

Problem (Problem 1): Let $U \subseteq \mathbb{C}$ be a nonempty open set.

Given a sequence $(z_n)_n \subseteq U$, we write $z_n \rightarrow \partial U$ if, for every compact subset $K \subseteq U$, there exists some $N = N(K) \in \mathbb{N}$ such that $z_n \notin K$ whenever $n \geq N$.

Given a function $u: U \rightarrow \mathbb{R}$, define

$$\limsup_{z \rightarrow \partial U} u(z) = \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{z \in U \setminus K} u(z).$$

(a) For each positive integer $n \in \mathbb{N}$, define

$$K_n := \left\{ z \in U \mid |z| \leq n, \text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{n} \right\}.$$

Show that:

- (i) each K_n is compact;
- (ii) $K_n \subseteq K_{n+1}^\circ$;
- (iii) $U = \bigcup_{n=1}^\infty K_n$.

(b) Let $L := \limsup_{z \rightarrow \partial U} u(z)$.

- (i) Show that for each $S > L$, there is a compact subset $K \subseteq U$ such that $u(z) \leq S$ for all $z \in U \setminus K$.
- (ii) Show that there exists a sequence $(z_n)_n$ in U with $z_n \rightarrow \partial U$ and $\limsup_{n \rightarrow \infty} u(z_n) \geq L$.

(c) Prove that

$$\limsup_{z \rightarrow \partial U} u(z) = \sup_{\substack{(z_n)_n \subseteq U \\ (z_n)_n \rightarrow \partial U}} \limsup_{n \rightarrow \infty} u(z_n),$$

where the supremum is over all sequences $(z_n)_n$ with $(z_n)_n \rightarrow \partial U$.

Solution:

(a) We claim that the set

$$C_n = \left\{ z \in U \mid \text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{n} \right\}$$

is closed. Then, we observe that $K_n = B(0, n) \cap C_n$ would then be an intersection of a closed set with a closed and bounded set, hence a closed and bounded set, hence compact. Towards this end, suppose $(w_k)_k \subseteq C_n$ converges to $w \in \mathbb{C}$. Then, for each k , we have

$$\inf_{z \in \mathbb{C} \setminus U} |w_k - z| \geq \frac{1}{n}.$$

Observe then that for any $z \in \mathbb{C} \setminus U$, we have

$$|w_k - z| \geq \frac{1}{n}$$

for each k , meaning that

$$\lim_{k \rightarrow \infty} |w_k - z| \geq \frac{1}{n},$$

or that

$$|w - z| \geq \frac{1}{n}.$$

In particular, it must be the case that $w \in U$, and that

$$\inf_{z \in \mathbb{C} \setminus U} |w - z| \geq \frac{1}{n},$$

so that $w \in C_n$, and thus C_n is closed, and K_n is compact.

To see that $K_n \subseteq K_{n+1}^\circ$, we show that $C_n \subseteq C_{n+1}^\circ$ by understanding the picture of C_n° . Towards this end, we see that $z \in C_n^\circ$ if and only if $z \in U$ and there is some $r > 0$ such that $\text{dist}_{\mathbb{C} \setminus U}(w) \geq \frac{1}{n}$ for all $w \in U(z, r)$.

Observe that if $\varepsilon > 0$, then if z satisfies $\text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{n} + \varepsilon$, then if $w \in \mathbb{C} \setminus U$ and $\zeta \in U(z, \varepsilon/2)$, we have

$$\begin{aligned} \frac{1}{n} + \varepsilon &\leq |z - w| \\ &\leq |z - \zeta| + |\zeta - w| \\ &< \varepsilon/2 + |\zeta - w|, \end{aligned}$$

meaning that $|\zeta - w| \geq \frac{1}{n} + \varepsilon/2$ for all $w \in \mathbb{C} \setminus U$, so that $\text{dist}_{\mathbb{C} \setminus U}(\zeta) \geq \frac{1}{n}$. In particular, this means that C_n° consists of all z for which there exists ε such that $\text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{n} + \varepsilon$, or more succinctly,

$$C_n^\circ = \left\{ z \in U \mid \text{dist}_{\mathbb{C} \setminus U}(z) > \frac{1}{n} \right\}.$$

