Section 4.1

Solution (Problem 4)**:** Evaluating with the initial conditions, we get

$$\begin{aligned} c_1 - c_2 &= 0 \\ -c_3 &= 2 \\ c_2 &= -1. \end{aligned}$$

We see that $c_1 = -1$, $c_2 = -1$, and $c_3 = -2$. This yields the particular solution of

$$y = -1 - \cos x - 2\sin x.$$

Solution (Problem 10)**:** The interval $(-\pi, \pi)$ contains a unique solution to the initial value problem.

Solution (Problem 14)**:**

(a) We have

$$\begin{aligned} c_1 + c_2 + 3 &= 0 \\ c_1 + c_2 + 3 &= 4, \end{aligned}$$

which is not possible.

(b) We have

$$\begin{aligned} 3 &= 0 \\ c_1 + c_2 + 3 &= 2, \end{aligned}$$

which is yet again not possible.

(c) We have

$$\begin{aligned} 3 &= 3 \\ c_1 + c_2 + 3 &= 0, \end{aligned}$$

meaning that the solution set is all pairs (c_1, c_2) such that $c_1 + c_2 = -3$.

(d) We have

$$\begin{aligned} c_1 + c_2 + 3 &= 3 \\ 4c_1 + 16c_2 + 3 &= 15, \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 &= 0 \\ 4c_1 + 16c_2 &= 12 \end{aligned}$$

meaning

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1. \end{aligned}$$

Solution (Problem 22)**:** Since

$$\sinh(x) = \frac{1}{2}(e^x + e^{-x}),$$

the functions are not linearly independent anywhere on $(-\infty, \infty)$.

Solution (Problem 28)**:** First, we verify that both solutions work.

$$\begin{aligned} x^2 \frac{d^2}{dx^2}(\cos(\ln(x))) + x\frac{d}{dx}(\cos(\ln(x))) + \cos(\ln(x)) &= x^2\left(-\frac{\cos(\ln(x))}{x^2} + \frac{\sin(\ln(x))}{x^2}\right) + x\left(-\frac{\sin(\ln(x))}{x}\right) + \cos(\ln(x)) \\ &= 0 \end{aligned}$$

$$\begin{aligned} x^2\frac{d^2}{dx^2}(\sin(\ln(x))) + x\frac{d}{dx}(\sin(\ln(x))) + \sin(\ln(x)) &= x^2\left(-\frac{\cos(\ln(x))}{x^2} - \frac{\sin(\ln(x))}{x^2}\right) + x\left(\frac{\cos(\ln(x))}{x}\right) + \sin(\ln(x)) \\ &= 0. \end{aligned}$$

Additionally, we find that

$$\begin{aligned} \det\begin{pmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ \frac{-\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{pmatrix} &= \frac{1}{x} \\ &\neq 0, \end{aligned}$$

so the solutions are linearly independent. Since the differential equation $x^2y'' + xy' + y' = 0$ is a second order equation, there are no other linearly independent solutions. Thus, we have the general solution of

$$y = \alpha\cos(\ln(x)) + \beta\sin(\ln(x)).$$

Solution (Problem 30)**:** I'm not checking the Wronskian on this one, they're clearly linearly independent. However, I will be doing the derivatives.

$$\begin{aligned} \frac{d^4}{dx^4}(1) + \frac{d^2}{dx^2}(1) &= 0 \\ \frac{d^4}{dx^4}(x) + \frac{d^2}{dx^2}(x) &= 0 \\ \frac{d^4}{dx^4}(\cos(x)) + \frac{d^2}{dx^2}(\cos(x)) &= \cos(x) - \cos(x) \\ &= 0 \\ \frac{d^4}{dx^4}(\sin(x)) + \frac{d^2}{dx^2}(\sin(x)) &= \sin(x) - \sin(x) \\ &= 0. \end{aligned}$$

Thus, since the solutions are linearly independent and have dimension 4, they form a basis for the general solution of $y^{(4)} + y'' = 0$. The general solution is

$$y(x) = c_1 + c_2x + c_3\cos(x) + c_4\sin(x).$$

Solution (Problem 36)**:**

(a) We have $y = 5$ is a particular solution to $y'' + 2y = 10$.

(b) We have $y = -2x$ is a particular solution to $y'' + 2y = 10$.

(c) Using linearity, we get that $y = -2x + 5$ is a particular solution to $y'' + 2y = -4x + 10$.

(d) Using a similar process, we have a particular solution of $y = 4x + \frac{5}{2}$.

(e) Neither of these linear combinations are general solutions of the differential equation, as the linearity principle only applies to solutions of the corresponding homogeneous equation.

Section 4.2

Solution (Problem 2)**:** Using the power of inspection, we find that our other solution is $y_2 = e^{-2x}$.

