

Observations on Excess Area Identities and Operator Symbols in Bergman Spaces

Wynesakia Akamah (Rose-Hulman), Avinash Iyer (Occidental College), Jennifer Yuan (NYU Abu Dhabi)
Mentor: Dr. Mehmet Çelik

Occidental College

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- 3 Results and Observations
- 4 Remarks and Future Directions
- 5 REU Experience
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Definition and Notations

- Ω : a region in \mathbb{C} e.g. \mathbb{D} , $D(0, r)$, $\mathbb{A}(0, r, 1)$, \mathbb{C}
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$: weight function

Definition (λ -weighted Square-Integrable Functions)

$$L^2(\Omega, \lambda) = \left\{ f : \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f(z)|^2 \lambda(z) dA(z) < \infty \right\}$$

- $L^2(\Omega, \lambda)$ forms a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega, \lambda)} = \int_{\Omega} f(z) \overline{g(z)} \lambda(z) dA(z)$$

inducing the norm

$$\|f\|_{L^2(\Omega, \lambda)}^2 = \int_{\Omega} |f(z)|^2 \lambda(z) dA(z)$$

Definitions and Notations

Definition (Holomorphic Function on Ω)

We say h is holomorphic on Ω , or $h \in \mathcal{O}(\Omega)$, if, for all $z \in \Omega$

$$\begin{aligned}\frac{\partial h(z)}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial h(z)}{\partial x} + i \frac{\partial h(z)}{\partial y} \right) \\ &= \frac{1}{2} \left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \\ &= 0.\end{aligned}$$

Definition (λ -weighted Bergman Space)

$$A^2(\Omega, \lambda) := \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

Definitions and Notations

Definition ($A^{1,2}(\Omega, \lambda)$)

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \mid \frac{\partial h}{\partial \bar{z}} \in A^2(\Omega, \lambda) \right\}$$

Definition (Weighted Image-Area)

Let $h \in A^{1,2}(\Omega, \lambda)$.

$$\begin{aligned} A_{\Omega, \lambda}(h) &= \int_{\Omega} \left| \frac{\partial h}{\partial \bar{z}} \right|^2 \lambda(z) dA(z) \\ &= \left\| \frac{\partial h}{\partial \bar{z}} \right\|_{L^2(\Omega, \lambda)}^2 \end{aligned}$$

Definitions and Notations

- $A^2(\Omega, \lambda)$ has a reproducing kernel i.e $\exists! K_{\Omega}^{\lambda}(\cdot, z) \in A^2(\Omega, \lambda) :$

$$h(z) = \langle h(\cdot), K_{\Omega}^{\lambda}(\cdot, z) \rangle_{L^2(\Omega, \lambda)}$$

- $A^2(\Omega, \lambda)$ is a closed subspace of $L^2(\Omega, \lambda)$.

Definition (Bergman Projection)

Let $P^{\Omega, \lambda} : L^2(\Omega, \lambda) \rightarrow A^2(\Omega, \lambda)$

$$\begin{aligned} (P^{\Omega, \lambda} h)(z) &:= \langle h(\cdot), K_{\Omega}^{\lambda}(\cdot, z) \rangle_{L^2(\Omega, \lambda)} \\ &= \int_{\Omega} h(w) \overline{K_{\Omega}^{\lambda}(w, z)} \lambda(w) dA(w) \end{aligned}$$

Definition (Multiplication Operator)

Let $M_\varphi : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda)$ where $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

Definition (Toeplitz Operator)

$T_\varphi^{\Omega, \lambda} : A^2(\Omega, \lambda) \rightarrow A^2(\Omega, \lambda)$, where $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega, \lambda} := P^{\Omega, \lambda} M_\varphi$$

Definitions and Notations

Definition (Commutator)

Let $[P^{\Omega,\lambda}, M_\varphi] : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda)$

$$[P^{\Omega,\lambda}, M_\varphi] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

Definition (Hankel Operator)

