4.7

Problem: Show that the function $P(x,y) = x^y$ is primitive recursive.

Solution: We have

$$\begin{aligned} P(x, y+1) &= M(x, P(x,y)), \\ P(x,0) &= 1 \\ &= S(C_0(x)) \end{aligned}$$

Thus, in the format of primitive recursion,

$$P(x, y+1) = g(x, y, P(x,y))$$

where

$$g(x,y,z) = M\left(P_1^{(3)}(x,y,z), P_3^{(3)}(x,y,z)\right)$$

and

$$\begin{aligned} P(x,0) &= f(x) \\ &= S(C_0(x)). \end{aligned}$$

It is the case that P is computed by primitive recursion on f and g, since $M(x,y)$ is primitive recursive by a previous result.

4.8

Problem: Given a primitive recursive function $p(x_1, \dots, x_n, y)$, show that the function

$$h(x_1, \dots, x_n, z) = \prod_{k=0}^{z} p(x_1, \dots, x_n, k)$$

is primitive recursive.

Solution: We have

$$\begin{aligned} h(x_1, \dots, x_n, z+1) &= M(p(x_1, \dots, x_n, z+1), h(x_1, \dots, x_n, z)), \\ h(x_1, \dots, x_n, 0) &= p(x_1, \dots, x_n, 0). \end{aligned}$$

Since h is a composition of primitive recursive functions, h is necessarily primitive recursive.

4.9

Problem: Show that the "less than or equal to" relation on $\mathbb{N} \times \mathbb{N}$ is primitive recursive.

Solution: We have

$$L(x,y) = 1 \dot{-} (x \dot{-} y)$$

computes 1 if $x \leqslant y$ and 0 if $x > y$. Thus, since L is a composition of primitive recursive functions, L is primitive recursive. Written in the format of proven primitive recursive functions, we have

$$L(x,y) = \text{sub}\left(S\left(C_0\left(P_1^{(2)}(x,y)\right)\right), \text{sub}(x,y)\right).$$

Extra Problem 1

Problem: Show that every primitive recursive function is total.

Solution (Proof 1)**:** Let f be primitive recursive. Then, f is obtained via $S, C_0, P_i^{(k)}$ by composition and primitive recursion. We want to show that composition and primitive recursion preserve totality.

Turning our attention to composition, we have

$$h\left(x_1, \dots, x_n\right) = f\left(g_1\left(x_1, \dots, x_n\right), \dots, g_m\left(x_1, \dots, x_n\right)\right).$$

We will use the established result that if f and $g_1, \dots, g_m$ are total functions, then h is total.

In primitive recursion, we have

$$\begin{aligned} h\left(x_1, \dots, x_n, y+1\right) &= g\left(x_1, \dots, x_n, y, h\left(x_1, \dots, x_n, y\right)\right) \\ h\left(x_1, \dots, x_n, 0\right) &= f\left(x_1, \dots, x_n\right). \end{aligned}$$

We will use the established result that if f, g are total, then h is total.

Additionally, we will use the established result that $S, C_0, P_i^{(k)}$ are total.

We say h has depth n if h is obtained by one application of composition or primitive recursion with one or more functions of depth less than or equal to $n-1$, one of which has depth exactly equal to $n-1$. We say h has depth 0 if $h = S, C_0, P_i^{(k)}$.

We then use induction on the depth of h to prove the totality of h.

$d = 0$: $h = S, C_0, P_i^{(k)}$, so by the previous result, h is total.

$d = n+1$: If every function with depth n is total, then for h with depth $n+1$, h is obtained by composition of functions with depth n, or

Solution (Proof 2)**:** Instead of using depth, we will let

$$\mathrm{PR}_0 = \left\{S, C_0, P_i^{(k)}\right\},$$

where $i \leqslant k \in \mathbb{N}$. We let

$$\mathrm{PR}_{n+1} = \left\{h \;\middle|\; h \text{ is obtained by one composition or one primitive recursion from functions in } \mathrm{PR}_n\right\} \cup \mathrm{PR}_n$$

Then,

$$\mathrm{PR} = \bigcup_{i=1}^{\infty} \mathrm{PR}_i$$

consists of all primitive recursive functions.

We want to show that if $h \in \mathrm{PR}_n$ is total, then $h \in \mathrm{PR}_{n+1}$ is total.

Extra Problem 2

Problem:

Solution: Let $g(n, x) = f(n) = A(n, n)$. Suppose toward contradiction that f is primitive recursive. Then, for all $n, x > M$

$$\begin{aligned} A(n, x) &> g(n, x) \\ &= f(n) \\ &= A(n, n). \end{aligned}$$

However, since $x > M$ and $n > M$, we can take $n = x$, meaning $A(n, n) > A(n, n)$, which is a contradiction.