

## Problem 1

Let  $\mathbb{F}$  be a field. Show that the following hold:

- (i)  $-1(a) = -a$
- (ii)  $-(-a) = a$
- (iii)  $-(a + b) = (-a) + (-b)$
- (iv)  $(-a)^{-1} = -(a^{-1})$
- (v)  $(ab)^{-1} = a^{-1}b^{-1}$

(i)

$$\begin{aligned}
 0 &= (1 + (-1)) \\
 0(a) &= (1 + (-1))a \\
 0 &= 1(a) + (-1)(a) \\
 0 &= a + (-1)(a) \\
 -a &= (-1)(a)
 \end{aligned}$$

(ii)

$$\begin{aligned}
 0 &= -(-a) + (-a) \\
 a &= -(-a) + ((-a) + a) \\
 a &= -(-a)
 \end{aligned}$$

(iii)

$$\begin{aligned}
 0 &= -(a + b) + (a + b) \\
 -b &= -(a + b) + a + (b - b) \\
 -a + (-b) &= -(a + b) + (a - a) \\
 (-a) + (-b) &= -(a + b)
 \end{aligned}$$

(iv)

$$\begin{aligned}
 1 &= (-a)^{-1}(-a) \\
 -1 &= (-a)^{-1}(a) \\
 -1(a^{-1}) &= (-a)^{-1} \\
 -(a^{-1}) &= (-a)^{-1}
 \end{aligned}$$

(v)

$$\begin{aligned}
 1 &= (ab)^{-1}(ab) \\
 b^{-1} &= (ab)^{-1}(a) \\
 a^{-1}b^{-1} &= (ab)^{-1}
 \end{aligned}$$

## Problem 2

Consider the set

$$K := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

Show that:

- (i)  $x, y \in K \Rightarrow x + y \in K \wedge xy \in K$
- (ii)  $x \neq 0 \Rightarrow x^{-1} \in K$

(i)

Let  $x, y \in K$ . Then,  $x = a + b\sqrt{2}$  and  $y = c + d\sqrt{2}$ , where  $a, b, c, d \in \mathbb{Q}$ .

$x + y = (a + c) + (b + d)\sqrt{2} \in K$ , as  $\mathbb{Q}$  is closed under addition.

$xy = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}$ , as  $\mathbb{Q}$  is closed under multiplication.

(ii)

Let  $x = a + b\sqrt{2} \neq 0 \in K$ . Thus, at least one of  $a, b \neq 0$ .

$$\begin{aligned}
 x^{-1} &= \frac{1}{a + b\sqrt{2}} \\
 &= \frac{a - b\sqrt{2}}{a^2 - 2b^2} \\
 &= \frac{a}{a^2 - 2b^2} + \frac{-b\sqrt{2}}{a^2 - 2b^2}
 \end{aligned}$$

Since  $a/(a^2 - 2b^2)$  and  $(-b)/(a^2 - 2b^2)$  are both in  $\mathbb{Q}$ ,  $x^{-1} \in K$ .

## Problem 3

Suppose  $\mathbb{F}$  is a field admitting  $P \subseteq F$  with the following properties:

- (C1) If  $x, y \in P$ , then  $x + y \in P$  and  $xy \in P$
- (C2) For all  $x \in \mathbb{F}$ ,  $x \in P$  or  $-x \in P$
- (C3) If  $x, -x \in P$ , then  $x = 0$ .

Show that there is an ordering on  $\mathbb{F}$  making it into an ordered field.

## Problem 4

Let  $a, b \in \mathbb{R}$ . Prove the following:

- (i) If  $0 \leq a \leq \varepsilon$  for all  $\varepsilon > 0$ , then  $a = 0$ .
- (ii) If  $a \leq b + \varepsilon$  for all  $\varepsilon > 0$ , then  $a \leq b$ .

(i)

Suppose toward contradiction that  $a \neq 0$ . Since  $a \geq 0$ , it must be that  $a > 0$ , so  $\frac{1}{2}a > 0$ . Let  $\varepsilon = \frac{1}{2}a$ . Therefore,  $0 < \frac{1}{2}a < a$ , which can't be true as  $a \leq \varepsilon$  for all  $\varepsilon > 0$ .  $\perp$

(ii)

Let  $a > b$ . Then,  $\exists \varepsilon > 0$  such that  $a \geq b + \varepsilon$ , where  $0 \leq \varepsilon \leq b - a$ . Therefore,  $a \not\leq b + \varepsilon$  for all  $\varepsilon \geq 0$ .

## Problem 5

If  $a, b \in \mathbb{R}$ , show that

$$\left(\frac{1}{2}(a+b)\right)^2 \leq \frac{1}{2}(a^2 + b^2)$$

$$\left(\frac{1}{2}(a+b)\right)^2 = \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}ab$$

WLOG, let  $a \geq b$ . There are three cases:  $a, b \in \mathbb{R}^+$ ,  $a \in \mathbb{R}^+$ ,  $-b \in \mathbb{R}^+$ , or  $-a, -b \in \mathbb{R}^+$ .

