Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $V := \left\{re^{i\theta} \in \mathbb{C} \,\middle|\, -\pi/4 < \theta < \pi/4, r > 0\right\}$. Fix $z_0 \in U$, and let $\mathcal{F} := \{f \in H(U) \mid f(z_0) = 1, \operatorname{im}(f) \subseteq V\}$. Show that $\mathcal{F}$ is normal.

Solution: We observe that a function $f \in H(U)$ if and only if $f(z_0) = 1$ and $\operatorname{im}(f) \subseteq V$, or equivalently, that $e^{i\pi/4}f(z_0) = e^{i\pi/4}$ and $\operatorname{im}(f)$ is a subset of the upper half-plane. Now, by composing with the Cayley Transform, $q(z) = \frac{z-i}{z+i}$, we find that the family

$$\mathcal{G} = \left\{q\left(e^{i\pi/4}f\right) \,\middle|\, f \in \mathcal{F}\right\}$$

is now locally bounded family of holomorphic functions (in fact, it is globally bounded, with every function in $\mathcal{G}$ being bounded above by 1).

Let $(f_n)_n \subseteq \mathcal{F}$. We observe then that $\left(q\left(e^{i\pi/4}f_n\right)\right)_n$ is a sequence in $\mathcal{G}$, meaning that there is a subsequence $\left(q\left(e^{i\pi/4}f_{n_k}\right)\right)_k \to g \colon U \to \mathbb{D}$ for some holomorphic function $g \colon U \to \mathbb{D}$. Since the Cayley Transform has a holomorphic inverse, it follows that $(f_{n_k})_k \to e^{-i\pi/4}q^{-1} \circ g \colon U \to \mathbb{C}$ is a subsequence of $(f_n)_n$ that converges on compact subsets to a holomorphic function, hence $\mathcal{F}$ is normal.

Problem (Problem 2)**:** Let $\mathcal{F} = \{f \in H(\mathbb{D}) \mid \operatorname{im}(f) \subseteq \mathbb{D}\}$. Fix $z_0 \in \mathbb{D}$. Show that there exists a holomorphic function $g \colon \mathbb{D} \to \mathbb{C}$ with $\operatorname{im}(g) \subseteq \mathbb{D}$, $|g'(z_0)| = \max_{f \in \mathcal{F}}|f'(z_0)|$, and $g(z_0) = 0$.

Solution: From Montel's Theorem, we know that the family $\mathcal{F}$ is normal.