Part 1

1.9, Problem 22

(a)

$$\begin{aligned}
\frac{dy}{dt} - a(t)y &= b(t)\\
\mu(t)\frac{dy}{dt} - \mu(t)a(t)y &= \mu(t)b(t)\\
e^{-\int_0^t \alpha(\tau)d\tau}\frac{dy}{dt} - e^{-\int_0^t \alpha(\tau)d\tau} &= e^{-\int_0^t \alpha(\tau)d\tau}b(t)\\
\frac{d}{dt}\left(-\alpha(t)e^{-\int_0^t \alpha(\tau)d\tau}y\right) &= e^{-\int_0^t \alpha(\tau)d\tau}b(t).
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{\mu(t)}\right) &= \frac{d}{dt}\left(e^{\int_0^t \alpha(\tau)d\tau}\right)\\
&= \alpha(t)e^{\int_0^t \alpha(\tau)d\tau}\\
&= \alpha(t)\left(\frac{1}{\mu(t)}\right).
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{dy_p}{dt} &= \frac{d}{dt}\left(\frac{1}{\mu(t)}\right)\int_0^t \mu(\tau)b(\tau)\,d\tau + \frac{1}{\mu(t)}\frac{d}{dt}\left(\int_0^t \mu(\tau)b(\tau)\,d\tau\right)\\
&= \alpha(t)\left(\frac{1}{\mu(t)}\int_0^t \mu(\tau)b(\tau)\,d\tau\right) + b(t)\\
&= \alpha(t)y_p(t) + b(t).
\end{aligned}$$

(d) The general solution is, thus,

$$y(t) = \frac{1}{\mu(t)}\int_0^t \mu(\tau)b(\tau)\,d\tau + C\frac{1}{\mu(t)}.$$

(e) These results are very similar to the case of

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt,$$

but instead of the indefinite integral, we use the equivalent expression using the definite integral.

(f)

$$\begin{aligned}
\mu(t) &= e^{-\int_0^t a(\tau)d\tau}\\
&= e^{t^2}\\
y(t) &= e^{-t^2}\int_0^t e^{\tau^2}\left(4e^{-\tau^2}\right)\,d\tau + Ce^{-t^2}\\
&= 4te^{-t^2} + Ce^{-t^2}.
\end{aligned}$$

1.9, Problem 24

$$
\begin{aligned}
\frac{dS}{dt} &= 2 - \frac{S}{15+t} \\
\frac{dS}{dt} + \frac{S}{15+t} &= 2 \\
\frac{d}{dt}\left((15+t)S\right) &= 2\,(15+t) \\
S &= \frac{30t+t^2}{15+t} + \frac{C}{15+t} \\
S(0) &= \frac{C}{15} \\
&= 6 \\
C &= 90 \\
S &= \frac{30t+t^2}{15+t} + \frac{90}{15+t} \\
S(15) &= 25.5
\end{aligned}
$$