Understanding Amenability in Discrete Groups

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Abstract

We provide a brief yet thorough overview of amenability in discrete groups by using techniques from functional analysis. We discuss the definition of a mean on a group, and provide some basic characterizations for amenability, including the interplay between means and invariant states on groups, paradoxical decompositions via Tarski's Theorem, and a more combinatorial approximation property via Følner sequences. We bridge important results in group theory and functional analysis in order to prove these results, and seek to provide proper scaffolding for understanding the results in higher analysis that relate to amenability in groups.

0 Preliminaries

Here, we overview some of the results we make liberal use of throughout this thesis. We assume that all the readers are familiar with real analysis and group theory, about at the level of Math 310 and Math 320, as well as their preliminaries. We also occasionally allude to results in topology.

0.1 More Group Theory

There's a bit more group theory that we need to cover. These groups will provide the backbone for Section 2

Here, we will discuss the archetypal (some might say universal) group that can be constructed from any set. This is known as the free group. The definitions and results in section are drawn from [Har00] and [Löh17].

Definition 0.1.1. Let S be a set. A group F containing S is said to be *freely generated* if, for every group G, and every set-map $\phi\colon S \to G$, there is a unique group homomorphism $\varphi\colon F \to G$ that extends φ. The following diagram, where ι denotes the inclusion of S into F, commutes:

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, hook, "\iota"'] & G \\ F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We say F is the *free group* generated by S.

Free groups do exist, and by definition, are unique up to isomorphism.

Theorem 0.1.1. If S is a set, we may define the formal inverse of elements of S, $S^{-1} := \left\{ s^{-1} \mid s \in S \right\}$. Let $W(S)$ be the set of words in the formal alphabet $S \cup S^{-1}$.

Let $F(S)$ be defined by $W(S)/{\sim}$, where $\sim$ is the equivalence relation generated by

$$\begin{aligned} xss^{-1}y &\sim xy \\ xs^{-1}sy &\sim xy. \end{aligned}$$

Then, $F(S)$ is freely generated by S.

Example 0.1.1. If we consider the set $S = \{a, b\}$, then the free group $F(a, b)$ is defined to be the set of all reduced words in the alphabet $\{a, b, a^{-1}, b^{-1}\}$.

The free group is an example of a more general construction — the free product of groups. We define the free product and its universal property, and leave it as an exercise for the reader to determine the specific family of groups for which $F(S)$ is the free product.

Definition 0.1.2 (Free Product)**.** Let A be a set, and set $W(A)$ to be the set of words in A equipped with the operation of concatenation. This turns $W(A)$ into a construction known as the *free monoid*.

If $\{\Gamma_i\}_{i\in I}$ is a family of groups, and $A = \coprod_{i\in I} \Gamma_i$ is the coproduct (or disjoint union) of the groups Γ_i, then we define the equivalence relation $\sim$ generated by

$$\begin{aligned} we_iw' &\sim ww' \text{ where } e_i \text{ is the neutral element of } \Gamma_i \text{ for some } i \in I \\ wabw' &\sim wcw' \text{ where } a, b, c \in \Gamma_i \text{ and } c = ab \text{ for some } i \in I. \end{aligned}$$

Then, the quotient $W(A)/\sim$ is known as the *free product* of the groups $\{\Gamma_i\}_{i\in I}$, and is denoted

$$\bigstar_{i\in I}\Gamma_i.$$

Predictably, the free group also admits a universal property.

Theorem 0.1.2. Let $\{\Gamma_i\}_{i\in I}$ be a family of groups, and let $h_i \colon \Gamma_i \to \Gamma$ be a family of homomorphisms for each Γ_i. Then, there is a unique homomorphism $h \colon \bigstar_{i\in I}\Gamma_i \to \Gamma$ such that the following diagram commutes for each Γ_{i_0}.

$$\begin{tikzcd} \Gamma_{i_0} \arrow[r, "h_{i_0}"] \arrow[d, hook, "\iota_{i_0}"'] & \Gamma_i \\ \bigstar_{i\in I}\Gamma_i \arrow[ur, "h"'] & \end{tikzcd}$$

One of the useful facts about the free product is that its properties allow us to find subgroups isomorphic to $F(a, b)$. This occurs through a special property of the action of a group on the set.

Theorem 0.1.3 (Ping Pong Lemma)**.** Let G be a group that acts on a set X, and let Γ_1, Γ_2 be subgroups of G, with $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$. Assume Γ_1 contains at least three elements and assume Γ_2 contains at least two elements.

Let $\varnothing \neq X_1, X_2 \subseteq X$ with $X_1 \triangle X_2 \neq \varnothing$. Suppose that for all $e_G \neq s \in \Gamma_1$ and for all $e_G \neq t \in \Gamma_2$, we have

$$\begin{aligned} s \cdot X_1 &\subseteq X_2 \\ t \cdot X_2 &\subseteq X_1. \end{aligned}$$

Then, Γ is isomorphic to the free product $\Gamma_1 \star \Gamma_2$.

Narrowing down, we may consider a "doubles" case that splits each of X_1 and X_2 and looks only at two elements of G.

Corollary 0.1.1 (Ping Pong Lemma for "Doubles")**.** Let G act on X, and let A_+, A_-, B_+, B_- be disjoint subsets of X whose union is not equal to X. Then, if

$$\begin{aligned} a \cdot (X \setminus A_-) &\subseteq A_+ \\ a^{-1} \cdot (X \setminus A_+) &\subseteq A_- \\ b \cdot (X \setminus B_-) &\subseteq B_+ \end{aligned}$$

$$b^{-1} \cdot (X \setminus B_+) \subseteq B_-,$$

then it is the case that $\langle a, b \rangle$ is isomorphic to $F(a, b)$.

0.2 Functional Analysis

In Section 3, we will begin discussing an alternative set of characterizations for amenability; in order to do that, we must cover some important concepts in functional analysis. Excellent resources to learn more include [Rud73] and [AB06].

We assume that all vector spaces are over the complex numbers.

First, we begin by discussing some important linear algebra concepts that are more geometric in nature.

Definition 0.2.1. Let X be a vector space.

- If $A, B \subseteq X$, then we define

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

  If $A = \{x_0\}$, we abbreviate $\{x_0\} + B$ as $x_0 + B$, which is called the translation of B by x_0.

- If $A \subseteq X$, and $\alpha \in \mathbb{C}$, then

$$\alpha A = \{\alpha x \mid x \in A\}$$

  is the scaling of A by α. We write $(-1)A = -A$.

- A subset $A \subseteq X$ is called *symmetric* if $-A = A$.
- A subset $A \subseteq X$ is called *balanced* if $\alpha A \subseteq A$ for all $|\alpha| \leqslant 1$.
- A subset $C \subseteq X$ is called *convex* if for all $t \in [0, 1]$ and $x_1, x_2 \in C$, $(1 - t)x_1 + tx_2 \in C$.

We define the *convex hull* of $A \subseteq X$ by

$$\begin{aligned} \operatorname{conv}(A) &:= \bigcap \{C \mid A \subseteq C \subseteq X, C \text{ is convex}\} \\ &= \left\{ \sum_{j=1}^{n} t_j a_j \;\middle|\; n \in \mathbb{N}, t_j \geqslant 0, \sum_{j=1}^{n} t_j = 1, a_j \in A \right\}. \end{aligned}$$

Definition 0.2.2. Let X be a vector space. A *seminorm* on X is a map $p \colon X \times X \to \mathbb{R}$ that satisfies

- $p(x) \geqslant 0$;
- $p(x, y) \leqslant p(x) + p(y)$;
- $p(\alpha x) = |\alpha| p(x)$;

for all $x, y \in X$ and $\alpha \in \mathbb{C}$. If p also satisfies

- $p(x) = 0$ if and only if $x = 0$;

then we say p is a *norm*. We usually write $\|\cdot\|$.

The pair $(X, \|[\|\cdot])$ is known as a normed vector space.

Remark 0.2.1. Naturally, norms induce a metric on the vector space, given by

$$d(x, y) = \|x - y\|.$$

It can be verified that the requirements for a metric are satisfied by this definition.

Example 0.2.1 (Some Normed Vector Spaces)**.**

(a) The space $\mathbb{R}^n$, equipped with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2},$$

is a normed vector space.

(b) The space of continuous functions, $f\colon [0,1] \to \mathbb{C}$, equipped with the norm

$$\|f\|_u = \sup_{x\in[0,1]} |f(x)|,$$

is also a normed vector space, typically denoted $C([0,1])$.

(c) In general, if Ω is any set, then the space $\ell_\infty(\Omega)$ is the space of all functions $f\colon \Omega \to \mathbb{C}$ such that

$$\begin{aligned}\|f\|_{\ell_\infty} &:= \sup_{x\in\Omega}|f(x)|\\ &< \infty.\end{aligned}$$

This is the space of bounded functions with domain Ω.

Definition 0.2.3 (Important Subsets of Normed Vector Spaces)**.** Let X be a normed vector space.

- We define the *open ball* centered at $x \in X$ with radius $\varepsilon > 0$ by

$$U(x,\varepsilon) := \{y \in X \mid \|x - y\| < \varepsilon\}.$$

The open unit ball of X is denoted $U_X := U(0,1)$.

- We define the *closed ball* centered at $x \in X$ with radius $\varepsilon > 0$ by

$$B(x,\varepsilon) := \{y \in X \mid \|x - y\| \leqslant \varepsilon\}.$$

The closed unit ball of X is denoted $B_X := B(0,1)$.

- We define the *sphere* centered at $x \in X$ with radius $\varepsilon > 0$ by

$$S(x,\varepsilon) := \{y \in X \mid \|x - y\| = \varepsilon\}.$$

The unit sphere of X is denoted $S_X := S(0,1)$.

Recall that if X and Y are vector spaces, then $\mathcal{L}(X,Y)$ is the vector space of all linear maps between X and Y when endowed with pointwise addition and scalar multiplication. If $Y = \mathbb{C}$, then $X' := \mathcal{L}(X,\mathbb{C})$ is the space of linear functionals on X.

However, when we deal with normed vector spaces, especially infinite-dimensional ones, we must take care to ensure the continuity of linear maps. We provide a brief overview of continuity in the context of normed vector spaces here, before moving on to one of the most important results related to continuity in normed vector spaces.

Definition 0.2.4. Let X and Y be normed vector spaces, and let $T\colon X \to Y$ be a map.

- The function T is *continuous* if, for all $c \in X$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\|x - c\| < \delta$, then $\|T(x) - T(c)\| < \varepsilon$.

- The function T is *uniformly continuous* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, if $\|x - y\| < \delta$, then $\|T(x) - T(y)\| < \varepsilon$.
- The function T is *Lipschitz continuous* if there exists some constant $C > 0$ such that, for all $x, y \in X$, $\|T(x) - T(y)\| \leqslant C\|x - y\|$.

Theorem 0.2.1. Let X and Y be normed vector spaces, and let $T\colon X \to Y$ be a linear map. Then, the following are equivalent:

- T is continuous at 0;
- T is continuous;
- T is uniformly continuous;
- T is Lipschitz continuous;
- there exists some $C > 0$ such that, for all $x \in X$,
$$\|T(x)\| \leqslant C\|x\|.$$

Definition 0.2.5.

- We say that a linear map $T\colon X \to Y$ is *bounded* if $T(B_X)$ is a bounded set in B_Y.
- The operator norm of T is defined by
$$\|T\|_{\mathrm{op}} := \sup_{x \in B_X} \|T(x)\|.$$
- We define the collection of all continuous (or bounded) linear maps between X and Y by
$$\mathbb{B}(X, Y) := \left\{ T \;\middle|\; T \in \mathcal{L}(X, Y), \|T\|_{\mathrm{op}} < \infty \right\}.$$
- The *continuous dual* of X is the space
$$X^* := \mathbb{B}(X, \mathbb{C}).$$

One of the most useful facts about continuity and uniform continuity is that, when our underlying metric space is complete,[1] then we are able to extend any uniformly continuous function defined on a dense subset to the whole set.

Theorem 0.2.2. Let X and M be complete metric spaces, and suppose $Y \subseteq X$ is a dense subset.

If $T\colon Y \to M$ is a uniformly continuous function, then there exists a unique extension $\widetilde{T}\colon X \to M$ such that $\widetilde{T}|_Y = T$.

The continuous dual, X^*, will feature prominently in our discussion of amenability in Section 3, so we expand upon it a little bit here. Specifically, we discuss some topologies on X^* and some prominent theorems related to the continuous dual.

Definition 0.2.6. Let X be a normed vector space, and let X^* denote the continuous dual. Let $(\varphi_\alpha)_\alpha$ be a net (or generalized sequence) in X^*.

- We say $(\varphi_\alpha)_\alpha \to \varphi$ in the *norm topology* if $\|\varphi_\alpha - \varphi\| \to 0$.
- We say $(\varphi_\alpha)_\alpha \to \varphi$ in the *weak* topology* if, for all $x \in X$, $(\varphi_\alpha)_\alpha \to \varphi(x)$. The weak* topology is the topology of pointwise convergence.

[1] All Cauchy sequences converge in the space.

Remark 0.2.2. Convergence in the norm topology implies convergence in the weak* topology, but not the other way around.

One of the central results relating to the weak* topology is the Banach–Alaoglu theorem, which we will use to prove the existence of particular continuous linear functionals in Section 3.

Theorem 0.2.3 (Banach–Alaoglu)**.** Let X be a normed vector space. Then, B_{X^*} is compact in the weak* topology.

The Banach–Alaoglu theorem provides information about the topological structure of X^*. Now, we turn our attention to understanding the analytic and geometric structure of X^*.

Consider the following problem from linear algebra: if X is a vector space, and $Y \subseteq X$ is a subspace, and $\varphi \in Y'$, is there a linear functional $\phi \in X$ such that $\phi|_Y = \varphi$?

The answer is yes. We may take a basis $\mathcal{B} = \{x_i\}_{i \in I}$ for Y, and extend it to a basis for X, $\mathcal{C}$. We may then define Φ on the basis elements $\left\{x_j\right\}_{j \in J}$ of X by

$$\Phi\left(x_j\right) = \begin{cases} \phi\left(x_j\right) & x_j \in \mathcal{B} \\ 0 & x_j \notin \mathcal{B}. \end{cases}$$

However, when X is a normed vector space, we also end up running into issues of continuity — if $\varphi \in Y^*$ is continuous, how do we know that there exists a continuous $\Phi \in X^*$ such that $\Phi|_Y = \varphi$. For that matter, how do we know that there are any nonzero elements in X^*?

This is the domain of the Hahn–Banach theorems. Both the extension and separation results will be eminently useful as we further study amenability.

Theorem 0.2.4 (Hahn–Banach Continuous Extension)**.** Let X be a normed vector space, and let $Y \subseteq X$ be a subspace. If $\varphi \in Y^*$ is a continuous linear functional, then there is a (not necessarily unique) continuous $\Phi \in X^*$ such that $\Phi|_Y = \varphi$.

One of the primary uses of the Hahn–Banach extension is to establish crucial separation results.

To provide some context for the separation results, consider two open, disjoint, convex subsets $A, B \subseteq \mathbb{R}^n$. The hyperplane separation theorem from convex optimization (see [BV04, Chapter 2.6]) states that there is a nonzero vector $m \in \mathbb{R}^n$ and some $b \in \mathbb{R}$ such that the map $\varphi \colon \mathbb{R}^n \to \mathbb{R}$, defined by $\varphi(x) = m^T x - b$, is strictly negative for all $x \in A$ and is strictly positive for all $x \in B$. The affine hyperplane defined by $\{x \mid \varphi(x) = b\}$ is known as a separating hyperplane for A and B.

A similar concept extends to normed vector spaces, strengthened by continuity.

Theorem 0.2.5 (Hahn–Banach Separation Theorems)**.** Let X be a normed vector space.

- Let $Y \subseteq X$ be a subspace. There is a continuous linear functional $\varphi \in X^*$ such that $\varphi|_Y = 0$ and $\varphi(x) = \mathrm{dist}_Y(x)$.
- If $C, K \subseteq X$ are closed and convex sets, with K compact, then there is a continuous linear functional $\varphi \in X^*$, with $\varphi = u + iv$, with $t \in \mathbb{R}$, and $\delta > 0$, such that

  $$u(x) \leqslant t \leqslant t + \delta \leqslant u(y)$$

  for all $x \in C$ and all $y \in K$.

1 What is Amenability?

The term "amenable" was coined by the mathematician M. M. Day, to refer to groups that John von Neumann termed "meßbar," or measurable. We will elaborate more on the relationship between the structure of groups themselves and group amenability in future sections, but going off definitions alone we may establish certain inheritance properties.

Definition 1.0.1. Let G be a group, and let $P(G)$ be the power set of the group.

An invariant *mean* on G is a set function $m\colon P(G) \to [0,1]$ which satisfies, for all $t \in G$ and $E, F \subseteq G$,

- $m(G) = 1$;
- $m(E \sqcup F) = m(E) + m(F)$;
- $m(tE) = m(E)$.

We say G is *amenable* if G admits a mean.

Proposition 1.0.1. Let G be an amenable group with $H \leqslant G$. Then, the following are true:

(1) H is amenable;

(2) if $H \trianglelefteq G$ is a normal subgroup, then G/H is amenable.

Proof.

(1) Let R be a right transversal for H, wherein we select one element of each right coset of H to make up R.

If m is a mean for G, we set $\lambda\colon P(H) \to [0,1]$ defined by

$$\lambda(E) = m(ER).$$

We have

$$\begin{aligned}\lambda(H) &= m(HR)\\ &= m(G)\\ &= 1.\end{aligned}$$

We claim that if $E \cap F = \emptyset$, then $ER \cap FR = \emptyset$. Suppose toward contradiction this is not the case. Then, $xr_1 = yr_2$ for some $x \in E$, $y \in F$, and $r_1, r_2 \in R$. Then, we must have $r_2 r_1^{-1} = y^{-1}x \in H$, meaning $r_1 = r_2$ as, by definition, R contains exactly one element of each right coset. Thus, $x = y$, so $E \cap F \neq \emptyset$.

