Amenability: A (Somewhat) Brief Introduction

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Outline

Contents

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

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then we call the pair $(A, \star)$ a *group*.

We abbreviate $a \star b$ as ab. If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* G/N.
- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

Some Groups

- The integers $\mathbb{Z}$ are a group under addition.
- The group of invertible $n \times n$ matrices over $\mathbb{C}$, $\mathrm{GL}_n(\mathbb{C})$, is a group under matrix multiplication.
- The subgroup $\mathrm{SO}(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under multiplication.

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X. We write $\rho(g, x) = g \cdot x$.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i\in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a σ-*algebra* of subsets.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

σ-Algebras and Measures, Cont'd

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then we say μ is a *finitely additive* measure.

If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say μ is a measure. If $\mu(X) = 1$, then we say μ is a probability measure.

Contents

Questions?

- If G is a group, is it possible to reconstruct G by using some subset of G?
- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

- We begin by considering a special group, known as $F(a, b)$ or the *free group on two generators*.

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- We define $F(a,b)$ to be the set of all "words" in the alphabet $\{a, b, a^{-1}, b^{-1}\}$, subject to the condition that, for $w, w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

A Curiosity

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- all words that start with a;
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- all words that start with b — think words that start with b^2 before you multiply b^{-1}.

Thus, all we need to do is add back $W\left(b^{-1}\right)$ to get $F(a,b)$ back.

$$F(a,b) = W\left(b^{-1}\right) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

We're able to take part of the group $F(a,b)$, take some translations, and, miraculously, obtain the entire group back.

Defining Paradoxical Decompositions

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$;

such that

$$G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

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$$G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Actions

If G acts on a set X, then a subset $A \subseteq X$ is G-*paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
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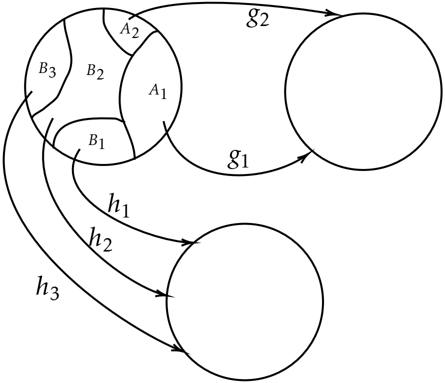
A paradoxical group is a paradoxical set under the action of left-multiplication.

Depiction



Examples

- The free group $F(a,b)$ is paradoxical.

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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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This is proven using the Ping-Pong lemma.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

- In other words, not all subsets of $\mathbb{R}^3$ have a definite "volume" invariant under isometry.

Equidecomposability

Let G be a group that acts on a set X, and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \ldots, A_n \subseteq A$, $B_1, \ldots, B_n \subseteq B$
- group elements $g_1, \ldots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say A and B are *G-equidecomposable*.

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Effectively, A and B are "equal" to each other up to the group action.

If A is G-paradoxical, then so too is B.

The Banach–Tarski Paradox: Proof Outline I

1. We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of SO(3) isomorphic to $F(a, b)$.

The Banach–Tarski Paradox: Proof Outline II

2. We use the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right) \end{aligned}$$

   to duplicate the unit sphere in $\mathbb{R}^3$, S^2, except for a countable subset D. (The *Hausdorff Paradox*.)
3. We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable — there is thus a paradoxical decomposition of S^2.
4. We show that the unit ball, $B(0,1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group E(3).

The Banach–Tarski Paradox: Proof Outline III

5. Define a relation $A \preceq B$ if A is G-equidecomposable with a subset of B, and show that if $A \preceq B$ and $B \preceq A$, then A and B are G-equidecomposable.
6. Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

Contents

Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped "create" the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu\colon F(a,b) \to [0,1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$1 = \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right)$$

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&= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\
&= \nu(F(a,b)) + \nu(F(a,b)) \\
&= 2.
\end{aligned}$$

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu \colon G \to [0,1]$ such that

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for all $t \in G$ and $E \subseteq G$.

If G admits a mean, we say G is *amenable*.

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- In other words, G is sufficiently "well-behaved."

Inheritance Properties of Amenability

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- If G is amenable, then quotient groups, G/N, are amenable.
- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

  Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

  is a mean.
- Abelian groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

Paradoxical Groups and Amenability

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Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

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More surprisingly, though, every *non*-paradoxical group is amenable.

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Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

Normed Vector Spaces

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Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|\colon V \to [0,\infty)$ satisfying:

- definiteness: $\|v\| \geq 0$, with equality if and only if $v = 0$;
- homogeneity: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$;
- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sup_{t\in\Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t\in\Gamma} |f(t)| < \infty \right\};$$

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$$\ell_2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sum_{t \in \Gamma} |f(t)|^2 < \infty \right\}.$$

They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t \in \Gamma} |f(t)|$;
- $\|f\|_{\ell_1} := \sum_{t \in \Gamma} |f(t)|$;
- $\|f\|_{\ell_2} := \left(\sum_{t \in \Gamma} |f(t)|^2 \right)^{1/2}$.

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

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We call the quantity on the left the *operator norm*, denoted $\|T\|_{\mathrm{op}}$.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi \colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a bounded linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}}$.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s \colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

for all $t \in \Gamma$ and fixed $s \in \Gamma$.

