Textbook Problems

Solution (8.1, Problem 6)**:** We have

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + e^{-t}\begin{pmatrix} \sin(2t) \\ 4\cos(2t) \\ -1 \end{pmatrix}.$$

Solution (8.1, Problem 12)**:** We have

$$\begin{aligned} \frac{dx}{dt} &= 5e^t\cos(t) - 5e^t\sin(t) \\ &= -10e^t\cos(t) + \left(15e^t\cos(t) - 5e^t\sin(t)\right) \\ &= -2x + 5y \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= 3e^t\cos(t) - e^t\sin(t) - 3e^t\sin(t) - e^t\cos(t) \\ &= 2e^t\cos(t) - 4e^t\sin(t) \\ &= \left(-10e^t\cos(t)\right) + 12e^t\cos(t) - 4e^t\sin(t) \\ &= -2x + 4y. \end{aligned}$$

Solution (8.1, Problem 20)**:** Writing our Wronskian, and using the power of Mathematica, we have

$$\begin{aligned} W(t) &= \det\begin{pmatrix} 1 & e^{-4t} & 2e^{3t} \\ 6 & -2e^{-4t} & 3e^{3t} \\ -13 & -e^{-4t} & -2e^{3t} \end{pmatrix} \\ &= -4e^{-t}, \end{aligned}$$

which is zero nowhere along $(-\infty, \infty)$, so the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ form a fundamental set of solutions.

Solution (8.1, Problem 26)**:** We start with the homogeneous equation:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{x}.$$

Finding the eigenvalues and eigenvectors, we have

$$\begin{aligned} \det\begin{pmatrix} -1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} &= \lambda^2 - 2 \\ \lambda_1 &= \sqrt{2} \\ \lambda_2 &= -\sqrt{2}, \end{aligned}$$

with associated eigenvectors of

$$\begin{aligned} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \pm\sqrt{2}\begin{pmatrix} a \\ b \end{pmatrix} \\ \mathbf{v}_1 &= \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}. \end{aligned}$$

Therefore, we have the homogeneous solution of

$$\mathbf{x}_h(t) = c_1\begin{pmatrix} -1 \\ -1-\sqrt{2} \end{pmatrix}e^{\sqrt{2}t} + c_2\begin{pmatrix} -1 \\ -1+\sqrt{2} \end{pmatrix}e^{-\sqrt{2}t}.$$

Note that the Wronskian gives

$$\begin{aligned} W(t) &= \det\begin{pmatrix} e^{\sqrt{2}t} & e^{-\sqrt{2}t} \\ \left(-1-\sqrt{2}\right)e^{\sqrt{2}t} & \left(-1+\sqrt{2}\right)\left(e^{-\sqrt{2}t}\right) \end{pmatrix} \\ &= 2\sqrt{2}, \end{aligned}$$

meaning that these solutions are linearly independent. Next, we examine the particular solution.

$$\begin{aligned} \mathbf{x}_p(t) &= \begin{pmatrix} t^2-2t+1 \\ 4t \end{pmatrix} \\ \frac{d\mathbf{x}_p}{dt} &= \begin{pmatrix} 2t-2 \\ 4 \end{pmatrix}. \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} t^2-2t+1 \\ 4t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} &= \begin{pmatrix} -t^2-2t-1+t^2+4t-1 \\ -t^2+2t-1+4t-6t+5 \end{pmatrix} \\ &= \frac{d\mathbf{x}_p}{dt}. \end{aligned}$$

Thus, this is the general solution of our equation on $(-\infty, \infty)$.

Solution (8.2, Problem 12)**:** Computing the characteristic polynomial, we get

$$p(x) = (6-\lambda)(3-\lambda)(-5-\lambda),$$

giving

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -5 \\ \lambda_3 &= 6. \end{aligned}$$

Plugging these into

$$\begin{pmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda_i\begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

we get

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_3 &= \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}e^{-5t} + c_3\begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}e^{6t}.$$

Solution (8.2, Problem 16)**:**

Solution (8.2, Problem 24)**:**

Solution (8.2, Problem 28)**:**

Solution (8.2, Problem 40)**:**

Solution (8.2, Problem 42)**:**

Solution (8.2, Problem 46)**:**

Extra Problems