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Collaborators: Gianluca Crescenzo, Noah Smith, Carly Venenciano

Problem 3

Problem. Let V be an \mathbb{F} -vector space.

- (a) Prove that an arbitrary intersection of subspaces of V is again a subspace of V .
- (b) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution.

- (a) Let $U, W \subseteq V$ be subspaces. Since U and W are subspaces, $0_V \in U$ and $0_V \in W$, meaning $U \cap W$ is nonempty.

Let $u, w \in U \cap W$, and let $\alpha \in \mathbb{F}$. Then, since $u \in U$ and $w \in U$, it is the case that $u + \alpha w \in U$. Similarly, since $u \in W$ and $w \in W$, it is the case that $u + \alpha w \in W$. Thus, $u + \alpha w \in U \cap W$, meaning $U \cap W$ is a subspace.

Having shown the base case, we let $\bigcap_{k=1}^N U_k$ be an intersection of subspaces U_k . By the inductive hypothesis, we have $W = \bigcap_{k=1}^N U_k$, where W is a subspace.

- (b) Let $U, W \subseteq V$ be subspaces.

In the reverse direction, if, without loss of generality, $U \subseteq W$, then it is the case that $U \cup W = W$, meaning that $U \cup W$ is a subspace of V .

In the forward direction, suppose toward contradiction that there exist subspaces $U, W \subseteq V$ such that $U \not\subseteq W$ and $W \not\subseteq U$, but $U \cup W$ is a subspace of V . Since $U \not\subseteq W$ and $W \not\subseteq U$, there exist non-trivial vectors $w \in W \setminus U$ and $u \in U \setminus W$. Since $w + u \in W \cup U$, it is the case that $w + u$ is contained either in U or in W . If $w + u \in U$, then $(w + u) - u \in U$ (as $u \in U$ and U is a subspace), meaning $w \in U$, which is a contradiction. Similarly, if $w + u \in W$, then $(w + u) - w \in W$, or $u \in W$, which is yet again a contradiction.

Thus, it must be the case that $W \subseteq U$ or $U \subseteq W$.

Problem 4

Problem. Let $T \in \text{Hom}_{\mathbb{F}}(\mathbb{F}, \mathbb{F})$. Prove there exists $\alpha \in \mathbb{F}$ such that $T(v) = \alpha v$ for all $v \in \mathbb{F}$.

Solution. Since $\dim_{\mathbb{F}}(\mathbb{F}) = 1$, we know that the basis of \mathbb{F} is $\{\beta\}$ for some $\beta \in \mathbb{F}$. For $v \in \mathbb{F}$, it is then the case that v is a linear combination of the basis of \mathbb{F} over \mathbb{F} , meaning $v = v_0\beta$ for some $v_0 \in \mathbb{F}$, implying $\beta = (v_0^{-1})v$.

Considering a linear transformation $T(v)$, we have

$$T(v) = T(v_0\beta).$$

Substituting $\beta = v_0^{-1}v$, and using the commutativity and associativity of multiplication under \mathbb{F} , we have

$$T(v) = T\left(v\left(v_0^{-1}\right)\right).$$

Using the fact that T is linear and $v \in \mathbb{F}$, we have

$$\begin{aligned} &= vT(v_0^{-1}v_0) \\ &= vT(1). \end{aligned}$$

Thus, $\alpha = T(1)$.

Problem 6

Problem. Let V be an \mathbb{F} -vector space. Prove that if $\{v_1, \dots, v_n\}$ is linearly independent, then so is the set $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$.

Solution. To prove that $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ is linearly independent, we consider the sum

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \dots + a_{n-1}(v_{n-1} - v_n) + a_nv_n,$$

and show that this sum equals zero if and only if $a_i = 0$ for each i . Rearranging the sum, we have

$$a_1v_1 + (a_2 - a_1)v_2 + \dots + (a_{n-1} - a_{n-2})v_{n-1} + (a_n - a_{n-1})v_n.$$

Since the set $\{v_1, \dots, v_n\}$ are linearly independent, this linear combination equals 0_V if and only if $a_1 = (a_2 - a_1) = \dots = a_n - a_{n-1} = 0$. In particular, since $a_1 = 0$, it must be the case that $a_2 = 0$, $a_3 = 0$, and so on.

Thus, $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ are linearly independent.

Problem 9

Problem. Let V be a finite-dimensional vector space and $T \in \text{Hom}_{\mathbb{F}}(V, V)$ with $T^2 = T$.

- Prove that $\text{im}(T) \cap \ker(T) = \{0\}$.
- Prove that $V = \text{im}(T) \oplus \ker(T)$.
- Let $V = \mathbb{F}^n$. Prove that there is a basis of V such that the matrix of T with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

Problem 13

Problem. Let p be a prime and V a dimension n vector space over \mathbb{F}_p . Show there are

$$(p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})$$

distinct bases of V .