Problem (Problem 1)**:** Let G be a Lie group, which is a topological group that is also a smooth manifold and where all group operations are smooth. For convenience, we will always assume that G is path-connected. Prove that the tangent bundle TG of G is trivial — i.e., TG composes as a direct product.

Solution: From Cayley's Theorem, we know that G acts on itself transitively by left-multiplication. That is, for any $g \in G$, there is a map $L_g \colon G \to G$ that takes $h \mapsto gh$. This is a diffeomorphism of smooth manifolds since L_g is smooth and admits the smooth inverse $L_{g^{-1}}$. In particular, this means that

$$D_e(L_g) \colon T_e G \to T_g G$$

is invertible as a linear map. Letting $T_e G \cong \mathbb{R}^n$ have a local basis $\mathcal{B}_e = \left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$, we then observe that $D_e(L_g)$ then maps this basis to a basis for $T_g G$ since $D_e(L_g)$ is a linear isomorphism, meaning that

$$\begin{aligned} TG &= \bigsqcup_{g \in G} T_g G \\ &= \bigsqcup_{g \in G} D_e(L_g)(T_e G) \\ &\cong \bigsqcup_{g \in G} \mathbb{R}^n \\ &\cong G \times \mathbb{R}^n. \end{aligned}$$

Thus, TG is trivial.

Problem (Problem 2)**:** Note that a Lie group can act on itself by left or right multiplication. A vector field on G is called *left-invariant* if it is invariant under the differential of left multiplication L_g for every $g \in G$. Prove that $T_e G$ can be identified with left invariant vector fields on G.

Solution: We observe that by definition, a left-invariant vector field X is one where $g \cdot X = X$ for every $g \in G$. In particular, this means that for any vector field $X_e \in T_e G$, there is a corresponding left-invariant vector field on G defined at each $g \in G$ by taking $X_g = D_e(L_g)(X_e)$; that such a vector field is left-invariant follows from the fact that L_g is a diffeomorphism of G onto itself. Thus, we get the correspondence between vector fields at $T_e G$ and left-invariant vector fields on G.

Problem (Problem 3)**:** Similar to invariant vector fields, invariant forms are ones for which $L_g^* \omega = \omega$. Prove that invariant forms are stable under taking d and under contraction by a left-invariant vector field.

Solution: Let ω be left-invariant. Then, by definition of the pullback,

$$\begin{aligned} L_g^*(d\omega) &= d(L_g^* \omega) \\ &= d\omega. \end{aligned}$$

Similarly, by definition of the contraction, if $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$ are a k-dimensional collection of vector fields, then

$$\begin{aligned} L_g^*(\iota_X(\omega))\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) &= L_g^*\left(\omega\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)\right) \\ &= \left(L_g^* \omega\right)\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \\ &= \omega\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \\ &= \iota_X(\omega)\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right). \end{aligned}$$

Problem (Problem 4)**:** Similar to left-invariant forms are right-invariant forms. Prove that a connected compact Lie group admits a bi-invariant volume form.

Solution: Let G be a compact connected Lie group, and define

$$\begin{aligned} L\colon G \times G &\to G \\ R\colon G \times G &\to G \end{aligned}$$

by

$$\begin{aligned} L(g, x) &= gx \\ R(g, x) &= xg^{-1}. \end{aligned}$$

Letting $L_g := L(g, \cdot)$ and $R_g := R(g, \cdot)$, we observe that L and R define faithful, free actions of G by regular permutations, with R following from the fact that

$$\begin{aligned} R_g \circ R_h(x) &= R_g\left(xh^{-1}\right) \\ &= xh^{-1}g^{-1} \\ &= x(gh)^{-1} \\ &= R_{gh}(x). \end{aligned}$$

For any $g \in G$, we may then define a linear map $T_g\colon T_eG \to T_eG$ by taking $x \mapsto \left(D_gR_g\right) \circ \left(D_eL_g\right)(x)$. We observe that T_g transforms the left-invariant vector fields in T_eG by right-multiplication, since if $X \in T_eG$, then

$$\begin{aligned} T_g(X) &= D_gR_g\left(\left(D_eL_g\right)(Y)\right) \\ &= \left(D_gR_g\right)(Y). \end{aligned}$$

