Problem (Problem 1)**:** Let I, J, K be ideals of R.

(a) Show that $(IJ)K = I(JK)$.

(b) Show that $(I + J)K = IK + JK$.

Solution:

(a) Let $u \in (IJ)K$. Then, u is of the form

$$u = \sum_{k=1}^{n} u_k z_k,$$

where the $u_k \in IJ$ and the $z_k \in K$. Since each u_k is an element of IJ, we may write

$$u_k = \sum_{i=1}^{m} x_{k_i} y_{k_i},$$

where the $x_{k_i} \in I$ and the $y_{k_i} \in J$. This yields an expression

$$u = \sum_{k=1}^{n} \left(\sum_{i=1}^{m} x_{k_i} y_{k_i} \right) z_k$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{m} x_{k_i} y_{k_i} z_k.$$

We observe that, for a fixed k, $y_{k_i} z_k \in JK$. So, $x_{k_i}(y_{k_i} z_k) \in I(JK)$ for a fixed k, meaning that $u \in I(JK)$. A similar argument holds in the reverse direction.

(b) Elements of $I + J$ are of the form $x_i + y_i$, where $x_i \in I$ and $y_i \in J$. This means that elements of $(I + J)K$ are of the form

$$u = \sum_{k=1}^{n} \sum_{i=1}^{m} (x_i + y_i) z_k$$

$$= \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{m} x_i \right)}_{=: x_k} z_k + \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{m} y_i \right)}_{=: y_k} z_k$$

$$= \sum_{k=1}^{n} x_k z_k + \sum_{k=1}^{n} y_k z_k.$$

Thus, we find that u is in $IK + JK$, and vice versa.

Problem (Problem 4)**:** Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and let $\iota_{S_i} : R \to S_i^{-1}R$ be the corresponding localization homomorphisms. Use the universal property of localization to show that there exists a unique ring homomorphism $\iota' : S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. Provide an explicit description of this ring homomorphism. Use this to show that if R is an integral domain and S an arbitrary multiplicative subset of R, then $S^{-1}R$ injects into the fraction field $K = \operatorname{frac}(R)$.

Solution: We observe that $\iota_{S_2} : R \to S_2^{-1}R$ maps elements of S_1 to units in $S_2^{-1}R$, as the units in $S_2^{-1}R$ are elements of the form $\frac{s}{s'}$ with $s, s' \in S_2$, so by the universal property, there is a unique ring homomorphism $\iota' : S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. In particular, this is the map $\left[\frac{r}{1} \right]_{S_1^{-1}R} \mapsto \left[\frac{r}{1} \right]_{S_2^{-1}R}$.

Since any arbitrary multiplicative subset $S \subseteq R$ of an integral domain is contained in $R \setminus \{0\}$, it follows that $S^{-1}R$ injects into $(R \setminus \{0\})^{-1}R =: \operatorname{frac}(R)$.

Problem (Problem 5): Let $R = \mathbb{Q} \times \mathbb{Q}$ and $S = \{(1,1)\} \cup (\mathbb{Q}^\times \times \{0\})$. The goal of this problem is to identify the localization $S^{-1}R$.

(a) Describe explicitly when $\frac{(a_1, a_2)}{(s_1, s_2)}$ is equal to $\frac{(b_1, b_2)}{(t_1, t_2)}$ in $S^{-1}R$.

(b) Use your result from part (a) to show that the localization $S^{-1}R$ is isomorphic to the localization $T^{-1}\mathbb{Q}$, where $T = \mathbb{Q} \setminus \{0\}$, hence is isomorphic to $\mathbb{R}$.

(c) Find the kernel of the localization homomorphism $\iota_S : R \to S^{-1}R$.

Solution:

(a) By the definition of the equivalence relation, we must have an element $(r_1, r_2) \in S$ such that

$$(r_1(a_1 t_1 - b_1 s_1), r_2(a_2 t_2 - b_2 s_2)) = (0, 0).$$

In particular, since $r_1 \in \mathbb{Q}^\times$, and we may always select $r_2 = 0$, it follows that

$$r_1(a_1 t_1 - b_1 s_1) = 0,$$

so that $a_1 t_1 - b_1 s_1 = 0$ (as $\mathbb{Q}$ is an integral domain).

