1.15

Problem. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(a, b) = 2^a 3^b$. Show that f is injective. Use the Cantor–Schröder–Bernstein theorem to deduce that $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Solution. Suppose $2^{a_1} 3^{b_1} = 2^{a_2} 3^{b_2}$. By the fundamental theorem of arithmetic, it must be the case that $a_1 = a_2$ and $b_1 = b_2$, meaning f is injective.

Since we have an injection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with $g(n) = (n, 0)$, it is the case that, by the Cantor–Schröder–Bernstein theorem, there exists some bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, meaning they have the same cardinality.

1.16

Problem. Let A be the set of all finite subsets of $\mathbb{N}$. Find injective functions from $\mathbb{N}$ to A and vice versa. Use the Cantor–Schröder–Bernstein theorem to deduce that A is countably infinite. Then, prove that the number of infinite subsets of $\mathbb{N}$ is uncountable.

Solution. There is a simple injection from $\mathbb{N}$ to $A = \mathcal{F}(\mathbb{N})$ by taking $f(n) = \{n\}$.

In the reverse direction, for some $X \in A$, define $X = \{x_1, \dots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$. Let p_i denote the ith prime number, and

$$f(X) = \prod_{i=1}^{n} p_i^{x_i}.$$

Suppose $f(X) = f(Y)$ for some $X, Y \in A$. Then, $X = \{x_1, \dots, x_m\}$ and $y = \{y_1, \dots, y_n\}$. Since $f(X) = f(Y)$, we have

$$\prod_{i=1}^{m} p_i^{x_i} = \prod_{i=1}^{n} p_i^{y_i}.$$

Suppose toward contradiction that $m \neq n$. Without loss of generality, we have $m > n$, implying that $p_m^{x_m} | f(X) = f(Y)$, meaning $p_m^{x_m} | p_1^{y_1} \cdots p_n^{y_n}$, but $p_m > p_1, \dots, p_n$, which is not possible.

Thus, we have

$$p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m} = p_1^{y_1} p_2^{y_2} \cdots p_m^{y_m},$$

which by the fundamental theorem of arithmetic, means $x_i = y_i$ for all i.

Since the set of all subsets of $\mathbb{N}$, $P(\mathbb{N})$, is uncountable, and $A = \mathcal{F}(\mathbb{N})$ is countable, it is the case that the set of infinite subsets of $\mathbb{N}$, $P(\mathbb{N}) \setminus \mathcal{F}(\mathbb{N})$, is uncountable. To show this, suppose toward contradiction that $P(\mathbb{N}) \setminus \mathcal{F}(\mathbb{N})$ is countable. Then, we would have $\mathcal{F}(\mathbb{N}) \cup (P(\mathbb{N}) \setminus \mathcal{F}(\mathbb{N}))$ is a countable union of countable sets, implying $P(\mathbb{N})$ is countable, which is a contradiction.

1.17

Problem. Let $\mathbb{R}^\times$ denote the set of nonzero real numbers. Use the Cantor–Schröder–Bernstein theorem to deduce that $|\mathbb{R}^\times| = |\mathbb{R}|$. Now, try to explicitly define a bijection between the sets.

Solution. The inclusion map $\iota : \mathbb{R}^\times \to \mathbb{R}$ is an injection, implying that $|\mathbb{R}^\times| \leqslant |\mathbb{R}|$. Additionally, the map $f : \mathbb{R} \to \mathbb{R}^\times$ defined by $f(x) = \arctan(x) + \pi/2$ is an injection from $\mathbb{R}$ into $\mathbb{R}^\times$, meaning $|\mathbb{R}| \leqslant |\mathbb{R}^\times|$. Thus, by Cantor–Schröder–Bernstein, there is a bijection from $\mathbb{R}$ to $\mathbb{R}^\times$.

The function

$$f : \mathbb{R} \to \mathbb{R}^{\times}$$

defined by

$$f(x) = \begin{cases} x+1 & x \in \mathbb{N} \\ x & x \notin \mathbb{N} \end{cases}$$

is a bijection from $\mathbb{R}$ to $\mathbb{R}^{\times}$.

