Notationally, we will use 1 to denote the identity operator.

Contents

Introduction and Preliminaries

We start by recalling some of the topologies on $B(H)$.

Definition: Let H be a Hilbert space, with $B(H)$ denoting the space of bounded operators on H.

The *strong operator topology*, or SOT, is the locally convex topology generated by the seminorms

$$\{\|Tv\| \mid T \in B(H), v \in H\}$$

The *weak operator topology*, or WOT, is the locally convex topology generated by the seminorms

$$\{|\langle Tv, w\rangle| \mid T \in B(H), v, w \in H\}$$

Theorem: Let $\phi\colon B(H) \to \mathbb{C}$ be a linear functional. The following are equivalent:

(i) there are $\xi_k, \eta_k \in H$ such that $\phi(T) = \sum_{k=1}^{n} \langle T\xi_k, \eta_k\rangle$;

(ii) ϕ is WOT-continuous;

(iii) ϕ is SOT-continuous.

Proof. The directions (i) implies (ii) implies (iii) are pretty much by definition. To see (iii) implies (i), we have $\xi_1, \dots, \xi_n$ such that, for all $T \in B(H)$, $\max\|T\xi_k\| \leq 1$ implies $\phi(T) \leq 1$. Then, we have

$$|\phi(T)| \leq \left(\sum_{k=1}^{n} \|T\xi_k\|^2\right)^{1/2}.$$

Let

$$H^{(n)} \coloneqq \bigoplus_{k=1}^{n} H$$
$$T^{(n)} \coloneqq \operatorname{diag}(T, \dots, T) \in B\left(H^{(n)}\right),$$

and let $\xi = (\xi_1, \dots, \xi_n) \in H^{(n)}$. We see then that the linear functional $\psi\colon H \to \mathbb{C}$ given by

$$\psi\left(T^{(n)}\xi\right) = \phi(T)$$

defines a linear functional on the closed subspace of K spanned by the vectors

$$\left\{T^{(n)}\xi \;\middle|\; T \in B(H)\right\},$$

and has

$$\left|\psi\left(T^{(n)}\xi\right)\right| \leq \left\|T^{(n)}\xi\right\|,$$

so by the Riesz Representation Theorem for Hilbert Spaces, it follows there is $\eta = (\eta_1, \dots, \eta_n)$ such that

$$\begin{aligned}\phi(x) &= \left\langle T^{(n)}\xi, \eta\right\rangle \\ &= \sum_{k=1}^{n} \langle T\xi_k, \eta_k\rangle.\end{aligned}$$

□

Corollary: Every SOT-closed convex subset of $B(H)$ is WOT-closed.

Proof. The closed convex subsets of a locally convex topological vector space are determined by the continuous linear functionals, as follows from an application of the Hahn–Banach separation. □

Theorem: The unit ball of $B(H)$ is WOT-compact.

Proof. Let $\overline{\mathbb{D}}$ denote the closed unit disk of $\mathbb{C}$, and consider the set

$$K = \prod_{x,y \in B_H} \overline{\mathbb{D}}.$$

This space is compact by Tychonoff's theorem. Define the embedding $\phi\colon B_{B(H)} \to K$ given by

$$\phi(T) = (\langle Tx, y\rangle)_{x,y}.$$

By Cauchy–Schwarz, we have

$$\begin{aligned}|\langle Tx, y\rangle| &\leq \|T\|_{\mathrm{op}}\|x\|\|y\| \\ &\leq 1,\end{aligned}$$

so ϕ is well-defined. We see that ϕ is WOT-continuous by definition and injective, so we only need to show that $\operatorname{im}(\phi)$ is closed. Let $(T_i)_i \subseteq B_{B(H)}$ be a net with

$$\lim_{i \in I} (\langle T_i x, y\rangle)_{x,y} = (z_{x,y})_{x,y}.$$

We have that $(z_{x,y})_{x,y} \in K$ since K is compact, and since the product topology is the topology of pointwise convergence, we have

$$\lim_{i \in I} \langle T_i x, y\rangle = z_{x,y}$$

defines a sesquilinear form $F(x,y)$. This means we may find $T \in B_{B(H)}$ such that $F(x,y) = \langle Tx, y\rangle$, and so $(T_i)_i \to T$ in WOT. □

Definition: A *partial isometry* is an operator $W \in B(H)$ such that for any $h \in (\ker(W))^{\perp}$, we have $\|Wh\| = \|h\|$. The space $(\ker(W))^{\perp}$ is called the *initial space* of W, and the space $\operatorname{im}(W)$ is called the final space of W.

Proposition: If $W \in B(H)$, the following are equivalent:

(i) W is a partial isometry;

(ii) W^* is a partial isometry;

(iii) W^*W is a projection (onto the initial space of W);

(iv) WW^* is a projection (onto the final space of W);

Proof. Let W be a partial isometry, meaning that W is an isometry from $(\ker(W))^{\perp}$ to $\operatorname{im}(W)$. Since $\operatorname{im}(W)$ is dense in $\ker(W^*)^{\perp}$, it follows that we only need to show that W^* is an isometry on $\operatorname{im}(W)$. Let $k \in \operatorname{im}(W)$, so there is $h \in (\ker(W))^{\perp}$ such that $Wh = k$. Then, we have

$$\langle Wh, Wh \rangle = \langle h, h \rangle$$

so

$$\langle W^*Wh - h, h \rangle = 0,$$

meaning that $W^*W - I$ is zero on $(\ker(W))^{\perp}$, so we have

$$\begin{aligned}\|W^*k\| &= \|W^*Wh\| \\ &= \|h\| \\ &= \|Wh\| \\ &= \|k\|,\end{aligned}$$

meaning W^* is a partial isometry.

By taking adjoints, we see that (i) and (ii) are equivalent. Let $x \in H$ have the decomposition $x = y + z$ where $y \in \ker(W)$ and $z \in (\ker(W))^{\perp}$. We will show that $W^*Wx = z$. Observe that $Wx = Wz$, meaning that

$$\begin{aligned}\langle z - W^*Wx, z \rangle &= \langle z - W^*Wz, z \rangle \\ &= \langle z, z \rangle - \langle W^*Wz, z \rangle \\ &= \langle z, z \rangle - \langle Wz, Wz \rangle \\ &= 0,\end{aligned}$$

since $\|Wz\| = \|z\|$ by definition. In particular, following from the polarization identity, this means that for all $v \in H$, we have $\langle z - W^*Wx, v \rangle = 0$, so that $z = W^*Wx$. This shows that (i) implies (iii). By replacing all instances of W with W^*, we see that (ii) implies that WW^* is a projection onto the initial space of W^*, which is equal to the final space of W. □

Theorem (Polar Decomposition)**:** Let $A \in B(H)$. Then, there is a partial isometry W with initial space $(\ker(A))^{\perp}$ and final space $\overline{\operatorname{im}(A)}$ such that $A = W|A|$. Moreover, if $A = UP$, where P is a positive operator and U is a partial isometry with $\ker(U) = \ker(P)$, then $P = |A|$ and $U = W$.

