The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of x).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

Some Functions and Sets

$$f(x, y) = x^2 - y^2$$

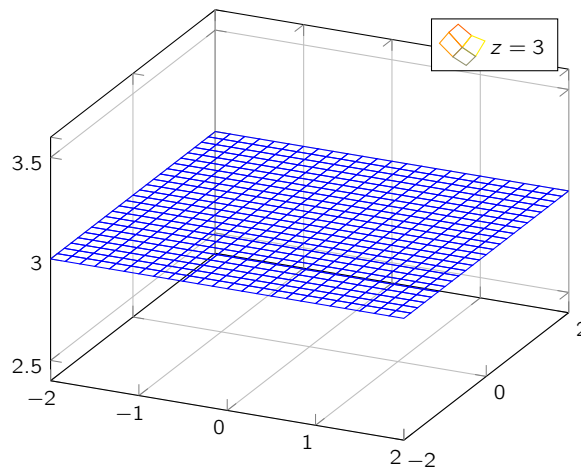Domain: $\{(x, y) \mid \exists f(x, y)\}$

Range: $\{f(x, y) \mid (x, y) \in \mathrm{Dom}(f)\} = \mathbb{R}$

Graph: $\mathrm{Graph}(f) = \{x, y.f(x, y) \mid x, y \in \mathrm{Dom}(f)\}$. For example, $(1, 3, 4) \notin \mathrm{Graph}(f)$ since $1^2 - 3^2 \neq 4$.
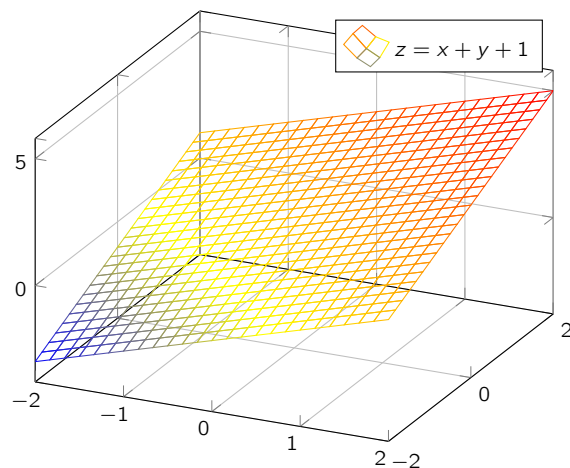

Examples

In $\mathbb{R}^3$, in x, y, z coordinates, $z = 3$ is a plane defined as follows:

- Parallel to the xy plane.

- Passes through the point $(0.0, 3)$.

Meanwhile, $y = 0$ would be a "wall" that passes through the origin that contains the line $y = 0$ in the xy plane.

Finally, $z = x + y + 1$ is a plane, as we can see below.
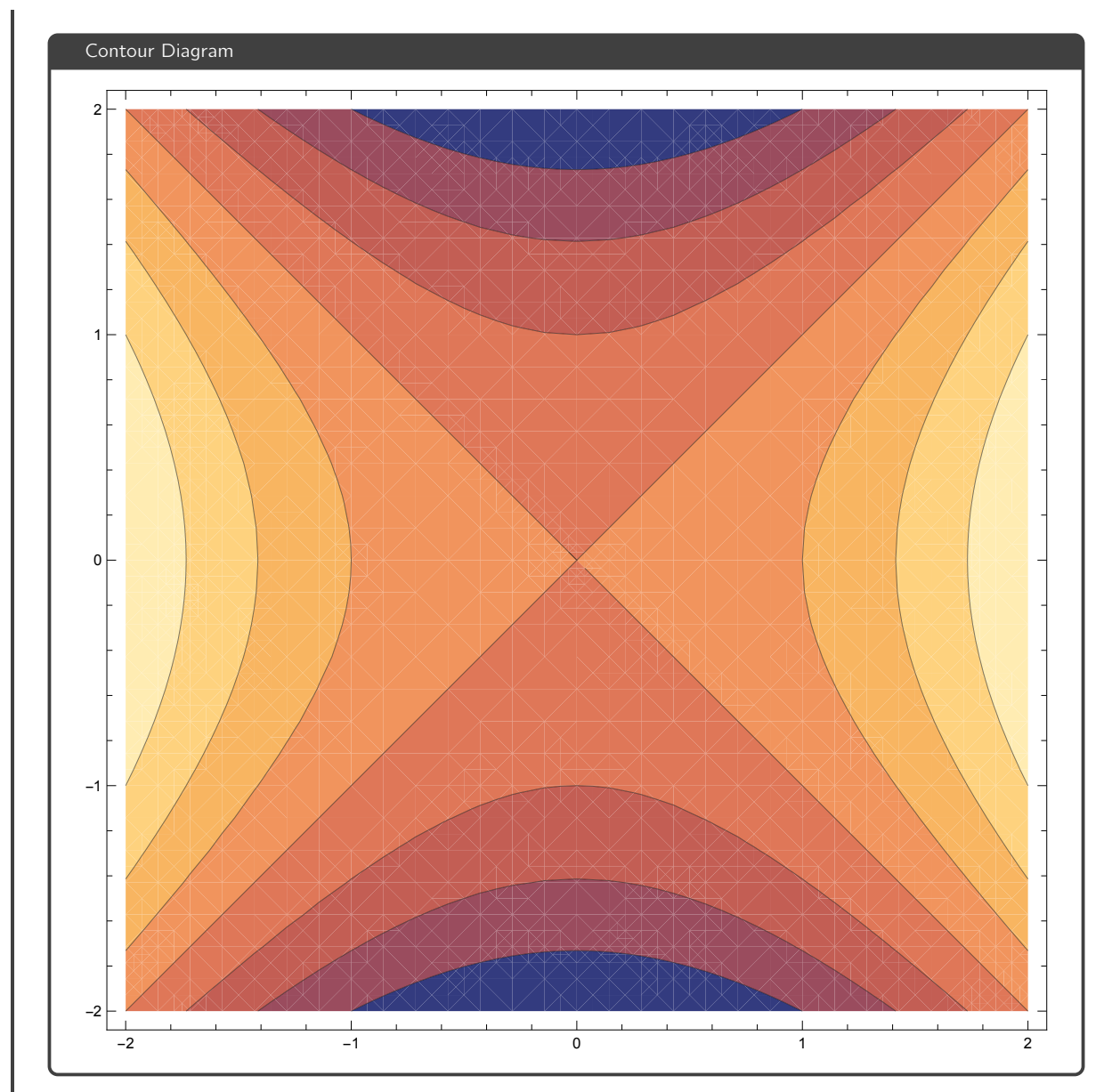
$z = x + y + 1$

