## Introduction: naive set theory

$$\begin{aligned}\mathbb{N} &= \{1,2,3,\dots,\}\\ \mathbb{Z} &= \{0,\pm 1,\pm 2,\dots,\}\\ \mathbb{Z}_{+} &= \{0,1,2,\dots,\}\\ \mathbb{Q} &= \left\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\right\}\\ \mathbb{C} &= \{a+bi \mid a,b\in\mathbb{R}\}\\ \mathbb{C}_q &= \{a+bi \mid a,b\in\mathbb{Q}\}\end{aligned}$$

Recall: given sets $X$ and $Y$, a relation from $X$ to $Y$ is a subset of $X\times Y$, where $\times$ denotes the cartesian product of $X$ and $Y$.

A relation $f\subseteq X\times Y$ is a function from $X$ to $Y$ such that $\forall x\in X$, $\exists! y\in Y$ such that $(x,y)\in f$. We write $f(x)=y$, and denote $f$ as $f: X\to Y$.

$X$ is the **domain** of $f$ and $Y$ is the **codomain**. The range $\text{Ran}(f)=\{f(x)\mid x\in X\}\subseteq Y$.

The graph of a function $\text{Graph}(f)=\{(x,f(x))\mid x\in X\}\subseteq X\times Y$.

### Examples

$$\text{id}_x : X\to X, \text{id}_X(x)=x$$

This is the identity function.

The Characteristic Function: If $A\subseteq X$

$$\mathbb{1}_A : X\to\mathbb{R},\ \mathbb{1}_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$$

### Algebra of Functions

Let $X$ be any set, and $(X;\mathbb{R})=\{f: X\to\mathbb{R}\}$ represent the function space of $X$ with codomain $\mathbb{R}$.

Let $f,g\in\mathcal{F}(X;\mathbb{R})$. Then, $(f+g)(x)=f(x)+g(x)$, and $(f\cdot g)(x)=f(x)\cdot g(x)$.

If $t\in\mathbb{R}$, then $(tf)(x)=tf(x)$ (scalar multiplication). If $g(x)\neq 0\forall x\in X$, then $\left(\frac{f}{g}\right)(x):=\frac{f(x)}{g(x)}$.

Finally, we have composition. If $f: X\to Y$ and $g: Y\to Z$ are functions, then $g\circ f(x)=g(f(x))$.

**Injective, Subjective, and Bijective**

A function $f : X \to Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.For example, the shift map $S : \mathbb{N} \to \mathbb{N},\ S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \to \mathbb{R}$, where $I$ is any interval, is injective.

A function $f$ is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\operatorname{ran}(f) = \mathbb{R}$.

$f$ is **bijective** if it is injective and surjective.

**Invertibility**

Let $f : X \to Y$ be a function. $f$ is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. $f$ is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$.

$f$ is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \mathrm{id}_Y$ and $k \circ f = \mathrm{id}_X$.

For example, the function $\operatorname{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

**Invertibility Definition**

$f$ is invertible if and only if $f$ is left and right invertible.

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Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose $g$ is a left-inverse of $f$, and $h$ is a right-inverse of $f$. Therefore, $g \circ f = \mathrm{id}_X$, and $f \circ h = \mathrm{id}_Y$. Observe that $g = g \circ \mathrm{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$.

**Injection and Surjection Invertibility**

If $f : X \to Y$ is a function:

1. $f$ is injective $\Leftrightarrow$ $f$ is left-invertible.
2. $f$ is surjective $\Leftrightarrow$ $f$ is right-invertible.
3. $f$ is bijective $\Leftrightarrow$ $f$ is invertible.

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We will prove the first proposition in the forward direction. Suppose $f$ is injective. Given $y \in \operatorname{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define $g$ as follows:

$$g(y) = \begin{cases} x_y & y \in \operatorname{ran}(f) \\ x_0 & y \notin \operatorname{ran}(f) \end{cases}$$

Where $x_0$ is an arbitrary point in $X$. We can see that $g \circ f = \mathrm{id}_X$.

## Cardinality and Finitude

Which set is "larger," $\{1,2,3\}$ or $\{1,2,3,4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets $A$ and $B$, we can show that $A$ is the same size as $B$ by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

### Cardinality

Sets $A$ and $B$ have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\text{card}(A) = \text{card}(B)$.

### Equivalent Cardinalities of Intervals

Given $a < b$ and $c < d$, we know that $\text{card}\left([a, b]\right) = \text{card}\left([c, d]\right)$.

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We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}\left([a, b]\right) = \text{card}\left([c, d]\right)$.

### Intervals and Real Numbers

$$\text{card}\left((0,1)\right) = \text{card}(\mathbb{R})$$

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- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - $\tan$ is strictly increasing (and thus injective)
  - $\lim_{x\to\infty} \tan(x) = \infty$ and $\lim_{x\to-\infty} \tan(x) = -\infty$, and by intermediate value theorem, $\tan$ is surjective
- $\ell : (0,1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0,1) \to \mathbb{R}$.

### Finitude

A set $F$ is **finite** if $F$ is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}\left(\{1, 2, \dots, n\}\right)$. A non-finite set is called infinite.

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We can *enumerate* $F$ by creating a function $\sigma : \{1, 2, \dots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

### Inequality of Finite Sets

If $m \neq n$, then $\text{card}\{1, 2, \dots, m\} = \text{card}\{1, 2, \dots, n\}$.

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WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is our bijection. This means there are $m$ "pigeons" and $n$ "holes."

One hole, $j$, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since $f$ is assumed to be injective, this is a contradiction.

**Infinitude of the Naturals**

$\mathbb{N}$ is infinite.

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Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m+1\} \to \mathbb{N}$. $i$ is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \to \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

**Proposition**

If $A$ is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

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$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

$\vdots$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of $A$.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. $f$ is injective as $a_n$ are distinct.

**Cardinality of Integers and Natural Numbers**

$$\operatorname{card}(\mathbb{Z}) = \operatorname{card}(\mathbb{N})$$

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$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

$f$ is a bijection as $g : \mathbb{N} \to \mathbb{Z}$, $g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of $f$.

**Power Set**

Given any set $X$, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of $X$.

$2^X := \{f \mid f : X \to \{0, 1\}\}$.

**Power Set and $2^X$**

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbf{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$.

**Cantor's Theorem**

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

Fact from calculus: $\forall \sigma \in (0,1)$, $\sigma$ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \dots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0, 1, \dots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0, 1, \dots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since $r$ is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

**Comparing Cardinalities**

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

**Transitive Property**

If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

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The composition of two injective functions is injective.

**Canonical Set Comparisons**

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

**Cardinality of the Power Set**

For any set $A$, $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

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Let us construct a function: $f : A \to \mathcal{P}(A)$, where $a \mapsto \{a\}$.

$f$ is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

**Claim** $\nexists g : A \to \mathcal{P}(A)$, $g$ is surjective.

Suppose toward contradiction that such a $g$ exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since $g$ is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. $\perp$

**Equivalent Propositions**

(i) $\text{card}(A) \leq \text{card}(B)$

(ii) $\exists f : A \hookrightarrow B$

(iii) $\exists g : B \to A$, $g$ surjection.

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By definition, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) If $f : A \hookrightarrow B$, $f$ is left-invertible, and thus $\exists g : B \to A$ with $g \circ f = id_A$. So, $g$ is right-invertible, so $g$ is surjective.

(iii) $\Rightarrow$ (ii) If $g : B \to A$ is surjective, then $g$ is right-invertible, so $\exists f : A \to B$ such that $g \circ f = id_B$. So, $f$ is left-invertible, so $f$ is injective.

**Corollary**

If $f : A \to B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$.

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Consider $g : A \to f(A)$, where $g(a) = f(a)$. So, $g$ is onto, so $\exists$ an injection $f(A) \hookrightarrow A$.

More Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, q(m,n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $H(m,n) = (h(m), n)$.

**Claim** $H$ is a bijection.

**Proof of Injection** If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since $h$ is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

**Proof of Surjection** Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m,n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m,n) = (k, \ell)$. Set $n = \ell$, and since $h$ is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m,n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, $\varphi$ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned} \text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\ &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \text{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \text{card}(\mathbb{N}) \end{aligned}$$

Cardinality Rules

(i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)

(ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)

(iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)

(iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

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**Proof of (iii)** We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

**Claim** We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, $g$ and $f$ are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1$. $\perp$

We let $A_n = g \circ f(A_{n-1})$ for arbitrary $n$, and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \to B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where $y_x$ is the unique element in $B$ with $g(y_x) = x$.

**Claim** We claim $h$ is the desired bijection.

**Proof of Injection** Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of $H$, $f(x_1) = f(x_2)$, $f$ is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y(x_2)) = x_2 \notin A_\infty$. This case is not possible.

Thus, $h$ is injective.

**Proof of Surjection** Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where $y_x$ is the unique element in $B$ with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since $g$ is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

**Countability**

A set $X$ is *countable* if $\exists f : x \hookrightarrow \mathbb{N}$ $(\text{card}(X) \leq \text{card}(\mathbb{N}))$. $\text{card}(\mathbb{N}) = \aleph_0$. If $X$ is countable and infinite, $X$ is *denumerable*.

**Corollary to Cantor-Schröder-Bernstein**

If $X$ is denumerable, then $\text{card}(X) = \aleph_0$.

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Since $X$ is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since $X$ is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

**Countability under Union**

The countable union of countable sets is countable. If $I$ is a countable indexing set and for each $i \in I$, $A_i$ is countable, then $\bigcup\limits_{i \in I} A_i$ is countable.

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Since each $A_i$ is countable, $\exists \pi_i : \mathbb{N} \twoheadrightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \to \bigcup_{i \in I} A_i$$

defined as $\pi(i,j) = \pi_i(j)$.

**Claim 1** $\pi$ is a surjection.

**Proof 1** Let $x \in \bigcup_{i \in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since $\pi_{i_0}$ is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, $\pi$ is surjective.

**Claim 2** $I \times \mathbb{N}$ is countable.

**Proof 2** We know $\exists f : I \hookrightarrow \mathbb{N}$ since $I$ is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i, n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m, n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since $\pi$ is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i \in I} A_i$ is countable.

**Continuum Hypothesis**

We saw that $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(2^{\mathbb{N}})$, where $2^{\mathbb{N}}\{f \mid f : \mathbb{N} \to \{0,1\}\}$.

**Theorem** $\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}})$, where $I$ is any non-degenerate interval.

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**Lemma 1** $\text{card}([0,1]) \leq \text{card}(2^{\mathbb{N}})$.

**Proof 1** Every $t \in [0,1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0,1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\varphi} [0,1]$, defined as $\phi(f) = \sum\limits_{k=1}^{\infty} \frac{f(k)}{2^k}$. Set $f : \mathbb{N} \to \{0,1\}$, $f(k) = \sigma_k$.

Therefore, $\varphi$ is surjective, so $\exists \{0,1\} \hookrightarrow 2^{\mathbb{N}}$, so $\text{card}([0,1]) \leq 2^{\mathbb{N}}$

**Lemma 2** $\text{card}([0,1]) = \text{card}(\mathbb{R})$.

**Proof 2** We have $[0,1] \overset{i}{\hookrightarrow} \mathbb{R}$ via inclusion, so $\text{card}([0,1]) \leq \text{card}(\mathbb{R})$.

Also, $\text{card}(\mathbb{R}) = \text{card}((0,1)) \leq \text{card}([0,1])$, so by Cantor-Schröder-Bernstein, $\text{card}(\mathbb{R}) = \text{card}([0,1])$.

**Lemma 3** Any two non-degenerate intervals $I$ and $J$ have the same cardinality.

**Proof 3** We can create injections $I \hookrightarrow J$ and vice-versa.

**Lemma 4** $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0,1])$.

**Proof 4** $\psi : 2^{\mathbb{N}} \to [0,1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

$\psi$ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim $\psi$ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$, $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} + \frac{1}{3^{k_0}} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$\begin{aligned} |t_f - t_g| &= |\sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k}| \\ &\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\ &\leq \sum_{k>k_0} \frac{1}{3^k} \\ &= \frac{(1/3)^{k_0+1}}{1-(1/3)} \\ &= \frac{1}{2} \cdot \frac{1}{3^{k_0}} \end{aligned}$$

$\perp$

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0,1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than $\aleph_0$ and strictly less than $2^{\aleph_0}$ is an axiom (i.e., it can be an assumption or not).

## Ordering

Let $X$ be a non-empty set. A relation on $X$ is a subset of $X \times X$.

- $R$ is *reflexive* if $\forall x \in X,\ (x,x) \in R$.
- $R$ is *transitive* if $(x,y),(y,z) \in R \to (x,z) \in R$.
- If $R$ is *antisymmetric* $(x,y),(y,x) \in R \to x = y$.

If $R$ is reflexive, transitive, and antisymmetric, then $R$ is an *ordering* of $X$.

If $R$ is an ordering of $X$, then we write:

$$(x,y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \ \forall x \in X$
- $x \leq_R y, \ y \leq_R z \rightarrow x \leq_R z$
- $x \leq_R y, \ y \leq_R x \rightarrow x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

**Algebraic ordering of $\mathbb{N}_0$**

$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$ such that $n + k = m$

**$\mathbb{N}$ ordered via division**

$$n \leq_D m \Leftrightarrow n | m$$

Under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

**Inclusion** Let $S$ be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_i B \Leftrightarrow A \subseteq B$.

**Containment** With $X$ defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

For $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \rightarrow \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x) \ \forall x \in X$.

**Types of Orderings**

- An ordering $\leq$ of $X$ is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.
- An ordering is *directed* if $\forall x, y \in X \ \exists z \in X$ such that $x \leq z$ and $y \leq z$.

If $X$ is a totally ordered set, $X$ is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D), \ (\mathcal{P}(S), \leq_i), \ (\mathcal{P}(S), \leq_c)$$

**Upper/Lower Bounds**

(i) Let $(X, \leq)$ be an ordered set, $A \subseteq X$. $A$ is bounded above if $\exists v \in X$ with $a \leq v \ \forall a \in A$. Such a $v$ is an upper bound.

(ii) $A$ is bounded below if $\exists \ell \in X$ such that $a \geq \ell \ \forall a \in A$. Such a $w$ is a lower bound.

(iii) If $v$ is an upper bound of $A$ and $v \in A$, then $v$ is the greatest element of $A$, or $\max(A) = v$.

(iv) If $\ell$ is a lower bound for $A$ and $\ell \in A$, then $\ell$ is the least element of $A$, or $\min(A) = \ell$.

(v) If $u$ is an upper bound for $A$, and $u \leq v$ for all other upper bounds $v$ of $A$, then $u$ is the *least upper bound* of $A$, or $\sup(A) = u$ (for *supremum*).

(vi) If $\ell$ is a lower bound for $A$, and $\ell \leq g$ for all other lower bounds $g$ of $A$, then $\ell$ is the *greatest lower bound* of $A$, or $\inf(A) = \ell$ (for *infimum*).

(vii) If $A$ is bounded above and below, then $A$ is bounded.

**Well-Ordering Principle**

With $(\mathbb{N}, \leq_a)$, every nonempty $A \subseteq \mathbb{N}$ has a least element.

**Examples**

**Example 1**

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \dots, 12\}$, we have the following:

**Bounded Above?** Yes.

**Upper Bounds** $12, 13, 14, \dots$

**Greatest Element** 12

**Example 2**

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \dots, 10\}$

**Bounded Above?** Yes.

**Upper Bounds** $10!$

**Greatest Element?** No.

**Supremum** $2^3 \cdot 3^2 \cdot 5 \cdot 7$

**Bounded Below?** Yes.

**Lower Bound** 1

**Least Element?** No.

**Infimum** 1

**Example 3**

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$.

**Supremum** $\bigcup_{i \in I} A_i$

**Infimum** $\bigcap_{i \in I} A_i$

**Complete Sets**

An ordered set $(X, \leq)$ is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, $\mathbb{Q}$ is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

**Ordering of $\mathbb{Z}$**

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0,\ n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

**Properties of $\mathbb{Z}^+$**

(i) $m, n \in \mathbb{Z} \Rightarrow m+n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

**Ordering of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$**

Recall the ordering of $\mathbb{Z}$:

$$n \leq_a m \overset{\text{def}}{\Longleftrightarrow} \exists k \in \mathbb{N}_0 \text{ with } n + k = m$$

**Claim** $\leq_a$ is an ordering of $\mathbb{Z}$

We claim that $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$. Thus, $\mathbb{Z}^+ = \mathbb{N}_0$.

**Properties of $\mathbb{Z}^+$**

(i) $m, n \in \mathbb{Z} \Rightarrow m+n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

**Other Properties of $\mathbb{Z}$**

(1) $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$

(2) $m \leq_a n$ and $p \leq_a q \Rightarrow m + p \leq_a n + q$

(3) $m \leq_a n$ and $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$

(4) $m \leq_a n \Rightarrow -m\ {}_a{\geq} n$

(5) $\leq_a$ is total.

(6) If $a\ {}_a{>}-$, and $ab\ {}_a{\geq} 0$, then $b\ {}_a{>} 0$

(7) If $a > 0$ and $ab\ {}_a{\geq} ac$, then $b \geq c$.

---

**Proof of (3):**

$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0$ with $m + k = n$.
$\Rightarrow pm + pk = pn$
$pk \in \mathbb{N}_0$ by the properties of $\mathbb{Z}^+$. So, $pm \leq_a pn$

**Proof of (5):**

Let $m, n \in \mathbb{Z}$. Consider $m - n$.
By (ii), $m - n \in \mathbb{Z}^+$ or $-(m-n) \in \mathbb{Z}^+$. Thus, $m - n = k$ for some $k \in \mathbb{Z}^+$, or $-(m-n) = \ell$ for some $\ell \in \mathbb{Z}^+$.
Thus, $n \leq_a m$ in the first case, or $m \leq_a n$ in the second case.

We now want an ordering on $Q$.

**Creating the Rationals**

Recall that $Q = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z},\ b \in \mathbb{Z},\ b \neq 0\}$. Consider the equivalence relation:

$$(a, b) \sim (c, d) \overset{\text{def}}{\iff} ad = bc$$

We will let $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in Q\}$ be the set of all equivalence classes in $Q$. We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined: $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{a'd' + c'b'}{b'd'} = \frac{ad+bc}{bd}$.

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make $\mathbb{Q}$ a **field**:

**Fields**

A ring is a nonempty set set $R$ equipped with two binary operations:

- $+ : R \times R \to R$, $(a, b) \mapsto a + b$ ("addition")
- $\cdot : R \times R \to R$, $(a, b) \mapsto a \cdot b$ ("multiplication")

such that the following hold:

(1) $(a + b) + c = a + (b + c)$

(2) $\exists z \in R$ such that $a + z = a = z + a\ \forall a \in R$; there is at most one such $z$. Set $z = 0_R$.

(3) $\forall a \in R, \exists b \in R$ such that $a + b = 0_R = b + a$; there is at most one such $b$. Set $b = -a$.

(4) $\forall a, b \in R,\ a + b = b + a$.

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If $(R, +, \cdot)$ satisfies $ab = ba$, $R$ is a commutative ring.

If there exists $u \in R$ such that $ua = au = a\ \forall a \in R$, $R$ is a unital ring; there is at most one unit. Set $u = 1_R$

An integral domain is a unital, commutative ring such that $ab = 0 \Rightarrow a = 0 \vee b = 0$. For example, $\mathbb{Z}$ is an integral domain. However, $c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ is a unital, commutative ring, but there exist two functions such that $f, g \neq \mathbf{0}$, but $f \cdot g = \mathbf{0}$.

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R,\ \text{with } ab = 1_R$$

There is only one such $b$. Set $b = a^{-1}$.

**Proof that $\mathbb{Q}$ is a Field:**

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Provided that $\frac{a}{b} \neq 0_{\mathbb{Q}}$.

Additionally, $\mathbb{Z} \overset{j}{\hookrightarrow} \mathbb{Q}$, $j(n) = \frac{n}{1}$ is injective.

**Ordering of $\mathbb{Q}$**

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

Prove that this ordering is well-defined.

**Order Embedding**

$\leq$ is a well-defined total ordering of $\mathbb{Q}$, and $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$, $j(n) = \frac{n}{1}$ is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

Properties of $\mathbb{Q}^+$

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

(i) $q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+,\ q_1 q_2 \in \mathbb{Q}^+$

(ii) $q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$

(iii) $\pm q \in \mathbb{Q}^+, q = 0$

(iv) $x \leq y, !u \leq v \Rightarrow x + u \leq y + v$

(v) $x \leq y,\ 0 \leq z \Rightarrow zx \leq zy$

Ordering of $\mathbb{R}$, cont'd

An **ordered field** is a field $F$ equipped with a total ordering $\leq_F$ such that:

(i) if $s \leq_F t$, and $x \leq_F y$, then $s + x \leq_F t + y$

(ii) if $s \leq_F t$ and $0 \leq_F z$, then $zs \leq_F zt$

For example, $\mathbb{Q}$ with its ordering is an ordered field.

**Proposition 1:** If $(F, \leq_F)$ is an ordered field, we define $F^+ = \{x \in F, x {}_F{\geq} 0\}$ with the following properties:

(1) $x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$

(2) $x \in F \Rightarrow x \in F^+, -x \in F^+$

(3) $\pm x \in F^+ \Rightarrow x = 0_F$

Proofs

(1) Let $x, y \in F^+$. Then, $x \geq 0$ and $y \geq 0$, so by property $(i)$ of an ordered field, $x + y \geq 0 + 0 = 0$, so $x + y \in F^+$. Additionally, we have $x \cdot y \geq x \cdot 0 = 0$, so $xy \in F^+$.

(2) Let $x \in F$. Since the ordering on $F$ is total, $x \geq 0$ or $0 \geq x$. In the first case, $x \in F^+$. In the second case, we add $-x$ to both sides, so by $(i)$, $-x \geq 0$, so $-x \in F^+$.

(3) We have $x \geq 0$ and $-x \geq 0$. So $x \geq 0$ and $x + (-x) \geq x + 0$, so $x \geq 0$ and $0 \geq x$. So, $x = 0$ by antisymmetry.

**Note:** $x \leq_F y \Leftrightarrow y - x \in F^+$.

**Proposition 2:** Let $F$ be an ordered field. Then, the following is true:

(1) $\forall a \in F,\ a^2 \in F^+$

(2) $0, 1 \in F^+$

(3) If $n \in \mathbb{N}$, $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$

(4) If $x \in F^+$, and $x \neq 0$, then $x^{-1} \in F^+$

(5) If $xy > 0$, then $x, y \in F^+$, or $-x, -y \in F^+$

(6) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$

(7) If $x \leq y$, then $-y \leq -x$

(8) $x \geq 1 \Rightarrow x^2 \geq x \geq 1$, and $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$.

