Contents

Introduction

Finally, the last part of my notes on C*-algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of C*-algebras, discussing ideas such as amenability and nuclearity in C*-algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of $C^*_\lambda(G)$ and $C^*(G)$.

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's C*-*Algebras and Finite-Dimensional Approximations*.

Review: Representations, the Reduced Group C^*-Algebra, and the Universal Group C^*-Algebra

Left-Regular Representation

Let Γ be a group. Consider the space $\ell_2(\Gamma)$. For every $s \in \Gamma$, we define the operator

$$\lambda_s(\xi)(t) = \xi\left(s^{-1}t\right).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned} \|\lambda_s(\xi)\|^2 &= \sum_{t\in\Gamma} |\lambda_s(\xi)(t)|^2 \\ &= \sum_{t\in\Gamma} \left|\xi\left(s^{-1}t\right)\right|^2 \\ &= \sum_{r\in\Gamma} |\xi(r)|^2 \\ &= \|\xi\|^2. \end{aligned}$$

Additionally, each λ_s admits an inverse, $\lambda_{s^{-1}} = \lambda_s^*$. Applying to the orthonormal basis $\{\delta_t\}_{t\in\Gamma}$, we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus, $\lambda_s \circ \lambda_r = \lambda_{sr}$, and we have the unitary representation of Γ, $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, where $\lambda(s) = \lambda_s$, for $s \in \Gamma$. This is the left-regular representation of Γ.

Note that the left regular representation is a faithful representation, hence injective.

Because the λ operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}\lambda_f(\xi)(t) &= \left(\sum_{s\in\Gamma} f(t)\lambda_s(\xi)\right)(t)\\ &= \sum_{s\in\Gamma} f(s)\xi\left(s^{-1}t\right)\end{aligned}$$

Note that the space of finitely supported functions on Γ, $\mathbb{C}[\Gamma]$,[1] is a $*$-algebra, where multiplication is given by convolution:

$$\begin{aligned}f * g(t) &= \sum_{s\in\Gamma} f(s)g\left(s^{-1}t\right)\\ &= \sum_{r\in\Gamma} f\left(tr^{-1}\right)g(r).\end{aligned}$$

Note that we are using $*$ both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on $\mathbb{C}[\Gamma]$ is given by

$$f^*(t) = \overline{f\left(t^{-1}\right)}.$$

A Bit on Representations and C^*-(Semi)norms

A C*-seminorm on a $*$-algebra is a seminorm such that defined by

- $\|ab\| \leqslant \|a\|\|b\|$;
- $\|a^*\| = \|a\|$;
- $\|a^*a\| = \|a\|^2$.

If A_0 is a $*$-algebra, then a representation of A_0 is a pair $(\pi_0, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi\colon A_0 \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism.

Additionally, if A_0 is a $*$-algebra with representation π_0, then we have C*-seminorm

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{op}.$$

If π_0 is injective, then $\|\cdot\|_{\pi_0}$ is a C*-norm. If π_0 is a C*-norm, then the completion of A_0 with respect to $\|\cdot\|_{\pi_0}$ is a C*-algebra.

The universal norm on A_0 is defined as

$$\|a\|_u = \sup_{p\in\mathcal{P}} p(a),$$

where $\mathcal{P}$ is the collection of all C*-seminorms on A_0. If $\|a\|_u < \infty$ for all $a \in A_0$, then $\|\cdot\|_u$ is a C*-seminorm on A_0. Note that if one of $p \in \mathcal{P}$ is a norm, then $\|\cdot\|_u$ defines a C*-norm on A_0.

If we have the unitary representation $u\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$, then

$$\pi_u(a) = \sum_{s\in\Gamma} u_s$$

[1] Also known as the free vector space over $\mathbb{C}$ with basis Γ.

is a representation of $\mathbb{C}[\Gamma]$. If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ is the left-regular representation, then the left-regular group C*-algebra is the group $*$-algebra with C*-norm defined by $\|a\| = \|\pi_\lambda(a)\|$.

The universal group C*-algebra is defined as the norm completion of

$$\|a\|_{\max} = \sup\left\{\|\pi(a)\|_{op} \;\middle|\; \pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H}_\pi) \text{ is a representation}\right\}.$$

Note that

$$\begin{aligned}\|\pi(a)\| &= \left\|\pi\left(\sum_{s\in\Gamma} a_s\delta_s\right)\right\| \\ &= \left\|\sum_{s\in\Gamma} a_s\pi(\delta_s)\right\| \\ &\leqslant \sum_{s\in\Gamma}\|a_s\pi(\delta_s)\| \\ &= \sum_{s\in\Gamma}|a_s|.\end{aligned}$$

Note that since $\|\cdot\|_\lambda$ is a norm, we must have $a = 0$ if and only if $\|a\|_{\max} = 0$. The full group C*-algebra admits a universal property.

Proposition: Let Γ be a discrete group. If $u\colon \Gamma \to \mathbb{B}(\mathcal{H})$, then there is a contractive $*$-homomorphism $\pi_u\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ that satisfies $\pi_u(\delta_s) = u(s)$.

Using the Left-Regular Representation to Establish Amenability

If $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\mathcal{H}$, then a vector $\xi \in \mathcal{H}$ is called invariant for π if $\pi(g)(\xi) = \xi$ for all $g \in \Gamma$.

Proposition: The left-regular representation for Γ admits an invariant vector if and only if Γ is finite.

Proof. Let Γ be finite. Since Γ is finite, all functions $a\colon \Gamma \to \mathbb{C}$ are square-summable. Thus, $\xi = \mathbb{1}_\Gamma$ is square-summable, and since $s\Gamma = \Gamma$ for all $s \in \Gamma$, we have $\mathbb{1}_\Gamma$ is invariant for λ.

Now, let $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ be the left-regular representation, and suppose there is $\xi \in \ell_2(\Gamma)$ such that for all $s \in \Gamma$, we have

$$\lambda_s(\xi) = \xi.$$

In particular, this means that for any $t \in \Gamma$, we have

$$\begin{aligned}\lambda_s(\xi)(t) &= \xi\left(s^{-1}t\right) \\ &= \xi(t).\end{aligned}$$

Since this holds for all $s \in \Gamma$, we have that $\xi = c\mathbb{1}_\Gamma$ for some $c \in \mathbb{C}$. However, since $\xi \in \ell_2(\Gamma)$, we must have that $\sum_{t\in\Gamma}|c|^2 < \infty$, which only holds if Γ is finite. □

An almost-invariant vector for a representation $\pi\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, as the name suggests,[II] a sequence (or net) of unit vectors $(\xi_i)_{i\in I}$ such that

$$\lim_{i\in I}\|\pi(g)(\xi_i) - \xi_i\| = 0.$$

[II] I'm only mostly being facetious here.

