Quasi-Review: Locally Compact Groups and the Banach $*$-algebra $L_1(G)$

Basic Properties of Topological Groups

A topological group is a group G equipped with a topology such that the operations

$$\begin{aligned}(x, y) &\mapsto xy\\ x &\mapsto x^{-1}\end{aligned}$$

are continuous. In general, we will let 1 denote the identity of G.

We call G a locally compact group if the topology of G is locally compact. Equivalently, the topology of G is locally compact if there there is a neighborhood system about 1 consisting of pre-compact open sets.

We will refer to the following subset operations in G regularly:

$$\begin{aligned}Ax &= \{ax \mid a \in A\}\\ xA &= \{xa \mid a \in A\}\\ A^{-1} &= \left\{a^{-1} \mid a \in A\right\}\\ AB &= \{ab \mid a \in A, b \in B\}.\end{aligned}$$

A subset V is called *symmetric* if $V = V^{-1}$.

These are some useful propositions.

Proposition: Let G be a topological group.

(i) The topology of G is invariant under translations and inversion.

(ii) For every neighborhood U of 1, there is a symmetric neighborhood V of 1 such that $VV \subseteq U$.

(iii) If H is a subgroup of G, then so is $\overline{H}$.

(iv) Every open subgroup of G is closed.

(v) If A and B are compact subsets of G, then so is AB.

Proposition: Suppose H is a subgroup of the topological group G.

(i) If H is closed, then G/H is Hausdorff.

(ii) If G is locally compact, then so is G/H.

(iii) If H is normal, then G/H is a topological group.

We will assume all the time that G is Hausdorff, via the following proposition.

Corollary: If G is a T1 topological group, then G is Hausdorff. If G is not T1, then $\overline{\{1\}}$ is a closed normal subgroup with $G/\overline{\{1\}}$ is a Hausdorff topological group.

> **Proposition:** Every locally compact group G has a subgroup G_0 that is open, closed, and σ-compact.

Considering various functions $f\colon G \to \mathbb{C}$, we define the left and right translates of f as

$$L_y f(x) = f\left(y^{-1}x\right)$$
$$R_y f(x) = f(xy),$$

and say that f is left (right) uniformly continuous if $\|L_y f - f\|_u \to 0$ ($\|R_y f - f\|_u \to 0$) as $y \to 1$.

> **Proposition:** If $f \in C_c(G)$, then f is left and right uniformly continuous.

A left *Haar measure* is a nonzero Radon measure μ on G such that $\mu(xE) = \mu(E)$ for every Borel subset $E \subseteq G$.

> **Proposition:** Every locally compact group G admits a left Haar measure λ. This Haar measure is unique up to a constant multiple.

If we have a left Haar measure λ, then if we define

$$\lambda_x(E) = \lambda(Ex),$$

we have that λ_x is again a left Haar measure, so there is some number $\Delta(x)$ such that $\lambda_x = \Delta(x)\lambda$, where $\Delta(x)$ is independent of the original choice of λ.

The function $\Delta\colon G \to (0,\infty)$ defined as such is known as the *modular function* of G.

> **Proposition:** The function Δ is a continuous homomorphism from G to $\mathbb{R}_{>0}$, and for any $f \in L_1(\lambda)$, we have
>
> $$\int R_y f\, d\lambda = \Delta\left(y^{-1}\right) \int f\, d\lambda.$$

We call G *unimodular* if $\Delta \equiv 1$.

> **Proposition:** If $G/[G,G]$ is compact, then G is unimodular.

Convolutions and $L_1(G)$

If G is a locally compact group, we let $M(G)$ denote the space of complex-valued Radon measures on G. The convolution of two measures $\mu, \nu \in M(G)$ is given as follows. If we let

$$I(\phi) = \iint \phi(xy)\, d\mu(x) d\nu(y),$$

then we observe that $I(\phi)$ is a linear functional on $C_0(G)$ that satisfies

$$|I(\phi)| \leq \|\phi\|_u \|\mu\| \|\nu\|,$$

meaning that it is given by a measure $\mu * \nu \in M(G)$ with $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$. We call $\mu * \nu$ the convolution of μ and ν.

Observe that if $\delta_x \in M(G)$ is the point mass at $x \in G$, then

$$\int \phi\, d(\delta_x * \delta_y) = \iint \phi(uv)\, d\delta_x(u)\delta_y(v)$$

$$\begin{aligned} &= \phi(xy) \\ &= \int \phi \, d\delta_{xy}, \end{aligned}$$

meaning that $\delta_x * \delta_y = \delta_{xy}$.

