Observations on Excess Area Identities and Operator Symbols in Bergman Spaces

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Summary

1. Definitions and Notations
2. Motivation and Problem
3. Results and Observations
4. Remarks and Future Directions
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Definition and Notations

- Ω : a region in $\mathbb{C}$ e.g. $\mathbb{D}, D(0,r), \mathbb{A}(0,r,1), \mathbb{C}$
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$: weight function

Definition (λ-weighted Square-Integrable Functions)

$$L^2(\Omega,\lambda) = \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(z)|^2 \lambda(z)\, dA(z) < \infty \right\}$$

- $L^2(\Omega,\lambda)$ forms a Hilbert space with inner product

$$\langle f, g\rangle_{L^2(\Omega,\lambda)} = \int_\Omega f(z)\overline{g(z)}\lambda(z)\, dA(z)$$

inducing the norm

$$\|f\|^2_{L^2(\Omega,\lambda)} = \int_\Omega |f(z)|^2 \lambda(z)\, dA(z)$$

Definitions and Notations

Definition (Holomorphic Function on Ω)

$$h \in \mathcal{O}(\Omega) \iff \frac{\partial h(z)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial h(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}\right) = 0,\ \forall z \in \Omega$$

Definition (λ-weighted Bergman Space)

$$A^2(\Omega, \lambda) := \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

Definitions and Notations

Definition ($A^{1,2}(\Omega, \lambda)$)

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \;\middle|\; \frac{\partial h}{\partial z} \in A^2(\Omega, \lambda) \right\}$$

Definition (Weighted Image-Area)

Let $h \in A^{1,2}(\Omega, \lambda)$.

$$A_{\Omega,\lambda}(h) = \int_\Omega \left| \frac{\partial h}{\partial z} \right|^2 \lambda(z)\, dA(z)$$

Definitions and Notations

- $A^2(\Omega,\lambda)$ has a reproducing kernel i.e $\exists!\ K_\Omega^\lambda(\cdot,z) \in A^2(\Omega,\lambda)$:

$$h(z) = \left\langle h(\cdot), K_\Omega^\lambda(\cdot,z)\right\rangle_{L^2(\Omega,\lambda)}$$

- $A^2(\Omega,\lambda)$ is a closed subspace of $L^2(\Omega,\lambda)$.

Definition (Orthogonal Projection)

Let $P^{\Omega,\lambda} : L^2(\Omega,\lambda) \longrightarrow A^2(\Omega,\lambda)$

$$\begin{aligned}\left(P^{\Omega,\lambda}h\right)(z) &:= \left\langle h(\cdot), K_\Omega^\lambda(\cdot,z)\right\rangle_{L^2(\Omega,\lambda)} \\ &= \int_\Omega h(w)\overline{K_\Omega^\lambda(w,z)}\lambda(w)\, dA(w)\end{aligned}$$

Definitions and Notations

Definition (Multiplication Operator)

Let $M_\varphi : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda)$ where $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

Definition (Toeplitz Operator)

$T_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$, where $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega,\lambda} := P^{\Omega,\lambda} M_\varphi$$

Definitions and Notations

Definition (Commutator)

Let $\left[P^{\Omega,\lambda}, M_\varphi\right] : L^2(\Omega,\lambda) \to L^2(\Omega,\lambda)$

$$\left[P^{\Omega,\lambda}, M_\varphi\right] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

Definition (Hankel Operator)

Let $H_\varphi^{\Omega,\lambda} : A^2(\Omega,\lambda) \to (A^2(\Omega,\lambda))^\perp$

$$\begin{aligned} H_\varphi^{\Omega,\lambda} &:= -\left[P^{\Omega,\lambda}, M_\varphi\right]\Big|_{A^2(\Omega,\lambda)} \\ &= \left(I - P^{\Omega,\lambda}\right) M_\varphi \\ &= M_\varphi - P^{\Omega,\lambda} M_\varphi \\ &= M_\varphi - T_\varphi^{\Omega,\lambda} \end{aligned}$$

