Contents

Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and C*-algebras. This section of notes will be focused on the essential results in functional analysis, starting from normed vector spaces, working our way up through C*-algebras.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

Normed Vector Spaces

Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over $\mathbb{C}$. Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

Definition (Vector Space)**.** A vector space V is a set closed under two operations

$$\begin{aligned} a &: V \times V \to V,\ (v_1, v_2) \mapsto v_1 + v_2 \\ m &: \mathbb{C} \times V \to V,\ (\lambda, v) \mapsto \lambda v. \end{aligned}$$

We refer to a as addition, and m as scalar multiplication; $(V, +)$ is an abelian ring.

Definition (Norm)**.** A norm is a function

$$\|\cdot\| : V \to \mathbb{R}^+,\ x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness: $\|v\| = 0$ if and only if $v = 0_V$.
- Triangle inequality: $\|v + w\| \leqslant \|v\| + \|w\|$.
- Absolute Homogeneity: $\|\lambda v\| = |\lambda| \|v\|$, for $\lambda \in \mathbb{C}$.

If a function $p : V \to \mathbb{R}^+$ satisfies the triangle inequality and absolute homogeneity, we say p is a semi-norm.

We say the pair $(V, \|\cdot\|)$ is a normed vector space.

Definition (Balls and Spheres)**.** Let X be a normed vector space, $x \in X$, and $\delta > 0$. Then,

$$\begin{aligned} U(x,\delta) &= \{y \in X \mid d(x,y) < \delta\}\\ B(x,\delta) &= \{y \in X \mid d(x,y) \leqslant \delta\}\\ S(x,\delta) &= \{y \in X \mid d(x,y) = \delta\}. \end{aligned}$$

For a normed vector space, we will use the following conventions for common sets:

$$\begin{aligned} U_X &= U(0,1)\\ B_X &= B(0,1)\\ S_X &= S(0,1)\\ \mathbb{D} &= U_{\mathbb{C}}\\ \mathbb{T} &= S_{\mathbb{C}}. \end{aligned}$$

Definition (Equivalent Norms)**.** Two norms on V, $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there are two constants C_1 and C_2 such that

$$\begin{aligned} \|v\|_a &\leqslant C_1 \|v\|_b\\ \|v\|_b &\leqslant C_2 \|v\|_a \end{aligned}$$

for all $v \in V$. We say $\|\cdot\|_a \sim \|\cdot\|_b$.

Examples

Example (Finite-Dimensional Vector Spaces)**.** The vector space $\mathbb{C}^n$ is with the p-norm is denoted ℓ_p^n, where for $p \in [1,\infty]$, the p-norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

In the case with $p = 2$, this gives the traditional Euclidean norm, and with $p = \infty$, this gives the sup norm:

$$\|x\|_\infty = \max_{1\leqslant i\leqslant n} |x_i|.$$

Example (A Sequence Space)**.** We let $\ell_p = \left\{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\right\}$ be the collection of sequences in $\mathbb{C}$ with finite p-norm. Here,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

In the case with $p = \infty$, this gives the sequence space ℓ_∞, which has norm

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

Example (A Function Space)**.** We let $\ell^{\infty}(\Omega)$ denote the set of all bounded functions $f : \Omega \to \mathbb{C}$, equipped with the norm

$$\|f\|_{\infty} = \sup_{x \in \Omega} |f(x)|.$$

If $\Omega = (\Omega, \mathcal{M}, \mu)$ is a measure space, then we let $L^{\infty}(\Omega)$ be the space of μ-a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_{\infty} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

Series Convergence and Completeness

Proposition (Criteria for Banach Spaces)**:** Let X be a normed vector space. The following are equivalent:

(i) X is a Banach space.[1]

(ii) If $(x_k)_k$ is a sequence of vectors such that $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

(iii) If $(x_k)_k$ is a sequence in X such that $\|x_k\| < 2^{-k}$, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. To show (i) implies (ii), for $n > m > N$, we have

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\leqslant \sum_{k=m+1}^{n} \|x_k\| \\ &< \epsilon, \end{aligned}$$

implying that s_n is Cauchy, and thus converges since X is complete.

Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let $(x_n)_n$ be a Cauchy sequence in X. We only need construct a convergent subsequence in order to show that $(x_n)_n$ converges.

Chose $n_1 \in \mathbb{N}$ such that for $n, m \geqslant n_1$, $\|x_m - x_n\| < \frac{1}{2^2}$, and inductively define $n_j > n_{j-1}$ such that $n, m \geqslant n_j$ implies $\|x_m - x_n\| < \frac{1}{2^{j+1}}$.

Let $y_1 = x_{n_1}$, $y_j = x_{n_j} - x_{n_{j-1}}$. Then,

$$\begin{aligned} \|y_j\| &= \left\|x_{n_j} - x_{n_{j-1}}\right\| \\ &< \frac{1}{2^j}, \end{aligned}$$

so $\sum_{j=1}^{\infty} y_j$ converges by our assumption. By telescoping, we see that $\sum_{j=1}^{k} y_j = x_{n_k}$, so $(x_{n_k})_k$ converges. □

Quotient Spaces

Let X be a normed vector space. Then, for $E \subseteq X$ a subspace, there is a quotient space X/E with the projection map $\pi : X \to X/E$, $x \mapsto x + E$. We want to make X/E into a normed space — in order to do this, we use the distance function:

$$\operatorname{dist}_E(x) = \inf_{y \in E} d(x, y),$$

[1]Complete normed vector space.

which is uniformly continuous. For E closed, then $\operatorname{dist}_E(x) = 0$ if and only if $x \in E$.

Proposition (Quotient Space Norm)**:** Let X be a normed vector space, and $E \subseteq X$ a subspace. Set

$$\|x + E\|_{X/E} = \operatorname{dist}_E(x).$$

Then,

(1) $\|\cdot\|_{X/E}$ is a well-defined seminorm on X/E.

(2) If E is closed, then $\|\cdot\|_{X/E}$ is a norm on X/E.

(3) $\|x + E\|_{X/E} \leqslant \|x\|$ for all $x \in X$.

(4) If E is closed, then $\pi : X \longrightarrow X/E$ is Lipschitz.

(5) If X is a Banach space and E is closed, then X/E is also a Banach space.

Proof.

(1) We will show that $\|\cdot\|_{X/E}$ is well-defined. If $x + E = x' + E$, $x' - x \in E$, so for every $y \in E$, $x' - x + y \in E$. Thus,

$$\begin{aligned} \|x - y\| &= \|x' - (x' - x + y)\| \\ &\geqslant \inf_{z \in E} \|x' - z\| \\ &= \|x' + E\|_{X/E}. \end{aligned}$$

Thus, $\|x + E\|_{X/E} \geqslant \|x' + E\|_{X/E}$, and vice versa.

