "multiply" at a faster rate than other players and eventually take over. This system converges to a state where all players choose the Nash equilibrium.

Existence of Nash Equilibria

The set of strategies played in Nash equilibrium is a subset of the strategies that survive IESDS. The Nash equilibrium isn't so strong such that there is no Nash equilibrium.

Nash Theorem: Every finite normal-form game has a mixed strategy Nash equilibrium.

Note: Pure strategy profiles are mixed strategy profiles with a degenerate probability distribution.

Intuition

Lemma: A mixed strategy profile σ^* is a Nash equilibrium if and only if it is a fixed point of the best response correspondence:

$$\sigma^* \in BR(\sigma^*)$$

where

$$BR(\sigma) = BR(\sigma_{-1}) \times BR(\sigma_{-2}) \times \cdots \times BR(\sigma_{-n})$$

If the mixed strategy profile is that which is the best response for every player in the mixed strategy profile, then that has to be a Nash equilibrium.

In one dimension, we know that every continuous function $f : [a, b] \to [a, b]$ must have a fixed point (by the Intermediate Value Theorem).

In $\mathbb{R}^n$, a continuous function from a compact (i.e., bounded and closed), convex (i.e., every two points in the set can be connected by a line) subset to itself must also have a fixed point (by Brouwer's Fixed Point Theorem).

Then, Shizuo Kakutani proved that Brouwer's result applied to particular types of correspondences.

A correspondence $r : X \to X$ has a fixed point such that $x \in r(x)$ if:

- X is a compact, convex, non-empty subset of $\mathbb{R}^n$
- $r(x)$ is nonempty for all x.
- $r(x)$ is convex for all x.
- r has a closed graph.

Nash showed that the best response correspondence satisfied these conditions.

Generic Properties

A property is *generic* if, given the number of players and strategies, the property is true with probability 1 for payoffs chosen at random.

For example, the property "In a Nash equilibrium players 1 and 2 obtain different payoffs" is generic (i.e., if you throw in random payoffs, players will almost certainly have different payoffs).

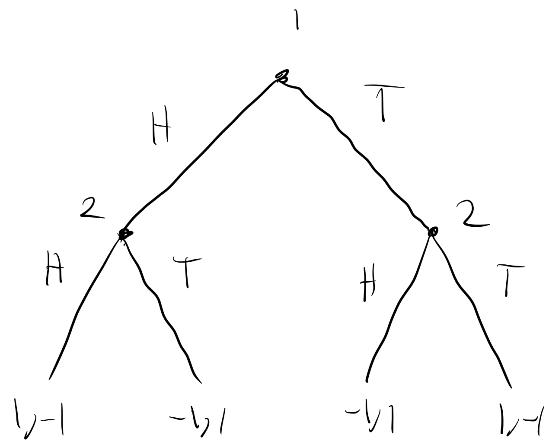
Additionally, the property "A finite strategic game has a finite and odd number of Nash equilibria." is generic.

- When looking for (mixed strategy) Nash equilibria, if you find an even number, you probably missed one.

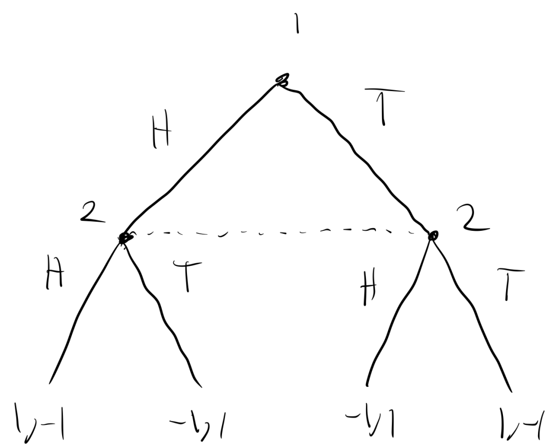
Sequential Games of Complete Information

Extensive Form Games

Consider a variant of the Matching Pennies in which Player 2 observes the choice made by Player 1 before making their choice.



We can model the simultaneous move version by connecting the Player 2 nodes with a dashed line.



Definition A *finite horizon extensive form game* consists of

(1) The set of players
(2) A set of *decision nodes* where at each node it is specified:
    (i) the player who moves
    (ii) the set of possible actions
    (iii) the successor node resulting from each action
(3) Payoffs upon reaching a *terminal node*
(4) What each player knows at each of the decision nodes, described by a collection of *information sets* that partition the set of decision nodes.

Definition A *information set* is a collection of decision nodes such that at every node in the set:

(i) The same player has the move
(ii) The same set of possible actions is available.

When the play of the game reaches a node in the information set, the player with the move does not know which node in the information set has or has not been reached.

A player can have multiple information sets, but each node must be in an information set.

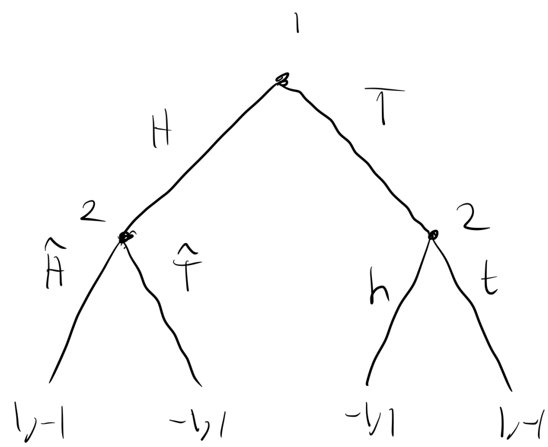
Perfect Information

A game of perfect information is a game in which all the information sets are singletons (but a player can have multiple information sets):

- In a game of perfect information, a player knows every move made by players before them.
- Sequential Matching Pennies is a game of perfect information, but Matching Pennies is a game of imperfect information.
- All normal form games can be thought of as extensive games with imperfect information.

Pure Strategies in Extensive Form Games

A pure strategy for a player in an extensive form game is a complete set of contingent actions (even if they are never played or never able to be played) for each player at each information set. In the sequential Matching Pennies, we label each contingency:

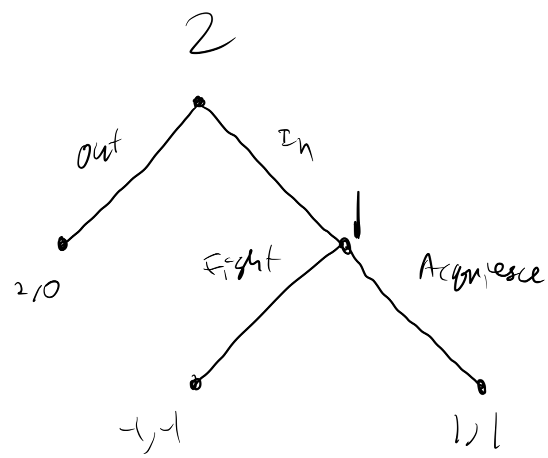


Then, we can create the normal form game as follows:

	$\hat{H}h$	$\hat{H}t$	$\hat{T}h$	$\hat{T}t$
H	$1,-1$	$1,-1$	$-1,1$	$-1,1$
T	$-1,1$	$1,-1$	$-1,1$	$1,-1$

where we denote Player 2's strategies by xy, where $x = s_2(H)$ and $y = s_2(T)$.

The Entry game (in Industrial Organization) has the following extensive form:



and the normal form game is as follows:

	Out	In
Fight	$2, 0$	$-1, -1$
Acquiesce	$2, 0$	$1, 1$

Notice that Player 1 still has to include the payoffs for Fight and Acquiesce *even if Player 2 plays Out.*

Behavioral Strategies

A behavioral strategy for player i is a function that assigns a probability distribution over the actions available to player i at each information set where player i moves.