The estimate $\|\mu * \nu\| \leq \|\mu\|\|\mu\|$ gives that convolution makes $M(G)$ a Banach algebra, which we call the *measure algebra* of G. Furthermore, $M(G)$ admits an involution

$$\mu^*(E) = \overline{\mu(E^{-1})},$$

so that

$$\int \phi \, d\mu^* = \int \phi\bigl(x^{-1}\bigr) \, d\overline{\mu(x)}.$$

We may identify the space $L_1(G)$ to be the subspace of $M(G)$ where a function f is identified with the measure $f(x)dx$. If $f, g \in L_1(G)$, then the convolution of f and g is the function

$$f * g(x) = \int f(y)g\bigl(y^{-1}x\bigr) \, dy.$$

With convolution and the involution given by

$$\begin{aligned} f^*(x)dx &= \overline{f(x^{-1})}d\bigl(x^{-1}\bigr) \\ f^*(x) &= \Delta\bigl(x^{-1}\bigr)\overline{f(x^{-1})}, \end{aligned}$$

we have that $L_1(G)$ is a Banach $*$-algebra known as the *group algebra* of G.

Now, we observe that if G is discrete, then if δ_e is the point mass at 1, we have that $f * \delta = \delta * f = f$ for any function f. If G is not discrete, we must use an *approximate identity* for G. In particular, we can select a family of mollifiers $\{\psi_U\}_{U \in \mathcal{U}}$ such that

$$\begin{aligned} \|\psi_U * f - f\| &\to 0 \\ \|f * \psi_U - f\| &\to 0 \end{aligned}$$

if f is uniformly continuous and $U \to \{1\}$ in a neighborhood system $\mathcal{U}$ of 1.

Homogeneous Spaces

If G is a locally compact group, then G can act on a locally compact Hausdorff space by homeomorphisms. Recall from algebra that the group action is transitive if there is one orbit. We call S a G-space.

The standard example of a transitive G-space is the quotient space G/H for some closed subgroup H of G. These are, to an extent, the only G-spaces, as follows from the orbit-stabilizer theorem. If S is a G-space, then we may define a map $\phi\colon G \to S$ by $\phi(x) = x \cdot s_0$, and take the quotient by the stabilizer subgroup

$$H = \{x \in G \mid x \cdot s_0 = s_0\},$$

so that $\Phi\colon G/H \to S$ has $\Phi \circ q = \phi$ for the quotient map $q\colon G \to G/H$ is a continuous bijection.

Proposition: If G is σ-compact, then Φ is a homeomorphism.

Proof. It suffices to show that ϕ maps open sets in G to open sets in S. Suppose U is open in G, $x_0 \in U$. Pick a compact symmetric neighborhood V of 1 such that $x_0VV \subseteq U$. Since G is σ-compact, there is a countable $\{y_n\}_{n\geq 1} \subseteq G$ such that $\{y_nV\}_{n\geq 1}$ covers G. Then, we have

$$S = \bigcup_{n=1}^{\infty} \phi(y_nV).$$

The sets $\phi(y_nV)$ are homeomorphic to $\phi(V)$ since the map $s \mapsto y_n \cdot s$ is a homeomorphism of S, and all the y_nV are compact, hence closed.

By Baire Category Theorem for LCH spaces, it follows that $\phi(V)$ has an interior point, which we call $\phi(x_1)$ for $x_1 \in V$. Then, $\phi(x_0)$ is an interior point of $\phi\bigl(x_0x_1^{-1}V\bigr)$, and $x_0x_1^{-1}V \in x_0VV \subseteq U$, so that $\phi(x_0)$ is an interior point of $\phi(U)$. Thus $\phi(U)$ is open. $\square$

If S is a transitive G-space that is isomorphic to a quotient space G/H, then will write $S \cong G/H$, and call S a *homogeneous space*. The identification is dependent on the choice of base point, but the identity $s_0' = x_0 \cdot x_0$ induces a map $H' = x_0Hx_0^{-1}$, inducing a G-equivariant homeomorphism $G/H \cong G/H'$.

We will address the question of whether there is a G-invariant Radon measure on G/H — that is, a radon measure μ such that $\mu(xE) = \mu(E)$ for every $x \in G$.

We assume that G is a locally compact group with left Haar measure dx, a H is a closed subgroup of G with left Haar measure $d\xi$, and $q\colon G \to G/H$ is the quotient map $q(x) = xH$, and Δ_G, Δ_H the corresponding modular functions.

