Contents

Introduction

This is going to be part of my notes for my Honors Thesis independent study focused on Amenability and C^*-algebras. This set of notes will be focused on the theory of Banach algebras and C^*-algebras. The primary source for this section of notes will be Timothy Rainone's *Functional Analysis: En Route to Operator Algebras*.

I do not claim any of this work to be original.

Algebras and $*$-Algebras

A lot of the structures we encounter in functional analysis, like $\mathbb{B}(X)$, are not only vector spaces, but also come with an algebraic structure with them. We will learn these more in depth before venturing into the study of Banach and C^*-algebras.

Definitions and Examples

We will let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition. An $\mathbb{F}$-algebra is a vector space A over the field $\mathbb{F}$ with a multiplication operation $(a, b) \mapsto a \cdot b$ satisfying the following for all $a, b, c \in A$ and $\alpha \in \mathbb{F}$;

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

- $a \cdot (b + c) = a \cdot b + a \cdot c$;
- $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$.

An algebra is called unital if there is a unique $1_A \in A$ such that $1_A \cdot a = a \cdot 1_A = a$.

We say the algebra is commutative if multiplication is commutative, else it is called noncommutative.

Remark: Usually, $\mathbb{F} = \mathbb{C}$ unless otherwise specified, and we drop the multiplication sign, writing ab for $a \cdot b$.

Remark: If A is an $\mathbb{F}$-vector space with basis B, we can always extend an associative map $B \times B \to B$ to multiplication in A by defining multiplication by the associative map on the basis elements.

Example (Functions)**.** Let Ω be any nonempty set. The function space $\mathcal{F}(\Omega, \mathbb{F})$, equipped with pointwise addition, scalar multiplication, and pointwise multiplication is an algebra.

In general, if A is an $\mathbb{F}$-algebra, then $\mathcal{F}(\Omega, A) = \{f \mid f\colon \Omega \to A\}$ is an $\mathbb{F}$-algebra. If A is unital, then the constant map $u(x) = 1_A$ is the unit for $F(\Omega, A)$.

Example (Linear Maps)**.** If X is a vector space over $\mathbb{F}$, then $\mathcal{L}(X)$, the space of all linear maps from X to itself, is a unital $\mathbb{F}$-algebra with multiplication as composition.

Example (Polynomials in One Variable)**.** If x is an abstract variable, then the linear space of all polynomials,

$$\mathbb{F}[x] = \left\{ \sum_{k=0}^{n} a_k x^k \,\middle|\, a_k \in \mathbb{F}, n \in \mathbb{Z}_+ \right\}$$

is an $\mathbb{F}$-algebra. We define multiplication via ordinary multiplication of polynomials,

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{m+n}\left(\sum_{i=0}^{k} a_i b_{k-i}\right)x^k.$$

If we let $x^0 = 1$, this space is a commutative unital algebra.

Example (General Polynomial)**.** If we have a set $S = \{x_i\}_{i \in I}$ of abstract symbols, then $\mathbb{F}\langle S\rangle$ is the space of all (not necessarily commuting) polynomials with symbols in S, where multiplication is defined by concatenation. This is a unital algebra.

If the symbols in S commute, then this is a commutative algebra, and we wrie $F[S]$.

Example. If x is an abstract symbol, then

$$\mathbb{F}(x) = \left\{ \frac{p(x)}{q(x)} \,\middle|\, p, q \in \mathbb{F}[x], q \neq 0 \right\}$$

is the unital commutative algebra of all rational functions.

Definition. Let A be an $\mathbb{F}$-algebera, and let $a \in A$ be fixed. For $p \in \mathbb{F}[x]$, we define

$$p(a) = \sum_{k \geqslant 0} \alpha_k a^k.$$

It is assumed that $\alpha_0 = 0$ when A does not have a unit.

Generally, if $a_1, \dots, a_n \in A$ and $p = \sum_I c_I x^I$ in $\mathbb{F}\langle x_1, \dots, x_n\rangle$, then

$$p(a_1, \dots, a_n) = \sum_I c_I a^I,$$

where $a^I = a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}$, and $(i_1, \dots, i_n) = I \in \mathbb{Z}_+^n$ is a multi-index.

Remark: The binomial theorem only holds for commutative algebras.

Definition. Let A be an algebra over $\mathbb{C}$. An involution on A is a self-map $*\colon A \longrightarrow A$ that satisfies the following, for all $a, b \in A$ and $\alpha \in C$:

(1) $(a + \alpha b)^* = a^* + \overline{\alpha} b^*$;

(2) $(ab)^* = b^* a^*$;

(3) $a^{**} = a$.

If A admits an involution, then A is known as a $*$-algebra.

Example. The complex numbers, $\mathbb{C}$, is a unital commutative $*$-algebra with the usual operations, where $z \overset{*}{\mapsto} \overline{z}$ is the involution.

Example. We can define an involution on $\mathcal{F}(\Omega, \mathbb{C})$ by taking $f^*(x) = \overline{f(x)}$.

If A is a $*$-algebra, we may define the involution as $f^*(x) = f(x)^*$.

Example (The Free $*$-Algebra)**.** Let $E = \{x_i\}_{i \in I}$ be a set of abstract symbols. We may add a set of symbols disjoint from E, called $E^* = \left\{x_i^*\right\}_{i \in I}$, and let $S = E \cup E^*$.

We consider $\mathbb{C}\langle S \rangle$, which is the set of general polynomials over S. The involution $*\colon \mathbb{C}\langle S \rangle \longrightarrow \mathbb{C}\langle S \rangle$ can be defined by

$$\left(\alpha x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_n}^{\epsilon_n}\right)^* = \overline{\alpha} x_{i_n}^{\delta_n} \cdots x_{i_2}^{\delta_2} x_{i_1}^{\delta_1},$$

where $\epsilon_j \in \{1, *\}$ for each $j = 1, \dots, n$, and

$$\delta_j = \begin{cases} * & \epsilon_j = 1 \\ 1 & \epsilon_j = * \end{cases}.$$

The $*$-algebra, $\mathbb{C}\langle E \cup E^* \rangle$ is referred to as the free $*$-algebra generated by E, denoted $\mathbb{A}^*(E)$.

Example (Matrix Algebra)**.** Let A be an algebra, and let

$$\operatorname{Mat}_n(A) = \left\{ \left(a_{ij}\right)_{ij} \;\middle|\; 1 \leqslant i, j \leqslant n, a_{ij} \in A \right\}.$$

This is an algebra with element-wise addition and scalar multiplication, as well as traditional matrix multiplication. If A is unital, then $\operatorname{diag}(1_A, \dots, 1_A)$ is the unit for $\operatorname{Mat}_n(A)$. When $n \geqslant 2$, this algebra is noncommutative. If A is a $*$-algebra, then $\operatorname{Mat}_n(A)$ is a $*$-algebra with the involution $\left(a_{ij}\right)_{ij}^* = \left(a_{ji}^*\right)_{ij}$.

Example. Let $(\Omega, \mathcal{M})$ be a measurable space, and let $L_0(\Omega, \mathcal{M})$ be the space of measurable functions. This is a $*$-algebra when equipped with pointwise operations and involution.

If μ is a measure, then $L(\Omega, \mathcal{M})$ of μ-equivalence classes is also a $*$-algebra when equipped with multiplication of equivalence classes and the involution

$$[f]_\mu^* = \left[\overline{f}\right]_\mu.$$

Algebraic Constructions

Algebras, like vector spaces and other algebraic objects, admit various sub-objects and super-objects.

Subalgebras, Ideals, Products, Sums, and Tensor Products

Definition. Let A be a $*$-algebra over $\mathbb{C}$, $B \subseteq A$.

(1) If $B \subseteq A$ is a linear subspace that is closed under multiplication, then B is known as a subalgebra. If $1_A \in B$, then B is unital.

(2) If $B \subseteq A$ is a subalgebra such that, for $b \in B$ and $a \in A$, then $ab, ba \in B$, then we say B is an ideal.

(3) If, for all $x \in B$, $x^* \in B$, then B is called self-adjoint or $*$-closed.

(4) If B is a subalgebra that is $*$-closed, then we say B is a $*$-subalgebra.

(5) If B is an ideal that is $*$-closed, then we say B is a $*$-ideal.

Example. If Ω is a nonempty set with $\mathcal{F}(\Omega, \mathbb{C})$ its corresponding $*$-algebra, then $\ell_\infty(\Omega) \subseteq \mathcal{F}(\Omega, \mathbb{C})$ is a unital $*$-subalgebra.

If $(\Omega, \mathcal{M})$ is a measurable space, then $B_\infty(\Omega, \mathcal{M})$,[I] is a $*$-subalgebra of $\ell_\infty(\Omega)$ and of $L_0(\Omega, \mathcal{M})$.

If μ is a measure on $(\Omega, \mathcal{M})$, then $L_\infty(\Omega, \mu)$ of μ-essentially bounded functions is a unital $*$-subalgebra of $L(\Omega, \mu)$. Moreover, $B_\infty(\Omega, \mu) \subseteq L_\infty(\Omega, \mu)$ is a unital $*$-subalgebra.

If Ω is a LCH[II] space, then the string of $*$-subalgbras is

$$C_c(\Omega) \subseteq C_0(\Omega) \subseteq C_b(\Omega) \subseteq \ell_\infty(\Omega) \subseteq \mathcal{F}(\Omega, \mathbb{C}).$$

It is also the case that $C_c(\Omega) \subseteq C_0(\Omega) \subseteq C_b(\Omega)$ is a string of $*$-ideals.

It is not the case that $C_b(\Omega) \subseteq B_\infty(\Omega)$ is a $*$-ideal.

If μ is a Radon measure, then the string of $*$-subalgebras is

$$C_c(\Omega, \mu) \subseteq C_0(\Omega, \mu) \subseteq C_b(\Omega, \mu) \subseteq B_\infty(\Omega, \mu) \subseteq L_\infty(\Omega, \mu).$$

Example. If $\Omega \subseteq \mathbb{C}$ is a compact subset of the complex plane, then the set $P(\Omega)$ of polynomials forms a unital subalgebra of $C(\Omega)$, but not a $*$-subalgebra. However,

$$\mathcal{Q}(\Omega) = \left\{ q\colon \Omega \to \mathbb{C} \;\middle|\; q(z) = \sum_{k,l=0}^{m} c_{k,l} z^k \overline{z}^l, c_{k,l} \in \mathbb{C} \right\},$$

the space of Laurent polynomials, is a unital $*$-subalgebra of $C(\Omega)$.

If $\Omega = \mathbb{T}$, then $\mathcal{Q}(\Omega)$ becomes the unital $*$-subalgebra of trigonometric polynomials,

$$\mathcal{T} = \left\{ \sum_{k=-n}^{n} c_k z^k \;\middle|\; n \in \mathbb{N}, c_k \in \mathbb{C} \right\}.$$

Definition. Let A be an algebra, and let $S \subseteq A$ be a subset. The subalgebra (or ideal) generated by S, denoted alg(S) or ideal(S), is the smallest subalgebra or ideal that contains S:

$$\begin{aligned} \operatorname{alg}(S) &= \bigcap \left\{ B \;\middle|\; B \supseteq S, B \subseteq A \text{ is a subalgebra} \right\} \\ \operatorname{ideal}(S) &= \bigcap \{ B \mid B \supseteq S, B \subseteq A \text{ is an ideal} \}. \end{aligned}$$

Similarly, we may define $*$-alg(S) and $*$-ideal(S).

[I] The space of bounded measurable functions.
[II] locally compact Hausdorff

Example. Let X be a vector space, and let $\mathcal{L}(X)$ be the unital algebra of linear operators on X. The collection $\mathbb{F}(X) \subseteq \mathcal{L}(X)$ of finite-rank operators forms an ideal. If $\dim(V) = \infty$, then this ideal is proper.

If X is infinite-dimensional, then $\mathbb{F}(X)$ is a non-commutative, non-unital subalgebra.

Definition. An algebra A is called simple if it has no nontrivial ideals.

Example. The algebra $\mathrm{Mat}_n(\mathbb{C})$ is simple.

To see this, if $I \subseteq \mathrm{Mat}_n(\mathbb{C})$ is a nontrivial ideal, and $0 \neq a \in I$, we select $a_{ij} \neq 0$. For every $k \in \{1, \dots, n\}$, we have

$$\begin{aligned} e_{kk} &= \frac{1}{a_{ij}}\left(a_{ij}e_{kk}\right) \\ &= \frac{1}{a_{ij}}\left(e_{ki}ae_{jk}\right) \\ &\in I, \end{aligned}$$

meaning $I_n = \sum_k e_{kk}$ is in I, so $I = \mathrm{Mat}_n(\mathbb{C})$ is not proper.

Definition. If

$$\mathcal{I}_p(A) = \{I \mid I \subsetneq A \text{ is an ideal}\}$$

is the collection of proper ideals ordered by inclusion, we call a maximal element in $\mathcal{I}_p(A)$ a maximal ideal.

Theorem: If A is a unital algebra, then every proper ideal $J \subseteq A$ is contained in some maximal ideal M.

Proof. Order $\mathcal{J} = \{I \mid J \subseteq I \subsetneq A, I \text{ is an ideal}\}$ by inclusion. If $(I_\lambda)_{\lambda\in\Lambda}$ is a chain in $\mathcal{J}$, then $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ is an ideal in A containing J. If $I = A$, then $1_A \in I_\lambda$ for some λ, which contradicts the definition. Thus, I is proper and belongs to $\mathcal{J}$, so by Zorn's lemma, there is some maximal element M in $\mathcal{J}$. □

We can characterize the maximal ideals of the space $C(\Omega)$, where Ω is compact. This will be very useful in the future.

Proposition: Let Ω be a compact Hausdorff space. If $I \subseteq C(\Omega)$ is a maximal ideal, then there is $x_0 \in \Omega$ such that

$$\begin{aligned} I &= N_{x_0} \\ &= \{f \in C(\Omega) \mid f(x_0) = 0\}. \end{aligned}$$

Moreover, for every $x \in \Omega$, N_x is a maximal ideal.

Proof. Suppose $I \neq N_x$ for every $x \in \Omega$. Since N_x is a proper ideal, and I is maximal, this implies that there is some $f_x \in I \setminus N_x$, meaning $f_x(x) \neq 0$.

Let $U_x = f_x^{-1}(\mathbb{C} \setminus \{0\})$. We must have $x \in U_x$ for all $x \in \Omega$, so

$$\Omega = \bigcup_{x\in\Omega} U_x.$$

Now, Ω is compact, so we select $\left\{x_1, \dots, x_j\right\} \subseteq \Omega$ such that

$$\Omega = \bigcup_{j=1}^{n} U_{x_j}.$$

Define

$$f = \sum_{j=1}^{n} \left|f_{x_j}\right|^2.$$

We have $f \in I$, and $f > 0$ on Ω by construction, so f is invertible in $C(\Omega)$. This implies that $\frac{1}{f} \in C(\Omega)$, so $\mathbb{1}_\Omega = f\frac{1}{f} \in I$, which means $I = C(\Omega)$, a contradiction.

Now, we fix $x \in \Omega$. If it is the case that N_x is not maximal, then there is some maximal ideal I such that $N_x \subseteq I$. We know that $I = N_y$ for some $y \in \Omega$, so $N_x \subseteq N_y$. This means any continuous function that vanishes at x must vanish at y. However, by Urysohn's lemma, this is only possible if $x = y$, so $N_x = I = N_y$, so N_x is maximal. □

Definition. Let A be an algebra, $J \subseteq A$ is an ideal. Then, A/J admits multiplication

$$(a + J) \cdot (b + J) = ab + J$$

that makes A/J into an algebra. If $1_A \in A$, then A/J has unit $1_A + J$, and if A is commutative, so too is A/J.

If A is a $*$-algebra, and J is a $*$-ideal, then A/J is a $*$-algebra with involution

$$(a + J)^* = a^* + J.$$

Definition. If $\{A_i\}_{i\in I}$ is a family of $*$-algebras, the product and coproduct are respectively defined by

$$\prod_{i\in I} A_i = \left\{ f \colon I \to \bigcup_{i\in I} A_i \;\middle|\; f(i) \in A_i \right\}$$
$$\bigoplus_{i\in I} A_i = \left\{ f \in \prod_{i\in I} A_i \;\middle|\; \operatorname{card}\big(\operatorname{supp}(f)\big) < \infty \right\}.$$

Note that $\bigoplus_{i\in I} A_i \subseteq \prod_{i\in I} A_i$ is a $*$-ideal.

Example (The Universal $*$-Algebra)**.** Let $E = \{x_i\}_{i\in I}$ be a collection of abstract symbols, and let $\mathbb{A}^*(E)$ be the free $*$-algebra generated by E. Given $R \subseteq \mathbb{A}^*(E)$, let $I(R)$ be the $*$-ideal generated by R. The quotient $*$-algebra

$$\mathbb{A}^*(E|R) = \mathbb{A}^*(E)/I(R)$$

is called the universal $*$-algebra generated by E with relations R. We write $\pi_R(x_i) = z_i$.

Proposition: Let A and B be $*$-algebras. The linear space $A \otimes B$ admits a multiplication

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

and an involution

$$(a \otimes b)^* = a^* \otimes b^*.$$

Proof. Fix $a \in A$ and $b \in B$. Consider the linear maps $L_a \colon A \to A$, given by $L_a(x) = ax$, and $L_b \colon B \to B$, given by $L_b(y) = by$.

The maps $a \mapsto L_a$ and $b \mapsto L_b$ are both linear, meaning the map

$$A \times B \to \mathcal{L}(A) \otimes \mathcal{L}(B),$$

given by $(a, b) \mapsto L_a \otimes L_b$, is bilinear. Thus, there is a linear map

$$L \colon A \otimes B \to \mathcal{L}(A) \otimes \mathcal{L}(\mathcal{B})$$

given by $a \otimes b \mapsto L_a \otimes L_b$. There is a linear embedding $\mathcal{L}(A) \otimes \mathcal{L}(\mathcal{B}) \hookrightarrow \mathcal{L}(A \otimes B)$, so we may identify the tensors in $\mathcal{L}(\mathcal{A}) \otimes \mathcal{L}(\mathcal{B})$ with the linear operators on $A \otimes B$.

We define

$$(A \otimes B) \times (A \otimes B) \to A \otimes B,$$

given by $(t, s) \mapsto t \cdot s = L(t)(s)$. This is a well-defined multiplication following from the fact that L is linear and $L(t)$ is linear for all $t \in A \otimes B$.

For all $a, a' \in A$ and $b, b' \in B$, we have

$$\begin{aligned}(a \otimes b)(a' \otimes b') &= L(a \otimes b)(a' \otimes b')\\ &= L_a \otimes L_b(a' \otimes b')\\ &= L_a(a') \otimes L_b(b')\\ &= aa' \otimes bb'.\end{aligned}$$

We write $\overline{A \otimes B}$ for the conjugate vector space. The map

$$A \times B \to \overline{A \otimes B},$$

given by $(a, b) \mapsto \overline{a' \otimes b'}$ is bilinear. Thus, there is a linear map $\psi\colon A \otimes B \to \overline{A \otimes B}$ given by $\psi(a \otimes b) = \overline{a' \otimes b'}$.

The map $\mu\colon \overline{A \otimes B} \to A \otimes B$, given by $\mu\left(\bar{t}\right) = t$ is conjugate linear. The composition, $\nu = \mu \circ \psi$, mapping $A \otimes B \to A \otimes B$ is conjugate linear, and sends $a \otimes b \mapsto a' \otimes b'$. We define the involution $t \mapsto t^* = \nu(t)$. We have

$$\begin{aligned}((a \otimes b)(c \otimes d))^* &= (ac \otimes bd)^*\\ &= (ac)^* \otimes (bd)^*\\ &= c^*a^* \otimes b^*d^*\\ &= (c^* \otimes d^*)(a^* \otimes b^*)\\ &= (c \otimes d)^*(a \otimes b)^*.\end{aligned}$$

□

The Group $*$-Algebra

Let Γ be a group, and let $\mathbb{C}[\Gamma]$ be the free vector space on Γ. For each $f, g \in \mathbb{C}[\Gamma]$, we define $f * g$ by convolution:

$$\begin{aligned}f * g(s) &= \sum_{t \in \Gamma} f(t) g\left(t^{-1}s\right)\\ &= \sum_{r \in \Gamma} f\left(sr^{-1}\right) g(r).\end{aligned}$$

This sum is finite since f and g have finite support.

This multiplication has the unit $1_{\mathbb{C}[\Gamma]} = \delta_e$.

The involution $f \mapsto f^*$ in $\mathbb{C}[\Gamma]$ is defined by

$$f^*(t) = \overline{f\left(t^{-1}\right)}.$$

We can verify that this forms an involution.

$$\begin{aligned}
(f \cdot g)^*(s) &= \overline{f \cdot g\left(s^{-1}\right)} \\
&= \overline{\sum_{t \in \Gamma} f(t) g\left(t^{-1} s^{-1}\right)} \\
&= \sum_{t \in \Gamma} \overline{f(t) g\left(t^{-1} s^{-1}\right)} \\
&= \sum_{r \in \Gamma} \overline{f\left(r^{-1}\right) g\left(r s^{-1}\right)} \\
&= \sum_{r \in \Gamma} \overline{g\left(\left(s r^{-1}\right)^{-1}\right) f\left(r^{-1}\right)} \\
&= \sum_{r \in \Gamma} g^*\left(s r^{-1}\right) f^*(r) \\
&= g^* \cdot f^*(s).
\end{aligned}$$

The $*$-algebra $\mathbb{C}[\Gamma]$ is known as the group $*$-algebra.

Distinguished Elements

Definition. Let A be a $*$-algebra.

(1) An element $e \in A$ is said to be idempotent if $e^2 = e$. We write $E(A)$ for the set of idempotents in A.

(2) If A is unital, then $x \in A$ is said to be invertible if there exists a unique $y \in A$ with $xy = yx = 1_A$. We call y the inverse of x, and write x^{-1}. We write $\mathrm{GL}(A)$ to be the set of all invertible elements in A.

(3) An element $x \in A$ is said to be Hermitian or self-adjoint if $x = x^*$. We write $A_{\text{s.a.}}$ for the set of self-adjoint elements in A.

(4) An element $a \in A$ is said to be positive if $a = b^*b$ for some $b \in A$. We write A_+ for the set of all positive elements in A.

(5) A projection in A is a self-adjoint idempotent — that is, $p^2 = p^* = p$. We write $\mathcal{P}(A)$ to be the set of projections in A.

(6) If A is unital, an element $u \in A$ is said to be unitary if $u^*u = uu^* = 1_A$. We write $\mathcal{U}(A)$ to be the set of all unitaries in A.

(7) An element $z \in A$ is called normal if $z^*z = zz^*$. We write $\mathrm{Nor}(A)$ for the collection of normal elements in A.

Fact. Let A be a $*$-algebra.

- The following inclusions hold:
  - $\mathcal{P}(A) \subseteq A_+ \subseteq A_{\text{s.a.}} \subseteq \mathrm{Nor}(A)$;
  - $\mathcal{U}(A) \subseteq \mathrm{Nor}(A)$.
- The linear span of $A_{\text{s.a.}}$ is A. If $x \in A$, then

$$\begin{aligned}
h &= \frac{1}{2}(x + x^*) \\
k &= \frac{i}{2}(x^* - x)
\end{aligned}$$

  are self-adjoint with $x = h + ik$.

- The self-adjoint elements of A form a real vector space.
- If A is unital, then $\mathrm{GL}(A)$ is $*$-closed, with $(x^*)^{-1} = \left(x^{-1}\right)^*$.
- If A is unital, then $\mathcal{U}(A) \subseteq \mathrm{GL}(A)$ is a subgroup with $u^{-1} = u^*$ for all $u \in \mathcal{U}(A)$.

Example. The spectral theorem from linear algebra states that if a matrix $a \in \mathrm{Mat}_n(\mathbb{C})$ is normal, then there is a unitary matrix u with $a = udu^*$, where $d = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix, and $\lambda_1, \dots, \lambda_n$ is a complete list of eigenvalues.

Self-adjoint elements in $\mathrm{Mat}_n(\mathbb{C})$ are matrices that are conjugate symmetric.

A square matrix a is invertible if and only if $\det(a) \neq 0$.

Example. Let $\mathcal{F}(\Omega)$ be the set of all $\mathbb{C}$-valued functions on Ω. Every element in $\mathcal{F}(\Omega)$ is normal. The following hold:

- $f \in \mathcal{F}(\Omega)_{\text{s.a.}}$ if and only if $f(\Omega) \subseteq \mathbb{R}$;
- $f \in \mathcal{F}(\Omega)_+$ if and only if $f(\Omega) \subseteq [0, \infty)$;
- $u \in \mathcal{U}(\mathcal{F}(\Omega))$ if and only if $u(\Omega) \subseteq \mathbb{T}$;
- $\mathcal{P}(\mathcal{F}(\Omega)) = \{\mathbb{1}_E \mid E \subseteq \Omega\}$.

Algebra Homomorphisms

Now, we can learn about morphisms in the category of algebras and $*$-algebras.

Definition. Let A and B be $\mathbb{F}$-algebras.

