Here, we overview and discuss some of the most important results related to projections in von Neumann algebras.

Comparison of Projections

Recall that if H is a Hilbert space, an element $w \in B(H)$ is called a partial isometry if, for any $h \in \ker(w)^{\perp}$, we have $\|Wh\| = \|h\|$. We call $\ker(w)^{\perp}$ the initial space of W and $\operatorname{im}(w)$ the final space of W.

There are a variety of equivalent definitions for partial isometries.

Proposition: If $w \in B(H)$, then the following are equivalent:

(i) w is a partial isometry;

(ii) w^* is a partial isometry;

(iii) w^*w is a projection onto the initial space of w;

(iv) ww^* is a projection onto the final space of w;

(v) $ww^*w = w$;

(vi) $w^*ww^* = w^*$.

Theorem (Polar Decomposition)**:** Let $a \in B(H)$. Then, there is a partial isometry $w \in B(H)$ with initial space $\ker(a)^{\perp}$ and final space $\overline{\operatorname{im}(a)}$ such that $a = w|a|$.

If $a \in M \subseteq B(H)$, where M is a von Neumann algebra, then both $|a|$ and w are in M.

Equivalence of Projections

If $M \subseteq B(H)$ is a von Neumann algebra, then we say two projections $p, q \in P(M)$, where $P(M)$ denotes the space of projections of M, are (Murray–von Neumann) *equivalent* in M if there is a partial isometry $v \in P(M)$ such that $v^*v = p$ and $vv^* = q$. We will write $p \sim q$.

Note that projections have an ordering by saying that $p \leq q$ if $pq = qp = p$, or $\operatorname{im}(p) \subseteq \operatorname{im}(q)$. This allows us to say that p is *sub-equivalent* to q (in M), written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$.[I]

The sub-equivalence relation in fact forms a partial order, and equivalence as projections forms an equivalence relation. We will first show that it is a preorder.

Proposition: In a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\preceq$ is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry follows from the fact that if v is a partial isometry, then so is v^*.

Now, we will show transitivity for $\preceq$, from which we will see that $\sim$ is transitive. Letting $p, q, r \in P(M)$ be such that $p \preceq q$ and $q \preceq r$, we have partial isometries $u, v \in M$ with

[I]We will say that the projection q majorizes p if $p \preceq q$, and we will say that q dominates p if $p \leq q$.

$u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. Then, we have

$$\begin{aligned} qu &= quu^*u \\ &= (quu^*)u \\ &= uu^*u \\ &= u, \end{aligned}$$

so that

$$\begin{aligned} (vu)^*(vu) &= u^*v^*vu \\ &= u^*qu \\ &= u^*u \\ &= p \\ (vu)(vu)^* &= vuu^*v^* \\ &\leq vqv^* \\ &= vv^*vv^* \\ &= vv^* \\ &\leq r. \end{aligned}$$

Therefore, $p \preceq r$, so $\preceq$ is a transitive relation. □

To see that $\preceq$ is a partial order, we need an analogue of the Cantor–Schröder–Bernstein theorem for projections. In fact, it can be proven in a similar manner. First, we discuss a simple lemma.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ are pairwise orthogonal families of projections with $p_i \preceq q_i$, then $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$.

Proof. Let u_i be the partial isometries with $u_i^*u_i = p_i$ and $r_i \coloneqq u_iu_i^* \leq q_i$. Then, the r_i are pairwise orthogonal since the q_i are pairwise orthogonal, and for any $i \neq j$,

$$\begin{aligned} u_i^*u_j &= u_i^*u_iu_i^*u_ju_j^*u_j \\ &= u_ir_ir_ju_j \\ &= 0 \\ u_iu_j^* &= u_iu_i^*u_iu_j^*u_ju_j^* \\ &= u_ip_ip_ju_j^* \\ &= 0. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \left(\sum_{i\in I} u_i^*\right)\left(\sum_{j\in I} u_j\right) &= \sum_{i\in I} u_i^*u_i \\ &= \sum_{i\in I} p_i \end{aligned}$$

$$
\begin{aligned}
\left(\sum_{i\in I} u_i\right)\left(\sum_{j\in I} u_j^*\right) &= \sum_{i\in I} u_i u_i^*\\
&\leq \sum_{i\in I} q_i.
\end{aligned}
$$

This gives $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$. ☐

Theorem: If $e \preceq f$ and $f \preceq e$, then $e \sim f$.

Proof. We will let $e_0 \coloneqq e$ and $f_0 \coloneqq f$. Let v and w be partial isometries with $v^*v = e$, $vv^* = f_1 \leq f$, $w^*w = f$, $ww^* = e_1 \leq e$. Inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of some projection dominated by f_1, it follows that we may write $f_2 \coloneqq ve_1(ve_1)^*$ with $f_2 \leq f_1$. Since w maps the range of f_1 onto the range of some projection dominated by e_1, it follows that we may write $wf_1(wf_1)^* \eqqcolon e_2$. Observe also that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

Inductively, we obtain decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1}, and w maps the range of f_n isometrically onto that of e_{n+1}. Defining $e_\infty \coloneqq \inf_n e_n$ and $f_\infty = \inf_n f_n$, we have that v maps the range of e_∞ onto that of f_∞, and w that of f_∞ onto the range of e_∞. Note that we have $e_\infty \sim f_\infty$.