Let $P\colon C_c(G) \to C_c(G/H)$ be defined by

$$Pf(xH) = \int_H f(x\xi)\, d\xi.$$

This is well-defined by left-invariance of $d\xi$. If $\phi \in C(G/H)$, then

$$P((\phi \circ q) \cdot f) = \phi \cdot Pf.$$

Lemma: If $E \subseteq G/H$ is compact, then there is a compact $K \subseteq G$ with $q(K) = E$.

Proof. Let V be an open neighborhood of 1 in G with compact closure. Since q is an open map, $q(xV)$ is an open cover of E, so there is a finite subcover $q(x_jV)$. Let $K = q^{-1}(E) \cap \bigcup_{j=1}^{n} x\overline{V}$. Then, since $q^{-1}(E)$ is closed, K is compact with $q(K) = E$. $\square$

Lemma: If $F \subseteq G/H$ is compact, then there is $f \geq 0$ in $C_c(G)$ with $Pf = 1$ on F.

Proof. Let E be a compact neighborhood of F in G/H. We find $K \subseteq G$ compact such that $q(K) = E$. Select positive $g \in C_c(G)$ with $g > 0$ on K, and $\phi \in C_c(G/H)$ supported in E with $\phi = 1$ on F. Set

$$f = \frac{\phi \circ q}{Pg \circ q} g,$$

with the fraction equal to zero whenever the numerator is zero. We have f is continuous, since

$Pg > 0$ on $\operatorname{supp}(\phi)$, has support contained in $\operatorname{supp}(g)$, and $Pf = \frac{\phi}{Pg}Pg = \phi$. $\square$

Proposition: If $\phi \in C_c(G/H)$, then there exists $f \in C_c(G)$ with $Pf = \phi$ and $q(\operatorname{supp}(f)) = \operatorname{supp}(\phi)$, and has $f \geq 0$ if $\phi \geq 0$.

Proof. If $\phi \in C_c(G)$, then by the previous lemma, then there exists $g \geq 0$ in $C_c(G)$ with $Pg = 1$ on $\operatorname{supp}(\phi)$. Letting $f = (\phi \circ q)g$, then $Pf = \phi(Pg) = \phi$. $\square$

Theorem: Let G be a locally compact group, H a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, we have

$$\begin{aligned}\int_G f(x)\,dx &= \int_{G/H} Pf\,d\mu\\ &= \int_{G/H}\int_H f(x\xi)\,d\xi\,d\mu(xH)\end{aligned}$$

for any $f \in C_c(G)$.

Proof. Suppose there is a G-invariant measure μ. Then, $f \mapsto \int Pf\,d\mu$ is a nonzero left-invariant positive linear functional on $C_c(G)$, so by the uniqueness of Haar measure, we have $\int Pf\,d\mu = c\int f(x)\,dx$ for some c.

This formula fully determines μ, meaning that μ is unique up to the arbitrary constant factor in Haar measure. We may assume that $c = 1$, so we have for any $\eta \in H$,

$$\begin{aligned}\Delta_G(\eta)\int_G f(x)\,dx &= \int_G f\left(x\eta^{-1}\right)dx\\ &= \int_{G/H}\int_H f\left(x\xi\eta^{-1}\right)d\xi\,d\mu(xH)\\ &= \Delta_H(\eta)\int_{G/H}\int_H f(x\xi)\,d\xi\,d\mu(xH)\\ &= \Delta_H(\eta)\int_G f(x)\,dx,\end{aligned}$$

so that $\Delta_G(\eta) = \Delta_H(\eta)$. $\square$

Unitary Representations

If G is a locally compact group, then a *unitary representation* of G is a homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$, where $\mathcal{U}(\mathcal{H}_\pi)$ denotes the unitary group of a Hilbert space $\mathcal{H}_\pi$. We call $\mathcal{H}_\pi$ the *representation space* of π, and the dimension of $\mathcal{H}_\pi$ is called the dimension (or degree) of the representation.

We do not require π to be continuous in the norm topology of $\mathcal{B}(\mathcal{H}_\pi)$, but as it turns out, both weak and strong continuity are equivalent as the WOT and SOT coincide on $\mathcal{U}(\mathcal{H}_\pi)$. If $(T_\alpha)_\alpha \to T$ is a net of unitary operators converging in WOT, then for any $u \in \mathcal{H}_\pi$, we have

$$\|(T_\alpha - T)u\|^2 = \|T_\alpha u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu\rangle + \|Tu\|^2$$

$$= \|u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu\rangle,$$

and the latter term converges to $2\|Tu\|^2 = 2\|u\|^2$, so that $\|T_\alpha u - Tu\| \to 0$.