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Motivations

- $\{z^n\}_{n=0}^{\infty}$ form a *complete orthogonal basis* for $A^2(\mathbb{D})$
- If $h \in \mathcal{O}(\mathbb{D})$, then h is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

  and

$$S_N := \sum_{n=0}^{N} h_n z^n$$

  converges uniformly on compact subsets.
- Relationship between L^2 norm of h to the ℓ^2 norm of $\{h_k\}_{k=0}^{\infty}$:

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 \, dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $\left[T_{\bar{z}}^{\mathbb{D}} M_z, DM_z\right](z^m) = 0$

Problems

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

Literature Review on Previous Results I

- D'Angelo's excess area identity [D'A19]

Let $h \in A^{1,2}(\mathbb{D})$. Then,

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial(zh)}{\partial z} \right\|^2_{L^2(\mathbb{D})} - \left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\mathbb{D})} \\ &= \frac{1}{2} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta \\ &= \pi \left\| Sh \right\|^2_{L^2(b\mathbb{D})} \end{aligned}$$

where Sh is the restriction of h to the unit circle.

Literature Review on Previous Results II

- Excess area identity with Blaschke product multiplier
- 'Excess area' identity for harmonic functions [BÇGH22]

- Generating symbols for Toeplitz operators for a given initial p and target polynomial q on unit disc and polydisc, $T_{\varphi}^{\mathbb{D}}(p) = q$ and $T_{\varphi}^{\mathbb{D}^n}(p) = q$ [ÇDTR+24]

- Substituted derivatives for Toeplitz operators in excess area identity [ÇDTR+24]

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Summary of Results

1. Results and Observations influenced by the Area Difference of the image of $\mathbb{D}$ between zh and h:
   i. On $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2})$, $A^2(\mathbb{D}, \lambda)$, $A^2(D(0,r))$
   ii. On convergence of identities on certain weighted discs.
2 Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
   i. On unweighted and weighted Toeplitz operators relation
   ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on $A^2(\mathbb{D})$

Methods Used

- Relation between L^2 norms of functions and ℓ^2 norms of Taylor series:

$$\|h\|^2_{L^2(\mathbb{D})} = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- Integration by parts via Stokes's theorem on forms:

$$\oint_{b\Omega} f\, dz = \int_{\Omega} \overline{\frac{\partial f}{\partial z}}\, d\bar{z} \wedge dz$$
$$\oint_{b\Omega} f\, d\bar{z} = \int_{\Omega} \frac{\partial f}{\partial z} dz \wedge d\bar{z}.$$

- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Special functions e.g. beta, gamma, hypergeometric

Using Integration by Parts to find Excess Area Identity: Wedge Product I

The area is integrated with respect to $dA = dx \wedge dy$. The wedge product has the following properties:

$$\begin{aligned}(a+b) \wedge c &= a \wedge c + b \wedge c \\ a \wedge b &= -b \wedge a \\ a \wedge a &= 0.\end{aligned}$$

With $z = x + iy$, $\bar{z} = x - iy$, the substitution $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ yields

$$\begin{aligned}dx \wedge dy &= \frac{1}{2i}\left(d\bar{z} \wedge dz\right) \\ &= -\frac{1}{2i}\left(dz \wedge d\bar{z}\right).\end{aligned}$$

Using Integration by Parts to find Excess Area Identity: Wedge Product II

The area integral is now rewritten as:

$$\begin{aligned}
\left\langle \frac{\partial h}{\partial z}, \frac{\partial h}{\partial z} \right\rangle_{L^2(\Omega,\lambda)} &= \int_\Omega \left(\overline{\frac{\partial h}{\partial z}}\right)\left(\frac{\partial h}{\partial z}\right)\lambda\left(|z|\right)\, dx \wedge dy \\
&= \frac{1}{2i}\int_\Omega \lambda\left(|z|\right)\, \left(\left(\overline{\frac{\partial h}{\partial z}}\right) d\bar{z}\right) \wedge \left(\left(\frac{\partial h}{\partial z}\right) dz\right)
\end{aligned}$$

