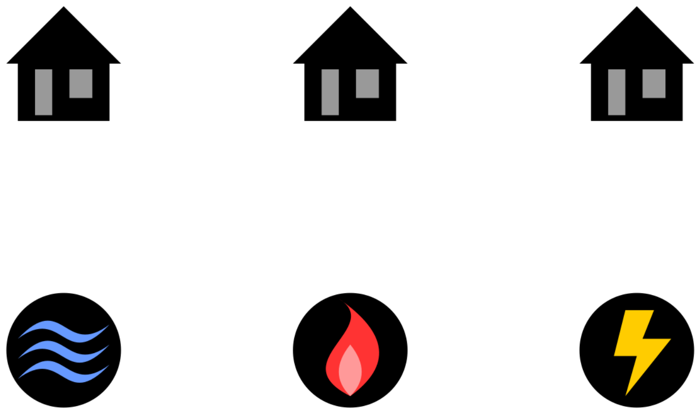
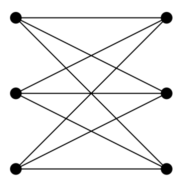
Graphs and the Three Utilities Problem

We can imagine trying to connect three houses below with three utilities without the utility lines crossing.



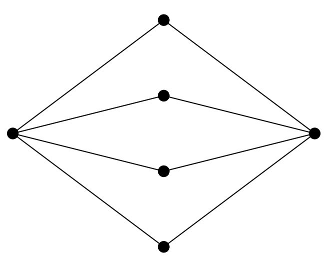
This problem is akin to the graph $K_{3,3}$ (the complete bipartite graph with three vertices in each partite set).



A *graph* is an ordered pair of sets (V, E), where $E \subseteq V \times V$.

For example, if $V = \{a, b, c\}$ and $E = \{(a, b), (a, c)\}$, then (V, E) is a graph. The goal of the three utilities puzzle is to draw $K_{3,3}$ in $\mathbb{R}^2$ without any edges crossing. A graph that can be drawn as such is *planar*.

- $K_{3,3}$ is not planar.
- $K_{2,4}$ is planar.



Euler's Theorem

Let $G \subseteq \mathbb{R}^2$ be a planar graph (i.e., drawn in $\mathbb{R}^2$ without edge crossings). Each disjoint subset of $\mathbb{R}^2 - G$ is a *face* of G.

For every graph G embedded in $\mathbb{R}^2$ (i.e., drawn without edge crossings) with V vertices, E edges, and F faces, the following is true:
$$V - E + F = 2$$

We will use this theorem to show that you cannot connect the three houses to the three utilities as follows:

Outline Proof (of $K_{3,3}$'s non-planarity)

Suppose toward contradiction that $K_{3,3}$ is planar. Then, by Euler's Theorem, we know that $V - E + F = 2$.

We know that $K_{3,3}$ has six vertices and nine edges, so we know that $6 - 9 + F = 2$. Therefore, we know that there must be 5 faces. In order to enclose a face, there must be at least four edges in $K_{3,3}$ (as there is no edge between two members of a partite set). Additionally, each edge encloses two faces. Therefore, $E \geq 2F$. However, since $E = 9$, and we assume that $F \geq 5$, we have reached a contradiction (as $9 < 10$). Thus, $K_{3,3}$ is not planar.

Four-Color Theorem

Every planar graph can be colored (adjacent vertices do not have the same color) with four colors. The planar graph can be colored by fewer colors.

Polynomial Example

Let $p(a,b,c,d) = ab + ac + ad + bc + bd + cd$. When we factor, we get $p(a,b,c,d) = a(b+c+d) + b(c+d) + cd$. In the first equation, we had to carry out 6 multiplications, while in the second equation we only had to carry out 3 multiplications. We could factor differently:

$$\begin{aligned} p(a,b,c,d) &= ab + ac + ad + bc + bd + cd \\ &= a(b+c+d) + b(c+d) + cd \\ &= (a+b)(c+d) + ab + cd \end{aligned}$$

We have a lower bound of three multiplications to carry out.

