Representations

Definition: If A is a C^*-algebra, a representation of A is a pair (π, H) where H is a Hilbert space and $\pi\colon A \to B(H)$ is a $*$-homomorphism. If A is unital, then we require $\pi(1) = I$.

Note that if A does not have an identity, we can extend to the unitization $A_1 = A \oplus \mathbb{C}$ and define $\widetilde{\pi}(a, \lambda) = \pi(a) + \lambda I$ for any $a \in A$ and $\lambda \in \mathbb{C}$.

Note that every representation is contractive and the range of any representation is closed.

Example:

(a) If A is a C^*-subalgebra of $B(H)$, then the inclusion map $A \hookrightarrow B(H)$ is a representation.

(b) If (X, Ω, μ) is a σ-finite measure space, then $\pi\colon L_\infty(\mu) = B(L_2(\mu))$, where $\pi(\phi) = M_\phi$, is a representation.

(c) If X is compact, and μ is a positive Borel measure on X, then $\pi_\mu\colon C(X) \to B(L_2(\mu))$ defined by $\pi_\mu(f) = M_f$ is a representation of $C(X)$.

Definition: Let A be a C^*-algebra.

(i) If d is a cardinal number, H a Hilbert space, we let $H^{(d)}$ denote the direct sum of H with itself over d. If $T \in B(H)$, we let $T^{(d)}$ be the direct sum of T with itself over d, which is known as the d-fold inflation of T.

Given a representation $\pi\colon A \to B(H)$, we have $\pi^{(d)}\colon A \to B\big(H^{(d)}\big)$, defined by $\pi^{(d)}(a) = \pi(a)^{(d)}$ is a representation, which is known as the inflation of π.

If $d = \aleph_0$, we will denote their respective inflations as $H^{(\infty)}$ and $\pi^{(\infty)}$.

(ii) If $\{(\pi_i, H_i)\}_{i \in I}$ is a collection of representations of A, the direct sum of these representations is the representation

$$\begin{aligned} \bigoplus_{i \in I} \pi_i\colon A &\to B\left(\bigoplus_{i \in I} H_i\right) \\ a &\mapsto \bigoplus_{i \in I} \pi_i(a). \end{aligned}$$

Note that since all representations are contractive, the direct sum is in fact a bounded operator. Furthermore, if π is isometric (hence injective), then so too is its inflation.

Example: If X is a compact topological space, and $(\mu_n)_n$ is a sequence of positive Borel measures for X, with corresponding representations $\pi_n\colon C(X) \to B(L_2(\mu_n))$ taking $f \mapsto M_f$, then $\bigoplus_{n \geq 1} \pi_n$ is also a representation.

Definition: Two representations (π_1, H_1) and (π_2, H_2) are called equivalent if there is a unitary $U\colon H_1 \to H_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$.

Definition: If A is a C^*-algebra, then a representation $\rho\colon A \to B(H)$ is *non-degenerate* if

$$\begin{aligned} [\rho(A)H] &= \overline{\{\rho(a)h \mid a \in A, h \in H\}} \\ &= H. \end{aligned}$$

That is, the reducing subspace $[\rho(A)H]$ for $\rho(A)$ "lives" on the entire Hilbert space, or that the only $g \in H$ for which $\rho(a)g = 0$ for all $a \in A$ is 0.

Definition: A representation $\rho\colon A \to B(H)$ is called *cyclic* if there is some $v \in H$ such that

$$H = [\rho(A)v].$$

We call the vector v a *cyclic vector* for ρ.

