

**Problem** (Problem 1): For all  $n \in \mathbb{N}$ , find the residue of  $f(z) = (1 - e^{-z})^n$  at  $z = 0$  via Cauchy's residue theorem.

**Solution:** Choose a square contour  $\gamma_R$  given by the rectangle with side length  $2R$  and height going from  $-\pi i$  to  $\pi i$ . This yields

$$\begin{aligned} 2\pi i \operatorname{Res}(f; 0) &= \oint_{\gamma_R} f(z) dz \\ &= \int_{-R}^R f(x - i\pi) d(x - i\pi) + \int_R^{-R} f(x + i\pi) d(x + i\pi) \\ &\quad + \int_{-\pi}^{\pi} f(R + iy) d(R + iy) + \int_{\pi}^{-\pi} f(-R + iy) d(-R + iy). \end{aligned}$$

Expanding each of these integrals, we get

$$\begin{aligned} \int_{-R}^R f(x - i\pi) d(x - i\pi) &= \int_{-R}^R \frac{1}{(1 + e^x)^n} dx \\ \int_R^{-R} f(x + i\pi) d(x + i\pi) &= \int_R^{-R} \frac{1}{(1 + e^x)^n} dx \\ \int_{\pi}^{-\pi} f(-R + iy) d(-R + iy) &= i \int_{\pi}^{-\pi} \frac{1}{(1 - e^{R-iy})^n} dy \\ \int_{-\pi}^{\pi} f(R + iy) d(R + iy) &= i \int_{-\pi}^{\pi} \frac{1}{(1 - e^{-R-iy})^n} dy. \end{aligned}$$

The first two integrals sum to zero. As for the third integral, we may compute

$$\begin{aligned} \left| i \int_{-\pi}^{\pi} \frac{1}{(1 - e^{R+iy})} dy \right| &\leq \frac{2\pi}{|1 - e^{R+iy}|^n} \\ &\leq \frac{2\pi}{(e^R - 1)^n} \\ &\rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ . Finally, for the fourth integral, we observe that

$$\lim_{R \rightarrow \infty} \frac{1}{(1 - e^{-R-iy})^n} = 1,$$

so our goal is to show that we can exchange limit and integral by showing that this convergence is uniform, in the sense that the sequence

$$\left( \frac{1}{(1 - e^{-k-iy})^n} \right)_{k=1}^{\infty} \rightarrow 1$$

uniformly. Towards this end, we observe that

$$\begin{aligned} \left| 1 - \frac{1}{(1 - e^{-k-iy})^n} \right| &= \left| \frac{(1 - e^{-k-iy})^n - 1}{(1 - e^{-k-iy})^n} \right| \\ &\leq \frac{|(1 - e^{-k-iy})^n - 1|}{\inf_{-\pi \leq y \leq \pi} |1 - e^{-k-iy}|^n} \\ &= \frac{|(1 - e^{-k-iy})^n - 1|}{|1 - e^{-k}|^n}. \end{aligned}$$

Observe that the numerator converges to 0 in  $k$  as  $(1 - e^{-k-iy})^n \xrightarrow{k \rightarrow \infty} 1$ , whence there is some  $K$  such that the above quantity is less than  $\varepsilon$  for arbitrary  $\varepsilon > 0$ . In particular, for  $k \geq K$ , we have

$$\sup_{-\pi \leq y \leq \pi} \left| 1 - \frac{1}{(1 - e^{-k-iy})^n} \right| < \varepsilon,$$

so that the convergence is uniform. In particular, this gives

$$\begin{aligned} 2\pi i \operatorname{Res}(f; 0) &= \lim_{R \rightarrow \infty} \int_{\gamma_R} f(z) dz \\ &= \lim_{R \rightarrow \infty} i \int_{-\pi}^{\pi} \frac{1}{(1 - e^{-R-iy})^n} dy \\ &= i \int_{-\pi}^{\pi} \lim_{R \rightarrow \infty} \frac{1}{(1 - e^{-R-iy})^n} dy \\ &= i \int_{-\pi}^{\pi} dy \\ &= 2\pi i, \end{aligned}$$

so  $\operatorname{Res}(f; 0) = 1$ .

