Remark: In all cases, we will use the following schema for the Fourier transform:

$$\begin{aligned}\hat{f}(k) &= \mathcal{F}[f(x)]\\ &= \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx\\ f(x) &= \mathcal{F}^{-1}\left[\hat{f}(k)\right]\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ikx}\,dk.\end{aligned}$$

Solution (Fourier Transform Problems)**:** (A) We have

$$\frac{\partial u}{\partial t} = 3\frac{\partial u}{\partial x},$$

with

$$u(x,0) = \sin(x).$$

Then, taking the Fourier transform with respect to x on both sides, we get the equation

$$\frac{d\hat{u}}{dt} = -3ik\hat{u},$$

so

$$\hat{u}(k,t) = \hat{u}(k,0)e^{-3ikt}.$$

Plugging this back into our system, we get

$$u(x,t) = \frac{1}{2\pi}\int_{0}^{\infty} \hat{u}(k,0)e^{-ik(x+3t)}\,dk.$$

We have the kernel of

$$K(x,t) = \mathcal{F}^{-1}\left[e^{-3ikt}\right](x,t),$$

and the solution of

$$u(x,t) = \sin(x+3t).$$

(B) We have

$$\frac{\partial u}{\partial t} = 2\frac{\partial u}{\partial x} + 3u(x,t)$$

with

$$u(x,0) = \sin(x).$$

Taking Fourier transforms with respect to x, we obtain

$$\begin{aligned}\frac{d\hat{u}}{dt} &= -2ik\hat{u} + 3\hat{u}\\ &= (3-2ik)\hat{u},\end{aligned}$$

meaning that

$$\hat{u}(k,t) = \hat{u}(k,0)e^{(3-2ik)t}.$$

and

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,0)e^{(3-2ik)t}e^{-ikx}\,dk.$$

We have a kernel of

$$K(x,t) = \mathcal{F}^{-1}\left[e^{(3-2ik)t}\right](x,t),$$

and a solution of

$$u(x,t) = e^{3t}\sin(x-2t).$$

Solution (4.6, Problem 32)**:** The Wronskian for

$$\frac{d^2y}{dx^2} - y = 0$$

is

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2. \end{aligned}$$

Therefore, the Green's Function is

$$\begin{aligned} G(x,t) &= \frac{e^t e^{-x} - e^x e^{-t}}{-2} \\ &= \frac{1}{2}\left(e^{-t}e^x - e^{-x}e^t\right). \end{aligned}$$

We may then evaluate

$$\begin{aligned} y_p(x) &= \frac{1}{2}\int \frac{1}{t}\left(e^{-t}e^x - e^{-x}e^t\right)\,dt \\ &= \frac{1}{2}e^x\int_a^x \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_b^x \frac{e^t}{t}\,dt, \end{aligned}$$

so

$$y(x) = a_1e^x + a_2e^{-x} + \frac{1}{2}e^x\int_a^x \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_b^x \frac{e^t}{t}\,dt$$

Solution (4.6, Problem 34)**:** Using the Green's function of

$$G(x,t) = \frac{1}{2}\left(e^x e^{-t} - e^{-x}e^t\right),$$

we evaluate

$$\begin{aligned} y_p(x) &= \int_0^x e^{2t}G(x,t)\,dt \\ &= \frac{1}{3}e^{2x} - \frac{1}{2}e^x + \frac{1}{6}e^{-x}. \end{aligned}$$

Solution (Green's Function Problems)**:**

(A) Solving for the Wronskian, we have

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^{4t} & e^{-2t} \\ 4e^{4t} & -2e^{-2t} \end{pmatrix} \\ &= -6e^{2t}. \end{aligned}$$

