We recall from linear algebra that a linear operator $T\colon V \to V$ is called diagonalizable if there is an orthonormal basis $\{e_j\}_{j\in J}$ and a bounded collection of elements $\{\lambda_j\}_{j\in J}$ such that for every $x \in V$, we have

$$Tx = \sum_{j\in J} \lambda_j \langle x, e_j\rangle e_j.$$

When V is a Hilbert space, there are a variety of generalizations. It will be useful to review the basic properties of compact and Fredholm operators.

Spectral Theory for Compact Normal Operators

The first, most basic version of the spectral theorem is the one for compact normal operators. We recall the different types of spectra.

Definition: Let $T \in B(X)$, where X is a Banach space.

(i) The *point spectrum* of T is the set

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \ker(T - \lambda I) \neq \{0\}\},$$

which are the eigenvalues of T.

(ii) The *approximate point spectrum* of T is the set

$$\pi(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not bounded below}\}.$$

(iii) The *compression spectrum* of T is

$$\gamma(T) = \{\lambda \in \mathbb{C} \mid \operatorname{im}(T - \lambda I) \text{ is not dense in } X\}.$$

There is a useful characterization of compact operators as follows.

Lemma: The following for $T \in B(H)$ are equivalent:

(i) T is compact;

(ii) $T|_{B_H}$ is a weak–norm continuous function from B_H into H.

Proof. Suppose T is compact. Then, if $(x_i)_{i\in I}$ is a weakly convergent net in B_H with limit x, and $\varepsilon > 0$, there is some finite-rank $S \in F(H)$ with $\|S - T\|_{\text{op}} < \varepsilon/3$. We have

$$\begin{aligned} \|Tx_i - Tx\| &\leq \|Tx_i - Sx_i\| + \|Sx_i - Sx\| + \|Sx - Tx\| \\ &\leq 2\|T - S\|_{\text{op}} + \|Sx_i - Sx\|. \end{aligned}$$

Every operator in $B(H)$ is weak–weak continuous, and since $\operatorname{im}(S)$ is finite-dimensional, all norms coincide, so that $Sx_i \to Sx$ in norm, giving that $\|Tx_i - Tx\| < \varepsilon/3$ for sufficiently large i. Thus, T is weak–norm continuous.

If T is weak–norm continuous, then since B_H is weakly compact, it follows that $T(B_H)$ is compact by continuity. □

Lemma: A diagonalizable operator T in $B(H)$ is compact if and only if its eigenvalues $\{\lambda_j \mid j \in J\}$ corresponding to an orthonormal basis $\{e_j \mid j \in J\}$ belongs to $c_0(J)$.

Proof. Since T is diagonalizable, we have

$$Tx = \sum_{j\in J} \lambda_j \langle x, e_j\rangle e_j.$$

If $T \in K(H)$, and $\varepsilon > 0$, then we set

$$J_\varepsilon = \{j \in J \mid |\lambda_j| \geq \varepsilon\}.$$

If J_ε is infinite, then since $\langle x, e_j \rangle \to 0$ by Parseval's identity, we have that the net $(e_j)_{j \in J_\varepsilon}$ converges weakly to zero. Yet, since $\|Te_j\| = |\lambda_j| \geq \varepsilon$ for any $j \in J_\varepsilon$, this contradicts the fact that T is weak–norm continuous. Thus, J_ε is finite for any $\varepsilon > 0$, so $(\lambda_j)_{j \in J}$ vanishes at infinity.