Proof. Let $h \in H$. Then,

$$\begin{aligned}\|Ah\| &= \langle A^*Ah, h \rangle \\ &= \langle |A|h, |A|h \rangle,\end{aligned}$$

so that

$$\|Ah\| = \||A|h\|.$$

We may thus define $W\colon \operatorname{im}(|A|) \to \operatorname{im}(A)$ by taking

$$W(|A|h) = Ah.$$

Then, from above, we know that W is an isometry, so it can be extended to an isometry from $\overline{\text{im}(|A|)}$ to $\overline{\text{im}(A)}$. We may then extend W to all of H by defining it to be 0 on $(\text{im}(|A|))^{\perp}$. This makes W a partial isometry with $W|A| = A$. We must verify that W has the correct initial space. That is, we must show that $\overline{\text{im}(|A|)} = (\ker(A))^{\perp}$.

Suppose $f = A^*g$ for some $g \in (\ker(A^*))^{\perp} = \overline{\text{im}(A)}$. Then, $\text{im}(A^*A)$ is dense in $(\ker(A))^{\perp}$. Yet, since $A^*Ak = |A|h$, where $h = |A|k$, it follows that $\text{im}(|A|)$ is dense in $(\ker(A))^{\perp}$.

For uniqueness, we have that $A^*A = PU^*UP$, but since U^*U is the projection onto the initial space, it follows that $(\ker(U))^{\perp} = (\ker(P))^{\perp} = \overline{\text{im}(P)}$, meaning $A^*A = P^2$, so $P = |A|$ by the uniqueness in the continuous functional calculus. For any $h \in H$, we have $W|A|h = Ah = U|A|h$, meaning that U and W agree on a dense subset of their initial space, so $U = W$. $\square$

Corollary: If $T = W|T|$ is the polar decomposition for $T \in B(H)$, then $|T^*| = W|T|W^*$, and $T^* = W^*|T^*|$.

Proof. We see that $W|T|W^*$ is positive, and

$$\begin{aligned} W|T|W^*W|T|W^* &= W|T|^2W^* \\ &= WTT^*W^* \\ &= TT^* \end{aligned}$$

Therefore, by uniqueness, we have $W|T|W = |T^*|$. Furthermore, we see that

$$\begin{aligned} W^*|T^*| &= W^*W|T|W^* \\ &= |T|W^* \\ &= (W|T|)^* \\ &= T^*. \end{aligned}$$

$\square$

Definition: If M is a von Neumann algebra in $B(H)$, then the *center* of M is given by

$$Z(M) \coloneqq M \cap M'.$$

If $Z(M) = \mathbb{C}1$, then we say M is a *factor*.

Structure of von Neumann Algebras

There are a variety of ways we will understand the structure of von Neumann algebras. We start with discussing the most basic characterization of von Neumann algebras (emerging from the Double Commutant Theorem), then go into more depth into the structure of abelian von Neumann algebras, and end with a discussion of a characterization of a von Neumann algebra as a dual space.

Double Commutant Theorem

Definition: Let $M \subseteq B(H)$. We define the *commutant* to be

$$M' \coloneqq \{S \in B(H) \mid TS = ST \text{ for all } T \in M\}.$$

The double commutant of M is denoted M'', and has $M \subseteq M''$.

We see that M' is a WOT-closed subalgebra, and if M' is self-adjoint, then M' is a C^*-algebra. Additionally, if $M_1 \subseteq M_2$, then $M_1' \supseteq M_1'$.

Theorem (Double Commutant Theorem)**:** Let M be a unital C^*-subalgebra of $B(H)$. The following are equivalent:

(i) $M = M''$;

(ii) M is WOT-closed;

(iii) M is SOT-closed.

Proof. The implications (i) implies (ii) follows from the discussion above, and (ii) if and only (iii) follow from the definitions (as subalgebras are convex). We focus on showing that (iii) implies (i).

For a fixed $\xi \in H$, let P be the projection onto the closure of the subspace $\{T\xi \mid T \in M\}$. We see that $P\xi = \xi$, since $1 \in M$. Additionally, $PTP = TP$ for each $T \in M$, so $P \in M'$. Letting $V \in M''$, we have that $PV = VP$, so $V\xi \in PH$. In particular, for each $\varepsilon > 0$, there is $S \in M$ such that $\|(V - S)\xi\| < \varepsilon$.

Let $\xi_1, \dots, \xi_n \in H$, and set $\xi = (\xi_1, \dots, \xi_n)$ in $H^{(n)}$. Letting $\rho\colon B(H) \hookrightarrow B\big(H^{(n)}\big)$ be the embedding defined by

$$T \mapsto T^{(n)},$$

we see that

$$\rho(M)' = \{S \in B(K) \mid S_{ij} \in M'\}.$$

Therefore, we have that $\rho(V) \in \rho(M)''$, meaning that using the same process as above in the amplified algebra, we have

$$\begin{aligned}\sum_{k=1}^{n} \|(V - T)\xi_k\|^2 &= \|(\rho(V) - \rho(T))\xi\|^2 \\ &< \varepsilon^2,\end{aligned}$$

meaning that we can approximate V in SOT from M, so $V \in M$. □

Definition: A *von Neumann algebra* is a unital SOT-closed (or WOT-closed) C^*-subalgebra of $B(H)$.

The double commutant theorem says that $M = M''$ is a characterization of a von Neumann algebra.

Observe that if $T \in M$ is a normal operator in a von Neumann algebra M, then if E denotes the spectral measure for T, and $S \in M'$, then $TS = ST$, so by Fuglede's Theorem, $T^*S = ST^*$, meaning that $Sf(T) = f(T)S$ for all $f \in B_\infty(\sigma(T))$. In particular, this means that $E(S) \in M'' = M$. Since the closed linear span of the characteristic functions χ_S is equal to $B_\infty(\sigma(T))$, it follows that, if M is a von Neumann algebra, then M is the (norm)-closed linear span of all of its projections.

To see this another way, let $a \in M_{\text{s.a.}}$, and consider a partition $-\|a\| = t_0 < t_1 < \cdots < t_n = \|a\|$, where $t_{j+1} - t_j < \varepsilon$ for each $j = 0, \dots, n-1$, and define projections

$$P_i = \chi_{[t_{j-1}, t_j)}$$

for $j = 1, \dots, n-1$, and $P_n = \chi_{[t_{n-1}, t_n]}$. Then, we necessarily have

$$\left\| a - \sum_{j=1}^{n} t_j P_j \right\|_{\text{op}} < \varepsilon,$$

so every self-adjoint operator is in the norm-closed linear span of the projections of M. Since every element of M can be written as a decomposition of self-adjoint operators, it follows that M is the norm-closed linear span of its projections.

Proposition: Let M be a von Neumann algebra, and let $A \in M$.

(a) If A is normal, and ϕ is a bounded Borel function on $\sigma(A)$, then $\phi(A) \in M$.

(b) The operator A is the linear combination of four unitaries in M.

(c) If E and F are the projections onto $\overline{\operatorname{im}(A)}$ and $\ker(A)$ respectively, then $E, F \in M$.

(d) If $A = W|A|$ is the polar decomposition for A, then W and $|A|$ are in M.

Abelian von Neumann Algebras

Definition: Two subsets $M_1 \subseteq B(H_1)$ and $M_2 \subseteq B(H_2)$ are said to be *spatially isomorphic* if there is an isomorphism $U\colon H_1 \to H_2$ such that $UM_1U^{-1} = M_2$.