**Problem (Problem 2):** Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2 + 1} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin^2(x)}{x^2 + 1} dx.$$

**Solution:** We compute

$$\int_{-R}^R \frac{\sin^2(x)}{x^2 + 1} dx = \frac{1}{2} \int_{-R}^R \frac{1}{x^2 + 1} dx - \frac{1}{2} \int_{-R}^R \frac{\cos(2x)}{x^2 + 1} dx.$$

Calling the latter integral  $I$ , we take

$$f(z) = \frac{e^{2iz}}{z^2 + 1},$$

close the contour  $\gamma$  in the upper half-plane with the half-circle  $C_R = \{Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$ . This gives

$$\begin{aligned} \operatorname{Re} \oint_{\gamma} f(z) dz &= \operatorname{Re}(I) + \operatorname{Re} \int_{C_R} f(z) dz \\ &= \operatorname{Re}(I) + \operatorname{Re} \int_0^{\pi} \frac{e^{2iRe^{i\theta}}}{R^2 e^{2i\theta} + 1} iRe^{i\theta} d\theta. \end{aligned}$$

Estimating the integrand on the second integral, we see that for  $R > 1$ ,

$$\begin{aligned} \left| \frac{iRe^{i\theta} e^{2iRe^{i\theta}}}{R^2 e^{2i\theta} + 1} \right| &\leq \frac{R}{R^2 - 1} |e^{2iR(\cos(\theta) + i\sin(\theta))}| \\ &\leq \frac{R}{(R^2 - 1)(e^{2R\sin(\theta)})} \\ &\leq \frac{R}{R^2 - 1} \end{aligned}$$

whence

$$\left| \int_{C_R} f(z) dz \right| \leq \pi \frac{R}{R^2 - 1}$$

$$\rightarrow 0.$$

Therefore, by Cauchy's residue theorem,

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{\cos(2x)}{x^2 + 1} dx &= \operatorname{Re}(2\pi i \operatorname{Res}(f; i)) \\ &= \operatorname{Re}\left(2\pi i \lim_{z \rightarrow i} \frac{(z-i)e^{2iz}}{(z-i)(z+i)}\right) \\ &= \frac{\pi}{e^2}. \end{aligned}$$

Thus, we find that

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2 + 1} dx = \frac{\pi}{2} - \frac{\pi}{2e^2}.$$

**Problem (Problem 3):** For  $\xi \in \mathbb{R}$ , evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2 + 4x + 5} dx = \lim_{R \rightarrow \infty} \frac{\cos(\xi x)}{x^2 + 4x + 5}.$$

**Solution:** First, if  $\xi = 0$ , then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} dx &= \int_{-\infty}^{\infty} \frac{1}{(x+2)^2 + 1} dx \\ &= \pi \end{aligned}$$

upon a  $u$ -substitution.

Now, let  $\xi > 0$ . Using  $f(z) = \frac{e^{i\xi z}}{z^2 + 4z + 5}$  and closing the contour

$$\gamma_R = [-R, R] + \{Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$$

in the upper half plane, we find that we get

$$\oint_{\gamma_R} f(z) dz = \underbrace{\int_{-R}^R f(x) dx}_{=: I} + \int_{C_R} f(z) dz.$$

Parametrizing the integral over  $C_R$  by  $z = Re^{i\theta}$ , we get

$$= I + \int_0^\pi \frac{e^{i\xi Re^{i\theta}}}{(Re^{i\theta} + 2)^2 + 1} iRe^{i\theta} d\theta.$$

Estimating the second integral, we see that for  $R > 5$ ,

$$\begin{aligned} \left| \frac{iRe^{i\theta} e^{i\xi Re^{i\theta}}}{(Re^{i\theta} + 2)^2 + 1} \right| &\leq \frac{R}{R^2 - 4R - 5} |e^{i\xi R(\cos(\theta) + i\sin(\theta))}| \\ &\leq \frac{R}{(R^2 - 4R - 5)(e^{\xi R \sin(\theta)})} \\ &\leq \frac{R}{R^2 - 4R - 5} \end{aligned}$$

