This is a collection of old real analysis qualifier exam solutions.

August 2019

Problem 1

(a) Recall that the Cantor set $\mathcal{C}$ is defined to consist of all $x \in [0,1]$ such that x only contains 0 and 2 in the ternary expansion of x. Writing $a \in [0,2]$ as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where $a_k \in \{0,1,2\}$, we may then find a_k at each ternary expansion slot for k as follows:

- if $a_k = 0$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = c_k = 0$
- if $a_k = 2$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = 2$ and $c_k = 0$ or vice versa.
- if $a_k = 1$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_{k+1} = c_{k+1} = 2$.

Therefore, since every digit of every ternary expansion in $[0,2]$ can be obtained from $\mathcal{C}$, we see that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) We may set B to be the union of all integer translates of $\mathcal{C}$, and set $A = \mathcal{C}$. This yields closed subsets of $\mathbb{R}$ with Lebesgue measure zero that sum to $\mathbb{R}$.

Problem 2

Consider the sequence of functions

$$f_n(x) = n\mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on $[0,1]$. This sequence is pointwise convergent everywhere to zero, as $f_n(0) = 0$ and the Archimedean property give that for any $x \in (0,1]$, there is some n large enough that gives $\frac{1}{n} < x$. Furthermore, we see that

$$\begin{aligned}\int f_n \, d\mu &= n\left(\frac{1}{n} - \frac{1}{n+1}\right)\\ &= \frac{1}{n+1}\\ &\to 0.\end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned}\int \Phi \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1}\\ &\to \infty.\end{aligned}$$

Problem 4

Suppose toward contradiction that both f and $1/f$ are in $L_1(\mathbb{R})$. Then, from Hölder's Inequality, we have

$$\begin{aligned}\infty &= \int 1 \, d\mu\\ &\leqslant \left(\int f \, d\mu\right)^{1/2}\left(\int \frac{1}{f} \, d\mu\right)^{1/2}\\ &< \infty,\end{aligned}$$

which is a contradiction.

Problem 5

(a) Let $f \in L_2([-1,1])$. We may find $g \in C([-1,1])$ such that $\|f-g\|_{L_2} < \varepsilon/2$. Similarly, we may find a polynomial p such that $\|g-p\|_u < \varepsilon/4$, meaning that $|p(x)-g(x)| < \varepsilon/4$ for all $x \in [-1,1]$. This yields

$$\begin{aligned}\|p-g\|_{L_2} &= \left(\int_{-1}^{1} |p(x)-g(x)|^2\,dx\right)^{1/2}\\ &< \left(\int_{-1}^{1} \left(\frac{\varepsilon}{4}\right)^2\,dx\right)^{1/2}\\ &= \left(\frac{\varepsilon^2}{8}\right)^{1/2}\\ &< \frac{\varepsilon}{2},\end{aligned}$$

so $\|f-p\|_{L_2} < \varepsilon$, meaning that the closed linear span of the monomials is dense in L_2, and the Legendre polynomials form an orthonormal system.

(b) We see that at every step in evaluating the expression

$$L_n(x) = c_n \frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{$*$}$$

the degree of the polynomial increases by 1, so each $L_n(x)$ has degree n. To verify that the polynomials generated from $(*)$ are orthogonal to each other, we let $n > m$ without loss of generality, and use integration by parts to obtain

$$\begin{aligned}\langle L_n, L_m\rangle &= \int_{-1}^{1}\left(\frac{d^n}{dx^n}\left(x^2-1\right)^n\right)\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx\\ &= \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^m}{dx^m}\left(x^2-1\right)^m\bigg|_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^{m+1}}{dx^{m+1}}\left(x^2-1\right)^m dx\\ &\vdots\\ &= (-1)^n\int_{-1}^{1}\frac{d^{m+n}}{dx^{m+n}}\left(x^2-1\right)^m dx\\ &= (-1)^n\int_{-1}^{1}\frac{d^n}{dx^n}\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx\\ &= (-1)^n\int\frac{d^n}{dx^n}L_m(x)\,dx\\ &= 0,\end{aligned}$$

seeing as we are taking n derivatives of a degree $m < n$ polynomial.

