Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a nonempty open set.

Given a sequence $(z_n)_n \subseteq U$, we write $z_n \to \partial U$ if, for every compact subset $K \subseteq U$, there exists some $N = N(K) \in \mathbb{N}$ such that $z_n \notin K$ whenever $n \geqslant N$.

Given a function $u\colon U \to \mathbb{R}$, define

$$\limsup_{z \to \partial U} u(z) = \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{z \in U \setminus K} u(z).$$

(a) For each positive integer $n \in \mathbb{N}$, define

$$K_n := \left\{ z \in U \;\middle|\; |z| \leqslant n, \operatorname{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{n} \right\}.$$

Show that:

(i) each K_n is compact;

(ii) $K_n \subseteq K_{n+1}^{\circ}$;

(iii) $U = \bigcup_{n=1}^{\infty} K_n$.

(b) Let $L := \limsup_{z \to \partial U} u(z)$.

(i) Show that for each $S > L$, there is a compact subset $K \subseteq U$ such that $u(z) \leqslant S$ for all $z \in U \setminus K$.

(ii) Show that there exists a sequence $(z_n)_n$ in U with $z_n \to \partial U$ and $\limsup_{n \to \infty} u(z_n) \leqslant L$.

(c) Prove that

$$\limsup_{z \to \partial U} u(z) = \sup_{\substack{(z_n)_n \subseteq U \\ z_n \to \partial U}} \limsup_{n \to \infty} u(z_n),$$

where the supremum is over all sequences $(z_n)_n$ with $(z_n)_n \to \partial U$.

Solution:

(a) We claim that the set

$$C_n = \left\{ z \in U \;\middle|\; \operatorname{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{n} \right\}$$

is closed. Then, we observe that $K_n = B(0, n) \cap C_n$ would then be an intersection of a closed set with a closed and bounded set, hence a closed and bounded set, hence compact. Towards this end, suppose $(w_k)_k \subseteq C_n$ converges to $w \in \mathbb{C}$. Then, for each k, we have

$$\inf_{z \in \mathbb{C} \setminus U} |w_k - z| \geqslant \frac{1}{n}.$$

Observe then that for any $z \in \mathbb{C} \setminus U$, we have

$$|w_k - z| \geqslant \frac{1}{n}$$

for each k, meaning that

$$\lim_{k \to \infty} |w_k - z| \geqslant \frac{1}{n},$$

or that

$$|w - z| \geqslant \frac{1}{n}.$$

In particular, it must be the case that $w \in U$, and that

$$\inf_{z\in\mathbb{C}\setminus U}|w-z| \geqslant \frac{1}{n},$$

so that $w \in C_n$, and thus C_n is closed, and K_n is compact.

To see that $K_n \subseteq K_{n+1}^\circ$, we show that $C_n \subseteq C_{n+1}^\circ$ by understanding the picture of C_n°. Towards this end, we see that $z \in C_n^\circ$ if and only if $z \in U$ and there is some $r > 0$ such that $\operatorname{dist}_{\mathbb{C}\setminus U}(w) \geqslant \frac{1}{n}$ for all $w \in U(z,r)$.

Observe that if $\varepsilon > 0$, then if z satisfies $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant \frac{1}{n} + \varepsilon$, then if $w \in \mathbb{C}\setminus U$ and $\zeta \in U(z, \varepsilon/2)$, we have

$$\begin{aligned}\frac{1}{n} + \varepsilon &\leqslant |z-w|\\ &\leqslant |z-\zeta| + |\zeta - w|\\ &< \varepsilon/2 + |\zeta - w|,\end{aligned}$$

meaning that $|\zeta - w| \geqslant \frac{1}{n} + \varepsilon/2$ for all $w \in \mathbb{C}\setminus U$, so that $\operatorname{dist}_{\mathbb{C}\setminus U}(\zeta) \geqslant \frac{1}{n}$. In particular, this means that C_n° consists of all z for which there exists ε such that $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant \frac{1}{n} + \varepsilon$, or more succinctly,

$$C_n^\circ = \left\{z \in U \,\middle|\, \operatorname{dist}_{\mathbb{C}\setminus U}(z) > \frac{1}{n}\right\}.$$