In particular, since $\frac{1}{n} > \frac{1}{n+1}$, it follows that $C_n \subseteq C_{n+1}^\circ$. Paired with the fact that $B(0, n) \subseteq U(0, n+1)$, we obtain that

$$\begin{aligned} K_n &= B(0, n) \cap C_n \\ &\subseteq U(0, n+1) \cap C_{n+1}^\circ \\ &= (B(0, n+1) \cap C_n)^\circ \\ &= K_{n+1}^\circ. \end{aligned}$$

Finally, to show that $U = \bigcup_{n=1}^\infty K_n$, we write

$$\begin{aligned} \bigcup_{n=1}^\infty K_n &= \bigcup_{n=1}^\infty (B(0, n) \cap C_n) \\ &= \left(\bigcup_{n=1}^\infty B(0, n) \right) \cap \left(\bigcup_{n=1}^\infty C_n \right), \end{aligned}$$

and since $\bigcup_{n=1}^\infty B(0, n) = \mathbb{C}$, it follows that we must show that

$$\bigcup_{n=1}^\infty C_n = U.$$

Towards this end, we prove that if $A \subseteq \mathbb{C}$ is any subset, then $\text{dist}_A(z) = 0$ if and only if $z \in \overline{A}$. Towards this end, if $\text{dist}_A(z) = 0$, then for any k , there is $w \in A$ such that $|w - z| < \frac{1}{k}$, so that we may construct a sequence $(w_n)_n$ in A such that $(w_n)_n \rightarrow z$, or that $z \in \overline{A}$. Similarly, if $z \in \overline{A}$, then if $(w_n)_n$ is a sequence in A converging to z , and $\varepsilon > 0$, it follows that $|w_n - z| < \varepsilon$ for sufficiently large n , so that $\inf_{w \in A} |w - z| = 0$.

Since U is open, it follows that for any $z \in \mathbb{C} \setminus U$, since $\mathbb{C} \setminus U = \overline{\mathbb{C} \setminus U}$, $\text{dist}_{\mathbb{C} \setminus U}(z) = 0$. Equivalently, if $z \in U$, we must have $\text{dist}_{\mathbb{C} \setminus U}(z) > 0$, so that there exists n sufficiently large such that $\text{dist}_{\mathbb{C} \setminus U}(z) \geq 1/n$; this means $z \in C_n$, so that

$$U \subseteq \bigcup_{n=1}^{\infty} C_n.$$

Meanwhile, if $z \in \bigcup_{n=1}^{\infty} C_n$, then there is some N such that $\text{dist}_{\mathbb{C} \setminus U}(z) \geq 1/N$, meaning that $\text{dist}_{\mathbb{C} \setminus U}(z) > 0$, meaning $z \notin \mathbb{C} \setminus U$, so that $z \in U$.

(b)

- (i) If $S = L + \varepsilon$ for $\varepsilon > 0$, it follows by the definition of the infimum that there exists a compact subset $K \subseteq U$ such that $\sup_{z \in U \setminus K} u(z) \leq S$. Therefore, for all $z \in U \setminus K$, $u(z) \leq S$.
- (ii) Let $z_n \in U \setminus K_n$, where the K_n are defined as above. Then, since the limit superior is an infimum over all compact subsets of U , we necessarily have

$$\limsup_{z \rightarrow \partial U} u(z) \leq u(z_n).$$

We claim that $(z_n)_n \rightarrow \partial U$. This follows from the fact that for any $K \subseteq U$ compact, there is some sufficiently large K_n such that $K \subseteq K_n$, as we have shown that $U = \bigcup_{n=1}^{\infty} K_n$, with the K_n nested. Thus, it follows that

$$\limsup_{z \rightarrow \partial U} u(z) \leq \limsup_{n \rightarrow \infty} u(z_n).$$

