Abstract

We detail the construction necessary to prove Urysohn's Lemma, which completely characterizes normal topological spaces via separation using continuous functions.

In this document, we will prove the following theorem.

Theorem (Urysohn's Lemma)**:** Let X be a topological space. Then, X is normal if and only if, for all closed, disjoint $A, B \subseteq X$, there is a continuous function $f\colon X \to [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Definition: A topological space X is normal if, for any closed, disjoint subsets $A, B \subseteq X$, there are open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

We may prove one direction of Urysohn's lemma already.

Proof of Reverse Direction. Suppose X is a topological space such that for all disjoint closed subsets $A, B \subseteq X$, there is a continuous $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then, by taking $U \coloneqq f^{-1}((-\infty, 1/2) \cap [0,1])$ and $V \coloneqq f^{-1}((1/2, \infty) \cap [0,1])$, we have $U \cap V = \emptyset$ and $A \subseteq U$, $B \subseteq V$. $\square$

The reverse direction is, unfortunately, quite a bit more difficult. To do this, we will construct a family of open sets that will allow us to define our continuous function afterward. This construction will follow similar proofs in *A Taste of Topology* by Runde and *Real Analysis* by Folland, although it will (probably) be more detailed.

Lemma: Let A and B be disjoint subsets of a normal topological space X, and let

$$\Delta \coloneqq \left\{ k2^{-n} \;\middle|\; n \geq 1, 0 < k < 2^n \right\}$$

be the set of dyadic rationals in $(0,1)$. Then, there is a family $\{U_r \mid r \in \Delta\} \subseteq \tau_X$ such that $A \subseteq U_r \subseteq B^c$ for all $r \in \Delta$, and $\overline{U_r} \subseteq U_s$ whenever $r < s$.

Proof. We start by showing that if $A \subseteq U$, then there is an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$. Note that if $A \subseteq U$, then A and U^c are disjoint closed sets, so since X is normal, there are disjoint open sets V and W such that $A \subseteq V$ and $U^c \subseteq W$. Note that since $V \subseteq W^c$, and W^c is closed, we have $A \subseteq V \subseteq \overline{V} \subseteq W^c \subseteq U$, which is our desired result.

Now, since B^c is open, and $A \subseteq B^c$, we have an open set $U_{1/2}$ such that $A \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq B^c$. Similarly, since $\overline{U}_{1/2} \subseteq B^c$, we have $U_{3/4} \subseteq B_c$ such that $\overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq B^c$, and similarly for $A \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{1/2}$.

Continuing in this process, we are able to construct a family $\{U_r\}_{r\in\Delta} \subseteq \tau_X$ such that $A \subseteq U_r \subseteq \overline{U}_r \subseteq U_s \subseteq \overline{U}_s \subseteq B^c$ whenever $r < s$. $\square$

Now, we may prove Urysohn's Lemma by using this family $\{U_r\}_{r\in\Delta}$.

Proof of Urysohn's Lemma. Let $\{U_r\}_{r\in\Delta}$ be our family with $U_1 \coloneqq X$.

For $x \in X$, we define $f(x) = \inf\{r \mid x \in U_r\}$. Since $A \subseteq U_r \subseteq B^c$ for $0 < r < 1$, we have $f(x) = 0$ for all $x \in A$, $f(x) = 1$ for all $x \in B$, and $0 \leq f(x) \leq 1$ for all $x \in X$. Now, all we need to show is that f is continuous.

Observe that $f(x) < \alpha$ if and only if $x \in U_r$ for some $r < \alpha$, which holds if and only if $x \in \bigcup_{r<\alpha} U_r$. Thus, $f^{-1}((-\infty, \alpha)) = \bigcup_{r<\alpha} U_r$ is open. Similarly, $f(x) > \alpha$ if and only if $x \notin U_r$ for some $r > \alpha$, which holds if and only if $x \notin \overline{U}_s$ for some $s > \alpha$, as $\overline{U}_s \subseteq U_r$ when $s < r$. Thus, this holds if and only if $x \in \bigcup_{s>\alpha} \left(\overline{U}_s\right)^c$, so $f((\alpha, \infty)) = \bigcup_{s>\alpha} \left(\overline{U}_s\right)^c$ is open.

Since the open half-lines generate the topology on $\mathbb{R}$, f is continuous. $\square$