Theorem: Let π be a representation for the C^*-algebra A. Then, there is a family of cyclic representations $\{\pi_i\}_{i\in I}$ for A such that

$$\pi \cong \bigoplus_{i\in I} \pi_i.$$

Proof. Let $\mathcal{E}$ be the family of sets of nonzero vectors in H such that $[\pi(A)e] \perp [\pi(a)f]$ for any $e \neq f \in E \in \mathcal{E}$. Ordering E by inclusion, using Zorn's Lemma gives us that $\mathcal{E}$ has a maximal element E_0. Define

$$H_0 = \bigoplus_{e\in E_0} [\pi(A)e].$$

Let $h \in H_0^{\perp}$, so that $0 = \langle \pi(a)e, h\rangle$ for all $a \in A$ and $e \in E_0$. Therefore, if we have $a, b \in A$ with $e \in E_0$, we have

$$\begin{aligned} 0 &= \langle \pi(b^*a)e, h\rangle \\ &= \langle \pi(b)^*\pi(a)e, h\rangle \\ &= \langle \pi(a)e, \pi(b)h\rangle. \end{aligned}$$

That is, $\pi(A)e \perp \pi(A)h$ for all $e \in E_0$, meaning that $E_0 \cup \{h\} \in \mathcal{E}$, meaning that by maximality of E_0, we must have that $h = 0$ and $H = H_0$.

Letting $H_e := [\pi(A)e]$, then for any $a \in A$, we have $\pi(a)H_e \subseteq H_e$, and $\pi(a)^* = \pi(a^*)$, so that H_e reduces $\pi(a)$. If we define $\pi_e \colon A \to B(H_e)$, we have that π_e is a representation of a, with

$$\pi = \bigoplus_{e\in E_0} \pi_e.$$

□

Definition: A representation π of a C^*-algebra A is called *irreducible* if the only invariant subspaces for $\pi(A)$ are 0 and H.

The best example of an irreducible representation is a cyclic representation.

Lemma: A representation of a C^*-algebra A is irreducible if and only if the only operators commuting with $\pi(A)$ are multiples of the identity.

Proof. If π has a nontrivial invariant subspace V, then P_V commutes with every $\pi(a)$ and is not a scalar. Conversely, if there is a non-scalar operator T commuting with $\pi(A)$, then either the real or imaginary part of T is a non-scalar self-adjoint operator S commuting with $\pi(A)$, so there is some spectral projection P for S that is neither the 0 projection or the identity that commutes with $\pi(A)$, meaning that $P(H)$ is an invariant subspace for $\pi(A)$. □

States

For now, we will assume that M is a unital self-adjoint subspace of a C^*-algebra A. If ρ is a linear functional on M, then the equation

$$\rho^*(a) = \overline{\rho(a^*)}$$

defines another linear functional; if $\rho = \rho^*$, then we call ρ hermitian. Equivalently, ρ is hermitian if $\rho(a^*) = \overline{\rho(a)}$. If ρ is a bounded hermitian functional on M, then we claim that

$$\|\rho\| = \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\}.$$

This follows from the fact that if $\varepsilon > 0$, then from the Riesz lemma, we may find a in the unit ball of M with $|\rho(a)| > \|\rho\| - \varepsilon$. For a suitable λ with $|\lambda| = 1$, we have

$$\|\rho\| - \varepsilon < |\rho(a)|$$

$$\begin{aligned}&= \rho(\lambda a)\\&= \overline{\rho(\lambda a)}\\&= \rho\big((\lambda a)^*\big).\end{aligned}$$

If $a_0 = \operatorname{Re}(\lambda a)$, we have $\|a_0\| \leq 1$ with $\rho(a_0) > \|\rho\| - \varepsilon$. Thus,

$$\|\rho\| \leq \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\},$$

with the reverse inequality being true by definition.

We say the linear functional ρ is *positive* if for any $a \in M_+$, $\rho(a) \geq 0$; if $\rho(1) = 1$, then we say ρ is a state. In fact, if ρ is positive, then ρ is hermitian (as we will see below).

We start by considering a version of the Cauchy–Schwarz inequality for states.

Proposition: If ρ is a positive linear functional on a C^*-algebra A, then

$$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b).$$

Proof. With $a \in A$, we have $a^*a \in A_+$, so $\rho(a^*a) \geq 0$. Then, since ρ is hermitian, we have that

$$\langle a, b\rangle = \rho(b^*a)$$

defines a positive sesquilinear form on A, so the traditional Cauchy–Schwarz inequality gives the desired result. □

Proposition: Let ρ be a bounded linear functional on a C^*-algebra A. The following are equivalent:

(i) ρ is positive;

(ii) for every approximate unit $(e_i)_{i\in I}$, $\|\rho\| = \lim_i \tau(e_i)$;

(iii) for some approximate unit $(e_i)_{i\in I}$, $\|\rho\| = \lim_i \tau(e_i)$.

