This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

Useful Results and Proofs

Analytic Functions

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A function $f\colon U \to \mathbb{C}$ is called *analytic* if, for any $z_0 \in U$, there is $r > 0$ and $(a_k)_k \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in U(z_0, r)$.

Analytic functions form a $\mathbb{C}$-algebra.

Theorem (Identity Theorem)**:** Let $f, g\colon U \to \mathbb{C}$ be analytic functions defined a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in U, then $f = g$ on U.

Proof. To begin, we show that if $f\colon U \to \mathbb{C}$ is an analytic function that is not uniformly zero, then for any $z_0 \in U$, there is $\rho > 0$ such that f is nonzero on $\dot{U}(z_0, \rho) \subseteq U$. Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all $z \in U(z_0, r)$, some $r > 0$, and since f is not uniformly zero, there is some minimal ℓ such that $a_\ell \neq 0$. This yields

$$f(z) = (z - z_0)^\ell \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function $h\colon U(z_0, r) \to \mathbb{C}$ given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as f and is not zero at z_0, so that g is not zero on some $U(z_0, \rho)$ as g is continuous.

Now, we let V_1 be the set of accumulation points of A in U, and let $V_2 = U \setminus V_1$.

If $z \in V_2$, then there is some $r_1 > 0$ such that $\dot{U}(z_0, r_1) \cap A = \emptyset$, or that $\dot{U}(z_0, r_1) \subseteq A^c$. Meanwhile, since U is open, there is some $r_2 > 0$ such that $U(z_0, r_2) \subseteq U$, meaning that if $r = \min\{r_1, r_2\}$, then $U(z_0, r) \subseteq U \setminus A$. Thus, V_2 is open.

Meanwhile, if $z \in V_1$, then since $V_1 \subseteq U$, it follows that there is $r > 0$ such that $U(z, r)$ and $(a_k)_k$ such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all $w \in U(z, r)$. We claim that $f(w) - g(w)$ is uniformly zero on $U(z, r)$. Else, if there were $w_0 \in U(z, r)$ such that $f(w_0) \neq g(w_0)$, then it would follow that there is $0 < s \leqslant r$ such that $f(w) \neq g(w)$ for all $w \in \dot{U}(w_0, s)$. Yet, this would contradict the assumption that z is an accumulation point, meaning that V_1 is open.

Since V_1 and V_2 are disjoint open sets whose union is equal to U, it follows that either $V_1 = U$ or $V_2 = U$. If $A \neq \emptyset$, then the identity theorem follows. □

Differentiability

Definition: If $U \subseteq \mathbb{C}$ is an open set, then we say f is differentiable at $z_0 \in U$ if

$$\lim_{w \to z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of f at z_0, and usually write $f'(z_0)$.

If f is differentiable at every $z_0 \in U$, we say f is differentiable on U.

If f is continuous and admits a continuous derivative, then we say f is *holomorphic*.

Note that the limit must be independent of direction. That is, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(w) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Now, given $U \subseteq \mathbb{C}$, write $z = x + iy$ and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Observe then that if f is differentiable at $x_0 + iy_0 \in U$, then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x + h, y) + iv(x + h, y)) - (u(x, y) + iv(x, y))}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x, y + h) + iv(x, y + h)) - (u(x, y) + iv(x, y))}{ih} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

Definition: The system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if f is differentiable, then the u and v in the definition of f satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly f is differentiable or holomorphic.

Proposition: If $f = u + iv$ is a holomorphic function such that u, v are in $C^2(U)$, then u and v are harmonic. That is, u and v satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We call u and v *harmonic conjugates* for each other. That is, if $u\colon U \to \mathbb{R}$ is a harmonic function, then $v \in C^1(U)$ is called a harmonic conjugate if the Cauchy–Riemann equations hold for u and v.

Theorem: Let $U \subseteq \mathbb{R}^2$ be a ball or all of $\mathbb{R}^2$. Then, every harmonic function on U has a harmonic conjugate. If $u \in C^3(U)$, then this conjugate is itself harmonic.

Lemma: Let $g\colon U((x_0, y_0), R) \to \mathbb{R}$ be such that g and $\frac{\partial g}{\partial x}$ are continuous. Then, $G\colon U((x_0, y_0), R) \to \mathbb{R}$, given by

$$G(x, y) = \int_{y_0}^{y} g(x, t)\, dt$$

satisfies

$$\frac{\partial G}{\partial x} = \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\, dt.$$

Proof of Lemma. Write

$$\frac{G(x + h, y) - G(x, y)}{h} - \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\, dt = \int_{y_0}^{y} \left(\frac{g(x + h, t) - g(x, t)}{h} - \frac{\partial g}{\partial x}(x, t)\right) dt.$$

By mean value theorem, the first term is equal to $\frac{\partial g}{\partial x}(x_1, t)$ for some x_1 between x and $x + h$. As $h \to 0$, $x_1 \to x$, as $\frac{\partial g}{\partial x}$ is uniformly continuous on a compact subset that contains x and $x + h$. We may exchange limit and integral to obtain the desired result. $\square$

Proof of Theorem. We prove for the case of $U = U((x_0, y_0), R)$. Define

$$v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt + \phi(x),$$

with $\phi(x)$ to be determined later. By the fundamental theorem of calculus, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

while by differentiating under the integral sign, and using the fact that u is harmonic, we have

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x, t)\, dt + \frac{d\phi}{dx} \\
&= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x, t)\, dt + \frac{d\phi}{dx} \\
&= -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \frac{d\phi}{dx}.
\end{aligned}$$

Defining $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)\, ds,$$

we see that v thus satisfies all the necessary requirements to be a harmonic conjugate.

Now, if u is C^3, then we defined v via the derivative of u, so that v is C^2, and thus v is harmonic. □

Cauchy's Integral Formula

Proposition: Fix $z_0 \in \mathbb{C}$, $R > 0$, and $f\colon U(z_0, R) \to \mathbb{C}$ holomorphic. For all $z \in U(z_0, R)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{S(z_0,R)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. It suffices to show that

$$\frac{1}{2\pi i}\int_{S(z_0,R)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = 0.$$

By using the chain rule and fundamental theorem of calculus, we find

$$\begin{aligned}\frac{1}{2\pi i}\int_{S(z_0,R)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int_{S(z_0,R)} \frac{\int_0^1 f'((1-t)z + t\zeta)(\zeta - z)\,dt}{\zeta - z}\,d\zeta\\ &= \frac{1}{2\pi i}\int_{S(z_0,R)} \int_0^1 f'((1-t)z + t\zeta)\,dt\,d\zeta\\ &= \frac{1}{2\pi i}\int_{S(z_0,R)} \frac{d}{d\zeta}\left(\frac{1}{t}f((1-t)z + t\zeta)\right)d\zeta\\ &= 0.\end{aligned}$$

□

Proposition: Let $f\colon U \to \mathbb{C}$ be a holomorphic function. The following all hold:

(i) f is analytic;

(ii) f is smooth with $f^{(n)}$ holomorphic;

(iii) for all $z_0 \in U$, if we let $R = \sup\{r > 0 \mid U(z_0, r) \subseteq U\}$, then there is $(a_n)_n \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the power series has radius of convergence R.

Proof.

(i) There exists $r < s$ with $U(z_0, s) \subseteq U$ and $r < r_1 < s$ such that $S(z_0, r_1) \subseteq U$. By Cauchy's Integral Formula, and a power series expansion of $\frac{1}{\xi - z}$ about z_0, this gives

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\oint_{S(z_0,r_1)} \frac{f(\xi)}{\xi - z}\,d\xi\\ &= \sum_{n=0}^{\infty}(z - z_0)^n \underbrace{\left(\frac{1}{2\pi i}\oint_{S(z_0,r_1)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi\right)}_{=:a_n}\\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n.\end{aligned}$$

(ii) Analytic functions are automatically smooth, hence complex-differentiable with continuous derivative.

(iii) If $r < r_1 < R$, then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{S(z_0,r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi \right),$$

and since the series converges uniformly, we have

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0,r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi.$$

Since r was arbitrary, this holds for any $0 < r_1 < R$, whence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for all $z \in U(z_0, R)$.

□

Corollary: Let $U \subseteq \mathbb{C}$ be open, let $z_0 \in U$, and $r > 0$ with $B(z_0, r) \subseteq U$. The following hold:

(i) for all $z \in U(z_0, r)$,

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi;$$

(ii) for all $n > 0$,

$$\left| f^{(n)}(z_0) \right| \leqslant \frac{n!}{r^n} \sup_{\zeta \in S(z_0,r)} |f(\zeta)|.$$

This particular result is known as the *Cauchy Estimate*.

Theorem (Liouville's Theorem)**:** If $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded in modulus, then f is constant.

Liouville's Theorem follows from applying Cauchy's estimate to f and using the fact that f is bounded to find that all higher derivatives of f vanish.

Theorem (Fundamental Theorem of Algebra)**:** If $p(z) = a_n z^n + \cdots + a_1 z + a_0$ has $n \geqslant 1$ and $a_n \neq 0$, then there is at least one z_0 such that $p(z_0) = 0$.

Proof. Suppose $p(z)$ were never zero. It would follow then that $\frac{1}{p(z)}$ is also an entire function.

Since $\lim_{|z|\to\infty} |p(z)| = \infty$, it follows that $\lim_{|z|\to\infty} \frac{1}{|p(z)|} = 0$, whence $\left|\frac{1}{p(z)}\right|$ is an entire function that is bounded (as all functions that vanish at infinity are bounded). This means that $\frac{1}{p(z)}$ is constant, so $p(z)$ is constant. □

Corollary: Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Then, $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Proof. Suppose there were $w \in \mathbb{C}$ and $r > 0$ such that $U(w, r) \cap f(\mathbb{C}) = \emptyset$. Then, $|f(z) - w| \geqslant r$ for all $z \in \mathbb{C}$, meaning that

$$g(z) = \frac{1}{f(z) - w}$$

is bounded and entire (the entirety following from the fact that $f(z) - w$ is nonvanishing). □

Cycles, Winding Numbers, and Homology

Now, we may generalize some of these results related to Cauchy's Integral Formula.

Proposition: Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \operatorname{im}(\gamma)$, we have

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\xi - z}\, d\xi \in \mathbb{Z}.$$

Proof. Let $\phi\colon [a,b] \to \mathbb{C}$ be defined by

$$\phi(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds.$$

Then, we observe

$$\phi(b) = \oint_{\gamma} \frac{1}{\xi - z}\, d\xi.$$

Then, define $\psi\colon [a,b] \to \mathbb{C}$ by

$$\psi(t) = \frac{e^{\phi(t)}}{\gamma(t) - z}.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}\phi'(t) &= \frac{\gamma'(t)}{\gamma(t) - z}\\ \psi'(t) &= \frac{\phi'(t)e^{\phi(t)}}{\gamma(t) - z} - \frac{e^{\phi'(t)}\gamma'(t)}{(\gamma(t) - z)^2}\\ &= 0,\end{aligned}$$

whence $\psi(t)$ is constant, and $\psi(t) = \psi(a)$, so

$$\psi(a) = \frac{1}{\gamma(a) - z}.$$

In particular, $\psi(b) = \psi(a)$, so

$$\begin{aligned}e^{\phi(b)} &= \psi(b)(\gamma(b) - z)\\ &= \psi(a)(\gamma(a) - z)\\ &= 1,\end{aligned}$$

so $\phi(b) = 2\pi i k$ for some $k \in \mathbb{Z}$. □

Definition: Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \operatorname{im}(\gamma)$, define

$$n(\gamma; z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\xi - z}\, d\xi$$

to be the *winding number* of γ about z.