This is different from a mixed strategy since a mixed strategy is a mixture over all the pure strategies, rather than a mixture at each action.

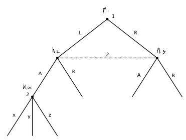
Given perfect recall, the mixed strategy profile can be created by multiplying the behavioral strategies together.

Extensive Form Games Example

Activity: Extensive-Form Games
Econ 305

Brandon Lehr

1 Characterizing an Extensive-Form Game



a. Is this a game of perfect or imperfect information? Why?
No, P2 has an information set of {n2, n3}

b. How many information sets does each player have in the above game?
P1: one information set: {n1}
P2: 2 information sets: {n2, n3}, {n4}

c. How many pure strategies does each player have in the above game?
P1: 2
P2: 6

1

2 Tic-Tac-Toe (Tadelis 7.3)

The extensive form representation of a game can be cumbersome even for very simple games. Consider the game of Tic-tac-toe where 2 players mark "X" or "O" in a 3×3 matrix. Player 1 moves first, then player 2, and so on. If a player gets three of his kind in a row, column, or one of the diagonals then he wins, and otherwise it is a tie. For this question assume that even after a winner is declared, the players *must completely* fill the matrix before the game ends.

a. Is this a game of perfect or imperfect information? Why?
Perfect: every player knows their exact position in relation to every other player

b. How many information sets does player 2 have after player 1's first move?
9

c. How many information sets does player 1 have after player 2's first move?
72

2

Equilibrium in Extensive Form Games

- A Nash equilibrium is still a strategy profile in which no player has a unilateral profitable deviation.
- The Nash Equilibrium of an extensive form game are the Nash equilibria of the game's normal form representation.
- We can apply the methods from the first third of the course to find the Nash equilibrium.

In the entry game, we have the following normal form:

	Out	In
Fight	$2, 0$	$-1, -1$
Acquiesce	$2, 0$	$1, 1$

There are an infinite number of Nash equilibria:

- (F, O): the challenger will stay out if it thinks the incumbent will fight; however, the challenger can call Firm 1's bluff and enter the market.
- (A, I): if the challenger enters the market, Firm 1 will acquiesce.
- $(pF + (1-p)A, O)$ for $p \geq 1/2$: Firm 1 needs to mix between fight and acquiesce such that Firm 2 stays out.

Given the fact that (F, O) is not a credible threat, we need a better concept for equilibrium that rules out non-credible threats.

For another example of a non-credible threat, consider Stackelberg Competition:

- Firm 1 chooses $q_1 \in [0, 1]$

- Firm 2 sees q_1, then chooses $q_2 \in [0, 1]$
- Let $c_1(q_1) = c_2(q_2) = 0$, and $P(q_1, q_2) = 1 - q_1 - q_2$
- We claim that for any $q_1' = [0, 1]$, the game has a Nash equilibrium in which firm 1 produces q_1'.

$$s_2(q_1) = \begin{cases} \frac{1-q_1'}{2} & q_1 = q_1' \\ 1 - q_1 & q_1 \neq q_1' \end{cases}$$

Note: The Nash equilibrium must be one for the game's strategy profile.

  - If firm 1 plays q_1', then player 2 plays its best response (anything off the equilibrium path).
  - Given firm 2's strategy, firm 1 makes positive profit by playing q_1', or else firm 2 floods the market, pushing the price to zero and yielding zero profits.
  - This is a Nash equilibrium — however, the second part of this statement is non-credible.

Subgame Perfect Equilibrium

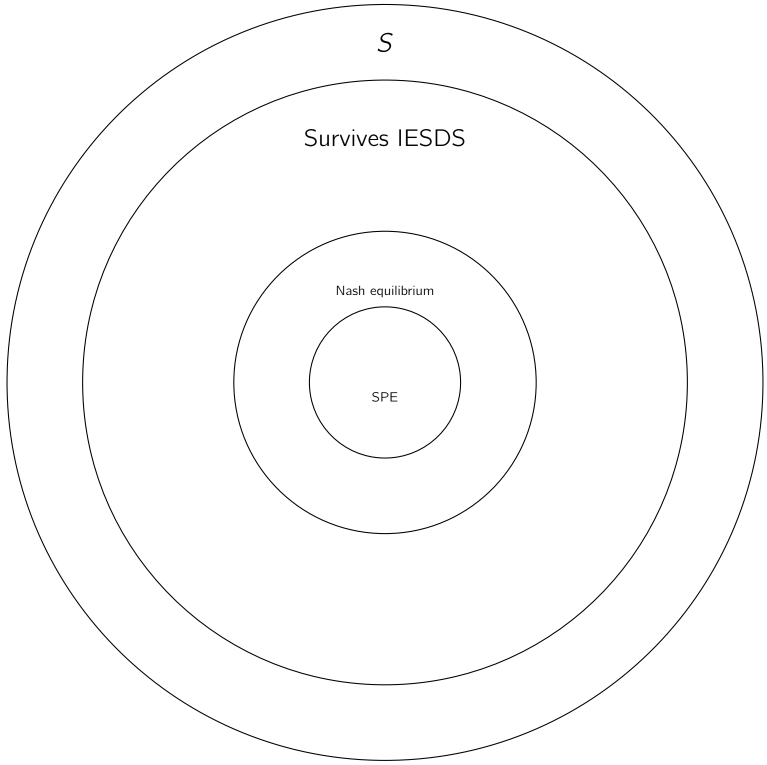
A **subgame** of an extensive form game has the following properties:

- Begins at a decision node n that is a singleton information set.
- Includes *all* the decision and terminal nodes following n in the game tree.
- Does not "cut through" any information sets (drawing a circle around the subgame doesn't cut a dashed line)

The game is always a subgame of itself.

A strategy profile s^* is a *subgame perfect equilibrium* if it is a Nash equilibrium of every subgame. It's important to remember that the payoffs in equilibrium are *not* a SPE; only the strategy profile.

- Every SPE is a Nash equilibrium because the game is a subgame of itself.
- SPE is a refinement of Nash equilibrium because it is a subset of the set of Nash equilibria.



In the Entry game, for example, there are an infinite number of Nash equilibria, but there is only one SPE: (A, I), as it is the Nash equilibrium of the subgame starting from the "in" node.

In Stackelberg Competition, we claim that the unique SPE is $s_1^* = \frac{1}{2}$ and $s_2^*(q_1) = \frac{1-q_1}{2}$.

Note: It would be wrong to write the proposed SPE as $(s_1^*, s_2^*) = (1/2, 1/4)$, even though this is the outcome along the equilibrium path. Recall that a strategy is a complete contingent plan — player 2's strategy has to be contingent on what player 1 does.

The subgames of Stackelberg competition are the game itself and the proper subgame that follows each possible choice of q_1. In each proper subgame, we have a 1-player game, so the Nash equilibrium is equivalent to maximizing profit.

$$\begin{aligned} s_2^*(q_1) &= \max_{q_2} q_2(1 - q_1 - q_2) \\ &= \frac{1 - q_1}{2} \end{aligned}$$

The Nash equilibrium of the entire game requires firm 1 to best respond to firm 2:

$$\begin{aligned} s_1^* &= \max_{q_1} q_1(1 - q_1 - s_2^*(q_1)) \\ &= 1/2 \end{aligned}$$

Note: Firm 1 produces more than firm 2 and has higher profits — this is emblematic of the first mover advantage. Firm 1 can find firm 2's best response function, then use that information to determine their own best response, as opposed to the Cournot game in which firm 1 and 2 have the same amount of information.

