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## Set Theory

### Naive Set Theory

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Z}_+ = \{0, 1, 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

$$\mathbb{C}_q = \{a + bi \mid a, b \in \mathbb{Q}\}$$

Recall: given sets  $X$  and  $Y$ , a relation from  $X$  to  $Y$  is a subset of  $X \times Y$ , where  $\times$  denotes the cartesian product of  $X$  and  $Y$ .

A relation  $f \subseteq X \times Y$  is a function from  $X$  to  $Y$  such that  $\forall x \in X, \exists! y \in Y$  such that  $(x, y) \in f$ . We write  $f(x) = y$ , and denote  $f$  as  $f : X \rightarrow Y$ .

$X$  is the **domain** of  $f$  and  $Y$  is the **codomain**. The range  $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$ .

The graph of a function  $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ .

## Function Examples

Identity Function:

$$\text{id}_X : X \rightarrow X, \text{id}_X(x) = x$$

The Characteristic Function: If  $A \subseteq X$

$$\mathbb{1}_A : X \rightarrow \mathbb{R}, \mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

## Function Operations

Let  $X$  be any set, and  $(X; \mathbb{R}) = \{f : X \rightarrow \mathbb{R}\}$  represent the function space of  $X$  with codomain  $\mathbb{R}$ .

**Addition:** Let  $f, g \in \mathcal{F}(X; \mathbb{R})$ . Then,  $(f + g)(x) = f(x) + g(x)$ , and  $(f \cdot g)(x) = f(x) \cdot g(x)$ .

**Scalar Multiplication:** If  $t \in \mathbb{R}$ , then  $(tf)(x) = tf(x)$ .

**Function Multiplication:** If  $g(x) \neq 0 \forall x \in X$ , then  $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$ .

**Composition:** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are functions, then  $g \circ f(x) = g(f(x))$ .

## Injective, Surjective, Bijective

A function  $f : X \rightarrow Y$  is a **injective** map, then, if  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$ . For example, the shift map  $S : \mathbb{N} \rightarrow \mathbb{N}$ ,  $S(n) = n + 1$  is injective.

Any strictly increasing function  $f : I \rightarrow \mathbb{R}$ , where  $I$  is any interval, is injective.

A function  $f$  is **surjective** if  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$ .

Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^3 - 2x + 1$ . We can show that this function is surjective because  $\lim_{x \rightarrow \infty} f(x) = \infty$ ,  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ . Due to the intermediate value theorem, we get that  $\text{ran}(f) = \mathbb{R}$ .

$f$  is **bijective** if it is injective and surjective.

## Invertibility

Let  $f : X \rightarrow Y$  be a function.  $f$  is **left-invertible** if  $\exists g : Y \rightarrow X$  such that  $g \circ f = \text{id}_X$ .  $f$  is **right-invertible** if  $\exists h : Y \rightarrow X$  such that  $f \circ h = \text{id}_Y$ .

$f$  is **invertible** if  $\exists k : Y \rightarrow X$  such that  $f \circ k = \text{id}_Y$  and  $k \circ f = \text{id}_X$ .

For example, the function  $\text{Sin}(x)$  defined as  $\sin(x)$  restricted to  $[-\pi/2, \pi/2]$  has an inverse,  $\arcsin(x) : [-1, 1] \rightarrow [-\pi/2, \pi/2]$ .

### Definition of Invertibility

**Statement:**  $f$  is invertible if and only if  $f$  is left and right invertible.

**Proof:**

$(\Rightarrow)$  This is via the definition of invertibility.

$(\Leftarrow)$  Suppose  $g$  is a left-inverse of  $f$ , and  $h$  is a right-inverse of  $f$ . Therefore,  $g \circ f = \text{id}_X$ , and  $f \circ h = \text{id}_Y$ .  
Observe that  $g = g \circ \text{id}_Y$ . Therefore,  $g = g \circ (f \circ h)$ . Via associativity,  $g = (g \circ f) \circ h = \text{id}_X \circ h = h$ .

### Injection and Surjection Invertibility

**Statement:** If  $f : X \rightarrow Y$  is a function:

- (1)  $f$  is injective  $\Leftrightarrow f$  is left-invertible.
- (2)  $f$  is surjective  $\Leftrightarrow f$  is right-invertible.
- (3)  $f$  is bijective  $\Leftrightarrow f$  is invertible.

**Proof:** (1),  $(\Rightarrow)$  — suppose  $f$  is injective. Given  $y \in \text{ran}(f)$ , we know that  $\exists! x_y \in X$  such that  $f(x_y) = y$ , by the definition of injective.

Let  $g : Y \rightarrow X$ . We will define  $g$  as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where  $x_0$  is an arbitrary point in  $X$ . We can see that  $g \circ f = \text{id}_X$ .

## Cardinality and Countability

### Introduction to Cardinality

Which set is “larger,”  $\{1, 2, 3\}$  or  $\{1, 2, 3, 4\}$ ?  $\mathbb{N}$  or  $\mathbb{N}_0$ ?  $\mathbb{Z}$  or  $\mathbb{Q}$ ?

In order to prove that one set is “the same size” as the other, we can create pairs. For two sets  $A$  and  $B$ , we can show that  $A$  is the same size as  $B$  by creating a function. For example, to show that  $\mathbb{N}$  and  $\mathbb{N}_0$  have the same size, we create  $s : \mathbb{N} \rightarrow \mathbb{N}_0$ ,  $s(n) = n + 1$ .

### Equivalent Cardinality

Sets  $A$  and  $B$  have the same **cardinality** if  $\exists$  bijection  $f : A \rightarrow B$ . We write  $\text{card}(A) = \text{card}(B)$ .

### Equivalent Cardinalities of Intervals

**Statement:** Given  $a < b$  and  $c < d$ , we know that  $\text{card}([a, b]) = \text{card}([c, d])$ .

**Proof:** We can create a linear function from  $[a, b]$  to  $[c, d]$ , and since linear functions are bijections, we know that  $\text{card}([a, b]) = \text{card}([c, d])$ .

## Intervals and Real Numbers

$$\text{card}((0, 1)) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$  is a bijection:
  - $\tan$  is strictly increasing (and thus injective)
  - $\lim_{x \rightarrow \infty} \tan(x) = \infty$  and  $\lim_{x \rightarrow -\infty} \tan(x) = -\infty$ , and by intermediate value theorem,  $\tan$  is surjective
- $\ell : (0, 1) \rightarrow (-\pi/2, \pi/2)$  is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is  $\tan \circ \ell : (0, 1) \rightarrow \mathbb{R}$ .

## Finitude and Infinitude

A set  $F$  is **finite** if  $F$  is empty or  $\exists n \in \mathbb{N}$  such that  $\text{card}(F) = \text{card}(\{1, 2, \dots, n\})$ . A non-finite set is called infinite.

We can *enumerate*  $F$  by creating a function  $\sigma : \{1, 2, \dots, n\} \rightarrow F$ , such that  $x_j = \sigma(j)$  for  $F = \{x_1, x_2, \dots, x_n\}$ .

### Inequality of Finite Sets

**Statement:** If  $m \neq n$ , then  $\text{card}\{1, 2, \dots, m\} \neq \text{card}\{1, 2, \dots, n\}$ .

**Proof:** WLOG, suppose  $m > n$ .

Suppose toward contradiction that  $f : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$  is our bijection. This means there are  $m$  “pigeons” and  $n$  “holes.”

One hole,  $j$ , must contain at least two pigeons (i.e.,  $f(i) = f(k) = j$  for some  $i \neq k \in \{1, 2, \dots, m\}$ ). Since  $f$  is assumed to be injective, this is a contradiction.

## Infinitude of the Naturals

**Statement:**  $\mathbb{N}$  is infinite.

**Proof:** Suppose toward contradiction that  $\mathbb{N}$  is finite. Thus,  $\exists m \in \mathbb{N}$  such that  $f : \mathbb{N} \rightarrow \{1, 2, \dots, m\}$  is a bijection.

Consider the inclusion  $i : \{1, 2, \dots, m+1\} \rightarrow \mathbb{N}$ .  $i$  is injective.

Then,  $f \circ i : \{1, 2, \dots, m+1\} \rightarrow \{1, 2, \dots, m\}$  is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

## Infinitude of a Set

**Statement:** If  $A$  is infinite,  $\exists i : \mathbb{N} \hookrightarrow A$ .



**Proof:**

$$\begin{array}{ll}
 \exists a_1 \in A & A \neq \emptyset \\
 \exists a_2 \in A \setminus \{a_1\} & A \setminus \{a_1\} \neq \emptyset \\
 \exists a_3 \in A \setminus \{a_1, a_2\} & A \setminus \{a_1, a_2\} \neq \emptyset \\
 \vdots &
 \end{array}$$

We thus get a sequence  $\{a_1, a_2, \dots\}$  of distinct elements of  $A$ .

Consider  $f : \mathbb{N} \rightarrow A$ ,  $f(n) = a_n$ .  $f$  is injective as  $a_n$  are distinct.

## Integers and Power Sets

### Cardinality of Integers and Natural Numbers

**Statement:**

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

**Proof:**

$$f : \mathbb{Z} \rightarrow \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

$f$  is a bijection as  $g : \mathbb{N} \rightarrow \mathbb{Z}$ ,  $g(n) = (-1)^{n+1} \lfloor \frac{n}{2} \rfloor$  is the inverse of  $f$ .

### Power Set and $2^X$

Given any set  $X$ ,  $\mathcal{P}(X) = \{A \mid A \subseteq X\}$  is the **power set** of  $X$ .

$$2^X := \{f \mid f : X \rightarrow \{0, 1\}\}.$$

**Statement:**

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

**Proof:** Let  $\varphi : \mathcal{P}(X) \rightarrow 2^X$ .

For  $A \subseteq X$ , put  $\varphi(A) := \mathbb{1}_A$ .

Consider  $\psi : 2^X \rightarrow \mathcal{P}(X)$ .  $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$ .

Then,  $\psi \circ \varphi(A) = \psi(\mathbb{1}_A) = \mathbb{1}_A^{-1}(\{1\}) = A$ ,

and, we claim  $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbb{1}_{f^{-1}(\{1\})} = f$ .

## Cantor's Theorem

### Statement:

$$\nexists \text{ surjection } \mathbb{N} \rightarrow (0, 1)$$

**Proof:** From calculus we know  $\forall \sigma \in (0, 1)$ ,  $\sigma$  can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where  $\sigma_k \in \{0, 1, \dots, 9\}$  and not terminating in 9s.

Suppose toward contradiction that  $\exists r : \mathbb{N} \rightarrow (0, 1)$  that is a surjection. Write  $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$ , and  $\sigma_j(n) \in \{0, 1, \dots, 9\}$ , and not terminating in 9s.

Consider  $\tau : \mathbb{N} \rightarrow \{0, 1, \dots, 9\}$ :

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let  $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$ . Since  $r$  is surjective,  $\exists m \in \mathbb{N}$  such that  $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$ .

This implies that  $\sigma_m(m) = \tau(m)$ , which is definitionally not true, which is our contradiction.

## Comparing Cardinality

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example,  $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$  because  $i : X \hookrightarrow Y, i(x) = x$  is an injection.

Since the composition of two injective functions is injective, if  $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$ , then  $\text{card}(A) \leq \text{card}(C)$ .

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

## Cardinality of the Power Set

**Statement:** For any set  $A$ ,  $\text{card}(A) < \text{card}(\mathcal{P}(A))$ .

**Proof:** Let us construct a function:  $f : A \rightarrow \mathcal{P}(A)$ , where  $a \mapsto \{a\}$ .

$f$  is injective, as if  $\{a\} = \{a'\}$ ,  $a = a'$ . So,  $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$ .

**Claim:**  $\nexists g : A \rightarrow \mathcal{P}(A)$ ,  $g$  is surjective.

Suppose toward contradiction that such a  $g$  exists. Consider  $S : \{a \in A \mid a \notin g(a)\}$ .

Since  $g$  is onto,  $\exists a_0 \in A$  with  $g(a_0) = S$ .  $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$ .  $\perp$

**Equivalent Cardinality Comparisons**

- (i)  $\text{card}(A) \leq \text{card}(B)$
- (ii)  $\exists f : A \hookrightarrow B$
- (iii)  $\exists g : B \rightarrow A$ ,  $g$  surjection.

**Proof:**

(ii)  $\Rightarrow$  (iii) If  $f : A \hookrightarrow B$ ,  $f$  is left-invertible, and thus  $\exists g : B \rightarrow A$  with  $g \circ f = \text{id}_A$ . So,  $g$  is right-invertible, so  $g$  is surjective.

(iii)  $\Rightarrow$  (ii) If  $g : B \rightarrow A$  is surjective, then  $g$  is right-invertible, so  $\exists f : A \rightarrow B$  such that  $g \circ f = \text{id}_B$ . So,  $f$  is left-invertible, so  $f$  is injective.

From the above, we can see that, if  $f : A \rightarrow B$  is any map,  $\text{card}(f(A)) \leq \text{card}(A)$ , by considering  $g : A \rightarrow f(A)$  defined as  $g(a) = f(a)$ , which is onto, meaning  $\exists$  an injection  $f(A) \hookrightarrow A$ .

**Cardinality Rules**

- (i)  $\text{card}(A) \leq \text{card}(A)$  (Reflexivity)
- (ii)  $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$  (Transitivity)
- (iii)  $\text{card}(A) \leq \text{card}(B)$  and  $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$  (Cantor-Schröder-Bernstein)
- (iv) Either  $\text{card}(A) \leq \text{card}(B)$  or  $\text{card}(B) \leq \text{card}(A)$ .

**Proof of (iii):** We have injections  $f : A \hookrightarrow B$  and  $g : B \hookrightarrow A$ .

Let  $A_0 \setminus \text{ran}(g)$ . Let  $A_1 = g \circ f(A_0)$ . Note that  $A_0 \cap A_1 = \emptyset$ . Let  $A_2 = g \circ f(A_1)$ . Note that  $A_0 \cap A_2 = \emptyset$ .

**Claim:** We claim  $A_1 \cap A_2 = \emptyset$ . If  $\exists z \in A_1 \cap A_2$ , then  $z = g(f(x_0))$  for some  $x_0 \in A_0$ , and  $z = g(f(x_1))$  where  $x_1 \in A_1$ . However,  $g$  and  $f$  are injective, so  $g \circ f$  is injective, so  $x_0 = x_1$ , but  $A_0 \cap A_1 = \emptyset$ .  $\perp$

We let  $A_n = g \circ f(A_{n-1})$  for arbitrary  $n$ , and  $A_\infty = \bigcup_{n \geq 0} A_n$ . If  $a \notin A_\infty$ , then  $a \notin A_0$ , so  $a \in \text{ran}(g)$ . Define  $h : A \rightarrow B$ .

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where  $y_x$  is the unique element in  $B$  with  $g(y_x) = x$ .

**Claim:** We claim  $h$  is the desired bijection.

**Proof of Injection:** Suppose  $h(x_1) = h(x_2)$ .

If  $x_1, x_2 \in A_\infty$ , then by the definition of  $H$ ,  $f(x_1) = f(x_2)$ ,  $f$  is injective, so  $x_1 = x_2$ .

Suppose  $x_1, x_2 \notin A_\infty$ . Then, by definition,  $h(x_1) = y_{x_1}$  and  $h(x_2) = y_{x_2}$ , then  $g(y_{x_1}) = g(y_{x_2})$ , so  $x_1 = x_2$ .

WLOG, suppose  $x_1 \in A_\infty$ , and  $x_2 \notin A_\infty$ .  $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$ . Then,  $g(f(x_1)) \in A_\infty = g(y_{x_2}) = x_2 \notin A_\infty$ . This case is not possible.

Thus,  $h$  is injective.

**Proof of Surjection:** Let  $y \in B$ . Set  $x := g(y)$ .

Suppose  $x \notin A_\infty$ . Then,  $h(x) = y_x$ , where  $y_x$  is the unique element in  $B$  with  $g(y_x) = x = g(y)$ , so  $y = y_x$ , so  $h(x) = y$ .

If  $x \in A_\infty$ . We know that  $x \notin A_0$ , as  $x \in \text{ran}(g)$ . So,  $x = g(f(z))$  for some  $z \in A_{m-1}$ . Since  $g$  is injective,  $y = f(z)$ ,  $z \in A_\infty$ . Thus,  $h(z) = f(z) = y$ .

### Cardinality of Canonical Sets

Consider the map  $q : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ ,  $q(m, n) = \frac{m}{n}$ . This map is *not* injective, as  $2/4 = 1/2$ . However, it is surjective, meaning  $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$ .

Earlier, we showed that  $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$ . Consider  $H : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ , defined as  $H(m, n) = (h(m), n)$ .

**Claim:**  $H$  is a bijection.

**Proof of Injection:** If  $H(m_1, n_1) = H(m_2, n_2)$ , then  $h(m_1) = h(m_2)$ , and  $n_1 = n_2$ , and since  $h$  is bijective,  $m_1 = m_2$ , and  $n_1 = n_2$ , so  $(m_1, n_1) = (m_2, n_2)$ .

**Proof of Surjection:** Let  $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ . We want to find  $(m, n) \in \mathbb{Z} \times \mathbb{N}$  such that  $H(m, n) = (k, \ell)$ . Set  $n = \ell$ , and since  $h$  is surjective, set  $m \in \mathbb{Z}$  such that  $h(m) = k$ .

Therefore  $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$ .

We claim that  $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$ . First, we need to find  $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ . Consider  $\varphi(m, n) = 2^m \cdot 3^n$ . By the Fundamental Theorem of Arithmetic,  $\varphi$  is injective.

Bringing together our inequalities, we have:

$$\begin{aligned} \text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\ &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \text{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \text{card}(\mathbb{N}) \end{aligned}$$

### Countability and the Continuum Hypothesis

A set  $X$  is *countable* if  $\exists f : X \hookrightarrow \mathbb{N}$  ( $\text{card}(X) \leq \text{card}(\mathbb{N})$ ).  $\text{card}(\mathbb{N}) = \aleph_0$ . If  $X$  is countable and infinite,  $X$  is *denumerable*.

#### Corollary to Cantor-Schröder-Bernstein

**Statement:** If  $X$  is denumerable, then  $\text{card}(X) = \aleph_0$ .

**Proof:** Since  $X$  is infinite,  $\exists f : \mathbb{N} \hookrightarrow X$ . Since  $X$  is countable,  $\exists g : X \hookrightarrow \mathbb{N}$ . By Cantor-Schröder-Bernstein,  $\text{card}(X) = \text{card}(\mathbb{N})$ , so  $\text{card}(X) = \aleph_0$ .

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

### Countability under Union

**Statement:** The countable union of countable sets is countable. If  $I$  is a countable indexing set and for each  $i \in I$ ,  $A_i$  is countable, then  $\bigcup_{i \in I} A_i$  is countable.

**Proof:** Since each  $A_i$  is countable,  $\exists \pi_i : \mathbb{N} \rightarrow A_i$ . Consider the function

$$\pi : I \times \mathbb{N} \rightarrow \bigcup_{i \in I} A_i$$

defined as  $\pi(i, j) = \pi_i(j)$ .

**Claim 1:**  $\pi$  is a surjection.

**Proof 1:** Let  $x \in \bigcup_{i \in I} A_i$ .  $\exists i_0$  such that  $x \in A_{i_0}$ . Since  $\pi_{i_0}$  is surjective,  $\exists k \in \mathbb{N}$  with  $\pi_{i_0}(k) = x$ .  $\pi_{i_0}(k) = \pi(i_0, k)$ . Therefore,  $\pi$  is surjective.

**Claim 2:**  $I \times \mathbb{N}$  is countable.

**Proof 2:** We know  $\exists f : I \hookrightarrow \mathbb{N}$  since  $I$  is countable. Thus,  $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$ ,  $(i, n) \mapsto (f(i), n)$ . Recall,  $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ ,  $(m, n) \mapsto 2^m \cdot 3^n$  is an injection. By composing these maps,  $I \times \mathbb{N} \hookrightarrow \mathbb{N}$ . Since  $\pi$  is onto, and  $I \times \mathbb{N}$  is countable,  $\bigcup_{i \in I} A_i$  is countable.

### Continuum Hypothesis

**Statement:**

$$\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}}),$$

where  $I$  is any non-degenerate interval.

**Proof:**

**Lemma 1:**  $\text{card}([0, 1]) \leq \text{card}(2^{\mathbb{N}})$ .

**Proof 1:** Every  $t \in [0, 1]$  has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where  $\sigma_k \in \{0, 1\}$ .

Consider  $2^{\mathbb{N}} \xrightarrow{\phi} [0, 1]$ , defined as  $\phi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$ . Set  $f : \mathbb{N} \rightarrow \{0, 1\}$ ,  $f(k) = \sigma_k$ .

Therefore,  $\phi$  is surjective, so  $\exists \{0, 1\} \hookrightarrow 2^{\mathbb{N}}$ , so  $\text{card}([0, 1]) \leq 2^{\mathbb{N}}$ .

**Lemma 2:**  $\text{card}([0, 1]) = \text{card}(\mathbb{R})$ .

**Proof 2:** We have  $[0, 1] \xrightarrow{i} \mathbb{R}$  via inclusion, so  $\text{card}([0, 1]) \leq \text{card}(\mathbb{R})$ .

Also,  $\text{card}(\mathbb{R}) = \text{card}((0, 1)) \leq \text{card}([0, 1])$ , so by Cantor-Schröder-Bernstein,  $\text{card}(\mathbb{R}) = \text{card}([0, 1])$ .

**Lemma 3:** Any two non-degenerate intervals  $I$  and  $J$  have the same cardinality.

**Proof 3:** We can create injections  $I \hookrightarrow J$  and vice-versa.

**Lemma 4:**  $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0, 1])$ .

**Proof 4:**  $\psi : 2^{\mathbb{N}} \rightarrow [0, 1]$ . Where  $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$ .

$\psi$  is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim  $\psi$  is injective. Suppose  $f \neq g$  in  $2^{\mathbb{N}}$ . Let  $k_0 = \min\{k \mid f(k) \neq g(k)\}$ . WLOG,  $f(k_0) = 0, g(k_0) = 1$ . Let  $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$ ,  $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$ .

Therefore,  $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$ , and  $\psi(g) = \sum_{k=1}^{k_0-1} \frac{1}{3^k} + t_g$ .

Suppose toward contradiction  $\psi(f) = \psi(g)$ . Then,  $t_f = \frac{1}{3^{k_0}} + t_g$ , or  $t_f - t_g = \frac{1}{3^{k_0}}$ .

$$\begin{aligned} |t_f - t_g| &= \left| \sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k} \right| \\ &\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\ &\leq \sum_{k>k_0} \frac{1}{3^k} \\ &= \frac{(1/3)^{k_0+1}}{1 - (1/3)} \\ &= \frac{1}{2} \cdot \frac{1}{3^{k_0}} \end{aligned}$$

$\perp$

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0, 1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than  $\aleph_0$  and strictly less than  $2^{\aleph_0}$  is an axiom (i.e., it can be an assumption or not).

## Field Ordering

### Ordering Relations

Let  $X$  be a non-empty set. A relation on  $X$  is a subset of  $X \times X$ .

- $R$  is *reflexive* if  $\forall x \in X, (x, x) \in R$ .
- $R$  is *transitive* if  $(x, y), (y, z) \in R \rightarrow (x, z) \in R$ .
- If  $R$  is *antisymmetric*  $(x, y), (y, x) \in R \rightarrow x = y$ .

If  $R$  is reflexive, transitive, and antisymmetric, then  $R$  is an *ordering* of  $X$ .

If  $R$  is an ordering of  $X$ , then we write:

$$(x, y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \ \forall x \in X$
- $x \leq_R y, y \leq_R z \rightarrow x \leq_R z$
- $x \leq_R y, y \leq_R x \rightarrow x = y$

Additionally,  $x <_R y$  means  $x \leq_R y$  and  $x \neq y$ .

## Examples of Orderings

Algebraic Ordering of  $\mathbb{N}_0$ :  $n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$  such that  $n + k = m$

$\mathbb{N}$  ordered via division:  $n \leq_D m \Leftrightarrow n|m$ ; under this definition, it is false that  $2 \leq_D 5$ , but it is true that  $4 \leq_D 20$ .

Inclusion: Let  $S$  be any set, and let  $X = \mathcal{P}(S)$ . For  $A, B \in \mathcal{P}(S)$ , we define  $A \leq_i B \Leftrightarrow A \subseteq B$ .

Containment: With  $X$  defined as above,  $A \leq_c B \Leftrightarrow A \supseteq B$ .

Functions: For  $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \rightarrow \mathbb{R}\}$ , we can define  $f \leq g \Leftrightarrow f(x) \leq g(x) \ \forall x \in X$ .

## Total and Directed Orderings

- An ordering  $\leq$  of  $X$  is *total* or *linear* if  $\forall x, y \in X, x \leq y$  or  $y \leq x$ .
- An ordering is *directed* if  $\forall x, y \in X \exists z \in X$  such that  $x \leq z$  and  $y \leq z$ .

If  $X$  is a totally ordered set,  $X$  is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D), (\mathcal{P}(S), \leq_i), (\mathcal{P}(S), \leq_c)$$

## Upper and Lower Bounds

- (i) Let  $(X, \leq)$  be an ordered set,  $A \subseteq X$ .  $A$  is bounded above if  $\exists v \in X$  with  $a \leq v \ \forall a \in A$ . Such a  $v$  is an upper bound.
- (ii)  $A$  is bounded below if  $\exists \ell \in X$  such that  $a \geq \ell \ \forall a \in A$ . Such a  $w$  is a lower bound.
- (iii) If  $v$  is an upper bound of  $A$  and  $v \in A$ , then  $v$  is the greatest element of  $A$ , or  $\max(A) = v$ .
- (iv) If  $\ell$  is a lower bound for  $A$  and  $\ell \in A$ , then  $\ell$  is the least element of  $A$ , or  $\min(A) = \ell$ .
- (v) If  $u$  is an upper bound for  $A$ , and  $u \leq v$  for all other upper bounds  $v$  of  $A$ , then  $u$  is the *least upper bound* of  $A$ , or  $\sup(A) = u$  (for *supremum*).
- (vi) If  $\ell$  is a lower bound for  $A$ , and  $\ell \leq g$  for all other lower bounds  $g$  of  $A$ , then  $\ell$  is the *greatest lower bound* of  $A$ , or  $\inf(A) = \ell$  (for *infimum*).
- (vii) If  $A$  is bounded above and below, then  $A$  is bounded.

An ordered set  $(X, \leq)$  is *complete* if for all  $A \subseteq X$  bounded,  $\inf(A)$  and  $\sup(A)$  exist.

For example,  $\mathbb{Q}$  is *not* complete, as there is not a largest rational number less than  $\sqrt{2}$ , for example.

**Well-Ordering Principle:** With  $(\mathbb{N}, \leq_a)$ , every nonempty  $A \subseteq \mathbb{N}$  has a least element.

**Examples**

For  $A \subseteq (\mathbb{N}, \leq_a)$ ,  $A = \{2, 3, \dots, 12\}$ , we have the following:

**Bounded Above?** Yes.

**Upper Bounds** 12, 13, 14, ...

**Greatest Element** 12

For  $A \subseteq (\mathbb{N}, \leq_D)$ ,  $A = \{2, 3, \dots, 10\}$

**Bounded Above?** Yes.

**Upper Bounds** 10!

**Greatest Element?** No.

**Supremum**  $2^3 \cdot 3^2 \cdot 5 \cdot 7$

**Bounded Below?** Yes.

**Lower Bound** 1

**Least Element?** No.

**Infimum** 1

For  $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$ ,  $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$ .

**Supremum**  $\bigcup_{i \in I} A_i$

**Infimum**  $\bigcap_{i \in I} A_i$

**Ordering of  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$** 

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0, n + k = m$$

This defines a total and complete ordering.

Define  $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

**Properties of  $\mathbb{Z}^+$** 

- (i)  $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, m \cdot n \in \mathbb{Z}^+$
- (ii)  $m \in \mathbb{Z}$ , then  $m \in \mathbb{Z}^+$  or  $-m \in \mathbb{Z}^+$
- (iii)  $m, -m \in \mathbb{Z}^+$ , then  $m = 0$
- (iv)  $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

**Statement:**

- (1)  $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$
- (2)  $m \leq_a n$  and  $p \leq_a q \Rightarrow m + p \leq_a n + q$
- (3)  $m \leq_a n$  and  $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$
- (4)  $m \leq_a n \Rightarrow -m \geq_a n$



(5)  $\leq_a$  is total.

(6) If  $a_a > 0$ , and  $ab_a \geq 0$ , then  $b_a \geq 0$

(7) If  $a > 0$  and  $ab_a \geq ac$ , then  $b \geq c$ .

**Proof of (3):**

$$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0 \text{ with } m + k = n.$$

$$\Rightarrow pm + pk = pn$$

$$pk \in \mathbb{N}_0 \text{ by the properties of } \mathbb{Z}^+. \text{ So, } pm \leq_a pn$$

**Proof of (5):**

Let  $m, n \in \mathbb{Z}$ . Consider  $m - n$ .

By (ii),  $m - n \in \mathbb{Z}^+$  or  $-(m - n) \in \mathbb{Z}^+$ . Thus,  $m - n = k$  for some  $k \in \mathbb{Z}^+$ , or  $-(m - n) = \ell$  for some  $\ell \in \mathbb{Z}^+$ .

Thus,  $n \leq_a m$  in the first case, or  $m \leq_a n$  in the second case.

**Creating the Rationals**

Recall that  $Q = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}^+, b \neq 0\}$ . Consider the equivalence relation:

$$(a, b) \sim (c, d) \stackrel{\text{def}}{\iff} ad = bc$$

We will let  $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in Q\}$  be the set of all equivalence classes in  $Q$ . We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined:  $\frac{a'}{b'} = \frac{a}{b}$  and  $\frac{c'}{d'} = \frac{c}{d}$ , then  $\frac{a'd' + c'b'}{b'd'} = \frac{ad + bc}{bd}$ .

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make  $\mathbb{Q}$  a **field**.

**Fields**

A ring is a nonempty set  $R$  equipped with two binary operations:

- $+: R \times R \rightarrow R, (a, b) \mapsto a + b$  ("addition")
- $\cdot: R \times R \rightarrow R, (a, b) \mapsto a \cdot b$  ("multiplication")

such that the following hold:

- (1)  $(a + b) + c = a + (b + c)$
- (2)  $\exists z \in R$  such that  $a + z = a = z + a \forall a \in R$ ; there is at most one such  $z$ . Set  $z = 0_R$ .
- (3)  $\forall a \in R, \exists b \in R$  such that  $a + b = 0_R = b + a$ ; there is at most one such  $b$ . Set  $b = -a$ .
- (4)  $\forall a, b \in R, a + b = b + a$ .
- (5)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (6)  $a \cdot (b + c) = a \cdot b + a \cdot c, (a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If  $(R, +, \cdot)$  satisfies  $ab = ba$ ,  $R$  is a commutative ring.

