

Basic Properties

Definition: A topological space M is called a *manifold* if it satisfies the following:

- M is Hausdorff (points can be separated by open sets);
- M is second countable (the basis for the topology of M is countable);
- M is locally Euclidean (every point in M has a neighborhood homeomorphic to \mathbb{R}^n for some n).

In particular, the third condition says that for every $p \in M$, there is $U \in \mathcal{O}_p$ and a homeomorphism $\varphi: U \rightarrow \mathbb{R}^n$. The value of n is called the *dimension* of the manifold M .

Definition: Let M be an n -manifold. A *chart* on M is a pair (U, φ) such that $U \subseteq M$ is open, $\varphi: U \rightarrow \mathbb{R}^n$ is a homeomorphism.

A family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is known as an *atlas* if

$$M = \bigcup_{i \in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point $p \in M$ and two charts (U, φ_U) and (V, φ_V) such that $p \in U$ and $p \in V$. The functions $\varphi_U: U \rightarrow \mathbb{R}^n$ and $\varphi_V: V \rightarrow \mathbb{R}^n$ are homeomorphisms, meaning that $\varphi_V \circ \varphi_U^{-1}: \varphi_U(U \cap V) \rightarrow \mathbb{R}^n$ defined on the (nonempty) $U \cap V$ is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs $\varphi_V \circ \varphi_U^{-1}$ are *diffeomorphisms*. To do this, we need to first develop the derivative in \mathbb{R}^n .

Definition: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function. We say f is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear map $L \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\frac{\|f(p+h) - f(p) - Lh\|}{\|h\|} \rightarrow 0$$

as $h \rightarrow 0$.

The *derivative* of f is the association $f \mapsto L$ for each $p \in \mathbb{R}^n$. We write $D_p f$ to denote this map. Note that we consider elements of $\text{Mat}_n(\mathbb{R})$ as points in \mathbb{R}^{n^2} with the standard topology on \mathbb{R}^{n^2} .

A function f is called a *diffeomorphism* if it is continuously differentiable and has a continuously differentiable inverse.

Definition: If (U, φ_U) and (V, φ_V) are charts such that $U \cap V \neq \emptyset$, the function $\varphi_V \circ \varphi_U^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is known as the *transition map* between φ_U and φ_V .

A smooth structure for M is an atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that for all i, j , the transition maps $\varphi_j \circ \varphi_i^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are diffeomorphisms where defined.

If $\{(U_i, \varphi_i)\}_{i \in I}$ is a *maximal* smooth atlas — i.e., any other smooth atlas that contains $\{(U_i, \varphi_i)\}_{i \in I}$ is equal to $\{(U_i, \varphi_i)\}_{i \in I}$ — then we call $\{(U_i, \varphi_i)\}_{i \in I}$ a *smooth structure* for M .

Note: From now on, we use “manifold” to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

Definition: A map $f: M \rightarrow N$ between manifolds is called *smooth* if for any chart (U, φ_U) in M and corresponding chart (V, φ_V) in N , the map $\varphi_V \circ f \circ \varphi_U^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is continuously differentiable.

The function f is a *diffeomorphism* if f is a smooth bijection with smooth inverse, and we say the manifolds M and N are diffeomorphic if they admit a diffeomorphism.

Examples

There are a couple special examples of (smooth) manifolds.

- (i) Open subsets of \mathbb{R}^n are always manifolds.
- (ii) The general linear group, $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices, viewed as a subset of $Mat_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, is a manifold. Furthermore, it is an open subset of \mathbb{R}^{n^2} , as considering the map $\det: Mat_n(\mathbb{R}) \rightarrow \mathbb{R}$ given by $A \mapsto \det(A)$, we see that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$.
- (iii) The special linear group, $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$, consisting of $n \times n$ matrices with determinant 1, is also a smooth manifold. Furthermore, this manifold is a closed subset of \mathbb{R}^{n^2} , as it is equal to $\det^{-1}(\{1\})$.
- (iv) The n -sphere, S^n , given by

$$S^n = \left\{ (x_0, \dots, x_n) \mid \sum_{i=0}^n x_i^2 = 1 \right\}$$

is a manifold in \mathbb{R}^n . That it is a smooth manifold is quite a bit less obvious.

Now, in low dimensions, we know that $S^2 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and that the continuously differentiable transformation $z \mapsto \frac{1}{z}$ takes the neighborhood basis of ∞ to deleted neighborhoods of 0, and takes the neighborhood basis of 0 to the neighborhood basis of ∞ . This is our desired smooth structure.

In the case of the general S^n , we use two stereographic projections to construct our smooth structure. The first stereographic projection is via the north pole, N_p , and maps points on $S^n \setminus \{N_p\}$ bijectively to \mathbb{R}^n ; this is a chart that is defined everywhere on S^n except N_p . Similarly, we may use a stereographic projection originating from the south pole, S_p , so as to create another chart defined everywhere except S_p . These two stereographic projections are our desired smooth structure, as these two charts are all that is necessary to cover S^n .