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and $x \in X$. Then,

$$\begin{aligned} \|\lambda (x + E)\|_{X/E} &= \|\lambda x + E\|_{X/E} \\ &= \inf_{y \in E} \|\lambda x - y\| \\ &= |\lambda| \inf_{y \in E} \left\|x - \lambda^{-1} y\right\| \\ &= |\lambda| \inf_{y' \in E} \|x - y\| \\ &= |\lambda| \|x + E\|_{X/E} \end{aligned}$$

Given $x, x' \in X$ and a fixed $\varepsilon > 0$, we have

$$\|x + E\| + \frac{\varepsilon}{2} > \|x - y\|$$

for some $y \in E$, and

$$\|x' + E\| + \frac{\varepsilon}{2} > \|x' - y'\|$$

for some $y' \in E$. Thus,

$$\begin{aligned} \|(x + x') - (y + y')\| &\leqslant \|x - y\| + \|x' - y'\| \\ &< \varepsilon + \|x + E\| + \|x' + E\|. \end{aligned}$$

Since $y + y' \in E$, we have

$$\begin{aligned} \|(x + E) + (x' + E)\|_{X/E} &= \|x + x' + E\|_{X/E} \\ &\leqslant \|(x + x') - (y + y')\| \\ &< \varepsilon + \|x + E\|_{X/E} + \|x' + E\|_{X/E}, \end{aligned}$$

meaning

$$\|(x + E) + (x' + E)\| \leqslant \|x + E\| + \|x' + E\|.$$

(2) If E is closed, and $\|x+E\| = 0$, then $x \in E$ so $x + E = 0_{X/E}$.

(3) For $x \in X$,

$$\begin{aligned}\|x+E\|_{X/E} &= \inf_{y\in E} \|x-y\| \\ &\leqslant \|x\|.\end{aligned}$$

(4) We have

$$\begin{aligned}\|(x+E)-(x'+E)\|_{X/E} &= \|x-x'+E\|_{X/E} \\ &\leqslant \|x-x'\|.\end{aligned}$$

(5) Let X be complete and $E \subseteq X$ be closed. Let $(x_k+E)_k$ be a sequence in X/E with $\|x_k+E\| < 2^{-k}$. We want to show that $\sum_{k=1}^{\infty}(x_k+E)$ converges.

For each k, since $\|x_k+E\| < 2^{-k}$, there exists $y_k \in E$ such that $\|x_k - y_k\| < 2^{-k}$. Since X is complete, $\sum_{k=1}^{\infty} x_k - y_k$ converges.

Let $\left(\sum_{k=1}^{n} x_k - y_k\right)_n \to x$ in X. Applying the canonical projection map, π, to both sides, we get

$$\begin{aligned}\sum_{k=1}^{n}(x_k+E) &= \sum_{k=1}^{n}\pi(x_k) \\ &= \pi\left(\sum_{k=1}^{n}(x_k-y_k)\right) \\ &\to \pi(x),\end{aligned}$$

implying that $\sum_{k=1}^{\infty}(x_k+E)$ converges.

□

Exercise: Consider ℓ_∞ and its closed subspace c_0. If $\pi : \ell_\infty \to \ell_\infty/c_0$ denotes the canonical quotient map, with $(z_k)_k \in \ell_\infty$, show that

$$\|(z_k)_k + c_0\| = \limsup_{k\to\infty} |z_k|$$

Solution: Let $z = (z_k)_k \in \ell_\infty$. We define the distance

$$\operatorname{dist}_{c_0}(z) = \inf_{t\in c_0} |z_k - t_k|.$$

Let $w \in c_c$ be defined by

$$w = (z_1, z_2, \dots, z_{n-1}, 0, 0, \dots).$$

Then,

$$\begin{aligned}\|z-w\|_\infty &= \sup_{k\in\mathbb{N}} |z_k - w_k| \\ &= \sup_{k\geqslant n} |z_k - w_k|,\end{aligned}$$

meaning that

$$\operatorname{dist}_{c_c}(z) \leqslant \sup_{k\geqslant n} |z_k|.$$

Since $c_0 \supseteq c_c$, we have

$$\begin{aligned}\operatorname{dist}_{c_0}(z) &\leqslant \operatorname{dist}_{c_c}(z)\\ &\leqslant \inf_{n\geqslant 1}\left(\sup_{k\geqslant n}|z_k|\right)\\ &= \limsup_{k\to\infty}|z_k|.\end{aligned}$$

Now, we show that $\limsup_{k\to\infty}|z_k| \leqslant \operatorname{dist}_{c_c}(z)$. Given $\varepsilon > 0$, there exists $w \in c_c$ such that

$$\|z - w\| < \operatorname{dist}_{c_c}(z) + \varepsilon.$$

Additionally, for w that terminates at $n-1$ (i.e., is equal to 0 for all $k \geqslant n$), we have

$$\sup_{k\geqslant n}|z_k - w_k| \leqslant \sup_{k\in\mathbb{N}}|z_k - w_k|,$$

meaning

$$\begin{aligned}\limsup_{k\to\infty}|z_k| &= \inf_{n\geqslant 1}\left(\sup_{k\geqslant n}|z_k|\right)\\ &\leqslant \sup_{k\geqslant n}|z_k - w_k|\\ &\leqslant \sup_{k\in\mathbb{N}}|z_k - w_k|\\ &= \|z - w\|\\ &< \operatorname{dist}_{c_c}(z) + \varepsilon,\end{aligned}$$

implying that

$$\limsup_{k\to\infty}|z_k| = \operatorname{dist}_{c_c}(z).$$

For $\varepsilon > 0$, let $w \in c_0$ be such that

$$\|z - w\| < \operatorname{dist}_{c_0}(z) + \varepsilon/2.$$

Additionally, let $\lambda \in c_c$ such that $\|\lambda - w\| < \varepsilon/2$. Then, we have

$$\begin{aligned}\operatorname{dist}_{c_0}(z) + \varepsilon &> \|z - \lambda\| + \|\lambda - w\|\\ &\geqslant \operatorname{dist}_{c_c}(z) + \varepsilon/2\\ &\geqslant \limsup_{k\to\infty}|z_k|.\end{aligned}$$

Thus, $\limsup_{k\to\infty}|z_k| \leqslant \operatorname{dist}_{c_0}(z)$, meaning $\limsup_{k\to\infty}|z_k| = \operatorname{dist}_{c_0}(z)$.

Bounded Linear Operators

Definition (Continuous Functions)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is called Lipschitz if there is a constant $C > 0$ such that

$$d_Y(f(x), f(x')) \leqslant C d_x(x, x')$$

for all $x, x' \in X$.

If $C \leqslant 1$, a Lipschitz map is known as a contraction.

If

$$d_Y(f(x), f(x')) = d_X(x, x')$$

for all $x, x' \in X$, then f is known as an isometry.

Proposition (Categorization of Continuous Linear Maps)**:** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map. The following are equivalent:

(i) T is continuous at 0.

(ii) T is continuous.

(iii) T is uniformly continuous.

(iv) T is Lipschitz.

(v) There exists a constant $C > 0$ such that $\|T(x)\| \leqslant C\|x\|$ for all $x \in X$.

Definition (Bounded Linear Operator)**.** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map.

(1) T is bounded if $T(B_X)$ is bounded in Y. Equivalently, T is bounded if and only if

$$\sup_{x \in B_X} \|T(x)\| < \infty,$$

or that $\exists r > 0$ such that $T(B_X) \subseteq B_Y(0, r)$.