If there exists  $u \in R$  such that  $ua = au = a \forall a \in R$ ,  $R$  is a unital ring; there is at most one unit. Set  $u = 1_R$

An integral domain is a unital, commutative ring such that  $ab = 0 \Rightarrow a = 0 \vee b = 0$ . For example,  $\mathbb{Z}$  is an integral domain. However,  $C(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  is a unital, commutative ring, but there exist two functions such that  $f, g \neq 0$ , but  $f \cdot g = 0$ .

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R, \text{ with } ab = 1_R$$

There is only one such  $b$ . Set  $b = a^{-1}$ .

### Ordering of $\mathbb{Q}$

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

$\leq$  is a well-defined total ordering of  $\mathbb{Q}$ , and  $j : \mathbb{Z} \hookrightarrow \mathbb{Q}, j(n) = \frac{n}{1}$  is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

### Properties of $\mathbb{Q}^+$

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

$$(i) \quad q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+, q_1 q_2 \in \mathbb{Q}^+$$

$$(ii) \quad q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$$

$$(iii) \quad \pm q \in \mathbb{Q}^+, q = 0$$

$$(iv) \quad x \leq y, u \leq v \Rightarrow x + u \leq y + v$$

$$(v) \quad x \leq y, 0 \leq z \Rightarrow zx \leq zy$$

### Ordered Fields and the Ordering of $\mathbb{R}$

An **ordered field** is a field  $F$  equipped with a total ordering  $\leq_F$  such that:

$$(i) \quad \text{if } s \leq_F t, \text{ and } x \leq_F y, \text{ then } s + x \leq_F t + y$$

$$(ii) \quad \text{if } s \leq_F t \text{ and } 0 \leq_F z, \text{ then } zs \leq_F zt$$

For example,  $\mathbb{Q}$  with its ordering is an ordered field.

**Statement:** If  $(F, \leq_F)$  is an ordered field, we define  $F^+ = \{x \in F \mid x_F \geq 0\}$  with the following properties:

$$(1) \quad x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$$

$$(2) \quad x \in F \Rightarrow x \in F^+, -x \in F^+$$

$$(3) \quad \pm x \in F^+ \Rightarrow x = 0_F$$

### Proofs:

(1) Let  $x, y \in F^+$ . Then,  $x \geq 0$  and  $y \geq 0$ , so by property (i) of an ordered field,  $x + y \geq 0 + 0 = 0$ , so  $x + y \in F^+$ . Additionally, we have  $x \cdot y \geq x \cdot 0 = 0$ , so  $xy \in F^+$ .

- (2) Let  $x \in F$ . Since the ordering on  $F$  is total,  $x \geq 0$  or  $0 \geq x$ . In the first case,  $x \in F^+$ . In the second case, we add  $-x$  to both sides, so by (i),  $-x \geq 0$ , so  $-x \in F^+$ .
- (3) We have  $x \geq 0$  and  $-x \geq 0$ . So  $x \geq 0$  and  $x + (-x) \geq x + 0$ , so  $x \geq 0$  and  $0 \geq x$ . So,  $x = 0$  by antisymmetry.

**Note:**  $x \leq_F y \Leftrightarrow y - x \in F^+$ .

**Statement:** Let  $F$  be an ordered field. Then, the following is true:

- (1)  $\forall a \in F, a^2 \in F^+$
- (2)  $0, 1 \in F^+$
- (3) If  $n \in \mathbb{N}$ ,  $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$
- (4) If  $x \in F^+$ , and  $x \neq 0$ , then  $x^{-1} \in F^+$
- (5) If  $xy > 0$ , then  $x, y \in F^+$ , or  $-x, -y \in F^+$
- (6) If  $0 < x \leq y$ , then  $0 < y^{-1} \leq x^{-1}$
- (7) If  $x \leq y$ , then  $-y \leq -x$
- (8)  $x \geq 1 \Rightarrow x^2 \geq x \geq 1$ , and  $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$ .

**Proof:**

- (1) Let  $a \in F$ . Then,  $a \in F^+$  or  $-a \in F^+$ .  
 Case 1 If  $a \in F^+$ , then by the previous proposition,  $a^2 \in F^+$ .  
 Case 2 If  $-a \in F^+$ , then by the previous proposition,  $(-a)(-a) = a^2 \in F^+$ .
- (2)  $0 \geq 0$ , so  $0 \in F^+$ .  $1 = 1 \cdot 1 = 1^2 \in F^+$  by the previous result.
- (3)  $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}} \in F^+$  by the previous proposition.
- (4) Let  $x \neq 0, x \in F^+$ . Suppose toward contradiction that  $x^{-1} \notin F^+$ , then  $-x^{-1} \in F^+$ . Thus,  $x \cdot (-x^{-1}) \in F^+$ , so  $-1 \in F^+$ , but  $1 \in F^+$ , so  $1 = 0$ .  $\perp$
- (5) Let  $xy > 0$ , meaning  $xy \in F^+$ . Suppose toward contradiction that  $x > 0$  and  $y < 0$ . So,  $x > 0$  and  $-y > 0$ , so  $(x)(-y) > 0$ , so  $-(xy) \in F^+$ , so  $xy = 0$ .  $\perp$
- (6) Let  $0 < x \leq y$ . We know  $x^{-1} \in F^+$ , so  $x^{-1}x \leq x^{-1}y$ . So  $1 \leq x^{-1}y$ . We also know  $y \in F^+$ , so  $y^{-1} \in F^+$ . So,  $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$ .
- (7) Let  $x \leq y$ . Then,  $0 \leq y - x$ , so  $-y \leq -x$ .
- (8) Let  $x \geq 1$ . Then,  $x \cdot x \geq 1 \cdot x \geq 1$ .

**Order Axiom:**  $\mathbb{R}$  is an ordered field. The injection  $\mathbb{Q} \hookrightarrow \mathbb{R}$ ,  $i(q) = q$  is an order embedding.

## Rational Orderings

**Statement:** If  $a \leq b$ , then  $a \leq \frac{1}{2}(a+b) \leq b$ .

**Proof:**  $2a = a + a \leq a + b \leq b + b$ , all by property (i) of an ordered field.

Therefore,  $2a \leq a + b \leq 2b$ . Since  $2 = 1 + 1 \in \mathbb{R}^+$ ,  $2^{-1} \in \mathbb{R}^+$ , so  $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$ , so  $a \leq \frac{1}{2}(a+b) \leq b$ .

**Statement:** If  $a \geq 0$  and  $(\forall \epsilon > 0), a \leq \epsilon$ , then  $a = 0$ .

**Proof:** Suppose toward contradiction that  $a \geq 0$  and  $a \neq 0$ , so  $a > 0$ . So, we have that  $\frac{1}{2}a < a$ . Let  $\epsilon = \frac{1}{2}a$ . We also have that  $a \leq \epsilon = \frac{1}{2}a < a$ , so  $a < a$ .  $\perp$

## Important Inequalities

### Arithmetic and Geometric Means

Given  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ :

#### Arithmetic Mean

$$= \frac{\sum_{i=1}^n a_i}{n}$$

#### Geometric Mean

$$= \sqrt[n]{a_1 a_2 \cdots a_n}$$

### Arithmetic Mean-Geometric Mean Inequality

**Statement:** Let  $a, b \geq 0$ .

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

**Proof:** If  $x, y \geq 0$ ,  $x \leq y \Leftrightarrow x^2 \leq y^2$ .

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y$$

by property (ii) of ordered fields

Therefore,

$$\begin{aligned} (ab)^{1/2} &\leq \frac{1}{2}(a+b) \\ ab &\leq \frac{1}{4}(a^2 + 2ab + b^2) \\ 4ab &\leq a^2 + 2ab + b^2 \\ 0 &\leq a^2 - 2ab + b^2 \\ 0 &\leq (a-b)^2 \end{aligned}$$

by definition

**Challenge:** Prove for  $m$ .

**Remark:** The harmonic mean is defined as:

$$\frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$$

## Bernoulli's Inequality

**Statement:** If  $x \geq -1$ , then  $(1+x)^n \geq 1+nx$ , for any  $n \in \mathbb{N}_0$ .

**Proof:** By induction, we know that for  $n=0$  and  $n=1$ , this holds.

Assume the inequality holds for some  $m \geq 1$ .

$$\begin{aligned}
 (1+x)^{m+1} &= (1+x)^m(1+x) \\
 &\geq (1+mx)(1+x) && \text{by the inductive hypothesis} \\
 &= 1+x+mx+mx^2 \\
 &= 1+(m+1)x+mx^2 \\
 &\geq 1+(m+1)x
 \end{aligned}$$

## Cauchy's Inequality

**Statement:** Let  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ . Then

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left( \sum_{j=1}^n a_j^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra language, this is equivalent to  $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$ .

**Proof:** Consider  $f: \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^n (a_i - b_i x)^2$$

We know that  $f(x) \geq 0$  for all  $x \in \mathbb{R}$

$$\begin{aligned}
 &= \sum_{i=1}^n (a_i^2 - 2a_i b_i x + b_i^2 x^2) \\
 &= \left( \sum_{j=1}^n b_j^2 \right) x^2 + \left( \sum_{j=1}^n -2a_j b_j \right) x + \sum_{j=1}^n a_j^2 \\
 &= Ax^2 + Bx + C
 \end{aligned}$$

Therefore,  $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned}
 \left( -2 \sum_{j=1}^n a_j b_j \right)^2 &\leq 4 \left( \sum_{j=1}^n a_j^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\
 \left| \sum_{j=1}^n a_j b_j \right| &= \left( \sum_{j=1}^n a_j^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2}
 \end{aligned}$$

As we know from linear algebra, the way we get equality is when  $\vec{v} = c\vec{w}$ , or that  $a_j = cb_j \forall j$  for some  $c \in \mathbb{R}$ .

## Triangle Inequality

**Statement:** Given  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left( \sum_{j=1}^n (a_j + b_j)^2 \right)^{1/2} \leq \left( \sum_{j=1}^n a_j^2 \right)^{1/2} + \left( \sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra, this is equivalent to  $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ .

**Proof:**

$$\begin{aligned} \sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\ &\leq \sum a_j^2 + 2 \left( \sum a_j^2 \right)^{1/2} \left( \sum b_j^2 \right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\ &= \left( \left( \sum a_j^2 \right)^{1/2} + \left( \sum b_j^2 \right)^{1/2} \right)^2 \end{aligned}$$

we take square roots to get our end result

## Metrics, Norms, and Bounds

### Metrics and Norms on $\mathbb{R}^n$

Consider  $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$ , defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

#### Theorems about Absolute Value:

- (i)  $|ab| = |a||b|$
- (ii)  $|a^2| = |a|^2$
- (iii)  $|-a| = |a|$
- (iv)  $|a| \in \mathbb{R}^+$
- (v)  $-|a| \leq a \leq |a|$
- (vi)  $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$  for  $\delta > 0$
- (vii)  $|a + b| \leq |a| + |b|$ ,  $|a - b| \leq |a| + |b|$ ,  $||a| - |b|| \leq |a - b|$

#### Proof of (i):

**Case 1:** If  $a, b \in \mathbb{R}^+$ , then  $|a| = a$ , and  $|b| = b$ , and  $ab \in \mathbb{R}^+$ , so  $|ab| = ab$

**Case 2:** If  $a, b \notin \mathbb{R}^+$ , then  $|a| = -a$ , and  $|b| = -b$ . Additionally,  $(-a)(-b) = ab \in \mathbb{R}^+$ , so  $|ab| = ab$ . The LHS =  $ab$ , and the RHS =  $ab$ .

**Case 3:**  $a \in \mathbb{R}^+$ ,  $-b \in \mathbb{R}^+$ . Then,  $|a||b| = (a)(-b) = -ab$ . Then, since  $a(-b) \in \mathbb{R}^+$ ,  $-ab \in \mathbb{R}^+$ , so  $|ab| = -ab$ . Therefore, the LHS and RHS are equal.

**Proof of (vii):** Having established that  $|a + b| \leq |a| + |b|$ , we will show that  $||a| - |b|| \leq |a - b|$ .

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a - b| + |b| \\ |a| - |b| &\leq |a - b| \end{aligned}$$

Similarly, by exchanging  $a$  for  $b$

$$|b| - |a| \leq |b - a|$$

$$|b| - |a| \leq |a - b|$$

Let  $t = |a| - |b|$ . We have shown that

$$\pm t \leq |a - b|$$

$$-|a - b| \leq t \leq |a - b|$$

$$|t| \leq |a - b|$$

## Bounded Sets

A subset  $A \subseteq \mathbb{R}$  is **bounded**  $\Leftrightarrow \exists c \geq 0$  such that  $\forall x \in A, |x| \leq c$ .

( $\Rightarrow$ ) Suppose  $A \subseteq \mathbb{R}$  is bounded. Then,  $\exists \ell, u \in \mathbb{R}$  such that  $\ell \leq x \leq u \forall x \in A$ . Let  $c := \max\{|\ell|, |u|\}$ .

Since  $|u| \leq c$ , we have that  $x \leq c$ .

Since  $|\ell| \leq c$ , and  $-\ell \leq x$ , we get that  $-x \leq |\ell| \leq c$ .

Since  $x \leq c$  and  $-x \leq c$ ,  $|x| \leq c$ .

( $\Leftarrow$ ) If such a  $c$  exists, then  $|x| \leq c$  if and only if  $-c \leq x \leq c$ . Thus,  $-c$  is the lower bound and  $c$  is the upper bound.

## Bounded Functions

Let  $D$  be any set. A function  $f : D \rightarrow \mathbb{R}$  is bounded if  $\text{Ran}(D) \subseteq \mathbb{R}$  is bounded.

For example, let  $f : [3, 7] \rightarrow \mathbb{R}$ ,  $f(x) = \frac{x^2+2x+1}{x-1}$ . We will show that  $f$  is bounded.

$$3 \leq x \leq 7 \Rightarrow 2 \leq x - 1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}.$$

$$\text{Also, } 4 \leq x + 1 \leq 8 \Rightarrow 16 \leq x^2 + 2x + 1 \leq 64 \Rightarrow |x^2 + 2x + 1| \leq 64.$$

So,  $|f(x)| \leq 32$ .

## Distance Metrics

For  $s, t \in \mathbb{R}$ , we will define  $d(s, t) = |s - t|$  to be the **distance** between  $s$  and  $t$ .

**Properties:**

(i)

$$d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$$

$$(s, t) \mapsto d(s, t) \geq 0$$

(ii)  $d(s, t) = d(t, s)$

(iii)  $d(s, r) \leq d(s, t) + d(t, r)$

(iv)  $d(s, s) = 0$

(v) If  $d(s, t) = 0$ , then  $s = t$ .

Let  $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ,  $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ .

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^n |x_j|$$

- $\infty$ -norm:

$$\|v\|_\infty = \max_{j=1}^n |x_j|$$

- 2-norm:

$$\|v\|_2 = \left( \sum_{j=1}^n x_j^2 \right)^{1/2}$$

### Properties of Norms

**Statement:** With  $v, w$  above, let  $p = 1, 2, \infty$ . The following are true:

- (1)  $\|v\|_p \geq 0$
- (2)  $\|v + w\|_p \leq \|v\|_p + \|w\|_p$
- (3)  $\|\vec{0}\|_p = 0$
- (4)  $\|v\|_p = 0 \Rightarrow v = \vec{0}$
- (5)  $\forall t \in \mathbb{R}, \|tv\|_p = |t|\|v\|_p$

**Proofs:** Let  $p = \infty$ . We will prove (2).

Say  $\|v\|_{\infty} = |x_i|$  and  $\|w\|_\infty = |y_k|$ . We want to show that  $\|v + w\|_\infty = \max_{j=1}^n |x_j + y_j| \leq |x_i| + |y_k|$ .

Note that  $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| && \text{Triangle Inequality} \\ &\leq |x_i| + |y_k| \\ &= \|v\|_\infty + \|w\|_\infty \end{aligned}$$

Therefore,  $\|v + w\|_\infty \leq \|v\|_\infty + \|w\|_\infty$ .

### Relating Distance Metrics and Norms

A **norm** on  $\mathbb{R}^n$  is a function  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$ ,  $v \mapsto \|v\|$ , satisfying the following properties for  $v \in \mathbb{R}^n$ :

- (1)  $\|v\| \geq 0$
- (2)  $\|v + w\| \leq \|v\| + \|w\|$
- (3)  $\|\vec{0}\| = 0$
- (4)  $\|v\| = 0 \Rightarrow v = \vec{0}$



$$(5) \forall t \in \mathbb{R}, \|tv\| = |t|\|v\|$$

If  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$  is a norm, we define  $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ , defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for  $v, w \in \mathbb{R}^n$ .

The properties of distance in  $\mathbb{R}$  still hold for distance in  $\mathbb{R}^n$ :

- (1)  $d(v, w) = d(w, v)$
- (2)  $d(u, w) \leq d(u, v) + d(v, w)$
- (3)  $d(v, v) = 0$
- (4)  $d(v, w) = 0 \Rightarrow v = w$

## Metric Spaces

A **metric space** is a nonempty set  $X$  equipped with a function  $d : X \times X \rightarrow \mathbb{R}^+$ ,  $(x, y) \mapsto d(x, y) \geq 0$ . The metric has the following properties:

- (1)  $d(x, y) = d(y, x) \forall x, y \in X$
- (2)  $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X$
- (3)  $d(x, x) = 0$
- (4)  $d(x, y) = 0 \Leftrightarrow x = y$

The map  $d$  is called a *metric* on  $X$ .

## Examples of Metric Spaces

- $\mathbb{R}$  with  $d(x, y) = |x - y|$ .
- $\mathbb{R}^n$  with the *Euclidean metric*:

$$\begin{aligned} d_2(v, w) &= \|v - w\|_2 \\ &= \left( \sum_{j=1}^n (x_j - y_j)^2 \right)^{1/2} \end{aligned}$$

- $\mathbb{R}^n$  with the 1-norm:

$$\begin{aligned} d_1(v, w) &= \|v - w\|_1 \\ &= \sum_{j=1}^n |x_j - y_j| \end{aligned}$$

- $\mathbb{R}^n$  with the  $\infty$ -norm:

$$\begin{aligned} d_\infty(v, w) &= \|v - w\|_\infty \\ &= \max_{j=1}^n |x_j - y_j| \end{aligned}$$

## Open and Closed Sets in Metric Spaces

Let  $(X, d)$  be a metric space.

- (1) The **open ball** centered at  $x_0 \in X$  with radius  $\delta$  is:

$$V(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

- (2) The **closed ball** centered at  $x_0 \in X$  with radius  $\delta$  is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

- (3) A set  $U \subseteq X$  is **open** if  $\forall x \in U, \exists \delta > 0$  such that  $V(x, \delta) \subseteq U$ .

- (4) A set  $C \subseteq X$  is **closed** if  $\overline{C} = X - C \subseteq X$  is open.

For example,

In  $\mathbb{R}$  with  $d(s, t) = |s - t|$ :

$$\begin{aligned} V(x_0, \delta) &= \{y \in \mathbb{R} \mid d(y, x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0, \delta) &= [x_0 - \delta, x_0 + \delta] \end{aligned}$$

The interval  $A = [1, \infty)$  is not open, as  $\forall \delta > 0, U(1, \delta) \not\subseteq [1, \infty)$ .

However,  $A$  is closed, as  $\overline{A} = (-\infty, 1)$  is open: given  $t \in \overline{A}$ , choose  $\delta = 1 - t$ . Let  $s \in V_\delta(t)$ . Then,  $s \in (t - \delta, t + \delta)$ , so  $s \in (t - (1 - t), t + (1 - t))$ , or  $s \in (2t - 1, 1)$ , so  $s < 1$ .

In  $(\mathbb{R}^2, d_2)$ ,  $B(0_{\mathbb{R}^2}, 1)$  is the **unit disc** centered at  $(0, 0)$ .

However, in  $(\mathbb{R}^2, d_\infty)$ :

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

## Supremum, Infimum, and Completeness

### Finding a Supremum

**Statement:** Let  $0 \neq A \subseteq \mathbb{R}$ . Let  $u \in \mathbb{R}$  be an upper bound for  $A$ . The following are equivalent:

- (i)  $u = \sup(A)$
- (ii) If  $t < u$ , then  $\exists a_t \in A$  such that  $a_t > t$
- (iii)  $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$  with  $u - \varepsilon < a_\varepsilon$

**Proof:**

(i)  $\Rightarrow$  (ii): Given  $t < u$ , if no such  $a \in A$  with  $t < a$  exists, then  $a \leq t \forall a \in A$ . Thus  $t$  would be an upper bound. However,  $t < u$  and  $u$  is the supremum of  $A$ .  $\perp$

(ii)  $\Rightarrow$  (iii): Given  $\varepsilon > 0$ , set  $t = u - \varepsilon < u$ . So, by (ii),  $\exists a_t \in A$  with  $t < a_t$ . Thus,  $u - \varepsilon < a_t$ . Set  $a_\varepsilon = a_t$ .

(iii)  $\Rightarrow$  (i): Let  $v$  be an upper bound for  $A$ . Suppose  $v < u$ . Then, set  $\varepsilon = u - v > 0$ . By (iii),  $\exists a_\varepsilon \in A$  with  $u - \varepsilon < a_\varepsilon$ . So  $u - (u - v) < a_\varepsilon$ , so  $v < a_\varepsilon$ , meaning  $v$  cannot be an upper bound.  $\perp$

**Supremum Example**

$\sup[0, 1) = 1$  : Certainly, 1 is an upper bound for  $[0, 1)$ . Let  $\varepsilon > 0$ .

If  $\varepsilon \geq 1$ , pick  $t = \frac{1}{2}$ . Then,  $1 - \varepsilon < 0 < \frac{1}{2}$

If  $0 < \varepsilon < 1$ , let  $t = (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$ . Then,  $t \in [0, 1)$ , and  $1 - \varepsilon < 1 - \varepsilon/2 = t$

**Finding an Infimum**

Let  $\emptyset \neq A \subseteq \mathbb{R}$ . Let  $\ell \in \mathbb{R}$  be a lower bound for  $A$ . The following are equivalent:

- (i)  $\ell = \inf(A)$
- (ii) If  $t > \ell$ ,  $\exists a_t$  such that  $t > a_t$
- (iii)  $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$  with  $\ell + \varepsilon > a_\varepsilon$

**Infimum Example**

$\inf \left\{ \frac{1}{n} \mid n \geq 1 \right\}$  : Clearly,  $0 < \frac{1}{n} \forall n \geq 1$ . Let  $\varepsilon > 0$ .

We need to find  $a \in \left\{ \frac{1}{n} \mid n \geq 1 \right\}$  with  $\varepsilon > a$ . By the Archimedean Property,  $\exists m \in \mathbb{N}$  such that  $\frac{1}{m} < \varepsilon$ .  
Let  $a_\varepsilon = \frac{1}{m}$ .

**Properties of Supremum and Infimum**

- If  $A \subseteq \mathbb{R}$  and  $\max(A) = u$ , then  $u = \sup(A)$ :  $u$  is an upper bound of  $A$  by the definition of  $\max$ , and if  $v \neq u$  is any upper bound of  $A$ , then  $u < v$  since  $u \in A$ .
- If  $\min(A) = \ell$ , then  $\ell = \inf(A)$  (by the same logic).
- If  $A$  is not bounded above,  $\sup(A) = +\infty$ , and if  $A$  is not bounded below, then  $\inf(A) = -\infty$ .
- If  $A \subseteq B$ , then  $\sup(A) \leq \sup(B)$ .
- If  $A \subseteq B$ , then  $\inf(A) \geq \inf(B)$ : Let  $\ell_A = \inf(A)$  and  $\ell_B = \inf(B)$ . By definition,  $\ell_B \leq b \forall b \in B$ . Since  $A \subseteq B$ ,  $\ell_B \leq a \forall a \in A$ . Thus,  $\ell_B$  is a lower bound for  $A$ . By definition of  $\ell_A$ ,  $\ell_B \leq \ell_A$ .

Let  $A, B \subseteq \mathbb{R}$  and  $t \in \mathbb{R}$ . Then, the following are also sets:

- (1)  $A + B = \{a + b \mid a \in A, b \in B\}$
- (2)  $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$
- (3)  $t \cdot A = \{ta \mid a \in A\}$
- (4)  $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

## Completeness Axiom

If  $\emptyset \neq A \subseteq \mathbb{R}$  is bounded above, then  $\sup(A)$  exists.

Well-Ordering Property: if  $\emptyset \neq S \subseteq \mathbb{N}$ , then  $\min(S)$  exists.

Therefore, we can prove that if  $F \subseteq \mathbb{Z}$  is bounded, then  $F$  has a least and greatest element.

## Archimedean Property

**Statement:** If  $x \in \mathbb{R}$ , then  $\exists n_x \in \mathbb{N}$  such that  $x \leq n_x$ .

**Proof:** Suppose there exists no natural number greater than  $x$ , then  $\mathbb{N}$  is bounded above by  $x$ . Let  $u = \sup(\mathbb{N})$ . By the Completeness Axiom,  $u \in \mathbb{R}$  exists. Let  $\varepsilon = 1$ . Then,  $\exists n \in \mathbb{N}$  with  $u - 1 < n$ . So,  $u < n + 1$ , but  $n + 1 \in \mathbb{N}$ , so  $u$  cannot be an upper bound.

## Corollary to the Archimedean Property

$$\forall t > 0 \exists n \in \mathbb{N} \text{ such that } \frac{1}{n} < t$$

## Corollary to the Corollary: Powers of 2

**Statement:**

$$\forall t > 0 \exists m \in \mathbb{N} \text{ such that } \frac{1}{2^m} < t$$

**Proof:** By the corollary to the Archimedean Property, we know that  $\exists n \in \mathbb{N}$  such that  $\frac{1}{n} < t$ . By Bernoulli's inequality with  $x = 1$ , we have  $2^n \geq n$ , so  $2^{-n} < n^{-1} < t$ .

## Corollary to the Corollary: In Between Integers

**Statement:**

$$\forall x \in \mathbb{R} \exists n_x \in \mathbb{Z} \text{ such that } n_x - 1 \leq x < n_x$$

**Proof:** Assume  $x \geq 0$ . Let  $S_x = \{n \mid n \in \mathbb{N}, x < n\}$ .

$S_x \neq \emptyset$  by the Archimedean Property. By the well-ordering property, let  $n_x = \min(S_x)$ .

Therefore,  $x < n_x$ . Also,  $n_x - 1 \notin S_x$ . Therefore  $n_x - 1 \leq x$ .

## Density

Let  $(X, d)$  be any metric space. A subset  $D \subseteq X$  is **dense** if  $\forall x \in X, \varepsilon > 0, U(x, \varepsilon) \cap D \neq \emptyset$ .

In the case of  $X = \mathbb{R}$  and  $d(s, t) = |s - t|$ ,  $D \subseteq \mathbb{R}$  is dense if given any open interval  $I$ ,  $I \cap D \neq \emptyset$ .

A metric space is **separable** if it admits a *countable* dense subset.

## Density of the Rationals

**Statement:**  $\mathbb{Q} \subseteq \mathbb{R}$  is dense.

**Proof:** Let  $I = (a, b)$  be an open interval. We may assume that  $a, b \in \mathbb{R}$  are finite.

Then,  $b - a > 0$ . By the Archimedean property corollary,  $\exists n \in \mathbb{N}$  such that  $\frac{1}{n} < b - a$ , meaning  $1 < nb - na$ .

There exists also an integer  $m$  such that  $m - 1 \leq na < m$ , implying that  $a \frac{m}{n}$ . Also,  $m \leq na + 1 < nb$ . Therefore,  $\frac{m}{n} < b$ .

So,  $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$ .

### Density of the Irrationals

**Statement:**  $\mathbb{R} \setminus \mathbb{Q}$  is dense.

**Proof:** Assume  $\sqrt{2}$  exists. Let  $I = (a, b)$  be any open interval. Then,  $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ .

Find  $q \in \mathbb{Q}$  such that  $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$ .

Then,  $a < q\sqrt{2} < b$ , where  $q\sqrt{2} \in \mathbb{R}$  and  $q\sqrt{2} \notin \mathbb{Q}$ .

### Uniqueness of $\sqrt{2}$

**Statement:**

$$\exists! x > 0 \text{ such that } x^2 = 2$$

**Proof:**

**Existence:** Let  $S = \{t \in \mathbb{R} \mid 0 < t, t^2 < 2\}$ .  $S$  is nonempty because  $1 \in S$ , and  $S$  is bounded above because  $y > 2 \Rightarrow y^2 > 4$ .

So, by the completeness axiom,  $x := \sup(S)$  exists, and  $x \geq 1$ .

**Claim:**  $x^2 = 2$

**Contradiction 1:** Assume  $x^2 < 2$ . We want to show that  $\exists n \in \mathbb{N}$  such that  $x + \frac{1}{n} \in S$ . By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2 \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x + 1) \end{aligned}$$

We want to find  $n \in \mathbb{N}$  with:

$$x^2 + \frac{1}{n}(2x + 1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

Therefore, by the Archimedean Property corollary, we know that  $n$  must exist.

**Contradiction 2:** We know that  $x^2 \geq 2$ . Since  $x = \sup(S)$ ,  $\forall m \in \mathbb{N}$ ,  $\exists t_m \in S$  with  $x - \frac{1}{m} < t_m$ , so  $(x - \frac{1}{m})^2 < t_m^2 < 2$ .

Therefore,  $x^2 - \frac{2x}{m} + \frac{1}{m^2} < 2$ , so  $x^2 - \frac{2x}{m} < 2$ , so  $0 \leq x^2 - 2 < \frac{2x}{m}$ .

So,  $0 \leq \frac{x^2 - 2}{2x} < \frac{1}{m}$ , so  $x^2 - 2 = 0$ , so  $x^2 = 2$ .

**Remark:** If we had set  $S' = \{t' \in \mathbb{Q} \mid t'^2 < 2, t' > 0\}$ , we would have still obtained  $\sup(S') = \sqrt{2}$ . This means that  $\mathbb{Q}$  is *not* complete.

## Intervals in $\mathbb{R}$

(\*) Given any interval  $I$ , if  $x_1, x_2 \in I$ , with  $x_1 < x_2$ , then  $[x_1, x_2] \in I$ .

This seems like an obvious property, but this is the *characteristic property* of intervals.

### Characterization of Intervals

**Statement:** Let  $S \in \mathbb{R}$  be any nonempty subset of cardinality at least 2. Suppose  $S$  satisfies (\*). Then,  $S$  is an interval.

