Solution (38.5)**:** Copying the template equation, we have

$$\frac{dv}{dt} = -\frac{c}{m}v^2 + g,$$

where c is some constant. We see that the terminal velocity is

$$v_t = \sqrt{\frac{mg}{c}}.$$

Separating variables, we have

$$\begin{aligned}\frac{dv}{-\frac{c}{m}v^2 + g} &= dt\\ \frac{1}{g}\left(\frac{dv}{1 - \frac{c}{mg}v^2}\right) &= dt\\ \frac{1}{g}\left(\frac{dv}{1 - (v/v_t)^2}\right) &= dt.\end{aligned}$$

Using the substitution $u := v/v_t$, we have $du = \frac{1}{v_t}dv$, meaning that

$$v_t \int \frac{1}{1-u^2}\,du = \int g\,dt.$$

The integral of $\frac{1}{1-u^2}$ is $\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = \operatorname{arctanh}(u)$. Therefore, we have

$$\begin{aligned}\frac{v}{v_t} &= \tanh\left(\frac{g}{v_t}t\right) + K\\ v &= v_t\tanh\left(\frac{g}{v_t}t\right) + v_0\\ &= \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{c}{mg}}t\right) + v_0.\end{aligned}$$

Solution (38.6)**:**

(a) Using the chain rule and letting $\frac{dm}{dt} = km^{2/3}$, we have

$$\begin{aligned}\frac{dv}{dt} &= km^{2/3}\frac{dv}{dm}\\ \frac{dv}{dm} + \frac{v}{m} &= -\frac{b}{km}v + \frac{g}{km^{2/3}}.\end{aligned}$$

With integrating factor $m^{1+\frac{b}{k}}$, we have

$$\begin{aligned}m^{1+\frac{b}{k}}v &= \frac{g}{k}\frac{m^{\frac{4}{3}+\frac{b}{k}}}{\frac{4}{3}+\frac{b}{k}} + C\\ v &= \frac{g}{k\left(\frac{4}{3}+\frac{b}{k}\right)}m^{\frac{1}{3}+\frac{b}{k}} + Cm^{-1-\frac{b}{k}}.\end{aligned}$$

We let $v(m_0) = 0$, so that

$$C = -\frac{g}{k\left(\frac{4}{3}+\frac{b}{k}\right)}m_0^{\frac{4}{3}+\frac{b}{k}},$$

so

$$v = \frac{g}{\frac{4}{3}k + b}m^{\frac{1}{3}}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{4}{3}+\frac{b}{k}}\right).$$

Thus,

$$\begin{aligned}\frac{dv}{dt} &= g - \frac{1}{m}\frac{dm}{dt}v \\ &= g - \frac{1}{m}\left(km^{2/3}\right)\left(\frac{g}{\frac{4}{3}k+b}m^{\frac{1}{3}}\left(1-\left(\frac{m_0}{m}\right)^{\frac{4}{3}+\frac{b}{k}}\right)\right).\end{aligned}$$

(b) Using $\frac{dm}{dt} = km^{2/3}v$, and $\frac{dv}{dt} = km^{2/3}v\frac{dv}{dm}$, we obtain

$$\begin{aligned}m\frac{dv}{dt} + v\frac{dm}{dt} &= -bm^{2/3}v^2 + mg \\ v\,dv + \left(\frac{v^2}{m}\left(1+\frac{b}{k}\right) - \frac{g}{km[2/3]}\right)dm &= 0.\end{aligned}$$

This gives $\alpha = v$ and $\beta = \frac{v^2}{m}\left(1+\frac{b}{k}\right) - \frac{g}{km^{2/3}}$. Solving for $p(m)$, we get

$$\begin{aligned}p(m) &= \frac{1}{v}\left(\frac{2v}{m}\left(1+\frac{b}{k}\right)\right) \\ &= \frac{2}{m}\left(1+\frac{b}{k}\right).\end{aligned}$$

Therefore, our integrating factor is

$$w(x) = m^{2+\frac{2b}{k}}.$$

This gives

$$\begin{aligned}\frac{\partial\Phi}{\partial v} &= \alpha \\ \Phi &= \frac{1}{2}m^{2+\frac{2b}{k}}v^2 + c_1(m) \\ \frac{\partial\Phi}{\partial m} &= \beta \\ \Phi &= \frac{1}{2}m^{2+\frac{2b}{k}}v^2 - \frac{g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}} + c_2(v).\end{aligned}$$

