

# Excess Area Identities and Operator Symbols in Bergman Spaces: A Multifaceted Analysis

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# Summary

- 1 Definitions and Notations
- 2 Motivation and Problem
- 3 Results and Observations
- 4 Remarks and Future Directions
- 5 REU Experience
- 6 Acknowledgements and References

# Contents

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# Definition and Notations

- $\Omega$  : a region in  $\mathbb{C}$  e.g.  $\mathbb{D}, D(0, r), \mathbb{A}(0, r, 1), \mathbb{C}$
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$ : weight function

## Definition ( $\lambda$ -weighted Square-Integrable Functions)

$$L^2(\Omega, \lambda) = \left\{ f \left| \int_{\Omega} |f(z)|^2 \lambda(z) \, dA(z) < \infty \right. \right\}$$

- $L^2(\Omega, \lambda)$  forms a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega, \lambda)} = \int_{\Omega} f(z) \overline{g(z)} \lambda(z) \, dA(z)$$

inducing the norm

$$\|f\|_{L^2(\Omega, \lambda)}^2 = \int_{\Omega} |f(z)|^2 \lambda(z) \, dA(z)$$

# Definitions and Notations

## Definition (Holomorphic Function on $\Omega$ )

$$h \in \mathcal{O}(\Omega) \iff \frac{\partial h(z)}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial h(z)}{\partial x} + i \frac{\partial h(z)}{\partial y} \right) = 0, \quad \forall z \in \Omega$$

## Definition ( $\lambda$ -weighted Bergman Space)

$$A^2(\Omega, \lambda) := \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

# Definitions and Notations

## Definition ( $A^{1,2}(\Omega, \lambda)$ )

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \mid \frac{\partial h}{\partial z} \in A^2(\Omega, \lambda) \right\}$$

## Definition (Weighted Image-Area)

Let  $h \in A^{1,2}(\Omega, \lambda)$ .

$$A_{\Omega, \lambda}(h) = \int_{\Omega} \left| \frac{\partial h}{\partial z} \right|^2 \lambda(z) \, dA(z)$$

# Definitions and Notations

- $A^2(\Omega, \lambda)$  has a reproducing kernel i.e  $\exists! K_{\Omega}^{\lambda}(\cdot, z) \in A^2(\Omega, \lambda)$  :

$$h(z) = \langle h(\cdot), K_{\Omega}^{\lambda}(\cdot, z) \rangle_{L^2(\Omega, \lambda)}$$

- $A^2(\Omega, \lambda)$  is a closed subspace of  $L^2(\Omega, \lambda)$ .

## Definition (Orthogonal Projection)

Let  $P^{\Omega, \lambda} : L^2(\Omega, \lambda) \rightarrow A^2(\Omega, \lambda)$

$$\begin{aligned} (P^{\Omega, \lambda} h)(z) &:= \langle h(\cdot), K_{\Omega}^{\lambda}(\cdot, z) \rangle_{L^2(\Omega, \lambda)} \\ &= \int_{\Omega} h(w) \overline{K_{\Omega}^{\lambda}(w, z)} \lambda(w) \, dA(w) \end{aligned}$$

# Definitions and Notations

## Definition (Multiplication Operator)

Let  $M_\varphi : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda)$  where  $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

## Definition (Toeplitz Operator)

$T_\varphi^{\Omega, \lambda} : A^2(\Omega, \lambda) \rightarrow A^2(\Omega, \lambda)$ , where  $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega, \lambda} := P^{\Omega, \lambda} M_\varphi$$



# Definitions and Notations

## Definition (Commutator)

Let  $[P^{\Omega,\lambda}, M_\varphi] : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda)$

$$[P^{\Omega,\lambda}, M_\varphi] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

