Contents

Introduction

Finally, the last part of my notes on C*-algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of C*-algebras, discussing ideas such as amenability and nuclearity in C*-algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of $C^*_\lambda(G)$ and $C^*(G)$.

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's *C*-Algebras and Finite-Dimensional Approximations*.

Review: Representations, the Reduced Group C^*-Algebra, and the Universal Group C^*-Algebra

Left-Regular Representation

Let Γ be a group. Consider the space $\ell_2(\Gamma)$. For every $s \in \Gamma$, we define the operator

$$\lambda_s(\xi)(t) = \xi\left(s^{-1}t\right).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned}\|\lambda_s(\xi)\|^2 &= \sum_{t\in\Gamma}|\lambda_s(\xi)(t)|^2\\ &= \sum_{t\in\Gamma}\left|\xi\left(s^{-1}t\right)\right|^2\\ &= \sum_{r\in\Gamma}|\xi(r)|^2\\ &= \|\xi\|^2.\end{aligned}$$

Additionally, each λ_s admits an inverse, $\lambda_{s^{-1}} = \lambda_s^*$. Applying to the orthonormal basis $\{\delta_t\}_{t\in\Gamma}$, we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus, $\lambda_s \circ \lambda_r = \lambda_{sr}$, and we have the unitary representation of Γ, $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, where $\lambda(s) = \lambda_s$, for $s \in \Gamma$. This is the left-regular representation of Γ.

Note that the left regular representation is a faithful representation, hence injective.

Because the λ operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}\lambda_f(\xi)(t) &= \left(\sum_{s\in\Gamma} f(t)\lambda_s(\xi)\right)(t)\\ &= \sum_{s\in\Gamma} f(s)\xi\left(s^{-1}t\right)\end{aligned}$$

Note that the space of finitely supported functions on Γ, $\mathbb{C}[\Gamma]$,[1] is a $*$-algebra, where multiplication is given by convolution:

$$\begin{aligned}f * g(t) &= \sum_{s\in\Gamma} f(s)g\left(s^{-1}t\right)\\ &= \sum_{r\in\Gamma} f\left(tr^{-1}\right)g(r).\end{aligned}$$

Note that we are using $*$ both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on $\mathbb{C}[\Gamma]$ is given by

$$f^*(t) = \overline{f\left(t^{-1}\right)}.$$

A Bit on Representations and C^*-(Semi)norms

A C*-seminorm on a $*$-algebra is a seminorm such that defined by

- $\|ab\| \leqslant \|a\|\|b\|$;
- $\|a^*\| = \|a\|$;
- $\|a^*a\| = \|a\|^2$.

If A_0 is a $*$-algebra, then a representation of A_0 is a pair $(\pi_0, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi\colon A_0 \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism.

Additionally, if A_0 is a $*$-algebra with representation π_0, then we have C*-seminorm

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{op}.$$

If π_0 is injective, then $\|\cdot\|_{\pi_0}$ is a C*-norm. If π_0 is a C*-norm, then the completion of A_0 with respect to $\|\cdot\|_{\pi_0}$ is a C*-algebra.

The universal norm on A_0 is defined as

$$\|a\|_u = \sup_{p\in\mathcal{P}} p(a),$$

where $\mathcal{P}$ is the collection of all C*-seminorms on A_0. If $\|a\|_u < \infty$ for all $a \in A_0$, then $\|\cdot\|_u$ is a C*-seminorm on A_0. Note that if one of $p \in \mathcal{P}$ is a norm, then $\|\cdot\|_u$ defines a C*-norm on A_0.

If we have the unitary representation $u\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$, then

$$\pi_u(a) = \sum_{s\in\Gamma} u_s$$

[1] Also known as the free vector space over $\mathbb{C}$ with basis Γ.

is a representation of $\mathbb{C}[\Gamma]$. If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ is the left-regular representation, then the left-regular group C*-algebra is the group $*$-algebra with C*-norm defined by $\|a\| = \|\pi_\lambda(a)\|$.