Backward Induction and Existence

The previous two examples show how to do *backward induction*, in which we can solve finite extensive form games by starting at the subgames furthest down the game tree and moving backwards, finding the Nash equilibria in each subgame.

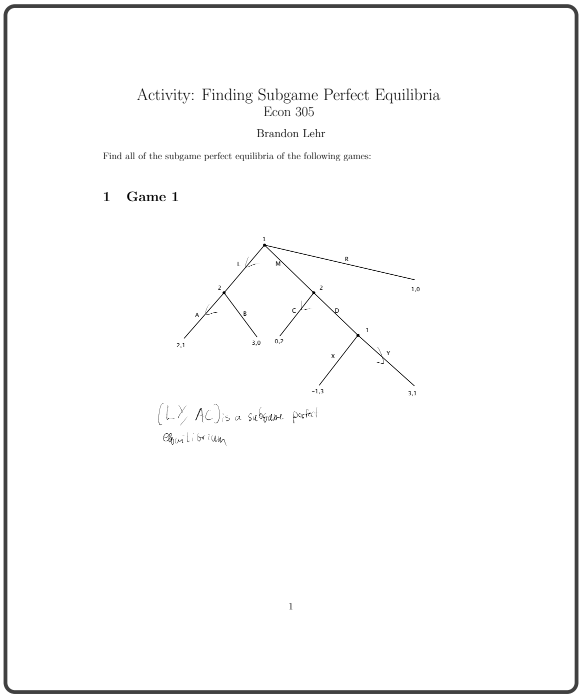
Every *finite* (finite number of terminal nodes) extensive form game of perfect information has a *pure strategy* SPE; generically, this SPE is unique.

- You solve the last rounds of the game, then replace these subgames with the outcome of the Nash equilibrium, and repeat the procedure.

Every finite extensive form game (not necessarily of perfect information) has a SPE.

- The refinement of Nash equilibrium is not too strong in the sense that a SPE always exists.
- Each subgame has a Nash equilibrium due to the Nash existence result.

Subgame Perfect Equilibria



Activity: Finding Subgame Perfect Equilibria
Econ 305

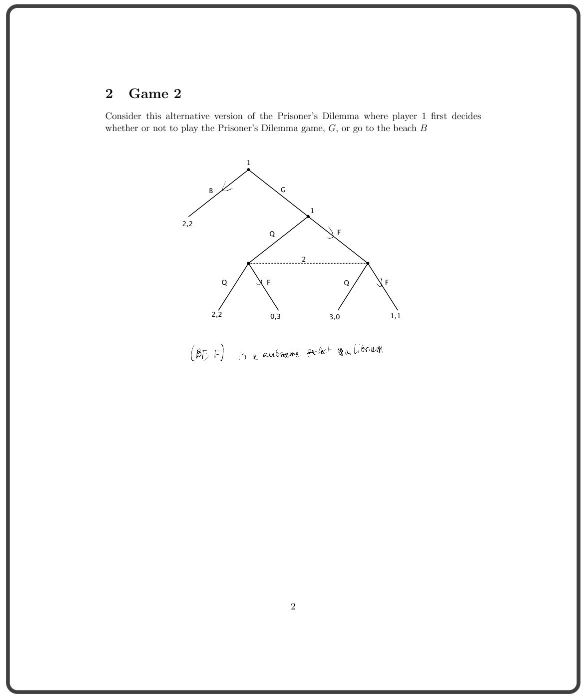
Brandon Lehr

Find all of the subgame perfect equilibria of the following games:

1 Game 1

(LY, AC) is a subgame perfect equilibrium

1

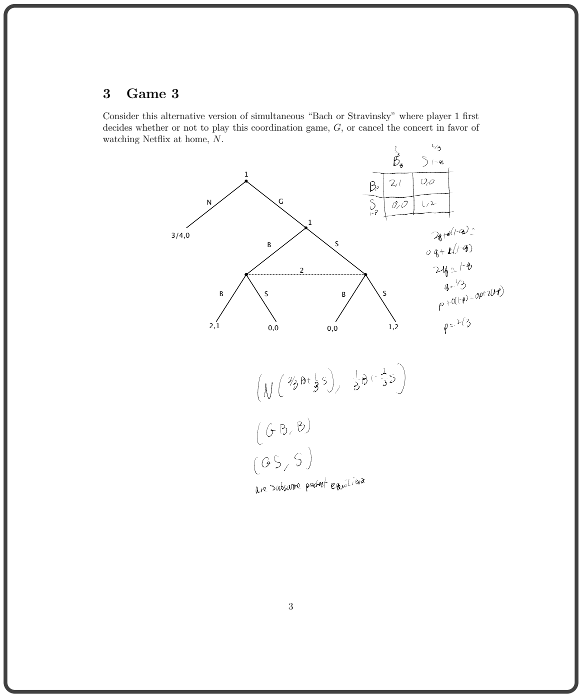


2 Game 2

Consider this alternative version of the Prisoner's Dilemma where player 1 first decides whether or not to play the Prisoner's Dilemma game, G, or go to the beach B

(BF, F) is a subgame perfect equilibrium

2



3 Game 3

Consider this alternative version of simultaneous "Bach or Stravinsky" where player 1 first decides whether or not to play this coordination game, G, or cancel the concert in favor of watching Netflix at home, N.

	B_q	S_{1-q}
B_p	2,1	0,0
S_{1-p}	0,0	1,2

$2q + 0(1-q) = 0q + 1(1-q)$

$2q = 1-q$

$q = 1/3$

$p + 0(1-p) = 0p + 2(1-p)$

$p = 2/3$

$\left(N\left(\frac{2}{3}B + \frac{1}{3}S\right), \frac{1}{3}B + \frac{2}{3}S\right)$

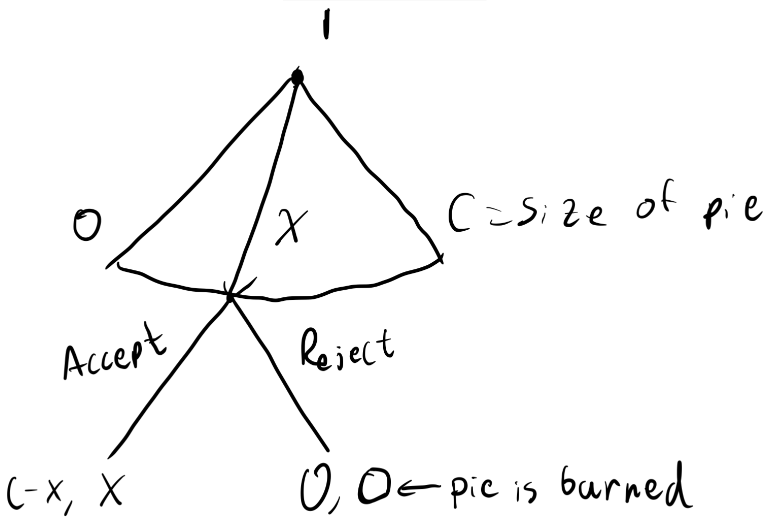
(GB, B)

(GS, S)

are subgame perfect equilibria

3

Ultimatum Game



To find the subgame perfect equilibrium, we go as follows:

- Player 2: infinite information sets.
  - Player 2 observes offer x.
  - Player 2's best response:

$$s_2(x) = \begin{cases} A, & \text{if } x > 0 \\ \alpha A + (1-\alpha)R, & \text{if } x = 0 \end{cases} \qquad \forall \alpha \in [0,1]$$

- Player 1: 1 information set.

