

Introduction: naive set theory

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Z}_+ = \{0, 1, 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

$$\mathbb{C}_q = \{a + bi \mid a, b \in \mathbb{Q}\}$$

Recall: given sets X and Y , a relation from X to Y is a subset of $X \times Y$, where \times denotes the cartesian product of X and Y .

A relation $f \subseteq X \times Y$ is a function from X to Y such that $\forall x \in X, \exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote f as $f : X \rightarrow Y$.

X is the **domain** of f and Y is the **codomain**. The range $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Examples

$$\text{id}_X : X \rightarrow X, \text{id}_X(x) = x$$

This is the identity function.

The Characteristic Function: If $A \subseteq X$

$$1_A : X \rightarrow \mathbb{R}, 1_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Algebra of Functions

Let X be any set, and $(X; \mathbb{R}) = \{f : X \rightarrow \mathbb{R}\}$ represent the function space of X with codomain \mathbb{R} .

Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$ (scalar multiplication). If $g(x) \neq 0 \forall x \in X$, then $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$.

Finally, we have composition. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions, then $g \circ f(x) = g(f(x))$.

Injective, Subjective, and Bijective

A function $f : X \rightarrow Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. For example, the shift map $S : \mathbb{N} \rightarrow \mathbb{N}$, $S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \rightarrow \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \rightarrow Y$ be a function. f is **left-invertible** if $\exists g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$. f is **right-invertible** if $\exists h : Y \rightarrow X$ such that $f \circ h = \text{id}_Y$.

f is **invertible** if $\exists k : Y \rightarrow X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

Invertibility Definition

f is invertible if and only if f is left and right invertible.

Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose g is a left-inverse of f , and h is a right-inverse of f . Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

Injection and Surjection Invertibility

If $f : X \rightarrow Y$ is a function:

1. f is injective $\Leftrightarrow f$ is left-invertible.
2. f is surjective $\Leftrightarrow f$ is right-invertible.
3. f is bijective $\Leftrightarrow f$ is invertible.

We will prove the first proposition in the forward direction. Suppose f is injective. Given $y \in \text{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = y$, by the definition of injective.

Let $g : Y \rightarrow X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X . We can see that $g \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \rightarrow [-\pi/2, \pi/2]$.

Cardinality and Finitude

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? \mathbb{N} or \mathbb{N}_0 ? \mathbb{Z} or \mathbb{Q} ?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets A and B , we can show that A is the same size as B by creating a function. For example, to show that \mathbb{N} and \mathbb{N}_0 have the same size, we create $s : \mathbb{N} \rightarrow \mathbb{N}_0$, $s(n) = n + 1$.

Cardinality

Sets A and B have the same **cardinality** if \exists bijection $f : A \rightarrow B$. We write $\text{card}(A) = \text{card}(B)$.

Equivalent Cardinalities of Intervals

Given $a < b$ and $c < d$, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

Intervals and Real Numbers

$$\text{card}((0, 1)) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ is a bijection:
 - \tan is strictly increasing (and thus injective)
 - $\lim_{x \rightarrow \infty} \tan(x) = \infty$ and $\lim_{x \rightarrow -\infty} \tan(x) = -\infty$, and by intermediate value theorem, \tan is surjective
- $\ell : (0, 1) \rightarrow (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0, 1) \rightarrow \mathbb{R}$.

Finitude

A set F is **finite** if F is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}(\{1, 2, \dots, n\})$. A non-finite set is called infinite.

We can *enumerate* F by creating a function $\sigma : \{1, 2, \dots, n\} \rightarrow F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

Inequality of Finite Sets

If $m \neq n$, then $\text{card}\{1, 2, \dots, m\} \neq \text{card}\{1, 2, \dots, n\}$.

WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$ is our bijection. This means there are m "pigeons" and n "holes."

One hole, j , must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since f is assumed to be injective, this is a contradiction.

Infinitude of the Naturals

\mathbb{N} is infinite.

Suppose toward contradiction that \mathbb{N} is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \rightarrow \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m+1\} \rightarrow \mathbb{N}$. i is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \rightarrow \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

Proposition

If A is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

\vdots

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of A .

Consider $f : \mathbb{N} \rightarrow A$, $f(n) = a_n$. f is injective as a_n are distinct.

Cardinality of Integers and Natural Numbers

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

$$f : \mathbb{Z} \rightarrow \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

f is a bijection as $g : \mathbb{N} \rightarrow \mathbb{Z}$, $g(n) = (-1)^{n+1} \lfloor \frac{n}{2} \rfloor$ is the inverse of f .

Power Set

Given any set X , $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of X .

$$2^X := \{f \mid f : X \rightarrow \{0, 1\}\}.$$

Power Set and 2^X

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Let $\varphi : \mathcal{P}(X) \rightarrow 2^X$.

For $A \subseteq X$, put $\varphi(A) := 1_A$.

Consider $\psi : 2^X \rightarrow \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(1_A) = 1^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = 1_{f^{-1}(\{1\})} = f$.

Cantor's Theorem

\nexists surjection $\mathbb{N} \rightarrow (0, 1)$

Fact from calculus: $\forall \sigma \in (0, 1)$, σ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \dots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \rightarrow (0, 1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0, 1, \dots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \rightarrow \{0, 1, \dots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since r is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

Comparing Cardinalities

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

Transitive Property

If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

The composition of two injective functions is injective.

Canonical Set Comparisons

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

Cardinality of the Power Set

For any set A , $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Let us construct a function: $f : A \rightarrow \mathcal{P}(A)$, where $a \mapsto \{a\}$.

f is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

Claim $\nexists g : A \rightarrow \mathcal{P}(A)$, g is surjective.

Suppose toward contradiction that such a g exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since g is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. \perp

Equivalent Propositions

(i) $\text{card}(A) \leq \text{card}(B)$

(ii) $\exists f : A \hookrightarrow B$

(iii) $\exists g : B \rightarrow A$, g surjection.

By definition, (i) \Leftrightarrow (ii).

(ii) \Rightarrow (iii) If $f : A \hookrightarrow B$, f is left-invertible, and thus $\exists g : B \rightarrow A$ with $g \circ f = \text{id}_A$. So, g is right-invertible, so g is surjective.

(iii) \Rightarrow (ii) If $g : B \rightarrow A$ is surjective, then g is right-invertible, so $\exists f : A \rightarrow B$ such that $g \circ f = \text{id}_B$. So, f is left-invertible, so f is injective.

Corollary

If $f : A \rightarrow B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$.

Consider $g : A \rightarrow f(A)$, where $g(a) = f(a)$. So, g is onto, so \exists an injection $f(A) \hookrightarrow A$.

More Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$, $q(m, n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, defined as $H(m, n) = (h(m), n)$.

Claim H is a bijection.

Proof of Injection If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since h is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

Proof of Surjection Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m, n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m, n) = (k, \ell)$. Set $n = \ell$, and since h is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m, n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, φ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned} \text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\ &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \text{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \text{card}(\mathbb{N}) \end{aligned}$$

Cardinality Rules

- (i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)
- (ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)
- (iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)
- (iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

Proof of (iii) We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

Claim We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, g and f are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1 = \emptyset$. \perp

We let $A_n = g \circ f(A_{n-1})$ for arbitrary n , and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \rightarrow B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where y_x is the unique element in B with $g(y_x) = x$.

Claim We claim h is the desired bijection.

Proof of Injection Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of H , $f(x_1) = f(x_2)$, f is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y_{x_2}) = x_2 \notin A_\infty$. This case is not possible.

Thus, h is injective.

Proof of Surjection Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where y_x is the unique element in B with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since g is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Therefore, we have $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$.

Countability

A set X is *countable* if $\exists f : X \hookrightarrow \mathbb{N}$ ($\text{card}(X) \leq \text{card}(\mathbb{N})$). $\text{card}(\mathbb{N}) = \aleph_0$. If X is countable and infinite, X is *denumerable*.

Corollary to Cantor-Schröder-Bernstein

If X is denumerable, then $\text{card}(X) = \aleph_0$.

Since X is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since X is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

(as shown earlier)

Countability under Union

The countable union of countable sets is countable. If I is a countable indexing set and for each $i \in I$, A_i is countable, then $\bigcup_{i \in I} A_i$ is countable.

Since each A_i is countable, $\exists \pi_i : \mathbb{N} \rightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \rightarrow \bigcup_{i \in I} A_i$$

defined as $\pi(i, j) = \pi_i(j)$.

Claim 1 π is a surjection.

Proof 1 Let $x \in \bigcup_{i \in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since π_{i_0} is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, π is surjective.

Claim 2 $I \times \mathbb{N}$ is countable.

Proof 2 We know $\exists f : I \hookrightarrow \mathbb{N}$ since I is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i, n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m, n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since π is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i \in I} A_i$ is countable.

Continuum Hypothesis

We saw that $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(2^{\mathbb{N}})$, where $2^{\mathbb{N}} = \{f \mid f : \mathbb{N} \rightarrow \{0, 1\}\}$.

Theorem $\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}})$, where I is any non-degenerate interval.

Lemma 1 $\text{card}([0, 1]) \leq \text{card}(2^{\mathbb{N}})$.

Proof 1 Every $t \in [0, 1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0, 1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\varphi} [0, 1]$, defined as $\varphi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Set $f : \mathbb{N} \rightarrow \{0, 1\}$, $f(k) = \sigma_k$.

Therefore, φ is surjective, so $\exists \{0, 1\} \hookrightarrow 2^{\mathbb{N}}$, so $\text{card}([0, 1]) \leq 2^{\mathbb{N}}$.

Lemma 2 $\text{card}([0, 1]) = \text{card}(\mathbb{R})$.

Proof 2 We have $[0, 1] \xrightarrow{i} \mathbb{R}$ via inclusion, so $\text{card}([0, 1]) \leq \text{card}(\mathbb{R})$.

Also, $\text{card}(\mathbb{R}) = \text{card}((0, 1)) \leq \text{card}([0, 1])$, so by Cantor-Schröder-Bernstein, $\text{card}(\mathbb{R}) = \text{card}([0, 1])$.

Lemma 3 Any two non-degenerate intervals I and J have the same cardinality.

Proof 3 We can create injections $I \hookrightarrow J$ and vice-versa.

Lemma 4 $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0, 1])$.

Proof 4 $\psi : 2^{\mathbb{N}} \rightarrow [0, 1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

ψ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim ψ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k > k_0} \frac{f(k)}{3^k}$, $t_g = \sum_{k > k_0} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} \frac{g(k)}{3^k} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$\begin{aligned} |t_f - t_g| &= \left| \sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k} \right| \\ &\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\ &\leq \sum_{k>k_0} \frac{1}{3^k} \\ &= \frac{(1/3)^{k_0+1}}{1 - (1/3)} \\ &= \frac{1}{2} \cdot \frac{1}{3^{k_0}} \end{aligned}$$

⊥

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0, 1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than \aleph_0 and strictly less than 2^{\aleph_0} is an axiom (i.e., it can be an assumption or not).

Ordering

Let X be a non-empty set. A relation on X is a subset of $X \times X$.

- R is *reflexive* if $\forall x \in X, (x, x) \in R$.
- R is *transitive* if $(x, y), (y, z) \in R \rightarrow (x, z) \in R$.
- If R is *antisymmetric* $(x, y), (y, x) \in R \rightarrow x = y$.

If R is reflexive, transitive, and antisymmetric, then R is an *ordering* of X .

If R is an ordering of X , then we write:

$$(x, y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \quad \forall x \in X$
- $x \leq_R y, y \leq_R z \rightarrow x \leq_R z$
- $x \leq_R y, y \leq_R x \rightarrow x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

Algebraic ordering of \mathbb{N}_0

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0 \text{ such that } n + k = m$$

\mathbb{N} ordered via division

$$n \leq_D m \Leftrightarrow n|m$$

Under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

Inclusion Let S be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_I B \Leftrightarrow A \subseteq B$.

Containment With X defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

For $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \rightarrow \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in X$.

Types of Orderings

- An ordering \leq of X is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.
- An ordering is *directed* if $\forall x, y \in X \exists z \in X$ such that $x \leq z$ and $y \leq z$.

If X is a totally ordered set, X is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D), (\mathcal{P}(S), \leq_i), (\mathcal{P}(S), \leq_c)$$

Upper/Lower Bounds

- Let (X, \leq) be an ordered set, $A \subseteq X$. A is bounded above if $\exists v \in X$ with $a \leq v \forall a \in A$. Such a v is an upper bound.
- A is bounded below if $\exists \ell \in X$ such that $a \geq \ell \forall a \in A$. Such a w is a lower bound.
- If v is an upper bound of A and $v \in A$, then v is the greatest element of A , or $\max(A) = v$.
- If ℓ is a lower bound for A and $\ell \in A$, then ℓ is the least element of A , or $\min(A) = \ell$.
- If u is an upper bound for A , and $u \leq v$ for all other upper bounds v of A , then u is the *least upper bound* of A , or $\sup(A) = u$ (for *supremum*).
- If ℓ is a lower bound for A , and $\ell \leq g$ for all other lower bounds g of A , then ℓ is the *greatest lower bound* of A , or $\inf(A) = \ell$ (for *infimum*).
- If A is bounded above and below, then A is bounded.

Well-Ordering Principle

With (\mathbb{N}, \leq_a) , every nonempty $A \subseteq \mathbb{N}$ has a least element.

Examples

Example 1

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \dots, 12\}$, we have the following:

Bounded Above? Yes.

Upper Bounds 12, 13, 14, ...

Greatest Element 12

Example 2

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \dots, 10\}$

Bounded Above? Yes.

Upper Bounds 10!

Greatest Element? No.

Supremum $2^3 \cdot 3^2 \cdot 5 \cdot 7$

Bounded Below? Yes.

Lower Bound 1

Least Element? No.

Infimum 1

Example 3

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$.

Supremum $\bigcup_{i \in I} A_i$

Infimum $\bigcap_{i \in I} A_i$

Complete Sets

An ordered set (X, \leq) is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, \mathbb{Q} is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

Ordering of \mathbb{Z}

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0, n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

Properties of \mathbb{Z}^+

- (i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, m \cdot n \in \mathbb{Z}^+$
- (ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$
- (iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$
- (iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

Ordering of \mathbb{Z} , \mathbb{Q} , and \mathbb{R}

Recall the ordering of \mathbb{Z} :

$$n \leq_a m \stackrel{\text{def}}{\Leftrightarrow} \exists k \in \mathbb{N}_0 \text{ with } n + k = m$$

Claim \leq_a is an ordering of \mathbb{Z}

We claim that $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$. Thus, $\mathbb{Z}^+ = \mathbb{N}_0$.

Properties of \mathbb{Z}^+

- (i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, m \cdot n \in \mathbb{Z}^+$
- (ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$
- (iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$
- (iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

Other Properties of \mathbb{Z}

- (1) $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$
- (2) $m \leq_a n$ and $p \leq_a q \Rightarrow m + p \leq_a n + q$
- (3) $m \leq_a n$ and $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$
- (4) $m \leq_a n \Rightarrow -m \geq_a n$
- (5) \leq_a is total.
- (6) If $a_a > -$, and $ab_a \geq 0$, then $b_a > 0$
- (7) If $a > 0$ and $ab_a \geq ac$, then $b \geq c$.

