4.11

Problem: Show that if A is a decidable set, then so is its complement. Then, show that if A and B are decidable sets, then so are $A \cup B$ and $A \cap B$.

Solution: Let f_A be the function that computes χ_A, and let f_B be the function that computes χ_B.

We define g_A, which computes $\mathbb{N} \setminus A$, by composing f_A with the partial function that computes 1 if the input is 0 and computes 0 if the input is 1.

To define $f_{A\cup B}$ and $f_{A\cap B}$, we take

$$\begin{aligned} f_{A\cup B} &= f_A + f_B - (f_A)(f_B) \\ f_{A\cap B} &= (f_A)(f_B), \end{aligned}$$

in which we use the multiplication and addition operations composed with f_A and f_B.

Extra Problem 1

Problem: Give an example of a relation that is not computable.

Solution: Let $\{T_m\}_{m\in\mathbb{N}}$ be a denumeration of the set of all Turing machines with one input. We define the relation $R \subseteq \mathbb{N} \times \mathbb{N}$ with the membership $(m, n) \in R$ if and only if $T_m(n)$ halts.

Since it is not possible to compute the halting problem, we know that the relation R is not computable.

Extra Problem 2

Problem: Suppose R, S, T are relations with $(a, b) \in T$ if and only if $(a, b) \in R$ or $(a, b) \in S$.

(a) Prove or disprove: if R and S are computable, then T is computable.

(b) Prove or disprove: if T and S are computable, then R is computable.

Solution:

(a) We can define a computation of T by saying

$$T(a, b) = R(a, b) + S(a, b) - M(S(a, b), R(a, b));$$

which is computable as addition and multiplication are computable.

(b) Consider R as the halting evaluation — that is, $(a, b) \in R$ if and only if $T_a(b)$ halts. If we let $S = \mathbb{N} \times \mathbb{N}$, then

$$\begin{aligned} T &= R \cup S \\ &= R \cup \mathbb{N} \times \mathbb{N} \\ &= \mathbb{N} \times \mathbb{N}, \end{aligned}$$

meaning T is computable, S is computable, but R is not computable.

Extra Problem 3

Problem: Prove that if $R \subseteq \mathbb{N} \times \mathbb{N}$ is computable, then so too is $\mathbb{N} \times \mathbb{N} \setminus R$.

Extra Problem 4

Problem: Show that the relation $a|b$ is primitive recursive.

Solution: The relation $a|b$ means there exists k such that $ak = b$.

So, we know that $a|b$ if and only if

$$((0)(a) = b) \vee ((1)(a) = b) \vee \cdots \vee (ba = b).$$

We then evaluate the product

$$d(a, b) = \prod_{i=0}^{b} (1 \dot{-} E(b \dot{-} M(i, a), 0))$$

to find the truth value of the statement.

Extra Problem 5

Problem: Define a function $\pi(n)$ by $\pi(n) = 1$ if n is prime and 0 otherwise. Use minimalization to show that π is computable.

Solution: From Wilson's Theorem, we know that $(n-1)! \equiv n-1$ modulo n if and only if n is prime. We start by defining the remainder function

$$\text{rem}(a, b) = \min_{z}((a \dot{-} bz) \dot{-} b = 0).$$

Since the factorial function is primitive recursive, we thus have

$$\pi(n) = E(\text{rem}(\text{fact}(n-1), n), n-1).$$

Since π is obtained by composition, primitive recursion, and minimalization, we must have that π is recursive (hence computable).

Extra Problem 7

Problem: Let

$$\pi(n) = \begin{cases} 1 & n \text{ prime} \\ 0 & \text{else} \end{cases}.$$

Show that π is primitive recursive.

Solution: We know that n is prime if and only if the only k with $k > 1$ and $k|n$ is $k = n$.

Take

$$\begin{aligned} \pi(n) &= (1 \dot{-} D(2, n))(1 \dot{-} D(3, n)) \cdots (1 \dot{-} D(n-1, n)) \\ &= \prod_{i=2}^{n-1} (1 \dot{-} D(i, n)). \end{aligned}$$