Proofs

(1) Let $a \in F$. Then, $a \in F^+$ or $-a \in F^+$.

Case 1 If $a \in F^+$, then by the previous proposition, $a^2 \in F^+$.

Case 2 If $-a \in F^+$, then by the previous proposition, $(-a)(-a) = a^2 \in F^+$.

(2) $0 \geq 0$, so $0 \in F+$. $1 = 1 \cdot 1 = 1^2 \in F^+$ by the previous result.

(3) $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots 1_F}_{n \text{ times}} \in F^+$ by the previous proposition.

(4) Let $x \neq 0, x \in F^+$. Suppose toward contradiction that $x^{-1} \notin F^+$, then $-x^{-1} \in F^+$. Thus, $x \cdot (-x^{-1}) \in F^+$, so $-1 \in F^+$, but $1 \in F^+$, so $1 = 0$. $\perp$

(5) Let $xy > 0$, meaning $xy \in F^+$. Suppose toward contradiction that $x > 0$ and $y < 0$. So, $x > 0$ and $-y > 0$, so $(x)(-y) > 0$, so $-(xy) \in F^+0$, so $xy = 0$. $\perp$

(6) Let $0 < x \leq y$. We know $x^{-1} \in F^+$, so $x^{-1}x \leq x^{-1}y$. So $1 \leq x^{-1}y$. We also know $y \in F^+$, so $y^{-1} \in F^+$. So, $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$.

(7) Let $x \leq y$. Then, $0 \leq y - x$, so $-y \leq -x$.

(8) Let $x \geq 1$. Then, $x \cdot x \geq 1 \cdot x \geq 1$.

Order Axiom

$\mathbb{R}$ is an ordered field. The injection $\mathbb{Q} \hookrightarrow \mathbb{R}$, $i(q) = q$ is an order embedding.

Rational Orderings

**Proposition 1:** If $a \leq b$, then $a \leq \frac{1}{2}(a + b) \leq b$

Proof

$2a = a + a \leq a + b \leq b + b$, all by property (i) of an ordered field.

Therefore, $2a \leq a + b \leq 2b$. Since $2 = 1 + 1 \in \mathbb{R}^+$, $2^{-1} \in \mathbb{R}^+$, so $(2a)/2 \leq \frac{1}{2}(a + b) \leq (2b)/2$, so $a \leq \frac{1}{2}(a + b) \leq b$.

**Proposition 2:** If $a \geq 0$ and $(\forall \varepsilon > 0), a \leq \varepsilon$.

Proof

If $a \geq 0$ and $a \neq 0$, then $a > 0$. So, we have that $\frac{1}{2}a < a$. Let $\varepsilon = \frac{1}{2}a$. We also have that $a \leq \varepsilon = \frac{1}{2}a < a$, so $a < a$. $\perp$

Arithmetic and Geometric Means

Given $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$:

**Arithmetic Mean**

$$= \frac{\sum_{i=1}^n a_i}{m}$$

**Geometric Mean**

$$= \sqrt[m]{a_1 a_2 \cdots a_m}$$

Arithmetic Mean-Geometric Mean Inequality

Let $a, b \geq 0$.

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

---

If $x, y \geq 0$, $x \leq y \Leftrightarrow x^2 \leq y^2$.

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y \qquad \text{by property (ii) of ordered fields}$$

Therefore,

$$\begin{aligned}
(ab)^{1/2} &\leq \frac{1}{2}(a+b) \\
ab &\leq \frac{1}{4}(a^2 + 2ab + b^2) \\
4ab &\leq a^2 + 2ab + b^2 \\
0 &\leq a^2 - 2ab + b^2 \\
0 &\leq (a-b)^2 && \text{by definition}
\end{aligned}$$

**Challenge:** Prove for $m$.

**Remark:** The harmonic mean is defined as:

$$\frac{n}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i}}$$

Bernoulli's Inequality

If $x \geq -1$, then $(1+x)^n \geq 1 + nx$, for any $n \in \mathbb{N}_0$

---

By induction, we know that for $n = 0$ and $n = 1$, this holds.

Assume the inequality holds for some $m \geq 1$.

$$\begin{aligned}(1+x)^{m+1} &= (1+x)^m(1+x) \\ &\geq (1+mx)(1+x) && \text{by the inductive hypothesis} \\ &= 1 + x + mx + mx^2 \\ &= 1 + (m+1)x + mx^2 \\ &\geq 1 + (m+1)x\end{aligned}$$

**Cauchy's Inequality**

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. Then

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

In linear algebra language, this is equivalent to $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$.

---

Consider $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^{n} (a_j - b_j x)^2$$

We know that $f(x) \geq 0$ for all $x \in \mathbb{R}$

$$\begin{aligned} &= \sum_{i=1}^{n} (a_j^2 - 2a_j b_j x + b_j^2 x^2) \\ &= \left(\sum_{j=1}^{n} b_j^2\right) x^2 + \left(\sum_{j=1}^{n} -2a_j b_j\right) x + \sum_{j=1}^{n} a_j^2 \\ &= Ax^2 + Bx + C\end{aligned}$$

Therefore, $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned}\left(-2\sum_{j=1}^{n} a_j b_j\right)^2 &\leq 4\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right) \\ \left|\sum_{j=1}^{n} a_j b_j\right| &= \left(\sum_{j=1}^{n} a_j\right)^{1/2} \left(\sum_{j=1}^{n} b_j\right)^{1/2}\end{aligned}$$

As we know from linear algebra, the way we get equality is when $\vec{v} = c\vec{w}$, or that $a_j = cb_j\ \forall j$ for some $c \in \mathbb{R}$.

**Triangle Inequality**

Given $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left(\sum_{j=1}^{n} (a_j + b_j)^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

In linear algebra, this is equivalent to $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

$$\begin{aligned}
\sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\
&\leq \sum a_j^2 + 2\left(\sum a_j^2\right)^{1/2}\left(\sum b_j^2\right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\
&= \left(\left(\sum a_j^2\right)^{1/2} + \left(\sum b_j^2\right)^{1/2}\right)^2
\end{aligned}$$

we take square roots to get our end result

**Metrics and Norms on $\mathbb{R}^n$**

Consider $|\cdot| : \mathbb{R} \to \mathbb{R}$, defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

**Theorems about Absolute Value:**

(i) $|ab| = |a||b|$

(ii) $|a^2| = |a|^2$

(iii) $|-a| = |a|$

(iv) $|a| \in \mathbb{R}^+$

(v) $-|a| \leq a \leq |a|$

(vi) $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$ for $\delta > 0$

(vii) $|a+b| \leq |a| + |b|$, $|a-b| \leq |a| + |b|$, $||a| - |b|| \leq |a-b|$

Proofs

Proof of (i)

**Case 1:** If $a, b \in \mathbb{R}^+$, then $|a| = a$, and $|b| = b$, and $ab \in \mathbb{R}^+$, so $|ab| = ab$

**Case 2:** If $a, b \notin \mathbb{R}^+$, then $|a| = -a$, and $|b| = -b$. Additionally, $(-a)(-b) = ab \in \mathbb{R}^+$, so $|ab| = ab$. The LHS $= ab$, and the RHS $= ab$.

**Case 3:** $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$. Then, $|a||b| = (a)(-b) = -ab$. Then, since $a(-b) \in \mathbb{R}^+$, $-ab \in \mathbb{R}^+$, so $|ab| = -ab$. Therefore, the LHS and RHS are equal.

Proof of (vii) Having established that $|a+b| \leq |a| + |b|$, we will show that $||a| - |b|| \leq |a - b|$.

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a-b| + |b| \\ |a| - |b| &\leq |a - b| \end{aligned}$$

Similarly, by exchanging $a$ for $b$

$$\begin{aligned} |b| - |a| &\leq |b - a| \\ |b| - |a| &\leq |a - b| \end{aligned}$$

Let $t = |a| - |b|$. We have shown that

$$\begin{aligned} \pm t &\leq |a - b| \\ -|a-b| \leq t &\leq |a - b| \\ |t| &\leq |a - b| \end{aligned}$$

Absolute Values, cont'd

Recall:

$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

If we want to find all $x \in \mathbb{R}$ such that $|x - 1| \leq |x|$, we would split up into cases:

$x \leq 0$ $x - 1 \leq -1$, so $|x| = -x$ and $|x-1| = 1 - x$, so $1 - x \leq -x$, so $0 \geq 1$. $\perp$

$0 < x \leq 1$ $|x| = x$ and $|x - 1| = 1 - x$, so $1 - x \leq x$, so $x \geq \frac{1}{2}$, so $\frac{1}{2} \leq x \leq 1$.

$1 < x$ $|x| = x$ and $|x-1| = x - 1$, so $x - 1 \leq x$, so $-1 \leq 0$, which is true $\forall \mathbb{R}$ in the interval, so $x > 1$.

Therefore, we have $x \in \left(\frac{1}{2}, \infty\right)$ as that which satisfies this inequality.

Bounded Sets

A subset $A \subseteq \mathbb{R}$ is **bounded** $\Leftrightarrow$ $\exists c \geq 0$ such that $\forall x \in A$, $|x| \leq c$.

$(\Rightarrow)$ Suppose $A \subseteq \mathbb{R}$ is bounded. Then, $\exists \ell, u \in \mathbb{R}$ such that $\ell \leq x \leq u$ $\forall x \in A$. Let $c := \max\{|\ell|, |u|\}$.

Since $|u| \leq c$, we have that $x \leq c$.

Since $|\ell| \leq c$, and $-|\ell| \leq x$, we get that $-x \leq |\ell| \leq c$.

Since $x \leq c$ and $-x \leq c$, $|x| \leq c$.

($\Leftarrow$) If such a $c$ exists, then $|x| \leq c$ if and only if $-c \leq x \leq c$. Thus, $-c$ is the lower bound and $c$ is the upper bound.

**Bounded Functions**

Let $D$ be any set. A function $f : D \to \mathbb{R}$ is bounded if $\text{Ran}(D) \subseteq \mathbb{R}$ is bounded.

**Example**

Let $f : [3,7] \to \mathbb{R}$, $f(x) = \frac{x^2+2x+1}{x-1}$. Show that $f$ is bounded.

---

$3 \leq x \leq 7 \Rightarrow 2 \leq x - 1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}$.

Also, $4 \leq x+1 \leq 8 \Rightarrow 16 \leq x^2 + 2x + 1 \leq 64 \Rightarrow |x^2 + 2x + 1| \leq 64$.

So, $|f(x)| \leq 32$.

**Distance Metrics**

For $s, t \in \mathbb{R}$, we will define $d(s,t) = |s-t|$ to be the **distance** between $s$ and $t$.

**Properties:**

(i)

$$\begin{aligned} d :& \mathbb{R} \times \mathbb{R} \to [0, \infty) \\ &(s,t) \mapsto d(s,t) \geq 0 \end{aligned}$$

(ii) $d(s,t) = d(t,s)$

(iii) $d(s,r) \leq d(s,t) + d(t,r)$

(iv) $d(s,s) = 0$

(v) If $d(s,t) = 0$, then $s = t$.

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^{n} |x_j|$$

- $\infty$-norm:

$$\|v\|_\infty = \max_{j=1}^{n} |x_j|$$

- 2-norm:

$$\|v\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

## Properties of the Norms

**Properties:** With $v, w$ above, let $p = 1, 2, \infty$. The following are true:

(1) $\|v\|_p \geq 0$
(2) $\|v + w\|_p \leq \|v\|_p + \|w\| + p$
(3) $\|\vec{0}\|_p = 0$
(4) $\|v\|_p = 0 \Rightarrow v = \vec{0}$
(5) $\forall t \in \mathbb{R},\ \|tv\|_p = |t|\|v\|_p$

### Proofs

Let $p = \infty$. We will prove (2).

Say $\|v\|_{infty} = |x_i|$ and $\|w\|_\infty = |y_k|$. We want to show that $\|v + w\|_\infty = \max_{j=1}^{n} |x_j + y_j| \leq |x_i| + |y_k|$.

Note that $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| && \text{Triangle Inequality} \\ &\leq |x_i| + |y_k| \\ &= \|v\|_\infty + \|w\|_\infty \end{aligned}$$

Therefore, $\|v + w\|_\infty \leq \|v\|_\infty + \|w\|_\infty$.

## Distances and Norms

A **norm** on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$, $v \mapsto \|v\|$, satisfying the following properties for $v \in \mathbb{R}^n$:

(1) $\|v\| \geq 0$

(2) $\|v + w\| \leq \|v\| + \|w\|$

(3) $\|\vec{0}\| = 0$

(4) $\|v\| = 0 \Rightarrow v = \vec{0}$

(5) $\forall t \in \mathbb{R},\ \|tv\| = |t|\|v\|$

If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$ is a norm, we define $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$, defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for $v, w \in \mathbb{R}^n$.

The properties of distance in $\mathbb{R}$ still hold for distance in $\mathbb{R}^n$:

(1) $d(v, w) = d(w, v)$

(2) $d(u, w) \leq d(u, v) + d(v, w)$

(3) $d(v, v) = 0$

(4) $d(v, w) = 0 \Rightarrow v = w$

**Metric Spaces**

A **metric space** is a nonempty set $X$ equipped with a function $d : X \times X \to \mathbb{R}^+$, $(x, y) \mapsto d(x, y) \geq 0$. The metric has the following properties:

(1) $d(x, y) = d(y, x)\ \forall x, y \in X$

(2) $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$

(3) $d(x, x) = 0$

(4) $d(x, y) = 0 \Leftrightarrow x = y$

The map $d$ is called a *metric* on $X$.

**Metric Spaces, Open Sets, and Closed Sets**

Examples of Metric Spaces:

- $\mathbb{R}$ with $d(x, y) = |x - y|$.
- $\mathbb{R}^n$ with the *Euclidean metric*:

$$\begin{aligned} d_2(v, w) &= \|v - w\|_2 \\ &= \left(\sum_{j=1}^{n}(x_j - y_j)^2\right)^{1/2} \end{aligned}$$

- $\mathbb{R}^n$ with the 1-norm:

$$\begin{aligned} d_1(v, w) &= \|v - w\|_1 \\ &= \sum_{j=1}^{n}|x_j - y_j| \end{aligned}$$

- $\mathbb{R}^n$ with the $\infty$-norm:

$$\begin{aligned} d_\infty(v, w) &= \|v - w\|_\infty \\ &= \max_{j=1}^{n}|x_j - y_j| \end{aligned}$$

Let $(X, d)$ be a metric space.

(1) The **open ball** centered at $x_0 \in X$ with radius $\delta$ is:

$$U(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

(2) The **closed ball** centered at $x_0 \in X$ with radius $\delta$ is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

(3) A set $U \subseteq X$ is **open** if $\forall x \in U$, $\exists \delta > 0$ such that $U(x, \delta) \subseteq U$.

(4) A set $C \subseteq X$ is **closed** if $\overline{C} = X - C \subseteq X$ is open.

**Examples**

In $\mathbb{R}$ with $d(s,t) = |s-t|$:

$$\begin{aligned} U(x_0,\delta) &= \{y \in \mathbb{R} \mid d(y,x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0,\delta) &= [x_0, \delta, x_0 + \delta] \end{aligned}$$

The interval $A = [1,\infty)$ is not open, as $\forall \delta > 0$, $U(1,\delta) \not\subseteq [1,\infty)$.

However, $A$ is closed, as $\overline{A} = (-\infty, 1)$ is open: given $t \in \overline{A}$, choose $\delta = 1 - t$. Let $s \in V_\delta(t)$. Then, $s \in (t-\delta, t+\delta)$, so $s \in (t - (1-t), t + (1-t))$, or $s \in (2t-1, 1)$, so $s < 1$.

**Exercises**

Show that the following are open:

- $(a,b)$
- $(a,\infty)$
- $(-\infty, b)$

and that the following are closed:

- $[a,b]$
- $[a,\infty)$
- $(-\infty, b]$

In $(\mathbb{R}^2, d_2)$, $B(0_{\mathbb{R}^2}, 1)$ is the **unit disc** centered at $(0,0)$.

However, in $(\mathbb{R}^2, d_\infty)$:

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

**Finding a Supremum**

Let $0 \neq A \subseteq \mathbb{R}$. Let $u \in \mathbb{R}$ be an upper bound for $A$. The following are equivalent:

(i) $u = \sup(A)$

(ii) If $t < u$, then $\exists a_t \in A$ such that $a_t > t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $u - \varepsilon < a_\varepsilon$

**Proofs**

(i) $\Rightarrow$ (ii): Given $t < u$, if no such $a \in A$ with $t < a$ exists, then $a \leq t\ \forall a \in A$. Thus $t$ would be an upper bound. However, $t < u$ and $u$ is the supremum of $A$. $\perp$

(ii) $\Rightarrow$ (iii): Given $\varepsilon > 0$, set $t = u - \varepsilon < u$. So, by (ii), $\exists a_t$ with $t < a_t$. Thus, $u - \varepsilon \leq a_t$. Set $a_\varepsilon = a_t$.

(iii) $\Rightarrow$ (i): Let $v$ be an upper bound for $A$. Suppose $v < u$. Then, set $\varepsilon = u - v > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u - \varepsilon < a_\varepsilon$. So $u - (u - v) < a_\varepsilon$, so $v < a_\varepsilon$, meaning $v$ cannot be an upper bound. $\perp$

**Supremum Example**

$\sup[0,1) = 1$**:** Certainly, 1 is an upper bound for $[0,1)$. Let $\varepsilon > 0$.

If $\varepsilon \geq 1$, pick $t = \frac{1}{2}$. Then, $1 - \varepsilon < 0 < \frac{1}{2}$

If $0 < \varepsilon < 1$, let $t = (1-\varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$. Then, $t \in [0,1)$, and $1 - \varepsilon < 1 - \varepsilon/2 = t$

**Finding an Infimum**

Let $\emptyset \neq A \subseteq \mathbb{R}$. Let $\ell \in \mathbb{R}$ be a lower bound for $A$. The following are equivalent:

(i) $\ell = \inf(A)$

(ii) If $t > \ell$, $\exists a_t$ such that $t > a_t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $\ell + \varepsilon > a_\varepsilon$

**Infimum Example**

$\inf\left\{\frac{1}{n} \mid n \geq 1\right\}$ : Clearly, $0 < \frac{1}{n}\ \forall n \geq 1$. Let $\varepsilon > 0$.

We need to find $a \in \left\{\frac{1}{n} \mid n \geq 1\right\}$ with $\varepsilon > a$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Let $a_\varepsilon = \frac{1}{m}$.

**More on Supremum/Infimum**

- If $A \subseteq \mathbb{R}$ and $\max(A) = u$, then $u = \sup(A)$: $u$ is an upper bound of $A$ by the definition of max, and if $v \neq u$ is any upper bound of $A$, then $u < v$ since $u \in A$.
- If $\min(A) = \ell$, then $\ell = \inf(A)$ (by the same logic).
- If $A$ is not bounded above, $\sup(A) = +\infty$, and if $A$ is not bounded below, then $\inf(A) = -\infty$.
- If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.
- If $A \subseteq B$, then $\inf(A) \geq \inf(B)$: Let $\ell_A = \inf(A)$ and $\ell_B = \inf(B)$. By definition, $\ell_B \leq b\ \forall b \in B$. Since $A \subseteq B$, $\ell_B \leq a\ \forall a \in A$. Thus, $\ell_B$ is a lower bound for $A$. By definition of $\ell_A$, $\ell_B \leq \ell_A$.

Let $A, B \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Then, the following are also sets:

(1) $A + B = \{a + b \mid a \in A, b \in B\}$

(2) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$

(3) $t \cdot A = \{ta \mid a \in A\}$

(4) $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

**Completeness Axiom**

If $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above, then $\sup(A)$ exists.

Well-Ordering Property: if $\emptyset \neq S \subseteq \mathbb{N}$, then $\min(S)$ exists.

Therefore, we can prove that if $F \subseteq \mathbb{Z}$ is bounded, then $F$ has a least and greatest element.

**Archimedean Property: Proof**

If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x \leq n_x$.

Suppose there exists no natural number greater than $x$, then $\mathbb{N}$ is bounded above by $X$. Let $u = \sup(\mathbb{N})$. By the Completeness Axiom, $u \in \mathbb{R}$ exists. Let $\varepsilon = 1$. Then, $\exists n \in \mathbb{N}$ with $u - 1 < n$. So, $u < n + 1$, but $n + 1 \in \mathbb{N}$, so $u$ cannot be an upper bound.

**Corollary to the Archimedean Property**

$$\forall t > 0 \ \exists n \in \mathbb{N} \ni \frac{1}{n} < t$$

**Corollary: Powers of 2**

$$\forall t > 0 \ \exists m \in \mathbb{N} \ni \frac{1}{2^m} < t$$

By the corollary to the Archimedean Property, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < t$. By Bernoulli's inequality with $x = 1$, we have $2^n \geq n$, so $2^{-n} < n^{-1} < t$.

**Corollary: In Between Integers**

$$\forall x \in \mathbb{R} \ \exists n_x \in \mathbb{Z} \ni n_x - 1 \leq x < n_x$$

Assume $x \geq 0$. Let $S_x = \{n \mid n \in \mathbb{N} \ x < n\}$.

$S_x \neq \emptyset$ by the Archimedean Property. By the well-ordering property, let $n_x = \min(S_x)$.

Therefore, $x < n_x$. Also, $n_x - 1 \notin S_x$. Therefore $n_x - 1 \leq x$.

### Density Theorems

Let $(X, d)$ be any metric space. A subset $D \subseteq X$ is **dense** if $\forall x \in X,\ \varepsilon > 0$, $U(x, \varepsilon) \cap D \neq \emptyset$.

In the case of $X = \mathbb{R}$ and $d(s, t) = |s - t|$, $D \subseteq \mathbb{R}$ is dense if given any open interval $I$, $I \cap D \neq \emptyset$.

A metric space is **separable** if it admits a *countable* dense subset.

#### Density of the Rationals

$\mathbb{Q} \subseteq \mathbb{R}$ is dense.

---

Let $I = (a, b)$ be an open interval. We may assume that $a, b \in \mathbb{R}$ are finite.

Then, $b - a > 0$. By the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, meaning $1 < nb - na$.

There exists also an integer $m$ such that $m - 1 \leq na < m$, implying that $a\frac{m}{n}$. Also, $m \leq na + 1 < nb$. Therefore, $\frac{m}{n} < b$.

So, $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$.

#### Density of the Irrationals

$\mathbb{R} \setminus \mathbb{Q}$ is dense.