Theorem: A group Γ is amenable if and only if the left-regular representation has an almost-invariant vector.

Proof. Let Γ be amenable, and let F_i be a Følner sequence, where $\frac{|sF_i \triangle F_i|}{|F_i|} \to 0$ for all $s \in \Gamma$.

Define $\xi_i = \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}$. Then,

$$\begin{aligned}
\|\lambda_s(\xi_i) - \xi_i\|^2 &= \sum_{t\in\Gamma} |\lambda_s(\xi_i)(t) - \xi_i(t)|^2 \\
&= \sum_{t\in\Gamma} \left|\lambda_s\left(\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right)(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right|^2 \\
&= \sum_{t\in\Gamma} \left|\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t)\right|^2 \\
&= \frac{|sF_i \triangle F_i|}{|F_i|}.
\end{aligned}$$

Thus, λ has an almost-invariant vector.

Suppose there exists an almost-invariant vector $(\xi_i)_i \in \ell_2(\Gamma)$. It is sufficient to construct an approximate mean. Since $\xi_i \in \ell_2(\Gamma)$, we have that $\xi_i^2 \in \ell_1(\Gamma)$. Setting $\mu_i = \xi_i^2$, we plug this into the expression for an approximate mean, and obtain

$$\begin{aligned}
\|\lambda_s(u_i) - u_i\|_{\ell_1} &= \sum_{t\in\Gamma} \left|\lambda_s\left(\xi_i^2\right)(t) - \xi_i^2(t)\right| \\
&= \sum_{t\in\Gamma} |(\lambda_s(\xi_i)(t) - \xi_i(t))(\lambda_s(\xi_i)(t) + \xi_i(t))| \\
&= \|(\lambda_s(\xi_i) - \xi_i)(\lambda_s(\xi_i) + \xi_i)\|_{\ell_1} \\
&\leqslant \|\lambda_s(\xi_i) - \xi_i\|_{\ell_2} \|\lambda_s(\xi_i) + \xi_i\| \\
&\leqslant 2\|\lambda_s(\xi_i) - \xi_i\| \\
&\to 0.
\end{aligned}$$

Thus, μ_i is an approximate mean. □

Using the criterion of almost invariant vectors, we may show that a group is amenable if and only if the trivial representation — defined by $1_\Gamma \colon \Gamma \to \mathbb{C}$, $1_\Gamma(g) = 1$ is what is known as weakly contained in the left-regular representation.

A representation $\pi \colon \Gamma \to \mathcal{U}(\mathcal{H})$ is weakly contained in another representation $\rho \colon \Gamma \to \mathcal{U}(\mathcal{H})$, denoted $\pi \prec \rho$, if for every $\xi \in \mathcal{H}$, finite $E \subseteq \Gamma$, and $\varepsilon > 0$, then there are $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that

$$\left|\langle \pi(g)(\xi), \xi\rangle - \sum_{i=1}^{n} \langle \rho(g)(\eta_i), \eta_i\rangle\right| < \varepsilon.$$

Theorem: A discrete group Γ is amenable if and only if $1_\Gamma \prec \lambda$, where λ is the left-regular representation.

Proof. We show that $1_\Gamma \prec \lambda$ is equivalent to the existence of an almost invariant vector for λ. We assume λ admits an almost-invariant vector. It is sufficient to show that for every $\varepsilon > 0$ and every finite set $E \subseteq \Gamma$, there are $\eta_1, \dots, \eta_n \in \ell_2(\Gamma)$ such that

$$\left|1 - \sum_{i=1}^{n} \langle \lambda_t(\eta_i), \eta_i\rangle\right| < \varepsilon$$

for every $t \in E$. If we take $n = 1$ and $\eta_1 = \xi$, where ξ is almost-invariant for all $g \in E$ — i.e., $\left\|\lambda_g(\xi) - \xi\right\|_{\ell_2} < \varepsilon$ for all $g \in E$. Note that we have

$$\begin{aligned}\left\|\lambda_g(\xi) - \xi\right\|^2 &= \left\langle \lambda_g(\xi) - \xi, \lambda_g(\xi) - \xi\right\rangle \\ &= \left\langle \lambda_g(\xi), \lambda_g(\xi)\right\rangle + \langle \xi, \xi\rangle - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &= 2 - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &= 2\operatorname{Re}\left(1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &\leqslant 2\left|1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right|.\end{aligned}$$

Additionally,

$$\begin{aligned}\left|1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right|^2 &= \left(1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right)\left(1 - \overline{\left\langle \lambda_g(\xi), \xi\right\rangle}\right) \\ &= 1 - \overline{\left\langle \lambda_g(\xi), \xi\right\rangle} - \left\langle \lambda_g(\xi), \xi\right\rangle + \left|\left\langle \lambda_g(\xi), \xi\right\rangle\right|^2 \\ &\leqslant 2 - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &= \left\|\lambda_g(\xi) - \xi\right\|^2.\end{aligned}$$

Thus, we have that

$$\begin{aligned}\left|1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right| &\leqslant \left\|\lambda_g(\xi) - \xi\right\| \\ &< \varepsilon.\end{aligned}$$

We start by showing that $1_\Gamma \prec \lambda$ if and only if for every finite $S \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\|\lambda_s(\xi) - \xi\|_{\ell_2} < \varepsilon.$$

In the forward direction, we see that there exists a unit vector ξ such that $|1 - \langle \lambda_s(\xi), \xi\rangle| < \varepsilon^2/2$, meaning $\|\lambda_s(\xi) - \xi\| < \varepsilon$ by above. Similarly, if $\|\lambda_s(\xi) - \xi\| < \varepsilon$, then $1_\Gamma \prec \lambda$.

Now, we assume $1_\Gamma \prec \lambda$. Thus, for a finite $E \subseteq \Gamma$ and $\varepsilon > 0$, then there exists $f \in \ell_2(\Gamma)$ with $\|f\|_{\ell_2} = 1$ such that $\|\lambda_s(f) - f\| < \varepsilon$ for all $s \in E$.

