Problem 1

Problem: An ordinal A is a successor ordinal if $A = B \cup \{B\}$ for some ordinal B. An element $m \in A$ is maximal if $\forall x \in A\,(x \in \vee x = m)$. Show that an ordinal is a successor ordinal if and only if it contains a maximal element.

Solution: Let y be an ordinal that contains a maximal element m. Then, for all $x \in y$, $x \in m$ or $x = m$. Thus, for $z = y \setminus \{m\}$, $\forall t \in z$, $t \in m$. Thus, $z \subseteq m$.

We claim that $m \subseteq z$. Since $m \in y$, $m \subset y$. Thus, $m \subset y \cup \{m\}$. However, since $m \notin m$, we have $m \subseteq z$. Thus, $z = m$, and $y = m \cup \{m\}$.

If y is a successor ordinal, then $y = m \cup \{m\}$ for some ordinal m, meaning that for all $x \in y$, $x \in m$ or $x \in \{m\}$, meaning $x \in m$ or $x = m$.

Problem 2

Problem: A limit ordinal is a nonzero ordinal that is not a successor ordinal. Prove that an ordinal A is a limit ordinal if and only if $A = \bigcup A$.

Solution: Let $A = \bigcup A$ and $A \neq 0$. Let $x \in A$. Then, $x \in \bigcup A = \bigcup_{y \in A} y$. Thus, $x \in y$ for some $y \in A$. Thus, $x < y$, so x is not maximal, meaning A has no maximal element. Thus, by problem 2, A is not a successor ordinal.

Problem 3

Problem: Prove that the following two versions of the Axiom of Choice are equivalent.

AC 1: Let T be a set such that every element of T is nonempty. Then, there exists a function f with domain T such that $\forall x \in T$, $f(x) \in x$.

AC 2: Let T be a set and F a function with domain T such that $\forall x \in T$, $F(x)$ is nonempty. Then, there exists a function f with domain T such that $\forall x \in T$, $f(x) \in F(x)$.

Solution: We can show that AC 2 implies AC 1 by taking $F(x) = x$ for each $x \in T$. Since T is nonempty, $F(x)$ is nonempty, meaning the particular choice function on T such that $f(x) \in x$ for each $x \in T$ also means $f(x) \in F(x)$ for each $x \in T$.

Let F be a function with domain T such that $F(x)$ is nonempty for all $x \in T$. Then, defining $T' = \{F(x) \mid x \in T\}$, T' is nonempty. Thus, by AC 1, there exists a choice function f' by taking $f'(y) = f(x')$, where $F(x') = y$. Thus, $f(x) \in F(x)$ for each $x \in T$.

Problem 4

Problem: Let $(S, \leqslant)$ be a partially ordered set where every chain in S has an upper bound in S. Prove that there is a maximal element in S.

Solution: Suppose toward contradiction that there does not exist a maximal element in S.

We start by showing that, if this assumption holds, then every chain in S has a strict upper bound.

Suppose towards contradiction[1] that some chain C in S does not have a strict upper bound. Since we are assuming that every chain in S has an upper bound, this implies that $u \in C$, where u is any upper bound for C. In particular, this means that there is no other s for which $s > u$, or else s would be a strict upper bound for C. Therefore, for any $s \in S$ with $s \geqslant u$, $s = u$, which means u is a maximal element.

Therefore, every chain in S has a strict upper bound.

Let H be the set of all chains in S. For any C, since C has a strict upper bound u, there is some C' with $C \subsetneq C'$ and $u \in C'$. Thus, we can create a choice function on H by selecting u for any $C \in H$, as $u \in C' \supsetneq C$, which means u is selected from C'.

Let Γ be an ordinal. We define $g : \Gamma \to H$ recursively by

$$g(\alpha) = \begin{cases} g(\beta) \cup f(g(\beta)) & \alpha = \beta \cup \{\beta\} \\ \bigcup_{\beta \in \alpha} g(\alpha) & \text{else} \end{cases}$$

We wish to prove that for all $\alpha \in \Gamma$, $g(\alpha)$ is a chain in S.

If $\alpha = 0$, then $g(\alpha) = \emptyset$, which is a chain (by vacuous truth).

Suppose toward contradiction there is some α such that $g(\alpha)$ is not a chain. Set $B = \{\alpha \in \Gamma \mid g(\alpha) \text{ is not a chain}\}$ Thus, $B \neq \emptyset$ by our assumption. Let α_0 be the least element of B (which exists because $B \subseteq \Gamma$ and Γ is well-ordered). Additionally, we know that $\alpha_0 \neq \emptyset$ since $g(\emptyset) = \emptyset$ is a chain.