Definition: A vector e_0 is said to be separating for $S \subseteq B(H)$ if the only operator $T \in S$ for which $Te_0 = 0$ is the 0 operator.

Proposition: If S is a subspace of $B(H)$, then every cyclic vector for S is separating for S'. If A is a C^*-algebra of operators, then a vector is cyclic for A if and only if it is separating for A'.

Proof. If e_0 is cyclic for S, and $T \in S'$ with $Te_0 = 0$, then for every $L \in S$, we have $TLe_0 = LTe_0 = 0$, meaning that $T[Se_0] = 0$. Since e_0 is cyclic, this means $T = 0$.

If A is a unital C^*-subalgebra of $B(H)$, with e_0 separating for A', we let P be the projection onto $N = [Ae_0]^{\perp}$. Since N reduces A, it follows that $P \in A'$, but since $e_0 \perp N$, we have $Pe_0 = 0$. Since e_0 is separating for A', it follows that $P = 0$, so e_0 is cyclic for A. □

Corollary: If A is an abelian algebra of operators, every cyclic vector for A is separating.

Theorem: If H is separable, and A is an unital, abelian C^*-subalgebra of $B(H)$, then the following are equivalent:

(a) A is a maximal abelian von Neumann algebra;

(b) $A = A'$;

(c) A is SOT-closed with a cyclic vector;

(d) there is a compact metric space X, a regular Borel measure μ supported on X, and an isomorphism $U\colon L_2(X,\mu) \to H$ such that $UA_\mu U^{-1} = A$, where A_μ is the representation of $L_\infty(X,\mu)$ as the space of multiplication operators acting on $L_2(X,\mu)$.

Proof. If A is a maximal abelian von Neumann algebra, then $A = A''$ and $A \subseteq A'$, or that $A' \supseteq A'' = A$, so $A = A'$. Similarly, if $A = A'$, then $A = A' = A''$, so that A is a maximal abelian von Neumann algebra. Thus, (a) and (b) are equivalent.

Now, assume $A = A'$, it follows that $A = A''$, so that A is SOT-closed and contains the identity. Let $\{e_n\}_{n\geq 1}$ be a maximal sequence of unit vectors with $[Ae_n] \perp [Ae_m]$ whenever $n \leq m$. Then, by maximality, we have

$$H = \bigoplus_{n\geq 1}[Ae_n].$$

Let $P_n = [Ae_n]$, and set $e_0 = \sum_{n=1}^{\infty} 2^{-n}e_n$. Since P_n reduces A, $P_n \in A'$, so from (b), $P_n \in A$, meaning that $e_n = 2^nPe_0 \in [Ae_0]$, and thus $[Ae_n] \subseteq [Ae_0]$ for each n. Thus, e_0 is cyclic for A. This shows (b) implies (c).

Now, since H is separable, B_A is WOT-compact, meaning there is a countable WOT-dense subset. Let A_1 be the C^*-algebra generated by this WOT-dense subset; then, A_1 is a separable C^*-algebra that is WOT-dense in A. Let X be the character space of A_1; since A_1 is separable, X is metrizable, and let $\rho\colon C(X) \to A_1 \subseteq A \subseteq B(H)$ be the inverse Gelfand transform. Then, ρ is a representation of $C(X)$, so there is a spectral measure E on X such that

$$\rho(f) = \int f\,dE.$$

For every bounded Borel function, we then have

$$\begin{aligned}\widetilde{\rho}(\phi) &= \int \phi\,dE\\ &\in A_1''\\ &= A''\end{aligned}$$

$$= A$$

by the Double Commutant Theorem.

Letting e_0 be a cyclic vector for A, set $\mu(B) = \langle E(B)e_0, e_0\rangle$ for any Borel $B \subseteq X$. We have

$$\langle \widetilde{\rho}(\phi)e_0, e_0\rangle = \int \phi\, d\mu$$

for every $\phi \in B_\infty(X)$, and

$$\begin{aligned}\|\widetilde{\rho}(\phi)e_0\|^2 &= \langle \widetilde{\rho}(\phi)^*\widetilde{\rho}(\phi)e_0, e_0\rangle\\ &= \int |\phi|^2\, d\mu.\end{aligned}$$

Therefore, $B_\infty(X)$, considered as a dense subspace of $L_2(X,\mu)$, admits the well-defined isometry $U\colon B_\infty(X) \to H$ given by $U\phi = \widetilde{\rho}(\phi)e_0$. We may extend U to be an isometry on all of $L_2(X,\mu)$.

Now, if $\phi \in B_\infty(X)$ and $\psi \in L_\infty(X,\mu)$, then

$$\begin{aligned}UM_\psi\phi &= U(\psi\phi)\\ &= \widetilde{\rho}(\psi\phi)e_0\\ &= \widetilde{\rho}(\psi)\widetilde{\rho}(\phi)e_0\\ &= \widetilde{\rho}(\psi)U\phi.\end{aligned}$$

That is, $UA_\mu U^{-1} = \widetilde{\rho}(L_\infty(X,\mu))$. Yet, since A_μ is WOT-closed in $B(L_2(X,\mu))$, we have $\widetilde{\rho}(L_\infty(X,\mu))$ is WOT-closed in $B(H)$. Furthermore, since $\widetilde{\rho}(L_\infty(X,\mu)) \supseteq \rho(C(X)) = A_1$, we have $UA_\mu U^{-1} = A$. This shows (c) implies (d).

Finally, to show (d) implies (b), we show that $A_\mu = A_\mu'$. Let $T \in A_\mu'$. Since X is compact and μ is regular, it follows that $\mu(X) < \infty$. Then, $1 \in L_2(X,\mu)$, so we may set $L_2(X,\mu) \ni \phi = T(1)$. For any $\psi \in L_\infty(X,\mu)$, then $\psi \in L_2(X,\mu)$, with $T\psi = TM_\psi 1 = M_\psi T(1) = \psi\phi$, with

$$\begin{aligned}\|\phi\psi\| &= \|T\psi\|\\ &\leq \|T\|_{\text{op}}\|\psi\|.\end{aligned}$$

Set $\Delta_n = \{x \in X \mid |\phi(x)| \geq n\}$. Setting $\psi = \chi_{\Delta_n}$, we have

$$\begin{aligned}\|T\|_{\text{op}}^2\mu(\Delta_n) &= \|T\|_{\text{op}}^2\|\psi\|^2\\ &\geq \|\phi\psi\|^2\\ &= \int_{\Delta_n} |\phi|^2\, d\mu\\ &\geq n^2\mu(\Delta_n).\end{aligned}$$

Yet, since T is bounded, for sufficiently large n it follows that $\mu(\Delta_n) = 0$, meaning $\phi \in L_\infty(\mu)$, and since $T = M_\phi$ on $L_\infty(\mu)$, we have $T = M_\phi$. □

Preduals and Duals of von Neumann Algebras

A theorem of Sakai says that a C^*-algebra $M \subseteq B(H)$ is a von Neumann algebra precisely whenever there exists a predual for M; we do not prove this here, but we start by seeking to understand the predual of certain von Neumann algebras. This will enable us to understand the dual space of a von Neumann algebra.