Setting $g = |f|^2$, we have $g \in \ell_1(\Gamma)$. From Hölder's inequality, we have

$$\begin{aligned}\|\lambda_s(g) - g\|_{\ell_1} &\leqslant \left\|\lambda_s\left(\overline{f}\right) + \overline{f}\right\|_{\ell_2} \|\lambda_s(f) - f\| \\ &\leqslant 2\|\lambda_s(f) - f\|_{\ell_2} \\ &< 2\varepsilon.\end{aligned}$$

Thus, Γ admits an approximate mean, hence is amenable. □

Having obtained some more resources on Kesten's criterion, we can now prove that.

Definition. Let $\lambda\colon \Gamma \to \mathbb{B}(\ell_2(\Gamma))$ be the left-regular representation. Then, for a finite set $E \subseteq \Gamma$, we define the Markov operator $M(E)$ by

$$M(E) = \sum_{t \in E} \lambda_t.$$

Note that since λ_t is an isometry for each t, we have

$$\|M(E)\|_{op} = \left\|\frac{1}{|E|}\sum_{t \in E} \lambda_t\right\|_{op}$$

$$\begin{aligned}
&= \frac{1}{|E|}\left\|\sum_{t\in E}\lambda_t\right\|_{op}\\
&\leqslant \frac{1}{|E|}\sum_{t\in E}\|\lambda_t\|_{op}\\
&= 1,
\end{aligned}$$

so the Markov operator is a bounded operator (indeed, a contraction).

Theorem (Kesten's Criterion)**:** Let Γ contain a finite symmetric generating set S. Then, Γ is amenable if and only if

$$\|M(S)\|_{op} = 1.$$

Proof. Let Γ be amenable. Then, λ admits an almost-invariant vector, $(\xi_n)_n \subseteq S_{\ell_2(\Gamma)}$, such that

$$\|\lambda_s(\xi_n) - \xi_n\|_{\ell_2} \to 0$$

for all $s \in \Gamma$. In particular, we have

$$\begin{aligned}
\left|\left(\left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n)\right\|_{\ell_2}\right) - \|\xi_n\|_{\ell_2}\right| &\leqslant \left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n) - \xi_n\right\|_{\ell_2}\\
&= \frac{1}{|S|}\left\|\left(\sum_{t\in S}\lambda_t\right)(\xi_n) - |S|\xi_n\right\|_{\ell_2}\\
&\leqslant \frac{1}{|S|}\sum_{t\in S}\|\lambda_t(\xi_n) - \xi_n\|_{\ell_2}\\
&\to 0,
\end{aligned}$$

meaning that

$$\sup_{\xi\in S_{\ell_2(\Gamma)}}\left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi)\right\| = \|\xi\|,$$

and so the norm of the Markov operator is 1.

Suppose

$$\left\|\frac{1}{|S|}\sum_{t\in S}\lambda_t\right\|_{op} = 1,$$

or

$$\left\|\sum_{t\in S}\lambda_t\right\|_{op} = |S|.$$

Proposition: If $T \in \mathbb{B}(\mathcal{H})$ is a self-adjoint operator, then

$$\|T\|_{op} = \sup_{x\in S_{\mathcal{H}}}|\langle T(x), x\rangle|.$$

Proof. We have that

$$|\langle T(x), x\rangle| \leqslant \|T(x)\|\|x\|$$

$$\begin{aligned} &\leqslant \|T\|_{op}\|x\|^2 \\ &= \|T\|_{op}. \end{aligned}$$

Now, we seek to establish the opposite direction. Note that since T is self-adjoint, we know that $\langle T(x), x\rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$, so by the polarization identity, we have that

$$\langle T(x), y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle).$$

Note that we know that

$$\|T\|_{op} = \sup_{x,y\in S_{\mathcal{H}}} |\langle T(x), y\rangle|.$$

Now, we set $\alpha = \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|$. Note that for any nonzero $x \in \mathcal{H}$, we have

$$\begin{aligned} \left|\left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle\right| &\leqslant \alpha \\ |\langle T(x), x\rangle| &\leqslant \alpha\|x\|^2. \end{aligned}$$

Now, for any $x, y \in \mathcal{H}$, we may assume that $\langle T(x), y\rangle \in \mathbb{R}$, as we may multiply $\langle T(x), y\rangle$ by $\operatorname{sgn}(\langle T(x), y\rangle)$. Thus, by the polarization identity and the fact that T is self-adjoint, we have

$$\begin{aligned} \langle T(x), y\rangle &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle) \\ |\langle T(x), y\rangle| &= \left|\frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)\right| \\ &\leqslant \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|) \\ &\leqslant \frac{\alpha}{4}\left(\|x+y\|^2 + \|x-y\|^2\right) \\ &= \frac{\alpha}{4}\left(2\|x\|^2 + 2\|y\|^2\right) \\ &= \alpha. \end{aligned}$$

Thus, we have $\|T\|_{op} \leqslant \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|$. □

Now, since S is symmetric, we have that $M(S)$ is self-adjoint. Therefore, we know that there is some $\xi_n \in S_{\mathcal{H}}$ such that

$$\begin{aligned} 1 - \frac{1}{n} &< \left\langle \left(\frac{1}{|S|}\sum_{t\in S} \lambda_t\right)(\xi_n), \xi_n \right\rangle \\ &\leqslant \left\langle \left(\frac{1}{|S|}\sum_{t\in S} \lambda_t\right)(|\xi_n|), |\xi_n| \right\rangle. \end{aligned}$$

Thus, rearranging, we have

$$1 - \left\langle \left(\frac{1}{|S|}\sum_{t\in S} \lambda_t\right)(|\xi_n|), |\xi_n| \right\rangle < \frac{1}{n}.$$

Since $M(S)$ is a self-adjoint operator, we have that $\operatorname{Re}\left(\left\langle \left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n), \xi_n\right\rangle\right) = \left\langle \left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n), \xi_n\right\rangle$. This gives

$$\left\|\left(\frac{1}{S}\sum_{t\in S}\lambda_t\right)(\xi) - \xi\right\| \leqslant \frac{1}{|S|}\sum_{t\in S}\|\lambda_t(\xi) - \xi\|$$

$$\begin{aligned}
&\leqslant \frac{1}{|S|}\sum_{t\in S}\sqrt{2}|1-\langle \lambda_t(\xi),\xi\rangle|\\
&= \sqrt{2}\left|1-\frac{1}{|S|}\sum_{t\in S}\langle \lambda_t(\xi),\xi\rangle\right|\\
&\to 0.
\end{aligned}$$

Thus, λ admits an almost-invariant vector. □

Next, we turn to Hulanicki's Criterion.

