

Chapter 2 Problems

2.3

Cylindrical Coordinates

Starting with our expression of \mathbf{r} , we have

$$\mathbf{r} = \rho \cos \phi \hat{\mathbf{i}} + \rho \sin \phi \hat{\mathbf{j}} + z \hat{\mathbf{k}}$$

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial z} dz.$$

Calculating each partial derivative,

$$\frac{\partial \mathbf{r}}{\partial \rho} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}$$

$$\hat{\rho} = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|}$$

$$= \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}},$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = \rho (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}})$$

$$\hat{\phi} = \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\|}$$

$$= -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}$$

implying

$$\frac{\partial \mathbf{r}}{\partial \phi} = \rho \hat{\phi},$$

and finally, we have

$$\frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}}.$$

The above calculations yield

$$d\mathbf{r} = (d\rho) \hat{\rho} + (\rho d\phi) \hat{\phi} + (dz) \hat{\mathbf{k}}.$$

Spherical Coordinates

Starting with our expression of \mathbf{x}^{I}

$$\mathbf{x} = r \sin \phi \sin \theta \hat{\mathbf{i}} + r \cos \phi \sin \theta \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$$

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial r} dr + \frac{\partial \mathbf{x}}{\partial \phi} d\phi + \frac{\partial \mathbf{x}}{\partial \theta} d\theta,$$

Evaluating each partial derivative, we have

$$\frac{\partial \mathbf{x}}{\partial r} = \sin \phi \sin \theta \hat{\mathbf{i}} + \cos \phi \sin \theta \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}}$$

$$\hat{\mathbf{r}} = \frac{\frac{\partial \mathbf{x}}{\partial r}}{\left\| \frac{\partial \mathbf{x}}{\partial r} \right\|}$$

^II am using \mathbf{x} instead of \mathbf{r} because \mathbf{r} is already used in the expression of the spherical coordinates.

$$\begin{aligned}
&= \sin \phi \sin \theta \hat{i} + \cos \phi \sin \theta \hat{j} + \cos \theta \hat{k}, \\
\frac{\partial \mathbf{x}}{\partial \phi} &= -r \sin \phi \sin \theta \hat{i} + r \cos \phi \sin \theta \hat{j} \\
\hat{\phi} &= \frac{\frac{\partial \mathbf{x}}{\partial \phi}}{\left\| \frac{\partial \mathbf{x}}{\partial \phi} \right\|} \\
&= -\sin \phi \sin \theta \hat{i} + \cos \phi \sin \theta \hat{j}
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \phi} = r \sin \theta \hat{\phi},$$

and finally, we have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \theta} &= r \cos \phi \cos \theta \hat{i} + r \sin \phi \cos \theta \hat{j} - r \sin \theta \hat{k} \\
\hat{\theta} &= \frac{\frac{\partial \mathbf{x}}{\partial \theta}}{\left\| \frac{\partial \mathbf{x}}{\partial \theta} \right\|} \\
&= \cos \phi \cos \theta \hat{i} + \sin \phi \cos \theta \hat{j} - \sin \theta \hat{k},
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \theta} = r \hat{\theta}.$$

The above calculations yield

$$d\mathbf{x} = (dr) \hat{r} + (r \sin \theta d\phi) \hat{\phi} + (r d\theta) \hat{\theta}.$$

2.8

Let

$$\begin{aligned}
\vec{a} &= r_a \cos \phi_a \sin \theta_a \hat{i} + r_a \sin \phi_a \sin \theta_a \hat{j} + r_a \cos \theta_a \hat{k} \\
\vec{b} &= r_b \cos \phi_b \sin \theta_b \hat{i} + r_b \sin \phi_b \sin \theta_b \hat{j} + r_b \cos \theta_b \hat{k}.
\end{aligned}$$

Then,

$$\begin{aligned}
\cos \gamma &= \frac{\vec{a} \cdot \vec{b}}{\left\| \vec{a} \right\| \left\| \vec{b} \right\|} \\
&= \frac{1}{r_a r_b} (r_a r_b (\sin \theta_a \sin \theta_b (\cos \phi_a \cos \phi_b + \sin \phi_a \sin \phi_b) + \cos \theta_a \cos \theta_b)) \\
&= \cos \theta_a \cos \theta_b + \sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b).
\end{aligned}$$