In the arbitrary case, we have the following. We want to find the lowest number of multiplications.

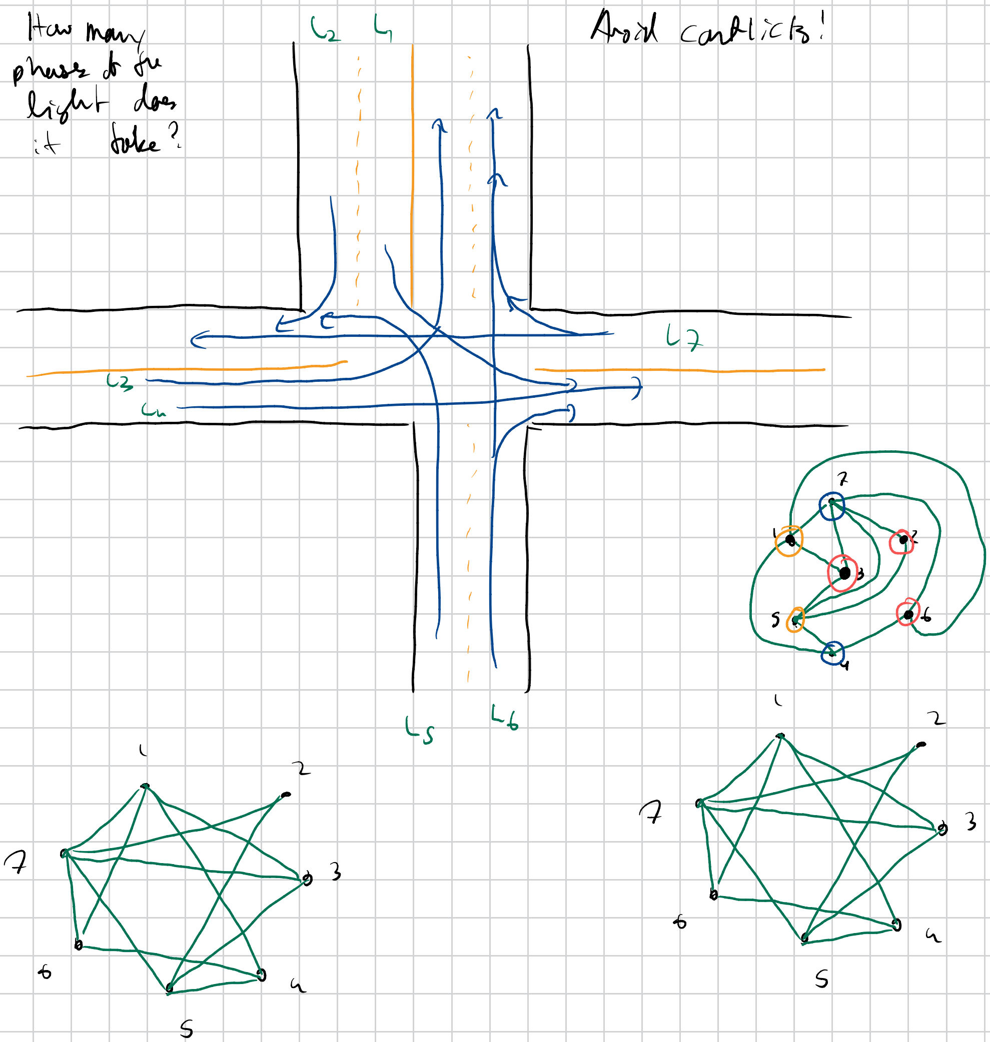
$$p(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j$$

The minimum number of multiplications we can do is $n-1$. We can find this via a graph with n vertices $\{x_1, \dots, x_n\}$, and for $x_i x_j$ in p, we have an edge from x_i to x_j. This is the complete graph on n vertices, K_n. Each complete bipartite subgraph represents a multiplication — so our question can be restated as follows:

> Given a complete graph on n vertices, K_n, partition its edges into as few complete graphs as possible.