Thus, $c_2(v) = 0$, and

$$\frac{1}{2}m^{2+\frac{2b}{k}}v^2 - \frac{g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}} = C.$$

Using $v(m_0) = 0$, we obtain the solution of

$$\frac{1}{2}m^{2+\frac{2b}{k}}v^2 = \frac{g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right).$$

Simplifying, this gives

$$v^2 = \frac{2g}{k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{\frac{1}{3}}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right).$$

Therefore,

$$2v\frac{dv}{dm} = \frac{2g}{3k\left(\frac{7}{3}+\frac{2b}{k}\right)}m^{-2/3}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right) + \frac{2g}{km}\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}},$$

and

$$\begin{aligned}\frac{dv}{dt} &= \frac{k}{2}m^{2/3}\left(2v\frac{dv}{dm}\right) \\ &= \frac{g}{3\left(\frac{7}{3}+\frac{2b}{k}\right)}\left(1-\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right) + \frac{g}{m^{\frac{1}{3}}}\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}.\end{aligned}$$

Solution (38.7)**:**

Solution (39.5)**:** We take the derivative of

$$\frac{du_p}{dx} = a_1(x)\frac{du_1}{dx} + a_2(x)\frac{du_2}{dx},$$

giving

$$\frac{d^2u_p}{dx^2} = a_1(x)\frac{d^2u_1}{dx^2} + \frac{da_1}{dx}\frac{du_1}{dx} + a_2(x)\frac{d^2u}{dx^2} + \frac{da_2}{dx}\frac{du_2}{dx}.$$

Note that we must have

$$\frac{d^2u_p}{dx^2} + p(x)\frac{du_p}{dx} + q(x)u_p = r(x),$$

so we have

$$\begin{aligned} r(x) &= a_1(x)\frac{d^2u_1}{dx^2} + \frac{da_1}{dx}\frac{du_1}{dx} + a_2(x)\frac{d^2u}{dx^2} + \frac{da_2}{dx}\frac{du_2}{dx} \\ &\quad + p(x)\left(a_1(x)\frac{du_1}{dx} + a_2(x)\frac{du_2}{dx}\right) \\ &\quad + q(x)(a_1(x)u_1(x) + a_2(x)u_2(x)). \end{aligned}$$

Reordering and simplifying, we get

$$\begin{aligned} r(x) &= a_1(x)\left(\frac{d^2u_1}{dx^2} + p(x)\frac{du_1}{dx} + q(x)u_1(x)\right) + a_2(x)\left(\frac{d^2u_2}{dx^2} + p(x)\frac{du_2}{dx} + q(x)u_2(x)\right) + \frac{da_1}{dx}\frac{du_1}{dx} + \frac{da_2}{dx}\frac{du_2}{dx} \\ &= \frac{da_1}{dx}\frac{du_1}{dx} + \frac{da_2}{dx}\frac{du_2}{dx}. \end{aligned}$$

Pairing this expression with

$$\frac{da_1}{dx}u_1(x) + \frac{da_2}{dx}u_2(x) = 0,$$

we may solve for $\frac{da_1}{dx}$ and $\frac{da_2}{dx}$, giving

$$\begin{aligned} \frac{da_1}{dx} &= -\frac{u_2(x)r(x)}{u_1(x)\frac{du_2}{dx} - u_2(x)\frac{du_1}{dx}} \\ \frac{da_2}{dx} &= \frac{u_1(x)r(x)}{u_1(x)\frac{du_2}{dx} - u_2(x)\frac{du_1}{dx}}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_1(x) &= -\int \frac{u_2(x)r(x)}{W(x)}\,dx \\ a_2(x) &= \int \frac{u_1(x)r(x)}{W(x)}\,dx. \end{aligned}$$

Solution (39.7)**:**

(a) We solve the homogeneous part to yield

$$\begin{aligned} u_1(x) &= e^{-x} \\ u_2(x) &= xe^{-x}. \end{aligned}$$

These give the Wronskian of

$$\begin{aligned} W(x) &= e^{-x}(e^{-x} - xe^{-x}) + xe^{-2x} \\ &= e^{-2x}. \end{aligned}$$