(1) An algebra homomorphism is a linear map $\varphi\colon A \to B$ that is multiplicative.

(2) A character on A is a nonzero homomorphism $h\colon A \to \mathbb{F}$. We write

$$\Omega(A) = \{h \mid h \text{ is a character on } A\}.$$

(3) An algebra isomorphism is a bijective algebra homomorphism.

(4) If A and B are $*$-algebras, $\varphi\colon A \to B$ is said to be $*$-preserving if $\varphi(a^*) = \varphi(a)^*$.

(5) If A and B are $*$-algebras, then a $*$-homomorphism (or $*$-isomorphism) is a homomorphism (or isomorphism) that is $*$-preserving.

(6) An automorphism of a $*$-algebra is a $*$-isomorphism $\alpha\colon A \to A$. We write

$$\mathrm{Aut}(A) = \left\{\alpha \,\middle|\, \alpha\colon A \to A \text{ is a } *\text{-automorphism}\right\}.$$

(7) If A and B are $*$-algebras, then $\phi\colon A \to B$ is said to be positive if $\varphi(A_+) \subseteq B_+$.

(8) A positive map between $*$-algebras is called faithful if $\ker(\phi) \cap A_+ = \{0\}$.

Theorem (First Isomorphism Theorem)**:** Let A, B be $*$-algebras, and let $I \subseteq A$ be a $*$-ideal. If $\varphi\colon A \to B$ is a $*$-homomorphism with $I \subseteq \ker(\varphi)$, then there exists a unique algebra $*$-homomorphism $\phi\colon A/I \to B$ such that $\phi \circ \pi = \varphi$.

If $I = \ker(\varphi)$, then ϕ is injective, and $\phi\colon A/\ker(\varphi) \to \mathrm{Ran}(\varphi)$ is a $*$-isomorphism.

If A, B, and φ are unital, then so is ϕ.

Example (Universal Property of the Universal $*$-Algebra)**.** Let $\mathbb{A}^*(E|R)$ be the universal $*$-algebra generated by $E = \{x_i\}_{i\in I}$ and $R \subseteq \mathbb{A}^*(E)$. Let B be a $*$-algebra admitting elements $\{b_i\}_{i\in I}$ indexed by the same set I that satisfies the relations in R.

The evaluation $*$-homomorphism, $\mathbb{A}^*(E) \to B$ defined by $x_i \mapsto b_i$ sends $I(R)$ to 0, so there is a unique $*$-homomorphism, $x_i + I(R) \to b_i$.

Corollary: If A is an algebra, and $h \in \Omega(A)$ is a character, then $\ker(h) \subseteq A$ is a maximal ideal, and $A/\ker(h) \cong \mathbb{C}$ are isomorphic as algebras.

Proof. Since $h \neq 0$, and $h\colon A \to \mathbb{C}$ is a linear functional, it is the case that $\operatorname{Ran}(h) = \mathbb{C}$.[III] By the first isomorphism theorem, we have $A/\ker(h) \cong \mathbb{C}$ as algebras.

Since $\mathbb{C}$ is simple, $A/\ker(h)$ is simple, so $\ker(h) \subseteq A$ is a maximal ideal. □

There are some algebras that do not admit characters.

Example. Let $A = \mathrm{Mat}_n(\mathbb{C})$ for $n \geqslant 2$. If $h\colon \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$ is a character, then $\ker(h) \subseteq \mathrm{Mat}_n(\mathbb{C})$ is a proper ideal. However, since $\mathrm{Mat}_n(\mathbb{C})$ is simple, $\ker(h) = 0$. However, this means

$$\begin{aligned} n^2 &= \mathrm{Mat}_n(\mathbb{C}) \\ &\leqslant \dim(\mathbb{C}) \\ &= 1, \end{aligned}$$

which is a contradiction. Thus, $\Omega(\mathrm{Mat}_n(\mathbb{C})) = \varnothing$.

Unitization

It is often the case that algebras lack a unit. However, we can create a "unitized" version of an algebra A, $\widetilde{A}$, such that $A \subseteq \widetilde{A}$ is an essential ideal.

Definition. Let A be an algebra, $J \subseteq A$ an ideal. We say J is essential if for any other ideal $I \subseteq A$, $I \cap J \neq \{0\}$.

Proposition: Let A be a complex algebra.

(1) The set $A \times \mathbb{C}$, equipped with

$$\begin{aligned} (a,\alpha) + (b,\beta) &= (a+b, \alpha+\beta) \\ z(a,\alpha) &= (za, z\alpha) \\ (a,\alpha)(b,\beta) &= (ab + \beta a + \alpha b, \alpha\beta) \end{aligned}$$

is a unital algebra, with unit $1_{\widetilde{A}} = (1,0)$. We denote this algebra $\widetilde{A}$.

(2) If A is a $*$-algebra, then $\widetilde{A}$ is a $*$-algebra, with

$$(a,\alpha)^* = (a^*, \alpha).$$

(3) The map $\iota_A : A \to \widetilde{A}$, given by $\iota_A(a) = (a,0)$ is an injective $*$-homomorphism, and $\pi_A : \widetilde{A} \to \mathbb{C}$ is a surjective $*$-homomorphism.

The image, $\iota_A(A) \subseteq \widetilde{A}$ is a maximal $*$-ideal.

This yields an exact sequence of $*$-algebras:

[III] Just find $f \in A$ such that $h(f) = k$, then take $1 = h(f/k)$.

$$0 \longrightarrow A \xrightarrow{\iota_A} \widetilde{A} \xrightarrow{\pi_A} \mathbb{C} \longrightarrow 0$$

(4) If A is nonunital, then $\iota_A(A) \subseteq \widetilde{A}$ is an essential ideal.

Proof. We will prove (3) and (4).

(3) From the definition, we see that ι_A is an injective $*$-homomorphism, and π_A is a surjective $*$-homomorphism, with $\mathrm{Ran}(\iota_A) = \ker(\pi_A)$. Thus, by the first isomorphism theorem, we have $\widetilde{A}/\mathrm{Ran}(\iota_A) \cong \mathbb{C}$, so the $*$-ideal, $\mathrm{Ran}(\iota_A)$, is maximal in A.

(4) Let $I \subseteq \widetilde{A}$ be a nonzero ideal, and let $0 \neq (a, \alpha) \in I$. If $\alpha = 0$, then $0 \neq (a, 0) \in \iota(A) \cap I$.

If $a - 0$, then $\alpha \neq 0$, so $1_{\widetilde{A}} = (0, 1) = \alpha^{-1}(0, \alpha) \in I$, so $I = \widetilde{A}$, so $\iota(A) \cap I = \iota(A)$.

We assume $a, \alpha \neq 0$. Multiplying by α^{-1}, setting $b = \alpha^{-1}a$, we get $(b, 1) \in I$, and since I is an ideal, we have $(xb + x, 0) \in I$ and $(bx + x, 0) \in I$. If $xb + x = bx + x = 0$, then $(-b)$ is a multiplicative unit for A, which contradicts the fact that A is nonunital. Thus, there must be $x \in A$ such that $xb + x \neq 0$ or $bx + x \neq 0$. Thus, $I \cap \iota(A) \neq \{0\}$, so $\iota(A)$ is an essential ideal.

□

When we talk about elements of $\widetilde{A}$, we write $(a, \alpha) = a + \alpha 1_{\widetilde{A}}$.

Proposition: Let A and B be $*$-algebras, and let $\phi\colon A \to B$ be a $*$-homomorphism.

(1) The map $\widetilde{\phi}(a, z) = (\phi(a), z)$ is a unital $*$-isomorphism that extends ϕ. Moreover, $\widetilde{\phi}$ is injective (or surjective) if and only if ϕ is injective (or surjective).

(2) If B is unital, the map $\overline{\phi}(a, z) = \phi(a) + z1_B$ is a unital $*$-homomorphism that extends ϕ. If A is nonunital, and ϕ is injective, then so is $\overline{\phi}$.

(3) If A is nonunital, and $h\colon A \to \mathbb{C}$ is a character on A, then $\overline{h}(a, \alpha) = h(a) + \alpha$ is a character on $\widetilde{A}$ extending h.

Proof. We will prove (3).

(3) If h is a character on A, then $|\ker(h) \neq A$. If it were the case that $\ker\left(\overline{h}\right) = \widetilde{A}$, then since $A \subseteq \widetilde{A}$ is an essential ideal, $\ker(h) = \ker\left(\overline{h}\right) \cap A = \widetilde{A} \cap A = A$, which is a contradiction. Thus, $\overline{h}$ is a character.

□

Proposition: Let X be a noncompact LCH space, and let X_∞ be the one-point compactification of X. There is a unital $*$-homomorphism $\varphi\colon \widetilde{C_0}(X) \to C(X_\infty)$ that maps $C_0(X)$ onto the ideal $I = \{f \mid f(\infty) = 0\} \subseteq C(X_\infty)$.

Proof. If $f \in C_0(X)$, we start by showing that $\phi\colon X_\infty \to \mathbb{C}$, given by

$$\phi(f)(x) = \begin{cases} f(x) & x \in X \\ 0 & x \in \infty \end{cases}$$

is continuous on X_∞. It is the case that $\phi(f)$ is continuous on X, since $\phi(f)|_X = f$, and $X \subseteq X_\infty$ is open. Let $(x_i)_i$ be a net in X_∞ converting to ∞, and let $\varepsilon > 0$. Since f vanishes at infinity, there is a compact subset $K \subseteq X$ such that $|f(x)| < \varepsilon$, for $x \notin K$. The set $X_\infty \setminus K$ is an open neighborhood of ∞, so $x_i \in X_\infty \setminus K$ for large i. Thus,

$$(\phi(f)(x_i))_i \to 0 = \phi(f)(\infty).$$

We can see that ϕ is a $*$-homomorphism by the way we have defined it, and that $0 = \phi(f)(x)$ if and only if $f = 0$ for all x, so $\phi(f)$ is an injective $*$-homomorphism.

We will show that $\operatorname{Ran}(\phi) = I$. Let $g \in I$. We have $g|_X \colon X \to \mathbb{C}$ vanishes at infinity. Given $\varepsilon > 0$, since $g(\infty) = 0$, there is a neighborhood V of ∞ with $|g| < \infty$ on V. This means we find compact $K \subseteq X$ such that $X \setminus K \subseteq V$, so $|g(x)| < \infty$ for $x \notin K$. Thus, $g|_X \in C_0(X)$. Thus, $g = \phi(g|_X)$, so $\operatorname{Ran}(\phi) = I$.

Since $C_0(X)$ is nonunital, the extension $\varphi \colon \widetilde{C_0}(X) \to C(X_\infty)$ is also injective. We will show that φ is onto. If $k \in C(X_\infty)$, then $g = k - k(\infty)\mathbb{1}_{X_\infty} \in I$, so there is $f \in C_0(X)$ with $\phi(f) = g$. Thus,

$$\begin{aligned}\varphi(k, k(\infty)) &= \phi(k) + k(\infty)\mathbb{1}_{X_\infty}\\ &= g + k(\infty)\mathbb{1}_{X_\infty}\\ &= k.\end{aligned}$$

□

We have seen the character space on $C(X)$ earlier when X is compact Hausdorff; now, we can see the character space on $C_0(X)$, where X is a LCH space.

Corollary: Let X be a LCH space. If $x \in X$, then $\delta_x \colon C_0(X) \to \mathbb{C}$, given by $\delta_x(f) = f(x)$ is a character on $C_0(X)$. Moreover, the map $\delta \colon X \to \Omega(C_0(X))$, given by $x \mapsto \delta_x$ is a bijection.

Proof. Each $\delta_x \colon C_0(X) \to \mathbb{C}$ is a character, and $\delta_x \neq 0$ by Urysohn's lemma.

Let $h \colon C_0(X) \to \mathbb{C}$ be a character. The unitization, $\overline{h} \colon \widetilde{C_0}(X) \to \mathbb{C}$ is a character. Let $\varphi \colon \widetilde{C_0}(X) \to C(X_\infty)$ be the $*$-isomorphism to the one-point compactification of X. Thus, there is a $\xi \in X_\infty$ with $\delta_\xi = \overline{h} \circ \varphi^{-1}$.

Thus, we see that $\delta_\xi \circ \phi = \delta_\xi \circ \varphi \circ \iota = \overline{h} \circ \iota = h$ on $C_0(X)$, where $\iota \colon C_0(X) \to \widetilde{C_0}(X)$ is the natural inclusion. Since $h \neq 0$ and $\phi(f)(\infty) = 0$ for every $f \in C_0(X)$, we must have $\xi = x \in X$, so

$$\begin{aligned}h(f) &= \delta_x \circ \phi(f)\\ &= f(x)\\ &= \delta_x(f)\end{aligned}$$

for every $f \in C_0(X)$, so δ is onto. Since $C_0(X)$ separates points, δ is injective. □

We are interested in identifying the character space of a nonunital algebra with a subset of the character space of its unitization. Note that the projection $\pi \colon \widetilde{A} \to \mathbb{C}$ given by $(a, \alpha) \mapsto \alpha$ is a character, but $\pi|_A$ is not a character on A. This is indeed the desired character that will extend the character space on A.

Proposition: Let A be a nonunital algebra. The map $\Omega(A) \to \Omega\left(\widetilde{A}\right)$, given by $h \mapsto \overline{h}$, is injective.

Moreover, $\Omega\left(\widetilde{A}\right) = \Omega(A) \cup \{\pi\}$, where $\pi \colon \widetilde{A} \to \mathbb{C}$ is given by $(a, \alpha) \mapsto \alpha$.

Proof. It is clear from the definition of $\overline{h}$ that $h \mapsto \overline{h}$ is well-defined and injective. Moreover, if $\phi \in \Omega\left(\widetilde{A}\right)$ is such that $h = \phi|_A \neq 0$, then $h \in \Omega(A)$ is such that $\overline{h} = \phi$. Since ϕ is unital, we have

$$\begin{aligned}\phi\left(a + \alpha 1_{\widetilde{A}}\right) &= \phi(a) + \alpha\\ &= h(a) + \alpha\\ &= \overline{h}\left(a + \alpha 1_{\widetilde{A}}\right).\end{aligned}$$

Thus, if $\phi \in \Omega\left(\widetilde{A}\right)$ is such that $\phi|_A = 0$, then $\phi = \pi$. □

Algebraic Spectrum

Definition. Let A be a unital $\mathbb{F}$-algebra. Let $a \in A$.

(1) The resolvent of a is the set
$$\rho(a) = \{\lambda \in \mathbb{F} \mid a - \lambda 1_A \in GL(A)\}.$$

(2) The spectrum of a is the complement of the resolvent, $\sigma(a) = \mathbb{F} \setminus \rho(a)$.

We may refine the definition for the case of a nonunital algebra.

Definition. Let A be a nonunital algebra, and let $a \in A$. We define
$$\rho(a) = \rho(\iota_A(a)),$$
where $\iota_A : A \hookrightarrow \widetilde{A}$ is the canonical embedding of A into its unitization. We define $\sigma(a) = \mathbb{F} \setminus \rho(a)$.

Exercise: Prove that similar elements in an algebra have the same resolvent.

Solution: Let $a = zbz^{-1}$ for some $z \in GL\left(\widetilde{A}\right)$. We will show that $\rho(a) = \rho(a)$.

Let $\lambda \in \rho(a)$. Then, $a - \lambda 1_A \in GL(A)$. Note that since $z \in GL(A)$, and $GL(A)$ is a group, we have $z(a - \lambda 1_A)z^{-1} \in GL(A)$, so $b - \lambda 1_A \in GL(A)$.

Example. If $A = \mathbb{C}$, then for every $z \in \mathbb{C}$, the spectrum $\sigma(z) = \{z\}$. This is because every non-zero number is invertible.

Example. Let $A = \mathrm{Mat}_n(\mathbb{C})$. Then, λ is an eigenvalue for a matrix $a \in A$ if and only if $a - \lambda I_n$ is not invertible. In this case, we say $\sigma(a) = \sigma_p(a)$, the point spectrum of a.

Example. Let $A = \ell_\infty(\Omega)$. We find
$$\begin{aligned}\lambda \in \rho(f) &\Leftrightarrow f - \lambda \mathbb{1}_\Omega \in GL(A)\\ &\Leftrightarrow \inf_{x\in\Omega} |f(x) - \lambda| > 0\\ &\Leftrightarrow \lambda \notin \overline{\mathrm{Ran}}(f).\end{aligned}$$

Thus, $\sigma(f) = \overline{\mathrm{Ran}}(f)$.

Example. Let Ω be compact Hausdorff, and let $A = C(\Omega)$. If $f \in A$, we have
$$\begin{aligned}\lambda \in \rho(f) &\Leftrightarrow f - \lambda \mathbb{1}_\Omega \in GL(A)\\ &\Leftrightarrow f(x) - \lambda \neq 0\\ &\Leftrightarrow \lambda \notin \mathrm{Ran}(f).\end{aligned}$$

Thus, $\sigma(f) = \mathrm{Ran}(f)$.

If Ω is a noncompact LCH space with one-point compactification Ω_∞, then if $A = C_0(\Omega)$, we find
$$\sigma(f) = \mathrm{Ran}(f) \cup \{0\}.$$

Example. Let $A = L_\infty(\Omega, \mu)$, where (Ω, μ) is a measure space. Fix $f \in A$. To understand $\sigma(f)$, we must understand the essential range of f.

Write $U(z, \varepsilon)$ for the open disk centered at $z \in \mathbb{C}$ with radius $\varepsilon > 0$. The essential range of f if
$$\operatorname{ess\,ran}(f) = \left\{\lambda \in \mathbb{F} \;\middle|\; \mu\left(f^{-1}(U(\lambda, \varepsilon))\right) > 0,\ \forall \varepsilon > 0\right\}.$$

The essential range can also be viewed as the support of the pushforward measure $f_*\mu$ on $\mathbb{C}$, where $f_*\mu(E) = \mu\left(f^{-1}(E)\right)$ for measurable $E \subseteq \mathbb{C}$.

A number $\lambda \notin \operatorname{ess\,ran}(f)$ if and only if $f - \lambda 1_A$ is essentially bounded below. Thus, we see that $\lambda \in \rho(f)$ if and only if $\lambda \notin \operatorname{ess\,ran}(f)$. Thus, $\sigma(f) = \operatorname{ess\,ran}(f)$.

Proposition: Let $\varphi\colon A \to B$ be a unital algebra homomorphism. Then, $\sigma(\varphi(a)) \subseteq \sigma(a)$, with equality if φ is bijective.

Proof. If $\lambda \in \rho(a)$, then $a - \lambda 1_A \in \mathrm{GL}(A)$. Thus, $\varphi(a) - \lambda 1_B \in \mathrm{GL}(B)$, so $\lambda \in \rho(\varphi(a))$. Thus, $\rho(a) \subseteq \rho(\varphi(a))$, so $\sigma(\varphi(a)) \subseteq \sigma(a)$.

If φ is bijective, then $\varphi^{-1}\colon B \to A$ is also a unital homomorphism, meaning

$$\begin{aligned}\sigma(a) &= \sigma\left(\varphi^{-1}(\varphi(a))\right)\\ &\subseteq \sigma(\varphi(a)).\end{aligned}$$

$\square$

Corollary: If A is an algebra, and $h \in \Omega(A)$, then $h(a) \in \sigma(a)$. We have

$$\{h(a) \mid h \in \Omega(A)\} \subseteq \sigma(a).$$

Proof. Assume A is unital. Taking $B = \mathbb{C}$ in the above proposition, and noting $\sigma(z) = \{z\}$, we prove the statement.

If A is nonunital, then $h(a) = \overline{h}(\iota_A(a))$. $\square$

Fact. Let A be a unital algebra with $a, b \in A$. Then, $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

Proof. If $\lambda \neq 0$ belongs to $\rho(ab)$, then $ab - \lambda 1_A \in \mathrm{GL}(A)$, meaning $\lambda^{-1}ab - 1_A \in \mathrm{GL}(A)$. Thus, $\lambda^{-1}ba - 1_A \in \mathrm{GL}(A)$, so $ba - \lambda 1_A \in \mathrm{GL}(A)$, so $\lambda \in \rho(ba)$. Thus, $\rho(ba) \setminus \{0\} \subseteq \rho(ab) \setminus \{0\}$, so, repeating this argument, we get $\rho(ba) \setminus \{0\} = \rho(ab) \setminus \{0\}$. $\square$

Proposition (Spectral Mapping)**:** Let A be a unital $\mathbb{C}$-algebra, and let $a \in A$ with $\sigma(a) \neq \emptyset$. If $p \in \mathbb{C}[x]$, then $\sigma(p(a)) = p(\sigma(a))$.

Proof. Splitting p, we take

$$p(z) = \alpha \prod_{j=1}^{n} \left(z - \alpha_j\right)$$

for $\alpha, \alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then,

$$p(a) = \alpha \prod_{j=1}^{n} \left(a - \alpha_j 1_A\right).$$

The factors $a - \alpha_j 1_A$ commute, and since $\mathrm{GL}(A)$ is a group, we get

$$\begin{aligned}p(a) \in \mathrm{GL}(A) &\Leftrightarrow a - \alpha_j 1_A \in \mathrm{GL}(A)\\ &\Leftrightarrow \alpha_j \in \rho(a)\\ &\Leftrightarrow \alpha_j \notin \sigma(a)\\ &\Leftrightarrow p(z) \neq 0\ \forall z \in \sigma(A)\\ &\Leftrightarrow 0 \notin p(\sigma(a)).\end{aligned}$$

Thus, $p(a) \notin \mathrm{GL}(A)$ if and only if $0 \in p(\sigma(a))$. $\square$

Banach and C^*-Algebras

In the notes on Hilbert space operators, we established the spectral theorem for compact normal operators. In order to establish the spectral theorem for all normal operators, we will study the unital C^*-algebra generated by the normal operator. This will hinge on understanding the abstract theory of Banach algebras and C^*-algebras.

We start with some examples of Banach and C^*-algebras, as well as discussing some constructions of and with C^*-algebras.

Examples

Definition. A Banach $*$-algebra is a Banach algebra A with an involution map $A \to A$, $a \mapsto a^*$, satisfying

$$\|a^*\| = \|a\|.$$

If A is a Banach $*$-algebra that satisfies the C^* property, $\|a^*a\| = \|a\|^2$, for every $a \in A$, then A is called a C^*-algebra.

We know that $*$-algebras admit a variety of distinguished elements. We can add two more to that list.

Definition. Let A be a C^*-algebra, and $w \in A$.

- We say w is an isometry if $w^*w = 1_A$.
- If w is an isometry, and $ww^* \neq 1_A$, then we say w is a proper isometry.

We may also speak of partial isometries.

Lemma: If A be a C^*-algebra with $v \in A$. The following are equivalent:

(i) v^*v is a projection;

(ii) $vv^*v = v$;

(iii) vv^* is a projection;

(iv) $v^*vv^* = v^*$.

Such an element is called a partial isometry.

Proof. We obtain the implication (i) implying (ii) through verifying

$$(vv^*v - v)^*(vv^*v - v) = 0,$$

meaning $vv^*v - v = 0$. Similarly, the implication (iii) implying (iv) is similar. □

Example. The complex numbers $\mathbb{C}$ with involution $z \mapsto \overline{z}$ and norm $z \mapsto |z|$ is a C^*-algebra.

Example. We know that $\mathbb{B}(\mathcal{H})$, the space of bounded linear operators on a Hilbert space, is a C^*-algebra.

Example. If $n \geqslant 2$, then $\text{Mat}_N(\mathbb{C})$ is a unital noncommutative $*$-algebra. We know that $\left(\text{Mat}_n(\mathbb{C}), \|\cdot\|_{\text{op}}\right)$ is a Banach space.

We want to show that $\left(\text{Mat}_n(\mathbb{C}), \|\cdot\|_{\text{op}}\right)$ is a C^*-algebra isomorphic to $\mathbb{B}\left(\ell_2^n\right)$.

We can establish a unital isomorphism $\text{Mat}_n(\mathbb{C}) \to \mathcal{L}(\mathbb{C}^n)$ by sending the matrix a to its corresponding transformation T_a.

Since $\mathbb{C}^n$ is a finite-dimensional Hilbert space, $\mathbb{B}\left(\ell_2^n\right) = \mathcal{L}(\mathbb{C}^n)$. We have a unital isomorphism of algebras $T(a) = T_a$ between $\mathrm{Mat}_n(\mathbb{C})$ and $\mathbb{B}\left(\ell_2^n\right)$.

By the definition of the operator norm, $\|a\|_{op} = \|T_a\|_{op}$, so $T\colon \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{B}\left(\ell_2^n\right)$ is an isometry.

If $a, b \in \mathrm{Mat}_n(\mathbb{C})$, then

$$\begin{aligned}\|ab\|_{op} &= \|T_{ab}\|_{op}\\ &= \|T_aT_b\|_{op}\\ &\leqslant \|T_a\|_{op}\|T_b\|_{op}\\ &= \|a\|_{op}\|b\|_{op}.\end{aligned}$$

Next, $\|I_n\|_{op} = \|T_{I_n}\|_{op} = \left\|\mathrm{id}_{\ell_2^n}\right\|_{op} = 1$, and

$$\begin{aligned}\|a^*\|_{op} &= \|T_{a^*}\|_{op}\\ &= \|T_a^*\|_{op}\\ &= \|T_a\|_{op}\\ &= \|a\|_{op}.\end{aligned}$$

Similarly,

$$\|a^*a\|_{op} = \|a\|_{op}^2.$$

Thus, $\left(\mathrm{Mat}_n(\mathbb{C}), \|\cdot\|_{op}\right)$ is a C^*-algebra.