Definition: A piecewise C^1 *cycle* is a formal sum

$$\Gamma = \gamma_1 + \cdots + \gamma_n,$$

where the $\gamma_j\colon \left[a_j, b_j\right] \to \mathbb{C}$ are piecewise C^1 loops. The *length* of Γ is the sum of the lengths of the respective γ_j.

Given a piecewise C^1 cycle Γ, define

$$\oint_{\Gamma} f(z)\,dz = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\,dz,$$

and

$$n(\Gamma; z) = \sum_{j=1}^{n} n\left(\gamma_j; z\right).$$

Proposition: The following hold for the winding number $n(\gamma; z)$:

(i) the function $n(\Gamma; \cdot)\colon \mathbb{C} \setminus \operatorname{im}(\gamma) \to \mathbb{Z}$ is continuous;

(ii) $n(\Gamma; z)$ is constant on each connected component of $\mathbb{C} \setminus \operatorname{im}(\Gamma)$;

(iii) there exists a unique unbounded connected component with $n(\Gamma; z) = 0$ for all z in this unbounded connected component.

Proof.

(i) Since $\operatorname{im}(\Gamma)$ is compact, any $z \notin \operatorname{im}(\Gamma)$ admits a strictly positive

$$\operatorname{dist}_{\operatorname{im}(\Gamma)}(z) = \inf_{w \in \operatorname{im}(\Gamma)} |w - z|.$$

Let $w \in \mathbb{C}$ be such that

$$|w - z| < \frac{1}{2} \operatorname{dist}_{\operatorname{im}(\Gamma)}(z),$$

so that $w \in \mathbb{C} \setminus \operatorname{im}(\Gamma)$. Observe then that

$$\begin{aligned} |n(\Gamma; z) - n(\Gamma; w)| &= \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\xi - z} - \frac{1}{\xi - w}\, d\xi \right| \\ &\leqslant \frac{1}{2\pi} \sum_{j=1}^{n} \oint_{\gamma_j} \left| \frac{1}{\xi - z} - \frac{1}{\xi - w} \right| |d\xi| \\ &= \frac{1}{2\pi} \sum_{j=1}^{n} \oint_{\gamma_j} \left| \frac{z - w}{(\xi - z)(\xi - w)} \right| |d\xi| \\ &\leqslant \frac{1}{2\pi} \left(\frac{2}{\operatorname{dist}_{\operatorname{im}(\Gamma)}(z)} \right)^2 \ell(\Gamma) |z - w|, \end{aligned}$$

whence $|n(\Gamma; z) - n(\Gamma; w)|$ is sufficiently small whenever $|z - w|$ is sufficiently small.

(ii) If C is a connected component of $\mathbb{C} \setminus \operatorname{im}(\Gamma)$, and $n(\Gamma; \cdot)\colon C \to \mathbb{Z}$ is continuous, then since $\mathbb{Z}$ is discrete, $n(\Gamma; \cdot)$ is constant on C.

(iii) For uniqueness, if there are unbounded connected components C_1 and C_2 of $\mathbb{C} \setminus \operatorname{im}(\Gamma)$, then there exists $M > \sup_{z \in \operatorname{im}(\Gamma)} |z|$ and $w_1 \in C_1, w_2 \in C_2$ such that $|w_1| > 2M$ and $|w_2| > 2M$. Since $\mathbb{C} \setminus U(0, 2M)$ is path connected, there exists $\gamma\colon [0,1] \to \mathbb{C}$ with $|\gamma(t)| \geqslant 2M$ and $\gamma(0) = w_1$, $\gamma(1) = w_2$. Therefore, w_1 and w_2 are in the same connected component.

Existence then follows from $\operatorname{im}(\Gamma)$ being compact.

Finally, let $(z_n)_n \subseteq C$, where C is the unbounded connected component, be such that $\lim_{n\to\infty}|z_n| = \infty$. For $M > \sup_{z\in \operatorname{im}(\gamma)}|z|$, there exists $m \in \mathbb{N}$ such that $|z_m| > M$. Then, we have

$$\begin{aligned} |n(\Gamma; z_m)| &= \left|\frac{1}{2\pi i}\oint_{\Gamma} \frac{1}{\xi - z}\,d\xi\right| \\ &\leqslant \frac{1}{2\pi}\sum_{j=1}^{k}\oint_{\gamma_j}\frac{1}{|\xi - z|}\,|d\xi| \\ &\leqslant \frac{1}{2\pi}\sum_{j=1}^{k}\oint_{\gamma_j}\frac{1}{|z_m| - M}\,|d\xi| \\ &= \frac{\ell(\Gamma)}{2\pi(|z_m| - M)}, \end{aligned}$$

whence $\lim_{m\to\infty} n(\Gamma; z_m) = 0$, meaning that there exists N such that $|n(\Gamma; z_m)| < 1$ for all $m \geqslant N$, meaning $n(\Gamma; z_m) = 0$ for all sufficiently large m. Since C is connected, it thus follows that $n(\Gamma; z) = 0$ for all $z \in C$.

□

Definition: Let $U \subseteq \mathbb{C}$ be open. A cycle Γ is *homologous to zero in* U if $\operatorname{im}(\Gamma) \subseteq U$ and for all $z \in \mathbb{C} \setminus U$, $n(\Gamma; z) = 0$.

Theorem (Cauchy's Integral Formula, General Case)**:** Let $\Gamma = \gamma_1 + \cdots + \gamma_k$ be a piecewise C^1 cycle homologous to zero in U, and $f\colon U \to \mathbb{C}$ holomorphic. Then, for all $z \in U \setminus \operatorname{im}(\Gamma)$,

$$n(\Gamma; z)f(z) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{\xi - z}\,d\xi$$

Theorem (Cauchy's Integral Theorem)**:** Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic, and Γ homologous to zero in U. Then,

$$\oint_{\Gamma} f(z)\,dz = 0.$$

Definition: A region $U \subseteq \mathbb{C}$ is called *simply connected* if its complement in the extended complex plane is connected.

Theorem: If $U \subseteq \mathbb{C}$ is simply connected, then every loop in U is homologous to zero.

Proof. Extend the function $n(\gamma; \cdot)$ to the extended complex plane by defining $n(\gamma; \infty) = 0$. This extended function is continuous on $\hat{\mathbb{C}} \setminus U$, as $n(\gamma; \cdot)$ is zero on the unique unbounded connected component of $\mathbb{C} \setminus \operatorname{im}(\gamma)$. It follows that $n(\gamma; z)$ is equal to zero on $\hat{\mathbb{C}} \setminus U$, whence γ is homologous to zero in U. □

Proposition: Let $U \subseteq \mathbb{C}$ be a region, $f\colon U \to \mathbb{C}$ holomorphic. The following are equivalent:

(i) there exists a holomorphic function $F\colon U \to \mathbb{C}$ such that $F'(z) = f(z)$;

(ii) for every piecewise C^1 loop γ with $\operatorname{im}(\gamma) \subseteq U$, we have

$$\oint_{\gamma} f(z)\,dz = 0.$$

Proof. The direction (i) ⇒ (ii) follows immediately from the fundamental theorem of calculus. In the reverse direction, we define $F\colon U \to \mathbb{C}$ by

$$f(z) = \int_{\sigma(z_0, z)} f(\xi)\,d\xi,$$

where $\sigma(z_0, z)\colon [0,1] \to U$ is a piecewise C^1 curve with $\sigma(0) = z_0$ and $\sigma(1) = z$. Such a curve always exists as U is open and connected (hence path-connected). The integral is well-defined, since if γ_1 and γ_2 are any two such paths, then $\Gamma = \gamma_1 \setminus \gamma_2$ is a piecewise C^1 loop. Additionally, F is continuous.

Now, we evaluate the derivative of F. Let $z \in U$, $r > 0$ such that $U(z, r) \subseteq U$, and $h \in \mathbb{C}$ be such that $z + h \in U(z, r)$. Then,

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\int_{\sigma(z_0, z_0+h)} f(\xi)\, d\xi - \frac{1}{h}\int_{\sigma(z_0,z)} f(\xi)\, d\xi \\ &= \frac{1}{h}\int_{\sigma(z,z+h)} f(\xi)\, d\xi. \end{aligned}$$

We may assume that $\sigma(z, z+h)$ is a straight line, so that

$$\int_{\sigma(z,z+h)} f(\xi)\, d\xi = hf(z),$$

meaning that

$$\begin{aligned} \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|\int_{\sigma(z,z+h)} f(\xi)\, d\xi - f(z)\right| \\ &\leqslant \sup_{w \in \operatorname{im}(\sigma(z,z+h))} |f(w) - f(z)|. \end{aligned}$$

Since f is continuous, it follows that the right hand side goes to zero as $|h|$ becomes small. Thus, F' is continuous, so f is holomorphic. □

Observe that $\mathbb{C} \setminus \{0\}$ is not simply connected, meaning that, for instance, the function

$$f(z) = \frac{1}{z}$$

does not have a holomorphic antiderivative defined on the entirety $\mathbb{C} \setminus \{0\}$, as

$$\int_{S^1} f(z)\, dz = 2\pi i.$$

Yet, if we restrict $f(z)$ to a simply connected subset of $\mathbb{C}$, there *is* a holomorphic antiderivative. Choosing such a simply connected subset of $\mathbb{C}$ is known as choosing a *branch* of the logarithm. In fact, there is more that we can say.

Corollary: Let $U \subseteq \mathbb{C}$ be simply connected, and let $f\colon U \to \mathbb{C} \setminus \{0\}$ be a nonvanishing holomorphic function. For each fixed pair $z_0 \in U$ and $w_0 \in \mathbb{C}$ for which $e^{w_0} = f(z_0)$, there exists a unique holomorphic function $g\colon U \to \mathbb{C}$ for which $g(z_0) = w_0$ and $e^{g(z)} = f(z)$.

We call g the logarithm of f, written $g(z) = \log(f(z))$.

Proof. Since f is nonvanishing and U is simply connected, it follows that $\frac{f'}{f}$ is holomorphic on U, meaning there is $\widetilde{g}\colon U \to \mathbb{C}$ such that $\widetilde{g}'(z) = \frac{f'(z)}{f(z)}$. Thus, there is some constant K such that

$$f(z) = Ke^{\widetilde{g}(z)}.$$

Define

$$g(z) = \log(K) + \widetilde{g}(z).$$

□

Theorem (Morera's Theorem)**:** Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ continuous. Suppose

$$\oint_T f(z)\,dz = 0$$

for all triangles $T \subseteq U$ homologous to zero. Then, f is holomorphic.