Proof of (3):

$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0$ with $m + k = n$.
 $\Rightarrow pm + pk = pn$
 $pk \in \mathbb{N}_0$ by the properties of \mathbb{Z}^+ . So, $pm \leq_a pn$

Proof of (5):

Let $m, n \in \mathbb{Z}$. Consider $m - n$.

By (ii), $m - n \in \mathbb{Z}^+$ or $-(m - n) \in \mathbb{Z}^+$. Thus, $m - n = k$ for some $k \in \mathbb{Z}^+$, or $-(m - n) = \ell$ for some $\ell \in \mathbb{Z}^+$.

Thus, $n \leq_a m$ in the first case, or $m \leq_a n$ in the second case.

We now want an ordering on \mathbb{Q} .

Creating the Rationals

Recall that $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$. Consider the equivalence relation:

$$(a, b) \sim (c, d) \stackrel{\text{def}}{\iff} ad = bc$$

We will let $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in \mathbb{Q}\}$ be the set of all equivalence classes in \mathbb{Q} . We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined: $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{a'd' + c'd'}{b'd'} = \frac{ad + bc}{bd}$.

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make \mathbb{Q} a **field**:

Fields

A ring is a nonempty set R equipped with two binary operations:

- $+: R \times R \rightarrow R, (a, b) \mapsto a + b$ ("addition")
- $\cdot: R \times R \rightarrow R, (a, b) \mapsto a \cdot b$ ("multiplication")

such that the following hold:

- (1) $(a + b) + c = a + (b + c)$
- (2) $\exists z \in R$ such that $a + z = a = z + a \forall a \in R$; there is at most one such z . Set $z = 0_R$.
- (3) $\forall a \in R, \exists b \in R$ such that $a + b = 0_R = b + a$; there is at most one such b . Set $b = -a$.
- (4) $\forall a, b \in R, a + b = b + a$.
- (5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (6) $a \cdot (b + c) = a \cdot b + a \cdot c, (a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If $(R, +, \cdot)$ satisfies $ab = ba$, R is a commutative ring.

If there exists $u \in R$ such that $ua = au = a \forall a \in R$, R is a unital ring; there is at most one unit. Set $u = 1_R$.

An integral domain is a unital, commutative ring such that $ab = 0 \Rightarrow a = 0 \vee b = 0$. For example, \mathbb{Z} is an integral domain. However, $C(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ is a unital, commutative ring, but there exist two functions such that $f, g \neq 0$, but $f \cdot g = 0$.

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R, \text{ with } ab = 1_R$$

There is only one such b . Set $b = a^{-1}$.

Proof that \mathbb{Q} is a Field:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Provided that $\frac{a}{b} \neq 0_{\mathbb{Q}}$.

Additionally, $\mathbb{Z} \hookrightarrow \mathbb{Q}, j(n) = \frac{n}{1}$ is injective.

Ordering of \mathbb{Q}

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

Prove that this ordering is well-defined.

Order Embedding

\leq is a well-defined total ordering of \mathbb{Q} , and $j : \mathbb{Z} \hookrightarrow \mathbb{Q}, j(n) = \frac{n}{1}$ is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

Properties of \mathbb{Q}^+

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

- (i) $q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+, q_1 q_2 \in \mathbb{Q}^+$
- (ii) $q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$
- (iii) $\pm q \in \mathbb{Q}^+, q = 0$
- (iv) $x \leq y, !u \leq v \Rightarrow x + u \leq y + v$
- (v) $x \leq y, 0 \leq z \Rightarrow zx \leq zy$

Ordering of \mathbb{R} , cont'd

An **ordered field** is a field F equipped with a total ordering \leq_F such that:

- (i) if $s \leq_F t$, and $x \leq_F y$, then $s + x \leq_F t + y$
- (ii) if $s \leq_F t$ and $0 \leq_F z$, then $zs \leq_F zt$

For example, \mathbb{Q} with its ordering is an ordered field.

Proposition 1: If (F, \leq_F) is an ordered field, we define $F^+ = \{x \in F, x_F \geq 0\}$ with the following properties:

- (1) $x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$
- (2) $x \in F \Rightarrow x \in F^+, -x \in F^+$
- (3) $\pm x \in F^+ \Rightarrow x = 0_F$

Proofs

- (1) Let $x, y \in F^+$. Then, $x \geq 0$ and $y \geq 0$, so by property (i) of an ordered field, $x + y \geq 0 + 0 = 0$, so $x + y \in F^+$. Additionally, we have $x \cdot y \geq x \cdot 0 = 0$, so $xy \in F^+$.
- (2) Let $x \in F$. Since the ordering on F is total, $x \geq 0$ or $0 \geq x$. In the first case, $x \in F^+$. In the second case, we add $-x$ to both sides, so by (i), $-x \geq 0$, so $-x \in F^+$.
- (3) We have $x \geq 0$ and $-x \geq 0$. So $x \geq 0$ and $x + (-x) \geq x + 0$, so $x \geq 0$ and $0 \geq x$. So, $x = 0$ by antisymmetry.

Note: $x \leq_F y \Leftrightarrow y - x \in F^+$.

Proposition 2: Let F be an ordered field. Then, the following is true:

- (1) $\forall a \in F, a^2 \in F^+$
- (2) $0, 1 \in F^+$
- (3) If $n \in \mathbb{N}$, $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$
- (4) If $x \in F^+$, and $x \neq 0$, then $x^{-1} \in F^+$

- (5) If $xy > 0$, then $x, y \in F^+$, or $-x, -y \in F^+$
- (6) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$
- (7) If $x \leq y$, then $-y \leq -x$
- (8) $x \geq 1 \Rightarrow x^2 \geq x \geq 1$, and $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$.

Proofs

- (1) Let $a \in F$. Then, $a \in F^+$ or $-a \in F^+$.
- Case 1 If $a \in F^+$, then by the previous proposition, $a^2 \in F^+$.
- Case 2 If $-a \in F^+$, then by the previous proposition, $(-a)(-a) = a^2 \in F^+$.
- (2) $0 \geq 0$, so $0 \in F^+$. $1 = 1 \cdot 1 = 1^2 \in F^+$ by the previous result.
- (3) $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}} \in F^+$ by the previous proposition.
- (4) Let $x \neq 0, x \in F^+$. Suppose toward contradiction that $x^{-1} \notin F^+$, then $-x^{-1} \in F^+$. Thus, $x \cdot (-x^{-1}) \in F^+$, so $-1 \in F^+$, but $1 \in F^+$, so $1 = 0$. \perp
- (5) Let $xy > 0$, meaning $xy \in F^+$. Suppose toward contradiction that $x > 0$ and $y < 0$. So, $x > 0$ and $-y > 0$, so $(x)(-y) > 0$, so $-(xy) \in F^+$, so $xy = 0$. \perp
- (6) Let $0 < x \leq y$. We know $x^{-1} \in F^+$, so $x^{-1}x \leq x^{-1}y$. So $1 \leq x^{-1}y$. We also know $y \in F^+$, so $y^{-1} \in F^+$. So, $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$.
- (7) Let $x \leq y$. Then, $0 \leq y - x$, so $-y \leq -x$.
- (8) Let $x \geq 1$. Then, $x \cdot x \geq 1 \cdot x \geq 1$.

Order Axiom

\mathbb{R} is an ordered field. The injection $\mathbb{Q} \hookrightarrow \mathbb{R}$, $i(q) = q$ is an order embedding.

Rational Orderings

Proposition 1: If $a \leq b$, then $a \leq \frac{1}{2}(a+b) \leq b$

Proof

$2a = a + a \leq a + b \leq b + b$, all by property (i) of an ordered field.

Therefore, $2a \leq a+b \leq 2b$. Since $2 = 1+1 \in \mathbb{R}^+$, $2^{-1} \in \mathbb{R}^+$, so $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$, so $a \leq \frac{1}{2}(a+b) \leq b$.

Proposition 2: If $a \geq 0$ and $(\forall \epsilon > 0), a \leq \epsilon$.

Proof

If $a \geq 0$ and $a \neq 0$, then $a > 0$. So, we have that $\frac{1}{2}a < a$. Let $\epsilon = \frac{1}{2}a$. We also have that $a \leq \epsilon = \frac{1}{2}a < a$, so $a < a$. \perp

Arithmetic and Geometric Means

Given $a_1, a_2, \dots, a_n \in \mathbb{R}^+$:

Arithmetic Mean

$$= \frac{\sum_{i=1}^n a_i}{n}$$

Geometric Mean

$$= \sqrt[n]{a_1 a_2 \cdots a_m}$$

Arithmetic Mean-Geometric Mean Inequality

Let $a, b \geq 0$.

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

If $x, y \geq 0$, $x \leq y \Leftrightarrow x^2 \leq y^2$.

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y$$

by property (ii) of ordered fields

Therefore,

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

$$ab \leq \frac{1}{4}(a^2 + 2ab + b^2)$$

$$4ab \leq a^2 + 2ab + b^2$$

$$0 \leq a^2 - 2ab + b^2$$

$$0 \leq (a-b)^2$$

by definition

Challenge: Prove for m .

Remark: The harmonic mean is defined as:

$$\frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$$

Bernoulli's Inequality

If $x \geq -1$, then $(1+x)^n \geq 1+nx$, for any $n \in \mathbb{N}_0$

By induction, we know that for $n=0$ and $n=1$, this holds.

Assume the inequality holds for some $m \geq 1$.

$$\begin{aligned} (1+x)^{m+1} &= (1+x)^m(1+x) \\ &\geq (1+mx)(1+x) \\ &= 1+x+mx+mx^2 \\ &= 1+(m+1)x+mx^2 \\ &\geq 1+(m+1)x \end{aligned}$$

by the inductive hypothesis

Cauchy's Inequality

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. Then

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left(\sum_{j=1}^n a_j^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra language, this is equivalent to $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$.

Consider $f: \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^n (a_i - b_i x)^2$$

We know that $f(x) \geq 0$ for all $x \in \mathbb{R}$

$$\begin{aligned} &= \sum_{i=1}^n (a_j^2 - 2a_j b_j x + b_j^2 x^2) \\ &= \left(\sum_{j=1}^n b_j^2 \right) x^2 + \left(\sum_{j=1}^n -2a_j b_j \right) x + \sum_{j=1}^n a_j^2 \\ &= Ax^2 + Bx + C \end{aligned}$$

Therefore, $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned} \left(-2 \sum_{j=1}^n a_j b_j \right)^2 &\leq 4 \left(\sum_{j=1}^n a_j^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \\ \left| \sum_{j=1}^n a_j b_j \right| &= \left(\sum_{j=1}^n a_j^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2} \end{aligned}$$

As we know from linear algebra, the way we get equality is when $\vec{v} = c\vec{w}$, or that $a_j = cb_j \forall j$ for some $c \in \mathbb{R}$.

Triangle Inequality

Given $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left(\sum_{j=1}^n (a_j + b_j)^2 \right)^{1/2} \leq \left(\sum_{j=1}^n a_j^2 \right)^{1/2} + \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra, this is equivalent to $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

$$\begin{aligned} \sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\ &\leq \sum a_j^2 + 2 \left(\sum a_j^2 \right)^{1/2} \left(\sum b_j^2 \right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\ &= \left(\left(\sum a_j^2 \right)^{1/2} + \left(\sum b_j^2 \right)^{1/2} \right)^2 \end{aligned}$$

we take square roots to get our end result

Metrics and Norms on \mathbb{R}^n

Consider $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$, defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

Theorems about Absolute Value:

- (i) $|ab| = |a||b|$
- (ii) $|a^2| = |a|^2$
- (iii) $|-a| = |a|$
- (iv) $|a| \in \mathbb{R}^+$
- (v) $-|a| \leq a \leq |a|$
- (vi) $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$ for $\delta > 0$
- (vii) $|a + b| \leq |a| + |b|$, $|a - b| \leq |a| + |b|$, $||a| - |b|| \leq |a - b|$

Proofs

Proof of (i)

Case 1: If $a, b \in \mathbb{R}^+$, then $|a| = a$, and $|b| = b$, and $ab \in \mathbb{R}^+$, so $|ab| = ab$

Case 2: If $a, b \notin \mathbb{R}^+$, then $|a| = -a$, and $|b| = -b$. Additionally, $(-a)(-b) = ab \in \mathbb{R}^+$, so $|ab| = ab$. The LHS = ab , and the RHS = ab .

Case 3: $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$. Then, $|a||b| = (a)(-b) = -ab$. Then, since $a(-b) \in \mathbb{R}^+$, $-ab \in \mathbb{R}^+$, so $|ab| = -ab$. Therefore, the LHS and RHS are equal.

Proof of (vii) Having established that $|a + b| \leq |a| + |b|$, we will show that $||a| - |b|| \leq |a - b|$.

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a - b| + |b| \\ |a| - |b| &\leq |a - b| \end{aligned}$$

Similarly, by exchanging a for b

$$\begin{aligned} |b| - |a| &\leq |b - a| \\ |b| - |a| &\leq |a - b| \end{aligned}$$

Let $t = |a| - |b|$. We have shown that

$$\begin{aligned} \pm t &\leq |a - b| \\ -|a - b| &\leq t \leq |a - b| \\ |t| &\leq |a - b| \end{aligned}$$

Absolute Values, cont'd

Recall:

$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

If we want to find all $x \in \mathbb{R}$ such that $|x - 1| \leq |x|$, we would split up into cases:

$x \leq 0$ $x - 1 \leq -1$, so $|x| = -x$ and $|x - 1| = 1 - x$, so $1 - x \leq -x$, so $0 \geq 1$. \perp

$0 < x \leq 1$ $|x| = x$ and $|x - 1| = 1 - x$, so $1 - x \leq x$, so $x \geq \frac{1}{2}$, so $\frac{1}{2} \leq x \leq 1$.

$1 < x$ $|x| = x$ and $|x - 1| = x - 1$, so $x - 1 \leq x$, so $-1 \leq 0$, which is true $\forall \mathbb{R}$ in the interval, so $x > 1$.

Therefore, we have $x \in (\frac{1}{2}, \infty)$ as that which satisfies this inequality.

Bounded Sets

A subset $A \subseteq \mathbb{R}$ is **bounded** $\Leftrightarrow \exists c \geq 0$ such that $\forall x \in A$, $|x| \leq c$.

(\Rightarrow) Suppose $A \subseteq \mathbb{R}$ is bounded. Then, $\exists \ell, u \in \mathbb{R}$ such that $\ell \leq x \leq u \forall x \in A$. Let $c := \max\{|\ell|, |u|\}$.

Since $|u| \leq c$, we have that $x \leq c$.

Since $|\ell| \leq c$, and $-\ell \leq x$, we get that $-x \leq |\ell| \leq c$.

Since $x \leq c$ and $-x \leq c$, $|x| \leq c$.

(\Leftarrow) If such a c exists, then $|x| \leq c$ if and only if $-c \leq x \leq c$. Thus, $-c$ is the lower bound and c is the upper bound.

Bounded Functions

Let D be any set. A function $f : D \rightarrow \mathbb{R}$ is bounded if $\text{Ran}(D) \subseteq \mathbb{R}$ is bounded.

