Introduction

It is my experience that proofs
involving matrices can be shortened
by 50% if one throws the matrices out.

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Essentially, the goal of this course is to prove a lot of the essential results of linear algebra without basis dependence (as in, using the properties of the linear transformations themselves rather than matrices).

Vector Spaces and Linear Transformations

Remark: We let $\mathbb{F}$ be either $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p$ (where p is a prime). Primarily, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Example (Our First Vector Space)**.** The primary vector space we study in lower-division linear algebra is

$$\begin{aligned} V &= \mathbb{R}^n \\ &= \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| a_1, \dots, a_n \in \mathbb{R} \right\} \end{aligned}$$

We know that for

$$\begin{aligned} v &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ w &= \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \end{aligned}$$

that

$$\begin{aligned} v + w &= \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \\ cv &= \begin{pmatrix} ca_1 \\ \vdots \\ ca_n \end{pmatrix}, \end{aligned}$$

where $c \in \mathbb{R}$ is some constant.

Definition (Vector Space)**.** Let V be a nonempty set with the following operations:

- $a : V \times V \to V$, $a(v, w) \mapsto v + w$ (vector addition);
- $m : F \times V \to V$, $m(c, v) \mapsto cv$ (scalar multiplication);

satisfying the following:

(1) there exists $0_v \in V$ such that $0_v + v = v = v + 0_v$ for all $v \in V$;

(2) for every $v \in V$, there exists $-v$ such that $v + (-v) = 0_v = (-v) + v$;

(3) for every $u, v, w \in V$, $(u + v) + w = u + (v + w)$;

(4) for every $v, w \in V$, $v + w = w + v$;

(5) for every $v, w \in V$ and $c \in \mathbb{F}$, $c(v + w) = cv + cw$;

(6) for every $c, d \in \mathbb{F}$, $v \in V$, $(c + d)v = cv + dv$;

(7) for every $c, d \in \mathbb{F}$, $v \in V$, $(cd)v = c(dv)$;

(8) for every $v \in V$, $(1_{\mathbb{F}})v = v$.

We say V is a $\mathbb{F}$-vector space.

Example ($\mathbb{F}^n$)**.** Let $\mathbb{F}$ be a field, $V = \mathbb{F}^n$.

$$V = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} | a_i \in \mathbb{F} \right\}.$$

Define:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$$
$$c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ \vdots \\ ca_n \end{pmatrix}.$$

We set

$$0_{\mathbb{F}^n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

$c, d \in \mathbb{F}$. We observe that

$$\begin{aligned} 0_{\mathbb{F}^n} + v &= \begin{pmatrix} 0 + v_1 \\ \vdots \\ 0 + v_n \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \end{aligned}$$

Define

$$-v = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}.$$

Then,

$$\begin{aligned} v + (-v) &= \begin{pmatrix} v_1 + (-v_1) \\ \vdots \\ v_n + (-v_n) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= 0_{\mathbb{F}^n}. \end{aligned}$$

Note that

$$\begin{aligned} (u + v) + w &= \begin{pmatrix} (u_1 + v_1) + w_1 \\ \vdots \\ (u_n + v_n) + w_n \end{pmatrix} \\ &= \begin{pmatrix} u_1 + (v_1 + w_1) \\ \vdots \\ u_n + (v_n + w_n) \end{pmatrix} \\ &= u + (v + w). \end{aligned}$$

We have

$$\begin{aligned} v + w &= \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} \\ &= \begin{pmatrix} w_1 + v_1 \\ \vdots \\ w_n + v_n \end{pmatrix} \\ &= w + v. \end{aligned}$$

Observe

$$\begin{aligned} c\left(v + w\right) &= c\begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} \\ &= \begin{pmatrix} c\left(v_1 + w_1\right) \\ \vdots \\ c\left(v_n + w_n\right) \end{pmatrix} \\ &= \begin{pmatrix} cv_1 + cw_1 \\ \vdots \\ cv_n + cw_n \end{pmatrix} \\ &= cv + cw, \\ (c + d)v &= (c + d)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= \begin{pmatrix} (c + d)v_1 \\ \vdots \\ (c + d)v_n \end{pmatrix} \\ &= \begin{pmatrix} cv_1 + dv_1 \\ \vdots \\ cv_n + dv_n \end{pmatrix} \\ &= cv + dv, \end{aligned}$$

and

$$\begin{aligned}(cd)v &= (cd)\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\\ &= \begin{pmatrix}(cd)v_1\\ \vdots\\ (cd)v_n\end{pmatrix}\\ &= \begin{pmatrix}c\,(dv_1)\\ \vdots\\ c\,(dv_n)\end{pmatrix}\\ &= c\,(dv)\,.\end{aligned}$$

