# Amenability in Discrete Groups

Conditions and Applications

Avinash Iyer

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# Paradoxical Decompositions

The primary goal of this section will be to introduce the idea of a paradoxical decomposition (and its effects on the analytic properties of $\mathbb{R}^3$) through the Banach–Tarski Paradox. The ultimate goal is to prove the following statement.

**Proposition 2.0.1 —** (General Banach–Tarski Paradox)

If $A$ and $B$ are bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of $A$ into finitely many disjoint subsets such a sequence of isometries applied to these subsets yields $B$.

The existence of the Banach–Tarski paradox throws a wrench into a major idea that we may have about subsets of $\mathbb{R}^3$ — namely, that they always have some "volume" to them that is invariant under isometry, similar to how "area" in $\mathbb{R}^2$ is invariant under isometry.

## 2.1 Group Action Essentials

We begin by discussing some of the basic properties of group actions.

**Definition 2.1.1 —** (Group Action)

Let $G$ be a group, and $A$ be a set. A left group action of $G$ onto $A$ is a map $\alpha : G \times A \to A$ that satisfies

- $\alpha(g_1, (g_2, a)) = \alpha(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$;
- $\alpha(e_G, a) = a$ for all $a \in A$.

For the sake of brevity, we write $(g, a) = g \cdot a$.

Every group action can be represented by a permutation on $A$.

**Definition 2.1.2 —** (Permutation Representation)

For each $g$, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ is a permutation of $A$. There is a homomorphism associated to these actions, $\varphi : G \to \mathrm{Sym}(A)$, where $\mathrm{Sym}(A)$ is the symmetric group on the elements of $A$.

The permutation representation can run in the opposite direction in the following sense: given a nonempty set $A$ and a homomorphism $\psi : G \to \mathrm{Sym}(A)$, we can take $g \cdot a = \psi(g)(a)$, where $\psi(g) = \sigma_g \in \mathrm{Sym}(A)$ is a permutation.

Just as we can pass group actions into permutation representations, and discuss ideas like the kernel of homomorphisms, we can also discuss the kernel of ajn action.

**Definition 2.1.3 —** (Kernel)

The **kernel** of the action of $G$ on $A$ is the set of elements in $g$ that act trivially on $A$:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}.$$

The kernel of the group action is the kernel of the permutation representation $\varphi : G \to \mathrm{Sym}(A)$.

**Definition 2.1.4 —** (Stabilizer)

For each $a \in A$, we define the **stabilizer** of $a$ under $G$ to be the set of elements in $G$ that fix $a$:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

**Remark 1**

The kernel of the group action is the intersection of the stabilizers of every element of $A$.

For each $a \in A$, $G_a$ is a subgroup of $G$.

**Definition 2.1.5 —** (Faithful Action)

An action is **faithful** if the kernel of the action is the identity, $e_G$. Equivalently, the permutation representation $\varphi : G \to \mathrm{Sym}(A)$ is injective.

The following definition will be useful in the future as we dig deeper into the idea of paradoxical groups.

**Definition 2.1.6 —** (Free Action)

For a set $X$ with $G$ acting on $X$, the action of $G$ on $X$ is free if, for every $x \in X$, $g \cdot x = x$ if and only if $g = e_G$.

The most important theorem relating to group actions is the orbit-stabilizer theorem. As we prove the following theorem, we will reveal the definition of an orbit as a type of equivalence class.

**Theorem 2.1.1 —** (Orbit-Stabilizer Theorem)

Let $G$ be a group that acts on a nonempty set $A$. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking the index of the stabilizer of $a$ in $G$, $|G : G_a|$.

**Proof:** We start by seeing that $a \sim a$, as $e_G \cdot a = a$. Similarly, if $a \sim b$, then there exists $g \in G$ such that $a = g \cdot b$. Thus,

$$\begin{aligned} g^{-1} \cdot a &= g^{-1} \cdot (g \cdot b) \\ &= g^{-1} g \cdot b \\ &= e \cdot b \\ &= b, \end{aligned}$$

meaning that $b \sim a$. Finally, if we have $a \sim b$ and $b \sim c$, we have $a = g \cdot b$ and $b = h \cdot c$ for some $g, h \in G$. Therefore,

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$. Thus, the relation $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

We claim there is a bijection between the left cosets of $G_a$ and the elements of $[a]_\sim$.

Define $C_a = \{g \cdot a \mid g \in G\}$, which is the set of elements in the equivalence class of $a$. Define the map $g \cdot a \mapsto gG_a$. Since $g \cdot a$ is always an element of $C_a$, this map is surjective. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, we have $h^{-1}g \in G_a$, which is only true if $gG_a = hG_a$. Thus, the map is injective.

Since there is a one to one map between the equivalence classes of $a$ under the action of $G$, and the number of left cosets of $G_a$, we know that the number of equivalence classes of $a$ under the action of $G$ is $|G : G_a|$. ■

**Definition 2.1.7 —** (Orbit)

Let $G$ act on $A$, and let $a \in A$. The **orbit** of $a$ under $G$ is the set

$$G \cdot a = \{g \cdot a \in A \mid g \in G\}$$

## 2.2 Paradoxical Decompositions in $\mathbb{R}^3$

With the essential facts about group actions in mind, we can turn our attention to "paradoxical" actions that seem to recreate a set by using some of its disjoint proper subsets.

**Definition 2.2.1 —** (Paradoxical Decompositions and Paradoxical Groups)

Let $G$ be a group that acts on a set $X$, with $E \subseteq X$. We say $E$ is $G$**-paradoxical** if there exist pairwise disjoint proper subsets $A_1, \dots, A_n, B_1, \dots, B_m \subset E$ and group elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

If $G$ acts on itself by left-multiplication, and $G$ satisfies these conditions, we say $G$ is a **paradoxical group**.

**Example 2.2.1**

The free group on two generators, $F(a, b)$, is a paradoxical group.

The free group is defined to be the set of all reduced words over the set $\left\{a, b, a^{-1}, b^{-1}, e_{F(a,b)}\right\}$, where $aa^{-1}$, $a^{-1}a$, $bb^{-1}$, and $b^{-1}b$ are replaced with the identity $e_{F(a,b)}$.

To see that $F(a, b)$ is a paradoxical group, we let $W(x) = \{w \in F(a, b) \mid w \text{ starts with } x\}$. For instance, $ba^2ba^{-1} \in W(b)$.

Since every word in $F$ is either the empty word, or starts with one of $a, b, a^{-1}, b^{-1}$, we see that

$$F(a, b) = \left\{e_{F(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For $w \in F(a, b) \setminus W(a)$, it is the case that $a^{-1}w \in W\left(a^{-1}\right)$, so $w \in aW\left(a^{-1}\right)$. Thus, for any $t \in F(a, b)$, $t \in W(a)$ or $t \in F(a, b) \setminus W(a) = aW\left(a^{-1}\right)$, so $F(a, b) = W(a) \sqcup aW\left(a^{-1}\right)$.
Similarly, for any $w \in F(a, b) \setminus W(b)$, it is the case that $b^{-1}w \in W\left(b^{-1}\right)$, so $w \in bW\left(b^{-1}\right)$. Thus, for any $t \in F(a, b)$, $t \in W(b)$ or $t \in F(a, b) \setminus W(b) = bW\left(b^{-1}\right)$. Thus, $F(a, b) = W(b) \sqcup bW\left(b^{-1}\right)$.

We have thus constructed

$$\begin{aligned} F(a, b) &= W(a) \sqcup aW\left(a^{-1}\right) \\ &= W(b) \sqcup bW\left(b^{-1}\right), \end{aligned}$$

a paradoxical decomposition of $F(a, b)$ with the action of left-multiplication.

Now that we understand a little more about paradoxical groups, we now want to understand the actions of paradoxical groups on sets.

**Proposition 2.2.1**

Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.

**Proof:** Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ be pairwise disjoint, and let $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

Let $M \subseteq X$ contain exactly one element from every orbit in X.

**Claim:** The set $\{g \cdot M \mid g \in G\}$ is a partition of X.

**Proof of Claim:** Since M contains exactly one element from every orbit in X, it is the case that $G \cdot M = X$, so

$$\bigcup_{g \in G} g \cdot M = X$$

Additionally, for $x, y \in M$, if $g \cdot x = h \cdot y$, then $\left(h^{-1} g\right) \cdot x = y$, meaning $y$ is in the orbit of $x$ and vice versa, implying $x = y$. Since G acts freely on X, we must have $h^{-1} g = e_G$.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$, implying $\{g \cdot M \mid g \in G\}$ is a partition of X. ■

We define

$$A_j^* = \bigcup_{g \in A_j} g \cdot M,$$

and similarly define

$$B_j^* = \bigcup_{h \in B_j} h \cdot M.$$

As a useful shorthand, we can also write $A_j^* = A_j \cdot M$, and similarly, $B_j^* = B_j \cdot M$, to denote the union of the elements of $A_j$ and $B_j$ respectively acting on M.

Since $\{g \cdot M \mid g \in G\}$ is a partition of X, and $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ are pairwise disjoint, it must be the case that $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^* \subset X$ are also pairwise disjoint.

For the original $g_1, \ldots, g_n, h_1, \ldots, h_m$ that defined the paradoxical decomposition of G, we thus have

$$\begin{aligned} \bigcup_{j=1}^{n} g_j \cdot A_j^* &= \bigcup_{j=1}^{n} \left(g_j A_j\right) \cdot M \\ &= G \cdot M \\ &= X, \end{aligned}$$

and

$$\begin{aligned} \bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} \left(h_j B_j\right) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Thus, X is G-paradoxical. ■

**Remark 2**

This proof requires the axiom of choice, as we invoked it to define $M$ to contain exactly one element from every orbit in $X$.

Now that we have established $F(a, b)$ as being a paradoxical group, we wish to use it to construct paradoxical decompositions of the unit sphere $S^2 \subseteq \mathbb{R}^3$.

**Definition 2.2.2 —** (Special Orthogonal Group)

For $n \in \mathbb{N}$, we define $SO(n)$ to be the group of all real $n \times n$ matrices $A$ such that $A^T = A^{-1}$ and $\det(A) = 1$.

In terms of an isometry of $\mathbb{R}^3$, the group $SO(3)$ denotes the set of all rotations about any line through the origin.

**Fact 2.2.1**

If $H$ is a paradoxical group, and $H \leqslant G$, then $G$ is a paradoxical group.

With this fact in mind, we will show that $SO(3)$ is a paradoxical group.

**Theorem 2.2.1**

There are rotations $A$ and $B$ that about lines through the origin in $\mathbb{R}^3$ that generate a subgroup of $SO(3)$ isomorphic to $F(a, b)$

**Proof :** We take

$$A = \begin{bmatrix} 1/3 & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/3 & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

We let $A^{\pm}$ denote $A$ and $A^{-1}$ respectively, and similarly for $B^{\pm}$.

Let $w$ be a reduced word in $\{A, A^{-1}, B, B^{-1}\}$ which is not the empty word. We claim that $w$ cannot be the identity.

Without loss of generality, we assume that $w$ ends in $A$ or $A^{-1}$ — this is because if $w$ is the identity, then $AwA^{-1}$ and $A^{-1}wA$ are also the identity.

We will show that there exist $a, b, c \in \mathbb{Z}$ with $b \not\equiv 0 \bmod 3$ such that

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}.$$

If $b \not\equiv 0 \bmod 3$, and $w$ is not empty, then $w$ cannot act as the identity.

We induct on the length of $w$. For $w = A^{\pm}$, we have

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix},$$

proving the base case.

Let $k > 0$, meaning $w = A^{\pm}w'$, or $w = B^{\pm}w'$, with $w'$ not equal to the empty. The inductive hypothesis says

$$w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}$$

for some $a', b', c' \in \mathbb{Z}$, and $b' \not\equiv 0 \bmod 3$. In particular,

$$A^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k}} \begin{pmatrix} a \mp 4b \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix}$$

$$B^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k}} \begin{pmatrix} 3a' \\ (b' \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix}.$$

Now, we set

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k}} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix},$$

meaning

$$a = \begin{cases} a' \mp 4b', & w = A^{\pm}w' \\ 3a', & w = B^{\pm}w' \end{cases}$$

$$b = \begin{cases} b' \pm 2a', & w = A^{\pm}w' \\ b' \mp 2c', & w = B^{\pm}w' \end{cases}$$

$$c = \begin{cases} 3c', & w = A^{\pm}w' \\ c' \pm 4b', & w = B^{\pm}w' \end{cases}$$

Let $w^*$ denote the word such that $w' = A^{\pm}w^*$ or $w' = B^{\pm}w^*$. We write

$$w^* = \frac{1}{3^{k-2}} \begin{pmatrix} a'' \\ b''\sqrt{2} \\ c'' \end{pmatrix},$$

where $a'', b'', c'' \in \mathbb{Z}$. Note that it may not be the case that $w^*$ is a non-empty word. We examine the following four cases.

**Case 1:** Suppose $w = A^{\pm}B^{\pm}w^*$. Then, $b = b' \mp 2a'$, where $a' = 3a''$. Since $b' \not\equiv 0 \bmod 3$ (by the inductive hypothesis), it is also the case $b \equiv 0 \bmod 3$.

**Case 2:** Suppose $w = B^{\pm}A^{\pm}w^*$. Then, $b = b' \mp 2c'$, where $c' = 3c''$. Since $b' \not\equiv 0 \bmod 3$ (by the inductive hypothesis), it is also the case that $b \not\equiv 0 \bmod 3$.

**Case 3:** Suppose $w = A^{\pm}A^{\pm}w^*$. Then, we have

$$\begin{aligned} b &= b' \pm 2a' \\ &= b' \pm 2\left(a'' \pm 4b''\right) \\ &= b' + \left(b'' \pm 2a''\right) - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Thus, regardless of the value of $b''$, since $b' \not\equiv 0 \bmod 3$ by the inductive hypothesis, it is the case that $b \not\equiv 0 \bmod 3$.