meaning that

$$\left| \int_{C_R} f(z) dz \right| \leq \pi \frac{R}{R^2 - 4R - 5}$$

$$\rightarrow 0.$$

Therefore, we find that

$$\begin{aligned} 2\pi i \operatorname{Res}(-2 + i) &= \lim_{R \rightarrow \infty} \oint_{\gamma_R} f(z) dz \\ &= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2 + 4x + 5} dx \\ &= 2\pi i \lim_{z \rightarrow -2+i} \frac{(z - (-2+i))e^{i\xi z}}{(z - (-2+i))(z - (-2-i))} \\ &= 2\pi i \frac{e^{i\xi(-2+i)}}{2i} \\ &= \frac{\pi}{e^\xi} e^{-2i\xi} \\ &= \frac{\pi}{e^\xi} (\cos(2\xi) - i \sin(2\xi)) \\ &= \frac{\pi}{e^\xi} \cos(2\xi) - i \frac{\pi}{e^\xi} \sin(2\xi). \end{aligned}$$

Therefore, we find

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2 + 4x + 5} dx &= \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2 + 4x + 5} dx \\ &= \frac{\pi}{e^\xi} \cos(2\xi). \end{aligned}$$

Now, let  $\xi < 0$ . We take  $\eta_R$  to be the contour

$$\eta_R = [-R, R] + \{Re^{-i\theta} \mid 0 \leq \theta \leq \pi\}.$$

We find that

$$\begin{aligned} \oint_{\eta_R} f(z) dz &= \int_{-R}^R f(x) dx + \int_{C_R} f(z) dz \\ &= I + \int_0^\pi \frac{e^{i\xi(Re^{-i\theta})}}{(Re^{-i\theta} + 2)^2 + 1} (-iRe^{-i\theta}) d\theta. \end{aligned}$$

Estimating the second integrand, we have for  $R > 5$

$$\begin{aligned} \left| \frac{-iRe^{i\theta} e^{i\xi(Re^{-i\theta})}}{(Re^{-i\theta} + 2)^2 + 1} \right| &\leq \frac{R}{R^2 - 4R - 5} |e^{i\xi R(\cos(\theta) - i \sin(\theta))}| \\ &\leq \frac{R}{R^2 - 4R - 5} e^{\xi R \sin(\theta)} \\ &\leq \frac{R}{R^2 - 4R - 5}. \end{aligned}$$

Thus,

$$\left| \int_{C_R} f(z) dz \right| \leq \pi \frac{R}{R^2 - 4R - 5},$$

whence the integral over  $C_R$  goes to zero as  $R \rightarrow \infty$ . Therefore, we have

$$-2\pi i \operatorname{Res}(f; -2 - i) = \lim_{R \rightarrow \infty} \int_{\eta_R} f(z) dz$$

$$\begin{aligned}
&= I + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz \\
&= I \\
&= -2\pi i \lim_{z \rightarrow -2-i} \frac{(z - (-2-i))e^{i\xi z}}{(z - (-2-i))(z - (-2+i))} \\
&= -2\pi i \frac{e^{i\xi(-2-i)}}{-2i} \\
&= \pi e^{i\xi(-2-i)} \\
&= \pi e^{\xi} (\cos(2\xi) - i \sin(2\xi)) \\
&= \pi e^{\xi} \cos(2\xi) - i\pi e^{\xi} \sin(2\xi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2 + 4x + 5} dx &= \operatorname{Re}(I) \\
&= \pi e^{\xi} \cos(2\xi).
\end{aligned}$$

**Problem (Problem 4):** Evaluate

$$\int_0^{\infty} \frac{(\log x)^2}{x^2 + 1} dx.$$

**Solution:** Select the branch of the logarithm that ignores  $[0, \infty)$ , so that  $\arg(z) \in (0, 2\pi)$  for all  $z \in \mathbb{C} \setminus [0, \infty)$ . Draw a keyhole contour  $\gamma_{\delta, \varepsilon, R}$  with an inner semicircle of radius  $\delta$ , an outer semicircle of radius  $R$ , and returning along the negative real axis to the start of the semicircle of radius  $\delta$ .

