## Problem 1

Let $v_1, \dots, v_n$ be mutually orthogonal vectors in an inner product space $V$. Show that

$$\left\|\sum_{k=1}^{n} v_k\right\|^2 = \sum_{k=1}^{n} \|v_k\|^2 .$$

**Proof:**

$$\begin{aligned}
\left\|\sum_{k=1}^{n} v_k\right\|^2 &= \left\langle \sum_{k=1}^{n} v_k, \sum_{k=1}^{n} v_k \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \sum_{k=1}^{n} v_k, v_i \right\rangle \\
&= \sum_{i=1}^{n} \langle v_i, v_i \rangle && \text{since for } i \neq j,\ \langle v_i, v_j \rangle = 0 \\
&= \sum_{i=1}^{n} \|v_i\|^2
\end{aligned}$$

## Problem 2

Let $V$ be an inner product space and fix $w \neq 0$ in $V$. We define the one-dimensional projection

$$P_w : V \to V; P_w(v) := \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

(i) Prove that $v - P_w(v) \perp P_w(v)$.

(ii) Show that $P_w : V \to V$ is a linear operator with $\|P_w\|_{\text{op}} = 1$.

(iii) Show that $P_w \circ P_w = P_w$.

## Problem 3

Let $V$ be an inner product space. Prove the reverse Cauchy-Schwarz Inequality which states

$$v, w \in V, \text{ and } |\langle v, w \rangle| = \|v\| \|w\| \Rightarrow v = \alpha w.$$

## Problem 4

Let $V$ be an inner product space. Then, for any $v, w \in V$, show that

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

**Proof:**

$$\begin{aligned}
\langle v + w, v + w \rangle + \langle v - w, v - w \rangle &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle + \langle v, v \rangle - \langle w, v \rangle - \langle v, w \rangle + \langle -w, -w \rangle \\
&= \langle v, v \rangle + \langle v, v \rangle + \langle w, w \rangle + \langle w, w \rangle \\
&= 2\|v\|^2 + 2\|w\|^2
\end{aligned}$$

## Problem 5

Let $\lambda = (\lambda_k)_k$ belong to $\ell_\infty$. Show that the map

$$D_\lambda : \ell_2 \to \ell_2; D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$

**Proof:**

Well-Defined: Let $(\zeta_k)_k = 0$ for all $k \in \mathbb{N}$. Then,

$$\begin{aligned} D_\lambda((\zeta_k)_k) &= (\lambda_k \zeta_k)_k \\ &= ((\lambda_k)(0))_k \\ &= 0 \end{aligned}$$

Linear:

$$\begin{aligned} D_\lambda((\alpha\xi_k)_k + (\beta\zeta_k)_k) &= D_\lambda((\alpha\xi_k + \beta\zeta_k)_k) \\ &= (\lambda_k(\alpha\xi_k + \beta\zeta_k))_k \\ &= (\alpha\lambda_k\xi_k + \alpha\lambda_k\zeta_k)_k \\ &= (\alpha\lambda_k\xi_k)_k + (\beta\lambda_k\zeta_k) \\ &= \alpha(\lambda_k\xi_k)_k + \beta(\lambda_k\zeta_k)_k \\ &= \alpha D_\lambda((\xi_k)_k) + \beta D_\lambda((\zeta_k)_k) \end{aligned}$$

Bounded:

$$\begin{aligned} \|D_\lambda\|_{\text{op}} &= \sup_{\|\xi_k\|_k \leq 1} \|D_\lambda((\xi_k)_k)\| \\ \|D_\lambda((\xi_k)_k)\| &= \left(\sum_{k=1}^{\infty} |\lambda_k\xi_k|^2\right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} \left|\sup_{k\in\mathbb{N}} |\lambda_k|\xi_k\right|^2\right)^{1/2} \\ &= \|\lambda\|_\infty \left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} \\ &= \|\lambda\|_\infty \|\xi_k\| \end{aligned}$$

## Problem 6

Consider the vector space $C([0, 2\pi])$ equipped with

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt.$$

(i) Show that this pairing defines an inner product on $C([0, 2\pi])$.

