Amenable Discrete Groups

Conditions and Applications

Avinash Iyer

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Chapter 1

Prelude

Chapter 2

Paradoxical Decompositions

The primary goal of this section will be to introduce the idea of a paradoxical decomposition (and its effects on the analytic properties of $\mathbb{R}^3$) through the Banach–Tarski Paradox. The ultimate goal is to prove the following statement.

Proposition 2.0.1 — (General Banach–Tarski Paradox)

If A and B are bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such a sequence of isometries applied to these subsets yields B.

The existence of the Banach–Tarski paradox throws a wrench into a major idea that we may have about subsets of $\mathbb{R}^3$ — namely, that they always have some "volume" to them that is invariant under isometry, similar to how "area" in $\mathbb{R}^2$ is invariant under isometry.

2.1 Group Action Essentials

We begin by discussing some of the basic properties of group actions.

Definition 1 — (Group Action)

Let G be a group, and A be a set. A left group action of G onto A is a map $\alpha : G \times A \to A$ that satisfies

- $\alpha(g_1, (g_2, a)) = \alpha(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$;
- $\alpha(e_G, a) = a$ for all $a \in A$.

For the sake of brevity, we write $(g, a) = g \cdot a$.

Every group action can be represented by a permutation on A.

Definition 2 — (Permutation Representation)

For each g, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ is a permutation of A. There is a homomorphism associated to these actions, $\varphi : G \to \mathrm{Sym}(A)$, where $\mathrm{Sym}(A)$ is the symmetric group on the elements of A.

The permutation representation can run in the opposite direction in the following sense: given a nonempty set A and a homomorphism $\psi : G \to \mathrm{Sym}(A)$, we can take $g \cdot a = \psi(g)(a)$, where $\psi(g) = \sigma_g \in \mathrm{Sym}(A)$ is a permutation.

Just as we can pass group actions into permutation representations, and discuss ideas like the kernel of homomorphisms, we can also discuss the kernel of ajn action.

Definition 3 — (Kernel)

The **kernel** of the action of G on A is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}.$$

The kernel of the group action is the kernel of the permutation representation $\varphi : G \to \mathrm{Sym}(A)$.

Definition 4 — (Stabilizer)

For each $a \in A$, we define the **stabilizer** of a under G to be the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

Remark 1

The kernel of the group action is the intersection of the stabilizers of every element of A.

For each $a \in A$, G_a is a subgroup of G.

Definition 5 — (Faithful Action)

An action is **faithful** if the kernel of the action is the identity, e_G. Equivalently, the permutation representation $\varphi : G \to \mathrm{Sym}(A)$ is injective.

The following definition will be useful in the future as we dig deeper into the idea of paradoxical groups.

Definition 6 — (Free Action)

For a set X with G acting on X, the action of G on X is free if, for every $x \in X$, $g \cdot x = x$ if and only if $g = e_G$.

The most important theorem relating to group actions is the orbit-stabilizer theorem. As we prove the following theorem, we will reveal the definition of an orbit as a type of equivalence class.

Theorem 2.1.1 — (Orbit-Stabilizer Theorem)

Let G be a group that acts on a nonempty set A. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking the index of the stabilizer of a in G, $|G : G_a|$.

Proof : We start by seeing that $a \sim a$, as $e_G \cdot a = a$. Similarly, if $a \sim b$, then there exists $g \in G$ such that $a = g \cdot b$. Thus,

$$\begin{aligned} g^{-1} \cdot a &= g^{-1} \cdot (g \cdot b) \\ &= g^{-1} g \cdot b \\ &= e \cdot b \\ &= b, \end{aligned}$$

meaning that $b \sim a$. Finally, if we have $a \sim b$ and $b \sim c$, we have $a = g \cdot b$ and $b = h \cdot c$ for some $g, h \in G$. Therefore,

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$. Thus, the relation $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

We claim there is a bijection between the left cosets of G_a and the elements of $[a]_\sim$.

Define $C_a = \{g \cdot a \mid g \in G\}$, which is the set of elements in the equivalence class of a. Define the map $g \cdot a \mapsto gG_a$. Since $g \cdot a$ is always an element of C_a, this map is surjective. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, we have $h^{-1}g \in G_a$, which is only true if $gG_a = hG_a$. Thus, the map is injective.

Since there is a one to one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we know that the number of equivalence classes of a under the action of G is $|G : G_a|$. ■

Definition 7 — (Orbit)

Let G act on A, and let $a \in A$. The **orbit** of a under G is the set

$$G \cdot a = \{g \cdot a \in A \mid g \in G\}$$

2.2 Paradoxical Decompositions in $\mathbb{R}^3$

With the essential facts about group actions in mind, we can turn our attention to "paradoxical" actions that seem to recreate a set by using some of its disjoint proper subsets.

Definition 8 — (Paradoxical Decompositions and Paradoxical Groups)

Let G be a group that acts on a set X, with $E \subseteq X$. We say E is G**-paradoxical** if there exist pairwise disjoint proper subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$ and group elements

$g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

If G acts on itself by left-multiplication, and G satisfies these conditions, we say G is a **paradoxical group**.

