This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

Useful Results and Proofs

Analytic Functions

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A function $f\colon U \to \mathbb{C}$ is called *analytic* if, for any $z_0 \in U$, there is $r > 0$ and $(a_k)_k \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in U(z_0, r)$.

Analytic functions form a $\mathbb{C}$-algebra.

Theorem (Identity Theorem)**:** Let $f, g\colon U \to \mathbb{C}$ be analytic functions defined a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in U, then $f = g$ on U.

Proof. To begin, we show that if $f\colon U \to \mathbb{C}$ is an analytic function that is not uniformly zero, then for any $z_0 \in U$, there is $\rho > 0$ such that f is nonzero on $\dot{U}(z_0, \rho) \subseteq U$. Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all $z \in U(z_0, r)$, some $r > 0$, and since f is not uniformly zero, there is some minimal ℓ such that $a_\ell \neq 0$. This yields

$$f(z) = (z - z_0)^\ell \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function $h\colon U(z_0, r) \to \mathbb{C}$ given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as f and is not zero at z_0, so that g is not zero on some $U(z_0, \rho)$ as g is continuous.

Now, we let V_1 be the set of accumulation points of A in U, and let $V_2 = U \setminus V_1$.

If $z \in V_2$, then there is some $r_1 > 0$ such that $\dot{U}(z_0, r_1) \cap A = \emptyset$, or that $\dot{U}(z_0, r_1) \subseteq A^c$. Meanwhile, since U is open, there is some $r_2 > 0$ such that $U(z_0, r_2) \subseteq U$, meaning that if $r = \min\{r_1, r_2\}$, then $U(z_0, r) \subseteq U \setminus A$. Thus, V_2 is open.

Meanwhile, if $z \in V_1$, then since $V_1 \subseteq U$, it follows that there is $r > 0$ such that $U(z, r)$ and $(a_k)_k$ such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all $w \in U(z, r)$. We claim that $f(w) - g(w)$ is uniformly zero on $U(z, r)$. Else, if there were $w_0 \in U(z, r)$ such that $f(w_0) \neq g(w_0)$, then it would follow that there is $0 < s \leqslant r$ such that $f(w) \neq g(w)$ for all $w \in \dot{U}(w_0, s)$. Yet, this would contradict the assumption that z is an accumulation point, meaning that V_1 is open.

Since V_1 and V_2 are disjoint open sets whose union is equal to U, it follows that either $V_1 = U$ or $V_2 = U$. If $A \neq \emptyset$, then the identity theorem follows. □

Differentiability

Definition: If $U \subseteq \mathbb{C}$ is an open set, then we say f is differentiable at $z_0 \in U$ if

$$\lim_{w \to z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of f at z_0, and usually write $f'(z_0)$.

If f is differentiable at every $z_0 \in U$, we say f is differentiable on U.

If f is continuous and admits a continuous derivative, then we say f is *holomorphic*.

Note that the limit must be independent of direction. That is, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(w) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Now, given $U \subseteq \mathbb{C}$, write $z = x + iy$ and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$. Observe then that if f is differentiable at $x_0 + iy_0 \in U$, then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x + h, y) + iv(x + h, y)) - (u(x, y) + iv(x, y))}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x, y + h) + iv(x, y + h)) - (u(x, y) + iv(x, y))}{ih} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

Definition: The system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if f is differentiable, then the u and v in the definition of f satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly f is differentiable or holomorphic.

Proposition: If $f = u + iv$ is a holomorphic function such that u, v are in $C^2(U)$, then u and v are harmonic. That is, u and v satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We call u and v *harmonic conjugates* for each other. That is, if $u\colon U \to \mathbb{R}$ is a harmonic function, then $v \in C^1(U)$ is called a harmonic conjugate if the Cauchy–Riemann equations hold for u and v.

Theorem: Let $U \subseteq \mathbb{R}^2$ be a ball or all of $\mathbb{R}^2$. Then, every harmonic function on U has a harmonic conjugate. If $u \in C^3(U)$, then this conjugate is itself harmonic.

Lemma: Let $g\colon U((x_0, y_0), R) \to \mathbb{R}$ be such that g and $\frac{\partial g}{\partial x}$ are continuous. Then, $G\colon U((x_0, y_0), R) \to \mathbb{R}$, given by

$$G(x, y) = \int_{y_0}^{y} g(x, t)\, dt$$

satisfies

$$\frac{\partial G}{\partial x} = \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\, dt.$$

Proof of Lemma. Write

$$\frac{G(x+h, y) - G(x, y)}{h} - \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\, dt = \int_{y_0}^{y} \left(\frac{g(x+h, t) - g(x, t)}{h} - \frac{\partial g}{\partial x}(x, t) \right) dt.$$

By mean value theorem, the first term is equal to $\frac{\partial g}{\partial x}(x_1, t)$ for some x_1 between x and $x + h$. As $h \to 0$, $x_1 \to x$, as $\frac{\partial g}{\partial x}$ is uniformly continuous on a compact subset that contains x and $x + h$. We may exchange limit and integral to obtain the desired result. $\square$

Proof of Theorem. We prove for the case of $U = U((x_0, y_0), R)$. Define

$$v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt + \phi(x),$$

with $\phi(x)$ to be determined later. By the fundamental theorem of calculus, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

while by differentiating under the integral sign, and using the fact that u is harmonic, we have

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x, t)\, dt + \frac{d\phi}{dx} \\
&= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x, t)\, dt + \frac{d\phi}{dx} \\
&= -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \frac{d\phi}{dx}.
\end{aligned}$$

Defining $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)\, ds,$$

we see that v thus satisfies all the necessary requirements to be a harmonic conjugate.

