Problem (Problem 1)**:** Prove that a space X is contractible if and only if the identity map is null-homotopic.

Solution: Suppose X is contractible, so there are functions $f\colon X \to *$ and $g\colon * \to X$ such that $f \circ g = \mathrm{id}_*$ and $g \circ f \simeq \mathrm{id}_X$. Since the domain of g is one-point, it follows that $g \circ f = c$ for some constant map c. Since homotopy is an equivalence relation, we have $\mathrm{id}_X \simeq c$, so the identity map on X is null-homotopic.

Now, suppose the identity map on X is null-homotopic. That is, there is a constant map c such that $c \simeq \mathrm{id}_X$. We let w be the point for which $c(x) = w$ for all $x \in X$. We may define the map $g\colon * \to X$ by taking $* \mapsto w$, so if $f\colon X \to *$ is the unique map that takes $x \in X$ to $*$, then $f \circ g = \mathrm{id}_*$ and $g \circ f = c \simeq \mathrm{id}_X$, meaning that X has the homotopy type of $*$.

Problem (Problem 2)**:** Prove that a retract of a contractible space is contractible.

Solution: Let $A \subseteq X$ be a retract, with $r\colon X \to X$ the relevant retraction. That is, $r|_A = \mathrm{id}_A$, and $r(X) = A$.

Now, since X is contractible, there are maps $f\colon X \to *$ and $g\colon * \to X$ such that $f \circ g = \mathrm{id}_*$ and $g \circ f \simeq \mathrm{id}_X$. We may now define

$$\begin{aligned} \bar{f}&\colon A \to * \\ \bar{g}&\colon * \to A \end{aligned}$$

by taking $\bar{f} = f|_A$ and $\bar{g} = r \circ g$. We observe that $\bar{f} \circ \bar{g} = \mathrm{id}_*$, and that

$$\begin{aligned} \bar{g} \circ \bar{f} &= r \circ (g \circ f|_A) \\ &= (r \circ g \circ f|_A)|_A \\ &\simeq (r \circ \mathrm{id}_X)|_A \\ &= r|_A \circ \mathrm{id}_X|_A \\ &= \mathrm{id}_A, \end{aligned}$$

so A is contractible.