Set  $f(z) = \frac{(\log z)^2}{z^2 + 1}$ , and observe that for  $0 < \varepsilon < \delta < 1 < R$ , we have

$$\begin{aligned}
\oint_{\gamma_{\delta, \varepsilon, R}} f(z) dz &= 2\pi i (\operatorname{Res}(f; i)) \\
&= 2\pi i \left( \lim_{z \rightarrow i} (z - i) \frac{(\log(z))^2}{(z - i)(z + i)} \right) \\
&= -\frac{\pi^3}{4}.
\end{aligned}$$

Meanwhile, we observe that in the limit as  $\varepsilon \rightarrow 0$ , we are left with a few integrals

$$\oint_{\gamma_{\delta, \varepsilon, R}} f(z) dz = \int_{\delta}^R \frac{(\log(x))^2}{x^2 + 1} dx + \int_{-R}^{-\delta} \frac{(\log(x))^2}{x^2 + 1} dx \quad (*)$$

$$+ \int_0^{\pi} \frac{\log(\delta e^{-i\theta})^2}{\delta^2 e^{-2i\theta} + 1} (-i\delta e^{-i\theta}) d\theta + \int_0^{\pi} \frac{\log(R e^{i\theta})^2}{R^2 e^{2i\theta} + 1} iR e^{i\theta} d\theta \quad (**)$$

We start by estimating the integrals in  $(**)$  by the circles  $\delta e^{-i\theta}$  and  $R e^{i\theta}$ . Towards this end, we observe that

$$\begin{aligned}
\left| \frac{-i\delta e^{-i\theta} (\ln(\delta) - i\theta)^2}{\delta^2 e^{-2i\theta} + 1} \right| &\leq \frac{\delta |\ln(\delta)|^2 + 2\theta \delta |\ln(\delta)| + \theta^2 \delta}{1 - \delta^2} \\
&\leq \frac{\delta |\ln(\delta)|^2 + 4\pi \delta |\ln(\delta)| + 4\pi^2 \delta}{1 - \delta^2} \\
&\rightarrow 0
\end{aligned}$$

as  $\delta \rightarrow 0$ . Thus,

$$\left| \int_0^{2\pi} \frac{-i\delta e^{-i\theta}(\ln(\delta) - i\theta)}{\delta^2 e^{2i\theta} + 1} d\theta \right| \leq \pi \frac{\delta |\ln(\delta)|^2 + 4\pi\delta |\ln(\delta)| + 4\pi^2 \delta}{1 - \delta^2} \rightarrow 0.$$

Similarly,

$$\begin{aligned} \left| \frac{Re^{i\theta}(\ln(R) + i\theta)^2}{R^2 e^{2i\theta} + 1} \right| &\leq \frac{R|\ln(R)|^2 + 2\theta R |\ln(R)| + \theta^2 R}{R^2 - 1} \\ &\leq \frac{R|\ln(R)|^2}{R^2 - 1} + \frac{2\pi R}{R^2 - 1} + \frac{4\pi^2}{R^2 - 1} \\ &= \frac{|\ln(R)|^2}{R - \frac{1}{R}} \frac{2\pi R}{R^2 - 1} + \frac{4\pi^2}{R^2 - 1} \\ &\rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ , so the corresponding integral also goes to zero.

Now, we turn our attention to (\*). We observe that by the coordinate change  $x \mapsto -x$ , we get

$$\int_{-\delta}^R \frac{\ln(x)}{x^2 + 1} dx + \int_{-\infty}^{-\delta} \frac{(\ln(x))^2}{x^2 + 1} dx = 2 \int_{\delta}^R \frac{(\ln(x))^2}{x^2 + 1} dx + 2\pi i \int_{\delta}^R \frac{\ln(x)}{x^2 + 1} dx - \pi^2 \int_{\delta}^R \frac{1}{x^2 + 1} dx.$$

As we take the limit as  $\delta \rightarrow 0$  and  $R \rightarrow \infty$ , we observe that we get the equation

$$\frac{\pi^3}{4} = 2 \underbrace{\int_0^\infty \frac{(\ln(x))^2}{x^2 + 1} dx}_{=: I_1} + 2\pi i \underbrace{\int_0^\infty \frac{\ln(x)}{x^2 + 1} dx}_{=: I_0}$$