(c) From part (b) subpart (ii), we have established

$$\limsup_{z \rightarrow \partial U} u(z) \leq \sup_{\substack{(z_n)_n \subseteq U \\ (z_n)_n \rightarrow \partial U}} \limsup_{n \rightarrow \infty} u(z_n).$$

It remains to show the other direction. Toward this end, fix a sequence $(z_n)_n \rightarrow \partial U$. If K is an arbitrary compact subset of U , then there is some N sufficiently large such that for all $n \geq N$, we have $z_n \in U \setminus K$, so that

$$u(z_n) \leq \sup_{z \in U \setminus K} u(z).$$

Therefore, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} u(z_n) &\leq \sup_{n \in \mathbb{N}} u(z_n) \\ &\leq \sup_{z \in U \setminus K} u(z), \end{aligned}$$

meaning that

$$\limsup_{n \rightarrow \infty} u(z_n) \leq \limsup_{z \rightarrow \partial U} u(z).$$

Therefore, by taking the supremum over all such compact subsets, we establish

$$\sup_{\substack{(z_n)_n \subseteq U \\ (z_n)_n \rightarrow \partial U}} u(z_n) \leq \limsup_{z \rightarrow \partial U} u(z).$$

Problem (Problem 2): Let

$$U = \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im}(z) > 0\}.$$

- (a) Construct a conformal map from U to $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$.
- (b) Construct an unbounded harmonic function $u: U \rightarrow (0, \infty)$ such that for all $(x_0, y_0) \in \partial U \setminus \{(1, 0)\}$, we have that $\lim_{(x,y) \rightarrow (x_0, y_0)} u(x, y) = 0$.
- (c) Suppose that $v: U \rightarrow (0, \infty)$ is an unbounded harmonic function such that for all $(x_0, y_0) \in \partial U \setminus \{(1, 0)\}$, we have that $\lim_{(x,y) \rightarrow (x_0, y_0)} v(x, y) = 0$. Show that there exists a sequence $((x_n, y_n))_n$ in U converging to $(1, 0)$ and $\lim_{n \rightarrow \infty} v(x_n, y_n) = \infty$.

Solution:

- (a) We start by taking the Cayley transform, mapping \mathbb{H} to \mathbb{D} , given by $\frac{z-i}{z+i}$. The inverse Cayley transform, which maps \mathbb{D} to \mathbb{H} , is then given by the inverse transform, which takes

$$Q(z) = i \frac{1+z}{1-z}.$$

By taking $a + bi \in U$ with $b > 0$ and $a^2 + b^2 \leq 1$, we find that

$$i \frac{1+(a+bi)}{1-a-bi} = \frac{1}{(1-a)^2 + b^2} (-2b + i(1-a^2-b^2)).$$

Therefore, we observe that the inverse transform maps U to the second quadrant, admitting arguments between $\frac{\pi}{2}$ and π . By squaring, we have

$$(Q(z))^2 = -\left(\frac{z+1}{1-z}\right)^2,$$

which maps to complex numbers with arguments between π and 2π . Multiplying by -1 , we get

$$H(z) = \left(\frac{z+1}{1-z}\right)^2$$

mapping from U to the upper half-plane. Since we composed a series of bijective holomorphic maps (and, within a correct domain for the case of square root, ones that have holomorphic inverse), it follows that H is a bijective holomorphic map with holomorphic inverse, hence conformal.

- (b) Consider the function

$$u(x, y) = \operatorname{Im}(H(x + yi)).$$

We observe that u is the imaginary part of a holomorphic function, so it is harmonic. Since H maps U conformally to the upper half-plane, it follows that u maps U to $(0, \infty)$, and that u is unbounded, as H is unbounded. It remains to show that u maps ∂U to 0 in limit save for $(1, 0)$. Towards this end, we split the case into two parts.

