**Abstract**

We discuss and prove the three big theorems of real analysis — the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

# Integration: An Introduction

In order to discuss integration, we need to start with the building blocks of all functions — simple functions.

**Definition:** Let $X$ be a measure space, and let $\phi\colon X \to [0,\infty]$ be a function. We say $\phi$ is a *simple function* if it has finite range (and does not take the value $+\infty$).

The *standard form* of a simple function $\phi$ is

$$\phi = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k},$$

where $\{c_1,\dots,c_n\} = \operatorname{ran}(\phi)$, and $E_k = \phi^{-1}(\{c_k\})$.

Recall that a function $f\colon X \to \mathbb{R}$, where $(X,\mathcal{M},\mu)$ is a measure space, is called Borel-measurable (or just measurable) if, for every $E \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(E) \in \mathcal{M}$.

**Definition:** If $\phi\colon X \to [0,\infty]$ is a simple, measurable function defined on a measure space $(X,\mathcal{M},\mu)$, then the *integral* of $\phi$ is defined to be

$$\int_X \phi\,d\mu = \sum_{k=1}^{n} c_k \mu(E_k). \tag{†}$$

**Proposition:** Let $\phi,\psi\colon X \to [0,\infty]$ be simple functions with standard forms

$$\begin{aligned}\phi &= \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}\\ \psi &= \sum_{k=1}^{m} b_k \mathbb{1}_{F_k}.\end{aligned}$$

Then, the following hold

(a) for all $c > 0$, $\displaystyle\int_X c\phi\,d\mu = c\int_X \phi\,d\mu$;

(b) $\displaystyle\int_X \phi+\psi\,d\mu = \int_X \phi\,d\mu + \int_X \psi\,d\mu$;

(c) if $\phi \leq \psi$ pointwise, then $\displaystyle\int_X \phi\,d\mu \leq \int_X \psi\,d\mu$.

*Proof.*

(a) We see that

$$\begin{aligned}\int_X c\phi\,d\mu &= \sum_{j=1}^{n}(c)(a_j)\mu(E_k)\\ &= c\sum_{k=1}^{n} a_j\mu(E_k)\\ &= c\int_X \phi\,d\mu.\end{aligned}$$

(b) Note that since

$$\begin{aligned} X &= \bigsqcup_{j=1}^{n} E_j \\ &= \bigsqcup_{k=1}^{m} F_k, \end{aligned}$$

we must have

$$\begin{aligned} E_j &= \bigsqcup_{k=1}^{m} E_j \cap F_k \\ F_k &= \bigsqcup_{j=1}^{m} F_k \cap E_j \end{aligned}$$

as a disjoint union. Therefore,

$$\begin{aligned} \int_X \phi \, d\mu + \int_X \psi \, d\mu &= \sum_{j=1}^{n}\sum_{k=1}^{m} (a_j + b_k)\mu(E_j \cap F_k) \\ &= \int_X \phi + \psi \, d\mu. \end{aligned}$$

(c) If $\phi \leq \psi$, $a_j \leq b_k$ whenever $E_j \cap F_k \neq \emptyset$. Therefore,

$$\begin{aligned} \int_X \phi \, d\mu &= \sum_{k=1}^{m}\sum_{j=1}^{n} a_j \mu(E_j \cap F_k) \\ &\leq \sum_{k=1}^{m}\sum_{j=1}^{n} b_k \mu(E_j \cap F_k) \\ &= \int_X \psi \, d\mu. \end{aligned}$$

□

Having established integrals for simple functions, we need to establish a convergence property for simple functions for all measurable functions.

**Theorem:** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \colon X \to [0, \infty]$ be a measurable function. Then, there is an increasing sequence $(\phi_n)_n$ of simple functions that converges pointwise to $f$. This sequence converges uniformly to $f$ on any bounded sets.

*Proof.* For each $n$, partition the interval $[0, 2^n]$ into subintervals of length $2^{-n}$. There are $2^{2n}$ subintervals, with

$$\begin{aligned} I_{n,0} &= \left[0, \frac{1}{2^n}\right] \\ I_{n,k} &= \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right], \end{aligned}$$

where $0 \leq k \leq 2^{2n} - 1$. We define $J_n = (2^n, \infty]$. Define

$$\begin{aligned} E_{n,k} &= f^{-1}(I_{n,k}) \\ F_n &= f^{-1}(J_n). \end{aligned}$$

Then, we may take

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n}\mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n}.$$

The family $\phi_n$ are simple, measurable, positive, and increasing.

Fix $x \in X$ such that $f(x) < \infty$, and find $N$ such that $f(x) \leq 2^N$. Then, for a fixed $n \geq N$, there is $0 \leq k \leq 2^{2n} - 1$ such that $x \in E_{n,k}$. Thus,

$$\begin{aligned} |\phi_n(x) - f(x)| &= \left|f(x) - \frac{k}{2^n}\right| \\ &\leq \frac{1}{2^n}. \end{aligned} \tag{$*$}$$

Thus, this family is pointwise convergent.

