Problem 2

The standard topology on $\mathbb{R}$ is generated by the metric on $\mathbb{R}$; therefore, we may show that $[a, b]$ is sequentially compact.

If $(x_n)_n \subseteq [a, b]$ is any sequence, then by Bolzano–Weierstrass, there is a subsequence $(x_{n_k})_k \to x \in \mathbb{R}$. Furthermore, since $[a, b]^c = (-\infty, a) \cup (c, \infty)$ is open, we must have $x \in [a, b]$, meaning that $[a, b]$ is sequentially compact, hence compact.

Problem 17

Write

$$\sum_{i=1}^{m} |x_i + y_i|^p = \sum_{i=1}^{m} |x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1}.$$

Then, by applying Hölder's Inequality, we have

$$\begin{aligned}
\sum_{i=1}^{m} |x_i||x_i + y_i|^{p-1} &\leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{m} |x_i + y_i|^{(p-1)q}\right)^{q} \\
\sum_{i=1}^{m} |y_i||x_i + y_i|^{p-1} &\leqslant \left(\sum_{i=1}^{m} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{m} |x_i + y_i|^{(p-1)q}\right)^{q}.
\end{aligned}$$

Since $(p-1)q = p$, we then have

$$\sum_{i=1}^{m} |x_i + y_i|^p \leqslant \left(\sum_{i=1}^{m} |x_i + y_i|^p\right)^{q} \left(\left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{m} |y_i|^p\right)^{1/p}\right),$$

and dividing, we get

$$\left(\sum_{i=1}^{m} |x_i + y_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{m} |y_i|^p\right)^{1/p}.$$

Problem 19

(i) We see that

$$\sup_{i \in \mathbb{N}} |x_i| = 0$$

if and only if $|x_i| \leqslant 0$ for all i, meaning that $(x_i)_i$ is the zero sequence. Similarly,

$$\begin{aligned}
\|\alpha x\| &= \sup_{i \in \mathbb{N}} |\alpha x_i| \\
&= |\alpha| \sup_{i \in I} |x_i| \\
&= |\alpha| \|x\|.
\end{aligned}$$

Finally,

$$\|x + y\| = \sup_{i \in \mathbb{N}} |x_i + y_i|$$

$$\begin{aligned} &\leqslant \sup_{i\in\mathbb{N}}(|x_i| + |y_i|) \\ &\leqslant \sup_{i\in\mathbb{N}}|x_i| + \sup_{j\in\mathbb{N}}|y_j| \\ &= \|x\| + \|y\|, \end{aligned}$$

meaning that $\|\cdot\|$ is a bona fide norm.

(ii) Let $B = \{x \in X \mid \|x\| \leqslant 1\}$. Let $(x_n)_n \subseteq B$ converge to $x \in \ell_\infty$ in the ℓ_∞ norm.

Note that for all n, $\sup_{i\in\mathbb{N}}|x_n(i)| \leqslant 1$, meaning that since

$$\sup_{i\in\mathbb{N}}|x(i) - x_n(i)| \to 0,$$

we have that

$$|x(i) - x_n(i)| \to 0$$

for each i, so

$$x_n(i) \to x(i)$$

for all i. Thus, $|x(i)| \leqslant 1$ for all i, meaning $\sup_{i\in\mathbb{N}}|x(i)| \leqslant 1$, so $\|x\| \in B$.

(iii) Let $\varepsilon = 1/2$, and consider the collection $(e_n)_n$ of sequences in ℓ_∞ consisting of 1 at position n and zero elsewhere. Then, $(e_n)_n \subseteq B$, but since $\sup_{i\in\mathbb{N}}|e_n(i) - e_m(i)| = 1$ for all $n \neq m$, we cannot have balls of radius $1/2$ cover the family $(e_n)_n$ with finitely many such balls, meaning that B is not totally bounded.

Problem 20

(i) We see that $d(x, y) = 0$ if and only if $x_n = y_n$ for each n, since each d_n is a metric; therefore, $d(x, y) = 0$ if and only if $x = y$.

Furthermore, we have that for all $x = (x_n)_n$, $y = (y_n)_n$, and $z = (z_n)_n$, $\frac{1}{2^n}d(x_n, z_n) \leqslant \frac{1}{2^n}d(x_n, y_n) + \frac{1}{2^n}d(y_n, z_n)$. Therefore, we get

$$\begin{aligned} d(x, z) &= \sum_{n=1}^{\infty} \frac{1}{2^n}d(x_n, z_n) \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n}d(x_n, y_n) + \frac{1}{2^n}d(y_n, z_n) \\ &\leqslant d(x, y) + d(y, z). \end{aligned}$$

Since $d(x_n, y_n), d(y_n, z_n) \leqslant 1$, these sums must converge, so $d(x, y)$ is indeed a metric.