Proof. We may assume that $\|\rho\| = 1$. To see that (i) implies (ii), we assume ρ is positive, and let $(e_i)_{i\in I}$ be an approximate unit for A. Then, $(\rho(e_i))_{i\in I}$ is an increasing net in $\mathbb{R}$ that converges to its supremum, which is not greater than 1, so $\lim_i \rho(e_i) \leq 1$.

Now, we let $a \in A$ be such that $\|a\| \leq 1$. We have

$$\begin{aligned}|\rho(e_i a)|^2 &\leq \rho\big(e_i^2\big)\rho(a^*a)\\&\leq \rho(e_i)\rho(a^*a)\\&\leq \lim_i \rho(e_i),\end{aligned}$$

so $|\rho(a)|^2 \leq \lim_i \rho(e_i)$, meaning $1 \leq \lim_i \rho(e_i)$.

Showing that (ii) implies (iii) is pretty much by definition. For (iii) implies (i), let $(e_i)_i$ be an approximate unit with $1 = \lim_i \rho(e_i)$. Let $a \in A_{\text{s.a.}}$ with $\|a\| \leq 1$, and write $\rho(a) = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. We may assume that $\beta \leq 0$, and we will show that $\beta = 0$. Letting n be any positive integer, we have

$$\begin{aligned}\|a - ine_i\|^2 &= \|(a + ine_i)(a - ine_i)\|\\&= \big\|a^2 + n^2 e_i^2 - in(ae_i - e_i a)\big\|\\&\leq 1 + n^2 + n\|ae_i - e_i a\|,\end{aligned}$$

so that

$$|\rho(a - ine_i)|^2 \leq 1 + n^2 + n\|ae_i - e_i a\|.$$

Yet, since $\lim_i \rho(a - ine_i) = \rho(a) - in$, with $\lim_i \|ae_i - e_i a\| = 0$, in the limit, we get

$$|\alpha + i\beta - in|^2 \leq 1 + n^2,$$

so by expanding, we have

$$-2n\beta \leq 1 - \beta^2 - \alpha^2.$$

Since $\beta \leq 0$, and this inequality holds for all positive n, it follows that $\beta = 0$. If a is positive with $\|a\| \leq 1$, we have $e_i - a$ is hermitian with $\|e_i - a\| \leq 1$, so $\rho(e_i - a) \leq 1$. Then, $1 - \rho(a) = \lim_i \rho(e_i - a) \leq 1$, meaning $\rho(a) \geq 0$. Thus, ρ is positive. □

Corollary: If ρ is a bounded linear functional on a unital C^*-algebra A, then ρ is positive if and only if $\rho(1) = \|\rho\|$.

Proof. The sequence consisting exclusively of 1 is an approximate unit for A. □

The best-known example of a state is that of the *vector state* on $B(H)$, given by $\rho_v \colon B(H) \to \mathbb{C}$,

$$\rho_v(T) = \langle Tv, v\rangle.$$

for a unit vector $v \in H$.

In fact, we will show in the next section that this is, to an extent, "every" state on a C^*-algebra.

The GNS Construction

The most important fact about states is that they allow us to represent any C^*-algebra as a subalgebra of $B(H)$.

Theorem (GNS Construction)**:** Let A be a C^*-algebra, and let $\rho\colon A \to \mathbb{C}$ be a state. Then, there is a representation $\pi_\rho\colon A \to B(H_\rho)$ with unit cyclic vector ξ_ρ such that

$$\langle \pi_\rho(a)\xi_\rho, \xi_\rho\rangle = \rho(a)$$

for all $a \in A$.

Furthermore, if ξ is a unit cyclic vector for a representation $\pi\colon A \to B(H_\pi)$, then the vector state

$$\begin{aligned} \tau\colon A &\to \mathbb{C} \\ a &\mapsto \langle \pi(a)\xi, \xi\rangle \end{aligned}$$

induces a unitary isomorphism of H_ρ onto H_π such that $\pi(a) = U\pi_\tau(a)U^{-1}$ for all $a \in A$.

