Problem (Problem 1)**:** A subset $A \subseteq \mathbb{R}^n$ is said to have *measure zero* if, for all $\varepsilon > 0$, the set A can be covered by open balls of total volume at most ε. Prove that a countable subset of $\mathbb{R}^n$ has measure zero, and that the standard middle-thirds cantor set in $[0,1] \subseteq \mathbb{R}$ has measure zero.

Solution: Let A be countable, and let $\{a_k\}_{k\geqslant 1}$ be an enumeration of the points in A. Let $\varepsilon > 0$. Let c_n be the constant dependent on n such that the volume of $U(x,r) = c_n r^n$. For each k, define

$$r_k = \left(\frac{1}{2^k c_n}\varepsilon\right)^{1/n}.$$

Then, we see that the family $\{U(a_k, r_k)\}_{k=1}^{\infty}$ has total volume no more than ε, seeing as if all the open balls are disjoint, their union has total volume ε. Thus, countable subsets of $\mathbb{R}^n$ have measure zero.

Now, let $\mathcal{C}$ denote the standard middle-thirds Cantor set determined by the intersection of the intervals

$$\mathcal{C} = \bigcap_{n=1}^{\infty}\bigcup_{k=0}^{3^{n-1}-1}\left[\frac{3k}{3^n}, \frac{3k+1}{3^n}\right]\bigcup\left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right],$$

where $\mathcal{C}_0 = [0,1]$. Now, let $\varepsilon > 0$. Before we move on, notice that at stage N of the construction, we have 2^N subintervals whose endpoints are distance 3^{-N} away from each other.

Find N large enough that

$$\left(\frac{2}{3}\right)^N < \varepsilon/3$$

Then, for each of the 2^N subintervals, we may find an open interval with length $\frac{1}{3^{N-1}}$ centered at the center of each subinterval; the total sum of the lengths of these intervals is less than ε.

Since $\mathcal{C} \subseteq \mathcal{C}_N$, and $\mathcal{C}_N$ is able to be covered by open intervals with total length less than ε, we find that $\mathcal{C}$ has measure zero.

Problem (Problem 2)**:** Prove that if $A \subseteq U \subseteq \mathbb{R}^n$ has measure zero (with U open), and $f\colon U \to \mathbb{R}^n$ is smooth, show that $f(A)$ has measure zero.

Solution: Let $f\colon U \to \mathbb{R}^n$ be smooth. Then, f is locally Lipschitz, as f' is continuous, hence attains a supremum on compact subsets. In particular, for any $a \in A$, we see that there is $r > 0$ such that $U(a,r) \subseteq B(a,r) \subseteq U$, meaning f has a Lipschitz constant C_a such that $|f(x) - f(y)| \leqslant C_a|x-y|$. In particular, we may show that $f(A)$ has measure zero if $f(A \cap U(a,r))$ has measure zero.

Since A has measure zero, so too does $A \cap U(a,r)$, so that we may cover $A \cap U(a,r)$ by a countable (since $\mathbb{R}^n$ is a second countable space) $\{U(x_k, r_k)\}_{k\geqslant 1}$ with $m\left(\bigcup_{k=1}^{\infty} U(x_k, r_k)\right) < \varepsilon$ for any $\varepsilon > 0$. Then, since f is Lipschitz on $A \cap U(a,r)$, we have that

$$\begin{aligned}
f(A \cap U(a,r)) &\subseteq f\left(\bigcup_{k=1}^{\infty} U(x_k, r_k)\right)\\
&\subseteq \bigcup_{k=1}^{\infty} U(x_k, r_k)\\
&\subseteq \bigcup_{k=1}^{\infty} U(f(x_k), C_a r_k),
\end{aligned}$$

meaning that

$$\begin{aligned} m(f(A \cap U(a, r))) &\leqslant m\left(\bigcup_{k=1}^{\infty} U(f(x-k), C_a r_k)\right) \\ &= C_a^n \varepsilon. \end{aligned}$$

Since C_a is a constant and n is fixed, we thus have that $m(f(A \cap U(a, r))) = 0$, meaning that $m(f(A)) = 0$.

Problem (Problem 3)**:** In this exercise, we will prove Sard's Theorem. Let $U \subseteq \mathbb{R}^m$ be open, and let $f\colon U \to \mathbb{R}^n$ be C^∞. Let $A \subseteq U$ be the set of points where Df has rank less than n. Then, $f(A)$ has measure zero in $\mathbb{R}^n$. Note that it need not be the case that A itself have measure zero.

We will let A_i be the set of points in U where all partial derivatives up to degree i vanish.

(a) Prove that $f(A \setminus A_1)$ has measure zero.