The universal group C*-algebra is defined as the norm completion of

$$\|a\|_u = \sup\left\{ \|\pi(a)\|_{op} \;\middle|\; \pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H}_\pi) \right\}.$$

Note that

$$\begin{aligned}\|\pi(a)\| &= \left\|\pi\left(\sum_{s\in\Gamma} a_s\delta_s\right)\right\| \\ &= \left\|\sum_{s\in\Gamma} a_s\pi(\delta_s)\right\| \\ &\leqslant \sum_{s\in\Gamma}\|a_s\pi(\delta_s)\| \\ &= \sum_{s\in\Gamma}|a_s|.\end{aligned}$$

Note that since $\|\cdot\|_\lambda$ is a norm, we must have $a = 0$ if and only if $\|a\|_u = 0$. The full group C*-algebra admits a universal property.

Proposition: Let Γ be a discrete group. If $u\colon \Gamma \to \mathcal{U}(\mathcal{H})$, then there is a contractive $*$-homomorphism $\pi_u\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ that satisfies $\pi_u(\delta_s) = u(s)$.

Using the Left-Regular Representation to Establish Amenability

If $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\mathcal{H}$, then a vector $\xi \in \mathcal{H}$ is called invariant for π if $\pi(g)(\xi) = \xi$ for all $g \in \Gamma$.

Proposition: The left-regular representation for Γ admits an invariant vector if and only if Γ is finite.

Proof. Let Γ be finite. Since Γ is finite, all functions $a\colon \Gamma \to \mathbb{C}$ are square-summable. Thus, $\xi = \mathbb{1}_\Gamma$ is square-summable, and since $s\Gamma = \Gamma$ for all $s \in \Gamma$, we have $\mathbb{1}_\Gamma$ is invariant for λ.

Now, let $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ be the left-regular representation, and suppose there is $\xi \in \ell_2(\Gamma)$ such that for all $s \in \Gamma$, we have

$$\lambda_s(\xi) = \xi.$$

In particular, this means that for any $t \in \Gamma$, we have

$$\begin{aligned}\lambda_s(\xi)(t) &= \xi\left(s^{-1}t\right) \\ &= \xi(t).\end{aligned}$$

Since this holds for all $s \in \Gamma$, we have that $\xi = c\mathbb{1}_\Gamma$ for some $c \in \mathbb{C}$. However, since $\xi \in \ell_2(\Gamma)$, we must have that $\sum_{t\in\Gamma}|c|^2 < \infty$, which only holds if Γ is finite. □

An almost-invariant vector for a representation $\pi\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, as the name suggests,[II] a sequence (or net) of unit vectors $(\xi_i)_{i\in I}$ such that

$$\lim_{i\in I}\|\pi(g)(\xi_i) - \xi_i\| = 0.$$

[II] I'm only mostly being facetious here.

Theorem: A group Γ is amenable if and only if the left-regular representation has an almost-invariant vector.

Proof. Let Γ be amenable, and let F_i be a Følner sequence — $\frac{|sF_i \triangle F_i|}{|F_i|} \to 0$ for all $s \in \Gamma$. Define $\xi_i = \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}$. Thus,

$$\begin{aligned}
\|\lambda_s(\xi_i) - \xi_i\|^2 &= \sum_{t\in\Gamma} |\lambda_s(\xi_i)(t) - \xi_i(t)|^2\\
&= \sum_{t\in\Gamma}\left|\lambda_s\left(\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right)(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right|^2\\
&= \sum_{t\in\Gamma}\left|\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t)\right|^2\\
&= \frac{|sF_i \triangle F_i|}{|F_i|}.
\end{aligned}$$

Thus, λ has an almost-invariant vector.