Proof. To start, we let ρ be a state on a C^*-algebra A, and define the subspace

$$N_\rho = \{a \in A \mid \rho(a^*a) = 0\}.$$

We see that $\rho(b^*a) = 0$ if either a or b are in N_ρ, meaning there is a well-defined inner product on A/N_ρ given by

$$\langle a + N_\rho, b + N_\rho\rangle = \rho(b^*a).$$

We may define H_ρ to be the Hilbert space completion of A/N_ρ. We will show that N_ρ is a left ideal, by taking, for $a \in A$ and $x \in N_\rho$, and using the identity $x^*a^*ax \leq \|a\|^2 x^*x$, to find

$$\begin{aligned} \langle ax, ax\rangle &= \phi\bigl((ax)^*ax\bigr) \\ &\leq \phi\Bigl(\|a\|^2 x^*x\Bigr) \\ &= 0, \end{aligned}$$

meaning that $ax \in N_\rho$. Furthermore, we see that

$$\begin{aligned} \|a(b + N_\rho)\|^2 &= \rho(b^*a^*ab) \\ &\leq \|a\|^2 \rho(b^*b) \end{aligned}$$

$$= \|a\|^2 \|b + N_\rho\|^2$$

Therefore, we may uniquely extend elements of a to bounded operators on H_ρ, which defines a representation $\pi_\rho \colon A \to B(H_\rho)$.

Now, if A is unital, we observe that $1 + N_\rho$ is cyclic for π_ρ, as

$$\begin{aligned} [\pi_\rho(A)\xi_\rho] &= \overline{\{\pi_\rho(a)\xi_\rho \mid a \in A\}} \\ &= \overline{\{a(1 + N_\rho) \mid a \in A\}} \\ &= \overline{A/N_\rho} \\ &= H_\rho, \end{aligned}$$

and we observe that

$$\begin{aligned} \langle \pi_\rho(a)\xi_\rho, \xi_\rho \rangle &= \langle a(1 + N_\rho), 1 + N_\rho \rangle \\ &= \langle a + N_\rho, 1 + N_\rho \rangle \\ &= \rho(a). \end{aligned}$$

Meanwhile, if A is not unital, then we may extend ρ to a state τ on the unitization $\widetilde{A}$, which induces an isometry V of H_ρ to H_τ mapping $a + N_\rho$ to $a + N_\tau$ that intertwines π_ρ and π_τ, in the sense that $V\pi_\rho(a) = \pi_\tau(a)V$. We may identify H_ρ with the subspace $VH_\rho \subseteq H_\tau$. This gives that $\pi_\tau|_A = \pi_\rho \oplus 0$ in $H_\rho \oplus H_\rho^\perp$. The projection of $1 + N_\tau$ onto H_ρ, which we may denote h_ρ, then satisfies

$$\begin{aligned} \pi_\rho(a)h_\rho &= \pi_\tau(a)(1 + N_\tau) \\ &= a + N_\tau, \end{aligned}$$

meaning that h_ρ is cyclic for π_ρ, with

$$\begin{aligned} \langle \pi_\rho(a)h_\rho, h_\rho \rangle &= \langle \pi_\tau(a)(1 + N_\tau), 1 + N_\tau \rangle \\ &= \tau(a) \\ &= \rho(a). \end{aligned}$$

Now, for the converse, we see that the linear functional τ defined by

$$\tau(a) = \langle \pi(a)\xi, \xi \rangle$$

is positive with norm at most 1; in fact, since for any approximate identity we have $\pi(e_i)\xi \to \xi$, it follows that τ in fact has norm 1. We have that

$$\begin{aligned} N_\tau &= \{a \in A \mid \langle \pi(a^*a)\xi, \xi \rangle\} \\ &= \{a \in A \mid \pi(a)\xi = 0\}, \end{aligned}$$

so there is a well-defined linear map $U_0 \colon A/N_\tau \to H_\pi$ defined by $U_0(a + N_\tau) = \pi(a)\xi$, which is isometric, since