---

Assume $\sqrt{2}$ exists. Let $I = (a, b)$ be any open interval. Then, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

Find $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$.

Then, $a < q\sqrt{2} < b$, where $q\sqrt{2} \in \mathbb{R}$ and $q\sqrt{2} \notin \mathbb{Q}$.

**Uniqueness of $\sqrt{2}$**

$$\exists ! x > 0 \; x^2 = 2$$

Existence: Let $S = \{t \in \mathbb{R} \mid 0 < t,\ t^2 < 2\}$. $S$ is nonempty because $1 \in \S$, and $S$ is bounded above because $y > 2 \Rightarrow y^2 > 4$.

So, by the completeness axiom, $x := \sup(S)$ exists, and $x \geq 1$.

Claim: $x^2 = 2$

Contradiction 1: Assume $x^2 < 2$. We want to show that $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$. By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x+1) \end{aligned}$$

We want to find $n \in \mathbb{N}$ with:

$$x^2 + \frac{1}{n}(2x+1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x+1}$$

Therefore, by the Archimedean Property corollary, we know that $n$ must exist.

Contradiction 2: We know that $x^2 \geq 2$. Since $x = \sup(S)$, $\forall m \in \mathbb{N}$, $\exists t_m \in S$ with $x - \frac{1}{m} < t_m$, so $\left(x - \frac{1}{m}\right)^2 < t_m^2 < 2$.

Therefore, $x^2 - \frac{2x}{m} + \frac{1}{m^2}$, so $x^2 - \frac{2x}{m} < 2$, so $0 \leq x^2 - 2 < \frac{2x}{m}$.

So, $0 \leq \frac{x^2-2}{2x} < \frac{1}{m}$, so $x^2 - 2 = 0$, so $x^2 = 2$.

**Remark:** If we had set $S' = \{t' \in \mathbb{Q} \mid t^2 < 2,\ t > 0\}$, we would have still obtained $\sup(S') = \sqrt{2}$. This means that $\mathbb{Q}$ is *not* complete.

**Intervals and Nested Intervals**

(*) Given any interval $I$, if $x_1, x_2 \in I$, with $x_1 < x_2$, then $[x_1, x_2] \in I$.

This seems like an obvious property, but this is the *characteristic property* of intervals.

Characterization of Intervals

Let $S \in \mathbb{R}$ be any nonempty subset of cardinality at least 2. Suppose $S$ satisfies (*). Then, $S$ is an interval.

**Case 1:** Suppose $S$ is bounded.

Let $a = \inf(S)$ and $b = \sup(S)$. Then, $S \subseteq [a, b]$. We will show that $(a, b) \subseteq S$. Once this is shown, $S = \{(a, b), [a, b], [a, b), (a, b]\}$.

Let $t \in (a, b)$. Since $a = \inf(S)$, $\exists x_1 \in S,\ x_1 \in (a, t)$. Similarly, since $b = \sup(S)$, $\exists x_2 \in S,\ x_1 \in (t, b)$.

By the hypothesis, $[x_1, x_2] \subseteq S$. Since $t \in [x_1, x_2]$, $t \in S$.

**Case 2:** Suppose $S$ is bounded above, but not below.

Let $b = \sup(S)$. Clearly, $S \subseteq (-\infty, b]$. We will show that $(-\infty, b) \subseteq S$. Once this is shown, $S = \{(-\infty, b), (-\infty, b]\}$.

Let $t \in (-\infty, b)$. Since $b = \sup(S)$, $\exists x_2 \in S,\ x_2 \in (t, b)$.

Since $S$ is not bounded below, $\exists x_1 \in S$ such that $x_1 < t$ (or else $t$ would be a lower bound).

By the hypothesis, $[x_1, x_2] \in S$, and $t \in [x_1, x_2]$, so $t \in S$.

**Case 3, 4:** Left as an exercise for the reader.

A sequence of intervals $(I_n)_{n\geq 1}$ is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in $\bigcap I_n$.

(a) $\bigcap_{n=1}[0, 1/n) = \{0\}$.

(b) $\bigcap_{n=1}(0, 1/n) = \emptyset$

(c) $\bigcap_{n=1}[n, \infty) = \emptyset$

Measure

The **measure** of an interval is basically its "size."

(a) $m([a, b]) = b - a$

(b) $m((a, b]) = b - a$

(c) $m((a, b)) = b - a$

(d) $m([a, b)) = b - a$

**Nested Intervals Theorem**

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested sequence of intervals.

(1) $\bigcap_{n\geq 1} I_n \neq \emptyset$

(2) If $\inf\{m(I_n) \mid n \geq 1\} = 0$, then $\bigcap_{n\geq 1} I_n = \{\xi\}$

---

**(a)**

Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, we have that $a_1 \leq a_2 \leq a_3, \dots$, and $b_1 \geq b_2 \geq b_3 \geq \cdots$.

We know that $\{a_n\}$ is bounded above and nonempty. Let $\xi = \sup\left(\{a_n\}_{n=1}^{\infty}\right)$.

We know that $\{b_n\}$ is bounded below. Let $\eta = \inf\left(\{b_n\}_{n=1}^{\infty}\right)$.

We claim that $\xi \leq b_n\ \forall n \geq 1$. Suppose toward contradiction that $\exists m$ such that $\xi > b_m$. Then, by the supremum property, $\exists a_i$ such that $\xi > a_i > b_m$. If $k \leq m$, $a_k \leq a_m \leq b_m < a_k$. If $m \leq k$, then $b_m \geq b_k \geq a_k > b_m$. $\perp$

Similarly, using the same argument, $a_n \leq \eta\ \forall n$.

Thus, $\xi \leq \eta$.

We claim that $\bigcap_{n\geq 1} I_n = [\xi, \eta]$. If $t \in [\xi, \eta]$, then $a_n \leq \xi \leq t \leq \eta \leq b_n$. So $t \in [a_n, b_n]\ \forall n$, so $t \in \bigcap_{n\geq 1}[a_n, b_n]$.

If $t \in \bigcap_{n\geq 1} I_n$, then $t \in [a_n, b_n]\ \forall n$, so $a_n \leq t \leq b_n\ \forall n$. So, $t$ is an upper bound on $a_n$, and a lower bound on $b_n$. So, $\xi \leq t \leq \eta$ by definition of $\xi$ and $\eta$.

**(b)**

We have $\forall n \geq 1$

$$\begin{aligned}
[\xi, \eta] &\subseteq [a_n, b_n] \\
\Rightarrow 0 \leq \eta - \xi &\leq b_n - a_n \\
&= m(I_n)
\end{aligned}$$

So, if $\inf\left(\{m(I_n) \mid n \geq 1\right) = 0$, then $0 \leq \eta - \xi \leq 0$, so $\eta = \xi$.

**Corollary to the Nested Intervals Theorem**

$[0, 1]$ is uncountable.

---

Suppose it is possible to denumerate the interval $[0, 1] = \{t_1, t_2, \dots, \}$.

We can find $[a_1, b_1] \subseteq [0, 1]$ with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$.

Then, we find $[a_2, b_2] \in [a_1, b_1]$ with $a_2 < b_2$, $t_2 \notin [a_2, b_2]$.

Recursively, we find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $a_n < b_n$, $t_n \notin [a_n, b_n]$.

So, $I_n = ([a_n, b_n])_0^\infty$ is a sequence of nested intervals.

Therefore, $\exists \xi \in \bigcap I_n \subseteq [0,1]$. However, $\xi \notin \{t_1, t_2, \dots\}$. $\perp$

## Sequences in Metric Spaces

A sequence in a metric space $M$ is a map

$$x : \mathbb{N} \to M \qquad M = \mathbb{R}, \text{ usually}$$
$$x = (x_n)_{n=1}^\infty$$

**I.** Sequences defined by a formula:

(i) $x_n = t$ (the constant sequence)
(ii) $x_n = 2n+1$
(iii) $x_n = \frac{1}{n-1}$ (here, $n \geq 2$)
(iv) $c_n = n \sin\left(\frac{1}{n}\right)$
(v) $d_n = \left(1 + \frac{1}{n}\right)^n$
(vi) Geometric *Sequence*: for $b \neq 0$, $(b^n)_{n\geq 0} = (1, b, b^2, \dots)$
(vii) $x_n = \frac{n!}{n^n}$
(viii) Given any function

$$f : [0,\infty) \to \mathbb{R}$$

we can set $x_n = f(n)$.

**II.** Sequences defined recursively:

(i) $a_1 = 1,\ a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
(ii) Fibonacci: $f_1 = 1,\ f_2 = 1,\ f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$. The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (1-\varphi)^n\right)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$
(iii) Let $f : M \to M$ be a self-map on a metric space. Fix $x_0 \in M$.

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

**III.** New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences, $t \in \mathbb{R}$. Then, we can do the following:
- $(a_n)_n + (b_n)_n + (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n(b_n)_n = (a_nb_n)_n$
- If $b_n \neq 0\ \forall n$, $\left(\frac{a_n}{b_n}\right)$

(ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

(iii) We can look at ratios for $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

(iv) We can look at partial sums, given $(a_k)_{k=1}^{\infty}$.

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^{n} a_k \end{aligned}$$

The sequence $(s_n)_n$ is called the sequence of partial sums. For example, the sequence of partial sums for $(b^k)_{k=0}^{\infty}$ is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for $b \neq 1$, $(1-b)(1+b+b^2+\cdots+b^n) = 1 - b^{n+1}$

Exercise

Let $a_k = \frac{1}{k(k+1)}$. Find $(s_n)_n$.

---

Via partial fraction decomposition, we get that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore, $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^{\infty}$

Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R},\ a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k| \qquad \text{Infinity Norm}$$

This norm has the traditional properties of the norm:

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty && \text{Triangle Inequality} \\ \|t(a_k)_k\|_\infty &= |t|\|(a_k)_k\|_\infty && \text{Scalar Multiplication} \\ \|(a_k)_k\|_\infty = 0 &\Leftrightarrow a_k = 0\ \forall k && \text{Zero Property} \\ \|(a_k)_k(b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty\|(b_k)_k\|_\infty && \text{Multiplication} \end{aligned}$$

**Proof**

Let $u = \|(a_k)_k\|_\infty$ and $v = \|(b_k)_k\|_\infty$.

Given any $k$,

$$\begin{aligned} |a_k + b_k| &\leq |a_k| + |b_k| && \text{Triangle Inequality on } |\cdot| \\ &\leq u + v && \text{definition of supremum} \\ \Rightarrow \sup_{k\geq 1} |a_k + b_k| &\leq u + v \end{aligned}$$

Thus,

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &= \|\left((a_k + b_k)_k\right)_k\|_\infty \\ &= \sup_{k\geq 1} |a_k + b_k| \\ &\leq u + v \end{aligned}$$

**Monotonicity**

A sequence $(x_n)_n$ is **increasing** if

$$x_1 \leq x_2 \leq \cdots \ \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \cdots \ \forall n$$

The sequence is *eventually* increasing if $\exists m \in \mathbb{N} \ni x_n \leq x_{n+1}$ for $n > m$.

Similarly, the sequence is eventually decreasing if $\exists m \in \mathbb{N} \ni x_n \geq x_{n+1}$ for $n > m$.

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

**Monotonicity Example**

The sequence

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \frac{1}{2}a_n + 2 \end{aligned}$$

is increasing and bounded above.

---

We will prove the first statement via induction:

**Base:** $a_1 = 1$, $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

**Inductive Hypothesis** $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+1}$

**Proof:**

$$\begin{aligned} a_n &\leq a_{n+1} \\ \frac{1}{2}a_n &\leq \frac{1}{2}a_{n+1} \\ \frac{1}{2}a_n + 2 &\leq \frac{1}{2}a_{n+1} + 2 \\ a_{n+1} &\leq a_{n+2} \end{aligned}$$

To prove the sequence is bounded above, we do the following:

$$\begin{aligned} a_1 = 1 &\leq 4 \\ \frac{1}{2}a_1 &\leq 2 \\ \frac{1}{2}a_1 + 2 &\leq 2 \\ a_2 &\leq 4 \end{aligned}$$

We claim that $\forall n,\ a_n \leq 4 \Rightarrow a_{n+1} \leq 4$, as we have shown the base case.

$$\begin{aligned} a_n &\leq 4 \\ \frac{1}{2}a_n &\leq 2 \\ \frac{1}{2}a_n + 2 &\leq 4 \\ a_{n+1} &\leq 4 \end{aligned}$$

Convergence of Sequences

Let $L \in \mathbb{R}$, $\varepsilon > 0$. Then, the $\varepsilon$-**neighborhood** of $L$ is $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$.

$$\begin{aligned} x &\in V_\varepsilon(L) \\ &\Leftrightarrow \\ |x - L| &< \varepsilon \\ L - \varepsilon < x &< L + \varepsilon \end{aligned}$$

With this in mind, we know the following:

Definition of Convergence

A real sequence $(x_n)_n$ converges to a number $x$ if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \ni n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such $L$ exists, then $(x_n)_n$ is said to **diverge**.

A sequence $(x_n)_n$ in a metric space $(X, d)$ converges to a point $x$ if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \ni d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an $N$ such that the $N$th tail (i.e., $x_N, x_{N+1}, \cdots$) are within $\varepsilon$ of $L$ for any $\varepsilon$.

**Note:** $N$ usually depends on $\varepsilon$ (the smaller the $\varepsilon$, the larger the $N$).

Convergence Proof

$$\left(\frac{1}{n}\right)_n \xrightarrow{n\to\infty} 0$$

We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given $\varepsilon > 0$, we want $\frac{1}{n} < \varepsilon$, meaning $n > \frac{1}{\varepsilon}$.

**Proof:** Let $\varepsilon > 0$. By the Archimedean property corollary, find $N \in \mathbb{N}$ large such that $\frac{1}{N} < \varepsilon$.

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

so, if $n \geq N$, then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

**Convergence Proof 2**

Show that

$$\left(\frac{5n-1}{3-n}\right)_{n\geq 4} \xrightarrow{n\to\infty} -5$$

---

$$\begin{aligned}
|x_n - L| &= \left|\frac{5n-1}{3-n} + 5\right| \\
&= \frac{14}{|3-n|} \\
&= \frac{14}{n-3} && < \varepsilon \\
\frac{14}{n-3} &< \varepsilon \\
n &> \frac{14}{\varepsilon} + 3
\end{aligned}$$

**Proof:** Let $\varepsilon > 0$. Find $N' \in \mathbb{N}$ so large that $\frac{1}{N'} < \frac{\varepsilon}{14}$ (which exists by the Archimedean property corollary). Let $N = N' + 3$. If $n \geq N$, then

$$\begin{aligned}
n - 3 &\geq \frac{1}{N'} \\
\frac{1}{n-3} &\leq \frac{1}{N'} \\
&< \frac{\varepsilon}{14}
\end{aligned}$$

whence

$$\begin{aligned}
|x_n - L| &= \frac{14}{n-3} \\
&< \frac{14\varepsilon}{14} \\
&= \varepsilon
\end{aligned}$$

**Sequences and their Limits, cont'd**

**Convergence and Distance**

Let $(X, d)$ be a metric space, and let $(x_n)_n$ be a sequence in the metric space. The following are equivalent:

(i) $(x_n)_n \to x$

(ii) $(d(x_n, x))_n \to 0$

---

(i) $\Rightarrow$ (b) Let $\varepsilon > 0$. Find $N_\varepsilon \in \mathbb{N}$ so large such that $d(x_n, x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

So, $|d(x_n, x) - 0| = d(x_n, x) < \varepsilon$ for all $\varepsilon > 0$. Whence, $(d(x_n, x))_n \to 0$.

(ii) $\Rightarrow$ (i) This direction is similar.

In $\mathbb{R}$, this implies that

$$\begin{aligned}(x_n)_n &\to x\\ &\Leftrightarrow\\ (|x_n - x|)_n &\to 0\end{aligned}$$

**Comparison Proposition**

Let $(X, d)$ be a metric space and let $x \in X$, and suppose $(x_n)_n$ is a sequence in $X$.

If $\exists c \geq 0$, $m \in \mathbb{N}$, and a sequence $(a_n)_n \in \mathbb{R}^+$ with $(a_n)_n \to 0$ and $d(x_n, x) \leq ca_n$ $\forall n > m$. Then, $(x_n)_n \to x$.

---

Let $\varepsilon > 0$. Note that $\frac{\varepsilon}{c} > 0$.

Find $N_1 \in \mathbb{N}$ large such that $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\varepsilon}{c}$, which is always possible since $(a_n)_n \to 0$.

Let $N = \max(N_1, m)$. Then, $n \geq N \Rightarrow n \geq N_1$ and $n \geq m$. So,

$$\begin{aligned}d(x_n, x) &\leq ca_n\\ &< c\frac{\varepsilon}{c}\\ &= \varepsilon\end{aligned}$$

so, $n \geq N \Rightarrow d(x_n, x) < \varepsilon$, whence $(x_n)_n \to x$

**Comparison Proposition, Example 1**

Prove

$$\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \to 0$$

---

$$\begin{aligned}\left|\frac{\sin(n^2-1)}{n^2+3} - 0\right| &= \frac{|\sin(n^2-1)|}{n^2+3}\\ &\leq \frac{1}{n^2+3}\\ &\leq \frac{1}{n^2}\\ &\leq \frac{1}{n}\end{aligned}$$

We know that $a_n = \frac{1}{n}$ converges to 0. So, by our comparison proposition, we are done.

**Comparison Proposition, Example 2**

Prove

$$\left(\frac{1}{2^n}\right)_n \to 0$$

---

$$\begin{aligned} 2^n &= (1+1)^n \\ &\geq 1+n && \text{Bernoulli's Inequality} \\ &> n \end{aligned}$$

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since $a_n = \frac{1}{n}$ converges, we know that $\frac{1}{2^n}$ must converge.

**Sequence Divergence**

A sequence $(x_n)_n$ is **divergent** if it does not converge. $(x_n)_n \to 0$ if and only if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni (\forall n \geq N_\varepsilon) d(x_n, x) < \varepsilon$$

So, $(x_n)_n$ diverges if and only if

$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \to d(x_n, x) \geq \varepsilon_0$$

**Diverging Sequence Proof**

Show that the following sequence diverges:

$$a_n = (-1)^n$$

---

**Step 1**

$$((-1)^n)_n \not\to 1$$

Take $\varepsilon_0 = 1/2$, given any $N \in \mathbb{N}$, we will find $n \geq N$ odd:

$$\begin{aligned} n &= 2N+1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

**Step 2**

$$((-1)^n)_n \not\to -1$$

by letting $\varepsilon_0 = 1/2$ and $n = 2N$.

**Diverging Sequence Proof 2**

Does

$$a_n = (\sin(n))_n$$

converge?

---

It is not the case that $(\sin(n))_n \to L$ for any $L \in \mathbb{R}$.

**Case 1** If $L > 1$, set $\varepsilon_0 = \frac{L-1}{2}$. Then, given any $N \in \mathbb{N}$, pick $n = N$.

$$\begin{aligned}|\sin(n) - L| &= L - \sin(n)\\ &\geq L - 1\\ &> \frac{L-1}{2}\\ &= \varepsilon_0\end{aligned}$$

**Case 2** Similarly for $L < -1$

**Case 3** Suppose $-1 < L < 1$.

**Case 3.1** Suppose $L > 0$. Set $\varepsilon_0 = \frac{L}{2}$. Given any $N$, find $n \geq N$ with $\sin(n) < 0$.

We find $k$ large such that $N < (2k+1)\pi$, which we can always do because we are finding $k > \frac{1}{2}\left(\frac{N}{\pi} - 1\right)$, which is always possible by the Archimedean property.

Note that $N < (2k+1)\pi < (2k+2)\pi$. Note that $\sin(x) < 0$ on the interval $((2k+1)\pi, (2k+2)\pi)$. Note that $|(2k+1)\pi - (2k+2)\pi| = \pi$. Let $n = \lceil (2k+1)\pi \rceil$. Then, $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

**Case 3.2** Suppose $L < 0$, set $\varepsilon_0 = \frac{-L}{2}$. Given $N$, find $n \geq N$ with $\sin(n) > 0$. Using the same strategy as above, we find $n$ such that $|L - \sin(n)| > \varepsilon_0$

**Case 3.3** Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given $N \in \mathbb{N}$, find $n \geq N$ with $\sin(n) \geq \frac{1}{2}$. Then, $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$.

Showing that a sequence diverges is not easy — later, we will divergence with non-uniqueness of convergent subsequences.

**Alternating Series**

Consider again

$$((-1)^n)_{n\geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to $-1$:

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating series diverges.

**Uniqueness of Limits**

A sequence $(x_n)_n$ can converge to at most one limit.

---

Suppose toward contradiction that $(x_n)_n$ converges to $L_1$ and $L_2$ with $L_1 \neq L_2$.

WLOG, let $L_2 > L_1$. Take $\varepsilon = \frac{L_2 - L_1}{3}$.

Since $(x_n)_n$ converges to $L_1$, $\exists N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \varepsilon$. Similarly, since $(x_n)_n$ converges to $L_2$, $\exists N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \varepsilon$.

Let $N = \max N_1, N_2$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

So, $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$. So, $x_n \in V_\varepsilon(L_1)$, and $x_n \in V_\varepsilon(L_2)$, meaning $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$, but $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. $\perp$

**Useful Lemmas for Convergence**

**Absolutely Convergent Sequences**

Let $(x_n)_n$ be a real sequence. If $x_n$ converges to $x$, then $|(x_n)_n| \to |x|$. However, the converse is not the case.

---

Note that since $(x_n)_n \to x$, $d(x_n, x) \to 0$.

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

**Convergence to Zero**

Let $a_n$ be a sequence.

$$\begin{aligned} (a_n)_n &\to 0 \\ &\Leftrightarrow \\ |(a_n)| &\to 0 \end{aligned}$$

---

($\Rightarrow$) If $(a_n)_n \to 0$, then we showed previously that $|(a_n)_n| \to |0| = 0$

($\Leftarrow$) Suppose $|(a_n)_n| \to 0$. Given $\varepsilon > 0$, then $\exists N$ such that $n \geq N$ implies

$$\begin{aligned} ||a_n| - 0| &< \varepsilon \\ ||a_n|| &< \varepsilon \\ |a_n| &< \varepsilon \\ |a_n - 0| &< \varepsilon \end{aligned}$$

So, $(a_n)_n \to 0$

**Geometric Sequence**

Let $b \in \mathbb{R}$. Consider

$$(b^n)_{n\geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided $-1 < b \leq 1$. Otherwise, the sequence is divergent.

---

If $b = 0$, then the sequence $(b^n)_n = (0, 0, 0, \dots)$ is convergent.

