Banach Limits

Theorem (Hahn–Banach–Minkowski)**:** Let X be a real vector space, and let $p\colon X \to \mathbb{R}$ be such that $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ and $p(tx) = tp(x)$ for all $t \geq 0$. If $f\colon Y \to \mathbb{R}$ is a linear functional defined on a subspace Y such that $f(x) \leq p(x)$ for all $x \in Y$, then there is an extension $F\colon X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for all $x \in X$ and $F|_Y = f$.

Furthermore, if $v \in X \setminus Y$, the value of $F(v)$ can be designated to be in the closed interval defined by

$$m = \sup_{w \in Y} (-p(-w - v) - f(w))$$

at the left endpoint, and

$$M = \inf_{w \in Y} (p(w + v) - f(w))$$

at the right endpoint.

Corollary: If X is a complex normed vector space with subspace $E \subseteq X$ and $\varphi \in E^*$, then there is $\Phi\colon X \to \mathbb{C}$ such that $\Phi|_E = \varphi$ and $\|\Phi\| = \|\varphi\|$.

Additionally, if there is $x_0 \in X \setminus E$, then there is $f \in X^*$ such that $f(x_0) = \operatorname{dist}_E(x_0)$ and $f|_E = 0$.

One of the most important vector spaces is the space ℓ_∞ of bounded sequences $x\colon \mathbb{N} \to \mathbb{C}$, which admits a subspace of convergent subspaces, often denoted c.

Proposition: There exists a linear functional $L\colon \ell_\infty \to \mathbb{C}$ with

(i) $\|L\| = 1$;

(ii) for any $x \in c$, $L(x) = \lim_{n\to\infty} x_n$;

(iii) for any $x \in \ell_\infty$ with $x_n \geq 0$ for each n, we have $L(x) \geq 0$;

(iv) for any $x \in \ell_\infty$, with $(S(x))_n \coloneqq x_{n+1}$, we have $L(S(x)) = L(x)$.

We will construct this linear functional using the Hahn–Banach theorem(s) by following the construction in Conway's book. We consider the real vector space $\Re(\ell_\infty)$, which we will write as ℓ_∞ for now.

To start we consider the subspace M of ℓ_∞ given by

$$M = \{x - S(x) \mid x \in \ell_\infty\}.$$

If $\mathbb{1}$ denotes the sequence of 1s in ℓ_∞, then we see that $\operatorname{dist}_M(\mathbb{1}) = 1$. First, $0 \in M$, so that $\operatorname{dist}_M(\mathbb{1}) \leq 1$. If there is n such that $(x - S(x))_n \leq 0$, then we see that

$$\begin{aligned} \|\mathbb{1} - (x - S(x))\| &\geq |\mathbb{1} - (x_n - (S(x))_n)| \\ &\geq 1. \end{aligned}$$

Else, if for all n, $0 \leq (x - S(x))_n = x_n - x_{n+1}$, then $x_{n+1} \leq x_n$ for all n. Since $x \in \ell_\infty$, there is $\alpha = \lim_{n\to\infty} x_n$. Therefore, $\lim_{n\to\infty}(x_n - x_{n+1}) = 0$, so $\|\mathbb{1} - (x - S(x))\| \geq 1$.

Thus, there is some linear functional $L\colon \ell_\infty \to \mathbb{R}$ such that $\|L\| = 1$ and $L(x) = L(S(x))$. This satisfies (i) and (iv) in the proposition.

Next, we show that $c_0 \subseteq \ker(L)$. Since (in our current focus) $c = c_0 + \mathbb{R}\mathbb{1}$, we would then obtain (b). To see this, let $x \in c_0$. Observe that $S^n(x) - x$ is contained in M, meaning that $L(x) = L(S^n(x))$ for each n. If $\varepsilon > 0$, there is some N such that $|x_m| < \varepsilon$ for all $m > N$. Therefore,

$$\begin{aligned} |L(x)| &= |L(S^n(x))| \\ &\leq \|S^n(x)\| \\ &< \varepsilon. \end{aligned}$$

Since ε is arbitrary, we thus have $x \in \ker(L)$.

Finally, to show (c), we let $x \in \ell_\infty$ be such that $x_n \geq 0$ for all n, and assume toward contradiction that $L(x) < 0$. Dividing out by $\|x\|$, we have that $L(x) < 0$ and $0 \leq x_n \leq 1$ for all n. Yet, this would imply that $\|\mathbb{1} - x\| \leq 1$ and $L(\mathbb{1} - x) = 1 - L(x) > 1$, contradicting (a).

To extend to $\mathbb{C}$, and rewriting the functional on $\mathbb{R}$ as L_1, we may write any element $x \in \ell_\infty$ as $x = x_1 + ix_2$. Observe then that

$$L(x) = L_1(x_1) + iL_1(x_2)$$

is a linear functional on ℓ_∞. Now, observe that $\|L\| \geq 1$ almost by design. Since L is a nonzero linear functional, we let x be such that $L(x) \neq 0$, and set

$$\alpha = \frac{|L(x)|}{L(x)}.$$

We have that $|\alpha| = 1$ and $\alpha L(x) = |L(x)|$. We may then compute

$$\begin{aligned}
|L(x)| &= L(\alpha x) \\
&= \operatorname{Re}(L(\alpha x)) \\
&= L_1(\alpha x) \\
&\leq \|L_1\| \|\alpha x\| \\
&= \|x\|.
\end{aligned}$$

In particular, this means $\|L\| \leq 1$, so $\|L\| = 1$.

Generalized Limits beyond Banach Limits

We have obtained one limit. However, there are lots of other extensions of the limit, each of which has norm 1. In fact, from our statement of the Hahn–Banach–Minkowski theorem, we claim that the following expression

$$M(x) := \lim_{n\to\infty} \sup_{j\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} x_{k+j}$$

is a sublinear functional.

First, we show that this is in fact a limit. Toward this end, we observe that, if we set

$$c_n = \sup_{j\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} x_{k+j},$$

then $c_{am} \leq c_m$ for each $a \geq 1$.