Using Integration by Parts to find Excess Area Identity: Stokes's Theorem I

In particular,

$$\overline{\frac{\partial}{\partial z}}\left((\lambda(|z|))\,\overline{h}\frac{\partial h}{\partial z}\right)d\overline{z}\wedge dz = \underbrace{(\lambda(|z|))\,\overline{\frac{\partial h}{\partial z}}d\overline{z}\wedge\frac{\partial h}{\partial z}dz}_{\text{area integrand}} + \left(\overline{\frac{\partial}{\partial z}}\lambda(|z|)\right)\overline{h}\wedge\frac{\partial h}{\partial z}dz$$

meaning

$$\frac{1}{2i}\int_\Omega \frac{\partial h}{\partial z}\overline{\frac{\partial h}{\partial z}}\lambda(|z|)\;d\overline{z}\wedge dz = \underbrace{\frac{1}{2i}\int_\Omega \overline{\frac{\partial}{\partial z}}\left(\lambda(|z|)\,\overline{h}\frac{\partial h}{\partial z}\right)\;d\overline{z}\wedge dz}_{\text{Integral } A}$$
$$-\frac{1}{2i}\int_\Omega \overline{h}\frac{\partial h}{\partial z}\left(\overline{\frac{\partial}{\partial z}}\lambda(|z|)\right)\;d\overline{z}\wedge dz.$$

Using Integration by Parts to find Excess Area Identity: Stokes's Theorem II

Turning our attention to Integral A,

$$\begin{aligned}\frac{1}{2i} &= \int_{\Omega} d\left(\lambda(|z|)\,\overline{h}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz \\ &= \underbrace{\int_{b\Omega} \lambda(|z|)\,\overline{h}\frac{\partial h}{\partial z}\, dz}_{=0}.\end{aligned}$$

With this, the area integral is now

$$\frac{1}{2i}\int \frac{\partial h}{\partial z}\overline{\frac{\partial h}{\partial z}}\lambda(|z|)\, d\overline{z}\wedge dz = -\frac{1}{2i}\int \overline{h}\frac{\partial h}{\partial z}\left(\overline{\frac{\partial}{\partial z}}\lambda(|z|)\right)\, d\overline{z}\wedge dz$$

Excess Area on Fock Spaces

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta = \pi \left\|Sh\right\|^2_{L^2(b\mathbb{D})}$$

Excess Area on Fock Space

Given $0 < \rho < 1$, $h \in \mathcal{F}^2$, let $h_\rho(z) := h(\rho z)$

$$\begin{aligned}
&A_{\mathcal{F}^2}\left(zh_\rho\right) - A_{\mathcal{F}^2}\left(h_\rho\right) \\
&= \pi \left\|zT_{\overline{z}}^{\mathcal{F}^2}\left(h_\rho\right)\right\|^2_{\mathcal{F}^2} + \pi \left\|T_{\overline{z}}^{\mathcal{F}^2}\left(h_\rho\right)\right\|^2_{\mathcal{F}^2} + \pi \left\|H_{\overline{z}}^{\mathcal{F}^2}\left(h_\rho\right)\right\|^2_{\mathcal{F}^2} \\
&= \pi \left\|z^2 h_\rho\right\|^2_{\mathcal{F}^2} - 2\pi \left\|zh_\rho\right\|^2_{\mathcal{F}^2} + \pi \left\|h_\rho\right\|^2_{\mathcal{F}^2}
\end{aligned}$$

Here, the restriction of h to the unit circle in D'Angelo's identity is replaced with the Bergman projection on $\mathbb{C}$.

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta = \pi \left\|Sh\right\|^2_{L^2(b\mathbb{D})}$$

Excess Area on $A^2(\mathbb{D}, \lambda)$

Let $h \in A^{1,2}(\mathbb{D}, \lambda)$, $\lambda(z) = 1 - |z|^2$. Then,

$$A_{\mathbb{D},\lambda}\left(z^{m+1}h\right) - A_{\mathbb{D},\lambda}\left(z^m h\right) = \pi \left\|z^m h\right\|^2_{L^2(\mathbb{D},\lambda)}.$$

Here, the restriction of h to the unit circle is replaced with the function itself.

