Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: Let $(f_n)_n$ be a Cauchy sequence in $C_0(\mathbb{R})$. Since each $f_k \in C_0(\mathbb{R})$, it must be the case that each f_k is uniformly continuous. For each $x \in \mathbb{R}$, it is thus the case that $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(f_n(x))_n \to f(x)$ for each $x \in \mathbb{R}$, and since each f_k is uniformly continuous, it must be the case that $f(x)$ is continuous.

For $\varepsilon > 0$, there must be N large such that for $m, n \geq N$ and $m \geq n$, it must be the case that $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Letting $m \to \infty$, we have $|f_n(x) - f(x)| < \varepsilon$, meaning $(f_n)_n \to f$. Thus, $f \in C_0(\mathbb{R})$.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $(x_n)_n$ be a Cauchy sequence in ℓ_2. That is, for $\varepsilon > 0$, there exists N large such that for $m, n \geq N$,

$$\|x_m - x_n\| < \varepsilon/2.$$

We wish to show that such $N' \in \mathbb{N}$ exists where for $n \geq N$, $\|x_n - x\| < \varepsilon$.

$$\begin{aligned}\|x_n - x\| &= \|x_n - x_m + x_m - x\| \\ &\leq \|x_n - x_m\| + \|x_m - x\| \\ &< \varepsilon/2 + \|x_m - x\|\end{aligned}$$

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned}d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq d(x_2, x_1)\left(\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}\right) \\ &= d(x_2, x_1)\theta^{n-1}\left(1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}\right) \\ &= d(x_2, x_1)\frac{\theta^{n-1}}{1-\theta}.\end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.

Problem 4

Let (X, d) be a complete metric space, and suppose $f : X \to X$ is a contractive map — i.e., there is a $\theta \in (0, 1)$ with

$$d(f(x), f(y)) \leq \theta d(x, y).$$

Prove that f has a unique fixed point.