4.11

Problem: Show that if A is a decidable set, then so is its complement. Then, show that if A and B are decidable sets, then so are $A \cup B$ and $A \cap B$.

Solution: Let f_A be the function that computes χ_A, and let f_B be the function that computes χ_B.

We define g_A, which computes $\mathbb{N} \setminus A$, by composing f_A with the partial function that computes 1 if the input is 0 and computes 0 if the input is 1.

To define $f_{A\cup B}$ and $f_{A\cap B}$, we take

$$\begin{aligned} f_{A\cup B} &= f_A + f_B - (f_A)(f_B) \\ f_{A\cap B} &= (f_A)(f_B), \end{aligned}$$

in which we use the multiplication and addition operations composed with f_A and f_B.

Extra Problem 1

Problem: Give an example of a relation that is not computable.

Solution: Let $\{T_m\}_{m\in\mathbb{N}}$ be a denumeration of the set of all Turing machines with one input. We define the relation $R \subseteq \mathbb{N} \times \mathbb{N}$ with the membership $(m, n) \in R$ if and only if $T_m(n)$ halts.

Since it is not possible to compute the halting problem, we know that the relation R is not computable.

Extra Problem 2

Problem: Suppose R, S, T are relations with $(a, b) \in T$ if and only if $(a, b) \in R$ or $(a, b) \in S$.

Extra Problem 3

Problem: Prove that if $R \subseteq \mathbb{N} \times \mathbb{N}$ is computable, then so too is $\mathbb{N} \times \mathbb{N} \setminus R$.

Extra Problem 7

Problem: Let

$$\pi(n) = \begin{cases} 1 & n \text{ prime} \\ 0 & \text{else} \end{cases}.$$

Show that π is primitive recursive.

Solution: Consider

$$f(x, z) = \begin{cases} 1 & z \text{ prime, no primes } p \text{ with } x < p < z \\ 0 & \text{else} \end{cases}.$$