Problem (Problem 1)**:** For two ideals $I, J \subseteq R$, prove the following.

(a) The intersection $I \cap J$ is an ideal of R.

(b) The product $IJ \subseteq I \cap J$.

(c) Let $f\colon R \to R/(IJ)$ be the canonical homomorphism. Then, for any $x \in I \cap J$, the image $f(x)$ is nilpotent.

(d) If $I + J = R$, then $IJ = I \cap J$.

Solution:

(a) If $x, y \in I \cap J$, then $x - y \in I \cap J$ since $x - y \in I$ and $x - y \in J$. Furthermore, if $r \in R$, then $rx \in I$ and $rx \in J$, so $rx \in I \cap J$, so $I \cap J$ is an ideal.

(b) We observe that for any $q \in IJ$, we may express

$$q = \sum_{k=1}^{n} x_k y_k,$$

where $x_k \in I$ and $y_k \in J$. In particular, each $x_k y_k \in I \cap J$, so $q \in I \cap J$, meaning $IJ \subseteq I \cap J$.

(c) Let $x \in I \cap J$. Then, following from the well-definedness of operations in the quotient ring, we see that $(x + IJ)^n = x^n + IJ$. In particular, if $n = 2$, then x^2 is a linear combination of an element of I multiplied by an element of J, so $x^2 \in IJ$, meaning that $(x + IJ)^2 = x^2 + IJ = IJ = 0 + IJ$, meaning that x is nilpotent.

(d) We will show that if $q \in I \cap J$, then q can be written as a linear combination of elements of I multiplied by elements of J. In particular, we start by letting $i \in I$ and $j \in J$ be such that $i + j = 1$. Then, $q(i + j) = q$, meaning that $qi + qj = q$, and since $q \in I \cap J$, we have expressed q as a linear combination of elements of I multiplied by elements of J. Thus, $I \cap J \subseteq IJ$, meaning $IJ = I \cap J$.