The answer for this is $n-1$, with a proof in linear algebra. However, there is no graph theory-specific proof for this question.

Light Cycles



Diestel book: Overview

A **graph** is an ordered pair $G = (V, E)$ of sets such that $\forall e \in E,\ e = \{v, w\}$ for some $v, w \in V$.

Paths and Cycles

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A **path** is a subgraph P of G such that $V(P) = \{v_0, \dots, v_k\}$ and $E(P) = \{v_0 v_1, \dots, v_{k-1} v_k\}$. We say the **length** of P is equal to $|E(P)|$.

If $v_k v_0 \in E(G)$, then $C = P + v_k v_0$ is a **cycle**. $V(C) = V(P)$ and $E(C) = E(P) \cup \{v_0 v_k\}$.

Abbreviations: $P = v_0 \dots v_k$, and $C = v_0 \dots v_k v_0$

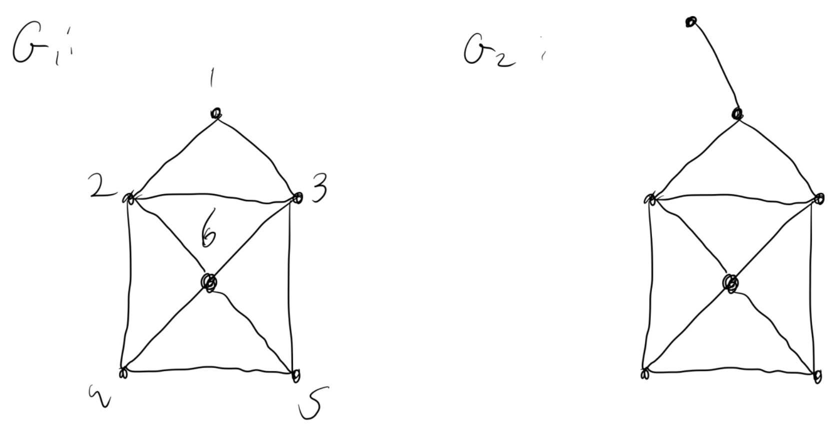
Degree, Order, and Size

Given $v \in V(G)$, the **degree** of v $d(v) = |\{vw \mid v \in E(G)\}|$. The edge vw is **incident** to v.

The **order** of G is $|V(G)|$, or $|G|$, and the **size** of G is $|E(G)|$, or $||G||$.

Hamiltonian Cycles

A cycle $C \subseteq G$ is **Hamiltonian** if $V(C) = V(G)$. A graph is Hamiltonian if it contains a Hamiltonian cycle.



For example, G_1 has a Hamiltonian cycle $\{1, 2, 4, 5, 6, 3, 1\}$, while G_2 does not have one as the stray vertex cannot be reached without going over an edge.

For example, the Knight's Tour (where you visit every square on a chess board) involves finding a particular kind of Hamiltonian cycle.

Dirac's Theorem

If G is a graph of order ≥ 3 such that every vertex has degree $\geq \left\lceil \frac{|G|}{2} \right\rceil$, then G is Hamiltonian.

Let P be a path in G with maximum length (i.e., *a* longest path). **Outline:**

Step 1 Show that $|V(P)| > \frac{|G|}{2}$

Step 2 Show $\exists C \subseteq G$ such that $V(C) = V(P)$.

Step 3 Show that C is a Hamiltonian cycle.