Example

Let $f : [3, 7] \rightarrow \mathbb{R}$, $f(x) = \frac{x^2+2x+1}{x-1}$. Show that f is bounded.

$$3 \leq x \leq 7 \Rightarrow 2 \leq x-1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}.$$

$$\text{Also, } 4 \leq x+1 \leq 8 \Rightarrow 16 \leq x^2+2x+1 \leq 64 \Rightarrow |x^2+2x+1| \leq 64.$$

$$\text{So, } |f(x)| \leq 32.$$

Distance Metrics

For $s, t \in \mathbb{R}$, we will define $d(s, t) = |s - t|$ to be the **distance** between s and t .

Properties:

(i)

$$d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$$

$$(s, t) \mapsto d(s, t) \geq 0$$

(ii) $d(s, t) = d(t, s)$

(iii) $d(s, r) \leq d(s, t) + d(t, r)$

(iv) $d(s, s) = 0$

(v) If $d(s, t) = 0$, then $s = t$.

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^n |x_j|$$

- ∞ -norm:

$$\|v\|_\infty = \max_{j=1}^n |x_j|$$

- 2-norm:

$$\|v\|_2 = \left(\sum_{j=1}^n x_j^2 \right)^{1/2}$$

Properties of the Norms

Properties: With v, w above, let $p = 1, 2, \infty$. The following are true:

- (1) $\|v\|_p \geq 0$
- (2) $\|v + w\|_p \leq \|v\|_p + \|w\|_p$
- (3) $\|\vec{0}\|_p = 0$
- (4) $\|v\|_p = 0 \Rightarrow v = \vec{0}$
- (5) $\forall t \in \mathbb{R}, \|tv\|_p = |t|\|v\|_p$

Proofs

Let $p = \infty$. We will prove (2).

Say $\|v\|_{\infty} = |x_i|$ and $\|w\|_{\infty} = |y_k|$. We want to show that $\|v + w\|_{\infty} = \max_{j=1}^n |x_j + y_j| \leq |x_i| + |y_k|$.

Note that $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| \\ &\leq |x_i| + |y_k| \\ &= \|v\|_{\infty} + \|w\|_{\infty} \end{aligned} \quad \text{Triangle Inequality}$$

Therefore, $\|v + w\|_{\infty} \leq \|v\|_{\infty} + \|w\|_{\infty}$.

Distances and Norms

A **norm** on \mathbb{R}^n is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$, $v \mapsto \|v\|$, satisfying the following properties for $v \in \mathbb{R}^n$:

- (1) $\|v\| \geq 0$
- (2) $\|v + w\| \leq \|v\| + \|w\|$
- (3) $\|\vec{0}\| = 0$
- (4) $\|v\| = 0 \Rightarrow v = \vec{0}$
- (5) $\forall t \in \mathbb{R}, \|tv\| = |t|\|v\|$

If $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a norm, we define $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$, defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for $v, w \in \mathbb{R}^n$.

The properties of distance in \mathbb{R} still hold for distance in \mathbb{R}^n :

- (1) $d(v, w) = d(w, v)$
- (2) $d(u, w) \leq d(u, v) + d(v, w)$
- (3) $d(v, v) = 0$
- (4) $d(v, w) = 0 \Rightarrow v = w$

Metric Spaces

A **metric space** is a nonempty set X equipped with a function $d : X \times X \rightarrow \mathbb{R}^+$, $(x, y) \mapsto d(x, y) \geq 0$. The metric has the following properties:

- (1) $d(x, y) = d(y, x) \quad \forall x, y \in X$
- (2) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$
- (3) $d(x, x) = 0$

$$(4) \ d(x, y) = 0 \Leftrightarrow x = y$$

The map d is called a *metric* on X .

Metric Spaces, Open Sets, and Closed Sets

Examples of Metric Spaces:

- \mathbb{R} with $d(x, y) = |x - y|$.
- \mathbb{R}^n with the *Euclidean metric*:

$$\begin{aligned} d_2(v, w) &= \|v - w\|_2 \\ &= \left(\sum_{j=1}^n (x_j - y_j)^2 \right)^{1/2} \end{aligned}$$

- \mathbb{R}^n with the 1-norm:

$$\begin{aligned} d_1(v, w) &= \|v - w\|_1 \\ &= \sum_{j=1}^n |x_j - y_j| \end{aligned}$$

- \mathbb{R}^n with the ∞ -norm:

$$\begin{aligned} d_\infty(v, w) &= \|v - w\|_\infty \\ &= \max_{j=1}^n |x_j - y_j| \end{aligned}$$

Let (X, d) be a metric space.

- (1) The **open ball** centered at $x_0 \in X$ with radius δ is:

$$U(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

- (2) The **closed ball** centered at $x_0 \in X$ with radius δ is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

- (3) A set $U \subseteq X$ is **open** if $\forall x \in U, \exists \delta > 0$ such that $U(x, \delta) \subseteq U$.

- (4) A set $C \subseteq X$ is **closed** if $\overline{C} = X - C \subseteq X$ is open.

Examples

In \mathbb{R} with $d(s, t) = |s - t|$:

$$\begin{aligned} U(x_0, \delta) &= \{y \in \mathbb{R} \mid d(y, x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0, \delta) &= [x_0 - \delta, x_0 + \delta] \end{aligned}$$

The interval $A = [1, \infty)$ is not open, as $\forall \delta > 0, U(1, \delta) \not\subseteq [1, \infty)$.

However, A is closed, as $\overline{A} = (-\infty, 1]$ is open: given $t \in \overline{A}$, choose $\delta = 1 - t$. Let $s \in V_\delta(t)$. Then, $s \in (t - \delta, t + \delta)$, so $s \in (t - (1 - t), t + (1 - t))$, or $s \in (2t - 1, 1)$, so $s < 1$.

Exercises

Show that the following are open:

- (a, b)
- (a, ∞)
- $(-\infty, b)$

and that the following are closed:

- $[a, b]$
- $[a, \infty)$
- $(-\infty, b]$

In (\mathbb{R}^2, d_2) , $B(0_{\mathbb{R}^2}, 1)$ is the **unit disc** centered at $(0, 0)$.

However, in (\mathbb{R}^2, d_∞) :

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

Finding a Supremum

Let $0 \neq A \subseteq \mathbb{R}$. Let $u \in \mathbb{R}$ be an upper bound for A . The following are equivalent:

- (i) $u = \sup(A)$
- (ii) If $t < u$, then $\exists a_t \in A$ such that $a_t > t$
- (iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $u - \varepsilon < a_\varepsilon$

Proofs

- (i) \Rightarrow (ii): Given $t < u$, if no such $a \in A$ with $t < a$ exists, then $a \leq t \forall a \in A$. Thus t would be an upper bound. However, $t < u$ and u is the supremum of A . \perp
- (ii) \Rightarrow (iii): Given $\varepsilon > 0$, set $t = u - \varepsilon < u$. So, by (ii), $\exists a_t \in A$ with $t < a_t$. Thus, $u - \varepsilon < a_t$. Set $a_\varepsilon = a_t$.
- (iii) \Rightarrow (i): Let v be an upper bound for A . Suppose $v < u$. Then, set $\varepsilon = u - v > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u - \varepsilon < a_\varepsilon$. So $u - (u - v) < a_\varepsilon$, so $v < a_\varepsilon$, meaning v cannot be an upper bound. \perp

Supremum Example

$\sup[0, 1) = 1$: Certainly, 1 is an upper bound for $[0, 1)$. Let $\varepsilon > 0$.

If $\varepsilon \geq 1$, pick $t = \frac{1}{2}$. Then, $1 - \varepsilon < 0 < \frac{1}{2}$

If $0 < \varepsilon < 1$, let $t = (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$. Then, $t \in [0, 1)$, and $1 - \varepsilon < 1 - \varepsilon/2 = t$

Finding an Infimum

Let $\emptyset \neq A \subseteq \mathbb{R}$. Let $\ell \in \mathbb{R}$ be a lower bound for A . The following are equivalent:

- (i) $\ell = \inf(A)$
- (ii) If $t > \ell$, $\exists a_t$ such that $t > a_t$
- (iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $\ell + \varepsilon > a_\varepsilon$

Infimum Example

$\inf \left\{ \frac{1}{n} \mid n \geq 1 \right\}$: Clearly, $0 < \frac{1}{n} \forall n \geq 1$. Let $\varepsilon > 0$.

We need to find $a \in \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ with $\varepsilon > a$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Let $a_\varepsilon = \frac{1}{m}$.

More on Supremum/Infimum

- If $A \subseteq \mathbb{R}$ and $\max(A) = u$, then $u = \sup(A)$: u is an upper bound of A by the definition of \max , and if $v \neq u$ is any upper bound of A , then $u < v$ since $u \in A$.
- If $\min(A) = \ell$, then $\ell = \inf(A)$ (by the same logic).
- If A is not bounded above, $\sup(A) = +\infty$, and if A is not bounded below, then $\inf(A) = -\infty$.
- If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.
- If $A \subseteq B$, then $\inf(A) \geq \inf(B)$: Let $\ell_A = \inf(A)$ and $\ell_B = \inf(B)$. By definition, $\ell_B \leq b \forall b \in B$. Since $A \subseteq B$, $\ell_B \leq a \forall a \in A$. Thus, ℓ_B is a lower bound for A . By definition of ℓ_A , $\ell_B \leq \ell_A$.

Let $A, B \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Then, the following are also sets:

- (1) $A + B = \{a + b \mid a \in A, b \in B\}$
- (2) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$
- (3) $t \cdot A = \{ta \mid a \in A\}$
- (4) $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

Completeness Axiom

If $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above, then $\sup(A)$ exists.

Well-Ordering Property: if $\emptyset \neq S \subseteq \mathbb{N}$, then $\min(S)$ exists.

Therefore, we can prove that if $F \subseteq \mathbb{Z}$ is bounded, then F has a least and greatest element.

Archimedean Property: Proof

If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x \leq n_x$.

Suppose there exists no natural number greater than x , then \mathbb{N} is bounded above by x . Let $u = \sup(\mathbb{N})$. By the Completeness Axiom, $u \in \mathbb{R}$ exists. Let $\varepsilon = 1$. Then, $\exists n \in \mathbb{N}$ with $u - 1 < n$. So, $u < n + 1$, but $n + 1 \in \mathbb{N}$, so u cannot be an upper bound.

Corollary to the Archimedean Property

$$\forall t > 0 \exists n \in \mathbb{N} \ni \frac{1}{n} < t$$

Corollary: Powers of 2

$$\forall t > 0 \exists m \in \mathbb{N} \ni \frac{1}{2^m} < t$$

By the corollary to the Archimedean Property, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < t$. By Bernoulli's inequality with $x = 1$, we have $2^n \geq n$, so $2^{-n} < n^{-1} < t$.

Corollary: In Between Integers

$$\forall x \in \mathbb{R} \exists n_x \in \mathbb{Z} \ni n_x - 1 \leq x < n_x$$

Assume $x \geq 0$. Let $S_x = \{n \mid n \in \mathbb{N} \ x < n\}$.

$S_x \neq \emptyset$ by the Archimedean Property. By the well-ordering property, let $n_x = \min(S_x)$.

Therefore, $x < n_x$. Also, $n_x - 1 \notin S_x$. Therefore $n_x - 1 \leq x$.

Density Theorems

Let (X, d) be any metric space. A subset $D \subseteq X$ is **dense** if $\forall x \in X, \varepsilon > 0, U(x, \varepsilon) \cap D \neq \emptyset$.

In the case of $X = \mathbb{R}$ and $d(s, t) = |s - t|$, $D \subseteq \mathbb{R}$ is dense if given any open interval I , $I \cap D \neq \emptyset$.

A metric space is **separable** if it admits a *countable* dense subset.

Density of the Rationals

$\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Let $I = (a, b)$ be an open interval. We may assume that $a, b \in \mathbb{R}$ are finite.

Then, $b - a > 0$. By the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, meaning $1 < nb - na$.

There exists also an integer m such that $m - 1 \leq na < m$, implying that $a \frac{m}{n}$. Also, $m \leq na + 1 < nb$. Therefore, $\frac{m}{n} < b$.

So, $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$.

Density of the Irrationals

$\mathbb{R} \setminus \mathbb{Q}$ is dense.

Assume $\sqrt{2}$ exists. Let $I = (a, b)$ be any open interval. Then, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

Find $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$.

Then, $a < q\sqrt{2} < b$, where $q\sqrt{2} \in \mathbb{R}$ and $q\sqrt{2} \notin \mathbb{Q}$.

Uniqueness of $\sqrt{2}$

$$\exists! x > 0 \ x^2 = 2$$

Existence: Let $S = \{t \in \mathbb{R} \mid 0 < t, \ t^2 < 2\}$. S is nonempty because $1 \in S$, and S is bounded above because $y > 2 \Rightarrow y^2 > 4$.

So, by the completeness axiom, $x := \sup(S)$ exists, and $x \geq 1$.

Claim: $x^2 = 2$

Contradiction 1: Assume $x^2 < 2$. We want to show that $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$. By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2 \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x + 1) \end{aligned}$$

We want to find $n \in \mathbb{N}$ with:

$$x^2 + \frac{1}{n}(2x + 1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

Therefore, by the Archimedean Property corollary, we know that n must exist.

Contradiction 2: We know that $x^2 \geq 2$. Since $x = \sup(S)$, $\forall m \in \mathbb{N}$, $\exists t_m \in S$ with $x - \frac{1}{m} < t_m$, so $(x - \frac{1}{m})^2 < t_m^2 < 2$.

Therefore, $x^2 - \frac{2x}{m} + \frac{1}{m^2} < 2$, so $x^2 - \frac{2x}{m} < 2$, so $0 \leq x^2 - 2 < \frac{2x}{m}$.

So, $0 \leq \frac{x^2 - 2}{2x} < \frac{1}{m}$, so $x^2 - 2 = 0$, so $x^2 = 2$.

Remark: If we had set $S' = \{t' \in \mathbb{Q} \mid t'^2 < 2, \ t' > 0\}$, we would have still obtained $\sup(S') = \sqrt{2}$. This means that \mathbb{Q} is *not* complete.

Intervals and Nested Intervals

(*) Given any interval I , if $x_1, x_2 \in I$, with $x_1 < x_2$, then $[x_1, x_2] \in I$.

This seems like an obvious property, but this is the *characteristic property* of intervals.

Characterization of Intervals

Let $S \subseteq \mathbb{R}$ be any nonempty subset of cardinality at least 2. Suppose S satisfies (*). Then, S is an interval.

Case 1: Suppose S is bounded.

Let $a = \inf(S)$ and $b = \sup(S)$. Then, $S \subseteq [a, b]$. We will show that $(a, b) \subseteq S$. Once this is shown, $S = \{(a, b), [a, b], [a, b), (a, b]\}$.

Let $t \in (a, b)$. Since $a = \inf(S)$, $\exists x_1 \in S$, $x_1 \in (a, t)$. Similarly, since $b = \sup(S)$, $\exists x_2 \in S$, $x_2 \in (t, b)$.

By the hypothesis, $[x_1, x_2] \subseteq S$. Since $t \in [x_1, x_2]$, $t \in S$.