Now, if u is C^3, then we defined v via the derivative of u, so that v is C^2, and thus v is harmonic. □

Cauchy's Integral Formula

Proposition: Fix $z_0 \in \mathbb{C}$, $R > 0$, and $f\colon U(z_0, R) \to \mathbb{C}$ holomorphic. For all $z \in U(z_0, R)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{S(z_0,R)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. It suffices to show that

$$\frac{1}{2\pi i}\int_{S(z_0,R)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = 0.$$

By using the chain rule and fundamental theorem of calculus, we find

$$\begin{aligned}
\frac{1}{2\pi i}\int_{S(z_0,R)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int_{S(z_0,R)} \frac{\int_0^1 f'((1-t)z + t\zeta)(\zeta - z)\,dt}{\zeta - z}\,d\zeta\\
&= \frac{1}{2\pi i}\int_{S(z_0,R)} \int_0^1 f'((1-t)z + t\zeta)\,dt\,d\zeta\\
&= \frac{1}{2\pi i}\int_{S(z_0,R)} \frac{d}{d\zeta}\left(\frac{1}{t} f((1-t)z + t\zeta)\right)d\zeta\\
&= 0.
\end{aligned}$$

□

Proposition: Let $f\colon U \to \mathbb{C}$ be a holomorphic function. The following all hold:

(i) f is analytic;

(ii) f is smooth with $f^{(n)}$ holomorphic;

(iii) for all $z_0 \in U$, if we let $R = \sup\{r > 0 \mid U(z_0, r) \subseteq U\}$, then there is $(a_n)_n \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the power series has radius of convergence R.

Proof.

(i) There exists $r < s$ with $U(z_0, s) \subseteq U$ and $r < r_1 < s$ such that $S(z_0, r_1) \subseteq U$. By Cauchy's Integral Formula, and a power series expansion of $\frac{1}{\xi - z}$ about z_0, this gives

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{S(z_0,r_1)} \frac{f(\xi)}{\xi - z}\,d\xi\\
&= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i}\oint_{S(z_0,r_1)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi\right)}_{=:a_n}\\
&= \sum_{n=0}^{\infty} a_n (z - z_0)^n.
\end{aligned}$$

(ii) Analytic functions are automatically smooth, hence complex-differentiable with continuous derivative.

(iii) If $r < r_1 < R$, then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi \right),$$

and since the series converges uniformly, we have

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi.$$

Since r was arbitrary, this holds for any $0 < r_1 < R$, whence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for all $z \in U(z_0, R)$.

□

Corollary: Let $U \subseteq \mathbb{C}$ be open, let $z_0 \in U$, and $r > 0$ with $B(z_0, r) \subseteq U$. The following hold:

(i) for all $z \in U(z_0, r)$,

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi;$$

(ii) for all $n > 0$,

$$\left| f^{(n)}(z_0) \right| \leqslant \frac{n!}{r^n} \sup_{\zeta \in S(z_0, r)} |f(\zeta)|.$$

This particular result is known as *Cauchy's Estimate*.

Theorem (Liouville's Theorem)**:** If $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded in modulus, then f is constant.

Liouville's Theorem follows from applying Cauchy's estimate to f and using the fact that f is bounded to find that all higher derivatives of f vanish.

Theorem (Fundamental Theorem of Algebra)**:** If $p(z) = a_n z^n + \cdots + a_1 z + a_0$ has $n \geqslant 1$ and $a_n \neq 0$, then there is at least one z_0 such that $p(z_0) = 0$.

Proof. Suppose $p(z)$ were never zero. It would follow then that $\frac{1}{p(z)}$ is also an entire function.

Since $\lim_{|z| \to \infty} |p(z)| = \infty$, it follows that $\lim_{|z| \to \infty} \frac{1}{|p(z)|} = 0$, whence $\left| \frac{1}{p(z)} \right|$ is an entire function that is bounded (as all functions that vanish at infinity are bounded). This means that $\frac{1}{p(z)}$ is constant, so $p(z)$ is constant. □

Corollary: Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Then, $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Proof. Suppose there were $w \in \mathbb{C}$ and $r > 0$ such that $U(w, r) \cap f(\mathbb{C}) = \emptyset$. Then, $|f(z) - w| \geqslant r$ for all $z \in \mathbb{C}$, meaning that

$$g(z) = \frac{1}{f(z) - w}$$

is bounded and entire (the entirety following from the fact that $f(z) - w$ is nonvanishing). □

Cycles, Winding Numbers, and Homology

Now, we may generalize some of these results related to Cauchy's Integral Formula.

