

**Problem (Problem 1):** Prove that if  $f: M \rightarrow N$  is smooth, and  $L$  is a  $k$ -codimensional submanifold of  $N$  that is transverse to  $f$ , then  $f^{-1}(L)$  is either empty or a submanifold of  $M$  with codimension  $k$ .

**Solution:** If  $L$  is not contained in  $f(M)$ , then  $f^{-1}(L)$  is clearly empty. Therefore, we focus on the case where  $f^{-1}(L)$  is not empty.

Let  $L$  be transverse to  $f$ ,  $q \in L$ , and  $p \in M$  such that  $f(p) = q$ . We observe that  $T_q L + D_p F(T_p M) = T_q N$ , so any vector in  $T_q N$  can be written (not necessarily uniquely) as an element of  $D_p F(T_p M)$  and  $T_q L$ . Next, we observe that, if we take a coordinate chart for  $q$  in  $U$  such that  $\varphi(U) \cong \mathbb{R}^k$ , then by the Regular Value Theorem, we may select  $\varphi$  such that  $L \cap U = \varphi^{-1}(0)$ . This follows from the assumption that  $L$  has codimension  $k$ .

Now, if we can show that  $0$  is a regular value for  $\varphi \circ f$ , then  $(\varphi \circ f)^{-1}(0) = f^{-1}(L) \cap f^{-1}(U)$ , meaning that  $f^{-1}(L)$  is a submanifold of  $M$  with codimension  $k$ . First, since  $0$  is a regular value for  $\varphi$ , it follows that if  $v \in T_0 \mathbb{R}^k$ , then there is some  $w \in T_q N$  such that  $D_q \varphi(w) = v$ . Since  $f$  is transverse to  $L$ , there is  $w_1 \in T_q L$  and  $w_2 \in T_p M$  such that  $w = w_1 + D_p F(w_2)$ . We observe that, since  $\varphi$  is constant on  $L$ , we have  $D_q \varphi(w_1) = 0$ , so that

$$\begin{aligned} D_p(\varphi \circ f)(w_2) &= D_q \varphi \circ D_p F(w_2) \\ &= D_q \varphi(w_1 + D_p F(w_2)) \\ &= D_q \varphi(w) \\ &= v, \end{aligned}$$

so  $0$  is a regular value for  $\varphi \circ f$ .

**Problem (Problem 2):** Let  $GL_n(\mathbb{R})$  denote the space of invertible  $n \times n$  matrices over  $\mathbb{R}$ , let  $SL_n(\mathbb{R})$  denote the matrices of determinant one, and let  $O(n)$  be the orthogonal group.

- Prove that we may identify the tangent space of  $GL_n(\mathbb{R})$  at the identity with  $n \times n$  matrices over  $\mathbb{R}$ .
- Prove that the tangent space of  $SL_n(\mathbb{R})$  at the identity consists of matrices of trace zero.
- Prove that the tangent space of  $O(n)$  at the identity consists of skew-symmetric matrices. What is the dimension of  $O(n)$ ?
- Show that  $SL_n(\mathbb{R})$  and  $O(n)$  do not intersect transversely at the identity.

**Solution:**

- Let  $A \in Mat_n(\mathbb{R})$ , and consider a path through the identity given by  $\gamma(t) = I + tA$ . Since the determinant is a smooth function, and  $\det(I) = 1$ , we have that for a small  $\varepsilon > 0$  there is  $\delta$ , such that  $|\det(I + tA) - 1| < \varepsilon$  whenever  $|t| < \delta$ . In particular, this means that the tangent space at the identity of  $GL_n(\mathbb{R})$  consists of all matrices.
- We let  $\gamma(t) = I + tA$  be a curve in  $SL_n(\mathbb{R})$ , so that  $\gamma'(0) = A$  is an element of the tangent space of  $SL_n(\mathbb{R})$  at the identity. We observe that  $\det(\gamma(t)) = 1$  for all (sufficiently small)  $t$ , so by chain rule, we find that

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \det(\gamma(t)) \\ &= D_{\gamma(0)} \det(\gamma'(0)) \\ &= D_I \det(A). \end{aligned}$$

Therefore, we must evaluate what  $\det'(I)(A)$  yields. Toward this end, we see that

$$D_I \det(A) = \lim_{t \rightarrow 0} \frac{\det(I - tA) - 1}{t}$$

$$= \lim_{t \rightarrow 0} \frac{t^n \det\left(\frac{1}{t}I - A\right) - 1}{t}.$$

Observe that the expression  $\det\left(\frac{1}{t}I - A\right)$  is the characteristic polynomial of  $A$  in  $\frac{1}{t}$ . This means that the  $\left(\frac{1}{t}\right)^{n-1}$  term is equal to  $\text{tr}(A)$ , so that  $D_I \det(A) = \text{tr}(A)$ . Thus, we find that  $A$  is traceless.

