

## Chapter 2 Problems

### 2.3

#### Cylindrical Coordinates

Starting with our expression of  $\mathbf{r}$ , we have

$$\mathbf{r} = \rho \cos \phi \hat{\mathbf{i}} + \rho \sin \phi \hat{\mathbf{j}} + z \hat{\mathbf{k}}$$

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial z} dz.$$

Calculating each partial derivative,

$$\frac{\partial \mathbf{r}}{\partial \rho} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}$$

$$\hat{\rho} = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|}$$

$$= \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}},$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = \rho (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}})$$

$$\hat{\phi} = \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\|}$$

$$= -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}$$

implying

$$\frac{\partial \mathbf{r}}{\partial \phi} = \rho \hat{\phi},$$

and finally, we have

$$\frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}}.$$

The above calculations yield

$$d\mathbf{r} = (d\rho) \hat{\rho} + (\rho d\phi) \hat{\phi} + (dz) \hat{\mathbf{k}}.$$

#### Spherical Coordinates

Starting with our expression of  $\mathbf{x}$ <sup>†</sup>

$$\mathbf{x} = r \sin \phi \sin \theta \hat{\mathbf{i}} + r \cos \phi \sin \theta \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$$

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial r} dr + \frac{\partial \mathbf{x}}{\partial \phi} d\phi + \frac{\partial \mathbf{x}}{\partial \theta} d\theta,$$

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<sup>†</sup>I am using  $\mathbf{x}$  instead of  $\mathbf{r}$  because  $\mathbf{r}$  is already used in the expression of the spherical coordinates.

Evaluating each partial derivative, we have

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial r} &= \sin \phi \sin \theta \hat{\mathbf{i}} + \cos \phi \sin \theta \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \\ \hat{\mathbf{r}} &= \frac{\frac{\partial \mathbf{x}}{\partial r}}{\left\| \frac{\partial \mathbf{x}}{\partial r} \right\|} \\ &= \sin \phi \sin \theta \hat{\mathbf{i}} + \cos \phi \sin \theta \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}}, \\ \frac{\partial \mathbf{x}}{\partial \phi} &= -r \sin \phi \sin \theta \hat{\mathbf{i}} + r \cos \phi \sin \theta \hat{\mathbf{j}} \\ \hat{\phi} &= \frac{\frac{\partial \mathbf{x}}{\partial \phi}}{\left\| \frac{\partial \mathbf{x}}{\partial \phi} \right\|} \\ &= -\sin \phi \sin \theta \hat{\mathbf{i}} + \cos \phi \sin \theta \hat{\mathbf{j}}\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \phi} = r \sin \theta \hat{\phi},$$

and finally, we have

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \theta} &= r \cos \phi \cos \theta \hat{\mathbf{i}} + r \sin \phi \cos \theta \hat{\mathbf{j}} - r \sin \theta \hat{\mathbf{k}} \\ \hat{\theta} &= \frac{\frac{\partial \mathbf{x}}{\partial \theta}}{\left\| \frac{\partial \mathbf{x}}{\partial \theta} \right\|} \\ &= \cos \phi \cos \theta \hat{\mathbf{i}} + \sin \phi \cos \theta \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}},\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \theta} = r \hat{\theta}.$$

The above calculations yield

$$d\mathbf{x} = (dr) \hat{\mathbf{r}} + (r \sin \theta d\phi) \hat{\phi} + (r d\theta) \hat{\theta}.$$

## 2.8

Let

$$\vec{\mathbf{a}} = r_a \cos \phi_a \sin \theta_a \hat{\mathbf{i}} + r_a \sin \phi_a \sin \theta_a \hat{\mathbf{j}} + r_a \cos \theta_a \hat{\mathbf{k}} \quad \vec{\mathbf{b}} = r_b \cos \phi_b \sin \theta_b \hat{\mathbf{i}} + r_b \sin \phi_b \sin \theta_b \hat{\mathbf{j}} + r_b \cos \theta_b \hat{\mathbf{k}}.$$

Then,

$$\begin{aligned}\cos \gamma &= \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\left\| \vec{\mathbf{a}} \right\| \left\| \vec{\mathbf{b}} \right\|} \\ &= \frac{1}{r_a r_b} (r_a r_b (\sin \theta_a \sin \theta_b (\cos \phi_a \cos \phi_b + \sin \phi_a \sin \phi_b) + \cos \theta_a \cos \theta_b)) \\ &= \cos \theta_a \cos \theta_b + \sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b).\end{aligned}$$

