

### Problem 1

Let  $X$  be a metric space and consider a subset  $Y \subseteq X$  viewed as a metric space. Show that  $C \subseteq Y$  is connected in  $Y$  if and only if it is connected as a subset of  $X$ .

**Proof:**  $C \subseteq Y$  is connected if and only if any splitting  $C \subseteq (Y \cap U) \sqcup (Y \cap V)$  in  $Y$  is trivial, for  $U, V \subseteq X$  open. Thus,  $C \subseteq Y \cap (U \sqcup V)$  is a trivial splitting, if and only if  $C \subseteq U \sqcup V$  is trivial.

### Problem 2

If  $X$  is a metric space, and  $Y \subseteq X$  is a connected subset of  $X$ , show that for every splitting  $X = X_1 \sqcup X_2$ ,  $X_i \subseteq X$  open, we must have  $Y \subseteq X_1$  or  $Y \subseteq X_2$ .

**Proof:** Let  $Y \subseteq X$  be connected. Then, for any splitting  $Y \subseteq X_1 \cup X_2$ , with  $X_1, X_2 \subseteq X$  open, it is the case that  $Y \cap X_1 \cap X_2 = \emptyset$ .

Since the splitting is trivial, it is the case that either  $Y \cap X_1 = \emptyset$  or  $Y \cap X_2 = \emptyset$ .

We also have that  $Y \cap (X_1 \cup X_2) = (Y \cap X_1) \cup (Y \cap X_2) = Y$ . Therefore, it must be the case that  $Y \cap X_1 = Y$  or  $Y \cap X_2 = Y$ , so  $Y \subseteq X_1$  or  $Y \subseteq X_2$ .

### Problem 3

For  $n = 0, 1, 2, 3, \dots$ , let  $X_n := [0, 1] \times \{2^{-n}\}$ , and consider the space

$$X = \{(0, 0), (1, 0)\} \cup \left( \bigcup_{n=1}^{\infty} X_n \right).$$

(i) List all the connected components of  $X$ .

(ii) If  $X = U \sqcup V$  is a nontrivial splitting of  $X$ , show that there is a finite subset  $F \subseteq \mathbb{N}$  with

$$U = \bigcup_{n \in F} X_n, \quad V = X \setminus U.$$

**Proof:**

- (i) Each of the  $X_n$  are connected components, as each  $X_n$  is connected, closed, and open in  $X$  (by selecting an open subsets of  $\mathbb{R}^2$  that splits two  $X_n$  segments). Therefore, we must have  $\{(0, 0)\}, \{(1, 0)\}, \{X_n\}_{n \geq 1}$  are the connected components of  $X$ .
- (ii) Let  $U' \sqcup V'$  be open sets in  $\mathbb{R}^2$  with  $U = X \cap U'$ ,  $V = X \cap V'$ , with  $U', V'$  non-empty and  $U' \cap V' = \emptyset$ . It must be the case that for any  $n$ ,  $X_n$  is wholly contained in either  $U'$  or  $V'$  — otherwise, we would have a non-trivial splitting for a connected component, which is a contradiction.

I don't know how to proceed from here.

### Problem 4

Show that the  $n$ -sphere,  $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\|_2 = 1\}$  is path-connected.

**Proof:** Let  $x, y \in S^{n-1}$ . Then,  $\|x\|_2 = \|y\|_2 = 1$ . Let  $\gamma : [0, 1] \rightarrow S^{n-1}$  be defined by  $\gamma(0) = x$ ,  $\gamma(1) = y$ , and  $\gamma(t) = \frac{(1-t)x + ty}{\|(1-t)x + ty\|}$  (for  $(1-t)x + ty \neq 0$ ). Since convex combinations and norms are continuous,  $\gamma(t)$  is continuous and  $\|\gamma(t)\| = 1$  for all  $t$ , meaning every element of  $\gamma(t)$  is an element of  $S^{n-1}$ , so  $\gamma(t)$  is a path.

