Solution (38.5)**:** Copying the template equation, we have

$$\frac{dv}{dt} = -\frac{c}{m}v^2 + g,$$

where c is some constant. We see that the terminal velocity is

$$v_t = \sqrt{\frac{mg}{c}}.$$

Separating variables, we have

$$\begin{aligned}\frac{dv}{-\frac{c}{m}v^2 + g} &= dt\\ \frac{1}{g}\left(\frac{dv}{1 - \frac{c}{mg}v^2}\right) &= dt\\ \frac{1}{g}\left(\frac{dv}{1 - (v/v_t)^2}\right) &= dt.\end{aligned}$$

Using the substitution $u \coloneqq v/v_t$, we have $du = \frac{1}{v_t}dv$, meaning that

$$v_t \int \frac{1}{1-u^2}\,du = \int g\,dt.$$

The integral of $\frac{1}{1-u^2}$ is $\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = \operatorname{arctanh}(u)$. Therefore, we have

$$\begin{aligned}\frac{v}{v_t} &= \tanh\left(\frac{g}{v_t}t\right) + K\\ v &= v_t \tanh\left(\frac{g}{v_t}t\right) + v_0\\ &= \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{c}{mg}}t\right) + v_0.\end{aligned}$$

Solution (38.6)**:**

(a) Using the chain rule and letting $\frac{dm}{dt} = km^{2/3}$, we have

$$\begin{aligned}\frac{dv}{dt} &= km^{2/3}\frac{dv}{dm}\\ \frac{dv}{dm} + \frac{v}{m} &= -\frac{b}{km}v + \frac{g}{km^{2/3}}.\end{aligned}$$

With integrating factor $m^{1+\frac{b}{k}}$, we have

$$\begin{aligned}m^{1+\frac{b}{k}}v &= \frac{g}{k}\frac{m^{\frac{4}{3}+\frac{b}{k}}}{\frac{4}{3}+\frac{b}{k}} + C\\ v &= \frac{g}{k\left(\frac{4}{3}+\frac{b}{k}\right)}m^{\frac{1}{3}+\frac{b}{k}} + Cm^{-1-\frac{b}{k}}.\end{aligned}$$

We let $v(m_0) = 0$, so that

$$C = -\frac{g}{k\left(\frac{4}{3}+\frac{b}{k}\right)}m_0^{\frac{4}{3}+\frac{b}{k}},$$

so

$$v = \frac{g}{\frac{4}{3}k + b}m^{\frac{1}{3}}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{4}{3}+\frac{b}{k}}\right).$$

Thus,

$$\begin{aligned}\frac{dv}{dt} &= g - \frac{1}{m}\frac{dm}{dt}v \\ &= g - \frac{1}{m}\left(km^{2/3}\right)\left(\frac{g}{\frac{4}{3}k + b}m^{\frac{1}{3}}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{4}{3}+\frac{b}{k}}\right)\right).\end{aligned}$$

(b) Using $\frac{dm}{dt} = km^{2/3}v$, and $\frac{dv}{dt} = km^{2/3}v\frac{dv}{dm}$, we obtain

$$\begin{aligned}m\frac{dv}{dt} + v\frac{dm}{dt} &= -bm^{2/3}v^2 + mg \\ v\,dv + \left(\frac{v^2}{m}\left(1 + \frac{b}{k}\right) - \frac{g}{km[2/3]}\right)dm &= 0.\end{aligned}$$

This gives $\alpha = v$ and $\beta = \frac{v^2}{m}\left(1 + \frac{b}{k}\right) - \frac{g}{km^{2/3}}$. Solving for $p(m)$, we get

$$\begin{aligned}p(m) &= \frac{1}{v}\left(\frac{2v}{m}\left(1 + \frac{b}{k}\right)\right) \\ &= \frac{2}{m}\left(1 + \frac{b}{k}\right).\end{aligned}$$