An operator $T \in B(H)$ is called *trace-class* if there exists an orthonormal basis $(e_i)_{i\in I}$ such that the quantity

$$\begin{aligned}\operatorname{tr}(|T|) &:= \sum_{i\in I}\langle |T|e_i, e_i\rangle \\ &< \infty.\end{aligned}$$

Similarly, an operator $T \in B(H)$ is called *Hilbert–Schmidt* if the quantity $\operatorname{tr}(T^*T) < \infty$. The set of all trace-class operators is denoted $L_1(B(H))$, while the set of Hilbert–Schmidt operators is denoted $L_2(B(H))$. We list some essential properties of trace-class operators. The proofs can be found in [Con00, Ch. 3, §18].

Proposition (Properties of trace-class and Hilbert–Schmidt operators)**:** Let $T_1 \in L_1(B(H))$ and $T_2 \in L_2(B(H))$. The following properties hold.

(i) The quantities

$$\begin{aligned}\|T_1\|_1 &:= \operatorname{tr}(|T_1|) \\ \|T_2\|_2 &:= \operatorname{tr}(T_2^*T_2)\end{aligned}$$

define norms for T_1 and T_2 respectively.

(ii) For any $A \in B(H)$, we have $\operatorname{tr}(AT_1) = \operatorname{tr}(T_1A)$, and $|\operatorname{tr}(AT_1)| \leq \|A\|_{\text{op}}\|T_1\|_1$.

(iii) Both $L_1(B(H))$ and $L_2(B(H))$ are ideals in $B(H)$ satisfying

$$\|AT_{1,2}\|_{1,2} \leq \|A\|_{\text{op}}\|T_{1,2}\|_{1,2}.$$

Furthermore, both $L_1(B(H))$ and $L_2(B(H))$ are subsets of $K(H)$.

(iv) The operator T_1 is the product of two Hilbert–Schmidt operators, and any operator S is trace-class if and only if it is the product of two Hilbert–Schmidt operators.

(v) The pairing $\langle A, B\rangle = \operatorname{tr}(B^*A)$ defines an inner product on $L_2(B(H))$, and $L_2(B(H))$ is a Hilbert space with respect to this inner product.

The main thing we are interested in is understanding the duality properties of trace-class operators. We observe that the following is an analogue of the duality $(c_0)^* = \ell_1$.

Theorem: For any $T \in L_1(B(H))$, define the linear functional $\phi_T\colon K(H) \to \mathbb{C}$ by $\phi_T(A) = \operatorname{tr}(TA) = \operatorname{tr}(AT)$. Then, the map $T \mapsto \phi_T$ is an isometric isomorphism between $L_1(B(H))$ and $(K(H))^*$.

Proof. We observe that

$$\sup\left\{|\operatorname{tr}(AC)| \;\middle|\; C \in K(H), \|C\|_{\text{op}} \leq 1\right\} \leq \|A\|_1,$$

so that Φ_A is a bounded linear functional on $K(H)$ satisfying $\|\Phi_A\| \leq \|A\|_1$. Defining $\rho\colon L_1(B(H)) \to K(H)$ by $\rho(A) = \Phi_A$, we have that ρ is a linear map with $\|\rho(A)\| \leq \|A\|_1$ for all $A \in L_1(B(H))$.

Now, we will show that ρ is surjective with $\|\rho(A)\| \geq \|A\|_1$ for any $A \in L_1(B(H))$. Define a sesquilinear form for $\Phi \in K(H)^*$ by $[g, h] = \Phi(\theta_{g,h})$, where $\theta_{g,h}$ is the rank-one bounded operator given by

$$\theta_{g,h}(k) = \langle k, h\rangle g.$$

We have that $|[g, h]| \leq \|\Phi\|\|g\|\|h\|$ for all g and h, so $[\cdot,\cdot]$ is bounded, so there is $A \in B(H)$ such that $[g, h] = \langle Ag, h\rangle$. We will show that $A \in L_1(B(H))$ with $\Phi = \Phi_A$.

Let $C \in F(H)$ be given by

$$C = \sum_{k=1}^{n}\theta_{g_k,h_k},$$

Then,

$$\begin{aligned}\Phi(C) &= \Phi\left(\sum_{k=1}^{n} \theta_{g_k \otimes h_k}\right)\\ &= \sum_{k=1}^{n} \langle A g_k, h_k\rangle\\ &= \sum_{k=1}^{n} \operatorname{tr}(A\theta_{g_k,h_k})\\ &= \operatorname{tr}(AC).\end{aligned}$$

If we can show that $A \in L_1(B(H))$, then both Φ and Φ_A are bounded linear functionals on $K(H)$ that agree on $F(H)$.

For this, let $A = W|A|$ be the polar decomposition of A, and let $(e_i)_{i \in I}$ be an orthonormal basis. For any finite subset $F \subseteq I$, we have

$$C_F := \left(\sum_{i\in F} \theta_{e_i,e_i}\right) W^*$$

is a contraction in $F(H)$ with

$$\begin{aligned}\|\Phi\| &\geq |\Phi(C_F)|\\ &= \left|\Phi\left(\sum_{i\in F} e_i \otimes W e_i\right)\right|\\ &= \sum_{i\in F} |\langle A e_i, W e_i\rangle|\\ &= \sum_{i\in F} \langle |A| e_i, e_i\rangle.\end{aligned}$$

Letting F grow arbitrarily gives $\|\Phi\| \geq \|A\|_1$, so $A \in L_1(B(H))$, and $\Phi = \Phi_A$. Yet, this means $\|\Phi_A\| \geq \|A\|_1$, so ρ is an isometry. □

Similarly, just as $(\ell_1)^* = \ell_\infty$, the following holds.

Theorem: Let $\Psi\colon L_1(B(H)) \to \mathbb{C}$ be given by

$$\Phi_B(A) = \operatorname{tr}(AB).$$

Then, the map $B \mapsto \Phi_B$ defines an isometric isomorphism of $B(H)$ onto $\ell_1(B(H))^*$.

Proof. That $\|\Psi_B\| \leq \|B\|$ follows from the fact that $|\operatorname{tr}(AB)| \leq \|A\|_1 \|B\|_{\text{op}}$. Defining $\rho(B) = \Psi_B$, we have ρ is linear. If $\varepsilon > 0$, we use the Riesz lemma to find a unit vector g such that $\|Bg\| > \|B\|_{\text{op}} - \varepsilon$. Find a unit vector h such that $\langle Bg, h\rangle = \|Bg\|$. Then, letting $C = \theta_{g,h}$, we have $C \in L_1(B(H))$ with $\|C\|_1 = 1$, with

$$\begin{aligned}\|\Psi_B\| &\geq |\operatorname{tr}(BC)|\\ &= \langle Bg, h\rangle\\ &= \|Bg\|\\ &> \|B\|_{\text{op}} - \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have $\|\Psi_B\| = \|B\|_{\text{op}}$, and ρ is an isometry.

Now, let $\Psi \in L_1(B(H))^*$. Then, there is an operator $B \in B(H)$ such that $\langle Bg, h\rangle = \Psi(\theta_{g,h})$ for all $g, h \in H$. Then, it follows that $\Psi(T) = \Psi_B(T)$ for every finite-rank operator T, so since $F(H)$ is dense in $L_1(B(H))$, we have that both Ψ and Ψ_B are bounded linear functionals with $\Psi = \Psi_B$. □

Therefore, we can talk about the weak* topology on $B(H)$ induced by $L_1(B(H))$. We discuss an alternative form of convergence known as σ-WOT and σ-SOT convergence.

