Problem 1

Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Show that the following are equivalent:

(i) c is a limit point of D.

(ii) There is a sequence $(x_n)_n$ in $D \setminus \{c\}$ with $(x_n)_n \to c$.

($\Rightarrow$) Let c be a limit point of D. Then, taking $\delta_n = 1/n$, let $x_n \in \dot{V}_{\delta_n}(c)$. Then, $(x_n)_n \to c$.

($\Leftarrow$) Let $(x_n)_n$ be a sequence in $D \setminus \{c\}$ with $(x_n)_n \to c$.

Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ with, $\forall n \geq N$, $|x_n - c| < \varepsilon$. Thus, $\forall \varepsilon > 0$, $\exists x_n$ such that $x_n \in \dot{V}_\varepsilon(c)$. Thus, c is a limit point.

Problem 2

Show that f can have at most one limit at c.

Suppose toward contradiction that $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, where $L_1 \neq L_2$. Then, $\exists \varepsilon_0 > 0$ such that $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$.

Let δ_1 be such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon_0$, and δ_2 be such that $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon_0$. Set $\delta = \min(\delta_1, \delta_2)$.

Then, $|x - c| < \delta \Rightarrow |f(x) - L_1| < \varepsilon_0$ and $|x - c| < \delta \Rightarrow |f(x) - L_2| < \varepsilon_0$. So, $\exists k$ such that $f(k) \in V_\varepsilon(L_1)$ and $f(k) \in V_\varepsilon(L_2)$. $\perp$

Problem 3

Show that the following are equivalent:

(i) $\lim_{x\to c} f(x) = L$

(ii) For every sequence $(x_n)_n$ in $D \setminus \{c\}$ such that $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

($\Rightarrow$) Let $\lim_{x\to c} f(x) = L$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

So, $\forall \varepsilon > 0$, $\exists f(x_k) \in V_\varepsilon(L)$, such that $x_k \in \dot{V}_\delta(c)$. So, we have a sequence $(x_n)_n \to c$ defined by $\delta(\varepsilon, c)$, where $(f(x_n))_n \to L$.

($\Leftarrow$) Assume toward contradiction that $\lim_{x\to c} f(x) \neq L$. Then, $\exists \varepsilon_0$ such that $\forall \delta > 0$, $\exists x \in \dot{V}_\delta(c) \cap D$ such that $|f(x) - L| > \varepsilon_0$.

Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in \dot{V}_{1/n}(c) \cap D$ with $|f(x_n) - L| > \varepsilon_0$.

Since $0 < |x - c| < 1/n$, $(x_n)_n \in D \setminus \{c\}$ and $(x_n)_n \to c$, meaning $(f(x_n))_n \to L$. However, $|f(x_n) - L| > \varepsilon_0$. $\perp$

Problem 4

If $\lim_{x\to c} f = L$ exists, show that there is a $\delta > 0$ such that

$$\sup_{x\in \dot{V}_\delta(c)} |f(x)| < \infty$$

Let $\varepsilon = 1$. Then, $\exists \delta > 0$ such that $\forall x \in \dot{V}_\delta(c)$, $|f(x) - L| < 1$. Therefore,

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| && \text{Triangle Inequality} \\ &< 1 + |L| \end{aligned}$$

So,

$$\sup_{x\in \dot{V}_\delta(c)} |f(x)| \leq 1 + |L|$$

Problem 5

Establish the following limits:

(a)

$$\lim_{x\to 1} \frac{3x}{1+x} = \frac{3}{2}$$

(b)

$$\lim_{x\to 6} \frac{x^2 - 3x}{x+3} = 2$$

(c)

$$\lim_{x\to 0} \mathbb{1}_{\mathbb{Q}} = 0$$

(d)

$$\lim_{x\to 0} \frac{x^2}{|x|} = 0$$

Problem 6

For which values of $k = 0, 1, 2, \dots$ does

$$\lim_{x\to 0} x^k \sin(1/x)$$

exist?

$k = 0$: Suppose $k = 0$. Let $(a_n)_n \in (0,1)$ be a sequence defined by $a_n = \frac{2}{(4n+1)\pi}$, and let $(b_n)_n \in (0,1)$

be a sequence defined by $\frac{1}{\pi n}$. Then,

$$(f(a_n))_n = (1, 1, 1, \dots),$$

and

$$(f(b_n))_n = (0, 0, 0, \dots),$$

meaning that $(f(a_n))_n \to 1$ and $(f(b_n))_n \to 0$. Let $(c_n)_n = (a_1, b_1, a_2, b_2, \dots)$. Then, $(f(c_n))_n$ has a subsequence $(f(a_n))_n \to 1$ and a subsequence $(f(b_n))_n \to 0$. Therefore, $(f(c_n))_n$ is divergent, meaning the limit does not exist.

$k \neq 0$: Suppose $k \neq 0$. Let $(x_n)_n$ be an arbitrary sequence in $D \setminus \{0\}$ such that $(x_n)_n \to 0$. Then,

$$\begin{aligned} |f(x_n)| &= \left| x_n \sin\left(\frac{1}{x_n}\right) \right| \\ &\leq |x_n| \\ &\to 0 \end{aligned}$$

meaning $(f(x_n))_n \to 0$.

Problem 7

Assume $f(x) \geq 0$ for all $x \in D$ and suppose $\lim_{x\to c} f =: L$ exists. Show that $L \geq 0$ and

$$\lim_{x\to c} \sqrt{f} = \sqrt{L}$$

Let $(x_n)_n \in D \setminus \{c\}$ such that $(x_n)_n \to c$. Then, $(f(x_n))_n \to L$, by the sequential definition of limits. Since $f(x_n) \geq 0$ for all x_n, by the properties of sequences, it must be the case that $L \geq 0$.

Similarly, it must be the case that $\left(\sqrt{f(x_n)}\right)_n \to \sqrt{L}$ by the properties of sequences — meaning that $\lim_{x\to c} \sqrt{f} = \sqrt{L}$.