Problem 1

Let V be a vector space and suppose $\{W_i\}$ is a family of subspaces of V.

(i) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of V contained in every W_i.

Proof: We will show that (a) $\bigcap_{i\in I} W_i$ is a subspace of V, and (b) there is is no larger subspace of V contained within every W_i.

(a) Let $v_i, v_j \in \bigcap_{i\in I} W_i$, $\alpha, \beta \in \mathbb{F}$. We want to show that $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$. Since $v_i \in \bigcap_{i\in I} W_i$, $v_i \in W_i$ for some W_i, and $v_j \in W_j$ for some W_j. Additionally, WLOG, $v_j \in W_i$, as both v_i and v_j are contained within their intersection. Therefore, $\alpha v_i + \beta v_j \in W_i$, so $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$.

(b) Suppose there is a subspace U of V such that every W_i is contained in U, and $U \supset \bigcap_{i\in I} W_i$.

(ii) Show that

$$\sum_{i\in I} W_i := \left\{\sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite}\right\}$$

is the smallest subspace containing each W_i.

Problem 2

Let V be a vector space and suppose $S \subseteq V$ is any subset. Show that

$$\text{span}(S) = \bigcap\{W \mid S \subseteq W,\ W \subseteq V \text{ subspace}\}$$

Deduce that $\text{span}(S)$ is the smallest subspace of V containing S.

Proof: Let W be a subspace containing S. Since W is a subspace, every linear combination of every element of S is inside W, as every element of S is an element of W. Therefore, for *every* subspace W such that $S \subseteq W$, any linear combination of every element in S is also in W — thus, $\text{span}(S) = W$.

From this, we can see that $\text{span}(S)$ can be no smaller than any subspace containing S, meaning $\text{span}(S)$ is the smallest subspace of V containing S.

Problem 3

Let V be a vector space with subspaces $W_i \subseteq V$ for $i = 1, 2$. If $W_1 \cup W_2 \subseteq V$ is a subspace, show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof: Suppose $W_1 \nsubseteq W_2$. Thus, $\exists u \in W_1$ such that $u \notin W_2$. Since W_1 is a subspace, it is not empty, and thus $\exists w \in W_1$. Since $W_1 \cup W_2$ is a subspace, and $u, w \in W_1 \cup W_2$, $u + w \in W_1 \cup W_2$. Additionally, $u + w \notin W_2$, as if it were the case, then $u + w - w \in W_2$, meaning $u \in W_2$, violating one of our assumptions. Therefore, $u + w \in W_1$, meaning $W_2 \subseteq W_1$.

Problem 4

Let V be a vector space over $\mathbb{F}$ and suppose $W \subseteq V$ is a subspace.

(i) Show that the quotient space $V/W = \{[v]_W \mid v \in V\}$ is a vector space with operations

$$[u]_W + [v]_W := [u + v]_W$$
$$\alpha[v]_W := [\alpha v]_W$$

Proof of Addition: Let $u_1 \sim u_2 \sim u$, $v_1 \sim v_2 \sim v$ under the equivalence relation $u \sim v \leftrightarrow u - v \in W$. Then,

$$\begin{aligned}
[u_1]_W + [v_1]_W &= (u_1 + W) + (v_1 + W)\\
&= (u + W) + (v + W)\\
&= (u + v) + W\\
&= [u + v]_W\\
[u_2]_W + [v_2]_W &= (u_2 + W) + (v_2 + W)\\
&= (u + W) + (v + W)\\
&= (u + v) + W\\
&= [u + v]_W
\end{aligned}$$

Proof of Scalar Multiplication: Let $\alpha \in \mathbb{F}$. Let $v_1 \sim v_2 \sim v$ under the equivalence relation $u \sim v \leftrightarrow u - v \in W$. Then,

$$\begin{aligned}
\alpha[v_1]_W &= \alpha\left(v_1 + W\right)\\
&= \alpha(v + W)\\
&= \alpha v + W\\
&= [\alpha v]_W\\
\alpha[v_2]_W &= \alpha\left(v_2 + W\right)\\
&= \alpha\left(v + W\right)\\
&= \alpha v + W\\
&= [\alpha v]_W.
\end{aligned}$$

(ii) Suppose $\|\cdot\|$ is a norm on V. Show that

$$\|[v]_W\|_{V/W} := \inf_{w \in W} \|v - w\|$$

is a seminorm on V/W.

