Problem 1

Let V be a vector space and suppose $\{W_i\}$ is a family of subspaces of V.

(i) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of V contained in every W_i.

Proof: We will show that (a) $\bigcap_{i\in I} W_i$ is a subspace of V, and (b) there is is no larger subspace of V contained within every W_i.

(a) Let $v_i, v_j \in \bigcap_{i\in I} W_i$, $\alpha, \beta \in \mathbb{F}$. We want to show that $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$. Since $v_i \in \bigcap_{i\in I} W_i$, $v_i \in W_i$ for some W_i, and $v_j \in W_j$ for some W_j. Additionally, WLOG, $v_j \in W_i$, as both v_i and v_j are contained within their intersection. Therefore, $\alpha v_i + \beta v_j \in W_i$, so $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$.

(b) Suppose there is a subspace U of V such that every W_i is contained in U, and $U \supset \bigcap_{i\in I} W_i$. Since U is a vector space, U has a basis B_u; additionally, since we have shown that $\bigcap_{i\in I} W_i$ is a subspace, $\bigcap_{i\in I} W_i$ has a basis, B_w.

(ii) Show that

$$\sum_{i\in I} W_i := \left\{ \sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite} \right\}$$

is the smallest subspace containing each W_i.