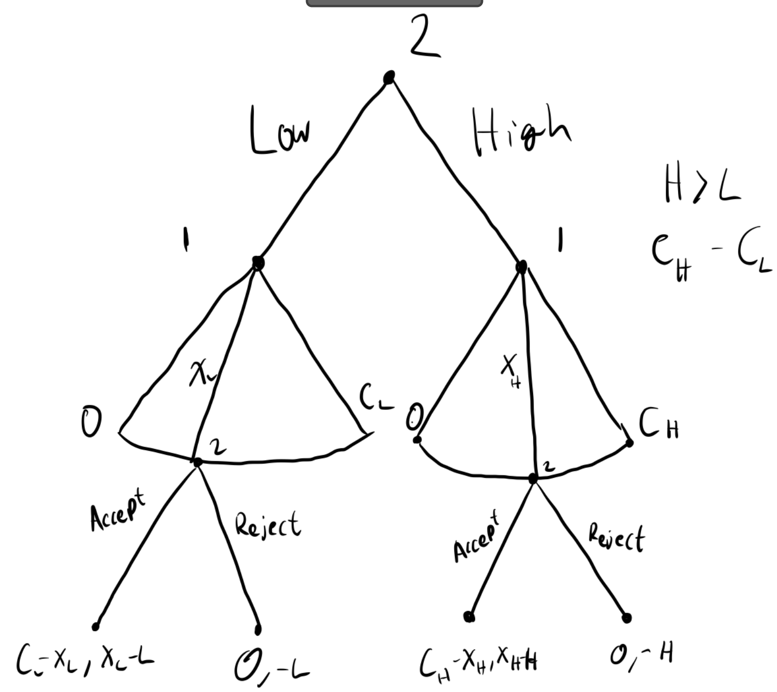
- Player 1's best response:

$$v_1(x, s_2(x)) = \begin{cases} c - x, & \text{if } x > 0 \\ \alpha c, & \text{if } x = 0 \end{cases}$$

- Player 1's best response *only* exists if $\alpha = 1$.

- Unique SPE:
  - $s_1 = 0$
  - $s_2(x) = A,\ \forall x$

Holdup Game



We know that in each ultimatum subgame, the subgame perfect equilibrium is

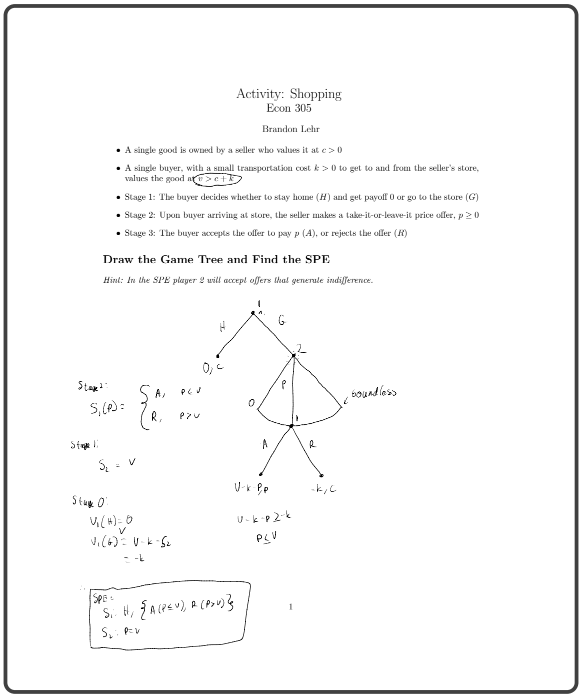
$$s_1 = 0$$
$$s_2(x) = A$$

If Player 2 plays Low, their payoff in the Low subgame is $-L$, while if Player 2 plays High, their payoff in the High subgame is $-H$.

Therefore, the subgame perfect equilibrium is:

$$s_1(E) = 0,\ \forall E$$
$$s_2(x) = (E = L, A),\ \forall x$$

Variations on the Ultimatum Game



Activity: Shopping
Econ 305

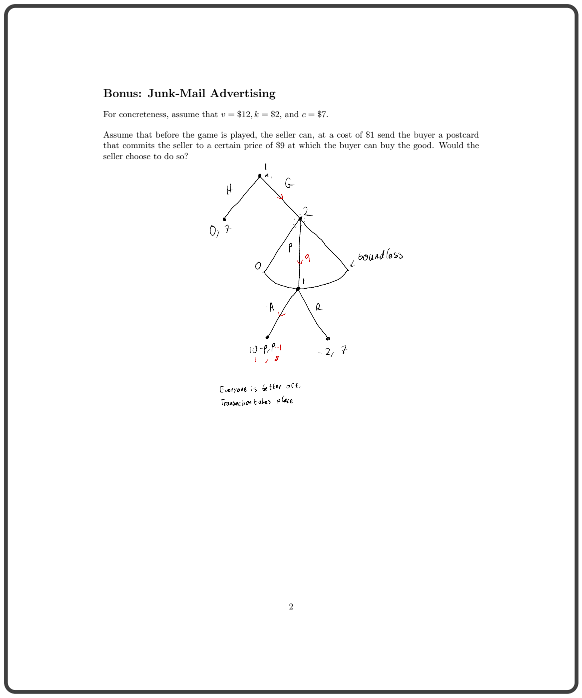
Brandon Lehr

- A single good is owned by a seller who values it at $c > 0$
- A single buyer, with a small transportation cost $k > 0$ to get to and from the seller's store, values the good at $v > c + k$
- Stage 1: The buyer decides whether to stay home (H) and get payoff 0 or go to the store (G)
- Stage 2: Upon buyer arriving at store, the seller makes a take-it-or-leave-it price offer, $p \geq 0$
- Stage 3: The buyer accepts the offer to pay p (A), or rejects the offer (R)

Draw the Game Tree and Find the SPE

Hint: In the SPE player 2 will accept offers that generate indifference.

1



Bonus: Junk-Mail Advertising

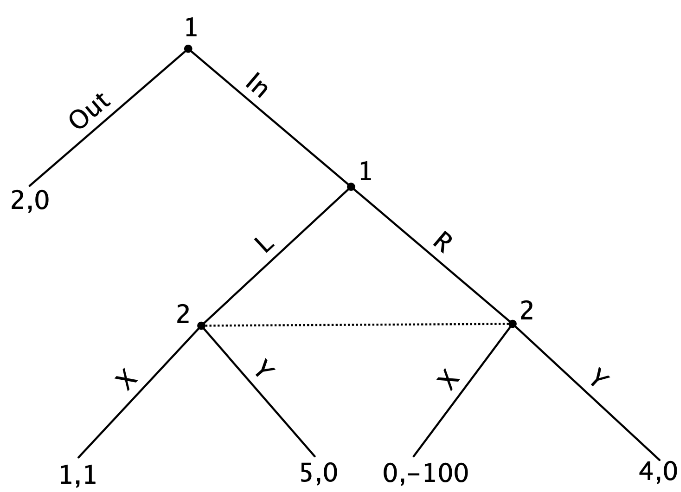
For concreteness, assume that $v = \$12, k = \$2$, and $c = \$7$.

Assume that before the game is played, the seller can, at a cost of \$1 send the buyer a postcard that commits the seller to a certain price of \$9 at which the buyer can buy the good. Would the seller choose to do so?

2

Critiques of SPE

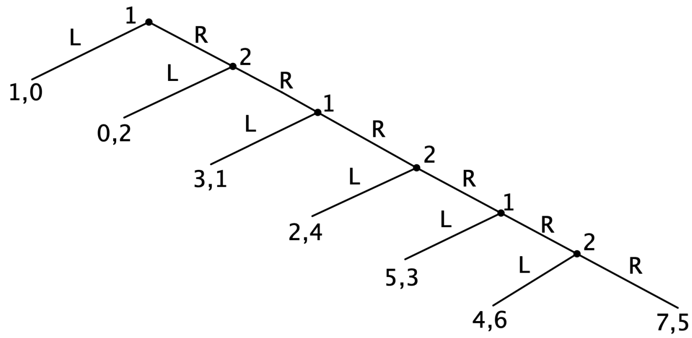
Consider the following extensive form game:



We can see that in the proper subgame, (L, X) is the Nash equilibrium, meaning that player 1 plays Out, meaning the subgame perfect equilibrium is (Out, L, X).