**Proof:**

**Case 1:** Suppose  $S$  is bounded.

Let  $a = \inf(S)$  and  $b = \sup(S)$ . Then,  $S \subseteq [a, b]$ . We will show that  $(a, b) \subseteq S$ . Once this is shown,  $S = \{(a, b), [a, b], [a, b), (a, b]\}$ .

Let  $t \in (a, b)$ . Since  $a = \inf(S)$ ,  $\exists x_1 \in S$ ,  $x_1 \in (a, t)$ . Similarly, since  $b = \sup(S)$ ,  $\exists x_2 \in S$ ,  $x_1 \in (t, b)$ .

By the hypothesis,  $[x_1, x_2] \subseteq S$ . Since  $t \in [x_1, x_2]$ ,  $t \in S$ .

**Case 2:** Suppose  $S$  is bounded above, but not below.

Let  $b = \sup(S)$ . Clearly,  $S \subseteq (-\infty, b]$ . We will show that  $(-\infty, b) \subseteq S$ . Once this is shown,  $S = \{(-\infty, b), (-\infty, b]\}$ .

Let  $t \in (-\infty, b)$ . Since  $b = \sup(S)$ ,  $\exists x_2 \in S$ ,  $x_2 \in (t, b)$ .

Since  $S$  is not bounded below,  $\exists x_1 \in S$  such that  $x_1 < t$  (or else  $t$  would be a lower bound).

By the hypothesis,  $[x_1, x_2] \in S$ , and  $t \in [x_1, x_2]$ , so  $t \in S$ .

**Case 3, 4:** Left as an exercise for the reader.

### Nested Intervals

A sequence of intervals  $(I_n)_{n \geq 1}$  is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in  $\bigcap I_n$ .

(a)  $\bigcap_{n=1} [0, 1/n] = \{0\}$ .

(b)  $\bigcap_{n=1} (0, 1/n) = \emptyset$

(c)  $\bigcap_{n=1} [n, \infty) = \emptyset$

## Measure

The **measure** of an interval is basically its “size.”

$$(a) \quad m([a, b]) = b - a$$

$$(b) \quad m((a, b]) = b - a$$

$$(c) \quad m((a, b)) = b - a$$

$$(d) \quad m([a, b)) = b - a$$

## Nested Intervals Theorem

Let  $I_n = [a_n, b_n]$  for  $n \in \mathbb{N}$  be a nested sequence of intervals.

$$(1) \quad \bigcap_{n \geq 1} I_n \neq \emptyset$$

$$(2) \quad \text{If } \inf \{m(I_n) \mid n \geq 1\} = 0, \text{ then } \bigcap_{n \geq 1} I_n = \{\xi\}$$

**Proof of (1):** Since  $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$ , we have that  $a_1 \leq a_2 \leq a_3, \dots$ , and  $b_1 \geq b_2 \geq b_3 \geq \dots$ .

We know that  $\{a_n\}$  is bounded above and nonempty. Let  $\xi = \sup(\{a_n\}_{n=1}^\infty)$ .

We know that  $\{b_n\}$  is bounded below. Let  $\eta = \inf(\{b_n\}_{n=1}^\infty)$ .

We claim that  $\xi \leq b_n \forall n \geq 1$ . Suppose toward contradiction that  $\exists m$  such that  $\xi > b_m$ . Then, by the supremum property,  $\exists a_i$  such that  $\xi > a_i > b_m$ . If  $k \leq m$ ,  $a_k \leq a_m \leq b_m < a_k$ . If  $m \leq k$ , then  $b_m \geq b_k \geq a_k > b_m$ .  $\perp$

Similarly, using the same argument,  $a_n \leq \eta \forall n$ .

Thus,  $\xi \leq \eta$ .

We claim that  $\bigcap_{n \geq 1} I_n = [\xi, \eta]$ . If  $t \in [\xi, \eta]$ , then  $a_n \leq \xi \leq t \leq \eta \leq b_n$ . So  $t \in [a_n, b_n] \forall n$ , so  $t \in \bigcap_{n \geq 1} [a_n, b_n]$ .

If  $t \in \bigcap_{n \geq 1} I_n$ , then  $t \in [a_n, b_n] \forall n$ , so  $a_n \leq t \leq b_n \forall n$ . So,  $t$  is an upper bound on  $a_n$ , and a lower bound on  $b_n$ . So,  $\xi \leq t \leq \eta$  by definition of  $\xi$  and  $\eta$ .

**Proof of (2):** We have  $\forall n \geq 1$

$$\begin{aligned} [\xi, \eta] &\subseteq [a_n, b_n] \\ \Rightarrow 0 &\leq \eta - \xi \leq b_n - a_n \\ &= m(I_n) \end{aligned}$$

So, if  $\inf(\{m(I_n) \mid n \geq 1\}) = 0$ , then  $0 \leq \eta - \xi \leq 0$ , so  $\eta = \xi$ .

## Corollary to the Nested Intervals Theorem

**Statement:**  $[0, 1]$  is uncountable.

**Proof:** Suppose it is possible to denumerate the interval  $[0, 1] = \{t_1, t_2, \dots\}$ .

We can find  $[a_1, b_1] \subseteq [0, 1]$  with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$ .

Then, we find  $[a_2, b_2] \subseteq [a_1, b_1]$  with  $a_2 < b_2$ ,  $t_2 \notin [a_2, b_2]$ .

Recursively, we find  $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$  with  $a_n < b_n$ ,  $t_n \notin [a_n, b_n]$ .

So,  $I_n = ([a_n, b_n])_0^\infty$  is a sequence of nested intervals.

Therefore,  $\exists \xi \in \bigcap I_n \subseteq [0, 1]$ . However,  $\xi \notin \{t_1, t_2, \dots\}$ .  $\perp$

## Sequences and Convergence

### Sequences in Metric Spaces

A sequence in a metric space  $M$  is a map

$$x : \mathbb{N} \rightarrow M$$

$M = \mathbb{R}$ , usually

$$x = (x_n)_{n=1}^\infty$$

I. Sequences defined by a formula:

- (i)  $x_n = t$  (the constant sequence)
- (ii)  $x_n = 2n + 1$
- (iii)  $x_n = \frac{1}{n-1}$  (here,  $n \geq 2$ )
- (iv)  $c_n = n \sin\left(\frac{1}{n}\right)$
- (v)  $d_n = \left(1 + \frac{1}{n}\right)^n$
- (vi) Geometric Sequence: for  $b \neq 0$ ,  $(b^n)_{n \geq 0} = (1, b, b^2, \dots)$
- (vii)  $x_n = \frac{n!}{n^n}$
- (viii) Given any function

$$f : [0, \infty) \rightarrow \mathbb{R}$$

we can set  $x_n = f(n)$ .

II. Sequences defined recursively:

- (i)  $a_1 = 1$ ,  $a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
- (ii) Fibonacci:  $f_1 = 1$ ,  $f_2 = 1$ ,  $f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$ . The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - (1 - \varphi)^n)$$

where  $\varphi = \frac{1+\sqrt{5}}{2}$



(iii) Let  $f : M \rightarrow M$  be a self-map on a metric space. Fix  $x_0 \in M$ .

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

### III. New sequences from old:

(i) Let  $(a_n)_n$  and  $(b_n)_n$  be sequences,  $t \in \mathbb{R}$ . Then, we can do the following:

- $(a_n)_n + (b_n)_n = (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n(b_n)_n = (a_nb_n)_n$
- If  $b_n \neq 0 \forall n$ ,  $\left(\frac{a_n}{b_n}\right)$

(ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

(iii) We can look at ratios for  $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

(iv) We can look at partial sums, given  $(a_k)_{k=1}^\infty$ .

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^n a_k \end{aligned}$$

The sequence  $(s_n)_n$  is called the sequence of partial sums. For example, the sequence of partial sums for  $(b^k)_{k=0}^\infty$  is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for  $b \neq 1$ ,  $(1-b)(1+b+b^2+\cdots+b^n) = 1-b^{n+1}$

### Finding a Sequence

**Statement:** Let  $a_k = \frac{1}{k(k+1)}$ . Find  $(s_n)_n$ .

**Solution:** Via partial fraction decomposition, we get that  $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ . Therefore,  $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^\infty$

### Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R}, a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k|$$

Infinity Norm

**Statement:** This norm has the traditional properties of the norm:

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty && \text{Triangle Inequality} \\ \|t(a_k)_k\|_\infty &= |t| \|(a_k)_k\|_\infty && \text{Scalar Multiplication} \\ \|(a_k)_k\|_\infty = 0 &\Leftrightarrow a_k = 0 \quad \forall k && \text{Zero Property} \\ \|(a_k)_k (b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty \|(b_k)_k\|_\infty && \text{Multiplication} \end{aligned}$$

**Proof:** Let  $u = \|(a_k)_k\|_\infty$  and  $v = \|(b_k)_k\|_\infty$ .

Given any  $k$ ,

$$\begin{aligned} |a_k + b_k| &\leq |a_k| + |b_k| && \text{Triangle Inequality on } |\cdot| \\ &\leq u + v && \text{definition of supremum} \\ \Rightarrow \sup_{k \geq 1} |a_k + b_k| &\leq u + v \end{aligned}$$

Thus,

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &= \|((a_k + b_k)_k)_k\|_\infty \\ &= \sup_{k \geq 1} |a_k + b_k| \\ &\leq u + v \end{aligned}$$

## Monotonicity

A sequence  $(x_n)_n$  is **increasing** if

$$x_1 \leq x_2 \leq \dots \quad \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \dots \quad \forall n$$

The sequence is *eventually* increasing if  $\exists m \in \mathbb{N}$  such that  $x_n \leq x_{n+1}$  for  $n > m$ .

Similarly, the sequence is eventually decreasing if  $\exists m \in \mathbb{N}$  such that  $x_n \geq x_{n+1}$  for  $n > m$ .

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

## Monotonicity Example

**Statement:** The sequence

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \frac{1}{2}a_n + 2 \end{aligned}$$

is increasing and bounded above.

**Proof:** We will prove the first statement via induction:

**Base:**  $a_1 = 1$ ,  $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

**Inductive Hypothesis**  $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+2}$

**Proof:**

$$\begin{aligned}
 a_n &\leq a_{n+1} \\
 \frac{1}{2}a_n &\leq \frac{1}{2}a_{n+1} \\
 \frac{1}{2}a_n + 2 &\leq \frac{1}{2}a_{n+1} + 2 \\
 a_{n+1} &\leq a_{n+2}
 \end{aligned}$$

To prove the sequence is bounded above, we do the following:

$$\begin{aligned}
 a_1 &= 1 \leq 4 \\
 \frac{1}{2}a_1 &\leq 2 \\
 \frac{1}{2}a_1 + 2 &\leq 2 \\
 a_2 &\leq 4
 \end{aligned}$$

We claim that  $\forall n, a_n \leq 4 \Rightarrow a_{n+1} \leq 4$ , as we have shown the base case.

$$\begin{aligned}
 a_n &\leq 4 \\
 \frac{1}{2}a_n &\leq 2 \\
 \frac{1}{2}a_n + 2 &\leq 4 \\
 a_{n+1} &\leq 4
 \end{aligned}$$

## Convergence of Sequences

Let  $L \in \mathbb{R}$ ,  $\varepsilon > 0$ . Then, the  $\varepsilon$ -neighborhood of  $L$  is  $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$ .

$$\begin{aligned}
 x &\in V_\varepsilon(L) \\
 &\Leftrightarrow \\
 |x - L| &< \varepsilon \\
 L - \varepsilon &< x < L + \varepsilon
 \end{aligned}$$

With this in mind, we know the following:

### Definition of Convergence

A real sequence  $(x_n)_n$  converges to a number  $x$  if

$$(\forall \varepsilon > 0) (\exists N_\varepsilon \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such  $L$  exists, then  $(x_n)_n$  is said to **diverge**.

A sequence  $(x_n)_n$  in a metric space  $(X, d)$  converges to a point  $x$  if

$$(\forall \varepsilon > 0) (\exists N_\varepsilon \in \mathbb{N}) \text{ such that } d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an  $N$  such that the  $N$ th tail (i.e.,  $x_N, x_{N+1}, \dots$ ) are within  $\varepsilon$  of  $L$  for any  $\varepsilon$ .

**Note:**  $N$  usually depends on  $\varepsilon$  (the smaller the  $\varepsilon$ , the larger the  $N$ ).

**Convergence Proof 1****Statement:**

$$\left(\frac{1}{n}\right)_n \xrightarrow{n \rightarrow \infty} 0$$

**Proof:** We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given  $\varepsilon > 0$ , we want  $\frac{1}{n} < \varepsilon$ , meaning  $n > \frac{1}{\varepsilon}$ .**Proof:** Let  $\varepsilon > 0$ . By the Archimedean property corollary, find  $N \in \mathbb{N}$  large such that  $\frac{1}{N} < \varepsilon$ .

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

so, if  $n \geq N$ , then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

**Convergence Proof 2****Statement:** Show that

$$\left(\frac{5n-1}{3-n}\right)_{n \geq 4} \xrightarrow{n \rightarrow \infty} -5$$

**Proof:**

$$\begin{aligned} |x_n - L| &= \left|\frac{5n-1}{3-n} + 5\right| \\ &= \frac{14}{|3-n|} \\ &= \frac{14}{n-3} \\ &< \varepsilon \\ \frac{14}{n-3} &< \varepsilon \\ n &> \frac{14}{\varepsilon} + 3 \end{aligned}$$

**Proof:** Let  $\varepsilon > 0$ . Find  $N' \in \mathbb{N}$  so large that  $\frac{1}{N'} < \frac{\varepsilon}{14}$  (which exists by the Archimedean property corollary). Let  $N = N' + 3$ . If  $n \geq N$ , then

$$\begin{aligned} n-3 &\geq \frac{1}{N'} \\ \frac{1}{n-3} &\leq \frac{1}{N'} \\ &< \frac{\varepsilon}{14} \end{aligned}$$

whence

$$\begin{aligned} |x_n - L| &= \frac{14}{n-3} \\ &< \frac{14\epsilon}{14} \\ &= \epsilon \end{aligned}$$

## Convergence and Distance

**Statement:** Let  $(X, d)$  be a metric space, and let  $(x_n)_n$  be a sequence in the metric space. The following are equivalent:

- (i)  $(x_n)_n \rightarrow x$
- (ii)  $(d(x_n, x))_n \rightarrow 0$

**Proof:**

(i)  $\Rightarrow$  (b) Let  $\epsilon > 0$ . Find  $N_\epsilon \in \mathbb{N}$  so large such that  $d(x_n, x) < \epsilon$  whenever  $n \geq N_\epsilon$ .

So,  $|d(x_n, x) - 0| = d(x_n, x) < \epsilon$  for all  $\epsilon > 0$ . Whence,  $(d(x_n, x))_n \rightarrow 0$ .

(ii)  $\Rightarrow$  (i) This direction is similar.

In  $\mathbb{R}$ , this implies that

$$\begin{aligned} (x_n)_n &\rightarrow x \\ &\Leftrightarrow \\ (|x_n - x|)_n &\rightarrow 0 \end{aligned}$$

## Comparison Proposition

**Statement:** Let  $(X, d)$  be a metric space and let  $x \in X$ , and suppose  $(x_n)_n$  is a sequence in  $X$ .

If  $\exists c \geq 0$ ,  $m \in \mathbb{N}$ , and a sequence  $(a_n)_n \in \mathbb{R}^+$  with  $(a_n)_n \rightarrow 0$  and  $d(x_n, x) \leq ca_n \forall n > m$ . Then,  $(x_n)_n \rightarrow x$ .

**Proof:** Let  $\epsilon > 0$ . Note that  $\frac{\epsilon}{c} > 0$ .

Find  $N_1 \in \mathbb{N}$  large such that  $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\epsilon}{c}$ , which is always possible since  $(a_n)_n \rightarrow 0$ .

Let  $N = \max(N_1, m)$ . Then,  $n \geq N \Rightarrow n \geq N_1$  and  $n \geq m$ . So,

$$\begin{aligned} d(x_n, x) &\leq ca_n \\ &< c \frac{\epsilon}{c} \\ &= \epsilon \end{aligned}$$

so,  $n \geq N \Rightarrow d(x_n, x) < \epsilon$ , whence  $(x_n)_n \rightarrow x$

**Comparison Example 1****Statement:**

$$\left( \frac{\sin(n^2 - 1)}{n^2 + 3} \right)_n \rightarrow 0$$

**Proof:**

$$\begin{aligned} \left| \frac{\sin(n^2 - 1)}{n^2 + 3} - 0 \right| &= \frac{|\sin(n^2 - 1)|}{n^2 + 3} \\ &\leq \frac{1}{n^2 + 3} \\ &\leq \frac{1}{n^2} \\ &\leq \frac{1}{n} \end{aligned}$$

We know that  $a_n = \frac{1}{n}$  converges to 0. So, by our comparison proposition, we are done.

**Comparison Example 2****Prove:**

$$\left( \frac{1}{2^n} \right)_n \rightarrow 0$$

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &\geq 1 + n \\ &> n \end{aligned}$$

Bernoulli's Inequality

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since  $a_n = \frac{1}{n}$  converges, we know that  $\frac{1}{2^n}$  must converge.

**Sequence Divergence**

A sequence  $(x_n)_n$  is **divergent** if it does not converge.  $(x_n)_n \rightarrow 0$  if and only if

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbb{N}) \text{ such that } (\forall n \geq N_\epsilon) d(x_n, x) < \epsilon$$

So,  $(x_n)_n$  diverges if and only if

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \rightarrow d(x_n, x) \geq \epsilon_0$$

**Sequence Divergence 1****Statement:** Show that the following sequence diverges:

$$a_n = (-1)^n$$

**Proof:**

**Step 1:**

$$((-1)^n)_n \not\rightarrow 1$$

Take  $\varepsilon_0 = 1/2$ , given any  $N \in \mathbb{N}$ , we will find  $n \geq N$  odd:

$$\begin{aligned} n &= 2N + 1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

**Step 2:**

$$((-1)^n)_n \not\rightarrow -1$$

by letting  $\varepsilon_0 = 1/2$  and  $n = 2N$ .

**Sequence Divergence 2**

**Statement:** Does

$$a_n = (\sin(n))_n$$

converge?

**Proof:** It is not the case that  $(\sin(n))_n \rightarrow L$  for any  $L \in \mathbb{R}$ .

**Case 1** If  $L > 1$ , set  $\varepsilon_0 = \frac{L-1}{2}$ . Then, given any  $N \in \mathbb{N}$ , pick  $n = N$ .

$$\begin{aligned} |\sin(n) - L| &= L - \sin(n) \\ &\geq L - 1 \\ &> \frac{L-1}{2} \\ &= \varepsilon_0 \end{aligned}$$

**Case 2** Similarly for  $L < -1$

**Case 3** Suppose  $-1 < L < 1$ .

**Case 3.1** Suppose  $L > 0$ . Set  $\varepsilon_0 = \frac{L}{2}$ . Given any  $N$ , find  $n \geq N$  with  $\sin(n) < 0$ .

We find  $k$  large such that  $N < (2k+1)\pi$ , which we can always do because we are finding  $k > \frac{1}{2}(\frac{N}{\pi} - 1)$ , which is always possible by the Archimedean property.

Note that  $N < (2k+1)\pi < (2k+2)\pi$ . Note that  $\sin(x) < 0$  on the interval  $((2k+1)\pi, (2k+2)\pi)$ . Note that  $|(2k+1)\pi - (2k+2)\pi| = \pi$ . Let  $n = \lceil (2k+1)\pi \rceil$ . Then,  $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

**Case 3.2** Suppose  $L < 0$ , set  $\varepsilon_0 = \frac{-L}{2}$ . Given  $N$ , find  $n \geq N$  with  $\sin(n) > 0$ . Using the same strategy as above, we find  $n$  such that  $|L - \sin(n)| > \varepsilon_0$

**Case 3.3** Suppose  $L = 0$ . Set  $\varepsilon_0 = 1/2$ . Given  $N \in \mathbb{N}$ , find  $n \geq N$  with  $\sin(n) \geq \frac{1}{2}$ . Then,  $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$ .

Showing that a sequence diverges is not easy — later, we will show divergence with non-uniqueness of convergent subsequences.

## Alternating Sequence

Consider again

$$((-1)^n)_{n \geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to  $-1$ :

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating sequence diverges.

## Uniqueness of Limits

**Statement:** A sequence  $(x_n)_n$  can converge to at most one limit.

**Proof:** Suppose toward contradiction that  $(x_n)_n$  converges to  $L_1$  and  $L_2$  with  $L_1 \neq L_2$ .

WLOG, let  $L_2 > L_1$ . Take  $\varepsilon = \frac{L_2 - L_1}{3}$ .

Since  $(x_n)_n$  converges to  $L_1$ ,  $\exists N_1 \in \mathbb{N}$  such that  $|x_n - L_1| < \varepsilon$ . Similarly, since  $(x_n)_n$  converges to  $L_2$ ,  $\exists N_2 \in \mathbb{N}$  such that  $|x_n - L_2| < \varepsilon$ .

Let  $N = \max N_1, N_2$ . If  $n \geq N$ , then  $n \geq N_1$  and  $n \geq N_2$ .

So,  $|x_n - L_1| < \varepsilon$  and  $|x_n - L_2| < \varepsilon$ . So,  $x_n \in V_\varepsilon(L_1)$ , and  $x_n \in V_\varepsilon(L_2)$ , meaning  $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$ , but  $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$ .  $\perp$

## Useful Lemmas for Convergence

### Absolutely Convergent Sequences

**Statement:** Let  $(x_n)_n$  be a real sequence. If  $x_n$  converges to  $x$ , then  $|(x_n)_n| \rightarrow |x|$ . However, the converse is not the case.

**Proof:** Note that since  $(x_n)_n \rightarrow x$ ,  $d(x_n, x) \rightarrow 0$ .

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

### Convergence to Zero

**Statement:** Let  $a_n$  be a sequence.

$$\begin{aligned} (a_n)_n &\rightarrow 0 \\ &\Leftrightarrow \\ |(a_n)| &\rightarrow 0 \end{aligned}$$

**Proof:**



( $\Rightarrow$ ) If  $(a_n)_n \rightarrow 0$ , then we showed previously that  $|(a_n)_n| \rightarrow |0| = 0$

( $\Leftarrow$ ) Suppose  $|(a_n)_n| \rightarrow 0$ . Given  $\varepsilon > 0$ , then  $\exists N$  such that  $n \geq N$  implies

$$||a_n| - 0| < \varepsilon$$

$$||a_n|| < \varepsilon$$

$$|a_n| < \varepsilon$$

$$|a_n - 0| < \varepsilon$$

So,  $(a_n)_n \rightarrow 0$

## Geometric Sequence

**Statement:** Let  $b \in \mathbb{R}$ . Consider

$$(b^n)_{n \geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided  $-1 < b \leq 1$ . Otherwise, the sequence is divergent.

**Proof:** If  $b = 0$ , then the sequence  $(b^n)_n = (0, 0, 0, \dots)$  is convergent.

If  $b = 1$ , then the sequence  $(b^n)_n = (1, 1, 1, \dots)$  is convergent.

If  $b = -1$ , then the sequence  $(b^n)_n = (1, -1, 1, \dots)$  is divergent.

**Case 1** Suppose  $0 < b < 1$ . Then,  $\frac{1}{b} > 1$ , so  $\frac{1}{b} = 1 + a$ .

So, by Bernoulli's Inequality,  $(\frac{1}{b})^n = (1 + a)^n \geq 1 + na > na$ , so  $b^n < \frac{1}{na}$ .

$$\begin{aligned} |b^n - 0| &= b^n \\ &< \frac{1}{na} \\ &= \frac{1}{a} \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

So,  $(b^n)_n \rightarrow 0$ .

**Case 2** Suppose  $-1 < b < 0$ . If we look at  $|b^n| = |b|^n$ , we know that  $(|b|^n)_n \rightarrow 0$  by our work above. By the previous lemma, we know that since  $|b^n| \rightarrow 0$ ,  $b^n \rightarrow 0$ .

**Case 3** Suppose  $b > 1$ . Then,  $b = 1 + a$  where  $a > 0$ .

$$\begin{aligned} b^n &= (1 + a)^n \\ &\geq 1 + na && \text{Bernoulli's Inequality} \\ &> na \end{aligned}$$

Suppose toward contradiction that  $(b^n)_n \rightarrow L$ . Let  $\varepsilon_0 = 1$ . Find  $N \in \mathbb{N}$  such that  $N > \frac{L+1}{a}$ .  $N$  must exist by the Archimedean property.

Therefore,  $(N)(a) > L + 1$ . If  $n \geq N$ , then  $(n)(a) > (N)(a) > L + 1$ , so  $|b^n - L| \geq na - L \geq \varepsilon_0$ .  $\perp$

**Case 4** Suppose  $b < -1$ , and suppose toward contradiction that  $(b^n)_n \rightarrow L$ . By the previous lemma, we know that  $|b^n| \rightarrow |L|$ . So,  $|b|^n \rightarrow |L|$ . But,  $|b| > 1$ , which means our assumption contradicts the result from above.  $\perp$

## **$n$ th Root Convergence**

**Statement:** If  $c > 0$ , then  $(c^{1/n})_n \rightarrow 1$ .

**Proof:**

**Case 1:** If  $c = 1$ , then we get  $(c^{1/n})_n = (1, 1, 1, \dots)$ , which clearly converges to one.

**Case 2:** Assume that  $c > 1$ . Then,  $c^{1/n} > 1$ , because if  $d = c^{1/n} \leq 1$ , then  $d^n \leq 1$ , so  $c \leq 1$ . We can write  $c^{1/n} = (1 + d_n)$ , where  $d_n > 0$ .

$$\begin{aligned} c &= c^n \\ &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned} \quad \text{Bernoulli's Inequality}$$

So,  $d_n \leq \frac{c}{n}$ . Remember,  $c^{1/n} = 1 + d_n$ .

$$\begin{aligned} |c^{1/n} - 1| &= c^{1/n} - 1 \\ &= d_n \\ &\leq c \cdot \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $c^{1/n} \rightarrow 1$ .

**Case 3:** Assume  $0 < c < 1$ . Then,  $c^{1/n} < 1$ , so  $\frac{1}{c^{1/n}} > 1$ .

So, we can write  $\frac{1}{c^{1/n}} = (1 + d_n)$ , where  $d_n > 0$ .

$$\begin{aligned} c^{1/n} &= \frac{1}{1 + d_n} \\ 1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\ &= \frac{d_n}{1 + d_n} \\ &\leq d_n \end{aligned}$$

Remember,  $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned} \frac{1}{c} &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned}$$

So,  $d_n \leq \frac{1}{cn}$

$$\begin{aligned} |1 - c^{1/n}| &= 1 - c^{1/n} \\ &\leq d_n \\ &\leq \frac{1}{c} \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $(c^{1/n})_n \rightarrow 1$ .

## Positive Sequence Convergence

**Statement:** Let  $(x_n)_n$  be a sequence with  $x_n \in \mathbb{R}^+ \forall n \in \mathbb{N}$ , with  $(x_n)_n \rightarrow x$ . Then,  $x$  is also positive, and  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ .

**Proof:** Suppose toward contradiction that  $x < 0$ . Let  $\varepsilon = \frac{|0-x|}{2}$ . Since  $(x_n)_n$  converges to  $x$ , we know that  $x_n \in V_\varepsilon(x)$  for large  $n$ . However, every member of  $V_\varepsilon(x) < 0$ , and  $x_n > 0$ .  $\perp$

**Case 1:** If  $x = 0$ , we will show that  $(\sqrt{x_n})_n \rightarrow 0$ .

Let  $\varepsilon > 0$ , find  $N \in \mathbb{N}$  large such that if  $n \geq N$ , we have

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ x_n &< \varepsilon^2 \\ \sqrt{x_n} &< \varepsilon \\ |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

**Case 2:** If  $x > 0$ , we will show that  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ .

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{1}{\sqrt{x}} |x_n - x| \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $|\sqrt{x_n} - \sqrt{x}| \rightarrow 0$ , so  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ .

## $n$ th Root of $n$ Convergence

**Show:**

$$(n^{1/n})_n \rightarrow 1$$

**Proof:** We will make use of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Note that  $n^{1/n} > 1$  for  $n$  past 1. So, we write

$$\begin{aligned} n^{1/n} &= 1 + d_n & d_n > 0 \\ n &= (1 + d_n)^n \\ &= \sum_{k=0}^n \binom{n}{k} d_n^k \\ &= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n \\ &\geq \binom{n}{0} + \binom{n}{2} d_n^2 & \text{as all terms are positive} \\ &= 1 + \frac{n(n-1)}{2} d_n^2 \end{aligned}$$

so

$$\begin{aligned} n-1 &\geq \frac{n(n-1)}{2} d_n^2 \\ \frac{2}{n} &\geq d_n^2 \\ \frac{\sqrt{2}}{\sqrt{n}} &\geq d_n \end{aligned}$$

So, we have

$$\begin{aligned} |n^{1/n} - 1| &= n^{1/n} - 1 \\ &= d_n \\ &\leq \sqrt{2} \frac{1}{\sqrt{n}} \\ &\rightarrow 0 \end{aligned}$$

by previous corollary

So,  $(n^{1/n})_n \rightarrow 1$ .

## Multiplication by Geometric Sequence

**Statement:** Let  $0 \leq b < 1$ . Show that

$$(nb^n)_n \rightarrow 0$$

**Proof:** If  $0 < b < 1$  (the 0 case is trivial). So,  $\frac{1}{b} > 1$ , meaning  $\frac{1}{b} = 1 + d$  for some  $d > 0$ .

$$\begin{aligned} \frac{1}{b^n} &= (1 + d)^n \\ &\geq \frac{n(n-1)}{2} d^2 \\ \frac{2}{d^2(n-1)} &\geq b^n \\ nb^n &\leq \frac{2}{d^2(n-1)} \\ &\rightarrow 0 \end{aligned}$$

by previous corollary

Therefore,  $(nb^n)_n \rightarrow 0$ .

## Boundedness and Convergence

**Statement:** If  $(x_n)_n$  is a convergent sequence,  $x_n$  is bounded. The converse is false in general.

**Proof:** Suppose  $(x_n)_n \rightarrow x$ . Let  $\varepsilon = 1$ .

Then,  $\exists N \in \mathbb{N}$  such that  $x_n \in V_\varepsilon(x)$  for all  $n \geq N$ .

Let  $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$ . If  $n \geq N$ , then  $|x_n| \leq c$ , because  $x_n \in V_\varepsilon(x)$ . If  $n < N$ , then  $|x_n| \leq c$ .

Together, we have  $|x_n| \leq c$  for all  $n$ .

To show the converse is not true, consider  $((-1)^n)_n$ . This sequence is bounded but not convergent.

