2.1

Problem: Recall that an ordered pair (a, b) can be defined as the set $\{\{a\}, \{a, b\}\}$. Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Solution. Let $L = \{\{a\}, \{a, b\}\}$ and $R = \{c, \{c, d\}\}$. Suppose $L = R$. Since $\{a\} \in L$, we have $\{a\} \in R$. Thus, $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$.

Case 1: If $\{a\} = \{c\}$, then $a \in \{c\}$, meaning $a = c$.

Case 2: If $\{a\} = \{c, d\}$, then $c \in \{a\}$, meaning $c = a$.

2.3

Problem: Show that the replacement schema implies the comprehension schema.

Solution. Let $\psi(u, v) = \phi(v) \wedge u = v$. Then, the replacement schema becomes

$$\begin{aligned}
\forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow \exists u\, (u \in a \wedge \psi(u, v))) \\
\forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow \exists u\, (u \in a \wedge \forall u\, (\phi(v) \wedge u = v))) \\
\forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow v \in a \wedge \phi(v))
\end{aligned}$$

2.4

Problem: In this question, we show how the pairing axiom follows from the replacement schema. Let sets a and b be given.

(a) We originally used the pairing axiom to construct the set $\{\varnothing, \{\varnothing\}\}$. Instead, us the power set axiom.

(b) Let $\psi(u, v)$ be the formula

$$(u = \varnothing \wedge v = a) \vee (u \neq \varnothing \wedge v = b)\,.$$

Show that this is a function-like formula.

(c) Use the replacement schema on the set $\{\varnothing, \{\varnothing\}\}$ and the function-like formula $\psi\,(u, v)$ to show the existence of the set with elements a and b.

Solution.

(a) Consider $\{\varnothing\}$. By the power set axiom, there exists a set c such that c consists of all subsets of $\{\varnothing\}$. Thus, $c = \{\varnothing, \{\varnothing\}\}$.

(b)

Extra Problem 2

Problem: Let s be a set. Use mathematical symbols exclusively to express t, the set of all singleton subsets of s.

Solution.

$$\forall s\, \exists t\, \forall x\, (x \in t \Leftrightarrow x \in s \wedge \forall a\, \forall b\, (a \in x \wedge b \in x \Rightarrow a = b))$$