Suppose $w = B^{\pm}B^{\pm}w^*$. Then, we have

$$\begin{aligned} b &= b' \mp 2c' \\ &= b' \mp 2\left(c'' \pm 4b''\right) \end{aligned}$$

$$
\begin{aligned}
&= b' + \left(b'' \mp 2c''\right) - 9b'' \\
&= 2b' - 9b''.
\end{aligned}
$$

Thus, regardless of the value of $b''$, since $b' \not\equiv 0 \bmod 3$ by the inductive hypothesis, it is the case that $b \not\equiv 0 \bmod 3$.

We have thus shown that any non-empty reduced word over $\left\{A, A^{-1}, B, B^{-1}\right\}$ does not act as the identity. The subgroup of SO(3) generated by $\left\{A, A^{-1}, B, B^{-1}\right\}$ is isomorphic to $F(a, b)$. ■

**Remark 3**

Since $SO(n)$ contains a subgroup isomorphic to SO(3) for all $n \geqslant 3$, it is the case that $SO(n)$ also contains a subgroup isomorphic to $F(a, b)$.

Since we have shown that SO(3) is paradoxical, as it contains a paradoxical subgroup, we can now begin to examine the action of SO(3) on subsets of $\mathbb{R}^3$.

**Theorem 2.2.2 —** (Hausdorff Paradox)

There is a countable subset $D$ of $S^2$ such that $S^2 \setminus D$ is SO(3)-paradoxical.

**Proof :** Let $A$ and $B$ be the rotations in SO(3) that serve as the generators of the subgroup isomorphic to $F(a, b)$.

Since $A$ and $B$ are rotations, so too is any reduced word over $\left\{A, A^{-1}, B, B^{-1}\right\}$. Thus, any such non-empty word contains two fixed points.

We let

$$
F = \left\{x \in S^2 \mid s \text{ is a fixed point for some word } w\right\}.
$$

Since the set of all reduced words in $\left\{A, A^{-1}, B, B^{-1}\right\}$ (henceforth $F(A, B)$) is countably infinite, so too is $F$. Thus, the union of all these fixed points under the action of all such words $w$ is countable.

$$
D = \bigcup_{w \in F(A,B)} w \cdot F.
$$

Therefore, $F(A, B)$ acts freely on $S^2 \setminus D$, so $S^2 \setminus D$ is SO(3)-paradoxical. ■

Unfortunately, the Hausdorff paradox is not enough for us to be able to prove the Banach–Tarski paradox. In order to do this, we need to be able to show that two sets are "similar" under the action of a group.

**Definition 2.2.3 —** (Equidecomposable Sets)

Let $G$ act on $X$, and let $A, B \subseteq X$. We say $A$ and $B$ are $G$-**equidecomposable** if there are partitions $\left\{A_j\right\}_{j=1}^{n}$ of $A$ and $\left\{B_j\right\}_{j=1}^{n}$ of $B$, and elements $g_1, \ldots, g_n \in G$, such that for all $j$,

$$
B_j = g_j \cdot A_j.
$$

We write $A \sim_G B$ if $A$ and $B$ are $G$-equidecomposable.

**Fact 2.2.2**

The relation $\sim_G$ is an equivalence relation.

**Proof :** Let $A$, $B$, and $C$ be sets.

To show reflexivity, we can select $g_1 = g_2 = \cdots = g_n = e_G$. Thus, $A \sim_G A$.

To show symmetry, let $A \sim_G B$. Set $\{A_j\}_{j=1}^n$ to be the partition of $A$, and set $\{B_j\}_{j=1}^n$ to be the partition of $B$, such that there exist $g_1, \ldots, g_n \in G$ with $g_j \cdot A_j = B_j$. Then,

$$\begin{aligned} g_j^{-1} \cdot (g_j \cdot A_j) &= g_j^{-1} \cdot B_j \\ A_j &= g_j^{-1} \cdot B_j, \end{aligned}$$

so $B_j \sim_G A_j$.

To show transitivity, let $A \sim_G B$ and $B \sim_G C$. Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be the partitions of $A$ and $B$ respectively and $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$. Let $\{B_j\}_{j=1}^m$ and $\{C_j\}_{j=1}^m$ be partitions of $B$ and $C$, and $h_1, \ldots, h_m \in G$, such that $h_j \cdot B_j = C_j$.

We refine the partition of $A$ to $A_{ij}$ by taking $A_{ij} = g_i^{-1}(B_i \cap B_j)$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then, $(h_j g_i) \cdot A_{ij}$ maps the refined partition of $A$ to a refined partition of $C$, meaning $A$ and $C$ are $G$-equidecomposable. ■

**Fact 2.2.3**

For $A \sim_G B$, there is a bijection $\phi : A \to B$ by taking $C_i = C \cap A_i$, and mapping $\phi(C_i) = g_i \cdot C_i$.

In particular, this means that for any subset $C \subseteq A$, it is the case that $C \sim \phi(C)$.

We can now use this equidecomposability to glean information about the existence of paradoxical decompositions.

**Proposition 2.2.2**

Let $G$ act on $X$, with $E, E' \subseteq X$ such that $E \sim_G E'$. Then, if $E$ is $G$-paradoxical, then so too is $E'$.

**Proof:** Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$ be pairwise disjoint, with $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$\begin{aligned} E &= \bigcup_{i=1}^n g_i \cdot A_i \\ &= \bigcup_{j=1}^m h_j \cdot B_j. \end{aligned}$$

We let

$$\begin{aligned} A &= \bigsqcup_{i=1}^n A_i \\ B &= \bigsqcup_{j=1}^m B_j. \end{aligned}$$

It follows that $A \sim_G E$ and $B \sim_G E$, since we can take the partition of $A$ to be $A_1, \ldots, A_n$, and partition $E$ by taking $g_i \cdot A_i$ for $i = 1, \ldots, n$, and similarly for $B$.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$. Thus, there is a paradoxical decomposition of $E'$ in $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. ■

We will now show that $S^2$ is $SO(3)$ paradoxical.

**Proposition 2.2.3**

Let $D \subseteq S^2$ be countable. Then, $S^2$ and $S^2 \setminus D$ are $SO(3)$-equidecomposable.

**Proof:** Let $L$ be a line in $\mathbb{R}^3$ such that $L \cap D = \varnothing$. Such an $L$ must exist since $S^2$ is uncountable.

Define $\rho_\theta \in SO(3)$ to be a rotation about L by an angle of $\theta$. For a fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{x \in D \mid \rho_\theta^n \cdot x \in D\right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

We define $W_n = \left\{\theta \mid R_{n,\theta} \neq \varnothing\right\}$. Since the map $\theta \mapsto \rho_\theta^n \cdot x$ into D is injective, it is the case that $W_n$ is countable. Therefore,

$$W = \bigcup_{n \in \mathbb{N}} W_n$$

is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. We define $\rho_\omega$ to be a rotation about L by $\omega$. Then, for every $n, m \in \mathbb{N}$, we have

$$\rho_\omega^n \cdot D \cap \rho_\omega^m \cdot D = \varnothing.$$

We define $\widetilde{D} = \bigsqcup_{n=0}^{\infty} \rho_\omega^n D$. Note that

$$\begin{aligned}
\rho_\omega \cdot \widetilde{D} &= \rho_\omega \cdot \bigsqcup_{n=0}^{\infty} \rho_\omega^n \cdot D \\
&= \bigsqcup_{n=1}^{\infty} \rho_\omega^n \cdot D \\
&= \widetilde{D} \setminus D,
\end{aligned}$$

meaning $\widetilde{D}$ and D are SO(3)-equidecomposable.

Thus, we have

$$\begin{aligned}
S^2 &= \widetilde{D} \sqcup \left(S^2 \setminus \widetilde{D}\right) \\
&\sim_{SO(3)} \left(\rho_\omega \cdot \widetilde{D}\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\
&= \left(\widetilde{D} \setminus D\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\
&= S^2 \setminus D,
\end{aligned}$$

establishing $S^2$ and $S^2 \setminus D$ as SO(3)-equidecomposable.

In particular, this means $S^2$ is also SO(3)-paradoxical. ■

To prove the Banach–Tarski paradox, we need a slightly larger group than SO(3) — one that includes translations in addition to the traditional rotations. This is the Euclidean group.

**Definition 2.2.4 —** (Euclidean Group)

The **Euclidean group**, $E(n)$, consists of all isometries of a Euclean space. An isometry of a Euclidean space consists of translations, rotations, and reflections.

We have $E(n) = T(n) \rtimes O(n)$, where $T(n)$ is the translation group, and $O(n)$ is the orthogonal group, which is the group of all rotations and reflections about the origin.

A further refinement of $E(n)$, $E_+(n)$, consists of all orientation-preserving isometries. In particular, $E_+(n) = T(n) \rtimes SO(n)$, where $SO(n)$ is the special orthogonal group, which is the group of all orientation-preserving rotations.

**Corollary 2.2.1 —** (Weak Banach–Tarski Paradox)

Every closed ball in $\mathbb{R}^3$ is $E(3)$-paradoxical.

**Proof:** We only need to show that $B(0,1)$ is $E(3)$-paradoxical. To do this, we start by showing that $B(0,1)\setminus\{0\}$ is $SO(3)$-paradoxical.

Since $S^2$ is $SO(3)$-paradoxical, there exists pairwise disjoint $A_1,\ldots,A_n,B_1,\ldots,B_m \subset S^2$ and elements $g_1,\ldots,g_n,h_1,\ldots,h_m \in SO(3)$ such that

$$\begin{aligned} S^2 &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

Define

$$\begin{aligned} A_i^* &= \{tx \mid t \in (0,1], x \in A_i\} \\ B_j^* &= \left\{ty \mid t \in (0,1], y \in B_j\right\}. \end{aligned}$$

Then, $A_1^*,\ldots,A_n^*,B_1^*,\ldots,B_m^* \subset B(0,1)\setminus\{0\}$ are pairwise disjoint, and

$$\begin{aligned} B(0,1)\setminus\{0\} &= \bigcup_{i=1}^{n} g_i \cdot A_i^* \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j^*. \end{aligned}$$

Thus, we have established that $B(0,1)\setminus\{0\}$ is $E(3)$-paradoxical.

Now, we want to show that $B(0,1)\setminus\{0\}$ and $B(0,1)$ are $E(3)$-equidecomposable. Let $x \in B(0,1)\setminus\{0\}$, and let $\rho$ be a rotation through $x$ by a line not through the origin such that $\rho^n \cdot 0 \neq \rho^m \cdot 0$ when $n \neq m$.

Let $D = \{\rho^n \cdot 0 \mid n \in \mathbb{N}\}$. We can see that $\rho \cdot D = D \setminus \{0\}$, and that $D$ and $\rho \cdot D$ are $E(3)$-equidecomposable. Thus,

$$\begin{aligned} B(0,1) &= D \sqcup (B(0,1)\setminus D) \\ &\sim_{E(3)} (\rho \cdot D) \sqcup (B(0,1)\setminus D) \\ &= (D\setminus\{0\}) \sqcup (B(0,1)\setminus D) \\ &= B(0,1)\setminus\{0\}. \end{aligned}$$

Therefore, $B(0,1)$ is $E(3)$-equidecomposable. ■

In order to prove the general case of the Banach–Tarski paradox, we need one more piece of mathematical machinery.

Our relation of $A \sim_G B$ is useful, but in order to show the general case, we want to refine the relation slightly.

**Definition 2.2.5**

Let $G$ act on a set $X$ with $A, B \subseteq X$. We write $A \preceq_G B$ if $A$ is equidecomposable with a subset of $B$.

**Fact 2.2.4**

The relation $\preceq_G$ is a reflexive and transitive relation.

**Proof:** To see reflexivity, we can see that since $A \sim_G A$, and $A \subseteq A$, $A \preceq_G A$.

To see transitivity, let $A \preceq_G B$ and $B \preceq_G C$. Then, there exist $g_1,\ldots,g_n \in G$ such that $g_i \cdot A_i = B_{\alpha,i}$ for each $i$, where $A \sim_G B_\alpha \subseteq B$. Similarly, there exist $h_1,\ldots,h_m \in G$ such that $h_j \cdot B_j = C_{\beta,j}$ for each $j$, where $B \sim_G C_\beta \subseteq C$.

We take a refinement of $B$ by taking intersections $B_{\alpha,ij} = B_{\alpha,i} \cap B_j$, with $i = 1,\ldots,n$ and $j = 1,\ldots,m$. We define $C_{\beta,\alpha,ij} = h_j \cdot B_{\alpha,ij}$ for each $j = 1,\ldots,m$. Then, $h_j g_i \cdot A_i = C_{\beta,\alpha,ij}$, meaning $A \sim_G C_{\beta,\alpha,ij} \subseteq C_\beta \subseteq C$, so $A \preceq_G C$. ■

We know from Fact 2.2.3 that $A \preceq_G B$ implies the existence of a bijection $\phi : A \to B' \subseteq B$, meaning $\phi : A \hookrightarrow B$ is an injection. Similarly, if $B \preceq_G A$, then Fact 2.2.3 implies the existence of an injection $\psi : B \hookrightarrow A$.

One may ask if an analogue of the Cantor–Schröder–Bernstein theorem exists in the case of the relation $\preceq_G$, implying that the preorder established in Fact 2.2.4 is indeed a partial order. The following theorem establishes this result.

**Theorem 2.2.3**

Let G act on X, and let $A, B \subseteq X$. If $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$.

**Proof:** Let $B' \subseteq B$ with $A \sim_G B'$, and let $A' \subseteq A$ with $B \sim_G A'$. Then, we know from Fact 2.2.3 that there exist bijections $\phi : A \to B'$ and $\psi : B \to A'$.

Define $C_0 = A \setminus A'$, and $C_{n+1} = \psi(\phi(C_n))$. We set

$$C = \bigcup_{n \geqslant 0} C_n.$$

Since $\psi^{-1}(\psi(\phi(C_n))) = \phi(C_n)$, we have

$$\psi^{-1}(A \setminus C) = B \setminus \phi(C).$$

Having established in Fact 2.2.3 that for any subset of $C \subseteq A$, $C \sim_G \phi(C)$, we also see that $A \setminus C \sim_G B \setminus \phi(C)$.