January 2020

Problem 1

(a) This is false. If $A \subseteq [0,1]$ is the "fat Cantor set" constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then $\mu(A) = \frac{1}{2}$, but A is nowhere dense, meaning that if $U \subseteq A$ is open, then $U = \emptyset$.

To see that A is nowhere dense, we see that A is closed, so if $x \in A \subseteq [0,1]$, and $\varepsilon > 0$, we may show that the interval $(x-\varepsilon, x+\varepsilon)$ is not contained in A. In the recursive construction of A, we may see that there is some step n_1 such that $\frac{1}{4^{n_1}} < 2\varepsilon$, implying that $(x-\varepsilon, x+\varepsilon)$ is not contained in the recursive construction at n_1. Therefore $A^\circ = \emptyset$.

(b) This is true. By the definition of the Lebesgue outer measure, for any $\varepsilon > 0$, there are $\{(a_k, b_k)\}_{k=1}^{\infty}$ such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^{\infty}(a_k, b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^{\infty}(a_k, b_k),$$

we have that U is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\left\{U \;\middle|\; A \subseteq U, U \text{ open}\right\}.$$

Remark: Part (a) can be solved by selecting $A = \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$.

Problem 3

(a) Consider the algebra of polynomials on $[0,1]$ without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and $f(x) = x$ separates points in $[0,1]$, this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at $x = 0$, the uniform closure of the algebra yields all the continuous functions on $[0,1]$ with $f(0) = 0$.

(b) In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra $\mathcal{A}$ to include the constant functions.

Problem 4

We consider the signed measure on $\mathcal{F}$ defined by

$$\nu(E) = \int_E f \, d\mu,$$

meaning that $\nu \ll \mu$, so the function $g := \frac{d\nu}{d\mu}$, where $\frac{d\nu}{d\mu}$ denotes the Radon–Nikodym derivative of ν with respect to μ (where we restrict μ to $\mathcal{F}$), is $\mathcal{F}$-measurable (by Radon–Nikodym) and in $L_1(\mathbb{R}, \mathcal{F}, \mu)$. This gives, for all $E \in \mathcal{F}$,

$$\begin{aligned}\int_E g \, d\mu &= \int_E \frac{d\nu}{d\mu} \, d\mu \\ &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f \, d\mu.\end{aligned}$$

Problem 5

Let $M = \mu(X)$.

Let $(f_n)_n \to f$ in measure, and let $\varepsilon > 0$. If we let

$$\begin{aligned} A &= \{x \mid |f_n(x) - f(x)| > \varepsilon/2M\} \\ B &= \{x \mid |f_n(x) - f(x)| \leqslant \varepsilon/2M\}, \end{aligned}$$

we have

$$\begin{aligned} \int_X \min(1, |f_n - f|)\, d\mu &= \int_A \min(1, |f_n - f|)\, d\mu + \int_B \min(1, |f_n - f|)\, d\mu \\ &\leqslant \mu(A) + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Meanwhile, if

$$\int_X \min(1, |f_n - f|)\, d\mu \to 0,$$

then by Chebyshev's Inequality, we have, for a fixed $0 < \varepsilon \leqslant 1$,

$$\begin{aligned} \mu(\{x \mid |f_n - f| \geqslant \varepsilon\}) &= \mu(\{x \mid \min(1, |f_n - f|) \geqslant \varepsilon\}) \\ &\leqslant \frac{1}{\varepsilon} \int_X \min(1, |f_n - f|)\, d\mu \\ &\to 0, \end{aligned}$$

so $(f_n)_n \to f$ in measure.

August 2020

Problem 1

This is false. To see this, let $\mathfrak{C}(x)$ denote the Cantor–Lebesgue function, and let

$$h(x) = \sum_{n=-\infty}^{\infty} \mathfrak{C}(x - n) + n.$$

Then, since $\mathfrak{C}(x)$ has derivative zero almost everywhere, the sum of a number of translates of $\mathfrak{C}(x)$ still has derivative zero almost everywhere. Then, setting

$$f(x) = h(x) + x,$$

we get that $f(x)$ has derivative equal to 1 almost everywhere. However, at the same time, $f(2) - f(1) = 2$.

