Problem (Problem 1)**:** A subset $A \subseteq \mathbb{R}^n$ is said to have *measure zero* if, for all $\varepsilon > 0$, the set A can be covered by open balls of total volume at most ε. Prove that a countable subset of $\mathbb{R}^n$ has measure zero, and that the standard middle-thirds cantor set in $[0,1] \subseteq \mathbb{R}$ has measure zero.

Solution: Let A be countable, and let $\{a_k\}_{k\geqslant 1}$ be an enumeration of the points in A. Let $\varepsilon > 0$. Let c_n be the constant dependent on n such that the volume of $U(x,r) = c_n r^n$. For each k, define

$$r_k = \left(\frac{1}{2^k c_n}\varepsilon\right)^{1/n}.$$

Then, we see that the family $\{U(a_k, r_k)\}_{k=1}^{\infty}$ has total volume no more than ε, seeing as if all the open balls are disjoint, their union has total volume ε. Thus, countable subsets of $\mathbb{R}^n$ have measure zero.

If $\mathcal{C} \subseteq [0,1]$ is the traditional middle-thirds Cantor set, then we calculate the measure of its complement by taking

$$\begin{aligned}\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k &= \frac{1}{3}\frac{1}{1-\left(\frac{2}{3}\right)}\\ &= 1,\end{aligned}$$

meaning that the Cantor set has measure zero.

Problem (Problem 2)**:** Prove that if $A \subseteq U \subseteq \mathbb{R}^n$ has measure zero (with U open), and $f\colon U \to \mathbb{R}^n$ is smooth, show that $f(A)$ has measure zero.

Solution: Let $f\colon U \to \mathbb{R}^n$ be smooth. Then, f is locally Lipschitz, as f' is continuous, hence attains a supremum on compact subsets. In particular, for any $a \in A$, we see that there is $r > 0$ such that $U(a,r) \subseteq U$, meaning f has a Lipschitz constant C_a such that $|f(x) - f(y)| \leqslant C_a|x-y|$. In particular, we may show that $f(A)$ has measure zero if $f(A \cap U(a,r))$ has measure zero.

Since A has measure zero, so too does $A \cap U(a,r)$, so that we may cover $A \cap U(a,r)$ by a countable (since $\mathbb{R}^n$ is a second countable space) $\{U(x_k,r_k)\}_{k\geqslant 1}$ with $m\left(\bigcup_{k=1}^{\infty} U(x_k,r_k)\right) < \varepsilon$ for any $\varepsilon > 0$. Then, since f is Lipschitz on $A \cap U(a,r)$, we have that

$$\begin{aligned}f(A\cap U(a,r)) &\subseteq f\left(\bigcup_{k=1}^{\infty} U(x_k,r_k)\right)\\ &\subseteq \bigcup_{k=1}^{\infty} U(x_k,r_k)\\ &\subseteq \bigcup_{k=1}^{\infty} U(f(x_k), C_a r_k),\end{aligned}$$

meaning that

$$\begin{aligned}m(f(A\cap U(a,r))) &\leqslant m\left(\bigcup_{k=1}^{\infty} U(f(x-k), C_a r_k)\right)\\ &= C_a^n\varepsilon.\end{aligned}$$

Since C_a is a constant and n is fixed, we thus have that $m(f(A \cap U(a,r))) = 0$, meaning that $m(f(A)) = 0$.

Problem (Problem 3)**:** In this exercise, we will prove Sard's Theorem. Let $U \subseteq \mathbb{R}^m$ be open, and let $f\colon U \to \mathbb{R}^n$ be C^{∞}. Let $A \subseteq U$ be the set of points where Df has rank less than n. Then, $f(A)$ has measure zero in $\mathbb{R}^n$. Note that it need not be the case that A itself have measure zero.

We will let A_i be the set of points in U where all partial derivatives up to degree i vanish.

(a) Prove that $f(A \setminus A_1)$ has measure zero.

(b) Prove that $f(A_k \setminus A_{k+1})$ has measure zero for all $k \geqslant 1$.

(c) Prove that $f(A_k)$ has measure zero for $k \gg 0$.