1.18

Problem. Let $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $B = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$. Find injective functions $f : A \to B$ and $g : B \to A$, and deduce that $|A| = |B|$. Try to define an explicit bijection between A and B.

Solution. The inclusion map $\iota : A \hookrightarrow B$ is an injection between $(0,1)$ and $[0,1]$. Additionally, $g : [0.1] \to (0,1)$ defined by $g(x) = \frac{1}{3}x + \frac{1}{3}$ is also an injection between $[0,1]$ and $(0,1)$. Thus, by Cantor–Schröder–Bernstein, there is a bijection between A and B.

We take

$$\left\{\frac{1}{n} \mid n \geqslant 2\right\},$$

and map $\frac{1}{2}$ to 0, $\frac{1}{3} \mapsto 1$, and $\frac{1}{n+2} \mapsto \frac{1}{n}$ for $n \geqslant 2$. For $x \notin \left\{\frac{1}{n} \mid n \geqslant 2\right\}$, we map $x \mapsto x$. This yields a bijection from $(0,1)$ to $[0,1]$.

Solution (Alternative using Chasing). We let $[0,1]$ be the set of dogs and $(0,1)$ be the set of cats, with $f(x) = x$ mapping $(0,1)$ into $[0,1]$, and $g(x) = \frac{1}{3} + \frac{1}{3}x$ mapping $[0,1]$ into $(0,1)$.

The first dog-sequence maps

$$\begin{aligned} g(0) &= \frac{1}{3} \\ g\left(\frac{1}{3}\right) &= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right) \\ g\left(\frac{1}{3} + \frac{1}{3^2}\right) &= \underbrace{\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3}}_{\sum_{i=1}^{3} \frac{1}{3^i}}. \end{aligned}$$

Inductively, we have $\underbrace{g \circ \cdots \circ g}_{n \text{ times}}(0)$ is

$$g^n(0) = \sum_{i=0}^{n} \frac{1}{3^i}.$$

For some cat $c \in (0,1)$, we have

$$h(c) = \begin{cases} f(c) & \text{otherwise} \\ g^{-1}(c) & x \text{ is in a dog sequence} \end{cases}.$$

In particular, our dog-sequence elements are the ones that are of the form

$$\sum_{i=0}^{n} \frac{1}{3^i} = \frac{1}{2}\left(1 - \frac{1}{3^i}\right)$$

for $n \in \mathbb{N}$, and the corresponding sequence that starts with $g(1)$.

1.19

Problem. Let $S = \{s_1, \dots, s_n\}$ be a nonempty set of finitely many symbols. Show that the number of finite strings consisting of elements of S is countably infinite. What happens if S is countably infinite?

Solution. We let S_i be the set of strings of length i; there are n^i elements of S_i, which is finite. The set of all finite strings in S is

$$\bigcup_{i=1}^{\infty} S_i.$$

Since the set S_i are disjoint, it is the case that the set of all finite strings in S is a countably infinite union of finite disjoint sets, which is countably infinite.

If S is countably infinite, then by ordering the finite subsets of S by length and lexicographical order, we find that the set of finite subsets of S is countably infinite.

1.20

Problem. The two questions below refer to Hilbert's Hotel, discussed at the end of the chapter.

(a) A fleet of countably infinite busses arives with countably infinite passengers. Describe a way to assign rooms to everyone, including those currently in the hotel, such that no rooms are left empty.

(b) There are now a countably infinite number of fleets of countably infinite buses with a countably infinite number of people. Find a way for the desk attendant to accommodate all guests.

Solution.

(a) Move every current resident of the hotel to 2 multiplied by their current room number. Use the Cantor pairing function to map $\mathbb{N} \times \mathbb{N}$ to map each of the countably infinite busses' countably infinite members to $\mathbb{N}$. Then, for each new resident, multiply their room number by 2 and add 1.

(b) Proceeding in a similar manner, we can compose the Cantor pairing function with itself to create a bijection from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, then multiply by 2 and add 1 to map every new resident to an odd room, while mapping every current resident to an even room.