Definition. Let $f \in \ell_1(\Gamma)$. Then, we define the bounded operator

$$\lambda_{f(t)} = \sum_{t\in\Gamma} f(t)\lambda_t.$$

Theorem: If Γ is a discrete group, then Γ is amenable if and only if for every positive finitely-supported $f\colon \Gamma \to \mathbb{C}$, we have

$$\sum f(t) \leqslant \left\|\lambda_{f(t)}\right\|_{\text{op}}.$$

Proof. Suppose Γ is amenable. Let $f \geqslant 0$ be a finitely supported function, and let $(F_n)_n$ be a Følner sequence such that for every $g \in \operatorname{supp}(f)$, we have

$$\frac{|gF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}.$$

Let $\xi_n = \frac{1}{\sqrt{|F_n|}}\mathbb{1}_{F_n}$. Note that $\|\xi_n\|_{\ell_2} = 1$.

We will use the fact that

$$\sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle| \leqslant \|T\|_{\text{op}}.$$

We see that

$$\begin{aligned}
\left|\left\langle\left(\sum_{t\in\Gamma} f(t)\lambda_t\right)(\xi_n),\xi_n\right\rangle\right| &= \left|\sum_{t\in\Gamma} f(t)\langle\lambda_t(\xi_n),\xi_n\rangle\right|\\
&= \left|\sum_{t,s\in\Gamma} f(t)\xi_n\left(t^{-1}s\right)\xi_n(s)\right|\\
&\leqslant \left\|\lambda_{f(t)}\right\|,
\end{aligned}$$

meaning

$$\lim_{n\to\infty}\left|\left\langle\left(\sum_{t\in\Gamma} f(t)\lambda_t\right)(\xi_n),\xi_n\right\rangle\right| \leqslant \left\|\lambda_{f(t)}\right\|.$$

Notice that ξ_n is an almost-invariant vector for λ, meaning that $\xi_n\left(t^{-1}s\right) \to \xi_n(s)$. Therefore, this means

$$\begin{aligned}
\lim_{n\to\infty}\left|\sum_{t,s\in\Gamma} f(t)\xi_n\left(t^{-1}s\right)\xi_n(s)\right| &= \lim_{n\to\infty}\left|\sum_{t,s\in\Gamma} f(t)|\xi_n(s)|^2\right|\\
&= \sum_{t\in\Gamma} f(t)\left|\sum_{s\in\Gamma}|\xi_n(s)|^2\right|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t\in\Gamma} f(t)\\
&\leqslant \left\|\lambda_{f(t)}\right\|_{op}.
\end{aligned}
$$

This establishes that there is some $C > 0$ such that

$$\sum_{t\in\Gamma} f(t) \leqslant C\left\|\lambda_{f(t)}\right\|_{op}.$$

To show that $C = 1$, we note that, by the definition of convolution, we must have

$$\left(\sum_{t\in\Gamma} f(t)\right)^{n} = \sum_{t\in\Gamma} (f * \cdots * f)(t),$$

and

$$
\begin{aligned}
\left(\lambda_{f(t)}\right)^{n} &= \left(\sum_{t\in\Gamma} f(t)\lambda)_{t}\right)^{n}\\
&= \sum_{t\in\Gamma} (f * \cdots * f)(t)\lambda_{t}\\
&= \lambda_{(f*\cdots*f)(t)}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\left(\sum_{t\in\Gamma} f(t)\right)^{n} &= \sum_{t\in\Gamma} (f * \cdots * f)(t)\\
&\leqslant C\left\|\lambda_{(f*\cdots*f)(t)}\right\|\\
&= C\left(\left\|\lambda_{f(t)}\right\|_{op}\right)^{n}.
\end{aligned}
$$

This means we have

$$\sum_{t\in\Gamma} f(t) \leqslant C^{1/n}\left\|\lambda_{f(t)}\right\|_{op}.$$

Since n is arbitrary, this means $C = 1$.

Now, if for all finitely supported f, we have

$$\sum_{t\in\Gamma} f(t) \leqslant \left\|\lambda_{f(t)}\right\|_{op}.$$

If $f = \mathbb{1}_{E}$ for some finite $E \subseteq \Gamma$, we see that

$$\left\|\sum_{t\in E} \lambda_{t}\right\|_{op} = |E|,$$

so by Kesten's criterion, we have that Γ is amenable. □

Completely [Property] Maps

We begin this section with an overview of positive maps, completely positive maps, and extensions. These will be useful for understanding the theorem that a group is amenable if and only if the reduced group

C*-algebra is nuclear. The ultimate goal here is to prove Arveson's extension theorem (i.e., that $\mathbb{B}(\mathcal{H})$ is injective with respect to completely positive maps). The primary text for this purpose will be Vern Paulsen's *Completely Bounded Maps and Operator Algebras*.

The idea behind completely positive maps is that they are positive when subjected to a certain amplification process related to the matrix algebras.

Definition. An element of a C*-algebra is positive if and only if it is self-adjoint and its spectrum is contained in the nonnegative reals. Alternatively, $b \in A$ is of the form $b = a^*a$ for some $a \in A$.

To introduce a norm such that $\text{Mat}_n(A)$ becomes a C*-algebra, we begin with the most basic C*-algebra, $\mathbb{B}(\mathcal{H})$, and consider the n-fold amplification of $\mathcal{H}$, $\mathcal{H}^{(n)}$. This is a Hilbert space equipped with inner product

$$\left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} \langle h_j, k_j \rangle.$$

Meanwhile, we may consider $\text{Mat}_n(\mathbb{B}(\mathcal{H}))$ as a linear map on $\mathcal{H}^{(n)}$, by taking

$$\left(T_{ij}\right)_{ij} = \begin{pmatrix} \sum_{j=1}^{n} T_{1j}(h_j) \\ \vdots \\ \sum_{j=1}^{n} T_{nj}(h_j) \end{pmatrix}.$$

This yields a $*$-isomorphism between $\text{Mat}_n(\mathbb{B}(\mathcal{H}))$ and $\mathbb{B}\left(\mathcal{H}^{(n)}\right)$.

Given any C*-algebra A, we may theorize $\text{Mat}_n(A)$ by first isometrically representing $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, letting A be a C*-subalgebra of $\mathbb{B}(\mathcal{H})$, and then identifying $\text{Mat}_n(A)$ as a $*$-subalgebra of $\text{Mat}_n(\mathbb{B}(\mathcal{H}))$.

