Abstract

We detail the construction and some of the properties of the Lebesgue measure.

Premeasures, Outer Measures, and Measures

Consider a set-function $\lambda \colon P(\mathbb{R}) \to [0, \infty]$ that satisfies

- $\lambda(\varnothing) = 0$;

- for any finite or infinite sequence of disjoint sets, $\{E_j\}_{j=1}^{\infty}$, we have

$$\lambda\left(\bigsqcup_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} \lambda(E_j);$$

- $\lambda(I) = b - a$, where I is an interval (either open, closed, or a half-interval);

- $\lambda(s + E) = \lambda(E)$.

Unfortunately, such a set-function doesn't exist.

In order to construct a set function on a restricted domain $\lambda \colon \mathcal{L} \to [0, \infty]$, we need to define a particular class of measurable subsets of $\mathbb{R}$. This is where the concept of an *outer measure* comes in.

Definition. Let X be a set, and let $\mu^* \colon P(X) \to [0, \infty]$ be a set function. We say μ^* is an *outer measure* if

- $\mu^*(\varnothing) = 0$;

- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$;

- $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j).$

We will obtain an outer measure on the entirety of $P(X)$ by defining a notion of measure on some "satisfactory" subfamily $\mathcal{E} \subseteq P(X)$, then by approximating other subsets using this family.

Proposition: Let $\mathcal{E} \subseteq P(X)$ be a family of subsets such that $\varnothing \in \mathcal{E}$ and $X \in \mathcal{E}$, and let $\rho \colon \mathcal{E} \to [0, \infty]$ be a set function such that $\rho(\varnothing) = 0$. For any $A \subseteq X$, define

$$\mu^*(E) = \inf\left\{\sum_{j \geq 1} \rho(E_j) \;\middle|\; E_j \in \mathcal{E}, A \subseteq \bigcup_{j \geq 1} E_j\right\}.$$

Then, μ^* is an outer measure.

Proof. We start by showing well-definedness, which stems from the fact that we may select $E_j = X$ for all j.

Since we may take $E_j = \varnothing$ for all j, we must have $\mu^*(\varnothing) = 0$. Furthermore, if $A \subseteq B$, since the set over which the infimum is taken for the definition of $\mu^*(A)$ includes the corresponding set for B, we must have $\mu^*(A) \leq \mu^*(B)$.

Finally, let $\{A_j\}_{j \geq 1} \subseteq P(X)$, and let $\varepsilon > 0$. For each j, there exists $\{E_{j,k}\}_{k \geq 1} \subseteq \mathcal{E}$ such that $A_j \subseteq \bigcup_{k \geq 1} E_{j,k}$ and $\sum_{k \geq 1} \rho(E_{j,k}) \leq \mu^*(A_j) + \varepsilon 2^{-j}$.

Then, if $A = \bigcup_{j \geq 1} A_j$, we have $A \subseteq \bigcup_{j,k \geq 1} E_{j,k}$, and $\sum_{j,k \geq 1} \rho(E_{j,k}) \leq \sum_{j \geq 1} \mu^*(A_j) + \varepsilon$, so that $\mu^*(A) \leq \sum_{j \geq 1} \mu^*(A_j) + \varepsilon$. Since ε is arbitrary, we are done. $\square$

Definition. A subset $A \subseteq X$ is said to be μ^*-*measurable* if for any $E \subseteq X$, A serves as a good "cookie cutter" for E, in that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Equivalently, due to subadditivity, we have A is measurable if and only if for all $E \subseteq X$,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition. Let $\mathcal{A}$ be an algebra of subsets of X. We call a set function $\mu_0 \colon \mathcal{A} \to [0, \infty]$ a *premeasure* if

- $\mu_0(\varnothing) = 0$;

- for a collection of disjoint elements of $\mathcal{A}$, $\{A_j\}_{j=1}^\infty$ where $\bigcup_{j \geq 1} A_j \in \mathcal{A}$, we have

$$\mu_0\left(\bigsqcup_{j \geq 1} A_j\right) = \sum_{j \geq 1} \mu_0(A_j).$$

Every premeasure gives rise to an outer measure by taking

$$\mu^*(E) = \inf\left\{\sum_{j \geq 1} \mu_0(A_j) \;\middle|\; A_j \in \mathcal{A}, E \subseteq \bigcup_{j \geq 1} A_j \right\}. \tag{$*$}$$

A remarkable result by Caratheodory allows us to extend premeasures from algebras to measures on σ-algebras. To start, there is a little bit of build-up.

Proposition: Let μ_0 be a premeasure on $\mathcal{A}$, with μ^* defined by $(*)$. Then,

(a) $\mu^*|_{\mathcal{A}} = \mu_0$;

(b) every set in $\mathcal{A}$ is μ^*-measurable.

Proof. Suppose $E \in \mathcal{A}$. If $E \subseteq \bigcup_{j \geq 1} A_j$ with $A_j \in \mathcal{A}$, we let $B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right)$. The B_n are disjoint members of $\mathcal{A}$ whose union is E, so

$$\mu_0(E) = \sum_{j=1}^\infty \mu_0(B_j)$$

$$\leq \sum_{j=1}^\infty \mu_0(A_j).$$

It follows that $\mu_0(E) \leq \mu^*(E)$. The reverse inequality is clear from the fact that we may specify $A_1 = E$ and $A_{j>1} = \varnothing$.

Meanwhile, if $A \in \mathcal{A}$, $E \subseteq X$, and $\varepsilon > 0$, then there is a collection $\{B_j\}_{j \geq 1} \subseteq \mathcal{A}$ with $E \subseteq \bigcup_{j \geq 1} B_j$ and $\sum_{j \geq 1} \mu_0(B_j) \leq \mu^*(E) + \varepsilon$. By additivity on $\mathcal{A}$, we get

$$\mu^*(E) + \varepsilon \geq \sum_{j=1}^\infty \mu_0(B_j \cap A) + \mu_0(B_j \cap A^c)$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

so A is measurable. $\qquad\square$

Theorem (Caratheodory's Theorem)**:** Let $\mathcal{A} \subseteq P(X)$ be an algebra, let μ_0 be a premeasure on $\mathcal{A}$, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 — namely, $\mu - \mu^*|_{\mathcal{M}}$, where μ^* is given by $(*)$.

If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$, with equality for all $\mu(E) < \infty$. Furthermore, if μ_0 is σ-finite, then μ is unique.