Proposition: Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \operatorname{im}(\gamma)$, we have

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\xi - z}\,d\xi \in \mathbb{Z}.$$

Proof. Let $\phi\colon [a,b] \to \mathbb{C}$ be defined by

$$\phi(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\,ds.$$

Then, we observe

$$\phi(b) = \oint_{\gamma} \frac{1}{\xi - z}\,d\xi.$$

Then, define $\psi\colon [a,b] \to \mathbb{C}$ by

$$\psi(t) = \frac{e^{\phi(t)}}{\gamma(t) - z}.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}
\phi'(t) &= \frac{\gamma'(t)}{\gamma(t) - z}\\
\psi'(t) &= \frac{\phi'(t)e^{\phi(t)}}{\gamma(t) - z} - \frac{e^{\phi'(t)}\gamma'(t)}{(\gamma(t) - z)^2}\\
&= 0,
\end{aligned}$$

whence $\psi(t)$ is constant, and $\psi(t) = \psi(a)$, so

$$\psi(a) = \frac{1}{\gamma(a) - z}.$$

In particular, $\psi(b) = \psi(a)$, so

$$\begin{aligned}
e^{\phi(b)} &= \psi(b)(\gamma(b) - z)\\
&= \psi(a)(\gamma(a) - z)\\
&= 1,
\end{aligned}$$

so $\phi(b) = 2\pi i k$ for some $k \in \mathbb{Z}$. □

Definition: Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \operatorname{im}(\gamma)$, define

$$n(\gamma; z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{\xi - z}\,d\xi$$

to be the *winding number* of γ about z.

Definition: A piecewise C^1 *cycle* is a formal sum

$$\Gamma = \gamma_1 + \cdots + \gamma_n,$$

where the $\gamma_j\colon [a_j, b_j] \to \mathbb{C}$ are piecewise C^1 loops. The *length* of Γ is the sum of the lengths of the respective γ_j.

Given a piecewise C^1 cycle Γ, define

$$\oint_{\Gamma} f(z)\,dz = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\,dz,$$

and

$$n(\Gamma; z) = \sum_{j=1}^{n} n(\gamma_j; z).$$

Proposition: The following hold for the winding number $n(\gamma; z)$:

(i) the function $n(\Gamma; \cdot)\colon \mathbb{C} \setminus \operatorname{im}(\gamma) \to \mathbb{Z}$ is continuous;

(ii) $n(\Gamma; z)$ is constant on each connected component of $\mathbb{C} \setminus \operatorname{im}(\Gamma)$;

(iii) there exists a unique unbounded connected component with $n(\Gamma; z) = 0$ for all z in this unbounded connected component.

Proof.

(i) Since $\operatorname{im}(\Gamma)$ is compact, any $z \notin \operatorname{im}(\Gamma)$ admits a strictly positive

$$\operatorname{dist}_{\operatorname{im}(\Gamma)}(z) = \inf_{w \in \operatorname{im}(\Gamma)} |w - z|.$$

Let $w \in \mathbb{C}$ be such that

$$|w - z| < \frac{1}{2} \operatorname{dist}_{\operatorname{im}(\Gamma)}(z),$$

so that $w \in \mathbb{C} \setminus \operatorname{im}(\Gamma)$. Observe then that

$$\begin{aligned}
|n(\Gamma; z) - n(\Gamma; w)| &= \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\xi - z} - \frac{1}{\xi - w}\,d\xi \right| \\
&\leqslant \frac{1}{2\pi} \sum_{j=1}^{n} \oint_{\gamma_j} \left| \frac{1}{\xi - z} - \frac{1}{\xi - w} \right| |d\xi| \\
&= \frac{1}{2\pi} \sum_{j=1}^{n} \oint_{\gamma_j} \left| \frac{z - w}{(\xi - z)(\xi - w)} \right| |d\xi| \\
&\leqslant \frac{1}{2\pi} \left(\frac{2}{\operatorname{dist}_{\operatorname{im}(\Gamma)}(z)} \right)^2 \ell(\Gamma) |z - w|,
\end{aligned}$$

whence $|n(\Gamma; z) - n(\Gamma; w)|$ is sufficiently small whenever $|z - w|$ is sufficiently small.

(ii) If C is a connected component of $\mathbb{C} \setminus \operatorname{im}(\Gamma)$, and $n(\Gamma; \cdot)\colon C \to \mathbb{Z}$ is continuous, then since $\mathbb{Z}$ is discrete, $n(\Gamma; \cdot)$ is constant on C.

(iii) For uniqueness, if there are unbounded connected components C_1 and C_2 of $\mathbb{C} \setminus \operatorname{im}(\Gamma)$, then there exists $M > \sup_{z \in \operatorname{im}(\Gamma)} |z|$ and $w_1 \in C_1, w_2 \in C_2$ such that $|w_1| > 2M$ and $|w_2| > 2M$. Since $\mathbb{C} \setminus U(0, 2M)$ is path connected, there exists $\gamma\colon [0,1] \to \mathbb{C}$ with $|\gamma(t)| \geqslant 2M$ and $\gamma(0) = w_1$, $\gamma(1) = w_2$. Therefore, w_1 and w_2 are in the same connected component.