Proof. Since U is open, if $z_0 \in U$, there is r such that $U(z_0, r) \subseteq U$. Define $F\colon U(z_0, r) \to \mathbb{C}$ by

$$F(z) = \int_{\sigma(z_0,z)} f(\xi)\,d\xi,$$

where σ is the straight line from z_0 to z. For $0 < |h| < r - |z - z_0|$, we construct the straight lines $\sigma(z, z+h)$ and $\sigma(z_0, z+h)$, such that

$$T = \sigma(z_0, z) + \sigma(z, z+h) - \sigma(z_0, z+h),$$

and

$$\begin{aligned}\oint_T f(z)\,dz = 0&\\ &= \int_{\sigma(z_0,z)} f(\xi)\,d\xi + \int_{\sigma(z,z+h)} f(\xi)\,d\xi - \int_{\sigma(z_0,z+h)} f(\xi)\,d\xi\\ &= F(z) - F(z+h) + \int_{\sigma(z,z+h)} f(\xi)\,d\xi,\end{aligned}$$

meaning

$$\begin{aligned}F(z+h) - F(z) &= \int_{\sigma(z,z+h)} f(\xi)\,d\xi\\ \frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\int_{\sigma(z,z+h)} f(\xi)\,d\xi\\ \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_{\sigma(z,z+h)} (f(\xi) - f(z))\,d\xi\right|\\ &\leqslant \frac{1}{|h|}|h| \sup_{w\in \operatorname{im}(\sigma(z,z+h))} |f(w) - f(z)|\\ &= \sup_{w\in \operatorname{im}(\sigma(z,z+h))} |f(w) - f(z)|.\end{aligned}$$

Since f is continuous, it follows that for sufficiently small $|h|$, the right-hand-side goes to zero, whence $F'(z) = f(z)$, meaning F is holomorphic, so F is analytic, meaning f is analytic, so f is holomorphic. □

Corollary: Let $U \subseteq \mathbb{C}$ be open, $\gamma\colon [a, b] \to U$ a piecewise C^1 curve, and $g\colon U \times \operatorname{im}(\gamma) \to \mathbb{C}$ continuous. Suppose that for each $w \in \operatorname{im}(\gamma)$, the function $g(\cdot, w)$ is holomorphic. Then,

$$f(z) := \int_\gamma g(z, w)\,dw$$

is holomorphic.

Proof. Let T be a triangle in U homologous to zero. Then, by Fubini's Theorem,

$$\oint_T f(z)\,dz = \oint_T \int_\gamma g(z, w)\,dw\,dz$$

$$= \int_{\gamma} \oint_{T} g(z,w)\, dz\, dw.$$

The interior integral vanishes for every w as $g(\cdot, w)$ is holomorphic. Thus, f is holomorphic. □

Definition: Let $U \subseteq \mathbb{C}$ be open, γ_1, γ_2 piecewise C^1 loops in U. We say γ_1 and γ_2 are homotopic in U if there is a continuous function

$$H\colon [a,b] \times [0,1] \to U$$

such that

$$\begin{aligned} H(s,0) &= \gamma_1(s) \\ H(s,1) &= \gamma_2(s) \\ H(a,t) &= H(b,t). \end{aligned}$$

For each t, $H(\cdot, t)$ is a continuous loop. We call H a homotopy between γ_0 and γ_1.

Theorem: If γ_0 and γ_1 are homotopic in U, then $\Gamma = \gamma_1 - \gamma_0$ is homologous to zero in U.

Theorem: If $K \subseteq U$ is compact and U is connected, then there is some cycle Γ homologous to zero in U such that $n(\Gamma; z) = 1$ for all $z \in K$.

Corollary: Let U be a region. The following are equivalent:

(i) U is simply connected;

(ii) for every nonvanishing holomorphic function $f\colon U \to \mathbb{C} \setminus \{0\}$, there is a holomorphic function $g\colon U \to \mathbb{C}$ such that $f(z) = e^{g(z)}$;

(iii) for all cycles Γ with $\operatorname{im}(\Gamma) \subseteq U$, Γ is homologous to zero in U.

Maximum Modulus Principle

Theorem (Mean Value Property)**:** Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic, with $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta.$$

Proof. By the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Parametrizing $\gamma(\theta) = z_0 + re^{i\theta}$, we get

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta. \end{aligned}$$

□

Corollary: If $u\colon \mathbb{R}^2 \supseteq U \to \mathbb{R}$ is harmonic, $(x_0, y_0) \in U$, and $r > 0$ is such that $B((x_0, y_0), r) \subseteq U$, then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta.$$

Proof. Take real parts of the mean value property for holomorphic $f = u + iv$. □

Observe then that the triangle inequality implies that

$$|u(x_0, y_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))|\, d\theta.$$

Functions that satisfy this weaker criterion are known as *subharmonic*. It is subharmonic functions for which the most general case of the *maximum modulus principle* hold.

Theorem (Maximum Modulus Principle)**:** Let $U \subseteq \mathbb{R}^2$ be open and connected, and let $u\colon U \to \mathbb{R}$ be subharmonic. Suppose there exists $(x_0, y_0) \in U$ such that $u(x_0, y_0) \geqslant u(x, y)$ for all $x, y \in U$. Then, u is constant.

Proof. Let $\lambda = u(x_0, y_0)$, and let $E = \{(x, y) \mid u(x, y) = \lambda\} = u^{-1}(\{\lambda\})$. We see immediately that E is closed; we claim that E is also open.

Fix $(x_1, y_1) \in E$. Then, $u(x_1, y_1) = \lambda$. Take $r > 0$ such that $U((x_1, y_1), r) \subseteq U$. Then, for all $0 < s < r$, we have $S((x_1, y_1), s) \subseteq U$, meaning that

$$\begin{aligned} \lambda &= u(x_1, y_1) \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + s\cos(\theta), y_1 + s\sin(\theta))\, d\theta \\ &\leqslant \lambda, \end{aligned}$$

with the latter inequality following from the fact that λ is a local maximum. Therefore, $u(x_1 + s\cos(\theta), y_1 + s\sin(\theta)) = \lambda$ for all $0 < s < r$, whence $U((x_1, y_1), r) \subseteq E$. Thus, E is open, so since U is connected, it follows that E is all of U, meaning u is constant. □

Corollary: If $U \subseteq \mathbb{R}^2$ is bounded and $u\colon \overline{U} \to \mathbb{R}$ is continuous with $u|_U$ subharmonic, then there exists $(x_0, y_0) \in \partial U$ such that $u(x_0, y_0) = \sup_{(x,y)\in U} u(x, y)$.

Corollary: If $U \subseteq \mathbb{C}$ is open and connected, with $f\colon U \to \mathbb{C}$ holomorphic, then if $|f|\colon U \to \mathbb{R}$ has a local maximum at $z_0 \in U$, then f is constant.

Proof. Let $r > 0$ be such that $U(z_0, r) \subseteq U$. Then, restricting $|f|$ to $U(z_0, r)$, we see that $|f|$ restricted to $U(z_0, r)$ is subharmonic viewed as a function on $U(z_0, r)$, hence $|f|$ is constant on $U(z_0, r)$.

Now, by the mean value property and triangle inequality, it follows that for all $0 < s < r$, we have

$$\begin{aligned} |f(z_0)| &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(z_0 + se^{i\theta}\right)\right| d\theta \\ &= |f(z_0)|, \end{aligned}$$

meaning that these are equalities. In particular, there exists some θ_s such that $e^{i\theta_s} f\left(z_0 + se^{i\theta}\right) \geqslant 0$, meaning that for this value of s, we have

$$\begin{aligned} |f(z_0)| &= e^{i\theta_s} \int_0^{2\pi} f\left(z_0 + se^{i\theta}\right) d\theta \\ &= e^{i\theta_s} f(z_0), \end{aligned}$$

with the latter equality following from the mean value property. Since this holds for any s, it follows that θ_s is independent of s, meaning that $f(z)e^{i\theta_s} \geqslant 0$ for all $z \in U(z_0, r)$, meaning that $\operatorname{Im}\left(e^{i\theta_s} f(z)\right) = 0$ on $U(z_0, r)$, whence $f(z)e^{i\theta_s}$ is constant, meaning f is constant on $U(z_0, r)$.

Finally, by the identity theorem, it follows that f is constant on U. □

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. We say a sequence $U \supseteq ((x_n, y_n))_n \to \partial U$ if, for every compact $K \subseteq U$, the set $\{n \in \mathbb{N} \mid (x_n, y_n) \in K\}$ is finite.

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. Given a function $u \colon U \to \mathbb{R}$, define

$$\limsup_{(x,y)\to\partial U} u(x,y) := \inf_{\substack{K\subseteq U \\ K \text{ compact}}} \sup_{(x,y)\in U\setminus K} u(x,y).$$

These definitions allow us to extend the maximum modulus principle for subharmonic functions even further.

Theorem: Let $U \subseteq \mathbb{C}$ be a region, $u \colon U \to \mathbb{R}$ a nonconstant subharmonic function. If $((x_n, y_n))_n \subseteq U$ is such that $u(x_n, y_n) \to \sup_{x,y\in U} u(x,y)$, then $((x_n, y_n))_n \to \partial U$. Moreover, $\limsup_{(x,y)\to\partial U} u(x,y) = \sup_{(x,y)\in U} u(x,y)$.

Proof. Suppose toward contradiction that $((x_n, y_n))_n \nrightarrow \partial U$, so there exists a compact subset $K \subseteq U$ and a subset $((x_{n_k}, y_{n_k}))_k$ wholly contained in K. Since K is compact, there is a subsequence of $((x_{n_k}, y_{n_k}))_k$ converging to $(x_0, y_0) \in U$. Therefore, $u(x_0, y_0) = \sup_{(x,y)\in U} u(x,y)$, so u is constant by the maximum modulus principle, which is a contradiction.

Finally, $\limsup_{(x,y)\to\partial U} u(x,y) \leqslant \sup_{(x,y)\in U} u(x,y)$, while if $((x_n, y_n))_n \to \partial U$ is such that $u(x_n, y_n)$ converges to $\sup_{(x,y)\in U} u(x,y)$, then $\sup_{(x,y)\in U} u(x,y) = \lim_{n\to\infty} u(x_n, y_n) \leqslant \limsup_{(x,y)\to\partial U} u(x,y)$. $\square$

Theorem (Open Mapping Principle)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f \colon U \to \mathbb{C}$ be a nonconstant holomorphic function. Then, $f(V) \subseteq \mathbb{C}$ is open.

Proof. Let $z_0 \in U$ and $r > 0$ be such that $B(z_0, r) \subseteq U$. We will show that there exists R such that $U(f(z_0), R) \subseteq f(U(z_0, r)) \subseteq U$, whence $f(U)$ is open.

Since U is a region and f is nonconstant, the zeros of $g(z) := f(z) - f(z_0)$ are isolated, so there exists some $0 < s < r$ such that

$$\delta = \inf_{|z-z_0|=s} |f(z) - f(z_0)|$$

is strictly greater than zero. We claim that $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$. Suppose this were not the case, meaning there would be some $\xi \in U(f(z_0), \delta/2) \setminus f(U(z_0, r))$, and define $h \colon B(z_0, s) \to \mathbb{C}$ by

$$h(z) = \frac{1}{f(z) - \xi}.$$

Since $\xi \notin f(U(z_0, r))$, this is holomorphic, while $\xi \in U(f(z_0), \delta/2)$ implies

$$\begin{aligned} \sup_{|z-z_0|=s} |h(z)| &= \sup_{|z-z_0|=s} \frac{1}{|f(z) - \xi|} \\ &\leqslant \sup_{|z-z_0|=s} \frac{1}{|f(z) - f(z_0)| - |f(z_0) - \xi|} \\ &\leqslant \frac{1}{\delta - \delta/2} \\ &= \frac{2}{\delta}. \end{aligned}$$

Yet,

$$\begin{aligned} |h(z_0)| &= \frac{1}{|f(z_0) - \xi|} \\ &> \frac{2}{\delta}, \end{aligned}$$

contradicting the maximum modulus principle. Thus, $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$. $\square$

In the proof of the Hadamard Three-Lines Theorem, we used the function $h_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}$ for this purpose.

