Problem 1

Using the definition of the derivative find $f'(c)$ where $c \in \mathbb{R}$ and $f(x) = \frac{1}{x}$.

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} \\ &= \lim_{x\to c} \frac{c - x}{(xc)(x-c)} \\ &= \lim_{x\to c} \frac{-1}{xc} \\ &= -\frac{1}{x^2} && c \neq 0 \end{aligned}$$

Problem 2

Let $n \in \mathbb{N}$ and consider the function

$$f(x) = \begin{cases} x^n, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

For which values of n is f differentiable at $x = 0$.

We have that on $(0, \infty)$, $f(x) = x^n$, meaning $f'(x)$ on $(0, \infty)$ is nx^{n-1}. Therefore, as $(x_n)_n \to 0$ for $x_n \in (0, \infty)$, $\left(\frac{f(x_n) - f(0)}{x_n - 0}\right)_n \to 0$, taking $f(0)$ as given above, assuming $n > 1$ — otherwise, $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 1$.

Problem 3

Consider the function

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

Show that f is differentiable at $x = 0$ and find $f'(0)$.

Let $(x_n)_n \to 0$, $x_n \neq 0$. Let $(x_{n_k})_k$ denote the sequence of irrational values of x_n, and let $(x_{m_l})_l$ denote the sequence of rational values of x_n. Then, $(f(x_n))_n \to 0$, regardless of whether $x_n \in (x_{m_l})_l$ or $x_n \in (x_{n_k})_k$. So, having established that the limit exists, we find that

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{x^2 - 0^2}{x - 0} \\ &= \lim_{x\to 0} x \\ &= 0 \end{aligned}$$