If  $x$  and  $y$  are antipodes, then there is some  $x^*$  in a  $\varepsilon$ -neighborhood of  $x$ , and a path from  $x^*$  to  $y$  found by the previous method, so by appending paths, we have a path from  $x$  to  $y$ .

### Problem 5

Let  $X$  be a metric space. We define a relation on  $X$ ,  $x \sim y$  if and only if there exists a path  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ . Show that this defines an equivalence relation on  $X$ . Equivalence classes are called path-connected components.

**Proof:** The relation is clearly reflexive.

For symmetry, if  $\gamma$  is a path from  $x$  to  $y$ , we define  $\gamma'$  as  $\gamma(1 - t)$ , which is a path from  $y$  to  $x$ .

If  $\gamma_1$  is a path from  $x$  to  $y$ , and  $\gamma_2$  is a path from  $y$  to  $z$ , we define  $\gamma : [0, 1] \rightarrow X$  as

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

This is a path from  $x$  to  $z$ , and thus the relation is transitive.

## Problem 6

Show that  $\mathbb{R}$  and  $\mathbb{R}^2$  are not homeomorphic.

**Proof:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  be a homeomorphism, meaning  $f$  is continuous.

Consider  $f(\mathbb{R} \setminus \{0\})$ . We have that  $\mathbb{R} \setminus \{0\} = f^{-1}(f(\mathbb{R} \setminus \{0\}))$  is disconnected. However,  $f(\mathbb{R} \setminus \{0\}) = f(\mathbb{R}) \setminus f(\{0\}) = \mathbb{R}^2 \setminus f(0)$ , but  $\mathbb{R}^2 \setminus f(0)$  is connected.  $\perp$

## Problem 7

Let  $V$  be a normed space and suppose  $Y \subseteq V$  is an open and connected subset. Fix a vector  $y_0 \in Y$ , and set

$$W := \{w \in Y \mid \text{there is a path from } y_0 \text{ to } w\}.$$

- (i) Show that  $W$  is open in  $Y$ .
- (ii) Show that  $W$  is closed in  $Y$ .
- (iii) Conclude that  $Y$  is path-connected.

**Proof:**

- (i) Let  $y \in W$ . Since  $Y$  is open,  $\exists \delta > 0$  with  $U(y, \delta) \subseteq Y$ . Letting  $w \in U(y, \delta)$ , we concatenate a path from  $y_0$  to  $y$ , then a path from  $y$  to  $w$  to find that  $w \in W$ . Thus,  $U(y, \delta) \subseteq W$ .
- (ii) For any  $w \in W$ , the sequence  $(w_n)_n = \frac{1}{n}y_0 + \left(1 - \frac{1}{n}\right)w$  converges to  $w$  — thus, any sequence in  $W$  converges to a point in  $W$ , as each of the  $w_n$  are path connected to  $y_0$  by restriction. Therefore,  $W$  is closed in  $Y$ .
- (iii) Since  $W$  is clopen in  $Y$ , and  $Y$  is connected, it must be the case that  $W = Y$ , so the set of all path-connected elements of  $Y$  is equal to the set of all elements of  $Y$ .

## Problem 8

A group is a nonempty set  $G$  with a binary operation  $G \times G \rightarrow G$ ,  $(s, t) \mapsto st$  satisfying

- $(st)r = s(tr)$ ;
- there is a unique neutral element  $e \in G$  with  $te = et$  for all  $t \in G$ ;
- for every  $t \in G$  there is a unique inverse  $t^{-1} \in G$  with  $t^{-1}t = tt^{-1} = e$ .

A subgroup of  $G$  is a nonempty subset  $H \subseteq G$  such that  $s, t \in H \Rightarrow st, t^{-1} \in H$ . The subgroup  $H$  is normal if  $t \in G, s \in H$  implies  $tst^{-1} \in H$ .

Consider a group  $G$  equipped with a metric so that the operations  $G \times G \rightarrow G$ ,  $(s, t) \mapsto st$  and  $G \rightarrow G$ ,  $t \mapsto t^{-1}$  are both continuous. Show that the connected component containing the neutral element  $e$ ,  $G_0$ , is a closed and normal subgroup of  $G$ .