Suppose there exists an almost-invariant vector $(\xi_i)_i \in \ell_2(\Gamma)$. It is sufficient to construct an approximate mean. Since $\xi_i \in \ell_2(\Gamma)$, we have that $\xi_i^2 \in \ell_1(\Gamma)$. Setting $\mu_i = \xi_i^2$, we plug this into the expression for an approximate mean, and obtain

$$\begin{aligned}
\|\lambda_s(u_i) - u_i\|_{\ell_1} &= \sum_{t\in\Gamma}\left|\lambda_s\left(\xi_i^2\right)(t) - \xi_i^2(t)\right|\\
&= \sum_{t\in\Gamma}|(\lambda_s(\xi_i)(t) - \xi_i(t))(\lambda_s(\xi_i)(t) + \xi_i(t))|\\
&= \|(\lambda_s(\xi_i) - \xi_i)(\lambda_s(\xi_i) + \xi_i)\|_{\ell_1}\\
&\leqslant \|\lambda_s(\xi_i) - \xi_i\|_{\ell_2}\|\lambda_s(\xi_i) + \xi_i\|\\
&\leqslant 2\|\lambda_s(\xi_i) - \xi_i\|\\
&\to 0.
\end{aligned}$$

Thus, μ_i is an approximate mean. □

Using the criterion of almost invariant vectors, we may show that a group is amenable if and only if the trivial representation — defined by $1_\Gamma\colon \Gamma \to \mathbb{C}$, $1_\Gamma(g) = 1$ is what is known as weakly contained in the left-regular representation.

A representation $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is weakly contained in another representation $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$, denoted $\pi \prec \rho$, if for every $\xi \in \mathcal{H}$, finite $E \subseteq \Gamma$, and $\varepsilon > 0$, then there are $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that

$$\left|\langle \pi(g)(\xi), \xi\rangle - \sum_{i=1}^{n}\langle \rho(g)(\eta_i), \eta_i\rangle\right| < \varepsilon.$$

Theorem: A discrete group Γ is amenable if and only if $1_\Gamma \prec \lambda$, where λ is the left-regular representation.

Proof. We show that $1_\Gamma \prec \lambda$ is equivalent to the existence of an almost invariant vector for λ. We assume λ admits an almost-invariant vector. It is sufficient to show that for every $\varepsilon > 0$ and every finite set $E \subseteq \Gamma$, there are $\eta_1, \dots, \eta_n \in \ell_2(\Gamma)$ such that

$$\left|1 - \sum_{i=1}^{n}\langle \lambda_t(\eta_i), \eta_i\rangle\right| < \varepsilon$$

for every $t \in E$. If we take $n = 1$ and $\eta_1 = \xi$, where ξ is almost-invariant for all $g \in E$ — i.e., $\left\|\lambda_g(\xi) - \xi\right\|_{\ell_2} < \varepsilon$ for all $g \in E$. Note that we have

$$\begin{aligned}\left\|\lambda_g(\xi) - \xi\right\|^2 &= \left\langle \lambda_g(\xi) - \xi, \lambda_g(\xi) - \xi\right\rangle \\ &= \left\langle \lambda_g(\xi), \lambda_g(\xi)\right\rangle + \langle \xi, \xi\rangle - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &= 2 - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &= 2\operatorname{Re}\left(1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &\leqslant 2\left|1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right|.\end{aligned}$$

Additionally,

$$\begin{aligned}\left|1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right|^2 &= \left(1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right)\left(1 - \overline{\left\langle \lambda_g(\xi), \xi\right\rangle}\right) \\ &= 1 - \overline{\left\langle \lambda_g(\xi), \xi\right\rangle} - \left\langle \lambda_g(\xi), \xi\right\rangle + \left|\left\langle \lambda_g(\xi), \xi\right\rangle\right|^2 \\ &\leqslant 2 - 2\operatorname{Re}\left(\left\langle \lambda_g(\xi), \xi\right\rangle\right) \\ &= \left\|\lambda_g(\xi) - \xi\right\|^2.\end{aligned}$$

