

Abstract

We detail the construction and some of the properties of the Lebesgue measure.

Premeasures, Outer Measures, and Measures

Consider a set-function $\lambda: P(\mathbb{R}) \rightarrow [0, \infty]$ that satisfies

- $\lambda(\emptyset) = 0$;
- for any finite or infinite sequence of disjoint sets, $\{E_j\}_{j=1}^\infty$, we have

$$\lambda\left(\bigsqcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \lambda(E_j);$$

- $\lambda(I) = b - a$, where I is an interval (either open, closed, or a half-interval);
- $\lambda(s + E) = \lambda(E)$.

Unfortunately, such a set-function doesn't exist.

In order to construct a set function on a restricted domain $\lambda: \mathcal{L} \rightarrow [0, \infty]$, we need to define a particular class of measurable subsets of \mathbb{R} . This is where the concept of an *outer measure* comes in.

Definition. Let X be a set, and let $\mu^*: P(X) \rightarrow [0, \infty]$ be a set function. We say μ^* is an *outer measure* if

- $\mu^*(\emptyset) = 0$;
- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$;
- $\mu^*\left(\bigcup_{j=1}^\infty A_j\right) \leq \sum_{j=1}^\infty \mu^*(A_j)$.

We will obtain an outer measure on the entirety of $P(X)$ by defining a notion of measure on some “satisfactory” subfamily $\mathcal{E} \subseteq P(X)$, then by approximating other subsets using this family.

Proposition: Let $\mathcal{E} \subseteq P(X)$ be a family of subsets such that $\emptyset \in \mathcal{E}$ and $X \in \mathcal{E}$, and let $\rho: \mathcal{E} \rightarrow [0, \infty]$ be a set function such that $\rho(\emptyset) = 0$. For any $A \subseteq X$, define

$$\mu^*(E) = \inf \left\{ \sum_{j \geq 1} \rho(E_j) \mid E_j \in \mathcal{E}, A \subseteq \bigcup_{j \geq 1} E_j \right\}.$$

Then, μ^* is an outer measure.

Proof. We start by showing well-definedness, which stems from the fact that we may select $E_j = X$ for all j .

Since we may take $E_j = \emptyset$ for all j , we must have $\mu^*(\emptyset) = 0$. Furthermore, if $A \subseteq B$, since the set over which the infimum is taken for the definition of $\mu^*(A)$ includes the corresponding set for B , we must have $\mu^*(A) \leq \mu^*(B)$.

Finally, let $\{A_j\}_{j \geq 1} \subseteq P(X)$, and let $\varepsilon > 0$. For each j , there exists $\{E_{j,k}\}_{k \geq 1} \subseteq \mathcal{E}$ such that $A_j \subseteq \bigcup_{k \geq 1} E_{j,k}$ and $\sum_{k \geq 1} \rho(E_{j,k}) \leq \mu^*(A_j) + \varepsilon 2^{-j}$.

Then, if $A = \bigcup_{j \geq 1} A_j$, we have $A \subseteq \bigcup_{j,k \geq 1} E_{j,k}$, and $\sum_{j,k \geq 1} \rho(E_{j,k}) \leq \sum_{j \geq 1} \mu^*(A_j) + \varepsilon$, so that $\mu^*(A) \leq \sum_{j \geq 1} \mu^*(A_j) + \varepsilon$. Since ε is arbitrary, we are done. \square

Definition. A subset $A \subseteq X$ is said to be μ^* -measurable if for any $E \subseteq X$, A serves as a good “cookie cutter” for E , in that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Equivalently, due to subadditivity, we have A is measurable if and only if for all $E \subseteq X$,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition. Let \mathcal{A} be an algebra of subsets of X . We call a set function $\mu_0: \mathcal{A} \rightarrow [0, \infty]$ a *premeasure* if

- $\mu_0(\emptyset) = 0$;
- for a collection of disjoint elements of \mathcal{A} , $\{A_j\}_{j=1}^\infty$ where $\bigcup_{j \geq 1} A_j \in \mathcal{A}$, we have

$$\mu_0\left(\bigcup_{j \geq 1} A_j\right) = \sum_{j \geq 1} \mu_0(A_j).$$

Every premeasure gives rise to an outer measure by taking

$$\mu^*(E) = \inf \left\{ \sum_{j \geq 1} \mu_0(A_j) \mid A_j \in \mathcal{A}, E \subseteq \bigcup_{j \geq 1} A_j \right\}. \quad (*)$$

A remarkable result by Caratheodory allows us to extend premeasures from algebras to measures on σ -algebras. To start, there is a little bit of build-up.

Proposition: Let μ_0 be a premeasure on \mathcal{A} , with μ^* defined by $(*)$. Then,

- (a) $\mu^*|_{\mathcal{A}} = \mu_0$;
- (b) every set in \mathcal{A} is μ^* -measurable.

Proof. Suppose $E \in \mathcal{A}$. If $E \subseteq \bigcup_{j \geq 1} A_j$ with $A_j \in \mathcal{A}$, we let $B_n = E \cap (A_n \setminus \bigcup_{j=1}^{n-1} A_j)$. The B_n are disjoint members of \mathcal{A} whose union is E , so

$$\begin{aligned} \mu_0(E) &= \sum_{j=1}^{\infty} \mu_0(B_j) \\ &\leq \sum_{j=1}^{\infty} \mu_0(A_j). \end{aligned}$$

It follows that $\mu_0(E) \leq \mu^*(E)$. The reverse inequality is clear from the fact that we may specify $A_1 = E$ and $A_{j>1} = \emptyset$.

