Compact Operators

Definition: A linear map $T\colon X \to Y$ between Banach spaces is called *compact* if $T(B_X) \subseteq Y$ has compact closure, where B_X denotes the closed unit ball of X. We denote the space of compact operators $K(X,Y)$.

The theory of compact operators (and the soon to arise Fredholm operators) arose from the analysis of integral equations. To start, let $I = [0,1]$, and consider the Banach space $C(I)$ with the supremum norm. Letting $k \in C(I \times I)$, we define $u \in B(X)$ by taking

$$Tf(x) = \int_0^1 k(x,y)f(y)\,dy.$$

The fact that $Tf \in X$ follows from an application of the Dominated Convergence Theorem and the fact that, since $k(x,y)$ is jointly continuous, it is also separately continuous (see [Fol99, Theorem 2.27]). In fact, we can show something even stronger: we claim that the family $T(B_X)$ is in fact equicontinuous. This follows from the fact that, I^2 is compact, so if $\varepsilon > 0$, there is δ such that whenever $\max\{|x-x'|, |y-y'|\} < \delta$, we have $|k(x,y) - k(x',y')| < \varepsilon$. Therefore,

$$\begin{aligned} |Tf(x) - Tf(x')| &= \left|\int_0^1 (k(x,y) - k(x',y))f(y)\,dy\right| \\ &\leq \int_0^1 |k(x,y) - k(x',y)||f(y)|\,dy \\ &\leq \sup_{y\in I}|k(x,y) - k(x',y)|\|f\|_u \\ &\leq \varepsilon\|f\|_u. \end{aligned}$$

Furthermore, since

$$|Tf(x)| \leq \|k\|_u\|f\|_u,$$

we have that $T(B_X)$ is pointwise bounded. Thus, by the Arzelà–Ascoli theorem, it follows that $T(B_X)$ is totally bounded, so T is a compact operator. We call the function k the *kernel* of the operator T.

Similarly, the operator $V \in B(X)$ given by

$$Vf(x) = \int_0^x f(y)\,dy$$

is such that $V(B_X)$ is totally bounded by Arzelà–Ascoli, so V is also compact. In fact, V has no eigenvalues as well. This follows from the fact that, if there were $\lambda \in \mathbb{C} \setminus \{0\}$ with $V(f) = \lambda f$, then $f(0) = 0$ and $f'(t) = 1/\lambda f(t)$, so that $f(t) = f(0)e^{t/\lambda} = 0$, meaning $f = 0$.

We call the operator V the *Volterra integral operator* on X.

We can see that $K(X)$ is in fact an algebraic ideal in $B(X)$ (by continuity). In fact, there is a topological dimension to $K(X) \subseteq B(X)$.

Proposition: If X, Y are Banach spaces, then $K(X,Y)$ is a closed subspace of $B(X,Y)$.

Proof. Let $(T_n)_n$ converge to $T \in B(X,Y)$. Let $\varepsilon > 0$, and select N such that $\|T_N - T\| < \varepsilon/3$. Since $T_N(B_X)$ is totally bounded, there are $x_1, \dots, x_n \in B_X$ such that for each $x \in S$, we have

$$\|T_N x - T_N x_j\| < \varepsilon/3$$

for some j. Therefore, we have

$$\begin{aligned} \|Tx - Tx_j\| &\leq \|Tx - T_N x\| + \|T_N x - T_N x_j\| + \|T_N x_j - Tx_j\| \\ &< \varepsilon. \end{aligned}$$

Therefore, $T(B_X)$ is totally bounded, so $T \in K(X,Y)$. $\square$

Therefore, we see that $\overline{F(X,Y)} \subseteq K(X,Y)$ is, where $F(X,Y)$ denotes the finite-rank operators, but this inclusion may be strict. In the cases where $\overline{F(X)} = K(X)$, we say the Banach space X has the approximation property. There are Banach spaces that do not have the approximation property.

Recall that if $T\colon X \to Y$ is a bounded linear map between Banach spaces, the transpose is defined by $T^*\colon Y^* \to X^*$, given by $T^*\varphi = \varphi \circ T$.

Theorem: If X and Y are Banach spaces with $T \in K(X,Y)$, then $T^* \in K(Y^*, X^*)$.