Now, to evaluate  $I_0$ , we use the same contour for  $g(z) = \frac{\ln(z)}{z^2 + 1}$ , giving

$$\begin{aligned} \int_{\gamma_{\delta, \varepsilon, R}} g(z) dz &= \int_{\delta}^R \frac{\ln(x)}{x^2 + 1} dx + \int_{-R}^{-\delta} \frac{\ln(x)}{x^2 + 1} dx \\ &\quad + \int_0^\pi \frac{\ln(Re^{i\theta})}{R^2 e^{2i\theta} + 1} iRe^{i\theta} d\theta + \int_0^\pi \frac{\ln(\delta e^{-i\theta})}{\delta^2 e^{2i\theta} + 1} (-i\delta e^{-i\theta}) d\theta. \end{aligned}$$

The circle integrands may be estimated by

$$\begin{aligned} \left| \frac{iRe^{i\theta} |\ln(R) + i\theta|}{R^2 e^{2i\theta}} \right| &\leq \frac{R \ln(R) + R\theta}{R^2 - 1} \\ &\leq \frac{R \ln(R) + \pi R}{R^2 - 1} \\ &\rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ , so that

$$\begin{aligned} \left| \int_0^\pi \frac{\ln(Re^{i\theta})}{R^2 e^{2i\theta} + 1} iRe^{i\theta} d\theta \right| &\leq \pi \frac{R \ln(R) + \pi R}{R^2 - 1} \\ &\rightarrow 0. \end{aligned}$$

Similarly,

$$\left| \frac{-i\delta e^{-i\theta} (\ln(\delta) - i\theta)}{\delta^2 e^{2i\theta} + 1} \right| \leq \frac{\delta |\ln(\delta)| + \pi\delta}{1 - \delta^2}$$

$$\rightarrow 0$$

as  $\delta \rightarrow \infty$ , so that

$$\left| \int_0^\pi \frac{\ln(\delta e^{-i\theta})}{\delta^2 e^{2i\theta} + 1} (-i\delta e^{-i\theta}) d\theta \right| \leq \pi \frac{\delta |\ln(\delta)| + \pi\delta}{1 - \delta^2} \rightarrow 0.$$

Thus, we must evaluate the first two integrals. Yet, by using the substitution  $x \mapsto -x$ , we see that

$$\int_\delta^R \frac{\ln(x)}{x^2 + 1} dx + \int_{-R}^{-\delta} \frac{\ln(x)}{x^2 + 1} dx = 2 \int_\delta^R \frac{\ln(x)}{x^2 + 1} dx + i\pi \int_\delta^R \frac{1}{x^2 + 1} dx.$$

Taking limits and evaluating residues gives

$$\begin{aligned} 2\pi i \operatorname{Res}(g; i) &= 2\pi i \left( \frac{i\pi/2}{2i} \right) \\ &= i \frac{\pi^2}{2} \\ &= 2 \int_0^\infty \frac{\ln(x)}{x^2 + 1} dx + i\pi \int_0^\infty \frac{1}{x^2 + 1} dx \\ &= 2 \int_0^\infty \frac{\ln(x)}{x^2 + 1} dx + i \frac{\pi^2}{2}, \end{aligned}$$

whence the integral for  $g(z)$  is zero.

Thus, we find that

$$\int_0^\infty \frac{(\ln(x))^2}{x^2 + 1} dx = \frac{\pi^3}{8}$$

**Problem (Problem 5):** For  $\xi \in \mathbb{R}$ , evaluate

$$\text{p. v. } \int_{-\infty}^\infty \frac{x^3}{(x^2 + 1)^2} e^{-2\pi i x \xi} dx.$$

**Solution:** We write

$$\int_{-\infty}^\infty \frac{x^3}{(x^2 + 1)^2} e^{-2\pi i x \xi} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^3}{(x - i)^2 (x + i)^2} e^{-2\pi i x \xi} dx.$$

$$\text{Write } f(z) = \frac{z^3}{(z^2 + 1)^2} e^{-2\pi i z \xi}.$$

Suppose  $\xi = 0$ . Then, we seek to compute

$$\int_{-\infty}^\infty \frac{x^3}{(x^2 + 1)^2} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^3}{(x^2 + 1)^2} dx.$$

Since the integral in  $R$  is an odd integrand over a symmetric interval, it follows that the principal value is 0.