(i) If $\alpha_0 = \alpha' \cup \{\alpha'\}$. Then $g(\alpha_0) = g(\alpha') \cup \{f(g(\alpha'))\}$.

We know that $g(\alpha')$ is a chain, since $\alpha' < \alpha_0$ and α_0 is the least element of B. Since $m = f(g(\alpha'))$ is a strict upper bound for $g(\alpha')$, $\forall x \in g(\alpha')$, $x < m$.

Let $a, b \in g(\alpha_0) = g(\alpha') \cup \{m\}$. Then, either $a, b \in g(\alpha')$, $a, b \in \{m\}$, or (without loss of generality), $a \in g(\alpha')$ and $b \in \{m\}$. If $a, b \in g(\alpha')$, then $a < b$, $b < a$, or $a = b$, since $g(\alpha')$ is totally ordered. If $a, b \in \{m\}$, then $a = b$. Else, if $a \in g(\alpha')$ and $b \in \{m\}$, then $a < b$.

Thus, $g(\alpha_0)$ is totally ordered. $\perp$

(ii) Let α_0 be a limit ordinal. Then,

$$g(\alpha_0) = \bigcup_{\beta \in \alpha_0} g(\beta).$$

We know that for each $\beta \in \alpha_0$, $g(\beta)$ is a chain. Since α_0 is well-ordered, for $a, b \in g(\alpha_0)$, there is some $\beta_0 \in \alpha_0$ such that $a, b \in g(\beta_0)$. Thus, either $a < b$, $b < a$, or $a = b$.

[1] Double contradiction, even.

Let $g : \Gamma \to H$. We have shown that $g(\alpha) \in H$ for any $\alpha \in H$.

Let $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. Without loss of generality, let $\alpha \subset \beta$. We will show that $g(\alpha) \subset g(\beta)$.

Suppose toward contradiction that $\alpha \subset \beta$ does not imply $g(\alpha) \subset g(\beta)$. Let β_0 be the smallest element of Γ such that there exists $\alpha_0 \subset \beta_0$ with $g(\alpha_0) \not\subset g(\beta_0)$. We know that $\beta_0 \neq 0$ because 0 has no proper subsets.

(i) If $\beta_0 = \beta' \cup \{\beta'\}$ for some $\beta' \in \Gamma$, then $g(\beta) = g(\beta') \cup \{f(g(\beta'))\}$.

Since $\alpha_0 \subset \beta_0$, $\alpha_0 \in \beta_0$, then $\alpha_0 \in \beta'$ or $\alpha_0 = \beta'$, meaning $\alpha_0 \in \beta'$ or $\alpha_0 = \beta'$.

If $\alpha_0 = \beta'$, then $g(\alpha_0) = g(\beta') \subset g(\beta_0)$ since $\{f(g(\beta'))\}$ is a strict upper bound.

If $\alpha_0 \subset \beta'$, then $g(\alpha_0) \subset g(\beta')$ since β' satisfies the assumption. Thus, $g(\alpha_0) \subset g(\beta_0)$.

(ii) If β_0 is a limit ordinal, then

$$\beta_0 = \bigcup_{\lambda \in \beta_0} \lambda.$$

If $\alpha \in \beta_0$, then $\alpha \in \lambda_0$ for some $\lambda_0 \in \beta_0$. Since $\lambda_0 \in \beta_0$, $\lambda_0 \subset \beta_0$, meaning $\alpha_0 \subset \lambda_0$, so $g(\alpha_0) \subset g(\lambda_0)$ for each $\lambda_0 \subset \beta_0$. Thus, $g(\alpha_0) \subset g(\beta_0)$.

Thus, we have shown that there exists a bijection from all ordinals Γ to H. $\perp$

Problem 5

Problem: Show that there exists an uncountable set T of countable subsets of $\mathbb{R}$.

Solution: Let S be the set of all countable subsets of $\mathbb{R}$, partially ordered by inclusion.

Let C be a chain in S. Since $C \subseteq S$, C consists of countable sets.

Suppose toward contradiction that there exists no chain with uncountable length. Then, C is countable, so

$$C' = \bigcup_{A \in C} A$$

is a countable union of countable sets, so C' is countable, and C' is an upper bound for C. It is thus the case that S has a maximal element M, which is a countable set.

However, since $\mathbb{R}$ is uncountable, there is some $q \in \mathbb{R} \setminus M$, meaning $M \subsetneq M \cup \{q\}$, contradicting the maximality of M.

Thus, there must be at least one uncountable chain, T, in S.