(ii) We will show that a sequence $(y_n)_n \subseteq X$ converges to $y \in X$ with the given distance metric if and only if it does so pointwise. This will show that the metric d induces the topology of pointwise convergence, which is exactly the topology τ_{prod}.

To start, let $(y_n)_n \to y$ in the given distance metric. Then, for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$d(y_n, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} d_j(y_n(j), y(j))$$

$$< \varepsilon,$$

so we see that for each j, $d(y_n(j), y(j)) < \varepsilon$, meaning that $y_n(j) \to y(j)$ for each j.

Let $(y_n)_n \to y$ pointwise. If $\varepsilon > 0$, convergence of series gives some J such that $\sum_{j=J+1}^{\infty} \frac{1}{2^j} < \varepsilon/2$, meaning that

$$\sum_{j=J+1}^{\infty} \frac{1}{2^j} d_j(y_n(j), y(j)) < \varepsilon/2$$

For $j = 1, \dots, J$, we find $N_1, \dots, N_J$ such that for all $n \geqslant N_j$, $d_j(y_n(j), y(j)) < \varepsilon/2$. Therefore, for $n \geqslant \max(N_1, \dots, N_J)$, we have

$$\begin{aligned}
d(y_n, y) &= \sum_{j=1}^{\infty} \frac{1}{2^j} d_j(y_n(j), y(j)) \\
&= \sum_{j=1}^{J} \frac{1}{2^j} d_j(y_n(j), y(j)) + \sum_{j=J+1}^{\infty} \frac{1}{2^j} d_j(y_n(j), y(j)) \\
&< \sum_{j=1}^{J} \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $(y_n)_n \to y$ in our given distance metric.

Since convergence of sequences in our given distance metric is given by pointwise convergence, the induced topologies must be equal, so $\tau_d = \tau_{\text{prod}}$.

(iii) We prove that X is complete if and only if X_n is complete for all n.

To see this, note that $(y_n)_n \subseteq X$ is Cauchy if and only if $(y_n(j))_n \subseteq X_j$ is Cauchy for each j, as for all $\varepsilon > 0$ and $m, n \geqslant N$ with $d(y_n, y_m) < \varepsilon$, then

$$\sum_{j=1}^{\infty} \frac{1}{2^j} d_j(y_n(j), y_m(j)) < \varepsilon,$$

meaning this holds for all j, and in the reverse direction, we use the same $\frac{\varepsilon}{2}$ method from part (ii).

The sequence $(y_n)_n$ thus converges in X if and only if every $y_n(j)$ converges in X_j (as τ_d is the topology of pointwise convergence), meaning that X is complete if and only if each X_j is complete.

Problem 21

Let X be complete, and let $(C_n)_n \subseteq P(X)$ be nonempty, decreasing, closed sets with $\operatorname{diam}(C_n) \to 0$.

Let $(x_n)_n$ be defined by $x_n \in C_n$ for each n. Then, for any $\varepsilon > 0$, we may find C_N such that $\operatorname{diam}(C_N) < \varepsilon$, meaning that for all $n, m \geqslant N$, we have that $x_n, x_m \in C_N$, so $d(x_n, x_m) < \varepsilon$, meaning that $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n \to x$ for some $x \in X$. This point must be in all such C_n, meaning that

$$\bigcap_{n=1}^{\infty} C_n = \{x\}.$$

Now, let X be a metric space such that for any $(C_n)_n \subseteq P(X)$ nonempty, decreasing, and closed with $\operatorname{diam}(C_n) \to 0$, there is some $x \in X$ with $\bigcap_{n=1}^{\infty} C_n = \{x\}$. Let $(x_n)_n$ be a Cauchy sequence in X.

Define a family of closed sets by

$$C_n = \overline{\{x_n, x_{n+1}, \dots\}}.$$

We note the following:

- each of the C_n is closed;
- $C_n \supseteq C_{n+1}$ by construction, since $\{x_n, x_{n+1}, \dots\} \supseteq \{x_{n+1}, x_{n+2}, \dots\}$, and closures respect set inclusion;
- $\operatorname{diam}(C_n) \to 0$, as $(x_n)_n$ is Cauchy, so if $\varepsilon > 0$, there is some N such that for all $n, m \geqslant N$, $d(x_n, x_m) < \varepsilon$, meaning that the diameter of the closure of the set $\{x_N, x_{N+1}, \dots\}$ is no more than ε.

Therefore, there is some $x \in X$ such that

$$\bigcap_{n=1}^{\infty} C_n = \{x\},$$

meaning that $(x_n)_n \to x$, and X is complete.