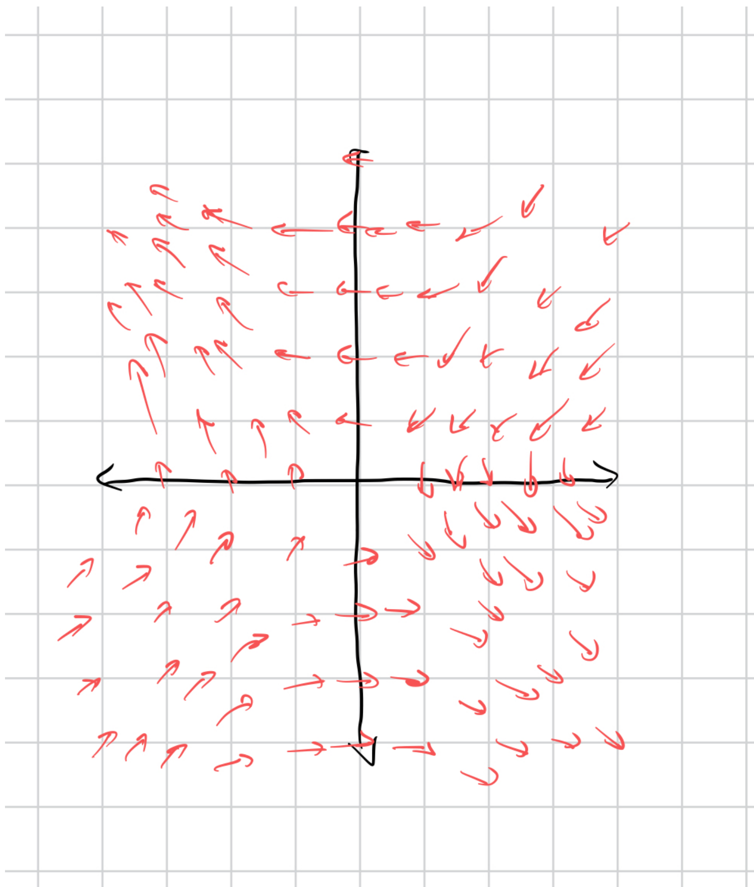
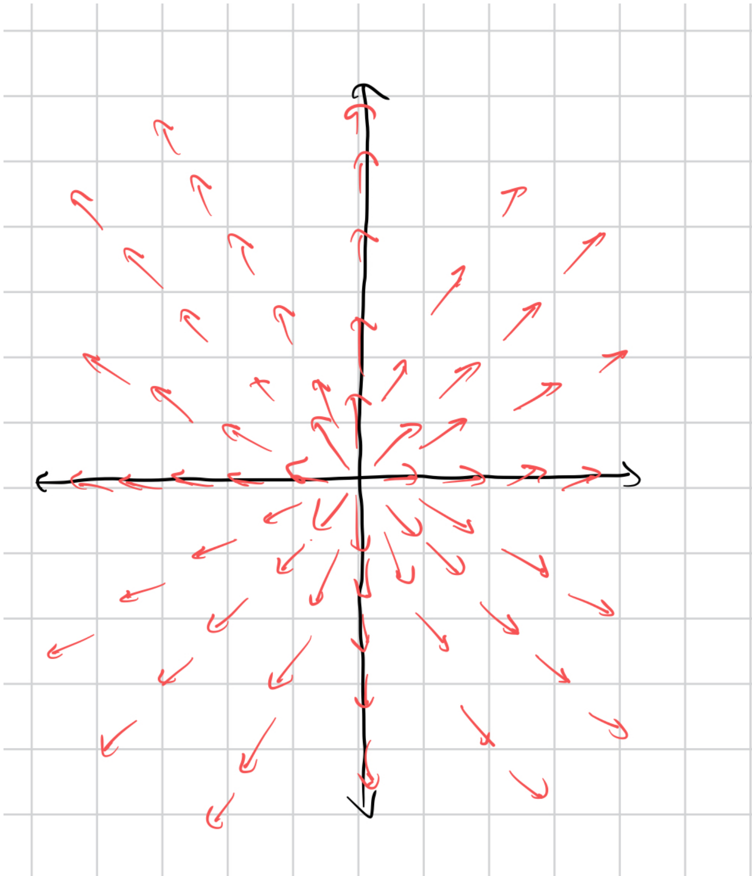
Chapter 11 Problems

Problem 1

(a) $\mathbf{F}(\mathbf{x}) = \frac{1}{\rho}\hat{\rho}$.



(b) $\mathbf{F}(\mathbf{x}) = y\hat{\imath} + x\hat{\jmath}$.



Problem 2

The parametrized streamlines for $\mathbf{v} = (-y, x)$ are of the form $r\cos t\hat{\imath} + r\sin t\hat{\jmath}$.

