Problem 1

Prove the following limits:

(i) $\left(\dfrac{2n}{n+2}\right)_n \to 2$

(ii) $\left(\dfrac{\sqrt{n}}{n+1}\right)_n \to 0$

(iii) $\left(\dfrac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0$

(iv) $\left(n^k b^n\right)_n \to 0$ where $0 \leq b < 1$ and $k \in \mathbb{N}$

(v) $\left(\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \to 3$

(i)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow \left|\frac{2n}{n+2} - 2\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned} \frac{2n}{n+2} &> 2 - \varepsilon \\ 2n &> (2n - \varepsilon n) - 2\varepsilon + 4 \\ n &> \frac{4-2\varepsilon}{\varepsilon} \end{aligned}$$

Proof Let $\varepsilon > 0$, $N = \left\lceil \dfrac{4-2\varepsilon}{\varepsilon} \right\rceil$. Then,

$$\begin{aligned} n &> \frac{4-2\varepsilon}{\varepsilon} \\ \varepsilon n &> 4 - 2\varepsilon \\ 0 &> 4 - 2\varepsilon - \varepsilon n \\ 2n &> 2n + 4 - \varepsilon(n+2) \\ 2n &> (2-\varepsilon)(n+2) \\ \frac{2n}{n+2} - 2 &> -\varepsilon \\ \left|\frac{2n}{n+2} - 2\right| &< \varepsilon && \frac{2n}{n+2} < 2\ \forall n \in \mathbb{N} \end{aligned}$$

(ii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \to \left| \left(\frac{\sqrt{n}}{n+1} \right) \right| < \varepsilon$$

Preliminary Work We will show that $\left(\frac{1}{\sqrt{n}}\right)_n \to 0$. Let $\varepsilon > 0$ and $N = 1 + \left\lceil \frac{1}{\varepsilon^2} \right\rceil$. Then,

$$\begin{aligned} n &\geq N \\ n &> \frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{n}} &< \varepsilon \\ \left| \frac{1}{\sqrt{n}} - 0 \right| &< \varepsilon \end{aligned}$$

Proof We know that $\forall n$, $\frac{\sqrt{n}}{n+1} > 0$ and $\frac{\sqrt{n}}{n+1} < \frac{1}{\sqrt{n}}$. Since we showed earlier that $\frac{1}{\sqrt{n}} \to 0$, it must be the case that $\frac{\sqrt{n}}{n+1} \to 0$.

(iii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow \left| \frac{(-1)^n}{\sqrt{n+7}} \right| < \varepsilon$$

Preliminary Work

$$\begin{aligned} \frac{1}{\sqrt{n+7}} &< \varepsilon \\ \frac{1}{\varepsilon} &< \sqrt{n+7} \\ n &> \frac{1}{\varepsilon^2} - 7 \end{aligned}$$

Proof Let $\varepsilon > 0$, $N = \left\lceil \frac{1}{\varepsilon^2} \right\rceil - 7$. Then,

$$\begin{aligned} n &> \frac{1}{\varepsilon^2} - 7 \\ n + 7 &> \frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{n+7}} &< \varepsilon \\ -\varepsilon &< \frac{-1}{\sqrt{n+7}} \\ \left| \frac{(-1)^n}{\sqrt{n+7}} \right| &< \varepsilon \end{aligned}$$

(iv)

If $b = 0$, then $n^k b^n = 0 \to 0$.

