Revised Problems

Problem (Homework 5, Problem 1)**:** If X is a connected space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy-equivalent to a wedge sum of 1-spheres and 2-spheres.

Solution: Let $A_1, \dots, A_n$ be the spheres in X; we will define a graph Γ where $V(\Gamma)$ denotes each of the spheres of X and $E(\Gamma)$ is the edge set defined defined by $\{v_i, v_j\} \in E(\Gamma)$ if and only if A_i and A_j are connected. Note that by our assumption, we have that Γ is a simple and connected graph.

First, we show that if Γ is a tree, then X is homotopy-equivalent to a wedge sum of 2-spheres. First, assign Γ a distinguished root vertex. Endow X with a CW complex structure by assigning

- 0-cells at each intersection point;
- extra 0-cells at each leaf and at the root of the tree, the latter of which we will denote a_0;
- 1-cells along the equator connecting all the 0-cells;
- 2-cells completing the respective spheres.

Since Γ is a tree, there is a unique path from the root of the vertex to each leaf. Traversing along a path from a_0 to the extra 0-cell on the leaf via 1-cells connecting to each intersection point on the intermediate spheres, we obtain a contractible subcomplex of X. Upon taking the quotient, we find that X is homotopy-equivalent to a wedge sum of all the spheres along this path (including the leaf) with the rest of X with connection point at a_0. Continuing in this fashion then gives that, if Γ is a tree, then X is homotopy-equivalent to a wedge sum of spheres.

If Γ is not a tree, then Γ admits spanning tree, and the rest of Γ is then given by a collection of edges that complete some number of cycles. Therefore, we show that if Γ is a cycle, then X is homotopy-equivalent to a wedge of 2-spheres (corresponding to each vertex) and a single 1-sphere. For this, observe that if Γ is a cycle, then X is homotopy-equivalent to a line of spheres connected by 0-cells at their equator with one extra 1-cell connecting 0-cells at the endpoints of the line. Collapsing along this equator will then give all the 2-spheres as a wedge sum identified with a single point,a swell as both ends of the extra 1-cell, meaning that we have that, in this case, X is a wedge sum of these 2-spheres with the 1-sphere corresponding to the extra 1-cell.

Therefore, in the general case, we may start by collapsing X along its spanning tree, which will necessarily collapse along every cycle, giving that X is a wedge sum of 1-spheres and 2-spheres.

Current Problems

Problem (Problem 1)**:** Consider the quotient space of S^2 obtained by identifying the north and south poles to a single point. Compute its fundamental group.

Solution: We observe in the figure below that the quotient can be expressed by homotopy-equivalent quotients of the sphere union a 1-cell A connecting between the north pole and the south pole. Therefore, this quotient space is homotopy-equivalent to $S^2 \vee S^1$, so that $\pi_1(X) = \mathbb{Z} * \{e\} = \mathbb{Z}$.