**Problem** (Problem 1): Let $M$ be a Riemannian manifold with metric $g$, $f$ a $C^\infty$ function on $M$, and let $X$ be a vector field on $M$. Find an expression in local coordinates for the gradient of $f$ and the divergence of $X$.

**Solution:** Let $p \in M$ and $U \subseteq M$ a chart for $p$ with coordinates $(x_1, \dots, x_n)$. Define

$$X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}.$$

We observe that the 1-form $\omega_X$ defined via the Riemannian metric, $\omega_X(Y) = g(X, Y)$, is given locally by

$$g(X, Y) = \sum_{i=1}^n f_i \sum_{j=1}^n a_{ij} g_j$$

$$= \sum_{i=1}^n f_i \sum_{j=1}^n a_{ij} \, dx_j \left( \sum_{k=1}^n g_k \frac{\partial}{\partial x_k} \right)$$

whence

$$\omega_X = \sum_{i=1}^n f_i \left( \sum_{j=1}^n a_{ij} \, dx_j \right).$$

Computing the divergence, which is given by $*d(*\omega_X)$, we find that

$$*\omega_X = \sum_{i=1}^n \sum_{j=1}^n (-1)^{j-1} f_i a_{ij} \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

$$d(*\omega_X) = \sum_{i=1}^n \sum_{j=1}^n (-1)^{j-1} d\big(f_i a_{ij}\big) \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \big(f_i a_{ij}\big)}{\partial x_j} \, dx_1 \wedge \cdots \wedge dx_n$$

$$*(d(*\omega_X)) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial f_i}{\partial x_j} + f_i \frac{\partial a_{ij}}{\partial x_j}.$$