If $f(x) = +\infty$, then $\phi_n(x) = 2^n$ for all $n$, meaning $\phi_n(x)$ also converges to $f(x)$.

If $f(x)$ is bounded, then for a sufficiently large $n$, $F_n = \emptyset$, and the construction in $(*)$ is valid for all $x \in X$, meaning $\|\phi_n - f\|_u \leq \frac{1}{2^n}$, and $\sup_n \|\phi_n\|_u \leq \|f\|_u$. $\square$

**Remark:** By decomposing any complex-valued function $f$ using the Cartesian decomposition to yield $f = (f_+ - f_-) + i(g_+ - g_-)$, the above theorem can be extended to all complex-valued functions. There, the modulus of the simple functions, $(|\phi_n|)_n$ can be taken to be pointwise increasing and bounded above by $|f|$, with uniform convergence on sets where $f$ is bounded in modulus.

# The Monotone Convergence Theorem

Since any measurable function $f\colon X \to [0,\infty]$ is a pointwise limit of simple functions, we may define the integral of a function as follows.

**Definition:** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f\colon X \to [0,\infty]$ be a measurable function. The *integral* of $f$ is defined to be

$$\int_X f\,d\mu = \sup\left\{\int_X \phi_n\,d\mu \;\middle|\; \phi \text{ simple}, 0 \leq \phi \leq f\right\}.$$

This definition of the integral agrees with the definition in (†) whenever $f$ is simple. Furthermore, it follows that, for all $c \in [0,\infty)$,

$$\int_X cf\,d\mu = c\int_X f\,d\mu,$$

and whenever $f \leq g$,

$$\int_X f\,d\mu \leq \int_X g\,d\mu.$$

Yet, the issue is that our family of simple functions is uncountable. In order to (more easily) establish this integral, we need to be able to extract a sequence.

**Theorem** (Monotone Convergence Theorem)**:** Let $(f_n)_n$ be a family of $[0,\infty]$-valued measurable functions on $X$ such that $f_j \leq f_{j+1}$ for all $j$. Define

$$f = \lim_{n\to\infty} f_n$$

$$= \sup_{n\in\mathbb{N}} f_n.$$

Then,

$$\int_X f\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu.$$

*Proof.* The sequence $\left(\int_X f_n\,d\mu\right)$ is an increasing sequence of real numbers, so it has a limit (which may be equal to $\infty$). Furthermore, $\int_X f_n\,d\mu \leq \int_X f\,d\mu$ for all $n$, meaning $\sup\left(\int_X f_n\,d\mu\right) \leq \int_X f\,d\mu$.

To establish the reverse inequality, let $\alpha \in (0,1)$, $0 \leq \phi \leq f$ a simple function, and let

$$E_n = \{x \mid f_n(x) \geq \alpha\phi(x)\}.$$

The family $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets whose union is $X$.[I] We have

$$\begin{aligned}\int_X f_n\,d\mu &\geq \int_{E_n} f_n\,d\mu\\ &\geq \alpha\int_{E_n}\phi\,d\mu.\end{aligned}$$

Since

$$\lim_{n\to\infty}\int_{E_n}\phi\,d\mu = \int_X \phi\,d\mu,$$

we have

$$\lim_{n\to\infty}\int_X f_n\,d\mu \geq \alpha\int_X\phi\,d\mu.$$

We may take the supremum over all $\alpha\in(0,1)$, meaning

$$\lim_{n\to\infty}\int_X f_n\,d\mu \geq \int_X\phi\,d\mu.$$

Taking the supremum over all simple $0\leq\phi\leq f$, we obtain

$$\lim_{n\to\infty}\int_X f_n\,d\mu \geq \int_X f\,d\mu.$$

□

There are a variety of applications of the Monotone Convergence Theorem when it comes to establishing properties of sequences and series of functions.

**Theorem:** Let $(f_n)_n$ be a sequence of $[0,\infty]$-valued measurable functions. Then,

$$\int_X \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty}\int_X f_n\,d\mu.$$

*Proof.* We start with functions $f_1, f_2\colon X \to [0,\infty]$. Let $(\phi_j)_j$ and $(\psi_j)_j$ be sequences of simple functions increasing to $f_1$ and $f_2$ respectively. Then,

$$\int_X f_1 + f_2\,d\mu = \lim_{n\to\infty}\int \phi_j + \psi_j\,d\mu$$

[I] To see that their union is equal to $X$, recall that $f$ is the pointwise limit of $f_n$.

$$= \lim_{n\to\infty} \int_X \phi_j \, d\mu + \lim_{n\to\infty} \int_X \psi_j \, d\mu \tag{$*$}$$

$$= \int_X f_1 \, d\mu + \int_X f_2 \, d\mu, \tag{$**$}$$

where in $(*)$, we used the linearity of integration for simple functions, and in $(**)$, we used the monotone convergence theorem.

