Abstract

Measures are just set functions that follow some particular basic properties, but we can expand them beyond the positive real numbers towards complex numbers; to conceptualize these signed and complex measures, we need to make use of results like the Lebesgue–Radon–Nikodym Theorem and the Hahn Decomposition Theorem that allow us to understand their structural properties.

Signed Measures and the Hahn Decomposition

We know that a measure is a set function $\mu\colon \mathcal{M} \to [0,\infty]$ on a σ-algebra such that

- $\mu(\emptyset) = 0$;
- for a family of disjoint sets $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$,

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

We may ask what happens if we change the codomain from $[0,\infty]$ to $\mathbb{R}$ or $\mathbb{C}$. This is where *signed measures* come in.

Definition: A *signed measure* μ is a real-valued countably additive set function such that $\mu(\emptyset) = 0$ and μ takes on at most one of $-\infty$ or ∞.

We begin by establishing some basic properties of signed measures (akin to the basic properties of measures).

Theorem: Let μ be a signed measure.

(a) If E and F are measurable sets with $E \subseteq F$ and $|\mu(F)| < \infty$, then $|\mu(E)| < \infty$.

(b) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ is a disjoint sequence of measurable subsets such that $\left|\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right)\right| < \infty$, then the series $\sum_{j=1}^{\infty} \mu(E_j)$ is absolutely convergent.

(c) If $\{E_j\}_{j=1}^{\infty}$ is a monotone sequence of measurable sets — and if decreasing, $|\mu(E_n)| < \infty$ for at least one such n — then

$$\mu\left(\lim_{j\to\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Proof.

(a) We see that $\mu(F) = \mu(F \setminus E) + \mu(E)$. If exactly one of the summands is infinite, then so is $\mu(F)$. If both are infinite, then since μ takes on at most one of $-\infty$ or ∞, they are equal and then $\mu(F)$ is infinite. Therefore, both summands must be finite.

(b) We set

$$E_j^+ = \begin{cases} E_j & \mu(E_j) \geq 0 \\ \emptyset & \mu(E_j) < 0 \end{cases}$$
$$E_j^- = \begin{cases} E_j & \mu(E_j) \leq 0 \\ \emptyset & \mu(E_j) > 0 \end{cases}.$$

Then,

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j^+\right) = \sum_{j=1}^{\infty} \mu(E_j^+)$$

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j^-\right) = \sum_{j=1}^{\infty} \mu(E_j^-).$$

Since the terms of both series have constant sign, and μ takes on at most one of $\pm\infty$, it follows that at least one of these series is convergent, and since $\sum_{j=1}^{\infty} \mu(E_j)$ is convergent, both series converge; therefore, the series is absolutely convergent.

(c) If $\{E_n\}_{n=1}^{\infty}$ is increasing, then we take

$$\begin{aligned}
\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigsqcup_{j=2}^{\infty} (E_j \setminus E_{j-1})\right)\\
&= \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1})\\
&= \lim_{n\to\infty} \sum_{j=2}^{n} \mu(E_j \setminus E_{j-1})\\
&= \lim_{n\to\infty} \mu\left(\bigsqcup_{j=2}^{n} (E_j \setminus E_{j-1})\right)\\
&= \lim_{j\to\infty} \mu(E_j),
\end{aligned}$$

and similarly for a decreasing sequence, using part (a) to ensure finiteness.

□

Now, we discuss the structure of positive-valued and negative-valued measurable sets.

Definition: Let μ be a signed measure on $(X, \mathcal{M})$. We call a set $E \in \mathcal{M}$ *positive* if, for every measurable $F \subseteq E$, $\mu(F) \geq 0$; similarly, we call $E \in \mathcal{M}$ *negative* if, for every measurable $F \subseteq E$, $\mu(F) \leq 0$.

Theorem (Hahn Decomposition Theorem)**:** If μ is a signed measure, then there exist two disjoint sets A and B such that $A \sqcup B = X$, A is positive with respect to μ, and B is negative with respect to μ. This decomposition unique up to μ-null symmetric difference.

Proof. Without loss of generality, we may assume that for all $E \in \mathcal{M}$, $-\infty < \mu(E) \leq \infty$.

We note that the difference of two negative sets is negative, and the disjoint, countable union of negative sets is negative, so every countable union of negative sets is negative. We let $\beta = \inf(\mu(B))$ for all negative $B \in \mathcal{M}$. We let $(B_j)_j \subseteq \mathcal{M}$ be a sequence of measurable negative sets such that $\lim_{j\to\infty} \mu(B_j) = \beta$, and set $B = \bigcup_{j=1}^{\infty} B_j$. We see then that B is a negative set for which $\mu(B)$ is minimal.