If $b = 1$, then the sequence $(b^n)_n = (1, 1, 1, \dots)$ is convergent.

If $b = -1$, then the sequence $(b^n)_n = (1, -1, 1, \dots)$ is divergent.

**Case 1** Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, so $\frac{1}{b} = 1 + a$.

So, by Bernoulli's Inequality, $\left(\frac{1}{b}\right)^n = (1+a)^n \geq 1 + na > na$, so $b^n < \frac{1}{na}$.

$$\begin{aligned}|b^n - 0| &= b^n \\ &< \frac{1}{na} \\ &= \frac{1}{a}\frac{1}{n} \\ &\to 0\end{aligned}$$

So, $(b^n)_n \to 0$.

**Case 2** Suppose $-1 < b < 0$. If we look at $|b^n| = |b|^n$, we know that $(|b|^n)_n \to 0$ by our work above. By the previous lemma, we know that since $|b^n| \to 0$, $b^n \to 0$.

**Case 3** Suppose $b > 1$. Then, $b = 1 + a$ where $a > 0$.

$$\begin{aligned}b^n &= (1+a)^n \\ &\geq 1 + na && \text{Bernoulli's Inequality} \\ &> na\end{aligned}$$

Suppose toward contradiction that $(b^n)_n \to L$. Let $\varepsilon_0 = 1$. Find $N \in \mathbb{N}$ such that $N > \frac{L+1}{a}$. $N$ must exist by the Archimedean property.

Therefore, $(N)(a) > L + 1$. If $n \geq N$, then $(n)(a) > (N)(a) > L + 1$, so $|b^n - L| \geq na - L \geq \varepsilon_0$. $\perp$

**Case 4** Suppose $b < -1$, and suppose toward contradiction that $(b^n)_n \to L$. By the previous lemma, we know that $|b^n| \to |L|$. So, $|b|^n \to |L|$. But, $|b| > 1$, which means our assumption contradicts the result from above. $\perp$

**$n$th Root Convergence**

If $c > 0$, then $\left(c^{1/n}\right)_n \to 1$.

---

**Case 1:** If $c = 1$, then we get $\left(c^{1/n}\right)_n = (1, 1, 1, \dots)$, which clearly converges to one.

**Case 2:** Assume that $c > 1$. Then, $c^{1/n} > 1$, because if $d = c^{1/n} \leq 1$, then $d^n \leq 1$, so $c \leq 1$. We can write $c^{1/n} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned}
c &= c^n \\
&= (1 + d_n)^n \\
&\geq 1 + nd_n && \text{Bernoulli's Inequality} \\
&> nd_n
\end{aligned}$$

So, $d_n \leq \frac{c}{n}$. Remember, $c^{1/n} = 1 + d_n$.

$$\begin{aligned}
|c^{1/n} - 1| &= c^{1/n} - 1 \\
&= d_n \\
&\leq c \cdot \frac{1}{n} \\
&\to 0
\end{aligned}$$

Therefore, $c^{1/n} \to 1$.

**Case 3:** Assume $0 < c < 1$. Then, $c^{1/n} < 1$, so $\frac{1}{c^{1/n}} > 1$.

So, we can write $\frac{1}{c^{1/n}} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned}
c^{1/n} &= \frac{1}{1 + d_n} \\
1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\
&= \frac{d_n}{1 + d_n} \\
&\leq d_n
\end{aligned}$$

Remember, $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned}
\frac{1}{c} &= (1 + d_n)^n \\
&\geq 1 + nd_n \\
&> nd_n
\end{aligned}$$

So, $d_n \leq \frac{1}{cn}$

$$\begin{aligned}
|1 - c^{1/n}| &= 1 - c^{1/n} \\
&\leq d_n \\
&\leq \frac{1}{c}\frac{1}{n} \\
&\to 0
\end{aligned}$$

Therefore, $\left(c^{1/n}\right)_n \to 1$.

**Positive Sequence Convergence**

Let $(x_n)_n$ be a sequence with $x_n \in \mathbb{R}^+$ $\forall n \in \mathbb{N}$, with $(x_n)_n \to x$. Then, $x$ is also positive, and $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

---

Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0-x|}{2}$. Since $(x_n)_n$ converges to $x$, we know that $x_n \in V_\varepsilon(x)$ for large $n$. However, every member of $V_\varepsilon(x) < 0$, and $x_n > 0$. $\perp$

**Case 1:** If $x = 0$, we will show that $\left(\sqrt{x_n}\right)_n \to 0$.

Let $\varepsilon > 0$, find $N \in \mathbb{N}$ large such that if $n \geq N$, we have

$$\begin{aligned}
|x_n - 0| &< \varepsilon^2 \\
x_n &< \varepsilon^2 \\
\sqrt{x_n} &< \varepsilon \\
|\sqrt{x_n} - 0| &< \varepsilon
\end{aligned}$$

**Case 2:** If $x > 0$, we will show that $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

$$\begin{aligned}
\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{\left(\sqrt{x_n} - \sqrt{x}\right)\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x_n}}\right| \\
&= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\
&\leq \frac{1}{\sqrt{x}}|x_n - x| \\
&\to 0
\end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \to 0$, so $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

***n*th Root of *n* Convergence**

$$\left(n^{1/n}\right)_n \to 1$$

---

We will make use of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Note that $n^{1/n} > 1$ for $n$ past 1. So, we write

$$\begin{aligned}
n^{1/n} &= 1 + d_n && d_n > 0\\
n &= (1+d_n)^n\\
&= \sum_{k=0}^{n} \binom{n}{k} d_n^k\\
&= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n\\
&\geq \binom{n}{0} + \binom{n}{2} d_n^2 && \text{as all terms are positive}\\
&= 1 + \frac{n(n-1)}{2} d_n^2
\end{aligned}$$

so

$$\begin{aligned}
n - 1 &\geq \frac{n(n-1)}{2} d_n^2\\
\frac{2}{n} &\geq d_n^2\\
\frac{\sqrt{2}}{\sqrt{n}} &\geq d_n
\end{aligned}$$

So, we have

$$\begin{aligned}
|n^{1/n} - 1| &= n^{1/n} - 1\\
&= d_n\\
&\leq \sqrt{2}\frac{1}{\sqrt{n}}\\
&\to 0 && \text{by previous corollary}
\end{aligned}$$

So, $\left(n^{1/n}\right)_n \to 0$.

**Multiplication by Geometric Sequence**

Let $0 \leq b < 1$. Show that

$$(nb^n)_n \to 0$$

---

If $0 < b < 1$ (the 0 case is trivial). So, $\frac{1}{b} > 1$, meaning $\frac{1}{b} = 1 + d$ for some $d > 0$.

$$\begin{aligned}
\frac{1}{b^n} &= (1+d)^n\\
&\geq \frac{n(n-1)}{2} d^2\\
\frac{2}{d^2(n)(n-1)} &\geq b^n\\
nb^n &\leq \frac{2}{d^2(n-1)}\\
&\to 0 && \text{by previous corollary}
\end{aligned}$$

Therefore, $(nb^n)_n \to 0$.

Boundedness and Convergence

If $(x_n)_n$ is a convergent sequence, $x_n$ is bounded. The converse is false in general.

Suppose $(x_n)_n \to x$. Let $\varepsilon = 1$.

Then, $\exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq N$.

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$. If $n \geq N$, then $|x_n| \leq c$, because $x_n \in V_\varepsilon(x)$. If $n < N$, then $|x_n| \leq c$.

Together, we have $|x_n| \leq c$ for all $n$.

To show the converse is not true, consider $((-1)^n)_n$. This sequence is bounded but not convergent.

Algebraic Operations on Sequences

Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$ be convergent sequences. Let $t \in \mathbb{R}$. Then, the following are all true:

(1) $(x_n \pm y_n)_n \to x \pm y$

(2) $(tx_n)_n \to tx$

(3) $(x_ny_n)_n \to xy$

(4) Assume $z_n \neq 0\ \forall n$, and $z \neq 0$. Then, $\left(\frac{1}{z_n}\right)_n \to \frac{1}{z}$, and $\left(\frac{x_n}{z_n}\right)_n \to \frac{x}{z}$.

**Proof of (1)** Let $\varepsilon > 0$. Since $x_n \to x$, $y_n \to y$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \to |x_n - x| < \frac{\varepsilon}{2}$, and $n \geq N_2 \to |x_n - x| \leq \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

$$\begin{aligned}|(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

**Proof of (3)** We have $(x_n)_n \to x$ and $(y_n)_n \to y$.

$$\begin{aligned}|x_ny_n - xy| &= |x_ny_n - x_ny + x_ny - xy| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |x_n - x||y|\end{aligned}$$

Since convergent sequences are bounded, $\exists c \in \mathbb{R}$ such that $|x_n| < c,\ \forall n$

$$\begin{aligned}&\leq c|y_n - y| + |x_n - x||y| \\ &\to 0\end{aligned}$$

Therefore, $|x_ny_n - xy| \to 0$, so $x_ny_n \to xy$.

**Proof of (4)** We have $z_n \neq 0$ and $z \neq 0$. Let $\varepsilon > 0$.

$$\begin{aligned}\left|\frac{1}{z_n} - \frac{1}{z}\right| &= \frac{|z - z_n|}{|z_n z|}\\ &= |z_n - z|\frac{1}{|z|}\frac{1}{|z_n|}\end{aligned}$$

Let $\varepsilon = \frac{|z|}{2}$. Since $(z_n)_n \to z$, we know that $z_n \in V_\varepsilon(z)$ for $n \geq N \in \mathbb{N}$. For $n \geq N$, $|z_n| > \frac{|z|}{2}$, so $\frac{1}{|z_n|} < \frac{2}{|z|}$.

$$\begin{aligned}&\leq |z_n - z|\frac{2}{|z|^2}\\ &\to 0\end{aligned}$$

So, $\left|\frac{1}{z_n} - \frac{1}{z}\right| \to 0$, so $\frac{1}{z_n} \to \frac{1}{z}$

Ordering of Limits

Let $(x_n)_n \to x$ and $(y_n)_n \to y$. If $x_n \leq y_n$ for all $n$, then $x \leq y$.

Suppose toward contradiction that $x > y$.

Let $\varepsilon = \frac{x-y}{2}$.

So, $\exists N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$.

Let $N = \max\{N_1, N_2\}$. Then, $x_N \in V_\varepsilon(x)$ and $y_N \in V_\varepsilon(y)$. But that means $x_N > y_N$. $\perp$

---

In particular, if $(x_n)_n \to x$, and $a \leq x_n \leq b$, then $a \leq x \leq b$.

Squeeze Theorem

Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$, where $x_n \leq y_n \leq z_n$ for all $n$.

If $L = x = z$, then $y = L$.

Let $\varepsilon > 0$. Find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow V_\varepsilon(L)$, and $n \geq N_2 \Rightarrow V_\varepsilon(L)$.

Let $N = \max\{N_1, N_2\}$. Then, $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$. Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so $y_n \in V_\varepsilon(L)$, so $(y_n)_n \to L$.

---

For example, let $a_n = \frac{\sin(n)}{n}$. Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero, $a_n \to 0$

---

As another example, consider $a_n = (2^n + 3^n)^{1/n}$. Then,

$$3^n \leq 2^n + 3^n \leq 2 \cdot 3^n$$
$$3 \leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3$$

Since $2^{1/n} \to 1$, we have $a_n \to 3$.

**Ratio Test**

Let $(x_n)$ be a sequence of strictly positive numbers, with $\left(\frac{x_{n+1}}{x_n}\right)_n \to r < 1$. Then, $(x_n)_n \to 0$.

---

Since $r < 1$, then $1 - r > 0$. Let $\rho = r + \frac{1-r}{2}$, and $\varepsilon = \rho - r = \frac{1-r}{2}$.

Since the sequence converges, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\left|\frac{x_{n+1}}{x_n} - r\right| < \varepsilon$$
$$\frac{x_{n+1}}{x_n} < \rho$$
$$x_{n+1} < \rho x_n$$

In particular, $x_{N+1} < \rho x_N$, and $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$. Inductively, one can show that $\forall k \geq 1,\ x_{N+k} < \rho^k x_N$.

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as $k \to \infty$, both sides of the inequality go to 0, implying that $x_n \to 0$

**Monotone Convergence Theorem**

Let $(x_n)_n$ be a monotone sequence. Then, $(x_n)_n$ is convergent if and only if it is bounded.

(a) If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n \to \sup(\{x_n \mid n \in \mathbb{N}\})$.

(b) If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n \to \inf(\{x_n \mid n \in \mathbb{N}\})$.

---

We have already shown that all convergent sequences are bounded.

Assume that $(x_n)_n$ is monotonic and bounded.

**Case 1:** Suppose $(x_n)_n$ is increasing. Let $\sup\{x_n \mid n \in \mathbb{N}\} := u$. We claim that $(x_n)_n \to u$.

Let $\varepsilon > 0$. By the definition of supremum, $\exists N \in \mathbb{N}$ such that $u - \varepsilon < x_N$. Note that $\forall n \geq N$, $u - \varepsilon < x_N \leq x_n \leq u$.

Therefore, if $n \geq N$, then $|x_n - u| < \varepsilon$.

**Case 2:** Suppose $(x_n)_n$ is decreasing. Let $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$. We claim that $(x_n)_n \to \ell$.

Let $\varepsilon > 0$. By the definition of infimum, $\exists N \in \mathbb{N}$ such that $\ell + \varepsilon > x_N$. Additionally, $\forall n \geq N$, $\ell \leq x_n \leq x_N < \ell + \varepsilon$.

Therefore, if $n \geq N$, $|x_n - \ell| < \varepsilon$.

Applications of the Monotone Convergence Theorem

Lemma

If $(x_n)_n$ is a convergent sequence, and $m \in \mathbb{N}$, the $m$-th tail, $x_{(m)} = (x_{m+k})_{k=1}^{\infty}$ is also convergent. If $(x_n)_n \to L$ then $x_{(m)} \to L$.

Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. If $k \geq N$, then $m + k \geq N$, so $|x_{m+k} - L| < \varepsilon$.

Thus, $(x_{m+k})_k \to L$

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that $x_n \geq 4\ \forall n$.

$$x_1 = 8 \geq 4$$

Suppose $x_k \geq 4$. We will show that $x_{k+1} \geq 4$.

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore, $(x_n)_n$ is bounded below by 4.

We claim that $(x_n)_n$ is decreasing. $\forall n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} &\leq x_n \Leftrightarrow \\ \frac{1}{2}x_n + 2 &\leq x_n \\ &\Leftrightarrow 4 \leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that $(x_n)_n \to L$.

To find $L$, we use the recursive relationship and the lemma.

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2}x_n + 2\right)_{n=1}^{\infty} \\ L &= \frac{1}{2}L + 2 \\ L &= 4 \end{aligned}$$

Consider the following sequence

$$x_1 = 1$$
$$x_2 = 1 + \frac{1}{4}$$
$$x_3 = 1 + \frac{1}{4} + \frac{1}{9}$$
$$x_k = \sum_{k=1}^{n} \frac{1}{k^2}$$

We will show that $(x_n)_n$, the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$\begin{aligned}
k^2 &\geq k(k-1) & k \geq 2\\
\frac{1}{k^2} &\leq \frac{1}{k(k-1)}\\
&= \frac{1}{k-1} - \frac{1}{k}\\
\sum_{k=2}^{n} \frac{1}{k^2} &\leq \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)\\
\sum_{k=1}^{n} \frac{1}{k^2} &\leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)
\end{aligned}$$

But

$$1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 2 - \frac{1}{n}$$

so, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2} &\leq 2 - \frac{1}{n}\\
&< 2
\end{aligned}$$

So, $(x_n)_n$ is bounded above.

**Nested Intervals Theorem, Alternative Proof**

Let $I_n = [a_n, b_n]$ be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

---

Since the intervals are nested, it must be the case that $a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq b_1$.

Similarly, $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$.

So, $(a_n)_n$ is an increasing sequence bounded above by $b_1$ and $(b_n)\, n$ is a decreasing sequence bounded below by $a_1$. So, $(b_n)_n \to r$ and $(a_n) \to \ell$
Note that $\ell = \sup(a_n)$ and $r = \inf(b_n)$.

Fix $n \in \mathbb{N}$, then for any $m \geq n$, $a_n \leq a_m \leq b_m \leq b_n$. So, $\sup(a_m) = \ell \leq b_n$. Unlocking $n$, we get that $\ell \leq \inf(b_n) = r$.

**Calculating Square Roots**

Let $a \in \mathbb{R}^+$. We will construct a sequence $(x_n)_n \to \sqrt{a}$.

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

---

We will prove that $x_n^2 \geq a$.

$$\begin{aligned} 2x_{n+1} &= x_n + \frac{a}{x_n} \\ 2x_{n+1}x_n &= x_n^2 + a \\ 0 &= x_n^2 - 2x_{n+1}x_n + a \end{aligned}$$

So, $x_n$ is a real root, meaning

$$\begin{aligned} \Delta &= 4x_{n+1}^2 - 4a \\ x_{n+1}^2 &\geq a && \forall n \end{aligned}$$

So, $\forall n \geq 2$

$$x_n^2 \geq a$$

---

We will show that $x_n$ is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &= \frac{1}{2}\underbrace{\left(\frac{x_n^2 - a}{x_n}\right)}_{\geq 0\ \forall n \geq 2} \end{aligned}$$

---

So, we have that $(x_n)_n$ is decreasing and bounded below, meaning $(x_n)_n \to x$ for some $x \in \mathbb{R}$.

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ x &= \frac{1}{2}\left(x + \frac{a}{x}\right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} && \text{remember that } x > 0 \end{aligned}$$

**Euler's Number**

Consider

$$\begin{aligned}(e_n)_n &= \left(1+\frac{1}{n}\right)^n\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}\end{aligned}$$

Similarly,

$$\begin{aligned}e_{n+1} &= \sum_{k=0}^{\infty}\left(\frac{1}{k!}\prod_{j=1}^{k-1}\left(1-\frac{j}{n+1}\right)\right)\\ e_{n+1} &\geq e_n && \forall n\end{aligned}$$

---

We claim that $(e_n)_n$ is bounded above.

$$\begin{aligned}e_1 &= \left(1+\frac{1}{1}\right)^1\\ 2 &\leq e_n\\ e_n &= \sum_{k=0}^{n}\left(\frac{1}{k!}\underbrace{\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)}_{\leq 1}\right)\\ 2^{k-1} &\leq k! && k\geq 2\\ \frac{1}{k!} &\leq \frac{1}{2^{k-1}}\\ e_n &= \sum_{k=0}^{n}\frac{1}{k!}\cdot\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)\\ &\leq \sum_{k=0}^{n}\frac{1}{k!}\\ &\leq 2+\sum_{\ell=1}^{n-1}\frac{1}{2^\ell}\\ &< 3\end{aligned}$$

so, we have

$$2\leq e_n\leq 3$$

so, by the monotone convergence theorem, $(e_n)_n$ converges

$$e := \sup_n\left(1+\frac{1}{n}\right)^n$$

**Monotone Divergence**

A sequence that is increasing and *unbounded* diverges to infinity.

Let $M > 0$. Since $(x_n)_n$ is unbounded, $\exists N \in \mathbb{N}$ such that $x_N > M$

Thus, if $n \geq N$, then $x_n \geq x_N > M$.

---

Consider

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

We can see that $h_n < h_{n+1}$. The primary question is as to whether $(h_n)_n$ is bounded.

$$\begin{aligned}
h_1 &= 1\\
&\geq 1\\
h_2 &= 1 + \frac{1}{2}\\
&\geq 1 + \frac{1}{2}\\
h_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\\
&\geq 1 + \frac{1}{2} + \frac{1}{2}\\
h_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\\
&\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}
\end{aligned}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^{k} \frac{1}{2}$$

Let $M$ be large. Find $n$ such that $n > 2(M-1)$. In this case, $n/2 + 1 > M$. Let $N = 2^n$. Then, for $m \geq N$, $h_m > M$.

Thus, $(h_n)_n$ diverges to infinity.

**Natural Sequences**

A **natural sequence** is a strictly increasing sequence of natural numbers, $(n_k)_{k=1}^{\infty}$

$$n_1 < n_2 < n_3 < \dots$$

where $\forall k \in \mathbb{N},\ n_k \in \mathbb{N}$.

**Natural Sequence Property**

Given $(n_k)_k$ natural sequence, show that $(n_k) \geq k$.

**Base Case:** We know that $n_1 \leq 1$, as $n_1 \in \mathbb{N}$.

**Inductive Step:** To be continued...

**Subsequences**

Let $(x_n)_n$ be a sequence. A subsequence $(x_{n_k})_{k=1}^{\infty}$, where $(n_k)_k$ is a natural sequence.

For example, if $(x_n)_n = (-1)^n$. If $(n_k) = 2k$, then, $(x_{n_k}) = \left((-1)^{2k}\right)_k = (1, 1, 1, \dots)$. But, if $(n_k) = 2k+1$, then $(x_{n_k}) = (-1, -1, -1, \dots)$.

If $(x_n) = (1/n)_n$, and $(n_k)_k = k^2$, then $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \dots)$.

If $(x_n)_n$ is a sequence, its $m$-th **tail** is $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \dots)$, where $n_k = m + k$.

**Convergence of Subsequence**

If $(x_n)_n \to x$, then for any natural sequence $(n_k)_k$,

$$(x_{n_k})_k \to x$$

---

Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ large such that $n \geq N$, $|x_n - x| < \varepsilon$.

Take $K = N$. Then,

$$\begin{aligned} n_k &\geq k \\ &\geq K \\ &= N \\ \Rightarrow |x_{n_k} - x| &< \varepsilon \end{aligned}$$

**Corollary to Convergence of Subsequences**

Given a sequence $(x_n)_n$, if there are two subsequences $(x_{n_k})_k \to x$, $(x_{n_\ell})_\ell \to x'$, where $x \neq x'$, then $(x_n)_n$ is divergent.

---

Recall the geometric sequence

$$(b^n)_{n=1}^{\infty} \to 0$$

if $0 < b < 1$.

The sequence $(1, b, b^2, \dots)$ is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem, $(b^n)_n \to \ell$.

Given $n = 2k$, we know that $(b^{2k})_k \to \ell$.

$$\begin{aligned} b^{2k} &= \left(b^k\right)^2 \\ \left(b^k\right)^2 &\to \ell^2 \\ \ell^2 &= \ell \\ \ell &= \{0, 1\} \end{aligned}$$

since $b < 1$

$$\ell = 0$$

**Divergence and Subsequence**

If $(x_n)_n \nrightarrow x$, then

$$(\exists \varepsilon_0 > 0)\,(\forall N \in \mathbb{N})\,(\exists n \geq N) \ni |x_n - x| \geq \varepsilon_0$$

We can use this to construct a sequence to show divergence.