Example. The space $\ell_\infty(\Omega)$ of bounded functions on Ω is a unital and commutative $*$-algebra under pointwise operations, which is also a Banach space under $\|\cdot\|_u$.

We can also see that $\|fg\|_u \leqslant \|f\|_u\|g\|_u$, and $\|f^*f\| = \|f\|_u$ for all $f, g \in \ell_\infty(\Omega)$, meaning $\ell_\infty(\Omega)$ is a unital and commutative Banach algebra.

Finally,

$$\begin{aligned}\|f^*f\|_u &= \sup_{x\in\Omega}|(f^*f)(x)|\\ &= \sup_{x\in\Omega}|f^*(x)f(x)|\\ &= \sup_{x\in\Omega}\left|\overline{f(x)}f(x)\right|\\ &= \sup_{x\in\Omega}|f(x)|^2\\ &= \|f\|_u^2.\end{aligned}$$

Lemma: Let B be a Banach algebra/Banach $*$-algebra/C^*-algebra. If $A \subseteq B$ is a norm closed subalgebra/$*$-subalgebra, then A is a Banach algebra/Banach $*$-algebra/C^*-algebra.

Definition. If B is a C^*-algebra, and $A \subseteq B$ is a norm-closed $*$-subalgebra, then A is a C^*-subalgebra of B.

If B is unital, then $A \subseteq B$ is a unital C^*-subalgebra if $1_B \in A$.

If $\mathcal{H}$ is a Hilbert space, then a C^*-subalgebra $A \subseteq \mathbb{B}(\mathcal{H})$ is sometimes called a concrete C^*-algebra.

Example. The compact operators, $\mathbb{K}(\mathcal{H})$ is an operator norm-closed $*$-subalgebra of $\mathbb{B}(\mathcal{H})$. It is unital if and only if $\dim(\mathcal{H}) < \infty$.

Example. Let $(\Omega, \mathcal{M})$ be a measurable space, The bounded measurable functions, $B_\infty(\Omega)$, is a unital $*$-subalgebra.

Equipped with the ∞ norm, $B_\infty(\Omega)$ is a Banach space, meaning $B_\infty(\Omega) \subseteq \ell_\infty(\Omega)$ is norm-closed, and is thus a unital commutative C^*-algebra.

Example. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. The essentially bounded functions, $L_\infty(\Omega, \mu)$, is a Banach space with the ess sup norm. It is also the case that $L_\infty(\Omega, \mu)$ is a unital commutative $*$-algebra. We can show that $\|f^*f\|_\infty = \|f\|_\infty^2$, so $L_\infty(\Omega, \mu)$ is a unital C^*-algebra.

Example. Let Ω be a LCH space. We know that $C_b(\Omega)$ and $C_0(\Omega)$ are $*$-subalgebras of $\ell_\infty(\Omega)$. We also know these are uniform norm-closed, meaning $C_b(\Omega)$ and $C_0(\Omega)$ are C^*-algebras. The C^*-algebra $C_b(\Omega)$ is always unital, but $C_0(\Omega)$ is unital if and only if Ω is compact.

Note that if Ω is given the discrete topology, then $\ell_\infty(\Omega) = C_b(\Omega)$, as any function on a discrete space is continuous.

The map $C_b(\Omega) \to C(\beta\Omega)$, given by $f \mapsto f^\beta$ is an isometric isomorphism of Banach spaces. We can also verify that this is a $*$-isomorphism, as $(fg)^\beta = f^\beta g^\beta$, as these agree on the dense subset $\Delta(\Omega) \subseteq \beta\Omega$. Similarly, $(f^*)^\beta = \left(f^\beta\right)^*$. Thus, $C_b(\Omega)$ and $C(\beta\Omega)$ are isomorphic as C^*-algebras.

We get the isometric $*$-isomorphism $\ell_\infty = C_b(\mathbb{N}) = C(\beta\mathbb{N})$.

Constructions

We are interested in constructing new C^*-algebras from old.

Generating Sets

We may start with closures.

Lemma: Let B be a Banach algebra/Banach $*$-algebra, and let $A \subseteq B$ be a subalgebra/$*$-subalgebra. The closure, $\overline{A} \subseteq B$, is a *Banach* subalgebra/$*$-subalgebra.

If B is a C^*-algebra with $A \subseteq B$ a $*$-subalgebra, then $\overline{A}$ is a C^*-subalgebra of B.

Given a collection of operators $S \subseteq \mathbb{B}(\mathcal{H})$, we are interested in constructing the picture of the smallest C^*-subalgebras of $\mathbb{B}(\mathcal{H})$ containing S. In a more general case, we may consider any C^*-algebra B and the subset $S \subseteq \mathbb{B}(\mathcal{H})$.

Definition. Let B be a Banach algebra/$*$-algebra, and let $S \subseteq B$ be any subset. The Banach algebra/$*$-algebra generated by S is the smallest Banach subalgebra/$*$-subalgebra containing S.

If B is a C^*-algebra, then the C^*-subalgebra generated by S is the smallest C^*-subalgebra of B containing S, denoted

$$C^*(S) = \bigcap \left\{A \mid S \subseteq A, A \subseteq B \text{ is a } C^*\text{-subalgebra}\right\}.$$

Notationally, we write $C^*(a_1, \dots, a_n)$ if $\{a_1, \dots, a_n\}$ is a finite subset of B.

Obviously, we need a more workable picture of the C^*-subalgebra generated by a set, at the very least we need something we can imagine.

Lemma: Let B be a Banach algebra and suppose $S \subseteq B$ is any subset.

(1) The Banach algebra generated by S is the closed span of the set of finite words in S. In other words, it is equal to $\overline{\operatorname{span}}(W)$, where

$$W = \left\{x_1x_2\cdots x_n \mid n \in \mathbb{N}, x_j \in S\right\}.$$

(2) If B is a Banach $*$-algebra or C^*-algebra, then the Banach $*$-algebra or C^*-algebra generated by S is the closed span of the set of finite words in S and S^*. In other words, it is equal to $\overline{\operatorname{span}}(W)$, where

$$W = \left\{x_1x_2\cdots x_n \;\middle|\; n \in \mathbb{N}, x_j \in S \cup S^*\right\}.$$

Proof. We will prove (2).

Note that S is closed under multiplication and involution, so $\operatorname{span}(W)$ is a $*$-algebra containing S, so $\overline{\operatorname{span}}(W)$ is a C^*-subalgebra of B containing S, so $C^*(S) \subseteq \overline{\operatorname{span}}(W)$.

In the reverse inclusion, any C^*-subalgebra of B containing S must contain $\operatorname{span}(W)$, so $\overline{\operatorname{span}}(W) \subseteq C^*(S)$. □

Proposition: Let B be a C^*-algebra, and suppose $a \in B$ is a normal element. The C^*-algebra generated by a, $C^*(a)$, is a commutative C^*-subalgebra. If B is unital, then $C^*(a, 1_B)$ is a unital commutative C^*-algebra.

Proof. We see that, in the notation of the lemma, if $S = \{a\}$ or $S = \{1_B, a\}$, if $w_1, w_2 \in W$, we have $w_1w_2 = w_2w_1$ (since $aa^* = a^*a$), so $\operatorname{span}(W)$ is a commutative $*$-subalgebra, hence $\overline{\operatorname{span}}(W)$ is commutative. □

Example. Let $\iota\colon [0,1] \to \mathbb{C}$ be the inclusion map, $\iota(t) = t$. By the Stone–Weierstrass theorem, we have

$$\begin{aligned}C^*\left(\iota, \mathbb{1}_{[0,1]}\right) &= C([0,1])\\ C^*(\iota) &= \{f \in C([0,1]) \mid f(0) = 0\}\\ &\cong C_0((0,1]).\end{aligned}$$

Note that if $\iota\colon \mathbb{T} \to \mathbb{C}$ is the inclusion $\iota(z) = z$, then $C^*(\iota) = C^*(\mathbb{T})$.

Exercise: Let Δ be the Cantor set. Let

$$\mathcal{C} = \left\{\mathbb{1}_C \;\middle|\; C \subseteq \Delta \text{ is clopen}\right\}.$$

Show that $C^*(\mathcal{C}) = C(\Delta)$.

Solution: Since $\mathcal{C}$ separates points and contains the constant function $\mathbb{1}_\Delta$, the Stone–Weierstrass theorem provides that $C^*(\mathcal{C}) = C(\Delta)$.

Definition. Recall that the definition of the right shift is such that $R^* = L$, where L is the left shift. We know that $R^*R = I$, but $RR^* \neq I$, since it has a nontrivial kernel.

The Toeplitz algebra is the C^*-algebra generated by the right shift. In other words,

$$\mathcal{T} = C^*(R).$$

Exercise: Prove that the Toeplitz algebra contains the compact operators.

Solution: We start by showing that the rank-one projection of e_j onto e_i, where $(e_n)_n$ are the canonical orthonormal basis of ℓ_2, is generated by the right shift as follows.

$$\theta_{e_i,e_j} = R^{i-1}(I - RR^*)(R^*)^{j-1}.$$

Note that we only need to show this equivalence when applied to e_n:

$$\begin{aligned}\theta_{e_i,e_j}(e_n) &= \langle e_n, e_j\rangle e_i\\ &= \delta_{nj}e_i.\end{aligned}$$

Applying in steps, we start with

$$\begin{aligned}R^{i-1}(I - RR^*)(R^*)^{j-1}(e_n) &= R^{i-1}(I - RR^*)\left(e_{n-j+1}\right)\\ &= \begin{cases}R^{i-1}\left(e_{n-j+1}\right) & n = j\\ R^{i-1}(0) & n \neq j\end{cases}\\ &= \delta_{nj}e_i.\end{aligned}$$

Thus, since the rank-one projections are contained in the Toeplitz algebra, the finite-rank operators are contained in the Toeplitz algebra, hence the compact operators are contained in the Toeplitz algebra.

Example. Consider the following isometries on ℓ_2:

$$\begin{aligned} V(\alpha_1,\alpha_2,\alpha_3,\dots) &= (\alpha_1,0,\alpha_2,0,\alpha_3,0,\dots)\\ W(\alpha_1,\alpha_2,\alpha_3,\dots) &= (0,\alpha_1,0,\alpha_2,0,\alpha_3,\dots). \end{aligned}$$

The operators V and W satisfy

$$\begin{aligned} V^*V &= I\\ W^*W &= I\\ VV^*+WW^* &= I. \end{aligned}$$

The Cuntz algebra, $\mathcal{O}_2$, is the $C^*(V,W)$.

Products, Sums, and Quotients

In the category of C^*-algebras, we can also look at products and coproducts.

Definition. Let $\{A_i\}_{i\in I}$ be a family of Banach algebras/Banach $*$-algebras/C^*-algebras. Then, we define the following two constructions with pointwise operations and the ∞ norm.

(1) The ℓ_∞ product is defined as

$$\prod_{i\in I} A_i = \left\{(a_i)_i \;\middle|\; a_i \in A_i, \|(a_i)_i\| = \sup_{i\in I}\|a_i\| < \infty\right\}.$$

(2) For the case of $I = \mathbb{N}$, we may consider the c_0 sum

$$\bigoplus_{n\in\mathbb{N}} A_n = \left\{a = (a_n)_n \;\middle|\; a_n \in A_n, \lim_{n\to\infty}\|a_n\|\right\}$$

as a subset of the ℓ_∞ product of $\{A_n\}_{n\in\mathbb{N}}$. This is a closed $*$-ideal.

(3) In the case where $I = \mathbb{N}$ and $A_n = A$ is fixed, we write $\ell_\infty(A) = \prod_{n\in\mathbb{N}} A_n$ and $c_0(A) = \bigoplus_{n\in\mathbb{N}} A_n$.

(4) For a finite family $\{A_n\}_{n=1}^N$, the c_0 sum equals the ℓ_∞ product. We decorate the notation to write $A_1 \oplus_\infty \cdots \oplus_\infty A_N$.

Example. For $n_1,\dots,n_r \in \mathbb{N}$, the C^*-algebra

$$M = \operatorname{Mat}_{n_1}(\mathbb{C}) \oplus_\infty \operatorname{Mat}_{n_2} \oplus_\infty \oplus_\infty \cdots \oplus_\infty \operatorname{Mat}_{n_r}(\mathbb{C})$$

is finite-dimensional. It is actually the case that every finite-dimensional C^*-algebra is of this form.

We can also take quotients.

Proposition: Let A be a normed $*$-algebra. Let $I \subseteq A$ be a closed $*$-ideal. The quotient space A/I equipped with the quotient norm is a normed $*$-algebra.

If A is complete, then so is A/I. If A is commutative or unital, then so is A/I.

Proof. We know that A/I with its quotient norm is a normed vector space, and that A/I is a $*$-algebra. We need to show that the quotient norm is submultiplicative and that the involution is isometric.

Let $a,b \in A$ and $\varepsilon > 0$. Then, there are x,y such that $\|a+I\|+\varepsilon \geqslant \|a-x\|$, and $\|b+I\|+\varepsilon \geqslant \|b-y\|$. Note that $ay+xb-xy \in I$, so

$$\begin{aligned} \|(a+I)(b+I)\| &= \|ab+I\|\\ &= \operatorname{dist}_I(ab) \end{aligned}$$

$$\begin{aligned}
&\leqslant \|ab - (ay + xb - xy)\| \\
&= \|(a - x)(b - y)\| \\
&\leqslant \|a - x\|\|b - y\| \\
&\leqslant (\|a + I\| + \varepsilon)(\|b + I\| + \varepsilon).
\end{aligned}$$

Sending $\varepsilon \to 0$, we get submultiplicativity. Regarding the involution, we get

$$\begin{aligned}
\|(a + I)^*\| &= \|a^* + I\| \\
&= \inf_{x \in I} \|a^* - x\| \\
&= \inf_{y \in I} \|a^* - y^*\| \\
&= \inf_{y \in I} \|(a - y)^*\| \\
&= \inf_{y \in I} \|a - y\| \\
&= \|a + I\|.
\end{aligned}$$

Completeness follows from the case of the quotient space in Banach spaces. □

Ideals in $C_0(\Omega)$

Earlier, we characterized the maximal ideal space of $C(\Omega)$, where Ω was compact Hausdorff. We are interested in applying this to characterizing the closed ideals of $C_0(\Omega)$, where Ω is a LCH space.

Definition. Let Ω be a LCH space.

(a) For a subset $K \subseteq \Omega$, we write N_K to be continuous hull of K, i.e.

$$N_K = \{f \in C_0(\Omega) \mid f(x) = 0,\ \forall x \in K\}.$$

If $K = \{x\}$, we write N_x.

(b) For any map $f \colon \Omega \to \mathbb{C}$, we denote the zero set of f by

$$Z(f) = f^{-1}(\{0\}).$$

(c) If $I \subseteq C_0(\Omega)$ is any subset, the kernel of I is

$$\ker(I) = \bigcap_{f \in I} Z(f).$$

Fact.

(1) If $K \subseteq \Omega$ is nonempty, then N_K is a closed proper $*$-ideal in $C_0(\Omega)$.

(2) If $I \subseteq C_0$ is any subset, then $\ker(I) \subseteq \Omega$ is closed.

(3) If $K \subseteq L \subseteq \Omega$, then $N_K \supseteq N_L$.

(4) If $I \subseteq J \subseteq C_0(\Omega)$, then $\ker(I) \supseteq \ker(J)$.

To show that every closed ideal in $C_0(\Omega)$ is of the form N_K for some closed $K \subseteq \Omega$, we start with the case of $C_c(\Omega)$. We will finish the proof by taking closures.

Lemma: Let Ω be a LCH space, and let $I \subseteq C_0(\Omega)$ be an ideal. If $g \in C_c(\Omega)$ with $\operatorname{supp}(g) \cap \ker(I) = \varnothing$, then $g \in I$.

Proof. Let $g \in C_c(\Omega)$ and $C = \operatorname{supp}(g)$. For each $x \in C$, define $h_x \in I$ such that $h_x(x) \neq 0$ on C, and let U_x be the open neighborhood on which $h_x \neq 0$. The open cover $\{U_x\}_{x \in C}$ admits a finite subcover,

$$C \subseteq \bigcup_{j=1}^{n} U_{x_j}.$$

We define the function

$$h = \sum_{j=1}^{n} \left|h_{x_j}\right|^2,$$

which belongs to I and is strictly positive on C by construction. Since C is compact, $\inf_C(h) > 0$. Let

$$f(x) = \begin{cases} \frac{g(x)}{h(x)} & x \in C \\ 0 & x \notin C \end{cases}.$$

Then, f is supported on C, and $g = fh$, so $g \in I$. □

Proposition: Let Ω be a LCH space. If $I \subseteq C_0(\Omega)$ is a closed proper ideal, then $K = \ker(I)$, and $I = N_K$.

Proof. Set $J = \left\{g \in C_c(\Omega) \;\middle|\; \operatorname{supp}(g) \cap K = \emptyset\right\}$. By the above lemma, we know that $J \subseteq I$. If K were empty, we would have $J = C_c(\Omega)$, implying

$$\begin{aligned} C_0(\Omega) &= \overline{C_c(\Omega)} \\ &= \overline{J} \\ &\subseteq \overline{I} \\ &= I, \end{aligned}$$

which would contradict the assumption that I is a proper ideal.

We can see that $I \subseteq N_K$ by the definition of N_K. We will now show that every function in N_K can be approximated arbitrarily by a member in J. We will establish the reverse inclusion, $J \subseteq N_K$.

Let $f \in N_K$, $\varepsilon > 0$, and set

$$C_\varepsilon = \{x \in \Omega \mid |f(x)| > \varepsilon\}.$$

Since f vanishes at infinity, C_ε is compact, and $C_\varepsilon \cap K = \emptyset$. By Urysohn's lemma, there is $g \in C_c(\Omega, [0,1])$ with $g|_{C_\varepsilon} = 1$ and $\operatorname{supp}(g) \subseteq K^c$. Thus, $h = fg \in J$, and $\|f - h\|_u \leqslant \varepsilon$. □

Proposition: Let Ω be a LCH space. If $K \subseteq \Omega$ is closed, then $K = \ker(N_K)$.

Proof. We can see that $K \subseteq \ker(N_K)$. If the inclusion is strict, then there is a point $x \in \ker(N_K) \setminus K$, and, by Urysohn's lemma, there is an $f \in C_c(\Omega, [0,1])$ with $f|_K = 0$ and $f(x) = 1$. Thus, $f \in N_K$.

Since $x \in \ker(N_K)$, we must also have $f(x) = 0$, which is a contradiction. Thus, $K = \ker(N_K)$. □

We arrive at the following characterization of the closed ideals of $C_0(\Omega)$.

Corollary: Let Ω be a LCH space. There is an order-reversing one-to-one correspondence between closed subsets of Ω and closed ideals of $C_0(\Omega)$, given by

$$\Omega \supseteq K \leftrightarrow N_K \subseteq C_0(\Omega).$$

Exercise: Show that every maximal ideal of $C_0(\Omega)$ is of the form N_x.

Solution: Via the containment ordering, we see that every maximal element of Ω with this ordering is of the form $\{x\}$, meaning that every ideal of the form N_x is maximal.

Indeed, we may go further. Letting Ω be a LCH space, and $\Lambda \subseteq \Omega$ be open, we know that both Λ and Λ^c are locally compact. We can identify $C_0(\Lambda)$ with the closed ideal N_K, where $K = \Lambda^c$.

Given $f \in C_0(\Lambda)$, define

$$f'(x) = \begin{cases} f(x) & x \in \Lambda \\ 0 & x \in \Lambda^c \end{cases}.$$

Clearly, $f' \in C_0(\Omega)$, and by definition, $f \in N_K$. Additionally, the inclusion map $\iota\colon C_0(\Omega) \to N_K$, defined by $f \mapsto f'$, is an isometric $*$-homomorphism.

Exercise: If $g \in N_K$, then $g|_\Lambda \in C_0(\Lambda)$, and $(g|_\Lambda)' = g$.

Solution: If $g \in N_K$, then $g = 0$ on Λ^c, so for all $\varepsilon > 0$, there is some compact $S \subseteq \Lambda$ such that $|g|_{S^c}| < \varepsilon$. Thus, $g \in C_0(\Lambda)$.

By the definition of ι, we must have $g \mapsto g'$ is an isometric $*$-homomorphism, and since g is 0 on Λ^c, we have that $(g|_\Lambda)' = g$.

Thus, we come to the conclusion that every closed ideal in $C_0(\Omega)$ is of the form $C_0(\Lambda)$, where $\Lambda \subseteq \Omega$ is open.

C^*-norms

We are interested in turning $*$-algebras into Banach $*$-algebras or C^*-algebras. To do this, we can actually use the Banach space completion, $\overline{\iota(A)}^{\|\cdot\|_{op}} \subseteq A^{**}$, where ι is the canonical injection.

Lemma: If A_0 is a normed $*$-algebra, then its Banach space completion is a Banach $*$-algebra, and the inclusion $A_0 \hookrightarrow A$ is an injective $*$-homomorphism.

Proof. We know that A is a Banach space, and the inclusion $A_0 \hookrightarrow A$ is an isometry. We show that A has an algebra structure that extends A_0, and the norm on A is submultiplicative.

Let $x, y \in A$, and let $(x_n)_n, (y_n)_n$ be sequences in A_0 converging to x and y respectively. Then, $\sup_n \|x_n\| = C_1 < \infty$ and $\sup_n \|y_n\| = C_2 < \infty$, since convergent sequences are bounded. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_n y_n - x_m y_m\| &= \|x_n y_n - x_n y_m + x_n y_m - x_m y_m\| \\ &\leqslant \|x_n(y_n - y_m)\| + \|(x_n - x_m)y_m\| \\ &\leqslant C_1\|y_n - y_m\| + C_2\|x_n - x_m\|,\end{aligned}$$

meaning $(x_n y_n)_n$ is Cauchy in A, and converges to $x \cdot y = \lim_{n\to\infty} x_n y_n$.

The map $(x, y) \mapsto x \cdot y$ extends the multiplication on A_0, and endows A with the structure of an algebra.

For $x, y \in A$, and $(x_n)_n, (y_n)_n$ sequences in A_0 converging to x and y respectively, we get

$$\begin{aligned}\|xy\| &= \left\|\lim_{n\to\infty} x_n y_n\right\| \\ &= \lim_{n\to\infty} \|x_n y_n\| \\ &\leqslant \lim_{n\to\infty} \|x_n\|\|y_n\| \\ &= \|x\|\|y\|.\end{aligned}$$

Thus, A is a Banach algebra.

To see that A is a Banach $*$-algebra, we show that A admits the involution defined by, for $x \in A$ and $(x_n)_n \subseteq A_0$ with $(x_n)_n \to x$,

$$x^* = \lim_{n\to\infty} x_n^*.$$

Similarly, we find that

$$\begin{aligned} \|x^*\| &= \lim_{n\to\infty} \|x_n^*\| \\ &= \lim_{n\to\infty} \|x_n\| \\ &= \|x\|, \end{aligned}$$

so A is a Banach $*$-algebra. □

Definition. Let A_0 be a $*$-algebra. A C^*-norm on A_0 is a norm satisfying

(i) $\|ab\| \leqslant \|a\|\|b\|$;

(ii) $\|a^*\| = a$;

(iii) $\|a^*a\| = \|a\|^2$

for all $a, b \in A_0$. We can define C^*-seminorms analogously.

On any given $*$-algebra, there can be many C^*-norms.

Example. Let $\mathcal{T}$ be the unital $*$-algebra of trigonometric polynomials in $C(\mathbb{T})$. For every closed infinite set $F \subseteq \mathbb{T}$, we have a C^*-*norm*, given by

$$\|p\|_F = \sup_{z\in F}|p(z)|.$$

This is pretty clearly a C^*-seminorm, but it isn't clear at first sight that this is a norm. We can show this as follows.

Suppose $\|p\|_F = 0$, meaning $p(z) = 0$ for all $z \in F$. Write

$$\begin{aligned} p(z) &= \sum_{k=-n}^{n} c_k z^k \\ q(z) &= z^n p(z). \end{aligned}$$

Then, $q(z)$ is a polynomial, that vanishes on F. However, since q is a polynomial with degree $2n$, q can have at most $2n$ distinct roots by the fundamental theorem of algebra. Thus, $q = 0$, so $p = 0$.

We can generate C^*-norms and seminorms via morphisms into C^*-algebras.

Lemma: Let A_0 be a $*$-algebra, and let $\phi\colon A_0 \to B$ be a $*$-homomorphism into a C^*-algebra B. Then,

$$\|a\|_\phi = \|\phi(a)\|$$

defines a C^*-seminorm on A_0. If ϕ is injective, then $\|\cdot\|_\phi$ is a norm.