However, this depends on on Player 2 trusting that player 1 is rational with near 100% certainty — if player 1 chooses to play In, then R in the simultaneous-move game, then player 2 has a large negative payoff. Player 2 may want to choose Y if they reason that player 1 may not be rational.

Similarly, in the centipede game, the SPE is to always play L; however, experimental evidence suggests most people play R until close to the end of the game. This is because people either don't backward induct or they reason that their opponent won't backward induct.



Multistage Games

- A multistage game is a sequence of normal-form stage games.
- The stage games are played sequentially by the same players.
- Payoffs are received at every stage, not just at the end of the game.
- After each stage, the outcome of the previous stage is common knowledge.

Specifically, we focus on repeated games. We need to be concerned with strategic interactions as players repeatedly play the same game.

For example, in the Prisoner's Dilemma, we can design a repeated game such that (Q, Q) is part of the SPE. More generally, we can create situations in which repeated play allows for cooperation and improvements in total payoff over one-shot play.

Let G be a n-player simultaneous move game. The repeated game $G(T, \delta)$ is the game where in periods $t = 1, 2, \dots, T$ the players simultaneously choose actions $(a_1^t, a_2^t, \dots, a_n^t)$ after observing all previous actions. We define payoffs in the game by

$$v_i\left(s_i, s_{-i}\right) = \sum_{t=1}^{t} \delta^{t-1} v_i(a_1^t, \dots, a_n^t)$$

where $(a_1^t, \dots, a_n^t)$ is the action profile taken in period t when players follow strategies $s_1, \dots, s_n$ and v_i are the payoffs for the players in each stage game.

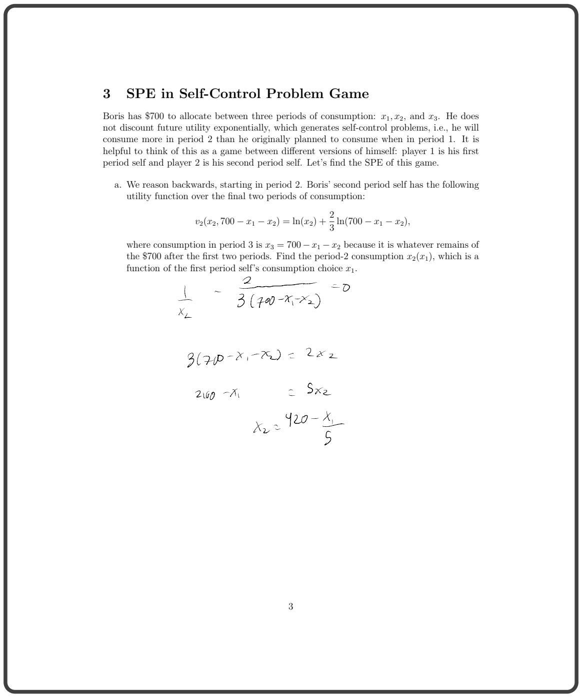
All strategies in the repeated games are complete contingent plans which specify the actiosn that each player will choose after observing all possible previous sequences of actions.

The parameter $\delta \in (0, 1]$ denotes the discount factor.

- δ can be akin to psychological time preferences — if δ is low, then players are impatient, while if δ is high, players are patient.
- If we define $\delta = \frac{1}{1+r}$, where r is the interest rate, then \$100 in the next period is worth \$$\delta$100 today since one must invest \$$\delta$100 into a bank account to get \$100 in the next period.

- δ can denote the probability of the end of the game; given that the game reaches stage t, the game continues to stage $t+1$ with probaiblity δ and ends with probability $1-\delta$. The expected number of periods is $\frac{1}{1-\delta}$ is finite.

Repeated Games and SPE Activity



3 SPE in Self-Control Problem Game

Boris has \$700 to allocate between three periods of consumption: x_1, x_2, and x_3. He does not discount future utility exponentially, which generates self-control problems, i.e., he will consume more in period 2 than he originally planned to consume when in period 1. It is helpful to think of this as a game between different versions of himself: player 1 is his first period self and player 2 is his second period self. Let's find the SPE of this game.

a. We reason backwards, starting in period 2. Boris' second period self has the following utility function over the final two periods of consumption:

$$v_2(x_2, 700 - x_1 - x_2) = \ln(x_2) + \frac{2}{3}\ln(700 - x_1 - x_2),$$

where consumption in period 3 is $x_3 = 700 - x_1 - x_2$ because it is whatever remains of the \$700 after the first two periods. Find the period-2 consumption $x_2(x_1)$, which is a function of the first period self's consumption choice x_1.

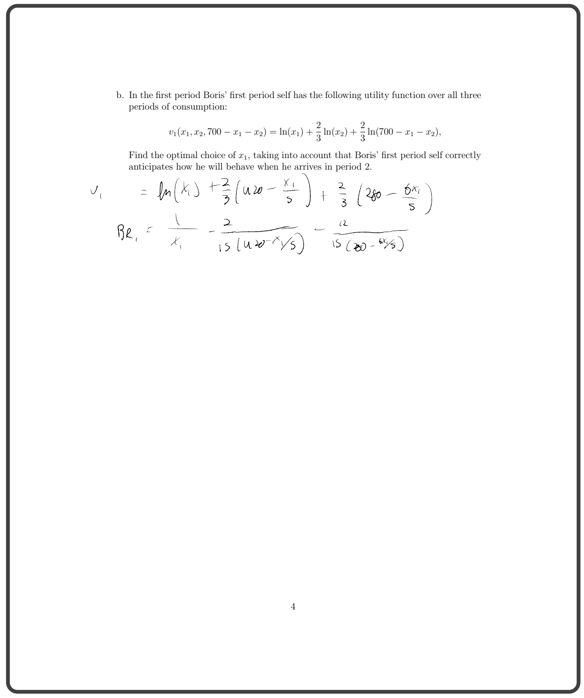
$$\frac{1}{x_2} - \frac{2}{3(700 - x_1 - x_2)} = 0$$

$$3(700 - x_1 - x_2) = 2x_2$$

$$2100 - x_1 = 5x_2$$

$$x_2 = 420 - \frac{x_1}{5}$$

3



b. In the first period Boris' first period self has the following utility function over all three periods of consumption:

$$v_1(x_1, x_2, 700 - x_1 - x_2) = \ln(x_1) + \frac{2}{3}\ln(x_2) + \frac{2}{3}\ln(700 - x_1 - x_2),$$

Find the optimal choice of x_1, taking into account that Boris' first period self correctly anticipates how he will behave when he arrives in period 2.

$$v_1 = \ln(x_1) + \frac{2}{3}\left(420 - \frac{x_1}{5}\right) + \frac{2}{3}\left(280 - \frac{6x_1}{5}\right)$$

$$BR_1 = \frac{1}{x_1} - \frac{2}{15(420 - x_1/5)} - \frac{12}{15(280 - 6x_1/5)}$$

4

Finitely Repeated Games: Unique Nash Equilibria

We find the SPE in a finitely repeated game via backward induction, as we would for a normal extensive-form game.

In the finitely repeated Prisoner's Dilemma has a unique SPE in which each player's *strategy* is to choose F in each period, regardless of the history.

