# Amenability: A (Somewhat) Brief Introduction

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# Outline

1. Definitions
2. Paradoxical Decompositions
3. From Paradoxical Decompositions to Amenability
4. Equivalent Definitions and Other Criteria
   - A Taste of Functional Analysis
   - Introducing Approximations
   - Approximations with Representations and Operators
5. Remarks and Acknowledgments

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5 Remarks and Acknowledgments

# Groups

If $A$ is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists $e_A$ such that $a \star e_A = e_A \star a = a$;
- for each $a$ there exists $a^{-1}$ such that $a \star a^{-1} = a^{-1} \star a = e_A$,

then we call the pair $(A, \star)$ a *group*.

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then we call the pair $(A, \star)$ a *group*.

We abbreviate $a \star b$ as $ab$. If $ab = ba$, then we say the group is *abelian*.

# Subgroups, Quotient Groups

Let $G$ be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say $H$ is a *subgroup*.

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- If $N \subseteq G$ is a subgroup that satisfies, for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say $N$ is a *normal subgroup*.

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* $G/N$.
- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

# Some Groups

- The integers $\mathbb{Z}$ are a group under addition.
- The group of invertible $n \times n$ matrices over $\mathbb{C}$, $\mathrm{GL}_n(\mathbb{C})$, is a group under matrix multiplication.
- The subgroup $\mathrm{SO}(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under multiplication.

# Group Actions

Let $G$ be a group, and $X$ a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say $\rho$ is an *action* of $G$ on $X$. We write $\rho(g, x) = g \cdot x$.

# $\sigma$-Algebras and Measures

If $X$ is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

If, for any countable collection, $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a *$\sigma$-algebra* of subsets.

# $\sigma$-Algebras and Measures, Cont'd

If $X$ is a set and $\mathcal{A}$ is a $\sigma$-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say $\mu$ is a *finitely additive* measure.

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then we say $\mu$ is a *finitely additive* measure.

If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if $\mu$ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say $\mu$ is a measure. If $\mu(X) = 1$, then we say $\mu$ is a probability measure.

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# Questions?

- If $G$ is a group, is it possible to reconstruct $G$ by using some subset of $G$?
- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

## Free Groups

- We begin by considering a special group, known as $F(a, b)$ or the *free group on two generators*.

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- We define $F(a,b)$ to be the set of all "words" in the alphabet $\left\{a,b,a^{-1},b^{-1}\right\}$, subject to the condition that, for $w,w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

# A Curiosity

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- all words that start with $a^{-1}$;
- all words that start with $b$ — think words that start with $b^2$ before you multiply $b^{-1}$.

Thus, all we need to do is add back $W\left(b^{-1}\right)$ to get $F(a, b)$ back.

$$F(a, b) = W\left(b^{-1}\right) \cup b^{-1}W(b).$$

# A Curiosity, Cont'd

Similarly, we can do this for $a$, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

We're able to take part of the group $F(a,b)$, take some translations, and, miraculously, obtain the entire group back.

# Defining Paradoxical Decompositions

Let $G$ be a group. A *paradoxical decomposition* of $G$ consists of

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$;

such that

$$G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

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$$G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

If $G$ admits a paradoxical decomposition, we say $G$ is *paradoxical*.

# Paradoxical Actions

If $G$ acts on a set $X$, then a subset $A \subseteq X$ is $G$-*paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$

such that

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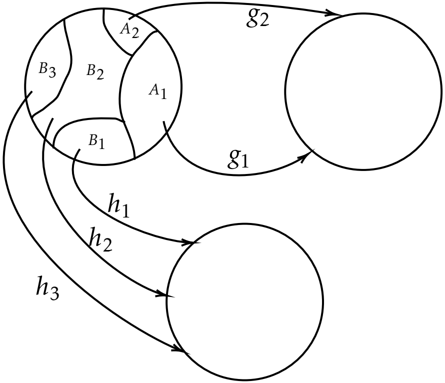
A paradoxical group is a paradoxical set under the action of left-multiplication.

# Depiction



# Examples

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- The free group $F(a,b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where $S$ is any nonempty set with more than two elements, is paradoxical.

# A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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This is proven using the Ping-Pong lemma.

# Introducing the Banach–Tarski Paradox

**Theorem (The Banach–Tarski Paradox)**

*Let $A$ and $B$ be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of $A$ into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields $B$.*

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- In other words, not all subsets of $\mathbb{R}^3$ have a definite "volume" invariant under isometry.

# Equidecomposability

Let $G$ be a group that acts on a set $X$, and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \ldots, A_n \subseteq A$, $B_1, \ldots, B_n \subseteq B$
- group elements $g_1, \ldots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say $A$ and $B$ are *G-equidecomposable*.

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such that $g_i \cdot A_i = B_i$, then we say $A$ and $B$ are *$G$-equidecomposable*.

Effectively, $A$ and $B$ are "equal" to each other up to the group action.