Example 2.2.1

The free group on two generators, $F(a, b)$, is a paradoxical group.

The free group is defined to be the set of all reduced words over the set $\left\{a, b, a^{-1}, b^{-1}, e_{F(a,b)}\right\}$, where aa^{-1}, $a^{-1}a$, bb^{-1}, and $b^{-1}b$ are replaced with the identity $e_{F(a,b)}$.

To see that $F(a, b)$ is a paradoxical group, we let $W(x) = \{w \in F(a, b) \mid w \text{ starts with } x\}$. For instance, $ba^2ba^{-1} \in W(b)$.

Since every word in F is either the empty word, or starts with one of a, b, a^{-1}, b^{-1}, we see that

$$F(a, b) = \left\{e_{F(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For $w \in F(a, b) \setminus W(a)$, it is the case that $a^{-1}w \in W\left(a^{-1}\right)$, so $w \in aW\left(a^{-1}\right)$. Thus, for any $t \in F(a, b)$, $t \in W(a)$ or $t \in F(a, b) \setminus W(a) = aW\left(a^{-1}\right)$, so $F(a, b) = W(a) \sqcup aW\left(a^{-1}\right)$. Similarly, for any $w \in F(a, b) \setminus W(b)$, it is the case that $b^{-1}w \in W\left(b^{-1}\right)$, so $w \in bW\left(b^{-1}\right)$. Thus, for any $t \in F(a, b)$, $t \in W(b)$ or $t \in F(a, b) \setminus W(b) = bW\left(b^{-1}\right)$. Thus, $F(a, b) = W(b) \sqcup bW\left(b^{-1}\right)$.

We have thus constructed

$$\begin{aligned} F(a, b) &= W(a) \sqcup aW\left(a^{-1}\right) \\ &= W(b) \sqcup bW\left(b^{-1}\right), \end{aligned}$$

a paradoxical decomposition of $F(a, b)$ with the action of left-multiplication.

Now that we understand a little more about paradoxical groups, we now want to understand the actions of paradoxical groups on sets.

Proposition 2.2.1

Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.

Proof: Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ be pairwise disjoint, and let $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

Let $M \subseteq X$ contain exactly one element from every orbit in X.

Claim: The set $\{g \cdot M \mid g \in G\}$ is a partition of X.

Proof of Claim: Since M contains exactly one element from every orbit in X, it is the case that $G \cdot M = X$, so

$$\bigcup_{g \in G} g \cdot M = X$$

Additionally, for $x, y \in M$, if $g \cdot x = h \cdot y$, then $\left(h^{-1} g\right) \cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Since G acts freely on X, we must have $h^{-1} g = e_G$.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$, implying $\{g \cdot M \mid g \in G\}$ is a partition of X. ■

We define

$$A_j^* = \bigcup_{g \in A_j} g \cdot M,$$

and similarly define

$$B_j^* = \bigcup_{h \in B_j} h \cdot M.$$

As a useful shorthand, we can also write $A_j^* = A_j \cdot M$, and similarly, $B_j^* = B_j \cdot M$, to denote the union of the elements of A_j and B_j respectively acting on M.

Since $\{g \cdot M \mid g \in G\}$ is a partition of X, and $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ are pairwise disjoint, it must be the case that $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^* \subset X$ are also pairwise disjoint.

For the original $g_1, \ldots, g_n, h_1, \ldots, h_m$ that defined the paradoxical decomposition of G, we thus have

$$\begin{aligned} \bigcup_{j=1}^{n} g_j \cdot A_j^* &= \bigcup_{j=1}^{n} \left(g_j A_j\right) \cdot M \\ &= G \cdot M \\ &= X, \end{aligned}$$

and

$$\begin{aligned}\bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} (h_j B_j) \cdot M \\ &= G \cdot M \\ &= X.\end{aligned}$$

Thus, X is G-paradoxical. ■

Remark 2

This proof requires the axiom of choice, as we invoked it to define M to contain exactly one element from every orbit in X.

Now that we have established $F(a, b)$ as being a paradoxical group, we wish to use it to construct paradoxical decompositions of the unit sphere $S^2 \subseteq \mathbb{R}^3$.

Definition 9 — (Special Orthogonal Group)

For $n \in \mathbb{N}$, we define $SO(n)$ to be the group of all real $n \times n$ matrices A such that $A^T = A^{-1}$ and $\det(A) = 1$.

In terms of an isometry of $\mathbb{R}^3$, the group $SO(3)$ denotes the set of all rotations about any line through the origin.

Fact 2.2.1

If H is a paradoxical group, and $H \leqslant G$, then G is a paradoxical group.

With this fact in mind, we will show that $SO(3)$ is a paradoxical group.

Theorem 2.2.1

There are rotations A and B that about lines through the origin in $\mathbb{R}^3$ that generate a subgroup of $SO(3)$ isomorphic to $F(a, b)$

Proof: We take

$$A = \begin{bmatrix} 1/3 & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/3 & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

We let $A^{\pm}$ denote A and A^{-1} respectively, and similarly for $B^{\pm}$.