Visualizing a function of multiple variables

Consider the function $f(x, y) = x^2 - y^2$. We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$

- $f(0, y) = -y^2$

- $f(2, y) = 4 - y^2$

- $f(x, -2) = x^2 + 4$

- $f(x, 0) = x^2$

- $f(x, 2) = x^2 + 4$

Alternatively, we can visualize via contour diagrams (i.e., everywhere that z is a certain value), as seen in mathematica as follows:

Contour Diagram

Contour Example

Consider the function $f(x, y) = y - 3x^2$. We want to find the contours.

For any c, we have that $c = y - 3x^3$, or $y = 3x^3 + c$. Therefore, every contour "looks like" $3x^3 + c$ for values of c. For example, in the following, we have $c = \{-2, -1, 0, 1, 2\}$
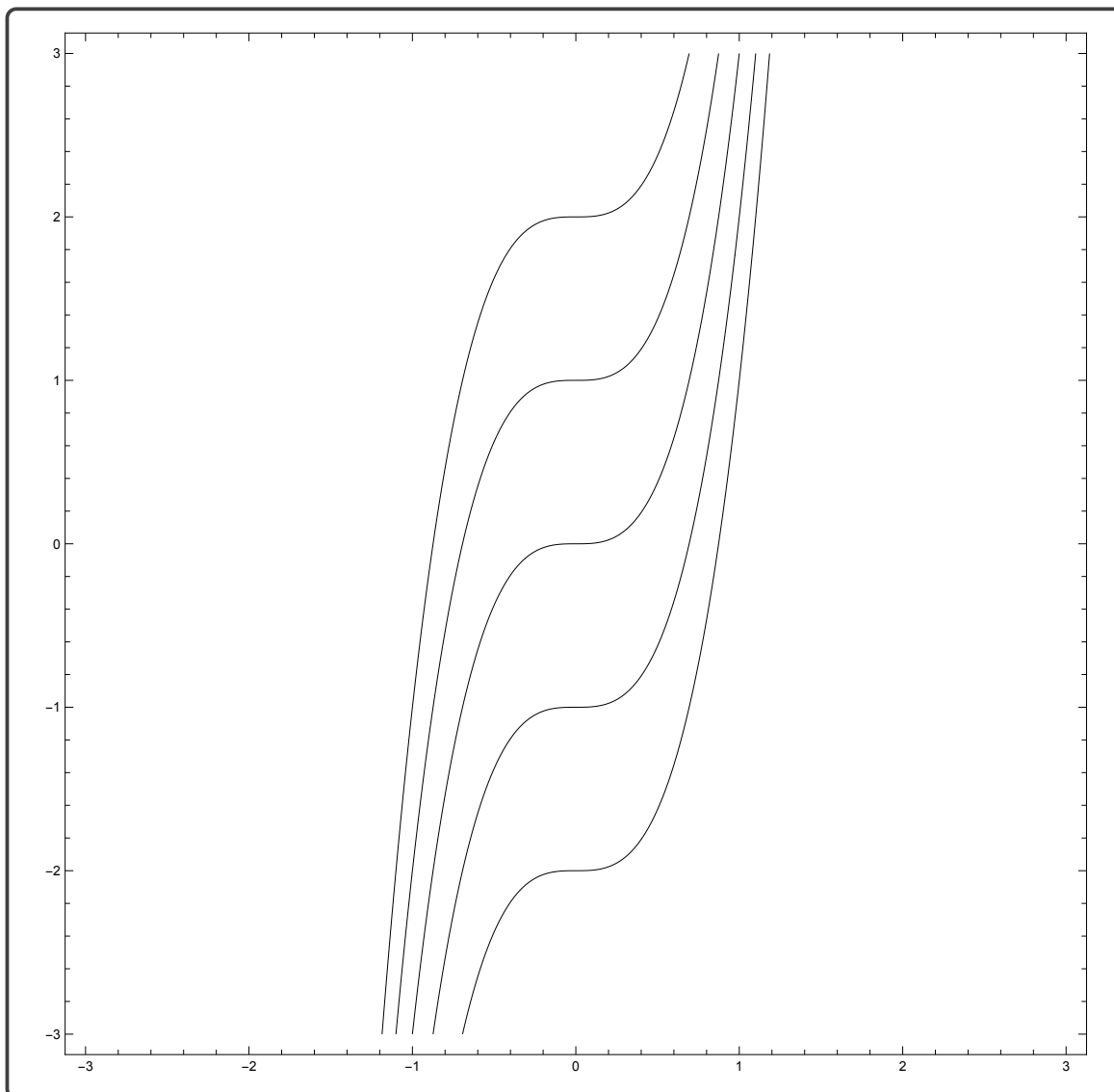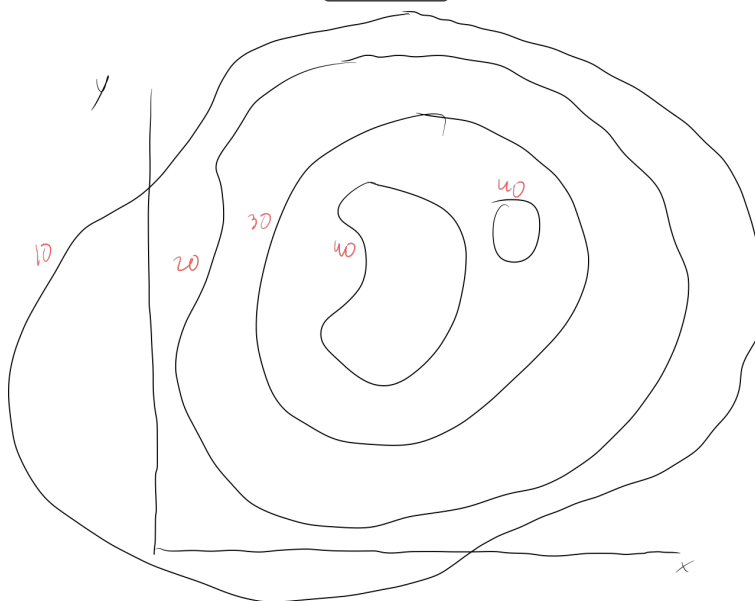
Distance

