Chapter 2 Problems

2.3

Cylindrical Coordinates

Starting with our expression of $\mathbf{r}$, we have

$$\begin{aligned}\mathbf{r} &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}\\ d\mathbf{r} &= \frac{\partial\mathbf{r}}{\partial\rho}d\rho + \frac{\partial\mathbf{r}}{\partial\phi}d\phi + \frac{\partial\mathbf{r}}{\partial z}dz.\end{aligned}$$

Calculating each partial derivative,

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\rho} &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\\ \hat{\rho} &= \frac{\frac{\partial\mathbf{r}}{\partial\rho}}{\left\|\frac{\partial\mathbf{r}}{\partial\rho}\right\|}\\ &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath},\\ \frac{\partial r}{\partial\phi} &= \rho\left(-\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\right)\\ \hat{\phi} &= \frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left\|\frac{\partial\mathbf{r}}{\partial\phi}\right\|}\\ &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\end{aligned}$$

implying

$$\frac{\partial\mathbf{r}}{\partial\phi} = \rho\hat{\phi},$$

and finally, we have

$$\frac{\partial\mathbf{r}}{\partial z} = \hat{k}.$$

The above calculations yield

$$d\mathbf{r} = (d\rho)\,\hat{\rho} + (\rho\,d\phi)\,\hat{\phi} + (dz)\,\hat{k}.$$

Spherical Coordinates

Starting with our expression of $\mathbf{x}$[1]

$$\mathbf{x} = r\sin\phi\sin\theta\hat{\imath} + r\cos\phi\sin\theta\hat{\jmath} + r\cos\theta\hat{k}$$

[1] I am using $\mathbf{x}$ instead of $\mathbf{r}$ because r is already used in the expression of the spherical coordinates.

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial r} dr + \frac{\partial \mathbf{x}}{\partial \phi} d\phi + \frac{\partial \mathbf{x}}{\partial \theta} d\theta,$$

Evaluating each partial derivative, we have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial r} &= \sin\phi \sin\theta \hat{\imath} + \cos\phi \sin\theta \hat{\jmath} + \cos\theta \hat{k} \\
\hat{r} &= \frac{\frac{\partial \mathbf{x}}{\partial r}}{\left\| \frac{\partial \mathbf{x}}{\partial r} \right\|} \\
&= \sin\phi \sin\theta \hat{\imath} + \cos\phi \sin\theta \hat{\jmath} + \cos\theta \hat{k}, \\
\frac{\partial \mathbf{x}}{\partial \phi} &= -r \sin\phi \sin\theta \hat{\imath} + r\cos\phi \sin\theta \hat{\jmath} \\
\hat{\phi} &= \frac{\frac{\partial \mathbf{x}}{\partial \phi}}{\left\| \frac{\partial \mathbf{x}}{\partial \phi} \right\|} \\
&= -\sin\phi \sin\theta \hat{\imath} + \cos\phi \sin\theta \hat{\jmath}
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \phi} = r\sin\theta \hat{\phi},$$

and finally, we have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \theta} &= r\cos\phi \cos\theta \hat{\imath} + r\sin\phi \cos\theta \hat{\jmath} - r\sin\theta \hat{k} \\
\hat{\theta} &= \frac{\frac{\partial \mathbf{x}}{\partial \theta}}{\left\| \frac{\partial \mathbf{x}}{\partial \theta} \right\|} \\
&= \cos\phi \cos\theta \hat{\imath} + \sin\phi \cos\theta \hat{\jmath} - \sin\theta \hat{k},
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \theta} = r\hat{\theta}.$$

The above calculations yield

$$d\mathbf{x} = (dr)\,\hat{r} + (r\sin\theta d\phi)\,\hat{\phi} + (r d\theta)\,\hat{\theta}.$$

2.8

Let

$$\begin{aligned}
\vec{a} &= r_a \cos\phi_a \sin\theta_a \hat{\imath} + r_a \sin\phi_a \sin\theta_a \hat{\jmath} + r_a \cos\theta_a \hat{k} \\
\vec{b} &= r_b \cos\phi_b \sin\theta_b \hat{\imath} + r_b \sin\phi_b \sin\theta_b \hat{\jmath} + r_b \cos\theta_b \hat{k}.
\end{aligned}$$

