Introduction

Consider the equations

$$y''(x) + y(x) = e^x \tag{1}$$

$$y^{(17)}(x) + \sin(y(x)) = (x^x)^x \tag{2}$$

Before we want to solve these equations, we need to understand what these equations *are*.

(1) This is a second order, inhomogeneous, linear ordinary differential equation.

(2) This is a 17th order, inhomogeneous, nonlinear ordinary differential equation.

Generally, when we have a nonlinear equation, we convert it (using the Jacobian) to the "nearest" corresponding linear equation using Taylor approximations. In this case, converting equation (2), we have

$$y^{(17)}(x) + y(x) = (x^x)^x. \tag{2'}$$

Now, equation (2′) is linear, so it is able to be solved. It may not be pretty,[1] but it can be solved, using Laplace Transforms or other methods.

Ordinary Differential Equations

Returning to our equation (1),

$$y''(x) + y(x) = e^x, \tag{1}$$

there is one more fact that we can see — this is an equation with constant coefficients. The most general form of a nth order linear ordinary differential equation is of the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x). \tag{†}$$

Specifically, we also require $a_k(x) \in C(I)$, where I is some interval (specifics will be detailed later).

Theorem (Existence and Uniqueness Theorem)**:** Any ordinary differential equation of the form (†) has unique solutions in the interval I.

There are n linearly independent solutions for $g(x) = 0$.

The corresponding homogeneous equation for (1) is

$$y''(x) + y(x) = 0. \tag{1'}$$

The equations (1) and (1′) are related by the linearity principle. In particular, if $y_0(x)$ is a solution to (1′), then we can add $\alpha y_0(x)$ to any solution $y_p(x)$ of (1), then we have all the solutions for (1). In particular, the solutions to (1′) are

$$y_1(x) = \sin(x)$$
$$y_2(x) = \cos(x).$$

To evaluate that these solutions are linearly independent, we consider the differential operator L from (†) defined by

$$L[y] = \sum_{k=0}^{n} a_k(x)y^{(k)}(x).$$

We rewrite (†) as

$$L[y] = g(x).$$

The operator L is linear, so L has the following properties:

[1] Citation needed.

- $L[y_1 + y_2]$;
- $L[cy] = cL[y]$.

Now, in (1) and (1′), if we set $L[y] = y''(x) + y(x)$, then evaluating our solutions y_1 and y_2 to (1′), we get

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= c_1L[y_1] + c_2L[y_2] \\ &= 0. \end{aligned}$$

Now, we get

$$y_0(x) = c_1 \sin(x) + c_2 \sin(x)$$

as our general solution to (1′). By the linearity principle, all we need is one solution to $L[y] = e^x$ to find all solutions to (1).

Evaluating (†) in the most general form, we have the general solution

$$y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{\text{homogeneous solution}} + y_p(x),$$

where $y_p(x)$ is the particular solution. In other words, our general solution is

$$y(x) = \operatorname{span}(y_1(x), y_2(x), \dots, y_n(x)) + y_p(x).$$

For this to work, we need the set $\{y_1, \dots, y_n\}$ to be linearly independent. To do this, we evaluate the Wronskian:

$$W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}.$$

Specifically, the set $\{y_1, \dots, y_n\}$ is linearly independent if $W(x) \neq 0$ for all $x \in I$.

Example. Consider the equation

$$y''(x) - y(x) = e^x \tag{1}$$

We want to find the general solution to this constant coefficient equation.

We start by finding two linearly independent homogeneous solutions to the equation, take their span, then add a particular solution.

The characteristic equation of the homogeneous equation for (1) is

$$r^2 - 1 = 0$$

We get $r = \pm 1$, which by the definition of the characteristic equation yields $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. To verify that this this solution set is linearly independent

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2 \end{aligned}$$

$$\neq 0.$$

Thus, our solutions are linearly independent. We get the general form of

$$y(x) = c_1 e^x + c_2 e^{-x} + y_p(x).$$

Now, we only have to find a particular solution. This is, unfortunately, the hard part.

We begin by guessing. But, in a way that doesn't suck. Specifically, we let $y_p(x) = Axe^x$. Evaluating, we get

$$\begin{aligned} y_p'(x) &= A(x+1)e^x \\ y_p''(x) &= A(x+2)e^x \\ y_p''(x) - y_p(x) &= A(x+2)e^x - Axe^x \\ &= 2Ae^x, \end{aligned}$$

so $2A = 1$, and $A = \frac{1}{2}$. Thus, we have the end result of

$$y(x) = c_1 e^x + c_2 e^x + \frac{1}{2} x e^x.$$

Evaluating in Mathematica, we take

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DSolve[y''[x] - y[x] == Exp[x], y[x], x]
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and we get

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{4}(2x-1)e^x,$$

corroborating our solution.[II]

Example. Consider the equation

$$y'''(x) - y(x) = 0.$$

The particular solution to this equation is $y(x) = 0$. The characteristic equation for this equation is

$$r^3 - 1 = 0.$$

Factoring, we get

$$\begin{aligned} (r-1)\left(r^2 + r + 1\right) &= 0 \\ (r-1)(r-\zeta_3)\left(r - \zeta_3^2\right) &= 0. \end{aligned}$$

Thus, we get

$$r = \left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}.$$

Thus, our solutions are of the form

$$y(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

[II]Only slightly different, but they're the same solution.