Using a faithful $*$-representation of A, we now have a way to turn $\text{Mat}_n(A)$ into a C*-algebra. However, since the norm is unique on a C*-algebra, the norm on $\text{Mat}_n(A)$ defined in this fashion is independent of the representation of A that we choose. Furthermore, since $*$-isomorphisms are positive maps, the positive elements of $\text{Mat}_n(A)$ are uniquely determined. This means that every C*-algebra carries with it a set of canonically defined norms and orders on each $\text{Mat}_n(A)$.

For example, consider $\text{Mat}_k(\mathbb{C})$, which can be identified with $\mathcal{L}\left(\mathbb{C}^k\right)$. We identify $\text{Mat}_n(\text{Mat}_k(\mathbb{C})) \cong \text{Mat}_{nk}(\mathbb{C})$. With this identification, the usual multiplication and involution on $\text{Mat}_n(\text{Mat}_k(\mathbb{C}))$ become multiplication and involution on $\text{Mat}_{nk}(\mathbb{C})$.

Now, let X be a compact Hausdorff space, and let $C(X)$ be the C*-algebra of continuous functions with $f^*(x) = \overline{f(x)}$, equipped with the norm $\|f\| = \sup_{x\in X}|f(x)|$. Then, an element $F = \left(f_{ij}\right)_{ij}$ of $\text{Mat}_n(C(X))$ can be considered as a continuous $\text{Mat}_n(\mathbb{C})$-valued function. Addition, multiplication, and involution in $\text{Mat}_n(C(X))$ are pointwise. Recalling that the norm on $\text{Mat}_n(C(X))$ is unique, we may let $\|F\| = \sup_{x\in X}\|F(x)\|$, where the latter norm is the canonical matrix norm on $\text{Mat}_n(C(X))$. The positive elements of $\text{Mat}_n(C(X))$ are those F for which $F(x)$ is a positive matrix for all x.

Now, given two C*-algebras A and B and a map $\phi \colon A \to B$, there are maps $\phi_n \colon \text{Mat}_n(A) \to \text{Mat}_n(B)$, given by

$$\phi_n\left(\left(a_{ij}\right)_{ij}\right) = \left(\phi\left(a_{ij}\right)\right)_{ij}.$$

In general, when we say that ϕ is completely [property], then we say that all the ϕ_n have that property. For instance, if ϕ is positive, in that it maps positive elements of A to positive elements of B, then we say

ϕ is completely positive if ϕ_n is a positive map for each n, where the positive elements of $\mathrm{Mat}_n(A)$ and $\mathrm{Mat}_n(B)$ are defined canonically.

Unfortunately, it's not always the case that (e.g.) positive maps are completely positive, or even that $\|\phi_n\|_{op} = \|\phi\|_{op}$ for each n.

There is an isomorphism between $\mathrm{Mat}_n(A)$ and the tensor product $\mathrm{Mat}_n(\mathbb{C}) \otimes A$. We detail it here. The proof is from Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*.

Theorem: Let A be an algebra, and let $\mathrm{Mat}_n(A)$ denote the matrix algebra of A. Then, there is a $*$-isomorphism

$$\mathrm{Mat}_n(A) \cong \mathrm{Mat}_n(\mathbb{C}) \otimes A.$$

Proof. Define $\varphi\colon \mathrm{Mat}_n(A) \to \mathrm{Mat}_n(\mathbb{C}) \otimes A$ by

$$\varphi\left(\left(a_{ij}\right)_{ij}\right) = \sum_{i,j=1}^{n} e_{ij} \otimes x_{ij}.$$

Recall that if A and B are two algebras, multiplication in $A \otimes B$ is defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

and if A and B are $*$-algebras, then the involution is defined by

$$(a \otimes b)^* = a^* \otimes b^*.$$

We start by showing that $\mathrm{Mat}_n(A) \cong \mathrm{Mat}_n(\mathbb{C}) \otimes A$ as vector spaces. By the definition of the tensor product, the map φ is linear.

Now, suppose

$$\begin{aligned}\varphi\left(\left(a_{ij}\right)_{ij}\right) &= \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij}\\ &= 0.\end{aligned}$$

Then, since $\left\{e_{ij}\right\}_{ij}$ is linearly independent, we know that $x_{ij} = 0$ for all i, j, so $\left(a_{ij}\right)_{ij} = 0$, so φ is injective.

Now, let $t \in \mathrm{Mat}_n(\mathbb{C}) \otimes A$ be given by

$$t = \sum_{k} m_k \otimes a_k,$$

where $m_k \in \mathrm{Mat}_n(\mathbb{C})$ and $a_k \in A$. Then, using the matrix units, we write each m_k as

$$m_k = \sum_{i,j=1}^{n} m_k(i,j) e_{ij}.$$

This gives

$$\begin{aligned}t &= \sum_{k}\left(\sum_{i,j=1}^{n} m_k(i,j)e_{ij}\right) \otimes a_k\\ &= \sum_{i,j=1}^{n} e_{ij} \otimes \left(\sum_{k} m_k(i,j)a_k\right).\end{aligned}$$

Defining $a_{ij} \coloneqq \sum_k m_k(i,j)a_k$, we get

$$t = \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij},$$

meaning that

$$\varphi\left(\left(x_{ij}\right)_{ij}\right) = t.$$

Thus, φ is surjective.

We will show now that φ is multiplicative and $*$-preserving. If $\left(a_{ij}\right)_{ij}$ and $\left(b_{ij}\right)_{ij}$ belong to $\mathrm{Mat}_n(A)$.

$$\begin{aligned}
\varphi((a_{ik})_{ik})\varphi\left(\left(b_{lj}\right)_{lj}\right) &= \left(\sum_{i,k=1}^{n} e_{ik} \otimes a_{ik}\right)\left(\sum_{l,j=1}^{n} e_{lj} \otimes b_{lj}\right)\\
&= \sum_{i,j,k,l=1}^{n} (e_{ik} \otimes a_{ik})\left(e_{lj} \otimes b_{lj}\right)\\
&= \sum_{i,j,k,l=1}^{n} e_{ik}e_{lj} \otimes a_{ik}b_{lj}\\
&= \sum_{i,j,k=1}^{n} e_{ik}e_{kj} \otimes a_{ik}b_{kj}\\
&= \sum_{ij,k=1}^{n} e_{ij} \otimes a_{ik}b_{kj}\\
&= \sum_{i,j=1}^{n} e_{ij} \otimes \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)\\
&= \varphi\left(\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{ij}\right)\\
&= \varphi\left(\left(a_{ij}\right)_{ij}\left(b_{ij}\right)_{ij}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi\left(\left(a_{ij}\right)_{ij}\right)^* &= \left(\sum_{i=1}^{n} e_{ij} \otimes a_{ij}\right)^*\\
&= \sum_{i,j=1}^{n} \left(e_{ij} \otimes a_{ij}\right)^*\\
&= \sum_{i,j=1}^{n} e_{ij}^* \otimes a_{ij}^*\\
&= \sum_{i,j=1}^{n} e_{ji} \otimes a_{ij}^*\\
&= \sum_{i,j=1}^{n} e_{ij} \otimes a_{ji}^*
\end{aligned}$$

$$\begin{aligned} &= \varphi\left(\left(a_{ji}^*\right)_{ij}\right)\\ &= \varphi\left(\left(a_{ij}\right)_{ij}^*\right). \end{aligned}$$

□

There are lots of useful results using amplification to the matrix algebras.