Let $H_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \rightarrow (A^2(\Omega, \lambda))^\perp$

$$\begin{aligned} H_\varphi^{\Omega,\lambda} &:= - [P^{\Omega,\lambda}, M_\varphi] \Big|_{A^2(\Omega,\lambda)} \\ &= (I - P^{\Omega,\lambda}) M_\varphi \\ &= M_\varphi - P^{\Omega,\lambda} M_\varphi \\ &= M_\varphi - T_\varphi^{\Omega,\lambda} \end{aligned}$$

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Motivations

- $\{z^n\}_{n=0}^{\infty}$ form a *complete orthogonal basis* for $A^2(\mathbb{D})$
- If h is holomorphic, then h is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

and

$$S_N := \sum_{n=0}^N h_n z^n$$

converges uniformly on compact subsets.

- Relationship between L^2 norm of h to the ℓ^2 norm of $\{h_k\}_{k=0}^{\infty}$:

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $[T_{\bar{z}}^{\mathbb{D}} M_z, DM_z](z^m) = 0$

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

Literature Review on Previous Results I

- D'Angelo's Excess Area identity [D'A19]

Let $h \in A^{1,2}(\mathbb{D})$. Then,

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial(zh)}{\partial z} \right\|_{L^2(\mathbb{D})}^2 - \left\| \frac{\partial h}{\partial z} \right\|_{L^2(\mathbb{D})}^2 \\ &= \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \\ &= \pi \|Sh\|_{L^2(b\mathbb{D})}^2 \end{aligned}$$

where Sh is the restriction of h to the unit circle.

Literature Review on Previous Results II

- Excess Area identity with Blaschke product multiplier
- 'Excess Area' identity for harmonic functions [BÇGH22]
- Generating symbols for Toeplitz operators for a given initial p and target polynomial q on unit disc and polydisc, $T_{\varphi}^{\mathbb{D}}(p) = q$ and $T_{\varphi}^{\mathbb{D}^n}(p) = q$ [ÇDTR⁺24]
- Substituted derivatives for Toeplitz operators in Excess Area identity [ÇDTR⁺24]

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Summary of Results

1. Results and Observations influenced by the Area Difference of the image of \mathbb{D} between zh and h :
 - i. On $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2}), A^2(\mathbb{D}, \lambda), A^2(D(0, r))$
 - ii. On convergence of identities on certain weighted discs.
- 2 Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
 - i. On unweighted and weighted Toeplitz operators relation
 - ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on $A^2(\mathbb{D})$

Methods Used

- Relation between L^2 norms of functions and ℓ^2 norms of Taylor series:

$$\|h\|_{L^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- Integration by parts via Stokes's theorem on forms:

$$\oint_{\partial\Omega} f \, dz = \int_{\Omega} \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz$$
$$\oint_{\partial\Omega} f \, d\bar{z} = \int_{\Omega} \frac{\partial f}{\partial z} \, dz \wedge d\bar{z}.$$

- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Beta, Gamma, and Hypergeometric functions

Using Integration by Parts to find Excess Area identity: Wedge Product I

The area is integrated with respect to $dA = dx \wedge dy$. The wedge product has the following properties:

$$(a + b) \wedge c = a \wedge c + b \wedge c$$

$$a \wedge b = -b \wedge a$$

$$a \wedge a = 0.$$

With $z = x + iy$, $\bar{z} = x - iy$, the substitution $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ yields

$$\begin{aligned} dx \wedge dy &= \frac{1}{2i} (d\bar{z} \wedge dz) \\ &= -\frac{1}{2i} (dz \wedge d\bar{z}). \end{aligned}$$

Using Integration by Parts to find Excess Area identity: Wedge Product II

The area integral is now rewritten as:

$$\begin{aligned}\left\langle \frac{\partial h}{\partial \bar{z}}, \frac{\partial h}{\partial z} \right\rangle_{L^2(\Omega, \lambda)} &= \int_{\Omega} \left(\overline{\frac{\partial h}{\partial z}} \right) \left(\frac{\partial h}{\partial z} \right) \lambda(|z|) \, dx \wedge dy \\ &= \frac{1}{2i} \int_{\Omega} \lambda(|z|) \left(\left(\overline{\frac{\partial h}{\partial z}} \right) d\bar{z} \right) \wedge \left(\left(\frac{\partial h}{\partial z} \right) dz \right)\end{aligned}$$