Now, if J_ε is finite for every $\varepsilon > 0$, we may define $T_\varepsilon \in F(H)$ by

$$T_\varepsilon = \sum_{j \in J_\varepsilon} \lambda_j \langle \cdot, e_j \rangle e_j,$$

and

$$\begin{aligned} \|(T - T_\varepsilon)x\|^2 &= \left\| \sum_{j \notin J_\varepsilon} \lambda_j \langle x, e_j \rangle e_j \right\|^2 \\ &= \sum_{j \in J_\varepsilon} |\lambda_j|^2 |\langle x, e_j \rangle|^2 \\ &\leq \varepsilon^2 \|x\|^2, \end{aligned}$$

so $\|T - T_\varepsilon\| \leq \varepsilon$, meaning that $T \in \overline{F(H)} = K(H)$. □

Note that by some basic computations, if T is diagonalizable, then we have

$$\begin{aligned} T^* &= \sum_{j \in J} \overline{\lambda_j} \langle \cdot, e_j \rangle e_j \\ T^*T &= \sum_{j \in J} |\lambda_j|^2 \langle \cdot, e_j \rangle e_j \\ &= TT^*. \end{aligned}$$

Thus, in particular, we have that every diagonalizable operator is normal.

Theorem: An operator $T \in B(H)$ is diagonalizable with eigenvalues vanishing at infinity if and only if it is a compact normal operator.

Proof. Now we only need to show that every compact normal operator is diagonalizable. Since T is compact, we know that the spectrum of T consists of 0 and a countable set of isolated points, and since T is normal, its spectral radius is equal to the operator norm, meaning that there is some λ such that $|\lambda| = \|T\|_{\text{op}}$. In particular, there is an eigenvector for T.

Let $\mathcal{Z}$ be the family of orthonormal systems of eigenvectors of T, ordered by inclusion. Since we have established that this family is nonempty, and the union provides an upper bound for any chain in $\mathcal{Z}$, there is some maximal orthonormal system $\{e_j\}_{j \in J}$ with corresponding eigenvalues $\{\lambda_j\}_{j \in J}$. We let P be the projection onto the closed subspace spanned by the e_j. For each $x \in H$, we have

$$\begin{aligned} TPx &= T\left(\sum_{j \in J} \langle x, e_j \rangle e_J \right) \\ &= \sum_{j \in J} \lambda_j \langle x, e_j \rangle e_j \\ &= \sum_{j \in J} \left\langle x, \overline{\lambda_j} e_j \right\rangle e_j \\ &= \sum_{j \in J} \langle x, T^* e_j \rangle e_j \\ &= \sum_{j \in J} \langle Tx, e_j \rangle e_j \\ &= PTx. \end{aligned}$$

Thus, the operator $(I - P)T$ is normal, and is also compact. If $P \neq I$, then either $(I - P)T = 0$, and every unit vector in $(I - P)(H)$ is an eigenvector for T (contradicting maximality), or else $(I - P)T \neq 0$, in which case there is $e_0 \in (I - P)(H)$ with $Te_0 = \lambda e_0$ and $|\lambda| = \|(I - P)T\|_{\text{op}}$, which once again contradicts maximality.

Thus, $P = I$, and we are done. □

Spectral Theory for Normal Operators

We now generalize from the special case of compact operators. Here, we cannot use the convenient properties of compact operators with respect to finite dimensionality/codimensionality.

First, we notice that if $T \in B(H)$ is a normal operator, then $C^*(T)$, the C^*-algebra generated by T, is abelian, so from the Gelfand isomorphism, we have that $C^*(T) \cong C(\sigma(T))$ are isometrically $*$-isomorphic.

We will generalize this in a moment, but first we will apply the continuous functional calculus to show an important commutation relation. In $\mathbb{M}_n(\mathbb{C})$, we know that an operator S commutes with a normal operator T if and only if all the eigenspaces for T are invariant under S; since T and T^* commute, it then follows that S commutes with T^*.

It turns out that this generalizes to infinite-dimensional spaces, but the proof requires the use of the continuous functional calculus.

Proposition (Fuglede's Theorem)**:** If S and T are operators in $B(H)$, and T is normal, then $ST = TS$ implies $ST^* = T^*S$.

Proof. Define

$$e^{\lambda T} = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}.$$

This is an element of $C^*(T)$ by the continuous functional calculus, and similarly, $e^{\lambda T^*} \in C^*(T)$, with

$$e^{\lambda T^*} = e^{\lambda T^* - \overline{\lambda} T} e^{\overline{\lambda} T}.$$

There is some self-adjoint operator R such that $\lambda T^* - \overline{\lambda} T = iR$, meaning that

$$U(\lambda) = e^{\lambda T^* - \overline{\lambda} T}$$

is a unitary operator in $C^*(T)$ with $U(\lambda)^* = U(-\lambda)$.

