

The Spectrum and Gelfand Transform

We first recall some definitions and results from the theory of Banach algebras.

Definition: Let A be a Banach algebra, $x \in A$. The spectrum of x in A is given by

$$\sigma_A(x) = \left\{ \lambda \in \mathbb{C} \mid x - \lambda 1 \text{ is not invertible in } \tilde{A} \right\},$$

where \tilde{A} denotes the unitization of A . If A does not have a unit, then $0 \in \sigma_A(x)$ for all $x \in A$. The complement of the spectrum is known as the resolvent, and is denoted $\rho_A(x)$.

Proposition: If A is a Banach algebra, then for all $a \in A$, we have that $\sigma(a) \subseteq \mathbb{C}$ is compact. Furthermore, we have $\sigma(A) \subseteq B(0, \|a\|)$.

Definition: The *spectral radius* of $a \in A$ is denoted $r(a)$, and is given by

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Theorem (Gelfand–Mazur): The only complex Banach division algebra is \mathbb{C} .

Theorem: Let A be a Banach algebra, and let $a \in A$. Then,

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

Corollary: If A is a C^* -algebra and $a \in A$ is a normal element, then $r(a) = \|a\|$.

Definition: A *character* on a commutative Banach algebra A is a nonzero unital algebra homomorphism. We denote the set of all characters on A by \hat{A} .

Theorem: Every character on \hat{A} corresponds to a maximal ideal $I \subseteq A$.

Proposition: If $\phi \in \hat{A}$, then $\phi(x) \in \sigma(x)$, and conversely, if $\lambda \in \sigma(x)$, there is $\phi \in \hat{A}$ with $\phi(x) = \lambda$. In particular, for every $\phi \in \hat{A}$, we have $|\phi(x)| \leq \|x\|$, and thus $\|\phi\| = 1$.

Proof. We have $\phi(x - \phi(x)1) = 0$, so $x - \phi(x)1$ is not invertible. Conversely, if I is a maximal ideal in A containing the ideal generated by $x - \lambda 1$, we have $A/I \cong \mathbb{C}$. \square

We observe that if A is a commutative unital Banach algebra, then we can identify \hat{A} with a closed subset of the unit ball of A^* , and so we can endow \hat{A} with the w^* -topology, which yields that \hat{A} is a compact Hausdorff space. If A is not unital, then \hat{A} is a locally compact Hausdorff space, and the one-point compactification of \hat{A} is the character space of \tilde{A} .

Then, in particular, $C(\hat{A})$ is a commutative C^* -algebra, which is equipped with a natural homomorphism $\Gamma: x \mapsto \hat{x}$ given by $\hat{x}(\phi) = \phi(x)$. The Gelfand transform is in fact a $*$ -homomorphism.

Theorem: If A is a commutative C^* -algebra, then the Gelfand transform is an isometric $*$ -isomorphism from A onto $C_0(\hat{A})$.

Proof. By passing to the unitization, we may assume that A is unital. We only need to show that Γ is isometric, since then the range will be closed, and hence equal to $C_0(\hat{A})$ by the Stone–Weierstrass theorem.

We know that there is $\lambda \in \sigma(x)$ such that $|\lambda| = \|x\|$, meaning that we can find ϕ with $|\hat{x}(\phi)| = \|x\|$, so that $\|\hat{x}\| \geq \|x\|$. \square

It has been well established that if X and Y are compact Hausdorff spaces, and $\phi: X \rightarrow Y$ is a continuous map, then we can define a $*$ -homomorphism $\hat{\phi}: C(Y) \rightarrow C(X)$ by taking $\hat{\phi}(f) = f \circ \phi$. This is a contravariant map, in that if ϕ is injective, then $\hat{\phi}$ is surjective, and vice versa.

Conversely, if $\psi: C(Y) \rightarrow C(X)$ is a $*$ -homomorphism, and $x \in X \cong \widehat{C(X)}$, then $\delta_x \circ \psi \in \widehat{C(Y)} \cong Y$, where $\delta_x(f) = f(x)$, and $\psi(x) = \delta_x \circ \psi$ defines an inverse to $\hat{\phi}$.

Corollary: Let A and B be C^* -algebras, $\phi: A \rightarrow B$ an injective $*$ -homomorphism. Then, ϕ is isometric.

Proof. By passing to the unitization (and using the fact that the injection into the unitization is isometric), we may assume without loss of generality that ϕ is isometric. If $a \in A$, then $x = a^*a$ is a normal element with a normal element as its image.

By restricting ϕ to $P = C^*(x, 1)$ and taking $Q = \phi(C)$, then we observe that the map defined by $\pi = \Gamma_Q \circ \phi \circ \Gamma_P^{-1}$ is an injective homomorphism between $C(\hat{P})$ and $C(\hat{Q})$. Thus, since Γ_C and Γ_D are isometric, as well as π , we must have ϕ is isometric. \square

References

- [RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^* -algebras*. Vol. 60. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. xiv+327. ISBN: 0-8218-0860-5. DOI: [10.1090/surv/060](https://doi.org/10.1090/surv/060). URL: <https://doi.org/10.1090/surv/060>.
- [Bla06] B. Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of C^* -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4; 3-540-28486-9. DOI: [10.1007/3-540-28517-2](https://doi.org/10.1007/3-540-28517-2). URL: <https://doi.org/10.1007/3-540-28517-2>.