Introduction to Game Theory

Game Theory analyzes the *interaction* among a *group* of *rational* agents who *behave strategically.*

- A group consists of at least two individuals who are free to make decisions.
- An interaction means that the decisions of at least one member of the group must affect at least one other member of the group.
- In strategic behavior, members of the group account for the interaction in their decision making process.
- Rational agents act in their best decisions based on their knowledge.

Keynes's Beauty Contest: Choose the face that is the most chosen in a newspaper contest.

In many games, we are not asked to pick *our* favorite, we are asked to pick *everyone else's* favorite.

Applications of Game Theory

- Labor Economics (compensation interactions, promotions)
- Industrial Organization (pricing, entry, exit, etc.)
- Public Finance (public goods games)
- Political Economy (strategic voting)
- Trade (tariff wars)
- Biology (hunting and mating)
- Linguistics

It's important to remember that game theory is a subfield of *mathematics*, not economics.

Static Games of Complete Information

We will begin by covering *static games of complete information.*

- Static: Play happens at once and payoffs are realized. Decisions are not necessarily made at the same time.
- Complete information: the following four are all common knowledge in the game
  (i) all possible actions of the players
  (ii) all possible outcomes
  (iii) how each combination of actions of all players affects which outcome will materialize
  (iv) the preferences of each and every player over outcomes
- An event, E, is common knowledge if everyone knows E, everyone knows everyone knows E, *ad infinitum.*

The Prisoner's Dilemma

- Two suspects are interrogated in separate rooms.
- There is enough evidence to convict each of them for a minor offense, but not enough to convict either of a major crime unless one finks (F).
- If they each stay quiet (Q), they only get 1 year in prison each.
- If only one finks, they are free, and the other gets 4 years in prison.
- If they both fink, they each will spend 3 years in prison.

We will try to write The Prisoner's Dilemma as a game. First, we can see this in a payoff matrix.

Player X \ Player Y	Q	F
Q	$(2,2)$	$(0,3)$
F	$(3,0)$	$(1,1)$

Normal-Form Game

The constituents of a *normal-form game* G consist of the following:

- A finite set of players: $N = \{1, 2, \dots, n\}$.
- For each player i, a set S_i denotes the *strategy space* of player i. We will let $S = S_1 \times S_2 \times \cdots S_n$ denote the strategy space of the entire game (i.e., the entire set of strategies possible).
  - Every element $s \in S$ is a *strategy profile*, where $s = (s_1, s_2, \dots, s_n)$.
  - We denote the strategy choices of all players except player i as $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.
- A payoff function: $v_i : S \to \mathbb{R}$. The payoff function depends on the strategies of *all players*.

Example

Let the following payoff matrix represent a game. Write the normal form.

	X	Y
A	$(5,1)$	$(2,6)$
B	$(0,9)$	$(3,2)$
C	$(4,4)$	$(4,7)$

- $n = 2$
- $S_1 = \{A, B, C\}$
$S_2 = \{X, Y\}$

Strategic Dominance

Recall the prisoner's dilemma.

		Player Y: Q	F
Player X	Q	$(2,2)$	$(0,3)$
	F	$(3,0)$	$(1,1)$

Suppose you were player 1. If player 2 stays quiet, it is more optimal for you to fink than to stay quiet. Similarly, if player 2 finks, then it is more optimal for you to fink than to stay quiet.

In a similar vein, for player 2, it is more optimal to fink in both cases. Therefore, the proper strategy is (F, F).

Dominated Strategy

A strategy s_i' if *strictly dominated* for i if there is one other strategy $s_i \in S_i$ such that $v_i(s_i, s_{-i}) > v_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$.

Essentially, a strategy is strictly dominated if there is another strategy that yields a strictly greater payoff regardless of the other strategies.

A rational player will *never* play a strictly dominated strategy.

In the prisoner's dilemma, Q is strictly dominated by F in both cases. Oddly, this yields the worst outcome from a social perspective (i.e., it has the lowest aggregate welfare).

	L	M	R
T	2,2	1,1	4,0
B	1,2	4,1	3,5

Through *iterated elimination of strictly dominated strategies* (IESDS), we start by removing M from the strategy profile of player 2 as playing L is strictly better. Then, Player 1 realizes that player 2 is rational, and thus does not play B (as B is strictly dominated by T once M is removed from the strategy space of player 2). Finally, Player 2 does not play R, as R is strictly dominated by L given that player 1 will play T. Thus, we get our answer of $\mathbf{T}, \mathbf{L}$.

A game is *dominance solvable* if it can be solved via iterated elimination of strictly dominated strategies. However, only a small number of games are not dominance solvable.

Strategic Dominance and Normal-Form Activity

Activity: Strategic Games and Dominance
Econ 305

Brandon Lehr

1 Strategic Games

For each of the games described below, determine the normal form of the game: number of players n, strategy space for each player S_i, and payoffs (as a matrix or function).

a. Matching Pennies (a zero-sum game). Two players simultaneously place a penny on a table. If the pennies match (e.g., both placed heads up), player 2 pays player 1 a dollar. If the pennies do not match, player 1 pays player 2 a dollar.

