Problem (Problem 1)**:** Let R be a Euclidean domain, $n \geq 2$ an integer.

(a) Use the proof of the Smith Normal Form to show that every matrix $A \in \mathrm{GL}_n(R)$ can be written as a product of elementary matrices $E_{ij}(\lambda)$, flip matrices F_{ij}, and a diagonal matrix D.

(b) Now show that the flip matrices can be eliminated from the product in (a), and one can assume that $D = \mathrm{diag}(d, 1, \dots, 1)$. That is, all diagonal entries of D except possibly the $(1, 1)$ entry are equal to 1.

(c) Deduce from (b) that $\mathrm{SL}_n(R)$ is generated by the elementary matrices $E_{ij}(\lambda)$.

Solution:

(a) Observe that a square matrix is in Smith normal form if and only if it is a diagonal matrix of the form

$$D = \begin{pmatrix} d_1 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & d_m & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$