Existence then follows from $\operatorname{im}(\Gamma)$ being compact.

Finally, let $(z_n)_n \subseteq C$, where C is the unbounded connected component, be such that $\lim_{n\to\infty}|z_n| = \infty$. For $M > \sup_{z\in \operatorname{im}(\gamma)}|z|$, there exists $m \in \mathbb{N}$ such that $|z_m| > M$. Then, we have

$$\begin{aligned}
|n(\Gamma; z_m)| &= \left|\frac{1}{2\pi i}\oint_{\Gamma} \frac{1}{\xi - z}\,d\xi\right| \\
&\leqslant \frac{1}{2\pi}\sum_{j=1}^{k}\oint_{\gamma_j} \frac{1}{|\xi - z|}\,|d\xi| \\
&\leqslant \frac{1}{2\pi}\sum_{j=1}^{k}\oint_{\gamma_j} \frac{1}{|z_m| - M}\,|d\xi| \\
&= \frac{\ell(\Gamma)}{2\pi(|z_m| - M)},
\end{aligned}$$

whence $\lim_{m\to\infty} n(\Gamma; z_m) = 0$, meaning that there exists N such that $|n(\Gamma; z_m)| < 1$ for all $m \geqslant N$, meaning $n(\Gamma; z_m) = 0$ for all sufficiently large m. Since C is connected, it thus follows that $n(\Gamma; z) = 0$ for all $z \in C$.

□

Definition: Let $U \subseteq \mathbb{C}$ be open. A cycle Γ is *homologous to zero in* U if $\operatorname{im}(\Gamma) \subseteq U$ and for all $z \in \mathbb{C} \setminus U$, $n(\Gamma; z) = 0$.

Theorem (Cauchy's Integral Formula, General Case)**:** Let $\Gamma = \gamma_1 + \cdots + \gamma_k$ be a piecewise C^1 cycle homologous to zero in U, and $f\colon U \to \mathbb{C}$ holomorphic. Then, for all $z \in U \setminus \operatorname{im}(\Gamma)$,

$$n(\Gamma; z)f(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\xi)}{\xi - z}\,d\xi$$

Theorem (Cauchy's Integral Theorem)**:** Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic, and Γ homologous to zero in U. Then,

$$\oint_{\Gamma} f(z)\,dz = 0.$$

Definition: A region $U \subseteq \mathbb{C}$ is called *simply connected* if its complement in the extended complex plane is connected.

Theorem: If $U \subseteq \mathbb{C}$ is simply connected, then every loop in U is homologous to zero.

Proof. Extend the function $n(\gamma; \cdot)$ to the extended complex plane by defining $n(\gamma; \infty) = 0$. This extended function is continuous on $\hat{\mathbb{C}} \setminus U$, as $n(\gamma; \cdot)$ is zero on the unique unbounded connected component of $\mathbb{C} \setminus \operatorname{im}(\gamma)$. It follows that $n(\gamma; z)$ is equal to zero on $\hat{\mathbb{C}} \setminus U$, whence γ is homologous to zero in U. □

Proposition: Let $U \subseteq \mathbb{C}$ be a region, $f\colon U \to \mathbb{C}$ holomorphic. The following are equivalent:

(i) there exists a holomorphic function $F\colon U \to \mathbb{C}$ such that $F'(z) = f(z)$;

(ii) for every piecewise C^1 loop γ with $\operatorname{im}(\gamma) \subseteq U$, we have

$$\oint_{\gamma} f(z)\,dz = 0.$$

Proof. The direction (i) ⇒ (ii) follows immediately from the fundamental theorem of calculus. In the reverse direction, we define $F\colon U \to \mathbb{C}$ by

$$f(z) = \int_{\sigma(z_0, z)} f(\xi)\,d\xi,$$

where $\sigma(z_0, z)\colon [0,1] \to U$ is a piecewise C^1 curve with $\sigma(0) = z_0$ and $\sigma(1) = z$. Such a curve always exists as U is open and connected (hence path-connected). The integral is well-defined, since if γ_1 and γ_2 are any two such paths, then $\Gamma = \gamma_1 \setminus \gamma_2$ is a piecewise C^1 loop. Additionally, F is continuous.