If $x_0 + iy_0 = e^{i\theta}$ for some $0 < \theta_0 < \pi$, then

$$\begin{aligned} \frac{e^{i\theta} + 1}{1 - e^{i\theta}} &= \frac{(1 + \cos(\theta) + i \sin(\theta))(1 - \cos(\theta) + i \sin(\theta))}{2 - 2 \cos(\theta)} \\ &= \frac{1}{2 - 2 \cos(\theta)} (1 - \cos^2(\theta) - \sin^2(\theta) + 2i \sin(\theta)) \end{aligned}$$

$$\begin{aligned}
&= \frac{2i \sin(\theta)}{2 - 2 \cos(\theta)} \\
&= i \cot(\theta/2).
\end{aligned}$$

Squaring, we then get

$$\begin{aligned}
\left(\frac{e^{i\theta} + 1}{1 - e^{i\theta}} \right)^2 &= -\cot^2(\theta/2) \\
&\in \mathbb{R},
\end{aligned}$$

so that $u(x_0, y_0) = 0$ whenever $x_0 + iy_0 = e^{i\theta}$ for some $0 < \theta_0 < \pi$.

Meanwhile, if $x_0 + iy_0 = x_0$ with $x_0 \neq 1$, then

$$\begin{aligned}
H(x_0 + iy_0) &= \left(\frac{x_0 + 1}{1 - x_0} \right)^2 \\
&\in \mathbb{R},
\end{aligned}$$

so that $u(x_0, y_0) = 0$ yet again.

- (c) We let $v \equiv u$, where u is defined as above. Since u is unbounded, it follows that for each $N \geq 1$, there is $(x_N, y_N) \in U$ such that $u(x_N, y_N) \geq N$. Inductively, this allows us to construct a sequence $(x_n, y_n) \subseteq U$ such that $u(x_n, y_n) \geq n$, meaning that $\lim_{n \rightarrow \infty} u(x_n, y_n) = \infty$.

Since u is harmonic, it is subharmonic, so by a previously established theorem, since $\sup(u) = \infty$, it follows that $((x_n, y_n))_n \rightarrow \partial U$. Yet, this sequence cannot converge to any element of $\partial U \setminus \{(1, 0)\}$, as otherwise, we would have $u(x_n, y_n) \rightarrow 0$, which would contradict the fact that u is continuous as it is harmonic. Therefore, we have $((x_n, y_n))_n \rightarrow (1, 0)$.

Problem (Problem 3): Let

$$U = \{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 1\}.$$

Let $f: \overline{U} \rightarrow \mathbb{C}$ be a continuous bounded function for which $f|_U$ is holomorphic. Suppose there exist constants $M_0 \geq 0$ and $M_1 \geq 0$ such that

$$\begin{aligned}
\sup_{\operatorname{Re}(z)=0} |f(z)| &\leq M_0 \\
\sup_{\operatorname{Re}(z)=1} |f(z)| &\leq M_1.
\end{aligned}$$

Show that for all $r \in [0, 1]$,

$$\sup_{\operatorname{Re}(z)=r} |f(z)| \leq M_0^{1-r} M_1^r.$$

Solution: Let $\varepsilon > 0$ be fixed. Define

$$f_\varepsilon(z) = f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)}.$$

We will show that $\sup_{z \in \overline{U}} |f_\varepsilon(z)| \leq 1$. Towards this end, if $\operatorname{Re}(z) = 0$, we have $z = bi$ for some $b \in \mathbb{R}$; since $M_0, M_1 \in \mathbb{R}_{\geq 0}$, we then get

$$\begin{aligned}
\left| f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)} \right| &= \left| f(z) M_0^{bi-1} M_1^{-bi} e^{-\varepsilon(b^2+1)} \right| \\
&= \left| f(z) M_0^{-1} e^{-\varepsilon(b^2+1)} \right|
\end{aligned}$$

$$\begin{aligned} &\leq |f(z)M_0^{-1}| \\ &\leq 1. \end{aligned}$$

Similarly, if $\operatorname{Re}(z) = 1$, then we have $z = 1 + bi$ for some $b \in \mathbb{R}$, and since $M_0, M_1 \in \mathbb{R}_{\geq 0}$, we have

$$\begin{aligned} \left| f(z)M_0^{z-1}M_1^{-z}e^{\varepsilon(z^2-1)} \right| &= \left| f(z)M_0^{bi}M_1^{-bi-1}e^{\varepsilon(-2bi-b^2)} \right| \\ &= \left| f(z)M_1^{-1}e^{-b^2\varepsilon} \right| \\ &\leq |f(z)M_1^{-1}| \\ &\leq 1. \end{aligned}$$

