

Problem 1

Let $(x_k)_k$ be a sequence of strictly positive numbers such that

$$(kx_k)_k \rightarrow L > 0.$$

Show that $\sum_k x_k$ diverges.

Since $(kx_k)_k \rightarrow L$, every subsequence of $(kx_k)_k$ converges to L . Let $n_k = 2^k$. Then,

$$(2^k x_{2^k})_k \rightarrow L > 0,$$

implying that

$$\sum_k 2^k x_{2^k} = \infty.$$

By the Cauchy Condensation test, this implies that $\sum_k x_k$ diverges.

Problem 2

Let $(x_k)_k$ be a sequence of strictly positive numbers. Show the following:

(i) If $\limsup_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} < 1$, then $\sum_k x_k$ converges.

(ii) If $\liminf_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} > 1$, then $\sum_k x_k$ diverges.

(a)

Let $\varepsilon > 0$.

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} &:= u < 1 \\ &= \inf_{n \geq 1} \left(\sup_{k \geq n} \frac{x_{k+1}}{x_k} \right) \end{aligned}$$

By the definition of \inf , we have that $\exists N \in \mathbb{N}$ large such that

$$\sup_{k \geq N} \frac{x_{k+1}}{x_k} < u + \varepsilon.$$

By the definition of \sup , we have that $\forall k \geq N$,

$$\begin{aligned} \frac{x_{k+1}}{x_k} &< u + \varepsilon \\ x_{k+1} &< (u + \varepsilon)x_k. \end{aligned}$$

Inductively on x_k , we have that

$$x_{k+m} < (u + \varepsilon)^m x_k,$$

and series-wise, we have

$$\sum_{k=N}^{\infty} x_k < x_N \sum_{m=1}^{\infty} (u + \varepsilon)^m.$$

For sufficiently small ε , the sum on the right-hand side converges, implying that the sum on the left-hand side must converge. Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{N-1} x_k + \sum_{k=N}^{\infty} x_k \\ &< \sum_{k=1}^{N-1} x_k + x_N \sum_{m=1}^{\infty} (u + \varepsilon)^m, \end{aligned}$$

meaning that $\sum_k x_k$ is bounded above by a convergent series, so it is convergent.

(b)

Let $\varepsilon > 0$.

$$\begin{aligned}\liminf_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} &:= \ell > 1 \\ &= \sup_{n \geq 1} \left(\inf_{k \geq n} \frac{x_{k+1}}{x_k} \right)\end{aligned}$$

By the definition of sup, we have that for large $N \in \mathbb{N}$, and for $k \geq N$,

$$\inf_{k \geq n} \frac{x_{k+1}}{x_k} > \ell - \varepsilon.$$

By the definition of inf, we also have that

$$\begin{aligned}\frac{x_{k+1}}{x_k} &> \ell - \varepsilon \\ x_{k+1} &> (\ell - \varepsilon)x_k\end{aligned}$$

Inductively, we have that

$$x_{k+m} > (\ell - \varepsilon)^m x_k,$$

and via series, we have

$$\sum_{k=N}^{\infty} x_k > x_N \sum_{m=1}^{\infty} (\ell - \varepsilon)^m.$$

For sufficiently small ε , the sum on the right-hand side diverges. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{N-1} x_k + \sum_{k=N}^{\infty} x_k \\ &> x_N \sum_{k=1}^{\infty} (\ell - \varepsilon)^m + \sum_{k=1}^{N-1} x_k,\end{aligned}$$

and since $\sum_k x_k$ is bounded below by a divergent series, the sum diverges.

Problem 3

Consider the sequence of functions

$$\begin{aligned}f_n &: \mathbb{R} \rightarrow \mathbb{R}; \\ f_n(x) &= \arctan(nx)\end{aligned}$$

- (i) Show that $(f_n)_n \rightarrow \frac{\pi}{2} \operatorname{sgn}$ point-wise.
- (ii) Show that the convergence in (i) is nonuniform on $(0, \infty)$.
- (iii) Show that the convergence in (i) is uniform on $[a, \infty)$ for a fixed $a > 0$.

(i)

Let $\varepsilon > 0$. We know that, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|\arctan(n) - \pi/2| < \varepsilon$.

Case 1: Let $x = 0$. Then,

$$\arctan(nx) = 0 \quad \forall n \geq 1$$

Case 2: Let $x > 0$. Then, set $N' = \lceil N/x \rceil$. So, for $n' \geq N'$, we have

$$\begin{aligned} |\arctan(nx) - \pi/2| &= |\arctan(n') - \pi/2| \\ &< \varepsilon \end{aligned}$$

implying that $\arctan(nx) \rightarrow \pi/2$ when $x > 0$.

Case 3: Let $x < 0$. Then, set $x^* = -x$, and we have the same result as in Case 2, where $\arctan(nx^*) \rightarrow \pi/2$.

Since $\arctan(nx^*) = \arctan(n(-x)) = -\arctan(nx)$, we have that $\arctan(nx) \rightarrow -\pi/2$.