We then have

$$\begin{aligned}\lambda(E \sqcup F) &= m((E \sqcup F)R)\\ &= m(ER \sqcup FR)\\ &= m(ER) + m(FR)\\ &= \lambda(E) + \lambda(F),\end{aligned}$$

and

$$\begin{aligned}\lambda(sE) &= m(sER)\\ &= m(ER)\\ &= \lambda(E).\end{aligned}$$

(2) Let $\pi\colon G \to G/H$ be the canonical projection, defined by $\pi(t) = tH$. We define

$$\lambda\colon P(G/H) \to [0,1]$$

by $\lambda(E) = m\big(\pi^{-1}(E)\big)$. We have

$$\begin{aligned}\lambda(G/H) &= m\Big(\pi^{-1}(G/H)\Big)\\ &= m(G)\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(E \sqcup F) &= m\Big(\pi^{-1}(E \sqcup F)\Big)\\ &= m\Big(\pi^{-1}(E) \sqcup \pi^{-1}(F)\Big)\\ &= m\Big(\pi^{-1}(E)\Big) + m\Big(\pi^{-1}(F)\Big)\\ &= \lambda(E) + \lambda(F).\end{aligned}$$

To show translation-invariance, we let $sH = \pi(s) \in G/H$, and $E \subseteq G/H$. Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for $r \in s\pi^{-1}(E)$, we have $r = st$ for $t \in \pi(E)$, so $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$.

Additionally, if $r \in \pi^{-1}(\pi(s)E)$, we have $\pi(r) \in \pi(s)E$, so $\pi\big(s^{-1}r\big) \in E$, meaning $s^{-1}r \in \pi^{-1}(E)$.

Thus,

$$\begin{aligned}\lambda(\pi(s)E) &= m\Big(\pi^{-1}(\pi(s)E)\Big)\\ &= m\Big(s\pi^{-1}(E)\Big)\\ &= m\Big(\pi^{-1}(E)\Big)\\ &= \lambda(E).\end{aligned}$$

□

The following proposition is, in a sense, a kind of converse to Proposition 1.0.1, in that if a subgroup is amenable, we can show that its parent group is also amenable, but this is only a sufficient condition if the subgroup has finite index.

Proposition 1.0.2. Let G be a group, and let $H \leqslant G$ be amenable, with $[G : H] = n < \infty$. Then, G is amenable.

Proof. Let $H \leqslant G$ be amenable with $[G : H] = n$. Let μ be the mean on H, and let $\{g_i H\}_{i=1}^{n}$ be a partition of G by the left cosets of H. We define the mean on G by taking, for $A \subseteq G$,

$$\lambda(A) = \frac{1}{n}\sum_{i=1}^{n} \mu\Big(g_i^{-1}A \cap H\Big).$$

We begin by verifying that this is well-defined. Specifically, we will show that this definition is independent of the coset representatives. Suppose $g_j H = h_j H$. Then, $h_j^{-1} g_j \in H$. Now, we have $g_j^{-1}A \cap H \subseteq H$, so by left-multiplication, we get $\left(h_j^{-1}g_j\right)g_j^{-1}A \cap H \subseteq H$, so $h_j^{-1}A \cap H \subseteq H$. Since $\{g_iH\}_{i=1}^n$ is a partition, we get that this definition of the mean on G is independent of the choice of coset representatives.

Next, we show that this is a finitely additive measure. Let $A, B \subseteq G$ be such that $A \cap B = \varnothing$. Then, we get

$$\begin{aligned}
\lambda(A \sqcup B) &= \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}(A \sqcup B) \cap H\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu\left(\left(g_i^{-1}A \cap H\right) \sqcup \left(g_i^{-1}B \cap H\right)\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} \mu\left(g_i^{-1}A \cap H\right) + \sum_{i=1}^{n} \mu\left(g_i^{-1}B \cap H\right)\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}A \cap H\right) + \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}B \cap H\right) \\
&= \lambda(A) + \lambda(B).
\end{aligned}$$

It is relatively simple to see that λ is a probability measure, as

$$\begin{aligned}
\lambda(G) &= \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}G \cap H\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu(G \cap H) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu(H) \\
&= 1.
\end{aligned}$$

Now, we must show that λ is translation-invariant.

Let $A \subseteq G$ and $t \in G$. Using the translation-invariance of μ, we get

$$\begin{aligned}
\lambda(tA) &= \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}tA \cap H\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}(t(A \cap H))\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu\left(g_i^{-1}A \cap H\right) \\
&= \lambda(A).
\end{aligned}$$

Thus, G is amenable. □

2 Paradoxical Decompositions and Amenability

Having established the inheritance properties of amenable groups, we will begin discussing the Banach–Tarski paradox, leading into a result known as Tarski's Theorem.

We will show a couple major results in this section:

- the weak Banach–Tarski paradox;
- the strong Banach–Tarski paradox;
- Tarski's theorem.

Each of these results will require a bit of machinery before we may prove them, but they are truly foundational in the history of amenability.

The exposition in this section will largely follow that of [Run20] and [Run02], with some clarity added to the more terse sections of those books.

2.1 The Banach–Tarski Paradox

In the Bible, one of the miracles of Jesus is known as the feeding of the five thousand.[II] Jesus is able to feed a large crowd that had only five loaves of bread and two fishes among themselves by praying to God, then breaking the food apart and passing it around the crowd. Unfortunately, such a miracle is not able to be performed in the physical world without some divine intervention, but mathematically, it is not only possible to recreate such a feat, but moreover, it is a fundamental feature of the group of Euclidean isometries of $\mathbb{R}^3$.

This is the substance of the most general form of the Banach–Tarski paradox.

> **Proposition 2.1.1** (Strong Banach–Tarski Paradox)**.** Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

The Banach–Tarski paradox throws a wrench into a common belief that we have about $\mathbb{R}^3$ — specifically, that every subset of $\mathbb{R}^3$ has a *finitely additive* "volume" that is invariant under rigid motion.[III] Soon, we will see that such a well-behaved measure *does* exist in $\mathbb{R}$ and $\mathbb{R}^2$ (though we will only use a nonconstructive proof for this purpose).

2.1.1 Paradoxical Decompositions

The paradox of the ship of Theseus asks, if on a voyage across the ocean, the crew of a ship repair the ship so that all the parts of the ship are replaced, which part is the real ship? Moreover, if the original parts of the ship are reconstituted into a ship, which one is the "original" ship, and which one is the "copy" ship?

Thankfully, in mathematics, we do not have to worry about the ramifications of such questions, but the idea of paradoxical actions, and paradoxical groups, borrows from the idea that it is possible, in some circumstances, to reconstitute the whole from only a subset of itself, through the miracle of group actions.

> **Definition 2.1.1** (Paradoxical Decompositions and Paradoxical Groups)**.** Let G be a group that acts on a set X, with $E \subseteq X$. We say E is G-*paradoxical* if there exist pairwise disjoint proper subsets $A_1, \dots, A_n$ and $B_1, \dots, B_m$ of E and group elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that
>
> $$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

[II] Fun fact: aside from the resurrection, the feeding of the five thousand is the only miracle of Jesus that is documented in all four gospels.

[III] Note that if we desire countable additivity, the axiom of choice shows that there does not exist a countably additive measure on $P(\mathbb{R})$ that is also translation-invariant (see [Fol84, Section 1.1]). Finite additivity is a weaker condition than countable additivity that allows for the existence of well-behaved measures on $P(\mathbb{R})$ and $P(\mathbb{R}^2)$, but even this fails in $\mathbb{R}^3$ and above.

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

If G acts on itself by left-multiplication, and G satisfies these conditions, we say G is a *paradoxical group*.

Example 2.1.1. The free group on two generators, $F(a, b)$, is a paradoxical group.

To see that $F(a, b)$ is a paradoxical group, we let $W(x)$ denote the set of words in $F(a, b)$ that start with $x \in \left\{a, b, a^{-1}, b^{-1}\right\}$. For instance, $ba^2ba^{-1} \in W(b)$.

Since every word in F is either the empty word, or starts with one of a, b, a^{-1}, b^{-1}, we see that

$$F(a, b) = \left\{e_{F(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

If $w \in F(a, b) \setminus W(a)$, we see that $a^{-1}w \in W(a^{-1})$. Thus, $w \in aW(a^{-1})$. For any $t \in F(a, b)$ either $t \in W(a)$ or $t \in F(a, b) \setminus W(a) = aW(a^{-1})$. Thus, $F(a, b)$ is equal to $W(a) \sqcup aW(a^{-1})$.

Similarly, if $t \in F(a, b)$ then either $t \in W(b)$ or $t \in bW(b^{-1})$, so $F(a, b) = W(b) \sqcup bW(b^{-1})$.

We have thus constructed

$$\begin{aligned} F(a, b) &= W(a) \sqcup aW\left(a^{-1}\right) \\ &= W(b) \sqcup bW\left(b^{-1}\right), \end{aligned}$$

a paradoxical decomposition of $F(a, b)$ with the action of left-multiplication.

Now that we understand a little more about paradoxical groups, we now want to understand the actions of paradoxical groups on sets. Recall that if G is a group that acts on a set X, we say the action is free if, for all $x \in X$, $g \cdot x = x$ if and only if $g = e_G$.

Proposition 2.1.2. Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.

Proof. Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ be pairwise disjoint, and let $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{i=1}^{n} g_i A_i \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

Let $M \subseteq X$ contain exactly one element from every orbit in X.

Claim. The set $\{g \cdot M \mid g \in G\}$ is a partition of X.

Proof of Claim: Since M contains exactly one element from every orbit in X, it is the case that $G \cdot M = X$, so

$$\bigcup_{g \in G} g \cdot M = X$$

Additionally, for $x, y \in M$, if $g \cdot x = h \cdot y$, then $\left(h^{-1}g\right) \cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Since G acts freely on X, we must have $h^{-1}g = e_G$.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$, implying $\{g \cdot M \mid g \in G\}$ is a partition of X. □

For any given i, we define

$$A_i^* = \bigcup_{g \in A_i} g \cdot M,$$

and similarly define, for any given j,

$$B_j^* = \bigcup_{h \in B_j} h \cdot M.$$

As a useful shorthand, we can also write $A_i^* = A_i \cdot M$, and similarly, $B_j^* = B_j \cdot M$, to denote the union of the elements of A_i and B_j respectively acting on M.

Since $\{g \cdot M \mid g \in G\}$ is a partition of X, and $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ are pairwise disjoint, it must be the case that $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^* \subset X$ are also pairwise disjoint.

For the original $g_1, \ldots, g_n, h_1, \ldots, h_m$ that defined the paradoxical decomposition of G, we thus have

$$\begin{aligned} \bigcup_{i=1}^{n} g_i \cdot A_i^* &= \bigcup_{i=1}^{n} (g_i A_i) \cdot M \\ &= G \cdot M \\ &= X, \end{aligned}$$

and

$$\begin{aligned} \bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} (h_j B_j) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Thus, X is G-paradoxical. □

Remark 2.1.1. This proof requires the axiom of choice, as we invoked it to define M to contain exactly one element from every orbit in X.

There is also a useful converse.

Proposition 2.1.3. Let G be a group that acts on a set X, and suppose that there is a free action of G on X such that X is G-paradoxical. Then, G is paradoxical with respect to the action of left-multiplication on itself.

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m \subseteq X$ be disjoint subsets of X and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} X &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

Fix a value $x_0 \in X$. Now, we define

$$\begin{aligned} A_i^* &= \{g \in G \mid g \cdot x_0 \in A_i\} \\ B_j^* &= \left\{g \in G \,\middle|\, g \cdot x_0 \in B_j\right\}. \end{aligned}$$

Notice that since the A_i and B_j are pairwise disjoint, we must also have the A_i^* and B_j^* are disjoint. Now, we consider the orbit of x_0, $G \cdot x_0$. Notice that

$$\begin{aligned} G \cdot x_0 &= X \cap G \cdot x_0 \\ &= \left(\bigcup_{i=1}^{n} g_i \cdot A_i\right) \cap G \cdot x_0 \\ &= \bigcup_{i=1}^{n} (g_i \cdot A_i \cap G \cdot x_0), \end{aligned}$$

and similarly,

$$G \cdot x_0 = \bigcup_{j=1}^{m} \left(h_j \cdot B_j \cap G \cdot x_0\right).$$

Now, this means that for any $g \in G$, we know that $g \cdot x_0 \in g_i \cdot A_i$ for some i, so $g \cdot x_0 = g_i \cdot a$ for some $a \in A_i$. This gives $\left(g_i^{-1} g\right) \cdot x_0 = a$, so $\left(g_i^{-1} g\right) \cdot x_0 \in A_i$, meaning $g_i^{-1} g \in A_i^*$. Therefore, we have $g \in g_i A_i^*$. Since g was arbitrary, we have

$$G = \bigcup_{i=1}^{n} g_i A_i^*.$$

By a similar process, we arrive at

$$G = \bigcup_{j=1}^{m} h_j B_j^*,$$

so G is a paradoxical group. □

2.1.2 The Weak Banach–Tarski Paradox

Now that we have established $F(a, b)$ as being a paradoxical group, we wish to use it to construct paradoxical decompositions of the unit sphere $S^2 \subseteq \mathbb{R}^3$. Specifically, we will show a weak version of the Banach–Tarski paradox — one where you can break apart the unit ball into finitely many pieces and reconstitute it into two copies of itself.

Fact 2.1.1. If H is a paradoxical group, and $H \leqslant G$, then G is a paradoxical group.

With this fact in mind, we will show that $SO(3)$ is a paradoxical group.

Theorem 2.1.1. There are rotations A and B that about lines through the origin in $\mathbb{R}^3$ that generate a subgroup of SO(3) isomorphic to $F(a, b)$

Proof. We let

$$a = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$a^{-1} = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}$$
$$b^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}.$$

We specify

$$X = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$A_+ = \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, x \equiv 3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\}$$
$$A_- = \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, x \equiv -3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\}$$
$$B_+ = \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, z \equiv 3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}$$
$$B_- = \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, z \equiv -3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}.$$

To verify that the conditions of Theorem 0.1.3 hold, we calculate

$$\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 3x + 4y \\ -4x + 3y \\ 5z \end{pmatrix} \tag{1}$$
$$\begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 3x - 4y \\ 4x + 3y \\ 5z \end{pmatrix} \tag{2}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 5x \\ 3y - 4z \\ 4y + 3z \end{pmatrix} \tag{3}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 5x \\ 3y + 4z \\ -4y + 3z \end{pmatrix}. \tag{4}$$

We verify that the conditions for Corollary 0.1.1 hold for each of these four conditions.

(1) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_-,$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x + 4y \equiv 3(-4x + 3y)$ modulo 5, and that $z' = 5z \equiv 0$ modulo 5.

(2) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_+,$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x - 4y \equiv -3(4x + 3y)$ modulo 5, and $z' = 5z \equiv 0$ modulo 5.

(3) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_-,$$

we see that $k+1 \in \mathbb{Z}$, $z' = 4y + 3z \equiv 3(3y - 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

(4) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_+,$$

we see that $k+1 \in \mathbb{Z}$, $z' = -4y + 3z \equiv -3(3y + 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

Thus, by Theorem 0.1.3 and Corollary 0.1.1, it is the case that $\langle a, b\rangle \cong F(a, b)$. □

Remark 2.1.2. Since SO(n) contains a subgroup isomorphic to SO(3) for all $n \geqslant 3$ (via block matrices), it is the case that SO(n) also contains a subgroup isomorphic to $F(a, b)$ for all $n \geqslant 3$.

Since we have shown that SO(3) is paradoxical, as it contains a paradoxical subgroup, we can now begin to examine the action of SO(3) on subsets of $\mathbb{R}^3$.

Theorem 2.1.2 (Hausdorff Paradox)**.** There is a countable subset D of S^2 such that $S^2 \setminus D$ is SO(3)-paradoxical.

Proof. Let A and B be the rotations in SO(3) that serve as the generators of the subgroup isomorphic to $F(a, b)$.

Since A and B are rotations, so too is any element of $\langle A, B\rangle$. Thus, any such non-empty word contains two fixed points.

We let

$$F = \left\{x \in S^2 \mid x \text{ is a fixed point for some word } w\right\}.$$

Since $\langle A, B\rangle$ is countably infinite, so too is F. Thus, the union of all these fixed points under the action of all such words w is countable.

$$D = \bigcup_{w \in \langle A,B \rangle} w \cdot F.$$

Therefore, $\langle A, B \rangle$ acts freely on $S^2 \setminus D$, so $S^2 \setminus D$ is SO(3)-paradoxical. □

Unfortunately, the Hausdorff paradox is not enough for us to be able to prove the Banach–Tarski paradox. In order to do this, we need to be able to show that two sets are "similar" under the action of a group.

Definition 2.1.2 (Equidecomposable Sets). Let G act on X, and let $A, B \subseteq X$. We say A and B are G-*equidecomposable* if there are partitions $\left\{A_j\right\}_{j=1}^{n}$ of A and $\left\{B_j\right\}_{j=1}^{n}$ of B, and elements $g_1, \ldots, g_n \in G$, such that for all j,

$$B_j = g_j \cdot A_j.$$

We write $A \sim_G B$ if A and B are G-equidecomposable.

Fact 2.1.2. The relation $\sim_G$ is an equivalence relation.

Proof. Let A, B, and C be sets.

To show reflexivity, we can select $g_1 = g_2 = \cdots = g_n = e_G$. Thus, $A \sim_G A$.

