Existence of Equilibria in Non-Cooperative Games of Strategy

Avinash Iyer

Occidental College

Word Count: 1292

Abstract

In this paper, I examine the background and the motivation behind the development of game theory, a discipline with broad applications to economics, but one which is still fundamentally based in mathematics. The most crucial set of papers that led to the development of game theory was John Nash's "Non-Cooperative Games," and the paper I am examining today, "Equilibrium Points in *N*-Person Games," published in the Proceedings of the National Academy of Sciences in 1950.

Games of Strategy: Background and Motivation

One of the earliest discussions of a game of strategy came in the 1838 book *Researches Into the Mathematical Principles of the Theory of Wealth* by Augustin Cournot[7] — wherein he modeled economic competition between two profit-maximizing producers in the creation of spring water, in which each producer is a price-taker from market demand. In it, he found that each producer determined their quantity strategically in response to the production of the other producer. Additionally, Cournot found that, in this determination, there was a point where every producer was maximizing their profits; essentially, there was an "equilibrium" of sorts.

Game theory as a discipline, however, was first examined in depth by John von Neumann and Oskar Morgenstern in the *Theory of Games and Economic Behavior*[4], in which the duo developed a theory of examining *zero-sum* games (where there is a fixed "pool" of payoffs that players compete over) and *cooperative* games (where players will collaborate to maximize their payoffs). John Nash, on the other hand, was instrumental in developing theories and understandings of games where one player maximizing their payoff does not necessarily mean all players are equally worse off or maximally better off.

The classic example of a non-cooperative game is the Prisoner's Dilemma. Two people are taken into custody for a crime they both committed — however, the police do not have enough evidence to convict either for the maximum sentence. The options for them are to either stay quiet (Q), in which case both are locked up on a lesser charge for 1 year, or "fink" (F), in which case the player who finks is set free and the player who stayed quiet is sentenced to 10 years. If both players fink, they are both sentenced to a term of 5 years. We can represent this non-cooperative game as a matrix — player 1's payoff is represented as the first number and their strategy on the left side of the matrix, while player 2's payoff is represented as the second number and their strategy on the top of the matrix.

	Q	F
Q	$-1, -1$	$-10, 0$
F	$-10, 0$	$-5, -5$

We can see from this matrix that an equilibrium point emerges — namely, both players choose to fink, as if player 1 moves from F to Q, their payoff is reduced by 5, and if player 2 moves from F to Q, their payoff is reduced by 5. Additionally, we can see that Q is not an equilibrium point, as either player improves their payoff by finking.

Nash further developed the theory of non-cooperative games by showing in his Ph.D. thesis (published in 1951), aptly titled "Non-Cooperative Games,"[2] that every finite, n-player game has an equilibrium point consisting of some mix of every player's strategies. In it, he cited his

1949 paper, "Equilibrium Points in N-Person Games,"[3] published in 1950 in the Proceedings of the National Academy of Sciences. I will provide an overview of the proof that Nash offered.

Existence of Equilibria

In an n-person game, there are, of course, the n players who make up the game. Each of these players chooses from a certain *strategy space*. A *pure strategy* is one in which a player exclusively plays one of their allowed strategies, while a *mixed strategy* is a probability distribution assigned over all the pure strategies; every player must play a strategy in any game. We will refer to this n-tuple of strategies as a "strategy profile."

A *payoff function* assigns each player a payoff based on their pure strategy in relation to the strategy profile. The payoff of a player's mixed strategy is the expected value of the pure strategies, weighted by the probability distribution placed on the pure strategies within the mixed strategy.

Nash refers to a strategy profile "counter[ing]" another in the paper as a mixed strategy for a certain player that maximizes expected payoff holding the other $n-1$ strategies in the n-tuple; in most modern game theory textbooks, this is known as a *best response*,[5] which is what I will refer to it as.

Just as with the prisoner's dilemma, we view an equilibrium in a game as one where no player is better off from changing their strategy. Equivalently, this means every player is engaged in the best response to every other player's strategy — in the language of functions, this means that a map from the set defined by all the strategy profiles to the space defined by the best responses to each strategy profile contains a fixed point (where a strategy is best-responding to itself). In the paper, Nash showed that the definition of an n-person game made it that such a fixed point must exist.