Meanwhile, if $A \in \mathcal{A}$, $E \subseteq X$, and $\varepsilon > 0$, then there is a collection $\{B_j\}_{j \geq 1} \subseteq \mathcal{A}$ with $E \subseteq \bigcup_{j \geq 1} B_j$ and $\sum_{j \geq 1} \mu_0(B_j) \leq \mu^*(E) + \varepsilon$. By additivity on \mathcal{A} , we get

$$\begin{aligned} \mu^*(E) + \varepsilon &\geq \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \mu_0(B_j \cap A^c) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c), \end{aligned}$$

so A is measurable. □

Theorem (Caratheodory's Theorem): Let $\mathcal{A} \subseteq P(X)$ be an algebra, let μ_0 be a premeasure on \mathcal{A} , and let \mathcal{M} be the σ -algebra generated by \mathcal{A} . There exists a measure μ on \mathcal{M} whose restriction to \mathcal{A} is μ_0 — namely, $\mu - \mu^*|_{\mathcal{M}}$, where μ^* is given by $(*)$.

If ν is another measure on \mathcal{M} that extends μ_0 , then $\nu(E) \leq \mu(E)$, with equality for all $\mu(E) < \infty$. Furthermore, if μ_0 is σ -finite, then μ is unique.

Proof. We start by showing that if μ^* is an outer measure, then if \mathcal{M}^* is the collection of μ^* -measurable sets, \mathcal{M}^* is a σ -algebra and $\mu^*|_{\mathcal{M}^*}$ is a complete measure.^I

By definition, \mathcal{M}^* is closed under complements, as the definition of μ^* -measurability is symmetric in A and A^c . To show finite additivity, if $A, B \in \mathcal{M}^*$ and $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) \\ &\quad + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

We note that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, so subadditivity gives

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B)).$$

Therefore,

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Therefore, $A \cup B \in \mathcal{M}^*$, so \mathcal{M}^* is an algebra. Moreover, if $A, B \in \mathcal{M}^*$ are disjoint, then

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) \\ &= \mu^*(A) + \mu^*(B).\end{aligned}$$

To show that \mathcal{M}^* is a σ -algebra, we show that \mathcal{M}^* is closed under countable *disjoint* unions. Let $\{A_j\}_{j \geq 1}$ be a sequence of disjoint sets in \mathcal{M}^* , and let $B_n = \bigsqcup_{j=1}^n A_j$, with $B = \bigsqcup_{j \geq 1} A_j$. Then, for any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}),\end{aligned}$$

so by induction, we have

$$\mu^*(E \cap B_n) = \sum_{j=1}^n \mu^*(E \cap A_j).$$

This gives

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &\geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c),\end{aligned}$$

and taking $n \rightarrow \infty$, we have

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c)$$

^IThis is Theorem 1.11 in Folland's *Real Analysis*.

$$\begin{aligned}
&\geq \mu^* \left(\bigsqcup_{j \geq 1} E \cap A_j \right) + \mu^*(E \cap B^c) \\
&= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\
&\geq \mu^*(E).
\end{aligned}$$

Therefore, $B \in \mathcal{M}^*$, and if we take $E = B$,

$$\mu^*(B) = \sum_{j=1}^{\infty} \mu^*(A_j),$$

and μ^* is countably additive on \mathcal{M}^* . Finally, if $\mu^*(A) = 0$, we have

$$\begin{aligned}
\mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A^c) \\
&\leq \mu^*(E),
\end{aligned}$$

so $A \in \mathcal{M}^*$, and $\mu^*|_{\mathcal{M}^*}$ is a complete measure.

Returning to our premeasure, μ_0 and the corresponding outer measure μ^* , we note that since every element of \mathcal{A} is μ^* -measurable, the σ -algebra of μ^* -measurable sets includes \mathcal{A} , so it includes $\mathcal{M} = \sigma(\mathcal{A})$.

Let ν be any other measure on \mathcal{M} that extends μ_0 . If $E \in \mathcal{M}$, and $E \subseteq \bigcup_{j \geq 1} A_j$ with $A_j \in \mathcal{A}$, then $\nu(E) \leq \sum_{j \geq 1} \nu(A_j) = \sum_{j \geq 1} \mu_0(A_j)$, so $\nu(E) \leq \mu(E)$.

If we set $A = \bigcup_{j \geq 1} A_j$, the properties of the premeasure give us

$$\begin{aligned}
\nu(A) &= \lim_{n \rightarrow \infty} \nu \left(\bigcup_{j=1}^n A_j \right) \\
&= \lim_{n \rightarrow \infty} \mu \left(\bigcup_{j=1}^n A_j \right) \\
&= \mu(A).
\end{aligned}$$

If $\mu(E) < \infty$, we may select the A_j such that $\mu(A) < \mu(E) + \varepsilon$, so $\mu(A \setminus E) < \varepsilon$, and

$$\begin{aligned}
\mu(E) &\leq \mu(A) \\
&= \nu(A) \\
&= \nu(E) + \nu(A \setminus E) \\
&\leq \nu(E) + \mu(A \setminus E) \\
&\leq \nu(E) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, $\mu(E) = \nu(E)$.

Now, if μ_0 is σ -finite, we write $X = \bigsqcup_{j \geq 1} A_j$, with $\mu_0(A_j) < \infty$ and the A_j are disjoint. For any $E \in \mathcal{M}$, we have

$$\begin{aligned}
\mu(E) &= \sum_{j \geq 1} \mu(E \cap A_j) \\
&= \sum_{j \geq 1} \nu(E \cap A_j) \\
&= \nu(E).
\end{aligned}$$

□