Positive Maps

We will start by focusing our discussion of positive maps on a subclass of linear subspaces of C^*-algebras.

Definition: Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{S} \subseteq \mathcal{A}$ be a self-adjoint linear subspace that contains 1. We call such an $\mathcal{S}$ an *operator system*.

Note that if h is a self-adjoint element of $\mathcal{S}$, then it is possible to write h as the difference of two positive elements in $\mathcal{S}$,

$$h = \frac{1}{2}(\|h\|1 + h) - \frac{1}{2}(\|h\|1 - h).$$

Definition: If $\mathcal{S} \subseteq \mathcal{A}$ is an operator system, $\mathcal{B}$ is a C^*-algebra, and $\phi\colon \mathcal{S} \to \mathcal{B}$ is a linear map, then we say ϕ is positive if it maps positive elements of $\mathcal{S}$ to positive elements of $\mathcal{B}$.

In the special case where the C^*-algebra $\mathcal{B}$ is the complex numbers (i.e., ϕ is a positive linear functional), then we know from results in C^*-algebra theory that $\|\phi\| = \phi(1)$. If $\mathcal{B}$ is an arbitrary C^*-algebra, it turns out that ϕ is still positive, but that the bound is different.

Proposition: If $\phi\colon \mathcal{S} \to \mathcal{B}$ is a positive map, then $\|\phi\| \leq 2\|\phi(1)\|$.

Proof. If p is positive, then since $0 \leq p \leq \|p\|1$, it follows that $0 \leq \phi(p) \leq \|p\|\phi(1)$, so that $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$.

If p_1 and p_2 are positive, then $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$, so if h is self-adjoint in $\mathcal{S}$, we have

$$\phi(h) = \frac{1}{2}\phi(\|h\|1 + h) - \frac{1}{2}\phi(\|h\|1 - h),$$

giving

$$\begin{aligned}\|\phi(h)\| &\leq \frac{1}{2}\max(\|\phi(\|h\|1 + h)\|, \|\phi(\|h\|1 - h)\|)\\ &\leq \|h\|\|\phi(1)\|.\end{aligned}$$

Finally, if a is an arbitrary element of $\mathcal{S}$, then we may write the Cartesian decomposition $a = h + ik$, and find

$$\begin{aligned}\|\phi(a)\| &\leq \|\phi(h)\| + \|\phi(k)\|\\ &\leq 2\|a\|\|\phi(1)\|.\end{aligned}$$

□

It turns out that this bound is strict.

Example: Consider the subspace $\mathcal{S} \subseteq C\left(S^1\right)$ spanned by $1, z, \bar{z}$. Then, we may define $\phi\colon \mathcal{S} \to \mathbb{M}_2$ given by

$$\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}.$$

It follows that an element of $\mathcal{S}$ is positive if and only if $c = \overline{b}$ and $a \geq 2|b|$, while a self-adjoint element of $\mathbb{M}_2$ is positive if and only if its diagonal entries and determinant are positive real numbers. Therefore, it follows that ϕ is a positive map.

Yet,

$$\begin{aligned}2\|\phi(1)\| &= 2\\ &= \|\phi(z)\|\\ &\leq \|\phi\|,\end{aligned}$$

meaning that $\|\phi\| = 2\|\phi(1)\|$.

Completely Positive Maps

Definition: If $\mathcal{B}$ is a C^*-algebra and $\phi\colon \mathcal{S} \to \mathcal{B}$ is a linear map, then we may define $\phi_n\colon \mathbb{M}_n(\mathcal{S}) \to \mathbb{M}_n(\mathcal{B})$ by $\phi_n\Big((a_{ij})_{i,j}\Big) = (\phi(a_{ij}))_{i,j}$.

We call ϕ n-positive if ϕ_n is positive, and we call ϕ completely positive if ϕ is n-positive for all n.

We call ϕ completely bounded if

$$\|\phi\|_{\text{cb}} \coloneqq \sup_n \|\phi_n\|$$

is finite. If $\|\phi\|_{\text{cb}} \leq 1$, then we call ϕ completely contractive.

Lemma: Let $\mathcal{A}$ be a unital C^*-algebra, and let $a, b \in \mathcal{A}$. Then,

(i) $\|a\| \leq 1$ if and only if

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$$

is positive in $\mathbb{M}_2(\mathcal{A})$.

(ii) The matrix

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$$

is positive in $\mathbb{M}_2(\mathcal{A})$ if and only if $a^*a \leq b$.

Proof. Faithfully represent $\mathcal{A}$ on $\mathcal{H}$ via $\pi\colon \mathcal{A} \to B(\mathcal{H})$, and set $A = \pi(a)$. Then, if $\|A\| \leq 1$, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, y\rangle + \langle Ay, x\rangle + \langle x, Ay\rangle + \langle y, y\rangle \\ &\geq \|x\|^2 - 2\|A\|\|x\|\|y\| + \|y\|^2 \\ &\geq 0. \end{aligned}$$

Conversely, if $\|A\| > 1<$ then there are unit vectors x and y such that $\langle Ay, x\rangle < -1$, so the inner product above would be negative.