Absolute Homogeneity: Let $\alpha \in \mathbb{F}$. Then,

$$\begin{aligned}
\|\alpha[v]_W\|_{V/W} &= \|[\alpha v]_W\|_{V/W}\\
&= \inf_{w \in W} \|\alpha v - w\|\\
&= \inf_{w \in W} \|\alpha(v - w)\| && W \text{ subspace}\\
&= |\alpha| \inf_{w \in W} \|v - w\|\\
&= |\alpha| \|[v]_W\|_{V/W}.
\end{aligned}$$

Triangle Inequality: Let $u, v \in V$. Then,

$$\begin{aligned}
\|[u]_W + [v]_W\|_{V/W} &= \|[u + v]_W\|_{V/W}\\
&= \inf_{w \in W} \|(u + v) - w\|\\
&= \inf_{w \in W} \|(u - w) + (v - w)\| && W \text{ subspace}\\
&\leq \inf_{w \in W} \left(\|u - w\| + \|v - w\|\right)\\
&= \|[u]_W\|_{V/W} + \|[v]_W\|_{V/W}.
\end{aligned}$$

Problem 5

Show that the quantity

$$\|f\|_1 := \int_0^1 |f(t)|dt$$

defines a norm on $C([0,1])$ with $\|f\|_1 \leq \|f\|_u$. Are $\|\cdot\|_1$ and $\|\cdot\|_u$ equivalent norms?

Non-Negativity: Since $|f(t)| \geq 0$ for $t \in [0,1]$ by the definition of absolute value, it is the case that $\int_0^1 |f(t)|dt \geq 0$.

Positive Definite: Clearly, $\|\mathbb{0}\|_1 = 0$. Additionally, since f is continuous, $|f|$ is continuous, and since $|f(t)| \geq 0$ for $t \in [0,1]$, it must be the case that $\int_0^1 |f(t)|dt = 0$ only when $f = \mathbb{0}$.

Absolute Homogeneity: Let $\alpha \in \mathbb{R}$

$$\begin{aligned}
\|\alpha f\|_1 &= \int_0^1 |\alpha f(t)|dt \\
&= \int_0^1 |\alpha||f(t)|dt \\
&= |\alpha| \int_0^1 |f(t)|dt \\
&= |\alpha|\|f\|_1
\end{aligned}$$

Triangle Inequality:

$$\begin{aligned}
\|f+g\|_1 &= \int_0^1 |f(t)+g(t)|dt \\
&\leq \int_0^1 (|f(t)| + |g(t)|)\, dt \\
&= \int_0^1 |f(t)|dt + \int_0^1 |g(t)|dt \\
&= \|f\|_1 + \|g\|_1
\end{aligned}$$

Norm Comparison:

$$\begin{aligned}
|f(x)| &\leq \|f\|_u && \text{Definition of Supremum} \\
\int_0^1 |f(x)| &\leq \int_0^1 \|f\|_u \\
\|f\|_1 &\leq \|f\|_u.
\end{aligned}$$

Problem 6

Show that all the p-norms, $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ on $\mathbb{F}^n$ are equivalent. Also, show that if $1 \leq p \leq q \leq \infty$, then $\ell_p \subseteq \ell_q$.