Since $|f_\varepsilon(z)| \leq 1$ holds on both $\operatorname{Re}(z) = 0$ and $\operatorname{Re}(z) = 1$, it follows by the maximum modulus principle that we must have $|f_\varepsilon(z)| \leq 1$ on the interior. In particular, this means that

$$\sup_{z \in \bar{U}} |f_\varepsilon(z)| \leq 1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$|f(z)M_0^{z-1}M_1^{-z}| \leq 1$$

for all $z \in \bar{U}$, so that

$$\begin{aligned} |f(z)| &\leq |M_0^{1-z}| |M_1^z| \\ &= M_0^{1-\operatorname{Re}(z)} M_1^{\operatorname{Re}(z)}. \end{aligned}$$

In particular, this means that for $\operatorname{Re}(z) = r$, we have

$$|f(z)| \leq M_0^{1-r} M_1^r,$$

meaning this holds for the supremum over all z with $\operatorname{Re}(z) = r$, yielding

$$\sup_{\operatorname{Re}(z)=r} |f(z)| \leq M_0^{1-r} M_1^r.$$

Problem (Problem 4): Let $U \subseteq \mathbb{C}$ be a region, $f: U \rightarrow \mathbb{C} \setminus \{0\}$ a holomorphic function. We say that f has a logarithm in U if there exists a holomorphic function $g: U \rightarrow \mathbb{C}$ satisfying $f(z) = e^{g(z)}$ for all $z \in U$.

(a) Show that f has a logarithm in U if and only if for every piecewise C^1 cycle Γ in U , we have

$$\oint_{\Gamma} \frac{f'}{f} dz = 0.$$

(b) If γ is a piecewise C^1 loop in U , show that

$$\oint_{\gamma} \frac{f'}{f} dz = 2\pi i n(f \circ \gamma; 0).$$

(c) Let $J \subseteq \mathbb{N}$ be a countably infinite set. Show that f has a logarithm in U if and only if it has k th roots for every $k \in J$, so that for each $k \in J$, there is a holomorphic function $h: U \rightarrow \mathbb{C} \setminus \{0\}$ satisfying $h(z)^k = f(z)$ for all $z \in U$.

Solution:

(a) Let $f(z)$ have the logarithm $e^{g(z)}$. If γ is any piecewise C^1 loop in U , we wish to show that

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 0,$$

meaning that for any piecewise C^1 cycle $\Gamma = \gamma_1 + \cdots + \gamma_n$, we would have this sum add up to zero.

Toward this end, we observe that

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = \oint_{\gamma} g'(z) dz,$$

meaning that, since we are taking the closed contour integral of the derivative of a holomorphic function, it is equal to zero.

Now, if

$$\int_{\Gamma} \frac{f'(z)}{f(z)} dz = 0$$

for all piecewise C^1 cycles Γ in U , then it certainly follows for a cycles consisting of a single C^1 loop, γ , where $\text{im}(\gamma) \subseteq U$. Therefore, by an established proposition, there is some holomorphic function $g: U \rightarrow \mathbb{C}$ such that g is an antiderivative for $\frac{f'}{f}$; that is,

$$g'(z) = \frac{f'(z)}{f(z)}.$$

Following a similar argument to the specific case of U being a simply connected region and f nonvanishing, we see that if we define

$$h(z) = f(z)e^{-g(z)},$$

that

$$\begin{aligned} h'(z) &= f'(z)e^{-g(z)} - f(z)g'(z)e^{-g(z)} \\ &= f'(z)e^{-g(z)} - f(z)\frac{f'(z)}{f(z)}e^{-g(z)} \\ &= 0, \end{aligned}$$

