

Positive Maps

We will start by focusing our discussion of positive maps on a subclass of linear subspaces of C^* -algebras.

Definition: Let \mathcal{A} be a C^* -algebra, and let $\mathcal{S} \subseteq \mathcal{A}$ be a self-adjoint linear subspace that contains 1. We call such an \mathcal{S} an *operator system*.

Note that if h is a self-adjoint element of \mathcal{S} , then it is possible to write h as the difference of two positive elements in \mathcal{S} ,

$$h = \frac{1}{2}(\|h\|1 + h) - \frac{1}{2}(\|h\|1 - h).$$

Definition: If $\mathcal{S} \subseteq \mathcal{A}$ is an operator system, \mathcal{B} is a C^* -algebra, and $\phi: \mathcal{S} \rightarrow \mathcal{B}$ is a linear map, then we say ϕ is positive if it maps positive elements of \mathcal{S} to positive elements of \mathcal{B} .

In the special case where the C^* -algebra \mathcal{B} is the complex numbers (i.e., ϕ is a positive linear functional), then we know from results in C^* -algebra theory that $\|\phi\| = \phi(1)$. If \mathcal{B} is an arbitrary C^* -algebra, it turns out that ϕ is still positive, but that the bound is different.

Proposition: If $\phi: \mathcal{S} \rightarrow \mathcal{B}$ is a positive map, then $\|\phi\| \leq 2\|\phi(1)\|$.

Proof. If p is positive, then since $0 \leq p \leq \|p\|1$, it follows that $0 \leq \phi(p) \leq \|p\|\phi(1)$, so that $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$.

If p_1 and p_2 are positive, then $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$, so if h is self-adjoint in \mathcal{S} , we have

$$\phi(h) = \frac{1}{2}\phi(\|h\|1 + h) - \frac{1}{2}\phi(\|h\|1 - h),$$

giving

$$\begin{aligned} \|\phi(h)\| &\leq \frac{1}{2} \max(\|\phi(\|h\|1 + h)\|, \|\phi(\|h\|1 - h)\|) \\ &\leq \|h\|\|\phi(1)\|. \end{aligned}$$

Finally, if a is an arbitrary element of \mathcal{S} , then we may write the Cartesian decomposition $a = h + ik$, and find

$$\begin{aligned} \|\phi(a)\| &\leq \|\phi(h)\| + \|\phi(k)\| \\ &\leq 2\|a\|\|\phi(1)\|. \end{aligned}$$

□

It turns out that this bound is strict.

Example: Consider the subspace $\mathcal{S} \subseteq C(S^1)$ spanned by $1, z, \bar{z}$. Then, we may define $\phi: \mathcal{S} \rightarrow \mathbb{M}_2$ given by

$$\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}.$$

It follows that an element of \mathcal{S} is positive if and only if $c = \bar{b}$ and $a \geq 2|b|$, while a self-adjoint element of \mathbb{M}_2 is positive if and only if its diagonal entries and determinant are positive real numbers. Therefore, it follows that ϕ is a positive map.

Yet,

$$\begin{aligned} 2\|\phi(1)\| &= 2 \\ &= \|\phi(z)\| \\ &\leq \|\phi\|, \end{aligned}$$

meaning that $\|\phi\| = 2\|\phi(1)\|$.

Completely Positive Maps

Definition: If \mathcal{B} is a C^* -algebra and $\phi: \mathcal{S} \rightarrow \mathcal{B}$ is a linear map, then we may define $\phi_n: \mathbb{M}_n(\mathcal{S}) \rightarrow \mathbb{M}_n(\mathcal{B})$ by $\phi_n((a_{ij})_{i,j}) = (\phi(a_{ij}))_{i,j}$.

We call ϕ n -positive if ϕ_n is positive, and we call ϕ completely positive if ϕ is n -positive for all n .

We call ϕ completely bounded if

$$\|\phi\|_{cb} := \sup_n \|\phi_n\|$$

is finite. If $\|\phi\|_{cb} \leq 1$, then we call ϕ completely contractive.