Finally,

$$\begin{aligned}1_{\mathbb{F}} &= 1_{\mathbb{F}}\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\\ &= \begin{pmatrix}1_{\mathbb{F}}v_1\\ \vdots 1_{\mathbb{F}}v_n\end{pmatrix}\\ &= \begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\\ &= v.\end{aligned}$$

Example (Polynomials)**.** Let $n \in \mathbb{Z}_{\geqslant 0}$. We define

$$\mathcal{P}_n(\mathbb{F}) = \{a_0 + a_1x + \cdots + a_nx^n \,|a_i \in \mathbb{F}\}\,.$$

For $f(x) = \sum_{j=0}^{n} a_jx^j$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $\mathcal{P}_n(\mathbb{F})$, we have

$$\begin{aligned}f(x) + g(x) &= \sum_{j=0}^{n}\left(a_j + b_j\right)x^j\\ cf(x) &= \sum_{j=0}^{n}\left(ca_j\right)x^j.\end{aligned}$$

Note that these are not functions *per se*, we are only $f(x)$ and $g(x)$ to represent elements of $\mathcal{P}_n(\mathbb{F})$. We can verify that $\mathcal{P}_n(\mathbb{F})$ is a $\mathbb{F}$-vector space.

We define

$$\mathbb{F}[x] = \bigcup_{n\geqslant 0}\mathcal{P}_n(\mathbb{F})\,,$$

which is also a $\mathbb{F}$-vector space.

Example (Matrices)**.** Let $m, n \in \mathbb{Z}_{>0}$. We set

$$V = \operatorname{Mat}_{m,n}(\mathbb{F}),$$

which is the set of $m \times n$ matrices with entries in $\mathbb{F}$. This is an $\mathbb{F}$-vector space with matrix addition and scalar multiplication.

In the case where $m = n$, we write $\operatorname{Mat}_n(\mathbb{F})$ to denote $\operatorname{Mat}_{n,n}(\mathbb{F})$.

Example (Complex Numbers)**.** Let $V = \mathbb{C}$. Then, V is a $\mathbb{C}$-vector space, an $\mathbb{R}$-vector space, and a $\mathbb{Q}$-vector space.

Note that the properties of a vector space change with the underlying scalar field.

Lemma (Basic Properties of Vector Spaces)**.** *Let V be a $\mathbb{F}$-vector space.*

(1) 0_V *is unique.*

(2) $0_{\mathbb{F}}v = 0_V$.

(3) $(-1_{\mathbb{F}})v = -v$.

Proof.

(1) Suppose toward contradiction that there exist $0, 0'$ both satisfy

$$0 + v = v \tag{$*$}$$
$$0' + v = v. \tag{$**$}$$

Then,

$$\begin{aligned} 0 + v &= v \\ 0 + 0' &= 0' && \text{by } (*) \text{ with } v = 0' \\ &= 0' + 0 \\ &= 0. && \text{by } (**) \text{ with } v = 0 \end{aligned}$$

(2) Note

$$\begin{aligned} 0_{\mathbb{F}}v &= (0_{\mathbb{F}} + 0_{\mathbb{F}})v \\ &= 0_{\mathbb{F}}v + 0_{\mathbb{F}}v. \end{aligned}$$

We subtract $0_{\mathbb{F}}v$ from both sides.

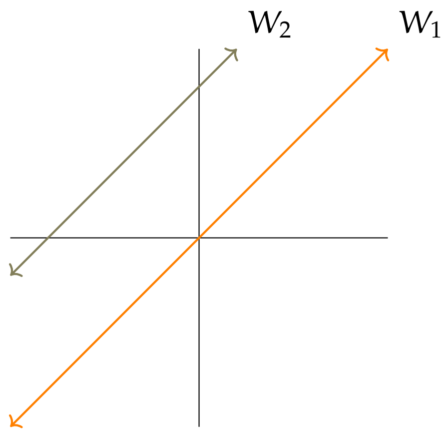
(3)

$$\begin{aligned} (-1_{\mathbb{F}})v + v &= (-1_{\mathbb{F}})v + 1_{\mathbb{F}}v \\ &= (-1_{\mathbb{F}} + 1_{\mathbb{F}})v \\ &= 0_{\mathbb{F}}v. \end{aligned}$$

□

Definition (Subspaces)**.** Let V be an $\mathbb{F}$-vector space. We say $W \subseteq V$ is an $\mathbb{F}$-subspace (henceforth subspace) if W is an $\mathbb{F}$-vector space under the same addition and scalar multiplication.

Example (Subspaces of $\mathbb{R}^2$)**.** Let $V = \mathbb{R}^2$.