Problem 2

We show the inverse problem, which is that every closed set in $\mathbb{R}^2$ is G_δ. To do this, we let $A \subseteq \mathbb{R}^2$ be closed, nonempty, and proper (if $A = \emptyset$ or $A = \mathbb{R}^2$ the answer is trivial).

Then, there is some $x \in A^c$, and specifically there is $x \in A^c$ with rational coordinates (else, select $y \in \mathbb{Q}^2$ within the ball of radius ε that allows A^c to be open). Furthermore, since $\mathbb{R}^2$ is a metric space, $\mathbb{R}^2$ is regular, so there are open U_x and V_x such that $A \subseteq U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$.

Therefore, we get

$$A = \bigcap \left\{U_x \;\middle|\; x \in \mathbb{Q}^2 \setminus A\right\},$$

meaning that A is G_δ. Taking complements, we thus get that every open set is F_σ.

Problem 3

(a) We see that

$$\begin{aligned}\left\langle Pf_i, f_j\right\rangle &= \delta_{i+1,j}\\ &= \delta_{i,j-1}\\ &= \left\langle f_i, f_{j-1}\right\rangle\\ &= \left\langle f_i, P^*f_j\right\rangle,\end{aligned}$$

so that $Pf_n = f_{n-1}$ if $n > 1$. Else, if $n = 1$, then $P^*f_n = 0$.

(b) We see that, acting on the orthonormal basis $(f_n)_n$, $P^*P(f_n) = f_n$, and

$$PP^*(f_n) = \begin{cases}0 & n = 1\\ 1 & \text{else},\end{cases}$$

so that $P^*P = I$ and PP^* is as above.

Problem 4

We see that

$$\mu(\{x \mid f_n(x) > t\}) = \mu(X) - \mu(\{x \mid f_n(x) \leqslant t\}),$$

so by taking limits, we find that

$$\lim_{n\to\infty} \mu(\{x \mid f_n(x) > t\}) = \begin{cases}1 & t < 0\\ 0 & t \geqslant 0\end{cases}.$$

So, if $\varepsilon > 0$, then

$$\begin{aligned}\mu(\{x \mid |f_n(x)| > \varepsilon\}) &= \mu(\{x \mid f_n(x) < -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\})\\ &\leqslant \mu(\{x \mid f_n(x) \leqslant -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\})\\ &\to 0.\end{aligned}$$

January 2021

Problem 1

This is not true. To see this, consider the family of functions defined by

$$\begin{aligned}f_1 &= \mathbb{1}_{[0,1]}\\ f_2 &= \mathbb{1}_{[0,1/2]}\\ f_3 &= \mathbb{1}_{[1/2,1]}\\ &\vdots\end{aligned}$$

where f_n is of width $\frac{1}{2^k}$ when $2^k \leqslant n < 2^{k+1}$, moving along $[0,1]$. Then, since $\mu(\{x \mid |f_n(x)| > 0\}) = \frac{1}{2^k}$, we have that $(f_n)_n \to 0$ in measure. Yet, since for any $x \in [0,1]$ there are infinitely many such n such that $f_n(x) = 1$, the family $(f_n)_n$ does not converge to 0 pointwise anywhere on $[0,1]$.

Problem 2

Note that the two-dimensional Lebesgue measure is the completion of $m \times m$, where $m \times m$ is the product measure on the product σ-algebra $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. If $B \in \mathcal{L}(\mathbb{R}^2)$, then $B = C \cup N$, where N is a μ-null set and $C \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. Therefore, if we show that $(m \times m)(C) = 0$, we then show that $\mu(B) = 0$.

To see that $(m \times m)(\mathbb{C}) = 0$, note that by our assumption, $B^x = \{y \in \mathbb{R} \mid (x, y) \in B\}$ is either finite or countable, so since $C^x \subseteq B^x$, we must have that C^x is either finite or countable. By Tonelli's Theorem, since $\mathbb{1}_C$ is positive, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbb{1}_C \, d(m \times m) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{C^x} \, dy \, dx \\ &= \int_{\mathbb{R}} m(C^x) \, dx \\ &= 0,\end{aligned}$$

so $(m \times m)(C^x) = 0$, meaning

$$\begin{aligned}\mu(B) &= \mu(C) + \mu(N) \\ &= (m \times m)(C) + \mu(N) \\ &= 0.\end{aligned}$$

