

Introduction: naive set theory

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Z}_+ = \{0, 1, 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

$$\mathbb{C}_q = \{a + bi \mid a, b \in \mathbb{Q}\}$$

Recall: given sets X and Y , a relation from X to Y is a subset of $X \times Y$, where \times denotes the cartesian product of X and Y .

A relation $f \subseteq X \times Y$ is a function from X to Y such that $\forall x \in X, \exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote f as $f : X \rightarrow Y$.

X is the **domain** of f and Y is the **codomain**. The range $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Examples

$$\text{id}_X : X \rightarrow X, \text{id}_X(x) = x$$

This is the identity function.

The Characteristic Function: If $A \subseteq X$

$$\mathbf{1}_A : X \rightarrow \mathbb{R}, \mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Algebra of Functions

Let X be any set, and $(X; \mathbb{R}) = \{f : X \rightarrow \mathbb{R}\}$ represent the function space of X with codomain \mathbb{R} .

Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$ (scalar multiplication). If $g(x) \neq 0 \forall x \in X$, then $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$.

Finally, we have composition. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions, then $g \circ f(x) = g(f(x))$.

Injective, Subjective, and Bijective

A function $f : X \rightarrow Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. For example, the shift map $S : \mathbb{N} \rightarrow \mathbb{N}$, $S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \rightarrow \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{Ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \rightarrow Y$ be a function. f is **left-invertible** if $\exists g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$. f is **right-invertible** if $\exists h : Y \rightarrow X$ such that $f \circ h = \text{id}_Y$.

f is **invertible** if $\exists k : Y \rightarrow X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

Proposition

f is invertible if and only if f is left and right invertible.

Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose g is a left-inverse of f , and h is a right-inverse of f . Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

Theorem

If $f : X \rightarrow Y$ is a function:

1. f is injective $\Leftrightarrow f$ is left-invertible.
2. f is surjective $\Leftrightarrow f$ is right-invertible.
3. f is bijective $\Leftrightarrow f$ is invertible.

We will prove the first proposition in the forward direction. Suppose f is injective. Given $y \in \text{Ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = y$, by the definition of injective.

Let $g : Y \rightarrow X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \text{Ran}(f) \\ x_0 & y \notin \text{Ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X . We can see that $g \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \rightarrow [-\pi/2, \pi/2]$.