Therefore, by induction, we get that

$$\int_X \sum_{n=1}^{N} f_n \, d\mu = \sum_{n=1}^{N} \int_X f_n \, d\mu.$$

Applying the monotone convergence theorem to the sequence of partial sums, we obtain

$$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

□

## Fatou's Lemma

Going deeper into our quest to find out when (pointwise) convergence of functions implies convergence of their integrals, we establish the "next best" option.

**Theorem** (Fatou's Lemma)**:** Let $(f_n)_n \colon X \to [0,\infty]$ be a sequence of measurable functions. Then,

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

*Proof.* For each $k \geq 1$ and for all $j \geq k$, we see that $\inf_{n\geq k} f_n \leq f_j$.

Since integration preserves relative order, this means $\int_X \inf_{n\geq k} f_n \, d\mu \leq \int_X f_j \, d\mu$ for all $j \geq k$.

By the definition of infimum, we thus get that $\int_X \inf_{n\geq k} f_n \, d\mu \leq \inf_{j\geq k} \int_X f_j \, d\mu$. Applying the monotone convergence theorem, we may take the supremum of both sides to obtain

$$\begin{aligned} \int_X \liminf_{n\to\infty} f_n \, d\mu &= \sup_{k\geq 1} \int_X \inf_{n\geq k} f_n \, d\mu \\ &\leq \liminf_{n\to\infty} \int_X f_n \, d\mu. \end{aligned}$$

□

## Dominated Convergence Theorem

Fatou's Lemma is primarily used to prove the Dominated Convergence Theorem, the latter of which is significantly more powerful (but also requires one more condition).

**Definition:** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \colon X \to \mathbb{R}$ be a measurable function. We define the integral of $f$ to be

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

where

$$\begin{aligned} f^+(x) &= \max\{0, f(x)\} \\ f^-(x) &= \max\{0, -f(x)\}. \end{aligned}$$

We define the integral of a measurable $f\colon X \to \mathbb{C}$ to be

$$\int_X f\, d\mu = \int_X \mathrm{Re}(f)\, d\mu + i \int_X \mathrm{Im}(f)\, d\mu.$$

We say $f$ is *integrable*, or a member of $L_1$, if

$$\int_X |f|\, d\mu < \infty.$$

**Proposition:** If $f \in L_1(X, \mu)$, then

$$\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu.$$

*Proof.* If $f$ is real-valued, then

$$\begin{aligned} \left|\int_X f\, d\mu\right| &= \left|\int_X f^+\, d\mu - \int_X f^-\, d\mu\right| \\ &\leq \int_X f^+\, d\mu + \int_X f^-\, d\mu \\ &= \int_X |f|\, d\mu. \end{aligned}$$

Now, if $f$ is complex-valued with $\int_X f\, d\mu \neq 0$, we define $\alpha = \mathrm{sgn}\left(\int_X f\, d\mu\right)$. Then,

$$\begin{aligned} \left|\int_X f\, d\mu\right| &= \alpha \int_X f\, d\mu \\ &= \int_X \alpha f\, d\mu. \end{aligned}$$

Note that $\int_X \alpha f\, d\mu$ is real-valued, so

$$\begin{aligned} \left|\int_X f\, d\mu\right| &= \mathrm{Re}\left(\int_X \alpha f\, d\mu\right) \\ &= \int_X \mathrm{Re}(\alpha f)\, d\mu \\ &\leq \int_X |\mathrm{Re}(\alpha f)|\, d\mu \\ &\leq \int_X |\alpha f|\, d\mu \\ &= \int_X |f|\, d\mu. \end{aligned}$$

□

Now that we have established some of the important properties of $L_1$, we may prove the Dominated Convergence Theorem.

**Theorem** (Dominated Convergence)**:** Let $(f_n)_n$ be a sequence in $L_1$ such that $f_n \to f$ almost everywhere. If there exists a nonnegative $g \in L_1$ such that $|f_n| \leq g$ almost everywhere for every $n$, then $f \in L_1$ and

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu.$$

*Proof.* Since $f$ is the pointwise limit of a sequence of measurable functions, $f$ is measurable, and since $|f| \leq g$ almost everywhere, we have $f \in L_1$. It is sufficient to assume that $f_n$ and $f$ are real-valued, meaning $g + f_n \geq 0$ and $g - f_n \geq 0$ almost everywhere.

Applying Fatou's Lemma, we have

$$\begin{aligned}\int_X g\,d\mu + \int_X f\,d\mu &\leq \liminf_{n\to\infty} \int_X (g + f_n)\,d\mu \\ &= \int_X g\,d\mu + \liminf_{n\to\infty} \int_X f_n\,d\mu,\end{aligned}$$

and

$$\begin{aligned}\int_X g\,d\mu - \int_X f\,d\mu &\leq \liminf \int_X (g - f_n)\,d\mu \\ &= \int_X g\,d\mu - \limsup_{n\to\infty} \int_X f_n\,d\mu,\end{aligned}$$

meaning

$$\begin{aligned}\liminf_{n\to\infty} \int_X f_n\,d\mu &\geq \int_X f\,d\mu \\ &\geq \limsup_{n\to\infty} \int_X f_n\,d\mu.\end{aligned}$$

□