(2) The operator norm of T is the value

$$\|T\|_{op} = \sup_{x \in B_X} \|T(x)\|.$$

Lemma: Let $T : X \to Y$ be a linear map between normed vector spaces. Then,

$$\|T\|_{op} = \sup_{x \in S_X} \|T(x)\|$$

and for all $x \in X$,

$$\|T(x)\| \leqslant \|T\|_{op} \|x\|.$$

Lemma: Let $T : X \to Y$ be a bounded linear map between normed vector spaces. Then, for any $x \in X$ and $r > 0$,

$$r\|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|$$

Proof. Let $C = \sup_{y \in B(x,r)} \|T(y)\|$. If $z \in B(0, r)$, then $z + x, z - x \in B(x, r)$, meaning

$$2T(z) = T(z + x) + T(z - x),$$

so by the triangle inequality, we get

$$\begin{aligned}
2\|T(z)\| &\leqslant \|T(z + x)\| + \|T(z - x)\| \\
&\leqslant 2\max\{\|T(z + x)\|, \|T(z - x)\|\} \\
&\leqslant 2C.
\end{aligned}$$

Thus,

$$\|T(z)\| \leqslant \sup_{y \in B(x,r)} \|T(y)\|,$$

meaning

$$r\|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|.$$

□

Remark: For a linear map $T : X \to Y$, the following are equivalent:

(1) T is continuous.
(2) T is bounded.
(3) $\|T\|_{op} < \infty$.

Definition. Let X and Y be normed spaces, $T : X \to Y$ a linear map.

(1) T is bounded below if there exists C_2 such that $\|T(x)\| \geqslant C_2 \|x\|$ for all $x \in X$.
(2) T is bicontinuous if T is bounded and bounded below.

$$C_2 \|x\| \leqslant \|T(x)\| \leqslant C_1 \|x\|$$

(3) T is a bicontinuous isomorphism if T is bijective, linear, and bicontinuous. We say X and Y are bicontinuously isomorphic.
(4) We say T is an isometric isomorphism if T is bijective, linear, and an isometry.

Example. Let ρ be the continuous surjective wrapping function $\rho : [0, 2\pi] \to \mathbb{T}$, $\rho(t) = e^{it}$. There is an induced isometry

$$T_\rho : C(\mathbb{T}) \to C([0, 2\pi]),$$

defined by $T_\rho(f)(t) = f \circ \rho(t) = f\left(e^{it}\right)$.

The range of T_ρ is $C = \{G \in C([0, 2\pi]) \mid g(0) = g(2\pi)\}$, which means that $C(\mathbb{T})$ and C are isometrically isomorphic Banach spaces.

Proposition: Let X and Y be normed spaces, and $T : X \to Y$ be a linear map. The following are equivalent.

(i) T is bicontinuous.
(ii) $T : X \to \text{Ran}(T)$ is a linear isomorphism and homeomorphism.

Proof. Let T be bicontinuous. Then, T is linear, injective, and surjective onto $\text{Ran}(T)$. Since T is continuous, T is bounded. Let $S : \text{Ran}(T) \to X$ be defined by $S(T(x)) = x$. We can see that S is well-defined, since $T : X \to \text{Ran}(T)$ is surjective, and so has a left inverse. Similarly, since $\|S(T(x))\| = \|x\| \leqslant \frac{1}{C_2} \|T(x)\|$, S is continuous.

Let $S : \text{Ran}(T) \to X$ be defined by $S(T(x)) = x$. Since T is continuous, it is bounded, so

$$\|T(x)\| \leqslant \|T\|_{op} \|x\|.$$

Since S is bounded,

$$\begin{aligned} \|x\| &= \|S(T(x))\| \\ &= \|S\|_{op} \|T(x)\|, \end{aligned}$$

so $\frac{1}{\|S\|_{op}} \|x\| \leqslant \|T(x)\|$. □

Corollary: Let X be a vector space with $\|\cdot\|$ and $\|\cdot\|'$ two norms. The following are equivalent:

(i) The norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.
(ii) The map $\text{id}_X : (X, \|\cdot\|) \to \left(X, \|\cdot\|'\right)$.

Proposition (Properties of Bounded Linear Operators)**:** Let X, Y, Z be normed spaces, $T : X \to Y$, $S : X \to Y$, and $R : Y \to Z$ be linear maps.

(1) $\|\alpha T\|_{op} = |\alpha| \|T\|_{op}$

(2) $\|T+S\|_{op} \leqslant \|T\|_{op} + \|S\|_{op}$

(3) $\|T\|_{op} = 0$ if and only if $T = 0$

(4) $\|R \circ T\|_{op} \leqslant \|R\|_{op} \|T\|_{op}$

(5) $\|\mathrm{id}_X\|_{op} = 1$

(6) If $E \subseteq X$ is a subspace, then $\|T|_E\|_{op} \leqslant \|T\|_{op}$

Proof. We will prove (4) here. For $x \in B_X$, we have

$$\begin{aligned}\|R \circ T(x)\| &= \|R(T(x))\| \\ &\leqslant \|R\|_{op} \|T(x)\| \\ &\leqslant \|R\|_{op} \|T\|_{op}.\end{aligned}$$

Taking the supremum, we obtain $\|R \circ T\|_{op} \leqslant \|R\|_{op} \|T\|_{op}$. □

Recall: $\mathcal{L}(X, Y)$ is the set of all linear operators with domain X and codomain Y.

Proposition: Let X and Y be normed spaces.

(1) The collection $\mathcal{B}(X, Y) = \left\{T \in \mathcal{L}(X, Y) \mid \|T\|_{op} < \infty\right\}$ equipped with the operator norm is a normed space known as the space of bounded linear operators between X and Y.

(2) If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

(3) The continuous dual space, $X^* = \mathcal{B}(X, \mathbb{C})$ is a Banach space.

Proof. We will prove (2). Let $(T_n)_n$ be Cauchy under $\|\cdot\|_{op}$. Since Cauchy sequences are bounded, there is some $C > 0$ such that $\|T_n\|_{op} \leqslant C$ for all $n \geqslant 1$. For $x \in X$,

$$\|T_n(x) - T_m(x)\| \leqslant \|T_n - T_m\|_{op} \|x\|,$$

meaning $(T_n(x))_n$ is Cauchy in Y. Since Y is complete, we define

$$T(x) = \lim_{n\to\infty} T_n(x)$$

in Y. If $x \in B_X$, we have

$$\begin{aligned}\|T(x)\| &= \left\|\lim_{n\to\infty} T_n(x)\right\| \\ &= \lim_{n\to\infty} \|T_n(x)\| \\ &\leqslant \limsup_{n\to\infty} \|T_n(x)\| \\ &\leqslant C\|x\|,\end{aligned}$$

meaning $\|T\|_{op} \leqslant C$.

Let $\varepsilon > 0$, and $N \in \mathbb{N}$ large such that $n, m \geqslant N$, $\|T_n - T_m\|_{op} \leqslant \varepsilon$. For $x \in B_X$,

$$\begin{aligned}\|T_n(x) - T(x)\| &= \lim_{m\to\infty} \|T_n(x) - T_m(x)\| \\ &\leqslant \limsup_{m\to\infty} \|T_n - T_m\|_{op} \|x\| \\ &< \varepsilon.\end{aligned}$$

Thus, $\|T - T_n\|_{op} < \varepsilon$ for all $n \geqslant N$. □

Definition (Algebras)**.** Let A be an algebra over $\mathbb{C}$.