Now, let $\nu \in \Lambda^n T_e^*G$ be a non-zero max-dimensional form at T_eG, and let $X_1, \dots, X_n$ be a basis for T_eG. We claim that $\omega \in \Lambda^n T_e^*G$ defined by

$$\omega(X_1, \dots, X_n)(e) := \nu\left(D_eR_g(X_1), \dots, D_eR_g(X_n)\right)\left(g^{-1}\right)$$

is a right-invariant form. Towards this end, we observe that

$$\begin{aligned} R_h^*\omega(X_1, \dots, X_n)(e) &= R_h^*\nu\left(D_eR_g(X_1), \dots, D_eR_g(X_n)\right)\left(g^{-1}\right) \\ &= \nu\left(D_{g^{-1}}R_h \circ D_eR_g(X_1), \dots, D_{g^{-1}}R_h \circ D_eR_g(X_n)\right)\left(g^{-1}h^{-1}\right) \\ &= \nu\left(D_e\left(R_h \circ R_g\right)(X_1), \dots, D_e\left(R_h \circ R_g\right)(X_n)\right)\left((gh)^{-1}\right) \\ &= \omega(X_1, \dots, X_n)(e). \end{aligned}$$

Having constructed now a right-invariant form, we observe that $L_g^*(\omega)$ is a map from $\Lambda^n T_e^*G$ to $\Lambda^n T_e^*G$, so by the definition of the determinant, it follows that $L_g^*(\omega) = \det\left(L_g^*\right)\omega$. Since L_g is a diffeomorphism, it follows that $\det\left(L_g^*\right) = \pm 1$, but since the family of L_g also form a group, and $\det\colon \operatorname{end}(T_eG) \to \mathbb{R}$ is a group homomorphism, we must have that $\det\left(L_g^*\right) = 1$, so ω is in fact bi-invariant.

Problem (Problem 5)**:** Fix a bi-invariant volume form μ on a compact connected lie group G whose total mass is 1. For ω an arbitrary k-form on G, and k left-invariant vector fields $X_1, \dots, X_k$, show that

$$\rho(\omega)(X_1, \dots, X_k) = \int_G L_g^*\omega(X_1, \dots, X_k)\, \mu$$

furnishes a left-invariant form on G, and that $\rho(\omega) = \omega$ if ω was already left-invariant.

Solution: Let $p \in G$, and consider

$$\begin{aligned}
L_p^* \rho(\omega)(X_1, \dots, X_k)(q) &= \rho(\omega)\big(D_p L_p(X_1), \dots, D_p L_p(X_k)\big)(pq) \\
&= \int_G L_g^* \omega\big(D_p L_p(X_1)(pq), \dots, D_p L_p(X_k)(pq)\big)\ \mu \\
&= \int_G L_p^* L_g^* \omega(X_1, \dots, X_k)(q)\ mu
\end{aligned}$$

Using the change of coordinates by the diffeomorphism $L_{p^{-1}}$, we get

$$= \int_G L_{pp^{-1}g}(X_1, \dots, X_k)(q)\ L_{p^{-1}}^* \mu$$

and since μ is bi-invariant, using the fact that L is a group action, we get

$$\begin{aligned}
&= \int_G L_g^* \omega(X_1, \dots, X_k)(q)\ \mu \\
&= \rho(\omega)(X_1, \dots, X_k)(q).
\end{aligned}$$

Since q is arbitrary, we are done.

Problem (Problem 6)**:** Show that ρ and d commute.

Solution: We observe that

$$\begin{aligned}
d(\rho(\omega)) &= d\left(\int_G L_g^* \omega\ \mu\right) \\
&= \int_G dL_g^* \omega\ \mu \\
&= \int_G L_g^* (d\omega)\ \mu \\
&= \rho(d\omega).
\end{aligned}$$