- (v) The real projective plane, consisting of lines through the origin in \mathbb{R}^{n+1} , can be expressed as

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \mathbb{R}^\times.$$

We will show that this is a manifold by constructing a family of charts mapping to \mathbb{R}^n .

Consider a point $(r_0, \dots, r_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. If $r_0 \neq 0$, then by dividing, we may associate this point's equivalence class in \mathbb{RP}^n to

$$(1, r_1/r_0, \dots, r_n/r_0) \in \{1\} \times \mathbb{R}^n,$$

so we may associate all points of the form $[(r_0, \dots, r_n)]$ with $r_0 \neq 0$ with a chart (U_0, φ_0) that maps \mathbb{RP}^n to \mathbb{R}^n .

Similarly, we may define U_k via

$$U_k = \{[(r_0, \dots, r_n)] \mid r_k \neq 0\}$$

with corresponding chart

$$\begin{aligned} \varphi_k: U_k &\rightarrow \mathbb{R}^n \\ [(r_0, \dots, r_n)] &\mapsto \frac{1}{r_k} (r_0, \dots, \widehat{r_k}, \dots, r_n), \end{aligned}$$

where $\widehat{r_k}$ denotes the exclusion of the r_k coordinate. Varying k from 0 to n , we see that

$$\mathbb{RP}^n = \bigcup_{k=0}^n U_k,$$

the chart functions $\varphi_k: U_k \rightarrow \mathbb{R}^n$ are homeomorphisms (as they are just division and projections). Furthermore, the transition maps $\varphi_j \circ \varphi_i^{-1}$ are coordinate-wise rational functions defined by

$$(u_1, \dots, u_n) \mapsto \left(\frac{u_1}{u_i}, \dots, \frac{1}{u_i}, \dots, \frac{u_n}{u_i} \right),$$

where the $\frac{1}{u_i}$ is at position j .

- (vi) We now turn to a very important example from algebraic geometry: the Grassmannian, $\text{Gr}(k, n)$, consisting of all the k -dimensional subspaces of \mathbb{R}^n .

This is a $k(n - k)$ -dimensional manifold; we need to understand what the smooth structure is. To do this, we let $\langle \cdot, \cdot \rangle$ be an inner product on \mathbb{R}^n , and for any $V \in \text{Gr}(k, n)$, we consider maps in $\text{Hom}(V, V^\perp)$, where V^\perp denotes the orthogonal complement of V .

Now, we see that if $W \in \text{Gr}(k, n)$ is any other k -dimensional subspace, the orthogonal $P_V: \mathbb{R}^n \rightarrow V$ restricted to W is a linear isomorphism if and only if $W \not\subseteq V^\perp$, or that $W \cap V^\perp = \{0\}$.

We see that if W is such that $P_V|_W: W \rightarrow V$ is a linear isomorphism, the inverse $(P_V|_W)^{-1}: V \rightarrow W$ is well-defined; so, we may make a correspondence between $\text{Hom}(V, V^\perp)$ and $\text{Gr}(k, n)$ by noting that any such $T \in \text{Hom}(V, V^\perp)$ has a corresponding graph $(v, T(v))$, so we take $v \mapsto P_V|_W^{-1}(v)$, then project onto V^\perp by taking $T(P_V|_W^{-1}(v)) = T(v)$. We depict it as a diagram below.

$$\begin{array}{ccccc} & & \text{id} & \xrightarrow{\quad} & V \\ & & \searrow & & \uparrow P_V \\ V & \xrightarrow{P_V|_W^{-1}} & W & \xhookrightarrow{\iota} & \mathbb{R}^n \\ & \searrow T & & & \downarrow P_{V^\perp} \\ & & & & V^\perp \end{array}$$

Defining $U_V = \{W \in \text{Gr}(k, n) \mid W \cap V^\perp = \{0\}\}$, we may define the chart from U_V onto $\text{Hom}(V, V^\perp)$ by $\varphi_V = P_{V^\perp} \circ P_V|_W^{-1}$. The family $\{(U_V, \varphi_V) \mid V \in \text{Gr}(k, n)\}$ is our smooth atlas.

Inverse and Implicit Function Theorems

In order to replace manifolds with linear maps, we need to understand smooth maps on \mathbb{R}^n . The most important theorems in this regard are the inverse function theorem and the implicit function theorem.

Theorem (Inverse Function Theorem): Let $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function. If $D_p f$ is invertible as a linear map, then f has a local, continuously differentiable inverse $f^{-1}: V \rightarrow W$, where $p \in W \subseteq U$ and $f(p) \in V \subseteq \mathbb{R}^n$.

The proof uses the contraction mapping theorem. Recall that if X is a complete metric space, and $f: X \rightarrow X$ is a strict uniform contraction — that is, there exists $0 \leq \lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$ — then f has a unique fixed point.

We begin with a technical lemma.