Thus, we can see that

$$\begin{aligned} A &= (A \setminus C) \sqcup C \\ &\sim_G (B \setminus \phi(C)) \sqcup \phi(C) \\ &\sim B. \end{aligned}$$

■

Finally, we are able to prove the general Banach–Tarski paradox. Recall that the paradox says the following.

**Proposition 2.2.4** — (Banach–Tarski Paradox)

Let A and B be bound subsets of $\mathbb{R}^3$ with nonempty interior. Then, $A \sim_{E(3)} B$.

**Proof:** By symmetry, it is enough to show that $A \preceq_{E(3)} B$.

Since A is bounded, there exists $r > 0$ such that $A \subseteq B(0, r)$.

Let $x_0 \in B^\circ$. Then, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq B$.

Since $B(0, r)$ is compact (hence totally bounded), there are translations $g_1, \dots, g_n$ such that

$$B(0, r) \subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup g_n \cdot B(x_0, \varepsilon).$$

We select translations $h_1, \dots, h_n$ such that $h_j \cdot B(x_0, \varepsilon) \cap h_k \cdot B(x_0, \varepsilon) = \emptyset$ for $j \neq k$. We set

$$S = \bigcup_{j=1}^{n} h_j \cdot B(x_0, \varepsilon).$$

Each $h_j \cdot B(x_0, \varepsilon) \subseteq S$ is E(3)-equidecomposable with any arbitrary closed ball subset of $B(x_0, \varepsilon)$, it is the case that $S \preceq B(x_0, \varepsilon)$.

Thus, we have

$$A \subseteq B(0, r)$$

$$
\begin{aligned}
&\subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup b_n \cdot B(x_0, \varepsilon) \\
&\preceq S \\
&\preceq B(x_0, \varepsilon) \\
&\preceq B.
\end{aligned}
$$

■

# Chapter 3

# Tarski's Theorem

Ultimately, the reason the Banach–Tarski paradox "works" is because the paradoxical group $F(a,b)$, lacks a property known as amenability. Readers may be surprised to hear that amenability and non-paradoxicality are equivalent — that is, a group is amenable if and only if it is non-paradoxical. This fact is formalized in Tarski's theorem.

**Theorem 3.0.1 —** (Tarski's Theorem)

Let $G$ be a group that acts on a set $X$, and let $E \subseteq X$ be nonempty. There is a finitely additive translation-invariant measure $\mu : P(X) \to [0,\infty]$ with $\mu(E) \in (0,\infty)$ if and only if $E$ is not $G$-paradoxical.

In fact, we can prove one of the directions of Tarski's theorem now.

**Proof of the Forward Direction of Tarski's Theorem :** Let $E$ be $G$-paradoxical. Suppose toward contradiction that such a translation-invariant finitely additive $\nu$ existed with $\nu(E) \in (0,\infty)$.

Let $A_1,\dots,A_n,B_1,\dots,B_m \subseteq E$ be pairwise disjoint, and let $t_1,\dots,t_n,s_1,\dots,f_m \in G$ such that

$$\begin{aligned} E &= \bigsqcup_{i=1}^{n} t_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} s_j \cdot B_j. \end{aligned}$$

Then, it would be the case that

$$\begin{aligned} \nu(E) &= \nu\left(\bigsqcup_{i=1}^{n} t_i \cdot A_i\right) \\ &= \sum_{i=1}^{n} \nu\left(t_i \cdot A_i\right) \\ &= \sum_{i=1}^{n} \nu\left(A_i\right), \end{aligned}$$

and

$$\nu(E) = \sum_{j=1}^{m} \nu\left(B_j\right).$$

However, this also yields

$$\nu(E) = \nu\left(\left(\bigsqcup_{i=1}^{n} A_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} B_j\right)\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \nu(A_i) + \sum_{j=1}^{m} \nu\left(B_j\right) \\
&= 2\nu(E),
\end{aligned}
$$

implying that $\nu(E) = 0$ or $\nu(E) = \infty$. ■

## 3.1 A Little Bit of Graph Theory

To prove the reverse direction of Tarski's theorem, we need to develop some machinery from graph theory that will allow us to prove that a certain semigroup we will construct in the next section satisfies the cancellation identity.

We start by defining graphs and paths, before proving a special case of Hall's theorem, ultimately extending to the infinite case with König's theorem.

**Definition 3.1.1** — (Graphs and Paths)

A **graph** is a triple $(V, E, \phi)$, with $V, E$ nonempty sets and $\phi : E \to P_2(V)$ a map from $E$ to the set of all unordered subset pairs of $V$.

For $e \in E$, if $\phi(e) = \{v, w\}$, then we say $v$ and $w$ are the **endpoints** of $e$, and $e$ is **incident** on $v$ and $w$.

A **path** in $(V, E, \phi)$ is a finite sequence $(e_1, \dots, e_n)$ of edges, with a finite sequence of vertices $(v_0, \dots, v_n)$, such that $\phi(e_k) = \{v_{k-1}, v_k\}$.

The **degree** of a vertex, $\deg(v)$, is the number of edges incident on $v$.

We define the **neighbors** of $S \subseteq V$ to be the set of all vertices $v \in V \setminus S$ such that $v$ is an endpoint to an edge incident on $S$. We denote this set $N(S)$.

**Definition 3.1.2** — (Bipartite Graphs and k-Regularity)

Let $(V, E, \phi)$ be a graph, with $k \in \mathbb{N}$.

(i) If $\deg(v) = k$ for each $v \in V$, we say $(V, E, \phi)$ is $k$-**regular**.

(ii) If $V = X \sqcup Y$, with each edge in $E$ having one endpoint in $X$ and one endpoint in $Y$, then we say $V$ is **bipartite**, and write $(X, Y, E, \phi)$.

**Definition 3.1.3** — (Perfect Matching)

Let $(X, Y, E, \phi)$ be a bipartite graph. Let $A \subseteq X$ and $B \subseteq Y$. A **perfect matching** of $A$ and $B$ is a subset $F \subseteq E$ with

(i) each element of $A \cup B$ is an endpoint of exactly one $f \in F$;

(ii) all endpoints of edges in $F$ are in $A \cup B$.

**Definition 3.1.4** — (Hall Condition)

We say a bipartite graph $(X, Y, E, \phi)$ satisfies the **Hall Condition** on $X$ if, for all $S \subseteq X$, $|N(S)| \geqslant |S|$.

Equivalently, we say a (finite) collection of not necessarily distinct finite sets $\mathcal{X} = \{X_i\}_{i=1}^{n}$ satisfies the marriage condition if and only if for all subcollections $\mathcal{Y}_k = \{X_{i_k}\}_{k=1}^m$,

$$|\mathcal{Y}_k| \leqslant \left|\bigcup_{k=1}^m X_{i_k}\right|.$$

**Remark 4**

These two formulations of the Hall condition are equivalent regarding an X-perfect matching.

**Theorem 3.1.1 —** (Hall's Theorem for Finite k-Regular Bipartite Graphs)

Let $(X, Y, E, \phi)$ be a $k$-regular bipartite graph for some $k \in \mathbb{N}$, and let $V = X \sqcup E$ be finite. Then, there is a perfect matching of $X$ and $Y$.

**Proof :** Note that since $|E| = k\,|K| = k\,|Y|$, it is the case that $|X| = |Y|$.

Let $M \subseteq V$ be any subset. We will show that $|N(M)| \geqslant |M|$ — that is, $(X, Y, E, \phi)$ satisfies the Hall condition.

Let $M_X = M \cap X$ and $M_Y = M \cap Y$, where $M = M_X \sqcup M_Y$. Let $[M_X, N(M_X)]$ be the set of edges with endpoints in $M_X$ and $N(M_X)$, and $[M_Y, N(M_Y)]$ be the set of edges with endpoints in $M_Y$ and $N(M_Y)$. We also let $[X, N(M_X)]$ denote the set of edges with endpoints in $X$ and $N(M_X)$, and similarly, $[Y, N(M_Y)]$ is the set of edges with endpoints in $Y$ and $N(M_Y)$.

We can see that $[M_X, N(M_X)] \subseteq [X, N(M_X)]$, and similarly, $[M_Y, N(M_Y)] \subseteq [Y, N(M_Y)]$.

Since $|[M_X, N(M_X)]| = k\,|M_X|$ and $|[X, N(M_X)]| = k\,|N(M_X)|$, we have

$$|M_X| \leqslant |N(M_X)|\,,$$

and similarly,

$$|M_Y| \leqslant |N(M_Y)|\,.$$

Thus, $|M| \leqslant |N(M)|$.

We will now show that there is an X-perfect matching. Suppose toward contradiction that $F$ is a maximal perfect matching on $A \subseteq X$ and $B \subseteq Y$ with $X \setminus A \neq \varnothing$.

Then, there is $x \in X \setminus A$. Consider $Z \subseteq V$ consisting of all vertices $z$ such that there exists a $F$-alternating path $(e_1, \ldots, e_n)$ between $z \in Z$ and $x$.

It cannot be the case that $Z \cap Y$ is empty, since the number of neighbors of $x$ is greater than or equal to 1 by the Hall condition — if it were the case that $Z \cap Y$ were empty, we could add an edge to $F$ consisting of $x$ and one element of $N(\{x\})$, which would contradict the maximality of $F$.

Consider a path traversing along $Z$, $(e_1, \ldots, e_n)$. It must be the case that $e_n \in F$, or else we would be able to "flip" the matching $F$ by exchanging $e_i$ with $e_{i+1}$ for $e_i \in F$, which would contradict the maximality of $F$ yet again. Thus, every element of $Z \cap Y$ is satisfied by $F$, so $Z \cap Y \subseteq B$.

Since each element in $Z \cap Y$ is paired with exactly one element of $Z \cap X$ (with one left over), it is the case that $|Z \cap X| = |Z \cap Y| + 1$.

Suppose toward contradiction that there exists $y \in N(Z \cap X)$ with $y \notin Z \cap Y$. Then, there exists $v \in Z \cap X$ and $e \in E$ such that $\phi(e) = \{v, y\}$. However, this means $v$ is connected via a path to $x$, meaning $y \in Z$, so $y \in Z \cap Y$. Thus, we must have $N(Z \cap X) = Z \cap Y$.

Therefore,

$$\begin{aligned}|Z \cap X| &= |Z \cap Y| + 1 \\ &= |N(Z \cap X)| + 1,\end{aligned}$$

which contradicts the fact that $(X, Y, E, \phi)$ satisfies the Hall condition. Therefore, $A = X$.

By symmetry, there is a perfect matching of $X$ and $Y$ in $(X, Y, E, \phi)$. ■

**Remark 5**

An equivalent formulation to Hall's theorem states that there is a *system of distinct representatives* on $\mathcal{X}$, which is a set $\{x_k\}_{k=1}^{n}$ such that $x_k \in X_k$ and $x_i \neq x_j$ for $i \neq j$.

This implies the existence of an injection $f : \mathcal{X} \hookrightarrow \bigcup_{k=1}^{n} X_k$, such that $f(X_k) \in X_k$.

**Theorem 3.1.2** — (Infinite Hall's Theorem)

Let $\mathcal{G} = \{X_i\}_{i\in I}$ be a collection of (not necessarily distinct) finite sets. If, for every finite subcollection $\mathcal{Y} = \{X_{i_k}\}_{k=1}^{n}$,

$$n \leqslant \left|\bigcup_{k=1}^{n} X_{i_k}\right|,$$

then there is a choice function on G.

**Proof:** We endow each $X_i \in \{X_i\}_{i\in I}$ with the discrete topology. Since each $X_i$ is finite, each $X_i$ is compact.

Thus, by Tychonoff's theorem, it is the case that $\prod_{i\in I} X_i$ is compact.

For every finite subset $Y \subseteq \mathcal{G}$, we define

$$S_Y = \left\{ f \in \prod_{i\in I} X_i \,\middle|\, f|_Y \text{ is injective} \right\}.$$

The injectivity of $f|_Y$ is equivalent to the existence of a system of distinct representatives on $Y$. Since $Y$ satisfies the Hall condition, each $S_Y$ is nonempty. Additionally, for any net of functions $f_\alpha \in S_Y$ with $\lim_\alpha f_\alpha = f$, it is the case that $f_\alpha|_Y$ is injective, so $f|_Y$ is injective, meaning $S_Y$ is closed.

We define $F = \{S_Y \mid Y \subseteq \mathcal{G} \text{ finite}\}$. For finite $Y_1, Y_2 \subseteq \mathcal{G}$, every system of distinct representatives in $Y_1 \cup Y_2$ is necessarily a system of distinct representatives on $Y_1$ and a system of distinct representatives on $Y_2$, meaning $S_{Y_1\cup Y_2} \subseteq S_{Y_1} \cap S_{Y_2}$. Thus, $F$ has the finite intersection property.

Since $\prod_{i\in I} X_i$ is compact, $\bigcap F$ is nonempty, where the intersection is taken over all finite subsets of $\mathcal{G}$. For any $f \in \bigcap F$, $f$ is necessarily a choice function. ■

**Remark 6**

This is equivalent to the existence of an injection $f : \mathcal{G} \hookrightarrow \bigcup_{i\in I} X_i$.

We will use this infinite case of Hall's theorem to prove König's theorem.

**Theorem 3.1.3** — (König's Theorem)

Let $(X, Y, E, \phi)$ be a k-regular bipartite graph (not necessarily finite). Then, there is a perfect matching of $X$ and $Y$.

**Proof:** If $k = 1$, it is clear that there is a perfect matching in $(X, Y, E, \phi)$ consisting of the edges in $(X, Y, E, \phi)$.

Let $k \geqslant 2$. Since any finite subset of X satisfies the Hall condition, as displayed in the proof of Theorem 3.1.1, there is some X-perfect matching in $(X, Y, E, \phi)$. We call this X-perfect matching F. There is an injection $f : X \hookrightarrow Y$ following the edges in F.

Similarly, since any finite subset of Y satisfies the Hall condition, there is some Y-perfect matching in $(X, Y, E, \phi)$. We call this Y-perfect matching G. There is an injection $g : Y \hookrightarrow X$ following the edges of G.