## Definition (Hankel Operator)

Let  $H_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \rightarrow (A^2(\Omega, \lambda))^\perp$

$$\begin{aligned} H_\varphi^{\Omega,\lambda} &:= - [P^{\Omega,\lambda}, M_\varphi] \Big|_{A^2(\Omega,\lambda)} \\ &= (I - P^{\Omega,\lambda}) M_\varphi \\ &= M_\varphi - P^{\Omega,\lambda} M_\varphi \\ &= M_\varphi - T_\varphi^{\Omega,\lambda} \end{aligned}$$

# Contents

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- 3 Results and Observations
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# Motivations

- $\{z^n\}_{n=0}^{\infty}$  form a *complete orthogonal basis* for  $A^2(\mathbb{D})$
- If  $h \in \mathcal{O}(\mathbb{D})$ , then  $h$  is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

and

$$S_N := \sum_{n=0}^N h_n z^n$$

converges uniformly on compact subsets.

- Relationship between  $L^2$  norm of  $h$  to the  $\ell^2$  norm of  $\{h_k\}_{k=0}^{\infty}$ :

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $[T_{\bar{z}}^{\mathbb{D}} M_z, D M_z](z^m) = 0$

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

# Literature Review on Previous Results I

- D'Angelo's excess area identity [D'A19]

Let  $h \in A^{1,2}(\mathbb{D})$ . Then,

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial(zh)}{\partial z} \right\|_{L^2(\mathbb{D})}^2 - \left\| \frac{\partial h}{\partial z} \right\|_{L^2(\mathbb{D})}^2 \\ &= \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \\ &= \pi \|Sh\|_{L^2(\partial\mathbb{D})}^2 \end{aligned}$$

where  $Sh$  is the restriction of  $h$  to the unit circle.

# Literature Review on Previous Results II

- Excess area identity with Blaschke product multiplier
- 'Excess area' identity for harmonic functions [BÇGH22]
- Generating symbols for Toeplitz operators for a given initial  $p$  and target polynomial  $q$  on unit disc and polydisc,  $T_{\varphi}^{\mathbb{D}}(p) = q$  and  $T_{\varphi}^{\mathbb{D}^n}(p) = q$  [ÇDTR<sup>+</sup>24]
- Substituted derivatives for Toeplitz operators in excess area identity [ÇDTR<sup>+</sup>24]

# Contents

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# Summary of Results

1. Results and Observations influenced by the Area Difference of the image of  $\mathbb{D}$  between  $zh$  and  $h$ :
  - i. On  $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2}), A^2(\mathbb{D}, \lambda), A^2(D(0, r))$
  - ii. On convergence of identities on certain weighted discs.
- 2 Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
  - i. On unweighted and weighted Toeplitz operators relation
  - ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on  $A^2(\mathbb{D})$



# Methods Used

- Relation between  $L^2$  norms of functions and  $\ell^2$  norms of Taylor series:

$$\|h\|_{L^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- Integration by parts via Stokes's theorem on forms:

$$\oint_{\partial\Omega} f \, dz = \int_{\Omega} \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz$$
$$\oint_{\partial\Omega} f \, d\bar{z} = \int_{\Omega} \frac{\partial f}{\partial z} \, dz \wedge d\bar{z}.$$

- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Special functions e.g. beta, gamma, hypergeometric

# Excess Area on Fock Spaces

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \|Sh\|_{L^2(b\mathbb{D})}^2$$

## Excess Area on Fock Space

Given  $0 < \rho < 1$ ,  $h \in \mathcal{F}^2$ , let  $h_\rho(z) := h(\rho z)$

$$\begin{aligned} A_{\mathcal{F}^2}(zh_\rho) - A_{\mathcal{F}^2}(h_\rho) \\ &= \pi \left\| z T_{\bar{z}}^{\mathcal{F}^2}(h_\rho) \right\|_{\mathcal{F}^2}^2 + \pi \left\| T_{\bar{z}}^{\mathcal{F}^2}(h_\rho) \right\|_{\mathcal{F}^2}^2 + \pi \left\| H_{\bar{z}}^{\mathcal{F}^2}(h_\rho) \right\|_{\mathcal{F}^2}^2 \\ &= \pi \left\| z^2 h_\rho \right\|_{\mathcal{F}^2}^2 - 2\pi \left\| zh_\rho \right\|_{\mathcal{F}^2}^2 + \pi \left\| h_\rho \right\|_{\mathcal{F}^2}^2 \end{aligned}$$