Proof. We will prove that this is a C^*-(semi)norm.

$$\begin{aligned} \|ab\|_\phi &= \|\phi(ab)\| \\ &= \|\phi(a)\phi(b)\| \\ &\leqslant \|\phi(a)\|\|\phi(b)\| \\ &= \|a\|_\phi\|b\|_\phi \end{aligned}$$

$$\begin{aligned}
\|a^*\|_{\phi} &= \|\phi(a^*)\| \\
&= \|\phi(a)^*\| \\
&= \|\phi(a)\| \\
&= \|a\|_{\phi}
\end{aligned}$$

$$\begin{aligned}
\|a^*a\|_{\phi} &= \|\phi(a^*a)\| \\
&= \|\phi(a)^*\phi(a)\| \\
&= \|\phi(a)\|^2 \\
&= \|a\|_{\phi}^2.
\end{aligned}$$

□

We can pass from seminorms to norms by modding out by the null set.

Lemma: Let p be a C^*-seminorm on the $*$-algebra A_0. The set

$$N_p = \{x \in A \mid p(x) = 0\}$$

is a $*$-ideal, and the map

$$\left\|a + N_p\right\|_{A/N_p} = p(a)$$

is a well-defined C^*-norm on A_0/N_p.

Now that we have defined a C^*-norm, we can extend this norm to the norm completion of the $*$-algebra A_0.

Lemma: Let $\|\cdot\|$ be a C^*-norm on a $*$-algebra A_0. The norm completion A is a C^*-algebra, and the inclusion $A_0 \hookrightarrow A$ is an isometric $*$-homomorphism.

Proof. We know that A is a Banach $*$-algebra, and the inclusion is an isometric $*$-homomorphism. We only need to check that the C^* property holds in A. Let $x \in A$, $(x_n)_n \to x$ in A_0. Then,

$$\begin{aligned}
\|x^*x\| &= \lim_{n\to\infty} \|x_n^* x_n\| \\
&= \lim_{n\to\infty} \|x_n\|^2 \\
&= \|x\|^2.
\end{aligned}$$

□

Definition. Let A_0 be a $*$-algebra equipped with C^*-seminorm p. The norm completion of the $*$-algebra A_0/N_p with respect to $\|\cdot\|_{A_0/N_p}$ is called the Hausdorff completion, or enveloping C^*-algebra, of the pair (A_0, p).

Universal C^*-Algebras

We are now interested in a sort of maximal Hausdorff completion of A_0.

Definition. Let A_0 be a $*$-algebra, and let $\mathcal{P}$ be the collection of all C^*-seminorms on A_0. For each $a \in A_0$, we set

$$\|a\|_u = \sup_{p \in \mathcal{P}} p(a).$$

If $\|a\|_u < \infty$ for all $a \in A_0$, then $\|\cdot\|_u$ defines a C^*-seminorm on A_0, called the universal C^*-seminorm. In this case, the universal enveloping C^*-algebra of A_0 is the enveloping algebra of $(A_0, \|\cdot\|_u)$.

Recall that given a set of generators $E = \{x_i\}_{i\in I}$ and relations $R \subseteq \mathbb{A}^*(E)$, we can construct the quotient $*$-algebra $\mathbb{A}^*(E|R) = \mathbb{A}(E)/I(R)$, where $I(R)$ is the $*$-ideal generated by R contained in the free $*$-algebra on E. We write $z_i = x_i + I(R)$.

We also saw that $\mathbb{A}^*(E|R)$ admits a universal property, wherein if B is any $*$-algebra admitting elements $\{b_i\}_{i\in I}$ that satisfy R, then there is a $*$-homomorphism $\phi_B : \mathbb{A}^*(E|R) \to B$, defined by $\phi_B(z_i) = b_i$.

We can define a universal C*-algebra by looking at the universal enveloping algebra of $\mathbb{A}^*(E|R)$, provided it exists.

Definition. Let E be a set of abstract symbols, and $R \subseteq \mathbb{A}^*(E)$ is a set of relations. If the universal C*-algebra of $\mathbb{A}(E|R)$ exists — i.e., if $\|a\|_u < \infty$ for all $a \in \mathbb{A}^*(E|R)$ — then we write $C^*(E|R)$ to denote this C*-algebra, and call it the universal C*-algebra generated by E with relations R.

Just as in the case of the universal $*$-algebra, we see that the universal C*-algebra admits an analogous universal property.

Proposition: Let $E = \{x_i\}_{i\in I}$ be a set of abstract symbols, and let $R \subseteq \mathbb{A}^*(E)$ be a collection of relations. Let $C^*(E|R)$ exist. If B is a C*-algebra admitting elements $\{b_i\}_{i\in I}$ that satisfy the relations, then there is a unique contractive $*$-homomorphism $\varphi_B : C^*(E|R) \to B$, defined by $\varphi_B(\nu_i) = b_i$, where $\nu_i = (x_i + I(R)) + N_u$.

Proof. By the universal property of $\mathbb{A}^*(E|R)$, we have $\phi_B : \mathbb{A}^*(E|R) \to B$, defined by $\phi_B(z_i) = b_i$, where $z_i = x_i + I(R)$.

We have the C*-seminorm given by $a \mapsto \|\phi_B(a)\|$, where $\|\phi_B(a)\| \leqslant \|a\|_u$ for all $a \in \mathbb{A}^*(E|R)$. Additionally, we must have that ϕ_B kills the $*$-ideal

$$N_u = \{a \in \mathbb{A}^*(E|R) \mid \|a\|_u = 0\}.$$

By the first isomorphism theorem, we get the $*$-homomorphism $\widetilde{\phi_B} : \mathbb{A}^*(E|R)/N_u \to B$, given by $z_i + N_u \mapsto b_i$. This map is still contractive, so we can continuously extend $\widetilde{\phi_B}$ to the desired contractive $*$-homomorphism, $\varphi_B : C^*(E|R) \to B$, mapping $z_i + N_u \mapsto b_i$.

Uniqueness follows from the fact that $\mathbb{A}^*(E|R)/N_u$ is dense in its completion. □

Example. It is sometimes the case that $C^*(E|R)$ doesn't exist. Consider $E = \{x\}$ and $R = \{x - x^*\}$. We write $z = x + I(R)$. For a $t > 0$, we find a C*-algebra B_t and a self-adjoint $b_t \in B_t$ with $\|b_t\| = t$.

For each $t > 0$, the universal property for $\mathbb{A}^*(E|R)$ gives a $*$-homomorphism $\phi_t : \mathbb{A}^*(E|R) \to B_t$, with $\phi_t(z) = b_t$. We get a C*-seminorm p_t on $\mathbb{A}^*(E|R)$ given by $p_t(a) = \|\phi_t(a)\| = t$, meaning that the universal C*-seminorm is

$$\begin{aligned}\|z\|_u &\geqslant \sup_{t>0} p_t(z)\\ &= \sup_{t>0}\|\phi_t(z)\|\\ &= \sup_{t>0}\|b_t\|\\ &= \sup_{t>0} t\\ &= \infty.\end{aligned}$$

To verify that the universal C*-seminorm is finite for every element in $\mathbb{A}^*(E|R)$, we can use a simpler characterization.

Lemma: Let $E = \{x_i\}_{i\in I}$ be a set of symbols and suppose $R \subseteq \mathbb{A}^*(E|R)$ is a collection of relations. Write $z_i = x_i + I(R)$. If there is a $C \geqslant 0$ for with $p(z_i) \leqslant C$ for every $i \in I$ and every C^*-seminorm p on $\mathbb{A}^*(E|R)$, then $C^*(E|R)$ exists.

We can consider the C^*-algebra of $n \times n$ matrices over $\mathbb{C}$, and construct this C^*-algebra using the universal C^*-algebra.

Example. Let $n \geqslant 1$, and let $E_n = \left\{x_{ij} \,\middle|\, 1 \leqslant i,j \leqslant n\right\}$. Let

$$R = \left\{x_{ij}^* - x_{ji}, x_{ij}x_{kl} - \delta_{jk}x_{il} \,\middle|\, i,j \in \{1,\dots,n\}\right\}$$

be our set of relations.[IV] Let $z_{ij} = x_{ij} + I(R)$. Then, if p is any C^*-seminorm on $\mathbb{A}^*(E_n|R)$, we have

$$\begin{aligned} p(z_{jj})^2 &= p\left(z_{jj}^* z_{jj}\right)\\ &= p(z_{jj}z_{jj})\\ &= p(z_{jj}), \end{aligned}$$

so $p(z_{jj}) \subseteq \{0,1\}$, and we also have

$$\begin{aligned} p(z_{ij})^2 &= p\left(z_{ij}^* z_{ij}\right)\\ &= p(z_{ji}z_{ij})\\ &= p(z_{jj})\\ &\in \{0,1\}. \end{aligned}$$

Thus, $C^*(E_n|R)$ exists. Write $v_{ij} = z_{ij} + N_u$. We will show that $C^*(E_n|R)$ is not trivial.

The matrix units $\left\{e_{ij} \,\middle|\, 1 \leqslant i,j \leqslant n\right\}$ satisfy the relations, so by the universal property of $C^*(E_n|R)$, we have a contractive $*$-homomorphism $\varphi\colon C^*(E_n|R) \to \mathrm{Mat}_n(\mathbb{C})$ given by $\varphi(v_{ij}) = e_{ij}$. Since $\operatorname{span}\left(\left\{e_{ij}\right\}_{i,j}\right) = \mathrm{Mat}_n(\mathbb{C})$, we must have $C^*(E_n|R) \cong \mathrm{Mat}_n(\mathbb{C})$.

Consequently, $C^*(E_n|R)$ is simple. Additionally, if B is any other C^*-algebra admitting elements $\left\{b_{ij} \,\middle|\, 1 \leqslant i,j \leqslant n\right\}$ with $b_{ij}^* = b_{ji}$ and $b_{ij}b_{kl} = \delta_{kl}b_{il}$, then there is a unique injective $*$-homomorphism between $\mathrm{Mat}_n(\mathbb{C})$ and B such that $\varphi(e_{ij}) \cong b_{ij}$.

We can also obtain $\mathrm{Mat}_n(\mathbb{C})$ another way. Consider $F_n = \{x_1,\dots,x_n\}$ and the relations

$$R' = \left\{x_i^* x_j - \delta_{ij}x_1 \,\middle|\, i,j = 1,\dots,n\right\}.$$

We write $z_i = x_i + I(R)$. If p is any C^*-seminorm on $\mathbb{A}^*(F_n|R')$, then

$$\begin{aligned} p(z_i)^2 &= p\left(z_i^* z_i\right)\\ &= p(z_1), \end{aligned}$$

so $p(z_i) \in \{0,1\}$ for all i. Thus, $C^*(F_n|R')$ exists.

Write $v_i = z_i + N_u$, and set

$$b_{ij} = v_i v_j^*.$$

[IV] The first set of relations denotes the (conjugate) transpose, $x_{ij}^* = x_{ji}$ and the second set of relations denotes $x_{ij}x_{kl} = \delta_{jk}x_{il}$, which is the index notation definition of matrix multiplication.

We have $b_{ij}^* = b_{ji}$, and since $v_i^* v_i = v_1$ is a projection for every i, each v_i is a partial isometry, meaning

$$\begin{aligned}
b_{ij}b_{kl} &= v_i\left(v_j^* v_k\right)v_l^* \\
&= v_i\left(\delta_{jk}v_1\right)v_l^* \\
&= v_i\left(\delta_{jk}v_i^*v_i\right)v_l^* \\
&= \delta_{jk}\left(v_iv_i^*v_i\right)v_l^* \\
&= \delta_{jk}v_iv_l^*.
\end{aligned}$$

Thus, there is a $*$-homomorphism between $\mathrm{Mat}_n(\mathbb{C})$ and $C^*(F_n|R')$, given by $\psi\left(e_{ij}\right) = b_{ij}$. Since $\mathrm{Mat}_n(\mathbb{C})$ is simple, ψ is injective.

We also have

$$\begin{aligned}
\psi(e_{i1}) &= b_{i1} \\
&= v_iv_1^* \\
&= v_iv_1 \\
&= v_iv_i^*v_i \\
&= v_i,
\end{aligned}$$

so ψ is onto. Thus $C^*(F_n|R') \cong \mathrm{Mat}_n(\mathbb{C})$.

Example. Let $E = \{1, x\}$ and

$$R = \left\{x^*x - 1, xx^* - 1, 1x - x, x1 - x, 1^2 - 1, 1^* - 1\right\}.$$

We see that $\mathbb{A}^*(E|R)$ is unital with unit $1 + I(R)$, and that $x + I(R)$ is invertible with inverse $x^* + I(R)$. Writing $z = x + I(R)$, we see that

$$\mathbb{A}^*(E|R) = \left\{\sum_{k\in\mathbb{Z}} \alpha_k z^k \;\middle|\; \alpha_k \in \mathbb{C}, \text{finitely many nonzero}\right\},$$

where $z^{-1} = z^*$ and $z^0 = 1$.

If p is any seminorm on $\mathbb{A}^*(E|R)$, we have

$$\begin{aligned}
p(1)^2 &= p(1^*1) \\
&= p\left(1^2\right) \\
&= p(1),
\end{aligned}$$

so $p(1) \in \{0, 1\}$, and

$$\begin{aligned}
p(z)^2 &= p(z^*z) \\
&= p(1) \\
&\in \{0, 1\}.
\end{aligned}$$

Thus, $C^*(E|R)$ exists. We write $u = z + N_u$. The universal property states that if w is a unitary in any unital C^*-algebra B, then there is a surjective $*$-homomorphism between $C^*(E|R)$ and $C^*(w) \subseteq B$, given by $u \mapsto w$.

Eventually, we wil show that $C^*(E|R) \cong C(\mathbb{T})$.

Representations and the Group C^*-algebra

We can realize $*$-algebras as $*$-subalgebras of bounded operators on a Hilbert space. This allows us to get a C^*-norm for free, and get a C^*-algebra by completion.

Definition. Let A_0 be a $*$-algebra. A representation of A_0 is a pair $(\pi_0, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi_0 \colon A \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism. We will refer to the representation by π_0 if the Hilbert space is understood.

If A_0 is unital, and $\pi(1_A) = I_{\mathcal{H}}$, then we say π is a unital representation.

Lemma: Let A_0 be a $*$-algebra, and suppose $(\pi_0, \mathcal{H})$ is a representation of A_0. Then,

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{op}$$

is a C^*-seminorm on A_0. If π_0 is injective, then $\|\cdot\|_{\pi_0}$ is a C^*-norm.

Lemma: Let A_0 and B_0 be normed $*$-algebras with respective completions A and B. If $\varphi_0 \colon A_0 \to B_0$ is a bounded $*$-homomorphism, then the continuous extension $\varphi \colon A \to B$ is a $*$-homomorphism.

Proof. Let $x, y \in A$ with $(x_n)_n \to x$ and $(y_n)_n \to y$ sequences in A_0. Then,

$$\begin{aligned}
\varphi(xy) &= \varphi\left(\lim_{n\to\infty} x_n y_n\right)\\
&= \lim_{n\to\infty} \varphi(x_n y_n)\\
&= \lim_{n\to\infty} \varphi_0(x_n y_n)\\
&= \lim_{n\to\infty} \varphi_0(x_n)\varphi_0(y_n)\\
&= \lim_{n\to\infty} \varphi(x_n)\varphi(y_n)\\
&= \varphi(x)\varphi(y).
\end{aligned}$$

A similar process, using the continuity of the involution, gives $\varphi(x^*) = \varphi(x)^*$. □

Corollary: Let A_0 be a $*$-algebra, and suppose $\pi \colon A_0 \to \mathbb{B}(\mathcal{H})$ is an injective representation. The completion A of A_0 with respect to the C^*-norm $\|\cdot\|_{\pi_0}$ is a C^*-algebra, and the continuous extension $\pi \colon A \to \mathbb{B}(\mathcal{H})$ is an isometric $*$-homomorphism.

The C^*-algebra that arises from a group is an important example of a C^*-algebra.[v]

Given a group Γ, we can construct the group $*$-algebra, $\mathbb{C}[\Gamma]$. An element $a \in \mathbb{C}[\Gamma]$ is a finitely supported complex-valued function on Γ, written as a finite sum

$$a = \sum_{s\in\Gamma} a(s)\delta_s,$$

where $\delta_s : \Gamma \to \mathbb{C}$ is the indicator function for s, $\delta_s(t) = \delta_{st}$.

Unit*ary* representations of Γ are related to representations of the group $*$-algebra $\mathbb{C}[\Gamma]$.

Proposition: Let Γ be a group, and let $\mathcal{H}$ be a Hilbert space.

(1) If $u \colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ, then the map $\pi_u : \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$ given by

$$\pi_u(a) = \sum_{s\in\Gamma} a(s)u_s$$

is a representation of $\mathbb{C}[\Gamma]$.

[v]It's partially the subject of my Honors thesis.

(2) If $\pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$ is a unit*al* representation, then the map $u\colon \Gamma \to \mathcal{U}(\mathcal{H})$, given by

$$u(s) = \pi(\delta_s)$$

is a unit*ary* representation of Γ.

Proof.

(1) The map $s \mapsto u_s \in \mathbb{B}(\mathcal{H})$ extends to a linear map $\pi_u\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$, satisfying $\pi_u(\delta_s) = u_s$ by the universal property of the free vector space.

For $s, t \in \Gamma$, we have

$$\begin{aligned}\pi_u(\delta_s\delta_t) &= \pi_u(\delta_{st})\\ &= u_{st}\\ &= u_s u_t\\ &= \pi_u(\delta_s)\pi_u(\delta_t)\end{aligned}$$

$$\begin{aligned}\pi_u(\delta_s^*) &= \pi_u\left(\delta_s^{-1}\right)\\ &= u_{s^{-1}}\\ &= u_s^*\\ &= \pi_u(\delta_s)^*.\end{aligned}$$

Using the linearity of π_u, we see that π_u is multiplicative and $*$-preserving.

(2) Every $\delta_s \in \mathbb{C}[\Gamma]$ is unitary, and since unital $*$-homomorphisms map unitaries to unitaries, we know that each $u(s)$ is unitary. Moreover, for $s, t \in \Gamma$, we have

$$\begin{aligned}u(st) &= \pi(\delta_{st})\\ &= \pi(\delta_s\delta_t)\\ &= \pi(\delta_s)\pi(\delta_t)\\ &= u(s)u(t),\end{aligned}$$

meaning u is a unitary representation.

□

For the group Γ is a group with neutral element e, we have defined the group $*$-algebra and the left-regular representation $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$. We thus get a representation of the group $*$-algebra

$$\pi_\lambda(a) = \sum_{s\in\Gamma} a(s)\lambda_s.$$

We claim that π_λ is injective. Suppose $\pi_\lambda(a) = 0$ for some $a = \sum_{s\in\Gamma} a(s)\delta_s \in \mathbb{C}[\Gamma]$. Evaluating δ_e, we have

$$\begin{aligned}0 &= \pi_\lambda(a)(\delta_e)\\ &= \left(\sum_{s\in\Gamma} a(s)\lambda_s\right)(\delta_e)\\ &= \sum_{s\in\Gamma} a(s)\lambda_s(\delta_e)\\ &= \sum_{s\in\Gamma}(\delta_s).\end{aligned}$$

Since the vectors $\{\delta_t\}_{t\in\Gamma}$ are linearly independent, we must have $a(s) = 0$ for all $s \in \Gamma$, so $a = 0$.

Thus, we have a C^*-norm on $\mathbb{C}[\Gamma]$ given by $\|a\|_\lambda = \|\pi_\lambda(a)\|_{\text{op}}$. The $\|\cdot\|_\lambda$-completion of $\mathbb{C}[\Gamma]$ is a C^*-algebra denoted by $C^*_\lambda(\Gamma)$. This is known as the left-regular group C^*-algebra.

Similarly, we may begin with the right-regular representation $\rho\colon \mathbb{C}[\Gamma] \to \mathcal{U}(\ell_2(\Gamma))$, and construct the representation

$$\pi_\rho(a) = \sum_{s\in\Gamma} a(s)\rho_s,$$

which induces the C^*-norm $\|\cdot\|_\rho$ on $\mathbb{C}[\Gamma]$, which gives rise to the right-regular group C^*-algebra, $C^*_\rho(\Gamma)$.

We often refer to $C^*_\lambda(\Gamma)$ as the reduced group C^*-algebra of Γ, often denoted $C^*_r(\Gamma)$.

There is also a full group C^*-algebra, with the full norm defined by

$$\|a\|_u = \sup\left\{\|\pi(a)\| \;\middle|\; \pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H}_\pi) \text{ is a representation}\right\}.$$

To see that this quantity is finite, note that for every representation $\pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H}_\pi)$, the elements $\pi(\delta_s)$ are unitaries in $\mathbb{B}(\mathcal{H})$, hence having norm 1. So, we have

$$\begin{aligned}
\|\pi(a)\| &= \left\|\pi\left(\sum_{s\in\Gamma} a(s)\delta_s\right)\right\| \\
&= \left\|\sum_{s\in\Gamma} a(s)\pi(\delta_s)\right\| \\
&\leqslant \sum_{s\in\Gamma} \|a(s)\delta_s\| \\
&= \sum_{s\in\Gamma} |a(s)|,
\end{aligned}$$

so $\|a\|_u \leqslant \sum_{s\in\Gamma}|a(s)| < \infty$. This is a C^*-norm, as if $\|a\|_u = 0$, then $\|a\|_\lambda = 0$, as π_λ is one of the representations, and since $\|\cdot\|_\lambda$ is a norm, we must have $a = 0$. Thus, completing $\mathbb{C}[\Gamma]$ with respect to $\|\cdot\|_u$ yields the full (or universal) group C^*-algebra, denoted $C^*(\Gamma)$.

The full group C^*-algebra admits a universal property.

Proposition: Let Γ be a discrete group. Given any unitary representation $u\colon \Gamma \to \mathcal{U}(\mathcal{H})$, there is a contractive $*$-homomorphism $\pi_u\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ satisfying $\pi_u(\delta_s) = u(s)$ for every $s \in \Gamma$.

Proof. We have a representation $\pi_u\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$ that extends $u\colon \Gamma \to \mathcal{U}(\mathcal{H})$. By definition, the universal norm provides $\|\pi(a)\|_u \leqslant \|a\|_u$.

The continuous extension $\pi_u\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ is contractive and a $*$-homomorphism. $\square$

Unitizations of C^*-Algebras

Given a non-unital algebra A, there is a unital algebra, $\widetilde{A}$, that contains A as a maximal and essential ideal. We will now examine the analytical component of unitization — given a Banach algebra or C^*-algebra, we want the resulting unitization to also be a Banach algebra or C^*-algebra.

Proposition: Let A be a Banach $*$-algebra. The unitzation $\widetilde{A}$ is a unital Banach $*$-algebra with the norm

$$\|(a,\alpha)\| = \|a\| + |\alpha|.$$

The inclusion $\iota_A\colon A \to \widetilde{A}$, given by $\iota(a) = (a, 0)$, is an isometric $*$-isomorphism.

Proof. Let A be a Banach $*$-algebra. We know that the unitization, $\widetilde{A}$, is a unital $*$-algebra, and ι_A is a $*$-homomorphism.

We can see that $\|\cdot\|$ is a norm on the vector space $\widetilde{A}$ from its definition. To verify that it is a norm on the algebra, we have

$$\begin{aligned}\|(a,\alpha)(b,\beta)\| &= \|(ab+\alpha b+\beta a,\alpha\beta)\|\\ &= \|ab+\alpha b+\beta a\| + |\alpha\beta|\\ &\leqslant \|a\|\|b\| + |\alpha|\|b\| + |\beta|\|a\| + |\alpha||\beta|\\ &= (\|a\|+|\alpha|)(\|b\|+|\beta|)\\ &= \|(a,\alpha)\|\|(b,\beta)\|.\end{aligned}$$

We also have

$$\begin{aligned}\|(a,\alpha)^*\| &= \|(a^*,\overline{\alpha})\|\\ &= \|a^*\| + |\overline{\alpha}|\\ &= \|a\| + |\alpha|\\ &= \|(a,\alpha)\|.\end{aligned}$$

To see that the norm on $\widetilde{A}$ is complete, recall that the projection $\pi\colon \widetilde{A} \to \mathbb{C}$, given by $(a,\alpha) \mapsto \alpha$, is a 1-quotient mapping, so $\widetilde{A}/A$ is isometrically isomorphic to $\mathbb{C}$, hence complete. Since A is also complete, we must have $\widetilde{A}$ is complete, as it is a two of three spaces property. □

Turning our attention to C^*-algebras, we know that the traditional unitization converts A into a Banach $*$-algebra. However, this norm is not a C^*-norm. Instead, we embed A isometrically into an algebra of bounded operators in order to obtain the unitization.