Classification of Singularities

The classification of singularities seeks to answer two fundamental questions: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, and $f\colon U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic,

- does f have a holomorphic extension to U including z_0;
- and what else can we say about the behavior of f at z_0?

Definition: Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f\colon U \setminus \{z_0\} \to \mathbb{C}$ holomorphic.

- If there exists a holomorphic $g\colon U \to \mathbb{C}$ with $g = f$ on $U \setminus \{z_0\}$, then we say z_0 is a *removable singularity*.
- If $\lim_{z\to z_0}|f(z)| = \infty$, then we say f has a *pole* at z_0.
- Else, we say f has an *essential singularity* at z_0.

Theorem (Riemann's Theorem on Removable Singularities)**:** Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and $f\colon U\setminus\{z_0\} \to \mathbb{C}$ holomorphic. Then, z_0 is a removable singularity if and only if $\lim_{z\to z_0} f(z) = 0$.

Proof. If z_0 is removable, then $g(z)$ is a holomorphic function with $g(z) = f(z)$ on $U \setminus \{z_0\}$, and since g is continuous, it follows that $\lim_{z\to z_0} g(z) = g(z_0)$, whence $\lim_{z\to z_0}(z - z_0)g(z) = \lim_{z\to z_0}(z - z_0)f(z) = 0$.

Now, if $\lim_{z\to z_0}(z - z_0)f(z) = 0$, then there is r such that $B(z_0, r) \subseteq U$, and since f is locally bounded around z_0, it follows that

$$f(z) = \frac{1}{2\pi i}\int_{S(z_0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

holds for all $z \in \dot{U}(z_0, r)$. Yet, the formula extends to z_0 as it is bounded, whence we may define the holomorphic extension for f by

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \frac{1}{2\pi i}\int_{S(z_0,r)} \frac{g(\zeta)}{\zeta-z}\, d\zeta & z = z_0 \end{cases}.$$

□

Proposition (Existence of Laurent Series)**:** Suppose $f\colon A(z_0, r, R) \to \mathbb{C}$ is holomorphic, with $0 \leqslant r < R$. Then, there exist holomorphic functions

$$g_1\colon U(z_0, R) \to \mathbb{C}$$
$$g_2\colon \mathbb{C} \setminus B(z_0, r) \to \mathbb{C}$$

such that $f = g_1 + g_2$ on $A(z_0, r, R)$. Moreover, there exists $(a_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

for all z, and the series converges uniformly on $A(z_0, \rho, s)$ with $r < \rho < s < R$.

Proof. Fix $z \in A(z_0, r, R)$. Then, for $r < \rho_1, \rho_2 < |z - z_0|$, the cycle

$$\Gamma_1 = S(z_0, \rho_1) - S(z_0, \rho_2)$$

is homologous to zero in $A(z_0, r, |z-z_0|)$. By Cauchy's Integral Theorem, it then follows that

$$\oint_{S(z_0,\rho_1)} \frac{f(\xi)}{\xi - z}\, d\xi = \oint_{S(z_0,\rho_2)} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Similarly, for $|z - z_0| < s_1, s_2 < R$, we have

$$\oint_{S(z_0,s_1)} \frac{f(\xi)}{\xi - z}\, d\xi = \oint_{S(z_0,s_2)} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Define $g_1 \colon U(z_0, R) \to \mathbb{C}$ by

$$g_1(z) = \frac{1}{2\pi i} \oint_{S(z_0,s)} \frac{f(\xi)}{\xi - z}\, d\xi,$$

where $|z - z_0| < s < R$. This function is holomorphic by Morera's Theorem. Similarly, we may define $g \colon \mathbb{C} \setminus B(z_0, r) \to \mathbb{C}$ by

$$g_2(z) = -\frac{1}{2\pi i} \oint_{S(z_0,\rho)} \frac{f(\xi)}{\xi - z}\, d\xi,$$

where $r < \rho < |z - z_0|$. We claim that $f = g_1 + g_2$ on $A(z_0, r, R)$.

For $z \in A(z_0, r, R)$, we may find, for any $r < \rho < |z - z_0| < s < R$, we let

$$\Gamma = S(z_0, s) - S(z_0, \rho),$$

homologous to zero in $A(z_0, r, R)$, whence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left(\oint_{S(z_0,s)} \frac{f(\xi)}{\xi - z}\, d\xi - \int_{S(z_0,\rho)} \frac{f(\xi)}{\xi - z}\, d\xi\right) \\ &= g_1(z) + g_2(z). \end{aligned}$$

□

Theorem: Let $U \subseteq \mathbb{C}$, $f \colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on $\dot{U}(z_0, R)$ for some R with $U(z_0, R) \subseteq U$. Then,

(i) f has a removable singularity at z_0 if and only if $a_n = 0$ for all $n < 0$;

(ii) f has a pole at z_0 if and only if

$$1 \leqslant |\{n < 0 \mid a_n \neq 0\}| < \infty.$$

(iii) f has an essential singularity at z_0 if and only if

$$|\{n < 0 \mid a_n \neq 0\}| = \infty.$$

Proof.

(i) If $a_n = 0$ for all $n < 0$, then $\lim_{z \to z_0} (z - z_0) f(z) = 0$, so f has a removable singularity at z_0.

Conversely, if f has a removable singularity at z_0, then for $n < 0$, we have

$$a_n = \frac{1}{2\pi i}\oint_{S(z_0,\rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$$

for any $0 < \rho < R$. Since $\lim_{z\to z_0}(z - z_0)f(z) = 0$, then for any $\varepsilon > 0$, there is sufficiently small ρ such that

$$\begin{aligned} |a_n| &= \left|\frac{1}{2\pi i}\oint_{S(z_0,\rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi\right| \\ &\leqslant \rho^{-1-n} \sup_{|\xi - z_0| = \rho} |(\xi - z_0)f(z)| \\ &\leqslant \varepsilon. \end{aligned}$$

Thus, $|a_n| = 0$ for all $n < 0$.

(ii) If $a_n \neq 0$ for a nonempty finite collection of $n < 0$, we let m be the largest number such that $a_{-m} < 0$, so that $f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n$. It follows that $\lim_{z\to z_0}(z - z_0)^m f(z) = a_{-m} \neq 0$. In particular, there is some small δ such that

$$\left|(z - z_0)^m f(z) - a_{-m}\right| < \frac{|a_{-m}|}{2},$$

whence for a sufficiently small ε,

$$\begin{aligned} |f(z)| &> \frac{|a_{-m}| - \left|(z - z_0)^m f(z) - a_{-m}\right|}{|z - z_0|^m} \\ &> \frac{|a_{-m}|}{2|z - z_0|^m} \\ &> \frac{1}{\varepsilon}, \end{aligned}$$

so f has a pole at z_0.

Conversely, if f has a pole at z_0, there is some $r > 0$ such that $|f(z)| \geqslant 1$ whenever $z \in \dot{U}(z_0, r)$. Define $g\colon \dot{U}(z_0, r) \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z)},$$

whence g is holomorphic and bounded on $\dot{U}(z_0, r)$. By the classification of singularities, z_0 is a removable singularity of g, so there exists a holomorphic function $h\colon U(z_0, r) \to \mathbb{C}$ that is equal to $\frac{1}{f(z)}$ for $z \neq z_0$ and $\lim_{z\to z_0} \frac{1}{f(z)}$ if $z = z_0$. We write

$$h(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n,$$

where we must have $b_0 = 0$. Let m be the smallest positive integer such that $h(z) = (z - z_0)^m \widetilde{h}(z)$, where $\widetilde{h}(z) = \sum_{n=0}^{\infty} b_{n+m}(z - z_0)^n$ and $b_m \neq 0$. The function $\widetilde{h}$ is holomorphic on $U(z_0, r)$ and nonzero on some $U(z_0, \rho)$ with $0 < \rho \leqslant r$, so that $\widetilde{f}(z) = \frac{1}{\widetilde{h}(z)}$ is holomorphic on $U(z_0, \rho)$, so

$$f(z) = (z - z_0)^{-m}\widetilde{f}(z)$$

on $\dot{U}(z_0, \rho)$. Writing

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

we deduce that

$$f(z) = \sum_{n=-m}^{\infty} c_{n+m} (z - z_0)^n.$$

(iii) Follows from (i) and (ii).

$\square$

Definition: Let $U \subseteq \mathbb{C}$ be open.

- If $f\colon U \setminus \{z_0\} \to \mathbb{C}$ has a pole or a removable singularity at z_0, then the order of the pole or singularity is the smallest $m \geqslant 0$ such that

  $$f(z) = (z - z_0)^{-m} g(z)$$

  with $g\colon U \to \mathbb{C}$ is holomorphic and has $g(z_0) \neq 0$.
- If $f\colon U \to \mathbb{C}$ has a zero at $z_0 \in U$, then the order of the zero at z_0 is the smallest $m \geqslant 0$ such that

  $$f(z) = (z - z_0)^m g(z)$$

  where $g\colon U \to \mathbb{C}$ is holomorphic and has $g(z_0) \neq 0$.

If $z_0 \in U$ is either a pole or a zero (or a removable singularity that is a zero when f is extended), then the *order* of f is the unique $m \in \mathbb{Z}$ such that

$$f(z) = (z - z_0)^m g(z)$$

with $g(z_0) \neq 0$.

Theorem (Casorati–Weierstrass)**:** Let $U \subseteq \mathbb{C}$ be an open set, and let $f\colon U \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function. If z_0 is an essential singularity of f, then for any $r > 0$ with $U(z_0, r) \subseteq U$, $f(\dot{U}(z_0, r)) \subseteq \mathbb{C}$ is dense.

Proof. Suppose $f(\dot{U}(z_0, r))$ is not dense in $\mathbb{C}$. Then, there exists $\varepsilon > 0$ and $w \in \mathbb{C}$ such that $|f(z) - w| \geqslant \varepsilon$ for all $z \in \dot{U}(z_0, r)$. We will show that z_0 is either removable or a pole.

Define

$$g\colon \dot{U}(z_0, r) \to \mathbb{C}$$

by

$$g(z) = \frac{1}{f(z) - w}.$$

Observe that g is definitionally bounded, so z_0 is a removable singularity of g. Thus, there is a holomorphic function $h\colon U(z_0, r) \to \mathbb{C}$ such that $h(z) = g(z)$ for $z \neq z_0$ and $h(z) = \lim_{z \to z_0} g(z)$. We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

and take

$$h(z) = (z - z_0)^m \widetilde{h}(z),$$

which exists as g is not uniformly zero, and where $\widetilde{h}(z_0) \neq 0$. Consequently, there is $0 < \rho \leqslant r$ such that $\widetilde{f}\colon U(z_0, \rho) \to \mathbb{C}$ given by

$$\widetilde{f}(z) = \frac{1}{\widetilde{h}(z)},$$

which is holomorphic. Thus,

$$f(z) = w + (z - z_0)^{-m}\widetilde{f}(z).$$

Thus, we get

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n,$$

whence z_0 is either removable or a pole. □

The Argument Principle and Rouché's Theorem

Definition: If $U \subseteq \mathbb{C}$ is open, and $V \subseteq U$ is an open subset such that $U \setminus V$ consists solely of isolated points, then a function $f\colon V \to \mathbb{C}$ is *meromorphic* if it is holomorphic on V and every $z_0 \in U \setminus V$ is either a pole or a removable singularity. We say f is meromorphic on U.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U. Let f be meromorphic on U with no poles or zeros on $\operatorname{im}(\Gamma)$.