Let w be a reduced word in $\{A, A^{-1}, B, B^{-1}\}$ which is not the empty word. We claim that w cannot be the identity.

Without loss of generality, we assume that w ends in A or A^{-1} — this is because if w is the identity, then AwA^{-1} and $A^{-1}wA$ are also the identity.

We will show that there exist $a, b, c \in \mathbb{Z}$ with $b \not\equiv 0 \bmod 3$ such that

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}.$$

If $b \not\equiv 0 \bmod 3$, and w is not empty, then w cannot act as the identity.

We induct on the length of w. For $w = A^{\pm}$, we have

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix},$$

proving the base case.

Let $k > 0$, meaning $w = A^{\pm}w'$, or $w = B^{\pm}w'$, with w' not equal to the empty. The inductive hypothesis says

$$w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}$$

for some $a', b', c' \in \mathbb{Z}$, and $b' \not\equiv 0 \bmod 3$. In particular,

$$A^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \mp 4b \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix}$$

$$B^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} 3a' \\ (b' \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix}.$$

Now, we set

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix},$$

meaning

$$a = \begin{cases} a' \mp 4b', & w = A^{\pm}w' \\ 3a', & w = B^{\pm}w' \end{cases}$$

$$b = \begin{cases} b' \pm 2a', & w = A^{\pm}w' \\ b' \mp 2c', & w = B^{\pm}w' \end{cases}$$

$$c = \begin{cases} 3c', & w = A^{\pm}w' \\ c' \pm 4b', & w = B^{\pm}w' \end{cases}$$

Let w^* denote the word such that $w' = A^{\pm}w^*$ or $w' = B^{\pm}w^*$. We write

$$w^* = \frac{1}{3^{k-2}} \begin{pmatrix} a'' \\ b''\sqrt{2} \\ c'' \end{pmatrix},$$

where $a'', b'', c'' \in \mathbb{Z}$. Note that it may not be the case that w^* is a non-empty word. We examine the following four cases.

Case 1: Suppose $w = A^{\pm}B^{\pm}w^*$. Then, $b = b' \mp 2a'$, where $a' = 3a''$. Since $b' \not\equiv 0 \bmod 3$ (by the inductive hypothesis), it is also the case $b \equiv 0 \bmod 3$.

Case 2: Suppose $w = B^{\pm}A^{\pm}w^*$. Then, $b = b' \mp 2c'$, where $c' = 3c''$. Since $b' \not\equiv 0 \bmod 3$ (by the inductive hypothesis), it is also the case that $b \not\equiv 0 \bmod 3$.

Case 3: Suppose $w = A^{\pm}A^{\pm}w^*$. Then, we have

$$\begin{aligned} b &= b' \pm 2a' \\ &= b' \pm 2(a'' \pm 4b'') \\ &= b' + (b'' \pm 2a'') - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Thus, regardless of the value of b'', since $b' \not\equiv 0 \bmod 3$ by the inductive hypothesis, it is the case that $b \not\equiv 0 \bmod 3$.

Suppose $w = B^{\pm}B^{\pm}w^*$. Then, we have

$$\begin{aligned} b &= b' \mp 2c' \\ &= b' \mp 2(c'' \pm 4b'') \\ &= b' + (b'' \mp 2c'') - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Thus, regardless of the value of b'', since $b' \not\equiv 0 \bmod 3$ by the inductive hypothesis, it is the case that $b \not\equiv 0 \bmod 3$.

We have thus shown that any non-empty reduced word over $\{A, A^{-1}, B, B^{-1}\}$ does not act as the identity. The subgroup of $SO(3)$ generated by $\{A, A^{-1}, B, B^{-1}\}$ is isomorphic to $F(a, b)$. ∎

Remark 3

Since $SO(n)$ contains a subgroup isomorphic to $SO(3)$ for all $n \geqslant 3$, it is the case that $SO(n)$ also contains a subgroup isomorphic to $F(a, b)$.

Since we have shown that $SO(3)$ is paradoxical, as it contains a paradoxical subgroup, we can now begin to examine the action of $SO(3)$ on subsets of $\mathbb{R}^3$.

Theorem 2.2.2 — (Hausdorff Paradox)

There is a countable subset D of S^2 such that $S^2 \setminus D$ is SO(3)-paradoxical.

Proof: Let A and B be the rotations in SO(3) that serve as the generators of the subgroup isomorphic to $F(a, b)$.

Since A and B are rotations, so too is any reduced word over $\{A, A^{-1}, B, B^{-1}\}$. Thus, any such non-empty word contains two fixed points.

We let

$$F = \{x \in S^2 \mid s \text{ is a fixed point for some word } w\}.$$

Since the set of all reduced words in $\{A, A^{-1}, B, B^{-1}\}$ (henceforth $F(A, B)$) is countably infinite, so too is F. Thus, the union of all these fixed points under the action of all such words w is countable.

$$D = \bigcup_{w \in F(A,B)} w \cdot F.$$

Therefore, $F(A, B)$ acts freely on $S^2 \setminus D$, so $S^2 \setminus D$ is SO(3)-paradoxical. ■

Unfortunately, the Hausdorff paradox is not enough for us to be able to prove the Banach–Tarski paradox. In order to do this, we need to be able to show that two sets are "similar" under the action of a group.