Using Integration by Parts to find Excess Area identity: Stokes's Theorem I

In particular,

$$\overline{\frac{\partial}{\partial z}} \left((\lambda(|z|)) \bar{h} \frac{\partial h}{\partial z} \right) d\bar{z} \wedge dz = \underbrace{(\lambda(|z|)) \overline{\frac{\partial h}{\partial z}} d\bar{z} \wedge \frac{\partial h}{\partial z} dz}_{\text{area integrand}} + \left(\overline{\frac{\partial}{\partial z}} \lambda(|z|) \right) \bar{h} \wedge \frac{\partial h}{\partial z} dz$$

meaning

$$\begin{aligned} \frac{1}{2i} \int_{\Omega} \frac{\partial h}{\partial \bar{z}} \overline{\frac{\partial h}{\partial z}} \lambda(|z|) d\bar{z} \wedge dz &= \underbrace{\frac{1}{2i} \int_{\Omega} \overline{\frac{\partial}{\partial z}} \left(\lambda(|z|) \bar{h} \frac{\partial h}{\partial z} \right) d\bar{z} \wedge dz}_{\text{Integral A}} \\ &\quad - \frac{1}{2i} \int_{\Omega} \bar{h} \frac{\partial h}{\partial z} \left(\overline{\frac{\partial}{\partial z}} \lambda(|z|) \right) d\bar{z} \wedge dz. \end{aligned}$$

Using Integration by Parts to find Excess Area identity: Stokes's Theorem II

Turning our attention to Integral A,

$$\begin{aligned}\frac{1}{2i} \int \overline{\frac{\partial}{\partial z}} \left(\lambda(|z|) \bar{h} \frac{\partial h}{\partial z} \right) d\bar{z} \wedge dz &= \frac{1}{2i} \int_{\Omega} d \left(\lambda(|z|) \bar{h} \frac{\partial h}{\partial z} \right) d\bar{z} \wedge dz \\ &= \frac{1}{2i} \underbrace{\int_{\partial\Omega} \lambda(|z|) \bar{h} \frac{\partial h}{\partial z} dz}_{=0}.\end{aligned}$$

With this, the area integral is now

$$\frac{1}{2i} \int \frac{\partial h}{\partial z} \overline{\frac{\partial h}{\partial z}} \lambda(|z|) d\bar{z} \wedge dz = -\frac{1}{2i} \int \bar{h} \frac{\partial h}{\partial z} \left(\overline{\frac{\partial}{\partial z}} \lambda(|z|) \right) d\bar{z} \wedge dz$$

Excess Area on Fock Spaces

D'Angelo's Excess Area identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \|Sh\|_{L^2(b\mathbb{D})}^2$$

Excess Area on Fock Space

Given $h \in \mathcal{F}^2$ with $\frac{\partial h}{\partial \bar{z}}$,

$$\begin{aligned} A_{\mathcal{F}^2}(zh) - A_{\mathcal{F}^2}(h) \\ = \pi \left\| z T_{\bar{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 + \pi \left\| T_{\bar{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 + \pi \left\| H_{\bar{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 \end{aligned}$$

Here, the restriction of h to the unit circle in D'Angelo's Excess Area identity is replaced with the Bergman projection on \mathbb{C} .

Excess Area on $A^2(\mathbb{D}, \lambda)$

D'Angelo's Excess Area identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \|Sh\|_{L^2(b\mathbb{D})}^2$$

Excess Area on $A^2(\mathbb{D}, \lambda)$

Let $h \in A^{1,2}(\mathbb{D}, \lambda)$, $\lambda(z) = 1 - |z|^2$. Then,

$$A_{\mathbb{D},\lambda}(z^{m+1}h) - A_{\mathbb{D},\lambda}(z^m h) = \pi \|z^m h\|_{L^2(\mathbb{D},\lambda)}^2.$$

Here, the restriction of h to the unit circle is replaced with the function itself.