In particular, since $\frac{1}{n} > \frac{1}{n+1}$, it follows that $C_n \subseteq C_{n+1}^\circ$. Paired with the fact that $B(0,n) \subseteq U(0, n+1)$, we obtain that

$$\begin{aligned}K_n &= B(0,n) \cap C_n\\ &\subseteq U(0,n+1) \cap C_{n+1}^\circ\\ &= (B(0,n+1) \cap C_n)^\circ\\ &= K_{n+1}^\circ.\end{aligned}$$

Finally, to show that $U = \bigcup_{n=1}^\infty K_n$, we write

$$\begin{aligned}\bigcup_{n=1}^\infty K_n &= \bigcup_{n=1}^\infty (B(0,n) \cap C_n)\\ &= \left(\bigcup_{n=1}^\infty B(0,n)\right) \cap \left(\bigcup_{n=1}^\infty C_n\right),\end{aligned}$$

and since $\bigcup_{n=1}^\infty B(0,n) = \mathbb{C}$, it follows that we must show that

$$\bigcup_{n=1}^\infty C_n = U.$$

Towards this end, we prove that if $A \subseteq \mathbb{C}$ is any subset, then $\operatorname{dist}_A(z) = 0$ if and only if $z \in \overline{A}$. Towards this end, if $\operatorname{dist}_A(z) = 0$, then for any k, there is $w \in A$ such that $|w - z| < \frac{1}{n}$, so that we may construct a sequence $(w_n)_n$ in A such that $(w_n)_n \to z$, or that $z \in \overline{A}$. Similarly, if $z \in \overline{A}$, then if $(w_n)_n$ is a sequence in A converging to z, and $\varepsilon > 0$, it follows that $|w_n - z| < \varepsilon$ for sufficiently large n, so that $\inf_{w\in Z}|w-z| = 0$.

Since U is open, it follows that for any $z \in \mathbb{C} \setminus U$, since $\mathbb{C} \setminus U = \overline{\mathbb{C} \setminus U}$, $\operatorname{dist}_{\mathbb{C}\setminus U}(z) = 0$. Equivalently, if $z \in U$, we must have $\operatorname{dist}_{\mathbb{C}\setminus U}(z) > 0$, so that there exists n sufficiently large such that $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant 1/n$; this means $z \in C_n$, so that

$$U \subseteq \bigcup_{n=1}^{\infty} C_n.$$

Meanwhile, if $z \in \bigcup_{n=1}^{\infty} C_n$, then there is some N such that $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant 1/N$, meaning that $\operatorname{dist}_{\mathbb{C}\setminus U}(z) > 0$, meaning $z \notin \mathbb{C} \setminus U$, so that $z \in U$.

(b)

(i) If $S = L + \varepsilon$ for $\varepsilon > 0$, it follows by the definition of the infimum that there exists a compact subset $K \subseteq U$ such that $\sup_{z\in U\setminus K} u(z) \leqslant S$. Therefore, for all $z \in U \setminus K$, $u(z) \leqslant S$.

(ii) Let $L_n = L + \frac{1}{n}$. We find $K_{j_n} \subseteq U$ that satisfies
- $u(z) \leqslant L_n$ for all $z \in U \setminus K_{j_n}$;
- $|z| \leqslant j_n$ for all $z \in K_{j_n}$;
- $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant \frac{1}{j_n}$.

The existence of such a K_{j_n} follows from the proof in (i) and the definitions in part (a). We may find $z_n \in U \setminus K_{j_n}$, so that $u(z_n) \leqslant L_n$.

The sequence $(z_n)_n$ thus escapes all the K_{j_n}, and since any $K \subseteq U$ is contained in some sufficiently large K_{j_n}, it follows that $(z_n)_n \to \partial U$. Furthermore, since $u(z_n) \leqslant L_n$ for each n, we have

$$\begin{aligned}\limsup_{n\to\infty} u(z_n) &\leqslant \limsup_{n\to\infty} L_n \\ &= L.\end{aligned}$$

Problem (Problem 2)**:** Let

$$U = \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im}(z) > 0\}.$$

(a) Construct a conformal map from U to $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$.