If $A$ is $G$-paradoxical, then so too is $B$.

# The Banach–Tarski Paradox: Proof Outline I

1. We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a, b)$.

# The Banach–Tarski Paradox: Proof Outline II

2. We use the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right) \end{aligned}$$

   to duplicate the unit sphere in $\mathbb{R}^3$, $S^2$, except for a countable subset $D$. (The *Hausdorff Paradox*.)
3. We show that $S^2$ and $S^2 \setminus D$ are SO(3)-equidecomposable — there is thus a paradoxical decomposition of $S^2$.
4. We show that the unit ball, $B(0,1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group E(3).

# The Banach–Tarski Paradox: Proof Outline III

5. Define a relation $A \preceq B$ if $A$ is $G$-equidecomposable with a subset of $B$, and show that if $A \preceq B$ and $B \preceq A$, then $A$ and $B$ are $G$-equidecomposable.
6. Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

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# Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped “create” the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu\colon F(a,b) \to [0,1]$ be a probability measure — we will show that $\nu$ *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

# Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant $\nu$ exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned}
1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\
&= \nu(F(a,b)) + \nu(F(a,b)) \\
&= 2.
\end{aligned}$$

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&= \nu(F(a,b)) + \nu(F(a,b)) \\
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\end{aligned}$$

Huh.

# Amenability

Let $G$ be a group. A *mean* is a finitely additive probability measure $\nu \colon G \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

If $G$ admits a mean, we say $G$ is *amenable*.

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If $G$ admits a mean, we say $G$ is *amenable*.

- In other words, $G$ is sufficiently “well-behaved.”

# Inheritance Properties of Amenability

- If $G$ is amenable, then any subgroup of $G$ is amenable.
- If $G$ is amenable, then quotient groups, $G/N$, are amenable.
- If $H \leq G$ is an amenable subgroup such that $[G:H] < \infty$, then $G$ is amenable.
- If $N \trianglelefteq G$ and $G/N$ are amenable, then $G$ is amenable.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

# Examples

- Finite groups are amenable: let $\delta_t$ be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

  Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

  is a mean.
- Abelian groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

# Paradoxical Groups and Amenability

Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a, b)$.

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**Theorem (Tarski's Theorem)**

*Let $G$ be a group. Then, $G$ is non-paradoxical if and only if $G$ is amenable.*

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Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a, b)$.

More surprisingly, though, every *non*-paradoxical group is amenable.

**Theorem (Tarski's Theorem)**

*Let $G$ be a group. Then, $G$ is non-paradoxical if and only if $G$ is amenable.*

Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

# Function Spaces

Functions are much easier to work with than sets and measures (or at least, they're more interesting to work with).

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We work with three special function spaces over the group $\Gamma$.

$$\ell_\infty(\Gamma) := \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sup_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) := \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_2(\Gamma) := \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t \in \Gamma} |f(t)|^2 < \infty \right\}.$$

# Work in $\ell_\infty(\Gamma)$

For a given $f \in \ell_\infty(\Gamma)$, we define

$$\lambda_s : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$$

by $\lambda_s(f)(t) = f\left(s^{-1}t\right)$.

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$$\lambda_s \colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$$

by $\lambda_s(f)(t) = f\left(s^{-1}t\right)$.

If there is some $\rho \colon \ell_\infty(\Gamma) \to \mathbb{C}$ such that

- $\rho(\mathbb{1}_\Gamma) = 1 = \left\|\rho\right\|_{\text{op}}$;
- $\rho(f) = \rho(\lambda_s(f))$,

then we call $\rho$ an *invariant state*.

# Work in $\ell_\infty(\Gamma)$, cont'd

Invariant states and means are interchangeable.

# Work in $\ell_\infty(\Gamma)$, cont'd

Invariant states and means are interchangeable.

Define

$$\mu(E) = \rho(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

# Motivating Følner's Condition

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Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was "significantly" "bigger" than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

# Følner's Condition

We can actually make this sense of "bigness" precise — if the symmetric difference of the translated set and our original set becomes "sufficiently small," then we can put a mean on the group.

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**Theorem (Følner's Theorem)**

*Let $\Gamma$ be a (countable, discrete) group. Then, $\Gamma$ is amenable if and only if there exists a sequence $(F_n)_n$ of finite sets such that for all $s \in \Gamma$,*

$$\lim_{n\to\infty} \frac{|sF_n \triangle F_n|}{|F_n|} = 0.$$

# Approximating Means

The Følner condition allows us to find an "approximate" version of a mean that works in $\ell_1(\Gamma)$ instead of $\ell_\infty(\Gamma)$.

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The Følner condition allows us to find an "approximate" version of a mean that works in $\ell_1(\Gamma)$ instead of $\ell_\infty(\Gamma)$.