Problem 3

Since $\mu \ll \nu$, and $\rho \ll \rho$, we have $\mu + \rho \ll \nu + \rho$, so by Radon–Nikodym, there is some measurable f such that

$$\mu(E) + \rho(E) = \int_E f \, d(\nu + \rho),$$

so

$$\mu(E) = \int_E f \, d\nu + \int_E (f - 1) \, d\rho.$$

Problem 4

Since all of f, g, a, b, c, d are positive, we may show

$$ac + bd + \int_0^1 f(x)g(x) \, dx \leqslant \left(a^3 + b^3 + \int_0^1 (f(x))^3 \, dx\right)^{1/3} \left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx\right)^{2/3}.$$

To do this, we use Hölder's Inequality three times:

$$\begin{aligned}ac + bd + \int_0^1 f(x)g(x) \, dx &\leqslant \left(a^3 + b^3\right)^{1/3} \left(c^{3/2} + d^{3/2}\right)^{2/3} + \int_0^1 f(x)g(x) \, dx \\ &\leqslant \left(a^3 + b^3\right)^{1/3} \left(c^{3/2} + d^{3/2}\right)^{2/3} + \left(\int_0^1 (f(x))^3 \, dx\right)^{1/3} \left(\int_0^1 (g(x))^{3/2} \, dx\right)^{2/3} \\ &\leqslant \left(a^3 + b^3 + \int_0^1 (f(x))^3 \, dx\right)^{1/3} \left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx\right)^{2/3}.\end{aligned}$$

Problem 5

We note that for each n,

$$\left(\frac{d}{dx}\right)^n\left(e^{-x^2}\right) = P_n(x)e^{-x^2}$$

where $P_n(x)$ is a degree n polynomial. To see this,

$$\begin{aligned}\frac{d}{dx}\left(e^{-x^2}\right) &= -2xe^{-x^2}\\ \frac{d}{dx}\left(P_n(x)e^{-x^2}\right) &= P_n'(x)e^{-x^2} - 2xP_n(x)e^{-x^2}\\ &:= P_{n+1}(x)e^{-x^2}.\end{aligned}$$

Therefore,

$$e^{x^2}\left(\frac{d}{dx}\right)^n\left(e^{-x^2}\right) = P_n(x).$$

Since each $P_n(x)$ is linearly independent (as they have different degrees of polynomials), the linear combinations of P_n form polynomials. By Stone–Weierstrass, there is some linear combination $\sum_{k=0}^{n} a_k P_k(x)$ such that $\left|f(x) - \sum_{k=0}^{n} a_k P_k(x)\right| < \varepsilon$ for all $x \in [0,1]$. Thus,

$$\left|f(x) - e^{x^2}\left(De^{-x^2}\right)\right| < \varepsilon$$

for the particular expression

$$D = \sum_{k=0}^{n} a_k\left(\frac{d}{dx}\right)^k.$$

August 2022

Problem 1

We note that

$$\begin{aligned}\left|\frac{n\sin(x/n)}{x(1+x^2)}\right| &\leqslant \left|\frac{n(x/n)}{x(1+x^2)}\right|\\ &= \frac{1}{1+x^2},\end{aligned}$$

and since $\frac{1}{1+x^2}$ is integrable, we may use Dominated Convergence to switch limit and integral, giving

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx &= \int_0^\infty \lim_{n\to\infty}\frac{n\sin(x/n)}{x(1+x^2)}\,dx\\ &= \int_0^\infty \lim_{h\to 0}\frac{\frac{1}{h}\sin(hx)}{x(1+x^2)}\,dx\\ &= \int_0^\infty \frac{x}{x(1+x^2)}\,dx\\ &= \frac{\pi}{2}.\end{aligned}$$

Problem 2

(a) Let f be Lipschitz, and let M denote the Lipschitz constant — i.e., $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [a, b]$. Set $\delta = \frac{\varepsilon}{M}$. Then, if $\left\{\left(a_j, b_j\right)\right\}_{j=1}^{k}$ is a partition such that $\sum_{j=1}^{k}\left|b_j - a_j\right| < \delta$, we have

$$\begin{aligned}\sum_{j=1}^{k}\left|f\left(b_j\right) - f\left(a_j\right)\right| &\leqslant M\sum_{j=1}^{k}\left|b_j - a_j\right| \\ &< \varepsilon.\end{aligned}$$