(b) Construct an unbounded harmonic function $u\colon U \to (0,\infty)$ such that for all $(x_0, y_0) \in \partial U \setminus \{(1,0)\}$, we have that $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = 0$.

(c) Suppose that $v\colon U \to (0,\infty)$ is an unbounded harmonic function such that for all $(x_0,y_0) \in \partial U \setminus \{(1,0)\}$, we have that $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = 0$. Show that there exists a sequence $((x_n,y_n))_n$ in U converging to $(1,0)$ and $\lim_{n\to\infty} v(x_n,y_n) = \infty$.

Solution:

(a) We start by taking the Cayley transform, mapping $\mathbb{H}$ to $\mathbb{D}$, given by $\frac{z-i}{z+i}$. The inverse Cayley transform, which maps $\mathbb{D}$ to $\mathbb{H}$, is then given by the inverse transform, which takes

$$Q(z) = i\frac{1+z}{1-z}.$$

By taking $a + bi \in U$ with $b > 0$ and $a^2 + b^2 \leqslant 1$, we find that

$$i\frac{1+(a+bi)}{1-a-bi} = \frac{1}{(1-a)^2+b^2}\left(-2b + i\left(1-a^2-b^2\right)\right).$$

Therefore, we observe that the inverse transform maps U to the second quadrant, admitting arguments between $\frac{\pi}{2}$ and π. By squaring, we have

$$(Q(z))^2 = -\left(\frac{z+1}{1-z}\right)^2,$$

which maps to complex numbers with arguments between π and 2π. Multiplying by -1, we get

$$H(z) = \left(\frac{z+1}{1-z}\right)^2$$

mapping from U to the upper half-plane. Since we composed a series of bijective holomorphic maps (and, within a correct domain for the case of square root, ones that have holomorphic inverse), it follows that H is a bijective holomorphic map with holomorphic inverse, hence conformal.

(b) Consider the function

$$u(x, y) = \operatorname{Im}(H(x + yi)).$$

We observe that u is the imaginary part of a holomorphic function, so it is harmonic. Since H maps U conformally to the upper half-plane, it follows that u maps U to $(0, \infty)$, and that u is unbounded, as H is unbounded. It remains to show that u maps ∂U to 0 in limit save for $(1, 0)$. Towards this end, we split the case into two parts.

If $x_0 + iy_0 = e^{i\theta}$ for some $0 < \theta_0 < \pi$, then

$$\begin{aligned}
\frac{e^{i\theta}+1}{1-e^{i\theta}} &= \frac{(1+\cos(\theta)+i\sin(\theta))(1-\cos(\theta)+i\sin(\theta))}{2-2\cos(\theta)}\\
&= \frac{1}{2-2\cos(\theta)}\left(1-\cos^2(\theta)-\sin^2(\theta)+2i\sin(\theta)\right)\\
&= \frac{2i\sin(\theta)}{2-2\cos(\theta)}.
\end{aligned}$$

Squaring, we then get

$$\begin{aligned}
\left(\frac{e^{i\theta}+1}{1-e^{i\theta}}\right)^2 &= -\frac{1}{2}\cot^2(\theta/2)\\
&\in \mathbb{R},
\end{aligned}$$

so that $u(x_0, y_0) = 0$ whenever $x_0 + iy_0 = e^{i\theta}$ for some $0 < \theta_0 < \pi$.

Meanwhile, if $x_0 + iy_0 = x_0$, then

$$\begin{aligned}
H(x_0 + iy_0) &= \left(\frac{x_0+1}{1-x_0}\right)^2\\
&\in \mathbb{R},
\end{aligned}$$

so that $u(x_0, y_0) = 0$ yet again.