Thus, we have that

$$\begin{aligned}\left|1 - \left\langle \lambda_g(\xi), \xi\right\rangle\right| &\leqslant \left\|\lambda_g(\xi) - \xi\right\| \\ &< \varepsilon.\end{aligned}$$

We start by showing that $1_\Gamma \prec \lambda$ if and only if for every finite $S \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\|\lambda_s(\xi) - \xi\|_{\ell_2} < \varepsilon.$$

In the forward direction, we see that there exists a unit vector ξ such that $|1 - \langle \lambda_s(\xi), \xi\rangle| < \varepsilon^2/2$, meaning $\|\lambda_s(\xi) - \xi\| < \varepsilon$ by above. Similarly, if $\|\lambda_s(\xi) - \xi\| < \varepsilon$, then $1_\Gamma \prec \lambda$.

Now, we assume $1_\Gamma \prec \lambda$. Thus, for a finite $E \subseteq \Gamma$ and $\varepsilon > 0$, then there exists $f \in \ell_2(\Gamma)$ with $\|f\|_{\ell_2} = 1$ such that $\|\lambda_s(f) - f\| < \varepsilon$ for all $s \in E$.

Setting $g = |f|^2$, we have $g \in \ell_1(\Gamma)$. From Hölder's inequality, we have

$$\begin{aligned}\|\lambda_s(g) - g\|_{\ell_1} &\leqslant \left\|\lambda_s\left(\overline{f}\right) + \overline{f}\right\|_{\ell_2}\|\lambda_s(f) - f\| \\ &\leqslant 2\|\lambda_s(f) - f\|_{\ell_2} \\ &< 2\varepsilon.\end{aligned}$$

Thus, Γ admits an approximate mean, hence is amenable. □

Having obtained some more resources on Kesten's criterion, we can now prove that.

Definition. Let $\lambda\colon \Gamma \to \mathbb{B}(\ell_2(\Gamma))$ be the left-regular representation. Then, for a finite set $E \subseteq \Gamma$, we define the Markov operator $M(E)$ by

$$M(E) = \sum_{t\in E} \lambda_t.$$

Note that since λ_t is an isometry for each t, we have

$$\|M(E)\|_{op} = \left\|\frac{1}{|E|}\sum_{t\in E}\lambda_t\right\|_{op}$$

$$\begin{aligned}
&= \frac{1}{|E|}\left\|\sum_{t\in E}\lambda_t\right\|_{op}\\
&\leqslant \frac{1}{|E|}\sum_{t\in E}\|\lambda_t\|_{op}\\
&= 1,
\end{aligned}$$

so the Markov operator is a bounded operator (indeed, a contraction).

Theorem (Kesten's Criterion)**:** Let Γ contain a finite symmetric generating set S. Then, Γ is amenable if and only if

$$\|M(S)\|_{op} = 1.$$

Proof. Let Γ be amenable. Then, λ admits an almost-invariant vector, $(\xi_n)_n \subseteq S_{\ell_2(\Gamma)}$, such that

$$\|\lambda_s(\xi_n) - \xi_n\|_{\ell_2} \to 0$$

for all $s \in \Gamma$. In particular, we have

$$\begin{aligned}
\left|\left(\left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n)\right\|_{\ell_2}\right) - \|\xi_n\|_{\ell_2}\right| &\leqslant \left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n) - \xi_n\right\|_{\ell_2}\\
&= \frac{1}{|S|}\left\|\left(\sum_{t\in S}\lambda_t\right)(\xi_n) - |S|\xi_n\right\|_{\ell_2}\\
&\leqslant \frac{1}{|S|}\sum_{t\in S}\|\lambda_t(\xi_n) - \xi_n\|_{\ell_2}\\
&\to 0,
\end{aligned}$$

meaning that

$$\sup_{\xi\in S_{\ell_2(\Gamma)}}\left\|\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi)\right\| = \|\xi\|,$$

and so the norm of the Markov operator is 1.