(b) Prove that $f(A_k \setminus A_{k+1})$ has measure zero for all $k \geqslant 1$.

(c) Prove that $f(A_k)$ has measure zero for $k \gg 0$.

Solution:

(a) Let $x \notin A_1$, meaning some partial derivative does not vanish at x. Letting $f = (f_1, \dots, f_n)$, by some rearrangement, we may assume that $\frac{\partial f_1}{\partial x_1}(x) \neq 0$. Let $h(x) = (f_1(x), x_2, \dots, x_m)$. Since h consists of identity coordinate maps and f_1, which has nonzero partial derivative with respect to x_1, the inverse function theorem means that $h\colon U_x \supseteq \mathbb{R}^m \to V_x \subseteq \mathbb{R}^m$ is a local diffeomorphism.

We observe that $g = f \circ h^{-1}$ has critical points at $h(A \cap U_x)$, meaning that $g(A \cap U_x) = f(A \cap U_x)$. Notice then that for $(t, x_2, \dots, x_m) \subseteq \{t\} \times \mathbb{R}^{m-1}$,

$$\begin{aligned} f(A \cap U_x) &= g(h(t, x_2, \dots, x_m)) \\ &= (t, f_2(t, x_2, \dots, x_m), \dots, f_n(t, x_2, \dots, x_m)), \end{aligned}$$

meaning that g maps hyperplanes of the form $\{t\} \times \mathbb{R}^{m-1}$ to hyperplanes of the form $\{t\} \times \mathbb{R}^{n-1}$.

If we restrict g to $\{t\} \times \mathbb{R}^{m-1}$, then $Dg|_{\{t\} \times \mathbb{R}^{m-1}}$ mapping $\mathbb{R}^{m-1} \to \mathbb{R}^{n-1}$ (with some translation) is not of full rank on $h(A \cap U_x)|_{\{t\} \times \mathbb{R}^{m-1}}$. Via the induction hypothesis, we may reduce this to the case of $m = 0$, and since $f(\{0\})$ has measure zero in $\mathbb{R}^k$ for all $k \geqslant 1$.

Via Fubini's Theorem, we observe that if $B \subseteq \mathbb{R}^n$ is such that $B \cap \left(\{t\} \times \mathbb{R}^{n-1}\right)$ has measure zero in $\mathbb{R}^{n-1}$, then

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbb{1}_B(t)\, dm &= \int_{\{t\} \times \mathbb{R}^{n-1}} \int \mathbb{1}_{B^t}\, dm_{n-1}\, dt \\ &= 0, \end{aligned}$$

so $f(A \setminus A_1)$ has measure zero.

Problem (Problem 4)**:** Prove that Sard's Theorem is not necessarily true if f is not C^k for sufficiently large k.

(a) Prove that there is a C^1 function $f(x)$ on the real line whose set of critical values contains the middle-thirds Cantor set.

(b) If $X \subseteq [0, 1]$ is the middle thirds Cantor set, then $X + X = [0, 2]$.

(c) Prove that $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = g(x) + g(y)$ is C^1 and the set of critical values does not have measure zero.

Solution:

(a) If $\mathcal{C}_n$ denotes the nth stage of the construction of the Cantor set, we define $g_n\colon \mathbb{R} \to \mathbb{R}$ by a finite sum of smooth bump functions such that g_n is supported on a compact subset of each of the subintervals

$$I_{k,n} = \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right).$$

We also multiply these bump functions by a constant value such that the integral from $[0,1]$ of these bump functions is equal to 1. Then, defining $g = \lim_{n\to\infty} g_n$, we see that g is continuous as g is the uniform limit of smooth functions. Finally, take

$$f(x) = \int_0^x g(t)\, dt,$$

which gives a continuous function on $\mathbb{R}$.

We observe that $\mathcal{C}$ is a subset of the critical points for f, as $f'(x) = g(x) = 0$ whenever $x \in \mathcal{C}$. Furthermore, we also observe that whenever $q \in \mathcal{C}$, the definition of f yields

$$q = \int_0^q g(t)\, dt,$$

implying that $q \in f^{-1}(\{q\})$, so that the set of critical values for f includes $\mathcal{C}$.

(b) If $\mathcal{C}$ is the standard middle-thirds Cantor set, we notice that a number in $[0,1]$ is an element of $\mathcal{C}$ if and only if it admits a base-3 expansion entirely in $\{0,2\}$.