Lemma: Let \mathcal{A} be a unital C^* -algebra, and let $a, b \in \mathcal{A}$. Then,

- (i) $\|a\| \leq 1$ if and only if

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$$

is positive in $\mathbb{M}_2(\mathcal{A})$.

- (ii) The matrix

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$$

is positive in $\mathbb{M}_2(\mathcal{A})$ if and only if $a^*a \leq b$.

Proof. Faithfully represent \mathcal{A} on \mathcal{H} via $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$, and set $A = \pi(a)$. Then, if $\|A\| \leq 1$, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, y \rangle + \langle Ay, x \rangle + \langle x, Ay \rangle + \langle y, y \rangle \\ &\geq \|x\|^2 - 2\|A\|\|x\|\|y\| + \|y\|^2 \\ &\geq 0. \end{aligned}$$

Conversely, if $\|A\| > 1$, then there are unit vectors x and y such that $\langle Ay, x \rangle < -1$, so the inner product above would be negative.

Now, if we let $B = \pi(b)$, then if we let $B \geq A^*A$, then $B - A^*A \geq 0$, so that $\langle By, y \rangle \geq \langle Ay, Ay \rangle$ for all $y \in \mathcal{H}$, meaning that for all $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle Ax, y \rangle + \langle A^*x, y \rangle + \langle By, y \rangle \\ &\geq \langle x, x \rangle + 2\operatorname{Re}\langle Ax, y \rangle + \langle Ay, Ay \rangle \\ &\geq \|x\|^2 + \|Ay\|^2 - 2\|Ay\|\|x\| \\ &\geq 0. \end{aligned}$$

In the case that $B \not\geq A^*A$, then there is some unit vector y such that $\langle By, y \rangle < \|Ay\|^2$, which would yield the analogous outcome as in the proof of (i). \square

Dilations and Extensions

Theorem (Stinespring's Dilation Theorem): Let \mathcal{A} be a unital C^* -algebra, and let $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely positive map. Then, there exists a Hilbert space \mathcal{K} , a unital $*$ -homomorphism $\pi: \mathcal{A} \rightarrow B(\mathcal{K})$,

and a bounded operator $V: \mathcal{H} \rightarrow \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that

$$\phi(a) = V^* \pi(a) V.$$

Proof. Let $A \odot \mathcal{H}$ be the algebraic tensor product, and define a symmetric bilinear map

$$(a \otimes x, b \otimes y) = \langle \phi(b^* a)x, y \rangle,$$

and extend linearly, where $\langle \cdot, \cdot \rangle$ is the inner product on \mathcal{H} . Then, since ϕ is completely positive, it follows that (\cdot, \cdot) is positive semidefinite, with

$$\begin{aligned} \left(\sum_{j=1}^n a_j \otimes x_j, \sum_{i=1}^n a_i \otimes x_i \right) &= \left\langle \phi_n \left((a_i^* a_j)_{i,j} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\ &\geq 0. \end{aligned}$$

Since positive semidefinite bilinear forms satisfy the Cauchy–Schwarz inequality, we may define the subspace

$$\begin{aligned} \mathcal{N} &= \{u \in A \odot \mathcal{H} \mid (u, u) = 0\} \\ &= \{u \in A \odot \mathcal{H} \mid (u, v) = 0 \text{ for all } v \in A \odot \mathcal{H}\}, \end{aligned}$$

with an induced bilinear form on the quotient space $A \odot \mathcal{H}/\mathcal{N}$ defined by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle = (u, v).$$

Define \mathcal{K} to be the Hilbert space completion of $A \odot \mathcal{H}/\mathcal{N}$. Now, define a linear map $\pi(a): A \odot \mathcal{H} \rightarrow A \odot \mathcal{H}$ by

$$\pi(a) \left(\sum_{i=1}^n a_i \otimes x_i \right) = \sum_{i=1}^n (aa_i) \otimes x_i.$$