In $\mathbb{R}^5$, let $p = (3, 1, 4, 1, 5)$, and $q = (1, 0, -2, 0, 2)$. Using the Euclidean metric, we can find the distance between p and q is $d(p, q) = ((3-1)^2 + (1-0)^2 + (4-(-2))^2 + (1-0)^2 + (5-2)^2)^{1/2} = (4 + 1 + 36 + 1 + 9)^{1/2} = \sqrt{51} = 7.14$. We can also call this the 2-norm.

$$d(p, q) = \left(\sum_{k=1}^{n} (p_k - q_k)^2 \right)^{1/2}$$

To denote a derivative, we can't talk about one value, we must use a *partial* derivative, $\frac{\partial f}{\partial x}$, or $\frac{\partial f}{\partial y}$. The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

x\y	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A "linear" approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and is an output in $z = f(x, y)$, and $m, n \in \mathbb{R}$.

For example, with the above table, we can see that the function is linear in x and y (i.e., the slope holding the other variable constant is constant).

Consider the following:

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow $y = mx$

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 - y^2} = \lim_{(x,y) \to (0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2}$$

$$= \frac{1 + m^2}{1 - m^2}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of m

m	$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours "approach" a particular point, we can see that the limit is defined.

Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$. These vectors are equivalent because they are components of $\mathbb{R}^3$.

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

Properties of Vectors

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

- $c\vec{u} = \langle cu_1, cu_2, \ldots, cu_k \rangle$

Additionally, we define $\vec{u} \cdot \vec{v}$ as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

Partial Derivatives

Consider $f(x, y) = x^2 y + xe^y$.

$$f_x := \frac{\partial f}{\partial x}$$
$$f_x(a, b) = \frac{\partial f}{\partial x}\bigg|_{(a,b)}$$

We know that $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, meaning f is endlessly differentiable.

Functions and Approximations

Let $f(x, y) = x^2 - y^2$, $g(x, y) = 2xy$

- $f_{xx} + f_{yy} = 0$

- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For $f(x, y)$ at $(a, b, f(a, b))$, we have the following:

$$\ell(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(y - b)$$

$$q(x, y) = \ell(x, y) + \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right)$$

In order to get a sense of the "derivative," we can use the following:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

Directional Derivative and Gradient

Given $f(x, y)$ and (a, b), where $f \in C^2(\mathbb{R}^2)$. Then, the quadratic approximation is:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_x(a, b)(y - b)$$

$$+ \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b)\right)$$

$$df = f_x(a, b)dx + f)y(a, b)dy \qquad\qquad \text{a differential}$$

$$\Delta f = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

Evaluating $f(x, y) = xe^y$ at $(a, b) = (-1, 0)$

$$f_x = e^y$$

$$f_y = xe^y$$

$$f_x(-1, 0) = 1$$

$$f_y(-1, 0) = -1$$

$$\Delta f = \Delta x - \Delta u$$

On a given contour map, let $\vec{u} = \langle u_1, u_2\rangle$ denote a *unit* vector in a direction that we want to find the derivative of f in.

$$f_{\vec{u}}(x, y) = \nabla f(a, b) \cdot \vec{u}$$

Where

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

The directional derivative for all vectors $\vec{v}$ is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