Consider the subgraph $(X, Y, F \cup G, \phi|_{F \cup G})$. The injections f and g still hold in this graph. By the Cantor–Schröder–Bernstein theorem, there is a bijection $h : X \to Y$ in $(X, Y, F \cup G, \phi|_{F \cup G})$. ■

## 3.2 Type Semigroups

**Definition 3.2.1**

Let G be a group that acts on a set X.

(i) We define $X^* = X \times \mathbb{N}_0$, and

$$G^* = \left\{(g, \pi) \mid g \in G, \pi \in \mathrm{Sym}\left(\mathbb{N}_0\right)\right\}.$$

(ii) If $A \subseteq X^*$, the values of $n$ for which there is an element of $A$ whose second coordinate is $n$ are called the **levels** of $A$.

**Fact 3.2.1**

If $E_1, E_2 \subseteq X$, then $E_1 \sim_G E_2$ if and only if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$ for all $m, n \in \mathbb{N}_0$

**Proof of Fact 3.2.1:** Let $E_1 \sim_G E_2$. Then, there exist pairwise disjoint $A_1, \ldots, A_n \subset E_1$, pairwise disjoint $B_1, \ldots, B_n \subset E_2$, and elements $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$. We select the permutation $\pi_i \in \mathrm{Sym}\left(\mathbb{N}_0\right)$ such that $\pi_i(n) = m$ and $\pi_i(m) = n$ for each $i$. Then,

$$(g_i, \pi_i) \cdot (A_i \cdot \{n\}) = B_i \cdot \{m\}.$$

Similarly, if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$, then of the pairwise disjoint subsets

$$A_1 \times \{n\}, \ldots, A_n \times \{n\} \subset E_1 \times \{n\}$$

and

$$B_1 \times \{m\}, \ldots, B_n \times \{m\} \subset E_2 \times \{m\},$$

we set $A_1, \ldots, A_n \subset E_1$ and $B_1, \ldots, B_n \subset E_2$. Similarly, for

$$(g_1, \pi_1), \ldots, (g_n, \pi_n) \in G^*$$

such that

$$(g_i, \pi_i) \cdot A_i \times \{n\} = B_i \times \{m\},$$

we select $g_1, \ldots, g_n \in G$. Then, by definition,

$$g_i \cdot A_i = B_i$$

for each $i$. Thus, $E_1 \sim_G E_2$. ■

**Definition 3.2.2**

**??** Let G be a group that acts on X, and let $G^*, X^*$ be defined as above.

(i) A set $A \subseteq X^*$ is said to be **bounded** if it has finitely many levels.

(ii) If $A \subseteq X^*$ is bounded, the equivalence class of $A$ with respect to $G^*$-equidecomposability is called the **type** of $A$, which is denoted $[A]$.

(iii) If $E \subseteq X$, we write $[E] = [E \times \{0\}]$.

(iv) Let $A, B \subseteq X^*$ be bounded with $k \in \mathbb{N}_0$ such that for

$$B' := \{(b, n+k) \mid (b, n) \in B\},$$

we have $B' \cap A = \varnothing$. Then, $[A] + [B] = [A \sqcup B']$. Note that $[B'] = [B]$.

(v) We define

$$\mathcal{S} = \{[A] \mid A \subseteq X^* \text{ bounded}\}$$

under the addition defined in (iv) to be the **type semigroup** of the action of $G$ on $X$.

**Fact 3.2.2**

Addition is well-defined in $(\mathcal{S}, +)$, and $(\mathcal{S}, +)$ is a well-defined commutative semigroup with identity $[\varnothing]$.

**Proof of Fact 3.2.2:** To show that addition is well-defined, we let $[A_1] = [A_2]$, and $[B_1] = [B_2]$. Without loss of generality, $A_1 \cap B_1 = \varnothing$ and $A_2 \cap B_2 = \varnothing$.

By the definition of the type, $A_1 \sim_{G^*} A_2$ and $B_1 \sim_{G^*} B_2$, meaning

$$A_1 \sqcup B_1 \sim_{G^*} A_2 \sqcup B_2,$$

so

$$\begin{aligned}[A_1] + [B_1] &= [A_1 \sqcup B_1] \\ &= [A_2 \sqcup B_2] \\ &= [A_2] + [A_2],\end{aligned}$$

meaning addition is well-defined.

Since addition is well-defined, and $A \sqcup B = B \sqcup A$, we can see that addition is also commutative. We also have

$$\begin{aligned}[A] + [\varnothing] &= [A \sqcup \varnothing] \\ &= [A],\end{aligned}$$

so $[\varnothing]$ is the identity on $\mathcal{S}$.

Finally, since for any $[A], [B] \in \mathcal{S}$, $A$ and $B$ have finitely many levels, it is the case that $A \cup B$ has finitely many levels for any $A$ and $B$, so $[A] + [B] \in \mathcal{S}$. ■

**Definition 3.2.3**

For any commutative semigroup $\mathcal{S}$ with $\alpha \in S$ and $n \in \mathbb{N}$, we define

$$n\alpha := \underbrace{\alpha + \cdots + \alpha}_{n \text{ times}}$$

**Definition 3.2.4**

For $\alpha, \beta \in \mathcal{S}$, if there exists $\gamma \in \mathcal{S}$ such that $\alpha + \gamma = \beta$, we write $\alpha \leqslant \beta$.

**Fact 3.2.3**

If $G$ is a group acting on $X$ with corresponding type semigroup $\mathcal{S}$, then the following are true.

(i) If $\alpha, \beta \in \mathcal{S}$ with $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$, then $\alpha = \beta$.

(ii) A set $E \subseteq X$ is $G$-paradoxical if and only if $[E] = 2[E]$.

**Proof of Fact 3.2.3:** Let $G$ act on $X$, and let $\mathcal{S}$ be the corresponding type semigroup.

(i) If $[A] \leqslant [B]$, then there exists $C_1 \in \mathcal{S}$ such that $[A] + [C_1] = [B]$. Without loss of generality, $C_1 \cap A = \emptyset$, meaning $[B] = [A \sqcup C_1]$. Thus, $A \sqcup C_1 \sim_{G^*} B$, meaning $B \preceq_{G^*} A$.

Similarly, if $[B] \leqslant [A]$, then $B \preceq_{G^*} A$. By Theorem 2.2.3, it is thus the case that $A \sim_{G^*} B$.

(ii) Let $E$ be $G$-paradoxical. Then, $E \sim_G \bigsqcup_{i=1}^{n} A_i$ and $E \sim_G \bigsqcup_{j=1}^{m} B_j$ for some disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subset E$. Thus, we have

$$\begin{aligned}[E] &= \left[\left(\bigsqcup_{i=1}^{n} A_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} B_j\right)\right] \\ &= \left[\bigsqcup_{i=1}^{n} A_i\right] + \left[\bigsqcup_{j=1}^{m} B_j\right] \\ &= 2[E].\end{aligned}$$

Similarly, if $[E] = 2[E]$, then there exist $A$ and $B$ such that

$$\begin{aligned}[E] &= [A] + [B] \\ &= [A \sqcup B],\end{aligned}$$

meaning $A$ and $B$ are each $G$-equidecomposable with $E$, so $E$ is $G$-paradoxical. ■

We can now prove the cancellation identity, which we will be useful as we construct our desired finitely additive measure.

**Theorem 3.2.1** — (Cancellation Identity on $\mathcal{S}$)

Let $\mathcal{S}$ be the type semigroup for some group action, and let $\alpha, \beta \in \mathcal{S}$, $n \in \mathbb{N}$ such that $n\alpha = n\beta$. Then, $\alpha = \beta$.

**Proof:** Let $n\alpha = n\beta$. Then, there are two disjoint bounded subsets $E, E' \subseteq X^*$ with $E \sim_{G^*} E'$, and pairwise disjoint subsets $A_1, \dots, A_n \subseteq E$, $B_1, \dots, B_n \subseteq E'$ such that

- $E = A_1 \cup \dots \cup A_n$, $E' = B_1 \cup \dots \cup B_n$
- $\left|A_j\right| = \alpha$ and $\left[B_j\right] = \beta$ for each $j = 1, \dots, n$.

Let $\chi : E \to E'$ be a bijection as in Fact 2.2.3, with $\phi_j : A_1 \to A_j$, $\psi_j : B_1 \to B_j$ also being bijections as in Fact 2.2.3; here we define $\phi_1$ and $\psi_1$ to be the identity map.

For each $a \in A_1$ and $b \in B_1$, we define

$$\begin{aligned}\overline{a} &= \{a, \phi_2(a), \dots, \phi_n(a)\} \\ \overline{b} &= \{b, \psi_2(b), \dots, \psi_n(b)\}.\end{aligned}$$

We construct a graph by letting $X = \{\overline{a} \mid a \in A_1\}$ and $Y = \left\{\overline{b} \mid b \in B_1\right\}$, and, for each $j$, define edges $\left\{\overline{a}, \overline{b}\right\}$ if $\chi\left(\phi_j(a)\right) \in \overline{b}$.

Since $\chi$ is a bijection, for each $j = 1, \dots, n$, $\chi\left(\phi_j(a)\right)$ must be an element of $B_k$ for some $k$, and since $\{B_k\}_{k=1}^{n}$ are disjoint, $\chi\left(\phi_j(a)\right)$ is an element of exactly one $B_k$. Thus, the graph is $n$-regular.

By König's Theorem (Theorem 3.1.3), this graph has a perfect matching $F$. As a result, for each $\overline{a} \in X$, there is a unique $\overline{b} \in Y$ and a unique edge $\left\{\overline{a}, \overline{b}\right\} \in F$ such that $\chi\left(\phi_j(a)\right) = \psi_k(b)$ for some $j, k \in \{1, \dots, n\}$.

We define

$$\begin{aligned} C_{j,k} &= \left\{a \in A_1 \mid \left\{\overline{a}, \overline{b}\right\} \in F,\ \chi\left(\phi_j(a)\right) = \psi_k(b)\right\} \\ D_{j,k} &= \left\{b \in B_1 \mid \left\{\overline{a}, \overline{b}\right\} \in F,\ \chi\left(\phi_j(a)\right) = \psi_k(b)\right\}. \end{aligned}$$

Therefore, we must have $\psi_k^{-1} \circ \chi \circ \phi_j$ is a bijection from $C_{j,k}$ to $D_{j,k}$, so $C_{j,k} \sim_{G^*} D_{j,k}$.

Since $C_{j,k}$ and $D_{j,k}$ are partitions of $A_1$ and $B_1$ respectively, it follows that $A_1 \sim_{G^*} B_1$, so $\alpha = \beta$. ■

**Corollary 3.2.1**

Let $\mathcal{S}$ be the type semigroup of some group action, and let $\alpha \in \mathcal{S}$ and $n \in \mathbb{N}$ such that $(n+1)\alpha \leqslant n\alpha$. Then, $\alpha = 2\alpha$.

**Proof:** We have

$$\begin{aligned} 2\alpha + n\alpha &= (n+1)\alpha + \alpha \\ &\leqslant n\alpha + \alpha \\ &= (n+1)\alpha \\ &\leqslant n\alpha. \end{aligned}$$

Inductively repeating this argument, we get $n\alpha \geqslant 2n\alpha$; since $n\alpha \leqslant 2n\alpha$ by definition, we must have $n\alpha = 2n\alpha$, so $\alpha = 2\alpha$. ■

**Remark 7**

We will call such an $\alpha$ a paradoxical element.

## 3.3 Two Results on Commutative Semigroups

Now that we are aware of paradoxical elements and the relationship between G-paradoxicality and the properties of the particular elements of the type semigroup (Fact 3.2.3), we will now relate these properties to finitely additive measures of sets by using the following lemma and theorem.

**Lemma 3.3.1**

Let $\mathcal{S}$ be a commutative semigroup, with $\mathcal{S}_0 \subseteq \mathcal{S}$ finite, and $\epsilon \in \mathcal{S}_0$ satisfying the following assumptions:

(a) $(n+1)\epsilon \nleqslant n\epsilon$ for all $n \in \mathbb{N}$ (i.e., that $\epsilon$ is non-paradoxical);

(b) for each $\alpha \in \mathcal{S}$, there is $n \in \mathbb{N}$ such that $\alpha \leqslant n\epsilon$.

Then, there is a set function $\nu : \mathcal{S}_0 \to [0, \infty]$ that satisfies the following conditions:

(i) $\nu(\epsilon) = 1$;

(ii) for $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathcal{S}_0$ with $\alpha_1 + \cdots + \alpha_n \leqslant \beta_1 + \cdots \beta_m$,

$$\sum_{j=1}^{n} \nu\left(\alpha_j\right) \leqslant \sum_{j=1}^{m} \nu\left(\beta_j\right).$$

**Proof :** We will prove this result by inducting on the cardinality of $\mathcal{S}_0$.

We start with $|\mathcal{S}_0| = 1$. In that case, we define $\nu(\epsilon) = 1$, satisfying condition (i). To satisfy condition (ii), we see that for $n, m \in \mathbb{N}$ with $n\epsilon \leqslant m\epsilon$, if $n \geqslant m+1$, then $(m+1)\epsilon \leqslant n\epsilon \leqslant m\epsilon$, implying that $\epsilon = 2\epsilon$, which contradicts assumption (a).

Let $\alpha_0 \in \mathcal{S}_0 \setminus \{\epsilon\}$. The induction hypothesis says there is a set function satisfying conditions (i) and (ii), $\nu : \mathcal{S}_0 \setminus \{\alpha_0\} \to [0, \infty]$.

For $r \in \mathbb{N}$, there are $\gamma_1, \ldots, \gamma_p, \delta_1, \ldots, \delta_q \in \mathcal{S} \setminus \{\alpha_0\}$ such that

$$\delta_1 + \cdots + \delta_q + r\alpha_0 \leqslant \gamma_1 + \cdots + \gamma_p. \tag{†}$$

Consider the set N defined as follows:

$$N = \left\{ \frac{1}{r} \left( \sum_{j=1}^{p} \nu(\gamma_j) - \sum_{j=1}^{q} \nu(\delta_j) \right) \mid \gamma_j, \delta_j \text{ satisfy (†)} \right\}. \tag{‡}$$

We define the extension of $\nu$ as follows:

$$\nu(\alpha_0) = \inf N.$$

This infimum is well-defined since, by assumption (b), there is some $n \in \mathbb{N}$ such that $\alpha_0 \leqslant n\epsilon$, and $\nu(\epsilon)$ is defined.