$$\begin{aligned} \langle U_0(a + N_\tau), U_0(b + N_\tau) \rangle &= \langle \pi(b^*a)\xi, \xi \rangle \\ &= \tau(b^*a) \\ &= \langle a + N_\tau, b + N_\tau \rangle, \end{aligned}$$

meaning that U_0 extends to an isometric linear map U on H_τ which surjects onto $[\pi(A)\xi] = H_\pi$ since ξ is cyclic. Therefore, U is unitary, and we have

$$\begin{aligned} U\pi_\tau(a)(b + N_\tau) &= U(ab + N_\tau) \\ &= \pi(ab)\xi \\ &= \pi(a)\pi(b)\xi \\ &= \pi(a)U(b + N_\tau). \end{aligned}$$

Therefore, $U\pi_\tau(a)U^{-1} = \pi(a)$. $\square$

Corollary: If π is an irreducible representation of a C^*-algebra A, and $\xi \in H_\pi$ is any unit vector, then π is unitarily equivalent to the GNS representation π_τ, where τ is the vector state $a \mapsto \langle \pi(a)\xi, \xi\rangle$.

Proof. We observe that $K = [\pi(A)\xi]$ is invariant under π, so by irreducibility we have that K is either 0 or H_π. Since π is non-degenerate, it follows that $\pi(e_i)\xi \to \xi$ for some approximate unit $(e_i)_i$, so ξ is a nonzero vector in K and K is all of H_π. □

Representations and the Extremal Structure of the State Space

The state space, $S(A)$, can be seen to be convex, as if A is unital with $\phi, \psi \in S(A)$, then

$$\begin{aligned}((1-t)\phi + t\psi)(1) &= (1-t)\phi(1) + t\psi(1)\\ &= 1.\end{aligned}$$

Furthermore, by taking a net $(\phi_i)_{i\in I} \subseteq S(A)$, we see that the state space is w^*-closed. Thus, from the Krein–Milman Theorem, it follows that the state space is equal to the w^*-closure of the convex hull of the extreme points of $S(A)$. The extreme points are known as *pure states*, and they have a relationship with representations of A.

Theorem: Let $\rho \in S(A)$. Then, the GNS representation π_ρ is irreducible if and only if ρ is a pure state.

Proof. Let $\pi \coloneqq \pi_\rho$ be non-irreducible. That is, there is an invariant subspace $K \subseteq H_\rho$ with corresponding projection P such that $P, I - P \neq 0$. We will write ρ as a nontrivial convex combination of states.

Since K is invariant, $\pi(a)P = P\pi(a) = P\pi(a)P$ for all $a \in A$. Since ξ_ρ is cyclic, it follows that $\|P\xi_\rho\| \neq 0$, as otherwise we would have

$$\begin{aligned}\pi(A)P\xi_\rho &= P\pi(A)\xi_\rho\\ &= PH_\rho\\ &= 0.\end{aligned}$$

Similarly, we have $\|(1-P)\xi_\rho\| \neq 0$, meaning that

$$\begin{aligned}\phi(a) &\coloneqq \frac{1}{\|P\xi_\rho\|^2}\langle \pi(a)P\xi_\rho, P\xi_\rho\rangle\\ \psi(a) &\coloneqq \frac{1}{\|(1-P)\xi_\rho\|^2}\langle \pi(a)(1-P)\xi_\rho, (1-P)\xi_\rho\rangle\end{aligned}$$

define states for A with $\rho(a) = (\lambda)\phi(a) + (1-\lambda)\psi(a)$, where $\lambda = \|P\xi_\rho\|^2$, and $0 < \lambda < 1$. Now, if we had $\rho = \phi$, then $P\pi(a)P = \pi(a)P$ would imply that

$$\langle \pi(a)\xi_\rho, \xi_\rho\rangle = \frac{1}{\|P\xi_\rho\|^2}\langle \pi(a)\xi_\rho, P\xi_\rho\rangle$$

for all $a \in A$, but this is only possible if $\|P\xi_\rho\|^2\xi_\rho = P\xi_\rho$, implying that $\|P\xi_\rho\|^2 P\xi_\rho = P\xi_\rho$, meaning $\|P\xi_\rho\|^2 = 1$, contradicting $0 < \lambda < 1$. Similarly, $\rho \neq \psi$, ρ thus admits a nontrivial convex decomposition, meaning ρ is not an extreme point of $S(A)$.

