Alternating Series and Conditional Convergence

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Table of Contents

Contents

A Series

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

A Series

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

This series appears to be related to the harmonic series, but also very different:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{Harmonic Series}$$

Divergence of the Harmonic Series

We can show that the harmonic series is divergent as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\ &= \infty\end{aligned}$$

Differences

However, our alternating harmonic series is different. Taking partial sums, we get the following sequence:

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{(-1)^{n+1}}{n}\\
s_1 &= 1\\
s_2 &= \frac{1}{2}\\
s_3 &= \frac{5}{6}\\
s_4 &= \frac{7}{12}\\
&\vdots
\end{aligned}
$$

Clearly, this sequence does not grow without bound — it is bounded above by 1, and doesn't seem to dip below $\frac{1}{2}$

Convergence

The alternating harmonic does converge. Specifically,

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...or does it?

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots$$

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$
\begin{aligned}
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots &= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\cdots \\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots
\end{aligned}
$$

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \\ &= \frac{1}{2} \ln 2 \end{aligned}$$

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Introduction to Conditional Convergence

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The sum should always equal 1.

Introduction to Conditional Convergence

- We saw that our alternating harmonic series converges to $\ln 2$, but should it not converge to $\ln 2$ all the time?
- For example, no matter how we arrange

$$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

The sum should always equal 1.
- Maybe we should redefine convergence?

Alternating Series

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- The answer is that the alternating harmonic series is *conditionally* convergent.
- We can always rearrange the terms of the alternating harmonic series to form whatever sum we want.
- In general, alternating series, of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

can be convergent, while at the same time

$$\sum_{n=1}^{\infty}a_n$$

is divergent.

Alternating Series Test

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- In general, we can find if an alternating series is *conditionally* convergent as follows:
    - The (absolute value) series terms are strictly positive and decreasing.

$$0 < a_{n+1} < a_n$$

    - The series terms tend to zero:

$$\lim_{n\to\infty} a_n = 0$$

Questions?

Thank you for listening. Any questions?