## Algebraic Operations on Sequences

**Statement:** Let  $(x_n)_n \rightarrow x$ ,  $(y_n)_n \rightarrow y$ , and  $(z_n)_n \rightarrow z$  be convergent sequences. Let  $t \in \mathbb{R}$ . Then, the following are all true:

$$(1) (x_n \pm y_n)_n \rightarrow x \pm y$$

$$(2) (tx_n)_n \rightarrow tx$$

$$(3) (x_n y_n)_n \rightarrow xy$$

$$(4) \text{ Assume } z_n \neq 0 \forall n, \text{ and } z \neq 0. \text{ Then, } \left(\frac{1}{z_n}\right)_n \rightarrow \frac{1}{z}, \text{ and } \left(\frac{x_n}{z_n}\right)_n \rightarrow \frac{x}{z}.$$

**Proof of (1):** Let  $\varepsilon > 0$ . Since  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $\exists N_1, N_2 \in \mathbb{N}$  such that  $n \geq N_1 \rightarrow |x_n - x| < \frac{\varepsilon}{2}$ , and  $n \geq N_2 \rightarrow |y_n - y| < \frac{\varepsilon}{2}$ .

Let  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then  $n \geq N_1$  and  $n \geq N_2$ .

$$\begin{aligned} |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

**Proof of (3):** We have  $(x_n)_n \rightarrow x$  and  $(y_n)_n \rightarrow y$ .

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |x_n - x||y| \end{aligned}$$

Since convergent sequences are bounded,  $\exists c \in \mathbb{R}$  such that  $|x_n| < c$ ,  $\forall n$

$$\begin{aligned} &\leq c|y_n - y| + |x_n - x||y| \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $|x_n y_n - xy| \rightarrow 0$ , so  $x_n y_n \rightarrow xy$ .

**Proof of (4):** We have  $z_n \neq 0$  and  $z \neq 0$ . Let  $\varepsilon > 0$ .

$$\begin{aligned} \left| \frac{1}{z_n} - \frac{1}{z} \right| &= \frac{|z - z_n|}{|z_n z|} \\ &= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|} \end{aligned}$$

Let  $\varepsilon = \frac{|z|}{2}$ . Since  $(z_n)_n \rightarrow z$ , we know that  $z_n \in V_\varepsilon(z)$  for  $n \geq N \in \mathbb{N}$ . For  $n \geq N$ ,  $|z_n| > \frac{|z|}{2}$ , so  $\frac{1}{|z_n|} < \frac{2}{|z|}$ .

$$\begin{aligned} &\leq |z_n - z| \frac{2}{|z|^2} \\ &\rightarrow 0 \end{aligned}$$

So,  $\left| \frac{1}{z_n} - \frac{1}{z} \right| \rightarrow 0$ , so  $\frac{1}{z_n} \rightarrow \frac{1}{z}$

## Ordering of Limits

**Statement:** Let  $(x_n)_n \rightarrow x$  and  $(y_n)_n \rightarrow y$ . If  $x_n \leq y_n$  for all  $n$ , then  $x \leq y$ .

**Proof:** Suppose toward contradiction that  $x > y$ .

Let  $\varepsilon = \frac{x-y}{2}$ .

So,  $\exists N_1 \in \mathbb{N}$  such that  $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$ , and  $\exists N_2 \in \mathbb{N}$  such that  $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$ .

Let  $N = \max\{N_1, N_2\}$ . Then,  $x_N \in V_\varepsilon(x)$  and  $y_N \in V_\varepsilon(y)$ . But that means  $x_N > y_N$ .  $\perp$

In particular, if  $(x_n)_n \rightarrow x$ , and  $a \leq x_n \leq b$ , then  $a \leq x \leq b$ .

## Squeeze Theorem

**Statement:** Let  $(x_n)_n \rightarrow x$ ,  $(y_n)_n \rightarrow y$ , and  $(z_n)_n \rightarrow z$ , where  $x_n \leq y_n \leq z_n$  for all  $n$ .

If  $L = x = z$ , then  $y = L$ .

**Proof:** Let  $\varepsilon > 0$ . Find  $N_1, N_2 \in \mathbb{N}$  such that  $n \geq N_1 \Rightarrow V_\varepsilon(L)$ , and  $n \geq N_2 \Rightarrow V_\varepsilon(L)$ .

Let  $N = \max\{N_1, N_2\}$ . Then,  $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$ . Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so  $y_n \in V_\varepsilon(L)$ , so  $(y_n)_n \rightarrow L$ .

## Squeeze Theorem Examples

For example, let  $a_n = \frac{\sin(n)}{n}$ . Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero,  $a_n \rightarrow 0$

As another example, consider  $a_n = (2^n + 3^n)^{1/n}$ . Then,

$$\begin{aligned} 3^n &\leq 2^n + 3^n \leq 2 \cdot 3^n \\ 3 &\leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3 \end{aligned}$$

Since  $2^{1/n} \rightarrow 1$ , we have  $a_n \rightarrow 3$ .

## Ratio Test

**Statement:** Let  $(x_n)$  be a sequence of strictly positive numbers, with  $\left(\frac{x_{n+1}}{x_n}\right)_n \rightarrow r < 1$ . Then,  $(x_n)_n \rightarrow 0$ .

**Proof:** Since  $r < 1$ , then  $1 - r > 0$ . Let  $\rho = r + \frac{1-r}{2}$ , and  $\varepsilon = \rho - r = \frac{1-r}{2}$ .

Since the sequence converges,  $\exists N \in \mathbb{N}$  such that for  $n \geq N$ ,

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} - r \right| &< \varepsilon \\ \frac{x_{n+1}}{x_n} &< \rho \\ x_{n+1} &< \rho x_n \end{aligned}$$

In particular,  $x_{N+1} < \rho x_N$ , and  $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$ . Inductively, one can show that  $\forall k \geq 1$ ,  $x_{N+k} < \rho^k x_N$ .

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as  $k \rightarrow \infty$ , both sides of the inequality go to 0, implying that  $x_n \rightarrow 0$ .

## Monotone Convergence Theorem

**Proof:** Let  $(x_n)_n$  be a monotone sequence. Then,  $(x_n)_n$  is convergent if and only if it is bounded.

(a) If  $(x_n)_n$  is increasing and bounded above, then  $(x_n)_n \rightarrow \sup(\{x_n \mid n \in \mathbb{N}\})$ .

(b) If  $(x_n)_n$  is decreasing and bounded below, then  $(x_n)_n \rightarrow \inf(\{x_n \mid n \in \mathbb{N}\})$ .

**Proof:** We have already shown that all convergent sequences are bounded.

Assume that  $(x_n)_n$  is monotonic and bounded.

**Case 1:** Suppose  $(x_n)_n$  is increasing. Let  $\sup\{x_n \mid n \in \mathbb{N}\} := u$ . We claim that  $(x_n)_n \rightarrow u$ .

Let  $\varepsilon > 0$ . By the definition of supremum,  $\exists N \in \mathbb{N}$  such that  $u - \varepsilon < x_N$ . Note that  $\forall n \geq N$ ,  $u - \varepsilon < x_N \leq x_n \leq u$ .

Therefore, if  $n \geq N$ , then  $|x_n - u| < \varepsilon$ .

**Case 2:** Suppose  $(x_n)_n$  is decreasing. Let  $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$ . We claim that  $(x_n)_n \rightarrow \ell$ .

Let  $\varepsilon > 0$ . By the definition of infimum,  $\exists N \in \mathbb{N}$  such that  $\ell + \varepsilon > x_N$ . Additionally,  $\forall n \geq N$ ,  $\ell \leq x_n \leq x_N < \ell + \varepsilon$ .

Therefore, if  $n \geq N$ ,  $|x_n - \ell| < \varepsilon$ .

## Applications of the Monotone Convergence Theorem

**Statement:** If  $(x_n)_n$  is a convergent sequence, and  $m \in \mathbb{N}$ , the  $m$ -th tail,  $x_{(m)} = (x_{m+k})_{k=1}^\infty$  is also convergent. If  $(x_n)_n \rightarrow L$  then  $x_{(m)} \rightarrow L$ .

**Proof:** Let  $\varepsilon > 0$ . Find  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |x_n - L| < \varepsilon$ . If  $k \geq N$ , then  $m+k \geq N$ , so  $|x_{m+k} - L| < \varepsilon$ .

Thus,  $(x_{m+k})_k \rightarrow L$

### Monotone Convergence Example 1

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that  $x_n \geq 4 \forall n$ .

$$x_1 = 8 \geq 4$$

Suppose  $x_k \geq 4$ . We will show that  $x_{k+1} \geq 4$ .

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore,  $(x_n)_n$  is bounded below by 4.

We claim that  $(x_n)_n$  is decreasing.  $\forall n \in \mathbb{N}$ ,

$$\begin{aligned} x_{n+1} \leq x_n &\Leftrightarrow \\ \frac{1}{2}x_n + 2 &\leq x_n \\ &\Leftrightarrow 4 \leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that  $(x_n)_n \rightarrow L$ .

To find  $L$ , we use the recursive relationship and the lemma.

$$\begin{aligned} x_{n+1} &= \left( \frac{1}{2}x_n + 2 \right)_{n=1}^{\infty} \\ L &= \frac{1}{2}L + 2 \\ L &= 4 \end{aligned}$$

## Monotone Convergence Example 2

Consider the following sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{4} \\ x_3 &= 1 + \frac{1}{4} + \frac{1}{9} \\ x_k &= \sum_{k=1}^n \frac{1}{k^2} \end{aligned}$$

We will show that  $(x_n)_n$ , the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$\begin{aligned} k^2 &\geq k(k-1) & k \geq 2 \\ \frac{1}{k^2} &\leq \frac{1}{k(k-1)} \\ &= \frac{1}{k-1} - \frac{1}{k} \\ \sum_{k=2}^n \frac{1}{k^2} &\leq \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) \\ \sum_{k=1}^n \frac{1}{k^2} &\leq 1 + \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) \end{aligned}$$



But

$$1 + \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{n}$$

so, we have

$$\sum_{k=1}^n \frac{1}{k^2} \leq 2 - \frac{1}{n} < 2$$

So,  $(x_n)_n$  is bounded above.

## Alternative Proof of the Nested Intervals Theorem

**Statement:** Let  $I_n = [a_n, b_n]$  be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

**Proof:** Since the intervals are nested, it must be the case that  $a_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq b_1$ .

Similarly,  $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$ .

So,  $(a_n)_n$  is an increasing sequence bounded above by  $b_1$  and  $(b_n)_n$  is a decreasing sequence bounded below by  $a_1$ . So,  $(b_n)_n \rightarrow r$  and  $(a_n)_n \rightarrow \ell$ .

Note that  $\ell = \sup(a_n)$  and  $r = \inf(b_n)$ .

Fix  $n \in \mathbb{N}$ , then for any  $m \geq n$ ,  $a_n \leq a_m \leq b_m \leq b_n$ . So,  $\sup(a_m) = \ell \leq b_n$ . Unlocking  $n$ , we get that  $\ell \leq \inf(b_n) = r$ .

## Calculating Square Roots

Let  $a \in \mathbb{R}^+$ . We will construct a sequence  $(x_n)_n \rightarrow \sqrt{a}$ .

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right).$$

We will prove that  $x_n^2 \geq a$ .

$$\begin{aligned} 2x_{n+1} &= x_n + \frac{a}{x_n} \\ 2x_{n+1}x_n &= x_n^2 + a \\ 0 &= x_n^2 - 2x_{n+1}x_n + a \end{aligned}$$

So,  $x_n$  is a real root, meaning

$$\begin{aligned} \Delta &= 4x_{n+1}^2 - 4a \\ x_{n+1}^2 &\geq a \end{aligned} \quad \forall n$$

So,  $\forall n \geq 2$

$$x_n^2 \geq a$$

We will show that  $x_n$  is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \\ &= \frac{1}{2} \underbrace{\left( \frac{x_n^2 - a}{x_n} \right)}_{\geq 0 \quad \forall n \geq 2} \end{aligned}$$

So, we have that  $(x_n)_n$  is decreasing and bounded below, meaning  $(x_n)_n \rightarrow x$  for some  $x \in \mathbb{R}$ .

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \\ x &= \frac{1}{2} \left( x + \frac{a}{x} \right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} \end{aligned}$$

remember that  $x > 0$

## Euler's Number

Consider

$$\begin{aligned} (e_n)_n &= \left( 1 + \frac{1}{n} \right)^n \\ &= \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} \end{aligned}$$

Similarly,

$$\begin{aligned} e_{n+1} &= \sum_{k=0}^{\infty} \left( \frac{1}{k!} \prod_{j=1}^{k-1} \left( 1 - \frac{j}{n+1} \right) \right) \\ e_{n+1} &\geq e_n \end{aligned}$$

$\forall n$

We claim that  $(e_n)_n$  is bounded above.

$$\begin{aligned}
 e_1 &= \left(1 + \frac{1}{1}\right)^1 \\
 2 &\leq e_n \\
 e_n &= \sum_{k=0}^n \left( \frac{1}{k!} \underbrace{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}_{\leq 1} \right) \\
 2^{k-1} &\leq k! & k \geq 2 \\
 \frac{1}{k!} &\leq \frac{1}{2^{k-1}} \\
 e_n &= \sum_{k=0}^n \frac{1}{k!} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right) \\
 &\leq \sum_{k=0}^n \frac{1}{k!} \\
 &\leq 2 + \sum_{\ell=1}^{n-1} \frac{1}{2^\ell} \\
 &< 3
 \end{aligned}$$

so, we have

$$2 \leq e_n \leq 3$$

so, by the monotone convergence theorem,  $(e_n)_n$  converges

$$e := \sup_n \left(1 + \frac{1}{n}\right)^n$$

## Monotone Divergence

A sequence that is increasing and *unbounded* diverges to infinity.

Let  $M > 0$ . Since  $(x_n)_n$  is unbounded,  $\exists N \in \mathbb{N}$  such that  $x_N > M$

Thus, if  $n \geq N$ , then  $x_n \geq x_N > M$ .

## Monotone Divergence Example

Consider

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We can see that  $h_n < h_{n+1}$ . The primary question is as to whether  $(h_n)_n$  is bounded.

$$\begin{aligned}
 h_1 &= 1 \\
 &\geq 1 \\
 h_2 &= 1 + \frac{1}{2} \\
 &\geq 1 + \frac{1}{2} \\
 h_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
 &\geq 1 + \frac{1}{2} + \frac{1}{2} \\
 h_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
 &\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}
 \end{aligned}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^k \frac{1}{2}$$

Let  $M$  be large. Find  $n$  such that  $n > 2(M-1)$ . In this case,  $n/2 + 1 > M$ . Let  $N = 2^n$ . Then, for  $m \geq N$ ,  $h_m > M$ .

Thus,  $(h_n)_n$  diverges to infinity.

## Subsequences and Bolzano-Weierstrass

### Natural Sequences

A **natural sequence** is a strictly increasing sequence of natural numbers,  $(n_k)_{k=1}^{\infty}$

$$n_1 < n_2 < n_3 < \dots$$

where  $\forall k \in \mathbb{N}$ ,  $n_k \in \mathbb{N}$ .

**Statement:** Given  $(n_k)_k$  natural sequence, show that  $(n_k) \geq k$ .

**Proof:**

**Base Case:** We know that  $n_1 \leq 1$ , as  $n_1 \in \mathbb{N}$ .

**Inductive Step:** To be continued...

### Subsequences

Let  $(x_n)_n$  be a sequence. A subsequence  $(x_{n_k})_{k=1}^{\infty}$ , where  $(n_k)_k$  is a natural sequence.

For example, if  $(x_n)_n = (-1)^n$ . If  $(n_k) = 2k$ , then,  $(x_{n_k}) = ((-1)^{2k})_k = (1, 1, 1, \dots)$ . But, if  $(n_k) = 2k + 1$ , then  $(x_{n_k}) = (-1, -1, -1, \dots)$ .

If  $(x_n) = (1/n)_n$ , and  $(n_k)_k = k^2$ , then  $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \dots)$ .

If  $(x_n)_n$  is a sequence, its  $m$ -th **tail** is  $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \dots)$ , where  $n_k = m + k$ .

## Convergence of Subsequences

**Statement:** If  $(x_n)_n \rightarrow x$ , then for any natural sequence  $(n_k)_k$ ,

$$(x_{n_k})_k \rightarrow x$$

**Proof:** Let  $\varepsilon > 0$ . Find  $N \in \mathbb{N}$  large such that  $n \geq N$ ,  $|x_n - x| < \varepsilon$ .

Take  $K = N$ . Then,

$$\begin{aligned} n_k &\geq k \\ &\geq K \\ &= N \\ \Rightarrow |x_{n_k} - x| &< \varepsilon \end{aligned}$$

## Corollary to Convergence of Subsequences

Given a sequence  $(x_n)_n$ , if there are two subsequences  $(x_{n_k})_k \rightarrow x$ ,  $(x_{n_\ell})_\ell \rightarrow x'$ , where  $x \neq x'$ , then  $(x_n)_n$  is divergent.

## Convergence of Subsequences Example

Recall the geometric sequence

$$(b^n)_{n=1}^\infty \rightarrow 0$$

if  $0 < b < 1$ .

The sequence  $(1, b, b^2, \dots)$  is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem,  $(b^n)_n \rightarrow \ell$ .

Given  $n = 2k$ , we know that  $(b^{2k})_k \rightarrow \ell$ .

$$\begin{aligned} b^{2k} &= (b^k)^2 \\ (b^k)^2 &\rightarrow \ell^2 \\ \ell^2 &= \ell \\ \ell &= \{0, 1\} \end{aligned}$$

since  $b < 1$

$$\ell = 0$$

## Divergence and Subsequences

If  $(x_n)_n \not\rightarrow x$ , then

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) (\exists n \geq N) \text{ such that } |x_n - x| \geq \varepsilon_0$$

We can use this to construct a sequence to show divergence.

**Statement:** Let  $(x_n)_n$  be a sequence, and  $x \in \mathbb{R}$ .

$$\begin{aligned} (x_n)_n &\not\rightarrow x \\ \Leftrightarrow \\ (\exists \varepsilon_0 > 0) &(\exists (x_{n_k})_k) \end{aligned}$$

with

$$|x_{n_k} - x| \geq \varepsilon_0$$

**Proof:**

( $\Rightarrow$ ) We know  $\exists \varepsilon_0 > 0$  as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \varepsilon_0$$

$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \varepsilon_0$$

$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \varepsilon_0$$

Assume we have  $n_1 < n_2 < \dots, n_k$  with

$$|x_{n_j} - x| \geq \varepsilon_0$$

$$j = 1, 2, \dots, k$$

$$N = n_k + 1 \Rightarrow n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \varepsilon_0$$

Iteratively, we have our desired subsequence  $(x_{n_k})_k$ .

( $\Leftarrow$ ) If  $(x_n)_n \rightarrow x$ , any subsequence converges to  $x$ .

By assumption,  $(\exists \varepsilon_0 > 0) (\exists (n_k)_k)$  with  $|x_{n_k} - x| \geq \varepsilon_0$ . Thus,  $(x_{n_k})_k \not\rightarrow x$ .

## Bolzano-Weierstrass Theorem

**Statement:** If  $(x_n)_n$  is a bounded sequence, then  $(x_n)_n$  admits a convergent subsequence.

**Proof:**

**Lemma:** Let  $(x_n)_n$  be any real sequence. Then,  $\exists n_k$  such that  $(x_{n_k})_k$  is monotone.

**Proof of Lemma:** A **peak** of a sequence  $(x_n)_n$  is an  $x_m$  such that  $x_m \geq x_n \forall n \geq m$ .

**Case 1:** There are infinitely many peaks,  $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ , where  $n_1 < n_2 < \dots$ .

Then,  $(x_{n_k})_k$  is decreasing.

**Case 2:** There are finitely many peaks. Let these peaks be  $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ .

Let  $n_1 = m_r + 1$ . Since  $x_{n_1}$  is not a peak,  $\exists n_2 > n_1$  such that  $x_{n_2} > x_{n_1}$ . Since  $x_{n_2}$  is not a peak,  $\exists n_3 > n_2$  such that  $x_{n_3} > x_{n_2}$ .

Iteratively, we have an increasing sequence of non-peaks  $(x_{n_k})_k$ .

Since  $(x_n)_n$  admits a monotone subsequence, and  $(x_{n_k})_k$  is bounded as  $(x_n)_n$  is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

## Limit Superior and Limit Inferior

### Limit Points

Let  $X = (x_n)_n$  be a bounded real sequence. By Bolzano-Weierstrass,  $(x_n)_n$  admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R}, t = \lim_{k \rightarrow \infty} x_{n_k} \right\} \quad \text{for any subsequence } (x_{n_k})_k$$

Then,  $t \in \overline{X}$  is called a **limit point** of  $X$ .

### Finding the Limit Points

Let  $u_1 = \sup_{n \geq 1} (x_n)$ ,  $\ell_1 = \inf_{n \geq 1} (x_n)$ . Clearly,  $\ell_1 \leq u_1$ , and  $\overline{X} \subseteq [\ell_1, u_1]$ .

Let  $u_2 = \sup_{n \geq 2} (x_n)$  and  $\ell_2 = \inf_{n \geq 2} (x_n)$ .

Since  $u_1$  is an upper bound for  $(x_n)_n$ , it is an upper bound for  $(x_n)_{n \geq 2}$ , so  $u_2 \leq u_1$ . Similarly, since  $\ell_1$  is a lower bound for  $(x_n)_n$ , it is a lower bound for  $(x_n)_{n \geq 2}$ , so  $\ell_2 \geq \ell_1$ .

As a result, we can see that  $\overline{X} \subseteq [\ell_2, u_2]$ .

We continue, letting  $u_m = \sup_{n \geq m} (x_n)$ , and  $\ell_m = \inf_{n \geq m} (x_n)$ . We get  $\ell_1 \leq \ell_2 \leq \dots$ , and  $u_1 \geq u_2 \geq \dots$ , and  $\overline{X} \subseteq [\ell_m, u_m]$ ,  $\forall m$ .

We get a nested sequence of intervals  $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \dots$ . By the Nested Intervals Theorem, we know that

$$\begin{aligned} \overline{X} &\subseteq \bigcap_{m \geq 1} [\ell_m, u_m] \\ &= [\ell, u] \end{aligned}$$

where  $\ell = \sup(\ell_m)$  and  $u = \inf(u_m)$ .

### Defining Limit Superior and Limit Inferior

Given a bounded sequence  $(x_n)_n = X$ ,

$$\begin{aligned} u &= \inf_{m \geq 1} (u_m) \\ &= \inf_{m \geq 1} \left( \sup_{n \geq m} x_n \right) \end{aligned}$$

called the **limit superior** of  $(x_n)_n$

$$u = \limsup_{n \rightarrow \infty} x_n$$

and

$$\begin{aligned} \ell &= \sup_{m \geq 1} (\ell_m) \\ &= \sup_{m \geq 1} \left( \inf_{n \geq m} (x_n) \right) \end{aligned}$$

called the **limit inferior** of  $(x_n)_n$

$$\ell = \liminf_{n \rightarrow \infty} x_n$$

## Fundamental Results in Limit Superior and Limit Inferior

**Statement:** Let  $(x_n)_n$  be bounded. Then,

- (1)  $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$
- (2)  $(x_n)_n \rightarrow x \Leftrightarrow \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$

**Proof of (1):** This was proven with the Nested Intervals Theorem

**Proof of (2):** Let  $\varepsilon > 0$ . Then,  $\exists N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$ .

We know that  $u_m = \sup_{n \geq m} x_n$ . If  $m \geq N$ , then  $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$ . Therefore,  $|u_m - x| \leq \varepsilon/2 < \varepsilon$ , so  $(u_m)_m \rightarrow x = \limsup_{n \rightarrow \infty} x_n$ .

Similarly, we know that  $\ell_m = \inf_{n \geq m} x_n$ . If  $m \geq N$ , then  $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$ . So,  $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$ , so  $(\ell_m)_m \rightarrow x = \liminf_{n \rightarrow \infty} x_n$ .

## Applying Limit Superior and Limit Inferior

Consider the sequence

$$x_n = \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases} \\ = (-1, 5/2, -1/3, 9/4, -1/5, \dots)$$

We begin by constructing the  $u_m$  sequence:  $(5/2, 5/2, 9/4, 9/4, \dots)$ . We can see that  $u_m \rightarrow 2$ .

Then, we construct the  $\ell_m$  sequence:  $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$ . We can see that  $\ell_m \rightarrow 0$ .

**Exercise:** If  $(x_n)_n$  and  $(y_n)_n$  are sequences with  $x_n \leq y_n \forall n$ , then  $\limsup x_n \leq \limsup y_n$  and  $\liminf x_n \leq \liminf y_n$ .

## Ratio Test and Root Test: Equivalent Convergence

**Statement:** If  $(a_n)_n$  is a sequence of strictly positive terms such that

$$\left( \frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$$

then,

$$\left( a_n^{1/n} \right)_{n=1}^{\infty} \rightarrow \rho$$



**Proof:** Let  $\varepsilon > 0$ . Then,  $\exists N$  large such that  $\forall n \geq N$ ,

$$\begin{aligned}
 \left| \frac{a_{n+1}}{a_n} - \rho \right| &< \varepsilon & \forall n \geq N \\
 \Rightarrow \frac{a_{n+1}}{a_n} &< \rho + \varepsilon & \forall n \geq N \\
 a_{n+1} &< a_n(\rho + \varepsilon) & \forall n \geq N \\
 a_n &< a_N(\rho + \varepsilon)^{n-N} & \forall n \geq N \\
 a_n &< (\rho + \varepsilon)^n \cdot \frac{a_N}{(\rho + \varepsilon)^N} \\
 a_n^{1/n} &< (\rho + \varepsilon) \left( \frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\
 \limsup a_n^{1/n} &\leq \limsup (\rho + \varepsilon) \left( \frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\
 \limsup_{n \rightarrow \infty} a_n^{1/n} &\leq \rho + \varepsilon
 \end{aligned}$$

**Case 1:** If  $\rho = 0$ , the case is trivial.

**Case 2:** Suppose  $\rho > 0$ . Find  $\varepsilon > 0$  small such that  $0 < \varepsilon < \rho$ .

Since  $\left( \frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$ , find  $N$  large such that  $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$ . So,  $\forall n \geq N$ ,

$$\begin{aligned}
 a_{n+1} &\geq a_n(\rho - \varepsilon) \\
 a_n &\geq a_N(\rho - \varepsilon)^{n-N} \\
 a_n^{1/n} &\geq (\rho - \varepsilon) \left( \frac{a_N}{(\rho - \varepsilon)^N} \right)^{1/n} \\
 \liminf a_n^{1/n} &\geq \rho - \varepsilon
 \end{aligned}$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together,  $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$ , so  $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$ , whence  $\left( a_n^{1/n} \right) \rightarrow \rho$

## Properties of $\overline{X}$

**Statement:** We found earlier that  $\overline{X} \subseteq [\ell, u]$ . We claim that

$$\begin{aligned}
 \sup \overline{X} &= u \\
 \sup \overline{X} &= \ell
 \end{aligned}$$

**Proof:** We have shown that  $u$  is an upper bound for  $\overline{X}$ . The goal is to show that  $u$  is the least upper bound.

Let  $\varepsilon > 0$ . We need to find a  $t \in \overline{X}$  with  $u - \varepsilon < t$ . Note that  $u - \varepsilon < u_m \forall m$ .

We know that  $u - \varepsilon < u_1$ . Since  $u_1 = \sup_{n \geq 1} x_n$ , we know  $\exists n_1 \in \mathbb{N}$  with  $u - \varepsilon < x_{n_1} < u_1$ .

Consider  $u_{n_1+1} = \sup_{n > n_1} x_n$ . We know that  $u - \varepsilon < u_{n_1+1}$ . Therefore,  $\exists x_{n_2}$  with  $n_2 > n_1$  and  $u - \varepsilon < x_{n_2} < u_{n_1+1}$ .

Then, we use  $u_{n_2+1}$ . Then,  $\exists n_3 > n_2$  with  $u - \varepsilon < x_{n_3} < u_{n_2+1}$ .

We get a subsequence from the natural sequence  $n_1, n_2, \dots$ , where  $u - \varepsilon < x_{n_k} \forall k$ .

Also,  $x_{n_k} < u_1 \forall k$ . Therefore,  $(x_{n_k})_k$  is a bounded sequence. By Bolzano-Weierstrass,  $\exists$  a convergent subsequence

$$(x_{n_{k_j}})_j \rightarrow t$$

We know that  $u - \varepsilon \leq t$ . Note that  $t \in \overline{X}$ .

**Exercise:** Show that  $\inf \overline{X} = \ell$ .

## Cauchy and Contractive Sequences

### Cauchy Sequences

A sequence  $(x_n)_n$  in a metric space  $(X, d)$  is Cauchy if

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \text{ such that } p, q \geq N \Rightarrow d(x_p, x_q) < \varepsilon$$

if  $(X, d) = (\mathbb{R}, |\cdot|)$ :

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \text{ such that } p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon$$

Consider the sequence  $(x_n)_n = \frac{1}{n}$ . Then,

$$\begin{aligned} |x_p - x_q| &= \left| \frac{1}{p} - \frac{1}{q} \right| \\ &= \frac{1}{q} - \frac{1}{p} \\ &\leq \frac{1}{q} \end{aligned}$$

Given  $\varepsilon > 0$ , find  $N$  large such that  $\frac{1}{N} < \varepsilon$ . Then,  $p, q \geq N$  implies

$$\begin{aligned} \left| \frac{1}{p} - \frac{1}{q} \right| &< \frac{1}{q} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

To show that any sequence is not Cauchy, we use the following negation of the definition:

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) \text{ such that } p, q \geq N \Rightarrow |x_p - x_q| \geq \varepsilon_0$$

### Boundedness of Cauchy Sequences

**Statement:** Cauchy sequences are bounded.

**Proof:** Let  $\varepsilon = 1$ . Then, by the Cauchy criterion,  $\exists N \in \mathbb{N}$  such that  $p, q \geq N \Rightarrow |x_p - x_q| < 1$ .

In particular,  $\forall n \geq N$ ,

$$\begin{aligned} |x_n| &= |x_n - x_N + x_N| \\ &\leq |x_n - x_N| + |x_N| \\ &< 1 + |x_N| \end{aligned}$$

Triangle Inequality

Let  $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$ . Then,  $x_n \leq c \forall n \geq 1$ . Thus,  $x_n$  is bounded.

## Convergent Subsequences and Cauchy Sequences

**Statement:** If  $(x_n)_n$  is Cauchy and  $(x_n)_n$  admits a convergent subsequence, then  $(x_n)_n$  is convergent.

**Proof:** Say  $(x_{n_k})_k \rightarrow x$  for some natural sequence  $(n_k)_k$ . We claim that  $(x_n)_n \rightarrow x$ .