Now, suppose $\pi = \pi_\rho$ is irreducible, with $\rho(a) = \langle \pi(a)\xi, \xi\rangle$ for some $\xi \in H_\pi$. Suppose $\rho = \lambda\phi + (1-\lambda)\psi$ for some states ϕ, ψ and some $0 < \lambda < 1$. Since ψ and ϕ are positive, we have that

$$\begin{aligned}N_\rho &= \{a \in A \mid \rho(a^*a) = 0\}\\ &\subseteq N_\phi\\ &= \{a \in A \mid \phi(a^*a) = 0\}.\end{aligned}$$

Since $\pi(a)h = \pi(b)h$ whenever $a - b \in N_\rho$, the Cauchy–Schwarz inequality implies that the sesquilinear form

$$(\pi(a)h, \pi(b)h) = \lambda\phi(b^*a)$$

is well-defined on the dense subspace $\pi(A)h \subseteq H_\pi$. By the polarization identity, $(\cdot,\cdot)$ is bounded on $\pi(A)h$, so it can be extended to a bounded sesquilinear form q on H_π, meaning there is some bounded operator $T \in B(H_\pi)$ such that $q(h,k) = \langle h, Tk\rangle$. In particular, we have

$$\begin{aligned}\langle \pi(a)h, T\pi(b)h\rangle &= (\pi(a)h, \pi(b)h)\\ &= \lambda\phi(b^*a).\end{aligned}$$

Since ϕ is positive and $\pi(A)h$ is dense, it follows that T is a positive operator with norm at most 1. Now, we claim that T commutes with $\pi(A)$. If $a,b,c \in A$, then

$$\begin{aligned}\langle \pi(a)h, T\pi(c)\pi(b)h\rangle &= \lambda\phi\big((cb)^*a\big)\\ &= \lambda\phi(b^*(c^*a))\\ &= \langle \pi(c^*a)h, T\pi(b)h\rangle\\ &= \langle \pi(a)h, \pi(c)T\pi(b)h\rangle,\end{aligned}$$

so since $\pi(A)h$ is dense, it follows that $\pi(c)T = T\pi(c)$. Since π is irreducible and T is positive, it follows that there is some $z \geq 0$ such that $T = zI$. For any approximate identity $(e_i)_{i\in I}$ for A and all $a \in A$, it then follows that

$$\begin{aligned}\lambda\phi(a) &= \lim_{i\in I} \lambda\phi(e_i a)\\ &= \lim_i \langle \pi(a)h, T\pi(e_i)h\rangle\\ &= z\langle \pi(a)h, h\rangle\\ &= z\rho(a),\end{aligned}$$

meaning that $\lambda = z$, so $\phi = \rho$. □

Remark: The operator T in the forward direction of the proof is akin to a non-commutative Radon–Nikodym derivative.

Lemma: For any $a \in A$, there is a *pure* state on A such that $\rho(a^*a) = \|a\|^2$.

Proof. Let Σ be the set of states satisfying $\rho(a^*a) = \|a\|^2$. This is nonempty as we established existence earlier, and the collection of all such ρ form a w^*-compact convex subset of A^*. By Krein–Milman, there is an extreme point ρ of Σ. We claim that ρ is a pure state.

Let $\phi, \psi \in S(A)$, $0 < \lambda < 1$, and $\rho = \lambda\phi + (1-\lambda)\psi$. Then, we have

$$\begin{aligned}\rho(a^*a) &= \lambda\phi(a^*a) + (1-\lambda)\psi(a^*a)\\ &\leq \|a\|^2\\ &= \rho(a^*a),\end{aligned}$$

so the inequality is equality, meaning that $\phi(a^*a) = \|a\|^2 = \psi(a^*a)$. Then, ϕ and ψ are in Σ, and since ρ is extreme in Σ, we have that $\rho = \phi = \psi$. □

Theorem: Let A be a C^*-algebra. Then, for each $a \in A$, there is an irreducible representation π of A with $\|\pi(a)\| = \|a\|$.

