

Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

$$d\mathbf{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}.$$

We let $\hat{e}_1 = \hat{\rho}$, $\hat{e}_2 = \hat{\phi}$, and $\hat{e}_3 = \hat{z}$, with $u_1 = \rho$, $u_2 = \phi$, and $u_3 = z$. Thus, we get

- Line element:

$$\begin{aligned} (ds)^2 &= \sum_{i,j} \frac{\partial \mathbf{r}}{\partial u_i} \frac{\partial \mathbf{r}}{\partial u_j} (\hat{e}_i \cdot \hat{e}_j) du_i du_j \\ &= \sum_{i=1} \left(\frac{\partial \mathbf{r}}{\partial u_i} \right) (du_i)^2 && \text{The } \hat{\rho}, \hat{\phi}, \hat{z} \text{ basis is orthogonal} \\ &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2. \end{aligned}$$

- Area element:

$$d\mathbf{a} = \left(\sum_k \epsilon_{ijk} \hat{e}_k \right) \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} du_i du_j$$

4.9

Without loss of generality, we have

$$\sum_{\ell} \epsilon_{mn\ell} \epsilon_{ij\ell} = \epsilon_{mn1} \epsilon_{ij1},$$

where $m, n, i, j = 2, 3$. If we have $m = i, n = j$, then $\epsilon_{mn1} \epsilon_{ij1} = 1$; if $m = j, n = i$, then $\epsilon_{mn1} \epsilon_{ij1} = -1$; else, $\epsilon_{mn1} \epsilon_{ij1} = 0$.

4.11

(a)

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k \\ &= - \sum_{i,j,k} \epsilon_{jik} B_j A_i \hat{e}_k \\ &= -(\mathbf{B} \times \mathbf{A}) \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= \sum_{i,j,k} (\epsilon_{ijk} A_i B_j \hat{e}_k) \cdot A_i \hat{e}_i \\ &= \sum_{i,j,k} \delta_{ik} (\epsilon_{ijk} A_i^2 B_j) \\ &= 0. \end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_\ell B_i C_j \\
&= \sum_{i,j,\ell} (\epsilon_{\ell ij} A_\ell B_i) C_j \\
&= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_\ell B_i C_j \\
&= \sum_{i,j,\ell} (\epsilon_{j\ell i} C_j A_i) B_i \\
&= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}).
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j} \epsilon_{ijk} A_i \left(\sum_{\alpha,\beta} \epsilon_{\alpha\beta j} B_\alpha C_\beta \right) \\
&= \sum_{i,j,\alpha,\beta} \epsilon_{ijk} \epsilon_{\alpha\beta j} A_i B_\alpha C_\beta \\
&= - \left(\sum_{i,j,\alpha,\beta} \epsilon_{ikj} \epsilon_{\alpha\beta j} A_i B_\alpha C_\beta \right) \\
&= - \left(\sum_{i,j,\alpha,\beta} (\delta_{i\alpha} \delta_{k\beta} - \delta_{i\beta} \delta_{k\alpha}) A_i B_\alpha C_\beta \right) \\
&= \sum_{i,j,\alpha,\beta} (\delta_{k\alpha} \delta_{i\beta} - \delta_{i\alpha} \delta_{k\beta}) A_i B_\alpha C_\beta \\
&= \sum_{i,j,\alpha,\beta} (B_\alpha \delta_{k\alpha}) (A_i C_\beta \delta_{i\beta}) - (C_\beta \delta_{k\beta}) (A_i B_\alpha \delta_{i\alpha}) \\
&= \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}).
\end{aligned}$$

Chapter 5 Problems

5.1

Let $f(x) = x^n$. We use linearity for the general case.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \rightarrow 0} \frac{x^n + n(hx^{n-1}) + \dots + nh^{n-1}x + h^n - x^n}{h} \\
&= \lim_{h \rightarrow 0} (nx^{n-1} + \dots + nh^{n-2}x + h^{n-1}) \\
&= nx^{n-1}.
\end{aligned}$$