Example (Dilating an Isometry)**.** Let V be an isometry, and let $P = I_{\mathcal{H}} - VV^*$ be the projection onto $\operatorname{Ran}(V)^{\perp}$. Define U on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ by

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix}.$$

We find that

$$\begin{aligned} U^* &= \begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix}\\ UU^* &= \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix}\begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix}\\ &= \begin{pmatrix} VV^* + P & PV \\ V^*P & V^*V \end{pmatrix}\\ &= \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{H}} \end{pmatrix}\\ &= I_{\mathcal{K}}\\ U^*U &= \begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix}\begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix}\\ &= I_{\mathcal{K}}. \end{aligned}$$

Thus, U is a unitary on $\mathcal{K}$. We may identify $\mathcal{H} \cong \mathcal{H} \oplus 0$, and take

$$V^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$$

for all $n \geqslant 0$. Thus, we are able to realize any isometry V as the restriction of some unitary to a subspace that respects powers.

Example (Dilating a Contraction)**.** Similarly, we may define the isometric dilation of a contraction. Let T be an operator on $\mathcal{H}$ with $\|T\| \leqslant 1$, and define $D_T = (I - T^*T)^{1/2}$. We see that

$$\begin{aligned} \|T(h)\|^2 + \|D_T(h)\|^2 &= \langle T^*T(h), h\rangle + \left\langle D_T^2(h), h\right\rangle\\ &= \|h\|^2. \end{aligned}$$

We consider now the sequence space

$$\ell_2(\mathcal{H}) = \left\{(h_n)_{n\in\mathbb{N}} \;\middle|\; h_n \in \mathcal{H}, \sum_{n=1}^{\infty} \|h_n\|^2 < \infty\right\}.$$

We have the norm

$$\|(h_n)_n\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2$$

and the inner product

$$\langle (h_n)_n, (k_n)_n \rangle = \sum_{n=1}^{\infty} \langle h_n, k_n \rangle.$$

We define the operator $V: \ell_2(\mathcal{H}) \to \ell_2(\mathcal{H})$ by

$$V((h_n)_n) = (T(h_1), D_T(h_1), h_2, \dots).$$

It then follows that V is an isometry on $\ell_2(\mathcal{H})$, and that if we identify $\mathcal{H} \cong \mathcal{H} \oplus 0 \oplus \cdots$, then $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$.

Theorem (Sz.-Nagy's Dilation Theorem)**:** Let T be a contraction operator on $\mathcal{H}$. There is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and a unitary operator U on $\mathcal{K}$ such that $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$.

Proof. Take $\mathcal{K} = \ell_2(\mathcal{H}) \oplus \ell_2(\mathcal{H})$, and identify $\mathcal{H}$ as $(\mathcal{H} \oplus 0 \oplus \cdots) \oplus 0$. Let V be the isometric dilation of T on $\ell_2(\mathcal{H})$, and let U be the unitary dilation of V on $\ell_2(\mathcal{H}) \oplus \ell_2(\mathcal{H})$. Then, since $\mathcal{H} \subseteq \ell_2(\mathcal{H}) \oplus 0$, we have that $P_{\mathcal{H}} U^n|_{\mathcal{H}} = P_{\mathcal{H}} V^n|_{\mathcal{H}} = T^n$ for all $n \geqslant 0$. □

Whenever Y is an operator on $\mathcal{K}$, $\mathcal{H}$ a (closed) subspace of $\mathcal{K}$, and $X = P_{\mathcal{H}} Y|_{\mathcal{H}}$, then we say X is a compression of Y.

Corollary (Von Neumann's Inequality)**:** Let T be a contraction on a Hilbert space. Then, for any polynomial p,

$$\|p(T)\| \leqslant \sup_{|z| \leqslant 1} |p(z)|.$$

Proof. Let U be a unitary dilation of T. Since $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$, linearity means we have $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$. Since U is defined on a larger space than T, then $\|p(T)\| \leqslant \|p(U)\|$. Furthermore, since unitaries are normal, we have

$$\|p(U)\| = \sup_{\lambda \in \sigma(U)} |p(\lambda)|,$$

where $\sigma(U)$ is the spectrum of U. Since U is unitary, $\sigma(U) \subseteq \mathbb{T}$, so von Neumann's inequality follows. □

Positive and Completely Positive Maps

Positive Maps

There are certain results on positive maps that are useful in the study of completely positive maps. We introduce them here.

Definition. If S is a subset of a C*-algebra A, we say S is an operator system if A is unital and S is a self-adjoint sub*space* of A with $1_A \in S$.

Note that if S is an operator system and $h \in S$ is self-adjoint, then though the values h_+ and h_-, defined by the continuous functional calculus with

$$\begin{aligned} f^+(x) &= \max\{0, x\} \\ f^-(x) &= \min_{0,-x} \end{aligned}$$

may not belong to S, we can write h as the difference of two positive elements in s by

$$h = \frac{1}{2}(\|h\| 1_A + h) - \frac{1}{2}(\|h\| 1_A - h).$$

Definition. If S is an operator system, B is a C*-algebra, and $\phi\colon S \to B$ is a linear map, then ϕ is called positive if it maps positive elements of S to positive elements of B.

Theorem: If ϕ is a positive linear functional on an operator system S, then $\|\phi\| = \phi(1_A)$.

When the range of ϕ is not $\mathbb{C}$, but rather a C*-algebra, then the situation is a bit different.

Proposition: Let S be an operator system, and let B be a C*-algebra. If $\phi\colon S \to B$ is a positive map, then ϕ is bounded, with

$$\|\phi\| \leqslant 2\|\phi(1_A)\|.$$

Proof. Note that if p is positive, then $0 \leqslant p \leqslant \|p\|1_A$, so $0 \leqslant \phi(p) \leqslant \|p\|\phi(1_A)$ since positive functions are order-preserving. Thus, we get $\|\phi(p)\| \leqslant \|p\|\|\phi(1)\|$ when $p \geqslant 0$.

