Positive Maps

We will start by focusing our discussion of positive maps on a subclass of linear subspaces of C^*-algebras.

Definition: Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{S} \subseteq \mathcal{A}$ be a self-adjoint linear subspace that contains 1. We call such an $\mathcal{S}$ an *operator system*.

Note that if h is a self-adjoint element of $\mathcal{S}$, then it is possible to write h as the difference of two positive elements in $\mathcal{S}$,

$$h = \frac{1}{2}(\|h\|1 + h) - \frac{1}{2}(\|h\|1 - h).$$

Definition: If $\mathcal{S} \subseteq \mathcal{A}$ is an operator system, $\mathcal{B}$ is a C^*-algebra, and $\phi\colon \mathcal{S} \to \mathcal{B}$ is a linear map, then we say ϕ is positive if it maps positive elements of $\mathcal{S}$ to positive elements of $\mathcal{B}$.

In the special case where the C^*-algebra $\mathcal{B}$ is the complex numbers (i.e., ϕ is a positive linear functional), then we know from results in C^*-algebra theory that $\|\phi\| = \phi(1)$. If $\mathcal{B}$ is an arbitrary C^*-algebra, it turns out that ϕ is still positive, but that the bound is different.

Proposition: If $\phi\colon \mathcal{S} \to \mathcal{B}$ is a positive map, then $\|\phi\| \leq 2\|\phi(1)\|$.

Proof. If p is positive, then since $0 \leq p \leq \|p\|1$, it follows that $0 \leq \phi(p) \leq \|p\|\phi(1)$, so that $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$.

If p_1 and p_2 are positive, then $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$, so if h is self-adjoint in $\mathcal{S}$, we have

$$\phi(h) = \frac{1}{2}\phi(\|h\|1 + h) - \frac{1}{2}\phi(\|h\|1 - h),$$

giving

$$\begin{aligned} \|\phi(h)\| &\leq \frac{1}{2}\max(\|\phi(\|h\|1 + h)\|, \|\phi(\|h\|1 - h)\|) \\ &\leq \|h\|\|\phi(1)\|. \end{aligned}$$

Finally, if a is an arbitrary element of $\mathcal{S}$, then we may write the Cartesian decomposition $a = h + ik$, and find

$$\begin{aligned} \|\phi(a)\| &\leq \|\phi(h)\| + \|\phi(k)\| \\ &\leq 2\|a\|\|\phi(1)\|. \end{aligned}$$

$\square$

It turns out that this bound is strict.

Completely Positive Maps

Dilations and Extensions

Nuclearity and Exactness

Application to Amenability

References

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