Case 2: Suppose S is bounded above, but not below.

Let $b = \sup(S)$. Clearly, $S \subseteq (-\infty, b]$. We will show that $(-\infty, b) \subseteq S$. Once this is shown, $S = \{(-\infty, b), (-\infty, b]\}$.

Let $t \in (-\infty, b)$. Since $b = \sup(S)$, $\exists x_2 \in S$, $x_2 \in (t, b)$.

Since S is not bounded below, $\exists x_1 \in S$ such that $x_1 < t$ (or else t would be a lower bound).

By the hypothesis, $[x_1, x_2] \subseteq S$, and $t \in [x_1, x_2]$, so $t \in S$.

Case 3, 4: Left as an exercise for the reader.

A sequence of intervals $(I_n)_{n \geq 1}$ is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in $\bigcap I_n$.

(a) $\bigcap_{n=1}^{\infty} [0, 1/n] = \{0\}$.

(b) $\bigcap_{n=1}^{\infty} (0, 1/n] = \emptyset$

(c) $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$

Measure

The **measure** of an interval is basically its "size."

(a) $m([a, b]) = b - a$

(b) $m((a, b]) = b - a$

(c) $m((a, b)) = b - a$

(d) $m([a, b)) = b - a$

Nested Intervals Theorem

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested sequence of intervals.

(1) $\bigcap_{n \geq 1} I_n \neq \emptyset$

(2) If $\inf \{m(I_n) \mid n \geq 1\} = 0$, then $\bigcap_{n \geq 1} I_n = \{\xi\}$

(a)

Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, we have that $a_1 \leq a_2 \leq a_3, \dots$, and $b_1 \geq b_2 \geq b_3 \geq \dots$.

We know that $\{a_n\}$ is bounded above and nonempty. Let $\xi = \sup(\{a_n\}_{n=1}^\infty)$.

We know that $\{b_n\}$ is bounded below. Let $\eta = \inf(\{b_n\}_{n=1}^\infty)$.

We claim that $\xi \leq b_n \forall n \geq 1$. Suppose toward contradiction that $\exists m$ such that $\xi > b_m$. Then, by the supremum property, $\exists a_i$ such that $\xi > a_i > b_m$. If $k \leq m$, $a_k \leq a_m \leq b_m < a_k$. If $m \leq k$, then $b_m \geq b_k \geq a_k > b_m$. \perp

Similarly, using the same argument, $a_n \leq \eta \forall n$.

Thus, $\xi \leq \eta$.

We claim that $\bigcap_{n \geq 1} I_n = [\xi, \eta]$. If $t \in [\xi, \eta]$, then $a_n \leq \xi \leq t \leq \eta \leq b_n$. So $t \in [a_n, b_n] \forall n$, so $t \in \bigcap_{n \geq 1} [a_n, b_n]$.

If $t \in \bigcap_{n \geq 1} I_n$, then $t \in [a_n, b_n] \forall n$, so $a_n \leq t \leq b_n \forall n$. So, t is an upper bound on a_n , and a lower bound on b_n . So, $\xi \leq t \leq \eta$ by definition of ξ and η .

(b)

We have $\forall n \geq 1$

$$\begin{aligned} [\xi, \eta] &\subseteq [a_n, b_n] \\ \Rightarrow 0 &\leq \eta - \xi \leq b_n - a_n \\ &= m(I_n) \end{aligned}$$

So, if $\inf(\{m(I_n) \mid n \geq 1\}) = 0$, then $0 \leq \eta - \xi \leq 0$, so $\eta = \xi$.

Corollary to the Nested Intervals Theorem

$[0, 1]$ is uncountable.

Suppose it is possible to denumerate the interval $[0, 1] = \{t_1, t_2, \dots\}$.

We can find $[a_1, b_1] \subseteq [0, 1]$ with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$.

Then, we find $[a_2, b_2] \subseteq [a_1, b_1]$ with $a_2 < b_2$, $t_2 \notin [a_2, b_2]$.

Recursively, we find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $a_n < b_n$, $t_n \notin [a_n, b_n]$.

So, $I_n = ([a_n, b_n])_0^\infty$ is a sequence of nested intervals.

Therefore, $\exists \xi \in \bigcap I_n \subseteq [0, 1]$. However, $\xi \notin \{t_1, t_2, \dots\}$. \perp

Sequences in Metric Spaces

A sequence in a metric space M is a map

$$\begin{aligned} x &: \mathbb{N} \rightarrow M \\ x &= (x_n)_{n=1}^\infty \end{aligned}$$

$M = \mathbb{R}$, usually

I. Sequences defined by a formula:

- (i) $x_n = t$ (the constant sequence)
- (ii) $x_n = 2n + 1$
- (iii) $x_n = \frac{1}{n-1}$ (here, $n \geq 2$)

- (iv) $c_n = n \sin\left(\frac{1}{n}\right)$
- (v) $d_n = \left(1 + \frac{1}{n}\right)^n$
- (vi) Geometric Sequence: for $b \neq 0$, $(b^n)_{n \geq 0} = (1, b, b^2, \dots)$
- (vii) $x_n = \frac{n!}{n^n}$
- (viii) Given any function

$$f : [0, \infty) \rightarrow \mathbb{R}$$

we can set $x_n = f(n)$.

II. Sequences defined recursively:

- (i) $a_1 = 1$, $a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
- (ii) Fibonacci: $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$. The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - (1 - \varphi)^n)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$

- (iii) Let $f : M \rightarrow M$ be a self-map on a metric space. Fix $x_0 \in M$.

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

III. New sequences from old:

- (i) Let $(a_n)_n$ and $(b_n)_n$ be sequences, $t \in \mathbb{R}$. Then, we can do the following:

- $(a_n)_n + (b_n)_n = (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n(b_n)_n = (a_nb_n)_n$
- If $b_n \neq 0 \forall n$, $\left(\frac{a_n}{b_n}\right)$

- (ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

- (iii) We can look at ratios for $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

- (iv) We can look at partial sums, given $(a_k)_{k=1}^\infty$.

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^n a_k \end{aligned}$$

The sequence $(s_n)_n$ is called the sequence of partial sums. For example, the sequence of partial sums for $(b^k)_{k=0}^\infty$ is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for $b \neq 1$, $(1-b)(1+b+b^2+\cdots+b^n) = 1-b^{n+1}$

Exercise

Let $a_k = \frac{1}{k(k+1)}$. Find $(s_n)_n$.

Via partial fraction decomposition, we get that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore, $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^\infty$

Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R}, a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k| \quad \text{Infinity Norm}$$

This norm has the traditional properties of the norm:

$$\|(a_k)_k + (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty \quad \text{Triangle Inequality}$$

$$\|t(a_k)_k\|_\infty = |t| \|(a_k)_k\|_\infty \quad \text{Scalar Multiplication}$$

$$\|(a_k)_k\|_\infty = 0 \Leftrightarrow a_k = 0 \quad \forall k \quad \text{Zero Property}$$

$$\|(a_k)_k (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty \|(b_k)_k\|_\infty \quad \text{Multiplication}$$

Proof

Let $u = \|(a_k)_k\|_\infty$ and $v = \|(b_k)_k\|_\infty$.

Given any k ,

$$\begin{aligned} |a_k + b_k| &\leq |a_k| + |b_k| \\ &\leq u + v \end{aligned} \quad \begin{array}{l} \text{Triangle Inequality on } |\cdot| \\ \text{definition of supremum} \end{array}$$

$$\Rightarrow \sup_{k \geq 1} |a_k + b_k| \leq u + v$$

Thus,

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &= \|((a_k + b_k)_k)_k\|_\infty \\ &= \sup_{k \geq 1} |a_k + b_k| \\ &\leq u + v \end{aligned}$$

Monotonicity

A sequence $(x_n)_n$ is **increasing** if

$$x_1 \leq x_2 \leq \dots \quad \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \dots \quad \forall n$$

The sequence is *eventually* increasing if $\exists m \in \mathbb{N} \ni x_n \leq x_{n+1}$ for $n > m$.

Similarly, the sequence is eventually decreasing if $\exists m \in \mathbb{N} \ni x_n \geq x_{n+1}$ for $n > m$.

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

Example

The sequence

$$a_1 = 1$$

$$a_{n+1} = \frac{1}{2}a_n + 2$$

is increasing and bounded above.

We will prove the first statement via induction:

Base: $a_1 = 1$, $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

Inductive Hypothesis $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+1}$

Proof:

$$a_n \leq a_{n+1}$$

$$\frac{1}{2}a_n \leq \frac{1}{2}a_{n+1}$$

$$\frac{1}{2}a_n + 2 \leq \frac{1}{2}a_{n+1} + 2$$

$$a_{n+1} \leq a_{n+2}$$

To prove the sequence is bounded above, we do the following:

$$a_1 = 1 \leq 4$$

$$\frac{1}{2}a_1 \leq 2$$

$$\frac{1}{2}a_1 + 2 \leq 2$$

$$a_2 \leq 4$$

We claim that $\forall n$, $a_n \leq 4 \Rightarrow a_{n+1} \leq 4$, as we have shown the base case.

$$a_n \leq 4$$

$$\frac{1}{2}a_n \leq 2$$

$$\frac{1}{2}a_n + 2 \leq 4$$

$$a_{n+1} \leq 4$$

Convergence of Sequences

Let $L \in \mathbb{R}$, $\varepsilon > 0$. Then, the ε -neighborhood of L is $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$.

$$x \in V_\varepsilon(L)$$

$$\Leftrightarrow$$

$$|x - L| < \varepsilon$$

$$L - \varepsilon < x < L + \varepsilon$$

With this in mind, we know the following:

Definition of Convergence

A real sequence $(x_n)_n$ converges to a number x if

$$(\forall \epsilon > 0) (\exists N_\epsilon \in \mathbb{N}) \ni n \geq N \Rightarrow |x_n - x| < \epsilon$$

If no such L exists, then $(x_n)_n$ is said to **diverge**.

A sequence $(x_n)_n$ in a metric space (X, d) converges to a point x if

$$(\forall \epsilon > 0) (\exists N_\epsilon \in \mathbb{N}) \ni d(x_n, x) < \epsilon$$

Essentially, we want to show that there always exists an N such that the N th tail (i.e., x_N, x_{N+1}, \dots) are within ϵ of L for any ϵ .

Note: N usually depends on ϵ (the smaller the ϵ , the larger the N).

Convergence Proof

$$\left(\frac{1}{n}\right)_n \xrightarrow{n \rightarrow \infty} 0$$

We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given $\epsilon > 0$, we want $\frac{1}{n} < \epsilon$, meaning $n > \frac{1}{\epsilon}$.

Proof: Let $\epsilon > 0$. By the Archimedean property corollary, find $N \in \mathbb{N}$ large such that $\frac{1}{N} < \epsilon$.

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \epsilon \end{aligned}$$

so, if $n \geq N$, then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \epsilon \end{aligned}$$

Convergence Proof 2

Show that

$$\left(\frac{5n-1}{3-n}\right)_{n \geq 4} \xrightarrow{n \rightarrow \infty} -5$$

$$\begin{aligned} |x_n - L| &= \left| \frac{5n-1}{3-n} + 5 \right| \\ &= \frac{14}{|3-n|} \\ &= \frac{14}{n-3} < \varepsilon \\ \frac{14}{n-3} &< \varepsilon \\ n &> \frac{14}{\varepsilon} + 3 \end{aligned}$$

Proof: Let $\varepsilon > 0$. Find $N' \in \mathbb{N}$ so large that $\frac{1}{N'} < \frac{\varepsilon}{14}$ (which exists by the Archimedean property corollary). Let $N = N' + 3$. If $n \geq N$, then

$$\begin{aligned} n-3 &\geq \frac{1}{N'} \\ \frac{1}{n-3} &\leq \frac{1}{N'} \\ &< \frac{\varepsilon}{14} \end{aligned}$$

whence

$$\begin{aligned} |x_n - L| &= \frac{14}{n-3} \\ &< \frac{14\varepsilon}{14} \\ &= \varepsilon \end{aligned}$$

Sequences and their Limits, cont'd

Convergence and Distance

Let (X, d) be a metric space, and let $(x_n)_n$ be a sequence in the metric space. The following are equivalent:

- (i) $(x_n)_n \rightarrow x$
- (ii) $(d(x_n, x))_n \rightarrow 0$

(i) \Rightarrow (b) Let $\varepsilon > 0$. Find $N_\varepsilon \in \mathbb{N}$ so large such that $d(x_n, x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

So, $|d(x_n, x) - 0| = d(x_n, x) < \varepsilon$ for all $\varepsilon > 0$. Whence, $(d(x_n, x))_n \rightarrow 0$.

(ii) \Rightarrow (i) This direction is similar.

In \mathbb{R} , this implies that

$$\begin{aligned} (x_n)_n &\rightarrow x \\ &\Leftrightarrow \\ (|x_n - x|)_n &\rightarrow 0 \end{aligned}$$

Comparison Proposition

Let (X, d) be a metric space and let $x \in X$, and suppose $(x_n)_n$ is a sequence in X .

If $\exists c \geq 0$, $m \in \mathbb{N}$, and a sequence $(a_n)_n \in \mathbb{R}^+$ with $(a_n)_n \rightarrow 0$ and $d(x_n, x) \leq ca_n \forall n > m$. Then, $(x_n)_n \rightarrow x$.

Let $\varepsilon > 0$. Note that $\frac{\varepsilon}{c} > 0$.

Find $N_1 \in \mathbb{N}$ large such that $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\varepsilon}{c}$, which is always possible since $(a_n)_n \rightarrow 0$.

Let $N = \max(N_1, m)$. Then, $n \geq N \Rightarrow n \geq N_1$ and $n \geq m$. So,

$$\begin{aligned} d(x_n, x) &\leq ca_n \\ &< c \frac{\varepsilon}{c} \\ &= \varepsilon \end{aligned}$$

so, $n \geq N \Rightarrow d(x_n, x) < \varepsilon$, whence $(x_n)_n \rightarrow x$

Comparison Proposition, Example 1

Prove

$$\left(\frac{\sin(n^2 - 1)}{n^2 + 3} \right)_n \rightarrow 0$$

$$\begin{aligned} \left| \frac{\sin(n^2 - 1)}{n^2 + 3} - 0 \right| &= \frac{|\sin(n^2 - 1)|}{n^2 + 3} \\ &\leq \frac{1}{n^2 + 3} \\ &\leq \frac{1}{n^2} \\ &\leq \frac{1}{n} \end{aligned}$$

We know that $a_n = \frac{1}{n}$ converges to 0. So, by our comparison proposition, we are done.

Comparison Proposition, Example 2

Prove

$$\left(\frac{1}{2^n} \right)_n \rightarrow 0$$

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &\geq 1 + n &> n \end{aligned}$$

Bernoulli's Inequality

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since $a_n = \frac{1}{n}$ converges, we know that $\frac{1}{2^n}$ must converge.