We evaluate

$$\begin{aligned}
a_1(x) &= -\int e^{x}(xe^{-x})(e^{-x})\,dx\\
&= -\int xe^{-x}\,dx\\
&= -(-xe^{-x} - e^{-x})\\
&= xe^{-x} + e^{-x}\\
a_2(x) &= \int e^{-x}\,dx\\
&= -e^{-x}.
\end{aligned}$$

Thus, we have the general solution of

$$u(x) = c_1 e^{-x} + c_2 xe^{-x} + e^{-2x}.$$

(b) Solving for the homogeneous solutions, we get

$$\begin{aligned}
u_1(x) &= e^{x}\\
u_2(x) &= e^{-x},
\end{aligned}$$

with Wronskian

$$W(x) = -2.$$

Setting up variation of parameters, we have

$$\begin{aligned}
a_1(x) &= -\int -\frac{1}{2}\,dx\\
&= \frac{1}{2}\\
a_2(x) &= -\frac{1}{2}\int e^{2x}\,dx\\
&= -\frac{1}{4}e^{2x}.
\end{aligned}$$

Thus, we have the general solution of

$$u(x) = c_1 e^{x} + c_2 e^{-x} + \frac{1}{4}e^{x}.$$

(c) Solving for the homogeneous solution, we get

$$\begin{aligned}
u_1(x) &= \cos(x)\\
u_2(x) &= \sin(x),
\end{aligned}$$

with Wronskian

$$W(x) = 1.$$

Setting up variation of parameters, we then get

$$\begin{aligned}
a_1(x) &= -\int \sin(x)\cos(x)\,dx\\
&= -\frac{1}{2}\cos(2x)\\
a_2(x) &= \int \sin^2(x)\,dx\\
&= \frac{1}{2}x + \frac{1}{2}\sin(2x).
\end{aligned}$$

Thus, we get the general solution of

$$u(x) = c_1\cos(x) + c_2\sin(x) + \frac{1}{2}(x + \sin(2x) - \cos(2x)).$$

Solution (39.8)**:** We have the particular solutions of

$$u_1(t) = e^{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t}$$
$$u_2(t) = e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t}$$

Evaluating the Wronskian, we get

$$W(t) = -2\sqrt{\beta^2 - \omega_0^2}e^{-2\beta t},$$

so with variation of parameters, we have

$$\begin{aligned} a_1(t) &= \frac{1}{2\sqrt{\beta^2 - \omega_0^2}} \int e^{\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t} \delta(t - t')\, dt \\ &= \frac{1}{2\sqrt{\beta^2 - \omega_0^2}} e^{\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t'} \\ a_2(t) &= -\frac{1}{2\sqrt{\beta^2 - \omega_0^2}} \int e^{\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t} \delta(t - t')\, dt \\ &= -\frac{1}{2\sqrt{\beta^2 - \omega_0^2}} e^{\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t'}. \end{aligned}$$

Thus, we get the particular solution of

$$u_p(t) = \frac{1}{2\sqrt{\beta^2 - \omega_0^2}}\left(\exp\left(\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t' + \left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t\right) - \exp\left(\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t' + \left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t\right)\right).$$

Solution (39.13)**:**

(a) Setting up our differential equation of

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega_0^2 t = F_0\cos(\omega t),$$

we have homogeneous solutions of

$$u_1(t) = e^{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t}$$
$$u_2(t) = e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t}.$$

Using the driving force $F_0\cos(\omega t)$, we use variation of parameters with the Wronskian of $W(t) = e^{-2\beta t}$ to get

$$a_1(t) = -F_0 \int e^{\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t} \cos(\omega t)\, dt$$
$$a_2(t) = F_0 \int e^{\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t} \cos(\omega t)\, dt$$

and a particular solution of

$$u_p(t) = F_0\left(\frac{\omega_0^2\cos(\omega t) + 2\beta\omega\sin(\omega t) - \omega^2\cos(\omega t)}{\left(\omega_0^2 - \omega^2\right)^2 + (2\beta\omega)^2}\right).$$