Here, the restriction of  $h$  to the unit circle in D'Angelo's identity is replaced with the Bergman projection on  $\mathbb{C}$ .

## Excess Area on $A^2(\mathbb{D}, \lambda)$

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \|Sh\|_{L^2(\partial\mathbb{D})}^2$$

### Excess Area on $A^2(\mathbb{D}, \lambda)$

Let  $h \in A^{1,2}(\mathbb{D}, \lambda)$ ,  $\lambda(z) = 1 - |z|^2$ . Then,

$$A_{\mathbb{D},\lambda}(z^{m+1}h) - A_{\mathbb{D},\lambda}(z^m h) = \pi \|z^m h\|_{L^2(\mathbb{D},\lambda)}^2.$$

Here, the restriction of  $h$  to the unit circle is replaced with the function itself.

## Excess Area on $A^2(D(0, r))$

### Excess Area on $A^2(D(0, r))$ , $0 < r < 1$

Let  $f_{r, a_k}(\zeta) = (rf_{a_k}) \circ (rf_b)^{-1}(\zeta)$  and  $|f_{r, a_k}(\zeta)| = r$  when  $|\zeta| = r$ ,  $a_k \neq b$ .

Let  $B_r = \prod_{k=1}^n f_{r, a_k}$  be a modified finite Blaschke product. Then,

$$A_{D(0, r)}(B_r h) - r^{2N} A_{D(0, r)}(h) = \pi r^{2(N-1)} \sum_{k=1}^n m_k \left\| \text{Sh} \left( f_{r, a_k}^{-1} \right) \right\|_{L^2(bD(0, r))}^2,$$

where  $m_k$  is the multiplicity of  $f_{r, a_k}$  and  $N = \sum_{k=1}^n m_k$ .

## Excess Area on $A^2(D(0, r))$ , cont'd

### Excess Area Identity for Harmonic Functions on $D(0, r)$ , $0 < r < 1$

For a harmonic function  $u \in L^2(D(0, r))$ ,  $\exists v \in L^2(D(0, r))$  harmonic conjugate [BÇGH22]. Let  $h = u + iv$  be the corresponding holomorphic function. Then,

$$\begin{aligned} & \left\| \frac{\partial(zu)}{\partial z} \right\|_{L^2(D(0, r))}^2 - r^2 \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D(0, r))}^2 \\ &= \frac{1}{4} \left( \underbrace{r^2 \pi \|Sh\|_{L^2(bD(0, r))}^2}_{A_{D(0, r)}(zh) - r^2 A_{D(0, r)}(h)} + 2r^2 \pi \Re(h_0^2) + \|h\|_{L^2(D(0, r))}^2 \right). \end{aligned}$$

## Dilation and Contraction from $A^2(D(0, r))$ to $A^2(\mathbb{D})$

Contracting  $h \in A^{1,2}(\mathbb{D})$  by taking  $h_r = h(rz)$  for some  $0 < r < 1$ ,

$$A_{\mathbb{D}}(zh_r) - A_{\mathbb{D}}(h_r) = \pi \|Sh_r\|_{L^2(b\mathbb{D})}^2 \quad (1)$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \|Sh\|_{L^2(bD(0,r))}^2. \quad (2)$$