(i) The set $\left\{z_0 \in U \;\middle|\; \operatorname{ord}_{z_0}(f) \neq 0, n(\Gamma; z_0) \neq 0\right\}$ is finite.

(ii) We have

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f'(z)}{f(z)}\,dz = \sum_{\substack{z_0 \in U \\ \operatorname{ord} z_0(f) \neq 0 \\ n(\Gamma; z_0)}} n(\Gamma; z_0)\operatorname{ord}_{z_0}(f).$$

Proof.

(i) Let $K = \{z \in U \mid n(\Gamma; z) \neq 0\} \cup \operatorname{im}(\Gamma)$. We know that for $R > \sup_{w \in \operatorname{im}(\Gamma)}|w|$, we have $n(\Gamma; z) = 0$ for all $z \in \mathbb{C} \setminus B(0, R)$, meaning that K is bounded. Furthermore, if $(z_n)_n \to z \in \mathbb{C}$, then either $z \in \operatorname{im}(\Gamma) \subseteq K$ or $z \in \mathbb{C} \setminus \operatorname{im}(\Gamma)$, so that $n(\Gamma; z) \neq 0$ by the continuity of the map $w \mapsto n(\Gamma; w)$. Thus, $z \in K$ as Γ is homologous to zero in U. It follows thus that K is compact.

Let $E = \left\{z_0 \in K \;\middle|\; \operatorname{ord}_{z_0}(f) \neq 0\right\}$, meaning that E consists of poles and zeros of f. These points are isolated, meaning E is a closed subset of K, hence compact. Since E is compact and contains isolated points only, it follows that E is finite.

(ii) For each $z \in K$, select $\delta_z > 0$ such that $U(z, \delta_z) \subseteq U$, and $U(z, \delta_z) \cap E \subseteq \{z\}$. Such a δ_z exists since z is isolated in E and U is open. The collection $\{U(z, \delta_z) \mid z \in K\}$ is an open cover of K, so there is a finite subcover $\{U(z_1, \delta_1), \dots, U(z_m, \delta_m)\}$. Define

$$V = \bigcup_{j=1}^{m} U\left(z_j, \delta_j\right),$$

so that V is open with $K \subseteq V \subseteq U$. Since $\{z \in U \mid n(\Gamma; z) \neq 0\} \subseteq K \subseteq V$, it follows that Γ is homologous to zero in V.

Define

$$g(z) = f(z) \prod_{z \in E} (z - z_0)^{-\operatorname{ord}_{z_0}(f)}.$$

Since $\operatorname{ord}_z(g) = 0$ for all $z \in V$, all the singularities of g are removable, and $\frac{g'}{g}$ is holomorphic on V and satisfies

$$\frac{g'}{g} = \frac{f'}{f} - \sum_{z_0 \in E} \frac{1}{z - z_0} \operatorname{ord}_{z_0}(f).$$

Cauchy's Integral theorem provides the desired result.

□

Theorem (Rouché's Theorem)**:** Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U. Let f and g be meromorphic on U with no poles or zeros on $\operatorname{im}(\Gamma)$. If $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in \operatorname{im}(\Gamma)$, then

$$\frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \oint_\Gamma \frac{g'(z)}{g(z)}\, dz.$$

Proof. Since $|f(z) - g(z)| < |f(z)| + |g(z)|$ on $\operatorname{im}(\Gamma)$, coupled with the fact that $\operatorname{ord}_z(f) = \operatorname{ord}_z(g) = 0$ on $\operatorname{im}(\gamma)$ implies that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$$

for all $z \in \operatorname{im}(\Gamma)$. This only holds if $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in \operatorname{im}(\Gamma)$. Since $\operatorname{im}(\Gamma)$ is compact, there exists some $\varepsilon > 0$ such that

$$\operatorname{dist}_{(-\infty,0]}\left|\frac{f}{g}(\operatorname{im}(\Gamma))\right| \geqslant \varepsilon.$$

Since $\frac{f}{g}$ is continuous and $\operatorname{im}(\Gamma)$ is compact, there also exists some $\delta > 0$ such that whenever $\operatorname{dist}_{\operatorname{im}(\Gamma)}(z) < \delta$, we have $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$.

Setting $V = \left\{z \in U \mid \operatorname{dist}_{\operatorname{im}(\Gamma)}(z) < \delta\right\}$, we let $h\colon V \to \mathbb{C}$ be defined by

$$h(z) = \log\left(\frac{f(z)}{g(z)}\right)$$

for the branch of the logarithm that excludes $(-\infty, 0]$, which is well-defined as $\frac{f}{g} \notin (-\infty, 0]$ on V. This satisfies

$$\frac{h'}{h} = \frac{f'}{f} - \frac{g'}{g},$$

whence by Cauchy's Integral Theorem,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_\Gamma \frac{h'(z)}{h(z)}\, dz \\ &= \frac{1}{2\pi i}\left(\oint_\Gamma \frac{f'(z)}{f(z)}\, dz - \oint_\Gamma \frac{g'(z)}{g(z)}\, dz\right). \end{aligned}$$

□

Remark: Most use cases for Rouché's Theorem involve finding $g(z)$ such that $|f(z) - g(z)| < |g(z)|$ on $\mathrm{im}(\Gamma)$, where both $f(z)$ and $g(z)$ have no zeros or poles on $\mathrm{im}(\Gamma)$.

It is possible to use Rouché's Theorem to prove the fundamental theorem of algebra.

Corollary: Let $P\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant polynomial. Then, there exists $z_0 \in \mathbb{C}$ with $P(z_0) = 0$.

Proof. Write $P(z) = a_n z^n + \cdots + a_1 z + a_0$, where $n \geqslant 1$, $a_n \neq 0$, and $a_1, \dots, a_n \in \mathbb{C}$.

Let $Q(z) = a_n z^n$, R sufficiently large, and $\Gamma = S(0, R)$. Then,

$$\begin{aligned} |P(z) - Q(z)| &= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right| \\ &\leqslant |a_n|R^n\left(\frac{|a_{n-1}|}{|a_n|R} + \cdots + \frac{|a_0|}{|a_n|R^n}\right) \\ &< |a_n|R^n \\ &\leqslant |P(z)| + |Q(z)|. \end{aligned}$$

Thus, P has the same number of zeros counted with multiplicity as Q. □

Corollary: Let $U \subseteq \mathbb{C}$ be an open set, $f\colon U \to \mathbb{C}$ be a holomorphic injective function. Then, $f'(z) \neq 0$ for all $z \in U$, whence f admits a holomorphic inverse.

Proof. Suppose there is some $z_0 \in U$ such that $f'(z_0) = 0$. By the identity theorem, there must be some $r > 0$ such that $B(z_0, r) \subseteq U$ and $f'(z) \neq 0$ for all $z \in \dot{U}(z_0, r)$, else f would be equal to a constant on some nonempty open set, hence not injective.

Let $g(z) = f(z) - f(z_0)$. Define $m := \mathrm{ord}_{z_0}(g)$. We observe that $m \geqslant 2$, as $g(z_0) = 0$ and $g'(z_0) = 0$. We may write

$$g(z) = \lambda(z - z_0)^m + (z - z_0)^{k+1}h(z)$$

for some holomorphic function $h\colon U \to \mathbb{C}$ and a constant $\lambda \in \mathbb{C} \setminus \{0\}$, meaning

$$f(z) = f(z_0) + \lambda(z - z_0)^m + (z - z_0)^{k+1}h(z)$$

for all $z \in U$.

Let $C = \sup_{z \in B(z-0,r)}|h(z)|$, which is finite as $B(z_0, r)$ is compact. Letting $\rho = \min\left(r, \frac{|\lambda|}{2C}\right)$ and $\eta = \frac{|\lambda|\rho^m}{2}$, and fixing $w \in \dot{U}(f(z_0,), \eta)$, we observe that if $z \in S(z-0, \rho)$,

$$\begin{aligned} \left|f(z) - w - \lambda(z - z_0)^m\right| &= \left|f(z_0) - w + (z - z_0)^{m+1}h(z)\right| \\ &\leqslant |f(z_0) - w| + |z - z_0|^{m+1}|h(z)| \\ &< \eta + \rho^{m+1}C \\ &< \rho^m|\lambda| \\ &= \left|(z - z_0)^m\lambda\right|. \end{aligned}$$

Therefore, by Rouché's theorem, the number of zeros in $U(z_0, \rho)$ for $f(z) - w$ is equal to the number of zeros counted with multiplicity in $U(z_0, \rho)$ of $(z - z_0)^m\lambda$. Since the latter $m \geqslant 2$, it follows that the former is also $m \geqslant 2$. Since $f'(z) \neq 0$ for all $z \in U(z_0, \rho)$, no zero of $f(z) - w$ can have order at least 2, meaning that there are at least two distinct zeros of $f(z) - w$ in $U(z_0, \rho)$, whence f is not injective. □

Residues

Definition: Let $U \subseteq \mathbb{C}$, $z_0 \in U$, $r > 0$ such that $U(z_0, r) \subseteq U$, and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on $\dot{U}(z_0, r)$. The *residue* of f at z_0 is the coefficient a_{-1} of $(z - z_0)^{-1}$. We write $\operatorname{Res}(f; z_0)$.

Proposition: Let $U \subseteq \mathbb{C}$ be an open set, and let f be meromorphic on U with a pole of order $m \geqslant 1$ at $z_0 \in U$. Then,

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z).$$

Proof. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Differentiating term by term, we find that

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = a_{-1}.$$

□

Proposition: Let $f, g \colon U \to \mathbb{C}$ be holomorphic with z_0 a simple zero for g that is not a zero or pole for f. Then,

$$\operatorname{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. We compute the residue directly, using the fact that $g(z_0) = 0$ to find

$$\begin{aligned} \lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} &= \frac{f(z_0)}{\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}} \\ &= \frac{f(z_0)}{g'(z_0)}. \end{aligned}$$

□

Example:

- If $f(z) = \frac{e^{iz}}{z}$, then

$$\operatorname{Res}(f; 0) = 1.$$

- If $f(z) = \pi \cot(\pi z)$, then

$$\begin{aligned} \operatorname{Res}(f; n) &= \frac{\pi \cos(\pi n)}{\frac{d}{dz}\big|_{z=n} \sin(\pi z)} \\ &= \frac{\pi \cos(\pi n)}{\pi \cos(\pi n)} \\ &= 1. \end{aligned}$$

- Let $f(z) = \frac{e^{3z}}{(z-2)^2}$. To compute the residue at $z = 2$, we may directly find

$$\begin{aligned}\operatorname{Res}(f;2) &= \lim_{z\to 2}\frac{d}{dz}(z-2)^2\frac{e^{3z}}{(z-2)^2}\\ &= 3e^6.\end{aligned}$$

Theorem: Let $f\colon \dot{U}(z_0, r) \to \mathbb{C}$ be holomorphic. Then, for all $0 < \rho < r$, we have

$$\oint_{S(z_0,\rho)} f(z)\,dz = 2\pi i \operatorname{Res}(f; z_0).$$

Proof. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

The series converges uniformly on compact sets, so we may exchange order of integration and summation to find

$$\begin{aligned}\oint_{S(z_0,\rho)} f(z)\,dz &= \sum_{n=-\infty}^{\infty} a_n \oint_{S(z_0,\rho)} (z-z_0)^n\,dz\\ &= 2\pi i a_{-1}.\end{aligned}$$