Recall that the most general second order constant-coefficient linear differential equation is

$$y'' + ay' + by = 0,$$

with characteristic equation

$$r^2 + ar + b = 0.$$

The solutions to the characteristic equation are

$$r = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}.$$

There are a few cases:

(1) $r_1 \neq r_2$ with $r_1, r_2 \in \mathbb{R}$;

(2) $r_1 = r_2$ with $r_1, r_2 \in \mathbb{R}$;

(3) $r_1 = c + id$, $r_2 = c - id$, where $c, d \in \mathbb{R}$.

The solutions are $y_1 = c_1 e^{r_1 x}$ and $y_2 = c_2 e^{r_2 x}$.

Example (Solving Second-Order Equations)**.**

(1) Let

$$y'' - 3y' + 2y = 0.$$

The characteristic equation is $r^2 - 3r + 2 = 0$, whose solutions are $r = 1, r = 2$. The general solution is, thus,

$$y(x) = c_1 e^x + c_2 e^{2x} \tag{†}$$

The Wronskian is

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \\ &= 2e^{3x} - e^{3x} \\ &= e^{3x} \\ &\neq 0. \end{aligned}$$

Thus, the solution is indeed (†).

(2) Let

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with solution $r = -3, -3$. Currently, we only have the solution $y_1(x) = c_1 e^{-3x}$.

Note that in an nth order linear ordinary differential equation, we always have n linearly independent solutions. Let's guess. Consider the equation $y_2(x) = c_2 x e^{-3x}$.

We can see that $y_2(x)$ is also a solution to this equation,[III] but we need to verify linear independence. Taking the Wronskian, we get

$$\begin{aligned}
W(x) &= \det\begin{pmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{pmatrix} \\
&= e^{-6x}\begin{pmatrix} 1 & x \\ -3 & -3x+1 \end{pmatrix} \\
&= e^{-6x}(-3x + 1 + 3x) \\
&= e^{-6x} \\
&\neq 0.
\end{aligned}$$

Thus, we have two linearly independent solutions, with the general solution of

$$y(x) = c_1e^{-3x} + c_2xe^{-3x}.$$

(3) Let

$$y'' + 4y' + 5 = 0.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with solutions of $r = -2 \pm i$. We then have the solutions

$$\begin{aligned}
y_1(x) &= e^{(-2+i)x} \\
y_2(x) &= e^{(-2-i)x}.
\end{aligned}$$

Unfortunately, we cannot just let these equations stand on their own, because we want *real* solutions. Let's use Euler's theorem, $e^{ix} = \cos x + i \sin x$. Then, we get

$$\begin{aligned}
y(x) &= c_1e^{(-2+i)x} + c_2e^{(-2-i)x} \\
&= e^{-2x}\left(c_1e^{ix} + c_2e^{-ix}\right).
\end{aligned}$$

Let $f(x) = c_1e^{ix} + c_2e^{-ix}$. Using the even/odd decomposition, we get

$$\begin{aligned}
f(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\
&= (c_1 + c_2)\cos(x) + i(c_1 - c_2)\sin(x).
\end{aligned}$$

We "real"-ize our solution by just dropping the value of i in $f(x)$. Thus, we get the full general solution

$$y(x) = e^{-2x}(d_1\cos(x) + d_2\sin(x)).$$

(4) If we have the equation

$$y^{(4)} - 25y'',$$

then using a similar process, we get the solution

$$y(x) = c_1 + c_2x + c_3e^{5x} + c_4e^{-5x}.$$

(5) Considering the equation

$$y^{(5)} + 4y''' + 4y' = 0,$$

we take the characteristic equation $r^5 + 4r^3 + 4r = 0$. Factoring, we get solutions of $r = 0, r = \pm i\sqrt{2}$. Thus, we get the solution of

$$y(x) = c_1 + c_2\cos\left(\sqrt{2}x\right) + c_3\sin\left(\sqrt{2}x\right) + c_4x\cos\left(\sqrt{2}x\right) + c_5x\sin\left(\sqrt{2}x\right).$$

[III] Exercise left for the reader.

Reducing our Orders

Let

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Suppose we know $y_1(x)$. Can we find $y_2(x)$? The answer is yes. We presume

$$y_2(x) = v(x)y_1(x).$$

Now, we have

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'', \end{aligned}$$

and inserting into the equation, we get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{=0} \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Now, we have

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p. \tag{$*$}$$

Integrating, we get

$$\ln(v') = -2\ln(y_1) - \int p(x)\,dx.$$

Taking powers, we get

$$\begin{aligned} v' &= e^{-2\ln(y_1) - \int p(x)\,dx} \\ &= y_1^{-2}e^{-\int p(x)\,dx} \\ &= \frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \\ v &= \int \frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\,dx \end{aligned}$$