2.9

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{d}{dt} (\dot{\rho} \hat{\rho}) + \frac{d}{dt} (\rho \dot{\phi} \hat{\phi}) \\
&= \dot{\rho} \ddot{\rho} + \dot{\rho} \frac{d\dot{\rho}}{dt} + \dot{\rho} \dot{\phi} \hat{\phi} + \rho \ddot{\phi} \hat{\phi} + \rho \dot{\phi} \frac{d\hat{\phi}}{dt} \\
&= \dot{\rho} \ddot{\rho} + \dot{\rho} \dot{\phi} \hat{\phi} + \rho \ddot{\phi} \hat{\phi} + \dot{\rho} \dot{\phi} \hat{\phi} + \left(\frac{\partial \hat{\phi}}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{\phi}}{\partial \phi} \frac{d\phi}{dt} \right) \\
&= (\ddot{\rho} - \rho \dot{\phi}^2) \hat{\rho} + (\rho \ddot{\phi} + 2\dot{\rho} \dot{\phi}) \hat{\phi}.
\end{aligned}$$

2.12

(a)

(i) $d\mathbf{a} = \rho \, d\phi \, dz$

(ii) $d\mathbf{a} = \rho \, dz$

(iii) $d\mathbf{a} = \rho \, \rho \, d\phi$

(b)

(i) $d\mathbf{a} = r^2 \sin \theta \, d\theta \, d\phi$

(ii) $d\mathbf{a} = r \sin \theta \, dr \, d\phi$

(iii) $d\mathbf{a} = r \, dr \, d\theta$

2.14

(a)

$$\begin{aligned}
 d\Phi &= \mathbf{E} \cdot \hat{\mathbf{n}} \, dA \\
 &= \|\mathbf{E}\| \|\hat{\mathbf{n}}\| \cos \theta \, dA \\
 &= \frac{q}{4\pi\epsilon_0 r^2} \cos \theta \, dA.
 \end{aligned}$$

Chapter 3 Problems

For all problems involving $\arg z$ (or equivalents), I will be using the principle branch, $\arg z \in (-\pi, \pi]$.

3.5

(a)

$$\begin{aligned}
 \sqrt{3} + i &= 2e^{i\frac{\pi}{3}} \\
 -\sqrt{3} + i &= 2e^{i\frac{2\pi}{3}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \sqrt{2}i &= \sqrt{2}e^{i\frac{\pi}{4}} \\
 \sqrt{2 + 2\sqrt{3}i} &= 2e^{i\frac{\pi}{6}}
 \end{aligned}$$

3.6

(a) Real:

$$\begin{aligned}
 (-1)^{1/i} &= \left(e^{i\pi}\right)^{-i} \\
 &= e^{\pi}.
 \end{aligned}$$

(b) Real:

$$\begin{aligned}
 \left(\frac{z}{z^*}\right)^i &= \left(e^{2i \arg z}\right)^i \\
 &= e^{-2 \arg z}.
 \end{aligned}$$

(c) Imaginary:

$$\begin{aligned}(z_1 z_2^* - z_1^* z_2)^* &= z_1^* z_2 - z_1 z_2^* \\ &= -(z_1 z_2^* - z_1^* z_2).\end{aligned}$$

(d) Complex:

$$\sum_{n=0}^N e^{in\theta} = \frac{1 - e^{iN\theta}}{1 - e^{i\theta}}.$$

(e) Real: for each $a \in \{1, 2, \dots, N\}$, $e^{ia\theta} + e^{-ia\theta} \in \mathbb{R}$.**3.9**

(a)

$$\begin{aligned}\cos(a+b) + \cos(a-b) &= \frac{1}{2} \left(e^{i(a+b)} + e^{-i(a+b)} \right) + \frac{1}{2} \left(e^{i(a-b)} + e^{-i(a-b)} \right) \\ &= \frac{1}{2} \left(e^{ia} (e^{ib} + e^{-ib}) + e^{-ia} (e^{ib} + e^{-ib}) \right) \\ &= \frac{1}{2} (e^{ia} + e^{-ia}) (e^{ib} + e^{-ib}) \\ &= 2 \cos a \cos b.\end{aligned}$$

(b)

$$\begin{aligned}\sin(a+b) + \sin(a-b) &= \frac{1}{2i} \left(e^{i(a+b)} - e^{-i(a+b)} \right) + \frac{1}{2i} \left(e^{i(a-b)} - e^{-i(a-b)} \right) \\ &= \frac{1}{2i} \left(e^{ia} (e^{ib} + e^{-ib}) - e^{-ia} (e^{ib} + e^{-ib}) \right) \\ &= \frac{1}{2i} (e^{ia} - e^{-ia}) (e^{ib} + e^{-ib}) \\ &= 2 \sin a \cos b.\end{aligned}$$

