Part 1

3.4, Problem 3

(a) Solving the eigenvalues, we find

$$\det\begin{pmatrix}-\lambda & 2\\ -2 & -\lambda\end{pmatrix} = \lambda^2 + 4,$$

so the eigenvalues are $\lambda = \pm 2i$.

(b) The origin is thus a center as each eigenvalue is pure imaginary.

(c) Since

$$e^{2it} = \cos(2t) + i\sin(2t),$$

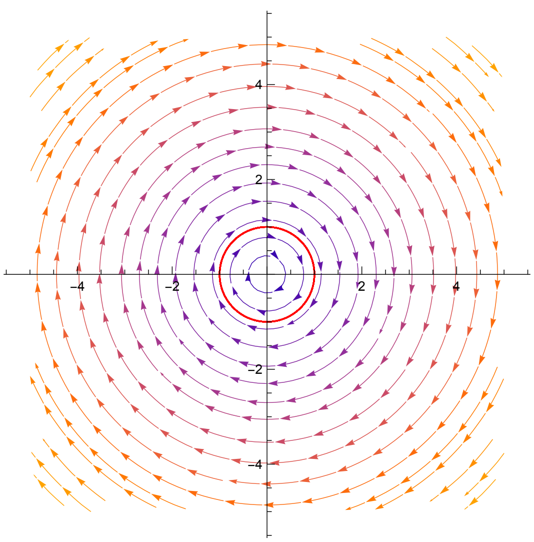
we have that the period of each oscillation is π and the frequency is $\frac{1}{\pi}$.

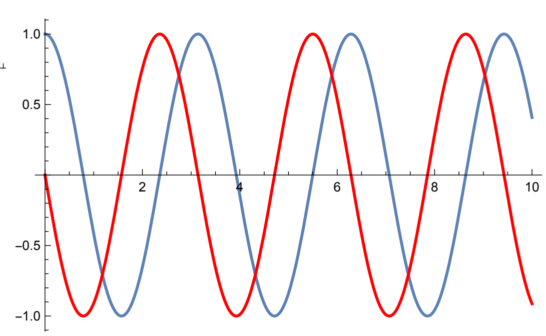
(d) Solving for the eigenvectors, we have

$$\begin{aligned}
\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} &= 2i\begin{pmatrix}x\\ y\end{pmatrix}\\
2y &= 2ix\\
y &= ix\\
\vec{v}_1 &= \begin{pmatrix}1\\ i\end{pmatrix}\\
\vec{v}_2 &= \begin{pmatrix}1\\ -i\end{pmatrix}\\
\vec{Y}(t) &= (\cos(2t) + i\sin(2t))\begin{pmatrix}1\\ i\end{pmatrix}\\
&= \begin{pmatrix}\cos(2t)\\ -\sin(2t)\end{pmatrix} + i\left(\sin(2t)\cos(2t)\right)\\
\vec{Y}_1(t) &= \begin{pmatrix}k_1\cos(2t) + k_2\sin(2t)\\ -k_1\sin(2t) + k_2\cos(2t)\end{pmatrix}.
\end{aligned}$$

Solving the initial condition, we get $k_1 = 1$ and $k_2 = 0$, so our solution must be counterclockwise.

(e)





3.4, Problem 4

(a) Solving the eigenvalues, we find

$$\begin{aligned}\det\begin{pmatrix}2-\lambda & 2\\ -4 & 6-\lambda\end{pmatrix} &= \left(\lambda^2-8\lambda+12\right)+8\\ \lambda^2-8\lambda+20 &= 0\\ \lambda &= 4\pm 2i.\end{aligned}$$

(b) The origin is a spiral source as the real part of λ is positive.

(c) The period of the oscillations is π, while the frequency is $\frac{1}{\pi}$.