We have that the inequality in $\text{IM}_n(\mathcal{A})$ given by

$$(a_i^* a^* aa_j)_{i,j} \leq \|a^* a\| (a_i^* a_j)_{i,j}$$

is satisfied, giving

$$\begin{aligned} \left(\pi(a) \left(\sum_{j=1}^n a_j \otimes x_j \right), \pi(a) \left(\sum_{i=1}^n a_i \otimes x_i \right) \right) &= \sum_{i,j=1}^n \langle \phi(a_i^* a^* aa_j) x_j, x_i \rangle \\ &\leq \|a^* a\| \sum_{i,j=1}^n \langle \phi(a_i^* a_j) x_j, x_i \rangle \\ &= \|a\|^2 \left(\sum_{j=1}^n a_j \otimes x_j, \sum_{i=1}^n a_i \otimes x_i \right). \end{aligned}$$

Therefore, $\pi(a)$ is invariant under \mathcal{N} , so induces a quotient map on $A \otimes \mathcal{H}/\mathcal{N}$, which we will also denote by $\pi(a)$. We have that (this new) $\pi(a)$ is bounded with $\|\pi(a)\| \leq \|a\|$, meaning that it extends to a bounded linear map on \mathcal{K} .

We define $V: \mathcal{H} \rightarrow \mathcal{K}$ by

$$V(x) = 1 \otimes x + \mathcal{N}.$$

Then,

$$\|Vx\|^2 = (1 \otimes x, 1 \otimes x)$$

$$\begin{aligned}
&= \langle \phi(1)x, x \rangle \\
&\leq \|\phi(1)\| \|x\|^2,
\end{aligned}$$

and $\|V\|^2 = \|\phi(1)\|$.

Finally, we observe that

$$\begin{aligned}
\langle V^* \pi(a) V x, y \rangle &= (\pi(a) 1 \otimes x, 1 \otimes y) \\
&= \langle \phi(a)x, y \rangle
\end{aligned}$$

for all x and y , so $V^* \pi(a) V = \phi(a)$. \square

We observe that if ϕ is unital, then V is an isometry, and we may identify \mathcal{H} with the subspace $V\mathcal{H}$ of \mathcal{K} , and that $\phi(a) = P\pi(a)P$, where P is the projection onto \mathcal{H} . In particular, this means that every unital completely positive map is the compression of a *-homomorphism.

This construction is very similar to the GNS representation, and we call the triple (π, V, \mathcal{K}) a Stinespring representation for ϕ .

Nuclearity and Exactness

Definition: We call a map $\theta: A \rightarrow B$ between C^* -algebras *nuclear* if there are contractive completely positive maps $\varphi_n: A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n: \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow B$ such that $\psi_n \circ \varphi_n \rightarrow \theta$ pointwise in the norm topology:

$$\|\psi_n \circ \varphi_n(a) - \theta(a)\| \rightarrow 0$$

for all $a \in A$.

Definition: If A is a C^* -algebra, and N is a von Neumann algebra, then a map $\theta: A \rightarrow N$ is called *weakly nuclear* if there exist contractive completely positive maps $\varphi_n: A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n: \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow N$ such that $\psi_n \circ \varphi_n \rightarrow \theta$ pointwise in the ultraweak topology:

$$\eta(\psi_n \circ \varphi_n(a)) \rightarrow \eta(\theta(a))$$

for all $a \in A$ and normal functionals $\eta \in N_*$.

By uniqueness of preduals, if $N \subseteq B(\mathcal{H})$ is a faithful normal representation, then it suffices to observe that $\psi_n \circ \varphi_n \rightarrow \theta$ pointwise in the ultraweak topology if and only if

$$\langle \psi_n \circ \varphi_n(a)v, w \rangle \rightarrow \langle \theta(a)v, w \rangle$$

for all $a \in A$ and $v, w \in \Omega$ for some collection Ω of vectors whose linear span is dense in \mathcal{H} .

Application to Amenability

References

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