Note that when p_1 and p_2 are positive, then $\|p_1 - p_2\| \leqslant \max\{\|p_1\|, \|p_2\|\}$. If h is self-adjoint, then we have

$$\|\phi(h)\| = \frac{1}{2}\phi(\|h\|1_A + h) - \frac{1}{2}\phi(\|h\|1_A - h),$$

which is the difference of two positive elements in B. Thus, we have

$$\begin{aligned}\|\phi(h)\| &\leqslant \frac{1}{2}\max\{\|\phi(\|h\|1_A + h)\|, \|\phi(\|h\|1_A - h)\|\}\\ &\leqslant \|h\|\|\phi(1)\|.\end{aligned}$$

Finally, if a is arbitrary then write $a = h + ik$ via the Cartesian decomposition, where $\|h\|, \|k\| \leqslant \|a\|$, and h, k are self-adjoint. Thus, we have

$$\begin{aligned}\|\phi(a)\| &\leqslant \|\phi(h)\| + \|\phi(k)\|\\ &\leqslant 2\|a\|\|\phi(1_A)\|.\end{aligned}$$

□

As it turns out, 2 is the best constant.

Example. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and $C(\mathbb{T})$ be the continuous functions on z. Let z be the cordinate function, and let $S \subseteq C(\mathbb{T})$ be the subspace spanned by $1, z, \overline{z}$. Defining

$$\phi(a + bz + c\overline{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix},$$

An element of S is positive if and only if $c = \overline{b}$ and $a \geqslant 2|b|$, and an element of $\text{Mat}_2(\mathbb{C})$ is positive if and only if its diagonal entries and determinant are nonnegative real numbers. Thus, it is the case that ϕ is a positive map, but also

$$\begin{aligned}2\|\phi(1)\| &= 2\\ &= \|\phi(z)\|\\ &\leqslant \|\phi\|,\end{aligned}$$

meaning $\|\phi\| = 2\|\phi(1)\|$.

We are interested in seeing when unital, positive maps are contractive.

Lemma: Let A be a C*-algebra, and let p_i be positive elements of A such that

$$\sum_{i=1}^{n} p_i \leqslant 1.$$

If λ_i are scalars with $|\lambda_i| \leqslant 1$, then

$$\left\|\sum_{i=1}^{n} \lambda_i p_i\right\| \leqslant 1.$$

Proof. Note that

$$\begin{pmatrix} \sum_{i=1}^{n}\lambda_i p_i & 0 & \cdots 0 & \\ 0 & 0 & \cdots 0 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} p_1^{1/2}\cdots & p_n^{1/2} & \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \operatorname{diag}(\lambda_1,\dots,\lambda_n) \begin{pmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_n^{1/2} & 0 & \cdots & 0 \end{pmatrix}.$$

The norm on the matrix on the left is $\left\|\sum_{i=1}^{n}\lambda_i p_i\right\|$, while the three matrices on the right have norm less than 1, using the fact that $\|a^*a\| = \|a\|^2$. □

Theorem: Let B be a C*-algebra, X a compact Hausdorff space, and C(X) the continuous functions on X. Let $\phi\colon C(X) \to B$ be a positive map. Then, $\|\phi\| = \|\phi(1)\|$.

Proof. We may assume $\phi(1) \leqslant 1$. Let $f \in C(X)$ with $\|f\| \leqslant 1$, and let $\varepsilon > 0$. Now, we may choose a finite open cover $\{U_i\}_{i=1}^{n}$ of X such that $|f(x) - f(x_i)| < \varepsilon$ for all $x \in U_i$, and let $\{p_i\}_{i=1}^{n}$ be a partition of unity subordinate to the cover. That is, $\{p_i\}_{i=1}^{n}$ are nonnegative continuous functions satisfying $\sum_{i=1}^{n} p_i = 1$ and $p_i(x) = 0$ for $x \notin U_i$.

Set $\lambda_i = f(x_i)$, and note that if $p_i(x) \neq 0$ for some i, then $x \in U_i$ and $|f(x) - \lambda_i| < \varepsilon$. Hence, for any x, we have

$$\begin{aligned} \left|f(x) - \sum_{i=1}^{n}\lambda_i p_i(x)\right| &= \left|\sum_{i=1}^{n}(f(x)-\lambda_i)p_i(x)\right| \\ &\leqslant \sum_{i=1}^{n}|f(x)-\lambda_i|p_i(x) \\ &< \sum_{i=1}^{n}\varepsilon p_i(x) \\ &= \varepsilon. \end{aligned}$$

By above, we know that $\left\|\sum_{i=1}^{n}\lambda_i p_i\right\| \leqslant 1$, we have

$$\begin{aligned} \|\phi(f)\| &\leqslant \left\|\phi\left(f - \sum_{i=1}^{n}\lambda_i p_i\right)\right\| + \left\|\sum_{i=1}^{n}\phi(p_i)\right\| \\ &< 1 + \varepsilon\|\phi\|. \end{aligned}$$

Since ε was arbitrary, we have $\|\phi\| \leqslant 1$. □

Lemma (Riesz–Fejér Theorem)**:** Let $\tau(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$ be a strictly positive function on $\mathbb{T}$. Then, there is a polynomial $p(z) = \sum_{n=0}^{n} p_n z^n$ such that

$$\tau\left(e^{i\theta}\right) = \left|p\left(e^{i\theta}\right)\right|^2.$$

Proof. Note that τ is real-valued, so $a_{-n} = \overline{a_n}$, and a_0 is real. Assuming $a_{-N} \neq 0$, we take $g(z) = \sum_{n=-N}^{N} a_n z^{n+N}$, so that g is a polynomial of degree $2n$, $g(0) \neq 0$.

We have $g(e^{i\theta}) = \tau(e^{i\theta})e^{iN\theta} \neq 0$, and that $\overline{g(1/\overline{z})} = z^{-2N}g(z)$.

We write the 2N zeros of g as $z_1, \dots, z_N, 1/\overline{z_1}, \dots, 1/\overline{z_N}$.