Then,

$$\begin{aligned}\cos\gamma &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\left\|\vec{b}\right\|}\\ &= \frac{1}{r_a r_b}\left(r_a r_b\left(\sin\theta_a\sin\theta_b\left(\cos\phi_a\cos\phi_b+\sin\phi_a\sin\phi_b\right)+\cos\theta_a\cos\theta_b\right)\right)\\ &= \cos\theta_a\cos\theta_b+\sin\theta_a\sin\theta_b\cos\left(\phi_a-\phi_b\right).\end{aligned}$$

2.9

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{d}{dt}\left(\dot{\rho}\hat{\rho}\right)+\frac{d}{dt}\left(\rho\dot{\phi}\hat{\phi}\right)\\ &= \hat{\rho}\ddot{\rho}+\dot{\rho}\frac{d\hat{\rho}}{dt}+\dot{\rho}\dot{\phi}\hat{\phi}+\rho\ddot{\phi}\hat{\phi}+\rho\dot{\phi}\frac{d\hat{\phi}}{dt}\\ &= \hat{\rho}\ddot{\rho}+\dot{\rho}\dot{\phi}\hat{\phi}+\rho\ddot{\phi}\hat{\phi}+\dot{\rho}\dot{\phi}\hat{\phi}+\rho\dot{\phi}\left(\frac{\partial\hat{\phi}}{\partial\rho}\frac{d\rho}{dt}+\frac{\partial\hat{\phi}}{\partial\phi}\frac{d\phi}{dt}\right)\\ &= \left(\ddot{\rho}-\rho\dot{\phi}^2\right)\hat{\rho}+\left(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}\right)\hat{\phi}.\end{aligned}$$

2.12

(a)

(i) $d\mathbf{a} = \rho\ d\phi dz$

(ii) $d\mathbf{a} = d\rho dz$

(iii) $d\mathbf{a} = \rho\ d\rho d\phi$

(b)

(i) $d\mathbf{a} = r^2\sin\theta\ d\theta d\phi$

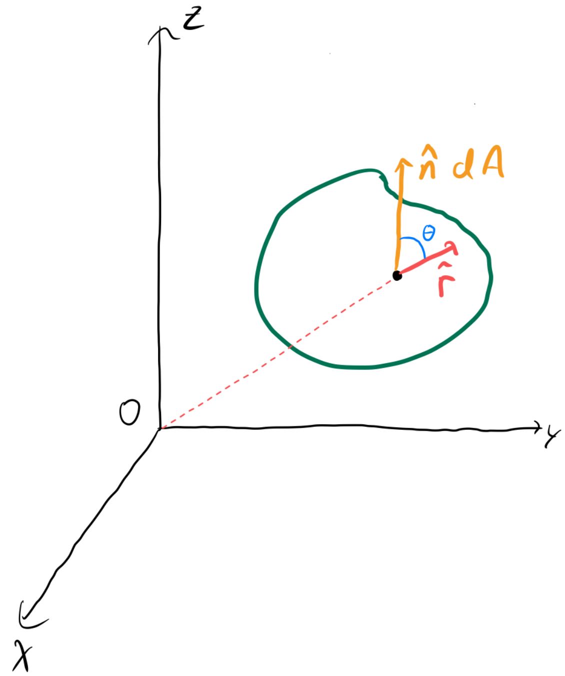
(ii) $d\mathbf{a} = r\sin\theta\ dr d\phi$

(iii) $d\mathbf{a} = r\ dr d\theta$

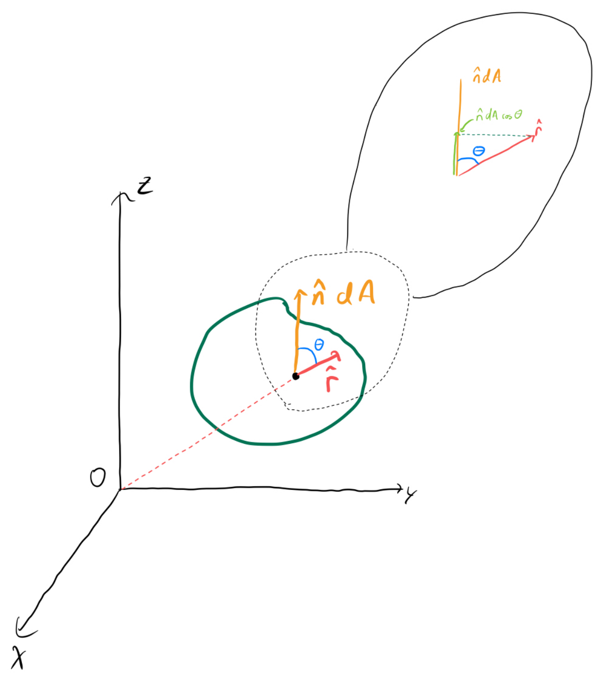
2.14

(a)