(d) Finding the eigenvectors, we take

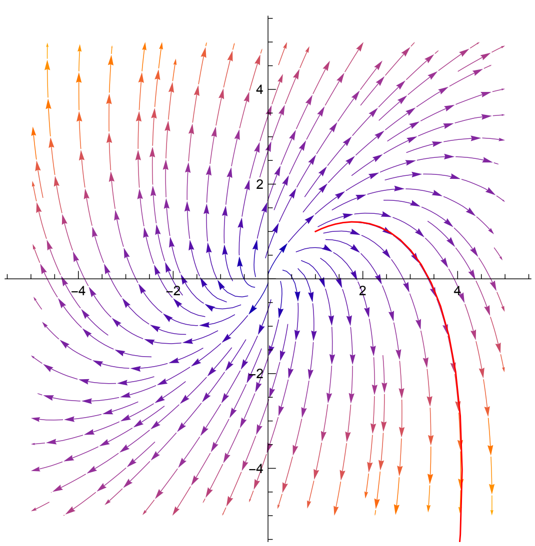
$$\begin{aligned}\begin{pmatrix}2 & 2\\ -4 & 6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} &= (4+2i)\begin{pmatrix}x\\ y\end{pmatrix}\\ 2x+2y &= (4+2i)x\\ 2y &= (2+2i)x\\ y &= (1+i)x\\ \vec{v} &= \begin{pmatrix}1\\ 1+i\end{pmatrix}\\ \vec{Y}(t) &= e^{4t}\left(\cos(2t)+i\sin(2t)\right)\begin{pmatrix}1\\ 1+i\end{pmatrix}\\ &= e^{4t}\begin{pmatrix}\cos(2t)+i\sin(2t)\\ (\cos(2t)+i\sin(2t))+i\cos(2t)-\sin(2t)\end{pmatrix}\\ &= e^{4t}\begin{pmatrix}\cos(2t)\\ \cos(2t)-\sin(2t)\end{pmatrix}+ie^{4t}\begin{pmatrix}\sin(2t)\\ \cos(2t)+\sin(2t)\end{pmatrix}.\end{aligned}$$

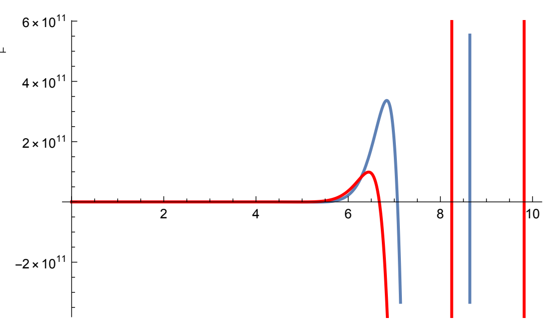
Thus, our general solution is

$$\vec{Y}_1(t) = e^{4t}\left(k_1\begin{pmatrix}\cos(2t)\\ \cos(2t)-\sin(2t)\end{pmatrix}+k_2\begin{pmatrix}\sin(2t)\\ \cos(2t)+\sin(2t)\end{pmatrix}\right).$$

With the initial condition of $\vec{Y}_0 = \begin{pmatrix}1\\ 1\end{pmatrix}$, we get $k_1 = k_2 = 1$. We find that the oscillations go clockwise.

(e)





3.4, Problem 16

$$\det\begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix} = (\lambda - a)^2 + b^2$$
$$\lambda = a \pm \sqrt{-b^2}.$$

Thus, $\lambda \in \mathbb{C}$.

3.4, Problem 23

(a) Let $v = \frac{dy}{dt}$. Then, we have

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

(b) The matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix},$$

meaning that for complex eigenvalues, we must have, for

$$\lambda^2 - p\lambda + q = 0,$$

that $q > \frac{p^2}{4}$.

(c) If $p > 0$, and $q > \frac{p^2}{4}$, then the origin is a spiral source. If $p < 0$ and $q > \frac{p^2}{4}$, then the origin is a spiral sink. If $p = 0$ and $q > 0$, then the origin is a center.

(d) I don't know how to do this problem.