The finite repetition of a stage game G that has a unique Nash equilibrium gives a unique SPE in which players play the Nash equilibrium in every stage, *regardless of what happened prior.*

Finitely Repeated Games: Multiple Nash Equilibria

Consider the game where the following stage game is repeated twice:

	L	C	R
T	$1,1$	$5,0$	$0,0$
M	$0,5$	$4,4$	$0,0$
B	$0,0$	$0,0$	$3,3$

To solve for the SPE of this game, we have to apply backward induction.

In the second round, a Nash equilibrium must be played, so we start by finding the Nash equilibria of the stage game.

- M is strictly dominated by a combination of T and B.
- C is strictly dominated by a combination of L and R.

	L	~~C~~	R
T	$1, 1$	~~$5, 0$~~	$0, 0$
~~M~~	~~$0, 5$~~	~~$4, 4$~~	~~$0, 0$~~
B	$0, 0$	~~$0, 0$~~	$3, 3$

- Therefore, we have three Nash equilibria:
  - (T, L) with payoff $(1, 1)$
  - (B, R) with payoff $(3, 3)$
  - $(3/4T + 1/4B, 3/4L + 1/4R)$ with payoff $(3/4, 3/4)$
- Example SPE: play a Nash equilibrium in the stage game in period 1, and any Nash equilibrium of the stage game in period 2, regardless of the history.
- However, we can also achieve outcomes in the first stage that are not Nash equilibria of the stage game by using the Nash equilibria in the second stage as reward and punishment.
- The following is a SPE where (M, C) is played in the first stage:

$$(s_1, s_2) = \begin{cases} (M, C) & \text{in stage 1} \\ (B, R) & \text{in stage 2 if } (M, C) \text{ in stage 1} \\ (T, L) & \text{in stage 2 otherwise} \end{cases}$$

- We can see that this is a SPE as we play a Nash equilibrium in every proper subgame; we can also see that player 1 and player 2 both have no profitable deviation (if player 1 plays T in stage 1, their total payoff is 6, as compared to 7 in this strategy profile).
- This exhibits the primary function of multiple Nash equilibria in a repeated stage game: using them as rewards and punishments for cooperation/non-cooperation.

Finitely Repeated Game

Activity: Finitely Repeated Game

Econ 305

Brandon Lehr

# of times

Consider the game $G(2, 1)$ where the stage game G is:

δ = discount factor

		q	$1-q$	
		L	C	~~R~~
p	T	3,1	0,0	~~5,0~~
$1-p$	M	2,1	1,2	~~3,1~~
	~~B~~	~~1,2~~	~~0,1~~	~~4,4~~

What are the Nash equilibria of G?

$3q = 2q + 1 - q$

$q = 1/2$

$p + 1 - p = 2(1-p)$

$p = \frac{1}{2}$

$4,4$

$4 + 3 = 7$

$4 + 1 = 5$

$(T, L), (M, C), \left(\frac{1}{2}T + \frac{1}{2}M, \frac{1}{2}L + \frac{1}{2}C\right)$

$\left(3/2, \frac{1}{2}\right)$

Construct a SPE of $G(2, 1)$ in which (B, R) is played in the first stage.

$$(s_1, s_2) = \begin{cases} (B, R) & \text{in stage 1} \\ \underline{(T, L)} & \text{in stage 2 if } \underline{(B, R)} \text{ played in stage 1} \\ \underline{(M, C)} & \text{in stage 2 otherwise} \end{cases}$$

Show no deviations in stage 1:

Player 1 has incentive to deviate from (B, R):

- Deviate to T ⇒ Total payoff = 6
- Deviate to M ⇒ Total payoff = 4

∴ no profitable deviation.

Bonus: Argue that there is no SPE of $G(2, 1)$ in which (B, C) is played in the first stage.

Infinitely Repeated Stage Games

Consider the repeated Prisoner's Dilemma, but infinitely so.

- The argument that the unique SPE of this repeated game was (F, F) was via backward induction.

- However, in an infinitely repeated game, we cannot make the argument from backward induction (where is the "final" stage game?).
- Can we sustain some cooperation in the infinitely repeated game?

One Stage Deviation Property

A strategy profile in an infinitely repeated game with $\delta < 1$ is a SPE if and only if no player can increase their payoff by changing their action at a single information set, given the other players' strategies and the rest of their own strategy.

- Any profitable deviation can be broken into a sequence of one-period changes — so, we only need to check single deviations
- For infinite horizon games, discounting is crucial to make the theorem true, as it allows payoffs in a far enough future to be worth little.
- We assume that all infinite horizon games are such that $\delta < 1$.

To check whether a strategy profile is a SPE:

- Classify all the information sets on and off the equilibrium path
- Apply the One Stage Deviation Property to each class separately

A strategy in which players play the same Nash Equilibrium of the stage game in each period, *regardless of the history*, is always a SPE.

Grim Trigger Strategy

The *discounted average* of any stream of payoffs $(v_i^1, v_i^2, \dots)$ for the discount factor δ is

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_i^t$$

If a player received a constant stream of payoffs $(c, c, c, \dots)$, then the discounted average would be c.

In the Prisoner's Dilemma, define the grim trigger strategy to be

$$s_i(a^1, \dots, a^t) = \begin{cases} Q & \text{if } (a^1, \dots, a^t) = ((Q,Q), \dots, (Q,Q)) \\ F & \text{otherwise} \end{cases}$$

Claim: The strategy profile in which both players use the grim trigger strategy is a SPE of the infinitely repeated Prisoner's Dilemma for $\delta \geq 1/2$.

Suppose there has been a defection at some point in the past to F.

- Player i's (current period) average discounted payoff by following the grim trigger: 1

- Player i's average discounted payoff by deviating to Q in the current period, then returning to s_i:

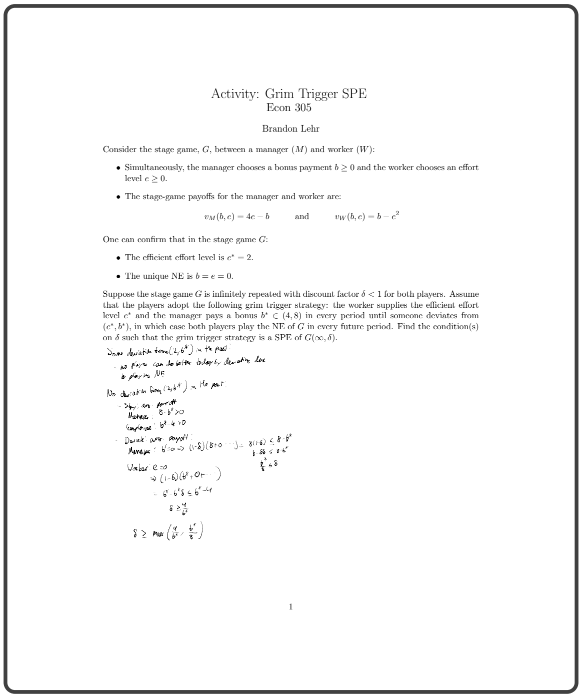
$$\begin{aligned}
v_i &= (1-\delta)(\overbrace{0}^{\substack{\text{deviation}\\\text{payoff}}} + \delta + \delta^2 + \dots) \\
&= (1-\delta)\left(\sum_{t=1}^{\infty}\delta^t\right) \\
&= \delta \\
&< 1
\end{aligned}$$

Suppose there has been no defection to F anytime in the past.