The most famous example of a theorem about fixed points is Brouwer's Fixed Point Theorem. The theorem states that a continuous mapping from a closed unit ball in a metric space, $B_1(p) = \{x \mid d(x,p) \leq 1\}$ to itself must contain a point where $f(x) = x$. However, rather than using Brouwer's Fixed Point Theorem, Nash used an expansion of it known as Kakutani's fixed point theorem. Kakutani's Fixed Point Theorem requires understanding correspondences, convexity, and compactness in a set.

Correspondence: A correspondence is a map from elements of a set D to the power set, $\mathcal{P}(D)$.

Convex: A convex set S is a set where, if p and q are elements of S, then any linear combination $tp + (1-t)q$ where $0 \leq t \leq 1$ is an element of S.[1]

Compact: In $\mathbb{R}^n$, this is a set that is closed and bounded.

Kakutani's Fixed Point Theorem states that any continuous correspondence from a compact, convex set to a convex set has a fixed point. To see how every game has a fixed point according to this theorem, we need to analyze the features of the correspondence between mixed strategy profiles and best responses.

In the set of mixed strategy profiles, we can see that the set is bounded by the set of pure strategy profiles. Since every pure strategy profile is a mixed strategy profile (with a full probability weight on one strategy), we can also see that the set is closed and convex. Additionally, any linear combination of mixed strategy profiles is also a mixed strategy profile, since the process of creating a linear combination merely shifts the probability distribution of the mixed strategies.

By the definition of best response, it must be the case that if p is a strategy profile, and $q_{i,1}, q_{i,2}, \ldots, q_{i,n}$ are best responses, the payoff to player i from $q_1, \ldots, q_n$ is identically maximal, meaning that a linear combination of these best responses is also a best response. Thus, the set of best responses is convex.

So, by Kakutani's Fixed Point Theorem, it must be the case that there is a fixed point, or equilibrium point — a mixed strategy profile in which every player is playing their best response to every other player.

Significance

Nash's finding in the PNAS paper, further developed in his Ph.D. thesis, helped set the stage for further developments in the field of game theory. The idea of a "Nash Equilibrium," which is the equilibrium point shown to exist above, is used heavily in the field of industrial organization to understand market entry and pricing decisions, as well as in modeling interactions in international relations.

However, at the same time, Nash equilibria are not the end-all and be-all of modeling games. Further along in the development of game theory, mathematical economists such as Reinhard Selten and John Harsanyi refined the concept of Nash equilibrium to applications within sequential games of complete and incomplete information.[6] However, it was Nash that set the stage for the development of the discipline of game theory into what it is now through his groundbreaking mathematical work.

References

[1] *Convex Sets - Definition, Convex Hull, Convex Combinations, Examples*. BYJUS. url: https://byjus.com/maths/convex-sets/.

[2] John Nash. "Non-Cooperative Games". In: *Annals of Mathematics* 54.2 (1951). Publisher: Annals of Mathematics, pp. 286–295. issn: 0003-486X. doi: 10.2307/1969529. url: https://www.jstor.org/stable/1969529.

[3] John F. Nash. "Equilibrium Points in n-Person Games". In: *Proceedings of the National Academy of Sciences of the United States of America* 36.1 (1950). Publisher: National Academy of Sciences, pp. 48–49. issn: 0027-8424. url: https://www.jstor.org/stable/88031.

[4] John von Neumann, Oskar Morgenstern, and Ariel Rubinstein. *Theory of Games and Economic Behavior (60th Anniversary Commemorative Edition)*. Princeton University Press, 1944. isbn: 978-0-691-13061-3. url: https://www.jstor.org/stable/j.ctt1r2gkx.

[5] Steven Tadelis. *Game Theory: An Introduction*. Princeton University Press, Jan. 10, 2023. 416 pp. isbn: 978-1-4008-4595-8.

[6] *The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994*. NobelPrize.org. url: https://www.nobelprize.org/prizes/economic-sciences/1994/summary/.

[7] *What Is the Cournot Competition Economic Model?* Investopedia. url: https://www.investopedia.com/terms/c/cournot-competition.asp.