3.10

(a)

$$\begin{aligned}e^{i\alpha} + e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} + e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)} \\ &= e^{i\frac{\alpha+\beta}{2}} \left(e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}} \right) \\ &= 2 \cos\left(\frac{\alpha-\beta}{2}\right) e^{i\frac{\alpha+\beta}{2}}.\end{aligned}$$

(b)

$$\begin{aligned}e^{i\alpha} - e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} - e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)} \\ &= e^{i\frac{\alpha+\beta}{2}} \left(e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}} \right) \\ &= 2i \sin\left(\frac{\alpha-\beta}{2}\right) e^{i\frac{\alpha+\beta}{2}}\end{aligned}$$

3.12

$$\begin{aligned}
\frac{1}{2i} \ln \left(\frac{a+ib}{a-ib} \right) &= \frac{1}{2i} (\ln(a+ib) - \ln(a-ib)) \\
&= \frac{1}{2i} \left(\ln|a+ib| + i \arctan \left(\frac{b}{a} \right) - \left(\ln|a+ib| + i \arctan \left(-\frac{b}{a} \right) \right) \right) \\
&= \arctan \left(\frac{b}{a} \right).
\end{aligned}$$

3.13

$$\begin{aligned}
\frac{d^n}{dt^n} (e^{at} \sin bt) &= \frac{1}{2i} \frac{d^n}{dt^n} (e^{(a+ib)t} - e^{(a-ib)t}) \\
&= \frac{1}{2i} ((a+ib)^n e^{(a+ib)t} - (a-ib)^n e^{(a-ib)t}) \\
&= \frac{1}{2i} e^{at} \left(\left((a^2 + b^2)^{n/2} e^{in \arctan(\frac{b}{a})} \right) e^{ibt} - \left((a^2 + b^2)^{n/2} e^{-in \arctan(\frac{b}{a})} \right) e^{-ibt} \right) \\
&= e^{at} \frac{1}{2i} (a^2 + b^2)^{n/2} (e^{i(b+n \arctan(\frac{b}{a}))t} - e^{i(b+n \arctan(\frac{b}{a}))t}) \\
&= e^{at} (a^2 + b^2)^{n/2} \sin \left(bt + n \arctan \left(\frac{b}{a} \right) \right)
\end{aligned}$$

3.20

Showing the equivalence between $C_1 \cos kx + C_2 \sin kx$ and $A \cos(kx + \alpha)$ and $B \sin(kx + \beta)$, we have

$$\begin{aligned}
A \cos(kx + \alpha) &= A \cos kx \cos \alpha - A \sin kx \sin \alpha \\
B \sin(kx + \beta) &= B \cos kx \sin \beta + B \sin kx \cos \beta
\end{aligned}$$

meaning (assuming $\alpha, \beta \neq \pi n, \pi/2 + \pi n$)

$$\begin{aligned}
A &= \frac{C_1}{\cos \alpha} \\
&= -\frac{C_2}{\sin \alpha} \\
B &= \frac{C_1}{\sin \beta} \\
&= \frac{C_2}{\cos \beta}.
\end{aligned}$$

Now, we show the equivalence between $C_1 \cos kx + C_2 \sin kx$ and $D_1 e^{ikx} + D_2 e^{-ikx}$.

$$\begin{aligned}
D_1 e^{ikx} + D_2 e^{-ikx} &= \frac{D_1 + D_2}{2} (e^{ikx} + e^{-ikx}) + \frac{D_1 - D_2}{2} (e^{ikx} - e^{-ikx}) \\
&= (D_1 + D_2) \cos kx + i(D_1 - D_2) \sin kx.
\end{aligned}$$

meaning

$$\begin{aligned}
C_1 &= D_1 + D_2 \\
C_2 &= i(D_1 - D_2).
\end{aligned}$$

Finally, we show the equivalence between $\text{Re}(Fe^{ikx})$ and $C_1 \cos kx + C_2 \sin kx$.

$$\begin{aligned}\text{Re}\left((a + ib)e^{ikx}\right) &= \text{Re}(a \cos kx + ia \sin kx + ib \cos kx - b \sin kx) \\ &= a \cos kx - b \sin kx,\end{aligned}$$

meaning

$$C_1 = a$$

$$C_2 = -b.$$