**CASE 1:** If  $a, b \in \mathbb{R}^+$ , then  $\frac{1}{2}ab \leq \frac{1}{2}a^2$ . Since  $a^2 \geq b^2$  (as  $a \geq b$ ), it must be that  $\frac{1}{2}a^2 \geq \frac{1}{4}a^2 + \frac{1}{4}b^2$ .

$$\begin{aligned} \left(\frac{1}{2}(a+b)\right)^2 &= \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}ab \\ &\leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \\ &= \frac{1}{2}(a^2 + b^2) \end{aligned}$$

**CASE 2:** If  $a \in \mathbb{R}^+$  and  $-b \in \mathbb{R}^+$ , then  $-\frac{1}{2}ab \in \mathbb{R}^+$ , or  $\frac{1}{2}ab < 0$ .

$$\begin{aligned} \left(\frac{1}{2}(a+b)\right)^2 &= \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}ab \\ &\leq \frac{1}{4}a^2 + \frac{1}{4}b^2 \\ &\leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \\ &= \frac{1}{2}(a^2 + b^2) \end{aligned}$$

**CASE 3:** If  $-a, -b \in \mathbb{R}^+$ , then  $\frac{1}{2}ab \in \mathbb{R}^+$ , so we use similar logic to Case 1.

## Problem 6

For  $x \in \mathbb{R}$ , show that  $\sqrt{x^2} = |x|$ .

Recall:

$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

Suppose  $x \in \mathbb{R}^+$ . Then, since  $\sqrt{x^2} \in \mathbb{R}^+$ , and  $y^2 = x^2 \Rightarrow y = \pm x$ , it must be the case that  $\sqrt{x^2} = x$ .

Suppose  $x \notin \mathbb{R}^+$ . Then,  $x^2 \in \mathbb{R}^+$ , so  $\sqrt{x^2} \in \mathbb{R}^+$ , so  $\sqrt{x^2} = -x$ .

Thus,  $\sqrt{x^2} = |x|$ .

## Problem 7

Let  $x, y, a, b \in \mathbb{R}$  and  $\varepsilon > 0$ .

- (i) Show that  $|x - a| < \varepsilon$  if and only if  $a - \varepsilon < x < a + \varepsilon$
- (ii) If  $a < x < b$  and  $a < y < b$ , show that  $|x - y| < b - a$ . What does this mean geometrically?

(i)

( $\Rightarrow$ ) Let  $|x - a| < \varepsilon$ . Then,  $x - a < \varepsilon$  and  $-(x - a) < \varepsilon$ . Thus,  $x < a + \varepsilon$  and  $-x < \varepsilon - a$ , so  $a - \varepsilon < x < a + \varepsilon$ .

( $\Leftarrow$ ) Let  $a - \varepsilon < x < a + \varepsilon$ . Then,  $-\varepsilon < (x - a) < \varepsilon$ . Therefore,  $|x - a| < \varepsilon$ .

(ii)

Let  $a < x < b$  and  $a < y < b$ . In the second case, we have that  $-b < -y < -a$  (by multiplying all the inequalities by  $-1$ ). Adding, we get  $a - b < x - y < b - a$ , or  $-(b - a) < x - y < b - a$ . Therefore,  $|x - y| < b - a$ .

## Problem 8

Find all  $x \in \mathbb{R}$  that satisfy:

$$4 < |x + 2| + |x - 1| < 5$$

**CASE 1:**  $x < -2$

$$\begin{aligned} 4 &< -(x + 2) - (x - 1) < 5 \\ -5 &< (x + 2) + (x - 1) < -4 \\ -5 &< 2x + 1 < -4 \\ -6 &< 2x < -5 \\ -3 &< x < -2.5 \end{aligned}$$

**CASE 2:**  $-2 \leq x < 1$

$$4 < (x+2) + -(x-1) < 5$$

$$4 < 2 < 5$$

⊥

**CASE 3:**  $1 \leq x$ 

$$4 < (x+2) + (x-1) < 5$$

$$4 < 2x+1 < 5$$

$$1.5 < x < 2$$

So the solution is:

$$x \in (-3, -2.5) \cup (1.5, 2)$$

**Problem 9**Let  $a, b \in \mathbb{R}$ . Show that

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|)$$

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|)$$

WLOG, let  $a > b$ . Then:

$$\begin{aligned} \frac{1}{2}(a + b + |a - b|) &= \frac{1}{2}(a + b + (a - b)) \\ &= a \\ \frac{1}{2}(a + b - |a - b|) &= \frac{1}{2}(a + b - (a - b)) \\ &= b \end{aligned}$$

Similarly, if  $a = b$ , then we have that  $\max(a, b) = \min(a, b) = a = b$ .**Problem 10**If  $x \neq y$  in  $\mathbb{R}$ , show that there is a  $\delta > 0$  such that  $V_\delta(x) \cap V_\delta(y) = \emptyset$ .Let  $\delta = \frac{1}{2}|x - y|$ . Then

$$V_\delta(x) \cap V_\delta(y) = \left(x - \frac{1}{2}|x - y|, x + \frac{1}{2}|x - y|\right) \cap \left(y - \frac{1}{2}|x - y|, y + \frac{1}{2}|x - y|\right) = \emptyset$$