Proof: We will show that for $x \in \mathbb{F}^n$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent norms.

$$\begin{aligned}
\|x\|_\infty &= \max_{1\leq j\leq n} |x_j| \\
&\leq \sum_{i=1}^{n} |x_i| \\
&= \|x\|_1. \\
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \\
&\leq \sum_{i=1}^{n} \max_{1\leq j\leq n} |x_j| \\
&= n\|x\|_\infty.
\end{aligned}$$

Now, we will show that any p norm is equivalent to the ∞ norm.

$$\begin{aligned}
\sum_{i=1}^{n} |x_j|^p &\leq \sum_{i=1}^{n} \left(\max_{1\leq j\leq n} |x_j|\right)^p \\
&= n\|x\|_\infty^p,
\end{aligned}$$

so

$$\|x\|_p \leq n^{1/p}\|x\|_\infty.$$

Since every $p \in (1, \infty)$ norm is equivalent to the ∞ norm, and the ∞ norm is equivalent to the 1 norm, every p norm is equivalent to every other p norm.

Let $1 \leq p \leq q \leq \infty$. Then, for $x \in \ell_p$, $\sum_{j=1}^{\infty} |x_j|^p$ is convergent.

Therefore, $\lim_{j\to\infty} |x_j|^p = 0$. So, $\exists J \in \mathbb{N}$ such that for $j \geq J$, $|x_j|^p \leq 1$.

For $j \geq J$, $|x_j|^q \leq |x_j|^p$, $\sum_{j=J}^{\infty} |x_j|^q \leq \sum_{j=J}^{\infty} |x_j|^p$. So, $\sum_{j=1}^{\infty} |x_j|^q$ is convergent, meaning $x \in \ell_q$, and $\ell_p \subseteq \ell_q$.

Problem 7

Let $\mathbb{M}_{m,n}(\mathbb{C})$ denote the linear space of all $m \times n$ matrices with coefficients from $\mathbb{C}$. For $a \in \mathbb{M}_{m,n}(\mathbb{C})$, set

$$\|a\|_{\text{op}} := \sup_{\xi \in B_{\ell_2}^n} \|a\xi\|_{\ell_2^m}.$$

Show that $\|\cdot\|_{\text{op}}$ is a norm on $\mathbb{M}_{m,n}(\mathbb{C})$. This is the operator norm.

Positive Definite: Since ℓ_2^m is a norm on $\mathbb{R}^m$, it must be the case that $\|a\|_{\text{op}} = 0$ if and only if the least upper bound on $\|a\xi\|$ is zero, occurring only when a is the zero operator.

Absolute Homogeneity: Let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned}
\|\alpha a\|_{\text{op}} &= \sup_{\xi \in B_{\ell_2^n}} \|(\alpha a)\xi\|_{\ell_2^m} \\
&= \sup_{\xi \in B_{\ell_2^n}} |\alpha|\|a\xi\|_{\ell_2^m} && \ell_2^m \text{ is a norm} \\
&= |\alpha| \sup_{\xi \in B_{\ell_2^n}} \|a\xi\|_{\ell_2^m} \\
&= |\alpha|\|a\|_{\text{op}}
\end{aligned}$$

Triangle Inequality: Let $a, b \in \mathbb{M}_{m,n}(\mathbb{C})$. Then,

$$\begin{aligned}
\|a+b\|_{\text{op}} &= \sup_{\xi \in B_{\ell_2^n}} \|(a+b)\xi\|_{\ell_2^m} \\
&= \sup_{\xi \in B_{\ell_2^n}} \|a\xi + b\xi\|_{\ell_2^m} \\
&\leq \sup_{\xi \in B_{\ell_2^n}} \left(\|a\xi\|_{\ell_2^m} + \|b\xi\|_{\ell_2^m}\right) && \ell_2^m \text{ is a norm} \\
&\leq \sup_{\xi \in B_{\ell_2^n}} \|a\xi\|_{\ell_2^m} + \sup_{\xi \in B_{\ell_2^n}} \|b\xi\|_{\ell_2^m} \\
&= \|a\|_{\text{op}} + \|b\|_{\text{op}}
\end{aligned}$$

Let $x \in \ell_p$, meaning $\|x\|_p < \infty$. Since $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent norms, $\exists c \in \mathbb{F}$ such that $\|x\|_q \leq c\|x\|_p$. Therefore, $\|x\|_q < \infty$, meaning $x \in \ell_q$. Therefore, $\ell_p \subseteq \ell_q$.

