

### Abstract

We detail the construction necessary to prove Urysohn's Lemma, which completely characterizes normal topological spaces via separation using continuous functions.

In this document, we will prove the following theorem.

**Theorem** (Urysohn's Lemma): Let  $X$  be a topological space. Then,  $X$  is normal if and only if, for all closed, disjoint  $A, B \subseteq X$ , there is a continuous function  $f: X \rightarrow [0, 1]$  such that  $f(a) = 0$  for all  $a \in A$  and  $f(b) = 1$  for all  $b \in B$ .

**Definition:** A topological space  $X$  is normal if, for any closed, disjoint subsets  $A, B \subseteq X$ , there are open sets  $U, V \subseteq X$  such that  $A \subseteq U$ ,  $B \subseteq V$ , and  $U \cap V = \emptyset$ .

We may prove one direction of Urysohn's lemma already.

*Proof of Reverse Direction.* Suppose  $X$  is a topological space such that for all disjoint closed subsets  $A, B \subseteq X$ , there is a continuous  $f: X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ . Then, by taking  $U := f^{-1}((-\infty, 1/2) \cap [0, 1])$  and  $V := f^{-1}((1/2, \infty) \cap [0, 1])$ , we have  $U \cap V = \emptyset$  and  $A \subseteq U$ ,  $B \subseteq V$ .  $\square$

The reverse direction is, unfortunately, quite a bit more difficult. To do this, we will construct a family of open sets that will allow us to define our continuous function afterward. This construction will follow similar proofs in *A Taste of Topology* by Runde and *Real Analysis* by Folland, although it will (probably) be more detailed.

**Lemma:** Let  $A$  and  $B$  be disjoint subsets of a normal topological space  $X$ , and let

$$\Delta := \{k2^{-n} \mid n \geq 1, 0 < k < 2^n\}$$

be the set of dyadic rationals in  $(0, 1)$ . Then, there is a family  $\{U_r \mid r \in \Delta\} \subseteq \tau_X$  such that  $A \subseteq U_r \subseteq B^c$  for all  $r \in \Delta$ , and  $\overline{U_r} \subseteq U_s$  whenever  $r < s$ .

*Proof.* We start by showing that if  $A \subseteq U$ , then there is an open set  $V$  such that  $A \subseteq V \subseteq \overline{V} \subseteq U$ . Note that if  $A \subseteq U$ , then  $A$  and  $U^c$  are disjoint closed sets, so since  $X$  is normal, there are disjoint open sets  $V$  and  $W$  such that  $A \subseteq V$  and  $U^c \subseteq W$ . Note that since  $V \subseteq W^c$ , and  $W^c$  is closed, we have  $A \subseteq V \subseteq \overline{V} \subseteq W^c \subseteq U$ , which is our desired result.

Now, since  $B^c$  is open, and  $A \subseteq B^c$ , we have an open set  $U_{1/2}$  such that  $A \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq B^c$ . Similarly, since  $\overline{U_{1/2}} \subseteq B^c$ , we have  $U_{3/4} \subseteq B^c$  such that  $\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq B^c$ , and similarly for  $A \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2}$ .

Continuing in this process, we are able to construct a family  $\{U_r\}_{r \in \Delta} \subseteq \tau_X$  such that  $A \subseteq U_r \subseteq \overline{U_r} \subseteq U_s \subseteq \overline{U_s} \subseteq B^c$  whenever  $r < s$ .  $\square$

Now, we may prove Urysohn's Lemma by using this family  $\{U_r\}_{r \in \Delta}$ .

*Proof of Urysohn's Lemma.* Let  $\{U_r\}_{r \in \Delta}$  be our family with  $U_1 := X$ .

For  $x \in X$ , we define  $f(x) = \inf\{r \mid x \in U_r\}$ . Since  $A \subseteq U_r \subseteq B^c$  for  $0 < r < 1$ , we have  $f(x) = 0$  for all  $x \in A$ ,  $f(x) = 1$  for all  $x \in B$ , and  $0 \leq f(x) \leq 1$  for all  $x \in X$ . Now, all we need to show is that  $f$  is continuous.

Observe that  $f(x) < \alpha$  if and only if  $x \in U_r$  for some  $r < \alpha$ , which holds if and only if  $x \in \bigcup_{r < \alpha} U_r$ . Thus,  $f^{-1}((-\infty, \alpha)) = \bigcup_{r < \alpha} U_r$  is open. Similarly,  $f(x) > \alpha$  if and only if  $x \notin U_r$  for some  $r > \alpha$ , which holds if and only if  $x \notin \overline{U_s}$  for some  $s > \alpha$ , as  $\overline{U_s} \subseteq U_r$  when  $s < r$ . Thus, this holds if and only if  $x \in \bigcup_{s > \alpha} (\overline{U_s})^c$ , so  $f^{-1}((\alpha, \infty)) = \bigcup_{s > \alpha} (\overline{U_s})^c$  is open.

Since the open half-lines generate the topology on  $\mathbb{R}$ ,  $f$  is continuous.  $\square$