Step 1 Let $P = (v_1, v_2, \dots, v_k)$ be a path in G with maximum length. Suppose toward contradiction that $|P| < n/2$, meaning $k < n/2$. Then, $\nexists v_i$ such that v_i is connected to any of $v_1, \dots, v_k$, or else we would be able to extend P. Thus, $\forall v \in \{v_1, \dots, v_k\}$, v is only adjacent to other members in $v_1, \dots, v_k$. However, this means that the maximum value v can take is $k-1$, and since $k < n/2$, this means $k - 1 < n/2$, or that v would not satisfy one of the conditions of G. $\perp$

Step 2 Let $P = v_0 \dots v_k$. It suffices to show that $\exists j \in \{2, \dots, k\}$ such that $v_1 \leftrightarrow v_j$ and $v_{j-1} \leftrightarrow v_k$. Since P has maximum length, v_1 has no neighbor outside P (or else P could be extended). Similarly, v_k has no neighbor outside P. However, every vertex has degree at least 2, meaning v_1 must have a neighbor in P. Suppose toward contradiction that $\nexists j - 1$ such that $v_{j-1} \leftrightarrow v_k$. Then, $N = \{v_{2-1}, \dots, v_{k-1-1}\} \geq \frac{n}{2}$ are not neighbors of v_k. This means $k \leq n$, so v_k has $k - 1 - N$ neighbors, implying $d(v_k) < \frac{n}{2}$, which is our contradiction.

Step 3 Let P is a path of maximum length in G, and C be a cycle in G such that $V(C) = V(P)$. Suppose toward contradiction that $|P| < n$. Then, $\exists v \in G$ such that $v \notin P$. Since $d(v) \geq \frac{n}{2}$, v is adjacent to at least one vertex $w \in P$ (as there are not enough vertices outside P for v to be adjacent to). Let $C = (v_{i_1}, \dots, v_{i_k}, v_{i_1})$. WLOG, v is adjacent to v_{i_1}. Then, $P' = v, v_{i_1}, \dots, v_{i_k}$ is a path that is longer than P, which is a contradiction.

Ore's Theorem

If $|G| \geq 3$ and $\forall v, w \in V(G)$ where $v \not\leftrightarrow w$ and $d(v) + d(w) \geq n$, then G is Hamiltonian.

We can use Ore's Theorem to prove Dirac's Theorem.

Vertex Deletion

Let $v \in G$. Then, $G - v$ is the subgraph of G with vertices $V(G) \setminus \{v\}$, and edges $E(G) \setminus \{vw \mid vw \in E(G)\}$.

Theorem 6.4

Let $v_1, \dots, v_k \in V(G)$. Then, $G - v_1 - v_2 - \cdots - v_k$ has at most k components.

Connectedness

A graph G is **connected** if $\forall v, w \in V(G)$, $\exists P : v \dots w$.

Distinct Representatives

Suppose we want to pick one student representative from every Oxy math class. No student should be chosen more than once. Say there are n classes: $c_1, \dots, c_n$, where $c_i = \{s_1, \dots, s_{k_i}\}$, where $1 \leq i \leq n$.

Obviously, there must be at least n students in all classes combined: i.e.,

$$\left|\bigcup c_i\right| \geq n$$

However, this goes deeper:

$$\begin{aligned} |c_1 \cup c_2| &\geq 2 \\ |c_3 \cup c_5 \cup c_6| &\geq 3 \\ &\vdots \\ |c_{i_1} \cup \cdots \cup c_{i_r}| &\geq r \ \forall r \end{aligned} \tag{*}$$

Obviously, condition (*) is necessary.

We want c_i and c_j to be distinct, (even when they are equal as sets).

Let $Z = (c_1, \dots, c_n)$ be a finite sequence. Then, $(c_{i_1}, \dots, c_{i_k})$ is a subsequence of Z if $i_1 < \dots, i_k$.

Hall's Theorem

Let $Z = (c_1, \dots, c_n)$ be a sequence of sets c_i. Suppose that for every subsequence Y of Z with $Y = (c_{i_1}, \dots, c_{i_k})$ such that $|c_{i_1} \cup \cdots \cup c_{i_k}| \geq k$. Then, $\exists$ pairwise distinct $s_1, \dots, s_n$ with $s_i \in c_i$.

