Problem (Problem 1)**:** Let $T\colon V \to W$ be a linear transformation between $\mathbb{F}$-vector spaces. Show that T is injective if and only if T maps $\mathbb{F}$-linearly independent subsets of V to $\mathbb{F}$-linearly independent subsets of W.

Solution: Let T be injective. We claim that if $\{v_1, \dots, v_n\}$ is linearly independent in V, then $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W. We see that if

$$\sum_{j=1}^{n} a_j T v_j = 0_W,$$

then

$$T\left(\sum_{j=1}^{n} a_j v_j\right) = 0_W,$$

meaning that

$$\sum_{j=1}^{n} a_j v_j \in \ker(T).$$

Now, since T is injective, $\ker(T) = \{0_V\}$, meaning that $\sum_{j=1}^{n} a_j v_j = 0_V$. Yet, since $\{v_1, \dots, v_n\}$ is linearly independent, this means $a_j = 0$ for each j, so $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W.

Now, let T map linearly independent subsets of V to linearly independent subsets of W.

Problem (Problem 2)**:** Let $P_{n+1}(\mathbb{R})$ be the space of polynomials with real coefficients of degree $\leqslant n+1$. Prove that for any n points $a_1, \dots, a_n \in \mathbb{R}$, there exists a nonzero polynomial $f \in P_{n+1}(\mathbb{R})$ such that $f(a_j) = 0$ for each j, and $\sum_{j=1}^{n} f'(a_j) = 0$.

Solution: Based on the first condition, we see that the product $\prod_{j=1}^{n}(x - a_j)$ must divide the polynomial f, and since f has degree at most $n+1$, we must have $f(x) = (Ax + B)\prod_{j=1}^{n}(x - a_j)$ for some $a, b \in \mathbb{R}$. Writing $f'(x)$, we see that

$$f'(x) = A\prod_{j=1}^{n}(x - a_j) + (Ax + B)\sum_{i=1}^{n}\prod_{j\neq i}(x - a_j),$$

Problem (Problem 7)**:**

(a) Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be a matrix such that $A^2 = I_n$. Show that A is diagonalizable.

(b) Give an example of of $A \in \mathrm{Mat}_2(\mathbb{C})$ satisfying $A^2 = \mathbf{0}_2$ (the zero matrix) which is not diagonalizable.

Solution:

(a) Since $A^2 - I_n = \mathbf{0}_n$, we see that the minimal polynomial of A is $m_A(t) = t^2 - 1$, which splits over $\mathbb{C}$ to yield $m_A(t) = (t-1)(t+1)$. In particular, since the minimal polynomial splits into a product of distinct linear factors, A is diagonalizable.

(b) The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfies $A^2 = \mathbf{0}_2$, but since $A \neq \mathbf{0}_2$, we see that $m_A(t) = t^2$. Since $m_A(t)$ does not split into distinct linear factors over $\mathbb{C}$, we see that A is necessarily not diagonalizable.

Problem (Problem 8)**:** Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that A^2 has n distinct eigenvalues. Show that A is diagonalizable.