Let $0 < b < 1$. To show that $(n^k b^n)_n \to 0$, we will find what the ratio of consecutive terms tends toward:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^k b^{n+1}}{n^k b^n}\\ &= b\left(\frac{n+1}{n}\right)^k\end{aligned}$$

We claim that $\left(\frac{n+1}{n}\right)^k \to 1$. For this, we need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow \left|\left(\frac{n+1}{n}\right)^k - 1\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}\left|\left(1 + \frac{1}{n}\right)^k - 1\right| &< \varepsilon\\ \left(1 + \frac{1}{n}\right)^k &< \varepsilon + 1\\ 1 + \frac{1}{n} &< (\varepsilon + 1)^{1/k}\\ n &> \frac{1}{(\varepsilon + 1)^{1/k} - 1}\end{aligned}$$

Proof Let $\varepsilon > 0$. Let $N = \left\lceil \frac{1}{(\varepsilon + 1)^{1/k} - 1} \right\rceil + 1$. Then, for $n \geq N$, we have

$$\begin{aligned}n &> \frac{1}{(\varepsilon + 1)^{1/k} - 1}\\ (\varepsilon + 1)^{1/k} &> 1 + \frac{1}{n}\\ \left(1 + \frac{1}{n}\right)^k - 1 &< \varepsilon\end{aligned}$$

whence $\left|\left(\frac{n+1}{n}\right)^k - 1\right| = \left(1 + \frac{1}{n}\right)^k - 1$.

Therefore, since $\left(\frac{n+1}{n}\right)^k \to 1$, the ratio converges to $b < 1$, meaning $n^k b^n \to 0$.

(v)

Preliminary Work

$$\begin{aligned}
\left|\frac{2^{n+1}+3^{n+1}}{2^n+3^n}-3\right| &< \varepsilon\\
3-\frac{2^{n+1}+3^{n+1}}{2^n+3^n} &< \varepsilon\\
\frac{3(2^n+3^n)-2^{n+1}-3^{n+1}}{2^n+3^n} &< \varepsilon\\
\frac{2^n}{2^n+3^n} &< \varepsilon\\
2^n &< (2^n+3^n)\varepsilon\\
(1-\varepsilon)2^n &< \varepsilon\cdot 3^n\\
\frac{1-\varepsilon}{\varepsilon} &< \left(\frac{3}{2}\right)^n\\
n &> \frac{\ln(1-\varepsilon)-\ln\varepsilon}{\ln 3-\ln 2}
\end{aligned}$$

Proof Let $\varepsilon > 0$ and $N = \left\lceil \frac{\ln(1-\varepsilon)-\ln\varepsilon}{\ln 3-\ln 2}\right\rceil + 1$. Then, for $n \geq N$, we have

$$\begin{aligned}
n &> \frac{\ln(1-\varepsilon)-\ln\varepsilon}{\ln 3-\ln 2}\\
n\ln\left(\frac{3}{2}\right) &> \ln\left(\frac{1-\varepsilon}{\varepsilon}\right)\\
\frac{3^n}{2^n} &> \frac{1-\varepsilon}{\varepsilon}\\
\varepsilon(3^n+2^n) &> 2^n\\
\frac{2^n}{2^n+3^n} &< \varepsilon
\end{aligned}$$

whence $\left|\frac{2^{n+1}+3^{n+1}}{2^n+3^n}-3\right| = \frac{2^n}{2^n+3^n}$.

Problem 2

Show that the sequence $(\cos(n))_n$ does not converge.

We will show that $(\cos(n))_n$ does not converge to L for any $L \in \mathbb{R}$

Case 1: Suppose $L > 1$. Set $\varepsilon_0 = \frac{L-1}{2}$. Then, for any $N \in \mathbb{N}$, let $n = N$.

$$\begin{aligned}
|\cos(n)-L| &= L-\cos(n)\\
&\geq L-1\\
&> \frac{L-1}{2}\\
&= \varepsilon_0
\end{aligned}$$

Case 2: Suppose $L < -1$. Set $\varepsilon_0 = \frac{1-L}{2}$. Then, for any $N \in \mathbb{N}$, let $n = N$.

$$\begin{aligned} |\cos(n) - L| &= \cos(n) - L \\ &\geq 1 - L \\ &> \frac{1-L}{2} \\ &= \varepsilon_0 \end{aligned}$$

Case 3: Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given any $N \in \mathbb{N}$, find $n \geq N$ with $\cos(n) \geq 1/2$. Then, $|\cos(n) - 0| \geq \varepsilon_0$.

