Problem 1

Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Show that the following are equivalent:

(i) c is a limit point of D.

(ii) There is a sequence $(x_n)_n$ in $D \setminus \{c\}$ with $(x_n)_n \to c$.

(⇒) Let c be a limit point of D. Then, taking $\delta_n = 1/n$, let $x_n \in \dot{V}_{\delta_n}(c)$. Then, $(x_n)_n \to c$.

(⇐) Let $(x_n)_n$ be a sequence in $D \setminus \{c\}$ with $(x_n)_n \to c$.

Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ with, $\forall n \geq N$, $|x_n - c| < \varepsilon$. Thus, $\forall \varepsilon > 0$, $\exists x_n$ such that $x_n \in \dot{V}_\varepsilon(c)$. Thus, c is a limit point.

Problem 2

Show that f can have at most one limit at c.

Suppose toward contradiction that $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, where $L_1 \neq L_2$. Then, $\exists \varepsilon_0 > 0$ such that $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$.

Let δ_1 be such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon_0$, and δ_2 be such that $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon_0$. Set $\delta = \min(\delta_1, \delta_2)$.

Then, $|x - c| < \delta \Rightarrow |f(x) - L_1| < \varepsilon_0$ and $|x - c| < \delta \Rightarrow |f(x) - L_2| < \varepsilon_0$. So, $\exists k$ such that $f(k) \in V_\varepsilon(L_1)$ and $f(k) \in V_\varepsilon(L_2)$. $\perp$

Problem 3

Show that the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$

(ii) For every sequence $(x_n)_n$ in $D \setminus \{c\}$ such that $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

(⇒) Let $\lim_{x \to c} f(x) = L$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

So, $\forall \varepsilon > 0$, $\exists f(x_k) \in V_\varepsilon(L)$, such that $x_k \in \dot{V}_\varepsilon(c)$. So, we have a sequence $(x_n)_n \to c$ defined by $\delta(\varepsilon, c)$, where $(f(x_n))_n \to L$.

(⇐) Suppose that for every sequence in $D \setminus \{c\}$ where $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

Then, $\forall \delta > 0$, $\exists N_1 \in \mathbb{N}$ such that $n_1 \geq N_1 \Rightarrow |x_{n_1} - c| < \delta$. Additionally, $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ such that $n_2 \geq N_2 \Rightarrow |f(x_{n_2}) - L| < \varepsilon$. Let $N = \max(N_1, N_2)$.

Then, $\forall \varepsilon > 0$, we have a $\delta > 0$, such that for all $n \geq N \Rightarrow |x_n - c| < \delta$, $|f(x_n) - L| < \varepsilon$. Thus, $\lim_{x \to c} f(x) = L$.