5.6

$$\begin{aligned}
\cos(N\phi) + i \sin(N\phi) &= (\cos \phi + i \sin \phi)^N \\
&= \sum_{k=0}^N \binom{N}{k} (\cos \phi)^k (\sin \phi)^{N-k} \left(e^{i\frac{\pi}{2}}\right)^{N-k} \\
&= \sum_{k=0}^N \binom{N}{k} (\cos \phi)^k (\sin \phi)^{N-k} \left(\cos\left((N-k)\frac{\pi}{2}\right) + i \sin\left((N-k)\frac{\pi}{2}\right)\right) \\
&= \sum_{k=0}^N \binom{N}{k} \cos\left((N-k)\frac{\pi}{2}\right) (\cos \phi)^k (\sin \phi)^{N-k} \\
&\quad + i \left(\sum_{k=0}^N \binom{N}{k} \sin\left((N-k)\frac{\pi}{2}\right) (\cos \phi)^k (\sin \phi)^{N-k}\right).
\end{aligned}$$

We get the final answer by equating real and imaginary parts.

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n}{n+1}} \right| \\
&= 1,
\end{aligned}$$

which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \quad \forall n \geq 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(b) • Ratio test:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1}\right) \left(\frac{1}{2}\right) \right| \\
&= \frac{1}{2} \\
&< 1,
\end{aligned}$$

meaning the series converges by the ratio test.

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$$\frac{1}{n2^n} < \frac{1}{2^n} \quad \text{for all } n \geq 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.

6.9

$$\begin{aligned}
\sum_{n=-N}^N e^{inx} &= 1 + \sum_{n=1}^N e^{-inx} + \sum_{n=1}^N e^{inx} \\
&= 1 + e^{-ix} \sum_{n=1}^N e^{i(n-1)x} + e^{ix} \sum_{n=1}^N e^{i(n-1)x} \\
&= 1 + e^{-ix} \sum_{n=0}^{N-1} e^{-inx} + e^{ix} \sum_{n=0}^{N-1} e^{inx} \\
&= 1 + e^{-ix} \frac{1 - e^{-iNx}}{1 - e^{-ix}} + e^{ix} \frac{1 - e^{iNx}}{1 - e^{ix}} \\
&= 1 + \frac{e^{-ix} - e^{-i(N+1)x}}{1 - e^{-ix}} + \frac{1 - e^{iNx}}{e^{-ix} - 1} \\
&= 1 + \frac{(e^{-ix} - 1) + e^{iNx} - e^{-i(N+1)x}}{1 - e^{-ix}} \\
&= \frac{e^{iNx} - e^{-i(N+1)x}}{1 - e^{-ix}} \\
&= \frac{e^{iNx} - e^{-i(N+1)x}}{e^{-i(\frac{x}{2})} (e^{i(\frac{x}{2})} - e^{-i(\frac{x}{2})})} \\
&= \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{-i(\frac{x}{2})} - e^{-i(\frac{x}{2})}} \\
&= \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})}.
\end{aligned}$$

6.13

(a)

$$\begin{aligned}
\frac{d}{dz} (\arctan(z)) &= 1 - z^2 + z^4 - z^6 + \dots \\
&= \sum_{i=0}^{\infty} (-1)^i z^{2i} \\
&= \frac{1}{1 + z^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\rho &= \limsup_{k \rightarrow \infty} \sqrt[k]{|(-1)^k|} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}$$

(c)

$$\rho = \limsup_{k \rightarrow \infty} \left(\left| \frac{(-1)^k}{2k+1} \right| \right)^{1/k}$$

$$\begin{aligned}
 &= \limsup_{k \rightarrow \infty} \frac{1}{(2k+1)^{1/k}} \\
 &= 1 \\
 r &= \frac{1}{\rho} \\
 &= 1.
 \end{aligned}$$

6.25

$$\begin{aligned}
 e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\
 &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \\
 &= \cos \theta + i \sin \theta.
 \end{aligned}$$

6.37**6.42**