(1) If A admits a norm $\|\cdot\|$ satisfying $\|ab\| \leqslant \|a\|\,\|b\|$, then A is a normed algebra. If A is unital, then $\|1_A\| = 1$.

(2) If A is complete with respect to its norm, then A is called a Banach algebra, and if A is unital, then A is a unital Banach algebra.

Lemma: In a normed algebra A, the map $\cdot : A \times A \to A, (a, b) \mapsto ab$ is continuous.

Proposition: Let X be a normed space. The set of bounded operators $\mathcal{B}(X, X) = \mathcal{B}(X)$ is a unital normed algebra. Moreover, if X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

Proposition: Let A be a unital Banach algebra, $a \in A$. The series

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges absolutely in A. We call $\exp(a)$ the exponential of a.

(1) $\exp(0) = 1_A$

(2) If A is commutative, then $\exp(a + b) = \exp(a)\exp(b)$.

(3) We have $\exp(a) \in GL(A)$ with $\exp(a)^{-1} = \exp(-a)$.

(4) $\left\|\exp(a)\right\| \leqslant \exp(\|a\|)$.

Quotient Maps

Definition. A map $f : X \to Y$ is called open if $U \subseteq X$ is open implies $f(U) \subseteq Y$ is open.

Proposition: Let X and Y be normed spaces, $T : X \to Y$ a linear map. The following are equivalent:

(i) T is surjective and open.

(ii) $T(U_X) \subseteq Y$ is open.

(iii) There exists $\delta > 0$ such that $\delta U_Y \subseteq T(U_X)$.

(iv) There exists δ such that $\delta B_Y \subseteq T(B_X)$.

(v) There exists $M > 0$ such that for all $y \in Y$, there exists $x \in X$ with $T(x) = y$ and $\|x\| \leqslant M\|y\|$.

Proof. To see (i) implies (ii), if T is surjective and open, then it is clear that $T(U_X)$, which is the image of an open set, is open.

To see (ii) implies (iii), if $T(U_X)$ is open, we have $0_Y \in T(U_X)$, so there is some δ such that $U(0, \delta) \subseteq T(U_X)$, meaning $\delta U_Y \subseteq T(U_X)$.

Assuming (iii), we see that $\frac{\delta}{2} B_Y \subseteq \delta U_Y \subseteq T(U_X) \subseteq T(B_X)$.

To see (iv) implies (v), let δ be such that $\delta B_Y \subseteq T(B_X)$, and set $M = \frac{1}{\delta}$. Note that for $y \in Y, y \neq 0$, $\frac{\delta}{\|y\|} y \in \delta B_Y$, meaning $\frac{\delta}{\|y\|} y = T(x)$ for some $x \in B_X$, implying that $T\left(\frac{\|y\|}{\delta} x\right) = y$. Finally, since $x \in B_X$, $\frac{\|y\|}{\delta}\|x\| \leqslant \frac{1}{\delta}\|y\| = M\|y\|$.

To see (v) implies (i), we can see that T is surjective by the assumption. Let $U \subseteq X$ be open, $y_0 \in T(U)$. Then, there exists x_0 such that $T(x_0) = y_0$, and $\delta > 0$ such that $U(x_0, \delta) \subseteq U$. Note that $U(x_0, \delta) = x_0 + \delta U_X$, so $x_0 + \delta U_X \subseteq U$. Applying T, we get $T(x_0 + \delta U_X) \subseteq T(U)$, or $y_0 + \delta T(U_X) \subseteq T(U)$. By assumption, since given $y \in U_Y$, there exists $x \in X$ such that $\|x\| \leqslant M\|y\|$, meaning $\|x\| \leqslant M$, we have $U_Y \subseteq T(MU_X)$. Thus, $\frac{1}{M} U_Y \subseteq T(U_X)$, meaning $y_0 + \frac{\delta}{M} U_Y \subseteq y_0 \delta T(U_X) \subseteq T(U)$, so $U_Y\left(y_0, \frac{\delta}{M}\right) \subseteq T(U)$. ◻

Definition. Let X and Y be normed vector spaces.

(1) A bounded linear map $T : X \to Y$ that is surjective and open is known as a quotient map.

(2) If $T(U_X) = U_Y$, then T is called a 1-quotient map.

Exercise: If $T(B_X) = B_Y$, show that $T(U_X) = U_Y$.

Solution: Since $T(B_X) = B_Y$, it is the case that $(T(B_X))^\circ = B_Y^\circ$. Since T is an open map, T is continuous, meaning $(T(B_X))^\circ = T\left(B_X^\circ\right)$. Thus, $T(U_X) = U_Y$.

Proposition: Let X and Y be normed vector spaces with $T : X \to Y$ a quotient map. If X is a Banach space, then Y is a Banach space.

Proof. We will show that Y is complete by showing that an absolutely convergent series converges.

Let $(y_k)_k$ be a sequence in Y with $\sum_{k=1}^{\infty} \|y_k\| < \infty$. Since T is a quotient map, there is a universal $M > 0$ such that for all k, there is $x_k \in X$ such that $T(x_k) = y_k$ and $\|x_k\| \leqslant M\|y_k\|$. Thus,

$$\begin{aligned}\sum_{k=1}^{\infty} &\leqslant M \sum_{k=1}^{\infty} \|y_k\| \\ &< \infty.\end{aligned}$$

Since X is complete, $\sum_{k=1}^{\infty} x_k$ converges. Let $\sum_{k=1}^{\infty} x_k = x$. Then, $\left(T\left(\sum_{k=1}^{n} x_k\right)\right)_n \xrightarrow{n\to\infty} T(x)$, meaning $\sum_{k=1}^{\infty} y_k = T(x)$. Thus, $\sum_{k=1}^{\infty} y_k$ converges in Y, so Y is a Banach space. □

Proposition: Let X be a normed vector space, $E \subseteq X$ a closed subspace. The canonical quotient map, $\pi : X \to X/E$ is a 1-quotient map.

Proof. We know that $\|\pi(x)\| \leqslant \|x\|$, meaning $\pi(U_X) \subseteq U_{X/E}$.

Let $\pi(x) = x + E \subseteq U_{X/E}$. Then, $\inf_{y\in E} \|x - y\| \leqslant 1$, meaning there exists some y such that $\|x - y\| < 1$, meaning $\pi(x - y) = \pi(x)$. □

Corollary: If X is a Banach space, $E \subseteq X$ a closed subspace, then X/E is a Banach space.

Corollary: Let X be a normed vector space and $E \subseteq X$ be closed. If two of $X, E, X/E$ are complete, the third is also complete.

Proof. We have shown that if X is complete, then E is necessarily complete (since E is closed) and X/E is complete as shown above.

Let E and X/E be complete. We now want to show that X is complete. Let $(x_k)_k$ be Cauchy in X.