Chain Rule

Let $f(x, y)$ be a function where $x - x(t)$ and $y = y(t)$. We want to find

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The chain rule works in higher dimensions too. Consider $k(x_1(t), x_2(t), \ldots, x_{152}(t))$. Then,

$$\frac{dk}{dt} = \sum_{i=1}^{152} \frac{\partial k}{\partial x_i}\frac{dx_i}{dt}$$

We can also view this as a vector. Let $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{152}(t) \end{pmatrix}$. Then, we can write $\frac{dk}{dt}$ more succinctly as follows:

$$\frac{dk}{dt} = \nabla k \cdot \frac{d\vec{x}}{dt}$$

For example, let $f(x, y, z) = 3x^2y + zx + 2$, where $x = x(t), y = y(t), z = z(t)$

$$\frac{df}{dt} = \begin{pmatrix} 6xy + z \\ 3x^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

$$= (6xy + z)\, x'(t) + 3x^2 y'(t) + xz'(t)$$

So, if we let $x(t) = \sin(t)$, $y(t) = e^t$, and $z(t) = t^2 + 1$. Then, we have

$$\frac{df}{dt} = 6\sin(t)\cos(t)e^t + t^2\cos(t) + \cos(t) + 3e^t\sin^2(t) + 2t\sin(t)$$

Alternatively, consider $f(x, y, z) = x^2 + yz + e^y$, where $x(s, t) = st, y = y(s, t) = t + s^2, z = z(s, t) = e^t$. Let

$$\vec{x} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix}$$

Then, we have

$$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t}$$

$$\frac{\partial f}{\partial s} = \nabla f \cdot \frac{\partial \vec{x}}{\partial s}$$

Evaluating the first expression, we have

$$\frac{\partial f}{\partial t} = \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix} \cdot \begin{pmatrix} s \\ 1 \\ e^t \end{pmatrix}$$

$$= 2s^2 t + 3^t + e^{t+s^2} + (t + s^2)e^t$$

Consider $f(x, y(x))$. Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

This is the technique we use to find implicit differentiation.

We know as a result that $\nabla f(a, b)$ is orthogonal to the contour curve at (a, b)

Recap

In $\mathbb{R}^3$, find the plane that contains $P = (P_1, P_2, P_3)$, Q, and R. We can find it by the following:

$$0 = \vec{n} \cdot \begin{pmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{pmatrix}$$

$$0 = n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3)$$

where

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

Differentiability

A function $f(x)$ of one variable is differentiable at $x = a$ if

$$f(a) = \lim_{h \to 0} f(a + h)$$

and

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

exists and is bounded

We can also linearize the function. f is differentiable if

$$f(x) = f(a) + f'(a)(x - a) + E(x)$$

where $\lim_{h \to 0} \frac{E(a+h)}{h} = 0$.

In the multiple dimensions example, we have $f(x, y)$ is differentiable if

$$f(x, y) = \ell(x, y) + E(x, y)$$

where $\lim_{h \to 0, k \to 0} \frac{E(a+h, b+k)}{\sqrt{h^2 + k^2}} = 0$

Local Maxima

Let $f(x, y) = x^2 + 2y^2$. We want to find (a, b) which are local maxima, minima, or other.

(a, b) is a local maximum if $f(a, b) \geq f(x, y) \ \forall (x, y) \in V_\varepsilon(a, b)$, where $\varepsilon > 0$.

(1) Find Critical Points for $f(x, y)$: $f_x(x, y), f_y(x, y) = 0$, $f_x(x, y), f_y(x, y)$ are undefined.

$$\begin{aligned}
f_x(x, y) &= 2x \\
f_y(x, y) &= 4y \\
f_x(0, 0) &= 0 \\
f_y(0, 0) &= 0 \\
f(0, 0) &= 0 \\
f(x, y) &> 0 \hspace{4cm} \forall (x, y) \neq (0, 0)
\end{aligned}$$

For all x, y, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 0$. Finally,

$$\begin{aligned}
D(x, y) &= f_{xx}(x, y) \cdot f_{yy}(x, y) + f_{xy}(x, y)^2 \\
&= 8 \\
&> 0
\end{aligned}$$

Since $D(x, y) > 0$, we look at the sign of f_{xx}. Since it is positive, $f(0, 0)$ has a local minimum.

Local Maxima and Minima Approach

Given $f(x, y)$, we want

(1) Find critical points:

$$\frac{\partial f}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} = 0$$

(2) Compute $f_{xx}, f_{yy}, f_{xy}, D = f_{xx}f_{yy} - (f_{xy})^2$

(3)

f_{xx}	D	Critical Point
+	+	Local Minimum
-	+	Local Maximum
$\pm$	-	Saddle Point
$\pm$	0	Nothing

Consider the function

$$f(x, y) = \ln(x^2 + y^2 + 1)$$
$$f(0, 0) = 0$$
$$f(x, y) > 0$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$
$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

Critical Points: $(0, 0)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = \frac{2(x^2 + y^2 + 1) - 4x^2}{(x^2 + y^2 + 1)^2}$$
$$= 2$$
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = \frac{2(x^2 + y^2 + 1) - 4y^2}{(x^2 + y^2 + 1)^2}$$
$$= 2$$
$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \frac{-4xy}{(x^2 + y^2 + 1)^2}$$
$$= 0$$

Now, consider the function

$$f(x, y) = x^2 - 2xy + y^2$$
$$\frac{\partial f}{\partial x} = 2x - 2y$$
$$\frac{\partial f}{\partial y} = -2x + 2y$$
$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial y^2} = 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2$$
$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$= 0$$

Therefore, the critical points of this function are indeterminate with the given approach. However, we know that $f(x, y) = (x - y)^2 = 0$ when $x = y$, so the line $y = x$ is a local minimum trough in 3-space.