It follows from the expression for $e^{\lambda T}$ that S commutes with $e^{\lambda T}$ for every λ, so that

$$e^{-\lambda T^*} S e^{\lambda T^*} = U(-\lambda) S U(\lambda),$$

with the operators uniformly bounded in norm by $\|S\|$.

Fixing $x, y \in H$, define $f \colon \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) = \left\langle e^{-\lambda T^*} S e^{\lambda T^*} x, y \right\rangle.$$

It follows that f is an entire function with $|f(\lambda)| \leq \|S\|$ for all λ, so that

$$\begin{aligned} \left\langle e^{-\lambda T^*} S e^{\lambda T^*} x, y \right\rangle - \langle Sx, y \rangle &= f(\lambda) - f(0) \\ &= 0, \end{aligned}$$

so that

$$e^{-\lambda T^*} S e^{\lambda T^*} - S = 0.$$

Thus, $ST^* - T^*S = 0$. □

In order to prove the spectral theorem for normal operators, we use the concept of a spectral measure.

Definition: Let Ω be a compact Hausdorff space, and H a Hilbert space. A *spectral measure* E relative to (ω, H) is a map E from the Borel σ-algebra of Ω to the set of projections on $B(H)$ satisfying

(i) $E(\emptyset) = 0$, $E(\Omega) = I_H$;

(ii) $E(S_1 \cap S_2) = E(S_1)E(S_2)$;

(iii) for all $x, y \in H$, the map $E_{x,y} \colon S \to \langle E(S)x, y\rangle$ is a regular complex Borel measure on Ω.

We will let $B_\infty(\Omega)$ be the set of bounded Borel functions on Ω, and $M(\Omega)$ the space of regular Borel complex measures with the total variation norm.

Example: Let Ω be a compact Hausdorff space, μ a positive regular Borel measure on Ω. Let $M_\varphi \in B(L_2(\Omega, \mu))$ be defined by

$$M_\varphi f = \varphi f.$$

We observe that

$$\begin{aligned} \|M_\varphi f\|^2 &= \int |\varphi f|^2 \, d\mu \\ &\leq \|\varphi\|_{L_\infty} \int |f|^2 \, d\mu. \end{aligned}$$

In particular, this means that $\|M_\varphi\|_{\text{op}} \leq \|\varphi\|_{L_\infty}$.

The map $L_\infty(\Omega, \mu) \to B(L_2(\Omega, \mu))$ is thus a $*$-homomorphism of C^*-algebras, where $M_\varphi^* = M_{\overline{\varphi}}$.

In fact, since this map is injective, it is in fact an isometric $*$-homomorphism of C^*-algebras, following from the continuous functional calculus.

Lemma: Let Ω be a compact Hausdorff space, H a Hilbert space. Let $\mu_{x,y} \in M(\Omega)$ for each $x, y \in H$. Suppose that for each Borel set S in Ω, the function $\pi_S \colon H \times H \to \mathbb{C}$ given by $(x, y) \mapsto \mu_{x,y}(S)$ is a sesquilinear form.

Then, for any $f \in B_\infty(\Omega)$, the function

$$\begin{aligned} \pi_f \colon H \times H &\to \mathbb{C} \\ (x, y) &\mapsto \int f \, d\mu_{x,y} \end{aligned}$$

is a sesquilinear form.

Proof. This is a standard boostrapping argument. We start by letting f be a simple function, so we may write

$$f = \sum_{j=1}^{n} \lambda_j \mathbb{1}_{S_j}$$

for pairwise disjoint Borel subsets $S_1, \ldots, S_j$ of Ω and complex numbers $\lambda_1, \ldots, \lambda_n$. Then,

$$\int f \, d\mu_{x,y} = \sum_{j=1}^{n} \lambda_j \mu_{x,y}(S_j),$$

and since the (bounded) sesquilinear forms on H are in one to one correspondence with $B(H)$, it follows that the case for simple functions follows.