$$\begin{aligned}d\Phi &= \mathbf{E}\cdot\hat{n}\ dA\\ &= \|\mathbf{E}\|\,\|\hat{n}\|\cos\theta dA\\ &= \frac{q}{4\pi\epsilon_0 r^2}\cos\theta dA.\end{aligned}$$



(b)



(c)

$$\oiint_S \, d\Phi = \oiint_S \frac{q}{4\pi\epsilon_0 r^2} \, d\mathbf{a}$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{\pi} \frac{q}{4\pi\epsilon_0 r^2} r^2 \sin\theta \, d\theta d\phi \\
&= (2\pi)\left(\frac{q}{4\pi\epsilon_0}\right)\left(-\cos\theta\big|_0^{\pi}\right) \\
&= \frac{q}{\epsilon_0}.
\end{aligned}
$$

Chapter 3 Problems

For all problems involving $\arg z$ (or equivalents), I will be using the principle branch, $\arg z \in (-\pi, \pi]$.

3.5

(a)

$$
\begin{aligned}
\sqrt{3} + i &= 2e^{i\frac{\pi}{3}} \\
-\sqrt{3} + i &= 2e^{i\frac{2\pi}{3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sqrt{2i} &= \sqrt{2}e^{i\frac{\pi}{4}} \\
\sqrt{2 + 2\sqrt{3}i} &= 2e^{i\frac{\pi}{6}}
\end{aligned}
$$

3.6

(a) Real:

$$
\begin{aligned}
(-1)^{1/i} &= \left(e^{i\pi}\right)^{-i} \\
&= e^{\pi}.
\end{aligned}
$$

(b) Real:

$$
\begin{aligned}
\left(\frac{z}{z^*}\right)^i &= \left(e^{2i\arg z}\right)^i \\
&= e^{-2\arg z}.
\end{aligned}
$$

(c) Imaginary:

$$
\begin{aligned}
\left(z_1 z_2^* - z_1^* z_2\right)^* &= z_1^* z_2 - z_1 z_2^* \\
&= -\left(z_1 z_2^* - z_1^* z_2\right).
\end{aligned}
$$

(d) Complex:

$$\sum_{n=0}^{N} e^{in\theta} = \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}.$$

(e) Real: for each $a \in \{1, 2, \dots, N\}$, $e^{ia\theta} + e^{-ia\theta} \in \mathbb{R}$.

3.9

(a)

$$\begin{aligned}
\cos(a+b) + \cos(a-b) &= \frac{1}{2}\left(e^{i(a+b)} + e^{-i(a+b)}\right) + \frac{1}{2}\left(e^{i(a-b)} + e^{-i(a-b)}\right) \\
&= \frac{1}{2}\left(e^{ia}\left(e^{ib} + e^{-ib}\right) + e^{-ia}\left(e^{ib} + e^{-ib}\right)\right) \\
&= \frac{1}{2}\left(e^{ia} + e^{-ia}\right)\left(e^{ib} + e^{-ib}\right) \\
&= 2\cos a \cos b.
\end{aligned}$$

(b)

$$\begin{aligned}
\sin(a+b) + \sin(a-b) &= \frac{1}{2i}\left(e^{i(a+b)} - e^{-i(a+b)}\right) + \frac{1}{2}\left(e^{i(a-b)} - e^{-i(a-b)}\right) \\
&= \frac{1}{2i}\left(e^{ia}\left(e^{ib} + e^{-ib}\right) - e^{-ia}\left(e^{ib} + e^{-ib}\right)\right) \\
&= \frac{1}{2i}\left(e^{ia} - e^{-ia}\right)\left(e^{ib} + e^{-ib}\right) \\
&= 2\sin a \cos b.
\end{aligned}$$

3.10

(a)

$$\begin{aligned}
e^{i\alpha} + e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} + e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)} \\
&= e^{i\frac{\alpha+\beta}{2}}\left(e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}}\right) \\
&= 2\cos\left(\frac{\alpha-\beta}{2}\right) e^{i\frac{\alpha+\beta}{2}}.
\end{aligned}$$