To show symmetry, let $A \sim_G B$. Set $\left\{A_j\right\}_{j=1}^{n}$ to be the partition of A, and set $\left\{B_j\right\}_{j=1}^{n}$ to be the partition of B, such that there exist $g_1, \ldots, g_n \in G$ with $g_j \cdot A_j = B_j$. Then,

$$\begin{aligned} g_j^{-1} \cdot \left(g_j \cdot A_j\right) &= g_j^{-1} \cdot B_j \\ A_j &= g_j^{-1} \cdot B_j, \end{aligned}$$

so $B_j \sim_G A_j$.

To show transitivity, let $A \sim_G B$ and $B \sim_G C$. Let $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ be the partitions of A and B respectively and $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$. Let $\left\{B_j\right\}_{j=1}^{m}$ and $\left\{C_j\right\}_{j=1}^{m}$ be partitions of B and C, and $h_1, \ldots, h_m \in G$, such that $h_j \cdot B_j = C_j$.

We refine the partition of A to A_{ij} by taking $A_{ij} = g_i^{-1}\left(B_i \cap B_j\right)$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then, $\left(h_j g_i\right) \cdot A_{ij}$ maps the refined partition of A to C, so A and C are G-equidecomposable. □

Fact 2.1.3. For $A \sim_G B$, there is a bijection $\phi \colon A \to B$ by taking $C_i = C \cap A_i$, and mapping $\phi(C_i) = g_i \cdot C_i$.

In particular, this means that for any subset $C \subseteq A$, it is the case that $C \sim \phi(C)$.

We can now use this equidecomposability to glean information about the existence of paradoxical decompositions.

Proposition 2.1.4. Let G act on X, with $E, E' \subseteq X$ such that $E \sim_G E'$. Then, if E is G-paradoxical, then so too is E'.

Proof. Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$ be pairwise disjoint, with $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$\begin{aligned} E &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

We let

$$A = \bigsqcup_{i=1}^{n} A_i$$
$$B = \bigsqcup_{j=1}^{m} B_j.$$

It follows that $A \sim_G E$ and $B \sim_G E$, since we can take the partition of A to be $A_1, \ldots, A_n$, and partition E by taking $g_i \cdot A_i$ for $i = 1, \ldots, n$, and similarly for B.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$. Thus, there is a paradoxical decomposition of E' in $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. □

We will now show that S^2 is SO(3) paradoxical.

Proposition 2.1.5. Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable.

Proof. Let L be a line in $\mathbb{R}^3$ such that $L \cap D = \emptyset$. Such an L must exist since S^2 is uncountable.

Define $\rho_\theta \in SO(3)$ to be a rotation about L by an angle of θ. For a fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{x \in D \mid \rho_\theta^n \cdot x \in D\right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

We define $W_n = \{\theta \mid R_{n,\theta} \neq \emptyset\}$. Since the map $\theta \mapsto \rho_\theta^n \cdot x$ into D is injective, it is the case that W_n is countable. Therefore,

$$W = \bigcup_{n \in \mathbb{N}} W_n$$

is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. We define ρ_ω to be a rotation about L by ω. Then, for every $n, m \in \mathbb{N}$, we have

$$\rho_\omega^n \cdot D \cap \rho_\omega^m \cdot D = \emptyset.$$

We define $\widetilde{D} = \bigsqcup_{n=0}^{\infty} \rho_\omega^n D$. Note that

$$\begin{aligned}
\rho_\omega \cdot \widetilde{D} &= \rho_\omega \cdot \bigsqcup_{n=0}^{\infty} \rho_\omega^n \cdot D \\
&= \bigsqcup_{n=1}^{\infty} \rho_\omega^n \cdot D \\
&= \widetilde{D} \setminus D,
\end{aligned}$$

meaning $\widetilde{D}$ and D are SO(3)-equidecomposable.

Thus, we have

$$\begin{aligned}
S^2 &= \widetilde{D} \sqcup \left(S^2 \setminus \widetilde{D}\right) \\
&\sim_{SO(3)} \left(\rho_\omega \cdot \widetilde{D}\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\
&= \left(\widetilde{D} \setminus D\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\
&= S^2 \setminus D,
\end{aligned}$$

establishing S^2 and $S^2 \setminus D$ as SO(3)-equidecomposable.

In particular, this means S^2 is also SO(3)-paradoxical. □

To prove the weak Banach–Tarski paradox, we need a slightly larger group than SO(3) — one that includes translations in addition to the traditional rotations.

Definition 2.1.3 (Euclidean Group)**.** The *Euclidean group*, $E(n)$, consists of all isometries of a Euclidean space. An isometry of a Euclidean space consists of translations, rotations, and reflections.

Corollary 2.1.1 (Weak Banach–Tarski Paradox)**.** Every closed ball in $\mathbb{R}^3$ is E(3)-paradoxical.

Proof. We only need to show that $B(0,1)$ is E(3)-paradoxical. To do this, we start by showing that $B(0,1) \setminus \{0\}$ is SO(3)-paradoxical.

Since S^2 is SO(3)-paradoxical, there exists pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subset S^2$ and elements $g_1, \dots, g_n, h_1, \dots, h_m \in SO(3)$ such that

$$\begin{aligned} S^2 &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

Define

$$\begin{aligned} A_i^* &= \{tx \mid t \in (0,1], x \in A_i\} \\ B_j^* &= \left\{ty \mid t \in (0,1], y \in B_j\right\}. \end{aligned}$$

Then, $A_1^*, \dots, A_n^*, B_1^*, \dots, B_m^* \subset B(0,1) \setminus \{0\}$ are pairwise disjoint, and

$$\begin{aligned} B(0,1) \setminus \{0\} &= \bigcup_{i=1}^{n} g_i \cdot A_i^* \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j^*. \end{aligned}$$

Thus, we have established that $B(0,1) \setminus \{0\}$ is E(3)-paradoxical.

Now, we want to show that $B(0,1) \setminus \{0\}$ and $B(0,1)$ are E(3)-equidecomposable. Let $x \in B(0,1) \setminus \{0\}$, and let ρ be a rotation through x by a line not through the origin such that $\rho^n \cdot 0 \neq \rho^m \cdot 0$ when $n \neq m$.

Let $D = \{\rho^n \cdot 0 \mid n \in \mathbb{N}\}$. We can see that $\rho \cdot D = D \setminus \{0\}$, and that D and $\rho \cdot D$ are E(3)-equidecomposable. Thus,

$$\begin{aligned} B(0,1) &= D \sqcup (B(0,1) \setminus D) \\ &\sim_{E(3)} (\rho \cdot D) \sqcup (B(0,1) \setminus D) \\ &= (D \setminus \{0\}) \sqcup (B(0,1) \setminus D) \\ &= B(0,1) \setminus \{0\}. \end{aligned}$$

Therefore, $B(0,1)$ is E(3)-paradoxical. □

2.1.3 The Strong Banach–Tarski Paradox

In order to prove the general case of the Banach–Tarski paradox, we need one more piece of mathematical machinery.

In Fact 2.1.2, we showed that the relation $A \sim_G B$ is an equivalence relation. Using the power of subsets,[IV] we may extend this to a preorder on any subsets A and B of X.

Definition 2.1.4. Let G act on a set X with $A, B \subseteq X$. We write $A \preceq_G B$ if A is equidecomposable with a subset of B.

Fact 2.1.4. The relation $\preceq_G$ is a reflexive and transitive relation.

Proof. To see reflexivity, we can see that since $A \sim_G A$, and $A \subseteq A$, $A \preceq_G A$.

To see transitivity, let $A \preceq_G B$ and $B \preceq_G C$. Then, there exist $g_1, \dots, g_n \in G$ such that $g_i \cdot A_i = B_{\alpha,i}$ for each i, where $A \sim_G B_\alpha \subseteq B$. Similarly, there exist $h_1, \dots, h_m \in G$ such that $h_j \cdot B_j = C_{\beta,j}$ for each j, where $B \sim_G C_\beta \subseteq C$.

We take a refinement of B by taking intersections $B_{\alpha,ij} = B_{\alpha,i} \cap B_j$, with $i = 1, \dots, n$ and $j = 1, \dots, m$. We define $C_{\beta,\alpha,ij} = h_j \cdot B_{\alpha,ij}$ for each $j = 1, \dots, m$. Then, $h_j g_i \cdot A_i = C_{\beta,\alpha,ij}$, meaning $A \sim_G C_{\beta,\alpha,ij} \subseteq C_\beta \subseteq C$, so $A \preceq_G C$. $\square$

We know from Fact 2.1.3 that $A \preceq_G B$ implies the existence of a bijection $\phi\colon A \to B' \subseteq B$, meaning $\phi\colon A \hookrightarrow B$ is an injection. Similarly, if $B \preceq_G A$, then Fact 2.1.3 implies the existence of an injection $\psi\colon B \hookrightarrow A$.

One may ask if an analogue of the Cantor–Schröder–Bernstein theorem exists in the case of the relation $\preceq_G$, implying that the preorder established in Fact 2.1.4 is indeed a partial order. The following theorem establishes this result.

Theorem 2.1.3. Let G act on X, and let $A, B \subseteq X$. If $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$.

Proof. Let $B' \subseteq B$ with $A \sim_G B'$, and let $A' \subseteq A$ with $B \sim_G A'$. Then, we know from Fact 2.1.3 that there exist bijections $\phi\colon A \to B'$ and $\psi\colon B \to A'$.

Define $C_0 = A \setminus A'$, and $C_{n+1} = \psi(\phi(C_n))$. We set

$$C = \bigcup_{n \geqslant 0} C_n.$$

Since $\psi^{-1}(\psi(\phi(C_n))) = \phi(C_n)$, we have

$$\psi^{-1}(A \setminus C) = B \setminus \phi(C).$$

Having established in Fact 2.1.3 that for any subset of $C \subseteq A$, $C \sim_G \phi(C)$, we also see that $A \setminus C \sim_G B \setminus \phi(C)$.

Thus, we can see that

$$\begin{aligned} A &= (A \setminus C) \sqcup C \\ &\sim_G (B \setminus \phi(C)) \sqcup \phi(C) \\ &= B. \end{aligned}$$

$\square$

Finally, we are able to prove Proposition 2.1.1. We restate the proposition here, followed by its proof.

[IV] But not the power set (or at least, not directly).

Proposition 2.1.1 (Strong Banach–Tarski Paradox)**.** Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Proof of Proposition 2.1.1: By symmetry, it is enough to show that $A \preceq_{E(3)} B$.

Since A is bounded, there exists $r > 0$ such that $A \subseteq B(0, r)$.

Let $x_0 \in B^\circ$. Then, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq B$.

Since $B(0, r)$ is compact (hence totally bounded), there are translations $g_1, \dots, g_n$ such that

$$B(0, r) \subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup g_n \cdot B(x_0, \varepsilon).$$

We select translations $h_1, \dots, h_n$ such that $h_j \cdot B(x_0, \varepsilon) \cap h_k \cdot B(x_0, \varepsilon) = \varnothing$ for $j \neq k$. We set

$$S = \bigcup_{j=1}^{n} h_j \cdot B(x_0, \varepsilon).$$

Each $h_j \cdot B(x_0, \varepsilon) \subseteq S$ is E(3)-equidecomposable with any arbitrary closed ball subset of $B(x_0, \varepsilon)$, it is the case that $S \preceq B(x_0, \varepsilon)$.

Thus, we have

$$\begin{aligned}
A &\subseteq B(0, r) \\
&\subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup b_n \cdot B(x_0, \varepsilon) \\
&\preceq S \\
&\preceq B(x_0, \varepsilon) \\
&\preceq B.
\end{aligned}$$

□

2.2 Tarski's Theorem

Ultimately, the reason the Banach–Tarski paradox "works" is because the paradoxical group $F(a, b)$, lacks a property known as amenability — specifically, that a group admitting a paradoxical decomposition is not amenable. Before we go further into the characterizations of amenability, we will show that this statement reverses.

Indeed, every amenable group is *non*-paradoxical.

Theorem 2.2.1 (Tarski's Theorem)**.** Let G be a group that acts on a set X, and let $E \subseteq X$ be nonempty.

There is a finitely additive measure $\mu\colon P(X) \to [0, \infty]$ with $\mu(E) \in (0, \infty)$ and $\mu(t \cdot E) = \mu(E)$ for all $t \in G$ if and only if E is not G-paradoxical.

We can prove one of the directions of Tarski's theorem now.

Proof of the Forward Direction of Theorem 2.2.1: Let E be G-paradoxical. Suppose toward contradiction that such a translation-invariant finitely additive ν existed with $\nu(E) \in (0, \infty)$.

Let $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ be pairwise disjoint, and let $t_1, \dots, t_n, s_1, \dots, f_m \in G$ such that

$$E = \bigsqcup_{i=1}^{n} t_i \cdot A_i$$

$$= \bigsqcup_{j=1}^{m} s_j \cdot B_j.$$

Then, it would be the case that

$$\begin{aligned}
\nu(E) &= \nu\left(\bigsqcup_{i=1}^{n} t_i \cdot A_i\right) \\
&= \sum_{i=1}^{n} \nu(t_i \cdot A_i) \\
&= \sum_{i=1}^{n} \nu(A_i),
\end{aligned}$$

and

$$\nu(E) = \sum_{j=1}^{m} \nu(B_j).$$

However, this also yields

$$\begin{aligned}
\nu(E) &= \nu\left(\left(\bigsqcup_{i=1}^{n} A_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} B_j\right)\right) \\
&= \sum_{i=1}^{n} \nu(A_i) + \sum_{j=1}^{m} \nu(B_j) \\
&= \sum_{i=1}^{n} \nu(t_i \cdot A_i) + \sum_{j=1}^{m} \nu(x_j \cdot B_j) \\
&= \nu(E) + \nu(E) \\
&= 2\nu(E).
\end{aligned}$$

implying that $\nu(E) = 0$ or $\nu(E) = \infty$. □

The opposite direction, unfortunately, will be significantly harder to prove. We will need to know some results from graph theory, understand the properties of the type semigroup of an action, and use some results on commutative semigroups to show the existence of a mean.

2.2.1 A Little Bit of Graph Theory

To prove the reverse direction of Tarski's theorem, we need to develop some machinery from graph theory that will allow us to prove that a certain semigroup we will construct in the next section satisfies the cancellation identity.

We start by defining graphs and paths, before proving a special case of Hall's theorem, ultimately extending to the infinite case with König's theorem.

Definition 2.2.1 (Graphs and Paths)**.** A *graph* is a triple (V, E, ϕ), with V, E nonempty sets and $\phi\colon E \to P_2(V)$ a map from E to the set of all unordered subset pairs of V.

For $e \in E$, if $\phi(e) = \{v, w\}$, then we say v and w are the *endpoints* of e, and e is *incident* on v and w.

A *path* in (V, E, ϕ) is a finite sequence $(e_1, \ldots, e_n)$ of edges, with a finite sequence of vertices $(v_0, \ldots, v_n)$, such that $\phi(e_k) = \{v_{k-1}, v_k\}$.

The *degree* of a vertex, $\deg(v)$, is the number of edges incident on v.

We define the *neighbors* of $S \subseteq V$ to be the set of all vertices $v \in V \setminus S$ such that v is an endpoint to an edge incident on S. We denote this set $N(S)$.

Definition 2.2.2 (Bipartite Graphs and k-Regularity)**.** Let (V, E, ϕ) be a graph, with $k \in \mathbb{N}$.

(i) If $\deg(v) = k$ for each $v \in V$, we say (V, E, ϕ) is k-*regular*.

(ii) If $V = X \sqcup Y$, with each edge in E having one endpoint in X and one endpoint in Y, then we say V is *bipartite*, and write (X, Y, E, ϕ).

Definition 2.2.3 (Perfect Matching)**.** Let (X, Y, E, ϕ) be a bipartite graph. Let $A \subseteq X$ and $B \subseteq Y$. A *perfect matching* of A and B is a subset $F \subseteq E$ with

(i) each element of $A \cup B$ is an endpoint of exactly one $f \in F$;

(ii) all endpoints of edges in F are in $A \cup B$.

Definition 2.2.4 (Hall Condition)**.** We say a bipartite graph (X, Y, E, ϕ) satisfies the *Hall condition* on X if, for all $S \subseteq X$, $|N(S)| \geqslant |S|$.

Equivalently, we say a (finite) collection of not necessarily distinct finite sets $\mathcal{X} = \{X_i\}_{i=1}^n$ satisfies the Hall condition if and only if for all subcollections $\mathcal{Y}_k = \{X_{i_k}\}_{k=1}^m$,

$$|\mathcal{Y}_k| \leqslant \left| \bigcup_{k=1}^{m} X_{i_k} \right|.$$

Remark 2.2.1. These two formulations of the Hall condition are equivalent regarding an X-perfect matching.

Theorem 2.2.2 (Hall's Theorem for Finite k-Regular Bipartite Graphs)**.** Let (X, Y, E, ϕ) be a k-regular bipartite graph for some $k \in \mathbb{N}$, and let $V = X \sqcup E$ be finite. Then, there is a perfect matching of X and Y.

Proof. Note that since $|E| = k|K| = k|Y|$, it is the case that $|X| = |Y|$.

Let $M \subseteq V$ be any subset. We will show that $|N(M)| \geqslant |M|$ — that is, (X, Y, E, ϕ) satisfies the Hall condition.

Let $M_X = M \cap X$ and $M_Y = M \cap Y$, where $M = M_X \sqcup M_Y$. Let $[M_X, N(M_X)]$ be the set of edges with endpoints in M_X and $N(M_X)$, and $[M_Y, N(M_Y)]$ be the set of edges with endpoints in M_Y and $N(M_Y)$. We also let $[X, N(M_X)]$ denote the set of edges with endpoints in X and $N(M_X)$, and similarly, $[Y, N(M_Y)]$ is the set of edges with endpoints in Y and $N(M_Y)$.