Problem 3

We can see that $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular by taking the standard inner product

$$\left\langle xy^2\hat{\imath} + x^2y\hat{\jmath}, x^2y\hat{\imath} - xy^2\hat{\jmath}\right\rangle = 0.$$

Additionally, for $\mathbf{E}$,

$$\begin{aligned}\frac{dy}{dt} &= x^2y\\ \frac{dx}{dt} &= xy^2\\ \frac{dy}{dx} &= \frac{x}{y}\\ y^2 &= x^2 + K,\end{aligned}$$

and for $\mathbf{B}$,

$$\frac{dy}{dt} = -xy^2$$

$$\begin{aligned}\frac{dx}{dt} &= x^2 y\\ \frac{dy}{dx} &= -\frac{y}{x}\\ y &= \frac{K}{x}.\end{aligned}$$

Problem 4

(a)

$$\begin{aligned}\int_V \mathbf{E}(\mathbf{r})\ d^3x &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^R \hat{r}\sin\theta\ dr d\phi d\theta\\ \int_V \mathbf{E}(\mathbf{r})\ d^3x &= \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_0^{\sqrt{R^2-x^2-y^2}} \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\ dz dy dx\end{aligned}$$

(b)

$$\int_V \mathbf{E}(\mathbf{r})\ d^3x = \int_0^{\pi/2}\int_0^{2\pi}\int_0^R \sin\theta\left(\cos\phi\sin\theta\hat{\imath}+\sin\phi\sin\theta\hat{\jmath}+\cos\theta\hat{k}\right)\ dr d\phi d\theta$$

This integral is more practical than the pure forms since the basis is position-independent and the integral is not a giant mess.

(c) Using symmetry, since $\cos\phi$ is integrated from 0 to 2π and $\sin\phi$ is integrated from 0 to 2π, both the $\hat{\imath}$ and $\hat{\jmath}$ components are 0.

$$\begin{aligned}\int_0^{\pi/2}\sin^2\theta\int_0^{2\pi}\cos\phi\int_0^R\ dr d\phi d\phi &= 0\\ \int_0^{\pi/2}\sin^2\theta\int_0^{2\pi}\sin\phi\int_0^R\ dr d\phi d\phi &= 0\end{aligned}$$

(d) Evaluating the $\hat{k}$ component,

$$\begin{aligned}\int_0^{\pi/2}\sin\theta\cos\theta\int_0^{2\pi}\int_0^R\ dr d\phi d\theta &= 2\pi R\int_0^{\pi/2}\sin\theta\cos\theta\ d\theta\\ &= \pi R.\end{aligned}$$

Problem 5

$$\begin{aligned}\mathbf{R}_{cm} &= \frac{1}{M}\int_S \mathbf{r}\, dm\\ &= \frac{\sigma}{M}\int_{-\ell/2}^{\ell/2}\int_0^{\pi}\left(R\cos\phi\hat{\imath}+R\sin\phi\hat{\jmath}+z\hat{k}\right)R\ d\phi dz\\ &= \frac{\sigma}{M}\left(2R^2\right)\hat{\jmath}.\end{aligned}$$

Chapter 12 Problems

Problem 1

(a) Letting $f(\mathbf{x}) = \rho$, we have

$$\nabla f = \hat{\rho}$$

in cylindrical coordinates, and

$$\nabla f = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath}$$

in Cartesian coordinates. These results are equal to each other by the definition of $\hat{\rho}$.

(b) Letting $f(\mathbf{x}) = y$, we have

$$\nabla f = \hat{\jmath}$$

in Cartesian coordinates, and

$$\nabla f = \sin\phi\hat{\rho} + \cos\phi\hat{\phi},$$

which yields $\hat{\jmath}$ under the coordinate conversion.

(c) Letting $f(\mathbf{x}) = z\rho^2$, we have

$$\nabla f = 2\rho z\hat{\rho} + \rho^2\hat{k}$$

in cylindrical coordinates, and

$$\nabla f = 2xz\hat{\imath} + 2yz\hat{\jmath} + \left(x^2+y^2\right)\hat{k},$$

which is equal under the coordinate conversion.

(d) Letting $f(\mathbf{x}) = \rho^2\tan\phi$, we have

$$\nabla f = 2\rho\tan\phi\hat{\rho} + \rho\sec^2\phi\hat{\phi}$$

and

$$\nabla f = \left(y - \frac{y^3}{x^2}\right)\hat{\imath} + \left(x + \frac{3y^2}{x}\right)\hat{\jmath},$$

which is equal under the coordinate conversion.