If $f \in B_\infty(\Omega)$ is arbitrary, then there is a sequence $(f_n)_n \to f$ of simple functions converging in the uniform norm. We observe that

$$\int |f_n - f| \, d|\mu_{x,y}| \leq \|f_n - f\|_{L_\infty} |\mu_{x,y}|(\Omega),$$

so we may exchange limit and integral by dominated convergence, giving

$$\int f\, d\mu_{x,y} = \lim_{n\to\infty} \int f_n\, d\mu_{x,y}$$

for every $x, y \in H$. Thus, π_f is a sesquilinear form on H. □

Theorem: Let Ω be a compact Hausdorff space, H a Hilbert space, and E a spectral measure on (Ω, H). Then, for any $f \in B_\infty(\Omega)$, the map $\pi_f \colon H \times H \to \mathbb{C}$ given by

$$(x, y) \mapsto \int f\, dE_{x,y}$$

is a bounded sesquilinear form, with $\|\pi_f\| \leq \|f\|_{L_\infty}$.

Proof. The previous lemma shows that π_f is a sesquilinear form. We only need to show that $\|\pi_f\| \leq \|f\|_{L_\infty}$. Let $\Omega = S_1 \cup \cdots \cup S_n$, with $S_1, \dots, S_n$ pairwise disjoint Borel sets. Then,

$$\begin{aligned}
\sum_{j=1}^{n} |\langle E(S_j)x, y\rangle| &= \sum_{j=1}^{n} |\langle E(S_j)x, E(S_j)y\rangle| \\
&\leq \left(\sum_{j=1}^{n} \|E(S_j)x\|^2\right)^{1/2} \left(\sum_{j=1}^{n} \|E(S_j)y\|^2\right)^{1/2} \\
&= \|E(\Omega)x\|\|E(\omega)y\| \\
&= \|x\|\|y\|.
\end{aligned}$$

Thus, $\|E_{x,y}\| \leq \|x\|\|y\|$, so

$$\begin{aligned}
\left|\int f\, dE_{x,y}\right| &\leq \|f\|_{L_\infty} \|E_{x,y}\| \\
&\leq \|f\|_{L_\infty} \|x\|\|y\|.
\end{aligned}$$

Thus, $\|\pi_f\| \leq \|f\|_{L_\infty}$. □

Thus, paired with the correspondence of sesquilinear forms and bounded operators on a Hilbert space, we obtain the following result.

Theorem: Let Ω be a compact Hausdorff space, H a Hilbert space, and E a spectral measure on (Ω, H). Then, for each $f \in B_\infty(\Omega)$, there is a unique bounded operator T on H such that

$$\langle Tx, y\rangle = \int f\, dE_{x,y}.$$

We will define the *integral* of $f \in B_\infty(\Omega)$ to be the (unique) operator such that for all $x, y \in H$,

$$\left\langle \left(\int f\, dE\right) x, y \right\rangle = \int f\, dE_{x,y}.$$

Proposition: If E is a spectral measure for (Ω, H), and we define

$$\begin{aligned}
\rho \colon B_\infty(\Omega) &\to B(H) \\
f &\mapsto \int f\, dE,
\end{aligned}$$

then ρ is a representation for the C^*-algebra $B_\infty(\Omega)$. That is, ρ is a unital $*$-homomorphism.

Proof. Linearity follows from a bootstrapping argument, and boundedness from the definition of the sesquilinear form.

Thus, we only need to show multiplicativity. Similarly from bootstrapping, we only need to show the case when f and g are simple. Suppose $f = \mathbb{1}_S$ and $g = \mathbb{1}_{S'}$. Then,

$$\begin{aligned}\rho(fg) &= \int \mathbb{1}_S \mathbb{1}_{S'}\,dE\\ &= E(S\cap S')\\ &= E(S)E(S')\\ &= \left(\int \mathbb{1}_S\,dE\right)\left(\int \mathbb{1}_{S'}\,dE\right)\\ &= \rho(f)\rho(g)\end{aligned}$$

and similarly projections are self-adjoint. □

Now, we've elucidated a lot of properties of spectral measures, but we still have not answered the question of their existence. This is the spectral theorem.