3.5, Problem 3

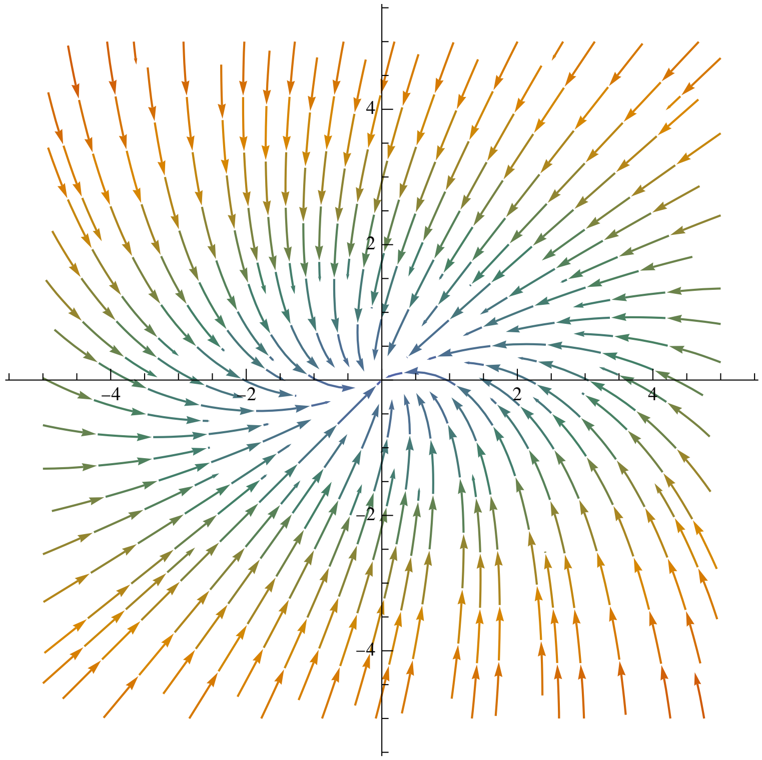
(a) The eigenvalue is found by

$$\lambda^2 + 6\lambda + 9 = 0$$
$$\lambda = -3.$$

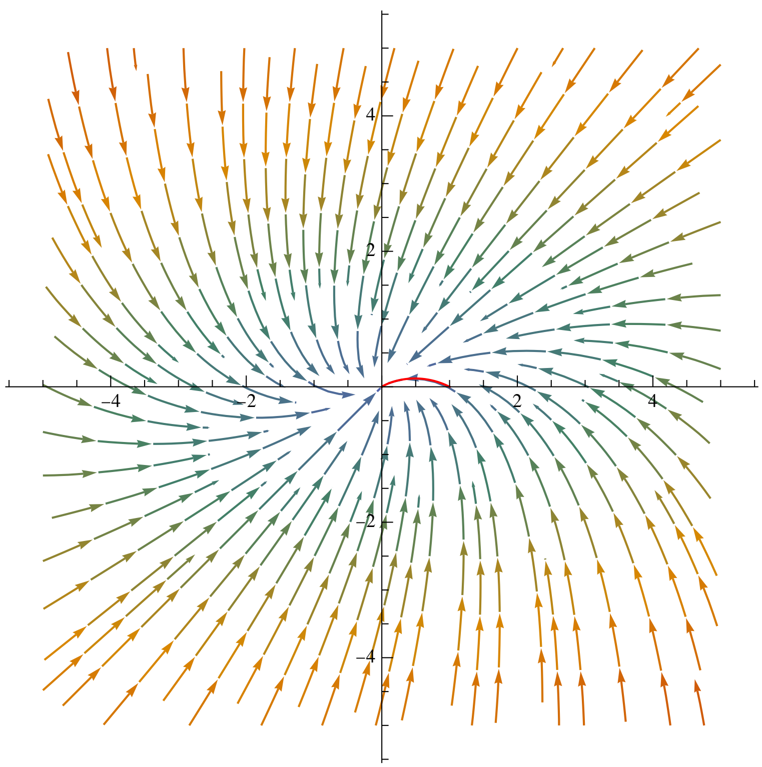
(b) An eigenvector is

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(c)



(d)



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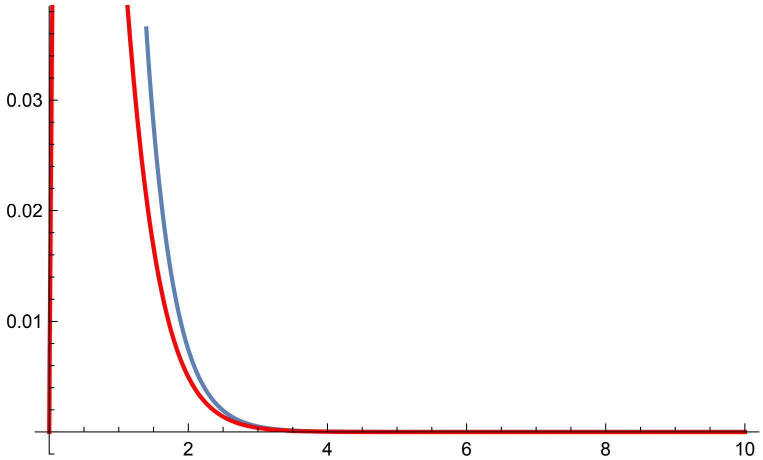