Definition: Let H be a Hilbert space. The σ-strong operator topology on $B(H)$ is the locally convex topology defined by the family of seminorms

$$p_\xi(T) = \|(T \otimes 1)\xi\|$$

for all $\xi \in H \otimes \ell_2$. The norm is defined by

$$\|(T \otimes 1)\xi\| = \left(\sum_{k=1}^{\infty} \|T\xi_k\|^2\right)^{1/2}.$$

The σ-weak operator topology on $B(H)$ is the locally convex topology defined by the family of seminorms

$$q_{\xi,\eta} = |\langle (T \otimes 1)\xi, \eta\rangle|$$

for all $\xi, \eta \in H \otimes \ell_2$. The inner product is defined by

$$|\langle (T \otimes 1)\xi, \eta\rangle| = \left|\sum_{k=1}^{\infty} \langle T\xi_k, \eta_k\rangle\right|.$$

We note that σ-WOT and WOT are equal on bounded subsets of $B(H)$. Furthermore, the following holds.

Proposition: The weak* topology on $B(H)$ induced by $L_1(B(H))$ and the σ-WOT are identical.

Proof. First, we observe that for any sequences $\xi, \eta \in H \otimes \ell_2$, we have that the operator

$$T = \sum_{k=1}^{\infty} \theta_{\xi_k, \eta_k} \tag{$*$}$$

is trace-class. Since multiplication by an element of $B(H)$ is continuous with respect to the trace-class norm, it follows that, whenever $(S_i)_i \to S$ is a w^*-convergent net, then

$$\begin{aligned}
\sum_{k=1}^{\infty} \langle S_i\xi_k, \eta_k\rangle &= \operatorname{tr}\left(\sum_{k=1}^{\infty} \theta_{S_i\xi_k, \eta_k}\right) \\
&= \operatorname{tr}(S_iT) \\
&\to \operatorname{tr}(ST) \\
&= \sum_{k=1}^{\infty} \langle S\xi_k, \eta_k\rangle.
\end{aligned}$$

Therefore, we have that each seminorm tends to 0 for all $\xi, \eta \in H \otimes \ell_2$, meaning $(S_i)_i \to S$ in σ-WOT.

Now, if $(S_i)_i \to S$ in σ-WOT, then since every trace-class operator is of the form in $(*)$, it follows that $\operatorname{tr}(S_iT) \to \operatorname{tr}(ST)$ for every $T \in L_1(B(H))$, so $(S_i)_i \to S$ is w^*-convergent. □

The existence of a predual for $B(H)$ extends to all von Neumann algebras.

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra. Then, there is a Banach space M_* such that M is isometrically isomorphic to $(M_*)^*$, where the w^* topology on M is the σ-weak topology.

Proof. Let $M^\perp$ be the annihilator of M in $B(H)$, in that

$$M^\perp = \{A \in L_1(B(H)) \mid \operatorname{tr}(AT) = 0 \text{ for all } T \in M\}.$$

Then, $M^\perp$ is a norm-closed subspace of $L_1(B(H))$, so we form the Banach space $M_* = L_1(B(H))/M^\perp$. Since M is σ-WOT closed (as it is WOT-closed), it follows that $M = \left(M^\perp\right)_\perp$, where $N_\perp$ denotes the preannihilator. The quotient map $Q\colon L_1(B(H)) \to M_*$ is thus an isometric embedding of $(M_*)^*$ onto M in $B(H)$. □

We specifically consider M_* to be the collection of σ-WOT continuous linear functionals on M.

Kaplansky Density Theorem and Pedersen's Up-Down Theorem

We start by discussing two extremely useful theorems.

Kaplansky's Density Theorem

Lemma: Let $(T_i)_{i\in I}, (S_i)_{i\in I} \subseteq B(H)$ be nets with $(T_i)_i \to T, (S_i)_i \to S$ in SOT. If $\sup_{i\in I}\|T_i\| < \infty$, then $(T_iS_i)_i \to TS$ in SOT.

Proof. Set $R = \sup_i\|T_i\|$. Then, for any $\xi \in H$,

$$\begin{aligned}\|TS\xi - T_iS_i\xi\| &\leq \|(T-T_i)S\xi\| + \|T_i(S-S_i)\xi\|\\ &\leq \|(T-T_i)\xi\| + R\|(S-S_i)\xi\|\\ &\to 0.\end{aligned}$$

□

Proposition: Let $f \in C(\mathbb{C})$. Then, the map $T \mapsto f(T)$ on normal operators in $B(H)$ is SOT-continuous on bounded subsets of $B(H)$.

Proof. Let $(T_i)_i$ be a uniformly bounded net of operators converging to T in SOT, with $R = \sup_i\|T_i\|$. By Stone–Weierstrass, we are able to approximate f uniformly $B(0,R)$ by a sequence of polynomials $(p_n)_n \subseteq \mathbb{C}[z,\overline{z}]$. Since multiplication is SOT-continuous on bounded subsets, it follows that $(p_n(T_i,T_i^*))_i \to p_n(T_i,T_i^*)$ in SOT.

Fix $\xi \in H$, $\varepsilon > 0$, and set N to be such that

$$\sup_{z\in B(0,R)}|f(z) - p_N(z,\overline{z})| < \frac{\varepsilon}{3\|\xi\|},$$

and i_0 to be such that for all $i \geq i_0$,

$$\|(p_N(T_i,T_i^*) - p_N(T,T^*))\xi\| < \varepsilon/3.$$

Then,

$$\begin{aligned}\|(f(T) - f(T_i))\xi\| &\leq \|(f(T) - p_N(T,T^*))\|\|\xi\| + \|(p_N(T,T^*) - p_N(T_i,T_i^*))\xi\| + \|(p_N(T_i,T_i^*) - f(T_i))\xi\|\\ &< \varepsilon.\end{aligned}$$

□

Now, we observe that if $T \in B(H)_{\text{s.a.}}$, then $\sigma(T) \subseteq \mathbb{R}$, meaning that $T + z1$ is invertible for any $z \in \mathbb{C}$ with $\operatorname{Im}(z) \neq 0$.

Definition: Let $T \in B(H)_{\text{s.a.}}$. Then, the *Cayley transform* of T is given by the operator

$$c(T) \coloneqq (T - i1)(T + i1)^{-1}.$$

Observe that the Cayley transform emerges from the continuous functional calculus on $c(z) = \frac{z-i}{z+i}$, meaning that $c(T)$ is a unitary operator, and $(T-i1)(T+i1)^{-1} = (T+i1)^{-1}(T-i1)$. This gives the following.

Proposition: The Cayley Transform is SOT-continuous on $B(H)_{\text{s.a.}}$.

Proof. Let $(T_j)_j \to T$ be a net of self-adjoint operators. By continuous functional calculus, we have $\left\|(T_j + i1)^{-1}\right\| \leq$

1 for all $i \in I$. If $\xi \in H$, we have

$$\begin{aligned}\|c(T)\xi - c(T_j)\xi\| &= \left\|c(T)\xi - (T_j + i1)^{-1}(T_j - i1)\xi\right\| \\ &= \left\|2i(T_j + i1)^{-1}(T - T_j)(T - i1)^{-1}\xi\right\| \\ &\leq 2\left\|(T - T_j)(T - i1)^{-1}\xi\right\|.\end{aligned}$$

Thus, SOT-convergence of $(T_j)_j$ to T implies SOT convergence of the Cayley transform. □

Corollary: If $f \in C_0(\mathbb{R})$, then the map $T \mapsto f(T)$ is SOT-continuous on $B(H)_{\text{s.a.}}$.