Therefore, our integrating factor is

$$w(x) = m^{2+\frac{2b}{k}}.$$

This gives

$$\begin{aligned}\frac{\partial \Phi}{\partial v} &= \alpha \\ \Phi &= \frac{1}{2}m^{2+\frac{2b}{k}}v^2 + c_1(m) \\ \frac{\partial \Phi}{\partial m} &= \beta \\ \Phi &= \frac{1}{2}m^{2+\frac{2b}{k}}v^2 - \frac{g}{k\left(\frac{7}{3} + \frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}} + c_2(v).\end{aligned}$$

Thus, $c_2(v) = 0$, and

$$\frac{1}{2}m^{2+\frac{2b}{k}}v^2 - \frac{g}{k\left(\frac{7}{3} + \frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}} = C.$$

Using $v(m_0) = 0$, we obtain the solution of

$$\frac{1}{2}m^{2+\frac{2b}{k}}v^2 = \frac{g}{k\left(\frac{7}{3} + \frac{2b}{k}\right)}m^{\frac{7}{3}+\frac{2b}{k}}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right).$$

Simplifying, this gives

$$v^2 = \frac{2g}{k\left(\frac{7}{3} + \frac{2b}{k}\right)}m^{\frac{1}{3}}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right).$$

Therefore,

$$2v\frac{dv}{dm} = \frac{2g}{3k\left(\frac{7}{3} + \frac{2b}{k}\right)}m^{-2/3}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right) + \frac{2g}{km}\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}},$$

and

$$\begin{aligned}\frac{dv}{dt} &= \frac{k}{2}m^{2/3}\left(2v\frac{dv}{dm}\right) \\ &= \frac{g}{3\left(\frac{7}{3} + \frac{2b}{k}\right)}\left(1 - \left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}\right) + \frac{g}{m^{\frac{1}{3}}}\left(\frac{m_0}{m}\right)^{\frac{7}{3}+\frac{2b}{k}}.\end{aligned}$$

Solution (38.7)**:** Via Newton's second law, we have

$$m\frac{dv}{dt} = -u\frac{dm}{dt} + F_{ext}.$$

Assuming $\frac{dm}{dt} = -k$ the rocket equation simplifies to

$$m\frac{dv}{dt} = uk + F_{ext}.$$

(a) Assume $F_{ext} = mg$. Then,

$$\begin{aligned} m\frac{dv}{dt} &= uk - mg \\ \frac{dv}{dt} &= \frac{uk}{m} - g \\ v &= \left(\frac{uk}{m} - g\right)t. \end{aligned}$$

(b) Assume $F_{ext} = bv$. Then,

$$\begin{aligned} m\frac{dv}{dt} &= uk - bv \\ \frac{m}{uk - bv}\,dv &= dt \\ t &= -\frac{m}{b}\ln(uk - bv) \\ \ln(uk - bv) &= -\frac{b}{m}t \\ v &= uk - Ce^{-\frac{b}{m}t}. \end{aligned}$$

Solution (39.5)**:** We take the derivative of

$$\frac{du_p}{dx} = a_1(x)\frac{du_1}{dx} + a_2(x)\frac{du_2}{dx},$$

giving

$$\frac{d^2u_p}{dx^2} = a_1(x)\frac{d^2u_1}{dx^2} + \frac{da_1}{dx}\frac{du_1}{dx} + a_2(x)\frac{d^2u}{dx^2} + \frac{da_2}{dx}\frac{du_2}{dx}.$$

Note that we must have

$$\frac{d^2u_p}{dx^2} + p(x)\frac{du_p}{dx} + q(x)u_p = r(x),$$

so we have

$$\begin{aligned} r(x) = a_1(x)\frac{d^2u_1}{dx^2} &+ \frac{da_1}{dx}\frac{du_1}{dx} + a_2(x)\frac{d^2u}{dx^2} + \frac{da_2}{dx}\frac{du_2}{dx} \\ &+ p(x)\left(a_1(x)\frac{du_1}{dx} + a_2(x)\frac{du_2}{dx}\right) \\ &+ q(x)(a_1(x)u_1(x) + a_2(x)u_2(x)). \end{aligned}$$