Case 4: Suppose $0 < L < 1$. Set $\varepsilon_0 = L/2$. Given any $N \in \mathbb{N}$, we want to find $n \geq N$ such that $\cos(n) < 0$.

Find k large such that $N < \frac{(4k+1)\pi}{2}$, which is always possible by the Archimedean property. Then, $N < \frac{(4k+1)\pi}{2} < \frac{(4k+3)\pi}{2}$. So, we find $n = \left\lceil \frac{(4k+1)\pi}{2} \right\rceil$, meaning $\cos(n) < 0$, so $|L - \cos(n)| \geq \varepsilon_0$.

Case 5: Suppose $-1 < L < 0$. Set $\varepsilon_0 = -L/2$. Given any $N \in \mathbb{N}$, we want to find $n \geq N$ such that $\cos(n) > 0$.

Find k large such that $N < \frac{(4k-1)\pi}{2}$. This is always possible by the Archimedean property. Then, $N < \frac{(4k-1)\pi}{2} < \frac{(4k+1)\pi}{2}$. So, we find $n = \left\lceil \frac{(4k-1)\pi}{2} \right\rceil$, meaning $\cos(n) > 0$, so $|L - \cos(n)| \geq \varepsilon_0$.

Problem 3

If $(x_n)_n$ is a real sequence converging to x, show that

$$(|x_n|)_n \to |x|$$

Is the converse true?

If $(x_n)_n \to x$, then $|x_n - x| \to 0$. So

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| && \text{Reverse Triangle Inequality} \\ &\to 0 \end{aligned}$$

So, $|x_n| \to |x|$.

The converse is not true. For example, the sequence $(|(-1)^n|)_n \to 1$, but $((-1)^n)_n$ does not converge.

Problem 4

If $(x_n)_n$ is a real sequence converging to $x > 0$, show that there is an $N \in \mathbb{N}$ and $c > 0$ such that

$$x_n \geq c \ \forall n \geq N$$

Since $(x_n)_n \to x$, we know that $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})$ such that $n \geq n \to x_n \in V_\varepsilon(x)$.

In particular, let $\varepsilon_0 = \frac{|0-x|}{3}$, $c = \frac{x}{3} < x$, and ε_1 small such that $V_{\varepsilon_1}(c) \cap V_{\varepsilon_0}(x) = \emptyset$.

Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in V_{\varepsilon_0}(x) > c$.

Problem 5

If $(x_n)_n$ is a real sequence of positive terms converging to x, show that $x \geq 0$ and

$$(\sqrt{x_n})_n \to \sqrt{x}$$

$x \geq 0$

Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0-x|}{2}$. Since $x_n \to x$, $\exists N \in \mathbb{N}$ large such that $x_n \in V_\varepsilon(x)$ for $n \geq N$. However, $\forall \ell \in V_\varepsilon(x)$, $\ell < 0$, meaning that $x_n < 0$ for large n. $\perp$

$\left(\sqrt{x_n}\right)_n \to \sqrt{x}$

Case 1: Suppose $x = 0$. Let $\varepsilon > 0$. Then,

$$\begin{aligned}|x_n - 0| &< \varepsilon^2\\ x_n &< \varepsilon^2\\ \sqrt{x_n} &< \varepsilon\\ |\sqrt{x_n} - 0| &< \varepsilon\end{aligned}$$

So, $\sqrt{x_n} \to 0$.