Now, we evaluate the derivative of F. Let $z \in U$, $r > 0$ such that $U(z, r) \subseteq U$, and $h \in \mathbb{C}$ be such that $z + h \in U(z, r)$. Then,

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\int_{\sigma(z_0, z_0+h)} f(\xi)\, d\xi - \frac{1}{h}\int_{\sigma(z_0,z)} f(\xi)\, d\xi \\ &= \frac{1}{h}\int_{\sigma(z,z+h)} f(\xi)\, d\xi.\end{aligned}$$

We may assume that $\sigma(z, z+h)$ is a straight line, so that

$$\int_{\sigma(z,z+h)} f(\xi)\, d\xi = hf(z),$$

meaning that

$$\begin{aligned}\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|\int_{\sigma(z,z+h)} f(\xi)\, d\xi - f(z)\right| \\ &\leqslant \sup_{w \in \operatorname{im}(\sigma(z,z+h))} |f(w) - f(z)|.\end{aligned}$$

Since f is continuous, it follows that the right hand side goes to zero as $|h|$ becomes small. Thus, F' is continuous, so f is holomorphic. □

Observe that $\mathbb{C} \setminus \{0\}$ is not simply connected, meaning that, for instance, the function

$$f(z) = \frac{1}{z}$$

does not have a holomorphic antiderivative defined on the entirety $\mathbb{C} \setminus \{0\}$, as

$$\int_{S^1} f(z)\, dz = 2\pi i.$$

Yet, if we restrict $f(z)$ to a simply connected subset of $\mathbb{C}$, there *is* a holomorphic antiderivative. Choosing such a simply connected subset of $\mathbb{C}$ is known as choosing a *branch* of the logarithm. In fact, there is more that we can say.

Corollary: Let $U \subseteq \mathbb{C}$ be simply connected, and let $f\colon U \to \mathbb{C} \setminus \{0\}$ be a nonvanishing holomorphic function. For each fixed pair $z_0 \in U$ and $w_0 \in \mathbb{C}$ for which $e^{w_0} = f(z_0)$, there exists a unique holomorphic function $g\colon U \to \mathbb{C}$ for which $g(z_0) = w_0$ and $e^{g(z)} = f(z)$.

We call g the logarithm of f, written $g(z) = \log(f(z))$.

Proof. Since f is nonvanishing and U is simply connected, it follows that $\frac{f'}{f}$ is holomorphic on U, meaning there is $\widetilde{g}\colon U \to \mathbb{C}$ such that $\widetilde{g}'(z) = \frac{f'(z)}{f(z)}$. Thus, there is some constant K such that

$$f(z) = Ke^{\widetilde{g}(z)}.$$

Define

$$g(z) = \log(K) + \widetilde{g}(z).$$

□

Theorem (Morera's Theorem)**:** Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ continuous. Suppose

$$\oint_T f(z)\,dz = 0$$

for all triangles $T \subseteq U$ homologous to zero. Then, f is holomorphic.

Proof. Since U is open, if $z_0 \in U$, there is r such that $U(z_0, r) \subseteq U$. Define $F\colon U(z_0, r) \to \mathbb{C}$ by

$$F(z) = \int_{\sigma(z_0,z)} f(\xi)\,d\xi,$$

where σ is the straight line from z_0 to z. For $0 < |h| < r - |z - z_0|$, we construct the straight lines $\sigma(z, z+h)$ and $\sigma(z_0, z+h)$, such that

$$T = \sigma(z_0, z) + \sigma(z, z+h) - \sigma(z_0, z+h),$$

and

$$\begin{aligned}\oint_T f(z)\,dz = 0&\\ &= \int_{\sigma(z_0,z)} f(\xi)\,d\xi + \int_{\sigma(z,z+h)} f(\xi)\,d\xi - \int_{\sigma(z_0,z+h)} f(\xi)\,d\xi\\ &= F(z) - F(z+h) + \int_{\sigma(z,z+h)} f(\xi)\,d\xi,\end{aligned}$$

meaning

$$\begin{aligned}F(z+h) - F(z) &= \int_{\sigma(z,z+h)} f(\xi)\,d\xi\\ \frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\int_{\sigma(z,z+h)} f(\xi)\,d\xi\\ \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_{\sigma(z,z+h)} (f(\xi) - f(z))\,d\xi\right|\\ &\leqslant \frac{1}{|h|}|h| \sup_{w \in \operatorname{im}(\sigma(z,z+h))} |f(w) - f(z)|\\ &= \sup_{w \in \operatorname{im}(\sigma(z,z+h))} |f(w) - f(z)|.\end{aligned}$$

Since f is continuous, it follows that for sufficiently small $|h|$, the right-hand-side goes to zero, whence $F'(z) = f(z)$, meaning F is holomorphic, so F is analytic, meaning f is analytic, so f is holomorphic. □

Definition: Let $U \subseteq \mathbb{C}$ be open, γ_1, γ_2 piecewise C^1 loops in U. We say γ_1 and γ_2 are homotopic in U if there is a continuous function

$$H\colon [a, b] \times [0, 1] \to U$$

such that

$$\begin{aligned}H(s, 0) &= \gamma_1(s)\\ H(s, 1) &= \gamma_2(s)\\ H(a, t) &= H(b, t).\end{aligned}$$

For each t, $H(\cdot, t)$ is a continuous loop. We call H a homotopy between γ_0 and γ_1.

Theorem: If γ_0 and γ_1 are homotopic in U, then $\Gamma = \gamma_1 - \gamma_0$ is homologous to zero in U.