Set $q(z) = (z - z_1)\cdots(z - z_N)$ and $h(z) = (z - 1/\overline{z_1})\cdots(z - 1/\overline{z_N})$. We have that

$$g(z) = a_N q(z)h(z),$$

where

$$\overline{h(z)} = \frac{(-1)^N \overline{z}^N q(1/\overline{z})}{z_1 \cdots z_N}.$$

Thus, we have

$$\begin{aligned}\tau\left(e^{i\theta}\right) &= e^{-iN\theta} g\left(e^{i\theta}\right)\\ &= \left|g\left(e^{i\theta}\right)\right|\\ &= \left|a_N q\left(e^{i\theta}\right)\overline{h}\left(e^{i\theta}\right)\right|\\ &= \frac{a_N}{z_1 \cdots z_N}\left|q\left(e^{i\theta}\right)\right|^2.\end{aligned}$$

□

Theorem: Let T be an operator on $\mathcal{H}$ with $\|T\| \leqslant 1$, and let $S \subseteq C(\mathbb{T})$ be the operator system defined by

$$S = \left\{ p\left(e^{i\theta}\right) + \overline{q(e^{i\theta})} \,\middle|\, p, q \text{ are polynomials} \right\}.$$

Then, $\phi\colon S \to \mathbb{B}(\mathcal{H})$, given by $\phi(p + \overline{q}) = p(T) + q(T)^*$ is positive.

Proof. It is enough to prove that $\phi(\tau)$ is positive for every *strictly* positive τ.

Let $\tau(e^{i\theta})$ be strictly positive in S, meaning $\tau(e^{i\theta}) = \sum_{\ell,k=0}^{n} \alpha_\ell \overline{\alpha_k} e^{i(\ell-k)\theta}$. We must prove that

$$\phi(\tau) = \sum_{\ell,k=0}^{n} \alpha_\ell \overline{\alpha_k} T(\ell - k),$$

where

$$T(j) = \begin{cases} T^j & j \geqslant 0 \\ (T^*)^{-j} & j < 0. \end{cases}$$

Fix $x \in \mathcal{H}$. Note that

$$\langle \phi(\tau)(x), x\rangle = \left\langle \begin{pmatrix} I & T^* & \cdots & (T^*)^n \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^n & \cdots & T & I \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} x \\ \overline{\alpha_2} x \\ \vdots \\ \overline{\alpha_n} x \end{pmatrix}, \begin{pmatrix} \overline{\alpha_1} x \\ \overline{\alpha_2} x \\ \vdots \\ \overline{\alpha_n} x \end{pmatrix} \right\rangle, \tag{$*$}$$

where our matrix operator acts on $\mathcal{H}^{(n)}$. Thus, we only need to show that this matrix operator is positive.

To that end, define the $n \times n$ matrix

$$R = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ T & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T & 0 \end{pmatrix},$$

and note that $R^{n+1} = 0$, with $\|R\|_{op} \leqslant 1$ (as T is a contraction).

We let I denote the identity operator on $\mathcal{H}^{(n)}$. The matrix operator $(*)$ can be written as

$$I + R + R^2 + \cdots + R^n + R^* + \cdots + (R^*)^n = (I - R)^{-1} + (I - R^*)^{-1} - I,$$

where we used the fact that $R^{n+1} = 0$ in the geometric series for $(I - R)^{-1}$ and $(I - R^*)^{-1}$. To see that this operator is positive, we let $h \in \mathcal{H}^{(n)}$, and let $h = (I - R)y$ for some $y \in \mathcal{H}^{(n)}$. Then,

$$\begin{aligned}\left\langle \left((I - R)^{-1} + (I - R^*)^{-1} - I\right)(h), h \right\rangle &= \langle y, (I - R)y\rangle + \langle (I - R)(y), y\rangle - \langle (I - R)(y), (I - R)(y)\rangle \\ &= \|y\|^2 - \|R(y)\|^2 \\ &\geqslant 0,\end{aligned}$$

since R is a contraction. $\square$

Now, we may prove von Neumann's inequality in a different way.

Theorem (von Neumann's Inequality)**:** Let T be an operator on a Hilbert space with $\|T\|_{op} \leqslant 1$. Then, for any polynomial p, we have

$$\|p(T)\|_{op} \leqslant \|p\|,$$

where $\|p\| = \sup_\theta \left|p\left(e^{i\theta}\right)\right|$.

Proof. The operator system defined by

$$S = \left\{ p\left(e^{i\theta}\right) + \overline{q(e^{i\theta})} \;\middle|\; p, q \text{ polynomials} \right\}$$

is a $*$-algebra that separates points, so by the Stone–Weierstrass theorem, S is dense in $C(\mathbb{T})$. We know that ϕ is bounded, so it extends $C(\mathbb{T})$. The extension to $\overline{S} = C(\mathbb{T})$ also positive, so ϕ is contractive. $\square$

Note that if $A(\mathbb{D})$ denotes the functions analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, we know that by the maximum modulus principle that the supremum of any function in $A(\mathbb{D})$ occurs on $\mathbb{T}$. We may thus consider $A(\mathbb{D})$ as a closed subalgebra of $C(\mathbb{T})$.

Furthermore, polynomials are dense in $A(\mathbb{D})$. Thus, the homomorphism $p \mapsto p(T)$ extends to a homomorphism $f \mapsto f(T)$ that satisfies $\|f(T)\|_{op} \leqslant \|f\|$ for all $f \in A(\mathbb{D})$.

Another consequence is that if a is an element of some unital C*-algebra A with $\|a\| \leqslant 1$, then there is a unital, positive map $\phi\colon C(\mathbb{T}) \to A$ such that $\phi(p) = p(a)$.

Corollary: Let B and C be unital C*-algebras. Let A be a unital subalgebra of B, and let $S = A + A^*$ be an operator space. If $\phi\colon S \to C$ is positive, then $\|\phi(a)\| \leqslant \|\phi(1)\|\|a\|$.

Proof. Let $a \in A$ with $\|a\| \leqslant 1$. We may extend ϕ to a positive map on $\overline{S}$. There is also a positive map $\psi\colon C(\mathbb{T}) \to B$ with $\psi(p) = p(a)$. Since A is an algebra, we must have $\operatorname{Ran}(\psi) \subseteq \overline{S}$.

The composition of positive maps is positive, so we have

$$\begin{aligned}\|\phi(a)\| &= \left\|\phi \circ \psi\left(e^{i\theta}\right)\right\| \\ &\leqslant \|\phi \circ \psi(1)\|\left\|e^{i\theta}\right\| \\ &= \|\phi(1)\|.\end{aligned}$$

$\square$

If $\phi(1) = 1$, then ϕ is a contraction on A, though ϕ may not be a contraction on all of S.

Corollary: Let A and B be unital C*-algebras with $\phi\colon A \to B$ a positive map. Then, $\|\phi\|_{op} = \|\phi(1)\|$.