We can see that $[M_X, N(M_X)] \subseteq [X, N(M_X)]$, and similarly, $[M_Y, N(M_Y)] \subseteq [Y, N(M_Y)]$.

Since $|[M_X, N(M_X)]| = k|M_X|$ and $|[X, N(M_X)]| = k|N(M_X)|$, we have

$$|M_X| \leqslant |N(M_X)|,$$

and similarly,

$$|M_Y| \leqslant |N(M_Y)|.$$

Thus, $|M| \leqslant |N(M)|$.

We will now show that there is an X-perfect matching. Suppose toward contradiction that F is a maximal perfect matching on $A \subseteq X$ and $B \subseteq Y$ with $X \setminus A \neq \emptyset$.

Then, there is $x \in X \setminus A$. Consider $Z \subseteq V$ consisting of all vertices z such that there exists a F-alternating path $(e_1, \ldots, e_n)$ between $z \in Z$ and x.

It cannot be the case that $Z \cap Y$ is empty, since the number of neighbors of x is greater than or equal to 1 by the Hall condition — if it were the case that $Z \cap Y$ were empty, we could add an edge to F consisting of x and one element of $N(\{x\})$, which would contradict the maximality of F.

Consider a path traversing along Z, $(e_1, \ldots, e_n)$. It must be the case that $e_n \in F$, or else we would be able to "flip" the matching F by exchanging e_i with e_{i+1} for $e_i \in F$, which would contradict the maximality of F yet again. Thus, every element of $Z \cap Y$ is satisfied by F, so $Z \cap Y \subseteq B$.

Since each element in $Z \cap Y$ is paired with exactly one element of $Z \cap X$ (with one left over), it is the case that $|Z \cap X| = |Z \cap Y| + 1$.

Suppose toward contradiction that there exists $y \in N(Z \cap X)$ with $y \notin Z \cap Y$. Then, there exists $v \in Z \cap X$ and $e \in E$ such that $\phi(e) = \{v, y\}$. However, this means v is connected via a path to x, meaning $y \in Z$, so $y \in Z \cap Y$. Thus, we must have $N(Z \cap X) = Z \cap Y$.

Therefore,

$$\begin{aligned} |Z \cap X| &= |Z \cap Y| + 1 \\ &= |N(Z \cap X)| + 1, \end{aligned}$$

which contradicts the fact that (X, Y, E, ϕ) satisfies the Hall condition. Therefore, $A = X$.

By symmetry, there is a perfect matching of X and Y in (X, Y, E, ϕ). □

Remark 2.2.2. An equivalent formulation to Hall's theorem states that there is a system of distinct representatives on the collection $\mathcal{X} = \{X_k\}_{k=1}^n$, which is a set $\{x_k\}_{k=1}^n$ such that $x_k \in X_k$ and $x_i \neq x_j$ for $i \neq j$.

This implies the existence of an injection $f\colon \mathcal{X} \hookrightarrow \bigcup_{k=1}^n X_k$, such that $f(X_k) \in X_k$.

We need some results in topology to prove the infinite case of Hall's theorem. The proof is inspired by one of the proofs in [Hal66].

Definition 2.2.5 (Choice Function)**.** Let $\mathcal{X} = \{X_i\}_{i \in I}$ be a collection of sets. A function $f\colon \mathcal{X} \to \bigcup_{i \in I} X_i$ is called a *choice function* if, for each $i \in I$, $f(X_i) \in Xi$.

We also say $f\colon \mathcal{X} \to \bigcup_{i \in I} X_i$ is a choice function if $f \in \prod_{i \in I} X_i$.

Theorem 2.2.3 (Tychonoff's Theorem)**.** If $\{X_i\}_{i \in I}$ is a family of compact topological spaces, then

$$X = \prod_{i \in I} X_i$$

is compact when endowed with the product topology.

Remark 2.2.3. The product topology is the coarsest topology on the set

$$X = \prod_{i \in I} X_i$$

such that the projection maps $\pi_i\colon X \to X_i$ are continuous.

Theorem 2.2.4 (Infinite Hall's Theorem)**.** Let $\mathcal{G} = \{X_i\}_{i\in I}$ be a collection of (not necessarily distinct) finite sets. If, for every finite subcollection $\mathcal{Y} = \{X_{i_k}\}_{k=1}^{n}$,

$$n \leqslant \left|\bigcup_{k=1}^{n} X_{i_k}\right|,$$

then there is a choice function on G.

Proof. We endow each $X_i \in \{X_i\}_{i\in I}$ with the discrete topology. Since each X_i is finite, each X_i is compact.

Thus, by Tychonoff's theorem, it is the case that $\prod_{i\in I} X_i$ is compact.

For every finite subset $Y \subseteq \mathcal{G}$, we define

$$S_Y = \left\{ f \in \prod_{i\in I} X_i \,\middle|\, f|_Y \text{ is injective} \right\}.$$

The injectivity of $f|_Y$ is equivalent to the existence of a system of distinct representatives on Y. Since Y satisfies the Hall condition, each S_Y is nonempty. Additionally, for any net of functions $f_\alpha \in S_Y$ with $\lim_\alpha f_\alpha = f$, it is the case that $f_\alpha|_Y$ is injective, so $f|_Y$ is injective, meaning S_Y is closed.

We define $F = \{S_Y \mid Y \subseteq \mathcal{G} \text{ finite}\}$. For finite $Y_1, Y_2 \subseteq \mathcal{G}$, every system of distinct representatives in $Y_1 \cup Y_2$ is necessarily a system of distinct representatives on Y_1 and a system of distinct representatives on Y_2, meaning $S_{Y_1\cup Y_2} \subseteq S_{Y_1} \cap S_{Y_2}$. Thus, F has the finite intersection property.

Since $\prod_{i\in I} X_i$ is compact, $\bigcap F$ is nonempty, where the intersection is taken over all finite subsets of $\mathcal{G}$. For any $f \in \bigcap F$, f is necessarily a choice function. □

Remark 2.2.4. This is equivalent to the existence of an injection $f\colon \mathcal{G} \hookrightarrow \bigcup_{i\in I} X_i$.

We will use this infinite case of Hall's theorem to prove König's theorem.

Theorem 2.2.5 (König's Theorem)**.** Let (X, Y, E, ϕ) be a k-regular bipartite graph (not necessarily finite). Then, there is a perfect matching of X and Y.

Proof. If $k = 1$, it is clear that there is a perfect matching in (X, Y, E, ϕ) consisting of the edges in (X, Y, E, ϕ).

Let $k \geqslant 2$. Since any finite subset of X satisfies the Hall condition, as displayed in the proof of Theorem 2.2.2, there is some X-perfect matching in (X, Y, E, ϕ). We call this X-perfect matching F. There is an injection $f\colon X \hookrightarrow Y$ following the edges in F.

Similarly, since any finite subset of Y satisfies the Hall condition, there is some Y-perfect matching in (X, Y, E, ϕ). We call this Y-perfect matching G. There is an injection $g\colon Y \hookrightarrow X$ following the edges of G.

Consider the subgraph $(X, Y, F \cup G, \phi|_{F\cup G})$. The injections f and g still hold in this graph. By the Cantor–Schröder–Bernstein theorem, there is a bijection $h\colon X \to Y$ in $(X, Y, F \cup G, \phi|_{F\cup G})$, which is equivalent to the existence of a perfect matching of X and Y. □

2.2.2 Type Semigroups

Definition 2.2.6. Let G be a group that acts on a set X.

(i) We define $X^* = X \times \mathbb{N}_0$, and

$$G^* = \left\{(g, \pi) \,\middle|\, g \in G, \pi \in \mathrm{Sym}(\mathbb{N}_0)\right\}.$$

(ii) If $A \subseteq X^*$, the values of n for which there is an element of A whose second coordinate is n are called the *levels* of A.

Fact 2.2.1. If $E_1, E_2 \subseteq X$, then $E_1 \sim_G E_2$ if and only if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$ for all $m, n \in \mathbb{N}_0$.

Proof. Let $E_1 \sim_G E_2$. Then, there exist pairwise disjoint $A_1, \dots, A_n \subset E_1$, pairwise disjoint $B_1, \dots, B_n \subset E_2$, and elements $g_1, \dots, g_n \in G$ such that $g_i \cdot A_i = B_i$. We select the permutation $\pi_i \in \mathrm{Sym}(\mathbb{N}_0)$ such that $\pi_i(n) = m$ and $\pi_i(m) = n$ for each i. Then,

$$(g_i, \pi_i) \cdot (A_i \cdot \{n\}) = B_i \cdot \{m\}.$$

Similarly, if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$, then of the pairwise disjoint subsets

$$A_1 \times \{n\}, \dots, A_n \times \{n\} \subset E_1 \times \{n\}$$

and

$$B_1 \times \{m\}, \dots, B_n \times \{m\} \subset E_2 \times \{m\},$$

we set $A_1, \dots, A_n \subset E_1$ and $B_1, \dots, B_n \subset E_2$. Similarly, for

$$(g_1, \pi_1), \dots, (g_n, \pi_n) \in G^*$$

such that

$$(g_i, \pi_i) \cdot A_i \times \{n\} = B_i \times \{m\},$$

we select $g_1, \dots, g_n \in G$. Then, by definition,

$$g_i \cdot A_i = B_i$$

for each i. Thus, $E_1 \sim_G E_2$. □

Definition 2.2.7. Let G be a group that acts on X, and let G^*, X^* be defined as in 2.2.6.

(i) A set $A \subseteq X^*$ is said to be *bounded* if it has finitely many levels.

(ii) If $A \subseteq X^*$ is bounded, the equivalence class of A with respect to G^*-equidecomposability is called the *type* of A, which is denoted $[A]$.

(iii) If $E \subseteq X$, we write $[E] = [E \times \{0\}]$.

(iv) Let $A, B \subseteq X^*$ be bounded with $k \in \mathbb{N}_0$ such that for

$$B' = \{(b, n + k) \mid (b, n) \in B\},$$

we have $B' \cap A = \varnothing$. Then, $[A] + [B] = [A \sqcup B']$. Note that $[B'] = [B]$.

(v) We define

$$\mathcal{S} = \{[A] \mid A \subseteq X^* \text{ bounded}\}$$

under the addition defined in (iv) to be the *type semigroup* of the action of G on X.

Fact 2.2.2. Addition is well-defined in $(\mathcal{S}, +)$, and $(\mathcal{S}, +)$ is a well-defined commutative semigroup with identity $[\varnothing]$.

Proof. To show that addition is well-defined, we let $[A_1] = [A_2]$, and $[B_1] = [B_2]$. Without loss of generality, $A_1 \cap B_1 = \varnothing$ and $A_2 \cap B_2 = \varnothing$.

By the definition of the type, $A_1 \sim_{G^*} A_2$ and $B_1 \sim_{G^*} B_2$, meaning

$$A_1 \sqcup B_1 \sim_{G^*} A_2 \sqcup B_2,$$

so

$$\begin{aligned}[A_1] + [B_1] &= [A_1 \sqcup B_1] \\ &= [A_2 \sqcup B_2] \\ &= [A_2] + [A_2],\end{aligned}$$

meaning addition is well-defined.

Since addition is well-defined, and $A \sqcup B = B \sqcup A$, we can see that addition is also commutative. We also have

$$\begin{aligned}[A] + [\varnothing] &= [A \sqcup \varnothing] \\ &= [A],\end{aligned}$$

so $[\varnothing]$ is the identity on $\mathcal{S}$.

Finally, since for any $[A], [B] \in \mathcal{S}$, A and B have finitely many levels, it is the case that $A \cup B$ has finitely many levels for any A and B, so $[A] + [B] \in \mathcal{S}$. □

Definition 2.2.8. For any commutative semigroup $\mathcal{S}$ with $\alpha \in S$ and $n \in \mathbb{N}$, we define

$$n\alpha = \underbrace{\alpha + \cdots + \alpha}_{n \text{ times}}$$

Definition 2.2.9. For $\alpha, \beta \in \mathcal{S}$, if there exists $\gamma \in \mathcal{S}$ such that $\alpha + \gamma = \beta$, we write $\alpha \leqslant \beta$.

Fact 2.2.3. If G is a group acting on X with corresponding type semigroup $\mathcal{S}$, then the following are true.

(i) If $\alpha, \beta \in \mathcal{S}$ with $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$, then $\alpha = \beta$.

(ii) A set $E \subseteq X$ is G-paradoxical if and only if $[E] = 2[E]$.

Proof. Let G act on X, and let $\mathcal{S}$ be the corresponding type semigroup.

(i) If $[A] \leqslant [B]$, then there exists $C_1 \in \mathcal{S}$ such that $[A] + [C_1] = [B]$. Without loss of generality, $C_1 \cap A = \varnothing$, meaning $[B] = [A \sqcup C_1]$. Thus, $A \sqcup C_1 \sim_{G^*} B$, meaning $B \preceq_{G^*} A$.

Similarly, if $[B] \leqslant [A]$, then $B \preceq_{G^*} A$. By Theorem 2.1.3, it is thus the case that $A \sim_{G^*} B$.

(ii) Let E be G-paradoxical.

Then, $E \sim_G \bigsqcup_{i=1}^n A_i$ and $E \sim_G \bigsqcup_{j=1}^m B_j$ for pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$. Thus, we have

$$\begin{aligned}[E] &= \left[\left(\bigsqcup_{i=1}^n A_i\right) \sqcup \left(\bigsqcup_{j=1}^m B_j\right)\right] \\ &= \left[\bigsqcup_{i=1}^n A_i\right] + \left[\bigsqcup_{j=1}^m B_j\right] \\ &= 2[E].\end{aligned}$$

Similarly, if $[E] = 2[E]$, then there exist A and B such that

$$\begin{aligned}[E] &= [A] + [B]\\ &= [A \sqcup B],\end{aligned}$$

meaning A and B are each G-equidecomposable with E, so E is G-paradoxical.

□

We can now prove the cancellation identity, which we will be useful as we construct our desired finitely additive measure.

Theorem 2.2.6 (Cancellation Identity on $\mathcal{S}$)**.** Let $\mathcal{S}$ be the type semigroup for some group action, and let $\alpha, \beta \in \mathcal{S}$, $n \in \mathbb{N}$ such that $n\alpha = n\beta$. Then, $\alpha = \beta$.

Proof. Let $n\alpha = n\beta$. Then, there are two disjoint bounded subsets $E, E' \subseteq X^*$ with $E \sim_{G^*} E'$, and pairwise disjoint subsets $A_1, \ldots, A_n \subseteq E$, $B_1, \ldots, B_n \subseteq E'$ such that

- $E = A_1 \cup \cdots \cup A_n$, $E' = B_1 \cup \cdots \cup B_n$
- $\left[A_j\right] = \alpha$ and $\left[B_j\right] = \beta$ for each $j = 1, \ldots, n$.

Let $\chi\colon E \to E'$ be a bijection as in Fact 2.1.3, with $\phi_j\colon A_1 \to A_j$, $\psi_j\colon B_1 \to B_j$ also being bijections as in Fact 2.1.3; here we define ϕ_1 and ψ_1 to be the identity map.

For each $a \in A_1$ and $b \in B_1$, we define

$$\begin{aligned}\overline{a} &= \{a, \phi_2(a), \ldots, \phi_n(a)\}\\ \overline{b} &= \{b, \psi_2(b), \ldots, \psi_n(b)\}.\end{aligned}$$

We construct a graph by letting $X = \{\overline{a} \mid a \in A_1\}$ and $Y = \left\{\overline{b} \,\middle|\, b \in B_1\right\}$, and, for each j, define edges $\left\{\overline{a}, \overline{b}\right\}$ if $\chi\left(\phi_j(a)\right) \in \overline{b}$.

Since χ is a bijection, for each $j = 1, \ldots, n$, $\chi\left(\phi_j(a)\right)$ must be an element of B_k for some k, and since $\{B_k\}_{k=1}^{n}$ are disjoint, $\chi\left(\phi_j(a)\right)$ is an element of exactly one B_k. Thus, the graph is n-regular.

By Theorem 2.2.5, this graph has a perfect matching F. As a result, for each $\overline{a} \in X$, there is a unique $\overline{b} \in Y$ and a unique edge $\left\{\overline{a}, \overline{b}\right\} \in F$ such that $\chi\left(\phi_j(a)\right) = \psi_k(b)$ for some $j, k \in \{1, \ldots, n\}$.

We define

$$\begin{aligned}C_{j,k} &= \left\{a \in A_1 \,\middle|\, \left\{\overline{a}, \overline{b}\right\} \in F,\ \chi\left(\phi_j(a)\right) = \psi_k(b)\right\}\\ D_{j,k} &= \left\{b \in B_1 \,\middle|\, \left\{\overline{a}, \overline{b}\right\} \in F,\ \chi\left(\phi_j(a)\right) = \psi_k(b)\right\}.\end{aligned}$$

Therefore, we must have $\psi_k^{-1} \circ \chi \circ \phi_j$ is a bijection from $C_{j,k}$ to $D_{j,k}$, so $C_{j,k} \sim_{G^*} D_{j,k}$.

Since $C_{j,k}$ and $D_{j,k}$ are partitions of A_1 and B_1 respectively, it follows that $A_1 \sim_{G^*} B_1$, so $\alpha = \beta$. □

Corollary 2.2.1. Let $\mathcal{S}$ be the type semigroup of some group action, and let $\alpha \in \mathcal{S}$ and $n \in \mathbb{N}$ such that $(n+1)\alpha \leqslant n\alpha$. Then, $\alpha = 2\alpha$.

Proof. We have

$$\begin{aligned}2\alpha + n\alpha &= (n+1)\alpha + \alpha\\ &\leqslant n\alpha + \alpha\end{aligned}$$

$$\begin{aligned} &= (n+1)\alpha \\ &\leqslant n\alpha. \end{aligned}$$

Inductively repeating this argument, we get $n\alpha \geqslant 2n\alpha$; since $n\alpha \leqslant 2n\alpha$ by definition, we must have $n\alpha = 2n\alpha$, so $\alpha = 2\alpha$. □

Remark 2.2.5. We will call such an α a paradoxical element.

