

## Positive Maps

We will start by focusing our discussion of positive maps on a subclass of linear subspaces of  $C^*$ -algebras.

**Definition:** Let  $\mathcal{A}$  be a  $C^*$ -algebra, and let  $\mathcal{S} \subseteq \mathcal{A}$  be a self-adjoint linear subspace that contains 1. We call such an  $\mathcal{S}$  an *operator system*.

Note that if  $h$  is a self-adjoint element of  $\mathcal{S}$ , then it is possible to write  $h$  as the difference of two positive elements in  $\mathcal{S}$ ,

$$h = \frac{1}{2}(\|h\|1 + h) - \frac{1}{2}(\|h\|1 - h).$$

**Definition:** If  $\mathcal{S} \subseteq \mathcal{A}$  is an operator system,  $\mathcal{B}$  is a  $C^*$ -algebra, and  $\phi: \mathcal{S} \rightarrow \mathcal{B}$  is a linear map, then we say  $\phi$  is positive if it maps positive elements of  $\mathcal{S}$  to positive elements of  $\mathcal{B}$ .

In the special case where the  $C^*$ -algebra  $\mathcal{B}$  is the complex numbers (i.e.,  $\phi$  is a positive linear functional), then we know from results in  $C^*$ -algebra theory that  $\|\phi\| = \phi(1)$ . If  $\mathcal{B}$  is an arbitrary  $C^*$ -algebra, it turns out that  $\phi$  is still positive, but that the bound is different.

**Proposition:** If  $\phi: \mathcal{S} \rightarrow \mathcal{B}$  is a positive map, then  $\|\phi\| \leq 2\|\phi(1)\|$ .

*Proof.* If  $p$  is positive, then since  $0 \leq p \leq \|p\|1$ , it follows that  $0 \leq \phi(p) \leq \|p\|\phi(1)$ , so that  $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$ .

If  $p_1$  and  $p_2$  are positive, then  $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$ , so if  $h$  is self-adjoint in  $\mathcal{S}$ , we have

$$\phi(h) = \frac{1}{2}\phi(\|h\|1 + h) - \frac{1}{2}\phi(\|h\|1 - h),$$

giving

$$\begin{aligned} \|\phi(h)\| &\leq \frac{1}{2} \max(\|\phi(\|h\|1 + h)\|, \|\phi(\|h\|1 - h)\|) \\ &\leq \|h\|\|\phi(1)\|. \end{aligned}$$

Finally, if  $a$  is an arbitrary element of  $\mathcal{S}$ , then we may write the Cartesian decomposition  $a = h + ik$ , and find

$$\begin{aligned} \|\phi(a)\| &\leq \|\phi(h)\| + \|\phi(k)\| \\ &\leq 2\|a\|\|\phi(1)\|. \end{aligned}$$

□

It turns out that this bound is strict.

**Example:** Consider the subspace  $\mathcal{S} \subseteq C(S^1)$  spanned by  $1, z, \bar{z}$ . Then, we may define  $\phi: \mathcal{S} \rightarrow \mathbb{M}_2$  given by

$$\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}.$$

It follows that an element of  $\mathcal{S}$  is positive if and only if  $c = \bar{b}$  and  $a \geq 2|b|$ , while a self-adjoint element of  $\mathbb{M}_2$  is positive if and only if its diagonal entries and determinant are positive real numbers. Therefore, it follows that  $\phi$  is a positive map.

Yet,

$$\begin{aligned} 2\|\phi(1)\| &= 2 \\ &= \|\phi(z)\| \\ &\leq \|\phi\|, \end{aligned}$$

meaning that  $\|\phi\| = 2\|\phi(1)\|$ .

## Completely Positive Maps

**Definition:** If  $\mathcal{B}$  is a  $C^*$ -algebra and  $\phi: \mathcal{S} \rightarrow \mathcal{B}$  is a linear map, then we may define  $\phi_n: \mathbb{M}_n(\mathcal{S}) \rightarrow \mathbb{M}_n(\mathcal{B})$  by  $\phi_n((a_{ij})_{i,j}) = (\phi(a_{ij}))_{i,j}$ .

We call  $\phi$   $n$ -positive if  $\phi_n$  is positive, and we call  $\phi$  completely positive if  $\phi$  is  $n$ -positive for all  $n$ .

