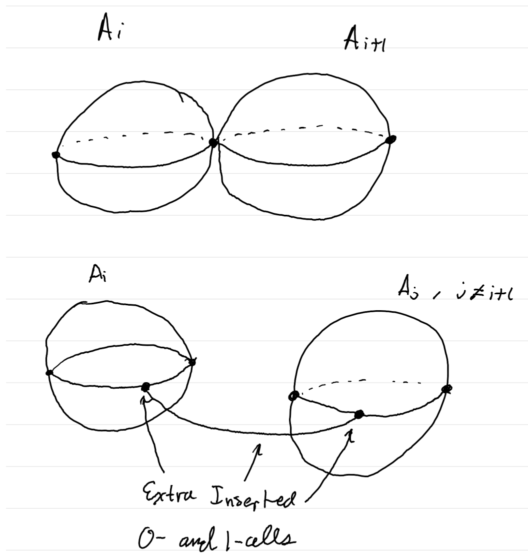
Revised Problems

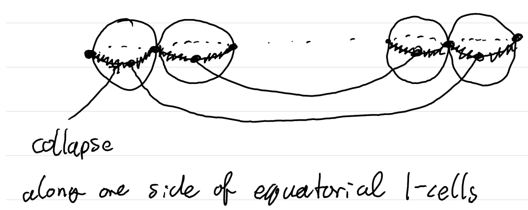
Problem (Homework 5, Problem 1)**:** If X is a connected space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy-equivalent to a wedge sum of 1-spheres and 2-spheres.

Solution: We start by constructing a space that is homotopy-equivalent to X as follows. Label the spheres in X by $A_1, \dots, A_n$, and equip each A_i with the cell complex structure consisting of two 0-cells, two 1-cells, and two 2-cells. Line up these spheres such that A_i and A_{i+1} have non-empty intersection for each $0 \leq i \leq n-1$, renumbering as necessary; such an arrangement must exist since X is connected. In this arrangement, we consider A_i and A_{i+1} to be adjoined at a common 0-cell.

Beyond this arrangement, if there are A_i and A_j adjacent to each other in X but not adjacent to each other in this line of spheres, insert a 0-cell along the equator of both A_i and A_j, then insert a 1-cell connecting them. Observe that upon completion of this process, this new space is homotopy-equivalent to X, as X is homeomorphic to the quotient by this collection of inserted 1-cells. Without loss of generality, we may assume that all these newly inserted 1-cells are connected along the same side of the equator. The figure below displays this idea.



Upon collapsing via the subcomplex described in the following figure, we observe that, since this subcomplex is contractible, as it is a collection of 0-cells and 1-cells that does not admit any sub-subcomplex homeomorphic to S^1, we get a homotopy equivalence where each of the spheres and each of the extra 1-cells are all identified with a single point, which is a wedge sum of 1-spheres and 2-spheres.



Current Problems

Problem (Problem 1)**:** Prove that for any space X, the following definitions are equivalent:

(a) any map $S_1 \to X$ is homotopic to a constant map;

(b) any map $S_1 \to X$ can be extended to a map $D^2 \to X$;

(c) $\pi_1(X, x_0) = 0$ for any $x_0 \in X$.

Solution: Let $F\colon S^1 \times [0,1] \to X$ be a homotopy between an arbitrary $f\colon S^1 \to X$ and a constant map c. Since D^2 is convex, we may use the straight-line homotopy $H(s,t) = (1-t)e^{is}$ that maps from S^1 to $0 \in D^2$.

The extension $\hat{f}\colon D^2 \to X$, given by

$$\hat{f}\left(re^{is}\right) = F(s, 1-r),$$

where $s \in [0, 2\pi)$ and $r \in [0,1]$, is a composition of two continuous functions, $F(s,t)$ and $r \mapsto 1-r$, so it is continuous with $\hat{f}\left(e^{is}\right) = F(s,0) = f\left(e^{is}\right)$.

Now, suppose any map $S^1 \to X$ has an extension to a map $D^2 \to X$. Consider a pointed map from S^1 to $x_0 \in X$ taking $1 = e^{i0} \mapsto x_0$, which we call f. Then, f has an extension to the closed unit disk, which we call $\hat{f}$, which maps from D^2 to X and takes $S_1 \mapsto f\left(S^1\right)$. Using the straight-line homotopy (and the triangle inequality), we have that $H\left(re^{is}, t\right) = t + (1-t)re^{is}$ is such that every point in D^2 maps to 1, so upon composing this homotopy with $\hat{f}$, we find that f is homotopic to the constant map at x_0, meaning that $\pi_1(X, x_0) = 0$.

If $\pi_1(X, x_0) = 0$ for any $x_0 \in X$, then for any $x_0 \in X$, it follows that any map $f\colon S^1 \to X$ has the same homotopy class as the constant map at x_0, giving (a).

Problem (Problem 2)**:** We have talked about how $\pi_1(X, x_0)$ can be thought of as homotopy classes of basepoint-preserving maps $\left(S^1, s_0\right) \to (X, x_0)$. Consider now the set of homotopy classes of non-basepoint-preserving maps $S^1 \to X$, denoted $\left[S^1, X\right]$. Forgetting basepoints, we obtain a map $\Psi\colon \pi_1(X, x_0) \to \left[S^1, X\right]$.

(a) Prove that if X is path-connected, then Ψ is onto.

(b) Prove that $\Psi[f] = \Psi[g]$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Solution:

(a) Let X be path-connected, and let $[f]$ be a homotopy class of a map $f\colon S^1 \to X$; define $y_0 = f(s_0)$. Since X is path-connected, there is a path γ from x_0 to y_0; the path defined by $\gamma \cdot f \cdot \overline{\gamma}$ is then a loop based at x_0, so it admits a homotopy class $[\gamma \cdot f \cdot \overline{\gamma}] \in \pi_1(X, x_0)$. We claim that, in fact, $[\gamma \cdot f \cdot \gamma] = [f]$.

To see this, consider the homotopy

$$H(s,t) = \begin{cases} \gamma(t) & 0 \leq s \leq t \\ \gamma(s) & t \leq s \leq 1 \end{cases},$$

which is continuous (since γ) is continuous, and has $[\gamma] = [c_{y_0}]$. By defining $\gamma_t \coloneqq H(\cdot, t)$, we then have a continuous 1-parameter family that yields the desired homotopy $[\gamma \cdot f \cdot \overline{\gamma}] \simeq [c_{y_0} \cdot f \cdot \overline{c_{y_0}}] = [f]$. Thus, when X is path-connected, Ψ is onto.

(b) Without loss of generality, we let $1 = e^{i0}$ be the basepoint for S^1. Suppose $\Psi[f] = \Psi[g]$, where f and g are pointed maps taking $1 \mapsto x_0$. Let H be a homotopy between f and g, considered in $\left[S^1, X\right]$; when we restrict our view to $H(1,t)$, we observe that $H(1,t)$ is a path in X, where $H(1,0) = f(1) = x_0$ and $H(1,1) = g(1) = x_0$, meaning that $H(1,t)$ is a loop based at x_0. Call it $\gamma(t)$.

I'm not exactly sure where to go from here, but I think the idea would be to show that either $f \cdot \gamma \simeq \gamma \cdot g$ or $\gamma \cdot f \simeq g \cdot \gamma$, which would show that $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.