2.2.3 Two Results on Commutative Semigroups

Now that we are aware of paradoxical elements and the relationship between G-paradoxicality and the properties of the particular elements of the type semigroup (Fact 2.2.3), we will now relate these properties to finitely additive measures of sets by using the following lemma and theorem.

Lemma 2.2.1. Let $\mathcal{S}$ be a commutative semigroup, with $\mathcal{S}_0 \subseteq \mathcal{S}$ finite, and $\epsilon \in \mathcal{S}_0$ satisfying the following assumptions:

(a) $(n+1)\epsilon \nleqslant n\epsilon$ for all $n \in \mathbb{N}$ (i.e., that ϵ is non-paradoxical);

(b) for each $\alpha \in \mathcal{S}$, there is $n \in \mathbb{N}$ such that $\alpha \leqslant n\epsilon$.

Then, there is a set function $\nu \colon \mathcal{S}_0 \to [0, \infty]$ that satisfies the following conditions:

(i) $\nu(\epsilon) = 1$;

(ii) for $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathcal{S}_0$ with $\alpha_1 + \cdots + \alpha_n \leqslant \beta_1 + \cdots \beta_m$,

$$\sum_{j=1}^{n} \nu(\alpha_j) \leqslant \sum_{j=1}^{m} \nu(\beta_j).$$

Proof. We will prove this result by inducting on the cardinality of $\mathcal{S}_0$.

We start with $|\mathcal{S}_0| = 1$. In that case, we define $\nu(\epsilon) = 1$, satisfying condition (i). To satisfy condition (ii), we see that for $n, m \in \mathbb{N}$ with $n\epsilon \leqslant m\epsilon$, if $n \geqslant m+1$, then $(m+1)\epsilon \leqslant n\epsilon \leqslant m\epsilon$, implying that $\epsilon = 2\epsilon$, which contradicts assumption (a).

Let $\alpha_0 \in \mathcal{S}_0 \setminus \{\epsilon\}$. The induction hypothesis says there is a set function satisfying conditions (i) and (ii), $\nu \colon \mathcal{S}_0 \setminus \{\alpha_0\} \to [0, \infty]$.

For $r \in \mathbb{N}$, there are $\gamma_1, \ldots, \gamma_p, \delta_1, \ldots, \delta_q \in \mathcal{S} \setminus \{\alpha_0\}$ such that

$$\delta_1 + \cdots + \delta_q + r\alpha_0 \leqslant \gamma_1 + \cdots + \gamma_p. \tag{†}$$

Consider the set N defined as follows:

$$N = \left\{ \frac{1}{r}\left(\sum_{j=1}^{p} \nu(\gamma_j) - \sum_{j=1}^{q} \nu(\delta_j) \right) \,\middle|\, \gamma_j, \delta_j \text{ satisfy } (\dagger) \right\}. \tag{‡}$$

We define the extension of ν as follows:

$$\nu(\alpha_0) = \inf N.$$

This infimum is well-defined since, by assumption (b), there is some $n \in \mathbb{N}$ such that $\alpha_0 \leqslant n\epsilon$, and $\nu(\epsilon)$ is defined.

Now, we must show that this extension of ν satisfies condition (ii).

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathcal{S}_0 \setminus \{\alpha_0\}$ and $s, t \in \mathbb{N}_0$ such that

$$\alpha_1 + \cdots + \alpha_n + s\alpha_0 \leqslant \beta_1 + \cdots + \beta_m + t\alpha_0. \tag{$*$}$$

We will verify condition (ii) in the three following cases.

Case 0: If $s = t = 0$, then the induction hypothesis states that ($*$) satisfies condition (ii).

Case 1: Let $s = 0$ and $t > 0$. We want to show that

$$\sum_{j=1}^{n} \nu(\alpha_j) \leqslant t\nu(\alpha_0) + \sum_{j=1}^{m} \nu(\beta_j),$$

which implies that

$$\nu(\alpha_0) \geqslant \frac{1}{t}\left(\sum_{j=1}^{n} \nu(\alpha_j) - \sum_{j=1}^{m} \nu(\beta_j)\right).$$

By the definition of infimum, it suffices to show that for $r \in \mathbb{N}$ and $\delta_1, \ldots, \delta_q, \gamma_1, \ldots, \gamma_p \in \mathcal{S} \setminus \{\alpha_0\}$ satisfying (†), it is the case that

$$\frac{1}{r}\left(\sum_{j=1}^{p} \nu(\gamma_j) - \sum_{j=1}^{q} \nu(\delta_j)\right) \geqslant \frac{1}{t}\left(\sum_{j=1}^{n} \nu(\alpha_j) - \sum_{j=1}^{m} \nu(\beta_j)\right).$$

Multiplying ($*$) by r on both sides, and adding $t\delta_1 + \cdots + t\delta_q$ to both sides, we have

$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q \leqslant r\beta_1 + \cdots + r\beta_m + t(r\alpha_0) + t\delta_1 + \cdots + t\delta_q.$$

Substituting (†), we find

$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q \leqslant r\beta_1 + \cdots + r\beta_m + t\gamma_1 + \cdots + t\gamma_p.$$

Applying the induction hypothesis, we have

$$r\sum_{j=1}^{n} \nu(\alpha_j) + t\sum_{j=1}^{q} \nu(\delta_j) \leqslant r\sum_{j=1}^{m} \nu(\beta_j) + t\sum_{j=1}^{p} \nu(\gamma_j),$$

yielding

$$\frac{1}{r}\left(\sum_{j=1}^{p} \nu(\gamma_j) - \sum_{j=1}^{q} \nu(\delta_j)\right) \geqslant \frac{1}{t}\left(\sum_{j=1}^{n} \nu(\alpha_j) - \sum_{j=1}^{m} \nu(\beta_j)\right).$$

Case 2: Let $s > 0$. For $z_1, \ldots, z_t \in N$ (‡), we need to show that

$$s\nu(\alpha_0) + \sum_{j=1}^{n} \nu(\alpha_j) \leqslant z_1 + \cdots + z_t + \sum_{j=1}^{n} \nu(\beta_j).$$

Without loss of generality, we can set $z_1, \ldots, z_n = z$, as for each $z \in N$, $z \geqslant \nu(\alpha_0)$.

As in Case 1, we multiply ($*$) by r, add $t\delta_1 + \cdots + t\delta_q$ to both sides, and substitute with (†), yielding

$$r\alpha_1 + \cdots + r\alpha_n + rs\alpha_0 + t\delta_1 + \cdots + t\delta_q \leqslant r\beta_1 + \cdots + r\beta_m + t(r\alpha_0) + t\delta_1 + \cdots + t\delta_q$$
$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q + rs\alpha_0 \leqslant r\beta_1 + \cdots + r\beta_m + t\gamma_1 + \cdots + t\gamma_p.$$

Defining

$$z = \frac{1}{r}\left(\sum_{j=1}^{p} \nu(\gamma_j) - \sum_{j=1}^{q} \nu(\delta_j)\right),$$

we get

$$\begin{aligned} s\nu(\alpha_0) + \sum_{j=1}^{n} \nu(\alpha_j) &\leqslant \sum_{j=1}^{n} \nu(\alpha_j) + \frac{s}{sr}\left(r\sum_{j=1}^{m} \nu(\beta_j) - r\sum_{j=1}^{n} \nu(\alpha_j) + t\sum_{j=1}^{p} \nu(\gamma_j) - t\sum_{j=1}^{q} \nu(\delta_j)\right) \\ &= tz + \sum_{j=1}^{m} \nu(\beta_j). \end{aligned}$$

Thus, we have shown that ν extends in a manner that satisfies conditions (i) and (ii). □

We can "upgrade" our finitely additive set function to a semigroup homomorphism as follows.

Theorem 2.2.7. Let $(\mathcal{S}, +)$ be a commutative semigroup with identity element 0, and let $\epsilon \in \mathcal{S}$. Then, the following are equivalent:

(i) $(n+1)\epsilon \leqslant n\epsilon$ for all $n \in \mathbb{N}$;

(ii) there is a semigroup homomorphism $\nu\colon (\mathcal{S}, +) \to ([0,\infty], +)$ such that $\nu(\epsilon) = 1$.

Proof. To show that (ii) implies (i), we let $\nu\colon (\mathcal{S}, +) \to ([0,\infty], +)$ be a semigroup homomorphism with $\nu(\epsilon) = 1$. Then,

$$\begin{aligned} \nu((n+1)\epsilon) &= (n+1)\nu(\epsilon) \\ &= n+1 \\ &> n \\ &= n\nu(\epsilon) \\ &= \nu(n\epsilon), \end{aligned}$$

meaning that $(n+1)\epsilon \not\leqslant n\epsilon$.

To show that (i) implies (ii), we suppose that for each $\alpha \in \mathcal{S}$, there is $n \in \mathbb{N}$ such that $\alpha \leqslant n\epsilon$ — for any such α for which this is not the case, we define $\nu(\alpha) = \infty$.

For a finite subset $\mathcal{S}_0 \subseteq \mathcal{S}$ with $\epsilon \in \mathcal{S}_0$, we define $M_{\mathcal{S}_0}$ to be the set of all $\kappa\colon \mathcal{S} \to [0,\infty]$ such that

- $\kappa(\epsilon) = 1$;
- $\kappa(\alpha + \beta) = \kappa(\alpha) + \kappa(\beta)$ for $\alpha, \beta, \alpha + \beta \in \mathcal{S}_0$.

Since we assume condition (i), we know that such a κ with $\kappa(\epsilon) = 1$ exists. Additionally, since

$$\alpha + \beta \leqslant (\alpha + \beta)$$

and

$$(\alpha + \beta) \leqslant \alpha + \beta,$$

it is the case that

$$\kappa(\alpha + \beta) \leqslant \kappa(\alpha) + \kappa(\beta) \leqslant \kappa(\alpha + \beta),$$

meaning $\kappa(\alpha + \beta) = \kappa(\alpha) + \kappa(\beta)$. Thus, $M_{\mathcal{S}_0}$ is nonempty. It is also the case that $M_{\mathcal{S}_0}$ is closed, since any net of functions $\kappa_p\colon \mathcal{S} \to [0,\infty]$ with $\kappa_p(\epsilon) = 1$ and $\kappa_p(\alpha + \beta) = \kappa_p(\alpha) + \kappa_p(\beta)$ will necessarily satisfy these conditions in the limit.

We let $[0,\infty]^{\mathcal{S}} = \{\kappa \mid \kappa : \mathcal{S} \to [0,\infty]\}$ be equipped with the product topology. By Tychonoff's theorem, $[0,\infty]^{\mathcal{S}}$ is compact.

Since, for any $\mathcal{S}_1, \ldots, \mathcal{S}_n$ finite, it is the case that

$$M_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n} \subseteq M_{\mathcal{S}_1} \cap \cdots \cap M_{\mathcal{S}_n},$$

since any such $\kappa \in M_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n}$ must necessarily be in every $M_{\mathcal{S}_i}$. Thus, the family

$$\{M_{\mathcal{S}_0} \mid \mathcal{S}_0 \subseteq \mathcal{S} \text{ finite}\}$$

has the finite intersection property. Thus, by compactness, there is some ν such that

$$\nu \in \bigcap \{M_{\mathcal{S}_0} \mid \mathcal{S}_0 \subseteq \mathcal{S} \text{ finite}\},$$

with $\nu(\epsilon) = 1$ and, for all $\alpha, \beta \in \mathcal{S}$, since $\nu \in M_{\{\alpha,\beta,\alpha+\beta\}}$, $\nu(\alpha+\beta) = \nu(\alpha) + \nu(\beta)$. □

2.2.4 Proof of Tarski's Theorem

Finally, we are able to prove the reverse direction of Tarski's Theorem. We restate the theorem before giving its proof.

Theorem 2.2.1 (Tarski's Theorem)**.** Let G be a group that acts on a set X, and let $E \subseteq X$ be nonempty.

There is a finitely additive measure $\mu \colon P(X) \to [0,\infty]$ with $\mu(E) \in (0,\infty)$ and $\mu(t \cdot E) = \mu(E)$ for all $t \in G$ if and only if E is not G-paradoxical.

Proof of the Reverse Direction of Theorem 2.2.1: Let $\mathcal{S}$ be the type semigroup of the action of G on X.

Suppose E is not G-paradoxical. Then, $[E] \neq 2[E]$ by Fact 2.2.3, meaning $(n+1)[E] \nleq n[E]$ for all $n \in \mathbb{N}$ by the contrapositive of Corollary 2.2.1.

Thus, by Theorem 2.2.7, there is a map $\nu \colon \mathcal{S} \to [0,\infty]$ with $\nu([E]) = 1$. The map $\mu \colon P(X) \to [0,\infty]$ defined by

$$\mu(A) = \nu([A])$$

is the desired finitely additive measure. □

Therefore, from Tarski's theorem and Proposition 2.1.3, we know that if G acts on itself by left-multiplication, there is a mean $m \colon P(G) \to [0,1]$ if and only if G is not paradoxical, which occurs only when G does not admit any paradoxical actions.

3 Amenability and Invariant States

Tarski's Theorem is one of our first criteria establishing amenability — that is, a group is amenable if and only if it is non-paradoxical. Tarski's Theorem, while informative about the nature of amenable groups, is unfortunately quite uninformative when it comes to establishing amenability for broader classes of groups. How might we know if a group admits a paradoxical decomposition, or if a group admits *no* paradoxical decompositions?

To establish the amenability of a large class of groups — as we will do with abelian and solvable groups in this chapter — we need tools from functional analysis. Rather than focusing on G, we will focus on the space $\ell_\infty(G)$, and prove the existence of a mean on G by proving the existence of an analogous construct on $\ell_\infty(G)$, known as an invariant state.

3.1 Invariant States: An Overview

Definition 3.1.1. Let G be a group.

(1) The space $\mathcal{F}(G)$ is defined by

$$\mathcal{F}(G) = \{f \mid f\colon G \to \mathbb{C} \text{ is a function}\}.$$

(2) A function $f \in \mathcal{F}(G)$ is called positive if $f(x) \geqslant 0$ for all $x \in G$.

(3) A function $f \in \mathcal{F}(G)$ is called simple if $\mathrm{Ran}(f)$ is finite. We let

$$\Sigma = \left\{f \in \mathcal{F}(G) \,\middle|\, f \text{ is simple}\right\}.$$

Fact 3.1.1. It is the case that $\Sigma \subseteq \mathcal{F}(G)$ is a linear subspace.

Definition 3.1.2. For $E \subseteq G$, we define

$$\mathbb{1}_E\colon G \to \mathbb{C}$$

by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

This is the characteristic function of E.

Fact 3.1.2. We have

$$\mathrm{span}\{\mathbb{1}_E \mid E \subseteq G\} = \Sigma.$$

Proof. We see that $\mathbb{1}_E \in \Sigma$ for any $E \subseteq G$, and that Σ is a subspace.

If $\phi \in \Sigma$ with $\mathrm{Ran}(\phi) = \{t_1, \dots, t_n\}$, where t_i are distinct, we set

$$E_i = \phi^{-1}(\{t_i\}),$$

yielding

$$\phi = \sum_{i=1}^{n} t_i \mathbb{1}_{E_i}.$$

□

Proposition 3.1.1. The space $\ell_\infty(G)$ is complete. Additionally, $\overline{\Sigma} = \ell_\infty(G)$.

Proof. Let $(f_n)_n$ be $\|\cdot\|$-Cauchy in $\ell_\infty(G)$. Then, for all $x \in G$, it is the case that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n - f_m)(x)| \\ &\leqslant \|f_n - f_m\|_{\ell_\infty}, \end{aligned}$$

meaning $(f_n(x))_n$ is Cauchy in $\mathbb{C}$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We must show that $f \in \ell_\infty(G)$, and $\|f_n - f\|_{\ell_\infty} \to 0$.

We have

$$
\begin{aligned}
|f(x)| &= \left|\lim_{n\to\infty} f_n(x)\right| \\
&= \lim_{n\to\infty} |f_n(x)| \\
&\leqslant \limsup_{n\to\infty} \|f_n\|_{\ell_\infty} \\
&\leqslant C,
\end{aligned}
$$

as Cauchy sequences are always bounded. Thus, $\sup_{x\in G}|f(x)| \leqslant C$.

Given $\varepsilon > 0$, we find N such that for all $m, n \geqslant N$, $\|f_n - f_m\|_{\ell_\infty} \leqslant \varepsilon$. Thus, for $x \in G$, we have

$$
\begin{aligned}
|f_n(x) - f_m(x)| &\leqslant \|f_n - f_m\|_{\ell_\infty} \\
&\leqslant \varepsilon.
\end{aligned}
$$

Taking $m \to \infty$, we get $|f_n(x) - f(x)| \leqslant \varepsilon$, for all $n \geqslant N$, so $\|f_n - f\|_{\ell_\infty} \leqslant \varepsilon$ for all $n \geqslant N$.

For real-valued $f \in \ell_\infty(G)$, let $|f| \subseteq [-M, M]$ for some $M > 0$. Let $\varepsilon > 0$. Since $[-M, M]$ is compact, it is totally bounded, so we can find intervals $I_1, \dots, I_n$ with $[-M, M] = \bigsqcup_{k=1}^{n} I_k$, with the length of each I_k less than ε.

Set $E_k = f^{-1}(I_k)$. Pick some $t_k \in I_k$. We set

$$
\phi = \sum_{i=1}^{n} t_k \mathbb{1}_{E_k}.
$$

Then, it is the case that $\|\phi - f\|_{\ell_\infty} < \varepsilon$.

