Problem 1

Problem: Determine whether each of the following statements is true or false. Prove your answers.

(a) If A is a limit ordinal, then $A + B$ is a limit ordinal.

(b) If B is a limit ordinal, then $A + B$ is a limit ordinal.

(c) If $A + B$ is a limit ordinal, then A is a limit ordinal.

(d) If $A + B$ is a limit ordinal, then B is a limit ordinal.

Solution:

(a) False — the ordinal $\omega + 1$ is a successor ordinal to ω, but ω is a limit ordinal.

(b) True — we consider $A + B \cong \{0\} \times A \cup \{1\} \times B = S$ with the lexicographical ordering. By a previous result, we know that B is a limit ordinal if and only if B has no maximal element. By the lexicographical ordering, we know that for all $x \in \{0\} \times A$ and $y \in \{1\} \times B$, $x < y$.

Thus, since $\{1\} \times B \cong B$, we know that $\{1\} \times B$ has no maximal element. $(*)$

Let $t \in S$. If $t \in \{0\} \times A$, then we know that $0 \in B$, so $t < (1, 0)$. If $t \in \{1\} \times B$, then by $(*)$, there is $t' \in \{1\} \times B$ with $t < t'$, so $t' \in S$ and $t < t'$. Thus, t is not a maximal element.

Thus, $A + B$ has no maximal element, so $A + B$ is a limit ordinal.

(c) False — the limit ordinal ω is equal to $2 + \omega$, but 2 is not a limit ordinal.

(d) True — by similar reasoning to (b), we see that there is no maximal element in $A + B$, and by the lexicographical ordering, this means there is no maximal element in $\{1\} \times B$, so there is no maximal element in B. Thus, B is a limit ordinal.

Problem 2

Problem: Let A, B, and C be nonzero ordinals. Determine whether each of the following is true or false. Prove your answers.

(a) $A < A + B$;

(b) $B < A + B$;

(c) if $A < B$, then $A + C < B + C$;

(d) if $A < B$, then $C + A < C + B$.

Solution:

(a) We know $A \cong \{0\} \times A$ are order isomorphic, and $\{0\} \times A \subsetneq \{0\} \times A \cup \{1\} \times B \cong A + B$. We wish to show that $\{0\} \times A$ is an "initial segment" of $\{0\} \times A \cup \{1\} \times B$.

Definition. Let S be a totally ordered set, $x \in S$. We define the initial segment S_x to be

$$S_x = \{y \in S \mid y \leqslant x\}.$$

We say S_x is the initial segment of S less than or equal to x.

For any $t \in \{0\} \times A$ and $s \in \{0\} \times A \cup \{1\} \times B$, it is either the case that $s \in \{1\} \times \{B\}$, in which case $t < s$, or $s \in \{0\} \times A$, in which case there exists t_s such that $t_s = s$. In particular, this means $\{0\} \times A$ is an initial segment of $\{0\} \times A \cup \{1\} \times B$.

Since $\{0\} \times A$ is an initial segment of $\{0\} \times A \cup \{1\} \times B$, and we know that $\{0\} \times A \subset \{0\} \times A \cup \{1\} \times B$, and $\{0\} \times A \cong A$ is well-ordered, $\in$-transitive subset of $\{0\} \times A \cup \{1\} \times B$, it is the case that $A < A + B$.

(b) Since $\omega \not< 2 + \omega$, this is false.

(c) If $A = 1$ and $B = 2$, then $1 < 2$, but $1 + \omega = \omega \not< 2 + \omega = \omega$.

(d) We know that $\{0\} \times C \cup \{1\} \times A \subsetneq \{0\} \times C \cup \{1\} \times B$. We want to show there exists a sequence

$$C + A \underset{\cong}{\xrightarrow{f}} \{0\} \times C \cup \{1\} \times A = \text{initial segment of } \{0\} \times C \cup \{1\} \times B \underset{\cong}{\xrightarrow{g}} C + B.$$

Problem 3

Problem: Prove that for all ordinals A, B, C, if $C + A = C + B$, then $A = B$.

Solution: Let A, B, C be ordinals, and let $C + A = C + B$. By trichotomy, we have either $A < B$, $A = B$, or $A > B$. We have $C + A < C + B$ (as established earlier) if $A < B$, and $C + B < C + A$ if $B < A$. Thus, since $C + A = C + B$, we must have $A = B$.

Question: True or false? If α and β are ordinals, and if $f : \alpha \hookrightarrow \beta$ is injective and preserves order, then $\alpha \leqslant \beta$.

Problem 4

Problem: Prove that for every infinite ordinal A, there exists a limit ordinal B and a natural number n such that $A = B + n$.

Solution: For infinite ordinals, the principle of induction says that $P(\alpha)$ holds if $P(\omega)$ holds and, we show that if $P(k)$ holds for all $k < \alpha$, then $P(\alpha)$ holds. We will use strong induction to prove this.

The induction hypothesis states that if $B < A$ and B is infinite, then $B = C + n$ for some limit ordinal C and natural number n.

If $A = \omega$, then $A = A + 0$.

If A is a limit ordinal, then $A = A + 0$.

If A is a successor ordinal, then there exists α such that $A = \alpha \cup \{\alpha\}$ for some ordinal α, meaning $A = \alpha + 1$. Since α is an infinite ordinal and $\alpha < A$, $\alpha = C + n$ for some limit ordinal C and natural number n. Thus, $A = \alpha + 1 = (C + n) + 1 = C + (n + 1)$.