For each k, let $x_k = s_k + y_k$, where $y_k \in E$ and $s_k + E = \pi(x_k)$. Notice that, since x_k is Cauchy, so too is s_k, as $\|s_k\| \leqslant \|x_k\|$ for all k. Additionally, for $m, n \geqslant N$, we have

$$\begin{aligned}\|x_m - x_n\| &= \|s_m + y_m - (s_n + y_n)\| \\ &\leqslant \|s_m - s_n\| + \|y_m - y_n\| \\ &< \varepsilon,\end{aligned}$$

implying that $(y_k)_k$ is Cauchy in E. Since X/E and E are complete, we define $x = \lim_{k\to\infty} s_k + \lim_{k\to\infty} y_k$. Finally, for $m, n \geqslant N$, we have

$$\begin{aligned}\|x - x_n\| &= \lim_{m\to\infty} \|x_m - x_n\| \\ &\leqslant \varepsilon,\end{aligned}$$

meaning $(x_k)_k \xrightarrow{k\to\infty} x$, so X is complete. □

Proposition: Let X and Y be normed spaces, $E \subseteq X$ a closed subspace, and $T : X \to Y$ bounded linear with $E \subseteq \ker(T)$. Then, there exists a unique bounded linear map $\overline{T} : X/E \to Y$ such that $\overline{T} \circ \pi = T$. Moreover, $\overline{T}$ is injective if and only if $E = \ker(T)$ and $\left\|\overline{T}\right\| = \|T\|$.

Proof. The existence and uniqueness of $\overline{T} : X/E \to Y$ such that $\overline{T} \circ \pi = T$ follows from the First Isomorphism Theorem for vector spaces, as does the fact that $\overline{T}$ is injective and only if $\ker(T) = E$.

Let $x + E \in X/E$. For $y \in E$, we have

$$\begin{aligned}\left\|\overline{T}(x+E)\right\| &= \left\|\overline{T}(x-y+E)\right\|\\ &= \|T(x-y)\|\\ &\leqslant \|T\|\,\|x-y\|.\end{aligned}$$

Taking infimum over all $y \in E$, we get $\left\|\overline{T}(x+E)\right\| \leqslant \|T\|\,\|x+E\|$, meaning $\left\|\overline{T}\right\| \leqslant \|T\|$. Additionally,

$$\begin{aligned}\|T\| &= \left\|\overline{T}\circ\pi\right\|\\ &\leqslant \left\|\overline{T}\right\|\|\pi\|\\ &= \left\|\overline{T}\right\|.\end{aligned}$$

□

Theorem (First Isomorphism Theorem for Normed Vector Spaces)**:** Let X and Y be normed vector spaces, $T \in \mathcal{B}(X, Y)$.

(1) T is a quotient map if and only if $\overline{T} : X/\ker(T) \to Y$ is a bicontinuous isomorphism.

(2) T is a 1-quotient map if and only if $\overline{T} : X/\ker(T) \to Y$ is an isometric isomorphism.

Proof.

(1) Let $\overline{T} : X/\ker(T) \to Y$ be a bicontinuous isomorphism. Since $\overline{T}$ is bicontinuous, it is a homeomorphism, meaning it is open and surjective. Since π is a quotient map, so too is $T : \overline{T} \circ \pi$.

Suppose T is a quotient map. Then, T is surjective, meaning $\overline{T}$ is an isomorphism. Since T is bounded below, $\overline{T}$ is also bounded. Let $\pi(x) = x + \ker(T) \in X/\ker(T)$, with $T(x) = y$. Let M be such that $\|x\| \leqslant M\|y\|$. There is an $x' \in X$ with $T(x') = y$, and $\|x'\| \leqslant M\|y\|$. Thus, $x - x' \in \ker(T)$, so $\pi(x) = \pi(x')$, meaning

$$\begin{aligned}\left\|\overline{T}\circ\pi(x)\right\| &= \|T\circ\pi(x')\|\\ &= \|y\|\\ &\geqslant M^{-1}\|x'\|\\ &\geqslant M^{-1}\|\pi(x')\|\\ &= M^{-1}\|\pi(x)\|,\end{aligned}$$

meaning T is bounded below.

(2) Suppose $\overline{T} : X/\ker(T) \to Y$ is an isometric isomorphism. Then, $\overline{T}$ is a 1-quotient map, and since π is a 1-quotient map, so too is $T = \overline{T} \circ \pi$.

Suppose T is a 1-quotient map. Since T is surjective, $\overline{T}$ is an isomorphism. Since T is a 1-quotient map, $\|T\| = \sup_{x \in U_X} \|T(x)\| \leqslant 1$, meaning $\left\|\overline{T}\right\| \leqslant \|T\| \leqslant 1$. Consider $S = \left(\overline{T}\right)^{-1} : Y \to X/\ker(T)$; S is also an isomorphism, so $S \circ \overline{T} == \mathrm{id}_{X/\ker(T)}$. We will now show S is a contraction, meaning $\overline{T}$ is an isometry.

Let $y \in U_Y$. Since T is a 1-quotient map, there exists $x \in U_X$ such that $T(x) = y$. Then, $\overline{T}(x + \ker(T)) = T(x) = y$, meaning $S(y) = x + \ker(T)$, and

$$\begin{aligned} \|S(y)\| &= \|x + \ker(T)\| \\ &\leqslant \|x\| \\ &\leqslant 1, \end{aligned}$$

meaning $\|S\| \leqslant 1$.

□

Proposition: Every separable Banach space is isometrically isomorphic to a quotient of ℓ_1.

Proof. Let X be a separable Banach space. Since X is separable, so too is S_X. Let $(z_n)_n$ be norm-dense in S_X, and define

$$\begin{aligned} T : \ell_1 &\to X \\ (\lambda_n)_n &\to \sum_{n=1}^{\infty} \lambda_n z_n. \end{aligned}$$

This series converges absolutely:

$$\begin{aligned} \sum_{n=1}^{\infty} \|\lambda_n z_n\| &= \sum_{n=1}^{\infty} |\lambda_n| \\ &< \infty, \end{aligned}$$

so this series converges in X. We can also see that T is linear; additionally, T is a contraction:

$$\begin{aligned} \|T((\lambda_n)_n)\| &= \left\|\sum_{n=1}^{\infty} \lambda_n z_n\right\| \\ &= \lim_{N\to\infty} \left\|\sum_{n=1}^{N} \lambda_n z_n\right\| \\ &\leqslant \lim_{N\to\infty} \sum_{n=1}^{N} \|\lambda_n z_n\| \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} |\lambda_n| \\ &= \|(\lambda_n)_n\|. \end{aligned}$$

Thus, $T(U_{\ell_1}) \subseteq U_X$. To show that $T(U_\ell) = U_X$, we will use the following fact (which follows from the density of z_n).

Fact. For $\delta > 0$ and $x \neq 0$ in X, and $k \in \mathbb{N}$, there exists $n > k$ such that

$$\begin{aligned} \left\|\frac{x}{\|x\|} - z_n\right\| &< \frac{\delta}{\|x\|} \\ \|x - (\|x\|) z_n\| &< \delta \end{aligned}$$

Let $x \in U_X$ with $x \neq 0$, and let $\varepsilon > 0$. Find n_1 such that

$$\|x - (\|x\|) z_{n_1}\| < \frac{\varepsilon}{2},$$

and set $\lambda_{n_1} = \|x\|$.