If A is an algebra, we let $L_a(x) = ax$ be left-multiplication by a. If A is normed, we can see that $L_a(x)$ is continuous:

$$\begin{aligned}\|L_a(x)\| &= \|ax\|\\ &\leqslant \|a\|\|x\|.\end{aligned}$$

Thus, we have a map $L\colon A \to \mathbb{B}(A)$ given by $a \mapsto L_a$. We can also see that $L_{a+\alpha b} = L_a + \alpha L_b$, and $L_{ab} = L_a \circ L_b$, so L is an algebra homomorphism. We may extend to the unitization, so we obtain the unital algebra homomorphism

$$\overline{L}(a,\alpha) = L_a + \alpha\,\mathrm{id}_A\,.$$

We know that if A is nonunital and L is injective, then $\overline{L}$ is injective. This will allow us to unitize a nonunital C^*-algebra.

Lemma: Let A be a normed algebra, and let $L\colon A \to \mathbb{B}(A)$ and $\overline{L}\colon \widetilde{A} \to \mathbb{B}(A)$ be as above.

(1) L is a contractive algebra homomorphism, and $\operatorname{Ran}(L) \subseteq \mathbb{B}(A)$ is a subalgebra.

(2) If A is a C^*-algebra, then L is isometric, and $\operatorname{Ran}(L) \subseteq \mathbb{B}(A)$ is closed in operator norm.

(3) If A is a nonunital C^*-algebra, then $\overline{L}$ is an injective algebra homomorphism, restricting to an isometry on A, and $\operatorname{Ran}\left(\overline{L}\right) \subseteq \mathbb{B}(A)$ is closed in operator norm.

Proof. We have proven (1) already, so we prove (2) and (3).

(2) If A is a C^*-algebra, then we see that

$$\begin{aligned}\|L_a\|_{op} &\geqslant \left\|L_a\left(\frac{a^*}{\|a\|}\right)\right\| \\ &= \frac{\|aa^*\|}{\|a\|} \\ &= \frac{\|a\|^2}{\|a\|} \\ &= \|a\|.\end{aligned}$$

Thus, $\|L_a\|_{op} = \|a\|$, so L is isometric. Since L is complete, and L is an isometry, $\operatorname{Ran}(L)$ is complete, so it is closed in $\mathbb{B}(A)$.

We have seen that L is isometric, hence injective. Since A is nonunital, $\overline{L}$ is injective too. Since $\operatorname{Ran}(L)$ is closed, the sum $\operatorname{Ran}(L) + \mathbb{C}\,\mathrm{id}_A$ is closed as well.

□

Proposition: Let A be a C^*-algebra, and let $L\colon A \to \mathbb{B}(A)$, $\overline{L}\colon A \to \mathbb{B}(A)$ be as above.

(1) The quantity

$$\|(a, \alpha)\|_L = \|L_a + \alpha\,\mathrm{id}_A\|_{op}$$

is a C^*-seminorm on $\widetilde{A}$.

(2) If A is nonunital, then $\|\cdot\|_L$ is a C^*-norm on $\widetilde{A}$, and $\left(\widetilde{A}, \|\cdot\|_L\right)$ is a unital C^*-algebra. The inclusion $\iota_A : A \hookrightarrow \left(\widetilde{A}, \|\cdot\|_L\right)$ is an isometric $*$-homomorphism.

(3) The quantity

$$\|(a, \alpha)\|_1 = \max(\|(a, \alpha)\|_L, |\alpha|)$$

is a C^*-norm on $\widetilde{A}$.

(4) $\left(\widetilde{A}, \|\cdot\|_1\right)$ is a unital C^*-algebra, and the inclusion, $\iota_A : A \hookrightarrow \left(\widetilde{A}, \|\cdot\|_1\right)$ is an isometric $*$-homomorphism.

Proof.

(1) Since $\overline{L}$ is an algebra homomorphism, we know that $\|\cdot\|_L$ is a seminorm. We will show the rest of the definitions simultaneously:

$$\begin{aligned}\|(a, \alpha)\|_L^2 &= \sup_{x \in B_A} \|ax + \alpha x\|^2 \\ &= \sup_{x \in B_A} \|(ax + \alpha x)^*(ax + \alpha x)\| \\ &= \sup_{x \in B_A} \left\|x^*a^*ax + \alpha x^*a^*x + \overline{\alpha}x^*ax + |\alpha|^2 x^*x\right\|\end{aligned}$$

Thus, $\|(a, \alpha)\|_L \leqslant \|(a, \alpha)^*\|_L$, and $\|(a, \alpha)^*\|_L \leqslant \|(a, \alpha)\|_L$, so $\|(a, \alpha)^*\|_L = \|(a, \alpha)\|$. This means all the inequalities above are indeed equalities, so we also recover the C^* identity.

(2) Since $\overline{L}\colon A \to \mathbb{B}(A)$ is injective, $\|\cdot\|_L$ is a norm.

Additionally, we know that $\overline{L}\colon \left(\widetilde{A}, \|\cdot\|_L\right) \to \left(\operatorname{Ran}\left(\overline{L}\right), \|\cdot\|_{op}\right)$ is an isometric isomorphism. Since $\left(\operatorname{Ran}\left(\overline{L}\right), \|\cdot\|_{op}\right)$ is a Banach algebra, so too is $\left(\widetilde{A}, \|\cdot\|_L\right)$, so $\left(\widetilde{A}, \|\cdot\|_L\right)$ is a C^*-algebra.

We can also see that ι_A is isometric, since

$$\begin{aligned}\|\iota(a)\|_L &= \|(a,0)\|_L\\ &= \left\|\overline{L}(a,0)\right\|_{op}\\ &= \|L_a\|_{op}\\ &= \|a\|.\end{aligned}$$

(3) That $\|\cdot\|_1$ is a C^*-seminorm follows from (1), and $a \mapsto |a|$ is a C^*-norm on $\mathbb{C}$. If $\|(a,\alpha)\|_1 = 0$, then $\alpha = 0$, so $\|(a,0)\|_L = 0$, meaning $\|L_a\|_{op} = 0$, so $a = 0$.

(4) Let $((a_n, \alpha_n))_n$ be a $\|\cdot\|_1$-Cauchy sequence in $\widetilde{A}$.

It follows that $(\alpha_n)_n$ is Cauchy in $\mathbb{C}$, and $(L_{a_n} + \alpha_n \operatorname{id}_A)_n$ is Cauchy in $\mathbb{B}(A)$. Thus, there are $\alpha \in \mathbb{C}$ and $T \in \mathbb{B}(A)$ that these sequences respectively converge to.

We see that $(\alpha_n \operatorname{id}_A)_n \to \alpha \operatorname{id}_A$, so $L_{a_n} \to T - \alpha \operatorname{id}_A$. Since Ran(L) is closed, $T - \alpha \operatorname{id}_A = L_a$ for some $a \in A$, meaning $T = L_a + \alpha \operatorname{id}_A$. Thus, $((a_n,\alpha_n))_n \xrightarrow{\|\cdot\|_1} (a,\alpha)$, so $\|\cdot\|_1$ is complete.

It is clear that ι_A is isometric, as

$$\begin{aligned}\|\iota(a)\|_1 &= \|(a,0)\|_1\\ &= \|(a,0)\|_L\\ &= \|a\|.\end{aligned}$$

$\square$

Definition. Let A be a C^*-algebra.

(1) If A is non-unital, then $\left(\widetilde{A}, \|\cdot\|_L\right)$ is known as the minimal C^*-unitization of A.

(2) The C^*-algebra $\left(\widetilde{A}, \|\cdot\|_1\right)$ is known as the forced unitization of A, referred to as A^1 or $A^\dagger$.

Proposition: Let A be a C^*-algebra.

(1) If A is nonunital, then $\|\cdot\|_1$ and $\|\cdot\|_L$ are equal.

(2) If A is unital, then there is an isometric $*$-isomorphism between $A \oplus \mathbb{C} \to A^1$.

Proof. We will prove (2).

(2) Consider the $*$-isomorphism $A \oplus \mathbb{C} \to \widetilde{A}$, given by $(a,z) = a + zp$, where $p = 1_{\widetilde{A}} - 1_A$. We only need to show that this is an isometry when $A \oplus \mathbb{C}$ is equipped with the infinity norm and $\widetilde{A}$ is equipped with $\|\cdot\|_1$.

Note that

$$\overline{L}(a + zp) = \overline{L}((a - z1_A, z))$$

$$
\begin{aligned}
&= L_{a-z1_A} + z\,\mathrm{id}_A\\
&= L_a - z\,\mathrm{id}_A + z\,\mathrm{id}_A\\
&= L_a.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|a + zp\|_1 &= \max\Big(\|L_a\|_{op}, |z|\Big)\\
&= \max(\|a\|, |z|)\\
&= \|(a, z)\|_\infty.
\end{aligned}
$$

□

Gelfand Theory

Diving deeper into the theory of C*-algebras, we are interested in characterizing C*-algebras, ultimately realizing commutative C*-algebras as continuous function algebras, and non-commutative C*-algebras as bounded operators on Hilbert spaces.

Properties of the Spectrum

We can create an analytic characterization of invertibility in a Banach algebra.

Proposition (Carl Neumann Series)**:** Let A be a unital Banach algebra. If $x \in A$ with $\|x\| < 1$, then $1_A - x \in \mathrm{GL}(A)$, and

$$
(1_A - x)^{-1} = \sum_{k=0}^{\infty} x^k.
$$

Moreover,

$$
\left\|(1_A - x)^{-1}\right\| \leqslant \frac{1}{1 - \|x\|}.
$$

Proof. Since A is a normed algebra, we have $\left\|x^k\right\| \leqslant \|x\|^k$, so

$$
\begin{aligned}
\sum_{k=0}^{\infty}\left\|x^k\right\| &\leqslant \sum_{k=0}^{\infty}\|x\|^{\infty}\\
&= \frac{1}{1 - \|x\|},
\end{aligned}
$$

so the series $\sum_{k=0}^{\infty} x^k$ converges absolutely, hence converges as A is complete. Note that this also means $\lim_{n\to\infty} x^n = 0$. We compute

$$
\begin{aligned}
(1_A - x)\left(\sum_{k=0}^{\infty} x^k\right) &= \lim_{n\to\infty}(1_A - x)\left(\sum_{k=0}^{n} x^k\right)\\
&= \lim_{n\to\infty}\left(\sum_{k=0}^{n} x^k - \sum_{k=0}^{n} x^{k+1}\right)\\
&= \lim_{n\to\infty}\left(1_A - x^{n+1}\right)\\
&= 1_A.
\end{aligned}
$$

Similarly, $\left(\sum_{k=0}^{\infty} x^k\right)(1_A - x) = 1_A$. □

Corollary: Let A be a unital Banach algebra, and let $a \in A$ with $\|1_A - a\| < 1$. Then, $a \in \mathrm{GL}(A)$, and

$$a^{-1} = \sum_{k=0}^{\infty} (1_A - a)^k$$
$$\left\|a^{-1}\right\| \leqslant \frac{1}{1 - \|1_A - a\|}.$$

Proposition: Let A be a unital Banach algebra.

(1) The group of invertible elements, $\mathrm{GL}(A) \subseteq A$, is open.

(2) The inverse map, $i\colon \mathrm{GL}(A) \to \mathrm{GL}(A)$, given by $i(a) = a^{-1}$, is a homeomorphism.

Proof.

(1) Let $a \in \mathrm{GL}(A)$, and set $\delta = \left\|a^{-1}\right\|^{-1}$. We will show that $U(a, \delta) \subseteq \mathrm{GL}(A)$. Let $b \in A$ be such that $\|a - b\| < \delta$. Then,

$$\begin{aligned}\left\|1_A - a^{-1}b\right\| &= \left\|a^{-1}(a - b)\right\| \\ &\leqslant \left\|a^{-1}\right\|\|a - b\| \\ &< 1.\end{aligned}$$

Thus, $a^{-1}b \in \mathrm{GL}(A)$ with

$$\left\|b^{-1}a\right\| \leqslant \frac{1}{1 - \left\|1 - a^{-1}b\right\|}.$$

Thus, we get $b = a\left(a^{-1}b\right) \in \mathrm{GL}(A)$.

(2) It suffices to show that i is continuous. Let $a, b \in \mathrm{GL}(A)$. Notice that

$$\begin{aligned}\left\|b^{-1} - a^{-1}\right\| &= \left\|b^{-1}\left(1_A - ba^{-1}\right)\right\| \\ &= \left\|b^{-1}(a - b)a^{-1}\right\| \\ &\leqslant \left\|b^{-1}\right\|\|a - b\|\left\|a^{-1}\right\|.\end{aligned}$$

We wish to control $\left\|b^{-1}\right\|$, so we find

$$\begin{aligned}\left\|b^{-1}\right\| &= \left\|b^{-1}aa^{-1}\right\| \\ &\leqslant \left\|b^{-1}a\right\|\left\|a^{-1}\right\| \\ &\leqslant \frac{\left\|a^{-1}\right\|}{1 - \left\|1_A - a^{-1}b\right\|}.\end{aligned}$$

Additionally, we have

$$\begin{aligned}\left\|1_A - a^{-1}b\right\| &= \left\|a^{-1}(a - b)\right\| \\ &\leqslant \left\|a^{-1}\right\|\|a - b\|,\end{aligned}$$

so

$$1 - \left\|1_A - a^{-1}b\right\| \geqslant 1 - \left\|a^{-1}\right\|\|a - b\|.$$

Combining, we get

$$\left\|b^{-1}\right\| \leqslant \frac{\left\|a^{-1}\right\|}{1 - \left\|a^{-1}\right\|\|a - b\|}.$$

Thus, we get

$$\left\|b^{-1} - a^{-1}\right\| \leqslant \frac{\left\|a^{-1}\right\|^2 \|a - b\|}{1 - \left\|a^{-1}\right\| \|a - b\|}.$$

For any sequence $(b_n)_n$ in $\mathrm{GL}(A)$ that converges to $a \in \mathrm{GL}(A)$, we see that $\|i(b_n) - i(a)\| \to 0$, so i is continuous.

□

We can now examine the analytic properties of the resolvent and spectrum.

Theorem: Let A be a Banach algebra with $a \in A$.

(1) The resolvent $\rho(a) \subseteq \mathbb{C}$ is open, and the spectrum $\sigma(a) \subseteq \mathbb{C}$ is closed.

(2) If $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$, then $\lambda \in \rho(a)$. Thus, $\sigma(a) \subseteq \overline{D(0, \|a\|)}$, meaning $\sigma(a)$ is compact.

(3) If A is unital, then $R_a \colon \rho(a) \to A$, given by $R_a(a - \lambda 1_A)^{-1}$ is holomorphic on $\rho(A)$.

(4) The spectrum $\sigma(a)$ is nonempty. If A is nonunital, $0 \in \sigma(a)$.

Proof.

(1) We assume A is unital. Let $t \colon \mathbb{C} \to A$ be given by $t(\lambda) = a - \lambda 1_A$, which is continuous. Since $t^{-1}(\mathrm{GL}(A)) = \rho(a)$, and $\mathrm{GL}(A)$ is open, $\rho(a)$ is open. The complement, $\sigma(a)$, is closed.

If A is not unital, then $\rho(a) = \rho(\iota_A(a))$ by definition, which is yet again open, so $\sigma(a)$ is yet again closed.

(2) We assume A is unital. If $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant \|a\|$, then $\left\|\lambda^{-1} a\right\| = |\lambda|^{-1}\|a\| < 1$, so $1 - \lambda^{-1} a \in \mathrm{GL}(A)$, meaning $a - \lambda 1_A = -\lambda\left(1_A - \lambda^{-1} a\right) \in \mathrm{GL}(A)$, so $\lambda \in \rho(a)$.

If A is nonunital, then the canonical inclusion ι_A is an isometry, so

$$\begin{aligned}
\sigma(a) &= \sigma(\iota_A(a)) \\
&\subseteq \overline{D(0, \|\iota_A(a)\|)} \\
&= \overline{D(0, \|a\|)}.
\end{aligned}$$

(3) The common denominator expansion

$$\begin{aligned}
R_a(\mu) - R_a(\lambda) &= (a - \mu 1_A)^{-1} - (a - \lambda 1_A)^{-1} \\
&= (a - \mu 1_A)((a - \lambda 1_A) - (a - \mu 1_A))(a - \lambda 1_A)^{-1} \\
&= R_a(\mu)(\mu - \lambda) 1_A R_a(\lambda).
\end{aligned}$$

When $\mu \neq \lambda$, we have

$$\frac{R_a(\mu) - R_a(\lambda)}{\mu - \lambda} = R_a(\mu) R_a(\lambda).$$

We note that $R_a = i \circ t \colon \rho(a) \to A$ is continuous, so for all $\lambda \in \rho(a)$, we have

$$\begin{aligned}
R_a(\lambda) &= \lim_{\mu \to \lambda} \frac{R_a(\mu) - R_a(\lambda)}{\mu - \lambda} \\
&= \lim_{\mu \to \lambda} R_a(\mu) R_a(\lambda) \\
&= R_a(\lambda)^2.
\end{aligned}$$

(4) Let A be unital, and assume $a \neq 0$. Suppose toward contradiction $\sigma(a) = \emptyset$, meaning $\rho(a) = \mathbb{C}$. Then, $R_a : \mathbb{C} \to A$ is entire, so for $\lambda \neq 0$, we have

$$\begin{aligned}
\|R_a(\lambda)\| &= \left\|(a - \lambda 1_A)^{-1}\right\| \\
&= \left\|\left(\lambda\left(\lambda^{-1}a - 1_A\right)\right)^{-1}\right\| \\
&= \left\|\lambda^{-1}\left(\lambda^{-1}a - 1_A\right)^{-1}\right\| \\
&= |\lambda|^{-1}\left\|\left(\lambda^{-1}a - 1_A\right)^{-1}\right\| \\
&\to 0.
\end{aligned}$$

Thus, R_a is also bounded, and specifically $R_a \in C_0(\mathbb{C})$, so by Liouville's theorem, $R_a = 0$. However, since $R_a(0) = a^{-1} \neq 0$, we have a contradiction. Thus, $\sigma(a) \neq \emptyset$.

If A is nonunital, then $\sigma(a) = \sigma(\iota_A(a)) \neq \emptyset$, and $0 \in \sigma(a)$ since A does not contain any invertible elements.

□

Definition. Let A be a Banach algebra. For each $a \in A$, we define the spectral radius to be

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Example. Let Ω be a compact Hausdorff space, and let $f \in C(\Omega)$. By compactness, there is $x \in \Omega$ with $|f(x)| = \|f\|_u$. Thus, for some $\omega \in \mathbb{T}$, we have $f(x) = \omega\|f\|_u$.

It follows that $f - \omega\|f\|\mathbb{1}_\Omega$ is noninvertible, so $\omega\|f\|_u \in \sigma(f)$. Thus, $\|f\|_u = r(f)$.

Corollary: If A is a Banach algebra with $a \in A$, then there is $\lambda \in \sigma(a)$ with $|\lambda| = r(a)$.

Proof. The spectrum is nonempty and compact, and the map $\lambda \mapsto |\lambda|$ is continuous, so its supremum is attained. □

What makes Banach algebras special is that we are able to relate the algebraic aspects of an element (like the spectrum) and the analytic aspects of that element (like the norm), through the following result.

Proposition: Let A be a Banach algebra, and let $a \in A$. Then,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. We assume A admits a unit, and assume $a \neq 0$.

Fix $\lambda \in \sigma(a)$. From spectral mapping of polynomials, we know that $\lambda^n \in \sigma(a^n)$. Thus, we get

$$\begin{aligned}
|\lambda|^n &= |\lambda^n| \\
&\leqslant \|a^n\|,
\end{aligned}$$

meaning

$$|\lambda| \leqslant \|a^n\|^{1/n}.$$

We see that the sequence $\left(\|a^n\|^{1/n}\right)_n$ is bounded, as

$$\|a^n\|^{1/n} \leqslant \left(\|a\|^n\right)^{1/n}$$

$$= \|a\|,$$

so

$$\begin{aligned}|\lambda| &\leqslant \liminf_{n\to\infty}\|a^n\|^{1/n}\\ &< \infty.\end{aligned}$$

Since this holds for all $\lambda \in \sigma(a)$, we have $r(a) \leqslant \liminf_{n\to\infty}\|a^n\|^{1/n}$.

To establish the opposite direction, let Ω be the set of all complex numbers with modulus strictly less than $r(a)^{-1}$, where we set $r(a)^{-1} = \infty$ if $r(a) = 0$. If $0 \neq z \in \Omega$, then $|z|^{-1} > r(a)$, so $z^{-1} \in \rho(a)$, and $a - z^{-1}1_A \in GL(A)$. Thus, we get that $1_A - za = -z\left(a - z^{-1}1_A\right) \in GL(A)$. If $z = 0$, then it is clear that $1_A - za = 1_A \in GL(A)$.

We consider the map $F\colon \Omega \to A$, given by $F(z) = (1 - za)^{-1}$. We can see that F is holomorphic on Ω. Fix $\varphi \in A^*$, and set $f = \varphi \circ F\colon \Omega \to \mathbb{C}$. Since f is the composition of holomorphic maps, f is holomorphic. From complex function theory, we have that f is analytic, so there is a unique sequence $(\alpha_n)_n \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty}\alpha_n z^n,$$

which converges absolutely and uniformly on compact subsets of Ω. Seeing as $r(a) \leqslant \|a\|$, we have $\frac{1}{\|a\|} \leqslant \frac{1}{r(a)}$, so the open disk $D\left(0, \|a\|^{-1}\right)$ is a subset of Ω. For any $w \in D\left(0, \|a\|^{-1}\right)$, we see that $\|wa\| = |w|\|a\| < 1$, so we get that

$$\begin{aligned}F(w) &= (1_A - wa)^{-1}\\ &= \sum_{n=0}^{\infty} w^n a^n.\end{aligned}$$

Applying φ, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\alpha_n w^n &= f(w)\\ &= \varphi(F(w))\\ &= \varphi\left(\sum_{n=0}^{\infty} w^n a^n\right)\\ &= \sum_{n=0}^{\infty} w^n \varphi(a^n),\end{aligned}$$

so, since Taylor expansions of holomorphic functions are unique, we have

$$f(z) = \sum_{n=0}^{\infty}\varphi(a^n)z^n.$$

Let $t > r(a)$, and set $\zeta = \frac{1}{t} \in \Omega$. Since the series for $f(z)$ converges for all z, we have that $|\varphi(a^n)\zeta^n| \to 0$ as $n \to \infty$, meaning $\sup_{n\geqslant 0}|\varphi(\zeta^n a^n)| < \infty$. Since this holds for all $\varphi \in A^*$, the uniform boundedness principle gives $\sup_{n\geqslant 0}\|\zeta^n a^n\| = C < \infty$. Thus, we have $\|a^n\| \leqslant \frac{C}{|\zeta|^n}$, so $\|a^n\|^{1/n} \leqslant \frac{C^{1/n}}{|\zeta|}$. Thus,

$$\limsup_{n\to\infty}\|a^n\|^{1/n} \leqslant \limsup_{n\to\infty}\frac{C^{1/n}}{|\zeta|}$$

$$\begin{aligned} &= \frac{1}{|\zeta|} \\ &= t. \end{aligned}$$

Since this holds for all $t > r(a)$, we get

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leqslant r(a).$$

Thus, we have

$$\begin{aligned} r(a) &\leqslant \liminf_{n\to\infty} \|a^n\|^{1/n} \\ &\leqslant \limsup_{n\to\infty} \|a^n\|^{1/n} \\ &\leqslant r(a). \end{aligned}$$

If A is nonunital, we consider the unitization, $\widetilde{A}$, and find

$$\begin{aligned} r(a) &= r(\iota_A(a)) \\ &= \lim_{n\to\infty} \left\|\iota_A(a)^n\right\|^{1/n} \\ &= \lim_{n\to\infty} \|\iota_A(a^n)\|^{1/n} \\ &= \lim_{n\to\infty} \|a^n\|^{1/n}. \end{aligned}$$

□

We can use the established fact that, for a normal element $b \in A$,

$$\|b\|^{2^k} = \left\|b^{2^k}\right\|$$

for all $k \geqslant 0$ to establish an important fact about the spectral radius of normal elements.

Proposition: Let A be a C^*-algebra. For any normal $b \in A$, we have

$$r(b) = \|b\|$$

Proof. We compute

$$\begin{aligned} r(b) &= \lim_{n\to\infty} \|b^n\|^{1/n} \\ &= \lim_{k\to\infty} \left\|b^{2^k}\right\|^{1/2^k} \\ &= \lim_{k\to\infty} (\|b\|)^{(2^k)(1/2^k)} \\ &= \|b\|. \end{aligned}$$

□

Proposition: Let A be a $*$-algebra with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that A is a C^*-algebra when equipped with both of these norms. Then, $\|\cdot\|_1 = \|\cdot\|_2$.

Proof. Let $a \in A$. We have

$$\begin{aligned} \|a\|_1^2 &= \|a^*a\|_1 \\ &= r(a^*a) \\ &= \|a^*a\|_2 \\ &= \|a\|_2^2. \end{aligned}$$

□

Exercise: Let a and b be similar elements in a unital C^*-algebra A. If A is normal, prove that $\|a\| \leqslant \|b\|$, with equality if b is normal as well.