Reordering and simplifying, we get

$$\begin{aligned} r(x) &= a_1(x)\left(\frac{d^2u_1}{dx^2} + p(x)\frac{du_1}{dx} + q(x)u_1(x)\right) + a_2(x)\left(\frac{d^2u_2}{dx^2} + p(x)\frac{du_2}{dx} + q(x)u_2(x)\right) + \frac{da_1}{dx}\frac{du_1}{dx} + \frac{da_2}{dx}\frac{du_2}{dx} \\ &= \frac{da_1}{dx}\frac{du_1}{dx} + \frac{da_2}{dx}\frac{du_2}{dx}. \end{aligned}$$

Pairing this expression with

$$\frac{da_1}{dx}u_1(x) + \frac{da_2}{dx}u_2(x) = 0,$$

we may solve for $\frac{da_1}{dx}$ and $\frac{da_2}{dx}$, giving

$$\begin{aligned}\frac{da_1}{dx} &= -\frac{u_2(x)r(x)}{u_1(x)\frac{du_2}{dx} - u_2(x)\frac{du_1}{dx}}\\ \frac{da_2}{dx} &= \frac{u_1(x)r(x)}{u_1(x)\frac{du_2}{dx} - u_2(x)\frac{du_1}{dx}}.\end{aligned}$$

Therefore,

$$\begin{aligned}a_1(x) &= -\int \frac{u_2(x)r(x)}{W(x)}\,dx\\ a_2(x) &= \int \frac{u_1(x)r(x)}{W(x)}\,dx.\end{aligned}$$

Solution (39.7)**:**

(a) We solve the homogeneous part to yield

$$\begin{aligned}u_1(x) &= e^{-x}\\ u_2(x) &= xe^{-x}.\end{aligned}$$

These give the Wronskian of

$$\begin{aligned}W(x) &= e^{-x}(e^{-x} - xe^{-x}) + xe^{-2x}\\ &= e^{-2x}.\end{aligned}$$

We evaluate

$$\begin{aligned}a_1(x) &= -\int e^{x}(xe^{-x})(e^{-x})\,dx\\ &= -\int xe^{-x}\,dx\\ &= -(-xe^{-x} - e^{-x})\\ &= xe^{-x} + e^{-x}\\ a_2(x) &= \int e^{-x}\,dx\\ &= -e^{-x}.\end{aligned}$$

Thus, we have the general solution of

$$u(x) = c_1e^{-x} + c_2xe^{-x} + e^{-2x}.$$

(b) Solving for the homogeneous solutions, we get

$$\begin{aligned}u_1(x) &= e^{x}\\ u_2(x) &= e^{-x},\end{aligned}$$

with Wronskian

$$W(x) = -2.$$

Setting up variation of parameters, we have

$$\begin{aligned}a_1(x) &= -\int -\frac{1}{2}\,dx\\ &= \frac{1}{2}\\ a_2(x) &= -\frac{1}{2}\int e^{2x}\,dx\end{aligned}$$

$$= -\frac{1}{4}e^{2x}.$$

Thus, we have the general solution of

$$u(x) = c_1 e^{x} + c_2 e^{-x} + \frac{1}{4}e^{x}.$$

(c) Solving for the homogeneous solution, we get

$$\begin{aligned} u_1(x) &= \cos(x) \\ u_2(x) &= \sin(x), \end{aligned}$$

with Wronskian

$$W(x) = 1.$$

Setting up variation of parameters, we then get

$$\begin{aligned} a_1(x) &= -\int \sin(x)\cos(x)\,dx \\ &= -\frac{1}{2}\cos(2x) \\ a_2(x) &= \int \sin^2(x)\,dx \\ &= \frac{1}{2}x + \frac{1}{2}\sin(2x). \end{aligned}$$

Thus, we get the general solution of

$$u(x) = c_1\cos(x) + c_2\sin(x) + \frac{1}{2}(x + \sin(2x) - \cos(2x)).$$

Solution (39.8)**:** We have the particular solutions of

$$\begin{aligned} u_1(t) &= e^{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t} \\ u_2(t) &= e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t} \end{aligned}$$