We find n_2 with $n_2 > n_1$ and

$$\|(x - \lambda_{n_1} z_{n_1}) - (\|x - \lambda_{n_1} z_{n_1}\|) z_{n_2}\| < \frac{\varepsilon}{2^2},$$

and set $\lambda_{n_2} = \|x - \lambda_{n_1} z_{n_1}\|$. We have

$$\|x - (\lambda_{n_1} z_{n_1} + \lambda_{n_2} z_{n_2})\| < \frac{\varepsilon}{2^2},$$

and $\lambda_{n_2} < \frac{\varepsilon}{2}$.

Inductively, we obtain the subsequence $(z_{n_k})_k$ in z_n and a sequence of scalars $(\lambda_{n_k})_k$ such that

$$\left\| x - \sum_{j=1}^{k} \lambda_{n_j} z_{n_j} \right\| < \frac{\varepsilon}{2^k}$$

and

$$\|\lambda_{n_k}\| < \frac{\varepsilon}{2^{k-1}}.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i = 0$ for $i \notin \{n_1, n_2, \dots\}$. We can see that

$$\begin{aligned} \|\lambda_{n_1}\| &= \left\| \lambda_{n_1} + \sum_{k=2}^{\infty} \lambda_{n_k} \right\| \\ &\leqslant \|x\| + \sum_{k=2}^{\infty} \frac{\varepsilon}{2^{k-1}} \\ &= \|x\| + \varepsilon. \end{aligned}$$

We choose ε such that $\|x\| + \varepsilon < 1$, meaning $\lambda \in U_{\ell_1}$.

We can also see that $\sum_{j=1}^{\infty} \lambda_{n_j} z_{n_j} = x$, meaning T is a 1-quotient map. □

Pillars of Functional Analysis

The five main theorems of functional analysis are:

- Baire Category Theorem;
- Open Mapping Theorem (and Bounded Inverse Theorem);
- Closed Graph Theorem;
- Uniform Boundedness Principle;
- and the Hahn Banach Theorems:
  - Hahn–Banach–Minkowski Theorem;
  - Hahn–Banach Extension Theorem;
  - Hahn–Banach Separation Theorem.

These theorems will appear time and again as we work through the fundamentals of functional analysis.

Baire Category Theorem

Definition (Baire Space)**.** Let $\{A_n\}_{n\geqslant 1}$ be a countable collection of open, dense subsets of a topological space X. We say X is a Baire space if

$$\bigcap_{n\geqslant 1} A_n$$

is dense for every such collection.

Definition (Meager Set)**.** If $X = \bigcup_{n\geqslant 1} F_n$, where $\left(\overline{F_n}\right)^{\circ} = \emptyset$ for each n, then we say X is meager.[II]

Proposition (Meager Spaces)**:** If X is a Baire space, then X is nonmeager.

Proof. Suppose toward contradiction that $X = \bigcup_{n\geqslant 1} F_n$, with F_n all nowhere dense. Then,

$$X = \bigcup_{n\geqslant 1} C_n,$$

where $C_n = \overline{F_n}$ are closed with $C_n^{\circ} = \emptyset$.

Let $A_n = C_n^c$. Then, A_n is open for all n, and $\overline{A_n} = \overline{C_n^c} = (C_n^c)^{\circ} = X$, meaning A_n are all open and dense.

Since X is a Baire space, we know that $\bigcap_{n\geqslant 1} A_n$ is dense. However, we also have

$$\begin{aligned}\emptyset &= X^c\\ &= \left(\bigcup_{n\geqslant 1} C_n\right)^c\\ &= \bigcap_{n\geqslant 1} C_n^c\\ &= \bigcap_{n\geqslant 1} A_n.\end{aligned}$$

□

Theorem (Baire Category Theorem)**:** If (X, d) is a complete metric space, then X is a Baire space.

Proof. Let $\{A_n\}_{n\geqslant 1}$ be a collection of open dense subsets of X. Let U_0 be any ball of radius $r > 0$, and set $B_0 = \overline{U_0}$. Since $A_1 \cap U_0$ is open and nonempty, it contains a closed ball B_1 with radius less than $r/2$.

Set $U_1 = B_1^{\circ}$. Similarly, we find a closed ball B_2 with radius less than $r/4$ such that $B_2 \subseteq A_2 \cap U_1$, and set $U_2 = B_2^{\circ}$.

Continuing in this manner, we find a closed ball B_n with radius less than $r/2^n$ with $B_n \subseteq A_n \cap U_{n-1}$, and the chain

$$B_0 \supseteq U_0 \supseteq B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \supseteq \cdots.$$

Letting $(x_n)_n$ be the center of B_n, we can see that x_n forms a Cauchy sequence in X, as the distance between x_m and x_n with $n > m$ is no more than $\frac{r}{2^{m-1}}$.

Since X is complete, $(x_n)_n \to x \in X$. We claim that x belongs to $\bigcap_{n\geqslant 1} B_n$.

[II] In other words, X is meager if X is a countable union of nowhere dense subsets.

Suppose toward contradiction that $x \notin B_N$ for some $N \in \mathbb{N}$. For $n \geqslant N$, we have $x \notin B_n$, so $d(x_n, x) \geqslant \operatorname{dist}_{B_n}(x) > 0$, which contradicts the fact that $(x_n)_n \to x$.

Thus, $x \in \bigcap_{n\geqslant 1} B_n \subseteq \bigcap_{n\geqslant 1} A_n$. Since $\bigcap_{n\geqslant 1} B_n \subseteq U_0$, we have $\left(\bigcap_{n\geqslant 1} A_n\right) \cap U_0 \neq \emptyset$, meaning $\bigcap_{n\geqslant 1} A_n$ is dense in X. □

Corollary: Let X be an infinite-dimensional Banach space. The cardinality of the Hamel basis of X is uncountable.

Proof. Suppose toward contradiction that $\{b_k\}_{k\in\mathbb{N}}$ is a Hamel basis for X. For each n, set $E_n = \operatorname{span}\{b_1, \dots, b_n\}$. Each E_n is closed, meaning $\overline{E_n} = E_n \neq X$ since X is infinite-dimensional.

Additionally, $E_n^\circ = \emptyset$ for each n, meaning the E_n are nowhere dense.

Since $\{b_k\}_{k\in\mathbb{N}}$ is a spanning set,

$$X = \bigcup_{n\geqslant 1} E_n,$$

implying that X is meager. □

Exercise: Let X be a Banach space, and $Z \subseteq X$ a subspace. Is it true that $\dim(Z) = \dim\left(\overline{Z}\right)$?

Solution: It is not the case that $\dim(Z) = \dim\left(\overline{Z}\right)$. For example, consider the subspace $c_c \subseteq \ell_\infty$. Then, the Hamel basis of c_c consists of e_n, which consists of 1 at index n and zero elsewhere, implying that $\dim(c_c) = \aleph_0$. However, $\overline{c_c} = c_0$, and c_0 is an infinite-dimensional Banach space, meaning that $\dim(\overline{c_c}) = 2^{\aleph_0} \neq \aleph_0$.

Open Mapping Theorem

A surjective continuous map between topological spaces is not necessarily an open map — however, if X and Y are Banach spaces, and $f : X \to Y$ is a surjective linear map. This is the Open Mapping theorem, which yields the result that a continuous linear bijection between Banach spaces always admits a bounded inverse.