We now prove that $A = X \setminus B$ is a positive set. Suppose toward contradiction that there is $E_0 \subseteq A$ such that $\mu(E_0) < 0$. The set E_0 cannot be a negative set, or else $B \cup E_0$ would be a negative set with a smaller measure than $\mu(B)$, which is not possible.

Let k_1 be the smallest natural number such that there is $E_1 \subseteq E_0$ with $\mu(E_1) \geq \frac{1}{k_1}$. Since

$$\begin{aligned}
\mu(E_0 \setminus E_1) &= \mu(E_0) - \mu(E_1)\\
&\leq \mu(E_0) - \frac{1}{k_1}\\
&< 0.
\end{aligned}$$

The argument applied to E_0 is now applicable to $E_0 \setminus E_1$. Letting k_2 be the smallest natural number such that $E_0 \setminus E_1$ contains $E_2 \subseteq E_0 \setminus E_1$ with $\mu(E_2) \geq \frac{1}{k_2}$, and proceeding ad infinitum, we see that since μ is finitely valued for measurable subsets of E_0, we have $\lim_{n\to\infty} \frac{1}{k_n} = 0$.

It follows that for every measurable subset F of

$$F_0 = E_0 \setminus \left(\bigcup_{j=1}^{\infty} E_j\right),$$

we have $\mu(F) \leq 0$ — i.e., F_0 is a measurable negative set. Since F_0 is disjoint from B, and

$$\begin{aligned}\mu(F_0) &= \mu(E_0) - \sum_{j=1}^{\infty} \mu(E_j)\\ &\leq \mu(E_0)\\ &< 0,\end{aligned}$$

this contradicts the minimality of B. Therefore, the hypothesis $\mu(E_0) < 0$ is untenable.

If $A_1 \sqcup B_1$ and $A_2 \sqcup B_2$ are two Hahn decompositions for X, then $A_1 \setminus A_2 \subseteq B_2$ and $A_1 \setminus A_2 \subseteq A_1$, meaning that $A_1 \setminus A_2$ is both positive and negative, hence null; similarly for $A_2 \setminus A_1$, so that $A_1 \triangle A_2$ is μ-null, and similarly for $B_1 \triangle B_2$. $\square$

Definition: We say two measures, μ and ν, are *mutually singular* if there exist $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$, $A \sqcup B = X$, A is μ-null, and B is ν-null. In other words, μ and ν "live on disjoint sets." We write $\mu \perp \nu$.

Definition: Let $X = A \sqcup B$ be a Hahn decomposition for the signed measure μ. We define

$$\begin{aligned}\mu^+(E) &= \mu(E \cap A)\\ \mu^-(E) &= -\mu(E \cap B)\end{aligned}$$

to be the *positive and negative variation* of μ. The *total variation* of μ is defined to be

$$|\mu| = \mu^+ + \mu^-.$$

Theorem (Jordan Decomposition)**:** Every signed measure is the difference of two mutually singular measures, at least one of which is finite.

Proof. Set $\mu = \mu^+ - \mu^-$. Then, by definition, $\mu(E) = \mu(A \cap E) + \mu(B \cap E)$ where $X = A \sqcup B$ is a Hahn decomposition. $\square$

Exercise: If μ and ν are signed measures, then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Solution: We use the decomposition $X = E \sqcup F$, where ν is concentrated on E and μ is concentrated on F. Then, we apply the Hahn decomposition to each of E and F.

Absolute Continuity and the Lebesgue–Radon–Nikodym Theorem

Definition: Let ν be a signed measure, μ a positive measure on $(X, \mathcal{M})$. We say ν is absolutely continuous with respect to μ, written

$$\nu \ll \mu,$$

if $\nu(E) = 0$ for all $E \in \mathcal{M}$ where $\mu(E) = 0$.

Exercise: If ν is a signed measure and μ a positive measure on $(X, \mathcal{M})$, then $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Solution: Let $\mu(E) = 0$. Then, $\nu(E) = \nu^+(E) - \nu^-(E) = 0$ for each E. Now, in particular, if $P \sqcup N$ is a Hahn decomposition for ν, then $\mu(E \cap P) = 0$, meaning $\nu(E \cap P) = 0 = \nu^+(E)$, and similarly for ν^-. Thus, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, and similarly, $|\nu| \ll \mu$.