We call  $\phi$  completely bounded if

$$\|\phi\|_{\text{cb}} := \sup_n \|\phi_n\|$$

is finite. If  $\|\phi\|_{\text{cb}} \leq 1$ , then we call  $\phi$  completely contractive.

**Lemma:** Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $a, b \in \mathcal{A}$ . Then,

(i)  $\|a\| \leq 1$  if and only if

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$$

is positive in  $\mathbb{M}_2(\mathcal{A})$ .

(ii) The matrix

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$$

is positive in  $\mathbb{M}_2(\mathcal{A})$  if and only if  $a^*a \leq b$ .

*Proof.* Faithfully represent  $\mathcal{A}$  on  $\mathcal{H}$  via  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$ , and set  $A = \pi(a)$ . Then, if  $\|A\| \leq 1$ , for any  $x, y \in \mathcal{H}$ , we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, y \rangle + \langle Ay, x \rangle + \langle x, Ay \rangle + \langle y, y \rangle \\ &\geq \|x\|^2 - 2\|A\|\|x\|\|y\| + \|y\|^2 \\ &\geq 0. \end{aligned}$$

Conversely, if  $\|A\| > 1$  then there are unit vectors  $x$  and  $y$  such that  $\langle Ay, x \rangle < -1$ , so the inner product above would be negative.

Now, if we let  $B = \pi(b)$ , then if we let  $B \geq A^*A$ , then  $B - A^*A \geq 0$ , so that  $\langle By, y \rangle \geq \langle Ay, Ay \rangle$  for all  $y \in \mathcal{H}$ , meaning that for all  $x, y \in \mathcal{H}$ , we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle Ax, y \rangle + \langle A^*x, y \rangle + \langle By, y \rangle \\ &\geq \langle x, x \rangle + 2\operatorname{Re}\langle Ax, y \rangle + \langle Ay, Ay \rangle \\ &\geq \|x\|^2 + \|Ay\|^2 - 2\|Ay\|\|x\| \\ &\geq 0. \end{aligned}$$

In the case that  $B \not\geq A^*A$ , then there is some unit vector  $y$  such that  $\langle By, y \rangle < \|Ay\|^2$ , which would yield the analogous outcome as in the proof of (i).  $\square$

## Dilations and Extensions

**Theorem** (Stinespring's Dilation Theorem): Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  be a completely positive map. Then, there exists a Hilbert space  $\mathcal{K}$ , a unital  $*$ -homomorphism  $\pi: \mathcal{A} \rightarrow B(\mathcal{K})$ ,

and a bounded operator  $V: \mathcal{H} \rightarrow \mathcal{K}$  with  $\|\phi(1)\| = \|V\|^2$  such that

$$\phi(a) = V^* \pi(a) V.$$

*Proof.* Let  $\mathcal{A} \odot \mathcal{H}$  be the algebraic tensor product, and define a symmetric bilinear map

$$(a \otimes x, b \otimes y) = \langle \phi(b^* a) x, y \rangle,$$

and extend linearly, where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{H}$ . Then, since  $\phi$  is completely positive, it follows that  $(\cdot, \cdot)$  is positive semidefinite, with

$$\begin{aligned} \left( \sum_{j=1}^n a_j \otimes x_j, \sum_{i=1}^n a_i \otimes x_i \right) &= \left\langle \phi_n \left( (a_i^* a_j)_{i,j} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\ &\geq 0. \end{aligned}$$

Since positive semidefinite bilinear forms satisfy the Cauchy–Schwarz inequality, we may define the subspace

$$\begin{aligned} \mathcal{N} &= \{u \in \mathcal{A} \odot \mathcal{H} \mid (u, u) = 0\} \\ &= \{u \in \mathcal{A} \odot \mathcal{H} \mid (u, v) = 0 \text{ for all } v \in \mathcal{A} \odot \mathcal{H}\}, \end{aligned}$$

with an induced bilinear form on the quotient space  $\mathcal{A} \odot \mathcal{H} / \mathcal{N}$  defined by

$$(u + \mathcal{N}, v + \mathcal{N}) = (u, v).$$

Define  $\mathcal{K}$  to be the Hilbert space completion of  $\mathcal{A} \odot \mathcal{H} / \mathcal{N}$ . Now, define a linear map  $\pi(a): \mathcal{A} \odot \mathcal{H} \rightarrow \mathcal{A} \odot \mathcal{H}$  by

$$\pi(a) \left( \sum_{i=1}^n a_i \otimes x_i \right) = \sum_{i=1}^n (a a_i) \otimes x_i.$$