As discussed earlier, we have that $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$, so since sums of pairwise orthogonal families of projections respects equivalence, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) &\sim \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2})\\
\sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) &\sim \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
e &= e_\infty + \sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2})\\
&\sim f_\infty + \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1})\\
&= f.
\end{aligned}
$$

☐

Central Projections and the Comparison Theorem

The projections in a von Neumann algebra form a complete lattice, as the collection of closed subspaces of H form a complete lattice under the operations

$$\bigvee_{i\in I} X_i := \overline{\sum_{i\in I} X_i}$$
$$\bigwedge_{i\in I} X_i := \bigcap_{i\in I} X_i.$$

If $S \subseteq H$ is any subset, then we will define the range projection of S by

$$[S] := P_{\overline{\operatorname{span}}(S)}.$$

Proposition: If $M \subseteq B(H)$ is a von Neumann algebra, and $x \in M$, then $[xH]$ and $[x^*H]$ are in M, with $[xH] \sim [x^*H]$ in M.

Proof. Let $x = v|x|$ be the polar decomposition. Note that $v \in M$. Now, vv^* is the projection onto $\overline{xH}$ and v^*v is the projection onto $\ker(x)^\perp = \overline{x^*H}$. Thus, these projections are equivalent in M. □

Definition: Let $x \in M$. We define the *central support* to be the projection

$$z(x) = \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say p and q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $M \subseteq B(H)$ is a von Neumann algebra, then the central support of any $p \in P(M)$ is given by

$$z(p) = [MpH].$$

Let $w = [MpH]$. Since M is unital, it follows that $p \leq w$, and since $\overline{MpH}$ is a reducing subspace for both M and M', we have $w \in M \cap M'$, so $z(p) \leq w$.

Conversely, if $x \in M$, then

$$\begin{aligned} xpH &= xz(p)pH \\ &= z(p)xpH, \end{aligned}$$

meaning that $[xpH] \leq z(p)$, so $w \leq z(p)$ as x was arbitrary.

Proposition: Let M be a von Neumann algebra, and let $p, q \in P(M)$ be projections. The following are equivalent:

(i) p and q are centrally orthogonal;

(ii) $pMq = \{0\}$;

(iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ with $p_0 \sim q_0$.

Proof. Let p and q be centrally orthogonal. Then, for any $x \in M$, we have

$$pxq = pz(p)xz(q)q$$

$$
\begin{aligned}
&= pxz(p)z(q)q \\
&= 0.
\end{aligned}
$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, so $p \leq 1 - z(q)$. Since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, meaning that $z(p)z(q) = 0$. Therefore, (i) and (ii) are equivalent.

Suppose (ii) is not the case. Let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$. Defining

$$
\begin{aligned}
p_0 &= [pxqH] \\
q_0 &= [qx^*pH],
\end{aligned}
$$

we have that $p_0 \leq p$, $q_0 \leq q$, and since $(pxq)^* = qx^*p$, we have $p_0 \sim q_0$.

Now, if there are nonzero projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim q_0$, then if v is a partial isometry with $v^*v = p_0$, $vv^* = q_0$, then $v^* = p_0v^*q_0$, meaning

$$
\begin{aligned}
pv^*q &= pp_0v^*q_0q \\
&= p_0v^*q_0 \\
&= v^* \\
&\neq 0,
\end{aligned}
$$

meaning $pMq \neq \{0\}$. □

Theorem (Comparison Theorem)**:** Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there is a central projection $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ of pairwise orthogonal projections with $p_i \sim q_i$ and, setting

$$
\begin{aligned}
p_0 &= \sum_{i\in I} p_i \\
q_0 &= \sum_{i\in I} q_i,
\end{aligned}
$$

we have $p_0 \preceq q_0$. From above, we have that $p_0 \sim q_0$.

Let $w := z(q - q_0)$. Since $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ are maximal, it follows that $z(q - q_0)$ and $z(p - p_0)$ are centrally orthogonal, yielding $(p - p_0)w = 0$, meaning $pw = p_0w$.

If we let v be a partial isometry implementing the equivalence $p_0 \sim q_0$, then we have that vw is a partial isometry implementing the equivalence $p_0w \sim q_0w$. Therefore, we have

$$
\begin{aligned}
pw &= p_0w \\
&\sim q_0w \\
&\leq q.
\end{aligned}
$$

Similarly, $p_0(1-w) \sim q_0(1-w)$, so since $q - q_0 \leq w$, we have $q(1-w) \preceq p(1-w)$. □

Recall that a factor is a von Neumann algebra M such that $Z(M) = \mathbb{C}1$.

Corollary: If M is a factor, then any two projections in M can be compared.

The Type Decomposition

Definition: Let M be a von Neumann algebra, and $p \in B(H)$ a projection not necessarily in M. The algebra pMp is known as a corner (or compression) of M.

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra, and let $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, and $(pMp)' = M'p$, $(M'p)' = pMp$.

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