(c) We let $v \equiv u$, where u is defined as above. Since u is unbounded, it follows that for each $N \geqslant 1$, there is $(x_N, y_N) \in U$ such that $u(x_N, y_n) \geqslant N$. Inductively, this allows us to construct a sequence $(x_n, y_n) \subseteq U$ such that $u(x_n, y_n) \geqslant n$, meaning that $\lim_{n\to\infty} u(x_n, y_n) = \infty$.

Since u is harmonic, it is subharmonic, so by a previously established theorem, it follows that $((x_n, y_n))_n \to \partial U$. Yet, this sequence cannot converge to any element of $\partial U \setminus \{(1, 0)\}$, as otherwise, we would have $u(x_n, y_n) \to 0$, which would contradict the fact that u is continuous as it is harmonic. Therefore, we have $((x_n, y_n))_n \to (1, 0)$.

Problem (Problem 3)**:** Let

$$U = \{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 1\}.$$

Let $f\colon \overline{U} \to \mathbb{C}$ be a continuous bounded function for which $f|_U$ is holomorphic. Suppose there exist constants $M_0 \geqslant 0$ and $M_1 \geqslant 0$ such that

$$\sup_{\operatorname{Re}(z)=0} |f(z)| \leqslant M_0$$
$$\sup_{\operatorname{Re}(z)=1} |f(z)| \leqslant M_1.$$

Show that for all $r \in [0,1]$,

$$\sup_{\operatorname{Re}(z)=r} |f(z)| \leqslant M_0^{1-r} M_1^{r}.$$

Solution: Let $\varepsilon > 0$ be fixed. Define

$$f_\varepsilon(z) = f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon\left(z^2-1\right)}.$$

We will show that $\sup_{z\in\overline{U}}|f_\varepsilon(z)| \leqslant 1$. Towards this end, if $\operatorname{Re}(z) = 0$, we have $z = bi$ for some $b \in \mathbb{R}$; since $M_0, M_1 \in \mathbb{R}_{\geqslant 0}$, we then get

$$\begin{aligned}\left|f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon\left(z^2-1\right)}\right| &= \left|f(z) M_0^{bi-1} M_1^{-bi} e^{-\varepsilon\left(b^2+1\right)}\right| \\ &= \left|f(z) M_0^{-1} e^{-\varepsilon\left(b^2+1\right)}\right| \\ &\leqslant \left|f(z) M_0^{-1}\right| \\ &\leqslant 1.\end{aligned}$$

Similarly, if $\operatorname{Re}(z) = 1$, then we have $z = 1 + bi$ for some $b \in \mathbb{R}$, and since $M_0, M_1 \in \mathbb{R}_{\geqslant 0}$, we have

$$\begin{aligned}\left|f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon\left(z^2-1\right)}\right| &= \left|f(z) M_0^{bi} M_1^{-bi-1} e^{\varepsilon\left(-2bi-b^2\right)}\right| \\ &= \left|f(z) M_1^{-1} e^{-b^2\varepsilon}\right| \\ &\leqslant \left|f(z) M_1^{-1}\right| \\ &\leqslant 1.\end{aligned}$$

Since $|f_\varepsilon(z)| \leqslant 1$ holds on both $\operatorname{Re}(z) = 0$ and $\operatorname{Re}(z) = 1$, it follows by the maximum modulus principle that we must have $|f_\varepsilon(z)| \leqslant 1$ on the interior. In particular, this means that

$$\sup_{z\in\overline{U}} |f_\varepsilon(z)| \leqslant 1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\left|f(z) M_0^{z-1} M_1^{-z}\right| \leqslant 1$$

for all $z \in \overline{U}$, so that

$$\begin{aligned}|f(z)| &\leqslant \left|M_0^{1-z}\right|\left|M_1^{z}\right| \\ &= M_0^{1-\operatorname{Re}(z)} M_1^{\operatorname{Re}(z)}.\end{aligned}$$

In particular, this means that for $\operatorname{Re}(z) = r$, we have

$$|f(z)| \leqslant M_0^{1-r} M_1^{r},$$

meaning this holds for the supremum over all z with $\operatorname{Re}(z) = r$, yielding

$$\sup_{\operatorname{Re}(z)=r} |f(z)| \leqslant M_0^{1-r} M_1^{r}.$$