Let  $\varepsilon > 0$ . Since  $(x_n)_n$  is Cauchy,  $\exists N \in \mathbb{N}$  such that  $p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon/2$ .

Also, since  $(x_{n_k})_k \rightarrow x$ , then  $\exists K \in \mathbb{N}$  and  $K \geq N$  with  $k \geq K \Rightarrow |x_{n_k} - x| < \varepsilon/2$ .

For all  $k \geq K$ ,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \end{aligned}$$

Let  $N_1 = \max\{N, K\}$ . Then,

$$\begin{aligned} n \geq N_1 &\Rightarrow n \geq N && \text{by max} \\ &\Rightarrow n_k \geq k \geq K \geq N && \text{def. of natural sequence} \\ |x_n - x| &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

## Cauchy Sequence Convergence in the Reals

**Statement:** Let  $(x_n)_n$  be any sequence in  $\mathbb{R}$ . The following are equivalent:

- (1)  $(x_n)_n$  converges.
- (2)  $(x_n)_n$  is Cauchy.

**Proof:**

(1)  $\Rightarrow$  (2) (Holds in any metric space). Suppose  $(x_n)_n \rightarrow x$ . Find  $N$  large such that  $n \geq N \rightarrow d(x_n, x) < \varepsilon/2$ .

Then,  $p, q \geq N \Rightarrow$

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

(2)  $\Rightarrow$  (1) If  $(x_n)_n$  is Cauchy, then  $(x_n)_n$  converges.

By Bolzano-Weierstrass,  $(x_n)_n$  admits a convergent subsequence, so by our previous lemma,  $(x_n)_n$  must converge.

**Note:** To show (2)  $\Rightarrow$  (1), we used Bolzano-Weierstrass, which requires the monotone convergence theorem, which itself requires the completeness axiom. This is why we cannot show (2)  $\Rightarrow$  (1) converges.

## Complete Metric Spaces

A metric space  $(X, d)$  is **complete** if every Cauchy sequence converges.

**Remark:** All convergent sequences are Cauchy, and all Cauchy sequences are bounded. We showed that  $\mathbb{R}$  under the absolute value metric is complete.

$\mathbb{Q}$  under  $d(s, t) = |s - t|$  is not complete; similarly,  $A = (0, 1)$  under the metric inherited from  $\mathbb{R}$  is not complete;  $x_n = \frac{1}{n}$  is Cauchy but not convergent in  $A$ .

## Finding Cauchy Sequences and Convergence in $\mathbb{R}$

### Cauchy Sequences and Convergence 1

Consider the harmonic sequence

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We claim that  $h_n$  is not convergent.

Let  $p > q$ . Then,

$$\begin{aligned} |h_p - h_q| &= \left| \sum_{k=1}^p \frac{1}{k} - \sum_{k=1}^q \frac{1}{k} \right| \\ &= \frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p} \\ &\geq \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} \\ &= \frac{p-q}{p} \\ &= 1 - \frac{q}{p} \end{aligned}$$

set  $p = 2q$ :

$$\begin{aligned} |h_{2q} - h_q| &\geq 1 - \frac{q}{2q} \\ &= 1/2 \end{aligned}$$

Therefore,  $h_n$  is not Cauchy, and thus not convergent.

### Cauchy Sequences and Convergence 2

Consider a sequence of partial sums

$$x_n = \sum_{k=0}^n \frac{(-1)^k}{k!}$$

We claim that  $(x_n)_n$  is Cauchy, and thus convergent. Let  $p > q$ . Then, we have

$$\begin{aligned} |x_p - x_q| &= \left| \sum_{k=q+1}^p \frac{(-1)^k}{k!} \right| \\ &\leq \sum_{k=q+1}^p \frac{1}{k!} \\ &\leq \sum_{k=q+1}^p \frac{1}{2^{k-1}} \\ &= \frac{1}{2^q} + \frac{1}{2^{q+1}} + \cdots + \frac{1}{2^{p-1}} \\ &= \frac{1}{2^q} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-q-1}} \right) \\ &\leq \frac{1}{2^{q-1}} \end{aligned}$$

Given  $\varepsilon > 0$ , choose  $N$  large such that  $\frac{1}{2^{N-1}} < \varepsilon$ . When  $p > q > N$ , then  $|x_p - x_q| \leq \frac{1}{2^{q-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$ .

Thus, the sequence is convergent.

## Contractive Sequences

A sequence  $(x_n)_n$  in a metric space  $(X, d)$  is **contractive** if

$$\exists 0 < \rho < 1 \text{ such that } d(x_{n+1}, x_n) \leq \rho d(x_n, x_{n-1}) \quad \forall n \geq 1$$

In  $\mathbb{R}$ , the definition is

$$|x_{n+1} - x_n| \leq \rho |x_n - x_{n-1}|$$

## Contractive and Cauchy

We claim that every contractive sequence is Cauchy.

From examination, we arrive at the following:

$$|x_n - x_{n-1}| \leq \rho^{n-2} |x_2 - x_1| \quad (*)$$

If  $p > q$ , then

$$\begin{aligned} |x_p - x_q| &= |x_p - x_{p-1} + x_{p-1} - x_{p-2} + \cdots + x_{q+1} - x_q| \\ &\leq |x_p - x_{p-1}| + \cdots + |x_{q+1} - x_q| && \text{Triangle Inequality} \\ &\leq |x_2 - x_1| (\rho^{p-2} + \rho^{p-3} + \cdots + \rho^{q-1}) \\ &= |x_2 - x_1| \rho^{q-1} (1 + \rho + \rho^2 + \cdots + \rho^{p-q-1}) \\ &= |x_2 - x_1| \rho^{q-1} \frac{1 - \rho^{p-q}}{1 - \rho} && \text{Finite Geometric Sequence} \\ &\leq |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} \end{aligned}$$

Given  $\varepsilon > 0$ , we can find  $N$  large such that

$$q \geq N \Rightarrow |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} < \varepsilon$$

Thus,  $p > q \geq N \Rightarrow |x_p - x_q| < \varepsilon$ .

## Applying Contractive Sequences 1

Consider  $(f_n)_n$  defined as follows:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_{n+1} &= f_n + f_{n-1} \end{aligned}$$

Consider  $x_n$  defined as follows:

$$x_n = \frac{f_{n+1}}{f_n}$$

We can rewrite  $x_n$  as:

$$\begin{aligned} x_n &= \frac{f_n + f_{n-1}}{f_n} \\ &= 1 + \frac{f_{n-1}}{f_n} \\ &= 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \\ &= 1 + \frac{1}{x_{n-1}} \end{aligned}$$

We claim that  $3/2 \leq x_n \leq 2 \forall n \geq 2$ .

$$x_2 = 2$$

Inductive Hypothesis: suppose  $3/2 \leq x_n \leq 2$

$$\begin{aligned} &: \frac{3}{2} \leq x_n \leq 2 \\ &\frac{2}{3} \geq \frac{1}{x_n} \geq \frac{3}{2} \\ 2 \geq \frac{5}{3} &\geq 1 + \frac{1}{x_n} \geq \frac{3}{2} \end{aligned}$$

We now claim that  $(x_n)_n$  is contractive.

$$\begin{aligned} |x_{n+1} - x_n| &= \left| \left(1 + \frac{1}{x_n}\right) - \left(1 + \frac{1}{x_{n-1}}\right) \right| \\ &= \left| \frac{1}{x_n} - \frac{1}{x_{n-1}} \right| \\ &= \left| \frac{x_{n-1} - x_n}{x_{n-1}x_n} \right| \\ &\leq \frac{4}{9} |x_n - x_{n-1}| \end{aligned}$$

Therefore,  $\rho = \frac{4}{9}$  is our constant of contraction. Thus,  $(x_n)_n$  is Cauchy, so it converges in  $\mathbb{R}$ .

$$\begin{aligned} x_{n+1} &= 1 + \frac{1}{x_n} & (n \rightarrow \infty, x_n \rightarrow \varphi) \\ \varphi &= 1 + \frac{1}{\varphi} \\ \varphi^2 - \varphi - 1 &= 0 \\ \varphi &= \frac{1 + \sqrt{5}}{2} \end{aligned}$$

## Applying Contractive Sequences 2

Let  $x_1 = 0$ ,  $x_2 = 1$ , and

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_n + x_{n-1}) \\ (x_n)_n &= (0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, \dots) \end{aligned}$$

While the sequence is not monotone, we can show that the sequence is contractive.

$$\begin{aligned} |x_{n+1} - x_n| &= \left| \frac{1}{2} (x_n + x_{n-1}) - x_n \right| \\ &= \left| \frac{1}{2} (x_{n-1} - x_n) \right| \\ &= \frac{1}{2} |x_n - x_{n-1}| \end{aligned}$$

Since the constant of contraction is equal to  $1/2$ , this sequence is Cauchy, and thus converges in the real numbers.

Since  $(x_n)_n \rightarrow x$ , every subsequence converges to  $x$ . Therefore,  $(x_{2k+1})_k \rightarrow x$ .

$$\begin{aligned} x_{2k+1} &= \sum_{j=1}^k \frac{1}{2^{2j-1}} \\ &= 2 \sum_{j=1}^k \frac{1}{4^j} \\ &= 2 \frac{1 - \frac{1}{4^{k+1}}}{1 - \frac{1}{4}} \\ &= \frac{2}{3} \end{aligned} \quad k \rightarrow \infty$$

## Sequence Divergence

### Properly Divergent Sequences

Let  $(x_n)_n$  be a real sequence.  $(x_n)_n$  properly diverges to  $+\infty$  if

$$(\forall \alpha > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow x_n \geq \alpha$$

We write that  $(x_n)_n \rightarrow +\infty$ . Similarly,  $(x_n)_n$  properly diverges to  $-\infty$  if

$$(\forall \beta < 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow x_n \leq \beta$$

and  $(x_n)_n \rightarrow -\infty$ . We say that  $(x_n)_n$  is properly divergent if  $(x_n)_n \rightarrow \pm\infty$ .

If  $(x_n)_n$  and  $(y_n)_n$  are sequences such that  $x_n \geq y_n \forall n$ , and  $(y_n)_n \rightarrow +\infty$ , then  $(x_n)_n \rightarrow +\infty$ .

### Divergence of the Geometric Sequence

In the geometric sequence, if  $b > 1$ , we can show that  $(b^n)_n \rightarrow +\infty$ .

Write  $b = 1 + a$  for some  $a > 0$ . Then, by Bernoulli's inequality, we have

$$\begin{aligned} b^n &= (1 + a)^n \\ &\geq 1 + na \\ &\geq na \end{aligned}$$

Given any  $\alpha > 0$ , find  $N$  large such that  $N > \frac{\alpha}{a}$ , which is always possible by the Archimedean property. Then, for  $Na \geq \alpha$ . If  $n \geq N$ , then  $na \geq Na > \alpha$ .

Since  $b^n > na$ , we have that  $(b^n)_n \rightarrow +\infty$ .

## Monotone Divergence

By the Monotone Convergence Theorem, we have that if  $(x_n)_n$  is monotone, then

$$(x_n)_n \rightarrow x \Leftrightarrow (x_n)_n \text{ bounded}$$

Negating, we have that if  $(x_n)_n$  is monotone, then

$$(x_n)_n \text{ divergent} \Leftrightarrow (x_n)_n \text{ unbounded}$$

However, we can make this statement stronger.

**Statement:** Let  $(x_n)_n$  be monotone.  $(x_n)_n$  is unbounded if and only if  $(x_n)_n$  is properly divergent.

**Proof:**

( $\Leftarrow$ ) If  $(x_n)_n$  is properly divergent, then  $(x_n)_n$  is divergent, and thus unbounded.

( $\Rightarrow$ ) Let  $(x_n)_n$  be unbounded and increasing. Then, given  $\alpha > 0$ ,  $\exists n_\alpha$  with  $x_{n_\alpha} > \alpha$ . If  $n \geq n_\alpha$ , then  $x_n \geq x_{n_\alpha} > \alpha$ , so  $(x_n)_n$  is properly divergent to  $+\infty$ .

## Sequence Comparison Test

Let  $(x_n)_n$  and  $(y_n)_n$  be sequences with  $x_n > 0$  and  $y_n > 0$ . Suppose that

$$\left(\frac{x_n}{y_n}\right)_n \rightarrow L > 0$$

Then,  $(x_n)_n \rightarrow +\infty \Leftrightarrow (y_n)_n \rightarrow \infty$ .

Let  $\varepsilon = L/2$ . Since

$$\left(\frac{x_n}{y_n}\right)_n \rightarrow L,$$

$\exists N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\begin{aligned} \frac{L}{2} &\leq \frac{x_n}{y_n} \leq \frac{3L}{2} \\ \frac{L}{2}y_n &\leq x_n \\ \frac{2}{3L}x_n &\leq y_n \end{aligned}$$

If  $(y_n)_n \rightarrow \infty$ , then so too does  $(L/2)(y_n)$ , so  $(x_n)_n \rightarrow \infty$ . Similarly, if  $(x_n)_n \rightarrow \infty$ , then so too does  $(2/3L)x_n$ , so  $(y_n)_n \rightarrow \infty$ .

## Applying the Sequence Comparison Test

**Problem:** Show that

$$\left(\sqrt{4n^2 - 3n + 1}\right)_n \rightarrow +\infty$$

**Solution:** We will compare to  $y_n = n$ . Then

$$\begin{aligned} \frac{x_n}{y_n} &= \frac{\sqrt{4n^2 - 3n + 1}}{n} \\ &= \sqrt{4 - \frac{3}{n} + \frac{1}{n^2}} \\ &\rightarrow 2 \geq 0 \end{aligned}$$

Since  $y_n$  is properly divergent to  $+\infty$ , so too is  $x_n$ .



## Series Convergence and Divergence

### Introduction to Infinite Series

An **infinite series** is a sequence of partial sums  $s_n$ , where  $s_n$  is formed from  $x_k$  as follows:

$$s_n = \sum_{k=1}^n x_k$$

alternatively,

$$\begin{aligned} s_1 &= x_1 \\ s_n &= s_{n-1} + x_n \end{aligned}$$

The limit of the sequence  $(s_n)_n$  is the value of

$$\sum_{n=1}^{\infty} x_n$$

The infinite series converges to  $s$  if  $(s_n)_n \rightarrow s$ .

If  $(s_n)_n$  diverges, then so too does the series. If  $(s_n)_n$  is properly divergent to  $\pm\infty$ , then we write that the series is equal to  $\pm\infty$ .

### Convergence of a Series of Positive Terms

**Statement:** Let  $(x_k)_k$  be a sequence of positive terms. The following are equivalent:

- (a)  $\sum x_k$  converges.
- (b) The sequence of partial sums  $(s_n)_n$  is bounded above.
- (c) A subsequence of the sequence of partial sums  $(s_{n_j})_j$  is bounded above.

**Proof:**

(1)  $\Rightarrow$  (2):  $\sum x_k$  is convergent  $\Rightarrow (s_n)_n$  is convergent  $\Rightarrow (s_n)_n$  is bounded.

(2)  $\Rightarrow$  (3): If  $(s_n)_n$  is bounded, so is any subsequence  $(s_{n_j})_j$ .

(3)  $\Rightarrow$  (2): Suppose  $s_{n_j} \leq c$ . If  $m$  is arbitrary,  $\exists j$  such that  $n_j \geq m$ . Take  $j = m$ . Then,  $s_m \leq s_{n_j} \leq c$ . Therefore,  $(s_n)_n$  is bounded above.

(2)  $\Rightarrow$  (1) Let  $(s_n)_n$  be bounded above. We know that  $(s_n)_n$  is increasing as  $x_k \geq 0$ . By the Monotone Convergence theorem,  $(s_n)_n$  converges, meaning  $\sum x_k$  converges.

### Corollary to Convergence of a Series of Positive Terms

Let  $(x_k)_k$  be a sequence with  $x_k \geq 0$ . Then,

$$\sum x_k \text{ properly diverges} \Leftrightarrow (s_n)_n \text{ is unbounded}$$

### Applying Convergence of a Series of Positive Terms 1

Recall that for  $x_k = 1/k$ , we proved that  $(s_n)_n$  is unbounded, and also that  $(s_n)_n$  is not Cauchy, meaning  $\sum_{k=1}^{\infty} 1/k$  is properly divergent.

### Applying Convergence of a Series of Positive Terms 2

Additionally, we saw that for  $x_k = 1/k^2$ ,  $(s_n)_n$  is increasing and bounded above.

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^n \frac{1}{k} - \frac{1}{k-1} \\ &= 2 - \frac{1}{n} \end{aligned}$$

### Applying Convergence of a Series of Positive Terms 3

Let  $b \in \mathbb{R}$ . Let  $x_k = b^k$ . Then, we have

$$\begin{aligned} s_n &= \sum_{k=0}^n b^k \\ &= \frac{1 - b^{n+1}}{1 - b} \end{aligned} \quad b \neq 1$$

Therefore, we know the end behavior of the series:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{1 - b^{n+1}}{1 - b} \\ &= \frac{1}{1 - b} \left( 1 - b \lim_{n \rightarrow \infty} b^n \right) \\ &= \begin{cases} \frac{1}{1-b} & |b| < 1 \\ \text{diverges} & |b| > 1 \end{cases} \end{aligned}$$

### Series Comparison Test

**Statement:** Let  $0 \leq x_k \leq y_k$ .

- If  $\sum y_k$  converges, then so too does  $\sum x_k$
- If  $\sum x_k$  diverges, then so too does  $\sum y_k$ .

**Proof:**

( $\Rightarrow$ ) If  $\sum y_k$  converges, then  $t_n = \sum_{k=1}^n y_k$  is bounded.

Setting  $s_n = \sum_{k=1}^n x_k$ , we see that  $0 \leq s_n \leq t_n$ . Seeing as  $t_n$  is bounded, so too is  $s_n$ . Therefore,  $\sum x_k$  is convergent.

For example, consider the series

$$\sum \frac{1}{k^2 + k}$$

Since  $\frac{1}{k^2} \geq \frac{1}{k^2 + k}$ , we know that, seeing as  $\frac{1}{k^2}$  converges, so does  $\frac{1}{k^2 + k}$ .

## Limit Comparison Test

**Statement:** Let  $x_k$  and  $y_k$  be strictly positive sequences. Suppose that

$$\lim_{k \rightarrow \infty} \frac{x_k}{y_k} = L$$

(a) If  $L > 0$ , then  $\sum x_k$  converges if and only if  $\sum y_k$  converges.

(b) If  $L = 0$ , then  $\sum y_k$  converges  $\Rightarrow \sum x_k$  converges.

**Proof:**

(a) Since

$$\frac{x_k}{y_k} \rightarrow L$$

Set  $\varepsilon = L$ . We know  $\exists K$  such that  $k \geq K \Rightarrow y_k \leq \frac{2}{L}x_k$ . Let  $s_n = \sum_{k=1}^n x_k$  and  $t_n = \sum_{k=1}^n y_k$ . Then,

$$\begin{aligned} t_n &= \sum_{k=1}^{K-1} y_k + \sum_{k=K}^n y_k \\ &\leq t_{K-1} + \frac{2}{L} \sum_{k=K}^n x_k \\ &\leq t_{K-1} + \frac{2}{L} s_n \\ &\leq t_{K-1} + c, \end{aligned}$$

implying that  $t_n$  is bounded, so  $\sum y_k$  converges.

(b) Since

$$\frac{x_k}{y_k} \rightarrow 0,$$

$\exists K$  such that  $\frac{x_k}{y_k} \leq 1 \forall k \geq K$ , meaning  $x_k < y_k \forall k \geq K$ .

Letting  $s_n = \sum_{k=1}^n x_k$  and  $t_n = \sum_{k=1}^n y_k$ . Thus,

$$\begin{aligned} s_n &= \sum_{k=1}^{K-1} x_k + \sum_{k=K}^n x_k \\ &= s_{K-1} + \sum_{k=K}^n y_k \\ &\leq s_{K-1} + t_n \\ &\leq s_{K-1} + c \end{aligned}$$

Thus,  $s_n$  is bounded, meaning  $\sum x_k$  is convergent.

## Applying the Limit Comparison Test

Consider

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}}$$

Letting  $x_n = \frac{1}{\sqrt{n^2-1}}$ , and  $y_n = \frac{1}{n}$ , we have

$$\begin{aligned}\frac{x_n}{y_n} &= \frac{n}{\sqrt{n^2-1}} \\ &\rightarrow 1 > 0\end{aligned}$$

Since  $\sum y_n$  diverges, so too does  $\sum x_n$ .

## **$n$ th Term Divergence Test**

If  $\sum x_k$  is convergent, then  $(x_k)_k \rightarrow 0$ . Conversely, if  $(x_k)_k \not\rightarrow 0$ , then  $\sum x_k$  diverges. Recall that  $s_n = s_{n-1} + x_n$ . If  $\sum x_k$  converges, then  $(x_n)_n \rightarrow 0$ . So,

$$\begin{aligned}x_n &= s_n - s_{n-1} \\ (s_n)_n &\rightarrow s \\ x_n &\rightarrow s - s \\ &= 0\end{aligned}$$

For example, we can find that

$$\sum_{k=1}^{\infty} \frac{1}{\arctan k}$$

diverges, as  $\lim_{k \rightarrow \infty} \frac{1}{\arctan k} = \frac{2}{\pi} \neq 0$

## **Cauchy Condensation Test**

**Statement:** Let  $(x_k)_k$  be a decreasing sequence of positive numbers. Then,

$$\sum_k x^k \text{ converges} \Leftrightarrow \sum_k 2^k x_{2^k} \text{ converges}$$

**Proof:** Look at the partial sum  $s_{2^n}$ ,

$$\begin{aligned}s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\leq x_1 + 2x_2 + 4x_4 + \cdots + 2^{n-1}x_{2^{n-1}} + x_{2^n} \\ &= \sum_{k=1}^{n-1} 2^k x_{2^k} + x_{2^n}\end{aligned}$$

If  $\sum_k 2^k x_{2^k}$  converges, then its partial sums are bounded, and we have that  $x_{2^n} \rightarrow 0$ . Then,  $s_{2^n}$  is bounded, and thus  $\sum x_k$  converges.

$$\begin{aligned}2s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\quad + x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &= (x_1 + x_1) + (x_2 + x_2) + (x_3 + x_3 + x_4 + x_4) + \cdots + (x_{2^{n-1}} + x_{2^{n-1}} + \cdots + x_{2^n} + x_{2^n}) \\ &\geq x_1 + 2x_2 + 4x_4 + \cdots + 2^n x_{2^n} \\ &= \sum_{k=0}^n 2^k x_{2^k}\end{aligned}$$

therefore, we get that

$$\frac{1}{2} \sum_{k=0}^n 2^k a_{2^k} \leq s_{2^n}$$

If  $\sum x_k$  converges, then  $s_n$  is bounded, so  $s_{2^n}$  is bounded, so  $\sum_{k=0}^n 2^k x_{2^k}$  is bounded, so the series  $\sum_{k=0}^{\infty} 2^k x_{2^k}$  is convergent.

## **p-Series**

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad p \in \mathbb{R}$$

By the Cauchy Condensation Test, we see that the series converges if and only if the following series converges:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{2^{np}} &= \sum_{n=1}^{\infty} \left( \frac{1}{2^{n(p-1)}} \right)^n \\ &\Leftrightarrow \frac{1}{2^{p-1}} < 1 \\ &\Leftrightarrow 2^{p-1} > 1 \\ &\Leftrightarrow p > 1 \end{aligned}$$

## **Sequences and Series of Functions**

### **Pointwise Convergence**

Fix a nonempty set  $\Omega$ . Then,

$$\mathcal{F}(\Omega, \mathbb{R}) = \{f \mid f : \Omega \rightarrow \mathbb{R}\}$$

A sequence of functions  $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$  converges pointwise to  $f \in \mathcal{F}(\Omega, \mathbb{R})$  if

$$\forall x \in \Omega, (f_n(x))_n \xrightarrow{n \rightarrow \infty} f(x)$$

Alternatively, using  $\varepsilon$ , we have:

$$\begin{aligned} (f_n)_n \rightarrow f \text{ pointwise } &\in \mathcal{F}(\Omega, \mathbb{R}) \\ \Leftrightarrow & \\ (\forall x \in \Omega)(\forall \varepsilon > 0)(\exists N_{x,\varepsilon} \in \mathbb{N}) \text{ such that } &n \geq N_{x,\varepsilon} \Rightarrow |f_n(x) - f(x)| < \varepsilon \end{aligned}$$

### **Applying Pointwise Convergence**

**Example 1:** Let  $f_n : [0, 1] \rightarrow \mathbb{R}$ , and  $f_n(x) = x^n$ . Note that  $(f_n)_n \rightarrow \delta_1$ , where

$$\delta_1(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

**Example 2:** Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ , where

$$f_n(x) = \frac{nx}{1 + n^2 x^2}$$

**Claim:**  $f_n \rightarrow \mathbf{o}$ .

If  $x = 0$ , then  $f_n(0) = \mathbf{o} \forall n \geq 1$ .

Otherwise, we have

$$\begin{aligned} |f_n(x) - \mathbf{o}(x)| &= \frac{n|x|}{1+n^2x^2} \\ &\leq \frac{n|x|}{n^2x^2} \\ &= \frac{1}{n|x|} \\ &\rightarrow 0 \end{aligned}$$

**Example 3:** Let  $h_n : [0, \infty) \rightarrow \mathbb{R}$ , where  $h_n(x) = x^{1/n}$ . We claim that

$$\begin{aligned} h_n &\rightarrow h \\ h(x) &= \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases} \\ &= \mathbb{1}_{(0, \infty)} \end{aligned}$$

Since, for any  $b > 0$ ,  $(b^{1/n}) \rightarrow 1$

**Example 4:** Let  $g_n : [0, \infty) \rightarrow \mathbb{R}$ , where  $g_n(x) = \frac{x^n}{1+x^n}$ . We claim that  $g_n \rightarrow g$ , where  $g : [0, \infty) \rightarrow \mathbb{R}$  defined as follows:

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}$$

When  $x > 1$ , we have

$$\begin{aligned} |g_n(x) - 1| &= \left| \frac{x^n}{1+x^n} - 1 \right| \\ &= \left| \frac{-1}{1+x^n} \right| \\ &= \frac{1}{1+x^n} \\ &\rightarrow 0 \end{aligned}$$

## Uniform Convergence

A sequence of functions  $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$  converges uniformly to  $f \in \mathcal{F}(\Omega, \mathbb{R})$  if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } (n \geq N_\varepsilon)(\forall x \in \Omega) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Equivalently,

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } n \geq N_\varepsilon \Rightarrow \sup_{x \in \Omega} |f_n(x) - f(x)| < \varepsilon.$$

## Applying Uniform Convergence

**Example 1:** Let  $f_n : [0, 4] \rightarrow \mathbb{R}$ .

$$f_n(x) = \frac{x}{x+n}$$

We claim that

$$f_n \rightarrow \mathbf{0} \text{ uniformly.}$$

We start by examining the maximum size of  $f_n(x)$ :

$$\begin{aligned} |f_n(x) - \mathbf{0}(x)| &= \frac{x}{x+n} \\ &\leq \frac{x}{n} \\ &\leq \frac{4}{n} \end{aligned}$$

so,

$$\sup_{x \in [0,4]} |f_n(x) - \mathbf{0}(x)| \leq \frac{4}{n}.$$

Given  $\varepsilon > 0$ , find  $N$  so large such that  $\frac{1}{N} < \frac{\varepsilon}{4}$ . Then, for  $n \geq N$ ,

$$\begin{aligned} \sup_{x \in \Omega} |f_n(x) - f(x)| &\leq \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \varepsilon \end{aligned}$$

## Negating Uniform Convergence

**Statement:**

$$\begin{aligned} (f_n)_n &\nrightarrow f \text{ uniformly} \\ \Leftrightarrow & \\ (\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N}) \text{ such that } &(\exists n_0 \geq N)(\exists x_0 \in \Omega) |f_{n_0}(x_0) - f(x_0)| \geq \varepsilon_0 \\ \Leftrightarrow & \\ (\exists \varepsilon_0 > 0)(\forall N)(\exists (x_k)_k \in \Omega)(\exists (f_{n_k})_k) \text{ such that } &|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \end{aligned}$$

**Proof:**

( $\Rightarrow$ ) We know  $\exists \varepsilon_0$  satisfying condition (1). Let  $N = 1$ . We know  $\exists n_1 \geq 1$  such that  $\exists x_1 \in \Omega$  with  $|f_{n_1}(x_1) - f(x_1)| \geq \varepsilon_0$ .

Now, set  $N = n_1 + 1$ . Then,  $\exists n_2 \geq N$  and  $x_2 \in \Omega$  satisfying condition (1).

Defining  $n_k$  and  $x_k$  recursively, we have a natural sequence  $(n_k)_k$ , and thus a subsequence of  $f_n$ , thereby satisfying condition (2).

### Negating Uniform Convergence 1

**Statement:** Does  $(f_n)_n \rightarrow f$  uniformly converge on  $[0, 1]$ , where  $f_n(x) = x^n$ ,  $f = \delta_1$ ?

**Proof:** Let  $x_k = (\frac{1}{2})^k$ ,  $n_k = k$ .

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= |f_{n_k}(x_k)| \\ &= \left(\frac{1}{2^{1/k}}\right)^k \\ &= \frac{1}{2} \end{aligned}$$

Setting  $\varepsilon_0 = 1/2$ , we have that it does *not* converge uniformly.

### Changing Domain and Uniform Convergence

Recall  $g_n : [0, \infty) \rightarrow \mathbb{R}$ , where

$$g_n(x) = \frac{nx}{1 + n^2 x^2}$$

We saw that  $(g_n)_n \rightarrow \mathbf{0}$  pointwise. However, it is *not* uniformly convergent. Take  $x_k = \frac{1}{k}$ , and  $n_k = k$ . Then,

$$\begin{aligned} |g_{n_k}(x_k) - \mathbf{0}(x_k)| &= \frac{k \cdot \frac{1}{k}}{1 + k^2 \cdot \frac{1}{k^2}} \\ &= 1/2 \\ &= \varepsilon_0. \end{aligned}$$

However,  $g_n \rightarrow g$  on  $[a, \infty)$  where  $a > 0$ . Let  $x \in [a, \infty)$

$$\begin{aligned} |g_n(x) - \mathbf{0}(x)| &= \frac{nx}{1 + n^2 x^2} \\ &\leq \frac{nx}{n^2 x^2} \\ &= \frac{1}{nx} \\ &\leq \frac{1}{na} \end{aligned}$$

therefore,

$$\sup_{x \in [a, \infty)} |g_n(x) - \mathbf{0}(x)| \leq \frac{1}{na}$$

### Negating Uniform Convergence 2

Consider the family of functions

$$\begin{aligned} f_n : [0, \infty) &\rightarrow \mathbb{R} \\ f_n(x) &= e^{-nx} \end{aligned}$$

Upon examination, we can see that:

$$f_n \xrightarrow{\text{p.w.}} \delta_0 = \mathbb{1}_{\{0\}}.$$

However, the convergence is *not* uniform.