Evaluating the Wronskian, we get

$$W(t) = -2\sqrt{\beta^2 - \omega_0^2}e^{-2\beta t},$$

so with variation of parameters, we have

$$\begin{aligned} a_1(t) &= \frac{1}{2\sqrt{\beta^2 - \omega_0^2}}\int e^{\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t}\delta(t-t')\,dt \\ &= \frac{1}{2\sqrt{\beta^2 - \omega_0^2}}e^{\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t'} \\ a_2(t) &= -\frac{1}{2\sqrt{\beta^2 - \omega_0^2}}\int e^{\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}\delta(t-t')\,dt \\ &= -\frac{1}{2\sqrt{\beta^2 - \omega_0^2}}e^{\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t'}. \end{aligned}$$

Thus, we get the particular solution of

$$u_p(t) = \frac{1}{2\sqrt{\beta^2-\omega_0^2}}\left(\exp\left(\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t' + \left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t\right) - \exp\left(\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t' + \left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t\right)\right).$$

Solution (39.13)**:**

(a) Setting up our differential equation of

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega_0^2 t = F_0\cos(\omega t),$$

we have homogeneous solutions of

$$\begin{aligned} u_1(t) &= e^{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t} \\ u_2(t) &= e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t}. \end{aligned}$$

Using the driving force $F_0\cos(\omega t)$, we use variation of parameters with the Wronskian of $W(t) = e^{-2\beta t}$ to get

$$\begin{aligned} a_1(t) &= -F_0\int e^{\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t}\cos(\omega t)\,dt \\ a_2(t) &= F_0\int e^{\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}\cos(\omega t)\,dt \end{aligned}$$

and a particular solution of

$$u_p(t) = F_0\left(\frac{\omega_0^2\cos(\omega t) + 2\beta\omega\sin(\omega t) - \omega^2\cos(\omega t)}{\left(\omega_0^2-\omega^2\right)^2 + (2\beta\omega)^2}\right).$$

Unfortunately, from here, the algebra gets way too messy for me to be able to evaluate this problem.

(b) Similarly, the algebra is too messy for me to be able to evaluate this problem.

Solution (39.17)**:**

(a) Using the power of the guess $e^{\lambda t}$, we find the solutions

$$\begin{aligned} u_1(t) &= e^{-2t} \\ u_2(t) &= e^{-t}. \end{aligned}$$

(b) We find the Wronskian

$$W(t) = -3e^{-3t},$$

from which we are able to find

$$\begin{aligned} a_1(t) &= \frac{1}{3}\int e^{2t}\cos(t)\,dt \\ &= \frac{1}{15}e^{2t}(\sin(t) + 2\cos(t)) \\ a_2(t) &= -\frac{1}{3}\int e^{t}\cos(t)\,dt \\ &= -\frac{1}{6}e^{t}(\sin(t)+\cos(t)). \end{aligned}$$

Thus, the particular solution is

$$u_p(t) = \frac{1}{15}(\sin(t)+2\cos(t)) - \frac{1}{6}(\sin(t)+\cos(t)).$$

(c) We find the full solution such that

$$\begin{aligned} c_1 + c_2 &= \frac{31}{30} \\ -2c_1 - c_2 &= \frac{1}{10}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} c_1 &= -\frac{34}{30} \\ c_2 &= \frac{13}{6}. \end{aligned}$$

Our solution is

$$-\frac{34}{30}e^{-2t} + \frac{13}{6}e^{-t} - \frac{1}{30}\cos(t) - \frac{1}{10}\sin(t).$$

Solution (39.18)**:** We start with the ansatz x^α. Plugging this into our homogeneous equation, we get

$$x^\alpha\left(\alpha^2 - \alpha - 2\right) = 0.$$

Therefore, we get that $\alpha = 2, -1$, giving the homogeneous solutions of

$$\begin{aligned} u_1(x) &= x^2 \\ u_2(x) &= \frac{1}{x}. \end{aligned}$$

We calculate the Wronskian to be $W(x) = -3$, so we use variation of parameters to obtain

$$\begin{aligned} a_1(x) &= \frac{1}{3}\int \left(\frac{1}{x}\right)x\,dt \\ &= \frac{1}{3}x \\ a_2(x) &= -\frac{1}{3}\int x^3\,dx \\ &= -\frac{1}{12}x^4. \end{aligned}$$

