Solution (21.20a)**:** There are poles at $2, 2e^{2i\pi/3}, 2e^{4i\pi/3}$, and since the integrand falls off with $1/r^2$, we may close the contour in the upper half-plane, with one of the poles on the contour. Thus, we get the solution

$$\oint_C \frac{z}{z^3-8}\,dz = 2\pi i\,\text{Res}\left[f(z), 2e^{2i\pi/3}\right] + \pi i\,\text{Res}[f(z), 2] = \frac{\pi}{2\sqrt{3}}.$$

Solution (21.25a)**:** Evaluating

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = -\sum_j \text{Res}\left[\frac{\pi\cot(\pi z)}{1+z^2}, z_j\right] = 2\pi\tanh(\pi).$$

Solution (21.26)**:** I don't know how to do this problem.

Solution (21.28)**:** If $t > 0$ is fixed, then we close the integral in the lower half-plane, which has the pole at $\omega = -i\varepsilon$ with residue $-2\pi i e^{-\varepsilon t}$. Taking $\varepsilon \to 0$, we get 1 for $\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + i\varepsilon}\,d\omega$.

If $t < 0$ is fixed, then we close the integral in the upper half-plane, which has no poles, so the integral $\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + i\varepsilon}\,d\omega$ is equal to zero. A similar case holds for $t = 0$.

Solution (21.31)**:**

(a) If $z = a$ is an nth order zero, then $w(z) = (z-a)^n g(z)$ for some $g(z) \neq 0$ on $z = a$. This gives

$$\frac{dw}{dz} = n(z-a)^{n-1}g(z) + g'(z)(z-a)^n$$

Note that $g'(z) \neq 0$ on $z = a$.

$$\frac{w'(z)}{w(z)} = \frac{n}{(z-a)} + \frac{g'(z)}{g(z)}.$$

Thus, we have a residue of n at $z = a$.

(b) If $z = a$ is a pth order pole, then $w(z) = (z-a)^{-p} g(z)$ for some $g(z) \neq 0$ at $z = p$. This gives

$$\begin{aligned}\frac{dw}{dz} &= -p(z-a)^{-p-1}g(z) + (z-a)^{-p}g'(z)\\ \frac{w'(z)}{w(z)} &= \frac{-p}{(z-a)} + \frac{g'(z)}{g(z)},\end{aligned}$$

so we have a residue of $-p$ at $z = a$.

(c) We have

$$\oint \frac{w'(z)}{w(z)}\,dz = 2\pi i \sum_i \text{Res}\left[\frac{w'(z)}{w(z)}, z_i\right] = 2\pi i\left(\sum_i n_i - \sum_i p_i\right).$$

Solution (21.32)**:**

(a) Inside $|z| = \frac{1}{2}$, there is an order 3 zero at $z = 0$, so the integral evaluates to $6\pi i$.

(b) Inside $|z| = 2$, there is a pole of order 3 at $z = 1$, a zero of order 1 at $z = -1$, and a zero of order 3 at $z = 0$. Thus, the integral evaluates to $2\pi i$.

(c) Inside $|z| = 9/2$, there is an order 2 pole at $z = -3$, an order 3 pole at $z = 1$, an order 3 zero at $z = 0$, an order 1 zero at $z = -1$, and an order 1 zero at $z = 4i$. Thus, the integral evaluates to 0.

Solution (21.33)**:**

(a) The phase change in $\arg(w)$ is equal to the amount of times that the contour crosses the branch cut along $(-\infty, 0]$. This gives

$$\begin{aligned}\oint_C \frac{w'(z)}{w(z)}\,dz &= \oint_C \frac{d}{dz}(\ln(w(z)))\,dz\\ &= i(\text{\# of times C crosses branch cut})\\ &= i\Delta_C \arg(w).\end{aligned}$$

In the case of $w(z) = \frac{1}{z}$, this yields the winding number of C.

(b) If w is nonvanishing on C, then since C crosses the branch cut three times, we have that

$$6\pi i = 2\pi i \sum_i n_i,$$

so there are three orders worth of zeros of w.

Solution (22.7)**:** Due to poor time management, I am unable to complete this problem with sufficient attention investment.