Lemma: Let X and Y be Banach spaces, and suppose $T \in \mathcal{B}(X, Y)$.

(1) If $U_Y \subseteq \overline{T(\delta U_X)}$ for some $\delta > 0$, then $U_Y \subseteq T(2\delta U_X)$.

(2) If $\delta U_Y \subseteq \overline{T(U_X)}$ for some $\delta > 0$, then $\frac{\delta}{2} U_Y \subseteq T(U_X)$.

Proof.

(1) Let $y \in U_Y$. By our assumption, there exists $x_1 \in \delta U_X$ such that $\|y - T(x_1)\| < 1/2$. Additionally,

$$\begin{aligned} y - T(x_1) &\in \frac{1}{2} U_Y \\ &\subseteq \frac{1}{2}\overline{T(\delta U_X)} \\ &= \overline{T\left(\frac{\delta}{2} U_X\right)}. \end{aligned}$$

Thus, there exists $x_2 \in \frac{\delta}{2} U_X$ such that $\|(y - T(x_1)) - T(x_2)\| < \frac{1}{4}$, implying that

$$\begin{aligned} y - T(x_1) - T(x_2) &\in \frac{1}{4} U_Y \\ &\subseteq \overline{T\left(\frac{\delta}{4} U_X\right)}. \end{aligned}$$

Inductively, we have a sequence $(x_k)_k \in \frac{\delta}{2^{k-1}}U_X$ for each k, and

$$\left\| y - \sum_{j=1}^{k} T\left(x_j\right) \right\| < 2^{-k}.$$

We consider $\sum_{j=1}^{\infty} x_j$. Since

$$\begin{aligned}\sum_{j=1}^{\infty} \left\|x_j\right\| &\leqslant \sum_{j=1}^{\infty} \frac{\delta}{2^{j-1}}\\ &= 2\delta\\ &< \infty,\end{aligned}$$

the series converges to $x \in X$ since X is complete.

Additionally, since $\|x\| \leqslant \sum_{j=1}^{\infty} \left\|x_j\right\| \leqslant 2\delta$, we have $x \in 2\delta U_X$, and $T(x) = y$ by the continuity of T.

(2) If $\delta U_y \subseteq \overline{T(U_X)}$, then $U_Y \subseteq \frac{1}{\delta}\overline{T(U_X)}$, so $U_Y \subseteq \overline{T\left(\frac{1}{\delta}U_X\right)}$, meaning $U_Y \subseteq T\left(\frac{2}{\delta}U_X\right)$, or $\frac{\delta}{2}U_Y \subseteq T(U_X)$.

□

Theorem (Open Mapping Theorem)**:** Let X and Y be Banach spaces, $T \in \mathcal{B}(X, Y)$ surjective. Then, T is open and thus a quotient mapping.

Proof. We will show that $\delta U_Y \subseteq T(U_X)$ for some $\delta > 0$. This is enough to show that T is a quotient mapping.

We can write

$$\begin{aligned}X &= n\bigcup_{n\geqslant 1} U_X\\ Y &= T(X)\\ &= \bigcup_{n\geqslant 1} T(nU_X)\end{aligned}$$

since T is onto. Since Y is nonmeager, there is an $m \geqslant 1$ such that $\overline{T(mU_X)}^{\circ} \neq \emptyset$. There exists $y_0 \in Y$ and $\varepsilon > 0$ such that $U_Y(y_0, \varepsilon) \subseteq \overline{T(mU_X)}$. We claim that

$$\begin{aligned}\varepsilon U_Y &= U_Y(0, \varepsilon)\\ &\subseteq T(mU_X).\end{aligned}$$

Let $z \in \varepsilon U_Y$. Note that $y_0 + z$ and $y_0 - z$ are in $U_Y(y_0, \varepsilon)$, and

$$\begin{aligned}2z &= (y_0 + z) - (y_0 - z)\\ &\in \overline{T(mU_X)} - \overline{T(mU_X)}.\end{aligned}$$

We write $2z = z_1 - z_2$, with $z_1, z_2 \in \overline{T(mU_X)}$. We can find sequences $(T(x_k))_k$ and $\left(T\left(x'_k\right)\right)_k$ with $(T(x_k))_k \to z_1$ and $\left(T\left(x'_k\right)\right)_k \to z_2$. Thus, we have

$$\begin{aligned}2z &= \lim_{k\to\infty} \left(T(x_k) - T\left(x'_k\right)\right)\\ &= \lim_{k\to\infty} T\left(x_k - x'_k\right),\end{aligned}$$

where $\left\|x_k - x'_k\right\| \leqslant 2m$. Thus, $2x \in \overline{T(mU_X)} = 2\overline{T(mU_X)}$, so $z \in \overline{T(mU_X)}$.

We now have

$$\frac{\varepsilon}{m}U_Y \subseteq \overline{T(U_X)},$$

so

$$\frac{\varepsilon}{2m}U_Y \subseteq T(U_X).$$

Setting $\delta = \frac{\varepsilon}{2m}$, we finish the proof. ◻

If $T : X \to Y$ is bijective linear, then $T^{-1} : Y \to X$ is linear. If $X = Y$, we say T is invertible in the unital algebra $\mathcal{L}(X)$. However, if X and Y are normed vector spaces, we also have to be concerned with the continuity of T^{-1}.

Corollary (Bounded Inverse Theorem)**:** Let X and Y be Banach spaces, $T : X \to Y$ is linear, bounded, and bijective. Then, $T^{-1} : Y \to X$ is also bounded.

Proof. Since T is surjective, T is open, so T^{-1} is continuous. ◻

Example. Consider the normed space $Y = (C([0,1]), \|\cdot\|_1)$. To show that Y is not complete, we let $X = (C([0,1]), \|\cdot\|_u)$, which we know is complete.

The identity function from X to Y is bijective and bounded linear since $\|\cdot\|_1 \leqslant \|\cdot\|_u$. If Y were to be complete, then it would imply that the inverse map is bounded. However, since there is no C such that $\|\cdot\|_u \leqslant C\|\cdot\|_1$, it is not the case that Y is complete.

Definition. Let X and Y be normed spaces. A bounded linear map $T \in \mathcal{B}(X, Y)$ is called invertible if there is a bounded linear map $S \in \mathcal{B}(Y, X)$ with $T \circ S = \mathrm{id}_Y$ and $S \circ T = \mathrm{id}_X$. We write $T^{-1} = S$.

Corollary: Let $T \in \mathcal{B}(X, Y)$ with X and Y Banach spaces. The following are equivalent.

(i) T is bounded below.

(ii) T is injective and $\mathrm{Ran}(T) \subseteq Y$ is closed.

(iii) $T : X \to \mathrm{Ran}(T)$ is a bicontinuous isomorphism.

Proof. For (i) to (ii), if T is bounded below, then $\ker T = \{0\}$, so T is injective. Additionally, since T is bounded below, if $(T(x_n))_n$ is a Cauchy sequence in $\mathrm{Ran}(T)$, then

$$\begin{aligned} C\|x_n - x_m\| &\leqslant \|T(x_n - x_m)\| \\ &= \|T(x_n) - T(x_m)\|, \end{aligned}$$

meaning $(x_n)_n$ is a Cauchy sequence in X. Since T is continuous, $(T(x_n))_n \to T(x) \in \mathrm{Ran}(T)$.