---

Let $(x_n)_n$ be a sequence, and $x \in \mathbb{R}$.

$$(x_n)_n \nrightarrow x$$
$$\Leftrightarrow$$
$$(\exists \varepsilon_0 > 0)(\exists (x_{n_k})_k)$$

with

$$|x_{n_k} - x| \geq \varepsilon_0$$

---

**($\Rightarrow$)** We know $\exists \varepsilon_0 > 0$ as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \varepsilon_0$$
$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \varepsilon_0$$
$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \varepsilon_0$$

Assume we have $n_1 < n_2 < \dots, n_k$ with

$$|x_{n_j} - x| \geq \varepsilon_0 \qquad j = 1, 2, \dots, k$$
$$N = n_k + 1 \Rightarrow n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \varepsilon_0$$

Iteratively, we have our desired subsequence $(x_{n_k})_k$.

($\Leftarrow$) If $(x_n)_n \to x$, any subsequence converges to $x$.

By assumption, $(\exists \varepsilon_0 > 0)\,(\exists (n_k)_k)$ with $|x_{n_k} - x| \geq \varepsilon_0$. Thus, $(x_{n_k})_k \nrightarrow x$.

**Bolzano-Weierstrass Theorem**

If $(x_n)_n$ is a bounded sequence, then $(x_n)_n$ admits a convergent subsequence.

**Lemma**

Let $(x_n)_n$ be any real sequence. Then, $\exists n_k$ such that $(x_{n_k})_k$ is monotone.

---

A **peak** of a sequence $(x_n)_n$ is an $x_m$ such that $x_m \geq x_n \ \forall n \geq m$.

**Case 1:** There are infinitely many peaks, $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$, where $n_1 < n_2 < \dots$. Then, $(x_{n_k})_k$ is decreasing.

**Case 2:** There are finitely many peaks. Let these peaks be $x_{m_1}, x_{m_2}, \dots, x_{m_r}$.

Let $n_1 = m_r + 1$. Since $x_{n_1}$ is not a peak, $\exists n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since $x_{n_2}$ is not a peak, $\exists n_3 > n_2$ such that $x_{n_3} > x_{n_2}$.

Iteratively, we have an increasing sequence of non-peaks $(x_{n_k})_k$.

Since $(x_n)_n$ admits a monotone subsequence, and $(x_{n_k})_k$ is bounded as $(x_n)_n$ is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

**Limit Superior and Limit Inferior**

Let $X = (x_n)_n$ be a bounded real sequence. By Bolzano-Weierstrass, $(x_n)_n$ admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R},\ t = \lim_{k\to\infty} x_{n_k} \right\} \qquad \text{for any subsequence } (x_{n_k})_k$$

Then, $t \in \overline{X}$ is called a **limit point** of $X$.

---

Let $u_1 = \sup_{n\geq 1}(x_n)$, $\ell_1 = \inf_{n\geq 1}(x_n)$. Clearly, $\ell_1 \leq u_1$, and $\overline{X} \subseteq [\ell_1, u_1]$.

Let $u_2 = \sup_{n\geq 2}(x_n)$ and $\ell_2 = \inf_{n\geq 2}(x_n)$.

Since $u_1$ is an upper bound for $(x_n)_n$, it is an upper bound for $(x_n)_{n\geq 2}$, so $u_2 \leq u_1$. Similarly, since $\ell_1$ is a lower bound for $(x_n)_n$, it is a lower bound for $(x_n)_{n\geq 2}$, so $\ell_2 \geq \ell_1$.

As a result, we can see that $\overline{X} \subseteq [\ell_2, u_2]$.

We continue, letting $u_m = \sup_{n\geq m}(x_n)$, and $\ell_m = \inf_{n\geq m}(x_n)$. We get $\ell_1 \leq \ell_2 \leq \cdots$, and $u_1 \geq u_2 \geq \cdots$, and $\overline{X} \in [\ell_m, u_m]$, $\forall m$.

We get a nested sequence of intervals $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \cdots$. By the Nested Intervals Theorem, we know that

$$\begin{aligned} \overline{X} &\subseteq \bigcap_{m\geq 1} [\ell_m, u_m] \\ &= [\ell, u] \end{aligned}$$

where $\ell = \sup(\ell_m)$ and $u = \inf(u_m)$.

---

Given a bounded sequence $(x_n)_x = X$,

$$\begin{aligned} u &= \inf_{m\geq 1}(u_m) \\ &= \inf_{m\geq 1}\left(\sup_{n\geq m} x_n\right) \end{aligned}$$

called the **limit superior** of $(x_n)_n$

$$u = \limsup_{n\to\infty} x_n$$

and

$$\begin{aligned} \ell &= \sup_{m\geq 1}(\ell_m) \\ &= \sup_{m\geq 1}\left(\inf_{n\geq m}(x_n)\right) \end{aligned}$$

called the **limit inferior** of $(x_n)_n$

$$\ell = \liminf_{n\to\infty} x_n$$

Applications of Limit Superior and Limit Inferior

Let $(x_n)_n$ be bounded. Then,

(1) $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$

(2) $(x_n)_n \to x \Leftrightarrow \liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n = x$

---

(1) This was proven with the Nested Intervals Theorem

(2) Let $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$.

We know that $u_m = \sup_{n\geq m} x_n$. If $m \geq N$, then $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$. Therefore, $|u_m - x| \leq \varepsilon/2 < \varepsilon$, so $(u_m)_m \to \varepsilon x \limsup_{n\to\infty} x_n$.

Similarly, we know that $\ell_m = \inf_{n\geq m} x_n$. If $m \geq N$, then $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$. So, $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$, so $(\ell_m)_m \to x = \liminf_{n\to\infty} x_n$.

---

Consider the sequence

$$\begin{aligned} x_n &= \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases} \\ &= (-1, 5/2, -1/3, 9/4, -1/5, \dots) \end{aligned}$$

We begin by constructing the $u_m$ sequence: $(5/2, 5/2, 9/4, 9/4, \dots)$. We can see that $u_m \to 2$.

Then, we construct the $\ell_m$ sequence: $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$. We can see that $\ell_m \to 0$.

---

**Exercise:** If $(x_n)_n$ and $(y_n)_n$ are sequences with $x_n \leq y_n$ $\forall n$, then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

Ratio Test and Root Test Equivalent Convergence

If $(a_n)_n$ is a sequence of strictly positive terms such that

$$\left(\frac{a_{n+1}}{a_n}\right)_n \to \rho$$

then,

$$\left(a_n^{1/n}\right)_{n=1}^{\infty} \to \rho$$

---

Let $\varepsilon > 0$. Then, $\exists N$ large such that $\forall n \geq N$,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n} - \rho\right| &< \varepsilon && \forall n \geq N \\ \Rightarrow \frac{a_{n+1}}{a_n} &< \rho + \varepsilon && \forall n \geq N \\ a_{n+1} n a_n &(\rho + \varepsilon) && \forall n \geq N \\ a_n &< a_N (\rho+\varepsilon)^{n-N} && \forall n \geq N \\ a_n &< (\rho+\varepsilon)^n \cdot \frac{a_N}{(\rho+\varepsilon)^N} \\ a_n^{1/n} &< (\rho+\varepsilon)\left(\frac{a_N}{(\rho+\varepsilon)^N}\right)^{1/n} \\ \limsup a_n^{1/n} &\leq \limsup (\rho+\varepsilon)\left(\frac{a_N}{(\rho+\varepsilon)^N}\right)^{1/n} \\ \limsup_{n\to\infty} a_n^{1/n} &\leq \rho + \varepsilon \end{aligned}$$

**Case 1:** If $\rho = 0$, the case his trivial.

**Case 2:** Suppose $\rho > 0$. Find $\varepsilon > 0$ small such that $0 < \varepsilon < \rho$.

Since $\left(\frac{a_{n+1}}{a_n}\right)_n \to \rho$, find $N$ large such that $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$. So, $\forall n \geq N$,

$$\begin{aligned} a_{n+1} &\geq a_n (\rho - \varepsilon) \\ a_n &\geq a_N (\rho-\varepsilon)^{n-N} \\ a_n^{1/n} &\geq (\rho-\varepsilon)\left(\frac{a_N}{(\rho-\varepsilon)^N}\right)^{1/n} \\ \liminf a_n^{1/n} &\geq \rho - \varepsilon \end{aligned}$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together, $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$, so $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$, whence $\left(a_n^{1/n}\right) \to \rho$

**Properties of $\overline{X}$**

We found earlier that $\overline{X} \subseteq [\ell, u]$. We claim that

$$\sup \overline{X} = u$$
$$\sup \overline{X} = \ell$$

We have shown that $u$ is an upper bound for $\overline{X}$. The goal is to show that $u$ is the least upper bound.

Let $\varepsilon > 0$. We need to find a $t \in \overline{X}$ with $u - \varepsilon < t$. Note that $u - \varepsilon < u_m \ \forall m$.

We know that $u - \varepsilon < u_1$. Since $u_1 = \sup_{n \geq 1} x_n$, we know $\exists n_1 \in \mathbb{N}$ with $u - \varepsilon < x_{n_1} < u_1$.

Consider $u_{n_1+1} = \sup_{n > n_1} x_n$. We know that $u - \varepsilon < u_{n_1+1}$. Therefore, $\exists x_{n_2}$ with $n_2 > n_1$ and $u - \varepsilon < x_{n_2} < u_{n_1+1}$.

Then, we use $u_{n_2+1}$. Then, $\exists n_3 > n_2$ with $u - \varepsilon < x_{n_3} < u_{n_2+1}$.

We get a subsequence from the natural sequence $n_1, n_2, \dots$, where $u - \varepsilon < x_{n_k} \ \forall k$.

Also, $x_{n_k} < u_1 \ \forall k$. Therefore, $(x_{n_k})_k$ is a bounded sequence. By Bolzano-Weierstrass, $\exists$ a convergent subsequence

$$\left(x_{n_{k_j}}\right)_j \to t$$

We know that $u - \varepsilon \leq t$. Note that $t \in \overline{X}$.

---

**Exercise:** Show that $\inf \overline{X} = \ell$.

**Cauchy Sequences**

A sequence $(x_n)_n$ in a metric space $(X, d)$ is Cauchy if

$$(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N}) \ni p, q \geq N \Rightarrow d(x_p, x_q) < \varepsilon$$

if $(X, d) = (\mathbb{R}, |\cdot|)$:

$$(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N}) \ni p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon$$

---

Consider the sequence $(x_n)_n = \frac{1}{n}$. Then,

$$\begin{aligned} |x_p - x_q| &= \left|\frac{1}{p} - \frac{1}{q}\right| \\ &= \frac{1}{q} - \frac{1}{p} \\ &\leq \frac{1}{q} \end{aligned}$$

Given $\varepsilon > 0$, find $N$ large such that $\frac{1}{N} < \varepsilon$. Then, $p, q \geq N$ implies

$$\begin{aligned} \left|\frac{1}{p} - \frac{1}{q}\right| &< \frac{1}{q} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

---

Show that $(-1)^n$ is not Cauchy.

$$(\exists \varepsilon_0 > 0)\,(\forall N \in \mathbb{N}) \ni p, q \geq N \Rightarrow |x_p - x_q| \geq \varepsilon_0$$

---

**Boundedness of Cauchy Sequences**

Cauchy sequences are bounded.

Let $\varepsilon = 1$. Then, by the Cauchy criterion, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < 1$.

In particular, $\forall n \geq N$,

$$\begin{aligned} |x_n| &= |x_n - x_N + x_N| \\ &\leq |x_n + x_N| + |x_N| && \text{Triangle Inequality} \\ &< 1 + |x_N| \end{aligned}$$

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$. Then, $x_n \leq c\ \forall n \geq 1$. Thus, $x_n$ is bounded.

**Convergent Subsequences of Cauchy Sequences**

If $(x_n)_n$ is Cauchy and $(x_n)_n$ admits a convergent subsequence, then $(x_n)_n$ is convergent.

Say $(x_{n_k}) \to x$ for some natural sequence $(n_k)_k$. We claim that $(x_n)_n \to x$.

Let $\varepsilon > 0$. Since $(x_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon/2$.

Also, since $(x_{n_k})_k \to x$, then $\exists K \in \mathbb{N}$ and $K \geq N$ with $k \geq K \Rightarrow |x_{n_k} - x| < \varepsilon/2$.

For all $k \geq K$,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \end{aligned}$$

Let $N_1 = \max\{N, K\}$. Then,

$$\begin{aligned} n \geq N_1 &\Rightarrow n \geq N && \text{by max} \\ &\Rightarrow n_k \geq k \geq K \geq N && \text{def. of natural sequence} \\ |x_n - x| &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

**Cauchy Sequence Convergence in the Reals**

Let $(x_n)_n$ be any sequence in $\mathbb{R}$. The following are equivalent:

(1) $(x_n)_n$ converges.

(2) $(x_n)_n$ is Cauchy.

---

(1) $\Rightarrow$ (2) (Holds in any metric space). Suppose $(x_n)_n \to x$. Find $N$ large such that $n \geq N \to d(x_n, x) < \varepsilon/2$.

Then, $p, q \geq N \Rightarrow$

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

(2) $\Rightarrow$ (1) If $(x_n)_n$ is Cauchy, then $(x_n)_n$ converges.

By Bolzano-Weierstrass, $(x_n)_n$ admits a convergent subsequence, so by our previous lemma, $(x_n)_n$ must converge.

**Note:** To show (2) $\Rightarrow$ (1), we used Bolzano-Weierstrass, which requires the monotone convergence theorem, which itself requires the completeness axiom. This is why we cannot show (2) $\Rightarrow$ (1) converges.

**Complete Metric Spaces**

A metric space $(X, d)$ is **complete** if every Cauchy sequence converges.

**Remark:** All convergent sequences are Cauchy, and all Cauchy sequences are bounded. We showed that $\mathbb{R}$ under the absolute value metric is complete.

$\mathbb{Q}$ under $d(s, t) = |s - t|$ is not complete; similarly, $A = (0, 1)$ under the metric inherited from $\mathbb{R}$ is not complete; $x_n = \frac{1}{n}$ is Cauchy but not convergent in $A$.

**Finding Cauchy Sequences and Convergence in $\mathbb{R}$**

Consider the harmonic sequence

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

We claim that $h_n$ is not convergent.

Let $p > q$. Then,

$$\begin{aligned}|h_p - h_q| &= \left|\sum_1^p \frac{1}{k} - \sum_1^q \frac{1}{k}\right| \\ &= \frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p} \\ &\geq \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} \\ &= \frac{p-q}{p} \\ &= 1 - \frac{q}{p}\end{aligned}$$

set $p = 2q$:

$$\begin{aligned}|h_{2q} - h_q| &\geq 1\frac{q}{2q} \\ &= 1/2\end{aligned}$$

Therefore, $h_n$ is not Cauchy, and thus not convergent.

---

Consider a sequence of partial sums

$$x_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

We claim that $(x_n)_n$ is Cauchy, and thus convergent. Let $p > q$. Then, we have

$$\begin{aligned}|x_p - x_q| &= \left|\sum_{k=q+1}^{p} \frac{(-1)^k}{k!}\right| \\ &\leq \sum_{k=q+1}^{p} \frac{1}{k!} \\ &\leq \sum k = q+1^p \frac{1}{2^{k-1}} \\ &= \frac{1}{2^q} + \frac{1}{2^{q+1}} + \cdots + \frac{1}{2^{p-1}} \\ &= \frac{1}{2^q}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-q-1}}\right) \\ &\leq \frac{1}{2^{q-1}}\end{aligned}$$

Given $\varepsilon > 0$, choose $N$ large such that $\frac{1}{2^{N-1}} < \varepsilon$. When $p > q > N$, then $|x_p - x_q| \leq \frac{1}{2^{q-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$.

Thus, the sequence is convergent.

Contractive Sequences

A sequence $(x_n)_n$ in a metric space $(X, d)$ is **contractive** if

$$\exists 0 < \rho < 1 \ni d(x_{n+1}, x_n) \leq \rho d(x_n, x_{n-1}) \qquad \forall n \geq 1$$

In $\mathbb{R}$, the definition is

$$|x_{n+1} - x_n| \leq \rho |x_n - x_{n-1}|$$

---

We claim that every contractive sequence is Cauchy.

From examination, we arrive at the following:

$$|x_n - x_{n-1}| \leq \rho^{n-2} |x_2 - x_1| \tag{$*$}$$

If $p > q$, then

$$\begin{aligned}
|x_p - x_q| &= |x_p - x_{p-1} + x_{p-1} - x_{p-1} + \cdots + x_{q+1} - x_q| \\
&\leq |x_p - x_{p-1}| + \cdots + |x_{q+1} - x_q| && \text{Triangle Inequality} \\
&\leq |x_2 - x_1| \left(\rho^{p-2} + \rho^{p-3} + \cdots + \rho^{q-1}\right) \\
&= |x_2 - x_1| \rho^{q-1} \left(1 + \rho + \rho^2 + \cdots + \rho^{p-q-1}\right) \\
&= |x_2 - x_1| \rho^{q-1} \frac{1 - \rho^{p-q}}{1 - x} && \text{Finite Geometric Sequence} \\
&\leq |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho}
\end{aligned}$$

Given $\varepsilon > 0$, we can find $N$ large such that

$$q \geq N \Rightarrow |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} < \varepsilon$$

Thus, $p > q \geq N \Rightarrow |x_p - x_q| < \varepsilon$.

**Application of Contractive Sequences**

Consider $(f_n)_n$ defined as follows:

$$\begin{aligned}
f_0 &= 1 \\
f_1 &= 1 \\
f_{n+1} &= f_n + f_{n-1}
\end{aligned}$$

Consider $x_n$ defined as follows:

$$x_n = \frac{f_{n+1}}{f_n}$$

We can rewrite $x_n$ as:

$$\begin{aligned}
x_n &= \frac{f_n + f_{n-1}}{f_n} \\
&= 1 + \frac{f_{n-1}}{f_n} \\
&= 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \\
&= 1 + \frac{1}{x_{n-1}}
\end{aligned}$$

We claim that $3/2 \leq x_n \leq 2 \ \forall n \geq 2$.

$$x_2 = 2$$

Inductive Hypothesis: suppose $3/2 \leq x_n \leq 2$

$$\begin{aligned} :\frac{3}{2} &\leq x_n \leq 2 \\ \frac{2}{3} &\geq \frac{1}{x_n} \geq \frac{3}{2} \\ 2 \geq \frac{5}{3} &\geq 1 + \frac{1}{x_n} \geq \frac{3}{2} \end{aligned}$$

We now claim that $(x_n)_n$ is contractive.

$$\begin{aligned} |x - n + 1 - x_n| &= \left|\left(1 + \frac{1}{x_n}\right) - \left(1 + \frac{1}{x_{n-1}}\right)\right| \\ &= \left|\frac{1}{x_n} - \frac{1}{x_{n-1}}\right| \\ &= \left|\frac{x_{n-1} - x_n}{x_{n-1}x_n}\right| \\ &\leq \frac{4}{9}|x_n - x_{n-1}| \end{aligned}$$

Therefore, $\rho = \frac{4}{9}$ is our constant of contraction. Thus, $(x_n)_n$ is Cauchy, so it converges in $\mathbb{R}$.

$$\begin{aligned} x_{n+1} &= 1 + \frac{1}{x_n} && (n \to \infty,\ x_n \to \varphi) \\ \varphi &= 1 + \frac{1}{\varphi} \\ \varphi^2 - \varphi - 1 &= 0 \\ \varphi &= \frac{1 + \sqrt{5}}{2} \end{aligned}$$

---

Let $x_1 = 0$, $x_2 = 1$, and

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_n + x_{n-1}) \\ (x_n)_n &= (0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, \dots) \end{aligned}$$

While the sequence is not monotone, we can show that the sequence is contractive.

$$\begin{aligned} |x_{n+1} - x_n| &= \left|\frac{1}{2}(x_n + x_{n-1}) - x_n\right| \\ &= \left|\frac{1}{2}(x_{n-1} - x_n)\right| \\ &= \frac{1}{2}|x_n - x_{n-1}| \end{aligned}$$

Since the constant of contraction is equal to $1/2$, this sequence is Cauchy, and thus converges in the real numbers.

Since $(x_n)_n \to x$, every subsequence converges to $x$. Therefore, $(x_{2k+1})_k \to x$.

$$\begin{aligned} x_{2k+1} &= \sum_{j=1}^{k} \frac{1}{2^{2j-1}} \\ &= 2\sum_{j=1}^{k} \frac{1}{4^j} \\ &= 2\frac{1-\frac{1}{4^{k+1}}}{1-\frac{1}{4}} \\ &= \frac{2}{3} && k \to \infty \end{aligned}$$

Properly Divergent Sequences

Let $(x_n)_n$ be a real sequence. $(x_n)_n$ *properly* diverges to $+\infty$ if

$$(\forall \alpha > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow x_n \geq \alpha$$

We write that $(x_n)_n \to +\infty$. Similarly, $(x_n)_n$ properly diverges to $-\infty$ if

$$(\forall \beta < 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow x_n \leq \beta$$

and $(x_n)_n \to -\infty$. We say that $(x_n)_n$ is properly divergent if $(x_n)_n \to \pm\infty$.

---

For example $(x_n)_n$ diverges to $n$.

If $\alpha > 0$, find $N \geq \alpha$ by the Archimedean property. Then, $n \geq N \Rightarrow n > \alpha$.

---

If $(x_n)_n$ and $(y_n)_n$ are sequences such that $x_n \geq y_n\ \forall n$, and $(y_n)_n \to +\infty$, then $(x_n)_n \to +\infty$.

Divergence of the Geometric Sequence

In the geometric sequence, if $b > 1$, we can show that $(b^n) \to +\infty$.

Write $b = 1 + a$ for some $a > 0$. Then, by Bernoulli's inequality, we have

$$\begin{aligned} b^n &= (1+a)^n \\ &\geq 1 + na \\ &\geq na \end{aligned}$$

Given any $\alpha > 0$, find $N$ large such that $N > \frac{\alpha}{a}$, which is always possible by the Archimedean property. Then, for $Na \geq \alpha$. If $n \geq N$, then $na \geq Na > \alpha$.

Since $b^n > na$, we have that $(b^n)_n \to +\infty$.

Monotone Divergence

By the Monotone Convergence Theorem, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \to x \Leftrightarrow (x_n)_n \text{ bounded}$$

Negating, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \text{ divergent} \Leftrightarrow (x_n)_n \text{ unbounded}$$

However, we can make this statement stronger.