Sequence Divergence

A sequence $(x_n)_n$ is **divergent** if it does not converge. $(x_n)_n \rightarrow 0$ if and only if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni (\forall n \geq N_\varepsilon) d(x_n, x) < \varepsilon$$

So, $(x_n)_n$ diverges if and only if

$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \rightarrow d(x_n, x) \geq \varepsilon_0$$

Diverging Sequence Proof

Show that the following sequence diverges:

$$a_n = (-1)^n$$

Step 1

$$((-1)^n)_n \not\rightarrow 1$$

Take $\varepsilon_0 = 1/2$, given any $N \in \mathbb{N}$, we will find $n \geq N$ odd:

$$\begin{aligned} n &= 2N + 1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

Step 2

$$((-1)^n)_n \not\rightarrow -1$$

by letting $\varepsilon_0 = 1/2$ and $n = 2N$.

Diverging Sequence Proof 2

Does

$$a_n = (\sin(n))_n$$

converge?

It is not the case that $(\sin(n))_n \rightarrow L$ for any $L \in \mathbb{R}$.

Case 1 If $L > 1$, set $\varepsilon_0 = \frac{L-1}{2}$. Then, given any $N \in \mathbb{N}$, pick $n = N$.

$$\begin{aligned} |\sin(n) - L| &= L - \sin(n) \\ &\geq L - 1 \\ &> \frac{L-1}{2} \\ &= \varepsilon_0 \end{aligned}$$

Case 2 Similarly for $L < -1$

Case 3 Suppose $-1 < L < 1$.

Case 3.1 Suppose $L > 0$. Set $\varepsilon_0 = \frac{L}{2}$. Given any N , find $n \geq N$ with $\sin(n) < 0$.

We find k large such that $N < (2k+1)\pi$, which we can always do because we are finding $k > \frac{1}{2} \left(\frac{N}{\pi} - 1 \right)$, which is always possible by the Archimedean property.

Note that $N < (2k+1)\pi < (2k+2)\pi$. Note that $\sin(x) < 0$ on the interval $((2k+1)\pi, (2k+2)\pi)$. Note that $|(2k+1)\pi - (2k+2)\pi| = \pi$. Let $n = \lceil (2k+1)\pi \rceil$. Then, $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

Case 3.2 Suppose $L < 0$, set $\varepsilon_0 = \frac{-L}{2}$. Given N , find $n \geq N$ with $\sin(n) > 0$. Using the same strategy as above, we find n such that $|L - \sin(n)| > \varepsilon_0$

Case 3.3 Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given $N \in \mathbb{N}$, find $n \geq N$ with $\sin(n) \geq \frac{1}{2}$. Then, $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$.

Showing that a sequence diverges is not easy — later, we will divergence with non-uniqueness of convergent subsequences.

Alternating Series

Consider again

$$((-1)^n)_{n \geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to -1 :

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating series diverges.

Uniqueness of Limits

A sequence $(x_n)_n$ can converge to at most one limit.

Suppose toward contradiction that $(x_n)_n$ converges to L_1 and L_2 with $L_1 \neq L_2$.

WLOG, let $L_2 > L_1$. Take $\epsilon = \frac{L_2 - L_1}{3}$.

Since $(x_n)_n$ converges to L_1 , $\exists N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \epsilon$. Similarly, since $(x_n)_n$ converges to L_2 , $\exists N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \epsilon$.

Let $N = \max N_1, N_2$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

So, $|x_n - L_1| < \epsilon$ and $|x_n - L_2| < \epsilon$. So, $x_n \in V_\epsilon(L_1)$, and $x_n \in V_\epsilon(L_2)$, meaning $x_n \in V_\epsilon(L_1) \cap V_\epsilon(L_2)$, but $V_\epsilon(L_1) \cap V_\epsilon(L_2) = \emptyset$. \perp

Useful Lemmas for Convergence

Absolutely Convergent Sequences

Let $(x_n)_n$ be a real sequence. If x_n converges to x , then $|x_n| \rightarrow |x|$. However, the converse is not the case.

Note that since $(x_n)_n \rightarrow x$, $d(x_n, x) \rightarrow 0$.

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

Convergence to Zero

Let a_n be a sequence.

$$(a_n)_n \rightarrow 0$$

$$\Leftrightarrow$$

$$|(a_n)_n| \rightarrow 0$$

(\Rightarrow) If $(a_n)_n \rightarrow 0$, then we showed previously that $|(a_n)_n| \rightarrow |0| = 0$

(\Leftarrow) Suppose $|(a_n)_n| \rightarrow 0$. Given $\varepsilon > 0$, then $\exists N$ such that $n \geq N$ implies

$$||a_n| - 0| < \varepsilon$$

$$||a_n|| < \varepsilon$$

$$|a_n| < \varepsilon$$

$$|a_n - 0| < \varepsilon$$

So, $(a_n)_n \rightarrow 0$

Geometric Sequence

Let $b \in \mathbb{R}$. Consider

$$(b^n)_{n \geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided $-1 < b \leq 1$. Otherwise, the sequence is divergent.

If $b = 0$, then the sequence $(b^n)_n = (0, 0, 0, \dots)$ is convergent.

If $b = 1$, then the sequence $(b^n)_n = (1, 1, 1, \dots)$ is convergent.

If $b = -1$, then the sequence $(b^n)_n = (1, -1, 1, \dots)$ is divergent.

Case 1 Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, so $\frac{1}{b} = 1 + a$.

So, by Bernoulli's Inequality, $(\frac{1}{b})^n = (1 + a)^n \geq 1 + na > na$, so $b^n < \frac{1}{na}$.

$$|b^n - 0| = b^n$$

$$< \frac{1}{na}$$

$$= \frac{1}{a} \frac{1}{n}$$

$$\rightarrow 0$$

So, $(b^n)_n \rightarrow 0$.

Case 2 Suppose $-1 < b < 0$. If we look at $|b^n| = |b|^n$, we know that $(|b|^n)_n \rightarrow 0$ by our work above. By the previous lemma, we know that since $|b^n| \rightarrow 0$, $b^n \rightarrow 0$.

Case 3 Suppose $b > 1$. Then, $b = 1 + a$ where $a > 0$.

$$b^n = (1 + a)^n$$

$$\geq 1 + na$$

$$> na$$

Bernoulli's Inequality

Suppose toward contradiction that $(b^n)_n \rightarrow L$. Let $\varepsilon_0 = 1$. Find $N \in \mathbb{N}$ such that $N > \frac{L+1}{a}$. N must exist by the Archimedean property.

Therefore, $(N)(a) > L + 1$. If $n \geq N$, then $(n)(a) > (N)(a) > L + 1$, so $|b^n - L| \geq na - L \geq \varepsilon_0$. \perp

Case 4 Suppose $b < -1$, and suppose toward contradiction that $(b^n)_n \rightarrow L$. By the previous lemma, we know that $|b^n| \rightarrow |L|$. So, $|b|^n \rightarrow |L|$. But, $|b| > 1$, which means our assumption contradicts the result from above. \perp

Sequences and Limits, Cont'd

 n th Root Convergence

If $c > 0$, then $(c^{1/n})_n \rightarrow 1$.

Case 1: If $c = 1$, then we get $(c^{1/n})_n = (1, 1, 1, \dots)$, which clearly converges to one.

Case 2: Assume that $c > 1$. Then, $c^{1/n} > 1$, because if $d = c^{1/n} \leq 1$, then $d^n \leq 1$, so $c \leq 1$. We can write $c^{1/n} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c &= c^n \\ &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned} \quad \text{Bernoulli's Inequality}$$

So, $d_n \leq \frac{c}{n}$. Remember, $c^{1/n} = 1 + d_n$.

$$\begin{aligned} |c^{1/n} - 1| &= c^{1/n} - 1 \\ &= d_n \\ &\leq c \cdot \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore, $c^{1/n} \rightarrow 1$.

Case 3: Assume $0 < c < 1$. Then, $c^{1/n} < 1$, so $\frac{1}{c^{1/n}} > 1$.

So, we can write $\frac{1}{c^{1/n}} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c^{1/n} &= \frac{1}{1 + d_n} \\ 1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\ &= \frac{d_n}{1 + d_n} \\ &\leq d_n \end{aligned}$$

Remember, $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned} \frac{1}{c} &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned}$$

So, $d_n \leq \frac{1}{cn}$

$$\begin{aligned} |1 - c^{1/n}| &= 1 - c^{1/n} \\ &\leq d_n \\ &\leq \frac{1}{c} \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore, $(c^{1/n})_n \rightarrow 1$.

Positive Sequence Convergence

Let $(x_n)_n$ be a sequence with $x_n \in \mathbb{R}^+ \forall n \in \mathbb{N}$, with $(x_n)_n \rightarrow x$. Then, x is also positive, and $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.

Suppose toward contradiction that $x < 0$. Let $\epsilon = \frac{|0-x|}{2}$. Since $(x_n)_n$ converges to x , we know that $x_n \in V_\epsilon(x)$ for large n . However, every member of $V_\epsilon(x) < 0$, and $x_n > 0$. \perp

Case 1: If $x = 0$, we will show that $(\sqrt{x_n})_n \rightarrow 0$.

Let $\epsilon > 0$, find $N \in \mathbb{N}$ large such that if $n \geq N$, we have

$$\begin{aligned} |x_n - 0| &< \epsilon^2 \\ x_n &< \epsilon^2 \\ \sqrt{x_n} &< \epsilon \\ |\sqrt{x_n} - 0| &< \epsilon \end{aligned}$$

Case 2: If $x > 0$, we will show that $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{1}{\sqrt{x}} |x_n - x| \\ &\rightarrow 0 \end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \rightarrow 0$, so $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.

nth Root of n Convergence

$$\left(n^{1/n}\right)_n \rightarrow 1$$

We will make use of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Note that $n^{1/n} > 1$ for n past 1. So, we write

$$n^{1/n} = 1 + d_n \quad d_n > 0$$

$$n = (1 + d_n)^n$$

$$= \sum_{k=0}^n \binom{n}{k} d_n^k$$

$$= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n$$

$$\geq \binom{n}{0} + \binom{n}{2} d_n^2$$

as all terms are positive

$$= 1 + \frac{n(n-1)}{2} d_n^2$$

so

$$n - 1 \geq \frac{n(n-1)}{2} d_n^2$$

$$\frac{2}{n} \geq d_n^2$$

$$\frac{\sqrt{2}}{\sqrt{n}} \geq d_n$$

So, we have

$$|n^{1/n} - 1| = n^{1/n} - 1$$

$$= d_n$$

$$\leq \sqrt{2} \frac{1}{\sqrt{n}}$$

$$\rightarrow 0$$

by previous corollary

So, $(n^{1/n})_n \rightarrow 1$.

Multiplication by Geometric Sequence

Let $0 \leq b < 1$. Show that

$$(nb^n)_n \rightarrow 0$$

If $0 < b < 1$ (the 0 case is trivial). So, $\frac{1}{b} > 1$, meaning $\frac{1}{b} = 1 + d$ for some $d > 0$.

$$\frac{1}{b^n} = (1 + d)^n$$

$$\geq \frac{n(n-1)}{2} d^2$$

$$\frac{2}{d^2(n)(n-1)} \geq b^n$$

$$nb^n \leq \frac{2}{d^2(n-1)}$$

$$\rightarrow 0$$

by previous corollary

Therefore, $(nb^n)_n \rightarrow 0$.

Boundedness and Convergence

If $(x_n)_n$ is a convergent sequence, x_n is bounded. The converse is false in general.

Suppose $(x_n)_n \rightarrow x$. Let $\varepsilon = 1$.

Then, $\exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq N$.

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$. If $n \geq N$, then $|x_n| \leq c$, because $x_n \in V_\varepsilon(x)$. If $n < N$, then $|x_n| \leq c$.

Together, we have $|x_n| \leq c$ for all n .

To show the converse is not true, consider $((-1)^n)_n$. This sequence is bounded but not convergent.

Algebraic Operations on Sequences

Let $(x_n)_n \rightarrow x$, $(y_n)_n \rightarrow y$, and $(z_n)_n \rightarrow z$ be convergent sequences. Let $t \in \mathbb{R}$. Then, the following are all true:

- (1) $(x_n \pm y_n)_n \rightarrow x \pm y$
- (2) $(tx_n)_n \rightarrow tx$
- (3) $(x_n y_n)_n \rightarrow xy$
- (4) Assume $z_n \neq 0 \forall n$, and $z \neq 0$. Then, $\left(\frac{1}{z_n}\right)_n \rightarrow \frac{1}{z}$, and $\left(\frac{x_n}{z_n}\right)_n \rightarrow \frac{x}{z}$.

Proof of (1) Let $\varepsilon > 0$. Since $x_n \rightarrow x$, $y_n \rightarrow y$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \rightarrow |x_n - x| < \frac{\varepsilon}{2}$, and $n \geq N_2 \rightarrow |y_n - y| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

$$\begin{aligned} |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Proof of (3) We have $(x_n)_n \rightarrow x$ and $(y_n)_n \rightarrow y$.

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |x_n - x||y| \end{aligned}$$

Since convergent sequences are bounded, $\exists c \in \mathbb{R}$ such that $|x_n| < c$, $\forall n$

$$\begin{aligned} &\leq c|y_n - y| + |x_n - x||y| \\ &\rightarrow 0 \end{aligned}$$

Therefore, $|x_n y_n - xy| \rightarrow 0$, so $x_n y_n \rightarrow xy$.

Proof of (4) We have $z_n \neq 0$ and $z \neq 0$. Let $\varepsilon > 0$.

$$\begin{aligned} \left| \frac{1}{z_n} - \frac{1}{z} \right| &= \frac{|z - z_n|}{|z_n z|} \\ &= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|} \end{aligned}$$

Let $\varepsilon = \frac{|z|}{2}$. Since $(z_n)_n \rightarrow z$, we know that $z_n \in V_\varepsilon(z)$ for $n \geq N \in \mathbb{N}$. For $n \geq N$, $|z_n| > \frac{|z|}{2}$, so $\frac{1}{|z_n|} < \frac{2}{|z|}$.

$$\begin{aligned} &\leq |z_n - z| \frac{2}{|z|^2} \\ &\rightarrow 0 \end{aligned}$$

So, $\left| \frac{1}{z_n} - \frac{1}{z} \right| \rightarrow 0$, so $\frac{1}{z_n} \rightarrow \frac{1}{z}$

Ordering of Limits

Let $(x_n)_n \rightarrow x$ and $(y_n)_n \rightarrow y$. If $x_n \leq y_n$ for all n , then $x \leq y$.

Suppose toward contradiction that $x > y$.

Let $\epsilon = \frac{x-y}{2}$.

So, $\exists N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow y_n \in V_\epsilon(y)$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow x_n \in V_\epsilon(x)$.

Let $N = \max\{N_1, N_2\}$. Then, $x_N \in V_\epsilon(x)$ and $y_N \in V_\epsilon(y)$. But that means $x_N > y_N$. \perp

In particular, if $(x_n)_n \rightarrow x$, and $a \leq x_n \leq b$, then $a \leq x \leq b$.

Squeeze Theorem

Let $(x_n)_n \rightarrow x$, $(y_n)_n \rightarrow y$, and $(z_n)_n \rightarrow z$, where $x_n \leq y_n \leq z_n$ for all n .

If $L = x = z$, then $y = L$.