Now, if we let $B = \pi(b)$, then if we let $B \geq A^*A$, then $B - A^*A \geq 0$, so that $\langle By, y\rangle \geq \langle Ay, Ay\rangle$ for all $y \in \mathcal{H}$, meaning that for all $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x\rangle + \langle Ax, y\rangle + \langle A^*x, y\rangle + \langle By, y\rangle \\ &\geq \langle x, x\rangle + 2\operatorname{Re}\langle Ax, y\rangle + \langle Ay, Ay\rangle \\ &\geq \|x\|^2 + \|Ay\|^2 - 2\|Ay\|\|x\| \\ &\geq 0. \end{aligned}$$

In the case that $B \ngeq A^*A$, then there is some unit vector y such that $\langle By, y\rangle < \|Ay\|^2$, which would yield the analogous outcome as in the proof of (i). $\square$

Dilations and Extensions

Theorem (Stinespring's Dilation Theorem)**:** Let $\mathcal{A}$ be a unital C^*-algebra, and let $\phi\colon \mathcal{A} \to B(\mathcal{H})$ be a completely positive map. Then, there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{K})$, and a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that

$$\phi(a) = V^* \pi(a) V.$$

Proof. Let $A \odot \mathcal{H}$ be the algebraic tensor product, and define a symmetric bilinear map

$$(a \otimes x, b \otimes y) = \langle \phi(b^* a)x, y \rangle,$$

and extend linearly, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Then, since ϕ is completely positive, it follows that $(\cdot, \cdot)$ is positive semidefinite, with

$$\begin{aligned}\left(\sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right) &= \left\langle \phi_n\left(\left(a_i^* a_j\right)_{i,j} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\ &\geq 0.\end{aligned}$$

Since positive semidefinite bilinear forms satisfy the Cauchy–Schwarz inequality, we may define the subspace

$$\begin{aligned}\mathcal{N} &= \{u \in \mathcal{A} \odot \mathcal{H} \mid (u, u) = 0\} \\ &= \{u \in \mathcal{A} \odot \mathcal{H} \mid (u, v) = 0 \text{ for all } v \in \mathcal{A} \odot \mathcal{H}\},\end{aligned}$$

with an induced bilinear form on the quotient space $A \odot \mathcal{H}/\mathcal{N}$ defined by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle = (u, v).$$

Define $\mathcal{K}$ to be the Hilbert space completion of $\mathcal{A} \odot \mathcal{H}/\mathcal{N}$. Now, define a linear map $\pi(a)\colon \mathcal{A} \odot \mathcal{H} \to \mathcal{A} \odot \mathcal{H}$ by

$$\pi(a)\left(\sum_{i=1}^{n} a_i \otimes x_i \right) = \sum_{i=1}^{n} (a a_i) \otimes x_i.$$

We have that the inequality in $\mathbb{M}_n(\mathcal{A})$ given by

$$\left(a_i^* a^* a a_j\right)_{i,j} \leq \|a^* a\| \left(a_i^* a_j\right)_{i,j}$$

is satisfied, giving

$$\begin{aligned}\left(\pi(a)\left(\sum_{j=1}^{n} a_j \otimes x_j \right), \pi(a)\left(\sum_{i=1}^{n} a_i \otimes x_i \right) \right) &= \sum_{i,j=1}^{n} \langle \phi(a_i^* a^* a a_j) x_j, x_i \rangle \\ &\leq \|a^* a\| \sum_{i,j=1}^{n} \langle \phi(a_i^* a_j) x_j, x_i \rangle \\ &= \|a\|^2 \left(\sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right).\end{aligned}$$

Therefore, $\pi(a)$ is invariant under $\mathcal{N}$, so induces a quotient map on $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$, which we will also denote by $\pi(a)$. We have that (this new) $\pi(a)$ is bounded with $\|\pi(a)\| \leq \|a\|$, meaning that it extends to a bounded linear map on $\mathcal{K}$.

We define $V\colon \mathcal{H} \to \mathcal{K}$ by

$$V(x) = 1 \otimes x + \mathcal{N}.$$

Then,

$$\|Vx\|^2 = (1 \otimes x, 1 \otimes x)$$

$$\begin{aligned} &= \langle \phi(1)x, x\rangle \\ &\leq \|\phi(1)\|\|x\|^2, \end{aligned}$$

and $\|V\|^2 = \|\phi(1)\|$.

