Abstract

We discuss and prove the three big theorems of real analysis — the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

Integration: An Introduction

In order to discuss integration, we need to start with the building blocks of all functions — simple functions.

Definition. Let X be a measure space, and let $\phi\colon X \to [0,\infty]$ be a function. We say ϕ is a *simple function* if it has finite range (and does not take the value $+\infty$).

The *standard form* of a simple function ϕ is

$$\phi = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k},$$

where $\{c_1, \dots, c_n\} = \operatorname{Ran}(\phi)$, and $E_k = \phi^{-1}(\{c_k\})$.

Recall that a function $f\colon X \to \mathbb{R}$, where $(X, \mathcal{M}, \mu)$ is a measure space, is called Borel-measurable (or just measurable) if, for every $E \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(E) \in \mathcal{M}$.

Definition. If $\phi\colon X \to [0,\infty]$ is a simple, measurable function defined on a measure space $(X, \mathcal{M}, \mu)$, then the *integral* of ϕ is defined to be

$$\int_X \phi \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k).$$

Proposition: Let $\phi, \psi\colon X \to [0,\infty]$ be simple functions with standard forms

$$\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$$
$$\psi = \sum_{k=1}^{m} b_k \mathbb{1}_{F_k}.$$

Then, the following hold

(a) for all $c > 0$, $\displaystyle\int_X c\phi \, d\mu = c \int_X \phi \, d\mu$;

(b) $\displaystyle\int_X \phi + \psi \, d\mu = \int_X \phi \, d\mu + \int_X \psi \, d\mu$;

(c) if $\phi \leq \psi$ pointwise, then $\displaystyle\int_X \phi \, d\mu \leq \int_X \psi \, d\mu$.