Thus, f is absolutely continuous. Now, if $x, x + h \in [a, b]$, we have that

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leqslant M,$$

meaning that

$$\begin{aligned}|f'(x)| &= \lim_{h\to 0}\left|\frac{f(x+h) - f(x)}{h}\right| \\ &\leqslant M,\end{aligned}$$

and since $f'(x)$ exists for a.e. $x \in [a, b]$, we have that $\operatorname{ess\,sup}_{x\in[a,b]}|f'(x)| \leqslant M$, so $f' \in L_\infty([a, b])$.

Let f be absolutely continuous with bounded derivative. Then, if M is the essential supremum of the f', the fundamental theorem of calculus gives

$$\begin{aligned}|f(y) - f(x)| &= \left|\int_x^y f'(t)\,dt\right| \\ &\leqslant \int_x^y |f'(t)|\,dt \\ &\leqslant \int_x^y M\,dx \\ &= M|y - x|,\end{aligned}$$

so f is Lipschitz.

(b) If f is such that $f'(x)$ exists, then for $x, x + h \in [a, b]$, we have

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leqslant \|f\|_{\mathrm{Lip}},$$

so by taking limits, we have

$$|f'(x)| \leqslant \|f\|_{\mathrm{Lip}}.$$

Thus, this ordering must respect essential suprema, meaning

$$\|f'\|_{L_\infty} \leqslant \|f\|_{\mathrm{Lip}}.$$

Furthermore, if $\varepsilon > 0$, there are $x, y \in [a, b]$ with $x < y$ such that

$$\begin{aligned}\|f\|_{\mathrm{Lip}} - \varepsilon &< \left|\frac{f(y) - f(x)}{y - x}\right| \\ &= \frac{1}{|y - x|}\left|\int_x^y f'(t)\,dt\right|\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{|y-x|}\int_x^y |f'(t)|\,dt\\
&\leqslant \frac{1}{|y-x|}\int_x^y \|f'\|_{L_\infty}\,dt\\
&= \|f'\|_{L_\infty},
\end{aligned}$$

and since ε is arbitrary, we have $\|f\|_{\mathrm{Lip}} \leqslant \|f'\|_{L_\infty}$.

Problem 3

We start by showing that

$$|a-b| = \int_0^\infty \left|\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)\right|\,dt$$

for all $a, b \in [0,\infty)$. Without loss of generality, $a \leqslant b$. To see this, note that there are three cases:

$$\left|1_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)\right| = \begin{cases} 0 & t < a, b \\ 1 & a \leqslant t < b \\ 0 & a, b \leqslant t \end{cases},$$

giving

$$\begin{aligned}
\int_0^\infty \mathbb{1}_{[a,b)}\,dt &= \mu([a,b))\\
&= b-a\\
&= |a-b|.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|f-g\|_{L_1} &= \int_X |f(x)-g(x)|\,d\mu(x)\\
&= \int_X\int_0^\infty \left|\mathbb{1}_{(t,\infty)}(f(x)) - \mathbb{1}_{(t,\infty)}(g(x))\right|\,dt\,d\mu(x),
\end{aligned}$$

and by Tonelli's Theorem, we have

$$\begin{aligned}
&= \int_0^\infty\int_X \left|\mathbb{1}_{f^{-1}((t,\infty))} - \mathbb{1}_{g^{-1}((t,\infty))}\right|\,d\mu(x)\,dt\\
&= \int_0^\infty\int_X \mathbb{1}_{f^{-1}((t,\infty))\triangle g^{-1}((t,\infty))}\,d\mu(x)\,dt\\
&= \int_0^\infty \mu\left(f^{-1}((t,\infty))\triangle g^{-1}((t,\infty))\right)\,dt.
\end{aligned}$$

Problem 4

(a) Since $|\mu| \perp |\nu|$, there are $U, V \subseteq X$ such that $|\mu|$ is concentrated on U and $|\nu|$ is concentrated on V, with $U \cap V = \emptyset$.

