Problem 1

Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function with $f(0) = f(1)$. Show that there is a $c \in [0,1/2]$ with $f(c) = f(c+1/2)$. Conclude that there are always antipodal points on the earth's equator with the same temperature.

Consider $g(x) = f(x) - f(x+1/2)$ on $[0,1/2]$. Then, $g(0) = f(0) - f(1/2)$, and $g(1/2) = f(1/2) - f(1)$. Since $f(0) = f(1)$, it must be the case that $g(0) = -g(1/2)$.

Therefore, on $[0,1/2]$, if $g(0) = k$ for $k \in \mathbb{R}$, then $g(1/2) = -k$, meaning that by the Intermediate Value Theorem, $\exists c \in [0,1/2]$ with $g(c) = 0$. This is equivalent to $f(c) = f(c+1/2)$ by the definition of g.

For any two antipodes on the earth's equator, let $t(x)$ be the temperature at point x. Then, moving from x to $-x$, where $-x$ denotes the opposite point on the earth's equator, it must be the case that the values of t at x and $-x$ flip. Therefore, there is a point where $t(c) = t(-c)$.

Problem 2

Suppose $f : [a,b] \to \mathbb{R}$ is injective and continuous. Show that f is strictly monotone.

Let $f : [a,b] \to \mathbb{R}$ be injective and continuous. WLOG, let $p < q \in [a,b]$. Then, since $p \neq q$, $f(p) \neq f(q)$, meaning that $f(p) < f(q)$ and $f(p) > f(q)$.

Since f is continuous, f by the Intermediate Value Theorem, $\forall x \in [f(p), f(q)]$ or $[f(q), f(p)]$, $\exists! x' \in [p,q]$ or $[q,p]$ such that $f(x') = x$. Therefore, $\forall p, q \in [a,b]$, $p < q \Rightarrow f(p) < f(q)$ or $f(p) > f(q)$, so f is strictly monotone.

Problem 3

Suppose $f : [0,1] \to \mathbb{R}$ is a map that takes on each of its values exactly twice. Show that f cannot be continuous at every point.

I don't know how to do this problem.

Problem 4

Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1,\infty)$ but not on $(0,\infty)$.

Let $f(x) = \frac{1}{x^2}$ defined on $[1,\infty)$. Let $\varepsilon > 0$.

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \left|\frac{x^2 - y^2}{x^2y^2}\right| \\
&= \frac{x+y}{x^2y^2}\,|x-y| \\
&\leq 2\,|x-y| \\
&< \varepsilon
\end{aligned}$$

Set $\delta = \frac{\varepsilon}{2}$.

On $(0,\infty)$, let $u_n = \frac{1}{\sqrt{n+1}}$ and $v_n = \frac{1}{\sqrt{n}}$. Then,

$$\begin{aligned}
|f(u_n) - f(v_n)| &= |n+1-n| \\
&= 1 \\
&= \varepsilon_0 \\
|u_n - v_n| &= \left|\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right| \\
&= \left|\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}\right| \\
&= \left|\frac{1}{\sqrt{n(n+1)}\left(\sqrt{n+1}+\sqrt{n}\right)}\right| \\
&\to 0.
\end{aligned}$$

Therefore, f is not uniformly continuous.

Problem 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p; that is,

$$f(x+p) = f(x) \qquad \forall x \in \mathbb{R}$$

If f is continuous, show that f is bounded and uniformly continuous on $\mathbb{R}$.

Let $x \in \mathbb{R}$. Since f is continuous on $\mathbb{R}$, f is continuous on $[x, x+p]$, and f takes every value on $[x, x+p]$ in all of $\mathbb{R}$, since if $q \in [x, x+p]$, then $f(q+np) = f(q)$.

Since f is continuous on $[x, x+p]$, f is bounded on $[x, x+p]$, and so is bounded on $\mathbb{R}$. Additionally, f is uniformly continuous on $[x, x+p]$, and so is uniformly continuous on $\mathbb{R}$.

Problem 6

Show that $f(x) = x$ and $g(x) = \sin(x)$ are both uniformly continuous on $\mathbb{R}$, but the product

$$h(x) = x\sin(x)$$

is not uniformly continuous on $\mathbb{R}$.

Let $f(x) = x$. Setting $\delta = \varepsilon$, we have that

$$\begin{aligned}
|x-y| &< \delta \\
|f(x) - f(y)| &< \delta \\
|f(x) - f(y)| &< \varepsilon.
\end{aligned}$$

Similarly, since $\sin(x)$ is periodic and continuous, it must be uniformly continuous.

Problem 7

If $f : D \to \mathbb{R}$ is uniformly continuous and $|f(x)| \geq k > 0$ for some k, show that $\frac{1}{f}$ is uniformly continuous on D.

Since $f : D \to \mathbb{R}$ is uniformly continuous, $\forall u_n, v_n \in D$ with $(u_n - v_n)_n \to 0$, $(f(u_n) - f(v_n))_n \to 0$.

Since $|f|$ is bounded away from 0, it must be the case that

$$\left(\frac{1}{f(u_n)} - \frac{1}{f(v_n)}\right)_n \to 0,$$

so $\frac{1}{f}$ is uniformly continuous.

Problem 8

If $D \subseteq \mathbb{R}$ is a bounded set and $f : D \to \mathbb{R}$ is uniformly continuous, show that f is bounded.

Since D is bounded, $\forall x \in D$, $|x| < M$ for some M. Let $\varepsilon > 0$, and $\delta > 0$ be the corresponding value such that $|x - y| < \delta$. Then,

$$\begin{aligned} |f(x)| &= |f(x) - f(y) + f(y)| \\ &\leq |f(x) - f(y)| + |f(y)| \\ &< \varepsilon + |f(y)| \end{aligned}$$

So, for all x, $|f(x)|$ is bounded above, meaning that f is bounded.

Problem 9

Suppose $f_n : D \to \mathbb{R}$ is a sequence of uniformly continuous functions such that $(f_n)_n \to f$ uniformly on D. Show that f is also continuous. Is this true with pointwise convergence?

Let $f_n : D \to \mathbb{R}$ be a sequence of uniformly continuous functions that uniformly converges to $f : D \to \mathbb{R}$.

Let $c \in D$. Since f_n is uniformly continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall y \in D$ where $|c - y| < \delta$, $|f_n(c) - f_n(y)| < \varepsilon$, for all $n \in \mathbb{N}$. Additionally, since $f_n \to f$ uniformly, if $|c - y| < \delta$, $|f(c) - f(y)| < \varepsilon$.

Therefore, f is continuous at c for any arbitrary $c \in D$.

This is not the case with pointwise convergence — for example, $f_n = x^n$ on $[0, 1]$ converges to the discontinuous function δ_1.

Problem 10

Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ with

$$\begin{aligned} f(\mathbb{Q}) &\subseteq \mathbb{R} \setminus \mathbb{Q} \\ f(\mathbb{R} \setminus \mathbb{Q}) &\subseteq \mathbb{Q}. \end{aligned}$$

Since f does not map an interval to an interval, f cannot be continuous.