- Player i's average discounted payoff by following the grim trigger: 2 (both players play (Q,Q) at all stages).
- Player i's average discounted payoff by deviating to F, then following the grim trigger:

$$\begin{aligned}
v_i &= (1-\delta)(\overbrace{3}^{\substack{\text{deviation}\\\text{payoff}}} + \underbrace{\delta + \delta^2 + \dots}_{\text{grim trigger for } F}) \\
&= (1-\delta)\left(3 + \sum_{t=1}^{\infty}\delta^t\right) \\
&= 3(1-\delta) + \delta \\
&= 3 - 2\delta \\
2 &\geq 3 - 2\delta && \text{Nash equilibrium condition} \\
\delta &\geq \frac{1}{2}
\end{aligned}$$

Grim Trigger Workthrough



Activity: Grim Trigger SPE
Econ 305

Brandon Lehr

Consider the stage game, G, between a manager (M) and worker (W):

- Simultaneously, the manager chooses a bonus payment $b \geq 0$ and the worker chooses an effort level $e \geq 0$.
- The stage-game payoffs for the manager and worker are:

$$v_M(b,e) = 4e - b \qquad \text{and} \qquad v_W(b,e) = b - e^2$$

One can confirm that in the stage game G:

- The efficient effort level is $e^* = 2$.
- The unique NE is $b = e = 0$.

Suppose the stage game G is infinitely repeated with discount factor $\delta < 1$ for both players. Assume that the players adopt the following grim trigger strategy: the worker supplies the efficient effort level e^* and the manager pays a bonus $b^* \in (4,8)$ in every period until someone deviates from (e^*, b^*), in which case both players play the NE of G in every future period. Find the condition(s) on δ such that the grim trigger strategy is a SPE of $G(\infty, \delta)$.

Some deviation from $(2, b^*)$ in the past:

- no player can do better today by deviating due to playing NE

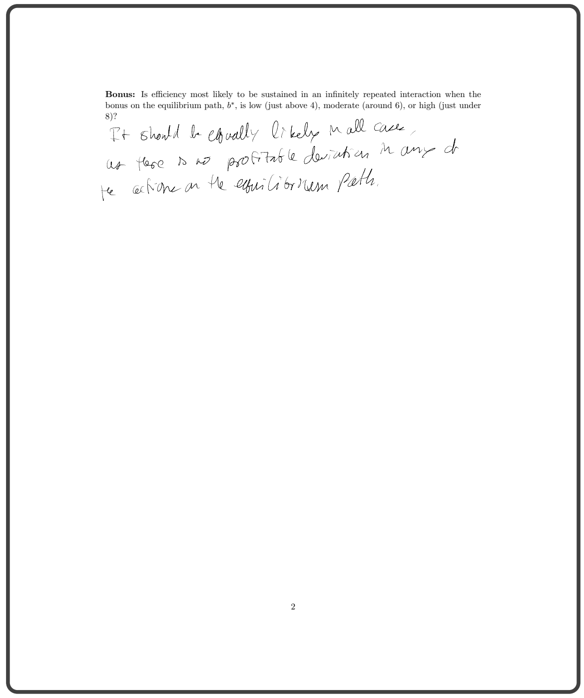
No deviation from $(2, b^*)$ in the past:

- Stay: avg. payoff
  Manager: $8 - b^* > 0$
  Employee: $b^* - 4 > 0$
- Deviate: avg. payoff:
  Manager: $b' = 0 \Rightarrow (1-\delta)(8 + 0 \cdots) = 8(1-\delta) \leq 8 - b^*$
  $8 - 8\delta \leq 8 - b^*$
  $\frac{b^*}{8} \leq \delta$

  Worker: $e = 0$
  $\Rightarrow (1-\delta)(b^* + 0 + \cdots)$
  $= b^* - b^*\delta \leq b^* - 4$
  $\delta \geq \frac{4}{b^*}$

$$\delta \geq \max\left(\frac{4}{b^*}, \frac{b^*}{8}\right)$$

1



Bonus: Is efficiency most likely to be sustained in an infinitely repeated interaction when the bonus on the equilibrium path, b^*, is low (just above 4), moderate (around 6), or high (just under 8)?

It should be equally likely in all cases, as there is no profitable deviation in any of the actions on the equilibrium path.

2

Rubinstein's Bargaining Game

Consider a game in which \$1 is to be divided between two players:

- In periods $1, 3, 5, \ldots$, player 1 offers the division (x_1, x_2). Player 2 either accepts the offer and the game ends, or rejects the offer and play continues
- In periods $2, 4, 6, \ldots$, player 2 offers the division (y_1, y_2). Player 1 either accepts the offer and the game ends, or rejects the offer and play continues

If the division $(z, 1-z)$ is agreed upon in period t, then the payoffs are $(v_1, v_2) = \left(\delta^{t-1}z, \delta^{t-1}(1-z)\right)$.

- Waiting to divide is costly
- This game is *infinite* horizon
- The game is not repeated

SPE of Rubinstein's Bargaining Game

Proposition The bargaining game has a unique SPE: in each period the player who proposes offer $\frac{1}{1+\delta}$ to themselves and the other player accepts any division giving them at least $\frac{\delta}{1+\delta}$, and rejects any offer giving them less.

Proof of Existence We want to show that the One Stage Deviation Property is satisfied by the proposed SPE.

There are two types of information sets: when the proposer makes an offer, and when the recipient makes a decision.

Proposer payoff from following:

$$v_p = \frac{1}{1+\delta}$$

Proposer payoff from deviation:

$$\begin{aligned} v_p &= \delta \frac{\delta}{1+\delta} && \text{accept following period offer} \\ &< \frac{1}{1+\delta} \end{aligned}$$

Recipient payoff from accepting an offer of z in the current period:

$$v_r = z$$

Recipient payoff from rejecting the offer:

$$v_r = \delta \frac{1}{1+\delta} \qquad \text{provide offer of } \tfrac{1}{1+\delta} \text{ in following period}$$

Therefore, recipient should accept offer of z if and only if

$$z \geq \delta \frac{1}{1+\delta}$$

Efficiency: On the equilibrium path, bargaining is immediate and therefore efficient.

Patience: Player 1 can earn a greater share of the pie if δ is small (players are impatient)

First-Mover Advantage: For all $\delta < 1$, player 1 earns more than player 2 in the SPE.

Proving a Bargaining Game SPE

Assume the current period.

Proposer Payoffs: (a) Find payoff from following.

(b) Find payoff from offering less, assuming the next period proposer's strategy holds constant.

Recipient Payoffs: (a) Find payoff from *accepting the proposer's offer* **Note:** the recipient does not "deviate", as they are not in charge of making the proposal.

(b) Find payoff from *rejecting the proposer's offer*, holding fixed the next period strategy (the current recipient becomes the proposer in the following period).

Bargaining Workthrough

Activity: Bargaining with Different Discount Factors

Econ 305

Brandon Lehr

Consider the Rubinstein bargaining game of alternating offers in which the players bargain over a pie of size 1. We will specify that the payoffs if $(z, 1-z)$ is accepted at date t are

$$(\delta_1^{t-1}z, \delta_2^{t-1}(1-z))$$

where $0 < \delta_i < 1$ is player i's discount factor.

a. Verify that the following strategy profile is an SPE of the above game:

- Player 1 always proposes $\left(\boxed{\frac{1-\delta_2}{1-\delta_1\delta_2}}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ and accepts a proposal y if and only if $y_1 \geq \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$
- Player 2 always proposes $\left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right)$ and accepts a proposal x if and only if $x_2 \geq \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$

First consider a subgame in which player 1 is the proposer.

- *Follow:* $v_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}$
- *Best Deviation (offer less to player 2):* $v_1 = \delta_1 \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$

↳ deviation ⇒ worse off

Now consider a subgame in which player 1 is responding to some offer z.