Here, we see that W_1 is a subspace, and W_2 is not a subspace (as W_2 does not contain 0_V).

Example (Subspaces of $\mathbb{C}$)**.** Let $V = \mathbb{C}$, $W = \{a + 0i \mid a \in \mathbb{R}\}$.

- If $\mathbb{F} = \mathbb{R}$, then W is a subspace of V.
- If $\mathbb{F} = \mathbb{C}$, then W is not a subspace; we can see that $2 \in W$, $i \in \mathbb{C}$, but $2i \notin W$.

Example (Matrices)**.** It is not the case that $\text{Mat}_2(\mathbb{R})$ is a subspace of $\text{Mat}_4(\mathbb{R})$, since $\text{Mat}_2(\mathbb{R})$ is not a subset of $\text{Mat}_4(\mathbb{R})$.

Example (Polynomials)**.** For the spaces $\mathcal{P}_m(\mathbb{F})$ and $\mathcal{P}_n(\mathbb{F})$, if $m \leqslant n$, then $\mathcal{P}_m(\mathbb{F})$ is a subspace of $\mathcal{P}_n(\mathbb{F})$.

Lemma (Proving Subspace Relation)**.** *Let V be a $\mathbb{F}$-vector space, $W \subseteq V$. Then, W is a subspace of V if*

(1) W is nonempty;

(2) W is closed under addition;

(3) W is closed under scalar multiplication.

Proof. The proof is an exercise. □

Definition (Linear Transformation)**.** Let V, W be $\mathbb{F}$-vector spaces. Let $T : V \to W$. We say T is a linear transformation (or linear map) if for every $v_1, v_2 \in V$, $c \in \mathbb{F}$, we have

$$T(v_1 + cv_2) = T(v_1) + cT(v_2).$$

Note that on the left side, addition is in V, and on the right side, addition is in W.

The collection of all linear maps from V to W is denoted $\text{Hom}_{\mathbb{F}}(V, W)$, or $\mathcal{L}(V, W)$.

Example (Identity Transformation)**.** Define

$$\mathrm{id}_V : V \to V,$$

where $\mathrm{id}_V(v) = v$. We can see that $\mathrm{id}_V \in \mathrm{Hom}_{\mathbb{F}}(V, V)$, since

$$\begin{aligned}\mathrm{id}_V(v_1 + cv_2) &= v_1 + cv_2\\ &= \mathrm{id}_V(v_1) + (c)(\mathrm{id}_V(v_2))\end{aligned}$$

Example (Complex Conjugation)**.** Let $V = \mathbb{C}$. Define $T : V \to V$ by $z \mapsto \overline{z}$.

We may ask whether $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ or $T \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.

$$\begin{aligned}T(z_1 + cz_1) &= \overline{z_1 + cz_2}\\ &= \overline{z_1} + (\overline{c})(\overline{z_2}).\end{aligned}$$

We can see that $T(z_1 + cz_2) = T(z_1)cT(z_2)$ if and only if $c = \overline{c}$, meaning c must be real. This means $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$, but $T \notin \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.

Example (Matrices)**.** Let $A \in \mathrm{Mat}_{m,n}(\mathbb{F})$. We define

$$\begin{aligned}T_A : \mathbb{F}^n &\to \mathbb{F}^m\\ x &\mapsto Ax.\end{aligned}$$

Then, $T_A \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^m)$.

Example (Linear Maps on Smooth Functions)**.** Let $V = C^{\infty}(\mathbb{R})$, which denotes the set of continuous functions with continuous derivatives at all orders. This is a vector space under pointwise addition and scalar multiplication.

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ (cf)(x) &= (c)(f(x)).\end{aligned}$$

Let $a \in \mathbb{R}$.

(1)

$$\begin{aligned}E_a : V &\to \mathbb{R}\\ f &\mapsto f(a).\end{aligned}$$

Then, $E_a \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

(2)

$$\begin{aligned}D : V &\to V\\ f &\mapsto f'.\end{aligned}$$

Then, $D \in \mathrm{Hom}_{\mathbb{R}}(V, V)$.

(3)

$$I_a : V \longrightarrow V f \mapsto \int_a^x f(t)\, dt.$$

Then, $I_a \in \operatorname{Hom}_{\mathbb{R}}(V, V)$.

(4) With a a function,

$$\tilde{E}_a : V \longrightarrow V f \mapsto f \circ a.$$

Then, $\tilde{E}_a \in \operatorname{Hom}_{\mathbb{R}}(V, V)$.

Additionally,

- $D \circ I_a = \mathrm{id}_V$;
- $I_a \circ D = \mathrm{id}_V - \tilde{E_a}$ for some $a \in \mathbb{R}$.