Let $\epsilon > 0$. Find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow V_\epsilon(L)$, and $n \geq N_2 \Rightarrow V_\epsilon(L)$.

Let $N = \max\{N_1, N_2\}$. Then, $n \geq N \Rightarrow x_n, z_n \in V_\epsilon(L)$. Thus,

$$L - \epsilon < x_n \leq y_n \leq z_n < L + \epsilon$$

so $y_n \in V_\epsilon(L)$, so $(y_n)_n \rightarrow L$.

For example, let $a_n = \frac{\sin(n)}{n}$. Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero, $a_n \rightarrow 0$

As another example, consider $a_n = (2^n + 3^n)^{1/n}$. Then,

$$\begin{aligned} 3^n &\leq 2^n + 3^n \leq 2 \cdot 3^n \\ 3 &\leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3 \end{aligned}$$

Since $2^{1/n} \rightarrow 1$, we have $a_n \rightarrow 3$.

Ratio Test

Let (x_n) be a sequence of strictly positive numbers, with $\left(\frac{x_{n+1}}{x_n}\right)_n \rightarrow r < 1$. Then, $(x_n)_n \rightarrow 0$.

Since $r < 1$, then $1 - r > 0$. Let $\rho = r + \frac{1-r}{2}$, and $\epsilon = \rho - r = \frac{1-r}{2}$.

Since the sequence converges, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} - r \right| &< \epsilon \\ \frac{x_{n+1}}{x_n} &< \rho \\ x_{n+1} &< \rho x_n \end{aligned}$$

In particular, $x_{N+1} < \rho x_N$, and $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$. Inductively, one can show that $\forall k \geq 1$, $x_{N+k} < \rho^k x_N$.

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as $k \rightarrow \infty$, both sides of the inequality go to 0, implying that $x_n \rightarrow 0$

Monotone Convergence Theorem

Let $(x_n)_n$ be a monotone sequence. Then, $(x_n)_n$ is convergent if and only if it is bounded.

(a) If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n \rightarrow \sup(\{x_n \mid n \in \mathbb{N}\})$.

(b) If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n \rightarrow \inf(\{x_n \mid n \in \mathbb{N}\})$.

We have already shown that all convergent sequences are bounded.

Assume that $(x_n)_n$ is monotonic and bounded.

Case 1: Suppose $(x_n)_n$ is increasing. Let $\sup\{x_n \mid n \in \mathbb{N}\} := u$. We claim that $(x_n)_n \rightarrow u$.

Let $\varepsilon > 0$. By the definition of supremum, $\exists N \in \mathbb{N}$ such that $u - \varepsilon < x_N$. Note that $\forall n \geq N$, $u - \varepsilon < x_N \leq x_n \leq u$.

Therefore, if $n \geq N$, then $|x_n - u| < \varepsilon$.

Case 2: Suppose $(x_n)_n$ is decreasing. Let $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$. We claim that $(x_n)_n \rightarrow \ell$.

Let $\varepsilon > 0$. By the definition of infimum, $\exists N \in \mathbb{N}$ such that $\ell + \varepsilon > x_N$. Additionally, $\forall n \geq N$, $\ell \leq x_n \leq x_N < \ell + \varepsilon$.

Therefore, if $n \geq N$, $|x_n - \ell| < \varepsilon$.

Applications of the Monotone Convergence Theorem

Lemma

If $(x_n)_n$ is a convergent sequence, and $m \in \mathbb{N}$, the m -th tail, $x_{(m)} = (x_{m+k})_{k=1}^\infty$ is also convergent. If $(x_n)_n \rightarrow L$ then $x_{(m)} \rightarrow L$.

Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. If $k \geq N$, then $m + k \geq N$, so $|x_{m+k} - L| < \varepsilon$.

Thus, $(x_{m+k})_k \rightarrow L$

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that $x_n \geq 4 \forall n$.

$$x_1 = 8 \geq 4$$

Suppose $x_k \geq 4$. We will show that $x_{k+1} \geq 4$.

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore, $(x_n)_n$ is bounded below by 4.

We claim that $(x_n)_n$ is decreasing. $\forall n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} &\leq x_n \Leftrightarrow \\ \frac{1}{2}x_n + 2 &\leq x_n \\ \Leftrightarrow 4 &\leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that $(x_n)_n \rightarrow L$.

To find L , we use the recursive relationship and the lemma.

$$x_{n+1} = \left(\frac{1}{2}x_n + 2 \right)_{n=1}^{\infty}$$

$$L = \frac{1}{2}L + 2$$

$$L = 4$$

Consider the following sequence

$$x_1 = 1$$

$$x_2 = 1 + \frac{1}{4}$$

$$x_3 = 1 + \frac{1}{4} + \frac{1}{9}$$

$$x_k = \sum_{k=1}^n \frac{1}{k^2}$$

We will show that $(x_n)_n$, the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$k^2 \geq k(k-1) \quad k \geq 2$$

$$\frac{1}{k^2} \leq \frac{1}{k(k-1)}$$

$$= \frac{1}{k-1} - \frac{1}{k}$$

$$\sum_{k=2}^n \frac{1}{k^2} \leq \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right)$$

$$\sum_{k=1}^n \frac{1}{k^2} \leq 1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right)$$

But

$$1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{n}$$

so, we have

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k^2} &\leq 2 - \frac{1}{n} \\ &< 2 \end{aligned}$$

So, $(x_n)_n$ is bounded above.

Nested Intervals Theorem, Alternative Proof

Let $I_n = [a_n, b_n]$ be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

Since the intervals are nested, it must be the case that $a_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq b_1$.

Similarly, $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$.

So, $(a_n)_n$ is an increasing sequence bounded above by b_1 and $(b_n)_n$ is a decreasing sequence bounded below by a_1 . So, $(b_n)_n \rightarrow r$ and $(a_n)_n \rightarrow \ell$

Note that $\ell = \sup(a_n)$ and $r = \inf(b_n)$.

Fix $n \in \mathbb{N}$, then for any $m \geq n$, $a_n \leq a_m \leq b_m \leq b_n$. So, $\sup(a_m) = \ell \leq b_n$. Unlocking n , we get that $\ell \leq \inf(b_n) = r$.

Calculating Square Roots

Let $a \in \mathbb{R}^+$. We will construct a sequence $(x_n)_n \rightarrow \sqrt{a}$.

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

We will prove that $x_n^2 \geq a$.

$$\begin{aligned} 2x_{n+1} &= x_n + \frac{a}{x_n} \\ 2x_{n+1}x_n &= x_n^2 + a \\ 0 &= x_n^2 - 2x_{n+1}x_n + a \end{aligned}$$

So, x_n is a real root, meaning

$$\begin{aligned} \Delta &= 4x_{n+1}^2 - 4a \\ x_{n+1}^2 &\geq a \end{aligned} \quad \forall n$$

So, $\forall n \geq 2$

$$x_n^2 \geq a$$

We will show that x_n is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \\ &= \frac{1}{2} \underbrace{\left(\frac{x_n^2 - a}{x_n} \right)}_{\geq 0 \quad \forall n \geq 2} \end{aligned}$$

So, we have that $(x_n)_n$ is decreasing and bounded below, meaning $(x_n)_n \rightarrow x$ for some $x \in \mathbb{R}$.

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \\ x &= \frac{1}{2} \left(x + \frac{a}{x} \right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} \end{aligned} \quad \text{remember that } x > 0$$

Euler's Number

Consider

$$\begin{aligned} (e_n)_n &= \left(1 + \frac{1}{n} \right)^n \\ &= \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} \end{aligned}$$

Similarly,

$$\begin{aligned} e_{n+1} &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n+1} \right) \right) \\ e_{n+1} &\geq e_n \end{aligned} \quad \forall n$$

We claim that $(e_n)_n$ is bounded above.

$$e_1 = \left(1 + \frac{1}{1}\right)^1$$

$$2 \leq e_n$$

$$e_n = \sum_{k=0}^n \left(\frac{1}{k!} \underbrace{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}_{\leq 1} \right)$$

$$2^{k-1} \leq k!$$

$$k \geq 2$$

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}}$$

$$e_n = \sum_{k=0}^n \frac{1}{k!} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)$$

$$\leq \sum_{k=0}^n \frac{1}{k!}$$

$$\leq 2 + \sum_{\ell=1}^{n-1} \frac{1}{2^\ell}$$

$$< 3$$

so, we have

$$2 \leq e_n \leq 3$$

so, by the monotone convergence theorem, $(e_n)_n$ converges

$$e := \sup_n \left(1 + \frac{1}{n}\right)^n$$

Monotone Divergence

A sequence that is increasing and *unbounded* diverges to infinity.

Let $M > 0$. Since $(x_n)_n$ is unbounded, $\exists N \in \mathbb{N}$ such that $x_N > M$

Thus, if $n \geq N$, then $x_n \geq x_N > M$.

Consider

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We can see that $h_n < h_{n+1}$. The primary question is as to whether $(h_n)_n$ is bounded.

$$h_1 = 1$$

$$\geq 1$$

$$h_2 = 1 + \frac{1}{2}$$

$$\geq 1 + \frac{1}{2}$$

$$h_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2}$$

$$h_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^k \frac{1}{2}$$

Let M be large. Find n such that $n > 2(M-1)$. In this case, $n/2 + 1 > M$. Let $N = 2^n$. Then, for $m \geq N$, $h_m > M$.

Thus, $(h_n)_n$ diverges to infinity.

Natural Sequences

A **natural sequence** is a strictly increasing sequence of natural numbers, $(n_k)_{k=1}^{\infty}$

$$n_1 < n_2 < n_3 < \dots$$

where $\forall k \in \mathbb{N}$, $n_k \in \mathbb{N}$.

Natural Sequence Property

Given $(n_k)_k$ natural sequence, show that $(n_k) \geq k$.

Base Case: We know that $n_1 \leq 1$, as $n_1 \in \mathbb{N}$.

Inductive Step: To be continued...

Subsequences

Let $(x_n)_n$ be a sequence. A subsequence $(x_{n_k})_{k=1}^{\infty}$, where $(n_k)_k$ is a natural sequence.

For example, if $(x_n)_n = (-1)^n$. If $(n_k) = 2k$, then, $(x_{n_k}) = ((-1)^{2k})_k = (1, 1, 1, \dots)$. But, if $(n_k) = 2k + 1$, then $(x_{n_k}) = (-1, -1, -1, \dots)$.

If $(x_n) = (1/n)_n$, and $(n_k)_k = k^2$, then $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \dots)$.

If $(x_n)_n$ is a sequence, its m -th **tail** is $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \dots)$, where $n_k = m + k$.

Convergence of Subsequence

If $(x_n)_n \rightarrow x$, then for any natural sequence $(n_k)_k$,

$$(x_{n_k})_k \rightarrow x$$

Let $\epsilon > 0$. Find $N \in \mathbb{N}$ large such that $n \geq N$, $|x_n - x| < \epsilon$.

Take $K = N$. Then,

$$\begin{aligned} n_k &\geq k \\ &\geq K \\ &= N \\ \Rightarrow |x_{n_k} - x| &< \epsilon \end{aligned}$$

Corollary to Convergence of Subsequences

Given a sequence $(x_n)_n$, if there are two subsequences $(x_{n_k})_k \rightarrow x$, $(x_{n_\ell})_\ell \rightarrow x'$, where $x \neq x'$, then $(x_n)_n$ is divergent.

Recall the geometric sequence

$$(b^n)_{n=1}^{\infty} \rightarrow 0$$

if $0 < b < 1$.

The sequence $(1, b, b^2, \dots)$ is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem, $(b^n)_n \rightarrow \ell$.

Given $n = 2k$, we know that $(b^{2k})_k \rightarrow \ell$.

$$\begin{aligned} b^{2k} &= (b^k)^2 \\ (b^k)^2 &\rightarrow \ell^2 \\ \ell^2 &= \ell \\ \ell &= \{0, 1\} \\ \ell &= 0 \end{aligned}$$

since $b < 1$

Divergence and Subsequence

If $(x_n)_n \not\rightarrow x$, then

$$(\exists \epsilon_0 > 0) (\forall N \in \mathbb{N}) (\exists n \geq N) \ni |x_n - x| \geq \epsilon_0$$

We can use this to construct a sequence to show divergence.

Let $(x_n)_n$ be a sequence, and $x \in \mathbb{R}$.

$$\begin{aligned} (x_n)_n &\not\rightarrow x \\ \Leftrightarrow \\ (\exists \epsilon_0 > 0) &(\exists (x_{n_k})_k) \end{aligned}$$

with

$$|x_{n_k} - x| \geq \epsilon_0$$

(\Rightarrow) We know $\exists \epsilon_0 > 0$ as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \epsilon_0$$

$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \epsilon_0$$

$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \epsilon_0$$

Assume we have $n_1 < n_2 < \dots, n_k$ with

$$|x_{n_j} - x| \geq \epsilon_0$$

$$j = 1, 2, \dots, k$$

$$N = n_k + 1 \Rightarrow \exists n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \epsilon_0$$

Iteratively, we have our desired subsequence $(x_{n_k})_k$.

(\Leftarrow) If $(x_n)_n \rightarrow x$, any subsequence converges to x .

By assumption, $(\exists \epsilon_0 > 0) (\exists (n_k)_k)$ with $|x_{n_k} - x| \geq \epsilon_0$. Thus, $(x_{n_k})_k \not\rightarrow x$.

Bolzano-Weierstrass Theorem

If $(x_n)_n$ is a bounded sequence, then $(x_n)_n$ admits a convergent subsequence.

Lemma

Let $(x_n)_n$ be any real sequence. Then, $\exists n_k$ such that $(x_{n_k})_k$ is monotone.

A **peak** of a sequence $(x_n)_n$ is an x_m such that $x_m \geq x_n \forall n \geq m$.

Case 1: There are infinitely many peaks, $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$, where $n_1 < n_2 < \dots$. Then, $(x_{n_k})_k$ is decreasing.

Case 2: There are finitely many peaks. Let these peaks be $x_{m_1}, x_{m_2}, \dots, x_{m_r}$.

Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, $\exists n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since x_{n_2} is not a peak, $\exists n_3 > n_2$ such that $x_{n_3} > x_{n_2}$.

Iteratively, we have an increasing sequence of non-peaks $(x_{n_k})_k$.

Since $(x_n)_n$ admits a monotone subsequence, and $(x_{n_k})_k$ is bounded as $(x_n)_n$ is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

Limit Superior and Limit Inferior

Let $X = (x_n)_n$ be a bounded real sequence. By Bolzano-Weierstrass, $(x_n)_n$ admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R}, t = \lim_{k \rightarrow \infty} x_{n_k} \right\} \quad \text{for any subsequence } (x_{n_k})_k$$

Then, $t \in \overline{X}$ is called a **limit point** of X .

Let $u_1 = \sup_{n \geq 1} (x_n)$, $\ell_1 = \inf_{n \geq 1} (x_n)$. Clearly, $\ell_1 \leq u_1$, and $\overline{X} \subseteq [\ell_1, u_1]$.

Let $u_2 = \sup_{n \geq 2} (x_n)$ and $\ell_2 = \inf_{n \geq 2} (x_n)$.