(c) If  $\gamma(t) = I + tA$  is a curve in  $O(n)$ , then then we have that

$$\begin{aligned} (I + tA)^T (I + tA) &= I \\ I + t(A^T + A) + t^2(A^T A) &= I, \end{aligned}$$

meaning that by taking an equivalence class of this tangent curve, we have

$$I + t(A^T + A) = I,$$

so that  $A^T = -A$ .

We observe that the function  $f: \text{Mat}_n(\mathbb{R}) \rightarrow \text{Mat}_n(\mathbb{R})_{\text{s.a.}}$ , given by

$$f(A) = A^T A,$$

has  $I_n$  as a regular value. To see this, observe that curves in  $T_I \text{Mat}_n(\mathbb{R})_{\text{s.a.}}$  are of the form  $\gamma(t) = I + tK$ , where  $K$  is a self-adjoint(/symmetric) matrix. Similarly,  $T_A \text{Mat}_n(\mathbb{R})$  is of the form  $\varepsilon(t) = A + tB$ , where  $B \in \text{Mat}_n(\mathbb{R})$  and  $t \in \mathbb{R}$ . Both of these follow from the fact that  $\text{Mat}_n(\mathbb{R})$  and  $\text{Mat}_n(\mathbb{R})_{\text{s.a.}}$  are isomorphic to Euclidean spaces. Therefore, we see that the image of  $\delta(t)$  is of the form  $A^T A + t(A^T B + B^T A)$ ; if  $A$  satisfies  $A^T A = I$ , we can put this in the form of  $I + tK$  by taking  $\delta(t) = A + \frac{1}{2}tAK$ . Therefore, by the Regular Value Theorem, the dimension of  $O(n)$  is  $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ .

(d) Since both skew-symmetric and traceless matrices have trace zero, it follows that the tangent spaces of  $SL_n(\mathbb{R})$  and  $O(n)$  cannot span the tangent space of  $GL_n(\mathbb{R})$ , as there are matrices with nonzero trace.

**Problem (Problem 4):** Let  $D$  be a distribution on a smooth manifold of dimension  $n$ . We write  $I(D)$  for the ideal of  $D$ , which consists of graded pieces  $I^k(D) \subseteq \mathcal{A}^k(M)$ , where  $I^k(D)$  consists of forms  $\omega$  such that  $\omega(X_1, \dots, X_k) = 0$  for all  $X_i \in D$ , and

$$I(D) = \bigoplus_{k=0}^n I^k(D).$$

The Frobenius Theorem says that  $D$  is involutive if and only if  $I$  is *differential* — i.e.,  $d(I) \subseteq I$ , where  $d$  is the exterior derivative.

- (a) Prove that  $I(D)$  is an ideal — i.e., if  $\omega \in I(D)$  and  $\eta$  is arbitrary, then  $\omega \wedge \eta \in I(D)$ .
- (b) Prove that  $I(D)$  is locally generated by  $s = n - r$  linearly independent 1-forms  $\omega_1, \dots, \omega_s$ , in the sense that for every point  $p \in M$ , there is a neighborhood  $U$  of  $p$  such that for any  $\omega \in I^k(D)$  with  $k$  arbitrary, we may write

$$\omega = \sum_{i=1}^s \theta_i \wedge \omega_i$$

for suitable forms  $\theta_1, \dots, \theta_s$ .

- (c) Prove that if  $D$  is involutive, then for all  $\omega \in I(D)$ , we have  $d\omega \in I(D)$ .

(d) Use this to show that if  $\omega$  is a 1-form, and  $X, Y$  are vector fields, then

$$d\omega(X, Y) = \frac{1}{2}(X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])).$$

Conclude that if  $\omega \in I^1(D)$ , and  $X, Y \in D$ , then  $\omega([X, Y]) = 0$ . Thus, if  $I$  is a differential ideal, then  $D$  is involutive.

(e) Show that if  $D$  is defined by the vanishing of linearly independent forms  $\omega_1, \dots, \omega_s$  near a point  $p$ , then  $D$  is involutive if and only if for each  $i$  there are 1-forms  $\omega_{i,j}$  such that

$$d\omega_i = \sum_{j=1}^s \omega_{i,j} \wedge \omega_j.$$

**Solution:**

(a) Write

$$\omega = \alpha_1 \wedge \dots \wedge \alpha_k$$

so that

$$\begin{aligned} \omega(X_1, \dots, X_k) &= \det\left((\alpha_i(X_j))_{i,j}\right) \\ &= 0. \end{aligned}$$

for  $X_1, \dots, X_k \in D$ . Then, if

$$\eta = \beta_1 \wedge \dots \wedge \beta_\ell,$$

we have the determinant of the block matrices

$$\begin{aligned} \omega \wedge \eta(X_1, \dots, X_k, \dots, X_{k+\ell}) &= \det\begin{pmatrix} \alpha_i(X_j) & \alpha_i(X_{\ell+j}) \\ \beta_i(X_j) & \beta_i(X_{\ell+j}) \end{pmatrix} \\ &= 0, \end{aligned}$$

so that  $\omega \wedge \eta$  is contained in  $I(D)$ .