Now, we must show that this extension of $\nu$ satisfies condition (ii).

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathcal{S}_0 \setminus \{\alpha_0\}$ and $s, t \in \mathbb{N}_0$ such that

$$\alpha_1 + \cdots + \alpha_n + s\alpha_0 \leqslant \beta_1 + \cdots + \beta_m + t\alpha_0. \tag{$*$}$$

We will verify condition (ii) in the three following cases.

Case 0: If $s = t = 0$, then the induction hypothesis states that $(*)$ satisfies condition (ii).

Case 1: Let $s = 0$ and $t > 0$. We want to show that

$$\sum_{j=1}^{n} \nu(\alpha_j) \leqslant t\nu(\alpha_0) + \sum_{j=1}^{m} \nu(\beta_j),$$

which implies that

$$\nu(\alpha_0) \geqslant \frac{1}{t} \left( \sum_{j=1}^{n} \nu(\alpha_j) - \sum_{j=1}^{m} \nu(\beta_j) \right).$$

By the definition of infimum, it suffices to show that for $r \in \mathbb{N}$ and $\delta_1, \ldots, \delta_q, \gamma_1, \ldots, \gamma_p \in \mathcal{S} \setminus \{\alpha_0\}$ satisfying (†), it is the case that

$$\frac{1}{r} \left( \sum_{j=1}^{p} \nu(\gamma_j) - \sum_{j=1}^{q} \nu(\delta_j) \right) \geqslant \frac{1}{t} \left( \sum_{j=1}^{n} \nu(\alpha_j) - \sum_{j=1}^{m} \nu(\beta_j) \right).$$

Multiplying $(*)$ by $r$ on both sides, and adding $t\delta_1 + \cdots + t\delta_q$ to both sides, we have

$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q \leqslant r\beta_1 + \cdots + r\beta_m + t(r\alpha_0) + t\delta_1 + \cdots + t\delta_q.$$

Substituting (†), we find

$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q \leqslant r\beta_1 + \cdots + r\beta_m + t\gamma_1 + \cdots + t\gamma_p.$$

Applying the induction hypothesis, we have

$$r\sum_{j=1}^{n} \nu(\alpha_j) + t\sum_{j=1}^{q} \nu(\delta_j) \leqslant r\sum_{j=1}^{m} \nu(\beta_j) + t\sum_{j=1}^{p} \nu(\gamma_j),$$

yielding

$$\frac{1}{r}\left(\sum_{j=1}^{p}\nu\left(\gamma_j\right)-\sum_{j=1}^{q}\nu\left(\delta_j\right)\right)\geqslant\frac{1}{t}\left(\sum_{j=1}^{n}\nu\left(\alpha_j\right)-\sum_{j=1}^{m}\nu\left(\beta_j\right)\right).$$

Case 2: Let $s > 0$. For $z_1, \dots, z_t \in \mathbb{N}$ (‡), we need to show that

$$s\nu\left(\alpha_0\right)+\sum_{j=1}^{n}\nu\left(\alpha_j\right)\leqslant z_1+\cdots+z_t+\sum_{j=1}^{n}\nu\left(\beta_j\right).$$

Without loss of generality, we can set $z_1, \dots, z_n = z$, as for each $z \in \mathbb{N}$, $z \geqslant \nu\left(\alpha_0\right)$.

As in Case 1, we multiply (∗) by $r$, add $t\delta_1 + \cdots + t\delta_q$ to both sides, and substitute with (†), yielding

$$\begin{aligned}
r\alpha_1+\cdots+r\alpha_n+rs\alpha_0+t\delta_1+\cdots+t\delta_q &\leqslant r\beta_1+\cdots+r\beta_m+t\left(r\alpha_0\right)+t\delta_1+\cdots+t\delta_q\\
r\alpha_1+\cdots+r\alpha_n+t\delta_1+\cdots+t\delta_q+rs\alpha_0 &\leqslant r\beta_1+\cdots+r\beta_m+t\gamma_1+\cdots+t\gamma_p.
\end{aligned}$$

Defining

$$z=\frac{1}{r}\left(\sum_{j=1}^{p}\nu\left(\gamma_j\right)-\sum_{j=1}^{q}\nu\left(\delta_j\right)\right),$$

we get

$$\begin{aligned}
s\nu\left(\alpha_0\right)+\sum_{j=1}^{n}\nu\left(\alpha_j\right) &\leqslant \sum_{j=1}^{n}\nu\left(\alpha_j\right)+\frac{s}{sr}\left(r\sum_{j=1}^{m}\nu\left(\beta_j\right)-r\sum_{j=1}^{n}\nu\left(\alpha_j\right)+t\sum_{j=1}^{p}\nu\left(\gamma_j\right)-t\sum_{j=1}^{q}\nu\left(\delta_j\right)\right)\\
&= tz+\sum_{j=1}^{m}\nu\left(\beta_j\right).
\end{aligned}$$

Thus, we have shown that $\nu$ extends in a manner that satisfies conditions (i) and (ii). ■

We can "upgrade" our finitely additive set function to a semigroup homomorphism as follows.

**Theorem 3.3.1**

Let $(\mathcal{S},+)$ be a commutative semigroup with identity element 0, and let $\epsilon \in \mathcal{S}$. Then, the following are equivalent:

(i) $(n+1)\,\epsilon \leqslant n\epsilon$ for all $n \in \mathbb{N}$;

(ii) there is a semigroup homomorphism $\nu : (\mathcal{S},+) \to ([0,\infty],+)$ such that $\nu(\epsilon) = 1$.

**Proof:** To show that (ii) implies (i), we let $\nu : (\mathcal{S},+) \to ([0,\infty],+)$ be a semigroup homomorphism with $\nu(\epsilon) = 1$. Then,

$$\begin{aligned}
\nu\left((n+1)\,\epsilon\right) &= (n+1)\,\nu\left(\epsilon\right)\\
&= n+1\\
&> n\\
&= n\nu\left(\epsilon\right)\\
&= \nu\left(n\epsilon\right),
\end{aligned}$$

meaning that $(n+1)\,\epsilon \nleqslant n\epsilon$.

To show that (i) implies (ii), we suppose that for each $\alpha \in \mathcal{S}$, there is $n \in \mathbb{N}$ such that $\alpha \leqslant n\epsilon$ — for any such $\alpha$ for which this is not the case, we define $\nu(\alpha) = \infty$.

For a finite subset $\mathcal{S}_0 \subseteq \mathcal{S}$ with $\epsilon \in \mathcal{S}_0$, we define $M_{\mathcal{S}_0}$ to be the set of all $\kappa : \mathcal{S} \to [0,\infty]$ such that

- $\kappa(\epsilon) = 1$;
- $\kappa(\alpha+\beta) = \kappa(\alpha) + \kappa(\beta)$ for $\alpha, \beta, \alpha+\beta \in \mathcal{S}_0$.

Since we assume condition (i), we know that such a $\kappa$ with $\kappa(\epsilon) = 1$ exists. Additionally, since

$$\alpha + \beta \leqslant (\alpha + \beta)$$

and

$$(\alpha + \beta) \leqslant \alpha + \beta,$$

it is the case that

$$\kappa(\alpha+\beta) \leqslant \kappa(\alpha) + \kappa(\beta) \leqslant \kappa(\alpha+\beta),$$

meaning $\kappa(\alpha+\beta) = \kappa(\alpha) + \kappa(\beta)$. Thus, $M_{\mathcal{S}_0}$ is nonempty. It is also the case that $M_{\mathcal{S}_0}$ is closed, since any net of functions $\kappa_p : \mathcal{S} \to [0,\infty]$ with $\kappa_p(\epsilon) = 1$ and $\kappa_p(\alpha+\beta) = \kappa_p(\alpha) + \kappa_p(\beta)$ will necessarily satisfy these conditions in the limit.

We let $[0,\infty]^{\mathcal{S}} = \{\kappa \mid \kappa : \mathcal{S} \to [0,\infty]\}$ be equipped with the product topology. By Tychonoff's theorem, $[0,\infty]^{\mathcal{S}}$ is compact.

Since, for any $\mathcal{S}_1, \ldots, \mathcal{S}_n$ finite, it is the case that

$$M_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n} \subseteq M_{\mathcal{S}_1} \cap \cdots \cap M_{\mathcal{S}_n},$$

since any such $\kappa \in M_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n}$ must necessarily be in every $M_{\mathcal{S}_i}$. Thus, the family

$$\left\{ M_{\mathcal{S}_0} \mid \mathcal{S}_0 \subseteq \mathcal{S} \text{ finite} \right\}$$

has the finite intersection property. Thus, by compactness, there is some $\nu$ such that

$$\nu \in \bigcap_{\mathcal{S}_0} \left\{ M_{\mathcal{S}_0} \mid \mathcal{S}_0 \subseteq \mathcal{S} \text{ finite} \right\},$$

with $\nu(\epsilon) = 1$ and, for all $\alpha, \beta \in \mathcal{S}$, since $\nu \in M_{\{\alpha,\beta,\alpha+\beta\}}$, $\nu(\alpha+\beta) = \nu(\alpha) + \nu(\beta)$. ■

## 3.4 Proof of Tarski's Theorem

Finally, we are able to prove Tarski's Theorem.

**Proof of Theorem 3.0.1:** Let $\mathcal{S}$ be the type semigroup of the action of G on X.

Suppose E is not G-paradoxical. Then, $[E] \neq 2[E]$, meaning $(n+1)[E] \nleqslant n[E]$ for all $n \in \mathbb{N}$.

Thus, there is a map $\nu : \mathcal{S} \to [0,\infty]$ with $\nu([E]) = 1$. The map

$$\mu : P(X) \to [0,\infty]$$

defined by

$$\nu(A) = \nu([A])$$

is the desired finitely additive measure. ■

## Chapter 4

# Invariant Means and Følner's Condition

# Chapter 5

# Day's Fixed Point Theorem

# Chapter 6

# Nuclear C*-Algebras

# Appendix A

# Group Theory

# Appendix B

# Point-Set Topology

We will need a bit of background in point-set topology in order to satisfactorily understand the functional analysis behind the results in Chapters 3, 4, and 5.

## B.1 Axioms of Set Theory

In order to garner sufficient understanding of point-set topology, we need to be able to comprehend some of the essential axioms behind the objects known as "sets." This is where the axioms of set theory come into play.

**Definition B.1.1** — (Zermelo–Fraenkel Axioms)

In Zermelo–Fraenkel set theory, all objects are sets. In order to maintain convention with the way the rest of this section will refer to sets, all sets will be referred to by capital letters, and all elements of sets by lowercase letters.

- Axiom of Existence: $\exists A\,(A = A)$. This axiom guarantees a nonempty universe.
- Axiom of Extensionality: $\forall x\,(x \in A \Leftrightarrow x \in B) \Rightarrow A = B$. This axiom states that if two sets share the same members, then the sets are equal.
- Axiom Schema of Comprehension: $\exists B\ \forall x\,(x \in B \Leftrightarrow x \in A \wedge \varphi(x))$. This axiom states that for any formula $\varphi(x)$, where $x$ is a free variable, there is a set $B$ such that the members of $B$ are the members of $A$ for which $\varphi$ holds.
- Pairing Axiom: $\forall A\ \forall B\ \exists C\,((A \in Z) \wedge (B \in Z))$. This axiom states that for any sets $A$ and $B$, there is a set $C = \{A, B\}$ that contains the sets $A$ and $B$ as elements.
- Power Set Axiom: $\forall A \exists P(A) \forall B\,(B \in P(A) \Leftrightarrow B \subseteq A)$. We use the shorthand $B \subseteq A$ to mean $\forall x\,(x \in B \Rightarrow x \in A)$. This axiom states that for any set $A$ there exists a set $P(A)$ such that any element of $P(A)$ is a subset of $A$, and any subset of $A$ is an element of $P(A)$.
- Union Axiom: $\forall \mathcal{A}\ \exists A\ \forall Y\ \forall x\,((x \in Y \wedge Y \in \mathcal{A}) \Rightarrow x \in A)$. This axiom states that for any collection $\mathcal{A}$, there is a set $A$, denoted $\bigcup \mathcal{A}$, that contains all the elements of all the sets in the collection $\mathcal{A}$.
- Axiom of Infinity: $\exists A\,(\varnothing \in A \wedge \forall x\,(x \in A \Rightarrow x \cup \{x\} \in A))$. This axiom states that there is a set, $A$, such that the empty set is in $A$ and, for any element $x$, if $x \in A$, then so too is the successor, $x \cup \{x\}$.
- Axiom of regularity: $\forall X\,(X \neq \varnothing \Rightarrow \exists Y\,(Y \in X \wedge Y \cap X = \varnothing))$. This axiom states that any nonempty set $X$ contains a set $Y$ such that $Y$ and $X$ are disjoint. As a consequence, any chain of sets descending in membership must terminate.
- Axiom Schema of Replacement: $\forall A\ \exists B\ \forall v\,(v \in B \Rightarrow \exists u\,(u \in A \wedge \psi\,(u, v)))$. The axiom schema of replacement says that for a function-like formula (a formula such that $\psi\,(u, v) \wedge \psi\,(u, w) \Rightarrow v = w$) $\psi\,(u, v)$, there is a set $A$ consisting of exactly those sets/elements $v \in B$ that correspond to $u \in A$.

The final axiom, the Axiom of Choice, is special, and as a result, we state it separately, for we will be using some of its consequences in the future sections. The following is one way of interpreting the axiom of choice.

**Definition B.1.2** — (Axiom of Choice)

Let $\{S_i\}_{i \in I}$ be an indexed collection of nonempty sets. Then, there exists an indexed set $\{x_i\}_{i \in I}$ such that $x_i \in S_i$ for each I.

Equivalently, if $\{S_i\}_{i \in I}$ is an indexed collection of nonempty sets, then there is some choice function

$$f \in \prod_{i \in I} S_i.$$

On its own, this formulation of the Axiom of Choice is not particularly useful. However, there is a statement of the Axiom of Choice which is just as useful.

