Homeomorphism of the Open-Ball Topology on $\mathbb{R}^2$ and the Rectangle Topology on $\mathbb{R}^2$

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Abstract

In this paper, I introduce the concept of metric spaces (including the definition of distance metrics, open sets, and continuity), topologies on sets, and homeomorphisms. With this knowledge, we can prove that two particular topologies on $\mathbb{R}^2$ are homeomorphic.

Metric on $\mathbb{R}$ and $\mathbb{R}^2$

Consider a set X, with $a, b \in X$. It's often useful to consider the idea of a distance between a and b, $d(a, b)$. This distance function must map every pair of points to some positive number.

$$d : X \times X \to \mathbb{R}^+$$

Additionally, it has to have the following properties:

- Zero distance property: $d(a, a) = 0$ and if $d(a, b) = 0$, then $a = b$.
- Commutativity: $d(a, b) = d(b, a)$.
- Triangle Inequality: $d(x, z) = d(x, y) + d(y, z)$.

This defines a **distance metric** on X. For instance, for $x, y \in \mathbb{R}^n$, we define the **Euclidean metric** as follows:

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

In $\mathbb{R}^1 = \mathbb{R}$, this is equivalent to $|x - y|$.

Open Balls and Open Sets

In $\mathbb{R}^n$ with the Euclidean metric, we define the **open ball** of radius $\varepsilon > 0$ about the point x as follows:

$$V_\varepsilon = \{p \mid d(x, p) < \varepsilon\}$$

In $\mathbb{R}$, this equivalent to the open interval $(x - \varepsilon, x + \varepsilon)$.

In $A \subseteq X$, if $\forall x \in A$, $\exists \varepsilon > 0$ such that $V_\varepsilon(x) \subseteq A$, then A is an **open set**. If $A = \emptyset$ or $A = X$, then A is open.

Suppose A and B are open sets in a metric space X. We claim that $A \cap B$ and $A \cup B$ are open sets in X. Let $a \in A$ and $b \in B$.

Then, $\exists \varepsilon_a > 0$ such that $V_{\varepsilon_a}(a) \subseteq A$, meaning that $V_{\varepsilon_a}(a) \subseteq A \cup B$, and similarly for b and ε_b.

If $A = \emptyset$, then $A \cap B = \emptyset$, meaning $A \cap B$ is open. Otherwise, suppose $A \cap B \neq \emptyset$. Let $x \in A \cap B$, meaning $x \in A$ and $x \in B$. So, $\exists \varepsilon_1 > 0$ such that $V_{\varepsilon_1}(x) \in A$.

Specifically, define ε_1 to be the maximum such value. Similarly, $\exists \varepsilon_2 > 0$ such that $V_{\varepsilon_2}(x) \in B$, where ε_2 is the maximum such value. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ Then, $V_\varepsilon \subseteq V_{\varepsilon_1}$ and $V_\varepsilon \subseteq V_{\varepsilon_2}$, meaning $V_\varepsilon \subseteq A$ and $V_\varepsilon \subseteq B$, so $V_\varepsilon \subseteq A \cap B$.

Continuity

We can consider functions between metric spaces. For $f : X \to Y$ where X and Y are metric spaces.If f is continuous, then $\forall \varepsilon > 0, \exists \delta > 0$ such that for $x_1, x_2 \in X$,

$$d(f(x_1), f(x_2)) < \varepsilon \Rightarrow d(x_1, x_2) < \delta$$

. Alternatively, $\forall \varepsilon > 0$, if $f(x_2) \in V_\varepsilon(x_1)$, then $\exists \delta > 0$ such that $x_2 \in V_\delta(x_1)$.

We claim that $f : X \to Y$ is a continuous function if and only if the preimage of every open set in Y is open in X.

($\Rightarrow$) Let $f : X \to Y$ be continuous, and let $B \subseteq Y$. We claim that $f^{-1}(B) = \{x \mid f(x) \in B\}$ is open. Let $f(b) \in B$. Then, since B is open, $\exists \varepsilon > 0$ such that $V_\varepsilon(f(b)) \subseteq B$. Since f is continuous, $\forall f(x) \in V_\varepsilon(f(b))$, $\exists \delta$ such that $x \in V_\delta(b)$.