Amenable Discrete Groups

Conditions and Applications

Avinash Iyer

19 September, 2024

Contents

Chapter 1

Prelude

Chapter 2

Paradoxical Decompositions

The primary goal of this section will be to introduce the idea of a paradoxical decomposition (and its effects on the analytic properties of $\mathbb{R}^3$) through the Banach–Tarski Paradox. The ultimate goal is to prove the following statement.

Proposition 2.0.1 (General Banach–Tarski Paradox)**:** If A and B are bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such a sequence of isometries applied to these subsets yields B.

The existence of the Banach–Tarski paradox throws a wrench into a major idea that we may have about subsets of $\mathbb{R}^3$ — namely, that they always have some "volume" to them that is invariant under isometry, similar to how "area" in $\mathbb{R}^2$ is invariant under isometry.

2.1 Prelude: Essential Group Actions

We begin by discussing some of the basic properties of group actions.

Definition (Group Action)**.** Let G be a group, and A be a set. A left group action of G onto A is a map $\alpha : G \times A \to A$ that satisfies

- $\alpha(g_1, (g_2, a)) = \alpha(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$;
- $\alpha(e_G, a) = a$ for all $a \in A$.

For the sake of brevity, we write $(g, a) = g \cdot a$.

Every group action can be represented by a permutation on A.

Definition (Permutation Representation)**.** For each g, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ is a permutation of A. There is a homomorphism associated to these actions, $\varphi : G \to \mathrm{Sym}(A)$, where $\mathrm{Sym}(A)$ is the symmetric group on the elements of A.

The permutation representation can run in the opposite direction in the following sense: given a nonempty set A and a homomorphism $\psi : G \to \mathrm{Sym}(A)$, we can take $g \cdot a = \psi(g)(a)$, where $\psi(g) = \sigma_g \in \mathrm{Sym}(A)$ is a permutation.

Just as we can pass group actions into permutation representations, and discuss ideas like the kernel of homomorphisms, we can also discuss the kernel of ajn action.

Definition (Kernel)**.** The kernel of the action of G on A is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}.$$

The kernel of the group action is the kernel of the permutation representation $\varphi : G \to \mathrm{Sym}(A)$.

Definition (Stabilizer)**.** For each $a \in A$, we define the stabilizer of a under G to be the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

Remark: The kernel of the group action is the intersection of the stabilizers of every element of A.

For each $a \in A$, G_a is a subgroup of G.

Definition (Faithful Action)**.** An action is faithful if the kernel of the action is the identity, e_G. Equivalently, the permutation representation $\varphi : G \to \text{Sym}(A)$ is injective.

The following definition will be useful in the future as we dig deeper into the idea of paradoxical groups.

Definition (Free Action)**.** For a set X with G acting on X, the action of G on X is free if, for every $x \in X$, $g \cdot x = x$ if and only if $g = e_G$.

The most important theorem relating to group actions is the orbit-stabilizer theorem. As we prove the following theorem, we will reveal the definition of an orbit as a type of equivalence class.

Theorem 2.1.1 (Orbit-Stabilizer Theorem)**:** Let G be a group that acts on a nonempty set A. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking the index of the stabilizer of a in G, $|G : G_a|$.

Proof. We start by seeing that $a \sim a$, as $e_G \cdot a = a$. Similarly, if $a \sim b$, then there exists $g \in G$ such that $a = g \cdot b$. Thus,

$$\begin{aligned} g^{-1} \cdot a &= g^{-1} \cdot (g \cdot b) \\ &= g^{-1} g \cdot b \\ &= e \cdot b \\ &= b, \end{aligned}$$

meaning that $b \sim a$. Finally, if we have $a \sim b$ and $b \sim c$, we have $a = g \cdot b$ and $b = h \cdot c$ for some $g, h \in G$. Therefore,

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$. Thus, the relation $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

We claim there is a bijection between the left cosets of G_a and the elements of $[a]_\sim$.

Define $C_a = \{g \cdot a \mid g \in G\}$, which is the set of elements in the equivalence class of a. Define the map $g \cdot a \mapsto gG_a$. Since $g \cdot a$ is always an element of C_a, this map is surjective. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, we have $h^{-1}g \in G_a$, which is only true if $gG_a = hG_a$. Thus, the map is injective.

Since there is a one to one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we know that the number of equivalence classes of a under the action of G is $|G : G_a|$.
□

Definition (Orbit)**.** Let G act on A, and let $a \in A$. The orbit of a under G is the set

$$G \cdot a = \{g \cdot a \in A \mid g \in G\}$$

Chapter 3

Tarski's Theorem