4.1

Problem: Write a Turing machine that computes the constant function $C_0(n) = 0$.

Solution: The Turing machine with the instructions

$$q_1 1 B q_2$$
$$q_2 B R q_3$$
$$q_3 1 B q_2$$

computes $C_0(n)$.

- The Turing machine deletes the first 1, then moves into state q_2.
- The Turing machine reads the blank, moves right, and enters state q_3.
- The Turing machine reads the next input in state q_3 — if 1, then blank, and go back into q_2, and if blank, it stops.

4.2

Problem: Write a Turing machine that computes the identity function $f(m) = m$ on $\mathbb{N}$. Then write a machine for the identity function on $\mathbb{N}^n$.

Solution: The Turing machine with instructions

$$q_1 1 B q_2$$

computes $f(m) = m$.

The Turing machine with instructions

$$q_1 1 B q_2$$
$$q_2 B R q_3$$
$$q_3 1 R q_3$$
$$q_3 B R q_4$$
$$q_4 B B q_5$$
$$q_4 1 B q_2$$

computes $\mathrm{id}_{\mathbb{N}^n}$.

4.3

Problem: Write a Turing machine that computes the function

$$Z(n) = \begin{cases} 1 & n = 0 \\ 0 & n \geqslant 1 \end{cases}.$$

Solution: The Turing machine with the instructions

$$
\begin{aligned}
q_1 1 R q_2 \\
q_2 1 B q_3 \\
q_3 B L q_4 \\
q_4 1 B q_5 \\
q_5 B R q_5 \\
q_5 1 1 q_6 \\
q_6 1 B q_7 \\
q_7 B R q_6
\end{aligned}
$$

computes $Z(n)$.

4.4

Problem: Write a Turing machine that computes $f(m, n) = |m - n|$.

Solution: The Turing machine with the instructions

$$
\begin{aligned}
q_1 1 B q_2 \\
q_2 B R q_3 \\
q_3 1 R q_3 \\
q_3 B R q_4 \\
q_4 1 R q_5 \\
q_5 1 R q_5 \\
q_5 B L q_6 \\
q_6 1 B q_7 \\
q_7 B L q_8 \\
q_8 1 L q_8 \\
q_8 B L q_9 \\
q_9 1 1 q_{10} \\
q_{10} 1 L q_{10} \\
q_{10} B R q_1
\end{aligned}
$$

computes $f(m, n) = |m - n|$. In short, for an input tape

$$
B, \underbrace{1, 1, \ldots, 1}_{m+1}, B, \underbrace{1, 1, \ldots, 1}_{n+1}, B
$$

the machine successively deletes the left-most 1 on the tape denoting m and deletes the right-most 1 on the tape denoting n, until it reaches the end of one of the domain elements.

4.6

Problem: Show that there is no Turing machine that can determine if a given Turing machine acting no a given input m will yield an output that contains the symbol s_k for a fixed $k \geqslant 1$.

Solution: Suppose V is computable. Define a Turing machine

$$G(n) = \begin{cases} 0 & V(n,n) = 0 \\ \text{undefined} & V(n,n) = 1 \end{cases}$$

Then, G is computable. I don't know how to do this problem.

Extra Problem 1

Problem: Write the partial function computed by each of the following Turing machines.

(a) $q_1 1 R q_2$, $q_2 B R q_3$, $q_2 1 1 q_2$.

(b) $q_1 1 R q_2$, $q_2 B 1 q_3$, $q_2 1 R q_4$, $q_4 1 1 q_1$.

Solution:

(a) This machine computes $f(0) = 2$, and does not halt for any other inputs.

(b) This machine computes $f(0) = 2$, and does not halt for any other inputs.

Extra Problem 2

Problem: Write a Turing machine that computes the function $f(n) = 0$ if n is even and $f(n) = 1$ if n is odd.

Solution:

$$\begin{gathered} q_1 1 R q_2 \\ q_2 1 B q_3 \\ q_3 B R q_4 \\ q_4 B B q_5 \\ q_5 B L q_5 \\ q_5 1 1 q_6 \\ q_4 1 B q_7 \\ q_7 B R q_8 \\ q_8 B B q_9 \\ q_9 B L q_9 \\ q_9 1 B q_{10} \\ q_8 1 1 q_2. \end{gathered}$$