Definition 10 — (Equidecomposable Sets)

Let G act on X, and let $A, B \subseteq X$. We say A and B are G-**equidecomposable** if there are partitions $\{A_j\}_{j=1}^n$ of A and $\{B_j\}_{j=1}^n$ of B, and elements $g_1, \dots, g_n \in G$, such that for all j,

$$B_j = g_j \cdot A_j.$$

We write $A \sim_G B$ if A and B are G-equidecomposable.

Fact 2.2.2

The relation $\sim_G$ is an equivalence relation.

Proof: Let A, B, and C be sets.

To show reflexivity, we can select $g_1 = g_2 = \cdots = g_n = e_G$. Thus, $A \sim_G A$.

To show symmetry, let $A \sim_G B$. Set $\{A_j\}_{j=1}^n$ to be the partition of A, and set $\{B_j\}_{j=1}^n$ to be the partition of B, such that there exist $g_1, \dots, g_n \in G$ with $g_j \cdot A_j = B_j$. Then,

$$g_j^{-1} \cdot (g_j \cdot A_j) = g_j^{-1} \cdot B_j$$

$$A_j = g_j^{-1} \cdot B_j,$$

so $B_j \sim_G A_j$.

To show transitivity, let $A \sim_G B$ and $B \sim_G C$. Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be the partitions of A and B respectively and $g_1, \dots, g_n \in G$ such that $g_i \cdot A_i = B_i$. Let $\{B_j\}_{j=1}^m$ and $\{C_j\}_{j=1}^m$ be partitions of B and C, and $h_1, \dots, h_m \in G$, such that $h_j \cdot B_j = C_j$.

We refine the partition of A to A_{ij} by taking $A_{ij} = g_i^{-1}(B_i \cap B_j)$, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Then, $(h_j g_i) \cdot A_{ij}$ maps the refined partition of A to a refined partition of C, meaning A and C are G-equidecomposable. ■

Fact 2.2.3

For $A \sim_G B$, there is a bijection $\phi : A \to B$ by taking $C_i = C \cap A_i$, and mapping $\phi(C_i) = g_i \cdot C_i$.

In particular, this means that for any subset $C \subseteq A$, it is the case that $C \sim \phi(C)$.

We can now use this equidecomposability to glean information about the existence of paradoxical decompositions.

Proposition 2.2.2

Let G act on X, with $E, E' \subseteq X$ such that $E \sim_G E'$. Then, if E is G-paradoxical, then so too is E'.

Proof: Let $A_1, \dots, A_n, B_1, \dots, B_m \subset E$ be pairwise disjoint, with $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} E &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

We let

$$A = \bigsqcup_{i=1}^{n} A_i$$
$$B = \bigsqcup_{j=1}^{m} B_j.$$

It follows that $A \sim_G E$ and $B \sim_G E$, since we can take the partition of A to be $A_1, \dots, A_n$, and partition E by taking $g_i \cdot A_i$ for $i = 1, \dots, n$, and similarly for B.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$. Thus, there is a paradoxical decomposition of E' in $A_1, \dots, A_n$ and $B_1, \dots, B_m$. ■

We will now show that S^2 is SO(3) paradoxical.

Proposition 2.2.3

Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \setminus D$ are $SO(3)$-equidecomposable.

Proof : Let L be a line in $\mathbb{R}^3$ such that $L \cap D = \varnothing$. Such an L must exist since S^2 is uncountable.

Define $\rho_\theta \in SO(3)$ to be a rotation about L by an angle of θ. For a fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{x \in D \mid \rho_\theta^n \cdot x \in D\right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

We define $W_n = \{\theta \mid R_{n,\theta} \neq \varnothing\}$. Since the map $\theta \mapsto \rho_\theta^n \cdot x$ into D is injective, it is the case that W_n is countable. Therefore,

$$W = \bigcup_{n \in \mathbb{N}} W_n$$

is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. We define ρ_ω to be a rotation about L by ω. Then, for every $n, m \in \mathbb{N}$, we have

$$\rho_\omega^n \cdot D \cap \rho_\omega^m \cdot D = \varnothing.$$

We define $\widetilde{D} = \bigsqcup_{n=0}^{\infty} \rho_\omega^n D$. Note that

$$\begin{aligned} \rho_\omega \cdot \widetilde{D} &= \rho_\omega \cdot \bigsqcup_{n=0}^{\infty} \rho_\omega^n \cdot D \\ &= \bigsqcup_{n=1}^{\infty} \rho_\omega^n \cdot D \\ &= \widetilde{D} \setminus D, \end{aligned}$$

meaning $\widetilde{D}$ and D are $SO(3)$-equidecomposable.