We have that the inequality in  $\mathbb{M}_n(\mathcal{A})$  given by

$$(a_i^* a^* a a_j)_{i,j} \leq \|a^* a\| (a_i^* a_j)_{i,j}$$

is satisfied, giving

$$\begin{aligned} \left( \pi(a) \left( \sum_{j=1}^n a_j \otimes x_j \right), \pi(a) \left( \sum_{i=1}^n a_i \otimes x_i \right) \right) &= \sum_{i,j=1}^n \langle \phi(a_i^* a^* a a_j) x_j, x_i \rangle \\ &\leq \|a^* a\| \sum_{i,j=1}^n \langle \phi(a_i^* a_j) x_j, x_i \rangle \\ &= \|a\|^2 \left( \sum_{j=1}^n a_j \otimes x_j, \sum_{i=1}^n a_i \otimes x_i \right). \end{aligned}$$

Therefore,  $\pi(a)$  is invariant under  $\mathcal{N}$ , so induces a quotient map on  $\mathcal{A} \odot \mathcal{H} / \mathcal{N}$ , which we will also denote by  $\pi(a)$ . We have that (this new)  $\pi(a)$  is bounded with  $\|\pi(a)\| \leq \|a\|$ , meaning that it extends to a bounded linear map on  $\mathcal{K}$ .

We define  $V: \mathcal{H} \rightarrow \mathcal{K}$  by

$$V(x) = 1 \otimes x + \mathcal{N}.$$

Then,

$$\|Vx\|^2 = (1 \otimes x, 1 \otimes x)$$

$$\begin{aligned}
&= \langle \phi(1)x, x \rangle \\
&\leq \|\phi(1)\| \|x\|^2,
\end{aligned}$$

and  $\|V\|^2 = \|\phi(1)\|$ .

Finally, we observe that

$$\begin{aligned}
\langle V^* \pi(a) V x, y \rangle &= (\pi(a) 1 \otimes x, 1 \otimes y) \\
&= \langle \phi(a)x, y \rangle
\end{aligned}$$

for all  $x$  and  $y$ , so  $V^* \pi(a) V = \phi(a)$ . □

We observe that if  $\phi$  is unital, then  $V$  is an isometry, and we may identify  $\mathcal{H}$  with the subspace  $V\mathcal{H}$  of  $\mathcal{K}$ , and that  $\phi(a) = P\pi(a)P$ , where  $P$  is the projection onto  $\mathcal{H}$ . In particular, this means that every unital completely positive map is the compression of a  $*$ -homomorphism.

This construction is very similar to the GNS representation, and we call the triple  $(\pi, V, \mathcal{K})$  a Stinespring representation for  $\phi$ .

## Nuclearity and Exactness

**Definition:** We call a map  $\theta: A \rightarrow B$  between  $C^*$ -algebras *nuclear* if there are contractive completely positive maps  $\varphi_n: A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$  and  $\psi_n: \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow B$  such that  $\psi_n \circ \varphi_n \rightarrow \theta$  pointwise in the norm topology:

$$\|\psi_n \circ \varphi_n(a) - \theta(a)\| \rightarrow 0$$

for all  $a \in A$ .

**Definition:** If  $A$  is a  $C^*$ -algebra, and  $N$  is a von Neumann algebra, then a map  $\theta: A \rightarrow N$  is called *weakly nuclear* if there exist contractive completely positive maps  $\varphi_n: A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$  and  $\psi_n: \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow N$  such that  $\psi_n \circ \varphi_n \rightarrow \theta$  pointwise in the ultraweak topology:

$$\eta(\psi_n \circ \varphi_n(a)) \rightarrow \eta(\theta(a))$$

for all  $a \in A$  and normal functionals  $\eta \in N_*$ .

By uniqueness of preduals, if  $N \subseteq B(\mathcal{H})$  is a faithful normal representation, then it suffices to observe that  $\psi_n \circ \varphi_n \rightarrow \theta$  pointwise in the ultraweak topology if and only if

$$\langle \psi_n \circ \varphi_n(a)v, w \rangle \rightarrow \langle \theta(a)v, w \rangle$$

for all  $a \in A$  and  $v, w \in \Omega$  for some collection  $\Omega$  of vectors whose linear span is dense in  $\mathcal{H}$ .

## Application to Amenability

### References

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