Math 395

Homework 3
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Problem 1

Let $\varphi : R \to S$ be a ring homomorphism. Let $\mathfrak{p} \in \text{Spec}(S)$. We will prove that $\varphi^{-1}(\mathfrak{p}) \subset R$ is an element of $\text{Spec}(R)$.

We will start by showing that $\varphi^{-1}(\mathfrak{p})$ is an ideal. Let $\mathfrak{p} \in \text{Spec}(S)$. Since $0_S \in \text{Spec}(S)$, and $0_R \in \varphi^{-1}(0_S)$, it is the case that $\varphi^{-1}(\mathfrak{p})$ is non-empty.

Let $a, b \in \varphi^{-1}(\mathfrak{p})$. Considering $\varphi(a-b)$, we have $\varphi(a-b) = \varphi(a) - \varphi(b)$, and since $\varphi(a), \varphi(b) \in \mathfrak{p}$, it is the case that $\varphi(a-b) \in \mathfrak{p}$, so $a - b \in \varphi^{-1}(\mathfrak{p})$.

Similarly, for $r \in R$, $a \in \varphi^{-1}(\mathfrak{p})$, $\varphi(ra) = \varphi(r)\varphi(a) \in \mathfrak{p}$, so $ra \in \varphi^{-1}(\mathfrak{p})$. Therefore, $\varphi^{-1}(\mathfrak{p})$ is non-empty, closed under subtraction, and closed under multiplication by elements of the ring, so $\varphi^{-1}(\mathfrak{p})$ is an ideal.
Let $ab \in \varphi^{-1}(\mathfrak{p})$. Then, $\varphi(ab) \in \mathfrak{p}$. So, $\varphi(a)\varphi(b) \in \mathfrak{p}$, meaning either $\varphi(a) \in \mathfrak{p}$ or $\varphi(b) \in \mathfrak{p}$. Therefore, $a \in \varphi^{-1}(\mathfrak{p})$ or $b \in \varphi^{-1}(\mathfrak{p})$. Therefore, $\varphi^{-1}(\mathfrak{p})$ is prime.

Problem 4

Let I, J be ideals of R with $I \subseteq J$. We will show that J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.

We know that J/I is non-empty, as it contains 0_R, so we will show that J/I is closed under subtraction and multiplication by elements of R/I. Thus, by the rules of subrings, for $j_1, j_2, j \in J$, we have

$$\begin{aligned}(j_1 + I) - (j_2 + I) &= (j_1 - j_2) + I \\ &\in J/I,\end{aligned}$$

and, since $j \in R$, by the properties of the quotient ring,

$$(r + I)(j + I) = (rj) + I,$$

and since $rj \in J$ as J is an ideal,

$$(rj + I) \in J/I.$$

Similarly,

$$\begin{aligned}(j + I)(r + I) &= (jr) + I \\ &\in J/I.\end{aligned}$$

Therefore, since J/I is non-empty, closed under subtraction, and closed under multiplication by elements of R/I, it is the case that J/I is an ideal R/I.

Let $\varphi : R/I \to R/J$, $r + I \mapsto r + J$. We will show that φ is a well-defined homomorphism with $\ker(\varphi) = J/I$.

Let $r_1 \sim_{R/I} r_2$. Then, $r_1 = r_2 + i$ for some $i \in I$. Then,

$$\begin{aligned}\varphi(r_1) &= r_1 + J \\ &= (r_2 + i) + J,\end{aligned}$$

and since $I \subseteq J$,

$$= r_2 + J.$$

Thus, φ is well-defined. Additionally,

$$\begin{aligned}\varphi((r_1+I)+(r_2+I)) &= \varphi((r_1+r_2)+I) \\ &= (r_1+r_2)+J \\ &= (r_1+J)+(r_2+J) \\ &= \varphi(r_1+I)+\varphi(r_2+I),\end{aligned}$$

and

$$\begin{aligned}\varphi((r_1+I)(r_2+I)) &= \varphi(r_1r_2+I) \\ &= r_1r_2+J \\ &= (r_1+J)(r_2+J) \\ &= \varphi(r_1+I)\varphi(r_2+I).\end{aligned}$$

Therefore, φ is a homomorphism. The elements that φ maps to $0+J$ are precisely the elements of $j+I$ where $j \in J$, as

$$\begin{aligned}\varphi(j+I) &= j+J \\ &= 0+J.\end{aligned}$$

Thus, $\ker(\varphi) = J/I$.

By the first isomorphism theorem, $(R/I)/(J/I) \cong R/J$.

Problem 5

Define $\varphi : \mathbb{F}_p \to \mathbb{F}_p$, where $\varphi(x) = x^p$ for $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We will show that φ is an isomorphism.

We will start by showing that φ is a well-defined homomorphism. Let $[a]_p = [b]_p$. Then, $a = b + kp$ for some $k \in \mathbb{Z}$. By Fermat's Little Theorem, $\varphi(a) = a^p \equiv [a]_p$, and $\varphi(b+kp) = b^p + p(\ell)$ for some ℓ, so $\varphi(b) \equiv [b]_p$ as well. Thus, $\varphi([a]_p) = \varphi([b]_p)$.

Since φ is well-defined, we find that, for $a, b \in \mathbb{F}_p$,

$$\begin{aligned}\varphi(a+b) &= ([a+b]_p)^p \\ &\equiv [a+b]_p \\ &= [a]_p + [b]_p \\ &\equiv ([a]_p)^p + ([b]_p)^p \\ &= \varphi(a)+\varphi(b),\end{aligned}$$

and

$$\begin{aligned}\varphi(ab) &= ([ab]_p)^p \\ &\equiv [ab]_p \\ &= [a]_p[b]_p \\ &\equiv ([a]_p)^p\,([b]_p)^p \\ &= \varphi(a)\varphi(b),\end{aligned}$$

meaning φ is a homomorphism.

Since, for all $x \in \mathbb{F}_p$, $x \equiv x^p$, it is the case that φ is surjective. Finally, since $\varphi(0) = 0$, and for $x \neq 0$, $\varphi(x) = x \neq 0$, it is the case that $\ker(\varphi) = \{0\}$, meaning φ is injective.

Since φ is a bijective homomorphism, φ is an isomorphism.

Problem 8

Let R be a commutative ring with $I, J \subseteq R$ ideals, and let $\mathfrak{p} \in \operatorname{Spec}(R)$, where $I \cap J \subset \mathfrak{p}$. We will show that either $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$.

Suppose toward contradiction that $I \nsubseteq \mathfrak{p}$ and $J \nsubseteq \mathfrak{p}$. Then, $\exists i \in I$ and $j \in J$ such that $i \notin \mathfrak{p}$ and $j \notin \mathfrak{p}$. However, $ij \in I$ and $ij \in J$ by the definition of ideal, meaning $ij \in I \cap J$. Since $I \cap J \subset \mathfrak{p}$, this means $ij \in \mathfrak{p}$. By the definition of prime ideal, this means $i \in \mathfrak{p}$ or $j \in \mathfrak{p}$, which is a contradiction of our assumption.