Problem (Problem 5)**:** A smooth map $f\colon M \to n$ is called a submersion if it induces surjections on tangent spaces. Prove that if M and N are smooth manifolds and $A \subseteq N$ is a smooth submanifold, then f is transverse to A.

Solution: Let $p \in f^{-1}(A)$. By the definition of the submersion, we have $T_{F(p)}N = D_pF(T_pM)$, meaning that $D_pF(T_pM) + T_{F(p)}A = T_{F(p)}N$.

Problem (Problem 6)**:** In this exercise, we will prove a version of the Transversality Theorem. Let M and N be smooth manifolds. The transversality theorem asserts that for all $1 \leqslant r \leqslant \infty$, the set of C^r maps $M \to N$ that are transverse to A is dense in any of the natural topologies $C^r(M, N)$.

We will restrict our attention to manifolds embedded in Euclidean space and prove a slightly weaker version of the transversality theorem.

(a) Let M, N, and A be as above, and let Y be an arbitrary smooth manifold. Let $F\colon Y \times M \to N$ be a smooth map transverse to A. For each $y \in Y$, let $f_y : M \to N$ be defined by $F(y, \cdot)$, and let $\pi\colon Y \times M \to Y$ be the projection.

Prove that for every regular value $y \in Y$ of π, the map f_y is transverse to A.

(b) Let $f\colon M \to \mathbb{R}^n$ be a smooth map, and let $A \subseteq \mathbb{R}^n$ be a smooth submanifold. Show that the set of $p \in \mathbb{R}^n$ for which $f_p(x) := f(x) + p$ is not transverse to A has measure zero.

(c) Prove that if M and N are smooth submanifolds of $\mathbb{R}^n$, then for all $p \in \mathbb{R}^n$ outside a set of measure zero, the manifolds $M + p$ and N intersect transversely.

(d) Prove that if $f\colon M \to N$ is smooth, and $A \subseteq N$ is a smooth submanifold, then f is smoothly homotopic to a map that is transverse to A.

Solution:

(a) Let $p \in A$, and let y be a regular value for π. Observe that, by the regular value theorem, we have that $\pi^{-1}(y) = \{y\} \times M$ is a smooth submanifold of $Y \times M$. It follows from the definition of the f_y that $F \circ \pi^{-1}(y) \equiv f_y$.

Since F is transverse to A, it follows that for any $(z, q) \in F^{-1}(p)$, we have

$$D_{(z,q)}F\big(T_{(z,q)}(Y \times M)\big) + T_pA = T_pN.$$

We have, by chain rule and the inverse function theorem (seeing as y is a regular value of π),

$$\begin{aligned} D_q f_y &= D_q\left(F \circ \pi^{-1}(y)\right) \\ &= D_{(y,q)}F \circ \left(D_{\pi^{-1}(y)}\pi\right)^{-1}(y) \\ &= D_{(y,q)}F, \end{aligned}$$

so that

$$\begin{aligned} D_q f_y\left(T_qM\right) + T_pA &= D_{(y,q)}F\big(T_{(y,q)}(Y \times M)\big) + T_pA \\ &= T_pN, \end{aligned}$$

meaning f_y is transverse to A for any regular value $y \in Y$ of π.

(b) If we let $Y \equiv \mathbb{R}^n$ in part (a), and let $F\colon \mathbb{R}^n \times M \to \mathbb{R}^n$ be defined by $F(p, x) = f(x) + p$, then we observe that for every regular value p of π, that $f(x) + p$ is transverse to A. In particular, since the set of critical values has measure zero in $\mathbb{R}^n$, it follows that for almost every p, $f(x) + p$ is transverse to A.