**Definition B.1.3** — (Preorders, Partial Orders, Total Orders, and Well-Orders)

Let X be a set, and $\leq$ be a relation on X. We say a relation is a preorder if it is reflexive and transitive:

- $a \leq a$
- $a \leq b \wedge b \leqslant c \Rightarrow a \leq c$.

We say X is a directed set if, for any $a, b \in X$, there is $c \in X$ such that $a \leq c$ and $b \leq c$.

If $\leq$ is also antisymmetric — that is, $a \leq b \wedge b \leq a \Rightarrow a = b$ — then, we say $\leq$ is a partial order.

We say $m \in X$ is a maximal element if, for any $x \in X$ with $m \leq x$, $m = x$.

If X is partially ordered by $\leq$ and, for any two elements $a, b \in X$, either $a \leq b$ or $b \leq a$, then we say $\leq$ is a total order on X.

If X is a totally ordered set that has the property that, for any nonempty $A \subseteq X$, there is some $x \in A$ such that for any $y \in A$, $x < y$ for all $y \in A$ with $y \neq x$, then we say $\leq$ is a well-order on X.

**Example B.1.1**

- The set $\mathbb{N}$ with the usual ordering is a well-ordered set.
- If A is a set, then $P(A)$ with the containment ordering, $A \leq B$ if $A \supseteq B$, is a partially ordered set.
- Similarly, if A is a set, then $P(A)$ with the inclusion ordering, $A \leq B$ if $A \subseteq B$, is a partially ordered set.
- A collection of functions $\{\varphi_i : Z_i \to Y\}_{i \in I}$ ordered by $\varphi_i \leq \varphi_j$ if $Z_i \subseteq Z_j$ and $\varphi_j|_{Z_i} = \varphi_i$, is a partially ordered set. This is often known as the extension ordering.

We can state an equivalent formulation of the Axiom of Choice as follows.

**Theorem B.1.1** — (Zorn's Lemma)

If $(X, \leq)$ is a partially ordered set with the property that for all $C \subseteq X$ with C totally ordered, C has an upper bound, then X has a maximal element.

There are many proofs of both Zorn's Lemma from the Axiom of Choice and the Axiom of Choice from Zorn's Lemma. However, we will mostly be using it for the purposes of proving other theorems. The following results can be proven using Zorn's Lemma.

**Example B.1.2**

- Every $\mathbb{F}$-vector space $V$ has a basis $B \subseteq V$ such that the set of all finite linear combinations of elements of $B$ over $\mathbb{F}$ is $V$.
- If $\varphi$ is a continuous linear functional defined on a subspace $W \subseteq V$, there is an extension $\Phi$ such that $\Phi|_W = \varphi$.
- The arbitrary product of compact spaces is compact.

# B.2 Metric Spaces

Building upon the basics of set theory, we move towards understanding metric spaces.

## B.2.1 Basics of Metric Spaces

**Definition B.2.1** — (Metrics)

Let $X$ be a set. A distance metric is a function

$$d : X \times X \longrightarrow [0, \infty)$$

such that the following three properties are satisfied:

- if $x, y \in X$ and $d(x, y) = 0$, then $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A function that satisfies the latter two properties is called a semimetric.

Two metrics $d$ and $\rho$ on $X$ are equivalent if there exist constants $c_1, c_2 \geqslant 0$ such that

$$\begin{aligned} d(x, y) &\leqslant c_1 \rho(x, y) \\ \rho(x, y) &\leqslant c_2 d(x, y) \end{aligned}$$

for all $x, y \in X$.

A metric space is a pair $(X, d)$, where $d$ is a metric.

**Example B.2.1** — (Some Distance Metrics)

- The discrete metric on any nonempty set is given by

$$d(x, y) \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

- The Euclidean metric between $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is

$$d_2(x, y) = \left( \sum_{j=1}^{n} \left| y_j - x_j \right|^2 \right)^{1/2}.$$

- Other metrics on $\mathbb{R}^n$ include

$$d_1(x, y) = \sum_{j=1}^{n} \left|y_j - x_j\right|$$
$$d_\infty(x, y) = \max_{j=1}^{n} \left|y_j - x_j\right|.$$

All of $d_1, d_2, d_\infty$ are equivalent metrics.

- The Hamming distance between two strings of bits is

$$d_H : \{0,1\}^n \times \{0,1\}^n \to [0, \infty)$$
$$d_H\left(\left(x_j\right)_{j=1}^{n}, \left(y_j\right)_{j=1}^{n}\right) = \left|\left\{j \mid x_j \neq y_j\right\}\right|.$$

- The set $C([0,1], \mathbb{R})$ consisting of continuous real-valued functions from $[0,1]$ to $\mathbb{R}$ can be equipped with

$$d_u(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|,$$

which is the uniform metric, or

$$d_1(f, g) = \int_0^1 |f(t) - g(t)| \, dt.$$

- All subsets of a metric space $X$ equipped with the same metric is also a metric space.

- If $\rho$ is a metric on $X$, then we can create a distance metric

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$$

that is bounded on $[0, 1]$.

- If $d_1, \ldots, d_n$ are metrics on $X$ and $c_1, \ldots, c_n > 0$ are constants, then

$$d(x, y) = \sum_{k=1}^{n} c_k d_k(x, y)$$

defines a metric on $X$.

- If $(\rho_k)_k$ is a family of separating semimetrics for $X$ — i.e., for $x, y \in X$ distinct, there is some $\rho_j$ such that $\rho_j(x, y) \neq 0$ — then, we can obtain bounded semimetrics by taking

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}$$

for each $k$. From each $d_k$, we define

$$d(x, y) = \sum_{k=1}^{n} 2^{-k} d_k(x, y),$$

which is a metric on $X$.

- If $(X_k, \rho_k)_{k\geqslant 1}$ is a sequence of metric spaces, then we can form the product space

$$X = \prod_{k\geqslant 1} X_k$$

with the metric

$$D(f,g) = \sum_{k\geqslant 1} d_k\left(f(k), g(k)\right).$$

Here, $d_k = \frac{\rho_k}{1+\rho_k}$ is the corresponding bounded metric to $\rho_k$.

**Definition B.2.2** — (Open and Closed Sets)

Let $(X, d)$ be a metric space.

(1) For $x \in X$ and $\delta > 0$, we define

(a) the open ball at $x$ with radius $\delta > 0$

$$U(x,\delta) = \{y \in X \mid d(y,x) < \delta\};$$

(b) the closed ball centered at $x$ with radius $\delta > 0$

$$B(x,\delta) = \{y \in X \mid d(y,x) \leqslant \delta\};$$

(c) the sphere centered at $x$ with radius $\delta > 0$

$$S(x,\delta) = \{y \in X \mid d(y,x) = \delta\}.$$

(2) A set $V \subseteq X$ is open if, for all $x \in V$, there is $\delta > 0$ such that $U(x,\delta) \subseteq V$.

A subset $C \subseteq X$ is closed if $C^c$ is open.

(3) If $x \in V$ and $V \subseteq X$ is open, then we say $V$ is an open neighborhood of $x$. A neighborhood of $x$ is any subset $N \subseteq X$ such that $N$ contains an open neighborhood of $x$.

(4) If $A \subseteq X$ is any subset, the interior of $A$ is

$$A^\circ := \bigcup \left\{V \mid V \text{ is open}, V \subseteq A\right\},$$

the closure of $A$ is

$$\overline{A} = \bigcap \{C \mid C \text{ is closed}, A \subseteq C\},$$

and the boundary of $A$ is

$$\partial A = \overline{A} \setminus A^\circ.$$

We can now talk about the topology of the metric space.

**Fact B.2.1**

Let $(X, d)$ be a metric space, and let

$$\mathcal{U} = \left\{V \mid V \subseteq X \text{ open}\right\}.$$

Then, the following are true.

- $\varnothing \in \mathcal{U}, X \in \mathcal{U}$.
- If $\{V_i\}_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} V_i \in \mathcal{U}$.
- If $\{V_i\}_{i=1}^{n}$ is a finite collection of open sets, then $\bigcap_{i=1}^{n} V_i \in \mathcal{U}$.

**Definition B.2.3**

Let $(X, d)$ be a metric space. Suppose $A \subseteq X$ is a nonempty subset.

(1) The distance from a point $x \in X$ to the set $A$ is defined by

$$\operatorname{dist}_A(x) = \inf_{a \in A} d(x, a).$$

(2) The diameter of $A$ is defined by

$$\operatorname{diam}(A) = \sup_{x, y \in A} d(x, y).$$

(3) If $\operatorname{diam}(A) < \infty$, then we say $A$ is bounded.

(4) If, for every $\delta > 0$, there is a finite subset $F_\delta \subseteq X$ such that

$$A \subseteq \bigcup_{x \in F_\delta} U(x, \delta).$$

(5) For $A, B \subseteq X$, we define the Hausdorff distance between $A$ and $B$ to be

$$d_H(A, B) = \max\left\{\sup_{x \in A} \operatorname{dist}_B(x), \sup_{y \in B} \operatorname{dist}_A(y)\right\}.$$

**Example B.2.2**

Let $\Omega$ be nonempty set, and $(X, d)$ be a metric space. A function $f : \Omega \to X$ is said to be bounded if $\operatorname{diam}(\operatorname{Ran}(f)) < \infty$.

The collection $\operatorname{Bd}(\Omega, X)$ denotes all bounded functions with domain $\Omega$ and codomain $X$.

On $\operatorname{Bd}(\Omega, X)$, we define the uniform metric by

$$D_u(f, g) = \sup_{x \in \Omega} d(f(x), g(x)).$$

## B.2.2 Convergence and Continuity in Metric Spaces

**Definition B.2.4** — (Crash Course on Sequences)

Let $(X, d)$ be a metric space.

(1) A sequence in $X$ is a map $x : \mathbb{N} \to X$, which we call $(x_n)_n$ or $(x_n)_{n \geqslant 1}$.

(2) A natural sequence is a strictly increasing sequence of natural numbers $(n_k)_{k \geqslant 1}$ with $n_k \geqslant k$ and $n_k < n_{k+1}$.

(3) If $(n_k)_k$ is a natural sequence, the sequence $(x_{n_k})_k$ is called a subsequence of $(x_n)_n$.

(4) We say $(x_n)_n \to x$ if $d(x_n, x)_n \xrightarrow{n\to\infty} 0$. We say $x$ is the limit of $(x_n)_n$.

**Example B.2.3** — (Convergence in Metric Spaces of Functions)

- If $\Omega$ is a nonempty set, and $(X, d)$ is a metric space, the sequence of functions $f_n : \Omega \to X$ is said to converge pointwise to $f : \Omega \to X$ if
$$f_n(x) \xrightarrow{n\to\infty} f(x)$$
for each $x \in \Omega$.
- If $(f_n)_n \in \mathrm{Bd}(\Omega, X)$ is a sequence, we say $(f_n)_n \to f$ converges uniformly if
$$D_u(f_n, f) \xrightarrow{n\to\infty} 0,$$
or, equivalently,
$$\sup_{x\in\Omega} d(f_n(x), f(x)) \xrightarrow{n\to\infty} 0.$$

**Definition B.2.5** — (Sequential Criteria for Closure)

If $(X, d)$ is a metric space, and $E \subseteq X$ is nonempty, then $E$ is closed if and only if, for all $(x_n)_n \to x$ with $x_n \in E$, $x \in E$.

If $E \subseteq X$ is any nonempty set, then $\overline{E}$ is precisely the set of all $x \in X$ such that $(x_n)_n \to x$ for some $(x_n)_n \subseteq E$.

**Definition B.2.6** — (Completeness)

Let $(X, d)$ be a metric space.

- If $(x_n)_n$ is a sequence in $X$ such that for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geqslant N$, $d(x_m, x_n) < \varepsilon$, then we say the sequence is called Cauchy.
- If, for any $(x_n)_n$ Cauchy, $(x_n)_n \to x$ in $X$, then we say $X$ is complete.
- If $(X, d)$ is complete, then for any $A \subseteq X$ closed, $A$ is also complete.
- If $A \subseteq X$ is complete as a metric space, then $A$ is closed.

**Example B.2.4**

The metric space $\mathbb{Q}$ with the metric inherited from $\mathbb{R}$ is not complete. For instance, there is a sequence of rational numbers $(2, 2.7, 2.71, 2.718, \dots)$ converging to $e$, but $e \notin \mathbb{Q}$.

The space $\mathrm{Bd}(\Omega, X)$ is complete if $X$ is complete.

**Definition B.2.7** — (Continuity)

- Let $(X, d)$ and $(Y, \rho)$ be metric spaces, and let $f : X \to Y$ be a function. We say $f$ is continuous at $x$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $z \in U(x, \delta) \Rightarrow \rho(f(x), f(z)) < \varepsilon$.
- If $f$ is continuous at every point in $X$, then we say $f$ is continuous.
- If $f$ is bijective, continuous, and $f^{-1}$ is continuous, then we say $f$ is a homeomorphism.

- We say $f$ is uniformly continuous on $X$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $y, z \in X$, $d(y,z) < \delta \Rightarrow \rho(f(y), f(z)) < \varepsilon$.
- We say $f$ is Lipschitz if there exists $C > 0$ such that $d(x,y) \leqslant C d(f(x), f(y))$ for all $x, y \in X$.
- We say $f$ is an isometry if $d(x,y) = d(f(x), f(y))$ for all $x, y \in X$.

**Fact B.2.2**

Let $f : X \to Y$ be a map between metric spaces. The following are equivalent:

(i) $f$ is continuous;

(ii) if $V \subseteq Y$ is open, then $f^{-1}(V) \subseteq X$ is open;

(iii) if $(x_n)_n \to x$ in $X$, then $(f(x_n))_n \to f(x)$ in $Y$.

**Fact B.2.3**

If $M$ and $N$ are metric spaces with $N$ complete, and $A \subseteq M$ is dense, then if $f : A \to N$ is uniformly continuous, then there is a unique uniformly continuous map $\tilde{f} : M \to N$.

# Appendix C

# Measure and Integration

In order to properly discuss amenability, we need a strong foundation in measure theory.