Problem 2

(a) Let $f(\mathbf{x}) = r\sin\theta\cos\phi$. Then,

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}\\ &= \sin\theta\cos\phi\hat{r} - \sin\phi\hat{\phi} + \cos\theta\cos\phi\hat{\theta},\end{aligned}$$

and

$$\nabla\cdot(\nabla f) = 0.$$

(b) Let $f(\mathbf{x}) = \ln\rho^2$. Then,

$$\begin{aligned}\nabla f &= \frac{2}{\rho}\hat{\rho}\\ \nabla\cdot(\nabla f) &= -\frac{2}{\rho^2}.\end{aligned}$$

(c) Let $f(\mathbf{x}) = x\cos y$. Then,

$$\nabla f = \cos y\hat{\imath} - x\sin y\hat{\jmath},$$

and

$$\nabla\cdot(\nabla f) = -x\cos y$$

(d) Let $f(\mathbf{x}) = x\left(y^2 - 1\right)$. Then,

$$\nabla f = \left(y^2 - 1\right)\hat{\imath} + 2xy\hat{\jmath},$$

and

$$\nabla\cdot(\nabla f) = 2x.$$

Problem 3

(a)

$$\begin{aligned}
\mathbf{r} &= \vec{r}\\
&= r\hat{r}\\
\nabla\cdot(r\hat{r}) &= 1\\
\nabla\times(r\hat{r}) &= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{r} &= \frac{\hat{r}}{r}\\
\nabla\cdot\left(\frac{\hat{r}}{r}\right) &= -\frac{1}{r^2}\\
\nabla\times\left(\frac{\hat{r}}{r}\right) &= 0.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{r} &= \frac{1}{r^2}\hat{\theta}\\
\nabla\cdot\left(\frac{1}{r^2}\hat{\theta}\right) &= 0\\
\nabla\times\left(\frac{1}{r^2}\hat{\theta}\right) &= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)\hat{\phi}\\
&= -\frac{1}{r^3}\hat{\phi}.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{r} &= \rho z\hat{\phi}\\
\nabla\cdot\left(\rho z\hat{\phi}\right) &= 0\\
\nabla\times\left(\rho z\hat{\phi}\right) &= -\rho\hat{\rho} + 2\rho z\hat{z}.
\end{aligned}$$

Problem 6

$$\begin{aligned}
\mathbf{B} &= \frac{1}{x^2+y^2}\left(-y\hat{\imath}+x\hat{\jmath}\right)\\
&= \frac{1}{\rho}\hat{\phi}\\
\nabla\times\mathbf{B} &= \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\hat{k}\\
&= \left(\frac{2}{x^2+y^2}-\frac{2x^2}{\left(x^2+y^2\right)^2}-\frac{2y^2}{\left(x^2+y^2\right)^2}\right)\\
&= 0\\
\nabla\times\mathbf{B} &= 0.
\end{aligned}$$

Problem 7

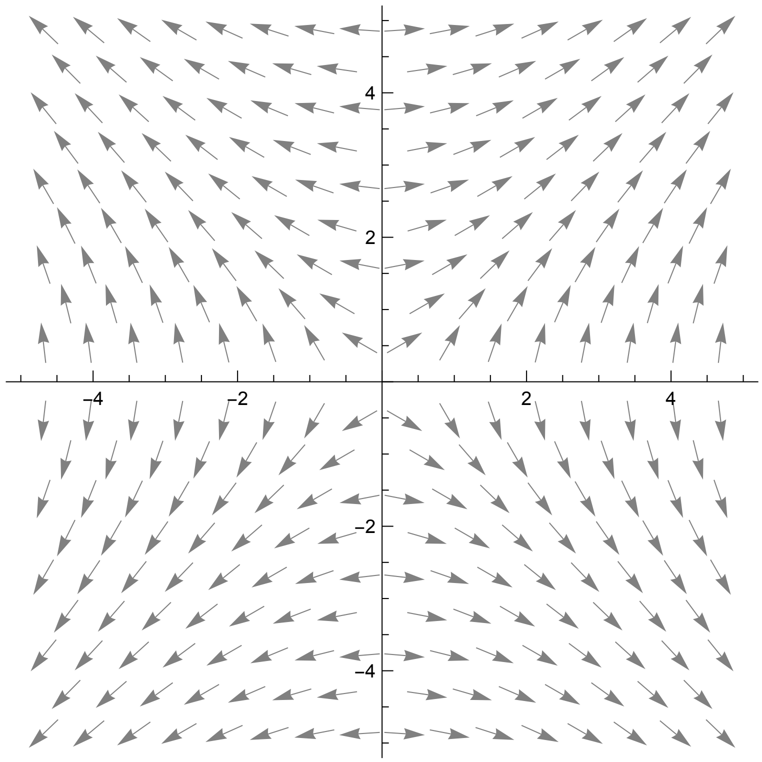
$$\begin{aligned}
\nabla\cdot\left(\nabla f(r)\right) &= \nabla\cdot\left(\frac{\partial f}{\partial r}\right)\hat{r}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{\partial f}{\partial r}\right)\right)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right).
\end{aligned}$$