Dilating  $h \in A^{1,2}(D(0, r))$  by taking  $h_{\frac{1}{r}} = h(\frac{z}{r})$  for some  $0 < r < 1$

$$A_{D(0,r)}(zh_{1/r}) - r^2 A_{D(0,r)}(h_{1/r}) = \pi r^2 \|Sh_{1/r}\|_{L^2(bD(0,r))}^2 \quad (3)$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \|Sh\|_{L^2(b\mathbb{D})}^2 \quad (4)$$

## Dilation from $A^2(D(0, r), \lambda_r)$ to $\mathcal{F}^2$

### Weighted Area on $D(0, r)$

Let  $\lambda_r(z) = \chi_{D(0, r)} \left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$  where  $r > 0$ . Then,

$$A_{D(0, r), \lambda_r}(h) = \int_{D(0, r)} |h'(z)|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} dA(z)$$

We find that, as  $r \rightarrow \infty$ ,  $A_{D(0, r), \lambda_r}(h) \rightarrow A_{\mathcal{F}^2}(h)$ .

Additionally, we know that

$$A_{\mathcal{F}^2}(h_\rho) = \left\| T_{\frac{\mathcal{F}^2}{z}} h_\rho \right\|_{\mathcal{F}^2}^2$$

# Berezin Transform Convergence

## Reproducing Kernel on $A^2(D(0, r), \lambda_r)$

$$K_{D(0,r)}^{\lambda_r}(w, z) = \frac{1}{\left(1 - \frac{\bar{z}w}{r^2}\right)^{r^2+2}}$$

$K_{D(0,r)}^{\lambda_r}(w, z)$  uniformly converges on compact subsets of  $D(0, r)$ .

## Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w, z) = e^{\bar{z}w}$$



# Berezin Transform Convergence, Cont'd

## Definition (Berezin Transform ([Zhu07]))

Let

$$k_z^{\Omega, \lambda}(w) := \frac{K_{\Omega}^{\lambda}(w, z)}{\sqrt{K_{\Omega}^{\lambda}(z, z)}}$$

Then, for some bounded operator  $T$  on  $L^2(\Omega, \lambda)$ , define  $\mathcal{B}^{\Omega, \lambda} : B(L^2(\Omega, \lambda)) \rightarrow L^2(\Omega, \lambda)$

$$(\mathcal{B}^{\Omega, \lambda} T)(z) := \left\langle T k_z^{\Omega, \lambda}, k_z^{\Omega, \lambda} \right\rangle_{L^2(\Omega, \lambda)}$$

# Berezin Transform Convergence, Cont'd

- For  $\varphi \in L^\infty(\Omega, \lambda)$ ,  $\mathcal{B}^{\Omega, \lambda} T_\varphi = \mathcal{B}^{\Omega, \lambda} M_\varphi$ . (see Axler and Zheng, [AZ98a]).
- $\varphi$  is harmonic if and only if  $\mathcal{B}^{\Omega, \lambda} M_\varphi = \varphi$  (proof by Engliš, [Eng94]).
- We find that, for  $T_\varphi^{D(0,r), \lambda_r} = P^{D(0,r), \lambda_r} M_\varphi$ , the Berezin transform  $\mathcal{B}^{D(0,r), \lambda_r} T_\varphi^{D(0,r), \lambda_r}$  converges pointwise to  $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$  as  $r \rightarrow \infty$  from Göğüş and Şahutoğlu ([Gc20])
- This convergence is uniform on compact subsets of  $\mathbb{C}$  (proof inspired by Göğüş and Şahutoğlu in [Gc20]).