Theorem: If $K \subseteq U$ is compact and U is connected, then there is some cycle Γ homologous to zero in U such that $n(\Gamma; z) = 1$ for all $z \in K$.

Corollary: Let U be a region. The following are equivalent:

(i) U is simply connected;

(ii) for every nonvanishing holomorphic function $f\colon U \to \mathbb{C} \setminus \{0\}$, there is a holomorphic function $g\colon U \to \mathbb{C}$ such that $f(z) = e^{g(z)}$;

(iii) for all cycles Γ with $\operatorname{im}(\Gamma) \subseteq U$, Γ is homologous to zero in U.

Proof. We have already shown the direction (i) $\Rightarrow$ (ii). To see (ii) $\Rightarrow$ (iii), we start by fixing $w \in \mathbb{C} \setminus U$. Let $g\colon U \to \mathbb{C}$ be a holomorphic function with $e^{g(z)} = z - w$. Taking derivatives, we have $g'(z) = \frac{1}{z-w}$, so for any cycle with $\operatorname{im}(\Gamma) \subseteq U$, we have

$$\begin{aligned} n(\Gamma; w) &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-w}\, dz \\ &= \frac{1}{2\pi i}\int_\Gamma g'(z)\, dz \\ &= 0. \end{aligned}$$

To see (iii) $\Rightarrow$ (i), we show the contrapositive. Toward this end, suppose U is not simply connected. Then, $\hat{\mathbb{C}} \setminus U$ is connected, so there are closed nonempty disjoint sets V_1 and V_2 such that $\hat{\mathbb{C}} \setminus U = V_1 \cup V_2$. One of these V_i contains ∞, so without loss of generality, suppose V_1 contains ∞, meaning V_2 is bounded (hence compact).

Let $\widetilde{U} = U \cup V_2$. $U \cup V_2$ is open in $\mathbb{C}$ as its complement in $\mathbb{C}$ is $V_1 \setminus \{\infty\}$, which is closed in $\mathbb{C}$. ◻

Maximum Modulus Principle

Theorem (Mean Value Property)**:** Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic, with $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. Then,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta.$$

Proof. By the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_{S(z_0,r)} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Parametrizing $\gamma(\theta) = z_0 + re^{i\theta}$, we get

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta. \end{aligned}$$

◻

Corollary: If $u\colon \mathbb{R}^2 \supseteq U \to \mathbb{R}$ is harmonic, $(x_0, y_0) \in U$, and $r > 0$ is such that $B((x_0, y_0), r) \subseteq U$, then

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta.$$

Proof. Take real parts of the mean value property for holomorphic $f = u + iv$. ◻

Observe then that the triangle inequality implies that

$$|u(x_0, y_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))|\, d\theta.$$

Functions that satisfy this weaker criterion are known as *subharmonic*. It is subharmonic functions for which the most general case of the *maximum modulus principle* hold.

Theorem (Maximum Modulus Principle)**:** Let $U \subseteq \mathbb{R}^2$ be open and connected, and let $u \colon U \to \mathbb{R}$ be subharmonic. Suppose there exists $(x_0, y_0) \in U$ such that $u(x_0, y_0) \geqslant u(x, y)$ for all $x, y \in U$. Then, u is constant.

Proof. Let $\lambda = u(x_0, y_0)$, and let $E = \{(x, y) \mid u(x, y) = \lambda\} = u^{-1}(\{\lambda\})$. We see immediately that E is closed; we claim that E is also open.

Fix $(x_1, y_1) \in E$. Then, $u(x_1, y_1) = \lambda$. Take $r > 0$ such that $U((x_1, y_1), r) \subseteq U$. Then, for all $0 < s < r$, we have $S((x_1, y_1), s) \subseteq U$, meaning that

$$\begin{aligned}\lambda &= u(x_1, y_1)\\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + s\cos(\theta), y_1 + s\sin(\theta))\, d\theta\\ &\leqslant \lambda,\end{aligned}$$

with the latter inequality following from the fact that λ is a local maximum. Therefore, $u(x_1 + s\cos(\theta), y_1 + s\sin(\theta)) = \lambda$ for all $0 < s < r$, whence $U((x_1, y_1), r) \subseteq E$. Thus, E is open, so since U is connected, it follows that E is all of U, meaning u is constant. □

Corollary: If $U \subseteq \mathbb{R}^2$ is bounded and $u \colon \overline{U} \to \mathbb{R}$ is continuous with $u|_U$ subharmonic, then there exists $(x_0, y_0) \in \partial U$ such that $u(x_0, y_0) = \sup_{(x,y)\in U} u(x, y)$.

Corollary: If $U \subseteq \mathbb{C}$ is open and connected, with $f \colon U \to \mathbb{C}$ holomorphic, then if $|f| \colon U \to \mathbb{R}$ has a local maximum at $z_0 \in U$, then f is constant.

Proof. Let $r > 0$ be such that $U(z_0, r) \subseteq U$. Then, restricting $|f|$ to $U(z_0, r)$, we see that $|f|$ restricted to $U(z_0, r)$ is subharmonic viewed as a function on $U(z_0, r)$, hence $|f|$ is constant on $U(z_0, r)$.