Since u_1 is an upper bound for $(x_n)_n$, it is an upper bound for $(x_n)_{n \geq 2}$, so $u_2 \leq u_1$. Similarly, since ℓ_1 is a lower bound for $(x_n)_n$, it is a lower bound for $(x_n)_{n \geq 2}$, so $\ell_2 \geq \ell_1$.

As a result, we can see that $\overline{X} \subseteq [\ell_2, u_2]$.

We continue, letting $u_m = \sup_{n \geq m} (x_n)$, and $\ell_m = \inf_{n \geq m} (x_n)$. We get $\ell_1 \leq \ell_2 \leq \dots$, and $u_1 \geq u_2 \geq \dots$, and $\overline{X} \subseteq [\ell_m, u_m]$, $\forall m$.

We get a nested sequence of intervals $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \dots$. By the Nested Intervals Theorem, we know that

$$\begin{aligned} \overline{X} &\subseteq \bigcap_{m \geq 1} [\ell_m, u_m] \\ &= [\ell, u] \end{aligned}$$

where $\ell = \sup(\ell_m)$ and $u = \inf(u_m)$.

Given a bounded sequence $(x_n)_n = X$,

$$\begin{aligned} u &= \inf_{m \geq 1} (u_m) \\ &= \inf_{m \geq 1} \left(\sup_{n \geq m} x_n \right) \end{aligned}$$

called the **limit superior** of $(x_n)_n$

$$u = \limsup_{n \rightarrow \infty} x_n$$

and

$$\begin{aligned}\ell &= \sup_{m \geq 1} (\ell_m) \\ &= \sup_{m \geq 1} \left(\inf_{n \geq m} (x_n) \right)\end{aligned}$$

called the **limit inferior** of $(x_n)_n$

$$\ell = \liminf_{n \rightarrow \infty} x_n$$

Applications of Limit Superior and Limit Inferior

Let $(x_n)_n$ be bounded. Then,

- (1) $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$
- (2) $(x_n)_n \rightarrow x \Leftrightarrow \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$

- (1) This was proven with the Nested Intervals Theorem
- (2) Let $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$.

We know that $u_m = \sup_{n \geq m} x_n$. If $m \geq N$, then $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$. Therefore, $|u_m - x| \leq \varepsilon/2 < \varepsilon$, so $(u_m)_m \rightarrow x = \limsup_{n \rightarrow \infty} x_n$.

Similarly, we know that $\ell_m = \inf_{n \geq m} x_n$. If $m \geq N$, then $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$. So, $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$, so $(\ell_m)_m \rightarrow x = \liminf_{n \rightarrow \infty} x_n$.

Consider the sequence

$$\begin{aligned}x_n &= \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases} \\ &= (-1, 5/2, -1/3, 9/4, -1/5, \dots)\end{aligned}$$

We begin by constructing the u_m sequence: $(5/2, 5/2, 9/4, 9/4, \dots)$. We can see that $u_m \rightarrow 2$.

Then, we construct the ℓ_m sequence: $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$. We can see that $\ell_m \rightarrow 0$.

Exercise: If $(x_n)_n$ and $(y_n)_n$ are sequences with $x_n \leq y_n \forall n$, then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

Ratio Test and Root Test Equivalent Convergence

If $(a_n)_n$ is a sequence of strictly positive terms such that

$$\left(\frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$$

then,

$$\left(a_n^{1/n} \right)_{n=1}^{\infty} \rightarrow \rho$$

Let $\varepsilon > 0$. Then, $\exists N$ large such that $\forall n \geq N$,

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} - \rho \right| &< \varepsilon & \forall n \geq N \\ \Rightarrow \frac{a_{n+1}}{a_n} &< \rho + \varepsilon & \forall n \geq N \\ a_{n+1} &< a_n(\rho + \varepsilon) & \forall n \geq N \\ a_n &< a_N(\rho + \varepsilon)^{n-N} & \forall n \geq N \\ a_n &< (\rho + \varepsilon)^n \cdot \frac{a_N}{(\rho + \varepsilon)^N} \\ a_n^{1/n} &< (\rho + \varepsilon) \left(\frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\ \limsup a_n^{1/n} &\leq \limsup (\rho + \varepsilon) \left(\frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\ \limsup_{n \rightarrow \infty} a_n^{1/n} &\leq \rho + \varepsilon \end{aligned}$$

Case 1: If $\rho = 0$, the case is trivial.

Case 2: Suppose $\rho > 0$. Find $\varepsilon > 0$ small such that $0 < \varepsilon < \rho$.

Since $\left(\frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$, find N large such that $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$. So, $\forall n \geq N$,

$$\begin{aligned} a_{n+1} &\geq a_n(\rho - \varepsilon) \\ a_n &\geq a_N(\rho - \varepsilon)^{n-N} \\ a_n^{1/n} &\geq (\rho - \varepsilon) \left(\frac{a_N}{(\rho - \varepsilon)^N} \right)^{1/n} \\ \liminf a_n^{1/n} &\geq \rho - \varepsilon \end{aligned}$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together, $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$, so $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$, whence $\left(a_n^{1/n} \right) \rightarrow \rho$

Properties of \overline{X}

We found earlier that $\overline{X} \subseteq [\ell, u]$. We claim that

$$\begin{aligned} \sup \overline{X} &= u \\ \inf \overline{X} &= \ell \end{aligned}$$

We have shown that u is an upper bound for \overline{X} . The goal is to show that u is the least upper bound.

Let $\varepsilon > 0$. We need to find a $t \in \overline{X}$ with $u - \varepsilon < t$. Note that $u - \varepsilon < u_m \forall m$.

We know that $u - \varepsilon < u_1$. Since $u_1 = \sup_{n \geq 1} x_n$, we know $\exists n_1 \in \mathbb{N}$ with $u - \varepsilon < x_{n_1} < u_1$.

Consider $u_{n_1+1} = \sup_{n > n_1} x_n$. We know that $u - \varepsilon < u_{n_1+1}$. Therefore, $\exists x_{n_2}$ with $n_2 > n_1$ and $u - \varepsilon < x_{n_2} < u_{n_1+1}$.

Then, we use u_{n_2+1} . Then, $\exists n_3 > n_2$ with $u - \varepsilon < x_{n_3} < u_{n_2+1}$.

We get a subsequence from the natural sequence n_1, n_2, \dots , where $u - \varepsilon < x_{n_k} \forall k$.

Also, $x_{n_k} < u_1 \forall k$. Therefore, $(x_{n_k})_k$ is a bounded sequence. By Bolzano-Weierstrass, \exists a convergent subsequence

$$(x_{n_{k_j}})_j \rightarrow t$$

We know that $u - \varepsilon \leq t$. Note that $t \in \overline{X}$.

Exercise: Show that $\inf \bar{X} = \ell$.

Cauchy Sequences

A sequence $(x_n)_n$ in a metric space (X, d) is Cauchy if

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \ni p, q \geq N \Rightarrow d(x_p, x_q) < \varepsilon$$

if $(X, d) = (\mathbb{R}, |\cdot|)$:

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \ni p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon$$

Consider the sequence $(x_n)_n = \frac{1}{n}$. Then,

$$\begin{aligned} |x_p - x_q| &= \left| \frac{1}{p} - \frac{1}{q} \right| \\ &= \frac{1}{q} - \frac{1}{p} \\ &\leq \frac{1}{q} \end{aligned}$$

Given $\varepsilon > 0$, find N large such that $\frac{1}{N} < \varepsilon$. Then, $p, q \geq N$ implies

$$\begin{aligned} \left| \frac{1}{p} - \frac{1}{q} \right| &< \frac{1}{q} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

Show that $(-1)^n$ is not Cauchy.

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) \ni p, q \geq N \Rightarrow |x_p - x_q| \geq \varepsilon_0$$

Boundedness of Cauchy Sequences

Cauchy sequences are bounded.

Let $\varepsilon = 1$. Then, by the Cauchy criterion, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < 1$.

In particular, $\forall n \geq N$,

$$\begin{aligned} |x_n| &= |x_n - x_N + x_N| \\ &\leq |x_n - x_N| + |x_N| \\ &< 1 + |x_N| \end{aligned} \quad \text{Triangle Inequality}$$

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$. Then, $x_n \leq c \forall n \geq 1$. Thus, x_n is bounded.

Convergent Subsequences of Cauchy Sequences

If $(x_n)_n$ is Cauchy and $(x_n)_n$ admits a convergent subsequence, then $(x_n)_n$ is convergent.

Say $(x_{n_k})_k \rightarrow x$ for some natural sequence $(n_k)_k$. We claim that $(x_n)_n \rightarrow x$.

Let $\varepsilon > 0$. Since $(x_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon/2$.

Also, since $(x_{n_k})_k \rightarrow x$, then $\exists K \in \mathbb{N}$ and $K \geq N$ with $k \geq K \Rightarrow |x_{n_k} - x| < \varepsilon/2$.

For all $k \geq K$,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \end{aligned}$$

Let $N_1 = \max\{N, K\}$. Then,

$$\begin{aligned} n \geq N_1 &\Rightarrow n \geq N \\ &\Rightarrow n_k \geq k \geq K \geq N \end{aligned}$$

by max
def. of natural sequence

$$\begin{aligned} |x_n - x| &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Cauchy Sequence Convergence in the Reals

Let $(x_n)_n$ be any sequence in \mathbb{R} . The following are equivalent:

- (1) $(x_n)_n$ converges.
- (2) $(x_n)_n$ is Cauchy.

(1) \Rightarrow (2) (Holds in any metric space). Suppose $(x_n)_n \rightarrow x$. Find N large such that $n \geq N \rightarrow d(x_n, x) < \varepsilon/2$.

Then, $p, q \geq N \Rightarrow$

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

(2) \Rightarrow (1) If $(x_n)_n$ is Cauchy, then $(x_n)_n$ converges.

By Bolzano-Weierstrass, $(x_n)_n$ admits a convergent subsequence, so by our previous lemma, $(x_n)_n$ must converge.

Note: To show (2) \Rightarrow (1), we used Bolzano-Weierstrass, which requires the monotone convergence theorem, which itself requires the completeness axiom. This is why we cannot show (2) \Rightarrow (1) converges.

Complete Metric Spaces

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Remark: All convergent sequences are Cauchy, and all Cauchy sequences are bounded. We showed that \mathbb{R} under the absolute value metric is complete.

\mathbb{Q} under $d(s, t) = |s - t|$ is not complete; similarly, $A = (0, 1)$ under the metric inherited from \mathbb{R} is not complete; $x_n = \frac{1}{n}$ is Cauchy but not convergent in A .

Finding Cauchy Sequences and Convergence in \mathbb{R}

Consider the harmonic sequence

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We claim that h_n is not convergent.

Let $p > q$. Then,

$$\begin{aligned}
 |h_p - h_q| &= \left| \sum_{k=1}^p \frac{1}{k} - \sum_{k=1}^q \frac{1}{k} \right| \\
 &= \frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p} \\
 &\geq \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} \\
 &= \frac{p-q}{p} \\
 &= 1 - \frac{q}{p}
 \end{aligned}$$

set $p = 2q$:

$$\begin{aligned}
 |h_{2q} - h_q| &\geq 1 - \frac{q}{2q} \\
 &= 1/2
 \end{aligned}$$

Therefore, h_n is not Cauchy, and thus not convergent.

Consider a sequence of partial sums

$$x_n = \sum_{k=0}^n \frac{(-1)^k}{k!}$$

We claim that $(x_n)_n$ is Cauchy, and thus convergent. Let $p > q$. Then, we have

$$\begin{aligned}
 |x_p - x_q| &= \left| \sum_{k=q+1}^p \frac{(-1)^k}{k!} \right| \\
 &\leq \sum_{k=q+1}^p \frac{1}{k!} \\
 &\leq \sum_{k=q+1}^p \frac{1}{2^{k-1}} \\
 &= \frac{1}{2^q} + \frac{1}{2^{q+1}} + \cdots + \frac{1}{2^{p-1}} \\
 &= \frac{1}{2^q} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-q-1}} \right) \\
 &\leq \frac{1}{2^{q-1}}
 \end{aligned}$$

Given $\varepsilon > 0$, choose N large such that $\frac{1}{2^{N-1}} < \varepsilon$. When $p > q > N$, then $|x_p - x_q| \leq \frac{1}{2^{q-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$.

Thus, the sequence is convergent.

Contractive Sequences

A sequence $(x_n)_n$ in a metric space (X, d) is **contractive** if

$$\exists 0 < \rho < 1 \ni d(x_{n+1}, x_n) \leq \rho d(x_n, x_{n-1}) \quad \forall n \geq 1$$

In \mathbb{R} , the definition is

$$|x_{n+1} - x_n| \leq \rho |x_n - x_{n-1}|$$

We claim that every contractive sequence is Cauchy.

From examination, we arrive at the following:

$$|x_n - x_{n-1}| \leq \rho^{n-2} |x_2 - x_1| \quad (*)$$

If $p > q$, then

$$\begin{aligned}
 |x_p - x_q| &= |x_p - x_{p-1} + x_{p-1} - x_{p-1} + \cdots + x_{q+1} - x_q| \\
 &\leq |x_p - x_{p-1}| + \cdots + |x_{q+1} - x_q| && \text{Triangle Inequality} \\
 &\leq |x_2 - x_1| (\rho^{p-2} + \rho^{p-3} + \cdots + \rho^{q-1}) \\
 &= |x_2 - x_1| \rho^{q-1} (1 + \rho + \rho^2 + \cdots + \rho^{p-q-1}) \\
 &= |x_2 - x_1| \rho^{q-1} \frac{1 - \rho^{p-q}}{1 - \rho} && \text{Finite Geometric Sequence} \\
 &\leq |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho}
 \end{aligned}$$

Given $\varepsilon > 0$, we can find N large such that

$$q \geq N \Rightarrow |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} < \varepsilon$$

Thus, $p > q \geq N \Rightarrow |x_p - x_q| < \varepsilon$.