Finally, we observe that

$$\begin{aligned} \langle V^*\pi(a)Vx, y\rangle &= (\pi(a)1\otimes x, 1\otimes y) \\ &= \langle \phi(a)x, y\rangle \end{aligned}$$

for all x and y, so $V^*\pi(a)V = \phi(a)$. □

We observe that if ϕ is unital, then V is an isometry, and we may identify $\mathcal{H}$ with the subspace $V\mathcal{H}$ of $\mathcal{K}$, and that $\phi(a) = P\pi(a)P$, where P is the projection onto $\mathcal{H}$. In particular, this means that every unital completely positive map is the compression of a $*$-homomorphism.

This construction is very similar to the GNS representation, and we call the triple $(\pi, V, \mathcal{K})$ a Stinespring representation for ϕ.

Nuclearity and Exactness

Definition: We call a map $\theta\colon A \to B$ between C^*-algebras *nuclear* if there are contractive completely positive maps $\varphi_n\colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n\colon \mathbb{M}_{k(n)}(\mathbb{C}) \to B$ such that $\psi_n \circ \varphi_n \to \theta$ pointwise in the norm topology:

$$\|\psi_n \circ \varphi_n(a) \to \theta(a)\| \to 0$$

for all $a \in A$.

Definition: If A is a C^*-algebra, and N is a von Neumann algebra, then a map $\theta\colon A \to N$ is called *weakly nuclear* if there exist contractive completely positive maps $\varphi_n\colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n\colon \mathbb{M}_{k(n)}(\mathbb{C}) \to N$ such that $\psi_n \circ \varphi_n \to \theta$ pointwise in the ultraweak topology:

$$\eta(\psi_n \circ \varphi_n(a)) \to \eta(\theta(a))$$

for all $a \in A$ and normal functionals $\eta \in N_*$.

By uniqueness of preduals, if $N \subseteq B(\mathcal{H})$ is a faithful normal representation, then it suffices to observe that $\psi_n \circ \varphi_n \to \theta$ pointwise in the ultraweak topology if and only if

$$\langle \psi_n \circ \varphi_n(a)v, w\rangle \to \langle \theta(a)v, w\rangle$$

for all $a \in A$ and $v, w \in \Omega$ for some collection Ω of vectors whose linear span is dense in $\mathcal{H}$.

One of the interesting aspects of nuclear maps is that whether a map is nuclear or not depends on the range.

Proposition: Let $M \subseteq B(H)$ be a von Neumann algebra. The natural inclusion map $M \hookrightarrow B(H)$ is always weakly nuclear.

Proof. Let $\{P_i\}_{i\in I}$ be a net of finite rank projections increasing to the identity. If P_i has rank $k(i)$, then we may define maps $\varphi_i\colon M \to \mathbb{M}_{k(i)}(\mathbb{C}) \cong P_iB(H)P_i$ by compression, and let $\psi_i\colon \mathbb{M}_{k(i)}(\mathbb{C}) \to B(H)$ be natural inclusion maps.

Since the predual of $B(H)$ is the trace class operators, we have that these maps converge weakly to the identity on $B(H)$, it follows that $M \hookrightarrow B(H)$ is weakly nuclear. □

Proposition: A map $\theta\colon A \to B$ is nuclear if and only if for every finite $F \subseteq A$ and every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and contractive completely positive maps $\varphi\colon A \to \mathbb{M}_n(\mathbb{C})$, $\psi\colon \mathbb{M}_n(\mathbb{C}) \to B$ such that $\|\theta(a) - \psi \circ \varphi(a)\| < \varepsilon$ for all $a \in F$.

Proof. Define the set $\mathcal{F} = \{(F, \varepsilon) \mid F \subseteq A \text{ finite}, \varepsilon > 0\}$, directed by

$$(F_1, \varepsilon_1) \preceq (F_2, \varepsilon_2) \Leftrightarrow F_1 \subseteq F_2 \text{ and } \varepsilon_2 \leq \varepsilon_1.$$

It can then be verified that convergence in the point-norm topology in the definition for nuclearity is equivalent to convergence via this directed set, and vice versa. $\square$

Definition: A C^*-algebra A is called *nuclear* if the identity map is nuclear.

Definition: A C^*-algebra A is called *exact* if there exists a faithful representation $\pi\colon A \to B(H)$ such that π is nuclear.

Definition: A von Neumann algebra M is called *semidiscrete* if the identity map is weakly nuclear.

We will show now that if the double dual of a C^*-algebra is semidiscrete, then the C^*-algebra is nuclear.

Lemma: Let A be a Banach space, and let $B(A)$ be the space of all bounded linear maps from A to A, and let $C \subseteq B(A)$ be a convex set. Then, the point-weak and point-norm closures of C coincide.

Application to Amenability

References

[Pau02] Vern Paulsen. *Completely bounded maps and operator algebras*. Vol. 78. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+300. ISBN: 0-521-81669-6.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+509. ISBN: 978-0-8218-4381-9; 0-8218-4381-8. DOI: 10.1090/gsm/088. URL: https://doi.org/10.1090/gsm/088.