Note (*) is a sufficient condition

Informally, we can restate the premise as follows: Let G be a bipartite graph. One set of vertices $c_1, \dots, c_n$, is the classes, and the other set $s_1, \dots, s_m$ is the set of all students. Each vertex c_i is connected by edges to its students.

Hall's Theorem (In Graphs)

Let G be a bipartite graph on vertices $C \sqcup S$, where $C = \{c_1, \dots, c_n\}$ and $S = \{s_1, \dots, s_m\}$. Then, G has a matching (i.e., a set of pairwise disjoint edges) if and only if $\forall r$ $1 \leq r \leq n$, any r vertices in C are connected to at least r vertices in S.

Proof of Hall's Theorem

Base Case: The theorem holds for $n = 1$. $S_1 \neq \emptyset$ by the theorem's hypothesis, as if $Y := (S_1)$, then $\left|\bigcup_{S \in Y} S\right| \geq 1$, so $|S_1| \geq 1$.

Induction Hypothesis Assume the theorem holds for $n - 1$ and every $m < n - 1$. We will show the theorem holds for n

Proof

Case 1: Assume every proper subsequence Y of Z is loose. Let $x_1 \in S_1$ ($S_1 \neq \emptyset$ as proved in the base case). Let $S_i' = S_i \setminus \{x_1\}$, where $2 \leq i \leq n$. Let $Z' = (S_2', \dots, S_n')$.

Let Y' be a subsequence of Z'. We want to show that

$$\left|\bigcup + S_i' \in Y' S_i'\right| \geq |Y'|$$

We know that Y consists of all S_i such that $S_i' \in Y'$. Since Y is loose (as $S_1 \notin Y$), and $\left|\bigcup_{S_i \in Y} S_i\right| \geq |Y|$.

$$\begin{aligned}\left|\bigcup_{S_i' \in Y'} S_i'\right| &\geq \left|\bigcup_{S_i \in Y} S_i\right| - 1\\ &> |Y| - 1\\ &\geq |Y|\\ &= |Y'|\end{aligned}$$

Case 2: Suppose $\exists$ a tight proper subsequence of Z, Y. Without loss of generality, $Y = (S_1, \dots, S_m)$, where $1 \leq m < n$. Since Y satisfies the theorem hypothesis, and $m < n$, so the induction hypothesis must hold.

For $m+1 \leq k \leq n$, let $S_k' = S_k \setminus \{x_1, \dots, x_m\}$. Let $Z' = (S_{m+1}', \dots, S_n')$. Let Y' be any subsequence of Z'. We want to show that $\left|\bigcup_{S_i' \in Y'} S_i'\right| \geq |Y'|$.

Let $\overline{Y}$ be the subsequence of Z corresponding to Y', i.e., $S_i \in \overline{Y} \Leftrightarrow S_i' \in Y'$.

Y	$\overline{Y}$
$S_1, \dots, S_m$	$S_{m+1}, \dots, S_n$

Let $W = Y + \overline{Y}$, where $+$ denotes concatenation. Since W is a subsequence of Z, and Z satisfies the Hall hypothesis, we have

$$\left|\bigcup W\right| \geq |W|$$

since

$$\begin{aligned}\bigcup W &= \bigcup Y \cup \bigcup \overline{Y}\\ &= \bigcup Y \cup \bigcup Y'\end{aligned}$$

as everything in $\overline{Y}$ is either in Y' or in Y, and due to double counting, we have

$$\left|\bigcup W\right| \leq \left|\bigcup Y\right| + \left|\bigcup Y'\right|$$

since

$$\begin{aligned}\left|\bigcup Y\right| &= |Y|\\ &= m\end{aligned}$$

as Y is tight, and

$$\begin{aligned}|W| &= |Y| + |Y'|\\ &= |Y| + |Y'|\end{aligned}$$

so, we have

$$\begin{aligned}|W| &\leq \left|\bigcup W\right|\\ m + |Y'| &\leq m + \left|\bigcup Y'\right|\\ |Y'| &\leq \left|\bigcup Y'\right|\end{aligned}$$

k-factorable Graphs

Let H be a subgraph of G. Let $k \in \mathbb{Z}^+$. H is a k-factor of G if

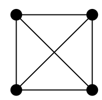
(i) H is k-regular (i.e., every vertex of H is of degree k)