(b) We consider the map $\pi_1 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, which maps $(a_1, a_2) \mapsto a_1$. Observe then that $S^{-1}R$ satisfies the universal property for localization, as we may write $S = (\mathbb{Q}^\times \times \{0\}) \cup (\mathbb{Q}^\times \cup \{1\})$, which maps to $\mathbb{Q}^\times \subseteq \mathbb{Q}$ under this projection map.

In particular, we see that the induced map $\widetilde{\pi_1} : S^{-1}R \to \mathbb{Q}$ is given by

$$\widetilde{\pi_1}\left(\frac{(a_1, a_2)}{(s_1, s_2)} \right) = a_1 s_1^{-1}$$

for $s_1 \in \mathbb{Q}^\times$ and $a_1 \in \mathbb{Q}$.

Now, we observe that the map $\mathrm{id} \circ \pi_1 = \pi_1$, and that $T^{-1}\mathbb{Q}$ satisfies the universal property for localization with respect to id, inducing the homomorphism $\widetilde{\mathrm{id}}$ that takes

$$\widetilde{\mathrm{id}}\left(\frac{a}{s} \right) = a s^{-1}$$

for $s \in \mathbb{Q}^\times$. Yet, we also observe that, if we set $\iota_T' = \iota_T \circ \widetilde{\pi_1} \circ \iota_S$, that

$$\begin{aligned}
\widetilde{\mathrm{id}} \circ \iota_T'(a_1, a_2) &= \widetilde{\mathrm{id}} \circ \iota_T \circ \widetilde{\pi_1} \circ \iota_S(a_1, a_2) \\
&= \widetilde{\mathrm{id}} \circ \iota_T \circ \widetilde{\pi_1}\left(\frac{(a_1, a_2)}{(1,1)} \right) \\
&= \widetilde{\mathrm{id}} \circ \iota_T(a_1) \\
&= \widetilde{\mathrm{id}}\left(\frac{a_1}{1} \right) \\
&= a_1 \\
&= \pi_1(a_1, a_2).
\end{aligned}$$

Thus, $T^{-1}\mathbb{Q}$ also satisfies the universal property for localization, implying that $T^{-1}\mathbb{Q}$ and $S^{-1}R$ are isomorphic.
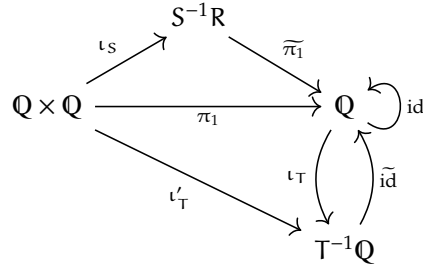
(c)

Problem (Problem 7): Let $S \subseteq R$ be a multiplicative subset, and let $\iota_S \colon R \to S^{-1}R$ be the corresponding localization homomorphism. Consider the map

$$\alpha \colon \left\{ P' \mid P' \text{ is a prime ideal of } S^{-1}R \right\} \to \left\{ P \mid P \text{ is a prime ideal of } R \text{ such that } S \cap P = \emptyset \right\}$$
$$P' \mapsto \iota_S^{-1}(P').$$

(a) Verify that α is well-defined.

(b) Define an inverse map β by $\beta(P) = P \cdot S^{-1}R$. Show that β is well-defined. That is, $\beta(P)$ is a prime ideal of $S^{-1}R$.

(c) Show that α and β are mutual inverses.

Solution:

(a) We observe that ι_S takes 1_R to $\frac{1}{1} \equiv 1_{S^{-1}R}$, the latter equality coming from the fact that $\frac{a}{1} \cdot \frac{1}{1} = \frac{a}{1}$, so that if P' is a prime ideal in $S^{-1}R$, then $\iota_S^{-1}(P')$ is a prime ideal in $S^{-1}R$.

(b) Let P be a prime ideal in R, and let $\frac{a}{s} \cdot \frac{b}{t} \in P \cdot S^{-1}R$. We desire to show that either $\frac{a}{s}$ or $\frac{b}{t}$ are in $P \cdot S^{-1}R$. Since elements of the form $\frac{s}{t}$ are units in $S^{-1}R$, it follows that $\frac{a \cdot b}{1} \in P \cdot S^{-1}R$.