2.1.22

Let T be an n-vertex tree with one vertex of each degree $2 \leq i \leq k$; the remaining $n-k+1$ vertices are leaves. Determine n in terms of k.

We will find the number of vertices in T by finding the number of edges in T and adding 1. For $2 \leq i \leq k$ corresponding to each of the non-leaf vertices, summation yields $\frac{k(k+1)}{2} - 1$ edges. However, this scheme double-counts each edge, so we have to subtract the $k-2$ edges connecting the $k-1$ non-leaf vertices, yielding $\frac{k(k+1)}{2} - k + 1$ edges. Finally, because T is a tree, we get that T has $\frac{k(k+1)}{2} - k + 2$ vertices.

2.1.27

Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \dots, d_n$ if and only if $\sum d_i = 2n - 2$.

($\Rightarrow$) Suppose that for some tree T, $d_1, \dots, d_n$ are the degrees of the vertices of the tree. Since T is a tree, this means $e(G) = n - 1$, and $\sum d_i = 2e(G)$, meaning $\sum d_i = 2(n-1) = 2n - 2$.

($\Leftarrow$) Suppose that $\sum d_i = 2n - 2$ for $d_1, \dots, d_n$ corresponding to the degrees of the vertices in G. By a previous result, we know that $\sum d_i = 2e(G)$, meaning that $\sum d_i = 2(n-1)$, implying that $e(G) = n - 1$. We can find a tree G with $n - 1$ edges by letting G be connected with $n - 1$ edges.

2.1.33

Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

($\Rightarrow$) Let G be a connected n-vertex graph with exactly one cycle. If we delete an edge from this cycle, then $G - e$ is acyclic, as well as connected (since e is not a cut-edge), so $G - e$ has $n - 1$ edges. Adding back e, we get that G has n edges.

($\Leftarrow$) Let G be a connected n-vertex graph with n edges. Then, G contains a spanning tree that contains all n vertices. Therefore, $T \subseteq G$ contains $n - 1$ edges. By adding another edge, we get that $e(G) = e(T) + 1 = n - 1 + 1$. Thus, G has exactly one cycle.

2.1.34

Let T be a tree with k edges, and let G be a n-vertex simple graph with more than $n(k-1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n > k$.

We will use induction to prove that $T \subseteq G$ as follows:

Base Case Suppose $n = k + 1$. Then, we can find the following:

$$\begin{aligned}
e(G) &> (k+1)(k-1) - \binom{k}{2} \\
e(G) &> (k^2 - 1) - \frac{k(k-1)}{2} \\
e(G) &> \frac{k^2-1}{2} + \frac{k^2 - 1 - (k^2 - k)}{2} \\
e(G) &> \frac{k^2+k}{2} - 1 \\
e(G) &> \frac{k(k+1)}{2} - 1
\end{aligned}$$

This means $e(G) = \frac{k(k+1)}{2}$ in the base case, meaning G is the complete graph on $k+1$ vertices, where $\delta(G) = k$. By Theorem 2.1.8, we know that $T \subseteq G$.

Inductive Hypothesis If $n > k + 1$, $e(G) > n(k-1) - \binom{k}{2}$, then either $\delta(G) \geq k$ or, if $\delta(G) < k$, then $e(G - x) > (n-1)(k-1) - \binom{k}{2}$ for $\delta(G) = d(x)$.

Proof If $\delta(G) \geq k$, then we know by Theorem 2.1.8 that $T \subseteq G$. Otherwise, suppose $\delta(G) < k$, and let $d(x) = \delta(G)$. Let $G' = G - x$.

$$\begin{aligned}
e(G') &= e(G) - \delta(G) \\
e(G') &\geq e(G) - (k-1) \\
e(G') &> n(k-1) - \binom{k}{2} - (k-1) \\
e(G') &> (n-1)(k-1) - \binom{k}{2}
\end{aligned}$$

Therefore, the inductive hypothesis is proven.