so that $h(z)$ is some constant $k \in \mathbb{C}$, following from the fundamental theorem of calculus as, if $z_0 \in U$, $r > 0$, and $z \in U(z_0, r)$, we get

$$\begin{aligned} h(z) - h(z_0) &= \int_0^1 h'((1-t)z_0 + tz)(z - z_0) dt \\ &= 0, \end{aligned}$$

meaning h is constant on $U(z_0, r)$, hence constant by the identity theorem.

Since f is never zero, it follows that $k \neq 0$, so that there is some $k^* \in \mathbb{C}$ with $e^{k^*} = k$, and

$$f(z) = e^{k^*g(z)}.$$

Since $k^*g(z)$ is also holomorphic, we thus find that f has a logarithm.

- (b) Let $\gamma: [a, b] \rightarrow U$ be a piecewise C^1 loop. We observe then that $f \circ \gamma$ is also a piecewise C^1 loop (as f is holomorphic on U), so that

$$\begin{aligned} \oint_{\gamma} \frac{f'(z)}{f(z)} dz &= \int_a^b \frac{\gamma'(t)f'(\gamma(t))}{f(\gamma(t))} dt \\ &= \int_a^b \frac{1}{(f \circ \gamma)'(t)} (f \circ \gamma)'(t) dt \\ &= \oint_{f \circ \gamma} \frac{1}{w} dw \\ &= 2\pi i n(f \circ \gamma; 0). \end{aligned}$$

- (c) If f has a logarithm in U , then we have a holomorphic $g(z)$ such that $f(z) = e^{g(z)}$. Then, if $k \in J$, we have

$$h(z) = e^{\frac{1}{k} g(z)}$$

satisfies $h(z)^k = \left(e^{\frac{1}{k} g(z)}\right)^k = e^{g(z)}$, so that $f(z) = h(z)^k$.

Suppose that for all $k \in J$, there is $h_k(z)$ such that $h_k(z)^k = f(z)$. Now, since J is a countably infinite subset of \mathbb{N} , J is necessarily unbounded. Letting $\gamma: [a, b] \rightarrow U$ be a satisfactory piecewise C^1 loop, we then observe that for arbitrary $k \in J$, we must have

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = k \oint_{\gamma} \frac{h'_k(z)}{h_k(z)} dz.$$

Since the integral on the left hand side is single-valued, and k is arbitrary and unbounded, it follows that both sides are necessarily equal to zero, so

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 0$$

for any piecewise C^1 loop. Thus, by part (a), we see that f has a logarithm.

Problem (Problem 5):

- (a) Let $U \subseteq \mathbb{C}$ be a region, $f: U \rightarrow \mathbb{C} \setminus \{0\}$ a holomorphic function, and let g be a logarithm of f . Suppose there exist $z, w \in U$ such that $f(z) = f(w)$. Show that for any piecewise C^1 curve $\gamma: [a, b] \rightarrow U$ for which $\gamma(a) = z$ and $\gamma(b) = w$, we have $g(z) - g(w) = 2\pi i n(f \circ \gamma; 0)$.
- (b) Let $V \subseteq \mathbb{C}$ be a simply connected region. Fix a finite collection of points $a_1, \dots, a_k \in V$ and define $U = V \setminus \{a_1, \dots, a_k\}$. Let $\delta > 0$ be sufficiently small such that $B(a_i, \delta) \cap B(a_j, \delta) = \emptyset$ for $i \neq j$, and let $\gamma_j: [0, 2\pi] \rightarrow U$ be given by $\gamma_j(\theta) = a_j + \delta e^{i\theta}$.

Show that a holomorphic function $f: U \rightarrow \mathbb{C} \setminus \{0\}$ admits a logarithm if and only if $n(f \circ \gamma_j; 0) = 0$ for all j .