The terminology "continuity" makes sense for absolute continuity of measures via the following theorem.

Theorem: Let ν be a finite signed measure, μ a positive measure on $(X, \mathcal{M})$. Then, $\nu \ll \mu$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Proof. Since $\nu \ll \mu$ if and only if $|\nu| \ll \mu$, and $|\nu(E)| \leq |\nu|(E)$, it suffices to assume that ν is positive. The ε-δ condition clearly implies that $\nu \ll \mu$.

In the other direction, if the ε-δ condition is not satisfied, then there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, we may find $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon_0$. Letting $F_k = \bigcup_{n=k}^{\infty} E_n$, and $F = \bigcap_{k=1}^{\infty} F_k$ (i.e., $F = \limsup_{n\to\infty} E_n$), then

$$\begin{aligned} \mu(F_k) &< \sum_{n=k}^{\infty} 2^{-n} \\ &= 2^{1-k}, \end{aligned}$$

so $\mu(F) = 0$.[I] Yet, since $\nu(F_k) \geq \varepsilon$ for all k, and ν is finite, $\nu(F) \geq \varepsilon$, meaning $\nu \not\ll \mu$. □

We may use the Hahn decomposition to prove a particular kind of dichotomy for finite measures.

Lemma: Suppose ν and μ are finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$ or there is $\varepsilon > 0$ and $E \in \mathcal{M}$ with $\mu(E) > 0$ with E a positive set for $\nu - \varepsilon\mu$.

Proof. Let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$, and let

$$\begin{aligned} P &= \bigcup_{n=1}^{\infty} P_n \\ N &= P^c. \end{aligned}$$

Then, N is a negative set for $\nu - \frac{1}{n}\mu$ for all n, meaning that $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$, for all n, or that $\nu(N) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$. Else, if $\mu(P) > 0$, then $\mu(P_n) > 0$ for some n, and P_n is positive for $\nu - \frac{1}{n}\mu$. □

Remark: If we define a signed measure by taking

$$\nu(E) = \int_E f\, d\mu,$$

where f is an L_1 function, then we will often write $d\nu = f\, d\mu$

Theorem (Radon–Nikodym Theorem)**:** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let ν be a σ-finite signed measure on $(X, \mathcal{M})$ with $\nu \ll \mu$. Then, there is an extended μ-integrable function $f\colon X \to \mathbb{R}$ such that $d\nu = f\, d\mu$.

Proof. Since X is a countable disjoint union of finite-measure sets, we may prove for the case of finite sets and take sums over the decomposition $\{E_j\}_{j=1}^{\infty} \subseteq X$. Similarly, since $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, we only need to show this for the case where both ν and μ are finite, positive measures.

Let $\mathcal{F}$ be the family of nonnegative μ-integrable functions f such that

$$\int_E f\, d\mu \leq \nu(E).$$

[I] We may also show this using the Borel–Cantelli Lemma. To see this, note that $\mu(\limsup_{n\to\infty} E_n) = \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} E_n\right)$, by continuity from above, meaning that $\mu(\limsup_{n\to\infty} E_n) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \mu(E_n) \to 0$.

Let

$$\alpha = \sup\left\{ \int_X f\,d\mu \;\middle|\; f \in \mathcal{F} \right\}.$$

Let $(f_n)_n$ be a sequence of functions in $\mathcal{F}$ such that

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \alpha.$$

Set

$$g_n = \max(f_1, \dots, f_n).$$

If E is any measurable set, then there is a union of disjoint measurable subsets $E = E_1 \sqcup \cdots \sqcup E_n$ such that

$$g_n(x) = f_j(x)$$

for each $x \in E_j$. Consequently,

$$\begin{aligned}\int_E g_n\,d\mu &= \sum_{j=1}^{n} \int_{E_j} f_j\,d\mu \\ &\leq \sum_{j=1}^{n} \nu(E_j) \\ &= \nu(E).\end{aligned}$$

Set $f_0(x) = \sup_n f_n(x)$ for each x, so that $f_0(x) = \lim_{n\to\infty} g_n(x)$. It follows that $f_0 \in \mathcal{F}$ and $\int_X f_0\,d\mu = \alpha$. Since f_0 is integrable, there is an everywhere-finite function f such that $f_0 = f$ μ-a.e. We will prove that if $\nu_0(E) = \nu(E) - \int_E f\,d\mu$, then $\nu_0(E)$ is identically zero.