## 2.9

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{d}{dt}(\dot{\rho}\hat{\rho}) + \frac{d}{dt}(\rho\dot{\phi}\hat{\phi}) \\
&= \hat{\rho}\ddot{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt} \\
&= \hat{\rho}\ddot{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \left(\frac{\partial\hat{\phi}}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\hat{\phi}}{\partial\phi}\frac{d\phi}{dt}\right) \\
&= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi}.
\end{aligned}$$

## 2.12

(a)

(i)  $d\mathbf{a} = \rho \, d\phi \, dz$

(ii)  $d\mathbf{a} = d\rho \, dz$

(iii)  $d\mathbf{a} = \rho \, d\rho \, d\phi$

(b)

(i)  $d\mathbf{a} = r^2 \sin \theta \, d\theta \, d\phi$

(ii)  $d\mathbf{a} = r \sin \theta \, dr \, d\phi$

(iii)  $d\mathbf{a} = r \, dr \, d\theta$

## 2.14

(a)

$$\begin{aligned}
d\Phi &= \mathbf{E} \cdot \hat{n} \, dA \\
&= \|\mathbf{E}\| \|\hat{n}\| \cos \theta \, dA \\
&= \frac{q}{4\pi\epsilon_0 r^2} \cos \theta \, dA.
\end{aligned}$$

## Chapter 3 Problems

For all problems involving  $\arg z$  (or equivalents), I will be using the principle branch,  $\arg z \in (-\pi, \pi]$ .

## 3.5

(a)

$$\begin{aligned}
\sqrt{3} + i &= 2e^{i\frac{\pi}{3}} \\
-\sqrt{3} + i &= 2e^{i\frac{2\pi}{3}}
\end{aligned}$$

(b)

$$\begin{aligned}
\sqrt{2}i &= \sqrt{2}e^{i\frac{\pi}{4} + n\pi} \\
\sqrt{2 + 2\sqrt{3}i} &= 2e^{i\frac{\pi}{6}}
\end{aligned}$$

## 3.6

(a) Real:

$$\begin{aligned} (-1)^{1/i} &= \left( e^{i\pi} \right)^{-i} \\ &= e^{\pi}. \end{aligned}$$

(b) Real:

$$\begin{aligned} \left( \frac{z}{z^*} \right)^i &= \left( e^{2i \arg z} \right)^i \\ &= e^{-2 \arg z}. \end{aligned}$$

(c) Imaginary:

$$\begin{aligned} (z_1 z_2^* - z_1^* z_2)^* &= z_1^* z_2 - z_1 z_2^* \\ &= - (z_1 z_2^* - z_1^* z_2). \end{aligned}$$

(d) Complex:

$$\sum_{n=0}^N e^{in\theta} = \frac{1 - e^{iN\theta}}{1 - e^{i\theta}}.$$

(e) Real: for each  $a \in \{1, 2, \dots, N\}$ ,  $e^{ia\theta} + e^{-ia\theta} \in \mathbb{R}$ .

## 3.9

(a)

$$\begin{aligned} \cos(a+b) + \cos(a-b) &= \frac{1}{2} \left( e^{i(a+b)} + e^{-i(a+b)} \right) + \frac{1}{2} \left( e^{i(a-b)} + e^{-i(a-b)} \right) \\ &= \frac{1}{2} \left( e^{ia} \left( e^{ib} + e^{-ib} \right) + e^{-ia} \left( e^{ib} + e^{-ib} \right) \right) \\ &= \frac{1}{2} \left( e^{ia} + e^{-ia} \right) \left( e^{ib} + e^{-ib} \right) \\ &= 2 \cos a \cos b. \end{aligned}$$

(b)

$$\begin{aligned} \sin(a+b) + \sin(a-b) &= \frac{1}{2i} \left( e^{i(a+b)} - e^{-i(a+b)} \right) + \frac{1}{2i} \left( e^{i(a-b)} - e^{-i(a-b)} \right) \\ &= \frac{1}{2i} \left( e^{ia} \left( e^{ib} + e^{-ib} \right) - e^{-ia} \left( e^{ib} + e^{-ib} \right) \right) \\ &= \frac{1}{2i} \left( e^{ia} - e^{-ia} \right) \left( e^{ib} + e^{-ib} \right) \\ &= 2 \sin a \cos b. \end{aligned}$$