Proof. Since a and b are similar elements, we must have that $\sigma(a) = \sigma(b)$, so $r(a) = r(b)$. Thus, we have

$$\begin{aligned} \|a\| &= r(a) \\ &= r(b) \\ &\leqslant \|b\|. \end{aligned}$$

□

Proposition: If $\varphi\colon A \to B$ is a $*$-homomorphism between C^*-algebras, then φ is contractive (i.e., $\|\varphi\|_{op} \leqslant 1$).

Proof. Assume A and B are unital, as well as φ. Let $x \in A$ be normal, so $\varphi(x)$ is normal too. Thus, we have $\sigma(\varphi(x)) \subseteq \sigma(x)$, so

$$\begin{aligned} \|\varphi(x)\| &= r(\varphi(x)) \\ &\leqslant r(x) \\ &= \|x\|. \end{aligned}$$

If $a \in A$ is arbitrary, then a^*a is normal, so

$$\begin{aligned} \|\varphi(a)\|^2 &= \|\varphi(a)^*\varphi(a)\| \\ &= \|\varphi(a^*a)\| \\ &\leqslant \|a^*a\| \\ &= \|a\|^2. \end{aligned}$$

In the general case, we know that a $*$-homomorphism extends to the unital $*$-homomorphism on the unitization, $\widetilde{\varphi}\colon \widetilde{A} \to \widetilde{B}$. Since A and B are C^*-algebras, the $*$-homomorphism $\widetilde{\varphi}\colon A^1 \to B^1$ is contractive, as

$$\begin{aligned} \|\varphi(a)\| &= \|(\varphi(a), 0)\|_1 \\ &= \|\widetilde{\varphi}(a, 0)\|_1 \\ &\leqslant \|(a, 0)\|_1 \\ &= \|a\|. \end{aligned}$$

□

Since the spectrum of any element in a Banach algebra is nonempty, we get the following structural result.

Theorem (Mazur's Theorem)**:** Let A be a unital Banach algebra such that $GL(A) = A \setminus \{0\}$. Then, $\varphi\colon \mathbb{C} \to A$, given by $\varphi(z) = z1_A$ is an isometric unital algebra isomorphism.

Proof. We only need show that φ is onto, as by construction the map is linear, unital, multiplicative, and isometric.

Let $a \in A$. We know that $\sigma(a) \neq \emptyset$, so there is some $\lambda \in \sigma(a)$ such that $a - \lambda 1_A \notin GL(A)$. Thus, $a - \lambda 1_A = 0$, so $a = \lambda 1_A = \varphi(\lambda)$. □

Corollary: If A is a unital, commutative, and simple Banach algebra, then $A \cong \mathbb{C}$.

Proof. Let $0 \neq a \in A$. Since A is commutative, $\text{ideal}(a) = \{xa \mid x \in A\}$. Since A is simple, $\text{ideal}(a) = A$, so $1_A \in A$, so a is invertible. By Mazur's theorem, we have $A \cong \mathbb{C}$. □

Remark: The above fails without a complete norm. The algebra of rational functions,

$$\mathbb{C}(z) = \left\{ \frac{p(z)}{q(z)} \;\middle|\; p, q \in \mathbb{C}[z], q \neq 0 \right\}$$

is unital, commutative, and every nonzero element is invertible (hence simple), but $\mathbb{C}(z)$ is not isomorphic to $\mathbb{C}$.

The Character Space

Recall that if A is a $\mathbb{C}$-algebra, a character on A is a nonzero algebra homomorphism $h\colon A \to \mathbb{C}$.[VI] We set $\Omega(A)$ to be the set of all characters on A.

The aim of this subsection will be to establish a correspondence between characters on a unital Banach algebra and maximal ideals in the algebra.

Proposition: Let A be a Banach algebra. Every character on A is bounded, with $\|h\|_{op} \leqslant 1$. If A is unital, then $\|h\|_{op} = 1$.

Proof. Suppose A is unital. Let $a \in A$. We know that $h(a) \in \sigma(a)$ for any $h \in \Omega(A)$. Since A is a Banach algebra, we know that $|h(a)| \leqslant \|a\|$. Thus, $\|h\|_{op} \leqslant 1$. Since $h(1_A) = 1$, we get $\|h\|_{op} = 1$.

If A is nonunital, then $\overline{h}\colon \widetilde{A} \to \mathbb{C}$ is also a character. We see that $\left\|\overline{h}\right\|_{op} = 1$, and for $a \in A$, we get

$$\begin{aligned} \|h(a)\| &= \left\|\overline{h}(a,0)\right\| \\ &\leqslant \|(a,0)\| \\ &= \|a\|, \end{aligned}$$

so $\|h\|_{op} \leqslant 1$. $\square$

We will endow the set $\Omega(A) \subseteq B_{A^*} \subseteq A^*$ with the weak* topology.

Definition. Let A be a Banach algebra with $\Omega(A) \neq \emptyset$. The pair $(\Omega(A), w^*)$ is called the character space of A.

Proposition: Let A be a Banach algebra.

(1) If A is unital, then the character space is compact Hausdorff.

(2) If A is nonunital, then the character space is LCH and its one-point compactification is homeomorphic to $\left(\Omega\left(\widetilde{A}\right), w^*\right)$.

Proof.

(1) We only need to show that $\Omega(A) \subseteq S_{A^*}$ is w^*-closed. Suppose $(h_\alpha)_\alpha \xrightarrow{w^*} h$, where $(h_\alpha)_\alpha \subseteq \Omega(A)$, and $h \in A^*$. If $a, b \in A$, then we have $(h_\alpha(ab))_\alpha \to h(ab)$, and also $(h_\alpha(ab)) = \left(h_\alpha(a)h_\beta(b)\right) \to h(a)h(b)$. We show the latter statement, using the fact that h is contractive. We have

$$\begin{aligned} \|h_\alpha(a)h_\alpha(b) - h(a)h(b)\| &= \|h_\alpha(a)h_\alpha(b) - h_\alpha(a)h(b)\| + \|h_\alpha(a)h(b) - h(a)h(b)\| \\ &\leqslant \|h_\alpha(a)\|\|h_\alpha(b) - h(b)\| + \|h_\alpha(a) - h(a)\|\|h(b)\| \\ &\leqslant \|a\|\|h_\alpha(b) - h(b)\| + \|h_\alpha(a) - h(a)\|\|h(b)\| \\ &\to 0. \end{aligned}$$

Since limits are unique, we have $h(ab) = h(a)h(b)$, so h is multiplicative. Since $h \neq 0$, as $h_\alpha(1_A) = 1 = h(1_A)$, we have $h \in \Omega(A)$. Thus, $\Omega(A) \subseteq S_{A^*}$ is w^*-closed, hence w^*-compact.

(2) The map $\Omega(A) \to \Omega\left(\widetilde{A}\right)$ that sends $h \mapsto \overline{h}$ is an injection. This map is a homeomorphism onto its range, as $(h_\alpha)_\alpha \xrightarrow{w^*} h$ if and only if $\left(\overline{h}_\alpha\right)_\alpha \xrightarrow{w^*} \overline{h}$. We know that $\Omega(A)$ is LCH. Since the one-point compactification of a LCH is unique up to homeomorphism, we have $\Omega(A)_\infty \cong \Omega\left(\widetilde{A}\right)$ are homeomorphic.

[VI]Such an algebra homomorphism is automatically surjective.

□

Remark: If A is nonunital, the character space is not necessarily w^*-closed. For instance, if $A = C_0(\mathbb{R})$, then the sequence of characters $\delta_n(f) = f(n)$ converges in w^* to 0, but 0 is not a character.

Proposition: Let A be a Banach algebra.

(1) If B is a subalgebra, then $\overline{B} \subseteq A$ is a subalgebra.

(2) If $I \subseteq A$ is an ideal, then $\overline{I} \subseteq A$ is an ideal.

(3) If A is unital, and $I \subsetneq A$ is a proper ideal, then $\overline{I} \subsetneq A$ is a proper ideal.

Proof. We will prove (3).

Suppose $I \subsetneq A$ is a proper ideal. Suppose toward contradiction that $\overline{I} = A$. Then, $1_A \in \overline{I}$, so there exists $x \in I$ with $\|1_A - x\| < 1$. We have $x = 1_A - (1_A - x) \in GL(A)$, so $1_A = x^{-1}x \in I$, so $I = A$, which is a contradiction. □

Remark: It is important for there to be a unit. For instance, $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R})$ is a proper ideal, but $\overline{C_c(\mathbb{R})} = C_0(\mathbb{R})$ is not proper.

Corollary: Let A be a unital Banach algebra, and suppose $M \subseteq A$ is a maximal ideal.

(1) M is closed.

(2) If A is commutative, then $A/M \cong C$ as Banach algebras.

Proof.

(1) Since $M \subseteq A$ is a proper ideal, as is $\overline{M}$, so $\overline{M} = M$ as M is maximal.

(2) Since A is unital and commutative, and M is closed, A/M is a unital and commutative Banach algebra. Since M is maximal, A/M is simple, so $GL(A/M) = A/M \setminus \{0\}$, so $A/M \cong \mathbb{C}$ by Mazur's theorem.

□

A unital Banach algebra may not have any nontrivial ideals. However, if A is commutative, and $A \neq \mathbb{C}$, then there are a lot of maximal ideals.

Theorem: Let A be a unital and commutative Banach algebra. There is a one to one correspondence between the collection of all maximal ideals in A,

$$\mathcal{M}(A) = \{M \subseteq A \mid M \text{ is a maximal ideal}\},$$

and the set of all characters on A, $\Omega(A)$, given by $h \leftrightarrow \ker(h)$.

Proof. We consider the map $\kappa\colon \Omega(A) \to \mathcal{M}(A)$, given by $h \mapsto \ker(h)$.

If $\ker(h_1) = \ker(h_2)$, then $h_1 = \alpha h_2$, but since characters are unital, we must have $h_1 = h_2$, hence the map is injective.

Let $M \in \mathcal{M}(A)$. Then, $A/M \cong \mathbb{C}$ as Banach algebras. Consider the sequence of mappings, $\pi\colon A \to A/M$, followed by $\varphi\colon A/M \to \mathbb{C}$, where π is the canonical quotient map and φ is the isometric isomorphism of Banach algebras. Then, the algebra homomorphism $h_M = \varphi \circ \pi$ is nonzero, as $h(a) \neq 0$ for all $a \in A/M \neq \emptyset$. Since $\ker(h_M) = M$, the map κ is onto. □

Remark: Some textbook authors refer to the space $(\Omega(A), w^*)$ as the maximal ideal space when A is a unital and commutative Banach algebra.

Theorem: Let A be a commutative Banach algebra.

(1) If A is unital, then $\Omega(A) \neq \emptyset$. Further, for each $a \in A$,

$$\sigma(a) = \{h(a) \mid h \in \Omega(A)\}.$$

(2) If A is nonunital, then for each $a \in A$,

$$\sigma(a) = \{h(a) \mid h \in \Omega(A)\} \cup \{0\}.$$

Proof.

(1) For any algebra A and character $h \in \Omega(A)$, we have $h(a) \in \sigma(a)$.

Let $\lambda \in \sigma(a)$, so $z = a - \lambda 1_A \notin GL(A)$. Consider the ideal generated by z, I, determined by

$$\text{ideal}(z) = \{xz \mid x \in A\}.$$

Since z is not invertible, $1_A \notin I$, so I is proper, and thus contained in a maximal ideal M. There is some $h \in \Omega(A)$ such that $\ker(h) = M$, so $h|_I = 0$, and $h(a - \lambda 1_A) = 0$, so $h(a) = \lambda$.

(2) If A is nonunital, then we know that

$$\Omega\left(\widetilde{A}\right) = \left\{\overline{h} \;\middle|\; h \in \Omega(A)\right\} \cup \{\pi\}.$$

If $a \in A$, we have

$$\begin{aligned} \sigma(a) &= \sigma(\iota_A(a)) \\ &= \{h(a) \mid h \in \Omega(A)\} \cup \{\pi(\iota_A(a))\} \\ &= \{h(a) \mid h \in \Omega(A)\} \cup \{0\}. \end{aligned}$$

□

Example. We have studied the characters of $C(X)$, where X is a compact Hausdorff space. We have shown that the map $x \overset{\delta}{\mapsto} \delta_x$, where $\delta_x(f) = f(x)$, is a bijection. We will show that δ is a homeomorphism when $\Omega(C(X))$ is endowed with the weak* topology.

If $(x_\alpha)_\alpha$ is a net in X, then since X is compact Hausdorff, it is completely regular, so for all $f \in C(X)$,

$$\begin{aligned} (x_\alpha)_\alpha \to x &\Leftrightarrow (f(x_\alpha))_\alpha \to f(x) \\ &\Leftrightarrow (\delta_{x_\alpha}(f)) \to \delta_x(f) \\ &\Leftrightarrow \delta_{x_\alpha} \xrightarrow{w^*} \delta_x. \end{aligned}$$

Now, we can classify all algebra homomorphisms $\varphi\colon C(X) \to C(Y)$, when X and Y are compact Hausdorff spaces.

Recall that if $\tau\colon Y \to X$ is a continuous function between compact Hausdorff spaces, we get a unital, contractive $*$-homomorphism $T_\tau\colon C(X) \to C(Y)$ given by $T_\tau(f) = f \circ \tau$, and if τ is injective, then T_τ is onto, and if τ is onto, then T_τ is isometric.

Proposition: Let X and Y be compact Hausdorff spaces, and suppose $\phi\colon C(X) \to C(Y)$ is a unital algebra homomorphism. Then, there exists a continuous function $\tau\colon Y \to X$ such that $\phi = T_\tau$. Consequently, ϕ is contractive and $*$-preserving. We also have ϕ is injective if and only if it is isometric.

Proof. Given $y \in Y$, the character δ_y composed with ϕ gives a character on $C(X)$, $\delta_y \circ \phi \colon C(X) \to \mathbb{C}$. Thus, there exists a unique $x \in X$ such that $\delta_y \circ \phi = \delta_x$. We have a map $\tau \colon Y \to X$ defined by $\tau(y) = x$ such that $\delta_y \circ \phi = \delta_{\tau(y)}$.

For all $f \in C(X)$ and $y \in Y$, we have

$$\begin{aligned} f \circ \tau(y) &= f(\tau(y)) \\ &= \delta_{\tau(y)}(f) \\ &= \delta_y \circ \phi(f) \\ &= \phi(f)(y), \end{aligned}$$

so $f \circ \tau = \phi(f)$.

To show that τ is continuous, let $(y_\alpha)_\alpha$ be a net in Y converging to y. Then, $\left(\delta_{y_\alpha}\right)_\alpha \xrightarrow{w^*} \delta_y$ in $(C(Y))^*$, so $\left(\delta_{y_\alpha} \circ \phi\right)_\alpha \xrightarrow{w^*} \delta_y \circ \phi$ in $(C(X))^*$. Thus, $\left(\delta_{\tau(y_\alpha)}\right)_\alpha \xrightarrow{w^*} \delta_{\tau(y)}$, so $(\tau(y_\alpha))_\alpha \to \tau(y)$ in X.

Since $\phi = T_\tau$, ϕ is a contractive $*$-homomorphism, and it is injective if and only if it is isometric. □

Example. We turn our attention to the algebra $\ell_1(\mathbb{Z})$ (where multiplication is convolution), and examine the character space of this algebra.

Let $z \in \mathbb{T}$. Consider $h_z(f) = \sum_{n \in \mathbb{Z}} f(n) z^n$. This series is absolutely convergent as $|z| = 1$ and $f \in \ell_1(\mathbb{Z})$. It is the case that h_z is linear, and $h_z \neq 0$, as $h_z(e_0) = 1$. If $f, g \in \ell_1(\mathbb{Z})$, then

$$\begin{aligned} h_z(f) h_z(g) &= \left(\sum_{k \in \mathbb{Z}} f(k) z^k\right)\left(\sum_{\ell \in \mathbb{Z}} g(l) z^l\right) \\ &= \sum_{k,l \in \mathbb{Z}} f(k) g(l) z^{k+l} \\ &= \sum_{n,l \in \mathbb{Z}} f(n-l) g(l) z^n && n = k+l \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} f(n-l) g(l)\right) z^n \\ &= \sum_{n \in \mathbb{Z}} f \cdot g(n) z^n \\ &= h_z(f \cdot g). \end{aligned}$$

Thus, h_z is a character. If $h_{z_1} = h_{z_2}$, then $z_1 = h_{z_1}(e_1) = h_{z_2}(e_1) = z_2$, so we have an injective map $\mathbb{T} \to \Omega(\ell_1(\mathbb{Z}))$, given by $z \mapsto h_z$.

We will show this map is onto. Let $h \in \Omega(\ell_1(\mathbb{Z}))$, and set $z = h(e_1)$. We claim that $|z| = 1$. Note that

$$\begin{aligned} 1 &= h(e_0) \\ &= h(e_1 \cdot e_{-1}) \\ &= h(e_1) h(e_{-1}). \end{aligned}$$

We have $|h(e_1)| \leqslant \|h\| \|e_1\| = 1$ and $|h(e_{-1})| \leqslant \|h\| \|e_{-1}\| = 1$, so $|h(e_1)| = 1$.

If $f \in \ell_1(\mathbb{Z})$, we write $f = \sum_{n \in \mathbb{Z}} f(n) e_1^n$ as a norm-convergent sum, so we have

$$h(f) = h\left(\sum_{n \in \mathbb{Z}} f(n) e_1^n\right)$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} f(n)h(e_1)^n \\
&= \sum_{n\in\mathbb{Z}} f(n)z^n \\
&= h_z(f).
\end{aligned}
$$

We claim the map $\mathbb{T} \to \Omega(\ell_1(\mathbb{Z}))$ is a homeomorphism. It suffices to show continuity, since $\mathbb{T}$ is compact and $\Omega(\ell_1(\mathbb{Z}))$ is Hausdorff.

Let $(z_n)_n \to z$ in $\mathbb{T}$. We show that $(h_{z_n})_n \xrightarrow{w^*} h_z$ in $\ell_1(\mathbb{Z})^*$. The set $\{e_k \mid k \in \mathbb{Z}\}$ is total, and $(h_{z_n})_n$ is bounded, so we only need show that $(h_{z_n}(e_k))_n \to h_z(e_k)$ for a fixed $k \in \mathbb{Z}$. Note that $h_w(e_k) = \sum_{n\in\mathbb{Z}} e_k(n)w^n = w^k$ for $w \in \mathbb{T}$, so

$$
\begin{aligned}
|h_{z_n}(e_k) - h_z(e_k)| &= \left|z_n^k \to z^k\right| \\
&\to 0.
\end{aligned}
$$

Thus, $\Omega(\ell_1(\mathbb{Z})) \cong \mathbb{T}$.

Example. Now, we turn our attention to the character space of the (full) group C^*-algebra, $C^*(\Gamma)$, where Γ is an abelian discrete group.

The unit circle $\mathbb{T}$ equipped with multiplication is an abelian group. A character on a discrete group Γ is a group homomorphism, $\chi\colon \Gamma \to \mathbb{T}$. We define

$$
\widehat{\Gamma} = \{\chi \mid \chi \text{ is a character on } \Gamma\}.
$$

We endow $\widehat{\Gamma}$ with the subspace topology inherited from $\prod_\Gamma \mathbb{T}$. We can see that $\widehat{\Gamma}$ is closed in this topology, hence compact.

If $\chi\colon \Gamma \to \mathbb{T}$ is a character on Γ, we set

$$
h_\chi(f) = \sum_{t\in\Gamma} f(t)\chi(t)
$$

as a finite sum. Note that $h_\chi(\delta_t) = \chi(t)$, where $t \in \Gamma$. We claim that h_χ is a character on the unital $*$-algebra $\mathbb{C}[\Gamma]$. It is the case that h_χ is linear. For $f, g \in \mathbb{C}[\Gamma]$, we compute

$$
\begin{aligned}
h_\chi(f\cdot g) &= \sum_{t\in\Gamma} (f\cdot g)(t)\chi(t) \\
&= \sum_{t\in\Gamma}\left(\sum_{s\in\Gamma} f(s)g\left(s^{-1}t\right)\right)\chi(t) \\
&= \sum_{s,t\in\Gamma} f(s)g\left(s^{-1}t\right)\chi\left(ss^{-1}t\right) \\
&= \sum_{s,t\in\Gamma} f(s)\chi(s)g\left(s^{-1}t\right)\chi\left(s^{-1}t\right) \\
&= \sum_{s,r\in\Gamma} f(s)\chi(s)g(r)\chi(r) \\
&= \left(\sum_{s\in\Gamma} f(s)\chi(s)\right)\left(\sum_{r\in\Gamma} g(r)\chi(r)\right) \\
&= h_\chi(f)h_\chi(g),
\end{aligned}
$$

and

$$\begin{aligned}
h_\chi(f^*) &= \sum_{t\in\Gamma} f^*(t)\chi(t)\\
&= \sum_{t\in\Gamma} \overline{f\left(t^{-1}\right)}\chi(t)\\
&= \sum_{s\in\Gamma} \overline{f(s)}\chi\left(s^{-1}\right)\\
&= \sum_{s\in\Gamma} \overline{f(s)\chi(s)}\\
&= \overline{\sum_{s\in\Gamma} f(s)\chi(s)}\\
&= \overline{h_\chi(f)}.
\end{aligned}$$

By the universal property of $C^*(\Gamma)$, the unital $*$-homomorphism h_χ extends to a unital $*$-homomorphism $h_\chi\colon C^*(\Gamma) \to \mathbb{C}$, a character on the C^*-algebra $C^*(\Gamma)$.

We have a map $\widehat{\Gamma} \to \Omega(C^*(\Gamma))$, given by $\chi \mapsto h_\chi$.

The map is injective, as if $\chi_1, \chi_2 \in \widehat{\Gamma}$ are characters on Γ with $h_{\chi_1} = h_{\chi_2}$, then $\chi_1(t) = h_{\chi_1}(\delta_t) = h_{\chi_2}(\delta_t) = \chi_2(t)$.

For surjectivity, let $h \in \Omega(C^*(\Gamma))$, and define $\chi_h\colon \Gamma \to \mathbb{C}$, given by $\chi_h(t) = h(\delta_t)$. Since h is multiplicative, χ_h is a character on Γ.

We claim that $h_{\chi_h} = h$. To see this, if $t \in \Gamma$, then

$$\begin{aligned}
h_{\chi_h}(\delta_t) &= \chi_h(t)\\
&= h(\delta_t),
\end{aligned}$$

so by linearity, $h_{\chi_h} = h$ on $\mathbb{C}[\Gamma]$, and by continuity, they agree on $C^*(\Gamma)$.

If $(\chi_\alpha)_\alpha$ is a net of characters in Γ converging to $\chi \in \widehat{\Gamma}$, then for all $t \in \Gamma$, we have $h_{\chi_\alpha}(\delta_t) = \chi_\alpha(t) \to \chi(t) = h_\chi(\delta_t)$. Thus, $\left(h_{\chi_\alpha}\right)_\alpha \xrightarrow{w^*} h_\chi$ in $\Omega(C^*(\Gamma))$, and since $\widehat{\Gamma}$ is compact and $\Omega(C^*(\Gamma))$ is Hausdorff.

Thus, we have that the character space $\Omega(C^*(\Gamma))$ is homeomorphic to $\widehat{\Gamma}$, the character space of Γ.

The Gelfand Transform

If A is unital and commutative, we know that the character space $\Omega(A)$ is nonempty and maps the spectrum of each element.

Definition. Let A be a commutative Banach algebra with $\Omega(A) \neq \emptyset$, and let $a \in A$. The map $\hat{a}\colon \Omega(A) \to \mathbb{C}$, given by $\hat{a}(h) = h(a)$ is known as the Gelfand transform of a.

Proposition: Let A be a commutative Banach algebra, with $\Omega(A) \neq \emptyset$. Let $a \in A$. We have the following:

(1) $\hat{a} \in C_0(\Omega(A))$;[VII]

(2) if A is unital, then $\operatorname{Ran}(\hat{a}) = \sigma(a)$, and if A is nonunital, $\operatorname{Ran}(\hat{a}) \cup \{0\} = \sigma(a)$;

(3) $\|\hat{a}\|_u = r(a) \leqslant \|a\|$;

[VII]Note that if $\Omega(A)$ is compact, then $C_0(\Omega(A)) = C(\Omega(A))$.

(4) the Gelfand transform of A, $\gamma_A \colon A \to C_0(\Omega(A))$, given by $\gamma_A(a) = \hat{a}$, is a contractive algebra homomorphism, and if A is unital, then γ_A is unital.

Proof.

(1) Continuity follows from the definitions. We see that

$$\begin{aligned}(h_\alpha)_\alpha \xrightarrow{w^*} h &\Rightarrow (h_\alpha(a))_\alpha \to h(a)\\ &\Rightarrow (\hat{a}(h_\alpha))_\alpha \to \hat{a}(h).\end{aligned}$$

If $\varepsilon > 0$, then the continuity of $\hat{a}$ implies that $K_\varepsilon = \{h \in \Omega(A) \mid |\hat{a}(h)| \geqslant \varepsilon\} \subseteq B_{A^*}$ is weak*-closed, so it is weak*-compact. If $h \in \Omega(A) \setminus K_\varepsilon$, then $|\hat{a}(h)| < \varepsilon$, so $\hat{a}$ vanishes at infinity.