**Proposition** Let $(x_n)_n$ be monotone. $(x_n)_n$ is unbounded if and only if $(x_n)_n$ is properly divergent.

**Proof:**

($\Leftarrow$) If $(x_n)_n$ is properly divergent, then $(x_n)_n$ is divergent, and thus unbounded.

($\Rightarrow$) Let $(x_n)_n$ be unbounded and increasing. Then, given $\alpha > 0$, $\exists n_\alpha$ with $x_{n_\alpha} > \alpha$. If $n \geq n_\alpha$, then $x_n \geq x_{n_\alpha} > \alpha$, so $(x_n)_n$ is properly divergent to $+\infty$.

Comparison Test

Let $(x_n)_n$ and $(y_n)_n$ be sequences with $x_n > 0$ and $y_n > 0$. Suppose that

$$\left(\frac{x_n}{y_n}\right)_n \to L > 0$$

Then, $(x_n)_n \to +\infty \Leftrightarrow (y_n)_n \to \infty$.

Let $\varepsilon = L/2$. Since

$$\left(\frac{x_n}{y_n}\right)_n \to L,$$

$\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} \frac{L}{2} \leq \frac{x_n}{y_n} &\leq \frac{3L}{2} \\ \frac{L}{2} y_n &\leq x_n \\ \frac{2}{3L} x_n &\leq y_n \end{aligned}$$

If $(y_n)_n \to \infty$, then so too does $(L/2)(y_n)$, so $(x_n)_n \to \infty$. Similarly, if $(x_n)_n \to \infty$, then so too does $(2/3L)x_n$, so $(y_n)_n \to \infty$.

---

Show that

$$\left(\sqrt{4n^2 - 3n + 1}\right)_n \to +\infty$$

We will compare to $y_n = n$. Then

$$\begin{aligned} \frac{x_n}{y_n} &= \frac{\sqrt{4n^2 - 3n + 1}}{n} \\ &= \sqrt{4 - \frac{3}{n} + \frac{1}{n^2}} \\ &\to 2 \geq 0 \end{aligned}$$

Since $y_n$ is properly divergent to $+\infty$, so too is $x_n$.

## Introduction to Infinite Series

An **infinite series** is a sequence of partial sums $s_n$, where $s_n$ is formed from $x_k$ as follows:

$$s_n = \sum_{k=1}^{n} x_k$$

alternatively,

$$s_1 = x_1$$
$$s_n = s_{n-1} + x_n$$

The limit of the sequence $(s_n)_n$ is the value of

$$\sum_{n=1}^{\infty} x_n$$

The infinite series converges to $s$ if $(s_n)_n \to s$.

If $(s_n)_n$ diverges, then so too does the series. If $(s_n)_n$ is properly divergent to $\pm\infty$, then we write that the series is equal to $\pm\infty$.

## Series of Positive Terms

Let $(x_k)_k$ be a sequence of positive terms. The following are equivalent:

(a) $\sum x_k$ converges.

(b) The sequence of partial sums $(s_n)_n$ is bounded above.

(c) A subsequence of the sequence of partial sums $(s_{n_j})_j$ is bounded above.

**Proof:**

**(1) $\Rightarrow$ (2):** $\sum x_k$ is convergent $\Rightarrow$ $(s_n)_n$ is convergent $\Rightarrow$ $(s_n)_n$ is bounded.

**(2) $\Rightarrow$ (3):** If $(s_n)_n$ is bounded, so is any subsequence $(s_{n_j})_j$.

**(3) $\Rightarrow$ (2):** Suppose $s_{n_j} \leq c$. If $m$ is arbitrary, $\exists j$ such that $n_j \geq m$. Take $j = m$. Then, $s_m \leq s_{n_j} \leq c$. Therefore, $(s_n)_n$ is bounded above.

(2) $\Rightarrow$ (1) Let $(s_n)_n$ be bounded above. We know that $(s_n)_n$ is increasing as $x_k \geq 0$. By the Monotone Convergence theorem, $(s_n)_n$ converges, meaning $\sum x_k$ converges.

## Corollary to Series of Positive Terms

Let $(x_k)_k$ be a sequence with $x_k \geq 0$. Then,

$$\sum x_k \text{ properly diverges} \Leftrightarrow (s_n)_n \text{ is unbounded}$$

Recall that for $x_k = 1/k$, we proved that $(s_n)_n$ is unbounded, and also that $(s_n)_n$ is not cauchy, meaning $\sum_{k=1}^{\infty} 1/k$ is properly divergent.

Additionally, we saw that for $x_k = 1/k^2$, $(s_n)_n$ is increasing and bounded above.

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{k-1} \\ &= 2 - \frac{1}{n} \end{aligned}$$

---

Let $b \in \mathbb{R}$. Let $x_k = b^k$. Then, we have

$$\begin{aligned} s_n &= \sum_{k=0}^{n} b^k \\ &= \frac{1-b^{n+1}}{1-b} && b \neq 1 \end{aligned}$$

Therefore, we know the end behavior of the series:

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{1-b^{n+1}}{1-b} \\ &= \frac{1}{1-b}\left(1 - b \lim_{b\to\infty} b^n\right) \\ &= \begin{cases} \frac{1}{1-b} & |b| < 1 \\ \text{diverges} & |b| > 1 \end{cases} \end{aligned}$$

**Series Comparison Test**

Let $0 \leq x_k \leq y_k$.

- If $\sum y_k$ converges, then so too does $\sum x_k$
- If $\sum x_k$ diverges, then so too does $\sum y_k$.

**Proof:**

$(\Rightarrow)$ If $\sum y_k$ converges, then $t_n = \sum_{k=1}^{n} y_k$ is bounded.

Setting $s_n = \sum_{k=1}^{n} x_k$, we see that $0 \leq s_n \leq t_n$. Seeing as $t_n$ is bounded, so too is $s_n$. Therefore, $\sum x_k$ is convergent.

For example, consider the series

$$\sum \frac{1}{k^2+k}$$

Since $\frac{1}{k^2} \geq \frac{1}{k^2+k}$, we know that, seeing as $\frac{1}{k^2}$ converges, so does $\frac{1}{k^2+k}$.

**Limit Comparison Test**

Let $x_k$ and $y_k$ be strictly positive sequences. Suppose that

$$\lim_{k\to\infty}\frac{x_k}{y_k} = L$$

(a) If $L > 0$, then $\sum x_k$ converges if and only if $\sum y_k$ converges.

(b) If $L = 0$, then $\sum y_k$ converges $\Rightarrow \sum x_k$ converges.

**Proof:**

(a) Since

$$\frac{x_k}{y_k} \to L$$

Set $\varepsilon = L$. We know $\exists K$ such that $k \geq K \Rightarrow y_k \leq \frac{2}{L}x_k$. Let $s_n = \sum_{k=1}^{n} x_k$ and $t_n = \sum_{k=1}^{n} y_k$. Then,

$$\begin{aligned}
t_n &= \sum_{k=1}^{K-1} y_k + \sum_{k=K}^{n} \\
&\leq t_{K-1} + \frac{2}{L}\sum_{k=K}^{n} \\
&\leq t_{K-1} + \frac{2}{L}s_n \\
&\leq t_{K-1} + c,
\end{aligned}$$

implying that $t_n$ is bounded, so $\sum y_k$ converges.

(b) Since

$$\frac{x_k}{y_k} \to 0,$$

$\exists K$ such that $\frac{x_k}{y_k} \leq 1\ \forall k \geq K$, meaning $x_k < y_k\ \forall k \geq K$.

Letting $s_n = \sum_{k=1}^{n} x_k$ and $t_n = \sum_{k=1}^{n} y_k$. Thus,

$$\begin{aligned}
s_n &= \sum_{k=1}^{K-1} x_k + \sum_{k=K}^{n} x_k \\
&= s_{K-1} + \sum_{k=K}^{n} y_k \\
&\leq s_{K-1} + t_n \\
&\leq s_{K-1} + c
\end{aligned}$$

Thus, $s_n$ is bounded, meaning $\sum x_k$ is convergent.

**Limit Comparison Test Examples**

Consider

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$$

Letting $x_n = \frac{1}{\sqrt{n^2-1}}$, and $y_n = \frac{1}{n}$, we have

$$\begin{aligned} \frac{x_n}{y_n} &= \frac{n}{\sqrt{n^2-1}} \\ &\to 1 > 0 \end{aligned}$$

Since $\sum y_n$ diverges, so too does $\sum x_n$.

***n*th Term Divergence Test**

If $\sum x_k$ is convergent, then $(x_k)_k \to 0$. Conversely, if $(x_k)_k \nrightarrow 0$, then $\sum x_k$ diverges. Recall that $s_n = s_{n-1} + x_n$. If $\sum x_k$ converges, then $(s_n)_n \to 0$. So,

$$\begin{aligned} x_n &= s_n - s_{n-1} \\ (s_n)_n &\to s \\ x_n &\to s - s \\ &= 0 \end{aligned}$$

For example, we can find that

$$\sum_{k=1}^{\infty} \frac{1}{\arctan k}$$

diverges, as $\lim_{k\to\infty} \frac{1}{\arctan k} = \frac{2}{\pi} \neq 0$

**Cauchy Condensation Test**

Let $(x_k)_k$ be a decreasing sequence of positive numbers. Then,

$$\sum_k x^k \text{ converges } \Leftrightarrow \sum_k 2^k x_{2^k} \text{ converges}$$

---

Look at the partial sum $s_{2^n}$,

$$\begin{aligned} s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_15) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\leq x_1 + 2x_2 + 4x_4 + \cdots + 2^{n-1}x_{2^{n-1}} + x_{2^n} \\ &= \sum_{k=1}^{n-1} 2^k x_{2^k} + x_{2^n} \end{aligned}$$

If $\sum_k 2^k x_{2^k}$ converges, then its partial sums are bounded, and we have that $x_{2^n} \to 0$. Then, $s_{2^n}$ is bounded, and thus $\sum x_k$ converges.

$$\begin{aligned}
2s_{2^n} &= \sum_{k=1}^{2^n} x_k \\
&= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_15) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n - 1}) + x_{2^n} \\
&+ x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_15) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n - 1}) + x_{2^n} \\
&= (x_1 + x_1) + (x_2 + x_2) + (x_3 + x_3 + x_4 + x_4) + \cdots (x_{2^{n-1}} + x_{2^{n-1}} + \cdots + x_{2^n} + x_{2^n}) \\
&\geq x_1 + 2x_2 + 4x_4 + \cdots + 2^n x_{2^n} \\
&= \sum_{k=0}^{n} 2^k x_{2^k}
\end{aligned}$$

therefore, we get that

$$\frac{1}{2}\sum_{k=0}^{n} 2^k a_{2^k} \leq s_{2^n}$$

If $\sum x_k$ converges, then $s_n$ is bounded, so $s_{2^n}$ is bounded, so $\sum_{k=0}^{n} 2^k x_{2^k}$ is bounded, so the series $\sum_{k=0}^{n} 2^k x_{2^k}$ is convergent.

**$p$-Series**

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \in \mathbb{R}$$

By the Cauchy Condensation Test, we see that the series converges if and only if the following series converges:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{np-1}}\right)^n \\
&\Leftrightarrow \frac{1}{2^{p-1}} < 1 \\
&\Leftrightarrow 2^{p-1} > 1 \\
&\Leftrightarrow p > 1
\end{aligned}$$

**Uniform Convergence**

Fix a nonempty set $\Omega$. Then,

$$\mathcal{F}(\Omega, \mathbb{R}) = \{f \mid f : \Omega \to \mathbb{R}\}$$

A sequence of functions $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$ converges pointwise to $f \in \mathcal{F}(\Omega, \mathbb{R})$ if

$$\forall x \in \Omega,\ (f_n(x))_n \xrightarrow{n\to\infty} f(x)$$

**Pointwise Convergence $\varepsilon$ Definition**

$$(f_n)_n \to f \text{ pointwise } \in \mathcal{F}(\Omega, \mathbb{R})$$
$$\Leftrightarrow$$
$$(\forall x \in \Omega)(\forall \varepsilon > 0)(\exists N_{x,\varepsilon} \in \mathbb{N}) \ni n \geq N_{x,\varepsilon} \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

**Pointwise Convergence Examples**

**Example 1:** Let $f_n : [0,1] \to \mathbb{R}$, and $f_n(x) = x^n$. Note that $(f_n)_n \to \delta_1$, where

$$\delta_1(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

**Example 2:** Let $f_n : \mathbb{R} \to \mathbb{R}$, where

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

**Claim:** $f_n \to \mathbf{0}$.

If $x = 0$, then $f_n(0) = \mathbf{0}$ $\forall n \geq 1$.

Otherwise, we have

$$\begin{aligned} |f_n(x) - \mathbf{0}(x)| &= \frac{n|x|}{1 + n^2x^2} \\ &\leq \frac{n|x|}{n^2x^2} \\ &= \frac{1}{n|x|} \\ &\to 0 \end{aligned}$$

**Example 3:** Let $h_n : [0, \infty) \to \mathbb{R}$, where $h_n(x) = x^{1/n}$. We claim that

$$\begin{aligned} h_n &\to h \\ h(x) &= \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases} \\ &= \mathbf{1}_{(0,\infty)} \end{aligned}$$

Since, for any $b > 0$, $\left(b^{1/n}\right) \to 1$

**Example 4:** Let $g_n : [0, \infty) \to \mathbb{R}$, where $g_n(x) = \frac{x^n}{1+x^n}$. We claim that $g_n \to g$, where $g : [0.\infty) \to \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}$$

When $x > 1$, we have

$$\begin{aligned} |g_n(x) - 1| &= \left|\frac{x^n}{1+x^n} - 1\right| \\ &= \left|\frac{-1}{1+x^n}\right| \\ &= \frac{1}{1+x^n} \\ &\to 0 \end{aligned}$$

Uniform Convergence

A sequence of functions $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$ converges uniformly to $f \in \mathcal{F}(\Omega, \mathbb{R})$ if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni (n \geq N_\varepsilon)\,(\forall x \in \Omega) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Equivalently,

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni n \geq N_\varepsilon \Rightarrow \sup_{x \in \Omega} |f_n(x) - f(x)| < \varepsilon.$$

Uniform Convergence Examples

**Example 1:** Let $f_n : [0,4] \to \mathbb{R}$.

$$f_n(x) = \frac{x}{x+n}$$

We claim that

$$f_n \to \mathbf{o} \text{ uniformly.}$$

We start by examining the maximum size of $f_n(x)$:

$$\begin{aligned} |f_n(x) - \mathbf{o}(x)| &= \frac{x}{x+n} \\ &\leq \frac{x}{n} \\ &\leq \frac{4}{n} \end{aligned}$$

so,

$$\sup_{x \in [0,4]} |f_n(x) - \mathbf{o}(x)| \leq \frac{4}{n}.$$

Given $\varepsilon > 0$, find $N$ so large such that $\frac{1}{N} < \frac{\varepsilon}{4}$. Then, for $n \geq N$,

$$\begin{aligned} \sup_{x \in \Omega} |f_n(x) - f(x)| &\leq \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \varepsilon \end{aligned}$$

**Negating Uniform Convergence**

$$(f_n)_n \nrightarrow f \text{ uniformly}$$
$$\Leftrightarrow$$
$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N}) \ni (\exists n_0 \geq N)(\exists x_0 \in \Omega)\,|f_{n_0}(x_0) - f(x_0)| \geq \varepsilon_0$$
$$\Leftrightarrow$$
$$(\exists \varepsilon_0 > 0)(\forall N)(\exists (x_k)_k \in \Omega)(\exists (f_{n_k})\, k) \ni |f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

---

($\Rightarrow$) We know $\exists \varepsilon_0$ satisfying condition (1). Let $N = 1$. We know $\exists n_1 \geq 1$ such that $\exists x_1 \in \Omega$ with $|f_{n_1}(x_1) - f(x_1)| \geq \varepsilon_0$.

Now, set $N = n_1 + 1$. Then, $\exists n_2 \geq N$ and $x_2 \in \Omega$ satisfying condition (1).

Defining $n_k$ and $x_k$ recursively, we have a natural sequence $(n_k)_k$, and thus a subsequence of $f_n$, thereby satisfying condition (2).

**Uniform Convergence: Contingency on Domain**

Does $(f_n)_n \to f$ uniformly converge on $[0, 1]$, where $f_n(x) = x^n$, $f = \delta_1$?

---

Let $x_k = \left(\frac{1}{2}\right)^k$, $n_k = k$.

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= |f_{n_k}(x_k)| \\ &= \left(\frac{1}{2^{1/k}}\right)^k \\ &= \frac{1}{2}\end{aligned}$$

Setting $\varepsilon_0 = 1/2$, we have that it does *not* converge uniformly.

---

Recall $g_n : [0, \infty) \to \mathbb{R}$, where

$$g_n(x) = \frac{nx}{1 + n^2x^2}$$

We saw that $(g_n)_n \to \mathbf{o}$ pointwise. However, it is *not* uniformly convergent. Take $x_k = \frac{1}{k}$, and $n_k = k$. Then,

$$\begin{aligned}|g_{n_k}(x_k) - \mathbf{o}(x_k)| &= \frac{k \cdot \frac{1}{k}}{1 + k^2 \cdot \frac{1}{k^2}} \\ &= 1/2 \\ &= \varepsilon_0.\end{aligned}$$

However, $g_n \to g$ on $[a, \infty)$ where $a > 0$. Let $x \in [a, \infty)$

$$\begin{aligned} |g_n(x) - \mathbf{0}(x)| &= \frac{nx}{1+n^2x^2} \\ &\leq \frac{nx}{n^2x^2} \\ &= \frac{1}{nx} \\ &\leq \frac{1}{na} \end{aligned}$$

therefore,

$$\sup_{x\in[a,\infty)} |g_n(x) - \mathbf{0}(x)| \leq \frac{1}{na}$$

**Further Examining Pointwise and Uniform Convergence**

Consider the family of functions

$$\begin{aligned} f_n &: [0,\infty) \to \mathbb{R} \\ f_n(x) &= e^{-nx} \end{aligned}$$

Upon examination, we can see that:

$$f_n \xrightarrow{\text{p.w.}} \delta_0 = \mathbb{1}_{\{0\}}.$$

However, the convergence is *not* uniform.

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Then, setting $\varepsilon_0 = e^{-1}$

$$\begin{aligned} |f_{n_k}(x_k) - \delta_0(x_k)| &= \left|f_k\left(\frac{1}{k}\right)\right| \\ &= e^{-1} \\ &\geq \varepsilon_0 \end{aligned}$$

**Uniform Norm**

For $f \in \mathcal{F}(\Omega, \mathbb{R})$, the **uniform norm** or **infinity norm** is defined as:

$$\|f\|_u = \sup_{x\in\Omega} |f(x)|.$$

Importantly, the value of the uniform norm is dependent on $\Omega$.

---

The uniform norm satisfies the rules we desire of any other norm:

- Scalar multiplication: $\forall t \in \mathbb{R}$, $\|tf\|_u = |t|\|f\|_u$
- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Zero Property: $\|f\|_u = 0 \Leftrightarrow f = \mathbf{0}_{\mathbb{R}}$
- Algebraic Property: $\|fg\|_u \leq \|f\|_u \cdot \|g\|_u$.

---

$$\ell_\infty(\Omega) = \{f \in \mathcal{F}(\Omega, \mathbb{R}) \mid \|f\|_u < \infty\}$$

is a normed vector space.

**A Different Definition of Uniform Convergence**

Given $(f_k)_k,\ f \in \ell_\infty(\Omega)$, we have

$$(f_k)_k \xrightarrow{\text{uniformly}} f \Leftrightarrow (\|f_k - f\|_u)_k \to 0$$

**Applying The New Definition**

Let

$$\begin{aligned} g_n &: [0,1] \to \mathbb{R} \\ g_n(x) &= x^n(1-x) \end{aligned}$$

Clearly, $(g_n)_n$ belongs to $\ell_\infty([0,1])$. We can see that

$$(g_n)_n \xrightarrow{\text{p.w.}} \mathbf{0}$$

To show that the convergence is uniform, we must find

$$\|g_n - \mathbf{0}\|_u \xrightarrow{n\to\infty} \mathbf{0},$$

or

$$\begin{aligned} \sup_{x\in[0,1]} x^n(1-x) &\to 0 \\ \frac{d}{dx}(x^n(1-x)) &= nx^{n-1} - (n+1)x^n \\ nx^{n-1} &= (n+1)x^n \\ x &= \frac{n}{n+1} \\ \sup_{x\in[0,1]} x^n(1-x) &= \left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) \\ &= \frac{1}{(1+1/n)^n}\left(\frac{1}{n+1}\right) \\ &\to 0 \end{aligned}$$

**Root Test and Series Convergence**

Let

$$\limsup_{k\to\infty} |x_k|^{1/k} = \rho.$$

If $\rho < 1$, then $\sum_k x_k$ converges absolutely. If $\rho > 1$, then $\sum_k x_k$ diverges.

---

Suppose $\rho < 1$. Let $\rho < r < 1$. By property of inf, $\exists N \in \mathbb{N}$ large such that $r \geq \sup_{k\geq N} |x_k|^{1/k}$.

Therefore, $\forall k \geq N$, we have

$$x_k^{1/k} \leq r$$
$$x_k \leq r^k \qquad \forall k \geq N$$

Therefore,

$$\sum_k x^k \leq \underbrace{\sum_{k=1}^{N-1} + \sum_{k \geq N} r^k}_{\text{converges: } r<1}$$

---

If $\limsup |x_k|^{1/k} = \rho > 1$, we can find a subsequence $(x_{k_\ell})^{1/k_\ell} \xrightarrow{\ell \to \infty} \rho$. We cannot have $((x_k)_k)^{1/k} \to 0$. Thus, the series diverges.

**Absolute Convergence**

A series $\sum_k x_k$ converges absolutely if $\sum_k |x_k|$ converges. If a series converges absolutely, then it always converges.

---

Let $s_n = \sum_{k=1}^n x_k$, $t_n = \sum_{k=1}^n |x_k|$. Let $m > n$. Then,

$$\begin{aligned} |s_m - s_n| &= \left| \sum_{k=n+1}^m x_k \right| \\ &\leq \sum_{k=n+1}^m |x_k| && \text{Triangle Inequality} \\ &= |t_m - t_n| \end{aligned}$$

By assumption, $(t_n)_n$ converges, and thus is Cauchy. By the above inequality, $(s_n)_n$ is Cauchy, and thus convergent.