Theorem (Spectral Theorem for Bounded Normal Operators)**:** Let Ω be a compact Hausdorff space, H a Hilbert space, and let $\varphi\colon C(\Omega) \to B(H)$ be a unital $*$-homomorphism. Then, there is a unique spectral measure E with respect to (Ω, H) such that

$$\varphi(f) = \int f\,dE$$

for all $f \in C(\Omega)$. Moreover, if $T \in B(H)$, then T commutes with $\varphi(f)$ for all $f \in C(\Omega)$ if and only if T commutes with $E(S)$ for all Borel $S \subseteq \Omega$.

Proof. For any $x, y \in H$, the function $\tau_{x,y}\colon C(\Omega) \to \mathbb{C}$ given by

$$f \mapsto \langle \varphi(f)x, y\rangle$$

is linear with $\|\tau_{x,y}\|_{\text{op}} \leq \|x\|\|y\|$. From the Riesz Representation Theorem, there is a unique measure $\mu_{x,y} \in M(\Omega)$ such that

$$\tau_{x,y}(f) = \int f\,d\mu_{x,y}$$

for all $f \in C(\Omega)$. We also have that $\|\mu_{x,y}\| = \|\tau_{x,y}\|_{\text{op}}$. The function

$$(x, y) \mapsto \langle \varphi(f)x, y\rangle$$

is a sesquilinear map from H to $M(\Omega)$ such that $x \mapsto \mu_{x,y}$ is linear and $y \mapsto \mu_{x,y}$ is conjugate-linear. Thus, for all $f \in B_\infty(\Omega)$, the map

$$(x, y) \mapsto \int f\,d\mu_{x,y}$$

is a sesquilinear form, with

$$\begin{aligned}\left|\int f\,d\mu_{x,y}\right| &\leq \|f\|_{L_\infty}\|\mu_{x,y}\|\\ &\leq \|f\|_{L_\infty}\|x\|\|y\|,\end{aligned}$$

so there is a unique bounded operator, $\psi(f) \in B(H)$ such that

$$\langle \psi(f)x, y\rangle = \int f\,d\mu_{x,y}$$

for all $x, y \in H$. If $f \in C(\Omega)$, then we have that

$$\langle \psi(f)x, y\rangle = \int f\,d\mu_{x,y}$$

$$\begin{aligned}&= \tau_{x,y}(f)\\&= \langle \varphi(f)x, y\rangle,\end{aligned}$$

so $\psi(f) = \varphi(f)$.

We now show that ψ is a $*$-homomorphism. If $f \in C(\Omega)$ with $\overline{f} = f$, then $\varphi(f)$ is self-adjoint, meaning that

$$\int f\, d\mu_{x,x} = \langle \varphi(f)x, x\rangle$$

is a real number, so $\mu_{x,x}$ is a real measure. Thus, if $f \in B_\infty(\Omega)$ is arbitrary, dominated convergence gives that

$$\langle \psi(f)x, x\rangle = \int f\, d\mu_{x,x}$$

is real. Thus, $\psi(f)$ is self-adjoint, so ψ preserves involutions.

If $f \in B_\infty$ and $x \in H$, then we claim that it is enough to show that

$$\langle \psi(fg)x, x\rangle = \langle \psi(f)\psi(g)x, x\rangle \tag{$*$}$$

holds for any $g \in C(\Omega)$. A way to rewrite $(*)$ is by

$$\int gf\, d\mu_{x,x} = \int g\, d\mu_{x,\psi(\overline{f})x},$$

so if $(*)$ holds for all $g \in C(\Omega)$, then the measures $f d\mu_{x,x}$ and $\mu_{x,\psi(\overline{f})x}$ are equal since their corresponding linear functionals are necessarily equal. In particular, this holds for all such g.