Problem 8

If $f : [a, b] \to \mathbb{R}$ is any function and $\mathcal{P} = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$ is a partition of $[a, b]$, we define the variation of f on $\mathcal{P}$ as

$$\text{Var}(f; \mathcal{P}) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

We say that f is of bounded variation if

$$\text{Var}(f) = \sup_{\mathcal{P}} \text{Var}(f; \mathcal{P}) < \infty,$$

where the supremum runs over all partitions of $[a, b]$. We define the space of all functions of bounded variation

$$\text{BV}([a, b]) := \{f : [a, b] \to \mathbb{R} \mid \text{Var}(f) < \infty\}$$

(i) Is the function $\mathbb{1}_{\mathbb{Q}} : [0, 1] \to \mathbb{R}$ of bounded variation?

Proof: The answer is no, $\mathbb{1}_{\mathbb{Q}}$ is not of bounded variation. Define $\mathcal{P}$ to be a partition where each alternating member of the partition is, respectfully, a member of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. Since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $[0, 1]$, this partition is valid. However, the variation of $\mathbb{1}_{\mathbb{Q}}$ over this partition is infinite, as there are infinitely many rational and irrational numbers in the space, and the difference between the image of each element of the infinite partition is 1, meaning the sum is infinite.

(ii) Show that $\text{BV}([a, b]) \subseteq \ell_\infty([a, b])$ is a subspace.

Proof: We will show that $\text{BV} \subseteq \ell_\infty$, and that any linear combination $f, g \in \text{BV}([a, b])$ is an element of $\text{BV}([a, b])$.

To show $\text{BV}([a, b]) \subseteq \ell_\infty([a, b])$, observe that for $x \in [a, b]$,

$$\begin{aligned}
|f(x)| &= |f(x) - f(a) + f(a)| \\
&\leq |f(x) - f(a)| + |f(a)| \\
&\leq \text{Var}(f) + |f(a)|,
\end{aligned}$$

meaning $|f(x)|$ is bounded above, so

$$\sup_{x \in [a,b]} |f(x)| < \infty.$$

Let $f, g \in \mathrm{BV}([a,b])$ and let $\alpha, \beta \in \mathbb{R}$. Then,

$$\begin{aligned}
\mathrm{Var}(\alpha f + \beta g; \mathcal{P}) &= \sum_{k=1}^{n} |(\alpha f(x_k) + \beta g(x_k)) - (\alpha f(x_{k-1}) + \beta g(x_{k-1}))| \\
&\leq \sum_{k=1}^{n} |\alpha f(x_k) - \alpha f(x)| + \sum_{k=1}^{n} |\beta g(x_k) - \beta g(x_{k-1})| \\
&= |\alpha| \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + |\beta| \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\
&\leq |\alpha| \mathrm{Var}(f) + |\beta| \mathrm{Var}(g),
\end{aligned}$$

meaning

$$\mathrm{Var}(\alpha f + \beta g) \leq |\alpha| \mathrm{Var}(f) + |\beta| \mathrm{Var}(g),$$

meaning $\alpha f + \beta g \in \mathrm{BV}([a,b])$.

(iii) Show that $\|f\|_{\mathrm{BV}} := |f(a)| + \mathrm{Var}(f)$ defines a norm on $\mathrm{BV}([a,b])$.

Proof: We will show that $\|f\|_{\mathrm{BV}}$ defines a norm on $\mathrm{BV}([a,b])$.