Proof. Since f vanishes at infinity, it follows that the function

$$g(z) := \begin{cases} 0 & z = 1 \\ f\left(i\frac{1+z}{1-z}\right) & \text{else} \end{cases}$$

defines a continuous function on S^1. Since any continuous function on $\mathbb{C}$ is SOT-continuous on bounded sets, it follows that g is SOT-continuous on unitary operators, so by composing g with the Cayley transform, it follows that f is SOT-continuous. □

For any subset $S \subseteq B(H)$, we define

$$(S)_1 := \{T \in S \mid \|T\| \leq 1\}.$$

Theorem (Kaplansky Density Theorem)**:** Let $A \subseteq B(H)$ be a $*$-subalgebra. Then,

$$\overline{A_{\text{s.a.}}}^{\text{SOT}} = \left(\overline{A}^{\text{SOT}}\right)_{\text{s.a.}}$$

and

$$\overline{(A)_1}^{\text{SOT}} = \left(\overline{A}^{\text{SOT}}\right)_1.$$

Proof. Denote $B = \overline{A}^{\text{SOT}}$. We start by showing that it suffices to show that A is (operator) norm-closed. This follows from the fact that norm convergence implies SOT convergence, meaning that if C denotes the norm closure of A, then $\overline{C}^{\text{SOT}} = \overline{A}^{\text{SOT}}$.

Since SOT convergence implies WOT convergence, it follows that $\overline{A_{\text{s.a.}}}^{\text{SOT}} \subseteq B_{\text{s.a.}}$. If $T \in B_{\text{s.a.}}$, then there exists a net $(T_i)_i \to T$ in SOT. Taking adjoints is WOT-continuous, so $\left(\frac{T_i + T_i^*}{2}\right)_i \subseteq A_{\text{s.a.}}$ converges to T in WOT. Therefore, $T \in \overline{A_{\text{s.a.}}}^{\text{WOT}}$, but since $A_{\text{s.a.}}$ is convex, $\overline{A_{\text{s.a.}}}^{\text{WOT}} = \overline{A_{\text{s.a.}}}^{\text{SOT}}$, meaning $B_{\text{s.a.}} = \overline{A_{\text{s.a.}}}^{\text{SOT}}$.

Now, to show that $\overline{(A)_1}^{\text{SOT}} = (B)_1$, we start by showing that the SOT closure of $(A_{\text{s.a.}})_1$ and $(B_{\text{s.a.}})_1$ coincide. Let $x \in (B_{\text{s.a.}})_1$, and let $(T_i)_i \subseteq A_{\text{s.a.}}$ converge to T in SOT. Let $f \in C_0(\mathbb{R})$ be a function with $\|f\|_u = 1$ and $f(t) = t$ for $|t| \leq 1$. Then, $(f(T_i))_i \subseteq (A_{\text{s.a.}})_1$, converging to $f(T) = T$ in SOT, meaning $(A_{\text{s.a.}})_1$ is SOT dense in $(B_{\text{s.a.}})_1$.

Next, we show that $\overline{\mathbb{M}_2(A)}^{\text{SOT}} = \mathbb{M}_2(B)$. Fixing elements

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathbb{M}_2(B)$$
$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in H \oplus H,$$

we use the fact that $B = \overline{A}^{\text{SOT}}$, so for each i, j, we can find $T_{ij} \in A$ such that $\|(T_{ij} - S_{ij})\xi_j\| < \varepsilon$. In particular, this gives

$$\begin{aligned}\left\|\begin{pmatrix}T_{11}-S_{11} & T_{12}-S_{12}\\ T_{21}-S_{21} & T_{22}-S_{22}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\|^2 &= \sum_{i=1}^{2}\|(T_{i1}-S_{i1})\xi_1+(T_{i2}-S_{i2})\xi_2\|^2\\ &< 8\varepsilon^2.\end{aligned}$$

Now, since we have $\overline{(A)_1}^{\text{SOT}} \subseteq (B)_1$, we then select $S \in (B)_1$, and consider

$$\begin{aligned}\overline{S} &= \begin{pmatrix}0 & S\\ S^* & \end{pmatrix}\\ &\in (\mathbb{M}_2(B))_1,\end{aligned}$$

which is self-adjoint. Therefore, by applying the earlier result replacing A and B with $\mathbb{M}_2(A)$ and $\mathbb{M}_2(B)$, we have a net $\left(\overline{S_i}\right)_i \subseteq (\mathbb{M}_2(A)_{\text{s.a.}})_1$ converging to $\overline{S}$ in SOT.

Now, if S_i denotes the $(1,2)$ entry of $\overline{S_i}$, then we observe that $\|S_i\| \leq 1$ and converges to S in SOT upon application to the vector $(0,\xi)$. ☐

Note that the choice of 1 for the operator norm bound in the KDT is arbitrary; by introducing some factors, we find that for any R, we have $\overline{(A)_R}^{\text{SOT}} = (B)_R$. The primary case will find use for is where $R = \|T\|$ for some $T \in B$.

Corollary: If $M \subseteq B(H)$ is a unital $*$-subalgebra, then the following are equal to each other:

- $\overline{M}^{\sigma\text{-SOT}}$;
- $\overline{M}^{\sigma\text{-WOT}}$;
- $\overline{M}^{\text{SOT}}$;
- $\overline{M}^{\text{WOT}}$;
- M''.

In particular, this means that M is a von Neumann algebra if and only if it is σ-SOT or σ-WOT closed.

Proof. The latter three equivalences follow from the Double Commutant Theorem. Now, since σ-SOT convergence implies σ-WOT convergence, which implies WOT convergence, it follows that all we need to show that $\overline{M}^{\text{SOT}} \subseteq \overline{M}^{\sigma\text{-SOT}}$. For $T \in \overline{M}^{\text{SOT}}$, we may find $(T_i)_i \to T$, where the net is contained in $(M)_{\|T\|}$, convergent in SOT. Since the net is uniformly bounded, and the σ-SOT and SOT coincide on bounded subsets, it follows that $\overline{M}^{\text{SOT}} \subseteq \overline{M}^{\sigma\text{-SOT}}$. ☐

Projections, Factors, and the Type Decomposition

From now on, we will use lower-case letters to denote elements of a von Neumann algebra, while retaining the use of upper-case letters to denote general operators in $B(H)$.

Comparison Theory of Projections

Recall that an element $P \in B(H)$ is called a projection if $P = P^2 = P^*$, and projects onto a unique closed subspace.

Proposition: If P and Q are projections in $B(H)$, then the following are equivalent:

(i) $QP = P$;

(ii) $\operatorname{im}(P) \subseteq \operatorname{im}(Q)$;

(iii) $P \leq Q$.

Proof. If we assume (i), then $Px = QPx$, so that $\operatorname{im}(Q) \supseteq Q(\operatorname{im}(P)) = \operatorname{im}(P)$, giving (ii). Similarly, (ii) implies (i) from the same definition.