Proof. Choose ρ as in the previous lemma. Then, π_ρ is irreducible, and

$$\begin{aligned}\|a\|^2 &= \rho(a^*a)\\ &= \langle \pi_\rho(a^*a)\xi_\rho, \xi_\rho\rangle\\ &= \|\pi_\rho(a)\xi_\rho\|^2\\ &\leq \|\pi_\rho(a)\|^2,\end{aligned}$$

which combined with the fact that $*$-homomorphisms are contractive, gives $\|\pi_\rho(a)\| = \|a\|$. □

The Universal Representation

To progress further, we must ensure that the state space separates the points of a C^*-algebra, so that we may use the states of a C^*-algebra to construct the universal representation.

Lemma: Let A be a C^*-algebra, and let $a \in A_{\text{s.a.}}$. Then, there is a state ρ of A such that $|\rho(a)| = \|a\|$. In particular, for any $a \in A$, there is a state ρ such that $\rho(a^*a) = \|a\|^2$.

Proof. We may assume that A has a unit. If we let $B = C^*(a)$, then there is an isometric isomorphism from $C^*(a) \cong C(\hat{B})$, where $\hat{B}$ is the character space of B. In particular, since $\hat{a}$ is a continuous map on a compact space, there is some $\phi \in \hat{B}$ such that

$$\begin{aligned} |\phi(a)| &= |\hat{a}(\phi)| \\ &= \|\hat{a}\| \\ &= \|a\|. \end{aligned}$$

Therefore, there is some $\rho \in A^*$ such that $\rho|_B = \phi$ and $\|\rho\| = \|\phi\| = 1$. Since $\phi(1) = 1$, we have $\rho(1) = 1$, so ρ is a state with $|\rho(a)| = \|a\|$. Thus, ρ is a state on A with the desired property. □

Proposition: Every C^*-algebra A has a faithful non-degenerate representation.

Proof. For each $a \neq 0$ in A, let ρ_a be a state such that $\rho_a(a^*a) = \|a\|^2$. Let π_{ρ_a} be the corresponding GNS representation with cyclic vector ξ_{ρ_a}. Then, $\pi_{\rho_a}(a) \neq 0$, as

$$\begin{aligned} 0 &< \|a\|^2 \\ &= \rho_a(a^*a) \\ &= \langle \pi_{\rho_a}(a^*a)\xi_{\rho_a}, \xi_{\rho_a} \rangle \\ &= \|\pi_{\rho_a}(a)\xi_{\rho_a}\|^2. \end{aligned}$$

Therefore, the representation

$$\pi := \bigoplus_{\substack{a \in A \\ a \neq 0}} \pi_{\rho_a}$$

is a nondegenerate faithful representation. □

Definition: The *universal representation* for A is the pair (H_u, π_u), where

$$\begin{aligned} H_u &= \bigoplus_{\rho \in S(A)} H_\rho \\ \pi_u &= \bigoplus_{\rho \in S(A)} \pi_\rho. \end{aligned}$$

We observe that, as we showed in the lemma, the state space of A separates the points of A, so this is in fact a faithful representation.

We will have more to discuss about the universal representation in the notes on von Neumann algebras.

References

[Mur90] Gerard J. Murphy. *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990, pp. x+286. ISBN: 0-12-511360-9.

[KR97] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I.* Vol. 15. Graduate Studies in Mathematics. Elementary theory, Reprint of the 1983 original. American Mathematical Society, Providence, RI, 1997, pp. xvi+398. ISBN: 0-8218-0819-2. DOI: 10.1090/gsm/015. URL: https://doi.org/10.1090/gsm/015.

[RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^*-algebras.* Vol. 60. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. xiv+327. ISBN: 0-8218-0860-5. DOI: 10.1090/surv/060. URL: https://doi.org/10.1090/surv/060.

[Con00] John B. Conway. *A course in operator theory.* Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: 10.1090/gsm/021. URL: https://doi.org/10.1090/gsm/021.