Positive Definite: Let $\|f\|_{\mathrm{BV}} = 0$. Then, since $\mathrm{Var}(f) \geq 0$, it must be the case that $\mathrm{Var}(f) = 0$. So, $\mathrm{Var}(f)$ is a constant function, $f(x) = c$ for $x \in [a,b]$.

Additionally, since $\|f\|_{\mathrm{BV}} = 0$, $|f(a)| = 0$, so $f(a) = 0$. Therefore, since $f(a) = 0$ and f is constant for $x \in [a,b]$, $f = \mathbb{0}$.

Absolute Homogeneity: Let $f \in \mathrm{BV}([a,b])$, and $\alpha \in \mathbb{R}$. Then,

$$\begin{aligned}
\|f\|_{\mathrm{BV}} &= |\alpha f(a)| + \mathrm{Var}(\alpha f) \\
&= |\alpha||f(a)| + \sup_{\mathcal{P}} \sum_{k=1}^{n} |\alpha f(x_k) - \alpha f(x_{k-1})| \\
&= |\alpha||f(a)| + |\alpha| \sup_{\mathcal{P}} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\
&= |\alpha| \left(|f(a)| + \mathrm{Var}(f)\right) \\
&= |\alpha| \|f\|_{\mathrm{BV}}.
\end{aligned}$$

Triangle Inequality: Let $f, g \in \mathrm{BV}([a,b])$. Then,

$$\|f + g\|_{\mathrm{BV}} = |f(a) + g(a)| + \mathrm{Var}(f + g)$$

$\alpha, \beta = 1$ from (ii), we have

$$\begin{aligned}
\|f + g\|_{\mathrm{BV}} &\leq |f(a)| + \mathrm{Var}(f) + |g(a)| + \mathrm{Var}(g) \\
&= \|f\|_{\mathrm{BV}} + \|g\|_{\mathrm{BV}}
\end{aligned}$$

Problem 9

Given any function $f : [0,1] \to \mathbb{C}$, we define

$$N(f) := \sup_{x \neq y, x, y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|}$$

and

$$\|f\|_{\Lambda} := |f(0)| + N(f).$$

Moreover, set

$$\Lambda[0,1] := \{f : [0,1] \to \mathbb{C} \mid \|f\|_{\Lambda} < \infty\}$$

(i) Show that $\Lambda[0,1]$ is precisely the set of Lipschitz continuous functions on $[0,1]$.

Proof: Let $f \in \Lambda[0,1]$. Then, $\|f\|_\Lambda = c$ for some finite c. Then, for $x, y \in [0,1]$

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|} &\leq N(f)\\ &\leq \|f\|_\Lambda\\ &= c.\end{aligned}$$

So,

$$|f(x)-f(y)| \leq c|x-y|,$$

which defines a Lipschitz continuous function.

(ii) Verify that $\Lambda[0,1]$ is a vector space with norm $\|f\|_\Lambda$, which is the Lipschitz norm.

Proof of Vector Space: Let $f, g \in \Lambda[0,1]$. Then, f and g are Lipschitz continuous. Let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned}|(\alpha f)(x) - (\alpha f)(y)| &= |\alpha||f(x)-f(y)|\\ &\leq |alpha|c|x-y|\\ &= h|x-y|,\end{aligned}$$

and

$$\begin{aligned}|(f+g)(x)-(f+g)(y)| &= |f(x)-f(y)+g(x)-g(y)|\\ &\leq |f(x)-f(y)| + |g(x)-g(y)|\\ &\leq c|x-y| + d|x-y|\\ &= \ell|x-y|,\end{aligned}$$

meaning that $\Lambda[0,1]$ is closed under addition and scalar multiplication.

