Problem 1

Fix a measure space $(\Omega, \mathcal{M}, \mu)$. If $\phi : \Omega \to [0, \infty)$ is a simple, positive, measurable function given by

$$\phi = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}, \quad a_i \geq 0; A_i \in \mathcal{M}$$

we define

$$\int_{\Omega} \phi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

Show that this is well-defined. That is, if there is another expression of ϕ

$$\phi = \sum_{j=1}^{m} b_j \mathbb{1}_{B_j}, \quad b_j \geq 0; B_j \in \mathcal{M}$$

then

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j).$$

Proof: Let $\{F_k\}_{k=1}^{\ell}$ be a refinement of disjoint subsets of Ω such that $A_i = \bigsqcup_{k \in I_i} F_k$ and $B_j = \bigsqcup_{j \in J_j} F_j$, where $I_i, J_j \subseteq \{1, \dots, \ell\}$.

Let $M_k = \{i \mid F_k \subseteq A_i\}$ and $N_k = \{j \mid F_k \subseteq B_j\}$. Then,

$$\begin{aligned} \sum_{i=1}^{n} a_i \mathbb{1}_{A_i} &= \sum_{k=1}^{\ell} \sum_{i \in M_k} a_i \mathbb{1}_{F_k} \\ &= \sum_{k=1}^{\ell} \sum_{j \in N_k} b_j \mathbb{1}_{F_k}, \\ &= \sum_{j=1}^{m} b_j \mathbb{1}_{B_j} \end{aligned}$$

so

$$\begin{aligned} \sum_{i=1}^{n} a_i \mu(A_i) &= \sum_{k=1}^{\ell} \sum_{i \in M_k} a_k \mu(F_k) \\ &= \sum_{k=1}^{\ell} \sum_{j \in N_k} b_j \mu(F_k) \\ &= \sum_{j=1}^{m} b_j \mu(B_j). \end{aligned}$$

Problem 2

Let Δ be a totally disconnected compact metric space (for example, the Cantor set). Also, suppose $\varphi : C(\Delta) \to \mathbb{R}$ is a state — φ is linear, continuous, positive ($f \geq 0 \Rightarrow \varphi(f) \geq 0$), and $\varphi(\mathbb{1}_{\Delta}) = 1$.

(i) Show that $\mathcal{C} := \{E \mid E \subseteq \Delta \text{ is clopen}\}$ is an algebra of subsets of Δ.

(ii) Show that

$$\mu_0 : \mathcal{C} \to [0,1]; \quad \mu_0(E) = \varphi(\mathbb{1}_E)$$

is a well-defined finitely additive measure.

(iii) Show that μ_0 is a pre-measure on $(\Delta, \mathcal{C})$.

(iv) Prove that there is a unique Borel probability measure μ on $(\Delta, \mathcal{B}_\Delta)$ such that

$$\int_\Delta f \, d\mu = \varphi(f) \quad \forall f \in C(\Delta).$$

Proof:

(i) Since the complement of any clopen set is clopen, and the finite union of clopen sets is clopen, $\mathcal{C}$ is an algebra of subsets of Δ.

(ii) We can see that $\varphi(\mathbb{1}_\emptyset) = 0$, meaning $\mu_0(\emptyset) = 0$, and for $E, F \in \mathcal{C}$ disjoint,

$$\begin{aligned}
\mu_0(E \sqcup F) &= \varphi(\mathbb{1}_{E \sqcup F}) \\
&= \varphi(\mathbb{1}_E + \mathbb{1}_F) \\
&= \varphi(\mathbb{1}_E) + \varphi(\mathbb{1}_F) \\
&= \mu_0(E) + \mu_0(F).
\end{aligned}$$

(iii) Let $\{E_k\}_{k\geq 1} \subseteq \mathcal{C}$ with $\bigsqcup_{k\geq 1} E_k \in \mathcal{C}$. Then,

$$\begin{aligned}
\mu_0\left(\bigsqcup_{k\geq 1} E_k\right) &= \varphi\left(\mathbb{1}_{\bigsqcup_{k\geq 1} E_k}\right) \\
&= \sum_{k=1}^{\infty} \varphi(\mathbb{1}_{E_k}) \\
&= \sum_{k=1}^{\infty} \mu_0(E_k).
\end{aligned}$$

Thus, μ_0 is a pre-measure.

(iv) We see that μ_0 extends to a measure on $(\Delta, \mathcal{B}_\Delta)$, as $\mathcal{C} \subseteq \mathcal{B}_\Delta$ (since every clopen set is contained in the Borel σ-algebra).

It is also known that $\operatorname{span}\{\mathbb{1}_E \mid E \subseteq \Delta \text{ clopen}\}$ is uniformly dense in $C(\Delta)$ (by the Stone–Weierstrass Theorem).