Solution (Problem 8)**:**

$$\begin{aligned} y_2(x) &= e^{x/3}\int\frac{e^{-x/6}}{e^{2x/3}}\,dx \\ &= -\frac{6}{5}e^{-x/2} \end{aligned}$$

Solution (Problem 16)**:**

$$\begin{aligned} y_2(x) &= \int e^{-\int \frac{2x}{1-x^2}\,dx}\,dx \\ &= \frac{1}{3}x^2 - x. \end{aligned}$$

Solution (Problem 20)**:** Using the power of inspection, the other homogeneous solution is $y_2(x) = e^{3x}$. Letting $y_p(x) = \nu(x)e^x$, we get

$$\begin{aligned} \left(\nu'' - 2\nu'\right)e^x &= x \\ \nu'' - 2\nu' &= xe^{-x} \\ \frac{d}{dx}\left(\nu'(x)\right) - 2\nu'(x) &= xe^{-x}. \end{aligned}$$

Using the integrating factor e^{-2x}, we have

$$\begin{aligned} \frac{d}{dx}\left(e^{-2x}\nu'(x)\right) &= xe^{-x} \\ e^{-2x}\nu'(x) &= \int xe^{-x}\,dx \\ &= -xe^{-x} - e^{-x} \\ \nu(x) &= -xe^x. \end{aligned}$$

Thus, we have the general solution of

$$y(x) = c_1e^x + c_2e^{3x} + c_3xe^{2x}.$$

Solution (Problem 22)**:** We know that

$$x\frac{d^2}{dx^2}(x) - x\frac{d}{dx}(x) + x = 0.$$

Dividing out, we have

$$y'' - y' + \frac{1}{x}y = 0,$$

and substituting into the expression for y_2, we have

$$\begin{aligned} y_2(x) &= x\int \frac{e^x}{x^2}\,dx \\ &= x\int \frac{1}{x^2}\sum_{k=0}^{\infty}\frac{x^k}{k!}\,dx \\ &= x\int \sum_{k=0}^{\infty}\frac{x^{k-2}}{k!}\,dx. \end{aligned}$$

Since the integrand is uniformly convergent for all $x \in \mathbb{R}$ by the Cauchy–Hadamard theorem, we get

$$\begin{aligned} &= x\left(\sum_{k=0}^{\infty}\frac{1}{k-1}\frac{x^{k-1}}{k!}\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{k-1}\frac{x^k}{k!}. \end{aligned}$$

A likely domain for y_2 is $(0,\infty)$.

Section 4.3

Solution (Problem 4)**:** Using the power of inspection, we have

$$y = c_1 e^x + c_2 e^{2x}.$$

Solution (Problem 6)**:** Using the power of inspection, we have

$$y = c_1 e^{5x} + c_2 x e^{5x}.$$

Solution (Problem 12)**:** Using the power of inspection+ε, we have

$$y(x) = e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)\right).$$

Solution (Problem 16)**:** Using the power of inspection+ε, we have

$$y(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

Solution (Problem 22)**:** Using the power of inspection+2ε, we have

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}.$$

Solution (Problem 36)**:** Using the power of inspection+ε, we have

$$y(x) = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x}.$$

Evaluating the initial conditions, we have

$$\begin{aligned} 1 &= c_1 + c_2 + c_3 \\ 1 &= -c_1 - 3c_2 + 2c_3 \\ 1 &= c_1 + 9c_2 + 4c_3, \end{aligned}$$

and using the power of inspection+ε, we get

$$\begin{aligned} c_1 &= \frac{2}{3} \\ c_2 &= -\frac{1}{5} \\ c_3 &= \frac{8}{15}. \end{aligned}$$

Solution (Problem 38)**:** Using the power of inspection, we know that the general solution is

$$y(x) = a_1 \cos(2x) + a_2 \sin(2x).$$

However, since $y(0) = y(\pi) = 0$, we have solutions of the form $y(x) = a \sin(2x)$.

Solution (Problem 50)**:** The corresponding homogeneous equation is

$$y''' - 11y'' + 14y' + 10y = 0.$$

Using Mathematica

(i) Using Mathematica with

```
DSolve[y'[x]^2 + y[x] == 2, y, x]
```

and simplifying, we get

$$y = \frac{1}{4}\left(8 - x^2 - 2c_1 x - c_1^2\right)$$

(ii) Using Mathematica with

```
DSolve[y''[x]*y'''[x] == 0, y, x]
```

and simplifying, we get

$$y = c_1 + c_2 x + c_3 x^2.$$

Solving and also Using Mathematica

(i) Using separation of variables and the power of inspection, we get a solution of

$$y = \tan(x + c)$$

Similarly, using Mathematica with

```
DSolve[y'[x] - y[x]^2 == 1, y, x]
```

and simplifying, we get

$$y = \tan(x + c)$$

(ii) Using the power of inspection, we get a solution of

$$y = \frac{1}{120}x^5 + c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

Similarly, using Mathematica with

```
DSolve[y''''[x] == x, y, x]
```

and simplifying, we get

$$y = \frac{1}{120}x^5 + c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

(iii) Using the power of inspection, we find a general form solution of

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x) + y_p(x).$$

Furthermore, using the power of inspection yet again, we find the particular solution $y_p(x) = -x^3$, so we find the general solution of

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x) - x^3.$$

Similarly, using Mathematica, with

```
DSolve[y''''[x] - y[x] == x^3, y, x]
```

and simplifying, we get

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x) - x^3.$$