**Series of Functions**

Given a sequence of functions $(f_k)_k \in \mathcal{F}(\Omega, \mathbb{R})$, we say that the series

$$\sum_k f_k$$

converges pointwise to $f$ in $\mathcal{F}(\Omega, \mathbb{R})$ if

$$s_n = \left( \sum_{k=1}^n f_k \right)_n$$

converges to $f$ pointwise.

$$\sum_{k=1}^{\infty} f_k(x) = f(x) \qquad \forall x \in \Omega$$

**Pointwise Convergence of Series of Functions, Example**

Let $f_k : (-1,1) \to \mathbb{R}$, where $f_k = x^k$. Then,

$$\sum_{k=0}^{\infty} f_k \to f(x) = \frac{1}{1-x}$$

**Uniform Convergence of Series**

$\sum f_k$ converges to $f$ **uniformly** if

$$s_n = \left(\sum_{k=1}^{n} f_k\right)_n$$

converges to $f$ uniformly.

**Uniform Convergence of Series**

We know that $\sum_{k=0}^{\infty} x_k$ converges pointwise to $s(x) = \frac{1}{1-x}$ on $(-1,1)$. Does it converge *uniformly* on the same interval?

---

**Claim:** The convergence is not uniform on $(-1,1)$, but convergence is uniform on $[a,b]$, where $-1 < a \leq b < 1$.

Let $s_n(x) = \sum_{k=0}^{n} x^k$.

$$\begin{aligned} |s_n(x) - s(x)| &= \left|\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}\right| \\ &= \frac{|x|^{n+1}}{1-x} \end{aligned}$$

Let $c = \max\{|a|,|b|\} < 1$

$$\begin{aligned} &\leq \frac{c^{n+1}}{1-b} && \forall a \leq x \leq b \\ \sup_{x\in[a,b]} |s_n(x) - s(x)| &\leq \frac{c^{n+1}}{1-b} \\ &\to 0 \end{aligned}$$

To show non-uniform convergence on $(-1,1)$, let $x_\ell = 1 - \frac{1}{\ell}$, and let $n_\ell = \ell$.

$$\begin{aligned} |s_{n_\ell}(x_\ell) - s(x_\ell)| &= \frac{\left(1-\frac{1}{\ell}\right)^{\ell+1}}{\frac{1}{\ell}} \\ &= \ell\left(1-\frac{1}{\ell}\right)^{\ell}\left(1-\frac{1}{\ell}\right) \\ &= (\ell-1)\left(1-\frac{1}{\ell}\right)^{\ell} \\ &\to \infty \end{aligned}$$

since $\left(1-\frac{1}{\ell}\right)^{\ell} \to \frac{1}{e}$.

**Weierstrass $M$-test**

Consider a sequence of functions $(f_k)_k$ in $\ell_\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}$.

If $\sum_{k=1}^{\infty} \|f\|_u$ converges, then $\sum_k f_k$ converges uniformly and absolutely on $\Omega$.

---

Set $M_k = \|f_k\|_u$. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\sum_{n+1}^{m} M_k < \varepsilon \qquad \forall m > n \geq N$$

since $\sum_{k=1}^{\infty} M_k$ is convergent, and thus Cauchy.

Let $s_n(x) = \sum_{k=1}^{n} f_k(x)$. So,

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left|\sum_{k=n+1}^{m} f_k(x)\right| \\ &\leq \sum_{k=n+1}^{m} |f_k(x)| \\ &\leq \sum_{k=n+1}^{m} M_k \\ &< \varepsilon & \text{whenever } m > n \geq N \end{aligned}$$

For every $x \in \Omega$, $s_n(x)$ is Cauchy. So, $\forall x \in \Omega$, $s(x) := \lim s_n(x)$ exists.

Additionally, $\forall x \in \Omega$,

$$|s_m(x) - s_n(x)| < \varepsilon.$$

Let $m \to \infty$. Then,

$$\begin{aligned} |s(x) - s_n(x)| &< \varepsilon & \forall x \in \Omega,\ \forall n \geq N \\ \sup_{x\in\Omega} |s(x) - s_n(x)| &< \varepsilon. & \forall n \geq N \end{aligned}$$

**Weierstrass $M$-test Examples**

Consider

$$\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2},$$

where $f_k : \mathbb{R} \to \mathbb{R}$. Then, $\|f_k\|_u \leq \frac{1}{k^2}$. So,

$$\begin{aligned} \sum \|f_k\|_u &\leq \sum \frac{1}{k^2} \\ &< \infty. \end{aligned}$$

Thus, $\sum \frac{1}{x^2+k^2}$ converges absolutely and uniformly.

---

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges $\forall x \in \mathbb{R}$, and converges *uniformly* on any closed and bounded interval $[a, b]$.

Power Series

A **power series** *centered* at $c$ in $\mathbb{R}$ is a formal series of functions

$$\sum_{k=0}^{\infty} a_k(x-c)^k.$$

---

We want to examine such things as the convergence and the uniformity of such convergence.

Given $\sum a_k(x-c)^k$, set $\rho = \limsup |a_k|^{1/k}$ and $r = 1/\rho$.

Cauchy-Hadamard Theorem

A power series

$$\sum_{k=1}^{\infty} a_k(x-c)^k$$

converges absolutely on $(c-r, c+r)$, diverges on $\overline{[c-r, c+r]}$, and uniformly convergent on $[a, b]$, $c-r < a \leq b < c+r$.

---

Let $\sum_{k=1}^{\infty} a_k(x-c)^k$, where $x_k = a_k(x-c)^k$.

$$|x_k|^{1/k} = |a_k|^{1/k}|x-c|$$

Root test:

$$\begin{aligned} \limsup_{k\to\infty} |x_k|^{1/k} &= |x-c| \limsup_{k\to\infty} |a_k|^{1/k} \\ &= |x-c|\rho \end{aligned}$$

Absolute Convergence:

$$\begin{aligned} |x-c|\rho &< 1 \\ |x-c| &< \frac{1}{\rho} \end{aligned}$$

Divergence:

$$\begin{aligned} |x-c|\rho &> 1 \\ |x-c| &> \frac{1}{\rho} \end{aligned}$$

Let $[a, b] \subset (c-r, c+r)$. Set $d = \max\{|a-c|, |b-c|\}$. So,

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left|\sum_{k=n+1}^{m} a_k(x-c)^k\right| \\ &\leq \sum_{k=n+1}^{m} |a_k||x-c|^k \\ &\leq \sum_{k=n+1}^{m} |a_k||d|^k \end{aligned}$$

we know that $d < r \Rightarrow d/r < 1 \Rightarrow d\rho < 1 \Rightarrow \rho < 1/d$. Pick $\rho < p < 1/d$. So, $\exists N \in \mathbb{N}$ with

$$\sup_{k\geq N} |a_k|^{1/k} < p$$
$$|a_k| < p^k$$

So, if $m > n \geq N$, we have

$$|s_m(x) - s_n(x)| \leq \sum_{n+1}^{m} (rd)^k$$
$$\sup_{x\in[a,b]} |s_m(x) - s_n(x)| \leq \sum_{n+1}^{m} (rd)^k$$

Given $\varepsilon > 0$, find $N_1 \in \mathbb{N}$ with $m > n \geq N_1$ meaning

$$\sup_{x\in[a,b]} |s_m(x) - s_n(x)| \leq \sum_{n+1}^{m} (rd)^k$$
$$< \varepsilon$$

Let $K = \max\{N, N_1\}$. With $m > n \geq K$, we have

$$\sup_{x\in[a,b]} |s_m(x) - s_n(x)| < \varepsilon$$

Letting $m \to \infty$, we have

$$\sup_{x\in[a,b]} |s(x) - s_n(x)| < \varepsilon.$$

So, $(s_n(x))_n \to s(x)$ uniformly on $[a, b]$.

## Limit Points

Recall: If $c \in \mathbb{R}$, and $\delta > 0$, then $V_\delta(x) = (c - \delta, c + \delta)$.

The *deleted neighborhood* $\dot{V}_\delta = (c - \delta, c) \cup (c, c + \delta) = V_\delta \setminus \{c\}$.

(i) $x \in V_\delta(c) \Leftrightarrow |x - c| < \delta$

(ii) $x \in \dot{V}_\delta(c) \Leftrightarrow 0 < |x - c| < \delta$

Let $D \subseteq \mathbb{R}$. A number $c \in \mathbb{R}$ is a *cluster point* or *limit point* of $D$ if

$$(\forall\delta > 0)(\exists x \in D \cap \dot{V}(c)) \Leftrightarrow \forall\delta > 0,\ \dot{V}(c) \cap D \neq \emptyset$$

**Remarks** If $c$ is a limit point of $D$, $c$ may or may not belong to $D$. If $c \in D$, then $c$ is not necessarily a limit point.

**Examples:**

- Let $D = (0, 1)$. Is $c = 0$ a limit point of $D$?

  Yes — given any $\delta > 0$, $\dot{V}_\delta(0) \cap (0, 1) = (0, \min(1.\delta))$. We have that $[0, 1]$ is the set of all limit points of $D$.
- Let $D = \mathbb{N}$. Then, $D$ admits no limit points.
- Additionally, all finite sets have no limit points.

- If $D = \mathbb{Q}$, then the set of limit points of $\mathbb{Q}$ is $\mathbb{R}$.

  Given any $t \in \mathbb{R}$, $\delta > 0$,

  $$\dot{V}_\delta \cap \mathbb{Q} \neq \emptyset$$

  because $\mathbb{Q}$ is dense.
- If $D = \left\{\frac{1}{n} \mid n \geq 1\right\}$, then $\{0\}$ is the set of limit points of $D$.

**Alternative Limit Point Definition**

Let $D \subseteq \mathbb{R}$, $c \in \mathbb{R}$. The following are equivalent:

(1) $c$ is a limit point of $D$.

(2) $\exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$

---

(2) $\Rightarrow$ (1) Let $\delta > 0$. Then, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $0 < |x_n - c| < \delta$. Thus $x_N \in \dot{V}_\delta(c) \cap D$.

(1) $\Rightarrow$ (2) Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in D \cap \dot{V}_{1/n}(c)$. So, $x_n \neq c$, $x_n \in D$, and $|x_n - c| < 1/n$. So, $(x_n)_n \to c$.

**Definition of a Limit**

Let $f : D \to \mathbb{R}$, and $c$ a limit point of $D$. Let $L \in \mathbb{R}$.

$$\lim_{x \to c} f(x) = L \overset{\text{defn.}}{\Longleftrightarrow} (\forall \varepsilon > 0)(\exists \delta > 0) \ni \forall x \in \dot{V}_\delta(c) \cap D,\ f(x) \in V_\varepsilon(L)$$

**Limit Proof Examples**

$$\lim_{x \to c} ax + b = ac + b \qquad a \neq 0$$

**Preliminary Work:**

$$\begin{aligned} |f(x) - L| &= |ax + b - (ac + b)| \\ &= |ax - ac| \\ &= |a||x - c| \end{aligned}$$

**Proof:** Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{|a|}$.

$$\begin{aligned} 0 < |x - c| &< \delta \\ 0 < |x - c| &< \frac{\varepsilon}{|a|} \\ |f(x) - L| &= |a||x - c| \\ &< |a|\frac{\varepsilon}{|a|} \\ &= \varepsilon \end{aligned}$$

---

$$\lim_{x \to c} x^2 = c^2$$

**Preliminary Work:**

$$\begin{aligned}|f(x)-L| &= |x^2-c^2|\\ &= |x-c||x+c|\end{aligned}$$

If $0<\delta<1$, and $|x-c|<\delta$, then $|x+c|\leq |x|+|c|\leq 2|c|+1$. In this case,

$$|f(x)-L|\leq (2|c|+1)\,|x-c|.$$

**Proof:** Given $\varepsilon>0$, let $\delta=\frac{1}{2}\min\left(1,\frac{\varepsilon}{2|c|+1}\right)$. This guarantees $\delta<1$. So, if $|x-c|<\delta$,

$$\begin{aligned}|f(x)-L| &\leq (2|c|+1)\,|x-c|\\ &< (2|c|+1)\,|x-c|\\ &< (2|c|+1)\,\frac{\varepsilon}{2|c|+1}\\ &= \varepsilon\end{aligned}$$

---

$$\lim_{x\to c}\frac{1}{x}=\frac{1}{c} \qquad c\neq 0$$

**Preliminary Work:**

$$\begin{aligned}|f(x)-L| &= \left|\frac{1}{x}-\frac{1}{c}\right|\\ &= \frac{1}{|x|}\frac{1}{|c|}|x-c|\end{aligned}$$

If $x\in\left(c-\frac{|c|}{2},c+\frac{|c|}{2}\right)$, then $|x|\geq |c|/2$, so $\frac{1}{|x|}\leq\frac{2}{|c|}$. So,

$$\frac{1}{|x|}\frac{1}{|c|}|x-c|\leq\frac{2}{|c|^2}|x-c|$$

**Proof:** Given $\varepsilon>0$, let $\delta=\frac{1}{2}\min\left(\frac{|c|}{2},\frac{|c|^2}{2}\varepsilon\right)$. If

$$\begin{aligned}0<|x-c|&<\delta\\ |f(x)-L| &\leq \frac{2}{|c|^2}|x-c|\\ &< \frac{2}{|c|^2}\frac{|c|^2}{2}\varepsilon\\ &= \varepsilon\end{aligned}$$

**Uniqueness of Limits**

Let $f:D\to\mathbb{R}$ with $c$ a limit point of $D$. Then, $f$ can have at most one limit.

Suppose toward contradiction that $\lim_{x\to c}f(x)=L_1$ and $\lim_{x\to c}f(x)=L_2$, where $L_1\neq L_2$.

Let $\varepsilon$ be small such that $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. So, $\exists \delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow f(x) \in V_\varepsilon(L_1),$$

and $\exists \delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow f(x) \in V_\varepsilon(L_2).$$

Set $\delta = \min(\delta_1, \delta_2)$. Then,

$$0 < |x - c| < \delta \Rightarrow f(x) \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$$

**Sequential Definition of a Limit**

Let $f : D \to \mathbb{R}$, $c$ a cluster point of $D$. The following are equivalent:

(i) $\lim_{x\to c} f = L$

(ii) $\forall (x_n)_n \in D \setminus \{c\}$ where $(x_n)_n \to c$, we have $(f(x_n))_n \to L$

---

**($\Leftarrow$)** Assume $\lim_{x\to c} f(x) \neq L$. Then, $(\exists \varepsilon_0)\,(\forall \delta > 0)(\exists x \in \dot{V}(c) \cap D)$ with $|f(x) - L| \geq \varepsilon_0$.

Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in \dot{V}_{1/n}(c) \cap D$, with $|f(x_n) - L| \geq \varepsilon_0$.

Note that $0 < |x - c| < 1/n$. So, $(x_n)_n \in D \setminus \{c\}$, and $(x_n)_n \to c$. By (ii), it must be the case that $(f(x_n))_n \to L$.

However, $|f(x_n) - L| \geq \varepsilon_0$. $\perp$

**Divergence and Limit Non-Existence**

Let $f : D \to \mathbb{R}$, and $c$ a cluster point of $D$. Let $L \in \mathbb{R}$. The following are true:

(1) $\lim_{x\to c} f(x) \neq L \Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$ but $f(x_n) \nrightarrow L$

(2) $\lim_{x\to c} f(x)$ DNE $\Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$ and $(f(x_n))_n$ divergent.

---

**(1)** This is a direct negation of the Sequential Definition.

**(2)**

($\Rightarrow$) Suppose toward contradiction, $\forall (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$, we have $(f(x_n))_n$ is convergent.

Pick any two such sequences, $(x_n)_n$ and $(y_n)_n$. We know $(f(x_n))_n \to L_1$, and $(f(y_n))_n \to L_2$.

Consider $(z_n)_n = (x_1, y_1, x_2, y_2, \dots)$. We know that $(z_n)_n \to c$, meaning $(f(z_n))_n \to M$.

The sequence $(f(z_n))_n$ admits two subsequences $(f(x_n))_n \to M$ and $(f(x_n))_n \to M$. Thus, $L_1 = L_2$.

We showed that, for any sequence $(x_n)_n \to c$, $(f(x_n))_n \to L$. Thus, $\lim_{x\to c} f(x)$ exists. $\perp$

## Limit Divergence using Sequences

We want to find $\lim_{x\to c} \mathbf{1}_{\mathbb{Q}}$. Consider two sequences $(r_n)_n \to c$, where $r_n \in \mathbb{Q}$ — this is always possible since the rationals are dense — and $(t_n)_n \to c$, where $t_n \notin \mathbb{Q}$.

Let $(x_n)_n = (r_1, t_1, r_2, t_2, \dots)$. Then, $(x_n) \to c$, but $\left(\mathbf{1}_{\mathbb{Q}}(x_n)\right)_n = (1, 0, 1, 0, \dots)$. Thus, $\lim_{x\to c} \mathbf{1}_{\mathbb{Q}}$ DNE.

## Bounded Functions

Recall that $f : D \to \mathbb{R}$ is bounded on $E \subseteq D$ if $\sup_{x\in E} |f(x)| < \infty$.

If $f : D \to \mathbb{R}$ and $c$ is a cluster point of $D$, if $\lim_{x\to c} f(x) = L$, then $\exists \delta > 0$ such that $f$ is bounded on $\dot{V}_\delta(c) \cap D$.

---

Let $\varepsilon = 1$. Then, $\exists \delta > 0$ such that $x \in \dot{V}_\delta(c) \cap D \Rightarrow |f(x) - L| < 1$. Then,

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L|, \end{aligned}$$

so,

$$\sup_{x\in \dot{V}_\delta(c)} |f(x)| \leq 1 + |L|$$

## Operations with Limits

Let $f, g : D \to \mathbb{R}$, and $c$ is a cluster point of $D$. Let $\alpha \in \mathbb{R}$.

(a) If $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then

  (i) $\lim_{x\to c}(f \pm g) = L \pm M$

  (ii) $\lim_{x\to c}(\alpha f) = \alpha L$

  (iii) $\lim_{x\to c}(fg) = LM$

  (iv) $\lim_{x\to c}\left(\frac{f}{g}\right) = \frac{L}{M}$ if $M \neq 0$

(b) $\lim_{x\to c} |f(x)| = |L|$

(c) $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$, provided $f(x) \geq 0$

(d) If $f(x)$ is a polynomial, then $\lim_{x\to c} f(x) = f(c)$.

(e) If $f(x)$ is rational, then $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, provided $q(c) \neq 0$.

---

Proof of (a)(iii): Let $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$. Then, $(f(x_n))_n \to L$, $(g(x_n))_n \to M$. Then,

$$\begin{aligned} (f \cdot g(x_n)) &= (f(x_n)g(x_n))_n \\ &\to LM && \text{by sequence properties} \end{aligned}$$

Proof of (a)(iv): Let $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$. Then, by the properties of sequences,

$$\begin{aligned} \left(\frac{f}{g}(x_n)\right) &= \left(\frac{f(x_n)}{g(x_n)}\right)_n \\ &\to \frac{L}{M} && \text{provided } M \neq 0 \end{aligned}$$

Proof of (d): Let $p(x) = \sum_{k=0}^{n} a_k x^k$. Then,

$$\begin{aligned}
\lim_{x\to c} p(x) &= \lim_{x\to c}\left(\sum_{k=0}^{n} a_k x^k\right) \\
&= \sum_{k=0}^{n} \lim_{x\to c} a_k x^k && \text{(a)(i)} \\
&= \sum_{k=0}^{n} a_k \lim_{x\to c} x^k && \text{(a)(ii)} \\
&= \sum_{k=0}^{n} a_k \left(\lim_{x\to c} x\right)^k && \text{(a)(i)} \\
&= p(c)
\end{aligned}$$

Proof of (b) Using the properties of sequence, we can show that $(|f(x_n)|)_n \to |L|$ for $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$

**Squeeze Theorem**

If $f : D \to \mathbb{R}$, $c$ is a cluster point of $D$.

(i) If $f(x) \leq b$ for $x$ in a deleted neighborhood of $c$, and if $\lim_{x\to c} f(x) = L$, then $L \leq b$.

(ii) If $f(x) \geq a$ for all $x$ in a deleted neighborhood of $c$, and if $\lim_{x\to c} f(x) = L$, then $L \geq a$.

(iii) If $f, g, h : D \to \mathbb{R}$, and $c$ is a cluster point of $D$. Suppose

$$g(x) \leq f(x) \leq h(x)$$

for all $x$ in some deleted neighborhood of $c$. Suppose $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then, $\lim_{x\to c} f(x) = L$.

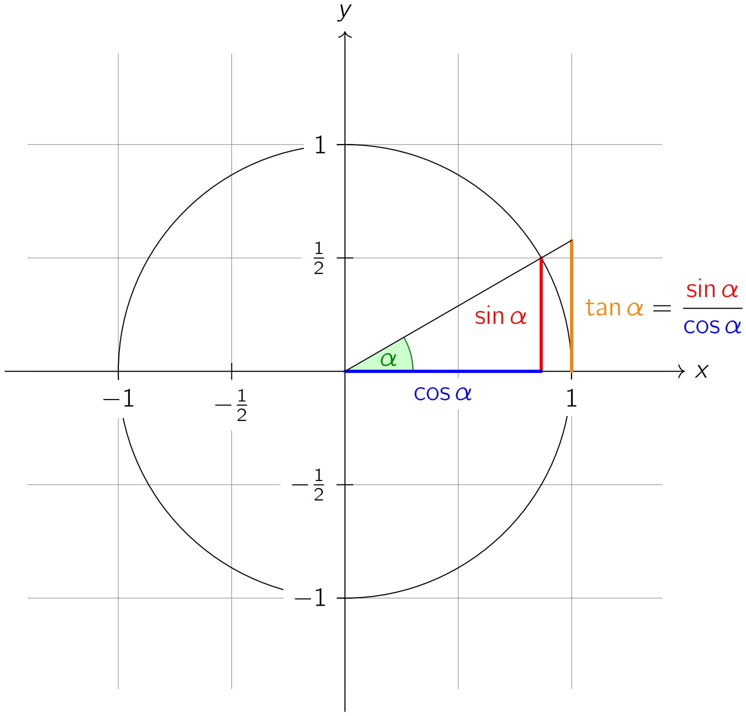
---

Proof of (iii) Let $(x_n)_n \in D \setminus \{c\}$, with $(x_n)_n \to c$. Then, as $n \to \infty$,

$$\begin{gathered}
g(x_n) \leq f(x_n) \leq h(x_n) \\
L \leq f(x_n) \leq L,
\end{gathered}$$

so $f(x_n)_n \to L$.