Now, by the mean value property and triangle inequality, it follows that for all $0 < s < r$, we have

$$\begin{aligned}|f(z_0)| &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(z_0 + se^{i\theta}\right)\right| d\theta\\ &= |f(z_0)|,\end{aligned}$$

meaning that these are equalities. In particular, there exists some θ_s such that $e^{i\theta_s} f\left(z_0 + se^{i\theta}\right) \geqslant 0$, meaning that for this value of s, we have

$$\begin{aligned}|f(z_0)| &= e^{i\theta_s} \int_0^{2\pi} f\left(z_0 + se^{i\theta}\right) d\theta\\ &= e^{i\theta_s} f(z_0),\end{aligned}$$

with the latter equality following from the mean value property. Since this holds for any s, it follows that θ_s is independent of s, meaning that $f(z)e^{i\theta_s} \geqslant 0$ for all $z \in U(z_0, r)$, meaning that $\operatorname{Im}\left(e^{i\theta_s} f(z)\right) = 0$ on $U(z_0, r)$, whence $f(z)e^{i\theta_s}$ is constant, meaning f is constant on $U(z_0, r)$.

Finally, by the identity theorem, it follows that f is constant on U. □

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. We say a sequence $U \supseteq ((x_n, y_n))_n \to \partial U$ if, for every compact $K \subseteq U$, the set $\{n \in \mathbb{N} \mid (x_n, y_n) \in K\}$ is finite.

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. Given a function $u: U \to \mathbb{R}$, define

$$\limsup_{(x,y)\to\partial U} u(x,y) := \inf_{\substack{K\subseteq U\\ K \text{ compact}}} \sup_{(x,y)\in U\setminus K} u(x,y).$$

These definitions allow us to extend the maximum modulus principle for subharmonic functions even further.

Theorem: Let $U \subseteq \mathbb{C}$ be a region, $u: U \to \mathbb{R}$ a nonconstant subharmonic function. If $((x_n,y_n))_n \subseteq U$ is such that $u(x_n,y_n) \to \sup_{x,y\in U} u(x,y)$, then $((x_n,y_n))_n \to \partial U$. Moreover, $\limsup_{(x,y)\to\partial U} u(x,y) = \sup_{(x,y)\in U} u(x,y)$.

Proof. Suppose toward contradiction that $((x_n,y_n))_n \nrightarrow \partial U$, so there exists a compact subset $K \subseteq U$ and a subset $((x_{n_k},y_{n_k}))_k$ wholly contained in K. Since K is compact, there is a subsequence of $((x_{n_k},y_{n_k}))_k$ converging to $(x_0,y_0) \in U$. Therefore, $u(x_0,y_0) = \sup_{(x,y)\in U} u(x,y)$, so u is constant by the maximum modulus principle, which is a contradiction.

Finally, $\limsup_{(x,y)\to\partial U} u(x,y) \leqslant \sup_{(x,y)\in U} u(x,y)$, while if $((x_n,y_n))_n \to \partial U$ is such that $u(x_n,y_n)$ converges to $\sup_{(x,y)\in U} u(x,y)$, then $\sup_{(x,y)\in U} u(x,y) = \lim_{n\to\infty} u(x_n,y_n) \leqslant \limsup_{(x,y)\to\partial U} u(x,y)$. □

Theorem (Open Mapping Principle)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f: U \to \mathbb{C}$ be a nonconstant holomorphic function. Then, $f(V) \subseteq \mathbb{C}$ is open.

Proof. Let $z_0 \in U$ and $r > 0$ be such that $B(z_0,r) \subseteq U$. We will show that there exists R such that $U(f(z_0),R) \subseteq f(U(z_0,r)) \subseteq U$, whence $f(U)$ is open.

Since U is a region and f is nonconstant, the zeros of $g(z) := f(z) - f(z_0)$ are isolated, so there exists some $0 < s < r$ such that

$$\delta = \inf_{|z-z_0|=s} |f(z) - f(z_0)|$$

is strictly greater than zero. We claim that $U(f(z_0),\delta/2) \subseteq f(U(z_0,r))$. Suppose this were not the case, meaning there would be some $\xi \in U(f(z_0),\delta/2) \setminus f(U(z_0,r))$, and define $h: B(z_0,s) \to \mathbb{C}$ by

$$h(z) = \frac{1}{f(z)-\xi}.$$

Since $\xi \notin f(U(z_0,r))$, this is holomorphic, while $\xi \in U(f(z_0),\delta/2)$ implies

$$\begin{aligned}\sup_{|z-z_0|=s} |h(z)| &= \sup_{|z-z_0|=s} \frac{1}{|f(z)-\xi|}\\ &\leqslant \sup_{|z-z_0|=s} \frac{1}{|f(z)-f(z_0)| - |f(z_0)-\xi|}\\ &\leqslant \frac{1}{\delta - \delta/2}\\ &= \frac{2}{\delta}.\end{aligned}$$

Yet,

$$\begin{aligned}|h(z_0)| &= \frac{1}{|f(z_0)-\xi|}\\ &> \frac{2}{\delta},\end{aligned}$$

contradicting the maximum modulus principle. Thus, $U(f(z_0),\delta/2) \subseteq f(U(z_0,r))$. □

Classification of Singularities

The classification of singularities seeks to answer two fundamental questions: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, and $f\colon U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic,

- does f have a holomorphic extension to U including z_0;
- and what else can we say about the behavior of f at z_0?

