

Problem (Problem 1): Let $T: V \rightarrow W$ be a linear transformation between \mathbb{F} -vector spaces. Show that T is injective if and only if T maps \mathbb{F} -linearly independent subsets of V to \mathbb{F} -linearly independent subsets of W .

Solution: Let T be injective. We claim that if $\{v_1, \dots, v_n\}$ is linearly independent in V , then $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W . We see that if

$$\sum_{j=1}^n a_j Tv_j = 0_W,$$

then

$$T\left(\sum_{j=1}^n a_j v_j\right) = 0_W,$$

meaning that

$$\sum_{j=1}^n a_j v_j \in \ker(T).$$

Now, since T is injective, $\ker(T) = \{0_V\}$, meaning that $\sum_{j=1}^n a_j v_j = 0_V$. Yet, since $\{v_1, \dots, v_n\}$ is linearly independent, this means $a_j = 0$ for each j , so $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W .

Now, let T map linearly independent subsets of V to linearly independent subsets of W . If $\mathcal{B}_V = \{v_i\}_{i \in I}$ is a basis for V , then since \mathcal{B}_V is linearly independent, $C = \{Tv_i\}_{i \in I}$ is a linearly independent subset of W , which can be extended to a basis \mathcal{B}_W . Since $C \subseteq \mathcal{B}_W$, we see that any linear combination in \mathcal{B}_W yields 0 if and only if every coefficient is zero, meaning that $\ker(T) = \{0_V\}$, so T is injective.

Problem (Problem 2): Let $P_{n+1}(\mathbb{R})$ be the space of polynomials with real coefficients of degree $\leq n+1$. Prove that for any n points $a_1, \dots, a_n \in \mathbb{R}$, there exists a nonzero polynomial $f \in P_{n+1}(\mathbb{R})$ such that $f(a_j) = 0$ for each j , and $\sum_{j=1}^n f'(a_j) = 0$.

Solution: Based on the first condition, we see that the product $\prod_{j=1}^n (x - a_j)$ must divide the polynomial f , and since f has degree at most $n+1$, we must have $f(x) = (Ax + B) \prod_{j=1}^n (x - a_j)$ for some $a, b \in \mathbb{R}$. Writing $f'(x)$, we see that

$$f'(x) = A \prod_{j=1}^n (x - a_j) + (Ax + B) \sum_{i=1}^n \prod_{j \neq i} (x - a_j),$$

Problem (Problem 3): Let $T: V \rightarrow W$ be a linear map of finite-dimensional vector spaces, and let $W_0 \subseteq W$ be a subspace.

- Show that $T^{-1}(W_0) = \{v \in V \mid Tv \in W_0\}$ is a subspace of V .
- Assuming T is surjective, express $\dim(T^{-1}(W_0))$ in terms of $\dim(W_0)$ and $\dim(\ker(T))$.

Solution:

- We see that if $v_1, v_2 \in T^{-1}(W_0)$ and $\alpha \in \mathbb{R}$, then since $Tv_1, \alpha Tv_2 \in W_0$, we have $Tv_1 + \alpha Tv_2 \in W_0$, so by linearity, $T(v_1 + \alpha v_2) \in W_0$, meaning $v_1 + \alpha v_2 \in T^{-1}(W_0)$, so $T^{-1}(W_0)$ is a subspace of V .
- First, since T is surjective, $T(T^{-1}(W_0)) = W_0$. Therefore, by restricting the map T , we get the surjective map $T': T^{-1}(W_0) \rightarrow W_0$, and since $\ker(T) \subseteq T^{-1}(W_0)$, the First Isomorphism Theorem gives $T^{-1}(W_0)/\ker(T) \cong W_0$, so by rank-nullity (as each of W_0 and $T^{-1}(W_0)$ are finite-dimensional), $\dim(T^{-1}(W_0)) = \dim(\ker(T)) + \dim(W_0)$.

Problem (Problem 4):

- (a) Do there exist invertible matrices
- $A, B \in \text{Mat}_2(\mathbb{R})$
- such that

$$ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

- (b) Do there exist matrices
- $A, B \in \text{Mat}_2(\mathbb{R})$
- such that

$$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

Solution:

- (a) There do not. This follows from the fact that
- $\det(ABA^{-1}B^{-1}) = 1$
- , while the determinant of the latter matrix is 2.