Players: $N = \{1,2\}$

Strategy Space: $S_i = \{H, T\}$

Payoff Functions: $u_1 = \begin{cases} 1, & s_1 = s_2 \\ -1, & s_1 \neq s_2 \end{cases}$

$u_2 = \begin{cases} 1, & s_1 \neq s_2 \\ -1, & s_1 = s_2 \end{cases}$

Player 2

Player 1	H	T
H	1,-1	-1,1
T	-1,1	1,-1

b. Bach or Stravinsky / Battle of the Sexes (a coordination game with some conflict). A couple wants to be together on their date night rather than alone, but they have different preferences over which type of concert they attend. They simultaneously — and without communication — choose to go to either the Bach or Stravinsky concert. Conditional on being together, player 1 prefers Bach and player 2 prefers Stravinsky.

Players: $N = \{1,2\}$

Strategy Sets: $S_i = \{B, S\}$

Payoffs:

Player 2

Player 1	B	S
B	5,3	2,2
S	0,0	3,5

c. Hawk vs. Dove / Chicken (an anti-coordination game). Two teenagers ride their bikes at high speed towards each other along a narrow ride. Neither of them wants to "chicken out" and lose their pride, but even worse is getting hurt by crashing into the oncoming biker.

Players: $n=2$

Strategy Space: $\{D, H\}$

Payoffs:

		Player 2	
		D	H
Player 1	D	1,1	1,2
	H	2,1	0,0

d. Cournot Competition (an industrial organization game). Two firms compete by simultaneously choosing how much to produce of a homogenous good (e.g., oil, soybeans) for a market.

Players: $n=2$

Strategy Space: $S_i = [0, \infty)$

$P = d^{-1}(q_1 + q_2)$

Payoff: $\pi_i = d^{-1}(q_1+q_2)\, q_i - c_i(q_i)$

2 Strict Dominance

Are the following games dominance solvable? Justify your answers.

a. A 4×4 game:

Yes, (B, X) is the result from IESDS on this game

	W	X	Y	Z
A	5,2	2,6	1,4	0,4
B	0,0	3,2	2,1	1,1
C	7,0	2,2	1,5	5,1
D	9,5	1,3	0,2	4,8

b. The beauty contest game, i.e., to win, come closest to guessing two-thirds the average of numbers between 0 and 100 selected by players.

Yes, every strategy is strictly dominated by 0 the game is dominance solvable

Nash Equilibrium: Definition

A strategy profile s^* is a *pure strategy Nash equilibrium* if and only if the following holds.

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i, s_{-i}^*)$$

for all players i and all strategies $s_i \in S_i$.

Given what all other players are doing, no single player has an incentive to deviate to another action. This does not inform us about how to get to the Nash equilibrium, it just tells us that it is one.

For example, the prisoner's dilemma has a pure strategy Nash equilibrium: (F, F)

- For any other strategy profile, there is a profitable deviation.
- Similarly, result corresponds to the outcome of IESDS.