Let  $(x_k)_k = \frac{1}{k}$  and  $n_k = k$ . Then, setting  $\varepsilon_0 = e^{-1}$

$$\begin{aligned} |f_{n_k}(x_k) - \delta_0(x_k)| &= \left| f_k\left(\frac{1}{k}\right) \right| \\ &= e^{-1} \\ &\geq \varepsilon_0 \end{aligned}$$

## Uniform Norm

For  $f \in \mathcal{F}(\Omega, \mathbb{R})$ , the **uniform norm** or **infinity norm** is defined as:

$$\|f\|_u = \sup_{x \in \Omega} |f(x)|.$$

Importantly, the value of the uniform norm is dependent on  $\Omega$ .

The uniform norm satisfies the rules we desire of any other norm:

- Scalar multiplication:  $\forall t \in \mathbb{R}, \|tf\|_u = |t|\|f\|_u$
- Triangle Inequality:  $\|f + g\|_u \leq \|f\|_u + \|g\|_u$
- Zero Property:  $\|f\|_u = 0 \Leftrightarrow f = \mathbf{0}_{\mathbb{R}}$
- Algebraic Property:  $\|fg\|_u \leq \|f\|_u \cdot \|g\|_u$ .

$$\ell_\infty(\Omega) = \{f \in \mathcal{F}(\Omega, \mathbb{R}) \mid \|f\|_u < \infty\}$$

is a normed vector space.

Given  $(f_k)_k$ ,  $f \in \ell_\infty(\Omega)$ , we have

$$(f_k)_k \xrightarrow{\text{uniformly}} f \Leftrightarrow (\|f_k - f\|_u)_k \rightarrow 0$$

## Applying Uniform Norm 1

Let

$$\begin{aligned} g_n &: [0, 1] \rightarrow \mathbb{R} \\ g_n(x) &= x^n(1-x) \end{aligned}$$

Clearly,  $(g_n)_n$  belongs to  $\ell_\infty([0, 1])$ . We can see that

$$(g_n)_n \xrightarrow{\text{p.w.}} \mathbf{0}$$

To show that the convergence is uniform, we must find

$$\|g_n - \mathbf{0}\|_u \xrightarrow{n \rightarrow \infty} \mathbf{0},$$

or

$$\begin{aligned}
 \sup_{x \in [0,1]} x^n(1-x) &\rightarrow 0 \\
 \frac{d}{dx}(x^n(1-x)) &= nx^{n-1} - (n+1)x^n \\
 nx^{n-1} &= (n+1)x^n \\
 x &= \frac{n}{n+1} \\
 \sup_{x \in [0,1]} x^n(1-x) &= \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \\
 &= \frac{1}{(1+1/n)^n} \left(\frac{1}{n+1}\right) \\
 &\rightarrow 0
 \end{aligned}$$

## Root Test and Series Convergence

**Statement:** Let

$$\limsup_{k \rightarrow \infty} |x_k|^{1/k} = \rho.$$

If  $\rho < 1$ , then  $\sum_k x_k$  converges absolutely. If  $\rho > 1$ , then  $\sum_k x_k$  diverges.

**Proof:** Suppose  $\rho < 1$ . Let  $\rho < r < 1$ . By property of  $\inf$ ,  $\exists N \in \mathbb{N}$  large such that  $r \geq \sup_{k \geq N} |x_k|^{1/k}$ .

Therefore,  $\forall k \geq N$ , we have

$$\begin{aligned}
 |x_k|^{1/k} &\leq r \\
 |x_k| &\leq r^k \quad \forall k \geq N
 \end{aligned}$$

Therefore,

$$\sum_k x_k \leq \underbrace{\sum_{k=1}^{N-1} x_k + \sum_{k \geq N} r^k}_{\text{converges: } r < 1}$$

If  $\limsup |x_k|^{1/k} = \rho > 1$ , we can find a subsequence  $(x_{k_\ell})^{1/k_\ell} \xrightarrow{\ell \rightarrow \infty} \rho$ . We cannot have  $((x_k)_k)^{1/k} \rightarrow 0$ . Thus, the series diverges.

## Absolute Convergence

**Statement:** A series  $\sum_k x_k$  converges absolutely if  $\sum_k |x_k|$  converges. If a series converges absolutely, then it always converges.

**Proof:** Let  $s_n = \sum_{k=1}^n x_k$ ,  $t_n = \sum_{k=1}^n |x_k|$ . Let  $m > n$ . Then,

$$\begin{aligned}
 |s_m - s_n| &= \left| \sum_{k=n+1}^m x_k \right| \\
 &\leq \sum_{k=n+1}^m |x_k| \quad \text{Triangle Inequality} \\
 &= |t_m - t_n|
 \end{aligned}$$

By assumption,  $(t_n)_n$  converges, and thus is Cauchy. By the above inequality,  $(s_n)_n$  is Cauchy, and thus convergent.

## Series of Functions

Given a sequence of functions  $(f_k)_k \in \mathcal{F}(\Omega, \mathbb{R})$ , we say that the series

$$\sum_k f_k$$

converges pointwise to  $f$  in  $\mathcal{F}(\Omega, \mathbb{R})$  if

$$s_n = \left( \sum_{k=1}^n f_k \right)_n$$

converges to  $f$  pointwise.

$$\sum_{k=1}^{\infty} f_k(x) = f(x) \quad \forall x \in \Omega$$

$\sum f_k$  converges to  $f$  **uniformly** if

$$s_n = \left( \sum_{k=1}^n f_k \right)_n$$

converges to  $f$  uniformly.

### Applying Pointwise Convergence of Series of Functions

Let  $f_k : (-1, 1) \rightarrow \mathbb{R}$ , where  $f_k = x^k$ . Then,

$$\sum_{k=0}^{\infty} f_k \rightarrow f(x) = \frac{1}{1-x}$$

### Applying Uniform Convergence of Series of Functions

**Statement:** We know that  $\sum_{k=0}^{\infty} x^k$  converges pointwise to  $s(x) = \frac{1}{1-x}$  on  $(-1, 1)$ . Does it converge *uniformly* on the same interval?

**Proof:**

We claim the convergence is not uniform on  $(-1, 1)$ , but convergence is uniform on  $[a, b]$ , where  $-1 < a \leq b < 1$ .

Let  $s_n(x) = \sum_{k=0}^n x^k$ .

$$\begin{aligned} |s_n(x) - s(x)| &= \left| \frac{1 - x^{n+1}}{1 - x} - \frac{1}{1 - x} \right| \\ &= \frac{|x|^{n+1}}{1 - x} \end{aligned}$$

Let  $c = \max\{|a|, |b|\} < 1$

$$\begin{aligned} &\leq \frac{c^{n+1}}{1 - b} \quad \forall a \leq x \leq b \\ \sup_{x \in [a, b]} |s_n(x) - s(x)| &\leq \frac{c^{n+1}}{1 - b} \\ &\rightarrow 0 \end{aligned}$$

To show non-uniform convergence on  $(-1, 1)$ , let  $x_\ell = 1 - \frac{1}{\ell}$ , and let  $n_\ell = \ell$ .

$$\begin{aligned} |s_{n_\ell}(x_\ell) - s(x_\ell)| &= \frac{\left(1 - \frac{1}{\ell}\right)^{\ell+1}}{\frac{1}{\ell}} \\ &= \ell \left(1 - \frac{1}{\ell}\right)^\ell \left(1 - \frac{1}{\ell}\right) \\ &= (\ell - 1) \left(1 - \frac{1}{\ell}\right)^\ell \\ &\rightarrow \infty \end{aligned}$$

since  $\left(1 - \frac{1}{\ell}\right)^\ell \rightarrow \frac{1}{e}$ .

## Weierstrass $M$ -test

**Statement:** Consider a sequence of functions  $(f_k)_k$  in  $\ell_\infty(\Omega)$ , where  $\Omega \subseteq \mathbb{R}$ .

If  $\sum_{k=1}^{\infty} \|f_k\|_u$  converges, then  $\sum_k f_k$  converges uniformly and absolutely on  $\Omega$ .

**Proof:** Set  $M_k = \|f_k\|_u$ . Given  $\varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that

$$\sum_{n+1}^m M_k < \varepsilon \quad \forall m > n \geq N$$

since  $\sum_{k=1}^{\infty} M_k$  is convergent, and thus Cauchy.

Let  $s_n(x) = \sum_{k=1}^n f_k(x)$ . So,

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left| \sum_{k=n+1}^m f_k(x) \right| \\ &\leq \sum_{k=n+1}^m |f_k(x)| \\ &\leq \sum_{k=n+1}^m M_k \\ &< \varepsilon \end{aligned} \quad \text{whenever } m > n \geq N$$

For every  $x \in \Omega$ ,  $s_n(x)$  is Cauchy. So,  $\forall x \in \Omega$ ,  $s(x) := \lim s_n(x)$  exists.

Additionally,  $\forall x \in \Omega$ ,

$$|s_m(x) - s_n(x)| < \varepsilon.$$

Let  $m \rightarrow \infty$ . Then,

$$\begin{aligned} |s(x) - s_n(x)| &< \varepsilon \\ \sup_{x \in \Omega} |s(x) - s_n(x)| &< \varepsilon. \end{aligned} \quad \begin{aligned} \forall x \in \Omega, \forall n \geq N \\ \forall n \geq N \end{aligned}$$

## Applying the Weierstrass $M$ -test

Consider

$$\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2},$$

where  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ . Then,  $\|f_k\|_u \leq \frac{1}{k^2}$ . So,

$$\sum \|f_k\|_u \leq \sum \frac{1}{k^2} < \infty.$$

Thus,  $\sum \frac{1}{x^2+k^2}$  converges absolutely and uniformly.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges  $\forall x \in \mathbb{R}$ , and converges *uniformly* on any closed and bounded interval  $[a, b]$ .

## Power Series

A **power series** centered at  $c$  in  $\mathbb{R}$  is a formal series of functions

$$\sum_{k=0}^{\infty} a_k(x - c)^k.$$

We want to examine the convergence and the uniformity of such convergence of these power series.

Given  $\sum a_k(x - c)^k$ , set  $\rho = \limsup |a_k|^{1/k}$  and  $r = 1/\rho$ .

## Cauchy-Hadamard Theorem

**Statement:** A power series

$$\sum_{k=1}^{\infty} a_k(x - c)^k$$

converges absolutely on  $(c - r, c + r)$ , diverges on  $\overline{[c - r, c + r]}$ , and uniformly convergent on  $[a, b]$ ,  $c - r < a \leq b < c + r$ .

**Proof:** Let  $\sum_{k=1}^{\infty} a_k(x - c)^k$ , where  $x_k = a_k(x - c)^k$ .

$$|x_k|^{1/k} = |a_k|^{1/k} |x - c|$$

Root test:

$$\begin{aligned} \limsup_{k \rightarrow \infty} |x_k|^{1/k} &= |x - c| \limsup_{k \rightarrow \infty} |a_k|^{1/k} \\ &= |x - c| \rho \end{aligned}$$

Absolute Convergence:

$$\begin{aligned} |x - c| \rho &< 1 \\ |x - c| &< \frac{1}{\rho} \end{aligned}$$

Divergence:

$$\begin{aligned} |x - c| \rho &> 1 \\ |x - c| &> \frac{1}{\rho} \end{aligned}$$

Let  $[a, b] \subset (c - r, c + r)$ . Set  $d = \max\{|a - c|, |b - c|\}$ . So,

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{k=n+1}^m a_k (x - c)^k \right| \\ &\leq \sum_{k=n+1}^m |a_k| |x - c|^k \\ &\leq \sum_{k=n+1}^m |a_k| d^k \end{aligned}$$

we know that  $d < r \Rightarrow d/r < 1 \Rightarrow dp < 1 \Rightarrow p < 1/d$ . Pick  $p < p < 1/d$ . So,  $\exists N \in \mathbb{N}$  with

$$\begin{aligned} \sup_{k \geq N} |a_k|^{1/k} &< p \\ |a_k| &< p^k \end{aligned}$$

So, if  $m > n \geq N$ , we have

$$\begin{aligned} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^m (rd)^k \\ \sup_{x \in [a, b]} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^m (rd)^k \end{aligned}$$

Given  $\varepsilon > 0$ , find  $N_1 \in \mathbb{N}$  with  $m > n \geq N_1$  meaning

$$\begin{aligned} \sup_{x \in [a, b]} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^m (rd)^k \\ &< \varepsilon \end{aligned}$$

Let  $K = \max\{N, N_1\}$ . With  $m > n \geq K$ , we have

$$\sup_{x \in [a, b]} |s_m(x) - s_n(x)| < \varepsilon$$

Letting  $m \rightarrow \infty$ , we have

$$\sup_{x \in [a, b]} |s(x) - s_n(x)| < \varepsilon.$$

So,  $(s_n(x))_n \rightarrow s(x)$  uniformly on  $[a, b]$ .

## Limits

### Cluster Points

Recall: If  $c \in \mathbb{R}$ , and  $\delta > 0$ , then  $V_\delta(x) = (c - \delta, c + \delta)$ .

The *deleted neighborhood*  $\dot{V}_\delta = (c - \delta, c) \cup (c, c + \delta) = V_\delta \setminus \{c\}$ .

$$(i) \quad x \in V_\delta(c) \Leftrightarrow |x - c| < \delta$$

$$(ii) \quad x \in \dot{V}_\delta(c) \Leftrightarrow 0 < |x - c| < \delta$$

Let  $D \subseteq \mathbb{R}$ . A number  $c \in \mathbb{R}$  is a *cluster point* or *limit point* of  $D$  if

$$(\forall \delta > 0)(\exists x \in D \cap \dot{V}_\delta(c)) \Leftrightarrow \forall \delta > 0, \dot{V}_\delta(c) \cap D \neq \emptyset$$

**Remarks** If  $c$  is a cluster point of  $D$ ,  $c$  may or may not belong to  $D$ . If  $c \in D$ , then  $c$  is not necessarily a cluster point.

**Examples:**

- Let  $D = (0, 1)$ . Is  $c = 0$  a cluster point of  $D$ ?

Yes — given any  $\delta > 0$ ,  $\dot{V}_\delta(0) \cap (0, 1) = (0, \min(1, \delta))$ . We have that  $[0, 1]$  is the set of all limit points of  $D$ .

- Let  $D = \mathbb{N}$ . Then,  $D$  admits no cluster points.
- Additionally, all finite sets have no cluster points.
- If  $D = \mathbb{Q}$ , then the set of cluster points of  $\mathbb{Q}$  is  $\mathbb{R}$ .

Given any  $t \in \mathbb{R}$ ,  $\delta > 0$ ,

$$\dot{V}_\delta \cap \mathbb{Q} \neq \emptyset$$

because  $\mathbb{Q}$  is dense.

- If  $D = \{\frac{1}{n} \mid n \geq 1\}$ , then  $\{0\}$  is the set of cluster points of  $D$ .

### Sequential Criterion of Cluster Points

**Statement:** Let  $D \subseteq \mathbb{R}$ ,  $c \in \mathbb{R}$ . The following are equivalent:

- (1)  $c$  is a limit point of  $D$ .
- (2)  $\exists (x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$

**Proof:**

(2)  $\Rightarrow$  (1) Let  $\delta > 0$ . Then,  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$ ,  $0 < |x_n - c| < \delta$ . Thus  $x_N \in \dot{V}_\delta(c) \cap D$ .

(1)  $\Rightarrow$  (2) Let  $\delta_n = \frac{1}{n}$ . Then,  $\exists x_n \in D \cap \dot{V}_{1/n}(c)$ . So,  $x_n \neq c$ ,  $x_n \in D$ , and  $|x_n - c| < 1/n$ . So,  $(x_n)_n \rightarrow c$ .

### Definition of a Limit

Let  $f : D \rightarrow \mathbb{R}$ , and  $c$  a limit point of  $D$ . Let  $L \in \mathbb{R}$ .

$$\lim_{x \rightarrow c} f(x) = L \xLeftrightarrow{\text{defn.}} (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } \forall x \in \dot{V}_\delta(c) \cap D, f(x) \in V_\varepsilon(L)$$

### Applying the Limit Definition: Linear Function

$$\lim_{x \rightarrow c} ax + b = ac + b \qquad a \neq 0$$

**Preliminary Work:**

$$\begin{aligned} |f(x) - L| &= |ax + b - (ac + b)| \\ &= |ax - ac| \\ &= |a||x - c| \end{aligned}$$

**Proof:** Given  $\varepsilon > 0$ , set  $\delta = \frac{\varepsilon}{|a|}$ .

$$\begin{aligned} 0 &< |x - c| < \delta \\ 0 &< |x - c| < \frac{\varepsilon}{|a|} \\ |f(x) - L| &= |a||x - c| \\ &< |a|\frac{\varepsilon}{|a|} \\ &= \varepsilon \end{aligned}$$

### Applying the Limit Definition: Quadratic Function

$$\lim_{x \rightarrow c} x^2 = c^2$$

**Preliminary Work:**

$$\begin{aligned} |f(x) - L| &= |x^2 - c^2| \\ &= |x - c||x + c| \end{aligned}$$

If  $0 < \delta < 1$ , and  $|x - c| < \delta$ , then  $|x + c| \leq |x| + |c| \leq 2|c| + 1$ . In this case,

$$|f(x) - L| \leq (2|c| + 1)|x - c|.$$

**Proof:** Given  $\varepsilon > 0$ , let  $\delta = \frac{1}{2} \min\left(1, \frac{\varepsilon}{2|c|+1}\right)$ . This guarantees  $\delta < 1$ . So, if  $|x - c| < \delta$ ,

$$\begin{aligned} |f(x) - L| &\leq (2|c| + 1)|x - c| \\ &< (2|c| + 1)|x - c| \\ &< (2|c| + 1)\frac{\varepsilon}{2|c| + 1} \\ &= \varepsilon \end{aligned}$$

### Applying the Limit Definition: Rational Function

$$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c} \quad c \neq 0$$

**Preliminary Work:**

$$\begin{aligned} |f(x) - L| &= \left| \frac{1}{x} - \frac{1}{c} \right| \\ &= \frac{1}{|x|} \frac{1}{|c|} |x - c| \end{aligned}$$

If  $x \in \left(c - \frac{|c|}{2}, c + \frac{|c|}{2}\right)$ , then  $|x| \geq |c|/2$ , so  $\frac{1}{|x|} \leq \frac{2}{|c|}$ . So,

$$\frac{1}{|x|} \frac{1}{|c|} |x - c| \leq \frac{2}{|c|^2} |x - c|$$

**Proof:** Given  $\varepsilon > 0$ , let  $\delta = \frac{1}{2} \min\left(\frac{|c|}{2}, \frac{|c|^2}{2}\varepsilon\right)$ . If

$$\begin{aligned} 0 &< |x - c| < \delta \\ |f(x) - L| &\leq \frac{2}{|c|^2} |x - c| \\ &< \frac{2}{|c|^2} \frac{|c|^2}{2} \varepsilon \\ &= \varepsilon \end{aligned}$$



## Uniqueness of Limits

**Statement:** Let  $f : D \rightarrow \mathbb{R}$  with  $c$  a limit point of  $D$ . Then,  $f$  can have at most one limit.

**Proof:** Suppose toward contradiction that  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} f(x) = L_2$ , where  $L_1 \neq L_2$ .

Let  $\varepsilon$  be small such that  $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$ . So,  $\exists \delta_1 > 0$  such that

$$0 < |x - c| < \delta_1 \Rightarrow f(x) \in V_\varepsilon(L_1),$$

and  $\exists \delta_2 > 0$  such that

$$0 < |x - c| < \delta_2 \Rightarrow f(x) \in V_\varepsilon(L_2).$$

Set  $\delta = \min(\delta_1, \delta_2)$ . Then,

$$0 < |x - c| < \delta \Rightarrow f(x) \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$$

## Sequential Criterion for Limits

**Statement:** Let  $f : D \rightarrow \mathbb{R}$ ,  $c$  a cluster point of  $D$ . The following are equivalent:

- (i)  $\lim_{x \rightarrow c} f = L$
- (ii)  $\forall (x_n)_n \in D \setminus \{c\}$  where  $(x_n)_n \rightarrow c$ , we have  $(f(x_n))_n \rightarrow L$

**Proof:**

$(\Leftarrow)$  Assume  $\lim_{x \rightarrow c} f(x) \neq L$ . Then,  $(\exists \varepsilon_0) (\forall \delta > 0) (\exists x \in \dot{V}(c) \cap D)$  with  $|f(x) - L| \geq \varepsilon_0$ .

Let  $\delta_n = \frac{1}{n}$ . Then,  $\exists x_n \in \dot{V}_{1/n}(c) \cap D$ , with  $|f(x_n) - L| \geq \varepsilon_0$ .

Note that  $0 < |x - c| < 1/n$ . So,  $(x_n)_n \in D \setminus \{c\}$ , and  $(x_n)_n \rightarrow c$ . By (ii), it must be the case that  $(f(x_n))_n \rightarrow L$ .

However,  $|f(x_n) - L| \geq \varepsilon_0$ .  $\perp$

## Limit Divergence and Non-Existence

**Statement:** Let  $f : D \rightarrow \mathbb{R}$ , and  $c$  a cluster point of  $D$ . Let  $L \in \mathbb{R}$ . The following are true:

- (1)  $\lim_{x \rightarrow c} f(x) \neq L \Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$  but  $f(x_n) \not\rightarrow L$
- (2)  $\lim_{x \rightarrow c} f(x)$  DNE  $\Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$  and  $(f(x_n))_n$  divergent.

**Proof:**

**(1)** This is a direct negation of the Sequential Definition.

**(2)**

$(\Rightarrow)$  Suppose toward contradiction,  $\forall (x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$ , we have  $(f(x_n))_n$  is convergent.

Pick any two such sequences,  $(x_n)_n$  and  $(y_n)_n$ . We know  $(f(x_n))_n \rightarrow L_1$ , and  $(f(y_n))_n \rightarrow L_2$ .

Consider  $(z_n)_n = (x_1, y_1, x_2, y_2, \dots)$ . We know that  $(z_n)_n \rightarrow c$ , meaning  $(f(z_n))_n \rightarrow M$ .

The sequence  $(f(z_n))_n$  admits two subsequences  $(f(x_n))_n \rightarrow M$  and  $(f(y_n))_n \rightarrow M$ . Thus,  $L_1 = L_2$ .

We showed that, for any sequence  $(x_n)_n \rightarrow c$ ,  $(f(x_n))_n \rightarrow L$ . Thus,  $\lim_{x \rightarrow c} f(x)$  exists.  $\perp$

### Applying Limit Divergence using Sequences

We want to find  $\lim_{x \rightarrow c} \mathbb{1}_{\mathbb{Q}}$ . Consider two sequences  $(r_n)_n \rightarrow c$ , where  $r_n \in \mathbb{Q}$  — this is always possible since the rationals are dense — and  $(t_n)_n \rightarrow c$ , where  $t_n \notin \mathbb{Q}$ .

Let  $(x_n)_n = (r_1, t_1, r_2, t_2, \dots)$ . Then,  $(x_n) \rightarrow c$ , but  $(\mathbb{1}_{\mathbb{Q}}(x_n))_n = (1, 0, 1, 0, \dots)$ . Thus,  $\lim_{x \rightarrow c} \mathbb{1}_{\mathbb{Q}}$  DNE.

### Bounded Functions and Cluster Points

**Statement:** Recall that  $f : D \rightarrow \mathbb{R}$  is bounded on  $E \subseteq D$  if  $\sup_{x \in E} |f(x)| < \infty$ .

If  $f : D \rightarrow \mathbb{R}$  and  $c$  is a cluster point of  $D$ , if  $\lim_{x \rightarrow c} f(x) = L$ , then  $\exists \delta > 0$  such that  $f$  is bounded on  $\dot{V}_\delta(c) \cap D$ .

**Proof:** Let  $\varepsilon = 1$ . Then,  $\exists \delta > 0$  such that  $x \in \dot{V}_\delta(c) \cap D \Rightarrow |f(x) - L| < 1$ . Then,

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L|, \end{aligned}$$

so,

$$\sup_{x \in \dot{V}_\delta(c)} |f(x)| \leq 1 + |L|$$

### Operations with Limits

**Statement:** Let  $f, g : D \rightarrow \mathbb{R}$ , and  $c$  is a cluster point of  $D$ . Let  $\alpha \in \mathbb{R}$ .

(a) If  $\lim_{x \rightarrow c} f(x) = L$ , and  $\lim_{x \rightarrow c} g(x) = M$ , then

- (i)  $\lim_{x \rightarrow c} (f \pm g) = L \pm M$
- (ii)  $\lim_{x \rightarrow c} (\alpha f) = \alpha L$
- (iii)  $\lim_{x \rightarrow c} (fg) = LM$
- (iv)  $\lim_{x \rightarrow c} \left( \frac{f}{g} \right) = \frac{L}{M}$  if  $M \neq 0$

(b)  $\lim_{x \rightarrow c} |f(x)| = |L|$

(c)  $\lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{L}$ , provided  $f(x) \geq 0$

(d) If  $f(x)$  is a polynomial, then  $\lim_{x \rightarrow c} f(x) = f(c)$ .

(e) If  $f(x)$  is rational, then  $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$ , provided  $q(c) \neq 0$ .

Proof of (a)(iii): Let  $(x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$ . Then,  $(f(x_n))_n \rightarrow L$ ,  $(g(x_n))_n \rightarrow M$ . Then,

$$\begin{aligned} (f \cdot g(x_n)) &= (f(x_n)g(x_n))_n \\ &\rightarrow LM \end{aligned} \quad \text{by sequence properties}$$

Proof of (a)(iv): Let  $(x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$ . Then, by the properties of sequences,

$$\begin{aligned} \left( \frac{f}{g}(x_n) \right) &= \left( \frac{f(x_n)}{g(x_n)} \right)_n \\ &\rightarrow \frac{L}{M} \end{aligned} \quad \text{provided } M \neq 0$$

Proof of (d): Let  $p(x) = \sum_{k=0}^n a_k x^k$ . Then,

$$\begin{aligned}
 \lim_{x \rightarrow c} p(x) &= \lim_{x \rightarrow c} \left( \sum_{k=0}^n a_k x^k \right) \\
 &= \sum_{k=0}^n \lim_{x \rightarrow c} a_k x^k && \text{(a)(i)} \\
 &= \sum_{k=0}^n a_k \lim_{x \rightarrow c} x^k && \text{(a)(ii)} \\
 &= \sum_{k=0}^n a_k \left( \lim_{x \rightarrow c} x \right)^k && \text{(a)(i)} \\
 &= p(c)
 \end{aligned}$$

Proof of (b) Using the properties of sequence, we can show that  $(|f(x_n)|)_n \rightarrow |L|$  for  $(x_n)_n \in D \setminus \{c\}$  with  $(x_n)_n \rightarrow c$

## Squeeze Theorem

**Statement:** If  $f : D \rightarrow \mathbb{R}$ ,  $c$  is a cluster point of  $D$ .

- (i) If  $f(x) \leq b$  for  $x$  in a deleted neighborhood of  $c$ , and if  $\lim_{x \rightarrow c} f(x) = L$ , then  $L \leq b$ .
- (ii) If  $f(x) \geq a$  for all  $x$  in a deleted neighborhood of  $c$ , and if  $\lim_{x \rightarrow c} f(x) = L$ , then  $L \geq a$ .
- (iii) If  $f, g, h : D \rightarrow \mathbb{R}$ , and  $c$  is a cluster point of  $D$ . Suppose

$$g(x) \leq f(x) \leq h(x)$$

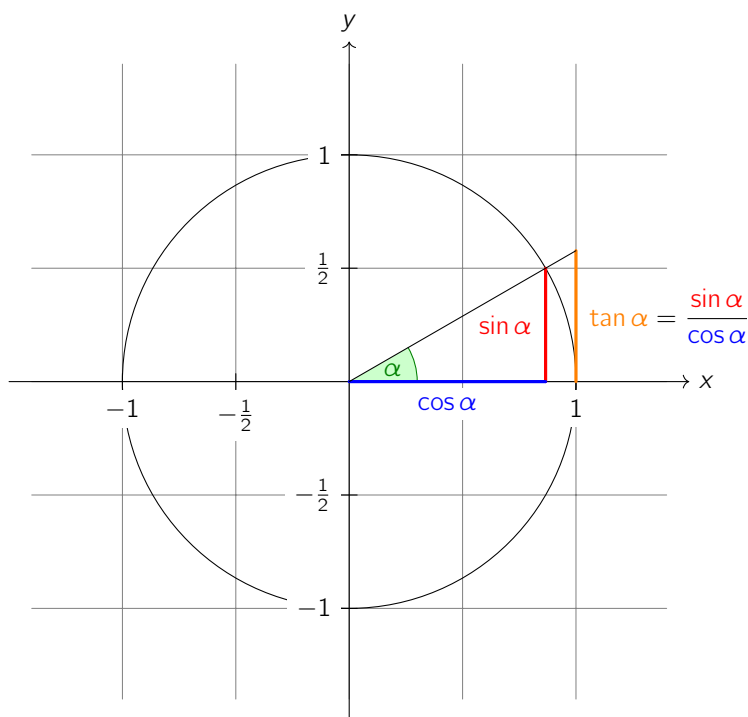
for all  $x$  in some deleted neighborhood of  $c$ . Suppose  $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$ . Then,  $\lim_{x \rightarrow c} f(x) = L$ .

Proof of (iii) Let  $(x_n)_n \in D \setminus \{c\}$ , with  $(x_n)_n \rightarrow c$ . Then, as  $n \rightarrow \infty$ ,

$$\begin{aligned}
 g(x_n) &\leq f(x_n) \leq h(x_n) \\
 L &\leq f(x_n) \leq L,
 \end{aligned}$$

so  $f(x_n)_n \rightarrow L$ .