Application of Contractive Sequences

Consider $(f_n)_n$ defined as follows:

$$\begin{aligned}
 f_0 &= 1 \\
 f_1 &= 1 \\
 f_{n+1} &= f_n + f_{n-1}
 \end{aligned}$$

Consider x_n defined as follows:

$$x_n = \frac{f_{n+1}}{f_n}$$

We can rewrite x_n as:

$$\begin{aligned}
 x_n &= \frac{f_n + f_{n-1}}{f_n} \\
 &= 1 + \frac{f_{n-1}}{f_n} \\
 &= 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \\
 &= 1 + \frac{1}{x_{n-1}}
 \end{aligned}$$

We claim that $3/2 \leq x_n \leq 2 \forall n \geq 2$.

$$x_2 = 2$$

Inductive Hypothesis: suppose $3/2 \leq x_n \leq 2$

$$\begin{aligned}
 &: \frac{3}{2} \leq x_n \leq 2 \\
 &\frac{2}{3} \geq \frac{1}{x_n} \geq \frac{3}{2} \\
 2 &\geq \frac{5}{3} \geq 1 + \frac{1}{x_n} \geq \frac{3}{2}
 \end{aligned}$$

We now claim that $(x_n)_n$ is contractive.

$$\begin{aligned}
 |x_{n+1} - x_n| &= \left| \left(1 + \frac{1}{x_n}\right) - \left(1 + \frac{1}{x_{n-1}}\right) \right| \\
 &= \left| \frac{1}{x_n} - \frac{1}{x_{n-1}} \right| \\
 &= \left| \frac{x_{n-1} - x_n}{x_{n-1}x_n} \right| \\
 &\leq \frac{4}{9} |x_n - x_{n-1}|
 \end{aligned}$$

Therefore, $\rho = \frac{4}{9}$ is our constant of contraction. Thus, $(x_n)_n$ is Cauchy, so it converges in \mathbb{R} .

$$\begin{aligned}x_{n+1} &= 1 + \frac{1}{x_n} & (n \rightarrow \infty, x_n \rightarrow \varphi) \\ \varphi &= 1 + \frac{1}{\varphi} \\ \varphi^2 - \varphi - 1 &= 0 \\ \varphi &= \frac{1 + \sqrt{5}}{2}\end{aligned}$$

Let $x_1 = 0$, $x_2 = 1$, and

$$\begin{aligned}x_{n+1} &= \frac{1}{2}(x_n + x_{n-1}) \\ (x_n)_n &= (0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, \dots)\end{aligned}$$

While the sequence is not monotone, we can show that the sequence is contractive.

$$\begin{aligned}|x_{n+1} - x_n| &= \left| \frac{1}{2}(x_n + x_{n-1}) - x_n \right| \\ &= \left| \frac{1}{2}(x_{n-1} - x_n) \right| \\ &= \frac{1}{2}|x_n - x_{n-1}|\end{aligned}$$

Since the constant of contraction is equal to $1/2$, this sequence is Cauchy, and thus converges in the real numbers.

Since $(x_n)_n \rightarrow x$, every subsequence converges to x . Therefore, $(x_{2k+1})_k \rightarrow x$.

$$\begin{aligned}x_{2k+1} &= \sum_{j=1}^k \frac{1}{2^{2j-1}} \\ &= 2 \sum_{j=1}^k \frac{1}{4^j} \\ &= 2 \frac{1 - \frac{1}{4^{k+1}}}{1 - \frac{1}{4}} \\ &= \frac{2}{3} & k \rightarrow \infty\end{aligned}$$

Properly Divergent Sequences

Let $(x_n)_n$ be a real sequence. $(x_n)_n$ *properly* diverges to $+\infty$ if

$$(\forall \alpha > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow x_n \geq \alpha$$

We write that $(x_n)_n \rightarrow +\infty$. Similarly, $(x_n)_n$ properly diverges to $-\infty$ if

$$(\forall \beta < 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow x_n \leq \beta$$

and $(x_n)_n \rightarrow -\infty$. We say that $(x_n)_n$ is properly divergent if $(x_n)_n \rightarrow \pm\infty$.

For example $(x_n)_n$ diverges to n .

If $\alpha > 0$, find $N \geq \alpha$ by the Archimedean property. Then, $n \geq N \Rightarrow n > \alpha$.

If $(x_n)_n$ and $(y_n)_n$ are sequences such that $x_n \geq y_n \forall n$, and $(y_n)_n \rightarrow +\infty$, then $(x_n)_n \rightarrow +\infty$.

Divergence of the Geometric Sequence

In the geometric sequence, if $b > 1$, we can show that $(b^n) \rightarrow +\infty$.

Write $b = 1 + a$ for some $a > 0$. Then, by Bernoulli's inequality, we have

$$\begin{aligned} b^n &= (1 + a)^n \\ &\geq 1 + na \\ &\geq na \end{aligned}$$

Given any $\alpha > 0$, find N large such that $N > \frac{\alpha}{a}$, which is always possible by the Archimedean property. Then, for $Na \geq \alpha$. If $n \geq N$, then $na \geq Na > \alpha$.

Since $b^n > na$, we have that $(b^n)_n \rightarrow +\infty$.

Monotone Divergence

By the Monotone Convergence Theorem, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \rightarrow x \Leftrightarrow (x_n)_n \text{ bounded}$$

Negating, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \text{ divergent} \Leftrightarrow (x_n)_n \text{ unbounded}$$

However, we can make this statement stronger.

Proposition Let $(x_n)_n$ be monotone. $(x_n)_n$ is unbounded if and only if $(x_n)_n$ is properly divergent.

Proof:

(\Leftarrow) If $(x_n)_n$ is properly divergent, then $(x_n)_n$ is divergent, and thus unbounded.

(\Rightarrow) Let $(x_n)_n$ be unbounded and increasing. Then, given $\alpha > 0$, $\exists n_\alpha$ with $x_{n_\alpha} > \alpha$. If $n \geq n_\alpha$, then $x_n \geq x_{n_\alpha} > \alpha$, so $(x_n)_n$ is properly divergent to $+\infty$.

Comparison Test

Let $(x_n)_n$ and $(y_n)_n$ be sequences with $x_n > 0$ and $y_n > 0$. Suppose that

$$\left(\frac{x_n}{y_n}\right)_n \rightarrow L > 0$$

Then, $(x_n)_n \rightarrow +\infty \Leftrightarrow (y_n)_n \rightarrow \infty$.

Let $\epsilon = L/2$. Since

$$\left(\frac{x_n}{y_n}\right)_n \rightarrow L,$$

$\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} \frac{L}{2} &\leq \frac{x_n}{y_n} \leq \frac{3L}{2} \\ \frac{L}{2} y_n &\leq x_n \\ \frac{2}{3L} x_n &\leq y_n \end{aligned}$$

If $(y_n)_n \rightarrow \infty$, then so too does $(L/2)(y_n)$, so $(x_n)_n \rightarrow \infty$. Similarly, if $(x_n)_n \rightarrow \infty$, then so too does $(2/3L)x_n$, so $(y_n)_n \rightarrow \infty$.

Show that

$$\left(\sqrt{4n^2 - 3n + 1}\right)_n \rightarrow +\infty$$

We will compare to $y_n = n$. Then

$$\begin{aligned}\frac{x_n}{y_n} &= \frac{\sqrt{4n^2 - 3n + 1}}{n} \\ &= \sqrt{4 - \frac{3}{n} + \frac{1}{n^2}} \\ &\rightarrow 2 \geq 0\end{aligned}$$

Since y_n is properly divergent to $+\infty$, so too is x_n .

Introduction to Infinite Series

An **infinite series** is a sequence of partial sums s_n , where s_n is formed from x_k as follows:

$$s_n = \sum_{k=1}^n x_k$$

alternatively,

$$\begin{aligned}s_1 &= x_1 \\ s_n &= s_{n-1} + x_n\end{aligned}$$

The limit of the sequence $(s_n)_n$ is the value of

$$\sum_{n=1}^{\infty} x_n$$

The infinite series converges to s if $(s_n)_n \rightarrow s$.

If $(s_n)_n$ diverges, then so too does the series. If $(s_n)_n$ is properly divergent to $\pm\infty$, then we write that the series is equal to $\pm\infty$.

Series of Positive Terms

Let $(x_k)_k$ be a sequence of positive terms. The following are equivalent:

- (a) $\sum x_k$ converges.
- (b) The sequence of partial sums $(s_n)_n$ is bounded above.
- (c) A subsequence of the sequence of partial sums $(s_{n_j})_j$ is bounded above.

Proof:

(1) \Rightarrow (2): $\sum x_k$ is convergent $\Rightarrow (s_n)_n$ is convergent $\Rightarrow (s_n)_n$ is bounded.

(2) \Rightarrow (3): If $(s_n)_n$ is bounded, so is any subsequence $(s_{n_j})_j$.

(3) \Rightarrow (2): Suppose $s_{n_j} \leq c$. If m is arbitrary, $\exists j$ such that $n_j \geq m$. Take $j = m$. Then, $s_m \leq s_{n_j} \leq c$. Therefore, $(s_n)_n$ is bounded above.

(2) \Rightarrow (1) Let $(s_n)_n$ be bounded above. We know that $(s_n)_n$ is increasing as $x_k \geq 0$. By the Monotone Convergence theorem, $(s_n)_n$ converges, meaning $\sum x_k$ converges.

Corollary to Series of Positive Terms

Let $(x_k)_k$ be a sequence with $x_k \geq 0$. Then,

$$\sum x_k \text{ properly diverges} \Leftrightarrow (s_n)_n \text{ is unbounded}$$

Recall that for $x_k = 1/k$, we proved that $(s_n)_n$ is unbounded, and also that $(s_n)_n$ is not Cauchy, meaning $\sum_{k=1}^{\infty} 1/k$ is properly divergent.

Additionally, we saw that for $x_k = 1/k^2$, $(s_n)_n$ is increasing and bounded above.

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^n \frac{1}{k} - \frac{1}{k-1} \\ &= 2 - \frac{1}{n} \end{aligned}$$

Let $b \in \mathbb{R}$. Let $x_k = b^k$. Then, we have

$$\begin{aligned} s_n &= \sum_{k=0}^n b^k \\ &= \frac{1 - b^{n+1}}{1 - b} \end{aligned} \quad b \neq 1$$

Therefore, we know the end behavior of the series:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{1 - b^{n+1}}{1 - b} \\ &= \frac{1}{1 - b} \left(1 - b \lim_{n \rightarrow \infty} b^n \right) \\ &= \begin{cases} \frac{1}{1-b} & |b| < 1 \\ \text{diverges} & |b| > 1 \end{cases} \end{aligned}$$

Series Comparison Test

Let $0 \leq x_k \leq y_k$.

- If $\sum y_k$ converges, then so too does $\sum x_k$
- If $\sum x_k$ diverges, then so too does $\sum y_k$.

Proof:

(\Rightarrow) If $\sum y_k$ converges, then $t_n = \sum_{k=1}^n y_k$ is bounded.

Setting $s_n = \sum_{k=1}^n x_k$, we see that $0 \leq s_n \leq t_n$. Seeing as t_n is bounded, so too is s_n . Therefore, $\sum x_k$ is convergent.

For example, consider the series

$$\sum \frac{1}{k^2 + k}$$

Since $\frac{1}{k^2} \geq \frac{1}{k^2 + k}$, we know that, seeing as $\frac{1}{k^2}$ converges, so does $\frac{1}{k^2 + k}$.

Limit Comparison Test

Let x_k and y_k be strictly positive sequences. Suppose that

$$\lim_{k \rightarrow \infty} \frac{x_k}{y_k} = L$$

- If $L > 0$, then $\sum x_k$ converges if and only if $\sum y_k$ converges.
- If $L = 0$, then $\sum y_k$ converges $\Rightarrow \sum x_k$ converges.

Proof:

(a) Since

$$\frac{x_k}{y_k} \rightarrow L$$

Set $\varepsilon = L$. We know $\exists K$ such that $k \geq K \Rightarrow y_k \leq \frac{2}{L} x_k$. Let $s_n = \sum_{k=1}^n x_k$ and $t_n = \sum_{k=1}^n y_k$. Then,

$$\begin{aligned} t_n &= \sum_{k=1}^{K-1} y_k + \sum_{k=K}^n y_k \\ &\leq t_{K-1} + \frac{2}{L} \sum_{k=K}^n x_k \\ &\leq t_{K-1} + \frac{2}{L} s_n \\ &\leq t_{K-1} + c, \end{aligned}$$

implying that t_n is bounded, so $\sum y_k$ converges.

(b) Since

$$\frac{x_k}{y_k} \rightarrow 0,$$

$\exists K$ such that $\frac{x_k}{y_k} \leq 1 \forall k \geq K$, meaning $x_k < y_k \forall k \geq K$.

Letting $s_n = \sum_{k=1}^n x_k$ and $t_n = \sum_{k=1}^n y_k$. Thus,

$$\begin{aligned} s_n &= \sum_{k=1}^{K-1} x_k + \sum_{k=K}^n x_k \\ &= s_{K-1} + \sum_{k=K}^n y_k \\ &\leq s_{K-1} + t_n \\ &\leq s_{K-1} + c \end{aligned}$$

Thus, s_n is bounded, meaning $\sum x_k$ is convergent.

Limit Comparison Test Examples

Consider

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$$

Letting $x_n = \frac{1}{\sqrt{n^2-1}}$, and $y_n = \frac{1}{n}$, we have

$$\begin{aligned} \frac{x_n}{y_n} &= \frac{n}{\sqrt{n^2-1}} \\ &\rightarrow 1 > 0 \end{aligned}$$

Since $\sum y_n$ diverges, so too does $\sum x_n$.

n th Term Divergence Test

If $\sum x_k$ is convergent, then $(x_k)_k \rightarrow 0$. Conversely, if $(x_k)_k \not\rightarrow 0$, then $\sum x_k$ diverges. Recall that $s_n = s_{n-1} + x_n$. If $\sum x_k$ converges, then $(s_n)_n \rightarrow 0$. So,

$$\begin{aligned} x_n &= s_n - s_{n-1} \\ (s_n)_n &\rightarrow s \\ x_n &\rightarrow s - s \\ &= 0 \end{aligned}$$

For example, we can find that

$$\sum_{k=1}^{\infty} \frac{1}{\arctan k}$$

diverges, as $\lim_{k \rightarrow \infty} \frac{1}{\arctan k} = \frac{2}{\pi} \neq 0$

Cauchy Condensation Test

Let $(x_k)_k$ be a decreasing sequence of positive numbers. Then,

$$\sum_k x^k \text{ converges} \Leftrightarrow \sum_k 2^k x_{2^k} \text{ converges}$$

Look at the partial sum s_{2^n} ,

$$\begin{aligned} s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\leq x_1 + 2x_2 + 4x_4 + \cdots + 2^{n-1}x_{2^{n-1}} + x_{2^n} \\ &= \sum_{k=1}^{n-1} 2^k x_{2^k} + x_{2^n} \end{aligned}$$

If $\sum_k 2^k x_{2^k}$ converges, then its partial sums are bounded, and we have that $x_{2^n} \rightarrow 0$. Then, s_{2^n} is bounded, and thus $\sum x_k$ converges.

$$\begin{aligned} 2s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\quad + x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &= (x_1 + x_1) + (x_2 + x_2) + (x_3 + x_3 + x_4 + x_4) + \cdots + (x_{2^{n-1}} + x_{2^{n-1}} + \cdots + x_{2^n} + x_{2^n}) \\ &\geq x_1 + 2x_2 + 4x_4 + \cdots + 2^n x_{2^n} \\ &= \sum_{k=0}^n 2^k x_{2^k} \end{aligned}$$

therefore, we get that

$$\frac{1}{2} \sum_{k=0}^n 2^k a_{2^k} \leq s_{2^n}$$

If $\sum x_k$ converges, then s_n is bounded, so s_{2^n} is bounded, so $\sum_{k=0}^n 2^k x_{2^k}$ is bounded, so the series $\sum_{k=0}^n 2^k x_{2^k}$ is convergent.

p series

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad p \in \mathbb{R}$$

By the Cauchy Condensation Test, we see that the series converges if and only if the following series converges:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{2^{np}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^n \\ &\Leftrightarrow \frac{1}{2^{p-1}} < 1 \\ &\Leftrightarrow 2^{p-1} > 1 \\ &\Leftrightarrow p > 1 \end{aligned}$$