Solution:

(a) Let $x \notin A_1$, so that some partial derivative does not vanish at x. Letting $f = (f_1, \dots, f_n)$, by some rearrangement, we may assume that $\frac{\partial f_1}{\partial x_1} \neq 0$. Let $h(x) = (f_1(x), x_2, \dots, x_m)$. Since h consists of identity coordinate maps and f_1, which has nonzero partial derivative with respect to x_1, the inverse function theorem means that $h\colon \mathbb{R}^m \to \mathbb{R}^m$ is a local diffeomorphism.

Let $g = f \circ h^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$. By chain rule, we see that $Dg = Df \circ (Dh)^{-1}$, since Dh is invertible, so g has the same critical points as f.

Problem (Problem 5)**:** Prove that $SL_2(\mathbb{R})$, the 2×2 real matrices of determinant one, is diffeomorphic to $\mathbb{R}^2 \times S^1$.

Solution: We consider the action of $SL_2(\mathbb{R})$ on the upper half-plane of $\mathbb{C}$, $\mathbb{H} = \{z \mid \operatorname{Im}(z) > 0\}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \colon z \mapsto \frac{az + b}{cz + d}.$$

In particular, if $z = x + iy$ with $y > 0$, then

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z &= \frac{(ax + b) + iay}{(cx + d) + icy} \\
&= \frac{1}{(cx + d)^2 + c^2y^2} \left(\left((ax + b)(cx + d) + acy^2 \right) + i(acxy - acxy + ady - bcy) \right) \\
&= \frac{1}{(cx + d)^2 + c^2y^2} \left(\left((ax + b)(cx + d) + acy^2 \right) + iy \right),
\end{aligned}$$

In particular, this is a fractional linear transformation on $\mathbb{C}$ that is an automorphism of $\mathbb{H}$, so by composing these fractional linear transformations, we can see that $SL_2(\mathbb{R})$ acting on $\mathbb{H}$ via this map is a group action.

This action is transitive, since for any $x + iy \in \mathbb{H}$, we may map $i \mapsto x + iy$ by using the transformation

$$\frac{ai + b}{ci + d} = i$$

which via multiplication and matching parts gives

$$\begin{aligned}
a &= cx + dy \\
b &= xd - yc
\end{aligned}$$

so by multiplying and back-substituting, we get

$$c^2 + d^2 = \frac{1}{y}.$$

By guessing that $c = 0$, we get

$$d = \frac{1}{\sqrt{y}}$$

$$\begin{aligned} a &= \sqrt{y} \\ b &= \frac{x}{\sqrt{y}}. \end{aligned}$$

Now, to understand the stabilizer of some $z \in \mathbb{H}$, we only need to understand the stabilizer of i. For this, we see that

$$\begin{aligned} \frac{ai + b}{ci + d} &= i \\ ai + b &= di - c \end{aligned}$$

so

$$\begin{aligned} a &= d \\ b &= -c, \end{aligned}$$

and by back-substituting into the determinant, we get

$$a^2 + c^2 = 1,$$

so the stabilizer of i is all matrices of the form

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Thus, by orbit-stabilizer, $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/P$, where P is the group of rotation matrices and the action is left-multiplication. In particular, since every rotation matrix corresponds one-to-one with an element of $S^1 \subseteq \mathbb{C}$, given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto e^{i\theta},$$

we find that $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/S^1$, or that $\mathbb{H} \times S^1 \cong \mathrm{SL}_2(\mathbb{R})$.

In particular, we may view $\mathbb{H}$ to consist of matrices of the form

$$\begin{aligned} h &= \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \\ &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \end{aligned}$$

that take i as their input. Since the former matrix is diffeomorphic to $\mathbb{R}$ via a series of projections and inverse projections, and the latter is diffeomorphic to $\mathbb{R}_{>0}$ via another series of projections and inverse projections, which itself is diffeomorphic to $\mathbb{R}$ by some tangents and arctangents, meaning that

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &\cong \mathbb{H} \times S^1 \\ &\cong \mathbb{R} \times \mathbb{R}_{>0} \times S^1 \\ &\cong \mathbb{R}^2 \times S^1. \end{aligned}$$

In particular, $\mathbb{R} \times \mathbb{R}_{>0} \times S^1$ has a corresponding element in $\mathrm{SL}_2(\mathbb{R})$ given by the map

$$(x, y, \theta) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$