

Solution (30.1): (a) This is a legal expression.

(b) This is a legal expression.

(c) This is not a legal expression; we should obtain a dual vector upon acting with B_{ij} on the vector C^i .

(d) This is not a legal expression assuming summation convention; we cannot have a repeated index on the same tensor.

(e) This is not a legal expression assuming summation convention; we cannot have a repeated index on the same tensor.

(f) This is a legal expression.

(g) This is not a legal expression; we should have a $(0, 2)$ tensor in the dual space, A_{ij} , rather than a $(1, 1)$ tensor of the form A_i^j .

(h) This is a legal expression.

Solution (30.3): Using the chain rule, we obtain

$$\begin{aligned} A^{j'} B_{j'} &= \frac{\partial u^{j'}}{\partial u^j} A^j \frac{\partial u^j}{\partial u^{j'}} B_j \\ &= \delta_j^{j'} A^j B_j \\ &= A^j B_j. \end{aligned}$$

Meanwhile,

$$\begin{aligned} A^{j'} B^{j'} &= \frac{\partial u^{j'}}{\partial u^j} A^j \frac{\partial u^{j'}}{\partial u^j} B^j \\ &= \frac{\partial u^{j'}}{\partial u^j} \frac{\partial u^{j'}}{\partial u^j} A^j B^j, \end{aligned}$$

which means $A^{j'} B^{j'}$ is a rank $(2, 0)$ tensor.

Solution (30.5): The matrix

$$g_{ab} = \begin{pmatrix} 1 & \cos(\phi) \\ \cos(\phi) & 1 \end{pmatrix}$$

has inverse

$$g^{ab} = \begin{pmatrix} \csc^2(\phi) & -\cot(\phi) \csc(\phi) \\ -\cot(\phi) \csc(\phi) & \csc^2(\phi) \end{pmatrix},$$

where $\cos(\phi) = \sin(\alpha + \beta)$. Therefore, we may calculate

$$\begin{aligned} \vec{e}^a &= g^{ab} \vec{e}_b \\ &= \frac{1}{\cos(\alpha + \beta)} \begin{pmatrix} \cos(\beta) \\ -\sin(\beta) \end{pmatrix} \\ \vec{e}^b &= g^{ab} \vec{e}_a \\ &= \frac{1}{\cos(\alpha + \beta)} \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}. \end{aligned}$$

Solution (30.6):

(a) We have the downstairs basis of $\{\hat{r}, r\hat{\phi}, \hat{z}\}$ for cylindrical coordinates.

(b) Using the metric of

$$g^{ab} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & 1 \end{pmatrix},$$

we calculate

$$\begin{aligned}\vec{e}^r &= g^{ab} \vec{e}_r \\ &= \hat{r} \\ \vec{e}^\phi &= g^{ab} \vec{e}_\phi \\ &= \frac{1}{r} \hat{\phi} \\ \vec{e}^z &= g^{ab} \vec{e}_z \\ &= \hat{z}.\end{aligned}$$

(c) We calculate

$$\begin{aligned}A_r \vec{e}^r &= A_r \hat{r} \\ &= A^r \hat{r} \\ A_\phi \vec{e}^\phi &= \frac{1}{r} A_\phi \hat{\phi} \\ &= A^\phi \hat{\phi} \\ A_z \vec{e}^z &= A_z \hat{z} \\ &= A^z \hat{z}.\end{aligned}$$

Thus,

$$\begin{aligned}A_r &= A^r \\ A_\phi &= \frac{1}{r} A^\phi \\ A_z &= A^z.\end{aligned}$$

(d) We have

$$\begin{aligned}A^r \vec{e}_r &= A^r \hat{r} \\ &= A_r \hat{r} \\ &= A_r \vec{e}^r \\ A^\phi \vec{e}_\phi &= A^\phi \hat{\phi} \\ &= r A_\phi \hat{\phi} \\ &= A_\phi \vec{e}^\phi. \\ A^z \vec{e}_z &= A^z \hat{z} \\ &= A_z \hat{z} \\ &= A_z \vec{e}_z.\end{aligned}$$

Solution (30.16):

(a) Let $g_{ab} = \text{diag}(1, r^2, r^2 \sin^2(\theta))$ be the metric for spherical coordinates. Then,

$$\begin{aligned}g_{rr} &= \left(\frac{\partial \rho}{\partial r}\right)^2 + \left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{\partial \theta}{\partial r}\right)^2 \\ &= \frac{z^2}{r^2} + 1 + \frac{1}{z^2} \\ g_{\phi\phi} &= 1 \\ g_{zz} &= \frac{r^2}{z^2} + 1 + \frac{1}{r^2}.\end{aligned}$$

(b) We convert from spherical to cylindrical by converting from spherical to Cartesian by taking g^{ab} on spherical

coordinates, summing over δ_{ij} , then multiplying by the cylindrical metric, with some relabeling, giving

$$\begin{aligned} g_{rr} &= \frac{z^2}{r^2} + 1 + \frac{1}{z^2} \\ g_{\phi\phi} &= 1 \\ g_{zz} &= \frac{r^2}{z^2} + 1 + \frac{1}{r^2}. \end{aligned}$$

This approach is simpler because we get to take advantage of the properties of the Cartesian metric (i.e., that it is independent of location).

Solution (30.20): Calculating

$$\sqrt{\det(g_{ab})} = \sqrt{1 - \cos^2(\varphi)},$$

we may calculate

$$\begin{aligned} \int_A d\tau' &= \int_0^b \int_0^a \sqrt{1 - \cos^2(\varphi)} du dv \\ &= ab \sqrt{1 - \cos^2(\varphi)}. \end{aligned}$$

Solution (30.21):

(a) We take

$$g_{ab} = \begin{pmatrix} R^2 & \\ & R^2 \sin^2(\theta) \end{pmatrix}.$$

(b) Calculating

$$\begin{aligned} \int_C ds &= \int_C \sqrt{|g_{ab} du^a du^b|} \\ &= R \int_0^{\theta_0} d\theta \\ &= R\theta_0. \end{aligned}$$

- (c) The circumference C of the circle $\theta = \theta_0$ is equal to $2\pi R \sin(\theta_0)$, which is not equal to $2\pi r$ because observers on the sphere do not perceive its curvature.
- (d) Without loss of generality we may assume we are on the north pole. Then, very close to the north pole, we have very small θ_0 , or that $\theta_0 \approx \sin(\theta_0)$.

Thus, a small neighborhood of the sphere with radius $r \ll R$ will appear as a disc of radius r rather than a curved section whose boundary has radius $r \sin(\theta_0)$.

Solution (30.22):

Solution (30.28):