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Introduction

This is going to be a part of my Honors thesis independent study, focused on amenability and C*-algebras. This section of notes will be a deeper dive into group amenability. These notes will be taken from the notes my professor has prepared on group amenability, with supplementation from Volker Runde's *Lectures on Amenability* and Pierre de la Harpe's *Topics in Geometric Group Theory*.

I do not claim any of this work to be original.

Amenable Groups and Subgroups

Let G be a group, with $P(G)$ denoting the power set.

Definition. An invariant mean on G is a set function $m : P(G) \to [0,1]$, which satisfies, for all $t \in G$ and $E, F \subseteq G$,

(1) $m(G) = 1$;

(2) $m(E \sqcup F) = M(E) + m(F)$;

(3) $m(tE) = m(E)$.

We say G is amenable if it admits a mean.

We can also say that m is a translation-invariant probability measure on $(G, P(G))$.

Proposition (Amenability of Subgroups and Quotient Groups)**:** Let G be amenable, with $H \leqslant G$.

(1) H is amenable;

(2) for $H \trianglelefteq G$, G/H is amenable.

Proof.

(1) Let R be a right transversal for H (i.e., selecting one element of each right coset of H to make up R).

If m is a mean for G, we set

$$\lambda : \mathcal{P}(H) \to [0,1]$$

by $\lambda(E) = m(ER)$. We have

$$\begin{aligned}\lambda(H) &= m(HR)\\ &= m(G)\\ &= 1.\end{aligned}$$

We claim that if $E \cap F = \emptyset$, then $ER \cap FR = \emptyset$, since if we suppose toward contradiction that $ER \cap FR \neq \emptyset$, then $xr_1 = yr_2$ for some $x \in E, y \in F$ and $r_1, r_2 \in R$. Then, we must have $r_2 r_1^{-1} = y^{-1}x \in H$, meaning $r_1 = r_2$ and $x = y$, which means $E \cap F \neq \emptyset$.

Thus, we have

$$\begin{aligned}\lambda(E \sqcup F) &= m((E \sqcup F)R)\\ &= m(ER \sqcup FR)\\ &= m(ER) + m(FR)\\ &= \lambda(E) + \lambda(F),\end{aligned}$$

and

$$\begin{aligned}\lambda(sE) &= m(sER)\\ &= m(ER)\\ &= \lambda(E).\end{aligned}$$

(2) For the canonical projection map $\pi : G \to G/H$ defined by $\pi(t) = tH$, we define

$$\lambda : P(G/H) \to [0,1]$$

by $\lambda(E) = m\left(\pi^{-1}(E)\right)$. We have

$$\begin{aligned}\lambda(G/H) &= m\left(\pi^{-1}(G/H)\right)\\ &= m(G)\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(E \sqcup F) &= m\left(\pi^{-1}(E \sqcup F)\right)\\ &= m\left(\pi^{-1}(E) \sqcup \pi^{-1}(F)\right)\\ &= m\left(\pi^{-1}(E)\right) + m\left(\pi^{-1}(F)\right)\\ &= \lambda(E) + \lambda(F).\end{aligned}$$

To show translation-invariance, we let $sH = \pi(s) \in G/H$, and $E \subseteq G/H$. Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for $r \in s\pi^{-1}(E)$, we have $r = st$ for $\pi(t) \in E$, so $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$.

Additionally, if $r \in \pi^{-1}(\pi(s)E)$, then $\pi(r) \in \pi(s)E$, so $\pi\left(s^{-1}r\right) \in E$, and $s^{-1}r \in \pi^{-1}(E)$. Thus, we have

$$\begin{aligned}\lambda(\pi(s)E) &= m\left(\pi^{-1}(\pi(s)E)\right)\\ &= m\left(s\pi^{-1}(E)\right)\\ &= m\left(\pi^{-1}(E)\right)\\ &= \lambda(E).\end{aligned}$$

□

Understanding Free Groups

In the Tarski's Theorem notes, we discussed a little bit the ramifications of the free group on two generators being paradoxical. In order to better understand free groups, we will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory*.