Now, set $M = \operatorname{im}(P)$. For any $x \in H$, write $x = y + z$ for $y \in M$ and $z \in M^\perp$. Then, we have $\langle Px, x\rangle = \|y\|^2$, and

$$\begin{aligned}
\langle Qx, x\rangle &= \langle Qy + Qz, y + z\rangle \\
&= \langle Qy, y\rangle + \langle Qz, z\rangle + \langle Qy, z\rangle + \langle Qz, y\rangle \\
&= \|y\|^2 + \langle Qy, z\rangle + \langle Qz, y\rangle \\
&= \|y\|^2 + \langle Qy, z\rangle + \langle z, Qy\rangle \\
&= \|y\|^2 + \langle Qy, z\rangle \\
&\geq \langle Px, x\rangle,
\end{aligned}$$

so $P \leq Q$.

Finally, assuming $P \leq Q$, if $Qx = 0$, then $0 \leq \langle Px, x\rangle \leq \langle Qx, x\rangle = 0$, so $Px = 0$, so $\ker(Q) \subseteq \ker(P)$, meaning $\operatorname{im}(P) \subseteq \operatorname{im}(Q)$. □

Projections form a complete lattice under the operations

$$\begin{aligned}
\bigwedge_{i\in I} P_{X_i} &= P_{\bigcap_{i\in I} X_i} \\
\bigvee_{i\in I} P_i &= P_{\overline{\sum_{i\in I} X_i}}.
\end{aligned}$$

Unfortunately, the primary issue here is that these operations are too restrictive; for instance, the matrix units e_{11} and e_{22} both have rank 1 in $\mathbb{M}_n(\mathbb{C})$, but project onto different subspaces and are not comparable in the traditional sense. We will introduce a different way to compare projections that successfully deals with this issue.

Definition: Let $M \subseteq B(H)$ be a von Neumann algebra, and let $P(M)$ be its projection lattice. We say that projections $p, q \in P(M)$ are *equivalent* if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. That is, the initial projection of v is p and the final projection of v is q. We write $p \sim q$.

We say p is *sub-equivalent* to q, written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. We will write $p \prec q$ if $p \preceq q$ and $p \nsim q$.

Note that the traditional ordering of projections yields that $p \leq q$ implies $p \preceq q$, but the reverse is not necessarily true.

Proposition: For a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\preceq$ is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry for $\sim$ follows from the fact that if v is a partial isometry, then so too is v^*.

We will show transitivity for $\preceq$, from which it will be clear that $\sim$ is transitive. Let $p, q, r \in P(M)$ with $p \leq q$ and $q \leq r$. Then, we have partial isometries $u, v \in M$ with $u^*u = p$, $uu^* \leq q$, $v^*v = q$ and $vv^* \leq r$. Then, from

$$\begin{aligned}
qu &= quu^*u \\
&= uu^*u \\
&= u,
\end{aligned}$$

meaning

$$\begin{aligned}
(vu)^*(vu) &= u^*v^*vu \\
&= u^*qu \\
&= u^*u
\end{aligned}$$

$$= p,$$

and

$$\begin{aligned}(vu)(vu)^* &= vuu^*v\\ &\leq v^*qv\\ &= v(v^*v)v\\ &= vv^*\\ &\leq r,\end{aligned}$$

so $p \preceq r$, and $\preceq$ is a transitive relation. □

In fact, the preorder is a partial order, but this requires a bit more work and a useful lemma.

Lemma: Suppose $\tau \colon L \to L$ is an order-preserving map on a complete lattice. Then, ϕ has a fixed point.

Proof. Let $T = \{x \in L \mid x \leq \tau(x)\}$, and set x_0 to be the supremum of T. For any $x \in T$, we have $x \leq x_0$, meaning $x \leq \tau(x) \leq \tau(x_0)$, meaning $x_0 \leq \tau(x_0)$ by the definition of the supremum. Yet, this means that $\tau(x_0) \leq \tau(\tau(x_0))$, so $\tau(x_0) \in T$ with $\tau(x_0) \leq x_0$. □

Theorem (Cantor–Schröder–Bernstein for Projections)**:** Let M be a von Neumann algebra, and let $p, q \in P(M)$. If $p \preceq q$ and $q \preceq p$, then $p \sim q$.

Proof. Suppose w and v are partial isometries with $w^*w = p$, $ww^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Let L be the collection of all projections $e \in M$ with $e \leq q$. Then, L is a complete lattice; defining $\tau \colon L \to L$ by

$$\tau(e) = q - w(p - vev^*)w^*.$$

Then, τ is the composition of two order-preserving maps ($*$-conjugation by a fixed element) and two order-reversing maps (subtraction), so τ is order-preserving on L. Thus, τ has a fixed point, which we will call f. That is, there is $f \in M$ such that $f \leq q$ and $f = q - w(p - vfv^*)w^*$.

Let $v_1 = fv^*$, so that

$$\begin{aligned}v_1v_1^* &= (fv^*)(fv^*)^*\\ &= fv^*vf\\ &= f\end{aligned}$$

as $f \leq q$, and

$$v_1^*v_1 = vfv^*.$$

Therefore, $f \sim vfv^*$. Now, setting $w_1 = (p - vfv^*)w^*$, we have

$$\begin{aligned}w_1^*w_1 &= q - p\\ w_1w_2^* &= p - vfv^*,\end{aligned}$$

so $q - f \sim p - f$, meaning $q \sim p$. □

This is perhaps too slick a proof, and there is in fact a more involved proof that is similar to the proof of the Cantor–Schröder–Bernstein theorem. For this proof, we will let e, f denote the projections in question.

Alternative Proof. Let v and w be partial isometries such that $v^*v = e$, $vv^* = f_1 \leq f$, $w^*w = f$, and $ww^* = e_1 \leq e$. We inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of $f_2 \leq f_1$, it follows that we may write $f_2 = ve_1(ve_1)^*$, and since w maps the range of f_1 onto the range of $e_2 \leq e_1$, we may write $wf_1(wf_1)^* = e_2$. Furthermore, observe that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

We obtain two decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1}, and w maps the range of f_n isometrically onto that of e_{n+1}. In particular, if we let $e_\infty = \inf_n(e_n)$ and $f_\infty = \inf_n(f_n)$, we have that v maps the range of e_∞ onto that of f_∞ and w that of f_∞ onto the range of e_∞.

Similarly, $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$ as discussed earlier, so by the lemma below relating to sums of pairwise orthogonal families of projections, we have

$$\sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) \sim \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2})$$
$$\sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) \sin \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}).$$

Thus, we have

$$\begin{aligned} e &= e_\infty + \sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) \\ &\sim f_\infty + \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}) \\ &= f. \end{aligned}$$

□

Proposition: Let $S \subseteq H$ be a subset, and let

$$[S] := P_{\overline{\operatorname{span}(S)}}.$$

If $M \subseteq B(H)$ is a von Neumann algebra, with $x \in M$, then $[xH], [x^*H] \in M$ with $[xH] \sim_M [x^*H]$.