Proof of Norm:

Positive Definiteness:

$$\begin{aligned}\|f\|_\Lambda &= 0\\ |f(0)| + \sup_{x,y\in[0,1],x\neq y}\frac{|f(x)-f(y)|}{|x-y|} &= 0,\end{aligned}$$

meaning that for $x, y \in [0,1]$ and $x \neq y$

$$f(x) = f(y)$$

and

$$f(0) = 0$$

so $f = \mathbb{0}_f$.

Absolute Homogeneity: Let $\alpha \in \mathbb{C}$.

$$\begin{aligned}\|\alpha f\| &= |\alpha f(0)| + N(\alpha f)\\ &= |\alpha||f(0)| + \sup_{x,y\in[0,1],x\neq y}\frac{|\alpha f(x)-\alpha f(y)|}{|x-y|}\\ &= |\alpha|\left(|f(0)| + \sup_{x,y\in[0,1],x\neq y}\frac{|f(x)-f(y)|}{|x-y|}\right)\\ &= |\alpha|\|f\|_\Lambda\end{aligned}$$

Triangle Inequality: Let $f, g \in \Lambda[0,1]$. Then,

$$\begin{aligned}\|f+g\| &= |f(0)+g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)+g(x)-(f(y)+g(y))|}{|x-y|}\\ &\leq \left(|f(0)| + \sup_{x,y\in[0,1],x\neq y}\frac{|f(x)-f(y)|}{|x-y|}\right) + \left(|g(0)| + \sup_{x,y\in[0,1],x\neq y}\frac{|g(x)-g(y)|}{|x-y|}\right)\\ &= \|f\|_\Lambda + \|g\|_\Lambda\end{aligned}$$

Therefore, $\Lambda[0,1]$ is a normed vector space with $\|\cdot\|_\Lambda$ as the Lipschitz norm.

(iii) Show that $\|f\|_u \leq \|f\|_\Lambda$ for every $f : [0,1] \to \mathbb{R}$.

Problem 10

Let p be a seminorm on a vector space V.

(i) Show that $N_p := \{w \in V \mid p(w) = 0\}$ is a subspace of V.

Proof: Let $v, w \in N_p$. Then, $p(v) = 0$ and $p(w) = 0$. Since p is a seminorm, for $\alpha, \beta \in \mathbb{F}$, we have:

$$\begin{aligned}p(\alpha v + \beta w) &\leq p(\alpha v) + p(\beta w)\\ &= |\alpha|p(v) + |\beta|p(w)\\ &= 0.\end{aligned}$$

Since p is definitionally non-negative, $p(\alpha v + \beta w) = 0$. Therefore, N_p is a vector space.

(ii) We form the quotient vector space V/N_p. Show that

$$\|[v]_{N_p}\|_p := p(v)$$

defines a norm on V/N_p.

Proof: Since p is a seminorm, we know that absolute homogeneity and the triangle inequality already hold for p. Therefore, what we need to show is that $\|\cdot\|_p$ is positive definite.

Let $\|[v]_{N_p}\|_p = 0$. Since $p(v) = 0$, $v \in N_p$, meaning $[v]_{N_p} = [\mathbb{0}]_{N_p}$.

(iii) If $(E, \|\cdot\|)$ is a normed space and $T : V \to E$ is a linear map, show that $p(v) := \|T(v)\|$ is a seminorm on V. In this case, what is N_p.

Absolute Homogeneity: Let $\alpha \in \mathbb{F}$. Then,

$$\begin{aligned}p(\alpha v) &= \|T(\alpha v)\|\\ &= \|\alpha T(v)\|\\ &= |\alpha|\|T(v)\|\\ &= |\alpha|p(v).\end{aligned}$$

Triangle Inequality: Let $v, w \in V$. Then,

$$\begin{aligned}p(v+w) &= \|T(v+w)\|\\ &= \|T(v) + T(w)\|\\ &\leq \|T(v)\| + \|T(w)\|\\ &= p(v) + p(w).\end{aligned}$$

The subspace N_p denotes the kernel of T.