If ν_0 is not identically zero, then there is $\varepsilon > 0$ and measurable A such that $\mu(A) > 0$ and

$$\begin{aligned}\varepsilon\mu(E\cap A) &\leq \nu_0(E \cap A) \\ &= \nu(E\cap A) - \int_{E\cap A} f\,d\mu\end{aligned}$$

for every measurable set E. If $g = f + \varepsilon\mathbb{1}_A$, then

$$\begin{aligned}\int_E g\,d\mu &= \int_E f\,d\mu + \varepsilon\mu(E\cap A) \\ &\leq \int_{E\setminus A} f\,d\mu + \nu(E\cap A) \\ &\leq \nu(E),\end{aligned}$$

yet

$$\begin{aligned}\int g\,d\mu &= \int f\,d\mu + \varepsilon\mu(A) \\ &> \alpha,\end{aligned}$$

which contradicts the maximality of f. $\square$

Theorem (Lebesgue Decomposition)**:** If ν is a general σ-finite signed measure with μ a positive σ-finite measure on $(X, \mathcal{M})$, then there are measures ν_0 and ν_1 such that $\nu_0 \perp \mu$, $\nu_1 \ll \mu$, and $\nu = \nu_0 + \nu_1$. This decomposition is essentially unique.

Proof. Similar to the Radon–Nikodym theorem, we may prove for the case of ν being a finite positive measure.

Note that $\nu \ll \mu + \nu$, meaning there is a measurable function f such that

$$\nu(E) = \int_E f\, d\mu + \int_E f\, d\nu,$$

where since $0 \leq \nu(E) \leq \mu(E) + \nu(E)$, we have $0 \leq f \leq 1$ almost everywhere with respect to $\mu + \nu$, or that $0 \leq f \leq 1$ ν-a.e. Letting $A = \{x \mid f(x) = 1\}$ and $B = \{x \mid 0 \leq f(x) < 1\}$, then

$$\begin{aligned}\nu(A) &= \int_A d\mu + \int_A d\nu \\ &= \mu(A) + \nu(A),\end{aligned}$$

so that $\mu(A) = 0$. Therefore, if we set $\nu_0(E) = \nu(E \cap A)$ and $\nu_1(E) = \nu(E \cap B)$ for every measurable subset E, then we see that $\nu_0 \perp \mu$.

Now, we must prove that $\nu_1 \ll \mu$. If $\mu(E) = 0$, then

$$\int_{E\cap B} d\nu = \int_{E\cap B} f\, d\nu,$$

or that

$$\int_{E\cap B} (1-f)\, d\nu = 0.$$

Since $1 - f \geq 0$ ν-a.e., it follows that $\nu_1(E) = \nu(E \cap B) = 0$.

Now, if $\nu = \nu_0 + \nu_1$ and $\nu = \widetilde{\nu_0} + \widetilde{\nu_1}$, then we see that $\nu_0 - \widetilde{\nu_0} = \widetilde{\nu_1} - \nu_1$. Since $\nu_0 - \widetilde{\nu_0}$ is singular with respect to μ and $\widetilde{\nu_1} - \nu_1$ is absolutely continuous with respect to μ, it follows what $\nu_0 = \widetilde{\nu_0}$ and $\nu_1 = \widetilde{\nu_1}$. $\square$

Definition: If $\nu \ll \mu$, the *Radon–Nikodym Derivative* of ν with respect to μ is defined to be

$$\frac{d\nu}{d\mu} = f,$$

where $d\nu = f\, d\mu$.

Proposition: Suppose ν is a σ-finite signed measure, μ and λ σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$ and $\mu \ll \lambda$.

(a) If $g \in L_1(\nu)$, then $g\frac{d\nu}{d\mu} \in L_1(\mu)$ and

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu}\, d\mu.$$

(b) We have $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}.$$

Proof. We may assume $\nu \geq 0$. Note that the equation

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu}\, d\mu$$

holds for $g = \mathbb{1}_E$ for all $E \in \mathcal{M}$ by the definition of $\frac{d\nu}{d\mu}$, meaning that it holds for simple functions by linearity, then nonnegative measurable functions by monotone convergence, then for functions in $L_1(\nu)$ by linearity.

Setting $g = \mathbb{1}_E \frac{d\nu}{d\mu}$, then we see that

$$\begin{aligned}\nu(E) &= \int_E \frac{d\nu}{d\mu}\,d\mu\\ &= \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\,d\lambda\end{aligned}$$

for all $E \in \mathcal{M}$, meaning $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$. □

Naturally, the Lebesgue Decomposition and Radon–Nikodym Theorem can be extended to the case of complex measures.