Note that by the Jordan decompositions, we have $|\mu| = \mu_1 + \mu_2 \geqslant \mu_{1,2}$ so $\mu_{1,2}$ are concentrated on U, and similarly $\nu_{1,2}$ are concentrated on V, so $\mu_i \perp \nu_j$.

(b) We show that the measures $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ are mutually singular. To see this, note the following:

- $\mu_1 = 0$ on $N_\mu \cup V$;
- $\nu_1 = 0$ on $N_\nu \cup U$;
- $\mu_2 = 0$ on $P_\mu \cup V$;
- $\nu_2 = 0$ on $P_\nu \cup U$,

so $\mu_1 + \nu_1 = 0$ on $A = \left(N_\mu \cup V\right) \cap (N_\nu \cup U)$, and $\mu_2 + \nu_2 = 0$ on $B = \left(P_\mu \cup V\right) \cap (P_\nu \cup U)$. Therefore, since

$$\begin{aligned}
A \cup B &= \left(N_\mu \cap N_\nu\right) \cup \left(N_\mu \cap U\right) \cup (N_\nu \cap V)\\
&\quad \cup \left(P_\mu \cap P_\mu\right) \cup \left(P_\mu \cap U\right) \cup (P_\nu \cap V)\\
&= X
\end{aligned}$$

$$\begin{aligned}
A \cap B &= \left(N_\mu \cup V\right) \cap (N_\nu \cup U)\\
&\quad \cap \left(P_\mu \cup V\right) \cap (P_\nu \cup U)\\
&= \varnothing,
\end{aligned}$$

the measures $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ are mutually singular, so $A \sqcup B$ forms a Hahn decomposition for $\mu + \nu$ with corresponding Jordan decomposition of $(\mu_1 + \nu_1) - (\mu_2 + \nu_2)$. Thus,

$$\begin{aligned}
\|\mu + \nu\|_{TV} &= |\mu + \nu|(X)\\
&= (\mu_1 + \nu_1)(X) + (\mu_2 + \nu_2)(X)\\
&= (\mu_1 + \mu_2)(X) + (\nu_1 + \nu_2)(X)\\
&= |\mu|(X) + |\nu|(X)\\
&= \|\mu\|_{TV} + \|\nu\|_{TV}.
\end{aligned}$$

Problem 5

(a) The conclusion of the Lebesgue differentiation theorem states that $\mu([0,1] \setminus L_f) = 0$.

(b) Let $x \in [0,1]$. We note that x must be in exactly one such interval $(j2^{-n}, (j+1)2^{-n}]$ since these intervals are disjoint. If we select $r > 0$ such that $\frac{1}{2^n} < r \leqslant \frac{1}{2^{n-1}}$, then we note the following:

- $I_{n,j} \subseteq U(x,r)$ for exactly one such j;
- $m(U(x,r)) \leqslant 4\mu\left(I_{n,j}\right)$.

If $x \in L_f$, then for any $\varepsilon > 0$, there is some $\delta > 0$ such that when $r < \delta$, then

$$\frac{1}{\mu(U(x,r))} \int_{U(x,r)} |f(t) - f(x)|\, dt < \varepsilon,$$

by the Lebesgue Differentiation Theorem. If n is such that $\frac{1}{2^{n-1}} < \delta$, then when $\frac{1}{2^n} < r \leqslant \frac{1}{2^{n-1}}$, then for any $x \in L_f$, we have

$$\begin{aligned}
|E_n f(x) - f(x)| &= \left|\frac{1}{m\left(I_{n,j}\right)} \int_{I_{n,j}} f(t)\, dt - f(x)\right|\\
&\leqslant \frac{1}{m\left(I_{n,j}\right)} \int_{I_{n,j}} |f(t) - f(x)|\, dt\\
&\leqslant \frac{1}{m\left(I_{n,j}\right)} \int_{U(x,r)} |f(t) - f(x)|\, dt\\
&\leqslant \frac{4}{U(x,r)} \int_{U(x,r)} |f(t) - f(x)|\, dt\\
&< 4\varepsilon,
\end{aligned}$$

so $\lim_{n\to\infty} E_n f(x) = f(x)$ for all $x \in L_f$, meaning that it holds for a.e. $x \in [0,1]$.