Proof. Let $\varepsilon > 0$. Since $T(B_X)$ is totally bounded, there exist elements $x_1, \dots, x_n$ such that if $x \in B_X$, then $\|Tx - Tx_i\| < \varepsilon/3$ for some i. Let $V \in B(Y^*, \mathbb{C}^n)$ be defined by $V\varphi = (\varphi(Tx_1), \dots, \varphi(Tx_n))$. Since V has finite rank, V is compact, so $V(B_{X^*})$ is totally bounded. Thus, there exist $\varphi_1, \dots, \varphi_m$ such that if $\varphi \in T$, then $\|V\varphi - V\varphi_j\| = \max_{i=1}^{n} |T^*\varphi(x_i) - T^*\varphi_j(x_i)|$.

Now, if $x \in B_X$, then $\|Tx - Tx_i\| < \varepsilon/3$ for some i, so thus $|T^*\varphi(x_i) - T^*\varphi_j(x_i)| < \varepsilon/3$. Thus,

$$\begin{aligned} |T^*\varphi(x) - T^*\varphi_j(x)| &\leq |T^*\varphi(x) - T^*\varphi(x_i)| + |T^*\varphi(x_i) - T^*\varphi_j(x_i)| + |T^*\varphi_j(x_i) - T^*\varphi_j(x)| \\ &< \varepsilon, \end{aligned}$$

whence $\|T^*\varphi - T^*\varphi_j\| \leq \varepsilon$, meaning $T^*(B_{X^*})$ is totally bounded, hence T^* compact. □

Recall that a linear map $T\colon X \to Y$ is called bounded below if there is $\delta > 0$ such that $\|Tx\| \geq \delta \|x\|$ for all x. In this case, $T(X) \subseteq Y$ is necessarily closed. Every invertible linear map is bounded below, as is every isometry.

Equivalently, a map $T\colon X \to Y$ is not bounded below if and only if there is a sequence of unit vectors $(x_n)_n \subseteq X$ such that $\lim_{n\to\infty} Tx_n = 0$.

Theorem: Let T be a compact operator on a Banach space X, and let $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) The space $\ker(T - \lambda \operatorname{id}_X)$ is finite-dimensional.

(ii) The space $(T - \lambda \operatorname{id}_X)(X)$ is closed and has finite codimension in X.

Proof. Let $Z = \ker(T - \lambda \operatorname{id}_X)$. Then, $T(Z) \subseteq Z$, and the restriction $T|_Z$ is in $K(Z)$. Since $T|_Z = \lambda \operatorname{id}_Z$ with $\lambda \neq 0$, it follows that $\operatorname{id}|_Z$ is compact, meaning Z is finite-dimensional.

Since Z is finite-dimensional, there is a closed subspace Y of X such that $X = Z \oplus Y$.

Observe that $(T - \lambda \operatorname{id}_X)X = (T - \lambda \operatorname{id}_X)Y$, so to show that $(T - \lambda \operatorname{id}_X)X$ is closed, it suffices to show that the restriction $(T - \lambda \operatorname{id}_X)|_Y$ is bounded below.

Suppose otherwise. Then, there is a sequence $(x_n)_n$ of unit vectors in Y such that $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$. We may assume without loss of generality that $(Tx_n)_n$ is convergent. It follows then that, since $x_n = \frac{1}{\lambda}(Tx_n - (T - \lambda \operatorname{id}_X)x_n)$, we have that $(x_n)_n \to x$ for some $x \in Y$, as Y is closed. Since $Tx = \lambda x$, we have $x \in Y \cap \ker(T - \lambda \operatorname{id}_X)$, meaning $x = 0$. Yet, x is the limit of unit vectors, and so is also a unit vector, which means we reach a contradiction. Thus, $(T - \lambda \operatorname{id}_X)|_Y$ is bounded below.

Let $W = X/(T - \lambda \operatorname{id}_X)X$. To show that $(T - \lambda \operatorname{id}_X)X$ has finite codimension, we show that W is finite-dimensional, by showing that W^* is finite-dimensional. Let $\pi\colon X \to W$ be the quotient map. Then, $\ker(\pi^*) \subseteq \ker(T^* - \lambda \operatorname{id}_{X^*})$. Letting $\sigma \in \ker(T^* - \lambda \operatorname{id}_{X^*})$, we have that σ annihilates $(T - \lambda \operatorname{id}_X)X$, so it induces a bounded linear functional $\tau\colon W \to \mathbb{C}$ such that $\sigma = \tau \circ \pi = \pi^*(\tau)$. Since T^* is compact, $\ker(T^* - \lambda \operatorname{id}_{X^*})$ is finite-dimensional, so π^* has finite-dimensional range, and since π^* is injective, W^* is thus finite-dimensional, so W is finite-dimensional. □

References

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