Math 395: Homework 6
Due: November 5, 2024
Name: Avinash Iyer
Collaborators: Carly Venenciano, Gianluca Crescenzo, Noah Smith, Ben Langer Weida, Chris Swanson

Problem 4

Problem: Let $t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $t(v, w) = v \times w$. Let $\mathcal{E}_3$ be the standard basis of $\mathbb{R}^3$, and let $\mathcal{B} = \left\{e_i \otimes e_j\right\}_{i,j=1}^{3}$. Let $T \in \operatorname{Hom}_F\left(\mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^3\right)$ be the linear map associated to t.

Calculate $[T]_{\mathcal{B}}^{\mathcal{E}_3}$.

Solution: Evaluating T at each element of $\mathcal{B}$, we get

- $e_1 \otimes e_1 \overset{T}{\mapsto} 0$
- $e_1 \otimes e_2 \overset{T}{\mapsto} e_3$
- $e_1 \otimes e_3 \overset{T}{\mapsto} -e_2$
- $e_2 \otimes e_1 \overset{T}{\mapsto} -e_3$
- $e_2 \otimes e_2 \overset{T}{\mapsto} 0$
- $e_2 \otimes e_3 \overset{T}{\mapsto} e_1$
- $e_3 \otimes e_1 \overset{T}{\mapsto} e_2$
- $e_3 \otimes e_2 \overset{T}{\mapsto} -e_1$
- $e_3 \otimes e_3 \overset{T}{\mapsto} 0$

Thus, the transformation matrix is

$$[T]_{\mathcal{B}}^{\mathcal{E}_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 5

Problem: Let V and W be F-vector spaces. Prove that $V \otimes_F W \cong W \otimes_F V$.

Solution: Define $t : W \times V \to V \otimes_F W$ by $t(w, v) = v \otimes w$.

By the definition of $v \otimes w$, t is necessarily a bilinear map, so the universal property gives a unique $T : W \otimes_F V \to V \otimes_F W$ by $T(w \otimes v) = v \otimes w$.

Let $s : V \times W \to W \otimes_F V$ be defined by $s(v, w) = w \otimes v$. By the definition of $w \otimes v$, it is the case that s is a bilinear map, so there is a unique map $S : V \otimes_F W \to W \otimes_F V$ given by $S(v \otimes w) = w \otimes v$.

Since $S \circ T(w \otimes v) = w \otimes v$ and $T \circ S(v \otimes w) = v \otimes w$, it is the case that T is an isomorphism, so $V \otimes_F W \cong W \otimes_F V$.

Problem 9

Problem: Let V_1, W_1, V_2, W_2 be F-vector spaces, with $T_1 \in \operatorname{Hom}_F(V_1, W_1)$ and $T_2 \in \operatorname{Hom}_F(V_2, W_2)$. Prove that there is a unique F-linear map $T_1 \otimes T_2$ from $V_1 \otimes_F V_2$ to $W_1 \otimes_F W_2$ satisfying $(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$.

Solution: Let $T_1 : V_1 \to W_1$ and $T_2 : V_2 \otimes W_2$ be fixed linear maps.

Define $t : V_1 \times V_2 \to W_1 \otimes_F W_2$ by $t(v_1, v_2) = T_1(v_1) \otimes T_2(v_2)$.

Since T_1 and T_2 are linear, t is a bilinear map, so by the universal property of tensor products, there is a unique $T_1 \otimes T_2 : V_1 \otimes_F V_2 \to W_1 \otimes_F W_2$.

Problem 12

Problem:

(a) Let $\varphi \in V'$ and $\psi \in W'$. Define a map

$$\begin{aligned} B_{\varphi,\psi} : V \times W &\to F \\ (v, w) &\mapsto \varphi(v)\psi(w). \end{aligned}$$

Show that $B_{\varphi,\psi}$ is a bilinear form.

(b) Prove that there is a natural isomorphism between $(V \otimes W)'$ and $V' \otimes W'$.

Solution:

(a) Letting $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\alpha \in F$, we have

$$\begin{aligned} B_{\varphi,\psi}(v, w_1 + \alpha w_2) &= \varphi(v)\psi(w_1 + \alpha w_2) \\ &= \varphi(v)(\psi(w_1) + \alpha\psi(w_2)) \\ &= \varphi(v)\psi(w_1) + \alpha\varphi(v)\psi(w_2) \\ &= B_{\varphi,\psi}(v, w_1) + \alpha B_{\varphi,\psi}(v, w_2) \end{aligned}$$

$$\begin{aligned} B_{\varphi,\psi}(v_1 + \alpha v_2, w) &= \varphi(v_1 + \alpha v_2)\psi(w) \\ &= (\varphi(v_1) + \alpha\varphi(v_2))\psi(w) \\ &= \varphi(v_1)\psi(w) + \alpha\varphi(v_2)\psi(w) \\ &= B_{\varphi,\psi}(v_1, w) + \alpha B_{\varphi,\psi}(v_2, w). \end{aligned}$$

(b) Since $B_{\varphi,\psi}$ is bilinear, we get a unique linear map, which we will call $\varphi \times \psi : V \otimes W \to F$ defined by $\varphi \times \psi(v \otimes w) = \varphi(v)\psi(w)$. Hence, $\varphi \times \psi \in (V \otimes W)'$.

Define

$$t : V' \times W' \to (V \otimes W)'$$

by $t(\varphi, \psi) = \varphi \times \psi$. We claim that t is bilinear. Let $\varphi, \varphi_1, \varphi_2 \in V'$, $\psi, \psi_1, \psi_2 \in W'$, and $\alpha \in F$. Then, for arbitrary $x \otimes y \in V \otimes W$, we have

$$\begin{aligned} t(\varphi, \psi_1 + \alpha\psi_2)(x \otimes y) &= (\varphi \times (\psi_1 + \alpha\psi_2))(x \otimes y) \\ &= \varphi(x)(\psi_1 + \alpha\psi_2)(y) \\ &= \varphi(x)\psi_1(y) + \alpha\varphi(x)\psi_2(y) \\ &= t(\varphi, \psi_1)(x \otimes y) + \alpha t(\varphi, \psi_2)(x \otimes y) \end{aligned}$$

$$\begin{aligned} t(\varphi_1 + \alpha\varphi_2, \psi)(x \otimes y) &= ((\varphi_1 + \alpha\varphi_2) \times \psi)(x \otimes y) \\ &= (\varphi_1 + \alpha\varphi_2)(x)\psi(y) \\ &= \varphi_1(x)\psi(y) + \alpha\varphi_2(x)\psi(y) \\ &= t(\varphi_1, \psi)(x \otimes y) + \alpha t(\varphi_2, \psi)(x \otimes y). \end{aligned}$$

Since t is bilinear, we have a unique linear map $T : V' \otimes W' \to (V \otimes W)'$ defined by $T(\varphi \otimes \psi) = \varphi \times \psi$.

We claim that T is an isomorphism. To see this, we define $S : (V \otimes W)' \to V' \otimes W'$ by $\varphi \times \psi \mapsto \varphi \otimes \psi$.

Since for any $\varphi \times \psi \in (V \otimes W)'$, we must have $B_{\varphi,\psi} : V \times W \to F$ such that $B_{\varphi,\psi}(v, w) = (\varphi \times \psi)(v \otimes w)$, it is the case that S is necessarily well-defined.

We must have

$$\begin{aligned} S \circ T(\varphi \otimes \psi) &= \varphi \otimes w \\ T \circ S(\varphi \times \psi) &= \varphi \times \psi, \end{aligned}$$

so T is an isomorphism, hence $(V \otimes W)' = V' \otimes W'$.