Background: Asymptotic Freeness and Large Deviations

We start by recalling the basic asymptotic freeness result discussed in class.

Proposition: Let $\left(A_1^N, \dots, A_r^N\right)$ be an independent r-tuple of GUE $N \times N$ matrices. Then, the family $A_1^N, \dots, A_r^N$ converge in distribution to r independent semicircular elements, $s_1, \dots, s_r \in B(\mathcal{F}(\mathbb{C}^r))$, in the sense that for all $m \geq 1$ and all $1 \leq i_1, \dots, i_m \leq r$, we have

$$\lim_{N \to \infty} E\left[\operatorname{tr}\left(A_{i_1}^N \cdots A_{i_m}^N\right)\right] = \varphi(s_{i_1} \cdots s_{i_m}),$$

where φ is the vacuum state, $\varphi(T) = \langle T\Omega, \Omega\rangle$.

In fact, this collection is *almost surely* asymptotically free, in the following sense. Suppose we have two random matrices A^N and B^N defined on probability spaces (X_N, μ_N). Define

$$X \coloneqq \prod_{N \in \mathbb{N}} X_N$$
$$\mu \coloneqq \prod_{N \in \mathbb{N}} \mu_N,$$

where the latter is the product measure on X. The matrices A^N and B^N are said to be almost surely asymptotically free if there exists a noncommutative probability space (A, φ) and $a, b \in A$, and for almost all $x = (x_N)_N \in X$, we have $A^N(x_N), B^N(x_N) \in (\mathbb{M}_N, \operatorname{tr})$ converge in distribution to a, b.

Now, from here, we may ask a seemingly simple question: as N grows large, how likely are we to encounter other distributions? To make this sense more precise, we consider a random $N \times N$ self-adjoint matrix A, and let

$$\mu_A = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_1}$$

be its empirical spectral distribution. This is a random probability measure on $\mathbb{R}$, and as $N \to \infty$, the semicircle law gives that μ_A converges weakly to the semicircle distribution; this can be strengthened to almost sure convergence by an application of the argument for asymptotic freeness. The question then becomes, how quickly does the deviation between μ_A and any other probability distribution ν decrease as N increases? This is where the theory of large deviations starts to take shape.

Much of this exposition related to the classical notions of entropy will be centered around results in [MS17, Ch. 7].

One-Dimensional Free Entropy

Microstates Free Entropy

Applications: Structural Properties of Free Group Factors

References

[MS17] James A. Mingo and Roland Speicher. *Free Probability and Random Matrices*. Vol. 35. Fields Institute Monographs. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017, pp. xiv+336. ISBN: 978-1-4939-6941-8; 978-1-4939-6942-5. DOI: 10.1007/978-1-4939-6942-5. URL: https://doi.org/10.1007/978-1-4939-6942-5.