Thus, we have

$$\begin{aligned} S^2 &= \widetilde{D} \sqcup \left(S^2 \setminus \widetilde{D}\right) \\ &\sim_{SO(3)} \left(\rho_\omega \cdot \widetilde{D}\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\ &= \left(\widetilde{D} \setminus D\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\ &= S^2 \setminus D, \end{aligned}$$

establishing S^2 and $S^2 \setminus D$ as $SO(3)$-equidecomposable.

In particular, this means S^2 is also $SO(3)$-paradoxical. ■

To prove the Banach–Tarski paradox, we need a slightly larger group than $SO(3)$ — one that includes translations in addition to the traditional rotations. This is the Euclidean group.

Definition 11 — (Euclidean Group)

The **Euclidean group**, $E(n)$, consists of all isometries of a Eucliean space. An isometry of a Euclidean space consists of translations, rotations, and reflections.

We have $E(n) = T(n) \rtimes O(n)$, where $T(n)$ is the translation group, and $O(n)$ is the orthogonal group, which is the group of all rotations and reflections about the origin.

A further refinement of $E(n)$, $E_+(n)$, consists of all orientation-preserving isometries. In particular, $E_+(n) = T(n) \rtimes SO(n)$, where $SO(n)$ is the special orthogonal group, which is the group of all orientation-preserving rotations.

Corollary 2.2.1 — (Weak Banach–Tarski Paradox)

Every closed ball in $\mathbb{R}^3$ is $E(3)$-paradoxical.

Proof : We only need to show that $B(0,1)$ is $E(3)$-paradoxical. To do this, we start by showing that $B(0,1) \setminus \{0\}$ is $SO(3)$-paradoxical.

Since S^2 is $SO(3)$-paradoxical, there exists pairwise disjoint $A_1, \ldots, A_n, B_1, \ldots, B_m \subset S^2$ and elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in SO(3)$ such that

$$\begin{aligned} S^2 &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

Define

$$\begin{aligned} A_i^* &= \{tx \mid t \in (0,1], x \in A_i\} \\ B_j^* &= \left\{ty \mid t \in (0,1], y \in B_j\right\}. \end{aligned}$$

Then, $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^* \subset B(0,1) \setminus \{0\}$ are pairwise disjoint, and

$$\begin{aligned} B(0,1) \setminus \{0\} &= \bigcup_{i=1}^{n} g_i \cdot A_i^* \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j^*. \end{aligned}$$

Thus, we have established that $B(0,1) \setminus \{0\}$ is $E(3)$-paradoxical.

Now, we want to show that $B(0,1) \setminus \{0\}$ and $B(0,1)$ are $E(3)$-equidecomposable. Let $x \in B(0,1) \setminus \{0\}$, and let ρ be a rotation through x by a line not through the origin such that $\rho^n \cdot 0 \neq \rho^m \cdot 0$ when $n \neq m$.

Let $D = \{\rho^n \cdot 0 \mid n \in \mathbb{N}\}$. We can see that $\rho \cdot D = D \setminus \{0\}$, and that D and $\rho \cdot D$ are $E(3)$-equidecomposable. Thus,

$$\begin{aligned} B(0,1) &= D \sqcup (B(0,1) \setminus D) \\ &\sim_{E(3)} (\rho \cdot D) \sqcup (B(0,1) \setminus D) \\ &= (D \setminus \{0\}) \sqcup (B\,(0,1) \setminus D) \\ &= B\,(0,1) \setminus \{0\}\,. \end{aligned}$$

Therefore, $B(0,1)$ is $E(3)$-equidecomposable. ■

In order to prove the general case of the Banach–Tarski paradox, we need one more piece of mathematical machinery.

Our relation of $A \sim_G B$ is useful, but in order to show the general case, we want to refine the relation slightly.

Definition 12

Let G act on a set X with $A, B \subseteq X$. We write $A \preceq_G B$ if A is equidecomposable with a subset of B.

Fact 2.2.4

The relation $\preceq_G$ is a reflexive and transitive relation.

Proof: To see reflexivity, we can see that since $A \sim_G A$, and $A \subseteq A$, $A \preceq_G A$.

To see transitivity, let $A \preceq_G B$ and $B \preceq_G C$. Then, there exist $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_{\alpha,i}$ for each i, where $A \sim_G B_\alpha \subseteq B$. Similarly, there exist $h_1, \ldots, h_m \in G$ such that $h_j \cdot B_j = C_{\beta,j}$ for each j, where $B \sim_G C_\beta \subseteq C$.

We take a refinement of B by taking intersections $B_{\alpha,ij} = B_{\alpha,i} \cap B_j$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We define $C_{\beta,\alpha,ij} = h_j \cdot B_{\alpha,ij}$ for each $j = 1, \ldots, m$. Then, $h_j g_i \cdot A_i = C_{\beta,\alpha,ij}$, meaning $A \sim_G C_{\beta,\alpha,ij} \subseteq C_\beta \subseteq C$, so $A \preceq_G C$. ■

We know from Fact 2.2.3 that $A \preceq_G B$ implies the existence of a bijection $\phi : A \to B' \subseteq B$, meaning $\phi : A \hookrightarrow B$ is an injection. Similarly, if $B \preceq_G A$, then Fact 2.2.3 implies the existence of an injection $\psi : B \hookrightarrow A$.