Dilation and Contraction from $A^2(D(0, r))$ to $A^2(\mathbb{D})$

Contracting $h \in A^{1,2}(\mathbb{D})$ by taking $h_r = h(rz)$ for some $0 < r < 1$,

$$A_{\mathbb{D}}(zh_r) - A_{\mathbb{D}}(h_r) = \pi \|Sh_r\|_{L^2(b\mathbb{D})}^2 \quad (1)$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \|Sh\|_{L^2(bD(0,r))}^2. \quad (2)$$

Dilating $h \in A^{1,2}(D(0, r))$ by taking $h_{\frac{1}{r}} = h(\frac{z}{r})$ for some $0 < r < 1$

$$A_{D(0,r)}(zh_{1/r}) - r^2 A_{D(0,r)}(h_{1/r}) = \pi r^2 \|Sh_{1/r}\|_{L^2(bD(0,r))}^2 \quad (3)$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \|Sh\|_{L^2(b\mathbb{D})}^2 \quad (4)$$

Approximation for Sequences of Berezin Transform

Weighted Area on $D(0, r)$

Let $\lambda_r(z) = \chi_{D(0,r)} \left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$ where $r > 0$. Then,

$$A_{D(0,r),\lambda_r}(h) = \int_{D(0,r)} |h'(z)|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} dA(z)$$

We find that, as $r \rightarrow \infty$, $A_{D(0,r),\lambda_r}(h) \rightarrow A_{\mathcal{F}^2}(h)$.

Separately,

$$A_{\mathcal{F}^2}(h) = \left\| T_{\bar{z}}^{\mathcal{F}^2} h \right\|_{\mathcal{F}^2}^2$$

Reproducing Kernel on $A^2(D(0, r), \lambda_r)$

$$\begin{aligned} K_{D(0,r)}^{\lambda_r}(w, z) &= \sum_{k=0}^{\infty} \frac{\bar{z}^k w^k}{\|w^k\|_{L^2(D(0,r), \lambda_r)}^2} \\ &= \frac{1}{\left(1 - \frac{\bar{z}w}{r^2}\right)^{r^2+2}} \end{aligned}$$

Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w, z) = e^{\bar{z}w}$$

Definition (Berezin Transform ([Zhu07]))

Let

$$k_z^{\Omega, \lambda}(w) := \frac{K_{\Omega}^{\lambda}(w, z)}{\sqrt{K_{\Omega}^{\lambda}(z, z)}}$$

Then, for some bounded operator T on $L^2(\Omega, \lambda)$, define $\mathcal{B}^{\Omega, \lambda} : B(L^2(\Omega, \lambda)) \rightarrow L^2(\Omega, \lambda)$

$$(\mathcal{B}^{\Omega, \lambda} T)(z) := \langle Tk_z^{\Omega, \lambda}, k_z^{\Omega, \lambda} \rangle_{L^2(\Omega, \lambda)}$$

Berezin Transform Convergence, Cont'd

Previous results:

- For $\varphi \in L^\infty(\Omega, \lambda)$, $\mathcal{B}^{\Omega, \lambda} T_\varphi = \mathcal{B}^{\Omega, \lambda} M_\varphi$. (see [AZ98a]).
- φ is harmonic if and only if $\mathcal{B}^{\Omega, \lambda} M_\varphi = \varphi$ (proof in [Eng94]).

New results:

- For $T_\varphi^{D(0,r), \lambda_r} = P^{D(0,r), \lambda_r} M_\varphi$, the Berezin transform $\mathcal{B}^{D(0,r), \lambda_r} T_\varphi^{D(0,r), \lambda_r}$ converges pointwise to $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$ as $r \rightarrow \infty$.
- By Dini's Theorem, this convergence is uniform on compact subsets of \mathbb{C} (proof inspired by [GŞ20]).