Suppose

$$\left\|\frac{1}{|S|}\sum_{t\in S}\lambda_t\right\|_{op} = 1,$$

or

$$\left\|\sum_{t\in S}\lambda_t\right\|_{op} = |S|.$$

Proposition: If $T \in \mathbb{B}(\mathcal{H})$ is a self-adjoint operator, then

$$\|T\|_{op} = \sup_{x\in S_{\mathcal{H}}}|\langle T(x), x\rangle|.$$

Proof. We have that

$$|\langle T(x), x\rangle| \leqslant \|T(x)\|\|x\|$$

$$\begin{aligned}&\leqslant \|T\|_{op}\|x\|^2\\&= \|T\|_{op}.\end{aligned}$$

Now, we seek to establish the opposite direction. Note that since T is self-adjoint, we know that $\langle T(x), x\rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$, so by the polarization identity, we have that

$$\langle T(x), y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle).$$

Note that we know that

$$\|T\|_{op} = \sup_{x,y\in S_{\mathcal{H}}} |\langle T(x), y\rangle|.$$

Now, we set $\alpha = \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|$. Note that for any nonzero $x \in \mathcal{H}$, we have

$$\begin{aligned}\left|\left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle\right| &\leqslant \alpha\\ |\langle T(x), x\rangle| &\leqslant \alpha\|x\|^2.\end{aligned}$$

Now, for any $x, y \in \mathcal{H}$, we may assume that $\langle T(x), y\rangle \in \mathbb{R}$, as we may multiply $\langle T(x), y\rangle$ by $\operatorname{sgn}(\langle T(x), y\rangle)$. Thus, by the polarization identity and the fact that T is self-adjoint, we have

$$\begin{aligned}\langle T(x), y\rangle &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)\\ |\langle T(x), y\rangle| &= \left|\frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)\right|\\ &\leqslant \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|)\\ &\leqslant \frac{\alpha}{4}\left(\|x+y\|^2 + \|x-y\|^2\right)\\ &= \frac{\alpha}{4}\left(2\|x\|^2 + 2\|y\|^2\right)\\ &= \alpha.\end{aligned}$$

Thus, we have $\|T\|_{op} \leqslant \sup_{x\in S_{\mathcal{H}}} |\langle T(x), x\rangle|$. □

Now, since S is symmetric, we have that $M(S)$ is self-adjoint. Therefore, we know that there is some $\xi_n \in S_{\mathcal{H}}$ such that

$$\begin{aligned}1 - \frac{1}{n} &< \left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n), \xi_n\right\rangle\\ &\leqslant \left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(|\xi_n|), |\xi_n|\right\rangle.\end{aligned}$$

Thus, rearranging, we have

$$1 - \left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(|\xi_n|), |\xi_n|\right\rangle < \frac{1}{n}.$$

Since $M(S)$ is a self-adjoint operator, we have that $\operatorname{Re}\left(\left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n), \xi_n\right\rangle\right) = \left\langle\left(\frac{1}{|S|}\sum_{t\in S}\lambda_t\right)(\xi_n), \xi_n\right\rangle$. This gives

$$\left\|\left(\frac{1}{S}\sum_{t\in S}\lambda_t\right)(\xi) - \xi\right\| \leqslant \frac{1}{|S|}\sum_{t\in S}\|\lambda_t(\xi) - \xi\|$$

$$
\begin{aligned}
&\leqslant \frac{1}{|S|}\sum_{t\in S}\sqrt{2}|1 - \langle\lambda_t(\xi),\xi\rangle| \\
&= \sqrt{2}\left|1 - \frac{1}{|S|}\sum_{t\in S}\langle\lambda_t(\xi),\xi\rangle\right| \\
&\to 0.
\end{aligned}
$$

Thus, λ admits an almost-invariant vector. □