One may ask if an analogue of the Cantor–Schröder–Bernstein theorem exists in the case of the relation $\preceq_G$, implying that the preorder established in Fact 2.2.4 is indeed a partial order. The following theorem establishes this result.

Theorem 2.2.3

Let G act on X, and let $A, B \subseteq X$. If $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$.

Proof: Let $B' \subseteq B$ with $A \sim_G B'$, and let $A' \subseteq A$ with $B \sim_G A'$. Then, we know from Fact 2.2.3 that there exist bijections $\phi : A \to B'$ and $\psi : B \to A'$.

Define $C_0 = A \setminus A'$, and $C_{n+1} = \psi(\phi(C_n))$. We set

$$C = \bigcup_{n \geqslant 0} C_n.$$

Since $\psi^{-1}(\psi(\phi(C_n))) = \phi(C_n)$, we have

$$\psi^{-1}(A \setminus C) = B \setminus \phi(C).$$

Having established in Fact 2.2.3 that for any subset of $C \subseteq A$, $C \sim_G \phi(C)$, we also see that $A \setminus C \sim_G B \setminus \phi(C)$.

Thus, we can see that

$$\begin{aligned} A &= (A \setminus C) \sqcup C \\ &\sim_G (B \setminus \phi(C)) \sqcup \phi(C) \\ &\sim B. \end{aligned}$$

■

Finally, we are able to prove the general Banach–Tarski paradox. Recall that the paradox says the following.

Proposition 2.2.4 — (Banach–Tarski Paradox)

Let A and B be bound subsets of $\mathbb{R}^3$ with nonempty interior. Then, $A \sim_{E(3)} B$.

Proof: By symmetry, it is enough to show that $A \preceq_{E(3)} B$.

Since A is bounded, there exists $r > 0$ such that $A \subseteq B(0, r)$.

Let $x_0 \in B^\circ$. Then, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq B$.

Since $B(0, r)$ is compact (hence totally bounded), there are translations $g_1, \ldots, g_n$ such that

$$B(0, r) \subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup g_n \cdot B(x_0, \varepsilon).$$

We select translations $h_1, \ldots, h_n$ such that $h_j \cdot B(x_0, \varepsilon) \cap h_k \cdot B(x_0, \varepsilon) = \varnothing$ for $j \neq k$. We set

$$S = \bigcup_{j=1}^{n} h_j \cdot B(x_0, \varepsilon).$$

Each $h_j \cdot B(x_0, \varepsilon) \subseteq S$ is $E(3)$-equidecomposable with any arbitrary closed ball subset of $B(x_0, \varepsilon)$, it is the case that $S \preceq B(x_0, \varepsilon)$.

Thus, we have

$$\begin{aligned} A &\subseteq B(0, r) \\ &\subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup b_n \cdot B(x_0, \varepsilon) \\ &\preceq S \\ &\preceq B(x_0, \varepsilon) \\ &\preceq B. \end{aligned}$$

■

Chapter 3

Tarski's Theorem

Ultimately, the reason the Banach–Tarski paradox "works" is because the paradoxical group $F(a, b)$, lacks a property known as amenability. Readers may be surprised to hear that amenability and non-paradoxicality are distinct — that is, a group is amenable if and only if it is non-paradoxical. This fact is formalized in Tarski's theorem.

Theorem 3.0.1 — (Tarski's Theorem)

Let G be a group that acts on a set X, and let $E \subseteq X$ be nonempty. There is a finitely additive translation-invariant measure $\mu : P(X) \to [0, \infty]$ with $\mu(E) \in (0, \infty)$ if and only if E is not G-paradoxical.

In fact, we can prove the converse now.

Proof of the Converse of Tarski's Theorem : Let E be G-paradoxical. Suppose toward contradiction that such a translation-invariant finitely additive ν existed with $\nu(E) \in (0, \infty)$.

Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq E$ be pairwise disjoint, and let $t_1, \ldots, t_n, s_1, \ldots, f_m \in G$ such that

$$\begin{aligned} E &= \bigsqcup_{i=1}^{n} t_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} s_j \cdot B_j. \end{aligned}$$

Then, it would be the case that

$$\begin{aligned} \nu(E) &= \nu\left(\bigsqcup_{i=1}^{n} t_i \cdot A_i\right) \\ &= \sum_{i=1}^{n} \nu\left(t_i \cdot A_i\right) \\ &= \sum_{i=1}^{n} \nu\left(A_i\right), \end{aligned}$$

and

$$\nu(E) = \sum_{j=1}^{m} \nu\left(B_j\right).$$

However, this yields

$$\begin{aligned}
\nu(E) &= \nu\left(\left(\bigsqcup_{i=1}^{n} t_i \cdot A_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} s_j \cdot B_j\right)\right) \\
&= \sum_{i=1}^{n} \nu(A_i) + \sum_{j=1}^{m} \nu(B_j) \\
&= 2\nu(E),
\end{aligned}$$

implying that $\nu(E) = 0$ or $\nu(E) = \infty$. ■

3.1 A Little Bit of Graph Theory

To prove the forward direction of Tarski's theorem, we need to develop some machinery from graph theory that will allow us to prove that a certain semigroup we will construct in the next section satisfies the cancellation identity.