## C.1 Constructing Measurable Spaces

Fix a set $\Omega$. We let $\mathcal{A} = \{A_i\}_{i \in I}$ be a collection of subsets of $\Omega$.

**Definition C.1.1** — (Algebra of Subsets)

The collection $\mathcal{A} = \{A_i\}_{i \in I}$ is known as an algebra of subsets for $\Omega$ if

- $\emptyset, \Omega \in \mathcal{A}$;
- for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

We can refine the concept of an algebra of subsets to consider countable unions rather than finite unions. This is known as a σ-algebra.

**Definition C.1.2** — (σ-Algebra of Subsets)

The collection $\mathcal{A} = \{A_i\}_{i \in I}$ is known as a σ-algebra of subsets for $\Omega$ if

- $\emptyset, \Omega \in \mathcal{A}$;
- for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- for any countable collection $\{A_n\}_{n \geqslant 1} \subseteq \mathcal{A}$, $\bigcup_{n \geqslant 1} A_n \in \mathcal{A}$.

**Definition C.1.3** — (Measurable Space)

A pair $(\Omega, \mathcal{A})$, where $\Omega$ is a set and $A \subseteq P(\Omega)$ is a σ-algebra, is called a measurable space. Elements in the measurable space are called $\mathcal{A}$-measurable sets.

**Definition C.1.4** — (Restriction of a σ-Algebra)

For a measurable space $(\Omega, \mathcal{A})$, with $E \in \mathcal{A}$, the family

$$\mathcal{A}_E = \{E \cap A \mid A \in \mathcal{A}\}$$

is a σ-algebra on $E$, known as the restriction of $\mathcal{A}$ to $E$.

**Definition C.1.5** — (Produced σ-Algebra)

Let $(\Omega, \mathcal{A})$ be a measurable space, and $f : \Omega \to \Lambda$ is a map. The σ-algebra produced by $f$ on $\Lambda$ is the collection

$$\mathcal{N} = \left\{ E \mid E \subseteq \Lambda,\ f^{-1}(E) \in \mathcal{A} \right\}.$$

**Definition C.1.6 —** (Generated σ-Algebra)

For a family $\mathcal{E} \subseteq P(\Omega)$, the σ-algebra generated by $E$ is the smallest σ-algebra that contains $E$.

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{E} \in \mathcal{M}_i \\ \mathcal{M}_i \ \sigma\text{-Algebra}}} \mathcal{M}_i$$

**Definition C.1.7 —** (Borel σ-Algebra)

If $\Omega$ is a topological space with topology $\tau \subseteq P(\Omega)$, we define

$$\mathcal{B}_\Omega = \sigma(\tau)$$

to be the Borel σ-algebra.

All open, closed, clopen, $F_\sigma$, and $G_\delta$ subsets of $\Omega$ are Borel.

We can now begin examining measurable functions.

**Definition C.1.8 —** (Measurable Functions)

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces.

(1) We say a map $f : \Omega \to \Lambda$ is $\mathcal{M}$-$\mathcal{N}$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

(2) We say a map $f : \Omega \to \mathbb{R}$ is measurable if it is $\mathcal{M}$-$\mathcal{B}_\mathbb{R}$-measurable.

(3) We say a map $f : \Omega \to \mathbb{C}$ is measurable if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.

The set of all measurable functions on $(\Omega, \mathcal{M})$ is denoted $L_0(\Omega, \mathcal{M})$.

The collection of all bounded measurable functions is the set

$$B_\infty(\Omega, \mathcal{M}) = \left\{ f \in L_0(\Omega, \mathcal{M}) \mid \sup_{x \in \Omega} |f(x)| < \infty \right\}.$$

**Example C.1.1**

If $f : \Omega \to \Lambda$ is a continuous map between topological spaces, then $f$ is $\mathcal{B}_\Omega$-$\mathcal{B}_\Lambda$-measurable, since

$$\mathcal{F} = \left\{ E \subseteq \Lambda \mid f^{-1}(E) \in \mathcal{B}_\Omega \right\}$$

is a σ-algebra containing every open set in $\Lambda$, so $\mathcal{F}$ contains $\mathcal{B}_\Lambda$.

**Example C.1.2**

If $(\Omega, \mathcal{M})$ is a measurable space, and $f : \Omega \to \Lambda$ is a map, the measurable space $(\Lambda, \mathcal{N})$ produced by $f$ is necessarily measurable.

**Fact C.1.1**

If $(\Omega, \mathcal{M})$, $(\Lambda, \mathcal{N})$, and $(\Sigma, \mathcal{L})$ are measurable spaces, with $f : \Omega \to \Lambda$ and $g : \Lambda \to \Sigma$ measurable, then $g \circ f$ is measurable.

**Proposition C.1.1**

Let $(\Omega, \mathcal{M})$ be a measurable space. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Suppose $f, g, h_n : \Omega \to \mathbb{F}$ are measurable for $n \geqslant 1$.

(1) If $\alpha \in \mathbb{F}$, then $f + \alpha g$ is measurable.

(2) $\overline{f}$ is measurable.

(3) $fg$ is measurable.

(4) $\frac{f}{g}$ is measurable assuming it is well-defined.

(5) if $h_n$ are $\mathbb{R}$-valued, and $(h_n(x))_n$ is bounded for each $x \in \Omega$, then $\sup h_n$ and $\inf h_n$ are measurable.

(6) If $f$ and $g$ are $\mathbb{R}$ valued, then $\max(f, g)$ and $\min(f, g)$ are measurable. In particular,

$$\begin{aligned} f_+ &= \max(f, 0) \\ f_- &= \max(0, -f) \end{aligned}$$

are measurable.

(7) $|f|$ is measurable.

(8) The pointwise limit of measurable functions is measurable — if $\lim_{n\to\infty} h_n(x)$ exists for all $x \in \Omega$, then $h = \lim_{n\to\infty} h_n$ is measurable.

**Definition C.1.9 —** (Simple Functions)

A simple function $s : \Omega \to \mathbb{F}$ is a function with finite range. In other words, $s$ is of the form

$$s = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k}$$

for $E_k \subseteq \Omega$ and $c_k \in \mathbb{F}$.

A simple function is measurable if and only if $E_k \in \mathcal{M}$ for each $k$.

## C.2 Constructing Measures

A measure assigns a nonnegative "length" or "volume" to measurable sets.

**Definition C.2.1 —** (Basics of Measures)

A measure on a measurable space $(\Omega, \mathcal{M})$ is a map $\mu : \mathcal{M} \to [0, \infty]$ that satisfies the following.

(i) $\mu(\varnothing) = 0$;

(ii) $\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

The triple $(\Omega, \mathcal{M}, \mu)$ is called a measure space.

A measure $\mu$ is finite if $\mu(\Omega) < \infty$

If $\mu(\Omega) = 1$, then $\mu$ is called a probability measure.

A measure $\mu$ is called finitely additive if $\mu(E \sqcup F) = \mu(E) + \mu(F)$.

A measure $\mu$ is called $\sigma$-finite if there is a countable family $\{E_n\}_{n \geqslant 1} \subseteq \mathcal{M}$ such that

$$\Omega = \bigcup_{n \geqslant 1} E_n$$

and $\mu(E_n) < \infty$.

A measure $\mu$ on $(\Omega, \mathcal{M})$ is called semi-finite if, for every $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$.

**Lemma C.2.1**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) If $E, F \in \mathcal{M}$ with $F \subseteq E$, then $\mu(F) \subseteq \mu(E)$.

(2) If $(E_n)_n$ is a sequence of measurable sets, then

$$\mu\left(\bigcup_{n \geqslant 1} E_n\right) \leqslant \sum_{n=1}^{\infty} \mu(E_n).$$

(3) If $(E_n)_{n \geqslant 1}$ is an increasing family of measurable sets, then

$$\mu\left(\bigcup_{n \geqslant 1} E_n\right) = \lim_{n \to \infty} \mu(E_n).$$

**Definition C.2.2** — (Pushforward Measure)

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $(\Lambda, \mathcal{N})$ be a measurable space. Let $f : \Omega \to \Lambda$ be measurable. The map

$$f_*\mu : \mathcal{N} \to [0, \infty]$$

defined by

$$f_*\mu(E) = \mu\left(f^{-1}(E)\right)$$

defines a measure on $(\Lambda, \mathcal{N})$. This is known as the pushforward measure of $\mu$.

If $\mathcal{N}$ on $\Lambda$ is produced by $f$, then the pushforward measure is necessarily defined on $\mathcal{N}$, and that any function $g : \Lambda \to \mathbb{F}$ is measurable if and only if $g \circ f$ is measurable.

**Definition C.2.3**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

A null set is a measurable set $N \in \mathcal{M}$ with $\mu(N) = 0$.

A property which holds for all $x \in \Omega \setminus N$ for some null set $N$ is said to hold $\mu$-almost everywhere, or $\mu$-a.e.

**Definition C.2.4**

If $(\Omega, \mathcal{M}, \mu)$ is a measure space, we can define an equivalence relation on the set $L_0(\Omega, \mathcal{M}, \mu)$, by

$$f \sim_\mu g \text{ if and only if } \mu(\{x \mid f(x) \neq g(x)\}) = 0.$$

We define the set of all classes of measurable functions by

$$\begin{aligned} L(\Omega, \mu) &= L_0(\Omega, \mathcal{M}) /\!\sim_\mu \\ &= \left\{ [f]_\mu \mid f \in L_0(\Omega, \mathcal{M}) \right\}. \end{aligned}$$

**Fact C.2.1**

The operations

- $[f]_\mu + [g]_\mu = [f+g]_\mu$;
- $[f]_\mu [g]_\mu = [fg]_\mu$;
- and $\alpha [f]_\mu = [\alpha f]_\mu$

are well-defined.

**Definition C.2.5 —** (Essentially Bounded Functions and Continuous Functions)

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and $f : \Omega \to \mathbb{C}$ be measurable. We say $f$ is $\mu$-essentially bounded if there is $C \geqslant 0$ such that

$$\mu(\{x \in \Omega \mid |f(x)| \geqslant C\}) = 0.$$

We say $C$ is an essential bound for $f$. The infimum of all essential bounds is the essential supremum, which gives

$$\begin{aligned} \|f\|_\infty &= \operatorname{ess\,sup}(f) \\ &= \inf \{C \geqslant 0 \mid \mu(\{x \in \Omega \mid |f(x)| \geqslant C\}) = 0\}. \end{aligned}$$

The collection of all $\mu$-essentially bounded functions is denoted

$$L_\infty(\Omega, \mu) = \left\{ [f]_\mu \in L(\Omega, \mu) \mid \|f\|_\infty < \infty \right\}.$$

Note that $B_\infty(\Omega, \mu) = L_\infty(\Omega, \mu)$ as sets.

If $\Omega$ is a topological space, with $\mathcal{B}_\Omega$ the Borel $\sigma$-algebra, we have $C(\Omega) \subseteq L_0(\Omega, \mathcal{B}_\Omega)$.

For $\mu$ a measure on $(\Omega, \mathcal{B}_\Omega)$, the $\mu$-equivalence classes of continuous functions are

$$C(\Omega, \mu) = \left\{ [f]_\mu \mid f \in C(\Omega) \right\}.$$

Members of $L(\Omega, \mu)$ and $L_\infty(\Omega, \mu)$ are equivalence classes of functions (rather than functions themselves), but we use the abuse of notation that $[f]_\mu = f$.

**Fact C.2.2**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $f, g : \Omega \to \mathbb{C}$ be measurable, and $\alpha \in \mathbb{C}$. Then, the following are true:

- $\|f+g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty$;
- $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$;
- if $\|f\|_\infty = 0$, then $f = 0$ $\mu$-a.e.;
- $\|f\|_\infty \leqslant \|f\|_u$;
- if $f$ is essentially bounded, then

$$\mu(\{x \mid |f(x)| \geqslant \|f\|_\infty\}) = 0.$$

**Definition C.2.6 — (Complete Measure Space)**

A measure space $(\Omega, \mathcal{M}, \mu)$ is said to be complete if all subsets of null sets are measurable (and null).

## C.3 Integration

**Definition C.3.1**

If $\phi : \Omega \to [0, \infty)$ is a positive, simple, and measurable function,

$$\phi = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k},$$

then the integral of $\phi$ is defined as

$$\int_\Omega \phi \; d\mu = \sum_{k=1}^{n} c_k \mu(E_k),$$

with the convention that $0 \cdot \infty = 0$.

The value of this integral is not dependent on the representation of $\phi$.

**Definition C.3.2**

If $f : \Omega \to \infty[0, \infty)$ is a positive measurable function, then

$$\int_\Omega f \; d\mu = \sup \left\{ \int_\Omega \phi \; d\mu \mid \phi \text{ measurable and simple, } 0 \leqslant \phi \leqslant f \right\}.$$

If $E \subseteq \Omega$ is measurable, we define

$$\int_E f \; d\mu = \int_\Omega f \mathbb{1}_E \; d\mu.$$

**Proposition C.3.1**

Let $(\Omega, \mathcal{M})$ be a measurable space, and let $f : \Omega \to \mathbb{C}$ be measurable. There is a sequence $(\phi_n)_n$ of simple, measurable functions with $(\phi_n(x))_n \xrightarrow{n \to \infty} f(x)$.

If $f \geqslant 0$, we can take $\phi_n$ to be positive and pointwise increasing.

If $f$ is bounded, then this convergence is uniform, and $\phi_n$ can be uniformly bounded.