Therefore,

$$u(x) = a_1x^2 + a_2x^{-1} + \frac{1}{4}x^3.$$

Plugging in the initial conditions, we have

$$\begin{aligned} 0 &= a_1 + a_2 + \frac{1}{4} \\ 0 &= 2a_1 - a_2 + \frac{3}{4}. \end{aligned}$$

This resolves to

$$\begin{aligned} a_1 &= -\frac{1}{3} \\ a_2 &= \frac{1}{12}, \end{aligned}$$

so we have the solution

$$u(x) = -\frac{1}{3}x^2 + \frac{1}{12}x^{-1} + \frac{1}{4}x^3.$$

Solution (39.21)**:** Let $\mathcal{L}_x$ be an operator with definite parity. Suppose $\mathcal{L}_x[u(x)] = 0$. Then,

$$\begin{aligned}\mathcal{L}_x[u(-x)] &= \mathcal{L}_{-t}[u(t)]\\ &= \pm\mathcal{L}_t[u(t)]\\ &= 0,\end{aligned}$$

so

$$\begin{aligned}u_+ &= \frac{u(x)+u(-x)}{2}\\ u_- &= \frac{u(x)-u(-x)}{2}\end{aligned}$$

are solutions of $\mathcal{L}_x[u] = 0$.

Solution (39.22 (b))**:** We start with

$$0 = \sum_{m=0}^{\infty} \frac{1}{m!}c_m\left((m+\alpha)^2 - \lambda^2\right)x^{m+\alpha} + \sum_{m=0}^{\infty}\frac{1}{m!}c_m x^{m+\alpha+2}.$$

Taking out the $m = 0, 1$ term in the first expression and reindexing, we get

$$= c_0\left(\alpha^2 - \lambda^2\right) + c_1\left((\alpha+1)^2 - \lambda^2\right) + \sum_{m=2}^{\infty}\left(\frac{c_m}{m!}\left((m+\alpha)^2 - \lambda^2\right) + \frac{c_{m-2}}{(m-2)!}\right)x^{m+\alpha}.$$

Solution (39.28)**:** We begin with the assumption that we have a power series of the form

$$u(x) = x^{\alpha}\sum_{k=0}^{\infty} c_k x^k.$$

Differentiating, we get

$$\begin{aligned}\frac{d^2u}{dx^2} &= \sum_{k=2}^{\infty} c_k(\alpha+k)(\alpha+k-1)x^{\alpha+k=2}\\ xu &= \sum_{k=1}^{\infty} c_{k-1}x^{\alpha+k}.\end{aligned}$$

Plugging this into Airy's equation, we are able to extract

$$c_2(\alpha+1)(\alpha+2) + \sum_{k=1}^{\infty}(c_{k+2}(\alpha+k+2)(\alpha+k+1) - c_{k-1})x^{\alpha+k} = 0.$$

Thus, we are left with the indicial equation of

$$c_2(\alpha+1)(\alpha+2) = 0$$

and recurrence relation of

$$c_{k+2} = \frac{c_{k-1}}{(\alpha+k+2)(\alpha+k+1)}.$$

Since c_2 is not the first term of the series, we are allowed to assume that $c_2 = 0$ and $\alpha = 0$. This gives chains $c_0 \to c_3 \to \cdots$ and $c_1 \to c_4 \to \cdots$ given by the recurrence relation. Therefore, we find the expressions

$$\begin{aligned}c_{3n} &= c_0\left(\prod_{j=1}^{n}(3j)(3j-1)\right)^{-1}\\ c_{3n+1} &= c_1\left(\prod_{j=1}^{n}(3j+1)(3j)\right)^{-1},\end{aligned}$$

whose corresponding series are linearly independent.