

## Problem 1

If  $F$  is a finite set and  $k : F \rightarrow F$  is a self-map, prove that  $k$  is injective if and only if  $k$  is surjective.

Let  $k$  be injective.

$$\begin{aligned} \text{card}(F) &= \text{card}(k(F)) && \text{definition of injection} \\ k(F) &\subseteq F && \text{definition of function} \\ k(F) &= F \end{aligned}$$

Let  $k$  be surjective.

$$\begin{aligned} k(F) &= F && \text{definition of surjection} \\ \text{card}(k(F)) &= \text{card}(F) && \text{cardinality definition} \end{aligned}$$

Therefore,  $k$  is injective, as  $F$  is finite.

## Problem 2

Prove that a set  $A$  is infinite if and only if there is a non-surjective injection  $f : A \hookrightarrow A$ .

( $\Rightarrow$ ) Let  $A$  be infinite. Then,  $\exists i : \mathbb{N} \hookrightarrow A$ ;  $\forall n \in \mathbb{N}, a_n := i(n)$ . Let  $f : A \rightarrow A$ ,  $f(a_i) = a_{i+1}$ . Then, for  $a_{i_1} \neq a_{i_2}$ ,  $f(a_{i_1}) = a_{i_1+1} \neq a_{i_2+1} = f(a_{i_2})$ . Therefore,  $f$  is injective.

( $\Leftarrow$ ) Suppose  $A$  is finite. Then, by the result in Problem 1,  $\forall f : A \hookrightarrow A$ ,  $f$  must be surjective.

## Problem 3

Let  $A$ ,  $B$ , and  $C$  be sets and suppose  $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$ . Prove that  $\text{card}(A) < \text{card}(C)$ .

## Problem 4

If  $A \subseteq B$  is an inclusion of sets with  $A$  countable and  $B$  uncountable, show that  $B \setminus A$  is uncountable.

Suppose toward contradiction that  $B \setminus A$  is countable.

Then,  $A \cup (B \setminus A)$  must be countable, by union of countable sets.

However,  $A \cup (B \setminus A) = B$ , and  $B$  is uncountable, meaning that  $B \setminus A$  must be uncountable.

## Problem 5

Is the set  $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$  countable?

Let  $q : \mathbb{Q} \rightarrow \mathbb{N}$  be the enumeration of the rationals; let  $f : \{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$  be defined as  $f(x) = q(x^2)$ .

$x > 0 \Rightarrow t(x) = x^2$  is a bijection;  $\mathbb{Q}$  countable  $\Rightarrow q$  is a bijection;  $f = q \circ t \Rightarrow f$  is a bijection, and thus  $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$  is countable.

## Problem 6

Consider the set  $\mathcal{F}(\mathbb{N})$  of all finite subsets of  $\mathbb{N}$ . Is  $\mathcal{F}(\mathbb{N})$  countable?

Let  $f : \mathcal{F} \rightarrow \mathbb{N}$  be defined as follows, where  $p_n$  denotes the  $n$ th prime number.

$$f(\{a_1, a_2, \dots, a_n\}) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_n^{a_n}$$

By the fundamental theorem of arithmetic, every natural number is equal to a unique product of powers of prime numbers, meaning that  $f$  is injective, so  $\mathcal{F}$  is countable.

## Problem 7

Let  $k \in \mathbb{N}$ .

- (i) Prove that  $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$  is countable.
- (ii) Show that the set  $\mathbb{N}^\infty := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}\}$  consisting of all sequences of natural numbers is uncountable.
- (iii) Prove that the set of **finitely-supported** natural sequences  $c_c(\mathbb{N}) := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}, n_k = 0 \text{ for all but finitely many } k\}$  is countable.

(i)

Let  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  be defined as follows, where  $p_n$  denotes the  $n$ th prime number in the sequence  $\{2, 3, 5, \dots\}$

$$f((a_1, a_2, \dots, a_k)) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$$

By the fundamental theorem of arithmetic, every natural number is equal to a unique product of powers of prime numbers, so  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  is an injection, meaning  $\mathbb{N}^k$  is countable

(ii)

Suppose toward contradiction that the set of all sequences of natural numbers is countable:  $f : A_n \rightarrow \mathbb{N}$  is surjective.

$$A_1 = \{a_{11}, a_{12}, a_{13}, \dots\}$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, \dots\}$$

$\vdots$

Create a new sequence  $N$  defined as follows:

$$n_k = a_{kk} + 1$$

Since  $f$  is surjective,  $\exists A_m = \{a_{m1}, a_{m2}, \dots, a_{mm}, \dots\} = \{n_1, n_2, \dots, n_m, \dots\}$ . However, since by definition,  $n_m \neq a_{mm}$ ,  $f$  must not be surjective. Thus,  $\mathbb{N}^\infty$  is not countable.

(iii)

Let  $f : c_c(\mathbb{N}) \rightarrow \mathbb{N}$  be defined as follows, where  $p_n$  denotes the  $n$ th prime number:

$$f(\{n_1, n_2, \dots, n_k\}) = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$$

Since every natural number is represented uniquely by a finite product of powers of primes by the fundamental theorem of arithmetic,  $f$  is injective, meaning  $c_c(\mathbb{N})$  is countable.

## Problem 8

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range  $\text{ran}(f)$  cannot contain any interval.

## Problem 9

Prove that the set

$$\mathcal{P} := \left\{ \sum_{k=0}^n a_k x^k \mid n \subseteq \mathbb{N}_0, a_k \in \mathbb{Q} \right\}$$

consisting of all polynomials with rational coefficients, is countable.

Let  $q : \mathbb{Q} \rightarrow \mathbb{N}$  be the enumeration of the rationals, and let  $p_n$  denote the  $n$ th element in the sequence of prime numbers, where  $p_1 = 2, p_2 = 3$ , etc.

Let  $f : \mathcal{P} \rightarrow \mathbb{N}^k$  be defined as follows:

$$f(a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots) = (q(a_0), q(a_1), \dots, q(a_k), \dots)$$

Since  $\mathbb{Q}$  is countable,  $\forall a \in \mathbb{Q}, q(a) \in \mathbb{N}$ , so the output of  $f$  is a bijection to  $\mathbb{N}^k$ , meaning  $\mathcal{P}$  is countable.

## Problem 10

A real number  $t$  is called **algebraic** if there is a nonzero polynomial  $p$  with rational coefficients such that  $p(t) = 0$ . If  $t \in \mathbb{R}$  is not algebraic, then it is called **transcendental**. For example,  $\sqrt{2}$  is algebraic, but  $\pi$  is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.

Because  $\mathcal{P}$  is countable, and there are  $k$  roots in a  $k$ -degree polynomial, there are a countable number of polynomial roots, so the algebraic numbers are countable.

Since  $\mathbb{R}$  is uncountable,  $\mathbb{R} \setminus \mathbb{A}$  is uncountable as shown by a previous result.