(ii) $V(H) = V(G)$ (H is a spanning subgraph)

k-factors are not necessarily connected subgraphs.

A graph G is k-factorable if its edges can be partitioned k-factors of G. If G has k-factors $H_1, \dots, H_m$ such that $\{E(H_1), \dots, E(H_m)\}$ is a partition of $E(G)$.

For example, K_4 is 1-factorable.



1-factorability of K_n

K_n is 1-factorable if and only if n is even.

($\Rightarrow$) The proof is trivial.

($\Leftarrow$) Number the vertices of K_n. Redraw the graph such that vertex 1 is in the center of a $n-1$-gon. Connect vertex 1 to vertex 2, and draw all the edges that are perpendicular to this edge. Let this 1-factor be denoted H_1.

Connect vertex 1 to vertex 3, and draw the edges perpendicular to that edge. This 1-factor is denoted H_2.

Continue until we finish connecting vertex 1 to vertex 10, and $H_1, \dots, H_{10}$ must partition the edges of K_n.

2-factorability of Graphs

A graph G is 2-factorable if and only if G k-regular for some even integer k.

An edge vw of G is a *bridge* if v and w are in different components of $G - vw$.

Chinese Postman Problem

- A **walk** is a sequence $(v_1, \dots, v_k)$ of vertices such that $\exists v_i v_{i+1} \in E(G)$.
- A **trail** is a walk that does not repeat edges.
- A **path** is a trail that does not repeat vertices.
- A closed path is a **cycle**, and a closed trail is a **circuit**.

A courier wants to deliver the mail on every street in a neighborhood. The goal is to minimize the number of streets to repeat.

We can represent this by letting vertices be intersections and edges to be streets. The goal is to create a closed walk with as few edges repeated as possible.

- A closed walk that minimizes the number of repeated edges is an *Eulerian* walk.

Finding an Eulerian closed walk:

(1) Let $v_1, \dots, v_{2k}$ be all the odd vertices.

(2) Let $G' = G + v_1v_2 + v_3v_4 + \cdots + v_{2k-1}v_{2k}$.

(3) Every vertex in G' has even degree, so G' contains within it an Eulerian circuit C.

(4) C contains every edge $v_1v_2, \dots, v_{2k-1}v_{2k}$. Replace each edge v_iv_{i+1} in this set with a shortest path in G from v_i to v_{i+1}. This gives a closed walk in G that contains all edges.

(5) Do steps (1)–(4) for all possible pairings of the odd vertices, and choose the shortest walk.

We can do the same problem on a weighted graph, where each edge is assigned a weight in a real number. In this case, an Eulerian closed walk is a closed walk that contains all edges and minimizes the total weight.

Proof of Euler's Theorem

Let G be a connected graph such that every vertex has even degree.

Let $v \in V(G)$. Since G is connected, $d(v) \neq 0$. Therefore, $\exists$ an edge incident on v. Therefore, v is in some trail, meaning $\{T \mid \text{trail containing } v\} \neq \emptyset$, and is finite.

Pick T_0 containing v such that T_0 is non-extendible; i.e., $\nexists T'$ such that $T_0 \subset T'$.

Claim T_0 is a circuit.