## Trigonometric Limits



We know that

$$0 \leq \sin(x) \leq x$$

so as  $x \rightarrow 0^+$ ,  $\sin(x) \rightarrow 0$ . Similarly, if  $x \rightarrow 0^-$ , then

$$\begin{aligned} \lim_{x \rightarrow 0^-} \sin(x) &= \lim_{y \rightarrow 0^+} \sin(-y) \\ &= - \lim_{y \rightarrow 0^+} \sin(y) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow 0^+} \cos(x) &= \lim_{x \rightarrow 0^+} \sqrt{1 - \sin^2(x)} \\ &= 1 \\ \lim_{x \rightarrow 0^-} \cos(x) &= \lim_{y \rightarrow 0^+} \cos(-y) \\ &= \lim_{y \rightarrow 0^+} \cos(y) \\ &= 1 \end{aligned}$$

**Claim:**

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

**Proof:** Let  $x \rightarrow 0$

$$\begin{aligned}\frac{\sin(x)}{2} &\leq \frac{x}{2} \leq \frac{\tan(x)}{2} \\ 0 &\leq \frac{\sin(x)}{x} \leq 1 \\ \cos(x) &\leq \frac{\sin(x)}{x} \\ \cos(x) &\leq \frac{\sin(x)}{x} \leq 1 \\ 1 &\leq \frac{\sin(x)}{x} \leq 1\end{aligned}$$

## Strictly Positive Limits

**Statement:** Let  $D \subseteq \mathbb{R}$ ,  $f : D \rightarrow \mathbb{R}$ . Let  $c$  be a cluster point of  $D$ . If  $\lim_{x \rightarrow c} f(x) = L > 0$ , then  $\exists \delta > 0$  and  $\exists t > 0$  such that  $f(x) > t$  for  $x \in \dot{V}_\delta(c) \cap D$ .

**Proof:** Let  $\varepsilon = \frac{L}{2}$ . Then,  $V_\varepsilon = (L/2, 3L/2)$ . So,  $\exists \delta > 0$  such that  $x \in \dot{V}_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$ . Set  $t = L/2$ .

## One-Sided Limits

Let  $f : D \rightarrow \mathbb{R}$ .

**Cluster Points:**

- (i) A number  $c \in D$  is a right cluster point if  $\forall \delta > 0$ ,  $\exists x \in (c, c + \delta) \cap D$
- (ii) A number  $c \in D$  is a left cluster point if  $\forall \delta > 0$ ,  $\exists x \in (c - \delta, c) \cap D$ .

**Limits:**

$$(i) \lim_{x \rightarrow c^+} f(x) = L \stackrel{\text{def}}{\iff}$$

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in (c, c + \delta) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

$$(ii) \lim_{x \rightarrow c^-} f(x) = L \stackrel{\text{def}}{\iff}$$

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in (c - \delta, c) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

**Sequential Criterion:**

- (i) Let  $c$  be a right cluster point of  $D$ .  $\lim_{x \rightarrow c^+} f(x) = L$  if and only if  $\forall (x_n)_n \in D \cap (c, \infty)$  with  $(x_n)_n \rightarrow c$ , we have  $(f(x_n))_n \rightarrow L$
- (ii) Let  $c$  be a left cluster point of  $D$ .  $\lim_{x \rightarrow c^-} f(x) = L$  if and only if  $\forall (x_n)_n \in (-\infty, c) \cap D$  with  $(x_n)_n \rightarrow c$ , we have  $(f(x_n))_n \rightarrow L$ .

## Limit Equality

Let  $f : D \rightarrow \mathbb{R}$ . Let  $c$  be a cluster point of  $D$ .

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

## Infinite Limits

Let  $f : D \rightarrow \mathbb{R}$ , and  $c$  be a limit point of  $D$ . Then,

$$\lim_{x \rightarrow c} f(x) = \infty \stackrel{\text{def}}{\iff} (\forall M \geq 0)(\exists \delta > 0) \text{ such that } x \in \dot{V}_\delta(c) \cap D \Rightarrow f(x) \geq M$$

We can also define

$$\begin{aligned} \lim_{x \rightarrow c} f(x) &= -\infty \\ \lim_{x \rightarrow c^\pm} f(x) &= \pm\infty \end{aligned}$$

## Applying Infinite Limits

**Statement:**

$$\lim_{x \rightarrow 1^-} \frac{1}{1-x} = -\infty$$

**Proof:** Let  $M \geq 0$  be large. We want  $f(x) \geq M$ .

$$\begin{aligned} \frac{1}{1-x} &\geq M \\ 1-x &\leq \frac{1}{M} \\ x &\geq 1 - \frac{1}{M} \end{aligned}$$

Set  $\delta = \frac{1}{M}$ . If  $x \in (1 - \delta, 1)$ , then  $x \geq 1 - \frac{1}{M}$ . So, by our work above,  $f(x) \geq M$ .

## Limits at Infinity

Let  $f : [a, \infty) \rightarrow \mathbb{R}$ ,  $L \in \mathbb{R}$ . Then,

$$\lim_{x \rightarrow \infty} f(x) = L \stackrel{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \geq a) \text{ such that } x \geq K \Rightarrow f(x) \in V_\varepsilon(L)$$

Similarly, we can define for  $f : (-\infty, b] \rightarrow \mathbb{R}$ ,  $L \in \mathbb{R}$

$$\lim_{x \rightarrow -\infty} f(x) = L \stackrel{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \leq b) \text{ such that } x \leq K \Rightarrow f(x) \in V_\varepsilon(L)$$

and for  $f : [a, \infty)$  where

$$\lim_{x \rightarrow \infty} f(x) = \infty \stackrel{\text{def}}{\iff} (\forall M \geq 0)(\exists K \geq a) \text{ such that } x \geq K \Rightarrow f(x) \geq M$$

and the respective sequential definitions.

## Applying Limits at Infinity 1

**Statement:** Let  $n \in \mathbb{N}$ .

$$\lim_{x \rightarrow \infty} x^n = \infty$$

**Proof:** Let  $M$  be large. We want  $x^n \geq M$ . Then,  $x \geq M^{1/n}$ . Set  $K = M^{1/n}$ .

## Applying limits at Infinity: Polynomials

$$\lim_{x \rightarrow -\infty} x^n = \begin{cases} +\infty, & n = 2k \\ -\infty, & n = 2k + 1 \end{cases}$$

$$p(x) = \sum_{k=1}^n a_k x^k$$

$$\lim_{x \rightarrow \infty} p(x) = \begin{cases} +\infty, & a_n > 0 \\ -\infty, & a_n < 0 \end{cases}$$

Let  $g(x) = x^n$ .

$$\frac{p(x)}{g(x)} = a_n + a_{n-1} \frac{1}{x} + \cdots + a_0 \frac{1}{x^n}$$

$$\lim_{x \rightarrow \infty} \frac{p(x)}{g(x)} = a_n$$

**Lemma:** If  $f, g : [a, \infty) \rightarrow \mathbb{R}$ , and  $g(x) > 0$ . If

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L \neq 0$$

- (1) If  $L > 0$ , then  $\lim_{x \rightarrow \infty} f(x) = \infty \Leftrightarrow \lim_{x \rightarrow \infty} g(x) = \infty$
- (2) If  $L < 0$ , then  $\lim_{x \rightarrow \infty} f(x) = -\infty \Leftrightarrow \lim_{x \rightarrow \infty} g(x) = +\infty$

Apply the lemma to  $p(x)$ ,  $x^n$ .

## Continuity and Uniform Continuity

### Continuity

Let  $D \subseteq \mathbb{R}$ ,  $f : D \rightarrow \mathbb{R}$ . Let  $c \in D$ . The function  $f$  is continuous at  $c$  if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in V_\delta(c) \cap D \Rightarrow f(x) \in V_\varepsilon(f(c))$$

**Remark:** Here,  $c$  may not be a cluster point of  $D$ .

For example, let

$$f(x) = \begin{cases} x & x = -1 \\ x^2 & x \geq 0 \end{cases}$$

$$D = \{-1\} \cup [0, \infty)$$

Here,  $f$  is continuous at  $c = -1$ . Given any  $\varepsilon > 0$ , let  $\delta = 1/2$ . Then, if  $x \in V_{1/2}(-1) \cap D$ ,  $x = -1$ , meaning  $|f(x) - f(-1)| = 0 < \varepsilon$

### Continuity and Limits

If  $f : D \rightarrow \mathbb{R}$ ,  $c \in D$  and  $c$  a cluster point of  $D$ , the following are equivalent:

- (i)  $f$  is continuous at  $c$
- (ii)  $\lim_{x \rightarrow c} f(x) = f(c)$

**Remark:** We are deign to use the second definition as *the* definition of continuity due to the fact that it removes the possibility of those mentioned above.

## Sequential Criterion of Continuity

Let  $f : D \rightarrow \mathbb{R}$ ,  $c \in D$ . The following are equivalent:

- (i)  $f$  is continuous at  $x = c$
- (ii)  $\forall (x_n)_n$  in  $D$  with  $(x_n)_n \rightarrow c$ , we have  $(f(x_n))_n \rightarrow f(c)$

## Left and Right Continuity

Let  $f : D \rightarrow \mathbb{R}$ ,  $c \in D$ .

- $f$  is left-continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } 0 \leq c - x < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\forall (x_n)_n \in D, x_n \leq c, (x_n)_n \rightarrow c \text{ we have } (f(x_n))_n \rightarrow f(c)$$

- $f$  is right-continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } 0 \leq x - c < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\forall (x_n)_n \in D, x_n \geq c, (x_n)_n \rightarrow c \text{ we have } (f(x_n))_n \rightarrow f(c)$$

## Continuity on Sets

Let  $f : D \rightarrow \mathbb{R}$ .

- (1)  $f$  is continuous on  $E \subseteq D$  if  $f$  is continuous at each  $c \in E$ .
- (2)  $f$  is continuous on  $[a, b]$  if
  - (i)  $f$  is continuous on  $(a, b)$
  - (ii)  $f$  is left-continuous at  $b$
  - (iii)  $f$  is right-continuous at  $a$

## Applying Continuity on Sets

- (1) Polynomials are continuous on  $\mathbb{R}$  because  $\lim_{x \rightarrow c} p(x) = p(c)$ .
- (2) Rational functions are continuous on their domain.
- (3)  $f : \mathbb{I}_{\mathbb{Q}}$  is continuous nowhere:

**Case 1:** Suppose  $c \in \mathbb{Q}$ . Let  $(t_n)_n \rightarrow c$  with  $t_n \in \mathbb{R} \setminus \mathbb{Q}$ . Then,  $(f(t_n))_n = 0 \rightarrow 0 \neq f(c) = 1$

**Case 2:** Let  $c \in \mathbb{R} \setminus \mathbb{Q}$ . Let  $(r_n)_n \rightarrow c$  with  $r_n \in \mathbb{Q}$ . Then,  $(f(r_n))_n = 1 \rightarrow 1 \neq f(c) = 0$

### Discontinuity

$f : D \rightarrow \mathbb{R}$  is not continuous at  $x = c$  if  $\exists (x_n)_n$  in  $D$  with  $(x_n)_n \rightarrow c$  and  $(f(x_n))_n \nrightarrow f(c)$

### Discontinuity of the Sign Function

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at  $x = 0$ , since  $(x_n)_n = \frac{1}{n} \rightarrow 0$  but  $(f(x_n))_n = 1 \neq 0$ .



## Thomae's Function

Let

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \in \mathbb{Q}, \quad b \in \mathbb{N}, \quad \gcd(a, b) = 1 \\ 1 & x = 0 \end{cases}$$

**Claim 1:**  $f$  is not continuous at  $x \in \mathbb{Q}$ : find a sequence  $(t_n)_n$  of irrationals with  $(t_n)_n \rightarrow x$ . Then,  $(f(t_n))_n = 0 \neq f(x) = \frac{1}{b}$

**Claim 2:**  $f$  is continuous at  $t \in \mathbb{R} \setminus \mathbb{Q}$ : let  $t \in \mathbb{R} \setminus \mathbb{Q}$ ,  $t > 0$ . Let  $n \in \mathbb{N}$ . Consider

$$A_n = \left\{ \frac{a}{b} \mid 1 \leq b \leq n \right\} \cap (t-1, t+1).$$

We claim that  $A_n$  is finite.

$$\begin{aligned} t-1 &< \frac{a}{b} < t+1 \\ b(t-1) &< a < b(t+1) \\ t-1 &< a < n(t+1), \end{aligned}$$

so there are finitely many values of  $a$  and finitely many values of  $b$  — therefore,  $A_n$  is finite. One can find  $\delta > 0$  such that  $(t-\delta, t+\delta) \cap A_n = \emptyset$

Given  $\varepsilon > 0$ , find  $n_0 \in \mathbb{N}$  with  $\frac{1}{n_0} < \varepsilon$ . Let  $\delta$  be such that  $(t-\delta, t+\delta) \cap A_{n_0} = \emptyset$ . If  $x \in (t-\delta, t+\delta)$ ,

$$\begin{aligned} |f(x) - f(t)| &= |f(x)| \\ &= \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \text{ lowest terms} \end{cases} \end{aligned}$$

but  $\frac{1}{b} < \varepsilon$  because  $x \notin A_{n_0}$ , meaning  $b > n_0$ .

## Extension of Function

Consider

$$g(x) = \sin\left(\frac{1}{x}\right) \quad x \neq 0$$

Assuming that  $g$  is continuous on its domain, can we find a  $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\tilde{g}(x) = g(x) \quad \forall x \in \mathbb{R} \setminus \{0\}$$

If such a  $\tilde{g}$  existed, we would expect that  $\lim_{x \rightarrow 0} \tilde{g}(x) = \tilde{g}(0)$ . But,  $\lim_{x \rightarrow 0} \tilde{g}(x) = \lim_{x \rightarrow 0} g(x)$ . However, since  $\lim_{x \rightarrow 0} g(x)$  DNE, so such an extension does not exist.

Therefore,  $x = 0$  is known as a non-removable discontinuity (i.e., we cannot create an extension of the function that “fills in” the function).

However, not all discontinuities involving  $\sin(1/x)$  are non-extendible:

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

### Jump Discontinuities

Suppose  $\lim_{x \rightarrow c^-} f(x) = L$ ,  $\lim_{x \rightarrow c^+} f(x) = R$ . If  $L \neq R$ , then  $x = c$  is a jump discontinuity.

### Lipschitz Functions

A function  $f : D \rightarrow \mathbb{R}$  is called Lipschitz if  $\exists c \geq 0$  with

$$|f(x) - f(y)| \leq c|x - y| \quad \forall x, y \in D$$

The linear function  $f(x) = ax + b$  is a Lipschitz function. Additionally, if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear, then  $\|T(\vec{v}) - T(\vec{w})\| \leq c\|\vec{v} - \vec{w}\|$  for any norm on  $\mathbb{R}^n$  and  $\mathbb{R}^m$ .

- If  $c < 1$ , then  $f$  is a contraction.
- If  $c = 1$  and  $|f(x) - f(y)| = |x - y|$ ,  $f$  is called an isometry.

Lipschitz functions are continuous on their domain:

**Proof:** Let  $c \in D$ , let  $\epsilon > 0$ . Set  $\delta = \epsilon/c$ .

$$\begin{aligned} |x - c| &< \delta \\ |f(x) - f(c)| &\leq c|x - c| \\ |f(x) - f(c)| &< c\delta \\ &= \epsilon \end{aligned}$$

If  $f(x) = \sin(x)$ , then

$$\begin{aligned} |\sin(x) - \sin(y)| &= \left| 2 \sin\left(\frac{1}{2}(x - y)\right) \cos\left(\frac{1}{2}(x + y)\right) \right| \\ &\leq 2 \frac{1}{2} |x - y| \\ &= |x - y| \end{aligned}$$

### Equality over a Dense Subset

Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Let  $E \subseteq \mathbb{R}$ . If  $f(x) = g(x) \forall x \in E$ , then  $f = g$ .

Let  $t \in \mathbb{R}$ . Since  $E$  is dense,  $\exists (x_n)_n \in E$  such that  $(x_n)_n \rightarrow t$ . So,  $(f(x_n))_n \rightarrow t$  because  $f$  is continuous, and  $(g(x_n))_n \rightarrow g(t)$  because  $g$  is continuous.

However, since  $f(x_n) = g(x_n) \forall x_n$ , it must be the case that  $f(t) = g(t)$ .

## Boundedness over a Dense Subset

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Suppose  $f|_E$  is bounded. That is,  $\exists c$  such that

$$|f(x)| \leq c, \quad \forall x \in E$$

Then,  $f$  is bounded.

Let  $t \in \mathbb{R}$ . Since  $E$  is dense,  $\exists (x_n)_n \in E$  such that  $(f(x_n))_n \rightarrow t$ . Then,

$$|f(x_n)_n| \leq c,$$

meaning that  $f(t) \leq c$ .

## Strictly Positive Continuity

If  $f$  is continuous at  $x = c$  and  $f(c) > 0$ , then  $\exists \delta > 0$  and  $\exists m > 0$  with  $f(x) \geq m \forall x \in V_\delta(c)$ . Similarly for the negative case.

Let  $\varepsilon = f(c)/2 > 0$ . Then,  $\exists \delta > 0$  such that  $\forall x \in V_\delta(c)$ ,  $f(x) \in V_\varepsilon(f(c)) = (f(c)/2, 3f(c)/2)$ . Set  $m = f(c)/2$ .

## Continuity Under Operations

Let  $f, g : D \rightarrow \mathbb{R}$ ,  $c \in D$ .

- (1) If  $f, g$  are continuous at  $x = c$ , then  $f \pm g$  are continuous at  $x = c$ . Similarly, if  $f, g$  are continuous on  $D$ , then  $f \pm g$  is continuous on  $D$ .
- (2) Let  $\alpha \in \mathbb{R}$ . If  $f$  is continuous at  $x = c$  or on  $D$ , then  $\alpha f$  is continuous at  $x = c$  or  $D$  respectively.
- (3) If  $f, g$  are continuous at  $x = c$  or on  $D$ , then  $f \cdot g$  is continuous on  $x = c$  or  $D$  respectively.
- (4) If  $f, g$  are continuous at  $x = c$ , and  $g(c) \neq 0$ , then  $\frac{f}{g}$  is continuous at  $c$ . Likewise, if  $f, g$  are continuous on  $D$  and  $g(x) \neq 0 \forall x \in D$ , then  $\frac{f}{g}$  is continuous.
- (5) If  $g$  is continuous at  $x = c$  and  $f$  is continuous at  $d = g(c)$ , then  $f \circ g$  is continuous at  $x = c$ . If  $\text{ran}(g) \subseteq \text{dom}(f)$ , with  $f, g$  continuous on their domain, then  $f \circ g$  is continuous.

Remark on (4): If  $g(c) \neq 0$ , then  $g \neq 0$  on a  $\delta$ -neighborhood of  $c$ .

Proof of (5): Let  $(x_n)_n \rightarrow c$ . Then,  $g(x_n)_n \rightarrow g(c)$ . So,  $(f(g(x_n)))_n \rightarrow f(g(c))$ .

## Further Operations

- (1) If  $f : D \rightarrow \mathbb{R}$  is continuous, and  $f(x) \geq 0$  on  $D$ , then  $\sqrt{f(x)}$  is continuous on  $D$ .
- (2) If  $f : D \rightarrow \mathbb{R}$  is continuous on  $D$ , then  $|f(x)|$  is continuous.
- (3) Polynomials and Rational functions are continuous on their domain.
- (4) If  $f(x), g(x)$  are continuous, then  $h(x) = \max(f(x), g(x))$  and  $k(x) = \min(f(x), g(x))$ .

Fundamental Theorem of Continuous Functions on  $[a, b]$ 

**Boundedness Theorem:** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $\|f\|_u < \infty$ .

**Proof:** Suppose it is not the case. Given any  $n \geq 1$ ,  $\exists x_n \in [a, b]$  with  $|f(x_n)| \leq n$ . We thus have a sequence  $(x_n)_n \in [a, b]$ .

By Bolzano-Weierstrass,  $\exists (x_{n_k})_k \rightarrow x \in [a, b]$ . So,  $f(x_{n_k}) \rightarrow f(x)$ . In particular,  $(f(x_{n_k}))_k$  is bounded; however,  $f(x_{n_k}) \geq k$ .  $\perp$

**Note:** It is possible for  $f$  to be bounded on an infinite interval where it does not attain the supremum or infimum.

Let  $f : D \rightarrow \mathbb{R}$ .

- (1)  $f$  has an absolute maximum on  $D$  if  $\exists x_M \in D$  with  $f(x) \leq f(x_M) \forall x \in D$ . Notably, this means  $\sup_{x \in D} f(x) = f(x_M)$ .
- (2)  $f$  has an absolute minimum on  $D$  if  $\exists x_m \in D$  with  $f(x_m) \leq f(x) \forall x \in D$ . Notably, this means  $\inf_{x \in D} f(x) = f(x_m)$ .

**Extreme Value Theorem (EVT):** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  admits an absolute minimum and absolute maximum.

**Proof:** We know that  $\sup_{x \in [a, b]} f(x) = u < \infty$  by the boundedness theorem. For each  $n \in \mathbb{N}$ ,  $\exists x_n \in [a, b]$  such that

$$u - \frac{1}{n} < f(x_n) \leq u.$$

Thus, there is a sequence  $(x_n)_n \in [a, b]$  — by Bolzano-Weierstrass,  $\exists (x_{n_k})_k \rightarrow x^*$  for some  $x^* \in [a, b]$ . So, for each  $k$ ,

$$\begin{aligned} u - \frac{1}{n_k} < f(x_{n_k}) &\leq u \\ u < f(x^*) &\leq u. \end{aligned} \quad \text{since } f \text{ is continuous}$$

So, by the squeeze theorem,  $f(x^*) = u$  is our absolute max.

**Corollary to the Extreme Value Theorem:** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous with  $f(x) > 0 \forall x \in [a, b]$ , then  $\exists \alpha > 0$  such that  $f(x) \geq \alpha \forall x \in [a, b]$ .

**Proof:** By the previous theorem, we know  $\exists x_m \in [a, b]$  such that  $f(x) \geq f(x_m) \forall x \in [a, b]$ . But  $\alpha := f(x_m) > 0$  by definition.

**Location of Roots:** We will use this to prove the Intermediate Value Theorem. Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Suppose  $f(a) < 0$  and  $f(b) > 0$ , or  $f(a) > 0$  and  $f(b) < 0$ . Then,  $\exists c \in (a, b)$  such that  $f(c) = 0$ .

**Proof:** Assume  $f(a) < 0$  and  $f(b) > 0$ . Let  $N = \{x \in [a, b] \mid f(x) \geq 0\}$ . Since  $b \in N$ ,  $N \neq \emptyset$ . Let  $z = \inf N$ . We claim that  $f(z) = 0$ .

We know that  $\exists (x_n)_n \in N$  with  $x_n \rightarrow z$ . Since  $(x_n)_n \in N$ ,  $f(x_n) \geq 0 \forall n \geq 1$ . However,  $f(x_n) \rightarrow f(z)$  since  $f$  is continuous. So,  $f(z) \geq 0$ .

Suppose toward contradiction that  $f(z) > 0$ . So,  $\exists \delta > 0$  such that  $f(x) \geq \frac{f(z)}{2}$  on  $(z - \delta, z + \delta)$ . Then,  $z - \frac{\delta}{2} \in N$ .  $\perp$

**Intermediate Value Theorem (IVT):** Let  $f : I \rightarrow \mathbb{R}$ , where  $I$  is any interval. Suppose  $\exists x_1, x_2 \in I$  and  $k \in \mathbb{R}$ , with  $f(x_1) < k < f(x_2)$ . Then,  $\exists \xi$  strictly between  $x_1$  and  $x_2$ , with  $f(\xi) = k$ .

**Proof:** Clearly,  $x_1 \neq x_2$ . Suppose  $x_1 < x_2$ . Consider  $g : [x_1, x_2] \rightarrow \mathbb{R}$ ,  $g(x) = f(x) - k$ . So,  $g$  is continuous (as  $f$  is continuous), and  $g(x_1) = f(x_1) - k < 0$ , and  $g(x_2) = f(x_2) - k > 0$ . Thus,  $\exists \xi \in [x_1, x_2]$  with  $g(\xi) = 0$ , whence  $f(\xi) = k$ .

**Corollary to IVT and EVT:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. If  $\inf_{[a,b]} f \leq k \leq \sup_{[a,b]} f$ , then  $\exists c \in [a, b]$  with  $f(c) = k$ .

**Proof:** We know that by EVT,  $\exists x_m, x_M$  with  $\inf_{[a,b]} f = f(x_m)$  and  $\sup_{[a,b]} f = f(x_M)$ . So,  $f(x_m) \leq k \leq f(x_M)$ . Apply IVT.

**Preservation of Intervals 1:** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f([a, b]) = [c, d]$ .

**Proof:** Set  $c = \inf_{[a,b]} f$  and  $d = \sup_{[a,b]} f$ . By definition,  $c \leq f(x) \leq d$ , meaning  $f([a, b]) \subseteq [c, d]$ . By the previous corollary, if  $k \in [c, d]$ , then  $\exists \xi \in [a, b]$  with  $f(\xi) = k$ . Thus,  $[c, d] \subseteq f([a, b])$ .

**Preservation of Intervals 2:** Let  $I$  be any interval, and  $f : I \rightarrow \mathbb{R}$  continuous. Then,  $f(I)$  is an interval.

**Proof:** Let  $\alpha, \beta \in f(I)$ . WLOG,  $\alpha < \beta$ . We will show that  $[\alpha, \beta] \subseteq f(I)$ . Say  $f(a) = \alpha$  and  $f(b) = \beta$  for some  $a, b \in I$ . Note that  $a \neq b$ . Let  $\alpha < k < \beta$ . By IVT,  $\exists \xi$  strictly between  $a$  and  $b$  with  $f(\xi) = k$ . If  $a < b$ , then  $[a, b] \subseteq I$ , and if  $b < a$ , then  $[b, a] \subseteq I$ . Thus,  $\xi \in I$ .

### Uniform Continuity

A function  $f : D \rightarrow \mathbb{R}$  is **uniformly continuous** on  $D$  if

$$(\forall \epsilon > 0)(\exists \delta > 0) \text{ such that } u, v \in D, |u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon$$

Uniform continuity is different from continuity in that  $f$  is continuous at a point  $x = c$  if

$$(\forall \epsilon > 0)(\exists \delta > 0) \text{ such that } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

In (non-uniform) continuity,  $\delta = \delta(\epsilon, c)$ .

For example, if  $f(x) = \frac{1}{x}$  and  $D = (0, \infty)$ , we will show that  $f$  is continuous at  $x = c > 0$ .

$$\begin{aligned} |f(x) - f(c)| &= \left| \frac{1}{x} - \frac{1}{c} \right| \\ &= \frac{1}{c} \frac{1}{x} |x - c| \end{aligned}$$

if  $0 < \delta < c/2$  and  $|x - c| < \delta$ , then  $x \geq c/2$ . Thus,

$$\begin{aligned} |f(x) - f(c)| &= \frac{1}{c} \frac{2}{c} |x - c| \\ &= \frac{2}{c^2} |x - c|. \end{aligned}$$

Given  $\epsilon > 0$ , pick  $\delta = \frac{1}{2} \min\left(\frac{c}{2}, \frac{2}{c^2} \epsilon\right)$ . Thus, if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \epsilon$

Specifically, we can see that on this domain, we require that  $\delta$  be a function of  $\varepsilon$  and  $c$ .

However, if we look at  $f(x) = \frac{1}{x}$  on  $[1, \infty)$ , we can see that for  $u, v \geq 1$ ,

$$\begin{aligned} |f(u) - f(v)| &= \left| \frac{1}{u} - \frac{1}{v} \right| \\ &= \frac{1}{uv} |v - u| \\ &\leq |v - u| \end{aligned}$$

Given  $\varepsilon > 0$ , set  $\delta = \varepsilon$ . If  $|u - v| < \delta$ , then  $|f(u) - f(v)| < \varepsilon$ .

Here, we see that  $\delta = \delta(\varepsilon)$ .

We will show that  $f(x) = x^2$  is uniformly continuous on  $[1, 4]$ .

$$\begin{aligned} |f(u) - f(v)| &= |u^2 - v^2| \\ &= |u - v||u + v| \\ &\leq |u - v|(|u| + |v|) && \text{Triangle Inequality} \\ &\leq 8|u - v| \end{aligned}$$

Given  $\varepsilon > 0$ , set  $\delta = \varepsilon/8$ . Whenever  $u, v \in [1, 4]$ , with  $|u - v| < \delta$ , then  $|f(u) - f(v)| < \varepsilon$

#### Lipschitz and Uniform Continuity

If  $f : D \rightarrow \mathbb{R}$  is Lipschitz, then  $f$  is uniformly continuous.

If  $f : D \rightarrow \mathbb{R}$  is Lipschitz, then  $\exists c > 0$  such that  $\forall u, v \in D$ ,

$$|f(u) - f(v)| \leq c|x - y|.$$

Given  $\varepsilon > 0$ , set  $\delta = \frac{\varepsilon}{c}$ . Whenever  $|u - v| < \delta$ , then  $|f(x) - f(y)| < \varepsilon$ .

#### Uniform Continuity and Continuity

If  $f : D \rightarrow \mathbb{R}$  is uniformly continuous, then  $f$  is continuous on  $D$ .

Let  $c \in D$ . Given  $\varepsilon > 0$ , by uniform continuity,  $\exists \delta > 0$  such that

$$\begin{aligned} |u - v| < \delta &\Rightarrow |f(u) - f(v)| < \varepsilon \\ |x - c| < \delta &\Rightarrow |f(x) - f(c)| < \varepsilon \end{aligned}$$

#### Negating Uniform Continuity

The following are equivalent for  $f : D \rightarrow \mathbb{R}$

- (i)  $f$  is *not* uniformly continuous
- (ii)  $\exists \varepsilon_0$  such that  $\forall \delta > 0$ ,  $\exists u_\delta, v_\delta$  such that  $|u_\delta - v_\delta| < \delta$  and  $|f(u_\delta) - f(v_\delta)| \geq \varepsilon_0$
- (iii)  $\exists \varepsilon_0$  such that  $\exists (u_n)_n, (v_n)_n \in D$  with  $(u_n - v_n)_n \rightarrow 0$  and  $|f(u_n) - f(v_n)| \geq \varepsilon_0$

(i)  $\Leftrightarrow$  (ii): Negating definition.

(ii)  $\Rightarrow$  (iii): Set  $\delta_n = 1/n$  in (ii). Given  $\delta_n$ , it must be the case that

$$|u_n - v_n| < \frac{1}{n}$$

so  $(u_n - v_n)_n \rightarrow 0$ , and

$$|f(u_n) - f(v_n)| \geq \varepsilon_0.$$

(iii)  $\Rightarrow$  (ii): Let  $\delta > 0$ . Then,  $\exists N \in \mathbb{N}$  large such that  $|u_N - v_N| < \delta$ , by the definition of sequence convergence. Set  $u_\delta = u_N$  and  $v_\delta = v_N$ .

#### Showing Non-Uniform Continuity

We will show that  $f(x) = \frac{1}{x}$  is not uniformly continuous on  $(0, 1)$ .

Set  $u_n = 1/n$ , and  $v_n = \frac{1}{n+1}$ . Then,

$$\begin{aligned} |f(u) - f(v)| &= |n - (n+1)| \\ &= 1 \\ &= \varepsilon_0 \\ |u_n - v_n| &= \left| \frac{1}{n} - \frac{1}{n+1} \right| \\ &= \frac{1}{n(n+1)} \\ &\rightarrow 0 \end{aligned}$$

Consider  $f(x) = x^2$  on  $[0, \infty)$ . We will show that  $f$  is not uniformly continuous.