January 2023

Problem 1

By using Fatou's Lemma, and assuming WLOG that $(f_n)_n \to f$ pointwise everywhere, we get

$$\begin{aligned}\int_X |f|^p \, d\mu &= \int_X \liminf_{n\to\infty} |f_n|^p \, d\mu \\ &\leqslant \liminf_{n\to\infty} \int_X |f_n|^p \, d\mu \\ &\leqslant 1,\end{aligned}$$

so $\|f\|_{L_p} \leqslant 1$.

Problem 2

Let

$$f(t) = \mu(E \cap (-\infty, t)),$$

and for any sequence $(t_n)_n$, define

$$E_n = E \cap (-\infty, t_n).$$

We will show that f is left- and right-continuous, hence continuous. To start, if $(t_n)_n \searrow t$, then

$$\bigcap_{n\in\mathbb{N}} E_n = E \cap (-\infty, t],$$

so

$$\begin{aligned}f(t) &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n \setminus \{t\}\right) \\ &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) - \mu(\{t\}).\end{aligned}$$

Since μ is atomless, we see that $\mu(\{t\}) = 0$, so since $\mu(E) < \infty$,

$$\begin{aligned}f(t) &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) \\ &= \lim_{n\to\infty} \mu(E_n) \\ &= \lim_{n\to\infty} f(t_n).\end{aligned}$$

Thus, f is right-continuous. Similarly, if f is left-continuous, and $(t_n)_n \nearrow t$, then

$$\bigcup_{n\in\mathbb{N}} E_n = E \cap (-\infty, t),$$

so by continuity from below,

$$\begin{aligned}f(t) &= \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \\ &= \lim_{n\to\infty} \mu(E_n)\end{aligned}$$

$$= \lim_{n\to\infty} f(t_n).$$

Therefore, f is continuous. Since

$$\begin{aligned}\lim_{t\to-\infty} f(t) &= 0\\ \lim_{t\to\infty} f(t) &= \mu(E)\\ &> 0,\end{aligned}$$

the intermediate value theorem gives some $t_0 \in \mathbb{R}$ such that

$$\begin{aligned}f(t_0) &= \mu(E \cap (-\infty, t_0))\\ &= \frac{1}{2}\mu(E).\end{aligned}$$

Problem 4

We start by showing that $\|\cdot\|_{W_p}$ is indeed a norm. To see that $\|\cdot\|_{W_p}$ is positive definite, if

$$\|f\|_{W_p} = 0,$$

then $|f(0)| = 0$ and $\|f'\|_{L_p} = 0$. Since $\|f'\|_{L_p} = 0$, $f' = 0$ a.e. as L_p is a Banach space. Note that, by the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t)\,dt,$$

so $f(x) = 0$ almost everywhere, hence $f(x) = 0$ in L_p.

Next, to see homogeneity, we have for all $\alpha \in \mathbb{C}$,

$$\begin{aligned}\|\alpha f\|_{W_p} &= |\alpha f(0)| + \left\|(\alpha f)'\right\|_{L_p}\\ &= |\alpha|\left(|\alpha| + \|f'\|_{L_p}\right)\\ &= |\alpha|\|f\|_{W_p},\end{aligned}$$

as $\|\cdot\|_{L_p}$ is a norm. Finally, we have

$$\begin{aligned}\|f+g\|_{W_p} &= |(f+g)(0)| + \left\|(f+g)'\right\|_{L_p}\\ &\leqslant |f(0)| + |g(0)| + \|f'\|_{L_p} + \|g'\|_{L_p}\\ &= \|f\|_{W_p} + \|g\|_{W_p},\end{aligned}$$

as $\|\cdot\|_{L_p}$ is a norm, so the triangle inequality holds. Thus, $\|\cdot\|_{W_p}$ is a norm.