If $f \in \ell_\infty(G)$ is complex-valued, we apply this process separately to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. □

Corollary 3.1.1. For any $f \in \ell_\infty(G)$, there is a sequence $(\phi_n)_n$ of simple functions with $\|\phi_n - f\|_{\ell_\infty} \to 0$. If $f \geqslant 0$, then we can select $\phi_n \geqslant 0$.

Now that we understand how simple functions relate to $\ell_\infty(G)$, we start by defining a translation action on $\ell_\infty(G)$, from which we will be able to convert the idea of means into invariant elements of the state space of the dual of $\ell_\infty(G)$.

Proposition 3.1.2. Let G be a group. There is an action

$$
\lambda\colon G \to \mathrm{Isom}(\ell_\infty(G)),
$$

where $\lambda(s) = \lambda_s$, defined by

$$
\lambda_s(f)(t) = f\left(s^{-1}t\right)
$$

Proof. We have

$$
\begin{aligned}
\lambda_s(f + \alpha g)(t) &= (f + \alpha g)\left(s^{-1}t\right) \\
&= f\left(s^{-1}t\right)\alpha g\left(s^{-1}t\right) \\
&= \lambda_s(f)(t) + \alpha\lambda_s(g)(t) \\
&= (\lambda_s(f) + \alpha\lambda_s(g))(t).
\end{aligned}
$$

Thus, λ_s is linear. Additionally,

$$\begin{aligned}\|\lambda_s(f)\|_{\ell_\infty} &= \sup_{t\in G}|\lambda_s(f)(t)| \\ &= \sup_{t\in G}\left|f\left(s^{-1}t\right)\right| \\ &= \|f\|_{\ell_\infty},\end{aligned}$$

and

$$\begin{aligned}\|\lambda_s(f) - \lambda_s(g)\|_{\ell_\infty} &= \|\lambda_s(f-g)\| \\ &= \|f-g\|_{\ell_\infty},\end{aligned}$$

meaning λ_s is an isometry.

We have

$$\begin{aligned}\lambda_s \circ \lambda_r(f)(t) &= \lambda_r(f)\left(s^{-1}t\right) \\ &= \lambda_r\left(r^{-1}s^{-1}t\right) \\ &= f\left((sr)^{-1}t\right) \\ &= \lambda_{sr}(f)(t),\end{aligned}$$

establishing that $\lambda_s \circ \lambda_r = \lambda_{sr}$.

By a similar process, we find that $\lambda_s(\mathbb{1}_E) = \mathbb{1}_{sE}$ for any $E \subseteq G$ and $s \in G$. □

Definition 3.1.3. A *state* on $\ell_\infty(G)$ is a continuous linear functional $\mu \in \ell_\infty(G)^*$ such that the following are true:

- μ is positive;
- $\mu(\mathbb{1}_G) = 1$.

A state is called left-invariant if

$$\mu(\lambda_s(f)) = \mu(f).$$

Example 3.1.1. The evaluation functional, $\delta_x : \ell_\infty \to \mathbb{R}$, defined by

$$\delta_x(f) = f(x),$$

is a state. However, it is not necessarily invariant, as

$$\begin{aligned}\delta_x(\lambda_s(f)) &= \lambda_s(f)(x) \\ &= f\left(s^{-1}x\right) \\ &\neq f(x).\end{aligned}$$

However, we can use the evaluation functional to create an invariant state. If G is finite, we define

$$\mu = \frac{1}{|G|}\sum_{x\in G}\delta_x,$$

which is indeed an invariant state.

We can characterize states slightly differently, which will enable us to show the equivalence between invariant states and means.

Lemma 3.1.1.

(1) If μ is a state on $\ell_\infty(G)$, then

$$\|\mu\|_{op} = 1.$$

(2) If $\mu \in \ell_\infty(G)^*$ is such that

$$\begin{aligned}\|\mu\|_{op} &= \mu(\mathbb{1}_G)\\ &= 1,\end{aligned}$$

then μ is positive and a state.

Proof.

(1) Let μ be a state. Given $f \in \ell_\infty(G)$, we have

$$\begin{aligned}\|f\|_{\ell_\infty}\mathbb{1}_G - f &\geqslant 0\\ \|f\|_{\ell_\infty}\mathbb{1}_G + f &\geqslant 0,\end{aligned}$$

so

$$\begin{aligned}0 &\leqslant \mu\left(\|f\|_{\ell_\infty}\mathbb{1}_G - f\right)\\ &= \|f\|_{\ell_\infty}\mu(\mathbb{1}_G) - \mu(f)\end{aligned}$$

meaning

$$\mu(f) \leqslant \|f\|_{\ell_\infty}.$$

Additionally,

$$\begin{aligned}0 &\leqslant \mu\left(\|f\|_{\ell_\infty}\mathbb{1}_G + f\right)\\ &= \|f\|_{\ell_\infty}\mu(\mathbb{1}_G) + \mu(f),\end{aligned}$$

meaning

$$-\mu(f) \leqslant \|f\|_{\ell_\infty}.$$

Thus, we have $|\mu(f)| \leqslant \|f\|_{\ell_\infty}$, so $\|\mu\|_{op} \leqslant 1$. However, since $\mu(\mathbb{1}_G) = 1$, we must have $\|\mu\|_{op} = 1$.

(2) Suppose $\|\mu\|_{op} = \mu(\mathbb{1}_G) = 1$. Let $f \geqslant 0$. Set $g = \frac{1}{\|f\|_{\ell_\infty}} f$.

Then, $\operatorname{Ran}(g) \subseteq [0, 1]$, and $\operatorname{Ran}(g - \mathbb{1}_G) \subseteq [-1, 1]$. Thus, $\|g - \mathbb{1}_G\|_{\ell_\infty} \leqslant 1$.

Since $\|\mu\|_{op} = 1$, we must have

$$\begin{aligned}|\mu(g - \mathbb{1}_G)| &\leqslant 1\\ |\mu(g) - 1| &\leqslant 1,\end{aligned}$$

and since $\mu(\mathbb{1}_G) = 1$, we have $\mu(g) \in [0, 2]$. Thus, $\mu(f) = \|f\|_{\ell_\infty}\mu(g) \geqslant 0$.

□

Corollary 3.1.2. The set of states in $\ell_\infty(G)^*$ forms a w^*-compact subset of $B_{\ell_\infty(G)^*}$.

Proof. From the Banach–Alaoglu Theorem (Theorem 0.2.3), we only need to show that the set of states, $S(\ell_\infty(G))$, is w^*-closed, as every element of $S(\ell_\infty(G))$ has norm 1.

Let $f \in \ell_\infty(G)$ be positive, and let $(\varphi_i)_i$ be a net in $S(\ell_\infty(G))$ with $(\varphi_i)_i \xrightarrow{w^*} \varphi \in \ell_\infty(G)^*$. From Lemma 3.1.1, we must show that φ is positive and $\varphi(\mathbb{1}_G) = 1$.

We start by seeing that, since each φ_i is a state, we have $\varphi_i(f) \geqslant 0$ for each $i \in I$, so we must have $\varphi(f) \geqslant 0$.

Similarly, since $\varphi_i(\mathbb{1}_G) = 1$ for each $i \in I$, and $(\varphi_i)_i \xrightarrow{w^*} \varphi$, we have $\varphi(\mathbb{1}_G) = 1$. Thus, by Lemma 3.1.1, we have that $S(\ell_\infty(G))$ is w^*-closed. □

Now, we may show the correspondence between invariant states and means.

Proposition 3.1.3. If $\mu \in \ell_\infty(G)^*$ is a state, then $m\colon P(G) \to [0,1]$ defined by $m(E) = \mu(\mathbb{1}_E)$ is a finitely additive probability measure on G.

Moreover, if μ is invariant, then m is a mean.

Proof. We have

$$\begin{aligned} m(G) &= \mu(\mathbb{1}_G) \\ &= 1 \end{aligned}$$

$$\begin{aligned} m(\varnothing) &= \mu(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} m(E \sqcup F) &= \mu(\mathbb{1}_{E\sqcup F}) \\ &= \mu(\mathbb{1}_E + \mathbb{1}_F) \\ &= \mu(\mathbb{1}_E) + \mu(\mathbb{1}_F) \\ &= m(E) + m(F). \end{aligned}$$

Additionally, since $0 \leqslant \mathbb{1}_E \leqslant \mathbb{1}_G$, we have $0 \leqslant \mu(\mathbb{1}_E) \leqslant 1$, so $0 \leqslant m(E) \leqslant 1$.

If μ is invariant, then

$$\begin{aligned} m(sE) &= \mu(\mathbb{1}_{sE}) \\ &= \mu(\lambda_s(\mathbb{1}_E)) \\ &= \mu(\mathbb{1}_E) \\ &= m(E). \end{aligned}$$

□

Proposition 3.1.4. If G admits a mean, then $\ell_\infty(G)^*$ admits an invariant state.

Proof. Let m be a mean. Define $\mu_0\colon \Sigma \to \mathbb{R}$ by

$$\mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right) = \sum_{k=1}^{n} t_k m(E_k).$$

Since m is finitely additive, it is the case that μ_0 is well-defined, linear, and positive, with $\mu_0(\mathbb{1}_G) = m(G) = 1$.

Additionally, since m is a mean, then for $f = \sum_{k=1}^{n} t_k \mathbb{1}_{E_k}$, we have

$$\begin{aligned}
\mu_0(\lambda_s(f)) &= \mu_0\left(\lambda_s\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right)\right) \\
&= \mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{sE_k}\right) \\
&= \sum_{k=1}^{n} t_k m(sE_k) \\
&= \sum_{k=1}^{n} t_k m(E_k) \\
&= \mu_0(f).
\end{aligned}$$

We see that

$$\begin{aligned}
|\mu_0(f)| &= \left|\sum_{k=1}^{n} t_k m(E_k)\right| \\
&\leqslant \sum_{k=1}^{n} |t_k| m(E_k) \\
&\leqslant \sum_{k=1}^{n} \|f\|_{\ell_\infty} \sum_{k=1}^{n} m(E_k) \\
&= \|f\|_{\ell_\infty} \sum_{k=1}^{n} m(E_k) \\
&\leqslant \|f\|_{\ell_\infty},
\end{aligned}$$

meaning μ_0 is continuous, so μ_0 is uniformly continuous.

Since $\overline{\Sigma} = \ell_\infty(G)$, uniform continuity provides that μ_0 extends to a continuous linear functional $\mu\colon \ell_\infty(G) \to \mathbb{R}$ with $\mu(\mathbb{1}_G) = \mu_0(\mathbb{1}_G) = 1$.

For $f \geqslant 0$, we find a sequence $(\phi_n)_n$ in Σ with $\phi_n \geqslant 0$ and $\|\phi_n - f\|_{\ell_\infty} \xrightarrow{n\to\infty} 0$. We set

$$\begin{aligned}
\mu(f) &= \lim_{n\to\infty} \mu(\phi_n) \\
&= \lim_{n\to\infty} \mu_0(\phi_n) \\
&\geqslant 0,
\end{aligned}$$

so μ is a state.

If $f \in \ell_\infty(G)$, $s \in G$, and $(\phi_n)_n$ a sequence in Σ with $(\phi_n)_n \to f$, then

$$\begin{aligned}
\|\lambda_s(\phi_n) - \lambda_s(f)\|_{\ell_\infty} &= \|\lambda_s(\phi_n - f)\|_{\ell_\infty} \\
&= \|\phi_n - f\|_{\ell_\infty} \\
&\to 0.
\end{aligned}$$

Thus, we have

$$\mu(\lambda_s(\phi_n)) = \mu_0(\lambda_s(\phi_n))$$

$$
\begin{aligned}
&= \mu_0(\phi_n) \\
&= \mu(\phi_n) \\
&\to \mu(f),
\end{aligned}
$$

so $\mu(f) = \mu(\lambda_s(f))$. Thus, $\mu \in \ell_\infty(G)^*$ is an invariant state. □

3.2 Establishing Amenability with Invariant States

Owing to the correspondence between invariant states and means, we are now able to establish amenability for large classes of groups.

Proposition 3.2.1. The group of integers, $\mathbb{Z}$, is amenable.

Proof. We define the left shift, $\lambda_1 \colon \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$, by

$$\lambda_1(f)(k) = f(k-1).$$

This is an action as in Proposition 3.1.2.

We set $Y = \operatorname{Ran}(\operatorname{id} - \lambda_1) \subseteq \ell_\infty(\mathbb{Z})$. We claim that $\operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$.

Suppose toward contradiction that there is $y \in Y$ with $\|\mathbb{1}_{\mathbb{Z}} - y\|_{\ell_\infty} = r < 1$. Then, $y = f - \lambda_1 f$ for some $f \in \ell_\infty(\mathbb{Z})$, so

$$\|\mathbb{1}_{\mathbb{Z}} - (f - \lambda_1(f))\|_{\ell_\infty} = r.$$

Thus, for all $k \in \mathbb{Z}$, we have

$$|1 - (f(k) - f(k-1))| \leqslant r,$$

so $|f(k) - f(k-1)| \geqslant 1 - r > 0$. However, such an f cannot be bounded.

Since $\operatorname{dist}_{\overline{Y}}(\mathbb{1}_{\mathbb{Z}}) = \operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}})$, the Hahn–Banach separation theorems provide $\mu \in (\ell_\infty(\mathbb{Z}))^*$ with $\|\mu\|_{op} = 1$, $\mu|_{\overline{Y}} = 0$, and $\mu(\mathbb{1}_{\mathbb{Z}}) = \operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$.

Since $\|\mu\|_{op} = 1$ and $\mu(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$, we must have $\mu(\mathbb{1}_{\mathbb{Z}}) = 1$.

Additionally, since $\|\mu\|_{op} = \mu(\mathbb{1}_{\mathbb{Z}}) = 1$, we have that μ is a state on $\ell_\infty(\mathbb{Z})$, and since $\mu(y) = 0$ for all $y \in Y$, we have

$$
\begin{aligned}
\mu(f - \lambda_1(f)) &= 0 \\
\mu(f) &= \mu(\lambda_1(f)).
\end{aligned}
$$

Inductively, this means that $\mu(f) = \mu(\lambda_k(f))$ for all $k \in \mathbb{Z}$, so μ is an invariant state on $\ell_\infty(\mathbb{Z})$. Thus, $\mathbb{Z}$ is amenable. □

Proposition 3.2.2. If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.

Proof. Let $\rho \in (\ell_\infty(G/N))^*$ be an invariant state, and let $p \colon P(N) \to [0,1]$ be a mean. For $E \subseteq G$, we define $f_E \colon G/N \to \mathbb{R}$ by

$$f_E(tN) = p\left(N \cap t^{-1}E\right).$$

We start by verifying that f_E is well-defined. For $tN = sN$, we have $s^{-1}t \in N$, so

$$p\left(N \cap t^{-1}E\right) = p\left(s^{-1}t\left(N \cap t^{-1}E\right)\right)$$

$$
\begin{aligned}
&= p\left(s^{-1}tN \cap s^{-1}E\right) \\
&= p\left(N \cap s^{-1}E\right).
\end{aligned}
$$

Since f_E is defined through p, we can see that f_E is bounded. Additionally,

$$
\begin{aligned}
f_{E \sqcup F}(tN) &= p\left(N \cap t^{-1}(E \sqcup F)\right) \\
&= p\left(N \cap \left(t^{-1}E \sqcup t^{-1}F\right)\right) \\
&= p\left(\left(N \cap t^{-1}E\right) \sqcup \left(N \cap t^{-1}F\right)\right) \\
&= p\left(N \cap t^{-1}E\right) + p\left(N \cap t^{-1}F\right) \\
&= f_E(tN) + f_F(tN) \\
&= (f_E + f_F)(tN),
\end{aligned}
$$

and

$$
\begin{aligned}
f(sE)(tN) &= p\left(N \cap t^{-1}sE\right) \\
&= f_E\left(s^{-1}tN\right) \\
&= \lambda_{sN}(f_E)(tN),
\end{aligned}
$$

so $f_{sE} = \lambda_{sN}(f_E)$. Finally,

$$
\begin{aligned}
f_G(tN) &= p\left(N \cap t^{-1}G\right) \\
&= p(N) \\
&= 1,
\end{aligned}
$$

meaning $f_G = \mathbb{1}_{G/N}$.

We define $m\colon P(G) \to [0,1]$ by

$$
m(E) = \rho(f_E).
$$

Then, we have

$$
m(E \sqcup F) = m(E) + m(F)
$$

$$
m(G) = 1
$$

$$
\begin{aligned}
m(sE) &= \rho(f_{sE}) \\
&= \rho(\lambda_{sN}(f_E)) \\
&= \rho(f_E) \\
&= m(E),
\end{aligned}
$$

so m is a mean. □

Corollary 3.2.1. The finite direct product of amenable groups is amenable.

Proof. If H and K are amenable, then $K \cong (H \times K)/H$ is amenable and H is amenable, so $H \times K$ is amenable by Proposition 3.2.2. Induction provides the general case. □

Corollary 3.2.2. Finitely generated abelian groups are amenable.

Proof. By the fundamental theorem of finitely generated abelian groups (Theorem **??**), all finitely generated abelian groups are isomorphic to $\mathbb{Z}^d \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$.

Since $\mathbb{Z}^d$ is a finite direct product of $\mathbb{Z}$, and the torsion subgroup $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$ is finite (hence amenable by Example 3.1.1), we see that a finitely generated abelian group is a direct product of two amenable groups, hence the finitely generated abelian group is amenable by Corollary 3.2.2. □

Corollary 3.2.3. If $\{G_i\}_{i\in I}$ is a directed family of amenable groups, then their union,

$$G = \bigcup_{i\in I} G_i,$$

is also amenable.

Proof. Let $\mu_i \in (\ell_\infty(G_i))^*$ be invariant states.