Now, consider the function

$$f(x, y) = (x - 1)^2(y + 2)$$
$$\frac{\partial f}{\partial x} = 2(x - 1)(y + 2)$$
$$\frac{\partial f}{\partial y} = (x - 1)^2$$

Critical points: $\{(1, y) \mid y \in \mathbb{R}\}$

$$\frac{\partial^2 f}{\partial x^2} = 2(y + 2)$$
$$\frac{\partial^2 f}{\partial y^2} = 0$$
$$\frac{\partial^2 f}{\partial x \partial y} = 2(x - 1)$$
$$D = 0 - (2(x - 1))^2$$
$$= 0 \qquad\qquad\qquad \text{Evaluating } D \text{ at critical points}$$

Finding Critical Points

Let $f(x, y) = (y^2 + 2) \sin(x)$. on $[-2, 2] \times [-2, 2]$

$$\frac{\partial f}{\partial x} = (y^2 + 2) \cos(x)$$
$$= 0$$
$$\frac{\partial f}{\partial y} = 2y \sin(x)$$
$$= 0$$
$$(x, y) = \left(\frac{(2n + 1)\pi}{2}, 0 \right)$$
$$= \{(\pi/2, 0), (-\pi/2, 0)\}$$
$$\frac{\partial^2 f}{\partial x^2} = -(y^2 + 2) \sin(x)$$
$$\frac{\partial^2 f}{\partial y^2} = 2 \sin(x)$$
$$\frac{\partial^2 f}{\partial x \partial y} = 2y \cos(x)$$
$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$
$$= -2(y^2 + 2) \sin^2(x) - 4y^2 \cos^2(x)$$
$$< 0$$

Therefore, the critical points are saddle points. If there is no domain restriction, we have a series of saddle points all along $y = 0$.

Why Finding Critical Points Works

We create the Taylor series of $f(x, y)$ at (x_0, y_0):

$$f(x, y) \approx \ell(x_0, y_0) + \frac{1}{2} \left(f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2 \right)$$

$$= f(x_0, y_0) + \underbrace{\nabla f(x, y) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}_{=0 \text{ at critical points}} + \frac{1}{2} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^T \underbrace{\begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}}_{\text{Hessian}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

If the Hessian is positive definite, then $\lambda_1, \lambda_2 > 0$ and the critical point is a local min. If the Hessian is negative definite, then $\lambda_1, \lambda_2 < 0$ and the critical point is a local max.

In any given 2×2 matrix, the eigenvalues λ_1, λ_2 are such that $\lambda_1 + \lambda_2 = \text{Tr}(A)$ and $\lambda_1 \lambda_2 = \text{Det}(A)$.

Optimization

Let $f(x, y) = 2x - y$. We want to optimize f with respect to $g(x, y) = x^2 - y^2 - 4 = 0$.

Define $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$. Given $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, then $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Then, we take

$$\nabla L = \nabla f = \lambda \nabla g$$
$$= 0 \qquad\qquad\qquad\qquad\qquad \text{critical points of } L$$

We find x, y, λ for each critical point.

$$\nabla f = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$2 = 2\lambda x$$

$$-3 = -2\lambda y$$

$$x^2 - y^2 = 4$$

$$\lambda = \frac{1}{x}$$

$$\lambda = \frac{3}{2y}$$

$$x = \frac{2y}{3}$$

$$\frac{4y^2}{9} - y^2 = 4$$

$$-\frac{5}{9}y^2 = 4$$

No Solution

However, if $g(x, y) = x^2 + y^2 - 4 = 0$, we have

$$\nabla f = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$2 = 2\lambda x$$

$$-3 = 2\lambda y$$

$$x^2 + y^2 = 4$$

$$\lambda = \frac{1}{x}$$

$$\lambda = \frac{-3}{2y}$$

$$x = \frac{-2y}{3}$$

$$\frac{4y^2}{9} + y^2 = 4$$

$$-\frac{13}{9}y^2 = 4$$

$$y = \pm\frac{6}{\sqrt{13}}$$

$$x = \mp\frac{4}{\sqrt{13}}$$

$$f_{\text{max}} = 2\sqrt{13}$$

$$f_{\text{min}} = -2\sqrt{13}$$

This system of Lagrange multipliers applies in the n dimensional case.