For (ii) to (i), since Y is complete and $\mathrm{Ran}(T) \subseteq Y$ is closed, $\mathrm{Ran}(T)$ is a Banach space, so $T^{-1} : \mathrm{Ran}(T) \to X$ is bounded. Thus,

$$\begin{aligned} \|x\| &= \left\|T^{-1}(T(x))\right\| \\ &\leqslant \left\|T^{-1}\right\|_{op} \|T(x)\|, \end{aligned}$$

meaning $\|T(x)\| \geqslant \left\|T^{-1}\right\|_{op}^{-1} \|x\|$ for all $x \in X$.

To show that (ii) is true if and only if (iii) is true, we can see that since T is bounded and T is bounded below, it is the case that T is a bicontinuous isomorphism. ◻

Corollary: Let X and Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then, T is invertible if and only if T is bounded below and surjective.

Complemented Subspaces and Direct Sums

For any normed vector spaces X and Y, we can form the product $X \oplus_p Y$ by defining $\|(x,y)\| = \left(\|x\|^p + \|y\|^p\right)^{1/p}$ for all $p \in [1,\infty)$.

A vector space Z with subspaces X and Y is called the direct sum of X and Y if

(a) for all $z \in Z$, there exist $x \in X$ and $y \in Y$ such that $z = x + y$;

(b) $X \cap Y = \{0\}$.

We write $Z = X \oplus Y$ for the internal direct sum.

Proposition: Let $(Z, \|\cdot\|_Z)$ be a Banach space, and suppose X and Y are closed subspaces of Z with $Z = X \oplus Y$. Then, $Z \cong X \oplus_p Y$ for all $p \in [1,\infty]$.

Proof. Let $p = 1$. Set $\phi : X \oplus_1 Y \to Z$ by taking $\phi((x,y)) = x + y$. Since $Z = X \oplus Y$, this is a bijection, hence an isomorphism. Additionally,

$$\begin{aligned}\|\phi((x,y))\|_Z &= \|x+y\|_Z \\ &\leqslant \|x\|_Z + \|y\|_Z \\ &= \|(x,y)\|_1,\end{aligned}$$

meaning ϕ is bounded. Thus, ϕ^{-1} is also bounded, meaning ϕ is bicontinuous. The proof is similar for all other $p \in (1,\infty]$. □

Definition. If Z is a normed space, X and Y are closed subspaces of Z such that $Z = X \oplus Y$, we say Z is the topological internal direct sum of X and Y.

Definition. Let Z be a normed space, and suppose X is a closed subspace of Z. We say X is complemented in Z if there is a closed $Y \subseteq Z$ with $X \oplus Y = Z$.

Not all closed subspaces are complemented.

Proposition: Let $T : X \to Y$ be a bounded linear map between Banach spaces. If $Z \subseteq Y$ is a closed subspace such that $Y = \operatorname{Ran}(T) \oplus Z$, then $\operatorname{Ran}(T)$ is closed (meaning the internal direct sum is topological).

Proof. Passing to the quotient

$$X/\ker(T) \to Y,\ x + \ker(T) \mapsto T(x),$$

we may assume that T is injective. The map $S : X \oplus_\infty Z \to Y$, $S(x,z) = T(x) + z$ is bounded and bijective. Thus, S is bounded below, so for some $C > 0$, we have

$$\begin{aligned}\|T(x)\| &= \|S(x,0)\| \\ &\geqslant C\,\|(x,0)\|_\infty \\ &= C\,\|x\|,\end{aligned}$$

meaning T is bounded below, and thus has closed range. □

Corollary: If X and Y are Banach spaces, and $T : X \to Y$ is bounded Fredholm,[III] then T has closed range.

Proof. There is a subspace $C \subseteq Y$ with C linearly isomorphic to $\operatorname{coker}(T)$, and $Y = \operatorname{Ran}(T) \oplus C$. Since T is Fredholm, $\dim(C)$ is finite, meaning C is closed. Thus, $\operatorname{Ran}(T)$ is closed. □

[III] A linear map is Fredholm if both $\ker(T)$ and $\operatorname{coker}(T)$ are finite. Here, $\operatorname{coker}(T) = Y/\operatorname{Ran}(T)$.

Closed Graph Theorem

Definition. If $f : A \to B$ is a map between arbitrary sets, then the graph of f is

$$\begin{aligned}\operatorname{graph}(f) &= \{(a, f(a)) \mid a \in A\}\\ &\subseteq A \times B.\end{aligned}$$

Proposition: If (X, d) and (Y, ρ) are metric spaces, and $f : (X, d) \to (Y, \rho)$ is continuous, then $\operatorname{graph}(f) \subseteq X \times Y$ is closed under the product topology.[IV]

Proof. Let $(x_n, f(x_n))_n$ be a sequence in $\operatorname{graph}(f)$ such that $(x_n, f(x_n))_n \to (x, y)$ in $X \times Y$. Then, $(x_n)_n \to x$ in X and $(f(x_n))_n \to y$ in Y.

By the continuity of f, we have $(f(x_n))_n \to f(x)$, and since limits are unique, we have $f(x) = y$. Thus,

$$\begin{aligned}(x, y) &= (x, f(x))\\ &\in \operatorname{graph}(f).\end{aligned}$$

□

Thus, we can see that the graph of any continuous function is closed in the product topology. However, the converse fails in the general case. For instance,

$$\begin{aligned}f &: \mathbb{R} \to \mathbb{R}\\ f(x) &= \begin{cases}\frac{1}{x} & x \neq 0\\ 0 & x = 0\end{cases}\end{aligned}$$

has a closed graph, but f is not continuous.

However, with linear maps between Banach spaces, the converse is actually true.

Theorem (Closed Graph Theorem)**:** Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear map. Then, T is continuous if and only if $\operatorname{graph}(T) \subseteq X \times Y$ is closed with respect to the product topology on $X \times Y$.

Proof. The forward direction follows from the previous proposition.

Suppose $\operatorname{graph}(T) \subseteq X \times Y$ is closed in the product topology. Note that the product topology coincides with the $\|\cdot\|_1$ topology, with $\|(x, y)\|_1 = \|x\| + \|y\|$. Thus, $\left(\operatorname{graph}(T), \|\cdot\|_1\right)$ is a Banach space.

Consider the projection map $P : \operatorname{graph}(T) \to X$ defined by $P((x, T(x))) = x$, which is bijective. We also have

$$\begin{aligned}\|P((x, T(x)))\| &= \|x\|\\ &\leqslant \|x\| + \|T(x)\|\\ &= \|(x, T(x))\|_1,\end{aligned}$$

meaning P is bounded. Thus, P is bicontinuous, meaning it is bounded below, so for some constant C, we have

$$\begin{aligned}\|x\| &= \|P((x, T(x)))\|\\ &\geqslant C\,\|(x, T(x))\|_1\\ &\geqslant C\,\|T(x)\|,\end{aligned}$$

meaning $\|T(x)\| \leqslant \frac{1}{C}\|x\|$, so T is bounded. □

[IV] The product topology is the coarsest topology on $X \times Y$ such that the projection maps π_X and π_Y are continuous.