Let $T_0 = (v_1, \dots, v_n)$. Suppose toward contradiction that $v_1 \neq v_n$. There must be an odd number of edges incident to v_1 in T_0, since, if $v_1v_2 \in T_0$, for each $1 < j < n$ such that $v_1 = v_j$, then $v_{j-1}v_j$ and v_jv_{j+1} are incident to v_1. But, $d(v_1)$ is even — so, $\exists e = v_1w \in E(G)$ such that $e \notin T_0$. Then, T_0 can be extended to $(w, v_1, \dots, v_n)$. $\perp$

Graph Decomposition

Let $a_1, \dots, a_n$ be distinct items. A **Steiner Triple System** on $a_1, \dots, a_n$ is a set S of triples $\{a, b, c\} \subseteq \{a_1, \dots, a_n\}$, such that every pair $\{a_i, a_j\}$, $i \neq j$ is a subset of exactly one element of S.

Example

$n = 4$, $a_1, \dots, a_4$ are distinct.

Let

$$S = \{\{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}\}$$

Is S a Steiner Triple System on $a_1, \dots, a_4$?

S is not a Steiner Triple System. $\{a_2, a_4\}$ is not a subset of any element of S, and $\{a_1, a_3\}$ appear in both elements.

Represent each a_i as a vertex, each pair a_ia_j as an edge, then a STS corresponds to K_n with $E(K_n)$

partitioned into 3-cycles.

Partitioning into 3-cycles

K_n can be decomposed into 3-cycles if and only if $n \equiv 1 \mod 6$ or $n \equiv 3 \mod 6$.

Decomposing into Trails

Let G be a connected graph with exactly four odd vertices. Show that G decomposes into two trails, T_1 and T_2. Furthermore, T_1 and T_2 contain exactly two of the odd vertices.

Let a, b, c, d be the odd vertices. Let $G' = G + ab + cd$. In G', every vertex has even degree, so G' has an Eulerian circuit C.

$$C = a\underbrace{b, \dots, c}_{T_1}\underbrace{d, \dots, a}_{T_2}$$

Trail Decomposition, Even Length

Find T_1' and T_2' such that T_1' and T_2' are of even length, given the same conditions as the previous problem.

We know that G has even size, and

$$|T_1| + |T_2| = |G|$$

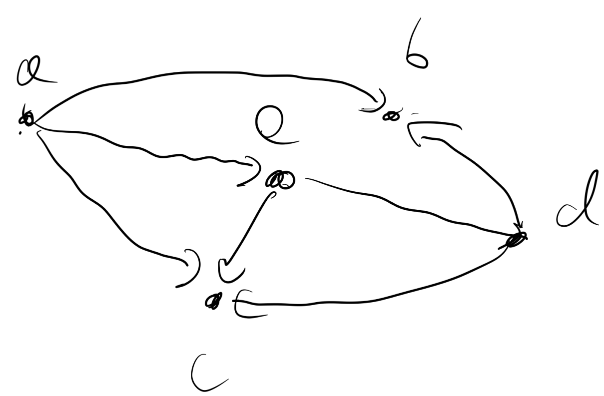
If T_1 and T_2 are of even length, then $T_1' = T_1$ and $T_2' = T_2$, and we are done.

Suppose T_1 and T_2 are both of odd length. If T_1 and T_2 do not share any vertices, $V(T_1) \sqcup V(T_2) = V(G)$, meaning G is disconnected.

Let $v \in V(T_1) \cap V(T_2)$. T_1 is from b to v to c, T_2 is from d to v to a. Let $T_i = R_i \cup S_i$.

Directed Graphs

A **directed graph** (also known as an oriented graph or digraph) is one which holds an arrow on every edge.



Definition 1: A directed graph is a pair (V, E), where $E \subseteq V \times V$. $(*)$

Definition 2: Let G be a graph. Let $f : E \to V \times V$ such that for each edge $e = \{v, w\} \in E(G)$, $f(e) = (v, w)$ or (w, v). Then, (G, f) is a directed graph.

A **tournament** is a directed complete graph.