Let $f(x, y, z) = x + 2y + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

$$\nabla f = \lambda \nabla g$$

$$\begin{pmatrix} 1 \\ 2 \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$2\lambda x = 1$$
$$2\lambda y = 2$$
$$2\lambda z = 2z \qquad\qquad (*)$$
$$x^2 + y^2 + z^2 = 1$$

Consider (*):

$$\lambda = 1$$
$$x = 1/2$$
$$y = 1$$
$$\frac{1}{4} + 1 + z^2 = 1$$

no solution

$$z = 0$$
$$x^2 + y^2 = 1$$
$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 1$$
$$\frac{5}{4\lambda^2} = 1$$
$$\lambda = \pm\frac{\sqrt{5}}{2}$$

Case 1:

$$\lambda = \frac{\sqrt{5}}{2}$$
$$x = \frac{1}{\sqrt{5}}$$
$$y = \frac{2}{\sqrt{5}}$$

Case 2:

$$\lambda = -\frac{\sqrt{5}}{2}$$
$$x = -\frac{1}{\sqrt{5}}$$
$$y = -\frac{2}{\sqrt{5}}$$

Evaluating f:

x	y	z	λ	$f(x, y, z)$
$\dfrac{1}{\sqrt{5}}$	$\dfrac{2}{\sqrt{5}}$	0	$\dfrac{\sqrt{5}}{2}$	$\sqrt{5}$
$-\dfrac{1}{\sqrt{5}}$	$-\dfrac{2}{\sqrt{5}}$	0	$-\dfrac{\sqrt{5}}{2}$	$-\sqrt{5}$

If we want to optimize f with respect to multiple constraint functions $g_1, g_2, g_3, \ldots, g_k$, we would do:

$$\nabla f = \sum_{i=1}^{k} \lambda_i \nabla g_i$$

Integration

Consider $f(x, y)$. We want to integrate along the rectangle $D = [0, 3] \times [0, 2]$. We can find this as follows:

$$\int_D f(x, y) = \int_0^3 \int_0^2 f(x, y) \, dy \, dx$$
$$= \int_0^3 dx \int_0^2 dy f(x, y)$$

For any two regions D_1 and D_2, we have:

$$\int_{D_1} f(x, y) + \int_{D_2} f(x, y) = \int_{D_1 \ominus D_2} f(x, y)$$
$$= \int_{D_1 \cup D_2 \setminus D_1 \cap D_2} f(x, y)$$

Multidimensional Integral Approximation

We want to find

$$\int_0^2 dy \int_0^3 dx f(x, y)$$

where $f(x, y)$ is expressed as below.

$y \backslash x$	0	1	2	3
0	1	2	5	4
1	2	1	2	0
2	1	−1	1	−2

Just as we can use the left/right endpoint method for evaluating integrals in one dimension, we can use left/right and top/bottom endpoints to approximate the integral.

Evaluating a Multidimensional Integral

$$\int\limits_{0}^{1}\underbrace{\int\limits_{0}^{1}}_{dy}\underbrace{}_{dx} xe^y \, dx \, dy = \left(\int\limits_{0}^{1} e^y \, dy\right)\left(\int\limits_{0}^{1} x \, dx\right)$$

$$= \left(e^y\Big|_0^1\right)\left(\frac{x^2}{2}\Big|_0^1\right)$$

$$= (e-1)\left(\frac{1}{2} - 0\right)$$

$$= \frac{e-1}{2}$$

This can scale up into multiple dimensions:

$$\int\limits_{0}^{1}\int\limits_{2}^{4}\int\limits_{-1}^{2} x + y + z^2 \, dx \, dy \, dz = \int\limits_{0}^{1}\int\limits_{2}^{4}\left(\int\limits_{-1}^{2} x + y + z^2 dx\right) dy \, dz$$

$$= \int\limits_{0}^{1}\int\limits_{2}^{4}\left(\frac{x^2}{2} + yx + xz^2\Big|_{x=-1}^{x=2}\right) dy \, dz$$

$$= \int\limits_{0}^{1}\int\limits_{2}^{4}\left(\left(2 - \frac{1}{2}\right) + (2y - (-y)) + (2z^2 - (-z^2))\right) dy \, dz$$

$$= \int\limits_{0}^{1}\int\limits_{2}^{4} 3z^2 + 3y + \frac{3}{2} dy \, dz$$

$$= \int\limits_{0}^{1}\left(\frac{3}{2}y^2 + \frac{3}{2}y + 3yz^2\Big|_{y=2}^{y=4}\right) dz$$

$$= \int\limits_{0}^{1} 6z^2 + 21 dz$$

$$= 2z^3 + 21z\Big|_{z=0}^{z=1}$$

$$= 23$$

Consider the integral below:

$$\int\int\limits_{D} xe^y \, dx \, dy = \int\limits_{0}^{1}\int\limits_{y}^{1} xe^y \, dx \, dy$$

$$= \int\limits_{0}^{1} e^y \, dy \left(\frac{x^2}{2}\Big|_{x=y}^{x=1}\right)$$

$$= \int\limits_{0}^{1} e^y \left(\frac{1}{2} - \frac{y^2}{2}\right) dy$$

$$= \frac{1}{2}\left(\int\limits_{0}^{1} e^y \, dy - \int\limits_{0}^{1} y^2 e^y \, dy\right)$$

$$= \frac{1}{2}\left((e-1) - (e-2)\right)$$

$$= \frac{1}{2}$$

Example Integrals

Consider the domain $D : \{(x, y) \mid 1 \leq x \leq 2,\ 0 \leq y \leq \ln 2,\ y = \ln x\}$. We are going to evaluate $f(x, y) = 1$.