Solution (39.17)**:**

(a) Using the power of the guess $e^{\lambda t}$, we find the solutions

$$\begin{aligned} u_1(t) &= e^{-2t} \\ u_2(t) &= e^{-t}. \end{aligned}$$

(b) We find the Wronskian

$$W(t) = -3e^{-3t},$$

from which we are able to find

$$\begin{aligned} a_1(t) &= \frac{1}{3}\int e^{2t}\cos(t)\,dt \\ &= \frac{1}{15}e^{2t}(\sin(t) + 2\cos(t)) \\ a_2(t) &= -\frac{1}{3}\int e^{t}\cos(t)\,dt \\ &= -\frac{1}{6}e^{t}(\sin(t) + \cos(t)). \end{aligned}$$

Thus, the particular solution is

$$u_p(t) = \frac{1}{15}(\sin(t) + 2\cos(t)) - \frac{1}{6}(\sin(t) + \cos(t)).$$

(c) We find the full solution such that

$$\begin{aligned} c_1 + c_2 &= \frac{31}{30} \\ -2c_1 - c_2 &= \frac{1}{10}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} c_1 &= -\frac{34}{30} \\ c_2 &= \frac{13}{6}. \end{aligned}$$

Our solution is

$$-\frac{34}{30}e^{-2t} + \frac{13}{6}e^{-t} - \frac{1}{30}\cos(t) - \frac{1}{10}\sin(t).$$

Solution (39.18)**:** We start with the ansatz x^{α}. Plugging this into our homogeneous equation, we get

$$x^{\alpha}\left(\alpha^2 - \alpha - 2\right) = 0.$$

Therefore, we get that $\alpha = 2, -1$, giving the homogeneous solutions of

$$\begin{aligned} u_1(x) &= x^2 \\ u_2(x) &= \frac{1}{x}. \end{aligned}$$

We calculate the Wronskian to be $W(x) = -3$, so we use variation of parameters to obtain

$$\begin{aligned} a_1(x) &= \frac{1}{3}\int \left(\frac{1}{x}\right)x\,dt \\ &= \frac{1}{3}x \\ a_2(x) &= -\frac{1}{3}\int x^3\,dx \end{aligned}$$

$$= -\frac{1}{12}x^4.$$

Therefore,

$$u(x) = a_1x^2 + a_2x^{-1} + \frac{1}{4}x^3.$$

Plugging in the initial conditions, we have

$$\begin{aligned} 0 &= a_1 + a_2 + \frac{1}{4} \\ 0 &= 2a_1 - a_2 + \frac{3}{4}. \end{aligned}$$

This resolves to

$$\begin{aligned} a_1 &= -\frac{1}{3} \\ a_2 &= \frac{1}{12}, \end{aligned}$$

so we have the solution

$$u(x) = -\frac{1}{3}x^2 + \frac{1}{12}x^{-1} + \frac{1}{4}x^3.$$

Solution (39.21)**:**

Solution (39.22 (b))**:**

Solution (39.28)**:** We begin with the assumption that we have a power series of the form

$$u(x) = x^\alpha \sum_{k=0}^{\infty} c_k x^k.$$

Differentiating, we get

$$\begin{aligned} \frac{d^2u}{dx^2} &= \sum_{k=2}^{\infty} c_k(\alpha+k)(\alpha+k-1)x^{\alpha+k=2} \\ xu &= \sum_{k=1}^{\infty} c_{k-1}x^{\alpha+k}. \end{aligned}$$

Plugging this into Airy's equation, we are able to extract

$$c_2(\alpha+1)(\alpha+2) + \sum_{k=1}^{\infty}(c_{k+2}(\alpha+k+2)(\alpha+k+1) - c_{k-1})x^{\alpha+k} = 0.$$

Thus, we are left with the indicial equation of

$$c_2(\alpha+1)(\alpha+2) = 0$$

and recurrence relation of

$$c_{k+2} = \frac{c_{k-1}}{(\alpha+k+2)(\alpha+k+1)}.$$

Since c_2 is not the first term of the series, we are allowed to assume that $c_2 = 0$ and $\alpha = 0$. This gives chains $c_0 \to c_3 \to \cdots$ and $c_1 \to c_4 \to \cdots$ given by the recurrence relation. Therefore, we find the expressions

$$c_{3n} = c_0\left(\prod_{j=1}^{n}(3j)(3j-1)\right)^{-1}$$

$$c_{3n+1} = c_1\left(\prod_{j=1}^{n}(3j+1)(3j)\right)^{-1},$$

whose corresponding series are linearly independent.