□

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U. Let f be meromorphic on U with no poles on $\operatorname{im}(\Gamma)$. Then,

$$\oint_{\Gamma} f(z)\,dz = 2\pi i \sum_{z_0\in E} n(\Gamma; z_0) \operatorname{Res}(f; z_0),$$

where

$$E = \left\{z_0 \in U \;\middle|\; \operatorname{ord}_{z_0}(f) < 0, n(\Gamma; z_0) \neq 0\right\}.$$

Proof. As in the proof of the argument principle, we see that E is finite, which we write $\{z_1, \dots, z_m\}$. Select $r > 0$ such that $B(z_j, r)$ are pairwise disjoint and contained in U. Set $\widetilde{\Gamma} = \Gamma - \sum_{j=1}^{n} S(z_j, r)$. Then, $\widetilde{\Gamma}$ is homologous to zero in $U \setminus \{z_1, \dots, z_m\}$, while f is holomorphic on $U \setminus \{z_1, \dots, z_m\}$, whence

$$\begin{aligned}0 &= \oint_{\widetilde{\Gamma}} f(z)\,dz\\ &= \oint_{\Gamma} f(z)\,dz - \sum_{j=1}^{m}\oint_{S(z_j,r)} f(z)\,dz\\ &= \oint_{\Gamma} f(z)\,dz - 2\pi i \sum_{z_0\in E} n(\Gamma; z_0)\operatorname{Res}(f; z_0).\end{aligned}$$

□

Conformal Maps and Spaces of Holomorphic Functions

Given an open subset $U \subseteq \mathbb{C}$, we define $H(U)$ to be the set of all holomorphic functions $f\colon U \to \mathbb{C}$. Similarly, we write $C(U)$ for the continuous functions $f\colon U \to \mathbb{C}$.

Definition: A sequence $(f_n)_n \subseteq C(U)$ converges uniformly on compacts to $f \in C(U)$ if, for every compact $K \subseteq U$, the sequence $(f_n|_K)_n \to f|_K$ uniformly.

Proposition: Let $U \subseteq \mathbb{C}$ be open. If a sequence $(f_n)_n \subseteq H(U)$ converges uniformly on compacts to $f \in C(U)$, then $f \in H(U)$. Moreover, the sequence $(f'_n)_n \to f'$.

Proof. Since any $(f_n)_n$ converges on compacts to f, it converges uniformly on any triangle T homologous to zero in U, so that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \oint_T f_n(z)\,dz \\ &= \oint_T \lim_{n\to\infty} f_n(z)\,dz \\ &= \oint_T f(z)\,dz. \end{aligned}$$

To show that $(f'_n)_n \to f'$ converges uniformly on compacts, we use Cauchy's integral formula to find that

$$f'_n(z) - f'(z) = \frac{1}{2\pi i}\oint_{S(z_0,R)} \frac{f_n(w)-f(w)}{(w-z)^2}\,dw,$$

where $z \in U(z_0, R)$ and $B(z_0, R) \subseteq U$. Cauchy's estimate then show that this tends to 0 as $n \to \infty$ uniformly for all $z \in U(z_0, R)$. Since compact sets are totally bounded, it thus follows that the convergence is uniform on compact subsets. □

Definition: Let $U \subseteq \mathbb{C}$ be an open set. An exhaustion of U is a collection of compacts $(K_m)_m$ for which $K_m \subseteq K^\circ_{m+1}$, and

$$U = \bigcup_{m=1}^{\infty} K_m.$$

The primary example we will use is

$$K_m := \left\{z \in U \;\middle|\; |z| \leqslant m, \operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant \frac{1}{m}\right\}.$$

Definition: Let $U \subseteq \mathbb{C}$ be an open set, and let $(K_m)_m$ be an Exhaustion of U. For $f, g \in C(U)$, we define

$$d(f,g) = \sum_{m=1}^{\infty} 2^{-m}\frac{\|f-g\|_{K_m}}{1+\|f-g\|_{K_m}}.$$

Now, despite the fact that the metric space $(C(U), d)$ depends on the choice of exhaustion, it can be shown that any two metrics based on exhaustions $(K_m)_m$ and $(K'_m)_m$ are uniformly equivalent.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, $(K_m)_m$ an exhaustion of U, and let

$$d(f,g) = \sum_{m=1}^{\infty} 2^{-m}\frac{\|f-g\|_{K_m}}{1+\|f-g\|_{K_m}}$$

for $f, g \in C(U)$. Then, $(H(U), d)$ is a complete metric space.

Proposition: The topology on $(H(U), d)$ is equal to the topology of uniform convergence on compact subsets.

Proof. Let $(K_m)_m$ be an exhaustion of U. Let $(X_m, d_m) = \left(C(K_m,), \|\cdot - \cdot\|_{K_m}\right)$, and set

$$X = \prod_{m=1}^{\infty} C(K_m).$$

The isometry $\iota\colon C(U) \to X$, given by $\iota(f) = (f|_{K_m})_m$ yields that a sequence $(f_n)_n \to f$ if and only if $(f_n|_{K_m})_n \to f|_{K_m}$ for each K_m. In particular, if $(f_n)_n \to f$ uniformly on compact subsets, then it converges uniformly on each K_m, hence $\lim_{n\to\infty} d(f_n, f) = 0$.

Conversely, if $K \subseteq \mathbb{C}$, $\{K_m^\circ\}_{m=1}^\infty$ is an open cover of K, so there exists a finite subcover. Since $K_m \subseteq K_{m+1}^\circ$, this means there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$. In particular, if $\lim_{n\to\infty} d(f_n, f) = 0$, it then follows that $(f_n)_n \to f$ uniformly on K. □

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A family $\mathcal{F} \subseteq H(U)$ is called *normal* if its closure $\overline{\mathcal{F}}$ is compact in $H(U)$.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

(i) $\mathcal{F}$ is normal;

(ii) for each $K \subseteq U$, the family $\mathcal{F}|_K$ has compact closure in $C(K)$:

(iii) for each $z \in U$, there exists a bounded open set $W_z \subseteq U$ containing z such that $\mathcal{F}|_{W_z}$ has compact closure in $C\left(\overline{W_z}\right)$.

Proof. We start by showing that (i) implies (iii). For $z \in U$, let $R = R_z > 0$ such that $B(z, R) \subseteq U$, and let $g_n \in \mathcal{F}|_{B(z,R)}$. Choose $f_n \in \mathcal{F}$ such that $f_n|_{B(z,R)} = g_n$. If $\mathcal{F}$ is normal, there exists a subsequence $(f_{n_k})_k \to f \in \overline{\mathcal{F}}$. In particular, $f_{n_k}|_{B(z,R)} \to f|_{B(z,R)}$. Thus, $\mathcal{F}|_{B(z,R)}$ has compact closure in $C(B(z, R))$.

Next, we show that (iii) implies (ii). Let W_z be as above. Given $K \subseteq U$, the collection $\{W_z\}_{z\in U}$ is an open cover of K that has a finite subcover $\{W_1, \dots, W_\ell\}$. Given $g_n \in \mathcal{F}|_K$, write $g_n = f_n|_K$ for some $f_n \in \mathcal{F}$. There then exists a subsequence $(f_{n_k})_k$ such that $\left(f_{n_k}|_{W_j}\right) \to f_j \in C\left(\overline{W_j}\right)$ for each j. The function $f := \lim_{k\to\infty} f_{n_k}(z)$ is well-defined and satisfies $f|_{\overline{W_j}} = f_j$ for each j. Moreover, since

$$\begin{aligned}\limsup_{k\to\infty} \|f_{n_k} - h\|_K &\leqslant \limsup_{k\to\infty} \max_{1\leqslant j\leqslant \ell} \left\|f_{n_k} - f_j\right\|_{\overline{W_j}}\\ &= 0,\end{aligned}$$

it follows that $\mathcal{F}|_K$ has compact closure in $C(K)$.

Finally, we show that (ii) implies (i). Let $(K_m)_m$ be an exhaustion of U. By Tychonoff's Theorem, $\prod_{m=1}^\infty \overline{\{f \mid {}_{K_m} | f \in \mathcal{F}\}}$ is compact. Thus, given a sequence $(f_n)_n$ in $\mathcal{F}$, there is a subsequence $(f_{n_k})_k$ such that for each $m \in \mathbb{N}$, there is some $g_m \in C(K_m)$ such that $f_{n_k}|_{K_m}$ converges uniformly to g_m. Thus, the function $f(z) = \lim_{k\to\infty} f_{n_k}(z)$ is well-defined and satisfies $f|_{K_m} = g_m$ for all m. Given a compact $K \subseteq U$, there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$, and consequently, $f_{n_k}|_K \to f|_K$ uniformly, so $\mathcal{F}$ is normal. □

We will now create a much more workable criterion for normality.

Definition: Let (X, d) be a compact metric space. A family $\mathcal{F} \subseteq C(X)$ is (uniformly) equicontinuous if, for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$.

Recall the Arzelà–Ascoli theorem.

Theorem (Arzelà–Ascoli)**:** Let $K \subseteq \mathbb{C}$ be compact, and let $\mathcal{F} \subseteq C(K)$ be a family of continuous functions. The following are equivalent:

(i) $\overline{\mathcal{F}}$ is compact;

(ii) $\mathcal{F}$ is bounded and, for all $z \in K$, $\overline{\{f(z) \mid f \in \mathcal{F}\}} \subseteq \mathbb{C}$ is compact.

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A family $\mathcal{F} \subseteq H(U)$ is called locally (uniformly) bounded on U if, for all $z_0 \in U$, there is some $\delta > 0$ such that $U(z_0, \delta) \subseteq U$ and there is some $C \geqslant 0$ such that $|f(z)| \leqslant C$ for all $z \in U(z_0, \delta)$ and all $f \in \mathcal{F}$.

Proposition: Let $U \subseteq \mathbb{C}$ be open, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

(i) $\mathcal{F}$ is locally bounded;

(ii) for every compact $K \subseteq U$, $\sup_{f\in\mathcal{F}}\|f\|_K$ is finite.

Proof. We start by showing that (i) implies (ii). For each $z \in U$, there is $\delta_z > 0$ and $C_z > 0$ such that $U(z, \delta_z) \subseteq U$ and $|f(w)| \leqslant C_z$ for all $w \in U(z, \delta_z)$. Let $K \subseteq U$ be compact. There is a finite subcover $\{U(z_1, \delta_1), \dots, U(z_k, \delta_k)\}$, with corresponding bounds $C_1, \dots, C_k$, so by defining $C = \max\{C_1, \dots, C_k\}$, it follows that $\sup_{f\in\mathcal{F}}\|f\|_K \leqslant C < \infty$.

Now, we show that (ii) implies (i). Given $z_0 \in U$, there is $r > 0$ such that $B(z_0, r) \subseteq U$. By taking $K = B(z_0, r)$, we are done. □

Theorem (Montel's Theorem)**:** Let $U \subseteq \mathbb{C}$ be open, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

(i) $\mathcal{F}$ is normal;

(ii) $\mathcal{F}$ is locally bounded.

Proof. We start by showing that (i) implies (ii). Given $K \subseteq U$, the map $\|\cdot\|_K$ is continuous, so $\sup_{f\in\mathcal{F}}\|f\|_K \leqslant \sup_{f\in\overline{\mathcal{F}}}\|f\|_K$, which is finite as $\overline{\mathcal{F}}$ is compact.