We start by defining graphs and paths, before proving a special case of Hall's theorem, ultimately extending to the infinite case with König's theorem.

Definition 13 — (Graphs and Paths)

A **graph** is a triple (V, E, ϕ), with V, E nonempty sets and $\phi : E \to P_2(V)$ a map from E to the set of all unordered subset pairs of V.

For $e \in E$, if $\phi(e) = \{v, w\}$, then we say v and w are the **endpoints** of e, and e is **incident** on v and w.

A **path** in (V, E, ϕ) is a finite sequence $(e_1, \ldots, e_n)$ of edges, with a finite sequence of vertices $(v_0, \ldots, v_n)$, such that $\phi(e_k) = \{v_{k-1}, v_k\}$.

The **degree** of a vertex, $\deg(v)$, is the number of edges incident on v.

We define the **neighbors** of $S \subseteq V$ to be the set of all vertices $v \in V \setminus S$ such that v is an endpoint to an edge incident on S. We denote this set $N(S)$.

Definition 14 — (Bipartite Graphs and k-Regularity)

Let (V, E, ϕ) be a graph, with $k \in \mathbb{N}$.

(i) If $\deg(v) = k$ for each $v \in V$, we say (V, E, ϕ) is k-**regular**.

(ii) If $V = X \sqcup Y$, with each edge in E having one endpoint in X and one endpoint in Y, then we say V is **bipartite**, and write (X, Y, E, ϕ).

Definition 15 — (Perfect Matching)

Let (X, Y, E, ϕ) be a bipartite graph. Let $A \subseteq X$ and $B \subseteq Y$. A **perfect matching** of A and B is a subset $F \subseteq E$ with

(i) each element of $A \cup B$ is an endpoint of exactly one $f \in F$;

(ii) all endpoints of edges in F are in $A \cup B$.

Definition 16 — (Hall Condition)

We say a bipartite graph (X, Y, E, ϕ) satisfies the **Hall Condition** on X if, for all $S \subseteq X$, $|N(S)| \geqslant |S|$.

Equivalently, we say a (finite) collection of not necessarily distinct finite sets $\mathcal{X} = \{X_i\}_{i=1}^n$ satisfies the marriage condition if and only if for all subcollections $\mathcal{Y}_k = \{X_{i_k}\}_{k=1}^m$,

$$|\mathcal{Y}_k| \leqslant \left|\bigcup_{k=1}^{m} X_{i_k}\right|.$$

Remark 4

These two formulations of the Hall condition are equivalent regarding an X-perfect matching.

Theorem 3.1.1 — (Hall's Theorem for Finite k-Regular Bipartite Graphs)

Let (X, Y, E, ϕ) be a k-regular bipartite graph for some $k \in \mathbb{N}$, and let $V = X \sqcup E$ be finite. Then, there is a perfect matching of X and Y.

Proof: Note that since $|E| = k\,|K| = k\,|Y|$, it is the case that $|X| = |Y|$.

Let $M \subseteq V$ be any subset. We will show that $|N(M)| \geqslant |M|$ — that is, (X, Y, E, ϕ) satisfies the Hall condition.

Let $M_X = M \cap X$ and $M_Y = M \cap Y$, where $M = M_X \sqcup M_Y$. Let $[M_X, N(M_X)]$ be the set of edges with endpoints in M_X and $N(M_X)$, and $[M_Y, N(M_Y)]$ be the set of edges with endpoints in M_Y and $N(M_Y)$. We also let $[X, N(M_X)]$ denote the set of edges with endpoints in X and $N(M_X)$, and similarly, $[Y, N(M_Y)]$ is the set of edges with endpoints in Y and $N(M_Y)$.

We can see that $[M_X, N(M_X)] \subseteq [X, N(M_X)]$, and similarly, $[M_Y, N(M_Y)] \subseteq [Y, N(M_Y)]$.

Since $|[M_X, N(M_X)]| = k\,|M_X|$ and $|[X, N(M_X)]| = k\,|N(M_X)|$, we have

$$|M_X| \leqslant |N(M_X)|,$$

and similarly,

$$|M_Y| \leqslant |N(M_Y)|.$$

Thus, $|M| \leqslant |N(M)|$.

We will now show that there is an X-perfect matching. Suppose toward contradiction that F is a maximal perfect matching on $A \subseteq X$ and $B \subseteq Y$ with $X \setminus A \neq \emptyset$.

Then, there is $x \in X \setminus A$. Consider $Z \subseteq V$ consisting of all vertices z such that there exists a F-alternating path $(e_1, \ldots, e_n)$ between $z \in Z$ and x.

It cannot be the case that $Z \cap Y$ is empty, since the number of neighbors of x is greater than or equal to 1 by the Hall condition — if it were the case that $Z \cap Y$ were empty, we could add an edge to F consisting of x and one element of $N(\{x\})$, which would contradict the maximality of F.