Since φ is a $*$-homomorphism, the equation $(*)$ holds for all $f, g \in C(\Omega)$, so it holds if $f \in C(\Omega)$ and $g \in B_\infty(\Omega)$ by density. Similarly, by replacing f and g with their conjugates, we have

$$\left\langle \psi(\overline{fg})x, x\right\rangle = \left\langle \psi(\overline{f})\psi(\overline{g})x, x\right\rangle,$$

so by taking conjugates and using the fact that ψ is a homomorphism, we get

$$\langle \psi(gf)x, x\rangle = \langle \psi(g)\psi(f)x, x\rangle \tag{$**$}$$

for all $g \in B_\infty(\Omega)$. Using $(*)$ by interchanging g and f, we obtain that $(**)$ holds for all $f, g \in B_\infty(\Omega)$. Since $x \in H$ was arbitrary, we have $\psi(gf) = \psi(g)\psi(f)$, so ψ is a homomorphism.

Now, if S is a Borel subset of Ω, we let $E(S) = \psi(\mathbb{1}_S)$. We see that $E(S)$ is a projection on H, and that the map $E\colon S \to E(S)$ is a spectral measure, with $E_{x,y} = \mu_{x,y} \in M(\Omega)$, as

$$\begin{aligned}E_{x,y}(S) &= \langle E(S)x, y\rangle\\&= \langle \psi(\mathbb{1}_S)x, y\rangle\\&= \int \mathbb{1}_S\, d\mu_{x,y}.\end{aligned}$$

If $f \in B_\infty(\Omega)$, then from a bootstrapping argument, we have

$$\begin{aligned}\left\langle \left(\int f\, dE\right)x, y\right\rangle &= \int f\, dE_{x,y}\\&= \int f\, d\mu_{x,y}\\&= \langle \psi(f)x, y\rangle,\end{aligned}$$

so that

$$\psi(f) = \int f\, dE,$$

and in particular, for all $f \in C(\Omega)$,

$$\varphi(f) = \int f\, dE.$$

Additionally, for all $x, y \in H$, if E' is another spectral measure that satisfies

$$\varphi(f) = \int f\, dE',$$

then we have for all $x, y \in H$,

$$\begin{aligned}\int f\, dE'_{x,y} &= \langle \varphi(f)x, y\rangle \\ &= \int f\, dE_{x,y},\end{aligned}$$

so $E'_{x,y} = E_{x,y}$ for all x, y, meaning that for all Borel $S \subseteq \Omega$,

$$\langle E'(S)x, y\rangle = \langle E(S)x, y\rangle,$$

meaning $E = E'$.

Finally, if T is an operator on H commuting with all the elements of the range of φ, then if $f \in C(\Omega)$, we have

$$\begin{aligned}\int f\, d\mu_{Tx,y} &= \langle \psi(f)Tx, y\rangle \\ &= \langle T\psi(f)x, y\rangle \\ &= \langle \psi(f)x, T^*y\rangle \\ &= \int f\, d\mu_{x,T^*y},\end{aligned}$$

so that $E_{Tx,y} = E_{x,T^*y}$, and $E(S)T = TE(S)$ for all Borel $S \subseteq \Omega$. Conversely, if T commutes with all the projections $E(S)$, then we have

$$\begin{aligned}\langle E(S)Tx, y\rangle &= \langle TE(S)x, y\rangle \\ &= \langle E(S)x, T^*y\rangle,\end{aligned}$$

or that $E_{Tx,y} = E_{x,T^*y}$, so for all $f \in C(\Omega)$,

$$\int f\, dE_{Tx,y} = \int f\, dE_{x,T^*y},$$

or that

$$\begin{aligned}\langle \varphi(f)Tx, y\rangle &= \langle \varphi(f)x, T^*y\rangle \\ &= \langle T\varphi(f)x, y\rangle,\end{aligned}$$

and since this holds for all $x, y \in H$, $\varphi(f)T = T\varphi(f)$. □

The most important case is when the $*$-homomorphism in question is a representation of the C^*-algebra generated by a normal operator, and is often known as *the* spectral theorem.