Solution:

- (a) Let $\gamma: [a, b] \rightarrow U$ be a piecewise C^1 curve with $\gamma(a) = z$ and $\gamma(b) = w$. Writing $f(z) = e^{g(z)}$, we observe that, by the fundamental theorem of calculus,

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = \int_{\gamma} g'(z) dz$$

$$\begin{aligned}
&= \int_a^b g'(\gamma(t))\gamma'(t) dt \\
&= \int_a^b (g \circ \gamma)'(t) dt \\
&= (g \circ \gamma)(b) - (g \circ \gamma)(a) \\
&= g(w) - g(z).
\end{aligned}$$

Meanwhile, by direct substitution, and using the fact that $f(z) = f(w)$, we obtain

$$\begin{aligned}
\int_{\gamma} \frac{f'(z)}{f(z)} dz &= \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt \\
&= \int_a^b \frac{1}{f(\gamma(t))} (f \circ \gamma)'(t) dt \\
&= \oint_{f \circ \gamma} \frac{1}{\zeta} d\zeta \\
&= 2\pi i n(f \circ \gamma; 0).
\end{aligned}$$

- (b) We use the condition in Problem 4 that f has a logarithm if and only if for every piecewise C^1 cycle in U , we have

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 0.$$

Since each of the γ_j are piecewise C^1 , it follows that if f has a logarithm in U , then

$$\begin{aligned}
\oint_{\gamma_j} \frac{f'(z)}{f(z)} dz &= 2\pi i n(f \circ \gamma_j; 0) \\
&= 0.
\end{aligned}$$

meaning that $n(f \circ \gamma_j; 0) = 0$ for each γ_j .

Now, suppose f satisfies $n(f \circ \gamma_j; 0) = 0$ for each j . Let γ be any piecewise C^1 loop in U , and define a piecewise C^1 cycle

$$\Gamma = \gamma - \sum_{j=1}^k n(\gamma; a_j) \gamma_j.$$

We claim that Γ is homologous to zero in U . Toward this end, we observe that $\mathbb{C} \setminus U$ consists of two disjoint sets: $\mathbb{C} \setminus V$ and $\{a_1, \dots, a_k\}$. We observe that Γ is a cycle in V as Γ is a cycle in U , so $n(\Gamma; z) = 0$ for any $z \in \mathbb{C} \setminus V$ as V is simply connected. Meanwhile, we have

$$\begin{aligned}
n(\Gamma; a_i) &= \oint_{\Gamma} \frac{1}{z - a_i} dz \\
&= \oint_{\gamma} \frac{1}{z - a_i} dz - \sum_{j=1}^k n(\gamma; a_j) \oint_{\gamma_j} \frac{1}{z - a_i} dz \\
&= n(\gamma; a_i) - n(\gamma; a_i) \\
&= 0,
\end{aligned}$$

where we use the fact that $n(\gamma_j; a_j) = 1$ by a previously established result.

Thus, we see that Γ is homologous to zero in \mathcal{U} . By the generalization of Cauchy's Integral Theorem, it follows that, since $f(z) \neq 0$ for all $z \in \mathcal{U}$, we have

$$\begin{aligned}
 0 &= \oint_{\Gamma} \frac{f'(z)}{f(z)} dz \\
 &= \oint_{\gamma} \frac{f'(z)}{f(z)} dz - \sum_{j=1}^k n(\gamma; a_j) \oint_{\gamma_j} \frac{f'(z)}{f(z)} dz \\
 &= \oint_{\gamma} \frac{f'(z)}{f(z)} dz - \sum_{j=1}^k n(\gamma; a_j) n(f \circ \gamma_j; 0) \\
 &= \oint_{\gamma} \frac{f'(z)}{f(z)} dz.
 \end{aligned}$$

Since γ is an arbitrary piecewise C^1 loop in \mathcal{U} , this holds for all such formal linear combinations of piecewise C^1 loops, hence for all piecewise C^1 cycles in \mathcal{U} . Thus, f admits a logarithm in \mathcal{U} by Problem 4 part (a).