Suppose  $\xi > 0$ . Let  $\gamma_R$  be the square contour in the lower half-plane with side length  $R$  sitting on the real axis. Then,

$$-2\pi i \operatorname{Res}(f; -i) = \oint_{\gamma_R} f(z) dz$$

$$\begin{aligned}
&= \int_{-R}^R f(x) dx + \int_0^R f(R - iy) d(R - iy) \\
&\quad + \int_R^{-R} f(x - iR) d(x - iR) + \int_R^0 f(-R - iy) d(-R - iy).
\end{aligned}$$

Writing each of the integrals not equal to the original integral, we get

$$\int_0^R f(R - iy) d(R - iy) = -i \int_0^R \frac{(R - iy)^3}{((R - iy)^2 + 1)^2} e^{-2\pi i \xi (R - iy)} dy \quad (1)$$

$$\int_R^{-R} f(x - iR) d(x - iR) = e^{-2\pi \xi R} \int_R^{-R} \frac{(x - iR)^3}{((x - iR)^2 + 1)^2} e^{-2\pi i \xi x} dx \quad (2)$$

$$\int_R^0 f(-R - iy) d(-R - iy) = -i \int_R^0 \frac{(-R - iy)^3}{((-R - iy)^2 + 1)^2} e^{-2\pi i \xi (-R - iy)} dy \quad (3)$$

Our goal now is to show that the remaining terms vanish. We start with (1), and observe that

$$\begin{aligned}
\left| \int_0^R -i \frac{(R - iy)^3}{((R - iy)^2 + 1)^2} e^{-2\pi i \xi R} e^{-2\pi \xi y} dy \right| &\leq \frac{\int_0^R e^{-2\pi \xi y} dy}{\inf_{0 \leq y \leq R} |(R - iy) + \frac{2}{(R - iy)} + \frac{1}{(R - iy)^3}|} \\
&\leq \frac{\frac{1}{2\pi \xi} (1 - e^{-2\pi \xi R})}{R - \frac{2}{R} - \frac{1}{R^3}} \\
&\rightarrow 0.
\end{aligned}$$

Next, regarding (2), we have

$$\begin{aligned}
\left| e^{-2\pi \xi R} \int_R^{-R} \frac{(x - iR)^3}{((x - iR)^2 + 1)^2} e^{-2\pi i \xi x} dx \right| &\leq e^{-2\pi \xi R} \frac{2R}{\inf_{-R \leq x \leq R} |(x - iR) + \frac{2}{(x - iR)} + \frac{1}{(x - iR)^3}|} \\
&\leq e^{-2\pi \xi R} \frac{1}{1 - \frac{2}{R^2} - \frac{1}{R^4}} \\
&\rightarrow 0.
\end{aligned}$$

Finally, (3) follows from the same argument as (1). Thus, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) dx &= \lim_{R \rightarrow \infty} \oint_{\gamma_R} f(z) dz \\
&= -2\pi i \operatorname{Res}(f; -i) \\
&= -2\pi i \left( \frac{d}{dz} \left( \frac{z^3}{(z - i)^2} e^{-2\pi i \xi z} \right) \Big|_{z=i} \right) \\
&= -2\pi i \left( \frac{3z^2}{(z - i)^2} e^{-2\pi i \xi z} - 2 \frac{z^3}{(z - i)^3} e^{-2\pi i \xi z} - 2\pi i \xi \frac{z^3}{(z - i)^2} e^{-2\pi i \xi z} \right) \Big|_{z=i} \\
&= -2\pi i \left( \frac{3(-i)^2}{(-2i)^2} e^{2\pi \xi} - 2 \frac{(-i)^3}{(-2i)^3} e^{2\pi \xi} - 2\pi i \xi \frac{(-i)^3}{(-2i)^2} e^{2\pi \xi} \right)
\end{aligned}$$

$$\begin{aligned}
&= -2\pi i \left( \frac{-3}{-4} e^{2\pi\xi} - 2 \frac{i}{8i} e^{2\pi\xi} - 2\pi i \xi \frac{i}{-4} e^{2\pi\xi} \right) \\
&= -2\pi i \left( \frac{1}{2} - \frac{1}{2}\pi\xi \right) e^{2\pi\xi} \\
&= i\pi(\pi\xi - 1) e^{2\pi\xi}.
\end{aligned}$$