**Proof:** We will show that $\langle f, g\rangle$ satisfies the axioms of the inner product.

Addition:

$$\begin{aligned} \langle f_1 + f_2, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (f_1(t) + f_2(t))\overline{g(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(f_1(t)\overline{g(t)} + f_2(t)\overline{g(t)}\right) dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f_1(t)\overline{g(t)}dt + \frac{1}{2\pi}\int_0^{2\pi} f_2(t)\overline{g(t)}dt \\ &= \langle f_1, g\rangle + \langle f_2, g\rangle. \end{aligned}$$

Scalar Multiplication:

$$\begin{aligned} \langle \alpha f, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (\alpha f(t))\overline{g(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \alpha\left(f(t)\overline{g(t)}\right) dt \\ &= \alpha\left(\frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt\right) \\ &= \alpha\langle f, g\rangle. \end{aligned}$$

Conjugation:

$$\begin{aligned}\overline{\langle g, f\rangle} &= \frac{1}{2\pi}\int_0^{2\pi}\overline{g(t)\overline{f(t)}}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt\\ &= \langle f, g\rangle.\end{aligned}$$

Positive Definition:

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt\\ &\geq 0.\end{aligned}$$

For $\langle f, f\rangle = 0$, we have that the integral equals zero — since $f$ is continuous, it means that if $|f(t)|^2 > 0$ for some $t_0 \in [0, 2\pi]$, then $|f(t)|^2 \neq 0$ on some interval $[t_0 - \delta, t_0 + \delta]$, meaning the integral can only equal zero if $f$ is $\mathbb{0}_f$ on $[0, 2\pi]$.

(ii) For $n \in \mathbb{Z}$, set $e_n(t) = \cos(nt) + i\sin(nt)$. Show that the family $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal.

**Proof:** We will show that $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal by showing that $\langle e_n, e_n\rangle = 1$ and $\langle e_n, e_m\rangle = 0$ for $m \neq n$.

$$\begin{aligned}\langle e_n, e_n\rangle &= \frac{1}{2\pi}\int_0^{2\pi}(\cos(nt) + i\sin(nt))(\cos(nt) - i\sin(nt))dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\cos^2(nt) + \sin^2(nt)\right)dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} dt\\ &= 1\\ \langle e_n, e_m\rangle &= \frac{1}{2\pi}\int_0^{2\pi}(\cos(nt) + i\sin(nt))(\cos(mt) - i\sin(mt))dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}(\cos(mt)\cos(nt) + i\sin(nt)\cos(mt) - i\sin(mt)\cos(nt) + \sin(nt)\sin(mt))\,dt\\ &= \frac{1}{2\pi}\left(\int_0^{2\pi}(\cos(mt)\cos(nt) + \sin(nt)\sin(mt))dt + i\int_0^{2\pi}(\sin(nt)\cos(mt) - \sin(mt)\cos(nt))dt\right)\\ &= 0.\end{aligned}$$

## Problem 7

Let $V$ be any normed space, $p \in [1, \infty]$, and suppose $T : \ell_p^n \to V$ is linear. Show that $T$ is bounded.

## Problem 8

Let $\mathbb{P}[0,1] = \left\{\sum_0^n a_k x^k \mid a_k \in \mathbb{C}\right\} \subseteq C([0,1])$ denote the linear subspace of all polynomial functions equipped with the uniform norm $\|\cdot\|_u$ inherited from $C([0,1])$. We define the map

$$\begin{aligned}D &: \mathbb{P}[0,1] \to \mathbb{P}[0,1]\\ D(p(x)) &= p'(x).\end{aligned}$$

Show that $D$ is unbounded.

**Proof:** Let $p(x) = x^n$. Then, in $\mathbb{P}[0,1]$,

$$\begin{aligned}\|p\|_u &= 1\\ \|D(p)\|_u &= n.\end{aligned}$$

For any $L \in \mathbb{R}$, we can find a $n \in \mathbb{N}$ sufficiently large such that $\|D(p)\|_u = n > L$, by the Archimedean property. Therefore, $D$ is unbounded.

## Problem 9

Let $V$ be an infinite-dimensional normed space. Show that there is a linear functional $\varphi : V \to \mathbb{F}$ that is unbounded.

**Proof:** Let $B = \{x_n\}$ be the basis for $V$. We define $\varphi : V \to \mathbb{F}$ as $\varphi(x) = \sum_n n\alpha_n$ for the $\alpha_n x_n$ component in $x$. Then, $\varphi$ is linear and unbounded, as the values $n$ takes are not bounded, seeing as $V$ is infinite-dimensional.

## Problem 10

Let $a, b \in \mathbb{M}_n$. Show the following properties of the operator norm.

(i) $\|a\|_{\text{op}} = \sup\left\{ |\langle a\xi, \eta\rangle| \mid \xi, \eta \in B_{\ell_2^n} \right\}$

(ii) $\|a^*\|_{\text{op}} = \|a\|_{\text{op}}$

(iii) $\|ab\|_{\text{op}} \leq \|a\|_{\text{op}} \|b\|_{\text{op}}$

(iv) $\|a^*a\|_{\text{op}} = \|a\|_{\text{op}}^2$

**Proof:**

(i)