Trigonometric Limits




We know that

$$0 \leq \sin(x) \leq x$$

so as $x \to 0^+$, $\sin(x) \to 0$. Similarly, if $x \to 0^-$, then

$$\begin{aligned}\lim_{x\to 0^-} \sin(x) &= \lim_{y\to 0^+} \sin(-y)\\ &= -\lim_{y\to 0^+}\sin(y)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 0^+} \cos(x) &= \lim_{x\to 0^+} \sqrt{1-\sin^2(x)}\\ &= 1\\ \lim_{x\to 0^-} \cos(x) &= \lim_{y\to 0^+} \cos(-y)\\ &= \lim_{y\to 0^+}\cos(y)\\ &= 1\end{aligned}$$

**Claim:**

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

**Proof:** Let $x \to 0$

$$\begin{aligned}
\frac{\sin(x)}{2} \leq \frac{x}{2} \leq \frac{\tan(x)}{2} \\
0 \leq \frac{\sin(x)}{x} \leq 1 \\
\cos(x) \leq \frac{\sin(x)}{x} \\
\cos(x) \leq \frac{\sin(x)}{x} \leq 1 \\
1 \leq \frac{\sin(x)}{x} \leq 1
\end{aligned}$$

**Strictly Positive Limits**

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$. Let $c$ be a cluster point of $D$. If $\lim_{x\to c} f(x) = L > 0$, then $\exists \delta > 0$ and $\exists t > 0$ such that $f(x) > t$ for $x \in \dot{V}_\delta(c) \cap D$.

---

Let $\varepsilon = \frac{L}{2}$. Then, $V_\varepsilon = (L/2, 3L/2)$. So, $\exists\delta > 0$ such that $x \in \dot{V}_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$. Set $t = L/2$.

**One-Sided Limits**

Let $f : D \to \mathbb{R}$.

**Cluster Points:**

(i) A number $c \in D$ is a right cluster point if $\forall\delta > 0$, $\exists x \in (c, c+\delta) \cap D$

(ii) A number $c \in D$ is a left cluster point if $\forall\delta > 0$, $\exists x \in (c-\delta, c) \cap D$.

**Limits:**

(i) $\lim\limits_{x\to c^+} f(x) = L \overset{\text{def}}{\Longleftrightarrow}$

$$(\forall\varepsilon > 0)(\exists\delta > 0) \ni x \in (c, c+\delta) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

(ii) $\lim\limits_{x\to c^-} f(x) = L \overset{\text{def}}{\Longleftrightarrow}$

$$(\forall\varepsilon > 0)(\exists\delta > 0) \ni x \in (c-\delta, c) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

**Sequential Definition:**

(i) Let $c$ be a right cluster point of $D$. $\lim_{x\to c^+} f(x) = L$ if and only if $\forall (x_n)_n \in D \cap (c, \infty)$ with $(x_n)_n \to c$, we have $(f(x_n))_n \to L$

(ii) Let $c$ be a left cluster point of $D$. $\lim_{x\to c^-} f(x) = L$ if and only if $\forall (x_n)_n \in (-\infty, c) \cap D$ with $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

**Limit Equality**

Let $f : D \to \mathbb{R}$. Let $c$ be a cluster point of $D$.

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

**Infinite Limits**

Let $f : D \to \mathbb{R}$, and $c$ be a limit point of $D$. Then,

$$\lim_{x\to c} f(x) = \infty \overset{\text{def}}{\iff} (\forall M \geq 0)(\exists \delta > 0) \ni x \in \dot{V}_\delta(c) \cap D \Rightarrow f(x) \geq M$$

---

We can also define

$$\lim_{x\to c} f(x) = -\infty$$
$$\lim_{x\to c^\pm} f(x) = \pm\infty$$

**Infinite Limits, Example**

$$\lim_{x\to 1^-} \frac{1}{1-x} = -\infty$$

**Proof:** Let $M \geq 0$ be large. We want $f(x) \geq M$.

$$\frac{1}{1-x} \geq M$$
$$1 - x \leq \frac{1}{M}$$
$$x \geq 1 - \frac{1}{M}$$

Set $\delta = \frac{1}{M}$. If $x \in (1-\delta, 1)$, then $x \geq 1 - \frac{1}{M}$. So, by our work above, $f(x) \geq M$.

**Limits at Infinity**

Let $f : [a, \infty) \to \mathbb{R}$, $L \in \mathbb{R}$. Then,

$$\lim_{x\to\infty} f(x) = L \overset{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \geq a) \ni x \geq K \Rightarrow f(x) \in V_\varepsilon(L)$$

---

Similarly, we can define for $f : (-\infty, b] \to \mathbb{R}$, $L \in \mathbb{R}$

$$\lim_{x\to-\infty} f(x) = L \overset{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \leq b) \ni x \leq K \Rightarrow f(x) \in V_\varepsilon(L)$$

and for $f : [a, \infty)$ where

$$\lim_{x\to\infty} f(x) = \infty \overset{\text{def}}{\iff} (\forall M \geq 0)(\exists K \geq a) \ni x \geq K \Rightarrow f(x) \geq M$$

and the respective sequential definitions.

**Limits at Infinity, Example**

Let $n \in \mathbb{N}$.

$$\lim_{x\to\infty} x^n = \infty$$

**Proof:** Let $M$ be large. We want $x^n \geq M$. Then, $x \geq M^{1/n}$. Set $K = M^{1/n}$.

---

$$\lim_{x\to -\infty} x^n = \begin{cases} +\infty, & n = 2k \\ -\infty, & n = 2k+1 \end{cases}$$

---

$$p(x) = \sum_{k=1}^{n} a_k x^k$$
$$\lim_{x\to\infty} p(x) = \begin{cases} +\infty, & a_n > 0 \\ -\infty & a_n < 0 \end{cases}$$

Let $g(x) = x^n$.

$$\frac{p(x)}{g(x)} = a_n + a_{n-1}\frac{1}{x} + \cdots + a_0\frac{1}{x^n}$$
$$\lim_{x\to\infty} \frac{p(x)}{g(x)} = a_n$$

**Lemma**

If $f, g : [a,\infty) \to \mathbb{R}$, and $g(x) > 0$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$$

(1) If $L > 0$, then $\lim_{x\to\infty} f(x) = \infty \Leftrightarrow \lim_{x\to\infty} g(x) = \infty$

(2) If $L < 0$, then $\lim_{x\to\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to\infty} g(x) = +\infty$

Apply the lemma to $p(x)$, $x^n$.

**Continuity**

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$. Let $c \in D$. The function $f$ is continuous at $c$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \ni x \in V_\delta(c) \cap D \Rightarrow f(x) \in V_\varepsilon(f(c))$$

**Remark:** Here, $c$ may not be a cluster point of $D$.

For example, let

$$f(x) = \begin{cases} x & x = -1 \\ x^2 & x \geq 0 \end{cases}$$
$$D = \{-1\} \cup [0,\infty)$$

Here, $f$ is continuous at $c = -1$. Given any $\varepsilon > 0$, let $\delta = 1/2$. Then, if $x \in V_{1/2}(-1) \cap D$, $x = -1$, meaning $|f(x) - f(-1)| = 0 < \varepsilon$

**Continuity and Limits**

If $f : D \to \mathbb{R}$, $c \in D$ and $c$ a cluster point of $D$, the following are equivalent:

(i) $f$ is continuous at $c$

(ii) $\lim_{x\to c} f(x) = f(c)$

**Remark:** We are deign to use the second definition as *the* definition of continuity due to the fact that it removes the possibility of those mentioned above.

**Sequential Definition of Continuity**

Let $f : D \to \mathbb{R}$, $c \in D$. The following are equivalent:

(i) $f$ is continuous at $x = c$

(ii) $\forall (x_n)_n$ in $D$ with $(x_n)_n \to c$, we have $(f(x_n))_n \to f(c)$

**Left and Right Continuity**

Let $f : D \to \mathbb{R}$, $c \in D$.

- $f$ is left-continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \ni 0 \leq c - x < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$
$$\forall (x_n)_n \in D,\ x_n \leq c,\ (x_n)_n \to c \text{ we have } (f(x_n))_n \to f(c)$$

**Continuity on Sets**

Let $f : D \to \mathbb{R}$.

(1) $f$ is continuous on $E \subseteq D$ if $f$ is continuous at each $c \in E$.

(2) $f$ is continuous on $[a, b]$ if

  (i) $f$ is continuous on $(a, b)$
  (ii) $f$ is left-continuous at $b$
  (iii) $f$ is right-continuous at $a$

---

(1) Polynomials are continuous on $\mathbb{R}$ because $\lim_{x\to c} p(x) = p(c)$.

(2) Rational functions are continuous on their domain.

(3) $f : \mathbf{1}_{\mathbb{Q}}$ is continuous nowhere:

**Case 1:** Suppose $c \in \mathbb{Q}$. Let $(t_n)_n \to c$ with $t_n \in \mathbb{R} \setminus \mathbb{Q}$. Then, $(f(t_n))_n = 0 \to 0 \neq f(c) = 1$
**Case 2:** Let $c \in \mathbb{R} \setminus \mathbb{Q}$. Let $(r_n)_n \to c$ with $r_n \in \mathbb{Q}$. Then, $(f(r_n))_n = 1 \to 1 \neq f(c) = 0$

**Discontinuity**

$f : D \to \mathbb{R}$ is not continuous at $x = c$ if $\exists (x_n)_n$ in $D$ with $(x_n)_n \to c$ and $(f(x_n))_n \nrightarrow f(c)$

**Discontinuity of the Sign Function**

$$\operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at $x = 0$, since $(x_n)_n = \frac{1}{n} \to 0$ but $(f(x_n))_n = 1 \neq 0$.

**Thomae's Function**

Let

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \in \mathbb{Q},\ b \in \mathbb{N},\ \gcd(a,b) = 1 \\ 1 & x = 0 \end{cases}$$

**Claim 1:** $f$ is not continuous at $x \in \mathbb{Q}$: find a sequence $(t_n)_n$ of irrationals with $(t_n)_n \to x$. Then, $(f(t_n))_n = 0 \neq f(x) = \frac{1}{b}$

**Claim 2:** $f$ is continuous at $t \in \mathbb{R} \setminus \mathbb{Q}$: let $t \in \mathbb{R} \setminus \mathbb{Q}$, $t > 0$. Let $n \in \mathbb{N}$. Consider

$$A_n = \left\{ \frac{a}{b} \mid 1 \leq b \leq n \right\} \cap (t-1, t+1).$$

We claim that $A_n$ is finite.

$$\begin{aligned} t - 1 &< \frac{a}{b} < t + 1 \\ b(t-1) &< a < b(t+1) \\ t - 1 &< a < n(t+1), \end{aligned}$$

so there are finitely many values of $a$ and finitely many values of $b$ — therefore, $A_n$ is finite. One can find $\delta > 0$ such that $(t - \delta, t + \delta) \cap A_n = \emptyset$

Given $\varepsilon > 0$, find $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \varepsilon$. Let $\delta$ be such that $(t-\delta, t+\delta) \cap A_{n_0} = \emptyset$. If $x \in (t-\delta, t+\delta)$,

$$\begin{aligned} |f(x) - f(t)| &= |f(x)| \\ &= \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \text{ lowest terms} \end{cases} \end{aligned}$$

but $\frac{1}{b} < \varepsilon$ because $x \notin A_{n_0}$, meaning $b > n_0$.

**Extension of Function**

Consider

$$g(x) = \sin\left(\frac{1}{x}\right) \qquad x \neq 0$$

Assuming that $g$ is continuous on its domain, can we find a $\tilde{g} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{g}(x) = g(x) \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

If such a $\tilde{g}$ existed, we would expect that $\lim_{x\to 0} \tilde{g}(x) = \tilde{g}(0)$. But, $\lim_{x\to 0} \tilde{g}(x) = \lim_{x\to 0} g(x)$.

However, since $\lim_{x\to 0} g(x)$ DNE, so such an extension does not exist.

Therefore, $x = 0$ is known as a non-removable discontinuity (i.e., we cannot create an extension of the function that "fills in" the function).

However, not all discontinuities involving $\sin(1/x)$ are non-extendible:

$$f(x) = x\sin\left(\frac{1}{x}\right)$$
$$\tilde{f}(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

**Jump Discontinuities**

Suppose $\lim_{x\to c^-} f(x) = L$, $\lim_{x\to c^+} f(x) = R$. If $L \neq R$, then $x = c$ is a jump discontinuity.

**Lipschitz Functions**

A function $f : D \to \mathbb{R}$ is called Lipschitz if $\exists c \geq 0$ with

$$|f(x) - f(y)| \leq c|x - y| \qquad \forall x, y \in D$$

The linear function $f(x) = ax + b$ is a Lipschitz function. Additionally, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $\|T(\vec{v}) - T(\vec{w})\| \leq c\|\vec{v} - \vec{w}\|$ for any norm on $\mathbb{R}^n$ and $\mathbb{R}^m$.

- If $c < 1$, then $f$ is a contraction.
- If $c = 1$ and $|f(x) - f(y)| = |x - y|$, $f$ is called an isometry.

Lipschitz functions are on their domain:

**Proof:** Let $c \in D$, let $\varepsilon > 0$. Set $\delta = \varepsilon/c$.

$$\begin{aligned} |x - c| &< \delta \\ |f(x) - f(c)| &\leq c|x - c| \\ |f(x) - f(c)| &< c\delta \\ &= \varepsilon \end{aligned}$$

If $f(x) = \sin(x)$, then

$$\begin{aligned} |\sin(x) - \sin(y)| &= \left|2\sin\left(\frac{1}{2}(x - y)\right)\cos\left(\frac{1}{2}(x + y)\right)\right| \\ &\leq 2\frac{1}{2}|x - y| \\ &= |x - y| \end{aligned}$$

**Equality over a Dense Subset**

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. Let $E \subseteq \mathbb{R}$. If $f(x) = g(x)$ $\forall x \in E$, then $f = g$.

---

Let $t \in \mathbb{R}$. Since $E$ is dense, $\exists (x_n)_n \in E$ such that $(x_n)_n \to t$. So, $(f(x_n))_n \to t$ because $f$ is continuous, and $(g(x_n))_n \to g(t)$ because $g$ is continuous.

However, since $f(x_n) = g(x_n)\ \forall x_n$, it must be the case that $f(t) = g(t)$.

**Boundedness over a Dense Subset**

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f|_E$ is bounded. That is, $\exists c$ such that

$$|f(x)| \leq c. \qquad \forall x \in E$$

Then, $f$ is bounded.

---

Let $t \in \mathbb{R}$. Since $E$ is dense, $\exists (x_n)_n \in E$ such that $(f(x_n))_n \to t$. Then,

$$|f(x_n)_n| \leq c,$$

meaning that $f(t) \leq c$.

**Strictly Positive Continuity**

If $f$ is continuous at $x = c$ and $f(c) > 0$, then $\exists \delta > 0$ and $\exists m > 0$ with $f(x) \leq m\ \forall x \in V_\delta(c)$. Similarly for the negative case.

---

Let $\varepsilon = f(c)/2 > 0$. Then, $\exists \delta > 0$ such that $\forall x \in V_\delta(c)$, $f(x) \in V_\varepsilon(f(c)) = (f(c)/2, 3f(c)/2)$. Set $m = f(c)/2$.

**Continuity Under Operations**

Let $f, g : D \to \mathbb{R}$, $c \in D$.

(1) If $f, g$ are continuous at $x = c$, then $f \pm g$ are continuous at $x = c$. Similarly, if $f, g$ are continuous on $D$, then $f \pm g$ is continuous on $D$.

(2) Let $\alpha \in \mathbb{R}$. If $f$ is continuous at $x = c$ or on $D$, then $\alpha f$ is continuous at $x = c$ or $D$ respectively.

(3) If $f, g$ are continuous at $x = c$ or on $D$, then $f \cdot g$ is continuous on $x = c$ or $D$ respectively.

(4) If $f, g$ are continuous at $x = c$, and $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at $c$. Likewise, if $f, g$ are continuous on $D$ and $g(x) \neq 0\ \forall x \in D$, then $\frac{f}{g}$ is continuous.

(5) If $g$ is continuous at $x = c$ and $f$ is continuous at $d = g(c)$, then $f \circ g$ is continuous at $x = c$. If $\operatorname{ran}(g) \subseteq \operatorname{dom}(f)$, with $f, g$ continuous on their domain, then $f \circ g$ is continuous.

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Remark on (4): If $g(c) \neq 0$, then $g \neq 0$ on a $\delta$-neighborhood of $c$.

Proof of (5): Let $(x_n)_n \to c$. Then, $g(x_n)_n \to g(c)$. So, $(f(g(x_n)))_n \to f(g(c))$.

**Further Operations**

(1) If $f : D \to \mathbb{R}$ is continuous, and $f(x) \geq 0$ on $D$, then $\sqrt{f(x)}$ is continuous on $D$.

(2) If $f : D \to \mathbb{R}$ is continuous on $D$, then $|f(x)|$ is continuous.

(3) Polynomials and Rational functions are continuous on their domain.

(4) If $f(x), g(x)$ are continuous, then $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$.

## Fundamental Theorem of Continuous Functions on $[a, b]$

**Boundedness Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous, then $\|f\|_u < \infty$.

**Proof:** Suppose it is not the case. Given any $n \geq 1$, $\exists x_n \in [a, b]$ with $|f(x_n)| \leq n$. We thus have a sequence $(x_n)_n \in [a, b]$.

By Bolzano-Weierstrass, $\exists (x_{n_k})_k \to x \in [a, b]$. So, $f(x_{n_k}) \to f(x)$. In particular, $(f(x_{n_k}))_k$ is bounded; however, $f(x_{n_k}) \geq k$. $\perp$

**Note:** It is possible for $f$ to be bounded on an infinite interval where it does not attain the supremum or infimum.

Let $f : D \to \mathbb{R}$.

(1) $f$ has an absolute maximum on $D$ if $\exists x_M \in D$ with $f(x) \leq f(x_M)$ $\forall x \in D$. Notably, this means $\sup_{x \in D} f(x) = f(x_M)$.

(2) $f$ has an absolute minimum on $D$ if $\exists x_m \in D$ with $f(x_m) \leq f(x)$ $\forall x \in D$. Notably, this means $\inf_{x \in D} f(x) = f(x_m)$.

**Extreme Value Theorem (EVT):** If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ admits an absolute minimum and absolute maximum.

**Proof:** We know that $\sup_{x \in [a,b]} f(x) = u < \infty$ by the boundedness theorem. For each $n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that

$$u - \frac{1}{n} < f(x_n) \leq u.$$

Thus, there is a sequence $(x_n)_n \in [a, b]$ — by Bolzano-Weierstrass, $\exists (x_{n_k})_k \to x^*$ for some $x^* \in [a, b]$. So, for each $k$,

$$\begin{aligned} u - \frac{1}{n_k} &< f(x_{n_k}) \leq u \\ u &< f(x^*) \leq u. && \text{since } f \text{ is continuous} \end{aligned}$$

So, by the squeeze theorem, $f(x^*) = u$ is our absolute max.

**Corollary to the Extreme Value Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous with $f(x) > 0$ $\forall x \in [a, b]$, then $\exists \alpha > 0$ such that $f(x) \geq \alpha$ $\forall x \in [a, b]$.

**Proof:** By the previous theorem, we know $\exists x_m \in [a, b]$ such that $f(x) \geq f(x_m)$ $\forall x \in [a, b]$. But $\alpha := f(x_m) > 0$ by definition.

**Location of Roots:** We will use this to prove the Intermediate Value Theorem. Let $f : [a, b] \to \mathbb{R}$ be continuous, Suppose $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. Then, $\exists c \in (a, b)$ such that $f(c) = 0$.

**Proof:** Assume $f(a) < 0$ and $f(b) > 0$. Let $N = \{x \in [a, b] \mid f(x) \geq 0\}$. Since $b \in N$, $N \neq \emptyset$. Let $z = \inf N$. We claim that $f(z) = 0$.

We know that $\exists (x_n)_n \in N$ with $x_n \to z$. Since $(x_n)_n \in N$, $f(x_n) \geq 0$ $\forall n \geq 1$. However, $f(x_n) \to f(z)$ since $f$ is continuous. So, $f(z) \geq 0$.

Suppose toward contradiction that $f(z) > 0$. So, $\exists \delta > 0$ such that $f(x) \geq \frac{f(z)}{2}$ on $(z-\delta, z+\delta)$. Then, $z - \frac{\delta}{2} \in N$. $\perp$

**Intermediate Value Theorem (IVT):** Let $f : I \to \mathbb{R}$, where $I$ is any interval. Suppose $\exists x_1, x_2 \in I$ and $k \in \mathbb{R}$, with $f(x_1) < k < f(x_2)$. Then, $\exists \xi$ strictly between $x_1$ and $x_2$, with $f(\xi) = k$.

**Proof:** Clearly, $x_1 \neq x_2$. Suppose $x_1 < x_2$. Consider $g : [x_1, x_2] \to \mathbb{R}$, $g(x) = f(x) - k$. So, $g$ is continuous (as $f$ is continuous), and $g(x_1) = f(x_1) - k < 0$, and $g(x_2) = f(x_2) - k > 0$. Thus, $\exists \xi \in [x_1, x_2]$ with $g(\xi) = 0$, whence $f(\xi) = k$.

**Corollary to IVT and EVT:** Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\inf_{[a,b]} f \leq k \leq \sup_{[a,b]} f$, then $\exists c \in [a, b]$ with $f(c) = k$.

**Proof:** We know that by EVT, $\exists x_m, x_M$ with $\inf_{[a,b]} f = f(x_m)$ and $\sup_{[a,b]} f = f(x_M)$. So, $f(x_m) \leq k \leq f(x_M)$. Apply IVT.

**Preservation of Intervals 1:** If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b]) = [c, d]$.

**Proof:** Set $c = \inf_{[a,b]} f$ and $d = \inf_{[c,d]} f$. By definition, $c \leq f(x) \leq d$, meaning $f([a, b]) \subseteq [c, d]$. By the previous corollary, if $k \in [c, d]$, then $\exists \xi \in [a, b]$ with $f(\xi) = k$. Thus, $[c, d] \subseteq f([a, b])$.

**Preservation of Intervals 2:** Let $I$ be any interval, and $f : I \to \mathbb{R}$ continuous. Then, $f(I)$ is an interval.

**Proof:** Let $\alpha, \beta \in f(I)$. WLOG, $\alpha < \beta$. We will show that $[\alpha, \beta] \in f(I)$. Say $f(a) = \alpha$ and $f(b) = \beta$ for some $a, b \in I$. Note that $a \neq b$. Let $\alpha < k < \beta$. By IVT, $\exists \xi$ strictly between $a$ and $b$ with $f(\xi) = k$. If $a < b$, then $[a, b] \subseteq I$, and if $b < a$, then $[b, a] \subseteq I$. Thus, $\xi \in I$.