Now, let  $\xi < 0$ . Then, we let  $\gamma_R$  be the same square with side length  $2R$ , but this time contained in the *upper* half-plane. This gives

$$\begin{aligned}
2\pi i \operatorname{Res}(f; i) &= \oint_{\gamma_R} f(z) dz \\
&= \int_{-R}^R f(x) dx + \int_0^R f(R+iy) d(R+iy) \\
&\quad + \int_R^{-R} f(x+iR) d(x+iR) + \int_R^0 f(-R+iy) d(-R+iy).
\end{aligned}$$

Similarly, we write out the three integrals not equal to our desired result, and find

$$\begin{aligned}
\int_0^R f(R+iy) d(R+iy) &= i \int_0^R \frac{(R+iy)^3}{((R+iy)^2 + 1)^2} e^{-2\pi i \xi (R+iy)} dy \\
\int_R^{-R} f(x+iR) d(x+iR) &= e^{2\pi \xi R} \int_R^{-R} \frac{(x+iR)^3}{((x+iR)^2 + 1)^2} e^{-2\pi i x} dx \\
\int_R^0 f(-R+iy) d(-R+iy) &= i \int_R^0 \frac{(-R+iy)^3}{((R+iy)^2 + 1)^2} e^{-2\pi i \xi (-R+iy)} dy
\end{aligned}$$

We use the same estimate process as with the case of  $\xi > 0$ .

$$\begin{aligned}
\left| \int_0^R i \frac{(R+iy)^3}{((R+iy)^2 + 1)^2} e^{-2\pi i \xi (R+iy)} dy \right| &\leq \frac{1}{\inf_{0 \leq y \leq R} |(R+iy) + \frac{2}{(R+iy)} + \frac{1}{(R+iy)^3}|} \int_0^R e^{2\pi \xi y} dy \\
&\leq \frac{1}{R - \frac{2}{R} - \frac{1}{R^3}} \left( -\frac{1}{2\pi\xi} + \frac{1}{2\pi\xi} e^{2\pi \xi R} \right) \\
&\rightarrow 0 \\
\left| e^{2\pi \xi R} \int_R^{-R} \frac{(x+iR)^3}{((x+iR)^2 + 1)^2} dx \right| &\leq e^{2\pi \xi R} \frac{2R}{\inf_{-R \leq x \leq R} |(x+iR) + \frac{2}{(x+iR)} + \frac{1}{(x+iR)^3}|} \\
&\leq e^{2\pi \xi R} \frac{1}{1 - \frac{2}{R^2} - \frac{1}{R^4}} \\
&\rightarrow 0,
\end{aligned}$$

and the third integral vanishes analogously to the first integral. Thus, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) dx &= \lim_{R \rightarrow \infty} \int_{\gamma_R} f(z) dz \\
&= 2\pi i \operatorname{Res}(f; i)
\end{aligned}$$

$$\begin{aligned}
&= 2\pi i \left( \frac{d}{dz} \left( \frac{z^3}{(z+i)^2} e^{-2\pi i \xi z} \right) \Big|_{z=i} \right) \\
&= 2\pi i \left( \frac{3z^2}{(z+i)^2} e^{-2\pi i \xi z} - 2 \frac{z^3}{(z+i)^3} e^{-2\pi i \xi z} - 2\pi i \xi \frac{z^3}{(z+i)^2} e^{-2\pi i \xi z} \right) \Big|_{z=i} \\
&= 2\pi i \left( \frac{3(i)^2}{(2i)^2} e^{2\pi \xi} - 2 \frac{(i)^3}{(2i)^3} e^{2\pi \xi} - 2\pi i \xi \frac{(i)^3}{(2i)^2} e^{2\pi \xi} \right) \\
&= 2\pi i \left( \frac{-3}{-4} e^{2\pi \xi} - 2 \frac{-i}{-8i} e^{2\pi \xi} - 2\pi i \xi \frac{-i}{-4} e^{2\pi \xi} \right) \\
&= i\pi(\pi\xi - 1)e^{2\pi\xi}.
\end{aligned}$$