Set

$$M_i = \{\mu \in S(\ell_\infty(G)) \mid \mu(\lambda_s(f)) = \mu(f) \text{ for all } s \in G_i\}.$$

We set $\mu(f) = \mu_i(f|_{G_i})$. Since each G_i is amenable, it is the case that each M_i is nonempty. Similarly, seeing as we have established the state space as w^*-closed in $B_{\ell_\infty(G)^*}$, it is the case that each M_i is w^*-closed in $B_{\ell_\infty(G)^*}$.

For $i_1, \dots, i_n$, we find $G_j \supseteq G_{i_1}, \dots, G_{i_n}$, which exists since $\{G_i\}_{i\in I}$ is directed. We have that $M_j \subseteq \bigcap_{k=1}^n M_{i_k}$, so $\{M_i\}_{i\in I}$ has the finite intersection property.

Since the set of states is w^*-compact, there is $\mu \in \bigcap_{i\in I} M_i$ which is necessarily invariant on G. □

Corollary 3.2.4. All abelian groups are amenable.

Proof. Every abelian group is the direct limit of its finitely generated subgroups. □

Corollary 3.2.5. All solvable groups are amenable.

Proof. Let $e_G = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n \leqslant G$ be such that $G_{j-1} \trianglelefteq G_j$ for $j = 1, \dots, n$, and G_i/G_j is abelian.

Since G_0 is abelian, it is amenable, as is G_1/G_0, so G_1 is amenable. We see then that G_2 is amenable as G_1 and G_2/G_1 are amenable.

Continuing in this fashion, we see that G is amenable. □

4 Følner's Condition and Amenability

Amenability, as stated earlier, is defined by a particular finitely additive, translation-invariant probability measure on the group. Of all the three conditions, the "finitely additive" and "probability measure" conditions are straightforward — we may define a measure δ_x on $P(G)$ by saying that $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ if $x \notin E$. This is a finitely additive probability measure — but it is not translation-invariant.

Indeed, the translation-invariance condition is what throws a wrench into our desire to establish means on various types of groups. For instance, we desired a translation-invariant, finitely additive probability measure on $F(a,b)$, but since, for instance $bW(b^{-1})$ is effectively equal to $F(a,b) \setminus W(b)$, we see that translating

$W(b^{-1})$ by b creates a "bigger" subset than we desire, closing off our ability to construct a mean.

As the reader may remark by now, this is an extremely nonspecific idea. What does it mean for a set to become "bigger" under translation, and how much "bigger" does it need to become in order to close off the possibility of establishing a mean on the group?

We can make the idea of "bigness" precise by considering the symmetric difference of a translated set and the original set — if such a symmetric difference is small (or tends to zero as our subsets become sufficiently "large" in the group), then we can expect that $m(E) \approx m(tE)$. Indeed, this is substance of the Følner condition.

4.1 Følner's Condition

Definition 4.1.1. A group is said to satisfy the *Følner condition* if, for every $\varepsilon > 0$ and $E \subseteq G$, there is a nonempty finite subset $F \subseteq G$ such that for all $t \in E$,

$$\frac{|tF \triangle F|}{|F|} \leqslant \varepsilon.$$

Equivalently, we can also say that the Følner condition is satisfied if and only if

$$\frac{|tF \cap F|}{|F|} \geqslant 1 - \varepsilon$$

for every $\varepsilon > 0$.

Lemma 4.1.1. A countable group G satisfies the Følner condition if and only if G admits a sequence $(F_n)_n$ with $F_n \subseteq G$ finite such that

$$\left(\frac{|tF_n \triangle F_n|}{|F_n|} \right)_n \xrightarrow{n \to \infty} 0$$

for all $t \in G$. Such a sequence is known as a *Følner sequence*.

Proof. Let G admit a Følner sequence, $(F_n)_n$. Given $\varepsilon > 0$ and $E \subseteq G$ finite, find N such that for all $s \in E$ and $n \geqslant N$,

$$\frac{|sF_n \triangle F_n|}{|F_n|} \leqslant \varepsilon.$$

We take $F = F_N$ in the definition of the Følner condition.

Let G satisfy the Følner condition. We write $G = \bigcup_{n \geqslant 1} E_n$, with $E_1 \subseteq E_2 \subseteq \cdots$, and define F_n such that for all $t \in E_n$,

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}.$$

Given $t \in G$, then $t \in E_N$ for some N, so $t \in E_n$ For all $n \geqslant N$, so

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}$$

for all $n \geqslant N$. Thus,

$$\left(\frac{|tF_n \triangle F_n|}{|F_n|} \right) \xrightarrow{n \to \infty} 0.$$

$\square$

What makes Følner sequences so powerful is that we only need to determine if they exist on a generating set for our group, assuming our group is finitely generated (and countable).

Lemma 4.1.2. Let G be a finitely generated group with generating set S (see Definition **??**). If $(F_n)_n$ is a sequence of finite subsets such that, for all $s \in S$,

$$\left(\frac{|sF_n \triangle F_n|}{|F_n|}\right)_n \to 0,$$

then $(F_n)_n$ is a Følner sequence for G.

Proof. Note that

- $s(A \triangle B) = sA \triangle sB$;
- $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

We see that for any $s \in S$,

$$\begin{aligned}\frac{\left|s^{-1}F_n \triangle F_n\right|}{|F_n|} &= \frac{\left|s^{-1}(F_n \triangle sF_n)\right|}{|F_n|} \\ &= \frac{|F_n \triangle sF_n|}{|F_n|} \\ &\to 0.\end{aligned}$$

Thus, we may assume that S is symmetric — i.e., that $\left\{s^{-1} \,\middle|\, s \in S\right\} = \{s \mid s \in S\}$.

For any $s, t \in S$, we have

$$\begin{aligned}\frac{|stF_n \triangle F_n|}{|F_n|} &\leqslant \frac{|stF_n \triangle F_n|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\ &= \frac{|s(tF_n \triangle F_n)|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\ &= \frac{|tF_n \triangle F_n|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\ &\to 0.\end{aligned}$$

We use induction to find the general case. □

Example 4.1.1. Consider the group $\mathbb{Z}$. Since $\mathbb{Z}$ is generated by the element $\{1\}$, we see that for the sets $F_n = \{-n, -n+1, \ldots, n-1, n\}$, that

$$\begin{aligned}\frac{|(F_n + 1) \triangle F_n|}{|F_n|} &= \frac{2}{2n+1} \\ &\to 0,\end{aligned}$$

meaning that $\mathbb{Z}$ satisfies the Følner condition.

4.2 Establishing Amenability through Følner Sequences

Now that we have a better understanding of Følner sequences, we will now see how to establish that the existence of a Følner sequences is equivalent to the group being amenable. By the end of this subsection, we will have established the following theorem, incorporating results from the previous sections.

Theorem 4.2.1. Let G be a group. The following are equivalent:

- G is non-paradoxical;
- G admits a mean;
- $\ell_\infty(G)$ admits an invariant state;
- G admits a Følner sequence;
- G satisfies the Følner condition;
- G admits an *approximate mean*.

Definition 4.2.1. For a group G, we define

$$\mathrm{Prob}(G) = \left\{ f\colon G \to [0,\infty) \,\middle|\, \mathrm{card}\big(\mathrm{supp}(f)\big) < \infty,\ \sum_{t \in G} f(t) = 1 \right\}.$$

Note that $\mathrm{Prob}(G) \subseteq B_{\ell_1(G)}$. For $f \in \mathrm{prob}(G)$, we set $\varphi_f\colon \ell_\infty(G) \to \mathbb{C}$ defined by

$$\varphi_f(g) = \sum_{t \in G} g(t)f(t).$$

Fact 4.2.1. For $f \in \mathrm{prob}(G)$, φ_f is a state on $\ell_\infty(G)$.

Proof. We can see that, by definition, φ_f is positive, linear, and has $\varphi_f(\mathbb{1}_G) = 1$.

We only need to show that $\|\varphi_f\|_{op} = 1$. We see that

$$\begin{aligned} |\varphi_f(g)| &= \left| \sum_{t \in G} g(t)f(t) \right| \\ &\leqslant \sum_{t \in G} |g(t)||f(t)| \\ &\leqslant \|g\|_{\ell_\infty} \sum_{t \in G} |f(t)| \\ &= \|g\|_{\ell_\infty}, \end{aligned}$$

so $\|\varphi_f\|_{op} \leqslant 1$. Since $\varphi_f(\mathbb{1}_G) = 1$, we must have $\|\varphi_f\|_{op} = 1$. ◻

Proposition 4.2.1. There is an action $\lambda\colon G \to \mathrm{Isom}(\ell_1(G))$ such that $\mathrm{prob}(G)$ is invariant.

Proof. Let $\lambda_s(f)(t) = f\big(s^{-1}t\big)$. Then,

$$\begin{aligned} \|\lambda_s(f)\|_{\ell_1} &= \sum_{t \in G} |\lambda_s(f)(t)| \\ &= \sum_{t \in G} \left| f\left(s^{-1}t\right) \right| \\ &= \sum_{r \in G} |f(r)| \\ &= \|f\|_{\ell_1}. \end{aligned}$$

Just as in Proposition 3.1.2, it is the case that λ_s is linear. Additionally,

$$\lambda_r \circ \lambda_s(f)(t) = \lambda_s(f)\left(r^{-1}t\right)$$

$$
\begin{aligned}
&= f\left(s^{-1}r^{-1}(t)\right) \\
&= f\left((rs)^{-1}t\right) \\
&= \lambda_{rs}(f)(t).
\end{aligned}
$$

We see that if $f \in \mathrm{prob}(G)$, then for $f \geqslant 0$, we have $\lambda_s(f) \geqslant 0$, and

$$
\begin{aligned}
\sum_{t \in G} \lambda_s(f)(t) &= \sum_{t \in G} f\left(s^{-1}t\right) \\
&= \sum_{r \in G} f(r) \\
&= 1
\end{aligned}
$$

for any $f \in \mathrm{prob}(G)$. □

Definition 4.2.2. For a countable group G, a sequence $(f_k)_k$ is called an approximate mean if, for all $s \in G$,

$$
\|f_k - \lambda_s(f_k)\|_{\ell_1} \xrightarrow{k \to \infty} 0.
$$

To begin the forward direction regarding the equivalence between the Følner condition, approximate means, and means, we begin by showing that the existence of a Følner sequence implies the existence of an approximate mean. Then, we will show that the existence of an approximate mean implies the existence of an invariant state (hence mean).

Proposition 4.2.2. If G admits a Følner sequence $(F_k)_k$, then G admits an approximate mean.

Proof. Set $f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k} \in \mathrm{prob}(G)$. Then,

$$
\begin{aligned}
\|f_k - \lambda_s(f_k)\|_{\ell_1} &= \frac{1}{|F_k|}\|\mathbb{1}_{F_k} - \lambda_s(\mathbb{1}_{F_k})\|_{\ell_1} \\
&= \frac{1}{F_k}\|\mathbb{1}_{F_k} - \mathbb{1}_{sF_k}\|_{\ell_1} \\
&= \frac{|F_k \Delta sF_k|}{|F_k|} \\
&\to 0.
\end{aligned}
$$

□

Proposition 4.2.3. If G admits an approximate mean, then G is amenable.

Proof. Let $(f_k)_k$ be an approximate mean. We define $\varphi_k = (\varphi_{f_k})_k$ (as in Definition 4.2.1) to be a sequence of states on $\ell_\infty(G)$.

Since the state space on $\ell_\infty(G)$ is w^*-compact, there is a state μ and a subnet $\left(\varphi_{k_j}\right)_j \xrightarrow{w^*} \mu$.

We only need to show that μ is invariant. Note that

$$
|\mu(g) - \mu(\lambda_s(g))| \leqslant \left|\mu(g) - \varphi_{k_j}(g)\right| + \left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| + \left|\varphi_{k_j}(\lambda_s(g)) - \mu(\lambda_s(g))\right|
$$

for all $g \in \ell_\infty(G)$, $s \in G$, and all j.

Given $\varepsilon > 0$, we find J such that for $j \geqslant J$,

$$
\left|\mu(g) - \varphi_{k_j}(g)\right| < \varepsilon/3
$$

$$\left|\mu(\lambda_s(g)) - \varphi_{k_j}(\lambda_s(g))\right| < \varepsilon/3.$$

We also see that

$$\begin{aligned}
\left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| &= \left|\sum_{t\in G} g(t)f_{k_j}(t) - \sum_{t\in G} g\left(s^{-1}t\right)f_{k_j}(t)\right| \\
&= \left|\sum_{t\in G} g(t)f_{k_j}(t) - \sum_{r\in G} g(r)f_{k_j}(sr)\right| && r = s^{-1}t \\
&= \left|\sum_{t\in G} g(t)\left(f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right)\right| \\
&\leqslant \|g\|_{\ell_\infty} \sum_{t\in G}\left|f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right| \\
&= \|g\|_{\ell_\infty}\left\|f_{k_j} - \lambda_{s^{-1}}\left(f_{k_j}\right)\right\|_{\ell_1} \\
&< \varepsilon/3
\end{aligned}$$

for large j. Thus, we have

$$|\mu(g) - \mu(\lambda_s(g))| < \varepsilon,$$

for all $\varepsilon > 0$, so $\mu(g) = \mu(\lambda_s(g))$. □

We will now show that amenability implies the existence of an approximate mean, after which we will show that the existence of an approximate mean implies Følner's condition.

Proposition 4.2.4. If G is amenable, then G admits an approximate mean.

Proof. Suppose G does not admit an approximate mean. Then, there exists a finite subset $E_0 \subseteq G$ and $\varepsilon_0 > 0$ such that for all $s \in E_0$ and all $f \in \mathrm{Prob}(G)$, we have $\|f - \lambda_s(f)\| \geqslant \varepsilon_0$.

Consider the set

$$X = \bigoplus_{|E_0|} \ell_1(G),$$

endowed with the norm

$$\begin{aligned}
\left\|(f_s)_{s\in E_0}\right\|_{\ell_1} &= \sum_{s\in E_0}\sum_{t\in G}|f_s(t)| \\
&= \sum_{s\in E_0}\|f_s\|_{\ell_1},
\end{aligned}$$

and let

$$C = \left\{(f - \lambda_s(f))_{s\in E_0} \;\middle|\; f \in \mathrm{Prob}(G)\right\}.$$

Since $\mathrm{Prob}(G)$ is convex, it is the case that C is convex, and since $|E_0|$ is finite, C is necessarily bounded. Note that $0 \notin \overline{C}$.

By the Hahn–Banach separation for convex sets (Theorem **??**), there is a real-valued $\varphi \in X^*$ such that $\varphi(C) \geqslant 1$. Here,

$$X^* \cong \bigoplus_{|E_0|} \ell_1(G)^*$$

$$\cong \sum_{|E_0|} \ell_\infty(G),$$

endowed with the norm

$$\begin{aligned}\left\|(g_s)_{s\in E_0}\right\|_{\ell_\infty} &= \max_{s\in E_0}\left(\sup_{t\in G}|g_s(t)|\right)\\ &= \max_{s\in E_0}\|g_s\|_{\ell_\infty}.\end{aligned}$$

We let $\varphi = \left(\varphi_{g_s}\right)_{s\in E_0}$, where $g_s \in \ell_\infty(G)$ is defined by the duality

$$\varphi_{g_s}(f) = \sum_{t\in G} f(t)g_s(t).$$

Thus, for all $f \in \operatorname{Prob}(G)$, we have

$$\begin{aligned}1 &\leqslant \varphi\left((f-\lambda_s(f))_{s\in E_0}\right)\\ &= \sum_{s\in E_0} \varphi_{g_s}(f-\lambda-s(f))\\ &= \sum_{s\in E_0}\sum_{t\in G}(f-\lambda_s(f))(t)g_s(t)\\ &= \sum_{s\in E_0}\left(\sum_{t\in G} f(t)g_s(t) - \sum_{t\in G} f\left(s^{-1}t\right)g_s(t)\right)\\ &= \sum_{s\in E_0}\left(\sum_{t\in G} f(t)g_s(t) - \sum_{r\in G} f(r)g_s(sr)\right)\\ &= \sum_{s\in E_0}\left(\sum_{r\in G} f(r)g_s(r) - \sum_{r\in G} f(r)\lambda_{s^{-1}}(g)(r)\right)\\ &= \sum_{s\in E_0}\sum_{r\in G} f(r)(g_s-\lambda_{s^{-1}}(g_s))(r).\end{aligned}$$

Note that this holds for any $f \in \operatorname{Prob}(G)$, including the case of $f = \delta_t$ for a given $t \in G$. We must have

$$\begin{aligned}&= \sum_{s\in E_0}\sum_{r\in G}\delta_t(r)(g_s(r)-\lambda_{s^{-1}}(g_s))(r)\\ &= \sum_{s\in E_0}(g_s-\lambda_{s^{-1}}(g_s))(t).\end{aligned}$$

This gives

$$\mathbb{1}_G \leqslant \sum_{s\in E_0}(g_s-\lambda_{s^{-1}}(g_s))(t).$$

Since G is amenable, there is a mean $\mu\colon \ell_\infty(G) \to \mathbb{C}$ with $\mu(g_s) = \mu(\lambda_{s^{-1}}(g_s))$, meaning

$$\begin{aligned}0 &= \mu\left(\sum_{s\in E_0}(g_s-\lambda_{s^{-1}}(g_s))(t)\right)\\ &\geqslant \mu(\mathbb{1}_G)\\ &= 1,\end{aligned}$$

which is a contradiction. □

To show that the existence of an approximate mean implies the Følner condition, we require the following lemma.

Lemma 4.2.1. Let $f\colon S \to \mathbb{R}$ be finitely supported with $\sum_{s\in S} f(s) = 1$. Then, there exist subsets $\{F_i\}_{i=1}^n$, where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$, and constants $\{c_i\}_{i=1}^n$, such that

$$f = \sum_{i=1}^n c_i \mathbb{1}_{F_i},$$

where

$$\sum_{i=1}^n c_i |F_i| = 1.$$

This is known as the layer cake representation for f.