Now, we show (ii) implies (i). Fix $z_0 \in U$, and choose $R_0 > 0$ such that $B(z_0, R_0) \subseteq U$. Let $W_0 = U(z_0, R_0/2)$. It suffices to show that $\mathcal{F}|_{\overline{W_0}}$ has compact closure in $C\left(\overline{W_0}\right)$, which by the Arzelà–Ascoli theorem, is equivalent to showing that $\mathcal{F}|_{\overline{W_0}}$ is equicontinuous. Since $\mathcal{F}$ is locally bounded, there is some $C_0 \geqslant 0$ such that $\sup_{f\in\mathcal{F}}\|f\|_{B(z_0,R_0)} \leqslant C_0$. By Cauchy's Integral Formula, we then have for all $z \in \overline{W_0}$ and all $f \in \mathcal{F}$,

$$\begin{aligned}
|f'(z)| &= \left|\frac{1}{2\pi i}\oint_{S(z_0,R_0)} \frac{f(\xi)}{(\xi - z)^2}\, d\xi\right| \\
&\leqslant \frac{C_0}{2\pi}\oint_{S(z_0,R_0)} \frac{1}{|\xi - z|^2}\, |d\xi| \\
&\leqslant \frac{4C_0}{R_0} \\
&=: A_0.
\end{aligned}$$

We have thus shown that $|f'(z)| \leqslant A_0$ for all $z \in \overline{W_0}$, meaning that each $f|_{\overline{W_0}} \in \mathcal{F}|_{\overline{W_0}}$ is A_0-Lipschitz, hence $\mathcal{F}|_{\overline{W_0}}$ is equicontinuous, hence normal. □

Worked Examples and Problem-Solving Methods

Example: Suppose U is a region in $\mathbb{C}$ that contains 0, and suppose $f\colon U \to \mathbb{C}$ is a holomorphic function satisfying

$$\left|f\left(\frac{1}{n}\right)\right| < e^n$$

for sufficiently large n. We will show that this means f is 0 everywhere.

Toward this end, since U is open, there is some $r > 0$ such that $U(0, r) \subseteq U$. Since f is holomorphic, on

$U(0,r)$, we may write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for some sequence $(a_n)_n \subseteq \mathbb{C}$. Now, we also observe that

$$\begin{aligned} |f(0)| &= \lim_{n\to\infty}\left|f\left(\frac{1}{n}\right)\right| \\ &\leqslant \lim_{n\to\infty} e^{-n} \\ &= 0. \end{aligned}$$

Suppose toward contradiction that f were nonconstant. Then, there would be some minimal positive value ℓ such that

$$f(z) = z^{\ell}\sum_{n=0}^{\infty} a_{n+\ell}z^n$$

has $a_\ell \neq 0$. Thus, defining

$$g(z) = \sum_{n=0}^{\infty} a_{n+\ell}z^n,$$

we observe that $g(0) \neq 0$, meaning that on some sufficiently small ball $U(0,\delta) \subseteq U(0,r)$, we have $|g(z)| > \left|\frac{a_\ell}{2}\right|$ for all $z \in U(0,\delta)$. In particular, this means that for n with $\frac{1}{n} < \delta$,

$$\begin{aligned} e^{-n} &\geqslant \left|f\left(\frac{1}{n}\right)\right| \\ &= n^{-\ell}\left|g\left(\frac{1}{n}\right)\right| \\ &\geqslant \frac{|a_\ell|}{2n^{\ell}}, \end{aligned}$$

whence

$$|a_\ell| \leqslant \frac{2n^{\ell}}{e^n}.$$

Yet, since n is arbitrary and ℓ is constant, this implies that $|a_\ell| = 0$, contradicting the assumption that there were such a g. Thus, in particular, we have that $f(z) = 0$ on $U(0,r)$, whence f is zero everywhere by the identity theorem.

Cauchy Estimate Problems

Example: Suppose f is an entire function, and suppose there exists a constant C such that for all $z \in \mathbb{C}$,

$$|f(z)| \leqslant C(1+|z|)^{1/2}.$$

We will show that f is then constant. Toward this end, we will be able to use the Cauchy estimate by taking

$$\left|f^{(n)}(z)\right| \leqslant \frac{n!}{R^n}\sup_{|z|=R}|f(z)|$$

$$\begin{aligned}
&\leqslant \frac{Cn!}{R^n} \sup_{|z|=R} (1+|z|)^{1/2} \\
&= \frac{Cn!}{R^n}(1+R)^{1/2},
\end{aligned}$$

whence for all $n \geqslant 1$, since R is arbitrary, we have $\left|f^{(n)}(z)\right| = 0$, so f is constant.

Maximum Modulus Principle Problems

Example: We show that if $f\colon U \to \mathbb{C}$ is holomorphic on a connected open set, and

$$u(x,y) = |f(x+iy)|$$

is harmonic on U, then f is constant.

Toward this end, we let $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. For any $0 < s < r$, the mean value property gives

$$\begin{aligned}
|f(z_0)| &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + se^{i\theta}\right)\right| \, d\theta \\
&= |f(z_0)|.
\end{aligned}$$

In particular, for any $0 < s < r$, we have the equality

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + se^{i\theta}\right)\right| \, d\theta.$$

Since f is continuous, there is some θ_s such that $\left|f\left(z_0 + se^{i\theta}\right)\right| = e^{i\theta_s} f\left(z_0 + se^{i\theta}\right)$, whence

$$\begin{aligned}
|f(z_0)| &= e^{i\theta_s} \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + se^{i\theta}\right) \, d\theta \\
&= e^{i\theta_s} f(z_0),
\end{aligned}$$

meaning that $\theta_s =: \theta_0$ is independent of s. Yet, this means that $e^{i\theta_0} f(z)$ is holomorphic on $U(z_0, r)$ and has $\operatorname{Im}\left(e^{i\theta} f(z)\right) = 0$, meaning that by the open mapping principle, $f(z)$ is constant on $U(z_0, r)$, and so f is constant on U by the identity theorem.

The Phragmén–Lindelöf Method

The maximum modulus principle is primarily useful in the case where f is continuous on the closure of a bounded open set U and holomorphic on the interior. Yet, this fails to be true if U is unbounded.

For instance, if

$$U = \left\{ z \in \mathbb{C} \,\middle|\, -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \right\},$$

and $f(z) = e^{e^z}$, then

$$f\left(x \pm \frac{\pi}{2} i\right) = e^{\pm i e^x},$$

whence $|f(z)| = 1$ for $z \in \partial U$. Yet, $f(z) \to \infty$ very rapidly along the positive real axis, which is contained in U.

Yet, all hope is not lost in the case that U is unbounded. If U is unbounded and there is $g: U \to \mathbb{C}$ such that $|f| < |g|$, and $g \to \infty$ "slowly" (so to speak) as $z \to \infty$, then it turns out that f is actually bounded in U, and we can use the maximum modulus principle to obtain other conclusions about f.

Finding such a g is part of the *Phragmén–Lindelöf* method, which we expand upon here.

Example: From the Cauchy estimates, we know that if f is entire and

$$|f(z)| \leqslant C\left(1 + |z|^{1/2}\right),$$

then f is constant.

Theorem (Hadamard Three-Lines Theorem)**:** Let $a, b \in \mathbb{R}$ be fixed with $a < b$. Let $U = \{z \mid a < \text{Re}(z) < b\}$. Suppose $|f(z)| < B$ for all $z \in U$ and some fixed $B < \infty$. Define

$$M(x) = \sup\left\{|f(z)| \;\middle|\; z \in \overline{U}\right\}.$$

Then,

$$M(x)^{b-a} \leqslant M(a)^{b-x} M(b)^{x-a}.$$

Proof. Suppose $M(a) = M(b) = 1$. Our task now is to show that $|f(z)| \leqslant 1$ for all $z \in U$. Toward this end, define

$$h_\varepsilon(z) = \frac{1}{1 + \varepsilon(z - a)}$$

for $z \in \overline{U}$. We have $|h_\varepsilon| \leqslant 1$ in $\overline{U}$, so that

$$|f(z)h_\varepsilon(z)| \leqslant 1$$

for all $z \in \partial U$. Furthermore, since $|1 + \varepsilon(z - a)| \geqslant \varepsilon|\text{Im}(z)|$, we have

$$|f(z)h_\varepsilon(z)| \leqslant \frac{B}{\varepsilon|\text{Im}(z)|}$$

for all $z \in \overline{U}$. Cut out a (closed) rectangle R from $\overline{U}$ via the lines $\text{Im}(z) = \pm\frac{B}{\varepsilon}$. Thus, along ∂R, we have $|f(z)h_\varepsilon(z)| \leqslant 1$, so that $|f(z)h_\varepsilon(z)| \leqslant 1$ on R by the maximum modulus principle.

Yet, since $|f(z)h_\varepsilon(z)| \leqslant \frac{B}{\varepsilon|\text{Im}(z)|}$ on the entirety of $\overline{U}$, and $\frac{B}{\varepsilon|\text{Im}(z)|} < 1$ outside R, it follows that $|fh_\varepsilon| \leqslant 1$ on $\overline{U}$, so $|f(z)h_\varepsilon(z)| \leqslant 1$ for all $z \in U$ and all $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0$, we obtain the desired result, that $|f(z)| \leqslant 1$.

In the general case, we define

$$g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)},$$

where for all $M > 0$ and complex w, we have $M^w = e^{w \ln(M)}$. Then, g is entire, g is always nonzero, $\frac{1}{g}$ is bounded on $\overline{U}$, and has

$$\begin{aligned}|g(a + iy)| &= M(a)\\ |g(b + iy)| &= M(b),\end{aligned}$$

meaning that $\frac{f}{g}$ satisfies the previous assumptions, so that $|f/g| \leqslant 1$ in U. □

In the Phragmén–Lindelöf method, we seek to find a particular ε-dependent function $h_\varepsilon : U \to \mathbb{C}$ such that the following hold:

- $|fh_\varepsilon(z)| \leqslant M$ for all $z \in \partial U$;
- $\lim_{\varepsilon\to 0} h_\varepsilon(z) = 1$;
- there exists a *bounded* $V \subseteq U$ such that $|fh_\varepsilon| \leqslant M$ on ∂V and on $U \setminus \overline{V}$.

Rouché's Theorem Problems

Example: We show that if f and g are holomorphic on a neighborhood of $B(0,1)$, and $f(z)$ has a simple zero at $z = 0$ and no other zero in $B(0,1)$, then $f_\varepsilon(z) = f(z) + \varepsilon g(z)$ has exactly one zero in $\mathbb{D}$ for sufficiently small ε.

To show this, we start by showing that the conditions of Rouché's Theorem are satisfied for both $f_\varepsilon(z)$ and $f(z)$. It is clear from the fact that f has no other zeros in $B(0,1)$ that f has no zeros on $S(0,1)$, while we may find ε small enough such that

$$\begin{aligned}|f_\varepsilon(z)| &\geqslant |f(z)| - \varepsilon|g(z)|\\ &> 0\end{aligned}$$

by selecting ε such that $\varepsilon \inf_{z\in S(0,1)}|g(z)| < \inf_{z\in S(0,1)}|f(z)|$. Thus, if we set $m_1 = \inf_{z\in S(0,1)}|f(z)|$ and $m_2 = \sup_{z\in S(0,1)}|g(z)|$, we have for $\varepsilon < \frac{m_1}{m_2}$ and all $z \in S(0,1)$,

$$\begin{aligned}|f_\varepsilon(z) - f(z)| &\leqslant |\varepsilon g(z)|\\ &\leqslant \varepsilon m_2\\ &< m_1\\ &< |f(z)|,\end{aligned}$$

whence $f(z)$ and $f_\varepsilon(z)$ have the same number of zeros in $\mathbb{D}$ counted with multiplicity.