If $\varphi \colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

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If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

Motivating Følner's Condition

There is actually one way that working with sets makes life easier.

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Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was "significantly" "bigger" than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

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But what does "bigger" actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n\to\infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

Approximate Means

The Følner condition allows us to find an “approximate” version of a mean.

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Keeping $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a "sufficient" finitely supported function g,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

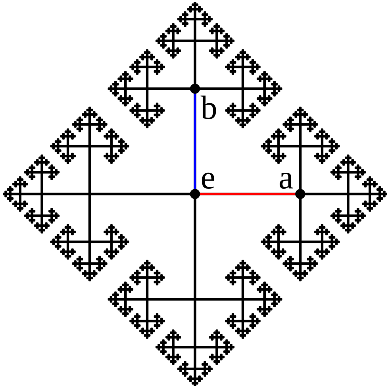
then use a "layer cake" decomposition to find our Følner sets:

$$g = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S, we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by "walking" along the generators.



Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to G (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

$$\inf\left\{ \frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), |S| \text{ finite} \right\} = 0.$$

- Essentially, the Cayley graph doesn't "get too big" "too fast."
- This is proven with the Følner condition.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ that satisfies

- $\langle x,x\rangle \geq 0$, with equality only when $x=0$;
- $\langle x_1+\alpha x_2, y\rangle = \langle x_1,y\rangle + \alpha\langle x_2,y\rangle$;
- $\langle x, y_1+\alpha y_2\rangle = \langle x,y_1\rangle + \overline{\alpha}\langle x,y_2\rangle$.

The inner product induces a norm $\|x\|^2 = \langle x,x\rangle$. If $\mathcal{H}$ is complete with respect to this norm, we call $\mathcal{H}$ a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that "plays nicely" with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

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$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

If $U : \mathcal{H} \to \mathcal{H}$ is such that

$$U^*U = I$$
$$UU^* = I,$$

then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$
$$\lambda\left(s^{-1}\right) = \lambda(s)^*$$

is called a *unitary representation* of Γ.

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is called a *unitary representation* of Γ.

All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma\colon \Gamma \to \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

One special representation is defined by $s \mapsto \lambda_s$, where $\lambda_s(f)(t) = f\left(s^{-1}t\right)$.

This is known as the *left-regular representation*, and is a very useful

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* if

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If $\lambda \colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T : \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\text{op}}$, is a vector space with the following properties:

- $\|TS\|_{\text{op}} \leq \|T\|_{\text{op}}\|S\|_{\text{op}}$;
- $\|T^*\|_{\text{op}} = \|T\|_{\text{op}}$;
- $\|T^*T\|_{\text{op}} = \|T\|_{\text{op}}^2$.

Additionally, the adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^*T^*$.

These make $\mathbb{B}(\mathcal{H})$ a C^**-algebra*. However, there are other C^*-algebras.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t)\delta_t,$$

where δ_t is the point mass at $t \in \Gamma$.

A Group C^*-Algebra

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$$x = \sum_{t\in\Gamma} x(t)\delta_t,$$

where δ_t is the point mass at $t \in \Gamma$.

This becomes a $*$-algebra when endowed with multiplication and involution:

$$f * g(s) = \sum_{t\in\Gamma} f(t)g\left(s^{-1}t\right)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^*-Algebra, cont'd

If we represent $\pi_\lambda : \mathbb{C}[\Gamma] \to \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\mathrm{op}},$$

we get the *reduced group C^*-algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

The $n \times n$ matrices, $\mathrm{Mat}_n(\mathbb{C})$, are also C^*-algebras.

Using a common tactic of finite approximations, we can define a special kind of finite-dimensional approximation for C^*-algebras using matrices.

Nuclearity

A C^*-algebra, A, is called *nuclear* if there exist two sequences of maps, $\varphi_n \colon A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n \colon \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$, such that

$$\left\| a - \psi_n \circ \varphi_n(a) \right\| \xrightarrow{n\to\infty} 0.$$

- Essentially, any $a \in A$ is "close enough" to a certain family of finite-dimensional analogues.

Nuclearity and Amenability

A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

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A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

- This is also proven using the Følner condition.

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's Theorem);
- Γ admits a finitely additive probability measure, $\mu\colon \Gamma \to [0,1]$ such that $\mu(E) = \mu(tE)$ (existence of means);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);
- there is a sequence $(f_k)_k \subseteq \ell_1(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_1} \to 0$ (Approximate Means);
- the Cayley graph of Γ satisfies $\inf\left\{ \frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), S \text{ finite} \right\} = 0$ (graph amenability);
- there is a sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_2} \to 0$ (almost-invariant vectors);
- the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear (nuclearity).

Contents

Final Remarks

Amenability is still a very active field of study.

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Nuclear C^*-algebras are classified, so active research areas primarily concern whether or not certain classes of C^*-algebras are nuclear (hence classifiable).

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Nuclear C^*-algebras are classified, so active research areas primarily concern whether or not certain classes of C^*-algebras are nuclear (hence classifiable).

There are also a lot of other directions that amenability can take the eager student, but I think this was a pretty nice overview of some of the ways that amenability touches all sorts of other fields of math.

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