Proof. We define $F_1 = \mathrm{supp}(f)$, and take $c_1 = \min(\mathrm{Ran}(f))$. Taking $E_1 = f^{-1}(c_1)$ (as a set-theoretic inverse), we define $F_2 = F_1 \setminus E_1$.

Take $d_1 = \min(f(F_2))$, and define $c_2 = d_1 - c_1$. Then, defining $E_2 = f^{-1}(d_1)$, $F_3 = F_2 \setminus E_2$, and $d_2 = \min(f(F_3))$, we define $c_3 = d_2 - c_2 - c_1$.

Continuing in this pattern, we find $d_{i-1} = \min(f(F_i))$, $E_i = f^{-1}(d_{i-1})$, and $c_i = d_{i-1} - \sum_{j=1}^{i-1} c_i$.

This yields a decomposition $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$, where $\sum_{i=1}^n c_i \mathbb{1}_{F_i} = f$ by construction.

We now verify that $\sum_{i=1}^n c_i |F_i| = 1$.

$$\begin{aligned} 1 &= \sum_{s\in S} f(s) \\ &= \sum_{s\in S} \sum_{i=1}^n c_i \mathbb{1}_{F_i}(s). \end{aligned}$$

By definition, if $s \in F_j$ for some j, we see that c_j is summed for $\left|F_j\right|$ many times. Thus, we obtain

$$= \sum_{i=1}^n c_i |F_i|.$$

□

Remark 4.2.1. Instead of using this construction where we take set-theoretic inverses and remove "residual" sets, there is an alternative method of construction that involves ordering the range as $r_1 < r_2 < \cdots < r_n$, and constructing the set $F_k = \{s \mid f(s) \geqslant r_k\}$.

We will use the layer cake decomposition to prove that if G admits an approximate mean, then G satisfies the Følner condition.

Proposition 4.2.5. Let G admit an approximate mean. Then, G satisfies the Følner condition.

Proof. Let $(f_k)_k$ be an approximate mean, as in Definition 4.2.2. Fix a finite nonempty set $S \subseteq G$. Then, by the definition of an approximate mean, there must exist some $N \in \mathbb{N}$ such that for all $k \geqslant N$ and all $s \in G$,

$$\|f_k - \lambda_s(f_k)\|_{\ell_1} \leqslant \frac{\varepsilon}{|S|}.$$

In particular, this holds for f_N and for all $s \in S$.

Since $f_N \in \text{Prob}(G)$ is finitely supported and $\sum_{s\in G} f_N(s) = 1$, we may use Lemma 4.2.1 to rewrite f_N as

$$f_N = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$, and $\sum_{i=1}^{n} c_i|F_i| = 1$.

For a given $1 \leqslant i \leqslant n$, for each $s \in S$ and $t \in sF_i \triangle F_i$, we have

$$f_N(t) - f_N\left(s^{-1}t\right) = \begin{cases} c_i & t \in F_i \setminus sF_i \\ -c_i & t \in sF_i \setminus F_i \end{cases}.$$

Thus, we see that $|f_N(t) - \lambda_s(f_N)(t)| \geqslant c_i$ on $sF_i \triangle F_i$. Thus, for each $s \in S$,

$$\begin{aligned} \sum_{i=1}^{n} c_i |sF_i \triangle F_i| &\leqslant \sum_{t\in S} |f_N(t) - \lambda_s(f)(t)| \\ &< \frac{\varepsilon}{|S|} \\ &= \frac{\varepsilon}{|S|} \sum_{i=1}^{n} c_i |F_i|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sum_{s\in S} \sum_{i=1}^{n} c_i |sF_i \triangle F_i| &< \frac{\varepsilon}{|S|} \sum_{s\in S} \sum_{i=1}^{n} c_i |F_i| \\ &= \varepsilon \sum_{i=1}^{n} c_i |F_i|. \end{aligned}$$

Thus, by the pigeonhole principle, there must exist some $1 \leqslant i \leqslant n$ for which

$$\sum_{s\in S} c_i |sF_i \triangle F_i| < \varepsilon c_i |F_i|.$$

Setting $F = F_i$, we find that, for all $s \in S$,

$$\begin{aligned} \frac{|sF \triangle F|}{|F|} &\leqslant \sum_{s\in S} \frac{|sF \triangle F|}{|F|} \\ &< \varepsilon. \end{aligned}$$

□

4.3 An Application of Følner's Condition: Groups of Subexponential Growth

Just as at the end of Section 3, we established the amenability of all the abelian and solvable groups, so too will we be able to use the Følner condition to establish the amenability of a wide class of groups. Specifically, we will establish that a certain class of groups commonly seen in the field of geometric group theory, the groups of subexponential growth, are amenable.

First, we construct a little bit of machinery to understand the growth rate of a group, then we prove that Følner's condition holds for these special classes of groups.

Definition 4.3.1. Let G be a group with finite symmetric generating set S (see Definition **??**). Then, we define the word length of $g \in G$ with respect to S to be

$$\ell_{G,S}(g) = \min\{n \mid g = s_1 \dots s_n,\ s_i \in S\},$$

taking $\ell_{G,S}(e_G) = 0$. We define the word metric on G with respect to S by taking

$$d_S(g,h) = \ell_{G,S}\left(g^{-1}h\right).$$

Fact 4.3.1. If S and T are finite symmetric generating sets for G, then the respective word metrics d_S and d_T are equivalent (as in the sense of Definition **??**).

Proof. We start by showing that d_S is indeed a metric. Notice that the following facts necessarily hold by our definition of the word length:

- $\ell_{G,S}(g) = \ell_{G,S}(g^{-1})$;
- $\ell_{G,S}(gh) \leqslant \ell_{G,S}(g) + \ell_{G,S}(h)$.

We thus have:

$$\begin{aligned} d_S(g,h) &= \ell_{G,S}\left(g^{-1}h\right) \\ &= \ell_{G,S}\left(h^{-1}g\right) \\ &= d_S(h,g) \end{aligned}$$

$$\begin{aligned} d_S(g,h) &= \ell_{G,S}\left(g^{-1}h\right) \\ &= \ell_{G.S}\left(g^{-1}kk^{-1}h\right) \\ &\leqslant \ell_{G,S}\left(g^{-1}k\right) + \ell_{G,S}\left(k^{-1}h\right) \\ &= d_S(g,k) + d_S(k,h) \end{aligned}$$

$$\begin{aligned} d_S(g,g) &= \ell_{G,S}\left(g^{-1}g\right) \\ &= \ell_{G,S}(e_G) \\ &= 0 \\ d_S(g,h) = 0 &\Leftrightarrow \ell_{G,S}\left(g^{-1}h\right) = 0 \\ &\Leftrightarrow g^{-1}h = e_G \\ &\Leftrightarrow g = h. \end{aligned}$$

Thus, d_S is indeed a metric.

Let S and T be finite symmetric generating sets for G. It is sufficient to show that there exists some $k \in \mathbb{N}$ such that, for all $g \in G$,

$$\frac{1}{k}\ell_{G,S}(g) \leqslant \ell_{G,T}(g) \leqslant k\ell_{G,S}(g).$$

Set

$$\begin{aligned} M &= \max\{\ell_{G,T}(s) \mid s \in S\} \\ N &= \max\{\ell_{G,S}(t) \mid t \in T\}. \end{aligned}$$

Now, let $n = \ell_{G,S}(g)$, such that $g = s_1 \cdots s_n$, where $s_i \in S$. Then, we have

$$\begin{aligned}\ell_{G,T}(g) &= \ell_{G,T}(s_1 \cdots s_n)\\ &\leqslant \ell_{G,T}(s_1) + \cdots + \ell_{G,T}(s_n)\\ &\leqslant M\ell_{G,S}(g),\end{aligned}$$

and similarly, $\ell_{G,S}(g) \leqslant N\ell_{G,T}(g)$. Setting $k = \max(M, N)$, we get

$$\frac{1}{k}\ell_{G,S}(g) \leqslant \ell_{G,T}(g) \leqslant k\ell_{G,S}(g).$$

□

Now, we may begin defining the growth rate of a group. We will use the fact that all word metrics with respect to a generating set are symmetric in order to show that the growth rate is well-defined (i.e., independent of the generating set for G).

Definition 4.3.2. Let G be a group with finite symmetric generating set S. We define

$$\begin{aligned}&B_{G,S}(n) = \{g \in G \mid \ell_{G,S}(g) \leqslant n\};\\ &\gamma_{G,S}(n) = |B_{G,S}(n)|.\end{aligned}$$

The following facts hold for γ.

Fact 4.3.2. Let G be a finitely generated group. The following facts hold:

(1) $\gamma_{G,S}(n)$ is an increasing function;

(2) $\gamma_{G,S}(n + m) \leqslant \gamma_{G,S}(n)\gamma_{G,S}(m)$;

(3) $\lim_{n\to\infty} (\gamma_{G,S}(n))^{1/n} = \rho_{G,S}$ exists;

(4) if S and T are finite symmetric generating sets for G, then there exists $k \in \mathbb{N}$ such that $\gamma_{G,T}(n) \leqslant \gamma_{G,S}(kn)$ for all $n \in \mathbb{N}$, and $\rho_{G,S} = \rho_{G,T}$.

Proof.

(1) Since $B_{G,S}(n) \subseteq B_{G,S}(n + 1)$, we have $\gamma_{G,S}(n) \leqslant \gamma_{G,S}(n + 1)$, so $\gamma_{G,S}$ is increasing.

(2) We start by showing that $B_{G,S}(n)B_{G,S}(m) = B_{G,S}(n + m)$. First, if $g \in B_{G,S}(n)$ and $h \in B_{G,S}(m)$, we know that $\ell_{G,S}(gh) \leqslant \ell_{G,S}(g) + \ell_{G,S}(h) \leqslant m + n$, so $B_{G,S}(n)B_{G,S}(n) \subseteq B_{G,S}(n + m)$. Additionally, if $g \in B_{G,S}(n + m)$, we may write

$$g = \underbrace{s_1 \cdots s_\ell}_{g_1}\,\underbrace{s_{\ell+1} \cdots s_k}_{g_2},$$

where $k \leqslant n + m$, $\ell \leqslant n$, and $k - \ell \leqslant m$, so $g_1 \in B_{G,S}(n)$ and $g_2 \in B_{G,S}(m)$. Thus, we have $B_{G,S}(n)B_{G,S}(m) = B_{G,S}(n + m)$.

Now, we have

$$\begin{aligned}\gamma_{G,S}(n + m) &= |B_{G,S}(n + m)|\\ &= |B_{G,S}(n)B_{G,S}(m)|\\ &\leqslant |B_{G,S}(n)||B_{G,S}(m)|\\ &= \gamma_{G,S}(n)\gamma_{G,S}(m).\end{aligned}$$

(3) From (2), we know that $\gamma_{G,S}(n) \leqslant \gamma_{G,S}(1)^n$. Inductively, we have

$$\gamma_{G,S}(n + 1) \leqslant \gamma_{G,S}(1)^{n+1},$$

and thus,

$$1 \leqslant \gamma_{G,S}(n)^{1/n} \leqslant \gamma_{G,S}(1).$$

(4) We know that there exists k such that $\frac{1}{k}\ell_{G,S} \leqslant \ell_{G,T} \leqslant k\ell_{G,S}$ by the proof of Fact 4.3.1. Thus, if $g \in B_{G,T}(n)$, then $\ell_{G,T}(g) \leqslant n$, so $\ell_{G,S}(g) \leqslant kn$, so $g \in B_{G,S}(kn)$ and $B_{G,T}(n) \subseteq B_{G,T}(kn)$. We have $\gamma_{G,T}(n) \leqslant \gamma_{G,S}(kn)$.

Similarly, if $g \in B_{G,S}(n)$, then $\ell_{G,S}(g) \leqslant n$, so $\ell_{G,T}(g) \leqslant kn$, and $g \in B_{G,T}(kn)$. Thus, we get $B_{G,S}(n) \subseteq B_{G,T}(kn)$, so $\gamma_{G,S}(n) \leqslant \gamma_{G,T}(kn)$.

It follows that

$$\gamma_{G,S}\left(\frac{n}{k}\right)^{1/n} \leqslant \gamma_{G,T}(n)^{1/n} \leqslant \left(\gamma_{G,S}(kn)^k\right)^{1/kn}.$$

Sending $n \to \infty$, we get $\rho_{G,S} \leqslant \rho_{G,T} \leqslant \rho_{G,S}$, so $\rho_{G,S} = \rho_{G,T}$.

□

Definition 4.3.3. Let G be a group with finite symmetric generating set S. The quantity

$$\rho_G = \limsup_{n\to\infty} \gamma_{G,S}(n)^{1/n}$$

is known as the growth rate of the group G. If we have $\rho = 1$, then we say G is of subexponential growth.

Fact 4.3.3. All finite groups are of subexponential growth.

Proof. Note that since ρ is independent of the generating set (as we proved in Fact 4.3.2), we can set $S = G$, and we have $\limsup_{n\to\infty}|G|^{1/n} = 1$. □

Fact 4.3.4. Let Γ be a finitely generated abelian group. Then, Γ is of subexponential growth.

Proof. We start by showing that $G = \mathbb{Z}^d$ is of subexponential growth. Notice that every element of $\mathbb{Z}^d$ is some linear combination of the set

$$S = \{e_1, e_2, \dots, e_d\}, \tag{$*$}$$

where

$$e_j = (0, 0, \dots, \underbrace{1}_{\text{position } j}, 0, 0, \dots).$$

Additionally, we see that any element of $B_{G,S}(n)$ is of the form $e_1^{i_1} e_2^{i_2} \dots e_d^{i_d}$, where $\sum_{j=1}^{d} i_j \leqslant n$. Thus, we must have $\gamma_{G,S}(n) \leqslant n^d$, meaning that

$$\begin{aligned}\rho &= \limsup_{n\to\infty} \gamma_{G,S}(n)^{1/n}\\ &= \limsup_{n\to\infty} n^{d/n}\\ &= 1,\end{aligned}$$

so $\mathbb{Z}^d$ is of subexponential growth.

Now, if $G' = \mathbb{Z}^d \times \mathbb{Z}/k_1\mathbb{Z} \times \cdots \times \mathbb{Z}/k_r\mathbb{Z}$, then since there is a torsion subgroup in G', we must have $\gamma_{G',S'}(n) \leqslant \gamma_{\mathbb{Z}^{d+r},T}(n)$ for any n, where T is a generating set for $\mathbb{Z}^{d+r}$ and S' is a generating set for G'.

Since

$$\rho_{\mathbb{Z}^{d+r}} = \limsup_{n\to\infty} \gamma_{\mathbb{Z}^{d+r},T}(n)^{1/n}$$
$$= 1,$$

and $1 \leqslant \gamma_{G',S'}(n)$, we must have $\rho_{G'} = 1$.

Since, by the fundamental theorem of finitely generated abelian groups (Theorem **??**), it is the case that $\Gamma \cong \mathbb{Z}^d \times \mathbb{Z}/k_1\mathbb{Z} \times \cdots \times \mathbb{Z}/k_r\mathbb{Z}$ for some $d, k_1, \ldots, k_r \in \mathbb{N}$, Γ is of subexponential growth. □

To prove that the groups of subexponential growth are amenable, we use the following lemma from real analysis.

Lemma 4.3.1. Let $(a_n)_n$ be a sequence such that $a_n > 0$ for each n. Then,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (a_n)^{1/n}.$$

Similarly,

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \limsup_{n\to\infty} (a_n)^{1/n}.$$

Theorem 4.3.1. Let Γ be a finitely generated group of subexponential growth. Then, Γ is amenable.

Proof. To prove that Γ is amenable, we show that it satisfies the Følner condition. From the results in Section **??**, we know that this implies that Γ is amenable. Let S be a finite symmetric generating set for Γ.

For any $\varepsilon > 0$, we see that there is some $k \in \mathbb{N}$ such that

$$|B_{\Gamma,S}(k)|^{1/k} \leqslant 1 + \varepsilon.$$

Thus, by the lemma above, we must have

$$\frac{|B_{\Gamma,S}(k+1)|}{|B_{\Gamma,S}(k)|} \leqslant 1 + \varepsilon.$$

Note that, by Lemma 4.1.2, we only need to verify that the Følner condition holds on S. For any $s \in S$, we have

$$\frac{|sB_{\Gamma,S}(k) \triangle B_{\Gamma,S}(k)|}{|B_{\Gamma,S}(k)|} \leqslant \frac{2(|B_{G,S}(k+1)| - |B_{\Gamma,S}(k)|)}{|B_{\Gamma,S}(k)|}$$
$$\leqslant 2\varepsilon.$$

Therefore, Γ satisfies the Følner condition, hence is amenable. □

Remark 4.3.1. The result in Theorem 4.3.1 can be used along with Fact 4.3.4 and Corollary 3.2.3 to prove Corollary 3.2.4.

5 Remarks and Notes

In [BO08, p. 48], the authors state that "amenable groups admit approximately $10^{10^{10}}$ characterizations." Unfortunately, despite my best efforts, I was not able to fit all $10^{10^{10}}$ characterizations in this thesis. However, we can provide some details on some of the more advanced characterizations that we did not have space to discuss in this thesis.

Any group Γ admits a family of representations $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is some Hilbert space. The most prominent representation is known as the left-regular representation, $\lambda\colon \Gamma \to \mathbb{B}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$. Here, $\lambda_s(\xi)(t) = \xi(s^{-1}t)$ is a particular group action on the space of square-summable functions, $\ell_2(\Gamma)$. There are various amenability criteria that use properties of the left-regular representation, such as Kesten's criterion, weak containment of the trivial representation, and Hulanicki's criterion. These are discussed more in depth in [Jus22, Appendix A].

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