Definition: A complex measure on $(X, \mathcal{M})$ is a map $\nu\colon \mathcal{M} \to \mathbb{C}$ such that

- $\nu(\emptyset) = 0$;
- for a sequence of disjoint sets $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ with $\sum_{j=1}^{\infty}|\nu(E_j)| < \infty$, then

$$\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty}\nu(E_j).$$

Note: Infinite values are not allowed.

Theorem: If ν is a complex measure, and μ is a σ-finite positive measure, then there is a complex measure λ and $f \in L_1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f\,d\mu$. This decomposition is essentially unique.

Definition: The *total variation* of a complex measure ν with Lebesgue–Radon–Nikodym decomposition $d\nu = d\lambda + f\,d\mu$ is given by

$$d|\nu| = d|\lambda| + |f|\,d\mu.$$

Proposition: Let ν be a complex measure on $(X, \mathcal{M})$. Then,

- $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$;
- $\nu \ll |\mu|$ and $\frac{d\nu}{d|\nu|}$ has modulus 1 $|\nu|$-a.e.

Proof. If $d\nu = f\,d\mu$, then

$$\begin{aligned}|\nu(E)| &= \left|\int_E f\,d\mu\right|\\ &\leq \int_E |f|\,d\mu\\ &= |\nu|(E).\end{aligned}$$

This shows that $\nu \ll |\nu|$. If $g = \frac{d\nu}{d|\nu|}$, then

$$\begin{aligned}f\,d\mu &= d\nu\\ &= g\,d|\nu|\\ &= g|f|\,d\mu.\end{aligned}$$

Thus, $g = \operatorname{sgn}(f)$ almost everywhere, and has modulus 1 almost everywhere. □

Proposition: The following are equivalent definitions for the total variation:

(i) $\mu_1(E) = \sup\left\{ \sum_{j=1}^{n} \nu(E_j) \;\middle|\; E_j \text{ disjoint, } E = \bigsqcup_{j=1}^{n} E_j \right\}$;

(ii) $\mu_2(E) = \sup\left\{ \sum_{j=1}^{\infty} \nu(E_j) \;\middle|\; E_j \text{ disjoint, } E = \bigsqcup_{j=1}^{\infty} E_j \right\}$;

(iii) $\mu_3(E) = \sup\left\{ \left|\int_E f\, d\nu\right| \;\middle|\; |f| \leq 1 \right\}$.

Proof. We can see immediately that $\mu_1(E) \leq \mu_2(E)$ by definition, as the latter includes the former by setting $E_j = \emptyset$ past a certain index.

Next, we observe that if $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ are disjoint and such that $E = \bigsqcup_{j=1}^{\infty} E_j$, then

$$\begin{aligned}
\sum_{j=1}^{\infty} |\nu(E_j)| &\leq \sum_{j=1}^{\infty} |\nu|(E_j) \\
&= \int_E \, d|\nu| \\
&= \int_E \left|\frac{d\nu}{d|\nu|}\right| d|\nu| \\
&= \int_E \overline{\frac{d\nu}{d|\nu|}} \frac{d\nu}{d|\nu|}\, d|\nu| \\
&= \int_E \overline{\frac{d\nu}{d|\nu|}}\, d\nu \\
&\leq \sup\left\{ \left|\int_E f\, d\nu\right| \;\middle|\; |f| \leq 1 \right\},
\end{aligned}$$

so that $\mu_2(E) \leq \mu_3(E)$. Next, we see that $\mu_3 = |\nu|$, as, since $f = \overline{\frac{d\nu}{d|\nu|}}$ has modulus 1, we have

$$\begin{aligned}
\int_E \overline{\frac{d\nu}{d|\nu|}}\, d\nu &= \int_E d|\nu| \\
&= |\nu|(E),
\end{aligned}$$

so that $\mu_3 = |\nu|$. Finally, we see that if $f = \sum_{k=1}^{n} a_k \mathbb{1}_{E_k}$ for some disjoint subsets $E_k \subseteq E$ and $|a_k| \leq 1$, then

$$\begin{aligned}
\left|\int_E f\, d\nu\right| &= \sum_{k=1}^{n} |a_k||\nu(E_k)| \\
&\leq \sum_{k=1}^{n} |\nu(E_k)| \\
&\leq \mu_1(E),
\end{aligned}$$

so $\mu_3 \leq \mu_1$, and thus these expressions are all equal to $|\nu|$. $\square$