Given $f(x)$, we want to find a value x' such that $f(x') = 0$.

(1) Pick a value x_0 such that $x_0 \in [f(a), f(b)]$, where $f(a) < 0$ and $f(b) > 0$.

(2) Take

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For example, take $f(x) = x^2 - 2$. We know that $f(1) < 0$ and $f(2) > 0$. Take $x_0 = 0$.

$$\begin{aligned} x_1 &= 1 - \frac{1^2 - 2}{2} \\ &= \frac{3}{2} = 1.5 x_2 && = \frac{3}{2} - \frac{\left(\frac{3}{2}\right)^2 - 2}{3} \\ &= \frac{17}{12} = 1.41\overline{6} \end{aligned}$$

Newton's method has quadratic convergence. However, we can look at an even better algorithm.

Via the Taylor series, we know that

$$\begin{aligned} f(x_{n+1}) &\approx f(x_n) + f'(x_n)(x_{n+1} - x_n) \\ 0 &\approx f(x_n) + f'(x_n)(x_{n+1} - x_n) \end{aligned}$$

However, we can make this better by adding another term, creating cubic convergence.

$$\begin{aligned} 0 &= f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)^2 \\ x_{n+1} - x_n &= \frac{-f'(x_n) \pm \sqrt{f'(x_n)^2 - 2f''(x_n)f(x_n)}}{f''(x_n)} \\ x_{n+1} &= x_n - \frac{f'(x_n)}{f''(x_n)} \mp \frac{\sqrt{f'(x_n)^2 - 2f''(x_n)f(x_n)}}{f''(x_n)} \end{aligned}$$

after tedious algebra,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2 f''(x_n)}{2f'(x_n)^3} \qquad \text{cubic convergence formula}$$