**Proof:** Let  $s, t \in G_0$ . We have some connected set  $C$  such that  $s, e \in C$ , and some connected set  $D$  such that  $t, e \in D$ . Notice that  $CD$  is connected, since  $C \times D$  is connected and  $CD$  is  $C \times D$  under a continuous map; since  $st \in CD$  and  $e \in CD$ ,  $CD$  is a connected set containing  $st$  and  $e$ , so  $st \in G_0$ .

By a similar argument, we see that if  $s, e \in D$ , then  $D \rightarrow D^{-1}$  is a connected set containing  $s^{-1}$  and  $e$  (as  $e^{-1} = e$ ), meaning  $s^{-1}$  is in a connected set with  $e$ . Thus,  $G_0$  is a subgroup.

Let  $g \in G$ . For  $C$  a connected component with  $e \in C$ , we have that  $gCg^{-1}$  is connected (by composing continuous mappings of connected sets), and  $gCg^{-1}$  contains  $e$ , as  $geg^{-1} = e$ . Thus  $gCg^{-1} = C$ , so  $G_0$  is normal.

Let  $C$  be a connected component with  $e \in C$ . Since  $C$  is not contained in any other connected set (as otherwise, it would not be a connected component), it must be the case that  $C = \overline{C}$ , so  $C$  is closed.

## Problem 9

Show that the Cantor set is totally disconnected.

**Proof:** Let  $a, b \in \mathcal{C}$ . We will show that the only components in  $\mathcal{C}$  are singletons.

Suppose  $a \neq b$ . Then, since  $\mathbb{R}$  is Hausdorff, there exists  $\epsilon$  so small such that  $(a - \epsilon, a + \epsilon) \cap (b - \epsilon, b + \epsilon) = \emptyset$ . We find  $n$  large such that  $\frac{1}{3^n} < \epsilon$ . Then, it is the case that  $[a - 3^{-n}, a + 3^{-n}] \cap [b - 3^{-n}, b + 3^{-n}] = \emptyset$ .

Since  $a, b \in \mathcal{C}_n$  for all  $n$ , we have that for all  $m \geq n$ ,  $[a - 3^{-m}, a + 3^{-m}] \cap [b - 3^{-m}, b + 3^{-m}] = \emptyset$ .

Therefore,  $(-\infty, a + 3^{-m}) \cup (b - 3^{-m}, \infty) \cap \mathcal{C}$  is a non-trivial splitting.

Since  $\{a\}$  and  $\{b\}$  are connected sets, it is the case that the only connected sets in  $\mathcal{C}$  are singletons.

## Problem 10

A metric space  $X$  is called zero-dimensional if for any  $x, y \in X$  with  $x \neq y$ , there are open subsets  $U, V \subseteq X$  with  $x \in U, y \in V$  and  $X = U \sqcup V$ .

(i) Show that every zero-dimensional metric space is totally disconnected.

(ii) If  $Y \subseteq \mathbb{R}$  is totally disconnected, show that  $Y$  is zero-dimensional.

(iii) Conclude that  $\mathbb{Q}$  and the Cantor set are zero-dimensional.

**Proof:**

(i) We know that for any  $x \in X$ ,  $\{x\}$  is connected, and for any  $x, y \in X$  with  $x \neq y$ , there is a non-trivial splitting for any set containing  $x$  and  $y$ , meaning the only connected components in  $X$  are singletons.

(ii) Without loss of generality, let  $x, y \in Y$  with  $x < y$ . Then,  $\exists z \in X$  with  $z \notin Y$  and  $x < z < y$  (as  $\{x\}$  and  $\{y\}$  are the connected components generated by  $x$  and  $y$ ). Thus,  $U = (-\infty, z) \cap Y$  and  $V = (z, \infty) \cap Y$  are disjoint, with  $U \sqcup V = Y$ ,  $x \in U$  and  $y \in V$ . Therefore,  $Y$  is zero-dimensional.

(iii) Since  $\mathbb{Q}$  and the Cantor set are totally disconnected, both  $\mathbb{Q}$  and the Cantor set are zero-dimensional.