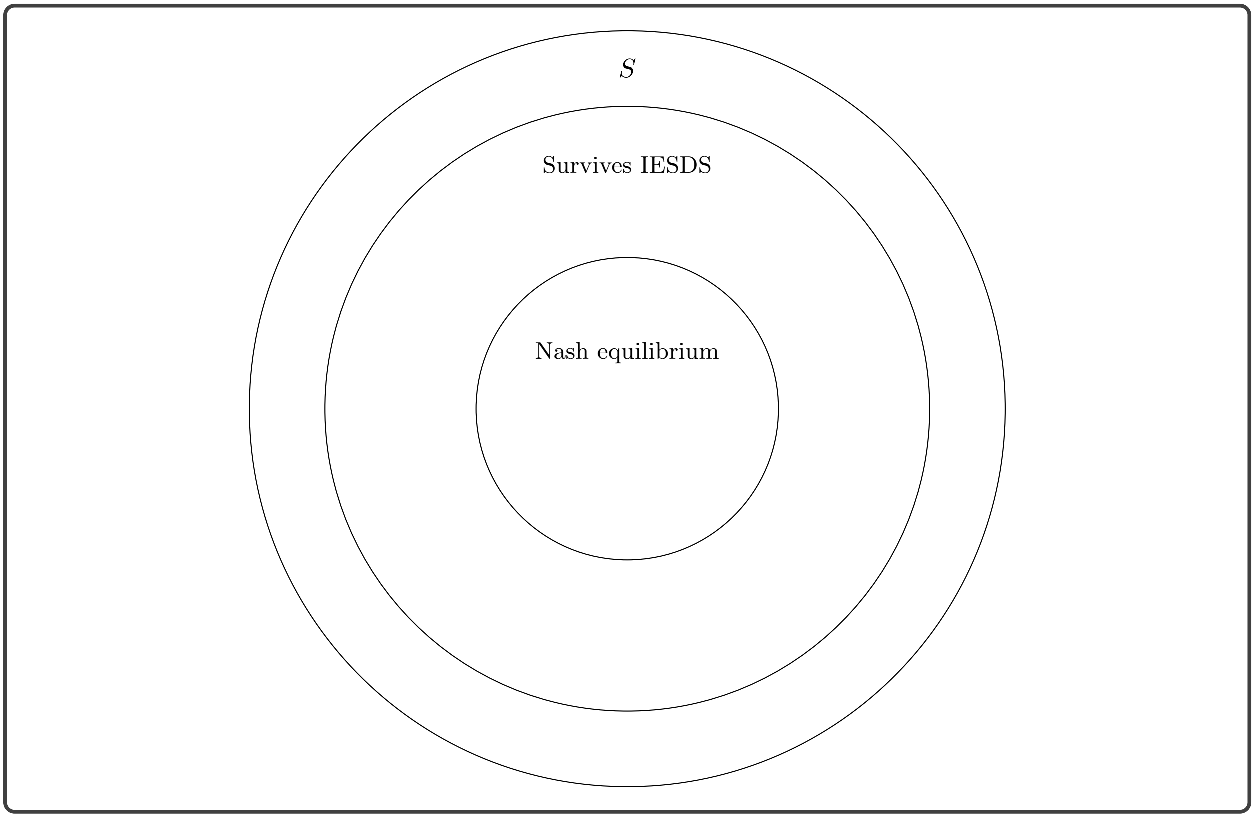
	L	M	R
T	2,2	1,1	4,0
B	1,2	4,1	3,5

In the above game, the pure strategy Nash equilibrium is (T, L). We can easily check that it is a Nash equilibrium, but in order to check that it is unique, we would need to look for deviations from the other strategy profiles. Or do we?

IESDS and Nash Equilibrium

- If s^* is a pure strategy Nash equilibrium of G, s^* survives IESDS.
- An action that is played in a Nash equilibrium is never eliminated in IESDS.
- If G is dominance solvable, then G has a unique Nash equilibrium:
  - The previous proposition tells us that G is dominance solvable $\Rightarrow$ there is at most one Nash equilibrium.

The relationship between the strategy set, S, the set of strategies that survive IESDS, and the Nash equilibrium can be seen below:


S
Survives IESDS
Nash equilibrium

Voter Participation and Nash Equilibrium Activity

Activity: Voter Participation
Econ 305

Brandon Lehr

Two candidates, Joe and Donald, compete in a national Presidential election. Of the 200 million registered voters in the U.S., 100 million support Joe and 100 million support Donald. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives a payoff of 2 if the candidate she supports wins, 1 if this candidate ties, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$. Find the (pure strategy) Nash equilibria.

We can do this by considering different types of strategy profiles. For each case, we need to check whether or not any single citizen has an incentive to deviate to another strategy, given the strategies of all other citizens. If not, we have a Nash equilibrium.

Case 1: All Citizens Vote

Follow: $1-c$

Deviate (abstain): 0

all vote

Case 2: Not All Citizens Vote; the Candidates Tie

Vote: $2-c$

Abstain: 1

WV → vote
WA → vote
LV → vote
LA → vote

Case 3: A Candidate Wins by One Vote

Loser	Winner
vote: $1-c$	vote: $2-c$
abstain: 0	abstain: 2

WV → abstain
WA → abstain
LV → vote
LA → vote

Case 4: A Candidate Wins by at Least Two Votes ← stable solution

Loser vote	Winner
vote: $-c$	vote: $2-c$
no vote: 0	no vote: 2

WV → abstain
WA → abstain
LV → abstain
LA → abstain

Bonus: Suppose that Joe has more supporters than Donald. What are the (pure strategy) Nash equilibria of this game?

Hint: Let n_J be the number of people who vote for Joe and n_D be the number of people who vote for Donald. Also, denote the number of Donald's supporters by $k < 100$ million. Proceed in cases as before.

There is no pure strategy nash equilibrium

2

Bertrand Competition

Assumptions We have the following:

Players Homogenous good produced by $n > 1$ firms (i.e., oil, soybeans)

Cost The cost of producing q_i units to $c_i(q_i)$.

Demand Total Market Demand is given by $D(p)$

Strategy Set $S_i = \mathbb{R}^+$, where $p_i \in S_i$ denotes the price.

Normal-Form Game We have the following:

Players $n = 2$

Cost Function $c_i(q_i) = cq_i$ for some $c \in \mathbb{R}^+$ and for $i = 1, 2$

Payoffs $v_i(p_i, p_j) = \begin{cases} 0, & p_i > p_j \\ (p_i - c)(D(p_i)), & p_i < p_j \\ \frac{1}{2}(p_i - c)(D(p_i)), & p_i = p_j \end{cases}$