"Excess Area" on $A^2(D(0,r))$

Excess Area Identity for Harmonic Functions on $D(0,r)$, $0 < r < 1$

For a harmonic function $u \in L^2(D(0,r))$, $\exists v \in L^2(D(0,r))$ harmonic conjugate [BÇGH22]. Let $h = u + iv$ be the corresponding holomorphic function. Then,

$$\left\| \frac{\partial(zu)}{\partial z} \right\|^2_{L^2(D(0,r))} - r^2 \left\| \frac{\partial u}{\partial z} \right\|^2_{L^2(D(0,r))}$$

$$= \frac{1}{4}\left(\underbrace{r^2\pi \|Sh\|^2_{L^2(bD(0,r))}}_{A_{D(0,r)}(zh)-r^2A_{D(0,r)}(h)} + 2r^2\pi\mathfrak{R}(h_0^2) + \|h\|^2_{L^2(D(0,r))} \right).$$

Dilation and Contraction from $A^2(D(0,r))$ to $A^2(\mathbb{D})$

Contracting $h \in A^{1,2}(\mathbb{D})$ by taking $h_r = h(rz)$ for some $0 < r < 1$,

$$A_{\mathbb{D}}(zh_r) - A_{\mathbb{D}}(h_r) = \pi \|Sh_r\|^2_{L^2(b\mathbb{D})} \tag{1}$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \|Sh\|^2_{L^2(bD(0,r))}. \tag{2}$$

Dilating $h \in A^{1,2}(D(0,r))$ by taking $h_{\frac{1}{r}} = h(\frac{z}{r})$ for some $0 < r < 1$

$$A_{D(0,r)}\left(zh_{1/r}\right) - r^2 A_{D(0,r)}\left(h_{1/r}\right) = \pi r^2 \left\|Sh_{1/r}\right\|^2_{L^2(bD(0,r))} \tag{3}$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \|Sh\|^2_{L^2(b\mathbb{D})} \tag{4}$$

Dilation from $A^2(D(0,r),\lambda_r)$ to $\mathcal{F}^2$

Weighted Area on $D(0,r)$

Let $\lambda_r(z) = \chi_{D(0,r)}\left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$ where $r > 0$. Then,

$$A_{D(0,r),\lambda_r}(h) = \int_{D(0,r)} |h'(z)|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} dA(z)$$

We find that, as $r \to \infty$, $A_{D(0,r),\lambda_r}(h) \to A_{\mathcal{F}^2}(h)$.

Additionally, we know that

$$A_{\mathcal{F}^2}\left(h_\rho\right) = \left\| T_{\overline{z}}^{\mathcal{F}^2} h_\rho \right\|_{\mathcal{F}^2}^2$$

Berezin Transform Convergence

Reproducing Kernel on $A^2\left(D(0,r),\lambda_r\right)$

$$K^{\lambda_r}_{D(0,r)}(w,z) = \frac{1}{\left(1 - \frac{\overline{z}w}{r^2}\right)^{r^2+2}}$$

$K^{\lambda_r}_{D(0,r)}(w,z)$ uniformly converges on compact subsets of $D(0,r)$.

Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w,z) = e^{\overline{z}w}$$

Berezin Transform Convergence, Cont'd

Definition (Berezin Transform ([Zhu07]))

Let

$$k_z^{\Omega,\lambda}(w) := \frac{K_\Omega^\lambda(w,z)}{\sqrt{K_\Omega^\lambda(z,z)}}$$

Then, for some bounded operator T on $L^2(\Omega,\lambda)$, define $\mathcal{B}^{\Omega,\lambda} : B(L^2(\Omega,\lambda)) \to L^2(\Omega,\lambda)$

$$(\mathcal{B}^{\Omega,\lambda}T)(z) := \left\langle Tk_z^{\Omega,\lambda}, k_z^{\Omega,\lambda} \right\rangle_{L^2(\Omega,\lambda)}$$