**3.10**

(a)

$$\begin{aligned}
 e^{i\alpha} + e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} + e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)} \\
 &= e^{i\frac{\alpha+\beta}{2}} \left( e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}} \right) \\
 &= 2 \cos\left(\frac{\alpha-\beta}{2}\right) e^{i\frac{\alpha+\beta}{2}}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 e^{i\alpha} - e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} - e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)} \\
 &= e^{i\frac{\alpha+\beta}{2}} \left( e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}} \right) \\
 &= 2i \sin\left(\frac{\alpha-\beta}{2}\right) e^{i\frac{\alpha+\beta}{2}}
 \end{aligned}$$

**3.12**

$$\begin{aligned}
 \frac{1}{2i} \ln\left(\frac{a+ib}{a-ib}\right) &= \frac{1}{2i} (\ln(a+ib) - \ln(a-ib)) \\
 &= \frac{1}{2i} \left( \ln|a+ib| + i \arctan\left(\frac{b}{a}\right) - \left( \ln|a+ib| + i \arctan\left(-\frac{b}{a}\right) \right) \right) \\
 &= \arctan\left(\frac{b}{a}\right).
 \end{aligned}$$

**3.13**

$$\begin{aligned}
 \frac{d^n}{dt^n} (e^{at} \sin bt) &= \frac{1}{2i} \frac{d^n}{dt^n} \left( e^{(a+ib)t} - e^{(a-ib)t} \right) \\
 &= \frac{1}{2i} \left( (a+ib)^n e^{(a+ib)t} - (a-ib)^n e^{(a-ib)t} \right) \\
 &= e^{at} \frac{1}{2i} \left( a^2 + b^2 \right)^{n/2} \left( e^{i(b+n \arctan(\frac{b}{a}))t} - e^{i(b+n \arctan(-\frac{b}{a}))t} \right) \\
 &= e^{at} \left( a^2 + b^2 \right)^{n/2} \sin\left(bt + n \arctan\left(\frac{b}{a}\right)\right)
 \end{aligned}$$

**3.20**

Showing the equivalence between  $C_1 \cos kx + C_2 \sin kx$  and  $A \cos(kx + \alpha)$  and  $B \sin(kx + \beta)$ , we have

$$\begin{aligned}
 A \cos(kx + \alpha) &= A \cos kx \cos \alpha - A \sin kx \sin \alpha \\
 B \sin(kx + \beta) &= B \cos kx \sin \beta + B \sin kx \cos \beta
 \end{aligned}$$

meaning (assuming  $\alpha, \beta \neq \pi n, \pi/2 + \pi n$ )

$$\begin{aligned} A &= \frac{C_1}{\cos \alpha} \\ &= -\frac{C_2}{\sin \alpha} \\ B &= \frac{C_1}{\sin \beta} \\ &= \frac{C_2}{\cos \beta}. \end{aligned}$$

Now, we show the equivalence between  $C_1 \cos kx + C_2 \sin kx$  and  $D_1 e^{ikx} + D_2 e^{-ikx}$ .

$$\begin{aligned} D_1 e^{ikx} + D_2 e^{-ikx} &= \frac{D_1 + D_2}{2} (e^{ikx} + e^{-ikx}) + \frac{D_1 - D_2}{2} (e^{ikx} - e^{-ikx}) \\ &= (D_1 + D_2) \cos kx + i(D_1 - D_2) \sin kx. \end{aligned}$$

meaning

$$\begin{aligned} C_1 &= D_1 + D_2 \\ C_2 &= i(D_1 - D_2). \end{aligned}$$

Finally, we show the equivalence between  $\text{Re}(F e^{ikx})$  and  $C_1 \cos kx + C_2 \sin kx$ .

$$\begin{aligned} \text{Re}((a + ib) e^{ikx}) &= \text{Re}(a \cos kx + ia \sin kx + ib \cos kx - b \sin kx) \\ &= a \cos kx - b \sin kx, \end{aligned}$$

meaning

$$\begin{aligned} C_1 &= a \\ C_2 &= -b. \end{aligned}$$