3.5, Problem 4

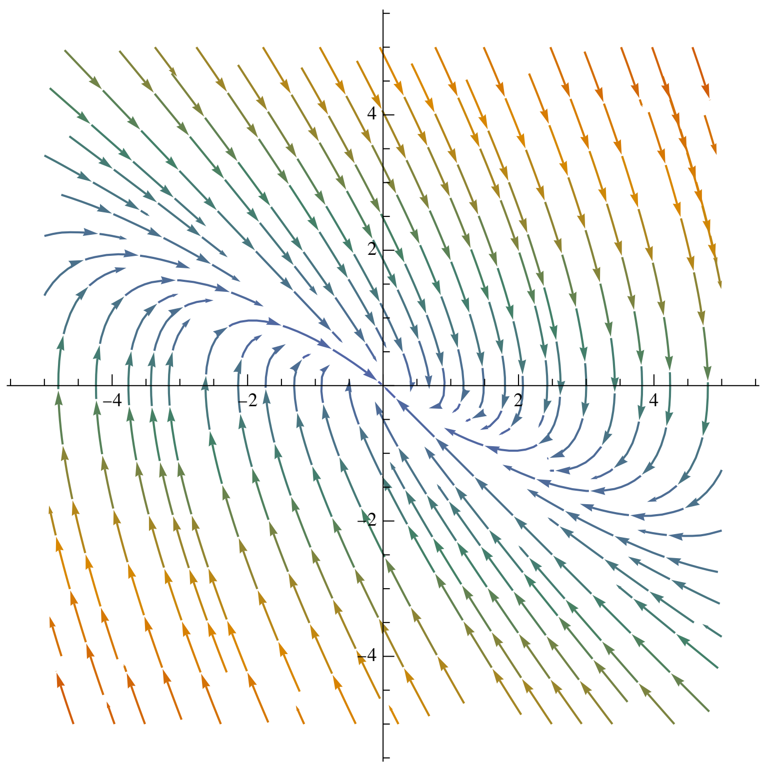
(a) The eigenvalue is found by

$$\begin{aligned}\lambda(\lambda+2)+1&=0\\ \lambda^2+2\lambda+1&=0\\ \lambda&=-1.\end{aligned}$$

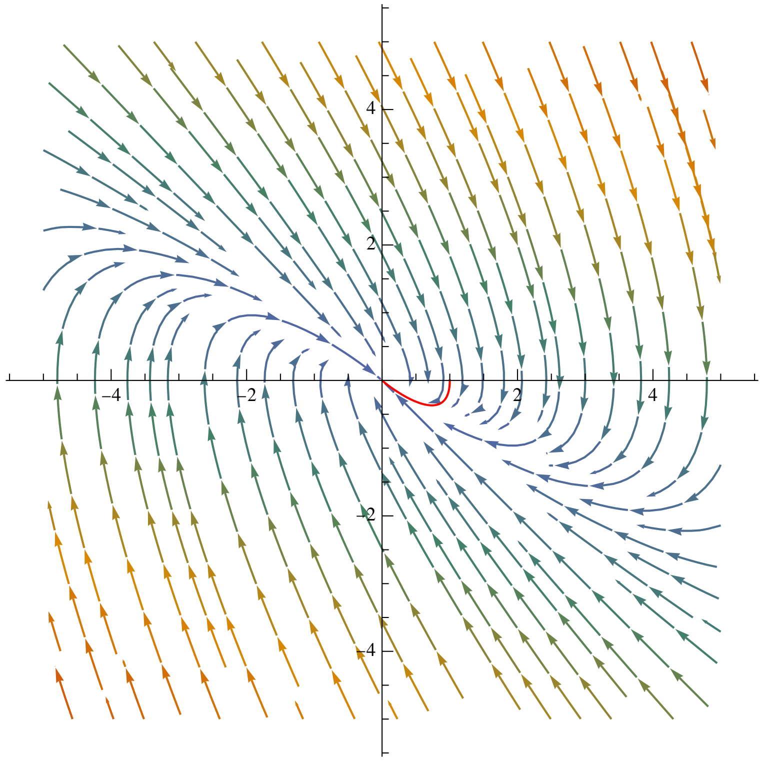
(b) An eigenvector is

$$\begin{aligned}y&=-x\\ \vec{v}&=\begin{pmatrix}1\\-1\end{pmatrix}.\end{aligned}$$