Case 2: Suppose $x > 0$. Let $\varepsilon > 0$. Then,

$$\begin{aligned}\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right|\\ &= \frac{1}{\sqrt{x_n} + \sqrt{x}}|x_n - x|\\ &\leq \frac{1}{\sqrt{x}}|x_n - x|\\ &\to 0\end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \to 0$, so $\sqrt{x_n} \to x$

Problem 6

If $(x_n)_n$ and $(y_n)_n$ are sequences with $(x_n)_n \to 0$ and $(y_n)_n$ bounded. Show that

$$(x_n y_n)_n \to 0$$

Let $y \in \mathbb{R}$ be an upper bound on $(y_n)_n$. Then,

$$\begin{aligned}|x_n y_n| &\leq |x_n||y|\\ &\to 0\end{aligned}$$

Therefore, $x_n y_n \to 0$.

Problem 7

If $(x_n)_n$ is a sequence of positive terms such that

$$\left(\frac{x_{n+1}}{x_n}\right)_n \to L > 1,$$

show that $(x_n)_n$ is not bounded, and thus not convergent. If $L = 1$, can we make any conclusions?

Since $L > 1$, $L = 1 + a$. Let $\varepsilon = \frac{a}{2}$. Then,

$$\begin{aligned}\left|\frac{x_{n+1}}{x_n} - (1+a)\right| &< \varepsilon \\ 1 + \frac{a}{2} < \frac{x_{n+1}}{x_n} &< 1 + \frac{3a}{2}\end{aligned}$$

so, $\forall n \in \mathbb{N}$,

$$\begin{aligned}x_{n+1} &> x_n\left(1 + \frac{a}{2}\right) \\ x_{n+2} &> x_{n+1}\left(1 + \frac{a}{2}\right) \\ &> x_n\left(1 + \frac{a}{2}\right)^2 \\ &\geq x_n\left(1 + \frac{(2)a}{2}\right) && \text{Bernoulli's Inequality}\end{aligned}$$

Inductively, we have

$$x_{n+k} > x_n\left(1 + \frac{(k)a}{2}\right)$$

Since $\left(1 + \frac{(k)a}{2}\right)_k \to \infty$ and $x_n > 0$ $\forall n \in \mathbb{N}$, we have that $(x_n)_n$ goes to infinity, meaning it is not bounded.

If $L = 1$, we cannot make any conclusions as to the boundedness or convergence of the sequence.

Problem 8

Let a, b be positive numbers. Show that

$$\left((a^n + b^n)^{1/n}\right)_n \to \max\{a, b\}$$

Suppose $a = b$. Then,

$$\begin{aligned}(a^n + b^n)^{1/n} &= (2 \cdot a^n)^{1/n} \\ &= 2^{1/n} a \\ &\to a && \text{sequence of roots converges to 1} \\ &= \max\{a, b\}\end{aligned}$$

Otherwise, without loss of generality, let $a > b$. Then,

$$\begin{aligned} b^n < a^n < a^n + b^n &< 2 \cdot a^n \\ b < a < (a^n + b^n)^{1/n} &< 2^{1/n} a \\ &\to \\ b < a < (a^n + b^n)^{1/n} &< a && \text{sequence of roots converges to 1} \end{aligned}$$

So, by the squeeze theorem, $(a^n + b^n)^{1/n} \to a = \max\{a, b\}$.

Problem 9

Let $(x_n)_n$ be a sequence of positive terms such that

$$\left(x_n^{1/n}\right)_n \to L < 1$$

Prove that $(x_n)_n \to 0$. If $L = 1$, can we make any conclusion? What about $L > 1$?

Let $\rho = L + \frac{1-L}{2}$, and $\varepsilon = \rho - L = \frac{1-L}{2}$.

Since $\left(x_n^{1/n}\right)_n$ converges, we know that

$$\begin{aligned} \left|(x_n)^{1/n} - L\right| &< \varepsilon \\ (x_n)^{1/n} &< \rho \\ x_n &< \rho^n \end{aligned}$$

Since $\rho < 1$, and as $n \to \infty$, $\rho^n \to 0$, therefore we know $(x_n)_n \to 0$.

We can't make any conclusions if $L = 1$, and if $L > 1$, we can assume that $(x_n)_n$ diverges, as we showed in the previous case with the ratio test.