Lemma: If $U(0, r) \subseteq V$ for some $r > 0$ where V is a normed vector space, $g: V \rightarrow V$ is a uniform contraction, and $f = \text{id} + g$, then the following hold:

- $(1 - \lambda)\|x - y\| \leq \|f(x) - f(y)\|$ (in particular, f is injective);
- if $g(0) = 0$, then

$$U(0, (1 - \lambda)r) \subseteq f(U(0, r)) \subseteq U(0, (1 + \lambda)r).$$

Proof of Lemma. To see the first item, we notice that by the triangle inequality,

$$\begin{aligned} \|x - y\| - \|f(x) - f(y)\| &\leq \|x - y\| - \|x - y\| + \|g(x) - g(y)\| \\ &\leq \lambda\|x - y\|, \end{aligned}$$

so $(1 - \lambda)\|x - y\| \leq \|f(x) - f(y)\|$, and f is injective. Furthermore, we see that if $g(0) = 0$, then

$$\begin{aligned} f(U(0, r)) &= U(0, r) + g(U(0, r)) \\ &\subseteq U(0, r) + \lambda U(0, r) \\ &= U(0, (1 + \lambda)r). \end{aligned}$$

Finally, if $y \in U(0, (1 - \lambda)r)$, then we want to find x such that $y = f(x) = x + g(x)$; equivalently, we see that we want x such that $x = y - g(x)$. Since the function $F(x) = y - g(x)$ is a translation of a uniform contraction, $F(x)$ is a contraction, so there is a fixed point, meaning $y \in f(U(0, r))$. \square

Note: We will use $|\cdot|$ to denote the norm on \mathbb{R}^n .

Proof of the Inverse Function Theorem. By using a series of affine maps — first by translating p to 0, then translating $f(p)$ to 0, then inverting D_0f as per our assumption, we may safely assume that $p = f(p) = 0$ and $D_0f = \text{Id}$.

Set $g = f - \text{Id}$. We will show that g is a contraction in a sufficiently small ball. Fixing $x, y \in \mathbb{R}^n$, consider the map $\mathbb{R} \rightarrow \mathbb{R}^n$ given by $t \mapsto g(x + t(y - x))$. Notice that by the Fundamental Theorem of Calculus,

$$|g(y) - g(x)| \leq |y - x| \sup_{0 \leq t \leq 1} |g'(x + t(y - x))|.$$

Furthermore, since $g'(0) = 0$ by the fact that $D_0f = \text{Id}$ and $(\text{Id})' = \text{Id}$, and since f is continuously differentiable, there is $r > 0$ such that

$$|g(y) - g(x)| \leq \frac{1}{2}|y - x|$$

for all $x, y \in U(0, r)$. Thus, g is a strict contraction on $U(0, r)$. By the previous lemma, we see that

$$U(0, r/2) \subseteq f(U(0, r));$$

by setting $U = U(0, r) \cap f^{-1}(U(0, r))$, we see that the map $f|_U: U \rightarrow V := U(0, r/2)$ is a bijection. The inverse function $f^{-1}: V \rightarrow U$ thus exists.

Now, we let $h = f^{-1}$, $x \in U$, $y \in V$ such that $h(x) = y$, and $A = D_x f$. We will show that $A^{-1} = D_y h$, which is enough to show that h is continuously differentiable, as we assume the map $x \mapsto D_x f$ is continuous, and inversion is continuous in $\text{GL}_n(\mathbb{R})$.

For sufficiently small vectors s and k , since f and h are bijections, we have

$$h(y + k) = x + s,$$

so

$$f(x + s) = y + k.$$

Furthermore, by unraveling the definitions of $f = g + \text{Id}$, s , and k , and the fact that g is a uniform contraction on \mathcal{U} , we get

$$\begin{aligned} |s - k| &= |(f(x + s) - f(x)) - s| \\ &= |(x + s + g(x + s)) - (x + g(x)) - s| \\ &= |g(x + s) - g(x)| \\ &\leq \frac{|s|}{2}. \end{aligned}$$

In particular, since

$$\begin{aligned} |s| &\leq |s - k| + |k| \\ &\leq |k| + \frac{|s|}{2}, \end{aligned}$$

we see that $|s|/2 \leq |k|$. We calculate

$$\begin{aligned} |h(y + k) - h(y) - A^{-1}k| &= |x + s - x - A^{-1}(f(x + s) - f(x))| \\ &= |s - A^{-1}(f(x + s) - f(x))| \\ &\leq \|A^{-1}\|_{\text{op}} |As - f(x + s) - f(x)|. \end{aligned}$$

Thus, since $|s|/2 \leq |k|$,

$$\begin{aligned} \frac{|h(y + k) - h(y) - A^{-1}k|}{|k|} &\leq \frac{2\|A^{-1}\|_{\text{op}} |As - f(x + s) - f(x)|}{|s|} \\ &\rightarrow 0, \end{aligned}$$

so $D_y h = A^{-1}$. □