Problem (Problem 1)**:** Let (X, x_0) be a pointed CW complex. Prove that $\Sigma X = X \wedge S^1$.

Solution: The reduced suspension yields the identifications

$$\begin{aligned}\Sigma X &= X \times [0,1]/(X \times \{0\} \sim \{x_0\}, X \times \{1\} \sim \{x_0\}, \{x_0\} \times [0,1] \sim \{x_0\}) \\ &= X \times [0,1]/(X \times \{1\} \sim \{x_0\} \times [0,1], X \times \{1\} \sim \{x_0\} \times [0,1]).\end{aligned}$$

Now, if we consider $\{0\} = \{1\}$ to be the basepoint of S^1, this yields our desired smash product

$$\begin{aligned}X \wedge S^1 &= X \times S^1/\bigl(X \times \{0\} \sim \{x_0\} \times S^1\bigr) \\ &= \Sigma X.\end{aligned}$$

Problem (Problem 2)**:** Prove that, if $f\colon X \to Y$ is a map, then Y is a deformation retract of the mapping cylinder M_f.

Solution: Recalling the definition of the mapping cylinder, we have

$$M_f = X \times [0,1] \coprod Y/((x,1) \sim f(x)).$$

In fact this allows us to define a homotopy from the identity on M_f into Y by taking

$$f_t(p) = \begin{cases} p & p \in Y \\ (p,t) & p \in X \end{cases}.$$

We observe that this is a constant homotopy when restricted to Y, that it is continuous in t as it is just a progression along $\{p\} \times [0,1]$, and that

$$f_1(p) = \begin{cases} p & p \in Y \\ f(p) & p \in X \end{cases},$$

so this is a deformation retract.

Problem (Problem 3)**:** Prove that, if $f\colon X \hookrightarrow Y$ is an inclusion map, then the mapping cone C_f is homotopy equivalent to the quotient $Y/f(X)$.

Solution: First, we observe that f is an inclusion map, so $X \subseteq Y$ and $f(x) = x$ for all $x \in X$. Additionally, the mapping cylinder M_f is given by

$$M_f = X \times [0,1] \sqcup Y/((x,1) \sim x),$$

and that Y is homotopy equivalent to M_f from Problem 2. We observe then that the mapping cone C_f is the quotient $M_f/X \times \{0\}$, while we see that

$$M_f/X \times \{1\} = (X \times [0,1] \sqcup Y/X)/((x,1) \sim x),$$

which we can view as an “inverted” mapping cone, of sorts. In particular, this “inverted mapping cone” deformation retracts via straight line homotopy to $[x] \in Y/X$ for each $x \in X$, which is a single point. Therefore, we may define a homotopy (with corresponding homotopy inverse)

$$H\colon C_f \times [0,1] \to Y/f(X)$$

given by

$$H_t(p) = [p],$$

where the equivalence class for p is considered in $M_f/(X \times \{t\})$. This map is continuous by the definition of the quotient topology, so this is our desired homotopy equivalence.