If G acts on a locally compact Hausdorff space S, then G acts on $C(S)$ by $(\pi(g)f)(s) = f(g^{-1}\cdot s)$. If S has a G-invariant Radon measure, then π defines a unitary representation on $L_2(\mu)$.

The most important representation is the representation on $L_2(G)$ induced by the action of G on itself by left-multiplication. This defines $(\lambda(g)f)(y) = f(x^{-1}y)$. Similarly, the action of G on itself by right-multiplication defines a representation $(\rho(g)f)(y) = f(yx)$. These are the *left-regular* and *right-regular* representations of G.

Any unitary representation π on $\mathcal{H}_\pi$ induces a representation $\overline{\pi}$ on the dual space $\overline{\mathcal{H}_\pi}$, determined by $\overline{\pi}(x) = \langle\cdot, \pi(x^{-1})\rangle$. We call $\overline{\pi}$ the contragradient of π.

If π_1 and π_2 are unitary representations of G, then an intertwining operator for π_1 and π_2 is a bounded linear map $T\colon H_1 \to H_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. We write $\mathcal{C}(\pi_1, \pi_2)$ for the space of intertwiners of π_1 and π_2. We say that π_1 and π_2 are *unitarily equivalent* if the set of intertwiners admits a unitary map.

Proposition: The left-regular and right-regular representations are unitarily equivalent.

Proof. Define the map $T\colon L_2(G) \to L_2(G)$ by

$$T\xi(x) = \Delta(x)^{-1/2}\xi\left(x^{-1}\right).$$

Then, since

$$\begin{aligned}
\langle T\xi, T\eta\rangle &= \int \Delta(x)^{-1/2}\xi\left(x^{-1}\right)\overline{\Delta(x)^{-1/2}\eta(x^{-1})}\, d\mu(x)\\
&= \int \Delta(x)^{-1}\xi\left(x^{-1}\right)\overline{\eta(x^{-1})}\, d\mu(x)\\
&= \int \xi(x)\overline{\eta(x)}\, d\mu(x)\\
&= \langle \xi, \eta\rangle.
\end{aligned}$$

□

If $\mathcal{M}$ is a closed subspace of $\mathcal{H}_\pi$, then we say $\mathcal{M}$ is invariant for π if $\pi(x)\mathcal{M} \subseteq \mathcal{M}$ for all $x \in G$. If $\mathcal{M}$ is a nontrivial invariant subspace, then $\pi|_{\mathcal{M}}$ defines a representation of G on $\mathcal{M}$, known as a subrepresentation of π. If π admits a nontrivial invariant subspace, then we say π is reducible; else, π is irreducible.

If $\{\pi_i\}_{i\in I}$ is a family of unitary representations, their direct sum $\bigoplus_{i\in I}\pi_i$ is the representation on $\mathcal{H} = \bigoplus_{i\in I}\mathcal{H}_i$ given by

$$\pi(x)((v_i)_i) = \sum_{i\in I}\pi_i(x)v_i.$$

We can see that subrepresentations always arise as direct summands.

Proposition: If $\mathcal{M}$ is invariant under π, then so is $\mathcal{M}^{\perp}$.

Proof. Let $u \in \mathcal{M}$ and $v \in \mathcal{M}^{\perp}$. Then,

$$\begin{aligned} \langle \pi(x)v, u\rangle &= \langle v, \pi\bigl(x^{-1}\bigr)u\rangle \\ &= 0, \end{aligned}$$

so $\pi(x)v \in \mathcal{M}^{\perp}$. □

If π is a unitary representation, and $u \in \mathcal{H}_\pi$, then the closed linear span of $\{\pi(x)u \mid x \in G\}$ is called the *cyclic subspace* generated by u. Observe that $\mathcal{M}_u$ is invariant under π; if $\mathcal{M}_u = \mathcal{H}_\pi$, then u is called a cyclic vector for π, and π is called a cyclic representation if it has a cyclic vector.

Proposition: Every unitary representation is a direct sum of cyclic representations.

Proof. Let π be a representation on $\mathcal{H}_\pi$. By Zorn's Lemma, there is a maximal collection $\{\mathcal{M}_i\}_{i\in I}$ of mutually orthogonal cyclic subspaces of $\mathcal{H}_\pi$. If there were some $u \in \mathcal{H}_\pi$ orthogonal to all the $\mathcal{M}_i$, then the cyclic subspace generated by u would also be orthogonal to the $\mathcal{M}_i$, which would contradict maximality. Therefore,

$$\begin{aligned} \mathcal{H}_\pi &= \bigoplus_{i\in I} \mathcal{M}_i \\ \pi &= \bigoplus_{i\in I} \pi|_{\mathcal{M}_i}. \end{aligned}$$

□

Proposition: Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}_\pi$, and let P be the orthogonal projection onto $\mathcal{M}$. Then, $\mathcal{M}$ is invariant under π if and only if P is an intertwiner for π.