Let $(f_n)_n$ be Cauchy in $W_p([0,1])$. Then, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geqslant N$,

$$\begin{aligned}\|f_n - f_m\|_{W_p} &= |f_n(0) - f_m(0)| + \|f_n' - f_m'\|_{L_p}\\ &< \varepsilon,\end{aligned}$$

meaning that both

$$\begin{aligned}|f_n(0) - f_m(0)| &< \varepsilon\\ \|f_n' - f_m'\|_{L_p} &< \varepsilon.\end{aligned}$$

Since $\mathbb{C}$ and $L_p([0,1])$ are complete, there is $c \in \mathbb{C}$ and $g \in L_p([0,1])$ such that

$$\begin{aligned} f_n(0) &\to c \\ f_n' &\to g. \end{aligned}$$

Define

$$f(x) = c + \int_0^x g(t)\, dt.$$

Then, we note that by the Fundamental Theorem of Calculus,

$$\begin{aligned} f'(x) &= g(x) \\ &\in L_p([0,1]), \end{aligned}$$

so $f \in W_p([0,1])$. Finally, we see that

$$\begin{aligned} \|f_n - f\|_{W_p([0,1])} &= |f_n(0) - f(0)| + \|f_n' - f'\|_{L_p} \\ &= |f_n(0) - c| + \|f_n' - g\|_{L_p} \\ &\to 0, \end{aligned}$$

so $(f_n)_n \to f$ in W_p, meaning W_p is complete.

Problem 5

(i) Letting $f\colon \Omega \to \mathbb{R}$ be defined by $f(\mathbb{1}_E) = m(E \cap [a,b])$, we have

$$\begin{aligned} |m(E \cap [a,b]) - m(F \cap [a,b])| &= \left|\int_a^b \mathbb{1}_E - \mathbb{1}_F\, dm\right| \\ &\leqslant \int_a^b |\mathbb{1}_E - \mathbb{1}_F|\, dm \\ &\leqslant \int_{\mathbb{R}} |\mathbb{1}_E - \mathbb{1}_F|\, dm \\ &= \|\mathbb{1}_E - \mathbb{1}_F\|_{L_1}, \end{aligned}$$

meaning that f is Lipschitz, hence continuous.

(ii) Let $\mathbb{1}_F \in \Omega$. Then, $0 \leqslant \mu(F \cap [a,b]) \leqslant b - a$. If these inequalities are strict, then $F \in U_{a,b}$. Else, we let $\varepsilon > 0$, and see two cases:

- if $\mu(F \cap [a,b]) = b-a$, then we may set $E = F \setminus ([a, a+\varepsilon/) \cup (b - \varepsilon/2, b])$, so that $0 < \mu(E \cap [a,b]) < b - a$, and $\|\mathbb{1}_E - \mathbb{1}_F\|_{L_1} = \mu(E \triangle F) \leqslant \varepsilon$;
- if $\mu(F \cap [a,b]) = 0$, then we may set $E = F \cup ([a, a+\varepsilon/2) \cup [b - \varepsilon/2, b))$, meaning that $0 < \mu(E \cap [a,b]) < b - a$, and $\mu(E \triangle F) \leqslant \varepsilon$.

Therefore, $U_{a,b}$ is dense in Ω. To see that $U_{a,b}$ is open, notice that for any $\mathbb{1}_E \in U_{a,b}$, we may find $\varepsilon > 0$ such that $0 < \mu(E \cap [a,b]) - \varepsilon < \mu(E \cap [a,b]) < \mu(E \cap [a,b]) + \varepsilon < b - a$, and for all F with $\|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} < \varepsilon$, we have

$$\begin{aligned} |\mu(F \cap [a,b]) - \mu(E \cap [a,b])| &\leqslant \|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} \\ &< \varepsilon, \end{aligned}$$

so $0 < \mu(F \cap [a,b]) < b - a$. Thus, $U_{a,b}$ is also open.

(iii) If $\{[a_k, b_k]\}$ is an enumeration of rational-endpoint intervals in $\mathbb{R}$, then for any interval I, there is some rational-endpoint interval $[a_k, b_k] \subseteq I$ by density and the characterization of an interval, meaning that U_{a_k,b_k} is such that for the given interval I, $0 < \mu(E \cap I) < \mu(I)$ for all $E \in U_{a_k,b_k}$. Thus, taking the intersection of all such U_{a_k,b_k}, we have

$$\bigcap_{k=1}^{\infty} U_{a_k,b_k} \subseteq D.$$