(b)

$$e^{i\alpha} - e^{i\beta} = e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} - e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)}$$

$$\begin{aligned}
&= e^{i\frac{\alpha+\beta}{2}}\left(e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}}\right)\\
&= 2i\sin\left(\frac{\alpha-\beta}{2}\right)e^{i\frac{\alpha+\beta}{2}}
\end{aligned}$$

3.12

$$\begin{aligned}
\frac{1}{2i}\ln\left(\frac{a+ib}{a-ib}\right) &= \frac{1}{2i}\left(\ln(a+ib) - \ln(a-ib)\right)\\
&= \frac{1}{2i}\left(\ln|a+ib| + i\arctan\left(\frac{b}{a}\right) - \left(\ln|a+ib| + i\arctan\left(-\frac{b}{a}\right)\right)\right)\\
&= \arctan\left(\frac{b}{a}\right).
\end{aligned}$$

3.13

$$\begin{aligned}
\frac{d^n}{dt^n}\left(e^{at}\sin bt\right) &= \frac{1}{2i}\frac{d^n}{dt^n}\left(e^{(a+ib)t} - e^{(a-ib)t}\right)\\
&= \frac{1}{2i}\left((a+ib)^n e^{(a+ib)t} - (a-ib)^n e^{(a-ib)t}\right)\\
&= \frac{1}{2i}e^{at}\left(\left(\left(a^2+b^2\right)^{n/2} e^{in\arctan\left(\frac{b}{a}\right)}\right)e^{ibt} - \left(\left(a^2+b^2\right)^{n/2} e^{-in\arctan\left(\frac{b}{a}\right)}\right)e^{-ibt}\right)\\
&= e^{at}\frac{1}{2i}\left(a^2+b^2\right)^{n/2}\left(e^{i\left(b+n\arctan\left(\frac{b}{a}\right)\right)t} - e^{i\left(b+n\arctan\left(\frac{b}{a}\right)\right)}\right)\\
&= e^{at}\left(a^2+b^2\right)^{n/2}\sin\left(bt + n\arctan\left(\frac{b}{a}\right)\right)
\end{aligned}$$

3.20

Showing the equivalence between $C_1\cos kx + C_2\sin kx$ and $A\cos(kx+\alpha)$ and $B\sin(kx+\beta)$, we have

$$\begin{aligned}
A\cos(kx+\alpha) &= A\cos kx\cos\alpha - A\sin kx\sin\alpha\\
B\sin(kx+\beta) &= B\cos kx\sin\beta + B\sin kx\cos\beta
\end{aligned}$$

meaning (assuming $\alpha, \beta \neq \pi n, \pi/2 + \pi n$)

$$\begin{aligned}
A &= \frac{C_1}{\cos\alpha}\\
&= -\frac{C_2}{\sin\alpha}
\end{aligned}$$

$$
\begin{aligned}
B &= \frac{C_1}{\sin\beta}\\
&= \frac{C_2}{\cos\beta}.
\end{aligned}
$$

Now, we show the equivalence between $C_1\cos kx + C_2\sin kx$ and $D_1e^{ikx} + D_2e^{-ikx}$.

$$
\begin{aligned}
D_1e^{ikx} + D_2e^{-ikx} &= \frac{D_1+D_2}{2}\left(e^{ikx}+e^{-ikx}\right) + \frac{D_1+D_2}{2}\left(e^{ikx}-e^{-ikx}\right)\\
&= (D_1+D_2)\cos kx + i\left(D_1-D_2\right)\sin kx.
\end{aligned}
$$

meaning

$$
\begin{aligned}
C_1 &= D_1 + D_2\\
C_2 &= i\left(D_1 - D_2\right).
\end{aligned}
$$

Finally, we show the equivalence between $\operatorname{Re}\left(Fe^{ikx}\right)$ and $C_1\cos kx + C_2\sin kx$.

$$
\begin{aligned}
\operatorname{Re}\left((a+ib)\,e^{ikx}\right) &= \operatorname{Re}\left(a\cos kx + ia\sin kx + ib\cos kx - b\sin kx\right)\\
&= a\cos kx - b\sin kx,
\end{aligned}
$$

meaning

$$
\begin{aligned}
C_1 &= a\\
C_2 &= -b.
\end{aligned}
$$