(2) If A is unital, we know that

$$\begin{aligned}\operatorname{Ran}(\hat{a}) &= \{\hat{a}(h) \mid h \in \Omega(A)\}\\ &= \{h(a) \mid h \in \Omega(A)\}\\ &= \sigma(a),\end{aligned}$$

and similarly for the case where A is nonunital.

(3) We compute

$$\begin{aligned}\|\hat{a}\|_u &= \sup_{h \in \Omega(A)} |\hat{a}(h)|\\ &= \sup_{h \in \Omega(A)} |h(a)|\\ &= \sup_{\lambda \in \sigma(a)} |\lambda|\\ &= r(a)\\ &\leqslant \|a\|.\end{aligned}$$

(4) From (2), we see that γ_A is contractive. To see multiplicativity, we have

$$\begin{aligned}\widehat{ab}(h) &= h(ab)\\ &= h(a)h(b)\\ &= \hat{a}(h)\hat{b}(h)\\ &= \hat{a}\hat{b}(h),\end{aligned}$$

so $\gamma_A(ab) = \gamma_A(a)\gamma_A(b)$.

$\square$

We are interested in seeing if the Gelfand map, γ_A, is $*$-preserving, in the case where A is a Banach $*$-algebra or C^*-algebra.

Proposition: Let A be a commutative Banach $*$-algebra with $\Omega(A) \neq \emptyset$. Let $\gamma_A \colon A \to C_0(\Omega(A))$ be the Gelfand map. The following are equivalent:

(i) γ_A is $*$-preserving;

(ii) if $a \in A_{\text{s.a.}}$, then $\operatorname{Ran}(\hat{a}) \subseteq \mathbb{R}$;

(iii) every $h \in \Omega(A)$ is $*$-preserving;

(iv) for every $h \in \Omega(A)$, $h(A_{\text{s.a.}}) \subseteq \mathbb{R}$.

If A satisfies any of these conditions, we say A is symmetric.

Proof. We see that γ_A is $*$-preserving if and only if $\gamma_A(A_{\text{s.a.}}) \subseteq C(\Omega(A))_{\text{s.a.}} = C(\Omega(A), \mathbb{R})$. The equivalence of (i) and (ii) follows, and similarly the equivalence between (iii) and (iv).

We only need to show the equivalence between (i) and (iii). Given $a \in A$, we have

$$\begin{aligned} \gamma_A(a^*) = \gamma_A(a)^* &\Leftrightarrow \widehat{a^*}(h) = \overline{\hat{a}(h)} \\ &\Leftrightarrow h(a^*) = \overline{h(a)}. \end{aligned}$$

□

Commutative C^*-algebras are symmetric, which enables us to realize commutative C^* algebras as continuous function spaces.

Proposition: If A is a commutative C^*-algebra, then A is symmetric.

Additionally, $\text{Ran}(\gamma_A) \subseteq C_0(\Omega(A))$ is a $*$-subalgebra. If A is unital, then $\text{Ran}(\gamma_A) \subseteq C(\Omega(A))$ is a unital $*$-subalgebra.

Proof. Let A be unital with $h \in \Omega(A)$, and let $a \in A_{\text{s.a.}}$. We will show that $h(a) \in \mathbb{R}$.

Write $h(a) = \alpha + i\beta$, and we will show that $\beta = 0$. Fix $t \in \mathbb{R}$, and define $z = a + it1_A$. Since $a = a^*$,

$$\begin{aligned} z^*z &= (a + it1_A)^*(a + it1_A) \\ &= a^*a + ita^* - ita + t^2 1_A \\ &= a^2 + t^2 1_A. \end{aligned}$$

Notice that

$$\begin{aligned} |h(z)|^2 &= |h(a) + it|^2 \\ &= |\alpha + i\beta + it|^2 \\ &= |\alpha|^2 + |\beta|^2 + 2\beta t + t^2, \end{aligned}$$

and

$$\begin{aligned} |h(z)|^2 &\leqslant \|z\|^2 \\ &= \|z^*z\| \\ &= \left\|a^2 + t^2 1_A\right\| \\ &\leqslant \|a\|^2 + t^2. \end{aligned}$$

Thus, we find

$$|\alpha|^2 + |\beta|^2 + 2\beta t \leqslant \|a\|^2.$$

Since this inequality holds for all $t \in \mathbb{R}$, this is only possible if $\beta = 0$.

If A is nonunital, we let $a \in A_{\text{s.a.}}$, and $h \in \Omega(A)$. Then, $\iota_A(a) \in \widetilde{A}_{\text{s.a.}}$, and $\overline{h} \in \Omega\left(\widetilde{A}\right)$, so

$$\begin{aligned} h(a) &= \overline{h}(\iota_A(a)) \\ &\in \mathbb{R}, \end{aligned}$$

so A is symmetric.

Since γ_A is a $*$-homomorphism, we see that $\text{Ran}(\gamma_A)$ is a $*$-subalgebra, which is unital if A is unital. □

Now, we are interested in understanding when the Gelfand map is isometric. Note that if $f \in C_0(\Omega)$ for some LCH space Ω, then for any $n \geqslant 1$, we have

$$\begin{aligned}\|f\|_u^n &= \left(\sup_{x\in\Omega}|f(x)|\right)^n \\ &= \sup_{x\in\Omega}(|f(x)|)^n \\ &= \sup_{x\in\Omega}\left|(f(x))^n\right| \\ &= \|f^n\|_u.\end{aligned}$$

Proposition: Let A be a commutative Banach algebra.

(1) If $a \in A$, then $\|\gamma_A(a)\|_u = \|a\|$ if and only if $\left\|a^{2^k}\right\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$.

(2) If A is a C^*-algebra, then γ_A is an isometry, and $\operatorname{Ran}(\gamma_A) \subseteq C_0(\Omega(A))$ is a C^*-subalgebra. If A is unital, then $\operatorname{Ran}(\gamma_A) \subseteq C(\Omega(A))$ is a unital C^*-subalgebra.

Proof.

(1) If $\|\gamma_A(a)\|_u = \|a\|$, then

$$\begin{aligned}\left\|a^{2^k}\right\| &\leqslant \|a\|^{2^k} \\ &= \|\gamma_A(a)\|_u^{2^k} \\ &= \left\|\gamma(a)^{2^k}\right\|_u \\ &= \left\|\gamma_A\left(a^{2^k}\right)\right\| \\ &\leqslant \left\|a^{2^k}\right\|,\end{aligned}$$

where we used the aforementioned property of the norm on $C_0(\Omega)$ and the contractiveness of γ_A. Thus, $\left\|a^{2^k}\right\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$.

Conversely, suppose $\left\|a^{2^k}\right\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$. By the spectral radius formula, we have

$$\begin{aligned}\|\gamma_A(a)\|_u &= r(a) \\ &= \lim_{n\to\infty}\|a^n\|^{1/n} \\ &= \lim_{k\to\infty}\left\|a^{2^k}\right\|^{2^{-k}} \\ &= \lim_{k\to\infty}\left(\|a\|^{2^k}\right)^{2^{-k}} \\ &= \|a\|.\end{aligned}$$

(2) Let $a \in A$, $b = a^*a \in A_{\text{s.a.}}$. Since A is symmetric, we get

$$\begin{aligned}\|a\|^2 &= \|a^*a\| \\ &= \|b\| \\ &= \|\gamma_A(b)\| \\ &= \|\gamma_A(a^*a)\| \\ &= \|\gamma_A(a)^*\gamma_A(a)\| \\ &= \|\gamma_A(a)\|^2,\end{aligned}$$

so γ_A is isometric. It is the case that the range of γ_A is a C^*-subalgebra of $C_0(\Omega(A))$.

□

Corollary: If A is a commutative C^*-algebra, then the characters on A separate points.

Proof. If $a_1, a_2 \in A$ with $h(a_1) = h(a_2)$ for all $h \in \Omega(A)$, then

$$\begin{aligned} \hat{a}_1(h) = \hat{a}_2(h) &\Leftrightarrow \hat{a}_1 = \hat{a}_2 \\ &\Leftrightarrow \gamma_A(a_1) = \gamma_A(a_2) \\ &\Leftrightarrow a_1 = a_2. \end{aligned}$$

□

Theorem (Gelfand–Naimark)**:** Let A be a commutative C^*-algebra. The gelfand map $\gamma_A : A \to C_0(\Omega(A))$ is an isometric $*$-isomorphism. If A is unital, then $\gamma_A : A \to C(\Omega(A))$ is an isometric unital $*$-isomorphism.

Proof. We only need to show that γ_A is onto. For this, we will use the Stone–Weierstrass theorem.

We start by claiming that $\operatorname{Ran}(\gamma_A)$ separates points. Suppose $h_1, h_2 \in \Omega(A)$ are distinct. Then, there exists some $a \in A$ such that $h_1(a) = h_2(a)$, so $\hat{a}(h_1) \neq \hat{a}(h_2)$.

Next, we show that $\operatorname{Ran}(\gamma_A)$ has no zeros. If $h \in \Omega(A)$, then $h \neq 0$, so there is some $a \in A$ with $h(a) \neq 0$, meaning $\hat{a}(h) \neq 0$.

Thus, by the Stone–Weierstrass theorem, we have $\operatorname{Ran}(\gamma_A) = C_0(\Omega(A))$. □

Example. Let Γ be a discrete abelian group. We have shown that every character χ on Γ gives rise to a character h_χ on the full group C^*-algebra, $C^*(\Gamma)$, and that the map $\chi \mapsto h_\chi$ is a homeomorphism.

Composing with the Gelfand map, $\gamma\colon C^*(\Gamma) \to C(\Omega(C^*(\Gamma)))$ gives the evaluation isomorphism,

$$\delta_t \mapsto \mathrm{ev}_t : \chi \mapsto \chi(t).$$

We can give a concrete example. Consider the homeomorphism $\mathbb{T} \to \widehat{\mathbb{Z}}$. We get an isometric $*$-isomorphism, $C\left(\widehat{\mathbb{Z}}\right) \to C(\mathbb{T})$. Composing with the evaluation isomorphism, we have the isometric $*$-isomorphism,

$$\sum_{n\in\mathbb{Z}} c_n\delta_n \mapsto \sum_{n\in\mathbb{Z}} c_n z^n,$$

where the sums are finite.

A useful consequence of the Gelfand–Naimark theorem is spectral permanence.

If A is a unital subalgebra of a unital algebra B, and $a \in A$, then we can specify $\sigma_A(a)$ and $\sigma_B(a)$, where

$$\begin{aligned} \sigma_B(a) &= \{\lambda \in \mathbb{C} \mid a - \lambda 1_B \notin GL(B)\} \\ \sigma_A(a) &= \{\lambda \in \mathbb{C} \mid a - \lambda 1_B \notin GL(A)\}. \end{aligned}$$

It can be seen that $\sigma_B(a) \subseteq \sigma_A(a)$, as $GL(A) \subseteq GL(B) \cap A$ always holds.

In a C^*-algebra, though, this becomes an equality.

Proposition (Spectral Permanence)**:** Let B be a unital C^*-algebra, and suppose $A \subseteq B$ is a unital C^*-algebra of B.

(1) If $z \in A \cap GL(B)$, then $z \in GL(A)$.

(2) For every $a \in A$, $\sigma_A(a) = \sigma_B(A)$.

Proof. Let $x \in A \cap GL(B)$ be self-adjoint. We need to show that $x^{-1} \in A$. We define $C = C^*\left(x, x^{-1}\right) \subseteq B$ and $D = C^*(1_B, x) \subseteq A$.

Since x is normal, both C and D are unital and commutative C^*-algebras with $D \subseteq C$. We will use the following commutative diagram to organize the argument.

$$\begin{tikzcd} D \arrow[r, "\iota"] \arrow[d, "\gamma_D"'] & C \arrow[d, "\gamma_C"] \\ C(\Omega(D)) \arrow[r, "\pi"] & C(\Omega(C)) \end{tikzcd}$$

We have ι is inclusion, γ_C and γ_D the isometric $*$-isomorphism Gelfand maps, and $\pi\colon C(\Omega(D)) \to C(\Omega(C))$, defined as $\pi = \gamma_C \circ \iota \circ \gamma_D^{-1}$, is a unital and isometric $*$-homomorphism. Consequently, $\operatorname{Ran}(\pi)$ is a unital C^*-subalgebra of $C(\Omega(C))$.

We will show that $\operatorname{Ran}(\pi)$ separates the points of $\Omega(C)$, so we may apply Stone–Weierstrass. Suppose $h_1, h_2 \in \Omega(C)$ are distinct characters. We claim that $\hat{x}(h_1) \neq \hat{x}(h_2)$. If this were not the case, then $h_1(x) = h_2(x)$, and

$$\begin{aligned} h_1\left(x^{-1}\right) &= \frac{1}{h_1(x)} \\ &= h_2\left(x^{-1}\right) \\ &= \frac{1}{h_2(x)}, \end{aligned}$$

so since C is generated by x and x^{-1}, multiplicativity and continuity of h_1, h_2 imply that $h_1 = h_2$, which is a contradiction. Since $\hat{x} \in \operatorname{Ran}(\pi)$, we have that $\operatorname{Ran}(\pi)$ separates points, so $\operatorname{Ran}(\pi) = C(\Omega(C))$. Since π is surjective, a diagram chase shows that ι is surjective, so $x^{-1} \in D \subseteq A$.

For the general case, let $z \in A \cap GL(B)$, and set $x = z^*z$. Since $GL(B)$ and A are both closed under involution and multiplication, we see that $x \in A \cap GL(B)$. Thus, $x^{-1} \in A$. Since $x^{-1}z^*$ is a left-inverse for z, and z is invertible, we must have $z^{-1} \in A$.

For the proof of (2), we know that $\sigma_B(a) \subseteq \sigma_A(a)$. If $\lambda \in \rho_B(a)$, then $a - \lambda 1_B \in GL(B)$, so $a - \lambda 1_B \in GL(A)$, so $\lambda \in \rho_A(a)$. Thus, we get $\rho_B(a) \subseteq \rho_A(a)$. Taking complements, we get $\sigma_A(a) \subseteq \sigma_B(a)$. ◻

Proposition: Let A and B be C^*-algebras. A $*$-homomorphism, $\phi\colon A \to B$, is injective if and only if it is isometric.

Proof. We assume A, B, and ϕ are unital. Let $a \in A$, and set $x = a^*a$. Then, $\phi(x)$ is a normal element in B. Set $C = C^*(x, 1_A) \subseteq A$, and $D = C^*(\phi(x), 1_B) \subseteq B$, which are unital and commutative C^*-algebras. By continuity, we see that $\phi|_C(C) \subseteq D$.

We will use the following commutative diagram to organize the argument.

$$\begin{tikzcd} C \arrow[r, "\phi|_C"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ C(\Omega(C)) \arrow[r, "\pi"] & C(\Omega(D)) \end{tikzcd}$$

Here, γ_C and γ_D are the isometric $*$-isomorphic Gelfand maps, and $\pi = \gamma_D \circ \phi|_C \circ \gamma_C^{-1}$. Since π is an injective unital homomorphism between continuous function spaces, π is isometric. We see that

$$\begin{aligned} \|x\| &= \|\gamma_C(x)\| \\ &= \|\pi(\gamma_C(x))\| \end{aligned}$$

$$\begin{aligned}
&= \|\gamma_D(\phi(x))\| \\
&= \|\phi(x)\|,
\end{aligned}$$

so

$$\begin{aligned}
\|a\|^2 &= \|a^*a\| \\
&= \|x\| \\
&= \|\phi(x)\| \\
&= \|\phi(a^*a)\| \\
&= \|\phi(a)^*\phi(a)\| \\
&= \|\phi(a)\|^2,
\end{aligned}$$

meaning ϕ is isometric.

In the non-unital case, we unitize ϕ to yield the unital $*$-homomorphism $\widetilde{\phi}\colon \widetilde{A} \to \widetilde{B}$. Since ϕ is injective, so is $\widetilde{\phi}$, and $\widetilde{\phi}$ is isometric. Since $\widetilde{\phi} \circ \iota_A = \phi$, and ι_A is isometric, it follows that ϕ is isometric. □

We may now consider the Gelfand transform of $\ell_1(\mathbb{Z})$ (where multiplication is given by convolution).

Example. We have identified the character space $\Omega(\ell_1(\mathbb{Z}))$ with $\mathbb{T}$. For a given z, the Gelfand transform is

$$\begin{aligned}
\gamma(f)(h_z) &= \hat{f}(h_z) \\
&= h_z(f) \\
&= \sum_{n\in\mathbb{Z}} f(n)z^n \\
&:= \hat{f}(z).
\end{aligned}$$

Recall that if $\varphi \in C(\mathbb{T})$, the Fourier transform of φ is the function

$$\check{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{it}\right)e^{-int}\,dt.$$

The formal Fourier series of φ is the power series

$$\varphi \sim \sum_{n\in\mathbb{Z}} \check{\varphi}(n)z^n.$$

Suppose $\varphi \in C(\mathbb{T})$ with $\varphi = \hat{f}$ for some $f \in \ell_1(\mathbb{Z})$. The Fourier coefficient of φ is

$$\begin{aligned}
\check{\varphi}(n) &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{it}\right)e^{-int}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \hat{f}\left(e^{it}\right)e^{-int}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k\in\mathbb{Z}} f(k)e^{ikt}\right)e^{-int}\,dt \\
&= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} f(k)\int_0^{2\pi} e^{i(k-n)t}\,dt \\
&= f(n).
\end{aligned}$$

Thus, $\check{f} = \check{\varphi} = f$ as functions on $\mathbb{Z}$.

We claim that the range of the Gelfand transform is

$$C = \left\{ \varphi \in C(\mathbb{T}) \;\middle|\; \sum_{n\in\mathbb{Z}} \check{\varphi}(n) z^n \text{ converges absolutely on } \mathbb{T} \right\}.$$

If $\varphi \in \gamma(\ell_1(\mathbb{Z}))$, then $\varphi = \hat{f}$ for some $f \in \ell_1(\mathbb{Z})$, and we computed that $\check{\varphi}(n) = f(n)$ for all $n \in \mathbb{Z}$, so $\sum_{n\in\mathbb{Z}} \check{\varphi}(n)z^n = \sum_{n\in\mathbb{Z}} f(n)z^n$ converges absolutely for all $z \in \mathbb{T}$.

Conversely, if the Fourier series of φ converges absolutely, then by taking $z = 1$, we get that $(\check{\varphi}(n))_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$. Set $f(n) = \check{\varphi}(n)$, and observe

$$\begin{aligned}\check{f}(z) &= \hat{f}(h_z)\\ &= h_z(f)\\ &= \sum_{n\in\mathbb{Z}} f(n)z^n\\ &= \sum_{n\in\mathbb{Z}} \check{\varphi}(n)z^n\\ &= \varphi(z).\end{aligned}$$

Continuous Functional Calculus

Now, we may use the Gelfand–Naimark theorem to apply continuous functions to elements of C^*-algebras.

Fix a commutative unital Banach algebra A. If $a \in A$, we know that the function $\hat{a}\colon \Omega(A) \to \sigma(a)$, given by $\hat{a}(h) = h(a)$ is continuous and onto when $\Omega(A)$ is endowed with the weak* topology. By duality, we get an injective unital $*$-homomorphism

$$T_{\hat{a}}\colon C(\sigma(a)) \to C(\Omega(A)),$$

given by $T_{\hat{a}} = f \circ \hat{a}$.

We are interested in the cases where $T_{\hat{a}}$ is surjective, which requires $\hat{a}$ to be injective.

Proposition: Let A be a commutative and unital Banach algebra. Let $a \in A$. The map $\hat{a}\colon \Omega(A) \to \sigma(a)$ is injective if any of the following conditions hold:

(i) A is generated by a and 1_A,

$$A = \overline{\{p(a) \mid p \in \mathbb{C}[x]\}}^{\|\cdot\|};$$

(ii) $a \in GL(A)$, and A is generated as a Banach algebra by a and a^{-1}, that is

$$A = \overline{\left\{ \sum_{n=-N}^{N} c_n a^n \;\middle|\; N \in \mathbb{N}, c_n \in \mathbb{C} \right\}}^{\|\cdot\|};$$

(iii) A is a C^*-algebra generated by a and 1_A, that is $A = C^*(a, 1_A)$.

Proof.

(i) If $\hat{a}(h_1) = \hat{a}(h_2)$, where $h_1, h_2 \in \Omega(A)$, then $h_1(a) = h_2(a)$. Since h_1, h_2 are multiplicative and linear, we have $h_1(p(a)) = h_2(p(a))$ for all $p \in \mathbb{C}[x]$. Since h_1, h_2 are continuous and agree on a dense subset, $h_1 = h_2$.

(ii) Assuming $h_1, h_2 \in \Omega(A)$ with $h_1(a) = h_2(a)$, since a is invertible, we know that $h_i(a) \neq 0$ and $h_i\left(a^{-1}\right) = h_i(a)^{-1}$ for each i. Thus, $h_1\left(a^{-1}\right) = h_2\left(a^{-1}\right)$. Since characters are multiplicative and linear, we see that $h_1(q(a)) = h_2(q(a))$ for all Laurent polynomials $q(z) = \sum_{n=-N}^{N} c_n z^n$. By continuity, $h_1 = h_2$.

(iii) Characters on a C*-algebra are always self-adjoint, so if $h_1(a) = h_2(a)$, then $\overline{h_1(a)} = \overline{h_2(a)}$. Multiplicativity means $h_1(w) = h_2(w)$ for all words w, and by linearity, $h_1 = h_2$ on $\operatorname{span}(W)$. By continuity, $h_1 = h_2 = C^*(a, 1_A) = \overline{\operatorname{span}}(W)$.

□

Corollary: Let A be a C*-algebra. Suppose $a \in A$ is normal.

(1) If A is unital, and $C^*(a, 1_A) = A$, then

$$T_{\hat{a}}(f) = f \circ \hat{a}$$

is a unital isometric $*$-isomorphism.

(2) If A is nonunital, and $C^*(a) = A$, then $C^*\left(a, 1_{\widetilde{A}}\right) = \widetilde{A}$, and the unital isometric $*$-isomorphism

$$T_{\hat{a}} \colon C(\sigma(a)) \to C\left(\Omega\left(\widetilde{A}\right)\right)$$

restricts to a $*$-isomorphism

$$\{f \in C(\sigma(a)) \mid f(0) = 0\} \to C_0(\Omega(A)).$$

Proof.

(1) Since the Gelfand transform $\hat{a} \colon \Omega(A) \to \sigma(a)$ is continuous and onto, $\hat{a}$ is injective, hence a homeomorphism. By duality, $T_{\hat{a}} \colon C(\sigma(a)) \to C(\Omega(A))$ is a unital isometric $*$-homomorphism.

(2) Since A is nonunital, and $\widetilde{A} = C^*\left(a, 1_{\widetilde{A}}\right)$, we have that $0 \in \sigma(a)$, and $\Omega\left(\widetilde{A}\right) = \Omega(A) \cup \{\pi\}$. We identify $C_0(\Omega(A))$ with the ideal $\left\{g \in C\left(\Omega\left(\widetilde{A}\right)\right) \,\middle|\, g(\pi) = 0\right\} \subseteq C\left(\Omega\left(\widetilde{A}\right)\right)$.

Note that if $f \in C(\sigma(a))$ with $f(0) = 0$, then

$$\begin{aligned} T_{\hat{a}}(f)(\pi) &= f \circ \hat{a}(\pi) \\ &= f(\pi(a)) \\ &= f(0) \\ &= 0, \end{aligned}$$

so $T_{\hat{a}} \in C_0(\Omega(A))$. Conversely, if $g(\pi) = 0$ and $g = T_{\hat{a}}(f)$ for some $f \in C(\sigma(a)q)$, then

$$\begin{aligned} f(0) &= f(\pi(a)) \\ &= f \circ \hat{a}(\pi) \\ &= T_{\hat{a}}(f)(\pi) \\ &= g(\pi) \\ &= 0. \end{aligned}$$

□

Definition (Continuous Functional Calculus)**.** Let B be a unital C*-algebra, and suppose $a \in B$ is normal. Let $A = C^*(a, 1_B)$ be the commutative unital C*-subalgebra B generated by a. The functional calculus at a is the isometric $*$-isomorphism $\phi_a \colon C(\sigma(a)) \to A$, given by $\phi_a = \gamma_A^{-1} \circ T_{\hat{a}}$.

We will write $f(a)$ for $\phi_a(f)$ for $f \in C(\sigma(a))$.