Problem (Problem 5):

- (a) Find the inverse matrix
- A^{-1}
- for the matrix

$$A = \begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}.$$

- (b) Prove that

$$\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x_n \end{vmatrix} = x_1 x_2 \cdots x_n \left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_n} \right).$$

Solution:

- (a) We may find
- A^{-1}
- by trying to find the sequence of elementary matrices
- E_1, \dots, E_n
- such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I.$$

First, we do row reduction on A , yielding

$$\begin{aligned} \begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix} &\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix} \\ &\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ a & a & a+1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2a & a+1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3a+1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 / (3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus, the product $E_n E_{n-1} \cdots E_2 E_1$ is our desired inverse, which we find by applying the elementary row operations to the identity matrix I , yielding

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & a & 1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & -a & 2a+1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix} \\ &\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ -a/(3a+1) & 1 - (a/(3a+1)) & -1 + ((2a+1)/(3a+1)) \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix} \\ &\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 - a/(3a+1) & -a/(3a+1) & -1 + (2a+1)/(3a+1) \\ -a/(3a+1) & 1 - (a/(3a+1)) & -1 + ((2a+1)/(3a+1)) \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}, \end{aligned}$$

which is our desired inverse.

(b) We show the case for $n = 2$, then use induction from then on. By raw calculation, we see that

$$\begin{aligned} \begin{vmatrix} a+x_1 & a \\ a & a+x_2 \end{vmatrix} &= (a+x_1)(a+x_2) - a^2 \\ &= x_1x_2 + ax_1 + ax_2 \\ &= x_1x_2 \left(1 + \frac{a}{x_1} + \frac{a}{x_2} \right). \end{aligned}$$

Now, for the general n case, we see that since determinants are multilinear,

$$\begin{aligned} \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x_n \end{vmatrix} &= \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix} \\ &= a \begin{vmatrix} a+x_1 & a & \cdots & 1 \\ a & a+x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix} \end{aligned}$$

and since determinants are alternating,

$$= a \begin{vmatrix} x_1 & 0 & \cdots & 1 \\ 0 & x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$

and by the cofactor expansion,

$$= a(x_1 x_2 \cdots x_{n-1}) + x_n \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_{n-1} \end{vmatrix}$$

and by the induction hypothesis,

$$\begin{aligned} &= a(x_1 x_2 \cdots x_{n-1}) + x_n (x_1 x_2 \cdots x_{n-1}) \left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}} \right) \\ &= x_1 x_2 \cdots x_n \left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}} + \frac{a}{x_n} \right), \end{aligned}$$

we obtain our desired result.

Problem (Problem 6): Let $A \in \text{Mat}_n(\mathbb{R})$, and $(a_{ij})_{ij}$ such that $|a_{ij}| < \frac{1}{n}$ for each i, j . Show that $\det(I_n - A) \neq 0$.

Solution: Let $\|x\| = \max_{i=1}^n |x_i|$. Let x_j be the component of x such that $|x_j| = \|x\|$. Then, we see that

$$\begin{aligned} |(Ax)_j| &= \left| \sum_{i=1}^n a_{ij} x_i \right| \\ &\leq \sum_{i=1}^n |a_{ij}| |x_i| \\ &< \sum_{i=1}^n \frac{1}{n} |x_i| \\ &\leq \sum_{i=1}^n \frac{1}{n} \|x\| \\ &= |x_j|, \end{aligned}$$

which means that $Ax \neq x$ at the component x_j , meaning $(I_n - A)x \neq 0$.

Problem (Problem 7):

- (a) Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that $A^2 = I_n$. Show that A is diagonalizable.
- (b) Give an example of $A \in \text{Mat}_2(\mathbb{C})$ satisfying $A^2 = \mathbf{0}_2$ (the zero matrix) which is not diagonalizable.

Solution:

- (a) Since $A^2 - I_n = \mathbf{0}_n$, we see that the minimal polynomial of A is $m_A(t) = t^2 - 1$, which splits over \mathbb{C} to yield $m_A(t) = (t - 1)(t + 1)$. In particular, since the minimal polynomial splits into a product of distinct linear factors, A is diagonalizable.

(b) The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfies $A^2 = \mathbf{0}_2$, but since $A \neq \mathbf{0}_2$, we see that $m_A(t) = t^2$. Since $m_A(t)$ does not split into distinct linear factors over \mathbb{C} , we see that A is necessarily not diagonalizable.

Problem (Problem 8): Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that A^2 has n distinct eigenvalues. Show that A is diagonalizable.