**Theorem C.3.1 — (Monotone Convergence Theorem)**

Let $(f_n : \Omega \to [0, \infty))$ be an inreasing sequence of positive, measurable functions converging pointwise to $f : \Omega \to [0, \infty)$. Then, $f$ is measurable, and

$$\lim_{n \to \infty} \int_\Omega f_n \; d\mu = \int_\Omega f \; d\mu.$$

**Definition C.3.3**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) A measurable function $f : \Omega \to [0, \infty)$ is integrable if

$$\int_\Omega f \, d\mu < \infty.$$

(2) A measurable function $f : \Omega \to \mathbb{R}$ is integrable if both $f_+$ and $f_-$ are integrable. We define

$$\int_\Omega f \, d\mu = \int_\Omega f_+ \, d\mu - \int_\Omega f_- \, d\mu.$$

(3) A measurable function $f : \Omega \to \mathbb{C}$ is said to be integrable if both $\mathrm{Re}(f)$ and $\mathrm{Im}(F)$ are integrable. We define

$$\int_\Omega f \, d\mu = \int_\Omega \mathrm{Re}(f) \, d\mu + i \int_\Omega \mathrm{Im}(f) \, d\mu.$$

**Fact C.3.1**

Let $f, g : \Omega \to \mathbb{C}$ be integrable functions, and $\alpha \in \mathbb{C}$. Then,

- $f + \alpha g$ is integrable, and $\displaystyle\int_\Omega (f + \alpha g) \, d\mu = \int_\Omega f \, d\mu + \alpha \int_\Omega g \, d\mu$;
- if $f$ and $g$ are real-valued, and $f \leqslant g$, then $\displaystyle\int_\Omega f \, d\mu \leqslant \int_\Omega g \, d\mu$;
- $\displaystyle\left| \int_\Omega f \, d\mu \right| \leqslant \int_\Omega |f| \, d\mu.$

**Fact C.3.2**

If $f = g$ $\mu$-a.e., then

$$\int_\Omega f \, d\mu = \int_\Omega g \, d\mu.$$

**Fact C.3.3**

If $f : \Omega \to \mathbb{C}$ is measurable, then $\int_\Omega |f| \, d\mu = 0$ if and only if $f = 0$ $\mu$-a.e.

**Fact C.3.4**

A measurable function $f : \Omega \to \mathbb{C}$ is integrable if and only if $|f|$ is integrable.

**Definition C.3.4** — (Integrable Functions)

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) We define the set of (classes of) integrable functions to be

$$L_1(\Omega, \mu) = \left\{ [f]_\mu \in L(\Omega, \mu) \mid f \text{ is integrable} \right\}.$$

(2) We define the set of (classes of) square-integrable functions to be

$$L_2(\Omega, \mu) = \left\{ [f]_\mu \in L(\Omega, \mu) \mid |f|^2 \text{ is integrable} \right\}.$$

**Definition C.3.5**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. If f and $(f_n)_n$ are integrable with $\|f - f_n\|_1 \xrightarrow{n\to\infty} 0$, we say $(f_n)_n$ converges in mean to f.

**Fact C.3.5**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) For $f \in L_1(\Omega, \mu)$, the maps

$$[f]_\mu \longmapsto \int_\Omega f \, d\mu$$
$$[f]_\mu \longmapsto \int_\Omega |f| \, d\mu$$

are well-defined.

(2) For $f \in L_1(\Omega, \mu)$, we define

$$\|f\|_1 = \int_\Omega |f| \, d\mu.$$

This is a well-defined norm.

$$\begin{aligned} \|f + g\|_1 &\leqslant \|f\|_1 + \|g\|_1 \\ \|\alpha f_1\| &= |\alpha| \, \|f\|_1 \\ \|f\| = 1 &\Leftrightarrow f = 0 \ \mu\text{-a.e.} \end{aligned}$$

(3) The form

$$d_1\left([f]_\mu, [g]_\mu\right) = \|f - g\|_1$$

is a metric on $L_1(\Omega, \mu)$.

**Theorem C.3.2** — (Dominated Convergence Theorem)

Let $(f_n : \Omega \to \mathbb{C})_n$ be a sequence of measurable functions converging pointwise to a measurable function $f : \Omega \to \mathbb{C}$. If there is an integrable $g : \Omega \to [0, \infty)$ with $|f_n| \leqslant g$ for all n, then

$$\int_\Omega f_n \, d\mu \xrightarrow{n\to\infty} \int_\Omega f \, d\mu.$$

**Corollary C.3.1**

If $f : \Omega \to \mathbb{C}$ is integrable, then there is a sequence of simple integrable functions $(\phi_n)_n$ with $\|f - \phi_n\|_1 \xrightarrow{n\to\infty} 0$.

**Corollary C.3.2**

If $f : \mathbb{R} \to \mathbb{C}$ is integrable, then there is a sequence $(f_n)_n$ of compactly supported integrable functions such that $\|f - f_n\|_1 \xrightarrow{n\to\infty} 0$.

**Theorem C.3.3**

If $f : \mathbb{R} \to \mathbb{C}$ is integrable, and $\varepsilon > 0$, there is a continuous, compactly supported function g with $\|f - g\|_1 < \varepsilon$.

**Proposition C.3.2**

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $(\Lambda, \mathcal{N})$ be a measurable space with $f : \Omega \to \Lambda$ a measurable map. Let $f_*\mu$ is the pushforward measure on $(\Lambda, \mathcal{N})$. For a measurable function $g : \Lambda \to [0, \infty)$, then

$$\int_\Lambda g \, d(f_*\mu) = \int_\Omega (g \circ f) \, d\mu.$$

Moreover, if $g : \Lambda \to \mathbb{F}$ is integrable with respect to $f_*\mu$, then so too is $g \circ f$ with respect to $\mu$.

## C.4 Complex Measures

**Example C.4.1**

If $(\Omega, \mathcal{M}, \mu)$ is a measure space, then the map $\mu_f(E) = \int_E f \, d\mu$ is a well-defined measure.

**Definition C.4.1**

Let $(\Omega, \mathcal{M}, \mu)$ be a measurable space.

(1) A complex number on $(\Omega, \mathcal{M}, \mu)$ is a map $\mu : \mathcal{M} \to \mathbb{C}$ satisfying the following conditions.

- $\mu(\emptyset) = 0$;
- $\mu\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$ for $\{E_k\}_{k\geqslant 1} \subseteq \mathcal{M}$.

(2) We write $M(\Omega)$ to be the set of all complex measures on $(\Omega, \mathcal{M})$.

(3) If $\mu \in M(\Omega)$, and $\mu(E) \in \mathbb{R}$ for all $E \in \mathcal{M}$, then we say $\mu$ is a real measure on $(\Omega, \mathcal{M})$.

(4) If $\mu \in M(\Omega)$ and $\mu(E) \geqslant 0$ for all $E \in \mathcal{M}$, then we say $\mu$ is a positive measure on $(\Omega, \mathcal{M})$.

(5) If $\mu$ is a positive measure on $(\Omega, \mathcal{M})$ with $\mu(\Omega) = 1$, we say $\mu$ is a probability measure on $(\Omega, \mathcal{M})$. We write $\mathcal{P}(\Omega, \mathcal{M})$ to be the collection of all probability measures on $(\Omega, \mathcal{M})$.

(6) If $\Omega$ is a locally compact Hausdorff space, we always let $M(\Omega)$ be the set of all complex Borel measures on $\Omega$.

**Definition C.4.2**

If $(\Omega, \mathcal{M})$ is a measurable space, and $x \in \Omega$, the Dirac measure at $x$ is defined by

$$\delta_x : \mathcal{M} \to [0, 1]$$

$$\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

If $x_1, \dots, x_n$ are distinct points in $\Omega$, and $t_1, \dots, t_n \in [0, 1]$ with $\sum_{j=1}^n t_j = 1$, then

$$\mu = \sum_{j=1}^{n} t_j \delta_{x_j}$$

is a probability measure on $(\Omega, \mathcal{M})$.

**Fact C.4.1**

If $\mu$ is a complex measure on $(\Omega, \mathcal{M})$, then $\overline{\mu}$, defined by $\overline{\mu}(E) = \overline{\mu(E)}$ for $E \in \mathcal{M}$, is also a complex measure. Additionally, $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$, defined by

$$\mathrm{Re}(\mu)(E) = \mathrm{Re}(\mu(E))$$
$$\mathrm{Im}(\mu)(E) = \mathrm{Im}(\mu(E))$$

are real measures.

**Definition C.4.3**

If $\mu \in M(\Omega)$, then the total variation of $\mu$ is the quantity

$$|\mu| : \mathcal{M} \to [0, \infty]$$

with

$$|\mu|(E) = \sup\left\{ \sum_{j=1}^{\infty} \left|\mu\left(E_j\right)\right| \middle| E = \sqcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{M} \right\}.$$

**Fact C.4.2**

If $\mu \in M(\Omega)$, then $|\mu|$ is a positive, finite measure. Additionally, if $\mu, \nu \in M(\Omega)$ with $\alpha \in \mathbb{C}$, then

(a) $|\mu(E)| \leqslant |\mu|(E)$

(b) $|\mu + \nu|(E) \leqslant |\mu|(E) + |\nu|(E)$

(c) $|\alpha\mu|(E) = |\alpha|\,|\mu|(E)$.

**Definition C.4.4** — (Absolute Continuity of Measures)

Let $(\Omega, \mathcal{M})$ be a measurable space, and let $\mu$ and $\nu$ be positive measures on this space. If $\mu(A) > 0$ implies $\nu(A) > 0$ for a given $A \in \mathcal{M}$, we say $\mu$ is absolutely continuous with respect to $\nu$. We write $\mu \ll \nu$.

**Theorem C.4.1** — (Radon–Nikodym Theorem)

If $\mu \ll \nu$ on $(\Omega, \mathcal{M})$, then there exists a measurable function $f : \Omega \to [0, \infty]$ such that

$$\nu(A) = \int_A f\, d\nu$$

for each $A \in \mathcal{M}$.

**Remark 8**

The Radon–Nikodym theorem extends to signed and complex measures.

**Fact C.4.3**

Let $(\Omega, \mathcal{M}, \lambda)$ be a measure space, and suppose $f \in L_1(\Omega, \lambda)$. Then, $\mu(E) = \int_E f\, d\lambda$ defines a complex measure. We write $f = \frac{d\mu}{d\lambda}$, which is the Radon–Nikodym derivative of $\mu$ with respect to $\lambda$.

It is also the case that

$$|\mu|(E) = \int_E |f|\ d\lambda.$$

**Fact C.4.4**

If $\mu \in M(\Omega)$, there exists a measurable function $f : \Omega \to \mathbb{C}$ such that $|f| = 1$ and $\mu(E) = \int_E f\, d|\mu|$ for all

$E \in \mathcal{M}$.

**Definition C.4.5**

Let $\Omega$ be a locally compact Hausdorff space equipped with the Borel $\sigma$-algebra, $\mathcal{B}_\Omega$.

(1) A Borel measure $\mu : \mathcal{B}_\Omega \to [0, \infty]$ is called

- inner regular on $E \in \mathcal{B}_\Omega$ if

$$\mu(E) = \sup \left\{\mu(K) \mid K \subseteq E, K \text{ compact}\right\};$$

- outer regular on $E \in \mathcal{B}_\Omega$ if

$$\mu(E) = \inf \left\{\mu(U) \mid U \supseteq E, U \text{ open}\right\};$$

- regular on $E$ if it is inner regular and outer regular on $E$;
- regular if it is regular on all $E \in \mathcal{B}_\Omega$;
- Radon if
  - $\mu(K) < \infty$ for all compact $K \subseteq \Omega$;
  - $\mu$ is inner regular on all open sets and outer regular on all Borel sets.

(2) A complex Borel measure $\mu : \mathcal{B}_\Omega \to \mathbb{C}$ is regular if $|\mu|$ is regular; $\mu$ is Radon if $|\mu|$ is Radon.

(3) We write $M_r(\Omega)$ to denote the set of all complex regular measures on $(\Omega, \mathcal{B}_\Omega)$.

**Fact C.4.5**

Every positive Radon measure is regular. Thus, every complex Borel measure is regular if and only if it is Radon.

Moreover, if $\Omega$ is a second countable locally compact Hausdorff space, then every complex Borel measure is regular.

**Definition C.4.6**

Let $(\Omega, \tau)$ be a topological space, and suppose $\mu : \mathcal{B}_\Omega \to [0, \infty]$ is a Borel measure.

(1) The kernel of $\mu$ is the set

$$N_\mu = \bigcup \{U \subseteq \Omega \mid U \in \tau,\ \mu(U) = 0\}.$$

(2) The support of $\mu$ is the complement of the kernel, $\operatorname{supp}(\mu) = N_\mu^c$.

**Fact C.4.6**

If $\mu$ is a Radon measure on a locally compact Hausdorff space $\Omega$, then $\mu\left(N_\mu\right) = 0$, meaning $\mu(\Omega) = \mu\left(\operatorname{supp}(\mu)\right)$.

**Theorem C.4.2** — (Hahn and Jordan Decomposition)

Let $(\Omega, \mathcal{M})$ be a measurable space, and let $\mu : \mathcal{M} \to \mathbb{R}$ be a real measure. Then, there is a measurable partition $\Omega = P \sqcup N$ such that for all $E \subseteq P$, $\mu(E) \geqslant 0$, and for all $E \subseteq N$, $\mu(E) \leqslant 0$. This partition is unique up to a $\mu$-null symmetric difference — that is, for any $P', N'$ satisfying the conditions, $\mu(P' \triangle P) = 0$ and $\mu(N' \triangle N) = 0$.

There is a unique decomposition $\mu = \mu_+ - \mu_-$, with $\mu_\pm$ that are positive such that if $E \subseteq P$ $\mu_-(E) = 0$ and if $E \subseteq N$, $\mu_+(E) = 0$.

**Definition C.4.7**

Let $(\Omega, \mathcal{M})$ be a measurable space, and let $f : \Omega \to \mathbb{C}$ be measurable.

(1) If $\mu : \mathcal{M} \to \mathbb{R}$ is a real measure with $\mu = \mu_+ - \mu_-$, we say that $f$ is $\mu$-integrable if it is both $\mu_+$ and $\mu_-$-integrable. We define

$$\int_\Omega f \, d\mu = \int_\Omega f \, d\mu_+ - \int_\Omega f \, d\mu_-.$$

(2) If $\mu : \mathcal{M} \to \mathbb{C}$ is a complex measure with $\mu_1 = \operatorname{Re}(\mu)$ and $\mu_2 = \operatorname{Im}(\mu)$, we say $f$ is $\mu$-integrable if it is both $\mu_1$ and $\mu_2$-integrable. We define

$$\int_\Omega f \, d\mu = \int_\Omega f \, d\mu_1 + i \int_\Omega f \, d\mu_2.$$

## Appendix D

# Functional Analysis

# Bibliography


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