Problem 9

$$\begin{aligned}
\nabla\cdot\left(\nabla\left(fg\right)\right) &= \nabla\cdot\left(g\nabla f+f\nabla g\right)\\
&= \nabla\cdot\left(g\nabla f\right)+\nabla\cdot\left(f\nabla g\right)\\
&= \nabla g\cdot\nabla f+g\left(\nabla\cdot\nabla f\right)+\nabla f\cdot\nabla g+f\left(\nabla\cdot\nabla g\right).
\end{aligned}$$

This expression is equal to $g\nabla^2 f+f\nabla^2 g$ if and only if $\nabla f\cdot\nabla g=0$ on the domain of f and g (i.e., that ∇f and ∇g are orthogonal to each other).

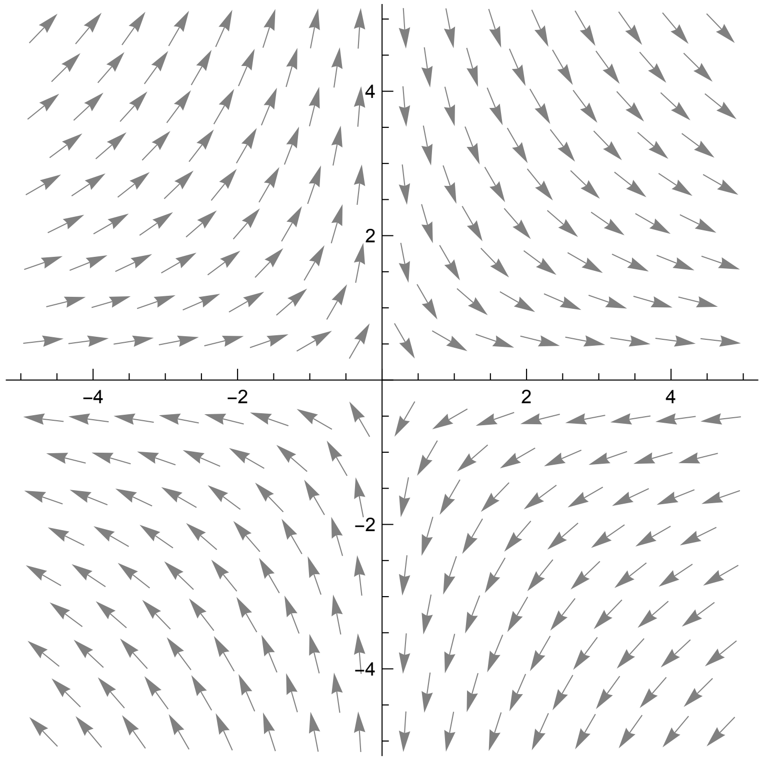
Problem 15

(a) 

Upon inspection, this field appears to have a significant amount of “surge,” but not any “swirl,” implying that its curl should be zero and its divergence positive.

$$\begin{aligned}\nabla \cdot \mathbf{E} &= y^2 + x^2\\ \nabla \times \mathbf{E} &= \left(\frac{\partial}{\partial x}\left(x^2 y\right) - \frac{\partial}{\partial y}\left(x y^2\right)\right)\hat{k}\\ &= 0.\end{aligned}$$

(b)



Upon inspection, this field appears to have a significant amount of "swirl," but not any "surge," implying that its divergence should be zero and its curl should be nonzero.

$$
\begin{aligned}
\nabla \cdot \mathbf{B} &= \frac{\partial}{\partial x}\left(x^2 y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\\
&= 0\\
\nabla \times \mathbf{B} &= \left(\frac{\partial}{\partial x}\left(-xy^2\right) - \frac{\partial}{\partial y}\left(x^2 y\right)\right)\hat{k}\\
&= -\left(x^2 + y^2\right)\hat{k}.
\end{aligned}
$$

Problem 19

(a)

$$
\begin{aligned}
\nabla \cdot (\phi \mathbf{A}) &= \sum_{i,j} \frac{\partial}{\partial i}\left(\phi A_j\right)\delta_{ij}\\
&= \sum_{i} \frac{\partial}{\partial i}\left(\phi A_i\right)\\
&= \sum_{i} \left(A_i \frac{\partial}{\partial i}\phi + \phi \frac{\partial}{\partial i} A_i\right)\\
&= \mathbf{A} \cdot (\nabla \phi) + \phi\left(\nabla \cdot \mathbf{A}\right).
\end{aligned}
$$

(b)

$$
\nabla \times (\phi \mathbf{A}) =
$$

Chapter 13 Problems

Problem 2