Theorem: Let T be a normal operator on a Hilbert space H. There is a unique spectral measure E relative to $(\sigma(T), H)$ such that

$$T = \int \iota \, dE,$$

where ι is the inclusion map of $\sigma(T)$ into $\mathbb{C}$.

Proof. Let $\varphi\colon C(\sigma(T)) \to B(H)$ be the functional calculus at T. There is then a unique spectral measure E relative to $(\sigma(T), H)$ such that

$$\varphi(f) = \int f \, dE$$

for all $f \in C(\sigma(T))$. In particular, we have

$$\begin{aligned} T &= \varphi(\iota) \\ &= \int \iota \, dE, \end{aligned}$$

and uniqueness following from the fact that 1 and ι generate $C(\sigma(T))$ as a C^*-algebra. □

We call the spectral measure in this special case the *resolution of the identity* for T. We have that for all $f \in B_\infty(\sigma(T))$, we may unambiguously define

$$f(T) = \int f \, dE.$$

We call the unital $*$-homomorphism taking $f \mapsto f(T)$ the *Borel functional calculus* at T.

Example: Let μ be a regular compactly supported Borel measure on $\mathbb{C}$. Define N_μ on $L_2(\mu)$ by $N_\mu f = zf$ for each $f \in L_2(\mu)$. Then, $N_\mu^* f = \overline{z} f$, and N_μ is normal.

Now, we claim that $\sigma(N_\mu) = \operatorname{supp}(\mu)$. This follows from the fact that if $\lambda \in \mathbb{C} \setminus \operatorname{supp}(\mu)$, then the operator S defined by

$$Sf = (z - \lambda)^{-1} f$$

has that $\|Sf\| < \infty$ for all $f \in L_2(\mu)$.

In particular, this means that for any bounded Borel function ϕ, we may define $M_\phi f = \phi f$ and we have $\phi(N_\mu) = M_\phi$.

Example: If (X, Ω, μ) is a σ-finite measure space, and $H = L_2(X, \mu)$, we may define, for any $\phi \in L_\infty(\mu)$, the operator $M_\phi f = \phi f$. Then, M_ϕ is normal with $M_\phi^* = M_{\overline{\phi}}$.

The *essential range* of ϕ is defined as

$$\operatorname{ess\,ran}(\phi) = \bigcap \left\{ \overline{\phi(S)} \;\middle|\; S \in \Omega, \mu(X \setminus S) = 0 \right\}.$$

Then, we have that $\sigma(M_\phi) = \operatorname{ess\,ran}(\phi)$. We see that if $\lambda \notin \operatorname{ess\,ran}(\phi)$, then there is a set S in Ω with $\mu(X \setminus S) = 0$ and $\lambda \notin \overline{\phi(S)}$, so there is $\delta > 0$ so $|\phi(x) - \lambda| \geq \delta$ for all $x \in S$. Therefore, we may define

$$M_\psi = (M_\phi - \lambda)^{-1}$$

with $\psi \in L_\infty(\mu)$.

Now, if $\lambda \in \operatorname{ess\,ran}(\phi)$, then for every n, there is $S_n \in \Omega$ with $0 < \mu(S_n) < \infty$ and $|\phi(x) - \lambda| < 1/n$ for all $x \in S_n$. Set

$$f_n = (\mu(S_n))^{-1/2} \mathbb{1}_{S_n},$$

so $f_n \in L_2(\mu)$ and $\|f_n\| = 1$. Yet, we have

$$\begin{aligned}\|(M_\phi - \lambda) f_n\|^2 &= \frac{1}{\mu(S_n)} \int_{S_n} |\phi - \lambda|^2 \, d\mu \\ &< \frac{1}{n^2},\end{aligned}$$

meaning that λ is an element of the approximate point spectrum of M_ϕ.

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