Now, if $c \in [0,2]$, we write

$$c = \sum_{n=0}^{\infty} \frac{c_n}{3^n},$$

where $c_n \in \{0,1,2\}$. For a fixed n, there are three cases that we may write c_n:

- if $c_n = 0$, we may find $a, b \in \mathcal{C}$ with $a_n, b_n = 0$;
- if $c_n = 2$, we may find $a, b \in \mathcal{C}$ with $a_n = 2$ and $b_n = 0$, or vice versa;
- if $c_n = 1$, we may fin $a, b \in \mathcal{C}$ with $a_{n+1} = 2$ and $b_{n+2} = 2$.

Since, for any element in the base-3 expansion of c, we may find two elements of the Cantor set such that $a + b = c$ at position n, we have that $\mathcal{C} + \mathcal{C} = [0,2]$.

(c) Letting $h\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $h(x,y) = f(x) + f(y)$, we see that h is the sum of two C^1 functions, meaning that h is C^1, yet since $\mathcal{C}$ is a subset of the critical values of f, we have that $\mathcal{C} + \mathcal{C}$ is a subset of the critical values of $f(x) + f(y)$, meaning $[0,2]$ is a subset of the critical values of $f(x) + f(y)$, or that the set of critical values of $f(x) + f(y)$ has nonzero measure.

Problem (Problem 5)**:** Prove that $SL_2(\mathbb{R})$, the 2×2 real matrices of determinant one, is diffeomorphic to $\mathbb{R}^2 \times S^1$.

Solution: We consider the action of $SL_2(\mathbb{R})$ on the upper half-plane of $\mathbb{C}$, $\mathbb{H} = \{z \mid \operatorname{Im}(z) > 0\}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \colon z \mapsto \frac{az+b}{cz+d}.$$

In particular, if $z = x + iy$ with $y > 0$, then

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z &= \frac{(ax+b) + iay}{(cx+d) + icy} \\ &= \frac{1}{(cx+d)^2 + c^2y^2}\left(\left((ax+b)(cx+d) + acy^2\right) + i(acxy - acxy + ady - bcy)\right) \\ &= \frac{1}{(cx+d)^2 + c^2y^2}\left(\left((ax+b)(cx+d) + acy^2\right) + iy\right),\end{aligned}$$

Thus, this is a fractional linear transformation on $\mathbb{C}$ that is an automorphism of $\mathbb{H}$. Therefore, by composing these fractional linear transformations, we see that $SL_2(\mathbb{R})$ acting on $\mathbb{H}$ via this map is a group action.

This action is transitive, since for any $x + iy \in \mathbb{H}$, we may calculate a map $i \mapsto x + iy$ in $SL_2(\mathbb{R})$ by using the transformation

$$\frac{ai + b}{ci + d} = i$$

which via multiplication and matching parts gives

$$\begin{aligned} a &= cx + dy \\ b &= xd - yc \end{aligned}$$

so by multiplying and back-substituting, we get

$$c^2 + d^2 = \frac{1}{y}.$$

By setting $c = 0$, we get

$$\begin{aligned} d &= \frac{1}{\sqrt{y}} \\ a &= \sqrt{y} \\ b &= \frac{x}{\sqrt{y}}. \end{aligned}$$

Now, to understand the stabilizer of some $z \in \mathbb{H}$, we only need to understand the stabilizer of i. For this, we see that

$$\begin{aligned} \frac{ai + b}{ci + d} &= i \\ ai + b &= di - c \end{aligned}$$

so

$$\begin{aligned} a &= d \\ b &= -c, \end{aligned}$$

and by back-substituting into the determinant, we get

$$a^2 + c^2 = 1,$$

so the stabilizer of i is all matrices of the form

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Thus, by orbit-stabilizer, $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/P$, where P is the group of rotation matrices and the action is left-multiplication. In particular, since every rotation matrix corresponds one-to-one with an element of $S^1 \subseteq \mathbb{C}$, given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto e^{i\theta},$$

we find that $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/S^1$, or that $\mathbb{H} \times S^1 \cong \mathrm{SL}_2(\mathbb{R})$.

In particular, we may view $\mathbb{H}$ to consist of matrices of the form

$$\begin{aligned} h &= \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \\ &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \end{aligned}$$

that take i as their input. Since the former matrix is diffeomorphic to $\mathbb{R}$ via a series of projections and inverse projections, and the latter is diffeomorphic to $\mathbb{R}_{>0}$ via another series of projections and inverse projections, which itself is diffeomorphic to $\mathbb{R}$ by exponents/logarithms, we find that

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &\cong \mathbb{H} \times S^1 \\ &\cong \mathbb{R} \times \mathbb{R}_{>0} \times S^1 \\ &\cong \mathbb{R}^2 \times S^1. \end{aligned}$$

In particular, $\mathbb{R} \times \mathbb{R}_{>0} \times S^1$ has a corresponding element in $\mathrm{SL}_2(\mathbb{R})$ given by the map

$$(x, y, \theta) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$