Definition: Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f\colon U \setminus \{z_0\} \to \mathbb{C}$ holomorphic.

- If there exists a holomorphic $g\colon U \to \mathbb{C}$ with $g = f$ on $U \setminus \{z_0\}$, then we say z_0 is a *removable singularity*.
- If $\lim_{z\to z_0}|f(z)| = \infty$, then we say f has a *pole* at z_0.
- Else, we say f has an *essential singularity* at z_0.

Theorem (Riemann's Theorem on Removable Singularities)**:** Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and $f\colon U\setminus\{z_0\} \to \mathbb{C}$ holomorphic. Then, z_0 is a removable singularity if and only if $\lim_{z\to z_0} f(z) = 0$.

Proof. If z_0 is removable, then $g(z)$ is a holomorphic function with $g(z) = f(z)$ on $U \setminus \{z_0\}$, and since g is continuous, it follows that $\lim_{z\to z_0} g(z) = g(z_0)$, whence $\lim_{z\to z_0}(z - z_0)g(z) = \lim_{z\to z_0}(z - z_0)f(z) = 0$.

Now, if $\lim_{z\to z_0}(z - z_0)f(z) = 0$, then there is r such that $B(z_0, r) \subseteq U$, and since f is locally bounded around z_0, it follows that

$$f(z) = \frac{1}{2\pi i}\int_{S(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

holds for all $z \in \dot{U}(z_0, r)$. Yet, the formula extends to z_0 as it is bounded, whence we may define the holomorphic extension for f by

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \frac{1}{2\pi i}\int_{S(z_0,r)} \frac{g(\zeta)}{\zeta - z}\,d\zeta & z = z_0 \end{cases}.$$

□

Proposition (Existence of Laurent Series)**:** Suppose $f\colon A(z_0, r, R) \to \mathbb{C}$ is holomorphic, with $0 \leqslant r < R$. Then, there exist holomorphic functions

$$\begin{aligned} g_1&\colon U(z_0, R) \to \mathbb{C} \\ g_2&\colon \mathbb{C} \setminus B(z_0, r) \to \mathbb{C} \end{aligned}$$

such that $f = g_1 + g_2$ on $A(z_0, r, R)$. Moreover, there exists $(a_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

for all z, and the series converges uniformly on $A(z_0, \rho, s)$ with $r < \rho < s < R$.

Proof. Fix $z \in A(z_0, r, R)$. Then, for $r < \rho_1, \rho_2 < |z - z_0|$, the cycle

$$\Gamma_1 = S(z_0, \rho_1) - S(z_0, \rho_2)$$

is homologous to zero in $A(z_0, r, |z - z_0|)$. By Cauchy's Integral Theorem, it then follows that

$$\oint_{S(z_0,\rho_1)} \frac{f(\xi)}{\xi - z}\,d\xi = \oint_{S(z_0,\rho_2)} \frac{f(\xi)}{\xi - z}\,d\xi.$$

Similarly, for $|z - z_0| < s_1, s_2 < R$, we have

$$\oint_{S(z_0,s_1)} \frac{f(\xi)}{\xi - z}\, d\xi = \oint_{S(z_0,s_2)} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Define $g_1 \colon U(z_0, R) \to \mathbb{C}$ by

$$g_1(z) = \frac{1}{2\pi i} \oint_{S(z_0,s)} \frac{f(\xi)}{\xi - z}\, d\xi,$$

where $|z - z_0| < s < R$. This function is holomorphic by Morera's Theorem. Similarly, we may define $g \colon \mathbb{C} \setminus B(z_0, r) \to \mathbb{C}$ by

$$g_2(z) = -\frac{1}{2\pi i} \oint_{S(z_0,\rho)} \frac{f(\xi)}{\xi - z}\, d\xi,$$

where $r < \rho < |z - z_0|$. We claim that $f = g_1 + g_2$ on $A(z_0, r, R)$.

For $z \in A(z_0, r, R)$, we may find, for any $r < \rho < |z - z_0| < s < R$, we let

$$\Gamma = S(z_0, s) - S(z_0, \rho),$$

homologous to zero in $A(z_0, r, R)$, whence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left(\oint_{S(z_0,s)} \frac{f(\xi)}{\xi - z}\, d\xi - \int_{S(z_0,\rho)} \frac{f(\xi)}{\xi - z}\, d\xi\right) \\ &= g_1(z) + g_2(z). \end{aligned}$$

□

Old Exams

Notation

- $U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leqslant r\}$
- $S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$
- $\dot{U}(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$
- $A(z_0, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$