$$\int_D f(x, y)\,dD = \int_0^{\ln 2} \int_{e^y}^2 1\ dx\ dy$$

$$= \int_0^{\ln 2} \left(x \Big|_{x=e^y}^{x=2} \right) dy$$

$$= \int_0^{\ln 2} (2 - e^y)\,dy$$

$$= 2\ln 2 - 1$$

Consider $A = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2 \leq (x, y) \leq \sqrt{x}\}$ and evaluating $f(x, y) = x + 2y$

$$\int_A x + 2y\,dA = \int_0^1 \int_{y^2}^{\sqrt{y}} x + 2y\ dx\ dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + 2xy \Big|_{x=y^2}^{x=\sqrt{y}} \right) dy$$

$$= \int_0^1 \frac{y}{2} + 2y^{3/2} - \left(\frac{y^4}{2} + 2y^3 \right) dy$$

$$= \frac{y^2}{4} + \frac{4}{5}y^{5/2} - \frac{y^5}{10} - \frac{y^4}{2} \Big|_{y=0}^{y=1}$$

$$= \frac{9}{20}$$

General Multivariable Integration

In any given area of integration A, we have the following general form:

$$\int_A f(x, y)\ dA = \underbrace{\int_{c_1}^{c_2}}_{\text{always constants}} \int_{g(y)}^{h(y)} !dx!dy$$

$$= \int_{d_1}^{d_2} \int_{p(x)}^{q(x)} f(x, y)\ dy\ dx$$

In the three-dimensional case, we have

$$\int_V f(x, y, z)\ dV = \int_{c_1}^{c_2} \int_{g(z)}^{h(z)} \int_{p(y,z)}^{q(y,z)} f(x, y, z)\ dx\ dy\ dz$$

Consider an integral with domain as follows:

$$D = \left\{ x, y, z \mid -1 \leq x \leq 1,\ 0 \leq y \leq 10,\ 0 \leq z \leq 1 - x^2 \right\}$$

For volume, we have that $f(x, y, z) = 1$

$$V = \int_{-1}^1 \int_0^{1-x^2} \int_0^{10} 1\ dy\ dz\ dx$$

$$= \int_{-1}^1 \int_0^{1-x^2} 10\ dz\ dx$$

$$= 10 \int_{-1}^1 1\ dx - 10 \int_{-1}^1 x^2\ dx$$

$$= \frac{40}{3}$$

For another example, consider the pyramid defined by the points $(0, 0, 0), (0, 2, 0), (1, 0, 0), (0, 0, 3)$. As established, we do $f(x, y, z) = 1$ integrated over the domain.

$$V = \int_0^3 \int_0^{g(y)} \int_0^{h(y,z)} f(x, y, z) \; dx \; dy \; dz$$

We find the cross product for the plane

$$\vec{n} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

$$6 = 6x + 3y + 2z$$

$$x = 1 - \frac{1}{2}y - \frac{1}{3}z$$

$$y = 2 - \frac{2}{3}z$$

$$V = \int_0^3 \int_0^{2 - \frac{2z}{3}} \int_0^{1 - \frac{1}{2}y - \frac{1}{3}z} 1 \; dx \; dy \; dz$$

$$= \int_0^3 \int_0^{2 - \frac{2z}{3}} \left(1 - \frac{1}{2}y - \frac{1}{3}z\right) dy \; dz$$

$$= \int_0^3 2 - \frac{2z}{3} - \frac{\left(2 - \frac{2z}{3}\right)^2}{4} - \frac{\left(2 - \frac{2z}{3}\right)z}{3} \; dz$$

u substitution:

$$u = 2 - \frac{2z}{3}$$

$$z = \frac{3(u - 2)}{-2}$$

$$dz = -\frac{3}{2} du$$

$$= \int_2^0 \left(u - \frac{u^2}{4} - \frac{3u(u - 2))}{-6}\right)\left(-\frac{3}{2} du\right)$$

$$= \int_2^0 \frac{u^2}{4}$$

$$= -\frac{3}{2} \int_2^0 \frac{u^2}{4}$$

$$= -\frac{3}{2} \frac{u^3}{12}\Big|_2^0$$

$$= 1$$

Let $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ be differentiable.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

If we seek to enact a change of variables, we compute $\det J$.

For example, let $x = r \cos \theta = x(r, \theta)$, $y = r \sin \theta = y(r, \theta)$.

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$\det J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r$$

Consider a circle of radius 5. We will integrate $f(x, y) = x^2 + y$ over this circle.

(1): Cartesian

$$\int_A f(x, y) dA = \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} (x^2 + y) \; dy \; dx$$

$$= \int_{-5}^{5} \left(x^2 y + \frac{y^2}{2} \Big|_{y=\sqrt{25-x^2}}^{y=\sqrt{25-x^2}} \right) \; dx$$

$$= \int_{-5}^{5} 2x^2 \sqrt{25 - x^2} \; dx$$

$$\vdots$$

(2): Polar

$$\int_A f(r, \theta) \; dr \; d\theta = \int_A f(r, \theta) \left(\begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \right) \; dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^5 (r^2 \cos^2 \theta + r \sin \theta) (r) \; dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^5 r^3 \cos^2 \theta \; dr \; d\theta + \int_0^{2\pi} \int_0^5 r^2 \sin \theta \; dr \; d\theta$$

$$= \left(\int_0^{2\pi} \cos^2 \theta \; d\theta \right) \left(\int_0^5 r^3 \; dr \right) + \left(\int_0^{2\pi} \sin \theta \; d\theta \right) \left(\int_0^5 r^2 \; dr \right)$$

$$= \frac{625}{4} \left(\int_0^{2\pi} \cos^2 \theta \; d\theta \right)$$

$$= \frac{625\pi}{4}$$

Cylindrical Coordinates

We can define a new coordinate base as follows:

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = z$$

Thus, any integral over the cylinder has the following form:

$$\int_V f(x, y, z) \; dV = \int_V f(r, \theta, z) \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \; dr \; d\theta \; dz$$

$$= \int_V f(r, \theta, z) r \; dr \; d\theta \; dz$$

Spherical Coordinates

If we have spherical symmetry, we will need to include a second angle φ, relative to the positive z axis:

$$x = \rho \cos\theta \sin\varphi$$
$$y = \rho \sin\theta \sin\varphi$$
$$z = \rho \cos\varphi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \rho^2 \sin\varphi$$

Integrating with Spherical and Cylindrical Coordinates

Consider a cylinder defined by $z \in [0, 4]$ and $r = 1$, and a half-sphere defined by $r = 1$ and $\varphi \in [0, \pi/2]$, located on "top" of the cylinder. We want to find

$$\int_V f(x, y, z) \ dV = \int_{V_1} f(x, y, z) \ dV_1 + \int_{V_2} f(x, y, z) \ dV_2$$

where $f(x, y, z = z)$. Then,

$$\int_V z \ dV = \int_{V_1} (z) r dr \ d\theta \ dz + \int_{V_2} (z)(\rho^2 \sin\varphi) d\rho \ d\theta \ d\varphi$$

$$I_c = \int_0^4 \int_0^{2\pi} \int_0^1 (z)(r) dr \ d\theta \ dz$$

$$= \int_0^4 \int_0^{2\pi} \left(\frac{z}{2} r^2 \Big|_0^1 \right) d\theta \ dz$$

$$= \int_0^4 \int_0^{2\pi} \frac{z}{2} \ d\theta \ dz$$

$$= \int_0^4 \pi z \ dz$$

$$= 8\pi$$

$$I_s = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (4 + \cos\varphi)(r^2 \sin\varphi) d\rho \ d\theta \ d\varphi$$

$$= 2\pi \int_0^{\pi/2} \int_0^1 (4 + \cos\varphi)(r^2 \sin\varphi) d\rho \ d\varphi$$

$$= 2\pi \int_0^1 r^2 \ dr \int_0^{\pi/2} 4 \sin\varphi + \cos\varphi \sin\varphi \ d\varphi$$

$$= \frac{2\pi}{3} \frac{9}{2}$$

$$= 3\pi$$

$$I = I_c + I_s$$
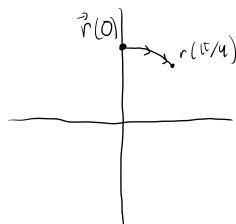
$$= 11\pi$$

Vector Valued Functions

Let

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \qquad\qquad 0 \le t \le \frac{\pi}{4}$$

$$\vec{r}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{r}\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

We can expand to $\mathbb{R}^3$:

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ 15 \end{pmatrix} \qquad\qquad \text{Circle at } z = 15$$

$$\vec{r}'(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ t\sin(t) \end{pmatrix} \qquad\qquad \text{Helix}$$

$$\vec{r}''(t) = \begin{pmatrix} t\sin(t) \\ t\cos(t) \\ t \end{pmatrix} \qquad\qquad \text{Conical Helix}$$

We can also add parameters:

$$\vec{r}(t) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \end{pmatrix} \qquad\qquad \text{Disc}$$

Differentiation in Vector Valued Functions

For

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

we have

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

For example, in uniform circular motion, we have

$$\vec{x}(t) = r \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

$$\frac{d\vec{x}}{dt} = \omega r \begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix}$$

$$\frac{d^2\vec{x}}{dt^2} = -\omega^2 r \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

$$= -\omega^2 r \vec{x}(t)$$

Examining the Helix

Let

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$= \begin{pmatrix} r \cos \omega t \\ r \sin \omega t \\ ct \end{pmatrix}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \begin{pmatrix} -\omega r \sin \omega t \\ \omega r \cos \omega t \\ c \end{pmatrix}$$

$$\vec{a}(t) = \begin{pmatrix} \omega^2 r \cos \omega t \\ \omega^2 r \sin \omega t \\ 0 \end{pmatrix}$$

$$\|\vec{r}(t)\| = \sqrt{r^2 + m^2 t^2}$$
$$\|\vec{v}(t)\| = \sqrt{\omega^2 r^2 + m^2}$$
$$\|\vec{a}(t)\| = \omega^2 r$$

Path Length of Helix

Along $\vec{r}(t)$, the total length traveled along $t \in [a, b]$ is

$$\ell = \int_a^b \|r'(t)\| \ dt$$

So, for the helix, we have

$$\ell = \int_a^b \sqrt{\omega^2 r^2 + m^2} \ dt$$

$$= \frac{2\pi \sqrt{\omega^2 r^2 + m^2}}{\omega}$$

Vector Fields

The following are vector fields:

$$\vec{F}(x, y) = \begin{pmatrix} x + y \\ y^2 - x \end{pmatrix}$$

$$\vec{F}(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{F}(x, y, z, w) = \begin{pmatrix} xy \\ yz \\ 0 \\ w \end{pmatrix}$$

For the vector field

$$\vec{F}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix},$$

we select various points and draw their resulting vectors.