Solution (29.5)**:**

(a) We have

$$\left(\vec{w}\cdot\overleftrightarrow{\mathsf{T}}\right)_k = \sum_{i,j,k} w_i T_{jk}\delta_{ij} = \sum_{i,k} w_i T_{ik},$$

which is a first-rank tensor.

(b) Since $\vec{w}\cdot\overleftrightarrow{\mathsf{T}}$ is a first-rank tensor, and we are taking the dot product of two first rank tensors the expression $\vec{w}\cdot\overleftrightarrow{\mathsf{T}}\cdot\vec{v}$ is a scalar (or rank zero tensor).

(c) We have

$$\overleftrightarrow{\mathsf{T}}\cdot\overleftrightarrow{\mathsf{U}} = \left(\sum_{i,j} T_{ij} e_i \otimes e_j\right)\cdot\left(\sum_{k,\ell} U_{k\ell} e_k \otimes e_\ell\right) = \sum_{i,j,k,\ell} T_{ij}U_{k\ell}(e_k\cdot e_i)(e_j\cdot e_\ell),$$

which is a scalar.

(d) The expression $\overleftrightarrow{\mathsf{T}}\vec{v}$ expresses the operation of

$$\overleftrightarrow{\mathsf{T}} = \sum_{i,j} T_{ij} e_i \otimes e_j$$

on

$$\vec{v} = \sum_i v_i e_i,$$

meaning that $\overleftrightarrow{\mathsf{T}}\vec{v}$ is a vector.

(e) The expression $\overleftrightarrow{\mathsf{T}}\overleftrightarrow{\mathsf{U}}$ is a composition of two linear maps on $V\otimes V$, so it is a rank 2 tensor (or another linear map on $V\otimes V$).

Solution (29.7)**:** We have 2^4 or 16 components in A_{ijkl}.

Solution (29.10)**:**

Solution (29.11)**:**

(a) We may write T_{ij} as $T = \frac{1}{2}\left(T + T^T\right) + \frac{1}{2}\left(T - T^T\right)$, which are the symmetric and antisymmetric components.

(b) Taking

$$S_{ij} = \sum_{k,\ell} R_{ik}R_{j\ell}S_{k\ell},$$

we have

$$S_{ji} = \sum_{k,\ell} R_{jk}R_{i\ell}S_{k\ell} = \sum_{k,\ell} R_{j\ell}R_{ik}S_{\ell k} = \sum_{k,\ell} R_{ik}R_{j\ell}S_{\ell k}$$

$$\begin{aligned} &= \sum_{k,\ell} R_{ik}R_{j\ell}S_{k\ell} \\ &= S_{ij}. \end{aligned}$$

Similarly,

$$\begin{aligned} A_{ij} &= \sum_{k,\ell} R_{ik}R_{j\ell}A_{k\ell} \\ A_{ji} &= \sum_{k,\ell} R_{jk}R_{i\ell}A_{k\ell} \\ &= -\sum_{k,\ell} R_{ik}R_{j\ell}A_{k\ell} \\ &= -A_{ij}. \end{aligned}$$

In matrix form, we have

$$\begin{aligned} S_{ji} &= S_{ij}^T \\ &= \left(RS_{k\ell}R^T\right)^T \\ &= RS_{k\ell}R^T. \end{aligned}$$

and similarly,

$$\begin{aligned} -A_{ji} &= \left(A_{ij}\right)^T \\ &= \left(RA_{k\ell}R^T\right)^T \\ &= RA_{k\ell}R^T. \end{aligned}$$

Solution (29.12)**:**

(a) Let $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Then,

$$\nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$

If we rotate $\nabla \mapsto R\nabla$, then

$$R\nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y},$$

which equals $\nabla \cdot \mathbf{f}$ by chain rule.

(b) We take

$$(R\nabla)_j = \sum_j R_{ij}\frac{\partial}{\partial x_j},$$

meaning we have

$$\begin{aligned} ((R\nabla)\cdot\mathbf{f})_j &= \sum_{j,\ell}\left(R_{ij}\frac{\partial}{\partial x_j}\right)f_\ell\delta_{j\ell} \\ &= \sum_{j,\ell} R_{ij}\frac{\partial f_\ell}{\partial x_j}\delta_{j\ell} \\ &= \sum_i \frac{\partial f_i}{\partial x_i} \end{aligned}$$

We use the fact that R is independent of x_i in the switch from line (2) to line (3).

Solution (29.14)**:** We have

$$\mathbf{u}(\mathbf{r} + d\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \overleftrightarrow{\varepsilon} \cdot d\mathbf{r} + \overleftrightarrow{\phi} \cdot d\mathbf{r}.$$

Applying Hooke's Law, and using the fact that k is described in different degrees of freedom than L and A, we have

$$\begin{aligned} \sigma_{ij} &= (Y\varepsilon)_{ij} & &= \left(\frac{k_{i,j} L_{k,\ell}}{A_{k,\ell}} \varepsilon_{i,j}\right)_{ij} \\ &= \sum_{k,\ell} Y_{ijk\ell} \varepsilon_{k\ell}. \end{aligned}$$

Solution (29.23)**:** We have

$$\begin{aligned} T_{ij} &= \sum_{k,\ell} R_{ik} R_{j\ell} \left(\frac{1}{3} \operatorname{tr}(T) \delta_{k\ell} + \frac{1}{2}(T_{k\ell} - T_{\ell k}) + \frac{1}{2}\left(T_{k\ell} + T_{\ell k} - \frac{2}{3} \operatorname{tr}(T) \delta_{ij}\right)\right) \\ &= \frac{1}{3}\delta_{ij} + \frac{1}{2}\left(T_{ij} - T_{ji}\right) + \frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3}\delta_{ij}\right). \end{aligned}$$

Solution (29.24)**:**

Solution (29.25)**:**