(ii)

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Set $\varepsilon_0 = \frac{\pi}{4}$. Then, we have that

$$\begin{aligned} |\arctan(n_k x_k) - \pi/2| &= \left| \arctan\left(k \frac{1}{k}\right) - \frac{\pi}{2} \right| \\ &= \left| \arctan(1) - \frac{\pi}{2} \right| \\ &= \left| \frac{\pi}{4} - \frac{\pi}{2} \right| \\ &= \frac{\pi}{4} \\ &\geq \varepsilon_0. \end{aligned}$$

(iii)

Since $\forall x \in [a, \infty)$, $x > 0$, and $\arctan(na) \rightarrow \frac{\pi}{2}$, it must be the case that $\arctan(nx) \rightarrow \frac{\pi}{2} \operatorname{sgn} x$ for $x \in [a, \infty)$.

Problem 4

Consider the sequence of functions

$$\begin{aligned} f_n &: [0, \infty) \rightarrow \mathbb{R}; \\ f_n(x) &= \frac{\sin(nx)}{1 + nx}. \end{aligned}$$

- (i) Show that $(f_n)_n \rightarrow 0$ pointwise.
- (ii) Show that the convergence in (i) is nonuniform on $[0, \infty)$.
- (iii) Show that the convergence in (i) is uniform on $[a, \infty)$ for a fixed $a > 0$.

(i)

We know that $f_n(0) = 0 \forall n \in \mathbb{N}$. For all $x > 0$, we have:

$$\begin{aligned} \left| \frac{\sin(nx)}{1+nx} - 0(x) \right| &\leq \frac{1}{1+nx} \\ &< \frac{1}{nx} \\ &\rightarrow 0. \end{aligned}$$

So,

$$f_n \xrightarrow{\text{p.w.}} 0.$$

(ii)

Let $n_k = k$ and $x_k = \frac{\pi}{2k}$. Set $\epsilon_0 = 1/4$. Then,

$$\begin{aligned} |f_{n_k}(x_k) - 0(x_k)| &= \frac{\sin\left(k \frac{\pi}{2k}\right)}{1 + k \frac{\pi}{2k}} \\ &= \frac{1}{1 + \frac{\pi}{2}} \\ &\geq \epsilon_0 \end{aligned}$$

(iii)

On $[a, \infty)$, we have

$$\begin{aligned} \left| \frac{\sin(nx)}{1+nx} - 0(x) \right| &\leq \frac{1}{1+nx} \\ &\leq \frac{1}{na} \\ \sup \left| \frac{\sin(nx)}{1+nx} - 0(x) \right| &\leq \frac{1}{na} \\ &\rightarrow 0 \end{aligned}$$

So, $\frac{\sin(nx)}{1+nx} \rightarrow 0$ on $[a, \infty)$ uniformly.

Problem 5

Show that the sequence of functions

$$\begin{aligned} f_n : [0, \infty) &\rightarrow \mathbb{R}; \\ f_n(x) &= x^2 e^{-nx} \end{aligned}$$

converges uniformly to 0.

We know that $\forall n \in \mathbb{N}$, $f_n(0) = 0$. Otherwise, we have that

$$\begin{aligned} \sup (x^2 e^{-nx}) &\Rightarrow \frac{df_n}{dx} = 0 \\ 2xe^{-nx} - nx^2 e^{-nx} &= 0 \\ xe^{-nx} (2 - nx) &= 0 \\ x &= \frac{2}{n} \\ f(x) &= \frac{4}{n^2 e^2}. \end{aligned}$$

Additionally, we have

$$\begin{aligned} n^2 &\geq n \\ \frac{e^2 n^2}{4} &\geq \frac{e^2 n}{4} \\ \frac{4}{e^2 n^2} &\leq \frac{4}{e^2 n}, \end{aligned}$$

so,

$$\sup(x^2 e^{-nx}) \rightarrow 0.$$

Therefore, $f_n(x)$ converges to 0 uniformly.

Problem 6

Let $f_n = \mathbf{1}_{[n, n+1]}$. Show that $(f_n)_n \rightarrow \mathbf{0}$ pointwise on \mathbb{R} . Is the convergence uniform?

$\forall x \in \mathbb{R}$, find $N \in \mathbb{N}$ so large such that $x < N$, which is always true by the Archimedean property. Then, $|f_n(x) - \mathbf{0}(x)| = 0 < \varepsilon$.

However, since $\sup(f_n) = 1 \forall n$, it must be the case that $(f_n)_n$ does not converge to $\mathbf{0}$ uniformly.

Problem 7

Let $(f_n)_n$ and $(g_n)_n$ be sequences in $\ell_\infty(\Omega)$ with $(f_n)_n \rightarrow f$ and $(g_n)_n \rightarrow g$ uniformly on Ω . Prove that $(f_n g_n)_n \rightarrow fg$ uniformly on Ω .

$$\begin{aligned} \|f_n(x)g_n(x) - f(x)g(x)\|_u &= \|f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)\|_u \\ &= \|f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))\|_u \\ &\leq \|f_n(x)\|_u \cdot \|g_n(x) - g(x)\|_u + \|g(x)\|_u \|f_n(x) - f(x)\|_u && \text{Triangle Inequality} \\ &\leq c \|f_n(x) - f(x)\|_u + d \|g_n(x) - g(x)\|_u && \text{Definition of Supremum} \\ &\rightarrow 0 \end{aligned}$$

Problem 8

Find a sequence of functions with $(f_n)_n$ defined on $[0, \infty)$ such that $|f_n|_u \geq n$, but $(f_n)_n \rightarrow 0$ pointwise.

Let f_n be defined as δ_n , where δ_n is defined as follows:

$$\delta_n(x) = \begin{cases} n & x = n \\ 0 & \text{otherwise} \end{cases}$$