Consider a path traversing along Z, $(e_1, \ldots, e_n)$. It must be the case that $e_n \in F$, or else we would be able to "flip" the matching F by exchanging e_i with e_{i+1} for $e_i \in F$, which would contradict the maximality of F yet again. Thus, every element of $Z \cap Y$ is satisfied by F, so $Z \cap Y \subseteq B$.

Since each element in $Z \cap Y$ is paired with exactly one element of $Z \cap X$ (with one left over), it is the case that $|Z \cap X| = |Z \cap Y| + 1$.

Suppose toward contradiction that there exists $y \in N(Z \cap X)$ with $y \notin Z \cap Y$. Then, there exists $v \in Z \cap X$ and $e \in E$ such that $\phi(e) = \{v, y\}$. However, this means v is connected via a path to x, meaning $y \in Z$, so $y \in Z \cap Y$. Thus, we must have $N(Z \cap X) = Z \cap Y$.

Therefore,

$$\begin{aligned} |Z \cap X| &= |Z \cap Y| + 1 \\ &= |N(Z \cap X)| + 1, \end{aligned}$$

which contradicts the fact that (X, Y, E, ϕ) satisfies the Hall condition. Therefore, $A = X$.

By symmetry, there is a perfect matching of X and Y in (X, Y, E, ϕ). ∎

Remark 5

An equivalent formulation to Hall's theorem states that there is a *system of distinct representatives* on $\mathcal{X}$, which is a set $\{x_k\}_{k=1}^{n}$ such that $x_k \in X_k$ and $x_i \neq x_j$ for $i \neq j$.

This implies the existence of an injection $f : \mathcal{X} \hookrightarrow \bigcup_{k=1}^{n} X_k$, such that $f(X_k) \in X_k$.

Theorem 3.1.2 — (Infinite Hall's Theorem)

Let $\mathcal{G} = \{X_i\}_{i \in I}$ be a collection of (not necessarily distinct) finite sets. If, for every finite subcollection $\mathcal{Y} = \{X_{i_k}\}_{k=1}^{n}$,

$$n \leqslant \left| \bigcup_{k=1}^{n} X_{i_k} \right|,$$

then there is a choice function on G.

Proof: We endow each $X_i \in \{X_i\}_{i\in I}$ with the discrete topology. Since each X_i is finite, each X_i is compact.

Thus, by Tychonoff's theorem, it is the case that $\prod_{i\in I} X_i$ is compact.

For every finite subset $Y \subseteq \mathcal{G}$, we define

$$S_Y = \left\{ f \in \prod_{i\in I} X_i \,\middle|\, f|_Y \text{ is injective} \right\}.$$

The injectivity of $f|_Y$ is equivalent to the existence of a system of distinct representatives on Y. Since Y satisfies the Hall condition, each S_Y is nonempty. Additionally, for any net of functions $f_\alpha \in S_Y$ with $\lim_\alpha f_\alpha = f$, it is the case that $f_\alpha|_Y$ is injective, so $f|_Y$ is injective, meaning S_Y is closed.

We define $F = \{S_Y \mid Y \subseteq \mathcal{G} \text{ finite}\}$. For finite $Y_1, Y_2 \subseteq \mathcal{G}$, every system of distinct representatives in $Y_1 \cup Y_2$ is necessarily a system of distinct representatives on Y_1 and a system of distinct representatives on Y_2, meaning $S_{Y_1 \cup Y_2} \subseteq S_{Y_1} \cap S_{Y_2}$. Thus, F has the finite intersection property.

Since $\prod_{i\in I} X_i$ is compact, $\bigcap F$ is nonempty, where the intersection is taken over all finite subsets of $\mathcal{G}$. For any $f \in \bigcap F$, f is necessarily a choice function. ■

Remark 6

This is equivalent to the existence of an injection $f : \mathcal{G} \hookrightarrow \bigcup_{i\in I} X_i$.

We will use this infinite case of Hall's theorem to prove König's theorem.

Theorem 3.1.3 — (König's Theorem)

Let (X, Y, E, ϕ) be a k-regular bipartite graph (not necessarily finite). Then, there is a perfect matching of X and Y.

Proof: If $k = 1$, it is clear that there is a perfect matching in (X, Y, E, ϕ) consisting of the edges in (X, Y, E, ϕ).

Let $k \geqslant 2$. Since any finite subset of X satisfies the Hall condition, as displayed in the proof of Theorem 3.1.1, there is some X-perfect matching in (X, Y, E, ϕ). We call this X-perfect matching F. There is an injection $f : X \hookrightarrow Y$ following the edges in F.

Similarly, since any finite subset of Y satisfies the Hall condition, there is some Y-perfect matching in (X, Y, E, ϕ). We call this Y-perfect matching G. There is an injection $g : Y \hookrightarrow X$ following the edges of G.

Consider the subgraph $(X, Y, F \cup G, \phi|_{F\cup G})$. The injections f and g still hold in this graph. By the Cantor–Schröder–Bernstein theorem, there is a bijection $h : X \to Y$ in $(X, Y, F \cup G, \phi|_{F\cup G})$. ■