Unweighted and Weighted Toeplitz Operators Relation

Using an extension of [ÇDTR⁺24, Lemma 2.1]

For weight $\lambda(z) = (1 - |z|^2)^\alpha$ ($\alpha \geq 0$) on the unit disc,
 $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

$$\frac{T_{\bar{z}^m}^{\mathbb{D}, \lambda_\alpha}(z^n)}{T_{\bar{z}^m}^{\mathbb{D}}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leq n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T_{\bar{z}^m}^{\mathbb{D}, \lambda_\alpha}(z^n) = s_{n,m,\alpha} T_{\bar{z}^m}^{\mathbb{D}}(z^n), \text{ and } \lim_{n \rightarrow \infty} s_{n,m,\alpha} = 1$$

Unweighted and Weighted Commutator on $A^2(\mathbb{D})$

Existence of Commutator Symbols

Given p and q are harmonic polynomials and $\frac{\partial}{\partial \bar{z}}(p) \neq 0$, there does not exist a polynomial symbol ϕ , such that $[P^{\mathbb{D}}, M_{\phi}](p) = q$ or $[P^{\mathbb{D}, \lambda}, M_{\phi}](p) = q$.

Compare to [ÇDTR⁺24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

Unweighted and Weighted Hankel Operator on $A^2(\mathbb{D})$

Existence of Hankel Operator Symbols

Given some holomorphic polynomials p, q where p is not constant, there does not exist a polynomial symbol ϕ such that $H_{\phi}^{\mathbb{D}}(p) = \bar{q}$ or

$$H_{\phi}^{\mathbb{D}, \lambda}(p) = \bar{q}$$

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Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0, r, 1))$

For all integers m and n ,

$$T_{\bar{z}^m}^{\mathbb{A}(0, r, 1)}(z^n) = \begin{cases} \frac{2mr^{2m} \ln(r)}{(r^{2m}-1)} z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m}-1}{2m \ln(r)} z^{-1} & \text{if } n = m-1 . \\ \frac{(n-m+1)(1-r^{2n+2})}{(n+1)(1-r^{2n-2m+2})} z^{n-m} & \text{else} \end{cases}$$

We attempted to find an algorithm to generate $\varphi \in L^\infty(\mathbb{A}(0, r, 1))$ such that $T_\varphi^{\mathbb{A}(0, r, 1)}(p) = q$ for given holomorphic Laurent polynomials p and q , but could not prove lack of existence if p has roots inside $\overline{\mathbb{A}(r, 0, 1)}$.

Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial p and target polynomial q on $\mathbb{A}(0, r, 1)$, $T_{\varphi}^{\mathbb{A}(0, r, 1)}(p) = q$
- Extension of 'Excess Area' identity to harmonic functions in $L^2(\mathbb{C}, e^{-|z|^2})$.
- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential,
 $(1 - |z|^2)^A e^{\frac{-B}{(1-|z|^2)^\alpha}} (A \geq 0, B > 0, \alpha > 0)$.

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What is Research Like?

- The complex analysis group consisted of myself, Jennifer Yuan (NYU Abu Dhabi), and Sakia Akamah (Rose–Hulman Institute of Technology).
- Weeks were 9am to 5pm, mostly doing various calculations and updating our collected results document.
- Every weekday morning, our mentor met with us to go over the previous day's calculations and suggest new ideas/directions.
- We rotated project directions every day after the morning's meeting so we could bring fresh perspectives and ideas to the work the previous person had done yesterday.
- We did not fully understand what we were doing a lot of the time (especially in the beginning), but that's honestly very expected; our mentor was always there to help us along the way and answer any questions we had.

Activities and Life Outside Research

- In addition to the research itself, we also received visitors from academia and industry to provide some guidance on our potential future paths.
- We also visited the University of North Texas in Denton to talk with PhD students and faculty who gave us important advice on how to succeed as a graduate student.
- We also had some fun — usually after work was over for the day we'd watch anime in the university housing common room, or play badminton at the gym, and occasionally played board games.

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