Problem 1

Prove the following limits:

(i) $\left(\frac{2n}{n+2}\right)_n \to 2$

(ii) $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0$

(iii) $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0$

(iv) $\left(n^k b^n\right)_n \to 0$ where $0 \leq b < 1$ and $k \in \mathbb{N}$

(v) $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \to 1/3$

(i)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \Rightarrow \left|\frac{2n}{n+2} - 2\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned} \frac{2n}{n+2} &> 2 - \varepsilon \\ 2n &> (2n - \varepsilon n) - 2\varepsilon + 4 \\ n &> \frac{4 - 2\varepsilon}{\varepsilon} \end{aligned}$$

Proof Let $N = \left\lceil \frac{4-2\varepsilon}{\varepsilon} \right\rceil$. Then,

$$\begin{aligned} n &> \frac{4 - 2\varepsilon}{\varepsilon} \\ \varepsilon n &> 4 - 2\varepsilon \\ 0 &> 4 - 2\varepsilon - \varepsilon n \\ 2n &> 2n + 4 - \varepsilon(n+2) \\ 2n &> (2 - \varepsilon)(n+2) \\ \frac{2n}{n+2} - 2 &> -\varepsilon \\ \left|\frac{2n}{n+2} - 2\right| &< \varepsilon \qquad \frac{2n}{n+2} < 2\ \forall n \in \mathbb{N} \end{aligned}$$

(ii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \to \left|\left(\frac{\sqrt{n}}{n+1}\right)\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}\frac{\sqrt{n}}{n+1} &< \frac{\sqrt{n}}{n}\\ &< \varepsilon\\ \frac{1}{\sqrt{n}} &< \varepsilon\\ n &> \frac{1}{\varepsilon^2}\end{aligned}$$

Proof Let $N = \left\lceil \frac{1}{\varepsilon^2} \right\rceil$. Then,

$$\begin{aligned}n &> \frac{1}{\varepsilon^2}\\ \frac{1}{\sqrt{n}} &< \varepsilon\\ \frac{\sqrt{n}}{n+1} &< \frac{\sqrt{n}}{n}\\ &< \varepsilon\end{aligned}$$

(iii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \Rightarrow \left|\frac{(-1)^n}{\sqrt{n+7}}\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}
\frac{1}{\sqrt{n+7}} &< \varepsilon \\
\frac{1}{\varepsilon} &< \sqrt{n+7} \\
n &> \frac{1}{\varepsilon^2} - 7
\end{aligned}$$

Proof Let $N = \left\lceil \frac{1}{\varepsilon^2} \right\rceil - 7$. Then,

$$\begin{aligned}
n &> \frac{1}{\varepsilon^2} - 7 \\
n + 7 &> \frac{1}{\varepsilon^2} \\
\frac{1}{\sqrt{n+7}} &< \varepsilon \\
-\varepsilon &< \frac{-1}{\sqrt{n+7}} \\
\left|\frac{(-1)^n}{\sqrt{n+7}}\right| &< \varepsilon
\end{aligned}$$

(iv)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \to \left|n^k b^n\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}
n^k b^n &< \varepsilon \\
b^n &< \frac{\varepsilon}{n^k} \\
n &< \frac{\ln \varepsilon - k \ln n}{\ln b} \\
n \ln b &> \ln \varepsilon - k \ln n \\
k \ln n + n \ln b &> \ln \varepsilon \\
kn + n \ln b &> k \ln n + n \ln b \\
&> \ln \varepsilon \\
n &> \frac{\ln \varepsilon}{k + \ln b}
\end{aligned}$$

Proof Let $k \in \mathbb{N}$ and $0 \leq b < 1$. For the trivial case $b = 0$, let $N = 1$. Then, $|(1)(0)| < \varepsilon \; \forall \varepsilon > 0$. Otherwise, let $N = \left\lceil \frac{\ln \varepsilon}{k + \ln b} \right\rceil$

$$\begin{aligned}
n &> \frac{\ln \varepsilon}{k + \ln b} \\
kn + n \ln b &> \ln \varepsilon \\
kn &> \ln \varepsilon - n \ln b \\
\frac{k \ln n}{\ln b} < \frac{kn}{\ln b} &< \frac{\ln \varepsilon - n \ln b}{\ln b} && \text{since } \ln b < 0 \\
\log_b \left(n^k\right) &< \log_b \left(\frac{\varepsilon}{b^n}\right) \\
n^k &< \frac{\varepsilon}{b^n} \\
b^n n^k &< \varepsilon
\end{aligned}$$