Theorem (Properties of the Continuous Functional Calculus)**:** Let B be a unital C^*-algebra, and let $a \in B$ be normal. Set $A = C^*(a, 1_B)$, and let

$$\phi_a \colon C(\sigma(a)) \to A$$

be the continuous functional calculus. The following hold:

(1) $\phi_a(p) = p(a)$ for all polynomials $\mathbb{C}[x]$ — in particular, $\phi_a(\iota) = a$ and $\phi_a(\mathbb{1}_{\sigma(a)}) = 1_B$, where $\iota \colon \sigma(a) \to \mathbb{C}$ is inclusion;

(2) $\phi_a(f) = f(a)$ is a normal element in A for every $f \in C(\sigma(a))$;

(3) $\sigma(f(a)) = f(\sigma(a))$ for all $f \in C(\sigma(a))$ (known as spectral mapping);

(4) if $f \in C(\sigma(a))$ and $g \in C(\sigma(f(a)))$, then $g \circ f \in C(\sigma(a))$, and $g(f(a)) = g \circ f(a)$;

(5) if $\pi \colon B \to C$ is a unital $*$-homomorphism between unital C^*-algebras, then
   (i) $\sigma(\pi(a)) \subseteq \sigma(a)$;
   (ii) $\pi(a)$ is normal and if $f \in C(\sigma(a))$, then $f(\pi(a)) = \pi(f(a))$;

(6) if $0 \in \sigma(a)$ and $f \in C(\sigma(a))$ with $f(0) = 0$, o4 if $0 \notin \sigma(a)$, then $\phi_a(f) \in C^*(a)$.

Proof.

(1) Since ϕ_a is a unital morphism, $\phi_a(\mathbb{1}_{\sigma(a)}) = 1_B$. Next, we note that $T_{\hat{a}}(\iota) = \iota \circ \hat{a} = \hat{a}$, and $\gamma_A^{-1}(\hat{a}) = a$. Thus, $\phi_a(\iota) = a$. Since ϕ_a is a homomorphism, we see that $\phi_a(p) = p(a)$.

(2) The image of any normal element under a $*$-homomorphism is normal.

(3) Since ϕ_a is an algebra isomorphism, we know that $\sigma(\phi_a(f)) = \sigma(f)$ for all $f \in C(\sigma(a))$. In particular, we have

$$\begin{aligned}\sigma(f(a)) &= \sigma(\phi_a(f))\\ &= \sigma(f)\\ &= \mathrm{Ran}(f)\\ &= f(\sigma(a)).\end{aligned}$$

(4) We note that for any character $h \in \Omega(A)$ and any continuous $k \in C(\sigma(a))$, that $h(k(a)) = k(h(a))$. This follows from the definition of the continuous functional calculus:

$$\begin{aligned}h(k(a)) &= h(\phi_a(k))\\ &= \widehat{\phi_a(k)}(h)\\ &= \gamma_A(\phi_a(k))(h)\\ &= (\gamma_A \circ \phi_a)(k)(h)\\ &= T_{\hat{a}}(k)(h)\\ &= k \circ \hat{a}(h)\\ &= k(h(a)).\end{aligned}$$

We know that $f(a)$ is normal, so $f(a)$ admits the continuous functional calculus, $\phi_{f(a)} \colon C(\sigma(f(a))) \to C^*(f(a), 1_B)$. Thus, if $g \colon \sigma(f(a)) \to \mathbb{C}$ is continuous, then $g(f(a)) = \phi_{f(a)}(g)$ is a valid expression. Since $\sigma(f(a)) = f(\sigma(a))$, we have $g \circ f \colon \sigma(a) \to \mathbb{C}$ is well-defined and continuous, defined by $g \circ f(a) = \phi_a(g \circ f)$.

Let $h \in \Omega(A)$ be arbitrary. Using the above expression with $k = g \circ f$, we get

$$h(g \circ f(a)) = g \circ f(h(a)).$$

Replacing a with $f(a)$ in the expression $h(k(a)) = k(h(a))$, and setting $k = g \in C(\sigma(f(a)))$, we get

$$\begin{aligned} h(g(f(a))) &= g(h(f(a))) \\ &= g(f(h(a))) \\ &= g \circ f(h(a)). \end{aligned}$$

Thus, we get $h(g(f(a))) = h(g \circ f(a))$. Since both $g \circ f(a)$ and $g(f(a))$ belong to A, and since characters separate points, we must have $g \circ f(a) = g(f(a))$.

(5) Since normal elements are preserved under $*$-homomorphisms, we have that $\pi(a)$ is normal in C. The C^*-subalgebra, $A_1 = C^*(\pi(a), 1_C) \subseteq C$ is thus unital and commutative. We thus have $\sigma(\pi(a)) \subseteq \sigma(a)$. Dualizing the inclusion map $\iota\colon \sigma(\pi(a)) \to \sigma(a)$, we get the restriction map $\rho\colon C(\sigma(a)) \to C(\sigma(\pi(a)))$.

The unital $*$-homomorphism $\pi\colon A \to A_1$ induces an adjoint map $\pi^*\colon \Omega(A_1) \to \Omega(A)$, given by $\pi^*(h) = h \circ \pi$. Since π^* consists of a character composed with a unital $*$-homomorphism, it is also a character. We see that π^* is continuous between these character spaces. If $(h_\alpha)_\alpha \to h$ in $\Omega(A_1)$, then

$$\begin{aligned} (h_\alpha)_\alpha \xrightarrow{w^*} h &\Rightarrow (h_\alpha(z))_\alpha \to h(z) && z \in A_1 \\ &\Rightarrow (h_\alpha(\pi(x)))_\alpha \to h(\pi(x)) && x \in A \\ &\Rightarrow (h_\alpha \circ \pi(x))_\alpha \to h \circ \pi(x) \\ &\Rightarrow \pi^*(h_\alpha) \xrightarrow{w^*} \pi^*(h). \end{aligned}$$

Dualizing $\pi^*\colon \Omega(A_1) \to \Omega(A)$, we get the unital $*$-homomorphism $T_{\pi^*}\colon C(\Omega(A)) \to C(\Omega(A_1))$. The following diagram expresses the results we have so far.

$$\begin{tikzcd} C(\sigma(a)) \arrow[r] \arrow[d] & C(\Omega(A)) \arrow[r] \arrow[d] & A \arrow[d] \\ C(\sigma(\pi(a))) \arrow[r] & C(\Omega(A_1)) \arrow[r] & A_1 \end{tikzcd}$$

Here, the top row expresses the continuous functional calculus on a, and the bottom row is the continuous functional calculus on $\pi(a)$. We claim that both squares in this diagram commute.

For $f \in C(\sigma(a))$ and $h \in \Omega(A_1)$, we have

$$\begin{aligned} T_{\pi^*} \circ T_{\hat{a}}(f)(h) &= T_{\pi^*}(f \circ \hat{a})(h) \\ &= f \circ \hat{a} \circ \pi^*(h) \\ &= f \circ \hat{a} \circ (h \circ \pi) \\ &= f(h(\pi(a))) \\ &= f \circ \widehat{\pi(a)}(h) \\ &= T_{\widehat{\pi(a)}} \circ \rho(f)(h), \end{aligned}$$

so the left square commutes. For the right square, it is enough to verify that $\gamma_{A_1} \circ \pi = T_{\pi^*} \circ \gamma_A$. We have

$$\begin{aligned} \gamma_{A_1} \circ \pi = G_{\pi^*} \circ \gamma_A &\Leftrightarrow \gamma_{A_1} \circ \pi(a) = T_{\pi^*} \circ \gamma_A(a) \\ &\Leftrightarrow \widehat{\pi(a)} = \hat{a} \circ \pi^* \\ &\Leftrightarrow \widehat{\pi(a)}(h) = \hat{a} \circ \pi^*(h) \\ &\Leftrightarrow h(\pi(a)) = \hat{a}(h \circ \pi) \end{aligned}$$

$$\Leftrightarrow h(\pi(a)) = h(\pi(a)).$$

Since the above diagram commutes, we have $\phi_{\pi(a)} \circ \rho = \pi \circ \phi_a$.

If $f \in C(\sigma(a))$, then

$$\begin{aligned} f(\pi(a)) &= \phi_{\pi(a)} \circ \rho(f) \\ &= \pi \circ \phi_a(f) \\ &= \pi(f(a)). \end{aligned}$$

(6) We may uniformly approximate f by a sequence of functions $p_n \colon \sigma(a) \to \mathbb{C}$, where each p_n is of the form

$$p_n(z) = \sum_{k,l=1}^{m} c_{k,l} z^k \bar{z}^l,$$

with no constant term. For each $n \geqslant 1$, we see that $p_n(a) \in C^*(a)$, as each p_n contains no constant term. Using the result from (1), and the fact that each ϕ_a is continuous, we get

$$\begin{aligned} \phi_a(f) &= \phi_a\left(\lim_{n\to\infty} p_n\right) \\ &= \lim_{n\to\infty} \phi_a(p_n) \\ &= \lim_{n\to\infty} p_n(a) \\ &\in C^*(a). \end{aligned}$$

□

When we want to carry out the continuous functional calculus in non-unital algebras, we need a slightly modified approach. If B is nonunital and $a \in B$ is normal, then $\phi_a \colon C(\sigma(a)) \to C^*\left(a, 1_{\widetilde{B}}\right)$ is our continuous functional calculus. Note that $0 \in \sigma(a)$, so $\phi_a(f) \in C^*(a)$ if $f \in C(\sigma(a))$ with $f(0) = 0$.

Corollary: If B is a C^*-algebra, and a is normal, then there is an isometric $*$-isomorphism

$$\psi_a \colon C_0(\sigma(a) \setminus \{0\}) \to C^*(a)$$

that satisfies $\psi_a(\iota) = a$, where $\iota \colon \sigma(a) \setminus \{0\} \to \mathbb{C}$ is inclusion.

Proof. If $0 \notin \sigma(a)$, then we set $\psi_a = \phi_a$, and apply the previous theorem.

Suppose $0 \in \sigma(a)$. We identify

$$C_0(\sigma(a) \setminus \{0\}) = \{f \in C(\sigma(a)) \mid f(0) = 0\}.$$

We restrict $\psi_a = \phi_a|_{C_0(\sigma(a)\setminus 0)}$, and apply the previous theorem. □

We can provide an alternative proof of a previous result.

Proposition: Let $\pi \colon A \to B$ be a $*$-homomorphism between C^*-algebras.

(1) If $a \in A$ is normal and π is injective, then $\sigma(\pi(a)) = \sigma(a)$.

(2) The map π is injective if and only if it is isometric.

Proof. We assume that A, B, π are unital.

(1) We already know that $\sigma(\pi(a)) \subseteq \sigma(a)$.

Suppose the inclusion is not strict. By Urysohn's lemma, we may find a nonzero $f \in C(\sigma(a))$ such that $f|_{\sigma(\pi(a))} = 0$. If $\phi_a \colon C(\sigma(a)) \to C^*(1, a)$ is the continuous functional calculus at a, then $\phi_a(f) = f(a) \neq 0$.

Since π is injective, we get $0 \neq \pi(f(a)) = f(\pi(a)) = 0$, which is a contradiction.

(2) It is clear that isometric maps are injective.

Suppose π is injective. Let $x \in A$, and set $a = x^*x$. Since a is normal, $\sigma(a) = \sigma(\pi(a))$, hence $r(a) = r(\pi(a))$. Thus, we have

$$\begin{aligned} \|\pi(a)\| &= r(\pi(a)) \\ &= r(a) \\ &= \|a\|. \end{aligned}$$

Thus, via the C^* identity, we get

$$\begin{aligned} \|x\|^2 &= \|x^*x\| \\ &= \|a\| \\ &= \|\pi(a)\| \\ &= \|\pi(x^*x)\| \\ &= \|\pi(x)^*\pi(x)\| \\ &= \|\pi(x)\|^2. \end{aligned}$$

□

We are interested in understanding the spectra of elements in C^*-algebras through the functional calculus.

Proposition: Let B be a C^*-algebra, and let $a \in B$ be normal. The following hold:

(1) $a \in B_{\text{s.a.}}$ if and only if $\sigma(a) \subseteq \mathbb{R}$;

(2) $a \in \mathcal{P}(B)$ if and only if $\sigma(a) \subseteq \{0, 1\}$;

(3) if B is unital, then $a \in \mathcal{U}(B)$ if and only if $\sigma(a) \subseteq \mathbb{T}$.

Proof. If B is not unital, we unitize and consider the calculus $\phi_a \colon C(\sigma(a)) \to C^*\left(a, 1_{\widetilde{B}}\right)$, recalling that $\sigma(a) = \sigma_{\widetilde{B}}(a)$. We assume B is unital, and let $f = \mathrm{id}_{\sigma(a)}$.

(1) Since ϕ_a is injective, self-adjoint, and since $\phi_a(f) = a$, we have

$$\begin{aligned} a^* = a &\Leftrightarrow \phi_a(f) = \phi_a(f)^* \\ &\Leftrightarrow \phi_a(f) = \phi_a(f^*) \\ &\Leftrightarrow f = f^* \\ &\Leftrightarrow f(\lambda) = \overline{f(\lambda)} \\ &\Leftrightarrow \lambda = \overline{\lambda} \\ &\Leftrightarrow \sigma(a) \subseteq \mathbb{R}. \end{aligned}$$

(2) Similarly,

$$a = a^2 \Leftrightarrow \phi_a(f) = \phi_a(f)^2$$

$$
\begin{aligned}
&\Leftrightarrow \phi_a(f) = \phi_a\left(f^2\right)\\
&\Leftrightarrow f = f^2\\
&\Leftrightarrow f(\lambda) = f(\lambda)^2\\
&\Leftrightarrow \lambda = \lambda^2\\
&\Leftrightarrow \sigma(a) \subseteq \{0,1\}.
\end{aligned}
$$

(3) Since ϕ_a is unital, we have

$$
\begin{aligned}
a^*a = 1_B &\Leftrightarrow \phi_a(f)^*\phi_a(f) = \phi_a\left(\mathbb{1}_{\sigma(a)}\right)\\
&\Leftrightarrow \phi_a(f^*f) = \phi_a\left(\mathbb{1}_{\sigma(a)}\right)\\
&\Leftrightarrow f^*f = \mathbb{1}_{\sigma(a)}\\
&\Leftrightarrow \overline{f(\lambda)}f(\lambda) = 1\\
&\Leftrightarrow |\lambda|^2 = 1\\
&\Leftrightarrow |\lambda| \in 1\\
&\Leftrightarrow \sigma(a) \subseteq \mathbb{T}.
\end{aligned}
$$

□

An important fact is that in any unital algebra, the inverse of an invertible element $a \in GL(A)$ can be obtained using the functional calculus.

Fact. Let A be a unital C^*-algebra. Suppose $a \in A$ is normal and invertible. Then, $a^{-1} = g(a)$, where $g(z) = z^{-1}$. Moreover,

$$
\sigma\left(a^{-1}\right) = \left\{z^{-1} \mid z \in \sigma(a)\right\}.
$$

Proof. Since a is invertible, $0 \notin \sigma(a)$, so g is defined and continuous on $\sigma(a)$. Then, where $\iota\colon \sigma(a) \hookrightarrow \mathbb{C}$ is inclusion, we have

$$
\begin{aligned}
\phi_a(g)a &= \phi_a(g)\phi_a(\iota)\\
&= \phi_a(g\iota)\\
&= \phi_a\left(\mathbb{1}_{\sigma(a)}\right)\\
&= 1_A.
\end{aligned}
$$

Similarly, $a\phi_a(g) = 1_A$. Since inverses are unique, we have $a^{-1} = \phi_a(g)$. □

Exercise: Let A be a unital C^*-algebra, and suppose $a \in A$ is normal and invertible. Prove that $a^{-1} \in C^*(a)$, and $C^*(a) = C^*(a, 1_A)$.

Solution: We only need to show the first statement, as the second statement follows directly.

Note that we have an isometric $*$-isomorphism $\psi_a\colon C_0(\sigma(a) \setminus \{0\}) \to C^*(a)$. Since $\sigma(a) \setminus \{0\} = \sigma(a)$, and $\sigma(a)$ is compact, we must have an isometric $*$-isomorphism between $C(\sigma(a))$ and $C^*(a)$.

Since $0 \notin \sigma(a)$, we have $g(z) = z^{-1} \in C(\sigma(a))$, so $a^{-1} \in C^*(a)$.

Exercise: Let $T \in \mathbb{B}(\mathcal{H})$ be a normal operator that is not a scalar multiple of the identity. Prove that there are self-adjoint operators $a, b \in C^*(T, I_{\mathcal{H}})$ such that $ab = 0$.

Another application of the continuous functional calculus is in the proof of the spectral theorem for compact normal operators.

Theorem: If T is a compact normal operator acting on an infinite-dimensional Hilbert space, there is a (possibly finite) sequence $(\lambda_n)_n$ in c_0 consisting of distinct nonzero scalars and a (possibly finite) sequence $(P_n)_n$ of mutually orthogonal projections such that

$$T = \sum_{n\geqslant 1} \lambda_n P_n$$

is an operator norm-convergent sum.

Proof. We know that $\sigma(T)$ is finite, or we have $\|T\|_{op} = |\lambda_1| > |\lambda_2| > \cdots \to 0 = \lambda_0$.

For $n \geqslant 1$, we have shown that λ_n is isolated, so $\delta_n = \mathbb{1}_{\{\lambda_n\}} \in C_0(\sigma(T)\{i\}nus\{0\})$. We have a uniformly convergent series

$$\iota = \sum_{n\geqslant 1} \lambda_n \delta_n$$

in $C_0(\sigma(T) \setminus \{0\})$, where ι is inclusion. Applying the isometric $*$-isomorphism $\psi_a\colon C_0(\sigma(T) \setminus \{0\}) \to C^*(T)$, we get the operator norm-convergent sum

$$T = \sum_{n\geqslant 1} \lambda_n P_n,$$

where $P_n = \psi_a(\delta_n) \in C^*(T) \subseteq \mathbb{K}(\mathcal{H})$. Note that compact projections are finite-rank, and for $n \neq m$, we have

$$\begin{aligned} P_n P_m &= \psi_a(\delta_n)\psi_a(\delta_m) \\ &= \psi_a(\delta_m \delta_n) \\ &= \psi_a(0) \\ &= 0. \end{aligned}$$

□

We can also use the continuous functional calculus to determine isomorphism classes of universal C^*-algebras.

Example. We have looked at the C^*-algebra generated by a unitary, $C^*(u)$. With the continuous functional calculus, we can show that $C^*(u)$ is $*$-isomorphic to any C^*-algebra generated by a unitary with full spectrum.

Let A be a unital C^*-algebra, and suppose $w \in \mathcal{U}(A)$, with $\sigma(w) = \mathbb{T}$.[VIII] Let $\phi_w\colon C(\mathbb{T}) \to C^*(w)$ and $\phi_u\colon C(\sigma(u)) \to C^*(u)$ be the continuous functional calculi at $w \in A$ and $u \in C^*(u)$ respectively.

Since $\sigma(u) \subseteq \mathbb{T}$, we have the restriction unital $*$-homomorphism $\rho\colon C(\mathbb{T}) \to C(\sigma(u))$, given by $\rho(f) = f|_{\sigma(u)}$. Moreover, by the universal property of $C^*(u)$, we have a surjective $*$-homomorphism $\psi\colon C^*(u) \to C^*(w)$.

The unital $*$-homomorphism $\varphi = \phi_u \circ \rho \circ \phi^{-1}(w)\colon C^*(w) \to C^*(u)$ sends $w \mapsto u$. Thus, $\varphi \circ \psi$ agrees on a dense $*$-subalgebra of $C^*(u)$, and by continuity, on $C^*(u)$, so ψ is injective. Thus, we have

$$\begin{aligned} C^*(u) &\cong C(\mathbb{T}) \\ &\cong C^*(V) \\ &\cong C^*(\mathbb{Z}), \end{aligned}$$

where V is the right bilateral shift on $\ell_2(\mathbb{Z})$.

[VIII] We call these Haar unitaries.

Ordering and Positive Elements

Recall that we say an element of a $*$-algebra $a \in A$ is positive (in the algebraic sense) if $a = x^*x$ for some $x \in A$. We denote $A_+ \subseteq A_{\text{s.a.}}$ to be the set of positive elements as a subset of the set of all self-adjoint elements.

However, if A is a C^*-algebra, this set of positive elements goes further, and forms a closed generating cone in $A_{\text{s.a.}}$ that induces an ordering on the self-adjoint elements in A. This will require some work.

Given a LCH space Ω, we have looked at the closed cone of positive elements, $C_0(\Omega)_+$ inside the $\mathbb{R}$-space of $C_0(\Omega, \mathbb{R})$. A function $f \in C_0(\Omega)$ is positively-valued if and only if it is positive in the algebraic sense.

Similarly, the set of positive operators

$$\mathbb{B}(\mathcal{H})_+ = \{T \in \mathbb{B}(\mathcal{H})_{\text{s.a.}} \mid \langle T(\xi), \xi\rangle \geqslant 0,\ \forall \xi \in \mathcal{H}\}$$

forms a norm-closed cone in $\mathbb{B}(\mathcal{H})_+$, which induces an ordering on the self-adjoint elements of $\mathbb{B}(\mathcal{H})_{\text{s.a.}}$. Positivity in $\mathbb{B}(\mathcal{H})_{\text{s.a.}}$ is described spatially, but it is not obvious that this definition agrees with the algebraic definition.

Any element in a $*$-algebra A can be written as $a = h + ik$, where $h, k \in A_{\text{s.a.}}$. If A is a normed $*$-algebra, then $\|h\| \leqslant a$ and $\|k\| \leqslant a$. This is the Cartesian decomposition of a. We are aware that every real-valued continuous function can be written as the difference of two positive continuous functions. We start by generalizing to C^*-algebras.

Proposition: Let A be a C^*-algebra and let $h \in A_{\text{s.a.}}$. There exist unique positive elements $p, q \in A_+$ such that

(a) $h = p - q$;

(b) $pq = 0$;

(c) $\sigma(p), \sigma(q) \subseteq [0, \infty)$.

Moreover, $\|h\| = \max(\|p\|, \|q\|)$. We denote these elements $h_= = p$ and $h_- = q$.

Proof. Assume A is unital. Let $\phi_h \colon C(\sigma(h)) \to C^*(h, 1_A)$ be the continuous functional calculus at h. Since $\sigma(h) \subseteq \mathbb{R}$, we consider the continuous functions

$$\begin{aligned} f(t) &= \max(t, 0) \\ g(t) &= \max(-t, 0). \end{aligned}$$

Note that $fg = 0$, and $\text{id}_{\sigma(h)} = f - g$. We set $p = \phi_h(f)$ and $q = \phi_h(g)$. Since f and g are positive, their $*$-homomorphic images are positive as well. Moreover,

$$\begin{aligned} \sigma(p) &= \sigma(f(h)) \\ &= f(\sigma(h)) \\ &\subseteq [0, \infty), \end{aligned}$$

and similarly for $\sigma(q)$. We also have

$$\begin{aligned} pq &= \phi_h(f)\phi_h(g) \\ &= \phi_h(fg) \\ &= \phi_h(0) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} h &= \phi_h\left(\mathrm{id}_{\sigma(h)}\right) \\ &= \phi_h(f - g) \\ &= \phi_h(f) - \phi_h(g) \\ &= p - q. \end{aligned}$$

Since ϕ_h is isometric, we also have

$$\begin{aligned} \|h\| &= \left\|\phi_h\left(\mathrm{id}_{\sigma(h)}\right)\right\| \\ &= \left\|\mathrm{id}_{\sigma(h)}\right\| \\ &= \max(\|f\|_u, \|g\|_u) \\ &= \max(\|\phi_h(f)\|, \|\phi_h(g)\|) \\ &= \max(\|p\|, \|q\|). \end{aligned}$$

We will now show uniqueness. Suppose $x, y \in A_+$ are such that $h = x - y$, $xy = 0$, and $\sigma(x), \sigma(y) \subseteq [0, \infty)$. By induction, we have $h^n = x^n + (-y)^n$ for all $n \geqslant 1$, so $p(h) = p(x) + p(-y)$ for all polynomials p without constant terms.

Since $f(0) = 0$, there is a sequence of polynomials converging uniformly to f on the compact set $K = \sigma(h) \cup \sigma(x) \cup \sigma(-y)$. Applying the functional calculus at h, x, and y, we get

$$\begin{aligned} p &= f(h) \\ &= \lim_{n \to \infty} p_n(h) \\ &= \lim_{n \to \infty} (p_n(x) + p_n(-y)) \\ &= f(x) + f(-y). \end{aligned}$$

Since $\sigma(x) \subseteq [0, \infty)$, and $f(t) = t$ on $[0, \infty)$, we must have $f(x) = x$. Also, since $\sigma(-y) = -\sigma(y) \subseteq (-\infty, 0]$, and $f(t) = 0$ on $(-\infty, 0]$, we must have $f(-y) = 0$. Thus, $p = x$, and $q = p - h = y$.

If A does not have a unit, we consider the functional calculus $\phi_h \colon C(\sigma(h)) \to C^*\left(h, 1_{\widetilde{A}}\right)$. Since $0 \in \sigma(h)$ and $f(0) = g(0) = 0$, we must have $p, q \in C^*(h) \subseteq A$. □

Lemma: Let A be a C^*-algebra with $y \in A_{\text{s.a.}}$. Then, $\sigma\left(y^2\right) \subseteq [0, \infty)$.

Proof. We assume A is unital. By spectral mapping, we have $\sigma\left(y^2\right) = (\sigma(y))^2 \subseteq [0, \infty)$, as $\sigma(y) \subseteq \mathbb{R}$. □