We let $\mathbb{C}^n$ be defined as follows:

$$\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \mid z_j \in \mathbb{C} \right\}$$

with vector addition and scalar multiplication. $\mathbb{C}^n$ is a **vector space**, in which there is an inner product, defined as follows:

$$\left\langle \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = \sum z_i \overline{w}_i$$

With this definition, we are able to have a **norm**, defined as follows:

$$\|v\|_2 = \langle v, v \rangle^{1/2}$$

with the given norm properties:

- $\|v + w\| \leq \|v\| + \|w\|$
- $\|\alpha v\| = |\alpha| \|v\|$
- $\|v\| = 0 \Rightarrow v = \vec{0}$

We also have the **Cauchy-Schwarz** inequality

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

With these defined, we have

$$(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$$

denoted ℓ_2^n.

With this settled, we will look at

$$\mathbb{M}_n(\mathbb{C}) = \{(a_{ij}) \mid a_{ij} \in \mathbb{C}\}$$

$\mathbb{M}_n(\mathbb{C})$ also has

- Matrix Addition
- Matrix Multiplication
- Scalar Multiplication

Remember that $AB \neq BA$, matrices in $\mathbb{C}$ are still non-commutative. There is also the **adjoint** operation, which occurs when you take transpose and complex conjugate.

$$\begin{aligned} A &= (a_{ij}) \\ A^* &= (\overline{a}_{ji})_{ij} \end{aligned}$$

The adjoint operation has the following properties:

- $(A + B)^* = A^* + B^*$
- $(\alpha A)^* = \alpha A^*$
- $(AB)^* = B^* A^*$
- $A^{**} = A$
- $\langle Av, w \rangle = \langle v, A^* w \rangle$

With these properties, $\mathbb{M}_n(\mathbb{C})$ is a ***-algebra**.

A matrix is a linear transformation:

$$\begin{aligned} T_A : \ell_2^n \to \ell_2^n \\ T_A(v) = Av \end{aligned}$$

for some matrix A.

Given $A \in \mathbb{M}_n$, we have $\|A\|_{\text{op}}$, for the **operator norm**, defined as:

$$\|A\|_{\text{op}} = \max \left\{ \|Av\|_2 \mid \|v\|_2 \leq 1 \right\}$$

Exercise: Show that $\|A\|_{\text{op}} \leq \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$

Properties of $\|\cdot\|_{\text{op}}$:

- $\|A + B\| \leq \|A\| + \|B\|$
- $\|\alpha A\| = |\alpha|\|A\|$
- $\|A\| = 0 \Rightarrow A = \mathbf{0}$
- $\|AB\| \leq \|A\|\|B\|$
- $\|I_n\| = 1$
- $\|A^*\| = \|A\|$
- $\|A^*A\| = \|A\|^2$ (known as the c^*-property)
- $\|A\| = \max |\langle Av, v\rangle|, \|v\| \leq 1$
- $\|Av\| \leq \|A\| \cdot \|v\|$
- A is **normal** if $AA^* = A^*A$.
- A is **self-adjoint** if $A = A^*$.
- A is **positive** if $\langle Av, v\rangle \geq 0$
- A is a **projection** if $A^2 = A^* = A$.
- A is an **isometry** if $A^*A = I$.
- A is a **unitary** if $A^*A = I$ and $AA^* = I$.
- A is a **contraction** if $\|A\|_{\text{op}} \leq 1$.

Why use the word "isometry?"

$$\begin{aligned}
\|Av\|^2 &= \langle Av, Av\rangle \\
&= \langle v, A^*Av\rangle \\
&= \langle v, Iv\rangle \\
&= \langle v, v\rangle \\
&= \|v\|^2
\end{aligned}$$

So,

$$\|Av\| = \|v\|$$

Spectral Theorem: $A \in \mathbb{M}_n(\mathbb{C})$ normal is always *diagonalizable* via a unitary (i.e., $\exists$ unitary matrix U with $U^*AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$), where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues (or the *point spectrum* $\sigma_p(A)$)

Therefore, $A = UDU^*$.

- $A^2 = (UDU^*)(UDU^*) = UD^2U^* = U\operatorname{diag}(\lambda_1^2, \ldots, \lambda_n^2)U^*$
- $A^m = U\operatorname{diag}(\lambda_1^m, \ldots, \lambda_n^m)U^*$
- $p(A) = U\operatorname{diag}(p(\lambda_1), \ldots, p(\lambda_n))U^*$ for any polynomial p

We have $f : \sigma_p(A) \to \mathbb{C}$, $f(A) \mapsto U\operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n))U^*$.

Von Neumann's Inequality Given $A \in \mathbb{M}_n$, $\|A\| \leq 1$, then $\|p(A)\|_{\text{op}} \leq \max_{z \leq 1} |p(z)|$, where p is any polynomial.