Thus,

$$G(x,t) = \frac{1}{6}\left(e^{-2x+2t} - e^{4x-4t}\right),$$

and

$$\begin{aligned} y_p(x) &= \frac{1}{6}\int_0^x (t+1)\left(e^{-2x+2t} - e^{4x-4t}\right)\,dt \\ &= \frac{1}{96}\left(-5e^{4x} - 4e^{-2x} + 12x - 9\right). \end{aligned}$$

(B) Evaluating the Wronskian, we have

$$W(x) = \begin{pmatrix} x & 1/x \\ 1 & -1/x^2 \end{pmatrix}$$

$$= -\frac{2}{x},$$

and a Green's function of

$$G(x,t) = \frac{x}{2} - \frac{t^2}{2x}.$$

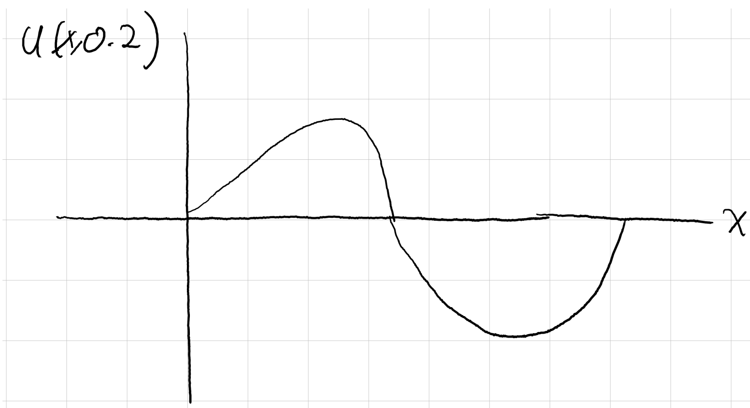
Evaluating $y_p(x)$, we get

$$\begin{aligned} y_p(x) &= \frac{1}{2}\int_0^x \left(x - \frac{t^2}{x}\right)\sin(t)\,dt \\ &= \frac{1}{x} + \frac{1}{2}x - \frac{1}{2x}(\cos(x) - 2x\sin(x)). \end{aligned}$$

Solution: We may write the equation as

$$0 = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{3}u^3\right).$$

In the sketch below, we see that the wave front moves forward in x as time moves forward, with differing speeds on the basis of the magnitude of u. This follows from the fact that one of the characteristic curves is $x_0 = x - u^2 t$.



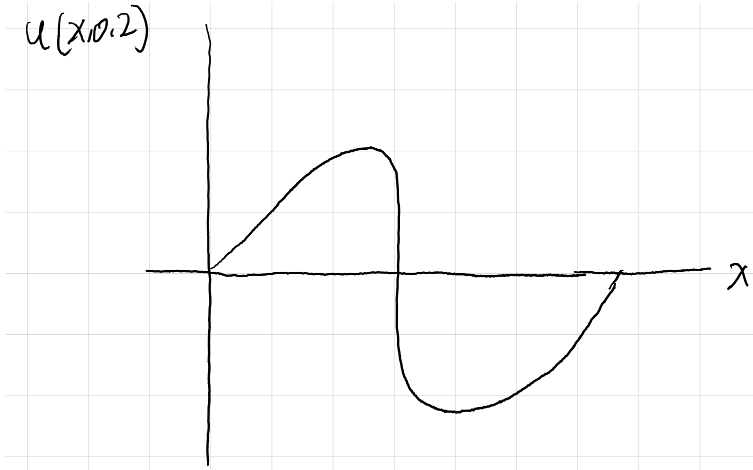
The solution curve is implicitly defined as

$$u(x,t) = \sin\left(x - u^2 t\right).$$

Solution: We may write the equation as

$$0 = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{4}u^4\right).$$

In the sketch below, we see that the wave front moves forward in x as time moves forward if u is positive, and moves backward in x if u is negative, eventually forming a shock. This follows from the fact that one of the characteristic curves is $x_0 = x - u^3 t$.



The solution curve is implicitly defined as

$$u(x, t) = \sin\left(x - u^3 t\right).$$