Keeping $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, we say $(f_k)_k \subseteq \ell_1(\Gamma)$ is an *approximate mean* if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0.$$

# Approximating Means, Cont'd

This is equal to Følner's condition; in one direction by defining

$$f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k},$$

# Approximating Means, Cont'd

This is equal to Følner's condition; in one direction by defining

$$f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k},$$

and in the other direction by using the "layer cake" decomposition

$$f = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

# Using Følner's Condition

If $S$ is a (finite) generating set for $G$, then letting

$$S^n :- \{g_1 \cdots g_n \mid g_i \in S\},$$

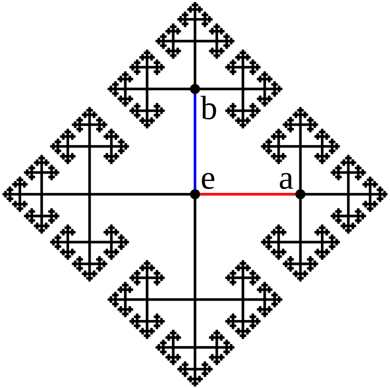
we call groups such that

$$\limsup_{n \to \infty} |S^n|^{1/n} = 1$$

*groups of subexponential growth*. Følner's condition is used to show that they are amenable.

# Graphs and Amenability

Given a group with generating set $S$, we may define a graph — known as the Cayley graph — by "walking" along the direction that leads us to an element of the group.



# Graphs and Amenability, cont'd

If the Cayley graph of $G$ has a "sufficiently slow-growing" neighbor vertex set, then $G$ is amenable — this is once again proven using the Følner condition.

# Introducing the Left-Regular Representation

Consider the map $s \mapsto \lambda_s$, where once again $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, this time when $f \in \ell_2(\Gamma)$.

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Then, if

$$\langle f, g\rangle = \sum_{t\in\Gamma} f(t)\overline{g(t)},$$

on $\ell_2(\Gamma)$, we have, for all $s \in \Gamma$

$$\langle \lambda_s(f), \lambda_s(g)\rangle = \langle f, g\rangle,$$
$$\lambda_s\lambda_s^* = I$$
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$$\lambda_s^*\lambda_s = I$$

This makes $\lambda_s$ a *unitary operator* on $\ell_2(\Gamma)$ — $\lambda_s \in \mathcal{U}(\ell_2(\Gamma))$.

# Introducing the Left-Regular Representation, cont'd

The map $s \mapsto \lambda_s$ is a group homomorphism that represents $\Gamma$ as a unitary operator on $\ell_2(\Gamma)$, known as the *left-regular representation*.

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The map $s \mapsto \lambda_s$ is a group homomorphism that represents $\Gamma$ as a unitary operator on $\ell_2(\Gamma)$, known as the *left-regular representation*.

Unitary representations like the left-regular representation are incredibly useful.

# The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

# The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

This looks very similar to the approximate mean — and its equivalence with amenability is proven using approximate means.

# Our First $C^*$-Algebra

If $\Gamma$ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t)\delta_t,$$

where $\delta_t$ is the point mass at $t \in \Gamma$.

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where $\delta_t$ is the point mass at $t \in \Gamma$.

This becomes a $*$-algebra when endowed with multiplication and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g\left(s^{-1} t\right)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

# Our First $C^*$-Algebra, cont'd

If we represent $\mathbb{C}[\Gamma]$ as an operator on $\ell_2(\Gamma)$ by mapping $\pi_\lambda(\delta_t) \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\text{op}},$$

we get the *reduced group $C^*$-algebra* on $\Gamma$ by completing with respect to this norm.

# Nuclear $C^*$-Algebras

A $C^*$-algebra $A$ is called *nuclear* if there exist certain maps $\varphi_n \colon A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n \colon \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$ such that, for all $a \in A$,

$$\left\| a - \psi_n \circ \varphi_n(a) \right\| \xrightarrow{n \to \infty} 0.$$

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$$\left\| a - \psi_n \circ \varphi_n(a) \right\| \xrightarrow{n \to \infty} 0.$$

This is a lot of symbols to say that $a$ can be asymptotically approximated by these special maps.

# Nuclearity and Amenability

A group $\Gamma$ is amenable if and only if the reduced group $C^*$-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

# Contents

1. Definitions
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3. From Paradoxical Decompositions to Amenability
4. Equivalent Definitions and Other Criteria
   - A Taste of Functional Analysis
   - Introducing Approximations
   - Approximations with Representations and Operators
5. Remarks and Acknowledgments

# Final Remarks

Amenability — and characterizations of amenability — is still a very active field of study.

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Nuclear $C^*$-algebras are classified, so active research areas primarily concern whether or not certain classes of $C^*$-algebras are nuclear (hence classifiable).

There are a lot of other classifications of amenability that I would have loved to learn more about and discuss here, but unfortunately time, while discrete, is also finite and limited.

# Acknowledgments

A large thank you goes to

- the professors of the math department;
- friends, family, and acquaintances both in the math major and outside;
- everyone in attendance.

# Questions?