- *Accept:* $v_1 = z$
- *Rejects:* $v_1 = \delta_1 \frac{(1-\delta_2)}{1-\delta_1\delta_2}$

accept if $z \geq \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$

Note that symmetric arguments apply for player 2.

b. What happens when $\delta_1 \to 1$ for fixed δ_2 and when $\delta_2 \to 1$ for fixed δ_1? Explain why there is a difference in the equilibrium shares of the pie.

$\delta_1 \to 1$ holding fixed δ_2 ⇒ Player 1 takes higher share (reduces to $\sim \frac{1-\delta_2}{1-\delta_2}$)

$\delta_2 \to 1$ holding fixed δ_1 ⇒ Player 2 takes higher share.

1

Bargaining Extensions and the Folk Theorem

Recall the Grim Trigger strategy for the infinitely repeated Prisoner's Dilemma is SPE with $\delta \geq \frac{1}{2}$.

However, this doesn't mean sufficiently patient players will always cooperate in repeated interactions.

There are many possible SPE with $\delta \geq \frac{1}{2}$.

(a) Play F in every period

(b) For $\delta \geq \frac{1}{2}$, there is a SPE in which players play F in the first period and Q in all future periods.

(c) For $\delta \geq \frac{1}{\sqrt{2}}$, there is a SPE in which players play F in every even period and Q in every odd period.

(d) For $\delta \geq \frac{1}{2}$, there is a SPE where the players play (Q, F) in every even period and (F, Q) in every odd period.

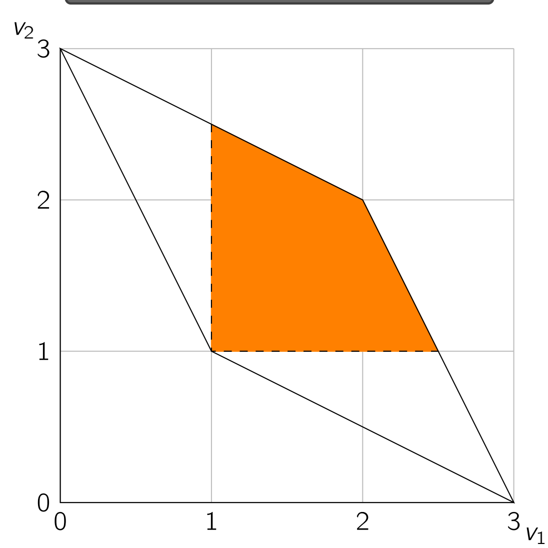
(e) For $\delta = 1/2$, there is a SPE where the players play the last-used action of the opponent.

We would like to characterize what is possible in SPE of infinitely repeated games.

- A payoff vector $v = (v_1, v_2, \dots, v_n)$ is *feasible* if it is a weighted average of the payoffs available in a strategic game.
- A payoff vector is *strictly individually rational* if each player obtains strictly more than their min-max payoff.
- The min-max payoff is the payoff a player can guarantee themself even if the other players try to punish them as badly as they can.

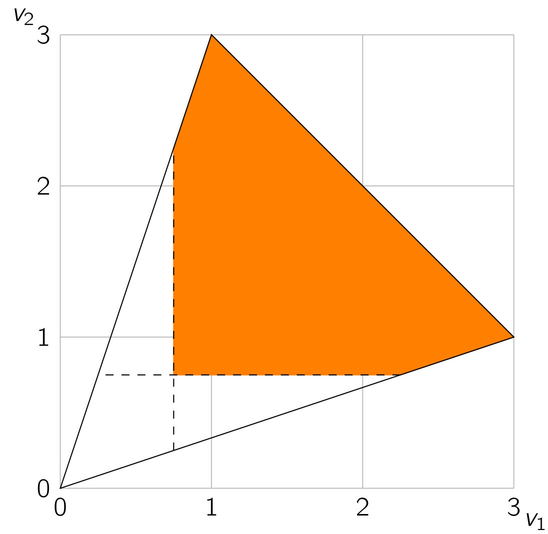
Folk Theorem: Suppose that the set of feasible payoffs of G is n-dimensional. Then, for any feasible and strictly individually rational payoff vector v, $\exists \overline{\delta} < 1$ such that $\forall \delta > \overline{\delta}$, $G(\infty, \delta)$ has a SPE with discount average payoffs v.

Folk Theorem for Prisoner's Dilemma



The feasible payoffs are defined by the outer diamond — the min-max payoff for each player is 1, since the worst punishment a rival can inflict is to choose F.

Folk Theorem for Bach or Stravinsky



The feasible payoffs are the triangle bounded by the outer triangle — the min-max payoff for each player is 3/4 since the worst punishment a rival can inflict is to choose $\frac{1}{4}B + \frac{3}{4}S$

Games of Incomplete Information

Narrowing the Scope

In real life economic situation, it's unlikely that players know:

- payoffs
- who other players are
- what moves are possible
- how outcomes depend on actions
- what players know

We will focus on the case where the players know who the other players are, and their available actions, but not how those actions impact payoffs. These are known as *Bayesian Games*.

Example I: Public Goods Game

	Clean	Don't
Clean	$1-c_1, 1-c_2$	$1-c_1, 1$
Don't	$1, 1-c_2$	$0, 0$

Assume that the actions are chosen simultaneously and that players don't know their own costs.

(I.a): Player 2 also knows player 1's cost, but player 1 believes that $c_2 \in \{\underline{c}, \overline{c}\}$ where $\underline{c} < \overline{c}$, and the low cost is true with probability p.

(I.b): Each player believes that the cost of the other roommate is equally likely to be any value between 0 and 2: $c_1, c_2 \sim U[0, 2]$

Example II: Auction

Consider a two-bidder **first-price sealed bid private value auction**:

- Sealed bid: announce b_1 and b_2 simultaneously
- First-price: the highest bidder wins and pays their bid
- Private value: valuation of the object, θ_i, is independent of how others view the object
- A coin is flipped if they choose the same bid

In this case, utilities are:

$$v_i(b_i, b_{-i}) = \begin{cases} \theta_i - b_i & b_i > b_{-i} \\ \frac{1}{2}(\theta_i - b_i) & b_i = b_{-i} \\ 0 & b_i < b_{-i} \end{cases}$$

The primary form of incomplete information here is that each player knows their own valuation, but does not know their rival's — assume each player believes that the valuation of the rival is uniform on $[0, 1]$

Characterization of Bayesian Games

A static game with incomplete information G consists of

- A set of types Θ_i for each player i. Player i's type is privately known to player i.
- A set of actions A_i for each player i.
- A joint probability distribution $\phi(\theta_1, \dots, \theta_n)$
- A payoff function $v_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n)$ for each player i

This information is common knowledge.

For example, in our examples, we had the type spaces:

(I.a): $\Theta_1 = \{c_1\}$, $\Theta_2 = \{\underline{c}, \overline{c}\}$

(I.b): $\Theta_1 = \Theta_2 = [0, 2]$

(II): $\Theta_1 = \Theta_2 = [0, 1]$

The payoffs depend on the action profile and the types of all players.

Harsanyi's Observation and Bayesian Pure Strategy

Games of incomplete information can be thought of as games of complete but imperfect information wher eNature is another player who moves first. Not everyone is informed about Nature's move — Nature chooses $\theta \in \Theta$ with probability $\phi(\theta)$, but only tells θ_i to player i.

Harsanyi's observation makes it possible to define strategies and equilibria just the same way we defined them in extensive form games. A strategy is (still) a function that assigns an action to each information set for each player.

- A *Bayesian pure strategy* is a function that assigns an action for each possible type. $s_i : \Theta_i \to A_i$