A **directed path** in a directed graph G is a sequence of vertices $(v_1, \ldots, v_n)$ where $(v_i, v_{i+1}) \in E(G)$, where $i = 1, 2, \ldots, n-1$. A **directed cycle** is a directed path $(v_1, \ldots, v_n)$ such that $v_n = v_1$.

A directed graph is **strongly oriented** if $\forall v, w \in V(G)$, $\exists$ a directed path from v to w and a directed path from w to v.

If a graph is strongly connected, then G is connected and bridgeless.

Robin's Theorem

Every connected bridgeless graph has a strong orientation.

Theorem 9.2

Every tournament has a directed Hamiltonian path.

Embedding Graphs

A function $f : G \to \mathbb{R}^2$ is an embedding if

- f is injective.
- f is continuous: we need either a metric or a topology on G in order to define continuity.
- $f^{-1}\big|_{f(G)}$ is continuous: the domain restriction of $f^{-1} : \mathbb{R} \to G$ to $f^{-1}\big|_{f(G)} : f(G) \to \mathbb{R}$

Embedding Graphs in $\mathbb{R}^n$

Any graph admits an embedding in $\mathbb{R}^3$.

A graph is **planar** if it admits an embedding in $\mathbb{R}^2$.

Every planar graph has at most $3n - 6$ edges, where $n = |V(G)|$.

- Add edges such that every face is a triangle.
- Use Euler's formula: $V - E + F = 2$.

Subdivisions

A graph H is a subdivision of a graph G if H is obtained from G by replacing one or more edges vw by the path $v, v_1, \ldots, v_k, w$.

Non-Planar Graphs and Kuratowski's Theorem

- If G is non-planar, then any subdivision of G is also non-planar.
- $K_{3,3}$ and K_5 are non-planar.
- If H is a non-planar graph, and H is a subgraph of G, then G is non-planar.
- If G has a subgraph H such that H is a subdivision of K_5 or $K_{3,3}$, then H is non-planar, and thus

G is non-planar.

- If G is non-planar, then G has a subgraph H that is or is a subdivision of K_5 or $K_{3,3}$.

Graph Minors

A graph H is a minor of a graph G if H is obtained by deleting 0 or more vertices, deleting 0 or more edges, and contracting 0 or more edges.

- Let vw be an edge in a graph G. Then, H with vw contracted, denoted G/vw, is defined by: $V(H) = (V(G) \setminus \{v, w\}) \cup \{x\}$, and for $a, b \in V(H)$
  (i) $a, b \in \{v, w\},\ ab \in E(G)$
  (ii) $a = x,\ vb \in E(G)$ or $b = x,\ wb = E(G)$
- If vw is contracted, where $v \leftrightarrow u$ and $w \leftrightarrow u$, then uv and uw become ux

Wagner's Theorem

A graph G is non-planar if and only if K_5 or $K_{3,3}$ is a minor of G.

Graph Minor Theorem

Every infinite set of finite graphs contains distinct graphs such that one is a minor of the other.

$$\exists G_i, G_j \in \{G_1, G_2, \dots, \} \ni G_i \prec G_j$$

Minor Minimals

A graph G is minor minimal with respect to a given property P (i.e., in a given set of graphs) if G has property P but no minor of G has property P.

For example, K_5 and $K_{3,3}$ are minor minimal with respect to non-planarity. Therefore, any non-planar graph must have K_5 or $K_{3,3}$ as a minor.

Minor-Closed

A graph property P is minor-closed if for every G that has property P, every minor of G has property P.

For example, planarity is minor-closed.

Corollary to Graph Minor Theorem

Given any property P, there are a finite quantity of minor-minimal graphs with property P.

Let S be the set of all minor-minimal graphs with property P. Suppose S is not finite. Then, $\exists G, H \in S$ distinct such that $G \prec H$ by the graph minor theorem. Then, H is not minor-minimal.