Problem (Problem 1)**:** For two ideals $I, J \subseteq R$, prove the following.

(a) The intersection $I \cap J$ is an ideal of R.

(b) The product $IJ \subseteq I \cap J$.

(c) Let $f: R \to R/(IJ)$ be the canonical homomorphism. Then, for any $x \in I \cap J$, the image $f(x)$ is nilpotent.

(d) If $I + J = R$, then $IJ = I \cap J$.

Solution:

(a) If $x, y \in I \cap J$, then $x - y \in I \cap J$ since $x - y \in I$ and $x - y \in J$. Furthermore, if $r \in R$, then $rx \in I$ and $rx \in J$, so $rx \in I \cap J$, so $I \cap J$ is an ideal.

(b) We observe that for any $q \in IJ$, we may express

$$q = \sum_{k=1}^{n} x_k y_k,$$

where $x_k \in I$ and $y_k \in J$. In particular, each $x_k y_k \in I \cap J$, so $q \in I \cap J$, meaning $IJ \subseteq I \cap J$.

(c) Let $x \in I \cap J$. Then, following from the well-definedness of operations in the quotient ring, we see that $(x + IJ)^n = x^n + IJ$. In particular, if $n = 2$, then x^2 is a linear combination of an element of I multiplied by an element of J, so $x^2 \in IJ$, meaning that $(x + IJ)^2 = x^2 + IJ = IJ = 0 + IJ$, meaning that x is nilpotent.

(d) We will show that if $q \in I \cap J$, then q can be written as a linear combination of elements of I multiplied by elements of J. In particular, we start by letting $i \in I$ and $j \in J$ be such that $i + j = 1$. Then, $q(i + j) = q$, meaning that $qi + qj = q$, and since $q \in I \cap J$, we have expressed q as a linear combination of elements of I multiplied by elements of J. Thus, $I \cap J \subseteq IJ$, meaning $IJ = I \cap J$.

Problem (Problem 3)**:** Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers.

(a) Show that every nonzero ideal $I \subseteq R$ contains a nonzero integer.

(b) Identify the quotient R/I where $I = (2 + i)$ is the principal ideal generated by $2 + i$.

Solution:

(a) Let $I \subseteq R$ be a nonzero ideal, and let $a + ib \in I$ with $a, b \in \mathbb{Z} \setminus \{0\}$. Since multiplication by any element of R yields another element in I, we see that

$$\begin{aligned}(a + ib)(a - ib) &= a^2 + b^2 \\ &\in R,\end{aligned}$$

and since $a, b \neq 0$, so too is $a^2 + b^2$, so any nonzero ideal of R contains a nonzero integer.

(b) Consider the map $\varphi: \mathbb{Z} \to R/I$ given by $z \mapsto z + I$. Since this is a composition of the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ and the projection map $\pi: \mathbb{Z}[i] \to \mathbb{Z}[i]/(2 + i)$, this is a ring homomorphism. We will show that this ring homomorphism is surjective.

Let $(a + bi) + I \in R/I$. We will show that there is some $k \in \mathbb{Z}$ such that $k - (a + bi) \in (2 + i)$. For this purpose, let

$$(x + yi)(2 + i) = (a - k) + bi,$$

so that

$$2x - y = (a - k)$$

$$2y + x = b.$$

We thus get that

$$\begin{aligned} 5x &= 2a + b - 2k \\ 5y &= 2b - a + k. \end{aligned}$$

Reducing modulo 5, we thus have that

$$\begin{aligned} 0 &\equiv 2a + b - 2k \\ &\equiv 2b - a + k, \end{aligned}$$

meaning that $k = 3b + a$ (modulo 5). We thus have that

$$\begin{aligned} (3b + a) - (a + bi) &= 3b - bi \\ &= b(3 - i) \\ &= b(1 - i)(2 + i), \end{aligned}$$

so $z \mapsto z + I$ is surjective. We observe furthermore that $5\mathbb{Z} \subseteq \ker(\varphi)$, and since 5 is prime, it is a subset of no other ideal, and since the homomorphism φ is nontrivial, we thus have that $\ker(\varphi) = 5\mathbb{Z}$, so by the first isomorphism theorem, $\mathbb{Z}[i]/(2 + i) \cong \mathbb{Z}/5\mathbb{Z}$.