Residue Integrals

Example: We will evaluate

$$I = \int_0^\infty \frac{1}{1+x^n}\,dx$$

via contour integration. Toward this end, let $f(z) = \frac{1}{1+z^n}$. We observe that $z^n + 1$ has roots for $e^{i\theta}$ at values $\theta = \frac{\pi+2\pi k}{n}$ for $0 \leqslant k < n$.

We take the closed contour γ_R given by

$$\oint_{\gamma_R} f(z)\,dz = \int_0^R f(x)\,dx + \int_0^{\frac{2\pi}{n}} f\left(Re^{i\theta}\right)\,d\left(Re^{i\theta}\right) + \int_R^0 f\left(xe^{i(2\pi/n)}\right)\,d\left(xe^{i(2\pi/n)}\right).$$

The left-hand side encloses the residue of f at $e^{i\pi/n}$, so by the Residue Theorem, since f has a simple pole at $e^{i\pi/n}$,

$$\begin{aligned}\oint_{\gamma_R} f(z)\,dz &= 2\pi i\,\mathrm{Res}\left(f; e^{i\pi/n}\right)\\ &= 2\pi i\frac{1}{n\left(e^{i\pi/n}\right)^{n-1}}\\ &= 2\pi i\frac{1}{n\left(e^{i\frac{(n-1)\pi}{n}}\right)}\end{aligned}$$

$$\begin{aligned} &= \frac{2\pi}{n}\frac{e^{i\pi/2}}{e^{i\pi\frac{n-1}{n}}}\\ &= \frac{2\pi}{n}e^{i\frac{\pi}{n}-\frac{\pi}{2}}. \end{aligned}$$

We now observe that the original integral expression equals

$$\begin{aligned} \oint_{\gamma_R} f(z)\,dz &= \int_0^R \frac{1}{1+x^n}\,dx + \int_0^{\frac{2\pi}{n}} \frac{1}{1+R^ne^{in\theta}}iRe^{i\theta}\,d\theta + e^{i(2\pi/n)}\int_R^0 \frac{1}{1+x^n}\,dx\\ &= \left(1-e^{i(2\pi/n)}\right)I + \int_0^{2\pi/n}\frac{1}{1+R^ne^{in\theta}}iRe^{i\theta}\,d\theta. \end{aligned}$$

Estimating the second integral, we get for $R > 1$,

$$\left|\int_0^R \frac{iRe^{i\theta}}{1+R^ne^{in\theta}}\,d\theta\right| \leqslant \frac{\frac{2\pi}{n}R}{R^n-1},$$

so that the integral tends to 0 as $R \to \infty$. Thus,

$$\begin{aligned} 2\pi i\operatorname{Res}\left(f; e^{i\pi/n}\right) &= \lim_{R\to\infty}\oint_{\gamma_R} f(z)\,dz\\ &= \left(1-e^{i(2\pi/n)}\right)I, \end{aligned}$$

whence

$$\begin{aligned} I &= \frac{2\pi}{n}\frac{e^{i\left(\frac{\pi}{n}-\frac{\pi}{2}\right)}}{1-e^{i(2\pi/n)}}\\ &= \frac{2\pi}{n}\frac{e^{-i\frac{\pi}{2}}}{e^{-i\pi/n}-e^{i\pi/n}}\\ &= \frac{2\pi}{n}\frac{-i}{-2i\sin\left(\frac{\pi}{n}\right)}\\ &= \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}. \end{aligned}$$

Old Exams

August 2019

Problem (Problem 1)**:** Let ξ be a nonnegative real number. Compute

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x^2+1}\,dx.$$

Solution: We consider

$$\begin{aligned} f(z) &= \frac{e^{i\xi z}}{z^2+1}\\ \int_{-\infty}^{\infty}\frac{e^{i\xi x}}{x^2+1}\,dx &= \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i\xi x}}{x^2+1}\,dx. \end{aligned}$$

We consider the contour γ_R given closing with the semicircle of radius R in the upper half-plane, parametrized by $\left\{Re^{i\theta} \mid 0 \leqslant \theta \leqslant \pi\right\}$. Then,

$$\begin{aligned}\oint_{\gamma_R} f(z)\,dz &= 2\pi i \operatorname{Res}(f;i)\\ &= \int_{-R}^{R} \frac{e^{i\xi x}}{x^2+1}\,dx + \int_0^{\pi} \frac{e^{i\xi Re^{i\theta}}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta.\end{aligned}$$

On the circular integral, we observe that for $R > 1$,

$$\begin{aligned}\left|\int_0^{\pi} \frac{iRe^{i\theta}e^{i\xi R(\cos(\theta)+i\sin(\theta))}}{R^2e^{2i\theta}+1}\,dz\right| &\leqslant \pi\frac{Re^{-R\xi\sin(\theta)}}{R^2-1}\\ &\to 0\end{aligned}$$

as $R \to \infty$. Therefore, since the pole at i is simple, we get

$$\begin{aligned}2\pi i \operatorname{Res}(f;i) &= 2\pi i\left(\lim_{z\to i} \frac{(z-i)e^{i\xi x}}{(z-i)(z+i)}\right)\\ &= \pi e^{-\xi},\end{aligned}$$

whence

$$\begin{aligned}\pi e^{-\xi} &= \lim_{R\to\infty} \oint_{\gamma_R} f(z)\,dz\\ &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\xi x}}{x^2+1}\,dx\\ &= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx.\end{aligned}$$

Problem (Problem 2)**:** Let f be an entire function, and suppose there is some $\alpha \in (0,\infty)$ such that

$$|f(z)| \leqslant C|z|^{\alpha}$$

for all $z \geqslant 1$. Show that f is a polynomial.

Solution: From the Archimedean property, we know that there is some natural number N such that $N > \alpha$. We observe then that, from Cauchy's estimates,

$$\begin{aligned}\left|f^{(N)}(z)\right| &\leqslant \frac{N!}{r^N}\sup_{|z|=r}|f(z)|\\ &\leqslant \frac{N!}{r^N}\sup_{|z|=r} C|z|^{\alpha}\\ &= \frac{CN!}{r^{N-\alpha}}\\ &\to 0\end{aligned}$$

as $r \to \infty$, whence the Taylor expansion for f about 0 terminates at some N. In particular, this means that f is a polynomial.

Problem (Problem 3)**:** Let f be an entire function. Suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

Solution: Consider the transformation $z \mapsto 1/z$, giving

$$\lim_{z\to 0} f(1/z) = \lim_{z\to\infty} f(z)$$

$$= \infty.$$

In particular, from the classification of singularities, this means that $f(1/z)$ has a pole at 0. This gives some n such that

$$f(1/z) = \sum_{k=0}^{n} a_k z^{-k},$$

whence

$$f(z) = \sum_{k=0}^{n} a_k z^{k},$$

so f is a polynomial.

Problem (Problem 4)**:** Let $\Omega = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. Suppose $f\colon \overline{\Omega} \to \mathbb{C}$ be continuous with $f|_{\Omega}$ holomorphic. Suppose $|f(iy)| \leqslant 1$ for all $y \in \mathbb{R}$ and $|f(z)| \leqslant 2$ for all $z \in \Omega$. Show that in fact $|f(z)| \leqslant 1$ for all $z \in \Omega$.

Solution: Consider the function

$$f_{\varepsilon}(z) = \frac{f(z)}{1 + \varepsilon z}.$$

We observe that

$$\begin{aligned} |f_{\varepsilon}(z)| &= \frac{|f(z)|}{|1 + \varepsilon z|} \\ &\leqslant \frac{2}{|1 + \varepsilon z|} \\ &\leqslant \frac{2}{\varepsilon |\operatorname{Im}(z)|}. \end{aligned}$$

Now, we observe that for z in the rectangle with corners $i2/\varepsilon$, $-i2/\varepsilon$, $1 + i2/\varepsilon$, and $1 - i2/\varepsilon$, that

$$|f_{\varepsilon}(z)| \leqslant 1$$

for all z on this rectangle, so by the maximum modulus principle, the inequality holds on the interior of the rectangle, and

$$|f_{\varepsilon}(z)| \leqslant 1$$

for all z in Ω outside this this rectangle, so that

$$\begin{aligned} |f(z)| &= \lim_{\varepsilon \to 0} |f(z)| \\ &\leqslant 1 \end{aligned}$$

for all $z \in \Omega$.

Problem (Problem 5)**:** Let $\mathbb{D} = \{z \mid |z| < 1\}$. Let $\mathcal{F}$ be a family of holomorphic functions on $\mathbb{D}$, and that $\sup_{f \in \mathcal{F}} |f(0)| < \infty$. Show that $\mathcal{F}$ is normal if and only if $\{f' \mid f \in \mathcal{F}\}$ is normal.

Solution: Call the family $\mathcal{G} = \{f' \mid f \in \mathcal{F}\}$. First, we observe that if $(f_n)_n \subseteq \mathcal{F}$ is a sequence with convergent subsequence $(f_{n_k})_k \to f\colon \mathbb{D} \to \mathbb{C}$ uniformly on compact sets, then it has been well-established that $\left(f'_{n_k}\right)_k \to f'$ uniformly on compact sets, whence $(f'_n)_n \subseteq \mathcal{G}$ admits a convergent subsequence.

Now, let $(f_n)_n \subseteq \mathcal{F}$, so that $(f'_n)_n \subseteq \mathcal{G}$. Then, $(f'_n)_n$ admits a subsequence $\left(f'_{n_k}\right)_k \to g\colon \mathbb{D} \to \mathbb{C}$.

First, we observe that since $\mathbb{D}$ is simply connected, g admits an antiderivative $f\colon \mathbb{D} \to \mathbb{C}$. We will show that $(f_{n_k})_k \to f$ uniformly on compacts.

Let $K \subseteq \mathbb{D}$ be compact, and let $z \in K$. Let $(K_m)_m$ be an exhaustion of $\mathbb{D}$ by closed balls of radius $\frac{m}{m+1}$. Then, there is some M such that $K \subseteq K_M^\circ$. We observe that the path $\gamma\colon [0,1] \to \mathbb{D}$ given by $\gamma(t) = tz$ is then contained wholly in K_M. Furthermore, we have

$$\begin{aligned} |f_{n_k}(z) - f(z)| &\leqslant \left|\int_0^1 z\left(f'_{n_k}(tz) - g(tz)\right)\,dt\right| \\ &\leqslant |z| \sup_{t\in[0,1]} \left|f'_{n_k}(tz) - g(tz)\right| \\ &\leqslant \sup_{z\in K_M} \left|f'_{n_k}(z) - g(z)\right| \\ &\to 0 \end{aligned}$$

whence

$$\begin{aligned} \sup_{z\in K}|f_{n_k}(z) - f(z)| &\leqslant \sup_{z\in K_M} |f_{n_k}(z) - f(z)| \\ &\leqslant \sup_{z\in K_M} \left|f'_{n_k}(z) - g(z)\right| \\ &\to 0 \end{aligned}$$

so that $(f_{n_k})_k \to f$ uniformly on K. Thus, $\mathcal{F}$ is normal.

Notation

- $U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leqslant r\}$
- $S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$
- $\dot{U}(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$
- $A(z_0, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$