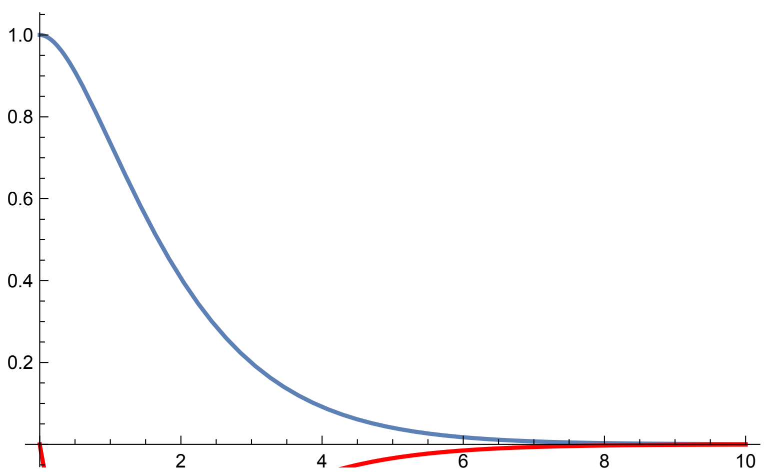
(c)



(d)



(e)



3.5, Problem 7

(a) We find the eigenvalue to be

$$\lambda = -3,$$

with corresponding eigenvector

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

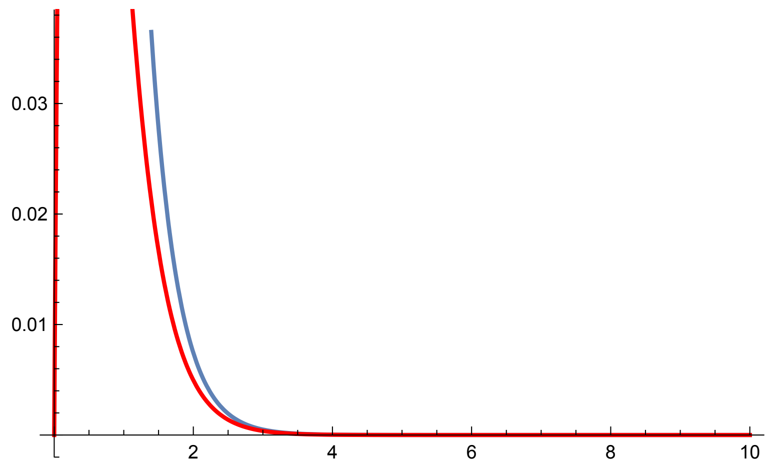
The general solution is, thus,

$$\vec{Y}(t) = te^{-t}\begin{pmatrix}-x_0 - y_0\\ x_0 - 3y_0\end{pmatrix} + e^{-t}\begin{pmatrix}x_0\\ y_0\end{pmatrix}.$$

(b) With initial condition $\vec{Y}(0) = \begin{pmatrix}1\\0\end{pmatrix}$, we have

$$\vec{Y}(t) = te^{-t}\begin{pmatrix}-1\\ 1\end{pmatrix} + e^{t}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

(c)



3.5, Problem 9

(a) If $\beta = \frac{\alpha^2}{4}$, then the quadratic has a double root.

(b) If $\beta = 0$, then the quadratic has zero as a root.

3.5, Problem 10

(a) If $\lambda > 0$, then $\lim_{t\to\infty} te^{\lambda t} = \infty$, as both t and $e^{\lambda t}$ are positive.

(b) If $\lambda < 0$, then $\lim_{t\to\infty} te^{\lambda t} = 0$ as, while $t > 0$, $e^{\lambda t}$ decreases at a faster rate than t increases.