Berezin Transform Convergence, Cont'd

- For $\varphi \in L^\infty(\Omega, \lambda)$, $\mathcal{B}^{\Omega,\lambda} T_\varphi = \mathcal{B}^{\Omega,\lambda} M_\varphi$. (see Axler and Zheng, [AZ98a]).
- φ is harmonic if and only if $\mathcal{B}^{\Omega,\lambda} M_\varphi = \varphi$ (proof by Engliš, [Eng94]).
- We find that, for $T_\varphi^{D(0,r),\lambda_r} = P^{D(0,r),\lambda_r} M_\varphi$, the Berezin transform $\mathcal{B}^{D(0,r),\lambda_r} T_\varphi^{D(0,r),\lambda_r}$ converges pointwise to $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$ as $r \to \infty$ from Göğüş and Şahutoğlu ([Gc20])
- This convergence is uniform on compact subsets of $\mathbb{C}$ (proof inspired by Göğüş and Şahutoğlu in [Gc20]).

Unweighted and Weighted Toeplitz Operators
Relation I

Using an extension of [ÇDTR+24, Lemma 2.1]

For weight $\lambda(z) = \left(1 - |z|^2\right)^{\alpha}$ ($\alpha \geqslant 0$) on the unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

$$\frac{T^{\mathbb{D},\lambda_\alpha}_{\overline{z}^m}(z^n)}{T^{\mathbb{D}}_{\overline{z}^m}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leqslant n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T^{\mathbb{D},\lambda_\alpha}_{\overline{z}^m}(z^n) = s_{n,m,\alpha} T^{\mathbb{D}}_{\overline{z}^m}(z^n), \text{ and } \lim_{n\to\infty} s_{n,m,\alpha} = 1$$

Unweighted and Weighted Commutator on $A^2(\mathbb{D})$

Existence of Commutator Symbols

Given p and q are harmonic polynomials and $\frac{\partial}{\partial z}(p) \neq 0$, there does not exist a polynomial symbol ϕ, such that $\left[P^{\mathbb{D}}, M_\phi\right](p) = q$ or $\left[P^{\mathbb{D},\lambda}, M_\phi\right](p) = q$.

Compare to [ÇDTR+24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

Existence of Hankel Operator Symbols

Given some holomorphic polynomials p, q where p is not constant, there does not exist a polynomial symbol ϕ such that $H_\phi^{\mathbb{D}}(p) = \overline{q}$ or $H_\phi^{\mathbb{D},\lambda}(p) = \overline{q}$

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Remarks on the Annulus

Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0,r,1))$

For all integers m and n,

$$T_{\bar{z}^m}^{\mathbb{A}(0,r,1)}(z^n) = \begin{cases} \frac{2m r^{2m} \ln(r)}{\left(r^{2m}-1\right)} z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m}-1}{2m \ln(r)} z^{-1} & \text{if } n = m-1 \\ \frac{(n-m+1)\left(1-r^{2n+2}\right)}{(n+1)\left(1-r^{2n-2m+2}\right)} z^{n-m} & \text{else} \end{cases}.$$

We attempted to find an algorithm to generate $\varphi \in L^\infty(\mathbb{A}(0,r,1))$ such that $T_\varphi^{\mathbb{A}(0,r,1)}(p) = q$ for given holomorphic Laurent polynomials p and q, but ran into trouble beyond the case where p has roots outside $\overline{\mathbb{A}(0,r,1)}$.

Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial p and target polynomial q on $\mathbb{A}(0, r, 1)$, $T_{\varphi}^{\mathbb{A}(0,r,1)}(p) = q$
- Extension of 'excess area' identity to harmonic functions in $L^2\left(\mathbb{C}, e^{-|z|^2}\right)$.
- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential, $(1 - |z|^2)^A e^{\frac{-B}{(1-|z|^2)^\alpha}}$ $(A \geqslant 0, B > 0, \alpha > 0)$.

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