Proof. Let $x = v|x|$ be the polar decomposition, and note that $v \in M$. Since vv^* is the projection onto $\overline{xH}$ and v^*v is the projection onto $\ker(x)^\perp = \overline{x^*H}$, it follows that these projections are equivalent in M. □

Definition: If $x \in M$, we define the *central support* of x in M to be the projection

$$z(x) := \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say $p, q \in P(M)$ are *centrally orthogonal* if $z(p)z(q) = 0$.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. The central support of any $p \in P(M)$ is given by

$$\begin{aligned} z(p) &= \sup_{x\in M}[xpH] \\ &= [MpH]. \end{aligned}$$

Proof. The second equality follows from the definition of the supremum. Suppose we have $w = [MpH]$. Since M is unital, we have that $p \leq w$. Since $\overline{MpH}$ is reducing for both M and M', we have that $w \in M \cap M' = Z(M)$, meaning that $z(p) \leq w$.

Conversely, if $x \in M$, then

$$\begin{aligned} xpH &= xz(p)pH \\ &= z(p)xpH, \end{aligned}$$

so that $[xpH] \leq z(p)$, meaning $w \leq z(p)$ as this holds for all $x \in M$. □

Proposition: Let M be a von Neumann algebra. For any $p, q \in P(M)$, the following are equivalent:

(i) p and q are centrally orthogonal;

(ii) $pMq = \{0\}$;

(iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ such that $p_0 \sim q_0$.

Proof. We start by showing that (i) and (ii) are equivalent. Let p and q be centrally orthogonal; then, for any $x \in M$, we have

$$\begin{aligned} pxq &= pz(p)xz(q)q \\ &= pxz(p)z(q)q \\ &= 0. \end{aligned}$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, meaning that $p \leq 1 - z(q)$, so since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, so that $z(p)z(q) = 0$.

Now, we will show that *(ii)* and (iii) are equivalent. If *(ii)* does not hold, we let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$, so if we define

$$\begin{aligned} p_0 &= [pxqH] \\ q_0 &= [qx^*pH], \end{aligned}$$

we have that p_0, q_0 are nonzero projections, with $p_0 \leq p$ and $q_0 \leq q$. Since $(pxq)^* = qx^*p$, it follows from the lemma above that $p_0 \sim q_0$.

Meanwhile, if (iii) does not hold, we let $p_0 \leq p$ and $q_0 \leq q$ be such that p_0, q_0 are nonzero and $p_0 \sim q_0$. If $v \in M$ is a partial isometry such that $v^*v = p_0$ and $vv^* = q_0$, we have that $v^* = p_0v^*q_0$, and

$$\begin{aligned} pv^*q &= pp_0v^*q_0q \\ &= p_0v^*q_0 \\ &= v^* \\ &\neq 0, \end{aligned}$$

so that $pMq \neq \{0\}$. □

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i \mid i \in I\}$ and $\{q_0 \mid i \in I\}$ are pairwise orthogonal families with $p_i \preceq q_i$, then $\sum_{i \in I} p_i \preceq \sum_{i \in I} q_i$.

Proof. Let u_i be partial isometries with $u_i^*u_i = p_i$ and $r_i := u_iu_i^* \leq q_i$. Note that the r_i are pairwise orthogonal since the q_i are pairwise orthogonal. Therefore, for $i \neq j$, we have

$$\begin{aligned} u_i^*u_j &= u_i^*u_iu_i^*u_ju_js^*u_j \\ &= u_ir_ir_ju_j \\ &= 0 \\ u_iu_j^* &= u_iu_i^*u_iu_j^*u_ju_j^* \\ &= u_ip_ip_ju_j^* \\ &= 0. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \left(\sum_{i \in I} u_i\right)\left(\sum_{j \in J} u_j\right) &= \sum_{i \in I} u_i^*u_i \\ &= \sum_{i \in I} p_i \end{aligned}$$

and

$$\begin{aligned}\left(\sum_{i\in I} u_i\right)\left(\sum_{j\in J} u_j^*\right) &= \sum_{i\in I} r_i\\ &\leq \sum_{i\in I} q_i,\end{aligned}$$

so that $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$. □

Theorem (Comparison Theorem)**:** Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there exists $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ of pairwise orthogonal projections with $p_i \sim q_i$, and

$$\begin{aligned}\underbrace{\sum_{i\in I} p_i}_{=:p_0} &\leq p\\ \underbrace{\sum_{i\in I} q_i}_{=:q_0} &\leq q.\end{aligned}$$

From the above lemma, we know that $p_0 \sim q_0$. Let $w := z(q - q_0)$. By maximality, we must have $z(p - p_0)w = 0$, meaning that $(p - p_0)w = 0$, or $pz = p_0z$. If v is a partial isometry such that $v^*v = p_0$ and $vv^* = q_0$, then the partial isometry vz implements the equivalence $p_0z \sim q_0z$. Therefore, we have $pz = p_0z \sim q_0z \leq qz$. Similarly, $p_0(1-z) \sim q_0(1-z)$ and we get $q(1-z) \preceq p(1-z)$. □

Corollary: If M is a factor, then any two projections can be compared.

Proof. There are no nontrivial central projections in a factor. □

Compressions

Definition: If M is a von Neumann algebra, and $P \in B(H)$ is a projection (not necessarily in M), then

$$PMP = \{PxP \mid x \in M\}$$

is called a *corner* (or *compression*) of M.

This terminology comes from the identification that, whenever $x \in M$, we have

$$\begin{aligned}PxP &\leftrightarrow \begin{pmatrix} PxP & 0\\ 0 & 0\end{pmatrix}\\ &\in B(PH \oplus (1-P)H).\end{aligned}$$

In fact, we have $PB(H)P = B(PH)$.

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra, with $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, with $(pMp)' = M'p$.

Type Decomposition

Definition: Let M be a von Neumann algebra. We say $p \in P(M)$ is

- *finite* if $q \leq p$ and $q \sim p$ implies $p = q$;
- *semi-finite* if there exists a family $\{p_i\}_{i\in I}$ of pairwise orthogonal finite projections such that $\sum_{i\in I} p_i = p$;
- *purely infinite* if $p \neq 0$ and there do not exist any nonzero finite projections $q \in P(M)$ with $q \leq p$;

- *properly infinite* if $p \neq 0$ and, for all nonzero $w \in P(Z(M))$, the projection wp is not finite.

We say M is finite, semi-finite, purely infinite, or properly infinite if the projection 1 has the corresponding property in M.

An equivalent criterion for semi-finiteness is that, if w is any central projection with $wp \neq 0$, then there is a finite projection $0 < q \leq wp$.

Observe that for a von Neumann algebra, we have that finite implies semi-finite, and that any semi-finite von Neumann algebra is not purely infinite; similarly, any purely infinite von Neumann algebra is properly infinite.

Proposition: A von Neumann algebra $M \subseteq B(H)$ is finite if and only if all isometries are unitaries.

Proof. Let M be finite, and let $v \in M$ be an isometry, $v^*v = 1$. Then, $vv^* \leq 1$, so by finiteness, $vv^* = 1$, meaning v is a unitary.

Conversely, suppose every isometry is a unitary, and suppose $p \leq 1$ satisfies $p \sim 1$. Then, if $v \in M$ is a partial isometry with $v^*v = 1$ and $vv^* = p$, we have that v is an isometry, hence a unitary, so $vv^* = 1$, so 1 is finite in M. $\square$

Tracial von Neumann Algebras

Two Fundamental von Neumann Algebras

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