# Unweighted and Weighted Toeplitz Operators

## Relation I

Using an extension of [ÇDTR<sup>+</sup>24, Lemma 2.1]

For weight  $\lambda(z) = (1 - |z|^2)^\alpha$  ( $\alpha \geq 0$ ) on the unit disc,  
 $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ :

$$\frac{T_{\bar{z}^m}^{\mathbb{D}, \lambda_\alpha}(z^n)}{T_{\bar{z}^m}^{\mathbb{D}}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leq n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T_{\bar{z}^m}^{\mathbb{D}, \lambda_\alpha}(z^n) = s_{n,m,\alpha} T_{\bar{z}^m}^{\mathbb{D}}(z^n), \text{ and } \lim_{n \rightarrow \infty} s_{n,m,\alpha} = 1$$

# Unweighted and Weighted Commutator on $A^2(\mathbb{D})$

## Existence of Commutator Symbols

Given  $p$  and  $q$  are harmonic polynomials and  $\frac{\partial}{\partial \bar{z}}(p) \neq 0$ , there does not exist a polynomial symbol  $\phi$ , such that  $[P^{\mathbb{D}}, M_{\phi}](p) = q$  or  $[P^{\mathbb{D}, \lambda}, M_{\phi}](p) = q$ .

Compare to [ÇDTR<sup>+</sup>24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

# Unweighted and Weighted Hankel Operator on $A^2(\mathbb{D})$

## Existence of Hankel Operator Symbols

Given some holomorphic polynomials  $p, q$  where  $p$  is not constant, there does not exist a polynomial symbol  $\phi$  such that  $H_{\phi}^{\mathbb{D}}(p) = \overline{q}$  or

$$H_{\phi}^{\mathbb{D}, \lambda}(p) = \overline{q}$$

# Contents

- 1 Definitions and Notations
- 2 Motivation and Problem
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- 4 Remarks and Future Directions**
- 5 REU Experience
- 6 Acknowledgements and References

## Remarks on the Annulus

### Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0, r, 1))$

For all integers  $m$  and  $n$ ,

$$T_{\bar{z}^m}^{\mathbb{A}(0, r, 1)}(z^n) = \begin{cases} \frac{2m r^{2m} \ln(r)}{(r^{2m} - 1)} z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m} - 1}{2m \ln(r)} z^{-1} & \text{if } n = m - 1 . \\ \frac{(n-m+1)(1-r^{2n+2})}{(n+1)(1-r^{2n-2m+2})} z^{n-m} & \text{else} \end{cases}$$

We attempted to find an algorithm to generate  $\varphi \in L^\infty(\mathbb{A}(0, r, 1))$  such that  $T_\varphi^{\mathbb{A}(0, r, 1)}(p) = q$  for given holomorphic Laurent polynomials  $p$  and  $q$ , but ran into trouble beyond the case where  $p$  has roots outside  $\overline{\mathbb{A}(0, r, 1)}$ .

# Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial  $p$  and target polynomial  $q$  on  $\mathbb{A}(0, r, 1)$ ,  $T_{\varphi}^{\mathbb{A}(0, r, 1)}(p) = q$
- Extension of 'excess area' identity to harmonic functions in  $L^2(\mathbb{C}, e^{-|z|^2})$ .
- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential,  
 $(1 - |z|^2)^A e^{\frac{-B}{(1-|z|^2)^\alpha}} (A \geq 0, B > 0, \alpha > 0)$ .



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- 4 Remarks and Future Directions
- 5 REU Experience**
- 6 Acknowledgements and References

# Experience at the REU

- Group consisted of me, Sakia Akamah (Rose–Hulman Institute of Technology), and Jennifer Yuan (NYU Abu Dhabi).
- Spent 7 weeks doing various calculations, final week spent preparing the final report and presentation.
- Other projects at the REU included machine learning, coding theory, and mathematical biology,

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- Work was not all that we did, though.



# Contents

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- 2 Motivation and Problem
- 3 Results and Observations
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- 5 REU Experience
- 6 Acknowledgements and References**

# Acknowledgments

This work was only possible by the guidance of Dr. Çelik.

We would also like to express our gratitude to Texas A&M-Commerce and its Mathematics Department for hosting the REU where we conducted this research.

This research is based upon work supported by the National Science Foundation under Grant DMS-2243991.

This presentation was originally given at the Texas A&M University-Commerce REU on 2 August, 2024.

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