Proof. If $P\pi(x) = \pi(x)P$, then for any $v \in \mathcal{M}$, we have $\pi(x)v = \pi(x)Pv = P\pi(x)v \in \mathcal{M}$, so $\mathcal{M}$ is invariant.

Conversely, if $\mathcal{M}$ is invariant, then we have $\pi(x)Pv = \pi(x)v = P\pi(x)v$ whenever $v \in \mathcal{M}$ and $\pi(x)Pv = 0 = P\pi(x)v$ whenever $v \in \mathcal{M}^{\perp}$, so $\pi(x)P = P\pi(x)$. □

Theorem (Schur's Lemma)**:**

(a) A unitary representation π of G is irreducible if and only if the intertwiners of π are scalar multiples of the identity.

(b) If π_1 and π_2 are equivalent irreducible representations of G, then the space of intertwiners for π_1 and π_2 is one-dimensional. Else, π_1 and π_2 only admit $\{0\}$ as their intertwiners.

Proof.

(a) If π is reducible, then $\mathcal{C}(\pi)$ has a nontrivial projection.

Conversely, suppose $T \in \mathcal{C}(\pi)$ is such that $T \neq cI$. Then, $A = \frac{1}{2}(T + T^*)$ is in $\mathcal{C}(\pi)$ and A is not a multiple of the identity. Since A is self-adjoint, every operator that commutes with A commutes with all the projections $\chi_E(A)$, so $\mathcal{C}(\pi)$ contains a nontrivial projection, so π is reducible.

(b) If $T \in \mathcal{C}(\pi_1, \pi_2)$, then $T^* \in \mathcal{C}(\pi_2, \pi_1)$, as

$$\begin{aligned} T^*\pi_2(x) &= \left(\pi_2\left(x^{-1}\right)T\right)^* \\ &= \left(T\pi_1\left(x^{-1}\right)\right)^* \\ &= \pi_1(x)T^*. \end{aligned}$$

Thus, $T^*T \in \mathcal{C}(\pi_1)$, and $TT^* \in \mathcal{C}(\pi_2)$, meaning that $T^*T = cI$ and $TT^* = cI$. In particular, either $T = 0$ or $c^{-1/2}T$ is unitary.

This means that $\mathcal{C}(\pi_1, \pi_2) = \{0\}$ whenever π_1 and π_2 are not equivalent, and $\mathcal{C}(\pi_1, \pi_2)$ consists of scalar multiples of unitary operators otherwise. If $T_1, T_2 \in \mathcal{C}(\pi_1, \pi_2)$, then $T_2^{-1}T_1 = T_2^*T_1 \in \mathcal{C}(\pi_1)$, so $T_1 = cT_2$ and thus $\dim(\mathcal{C}(\pi_1, \pi_2)) = 1$.

□

Representations of a Group and its Group Algebra

We will show that any unitary representation of G corresponds uniquely with the non-degenerate $*$-representations of $L_1(G)$. If π is a unitary representation of G, it determines a representation of $L_1(G)$ by defining the bounded operator $\pi(f)$ on $\mathcal{H}_\pi$ by

$$\pi(f) = \int f(x)\pi(x)\,dx$$

in the weak sense — that is, for any $u, v \in \mathcal{H}_\pi$, we define $\pi(f)$ to be the operator corresponding to the sesquilinear form

$$\langle \pi(f)u, v\rangle = \int f(x)\langle \pi(x)u, v\rangle\,dx.$$

If λ is the left-regular representation of G, then $\lambda(f)$ is left-convolution with f,

$$\begin{aligned} \lambda(f)g &= \int f(x)\lambda(x)g\,dx \\ &= f * g. \end{aligned}$$

Theorem: Let π be a unitary representation of G. Then, the map $f \mapsto \pi(f)$ is a nondegenerate $*$-representation of $L_1(G)$ on $\mathcal{H}_\pi$. Moreover, for any $x \in G$ and $f \in L_1(G)$,

$$\begin{aligned} \pi(x)\pi(f) &= \pi(L_x f) \\ \pi(f)\pi(x) &= \Delta\left(x^{-1}\right)\pi(R_{x^{-1}} f). \end{aligned}$$

Functions of Positive Type

References

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