Let  $u_n = n$  and  $v_n = n + \frac{1}{n}$ . Clearly,  $(u_n - v_n)_n \rightarrow 0$ .

$$\begin{aligned} |f(u_n) - f(v_n)| &= \left| n^2 - \left( n + \frac{1}{n} \right)^2 \right| \\ &= \left| n^2 - n^2 - 2 - \frac{1}{n^2} \right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2 \end{aligned}$$

#### Uniform Continuity Theorem

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  is uniformly continuous.

Suppose toward contradiction that  $f$  is not uniformly continuous. Then,  $\exists (u_n)_n, (v_n)_n \in [a, b]$  and  $\varepsilon_0 > 0$  such that  $(u_n - v_n)_n \rightarrow 0$  and  $|f(u_n) - f(v_n)| \geq \varepsilon > 0$ .

Since  $(u_n)_n$  is bounded,  $\exists n_k$  such that  $(u_{n_k})_k \rightarrow z$  by the Bolzano-Weierstrass. We claim that  $(v_{n_k})_k \rightarrow z$ :

$$\begin{aligned} |v_{n_k} - z| &= |v_{n_k} - u_{n_k} + u_{n_k} - z| \\ &\leq |v_{n_k} - u_{n_k}| + |u_{n_k} - z| \\ &\rightarrow 0. \end{aligned}$$

So,

$$\begin{aligned} 0 < \varepsilon_0 &\leq |f(u_{n_k}) - f(v_{n_k})| \\ &\rightarrow 0 \end{aligned}$$

since  $(f(u_k))_k \rightarrow f(z)$  and  $(f(v_k))_k \rightarrow f(z)$ .

### Uniform Continuity and Lipschitz (or lack thereof)

The function  $f(x) = \sqrt{x}$  on  $[0, 1]$  is uniformly continuous. However,  $f(x) = \sqrt{x}$  is not Lipschitz:

Suppose toward contradiction that  $f$  is Lipschitz.

$$|f(x) - f(y)| \leq c|x - y| \quad \forall x, y \in [0, 1]$$

Take  $y = 0$ .

$$\begin{aligned} \sqrt{x} &\leq cx \\ 0 < \frac{1}{c} &\leq \sqrt{x} \end{aligned}$$

### Lemma: Uniform Continuity and Cauchy Sequences

Let  $f : D \rightarrow \mathbb{R}$  be uniformly continuous. If  $(x_n)_n \in D$  is Cauchy, then  $(f(x_n))_n$  is Cauchy.

This is not true for mere continuity. For example, for  $f(x) = \frac{1}{x}$  in  $(0, \infty)$ ,  $(x_n)_n = \frac{1}{n}$  is Cauchy in  $(0, \infty)$ , but  $f(x_n) = n$  is not Cauchy.

Let  $(x_n)_n$  be Cauchy. Let  $\varepsilon > 0$ . Then,  $\exists \delta > 0$  such that  $\forall u, v \in D$  with  $|u - v| < \delta$ , we have  $|f(u) - f(v)| < \varepsilon$ .

Since  $(x_n)_n$  is Cauchy,  $\exists N \in \mathbb{N}$  such that for  $p, q \geq N$ ,  $|x_p - x_q| < \delta$ . So,  $|f(x_p) - f(x_q)| < \varepsilon$ . So,  $(f(x_n))_n$  is Cauchy.

### Continuous Extension Theorem

Let  $f : (a, b) \rightarrow \mathbb{R}$  be a map. The following are equivalent:

- (1)  $f$  is uniformly continuous.
- (2)  $\exists \tilde{f} : [a, b] \rightarrow \mathbb{R}$  such that
  - $\tilde{f}$  is continuous
  - $\tilde{f}(x) = f(x) \forall x \in (a, b)$

(2)  $\Rightarrow$  (1): Since  $\tilde{f}$  is continuous on  $[a, b]$ ,  $\tilde{f}$  is uniformly continuous on  $[a, b]$ . So,  $\tilde{f}$  is uniformly continuous on  $(a, b)$ . But,  $\tilde{f} = f$  on  $(a, b)$ . So,  $f$  is uniformly continuous.



(1)  $\Rightarrow$  (2): Let  $f : (a, b) \rightarrow \mathbb{R}$  be uniformly continuous.

**Claim:**  $\lim_{x \rightarrow a^+} f(x)$  exists. Let  $(x_n)_n$  be any sequence where  $x_n > a$  and  $(x_n)_n \rightarrow a$ . Then,  $(x_n)_n$  is Cauchy. So, by the lemma,  $(f(x_n))_n$  is Cauchy. Since  $\mathbb{R}$  is complete,  $\exists L \in \mathbb{R}$  such that  $(f(x_n))_n \rightarrow L$ .

We claim that the limit is  $L$ . Let  $(y_n)_n$  be any sequence with  $y_n > a$ ,  $(y_n)_n \rightarrow a$ . By our work above,  $(f(y_n))_n \rightarrow L'$  for some  $L' \in \mathbb{R}$ . Consider  $z_n = (x_1, y_1, x_2, y_2, \dots)$ . Then,  $z_n > a$  with  $(z_n)_n \rightarrow a$ . By our work above,  $(f(z_n))_n \rightarrow L''$ , for some  $L'' \in \mathbb{R}$ . Since  $(f(x_n))_n$  is a subsequence of  $(f(z_n))_n$ ,  $(f(x_n))_n \rightarrow L''$ , so  $L = L''$ , and similarly,  $L' = L''$ .

Therefore,  $L = L'$ . So, we have  $\lim_{x \rightarrow a^+} f(x) = L$ .

Similarly,  $\lim_{x \rightarrow b^-} f(x) = R$  exists. Set  $\tilde{f} : [a, b] \rightarrow \mathbb{R}$  such that

$$\tilde{f}(x) = \begin{cases} f(x) & x \in (a, b) \\ L & x = a \\ R & x = b \end{cases}$$

Then,  $\tilde{f}$  is the desired continuous extension.

#### Applying the Continuous Extension Theorem

If  $f(x) = \sin(1/x)$ , then  $f(x)$  is not uniformly continuous on  $(0, 1)$ . This is because  $\lim_{x \rightarrow 0^+} f(x)$  does not exist.

Meanwhile,  $g(x) = x \sin(1/x)$  is uniformly continuous on  $(0, 1)$ , since we can define  $\tilde{g}(x)$  as follows:

$$\tilde{g}(x) = \begin{cases} 0 & x = 0 \\ g(x) & 0 < x < 1 \end{cases}$$

#### Approximations by Step Function

A map  $s : [a, b] \rightarrow \mathbb{R}$  is called a step function if

- (1)  $[a, b] = \bigsqcup_{j=1}^n I_j$  where  $I_j$  are intervals.
- (2)  $\exists c_1, \dots, c_n \in \mathbb{R}$  such that  $s(x) = c_j \forall x \in I_j$ .

Alternatively, this is equivalent to:

$$s = \sum_{j=1}^n c_j \mathbb{1}_{I_j}$$

**Theorem:** If  $f : [a, b] \rightarrow \mathbb{R}$  is uniformly continuous and  $\varepsilon > 0$ , then  $\exists s : [a, b] \rightarrow \mathbb{R}$  with  $\|f - s\|_u < \varepsilon$ .

We know that  $f$  is uniformly continuous. Given  $\varepsilon > 0$ ,  $\exists \delta > 0$  with  $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$ .

Choose  $N$  large such that

$$\Delta_n = \frac{b-a}{N} < \delta.$$

Set  $x_j = j\Delta_N$ . Set  $I_j = [x_j, x_{j+1})$  with  $0 \leq j \leq N-1$ .

Set  $c_j = f(x_j)$ ,

$$s = \sum_{j=0}^{N-1} c_j \mathbb{1}_{I_j}.$$

If  $x \in [a, b]$ ,  $x \in I_k$  for some  $k = 0, \dots, N-1$ . Then,

$$\begin{aligned} |f(x) - s(x)| &= |f(x) - c_k| \\ &\leq |f(x) - f(x_k)| \\ &< \varepsilon \end{aligned}$$

since

$$\begin{aligned} |x - x_k| &< \Delta_N \\ &< \delta \end{aligned}$$

so,

$$\|f - s\|_u < \varepsilon$$

### Approximation by Piecewise Linear Function

A function  $g$  is piecewise linear if

(a)  $[a, b] = \bigcup_{j=1}^n I_j$ , where  $I_j$  are intervals.

(b)  $g|_{I_j}$  is linear;  $\exists a_1, b_1, \dots, a_n, b_n$  with  $g(x) = a_j + b_j x \forall x \in I_j$ .

**Theorem:** If  $f : [a, b] \rightarrow \mathbb{R}$  is uniformly continuous and  $\varepsilon > 0$ , then there is a continuous piecewise linear  $g : [a, b] \rightarrow \mathbb{R}$  with  $\|f - g\|_u < \varepsilon$ .

We know that  $f$  is uniformly continuous. Given  $\varepsilon > 0$ ,  $\exists \delta > 0$  with  $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon/2$ . Choose  $N$  large such that

$$\Delta_n = \frac{b-a}{N} < \delta.$$

Set  $x_j = j\Delta_N$ . Set  $I_j = [x_j, x_{j+1})$  with  $0 \leq j \leq N-1$ .

Set  $g(x) = \sum_{k=0}^{N-1} g_k(x) \mathbb{1}_{I_k}$ , where

$$g_k(x) = f(x_k) + \left( \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right) (x - x_k)$$

We observe that if  $x \in I_k$ , then

$$\begin{aligned} |f(x) - g(x)| &= \left| f(x) - f(x_k) - \left( \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right) (x - x_k) \right| \\ &\leq |f(x) - f(x_k)| + |f(x_{k+1}) - f(x_k)| \frac{|x - x_k|}{|x_{k+1} - x_k|} \\ &\leq |f(x) - f(x_k)| + |f(x_{k+1}) - f(x_k)| \\ &< \varepsilon \end{aligned}$$

so,

$$\|f - g\| < \varepsilon$$

### Monotone Functions

Let  $D \subseteq \mathbb{R}$ ,  $f : D \rightarrow \mathbb{R}$ .

- (1)  $f$  is increasing if  $x_1, x_2 \in D$  with  $x_1 \leq x_2$  implies  $f(x_1) \leq f(x_2)$ .
- (2)  $f$  is strictly increasing if  $x_1, x_2 \in D$  with  $x_1 < x_2$  implies  $f(x_1) < f(x_2)$ .
- (3)  $f$  is monotone if  $f$  is increasing or decreasing.
- (4)  $f$  is strictly monotone if  $f$  is strictly increasing or strictly decreasing.

If  $f : D \rightarrow \mathbb{R}$  is increasing or strictly increasing, then  $-f : D \rightarrow \mathbb{R}$  is decreasing or strictly decreasing (respectively).

Additionally, monotone functions are not always continuous. However, one-sided limits always exist.

Let  $I$  be an interval,  $f : I \rightarrow \mathbb{R}$  increasing. Let  $c \in I$ , where  $c$  is not an endpoint. Then,

- (1)  $\lim_{x \rightarrow c^-} f(x) = \sup_{x \in I, x < c} f(x)$
- (2)  $\lim_{x \rightarrow c^+} f(x) = \inf_{x \in I, x > c} f(x)$

are both existent and finite:

**Proof of (1):** Since  $c$  is not an endpoint,  $\{x \mid x \in I, x < c\} \neq \emptyset$  and is bounded above by  $c$ . Therefore,  $\{f(x) \mid x \in I, x < c\}$  is nonempty and bounded above by  $f(c)$  (since  $f$  is increasing). So,  $u = \sup_{x \in I, x < c} f(x)$  exists.

Let  $\varepsilon > 0$ .  $\exists x_\varepsilon \in I$  with  $x_\varepsilon < c$  such that  $u - \varepsilon < f(x_\varepsilon)$ . Set  $\delta = c - x_\varepsilon > 0$ . If  $x \in I$ ,  $c - x < \delta$ , then  $x_\varepsilon < x < c$ , so  $f(x_\varepsilon) \leq f(x) \leq f(c)$ . So,  $u - f(x) \leq u - f(x_\varepsilon) < \varepsilon$ . But,  $u \geq f(x)$ , so  $u - f(x) = |u - f(x)|$ . Thus,  $0 < c - x < \delta \Rightarrow |u - f(x)| < \varepsilon$ . Thus,  $u = \lim_{x \rightarrow c^-} f(x)$ .

Let  $I$  be an interval,  $f : I \rightarrow \mathbb{R}$  increasing. Suppose  $c \in I$  is not an endpoint. The following are equivalent:

- (1)  $f$  is continuous at  $x = c$ .
- (2)  $\lim_{x \rightarrow c} f(x) = f(c)$ .
- (3)  $\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$ .
- (4)  $\sup_{x \in I, x < c} f(x) = f(c) = \inf_{x \in I, x > c} f(x)$ .

Suppose  $c$  is a right endpoint of  $I$ . The following are equivalent:

- (1)  $f$  is continuous at  $x = c$ .
- (2)  $\lim_{x \rightarrow c^-} f(x) = f(c)$ .
- (3)  $\sup_{x \in I, x < c} f(x) = f(c)$ .

Suppose  $c$  is a left endpoint of  $I$ . The following are equivalent:

- (1)  $f$  is continuous at  $x = c$ .
- (2)  $\lim_{x \rightarrow c^+} f(x) = f(c)$ .
- (3)  $\inf_{x \in I, x > c} f(x) = f(c)$ .

We can make a similar set of corollaries with decreasing functions.

### Jump of a Function

Let  $I$  be an interval,  $f : I \rightarrow \mathbb{R}$  increasing.

- (1) If  $c$  is not an endpoint of  $I$ , we define the jump of  $f$  at  $x = c$  as:

$$j_f(c) = \lim_{x \rightarrow c^+} f(x) - \lim_{x \rightarrow c^-} f(x)$$

- (2) If  $c$  is a left endpoint of  $I$ , we define the jump of  $f$  at  $x = c$  as:

$$j_f(c) = \lim_{x \rightarrow c^+} f(x) - f(c)$$

- (3) If  $c$  is a right endpoint of  $I$ , we define the jump of  $f$  at  $x = c$  as:

$$j_f(c) = f(c) - \lim_{x \rightarrow c^-} f(x)$$

We claim that  $f$  is continuous at  $c \in I$  if and only if  $j_f(c) = 0$ .

If  $c$  is not an endpoint, then  $f$  is continuous at  $x = c$  if and only if  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$ .

If  $c$  is a left endpoint, then  $f$  is continuous at  $x = c$  if and only if  $f(c) = \lim_{x \rightarrow c^+} f(x)$ , if and only if  $j_f(c) = 0$ .

### Countability of Monotone Function Discontinuities

Let  $I \subseteq \mathbb{R}$  be any interval. Let  $f : I \rightarrow \mathbb{R}$  be monotone. Then,  $D = \{x \in I \mid f \text{ not continuous at } x = c\}$  is countable.

For the sake of simplicity, we will assume that  $f$  is monotone increasing.

**Lemma:** Let  $\{x_1, x_2, \dots, x_n\}$  be a partition of  $I = [a, b]$ , where  $a \leq x_1 < x_2 < \dots < x_n \leq b$ . Then,  $f(a) + \sum_{i=1}^n j_f(x_i) \leq f(b)$ .

**Proof of Lemma:** By induction on  $n$ , if  $x_1 = a$ , then

$$\begin{aligned} f(a) + j_f(x_1) &= f(a) + j_f(a) \\ &= f(a) + \lim_{x \rightarrow a^+} f(x) - f(a) \\ &= \lim_{x \rightarrow a^+} f(x) \\ &\leq f(b). \end{aligned}$$

If  $x_1 = b$ , then

$$\begin{aligned} f(a) + j_f(x_1) &= f(a) + j_f(b) \\ &= f(a) + f(b) - \lim_{x \rightarrow b^-} f(b) \\ &= f(b) - (\lim_{x \rightarrow b^-} f(x) - a) \\ &\leq f(b). \end{aligned}$$

If  $a < x_1 < b$ , then

$$\begin{aligned} f(a) + j_f(x_1) &= f(a) + \lim_{x \rightarrow x_1^+} f(x) - \lim_{x \rightarrow x_1^-} f(x) \\ &\leq f(a) - \lim_{x \rightarrow x_1^-} f(x) + f(b) \\ &\leq f(b) \end{aligned}$$

Assume the formula holds for  $n$ . Then, for the  $n+1$  case:

$$\begin{aligned} f(a) + \sum_{i=1}^{n+1} j_f(x_i) &= f(a) + \sum_{i=1}^n f(x_i) + j_f(x_{n+1}) \\ &\leq f(x_n) + j_f(x_{n+1}) \\ &\leq f(b) \end{aligned}$$

**Case 1:** Suppose  $I = [a, b]$ . Consequently,

$$\sum_{i=1}^n j_f(x_i) \leq f(b) - f(a)$$

Let  $G_k = \left\{ x \in [a, b] \mid j_f(x) \geq \frac{f(b)-f(a)}{k} \right\}$ . By the lemma,  $|G_k| \leq k$ . This is because, if  $x_1, \dots, x_n \in G_k$  with  $n > k$ , then

$$\begin{aligned} \sum_{i=1}^n j_f(x_i) &\geq \frac{n(f(b) - f(a))}{k} \\ &> f(b) - f(a) \end{aligned}$$

contradicting the lemma.

Recall that  $f$  is discontinuous at  $x = c$  if and only if  $j_f(c) > 0$ . Therefore, we have that

$$D = \bigcup_{k=1}^{\infty} G_k,$$

So for  $k$  large enough,  $j_f(x) \geq \frac{f(b)-f(a)}{k}$ . Since each  $G_k$  is a finite set,  $D$  is a countable union of countable sets, and is thus countable.

**Case 2:**  $I = (a, b]$ . Write  $I$  as

$$I = \bigcup_{n=1}^{\infty} [a + 1/n, b].$$

Let  $D_n = \{x \in [a + 1/n, b] \mid f \text{ discontinuous at } x\}$ . By case 1,  $D_n$  is countable. Let  $D = \{x \in (a, b] \mid f \text{ discontinuous at } x\}$ . Note that  $D = \bigcup D_n$ . Therefore,  $D$  is countable.

**Case 3:**  $I = [a, b)$ . Write  $I$  as

$$I = \bigcup_{n \geq 1} [a, b - 1/n].$$

Proceed as with case 2.

**Case 4:**  $I = (a, b)$ . Write  $I$  as

$$I = (a, b - \delta] \cup [b - \delta, b).$$

Apply case 2 and case 3.

**Case 5:**  $I = (-\infty, b)$  or  $I = (-\infty, b]$ . Write  $I$  as

$$I = \bigcup_{n \geq 1} (b - n, b)$$

or

$$I = \bigcup_{n \geq 1} (b - n, b].$$

Proceed via the countable union of countable sets.

**Case 6:**  $I = [a, \infty)$  or  $I = [a, \infty]$ . Write  $I$  as

$$I = \bigcup_{n \geq 1} (a, a + n)$$

or

$$I = \bigcup_{n \geq 1} [a, a + n].$$

Proceed via the countable union of countable sets.

**Case 7:**  $I = \mathbb{R}$ . Write  $I$  as

$$I = \bigcup_{n \geq 1} [-n, n].$$

Proceed via the countable union of countable sets.

### Continuous Inverse Theorem

Let  $I \in \mathbb{R}$  be an interval, and let  $f : I \rightarrow \mathbb{R}$  be continuous and strictly monotone. Then,

- (1)  $J = f(I)$  is an interval. (Proved earlier.)
- (2)  $f : I \rightarrow J$  is bijective and thus invertible.
- (3)  $f^{-1} : J \rightarrow I$  is continuous and strictly monotone.

Assume  $f$  is continuous and strictly increasing.

**Proof of (3):** First, we prove  $g : J \rightarrow I$  is also strictly increasing. To see this, let  $y_1, y_2 \in J$ , with  $y_1 < y_2$ . If

$$g(y_1) \geq g(y_2)$$

then,

$$f(g(y_1)) \geq f(g(y_2))$$

$$y_1 \geq y_2,$$

⊥

So  $g(y_1) < g(y_2)$ .

Now, we will show that  $g$  is continuous. Note that since  $f(I) = J$ , it must be the case that  $g(J) = I$ . Suppose toward contradiction that  $g$  is discontinuous at  $x = c \in J$ . Then,  $j_g(c) = \lim_{x \rightarrow c^+} g(x) - \lim_{x \rightarrow c^-} g(x) > 0$ .

So, we find  $x \in I$  with  $\lim_{x \rightarrow c^-} g(x) < x < \lim_{x \rightarrow c^+} g(x)$ . However, since  $g$  is strictly increasing, it follows that  $x \notin \text{Rang}$ . If  $y < c$ , then  $g(y) \leq \lim_{x \rightarrow c^-} g(x)$ , and if  $z > c$ , then  $g(z) \geq \lim_{x \rightarrow c^+} g(x)$ . However, we know that  $g(J) = I$ . ⊥

### The $n$ th Root Function

Let  $n$  be even,  $f : [0, \infty) \rightarrow \mathbb{R}$  where  $f(x) = x^n$ . Clearly,  $f$  is continuous, and  $f$  is also strictly increasing.

- $\text{Ran}(f) = [0, \infty)$ . To see this, we see that  $f(0) = 0$  and  $\lim_{x \rightarrow +\infty} f(x) = +\infty$ . By the Intermediate Value Theorem,  $f$  must obtain every value in  $[0, \infty)$ .

Thus,  $f : [0, \infty) \rightarrow [0, \infty)$  is invertible, and we write  $g : [0, \infty) \rightarrow [0, \infty)$ , where  $g(x) = x^{1/n}$ .

If  $x, y > 0$ , then  $(xy)^{1/n} = x^{1/n}y^{1/n}$ . Note that  $f(uv) = f(u)f(v)$ .

If  $x = f(u)$  and  $y = f(v)$ , then  $f((xy)^{1/n}) = f(g(xy)) = xy = f(g(x))f(g(y)) = f(x^{1/n})f(y^{1/n}) = f(x^{1/n}y^{1/n})$ .

If  $x > 0$ , then  $(x^n)^{1/n} = x = (x^{1/n})^n$ , following from the fact that  $g \circ f(x) = x = f \circ g(x)$ . If  $x < 0$ , then  $(x^n)^{1/n} = |x|$ .

Since  $x < 0$ , we can write

$$\begin{aligned} (x^n)^{1/n} &= ((-|x|)^n)^{1/n} \\ &= ((-1)^n |x|^n)^{1/n} \\ &= |x| \end{aligned}$$

Note that if  $x < 0$ ,  $(x^{1/n})^n$  is not defined.

If  $n$  is odd, then  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^n$  is continuous and strictly increasing with range  $\mathbb{R}$ . By the continuous inverses theorem,  $f^{-1} = g$  is continuous and strictly increasing. We write  $g(x) = x^{1/n}$ .

Similarly as to the even case, we can show that

- $(xy)^{1/n} = x^{1/n}y^{1/n}$

- $\forall x \in \mathbb{R}, (x^{1/n})^n = x = (x^n)^{1/n}$

---

Recall that if  $x \neq 0$  in  $\mathbb{R}$ , then  $x^{-1}$  is defined as the unique value such that  $xx^{-1} = 1$ .

If  $x \neq 0$  and  $n \in \mathbb{N}$ , then  $(x^n)^{-1} = (x^{-1})^n$ . We write  $x^{-n}$  as the common value.

(1) If  $n$  is even and  $x > 0$ , then  $(x^{1/n})^{-1} = (x^{-1})^{1/n}$

(2) If  $n$  is odd, and  $x \neq 0$ , then  $(x^{1/n})^{-1} = (x^{-1})^{1/n}$ .

**Proof:** If  $x > 0$ , then  $x^{1/n} > 0$ . So,

$$\begin{aligned} x^{1/n} (x^{-1})^{1/n} &= (x \cdot x^{-1})^{1/n} \\ &= 1 \end{aligned}$$

So by the uniqueness of inverses, the theorem follows.

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Let  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}$ .

(1) If  $n$  is even,  $x > 0$ , then  $(x^m)^{1/n} = (x^{1/n})^m$

(2) If  $n$  is odd,  $x \neq 0$ , then  $(x^m)^{1/n} = (x^{1/n})^m$

We define the unique values as  $x^{m/n}$ .

## Derivatives

### Differentiation

In this context,  $I$  always refers to an interval, and  $c \in I$ .

A function  $f$  is differentiable at  $c$  if

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite.

In that case, we denote the limit as  $f'(c)$ . The value  $f'(c)$  is called the derivative of  $f$  at  $c$ .

Like with continuity,  $f$  is differentiable on  $I$  if  $f'(c)$  exists  $\forall c \in I$ .

---

Let  $f(x) = ax + b$ ,  $c \in \mathbb{R}$ . Then,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \frac{(ax + b) - (ac + b)}{x - c} \\ &= \frac{a(x - c)}{x - c} \\ &= a \end{aligned}$$



Let  $f(x) = x^2$ ,  $c \in \mathbb{R}$ . Then,

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \rightarrow c} x + c \\ &= 2c\end{aligned}$$

Let  $f(x) = \sqrt{x}$ ,  $c \geq 0$ . Then,

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{\sqrt{x} - \sqrt{c}}{x - c} \\ &= \lim_{x \rightarrow c} \frac{1}{\sqrt{x} + \sqrt{c}} \\ &= \begin{cases} \frac{1}{2\sqrt{c}} & c \neq 0 \\ +\infty & c = 0 \end{cases}\end{aligned}$$

Therefore,  $f'(c)$  exists only when  $c \geq 0$ .

For example,  $f(x) = |x|$  is *not* differentiable at  $c = 0$ .

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{|x|}{x}$$

Let  $(x_n)_n = \frac{(-1)^n}{n}$ . Then,  $(x_n)_n \rightarrow 0$ . However,  $\frac{|x_n|}{x_n} = (-1)^n$ , which diverges. Therefore, the limit does not exist.

Let

$$g(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then,

$$\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \sin(1/x).$$

Let  $(x_n)_n = \frac{2}{\pi n}$ . Then,  $(x_n)_n \rightarrow 0$ , but  $\sin(1/x_n)$  is divergent.

Let  $f(x) = \sin(x)$ ,  $c \in \mathbb{R}$ . Then,

$$f'(c) = \lim_{x \rightarrow c} \frac{\sin(x) - \sin(c)}{x - c}$$

Let  $h = x - c$ . Then,  $x \rightarrow c \Leftrightarrow h \rightarrow 0$ . Then,

$$\begin{aligned}f'(c) &= \lim_{h \rightarrow 0} \frac{\sin(h + c) - \sin(c)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(h) \cos(c) + \cos(h) \sin(c) - \sin(c)}{h} \\ &= \lim_{h \rightarrow 0} \cos(c) \frac{\sin(h)}{h} + \sin(c) \frac{\cos(h) - 1}{h} \\ &= \cos(c)\end{aligned}$$

## Differentiability and Continuity

If  $f : I \rightarrow \mathbb{R}$  is differentiable at  $x = c$ , then  $f$  is continuous at  $x = c$ .

$$\begin{aligned}\lim_{x \rightarrow c} (f(x) - f(c)) &= \lim_{x \rightarrow c} \left( (x - c) \frac{f(x) - f(c)}{x - c} \right) \\ &= \lim_{x \rightarrow c} (x - c) f'(c) \\ &= 0\end{aligned}$$

Thus,  $\lim_{x \rightarrow c} f(x) = f(c)$ , and  $f$  is continuous.

## Operations with Differentiation

Let  $I \in \mathbb{R}$  be an interval,  $c \in I$ . Let  $f, g : I \rightarrow \mathbb{R}$  be differentiable at  $x = c$ . Let  $\alpha \in \mathbb{R}$ . Then,

- (1)  $(\alpha f)'(c) = \alpha f'(c)$
- (2)  $(f + g)'(c) = f'(c) + g'(c)$
- (3)  $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
- (4)  $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$ , provided  $g(c) \neq 0$ .

**Proof of (4):**

$$\begin{aligned}\left(\frac{f}{g}\right)'(c) &= \lim_{x \rightarrow c} \frac{(f/g)(x) - (f/g)(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(c) - f(c)g(x)}{(x - c)g(x)g(c)} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{(x - c)g(x)g(c)} \\ &= \lim_{x \rightarrow c} \frac{g(c)(f(x) - f(c))}{(x - c)g(x)g(c)} - \lim_{x \rightarrow c} \frac{f(c)(g(x) - g(c))}{(x - c)g(x)g(c)} \\ &= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad \text{since } \lim_{x \rightarrow c} g(x) = g(c)\end{aligned}$$

## Rules with Differentiation

Let  $f_n(x) = x^n$ , where  $n \in \mathbb{Z}$ . Then,  $f'_n(x) = nx^{n-1}$ .

**Proof:** Let  $n \geq 1$ . We have already proved the linear case ( $n = 1$ ). Inductively assume true for  $n$ .

Then,

$$\begin{aligned}f'_{n+1}(x) &= (x \cdot f_n(x))' \\ &= f_n(x) + x f'_n(x) \\ &= x^n + x \cdot n x^{n-1} \\ &= (n+1)x^n\end{aligned}$$

Similarly, the proof is clear for  $n = 0$ . Using the quotient rule, we can show the similar case for

$$n < 0.$$

$$f_{-n}(x) = \frac{1}{f_n(x)} \quad n = 1, 2, 3, \dots$$

### Carathéodory's Theorem

If  $f : I \rightarrow \mathbb{R}$ ,  $c \in I$ .  $f$  is differentiable at  $x = c$  if and only if  $\exists \varphi : I \rightarrow \mathbb{R}$  continuous at  $c$  such that  $\forall x \in I$ ,  $f(x) - f(c) = \varphi(x) \cdot (x - c)$ . In this case,  $f'(c) = \varphi(c)$ .

For example, if  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^3$ . Fix  $c \in \mathbb{R}$ . Then,  $f(x) - f(c) = (x - c)(x^2 + cx + c^2)$ . Let  $\varphi(x) = x^2 + cx + c$ . Then,  $\varphi(c) = 3c^2$ .

( $\Rightarrow$ ): Suppose  $\exists \varphi : I \rightarrow \mathbb{R}$  such that  $f(x) - f(c) = \varphi(x)(x - c) \forall x \in I$ . Then,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \varphi(x) \\ &= \varphi(c) \end{aligned}$$

So,  $f$  is differentiable and  $f'(c) = \varphi(c)$ .

( $\Leftarrow$ ) Assume  $f$  is differentiable at  $x = c$ . Let  $\varphi : I \rightarrow \mathbb{R}$

$$\varphi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & x \neq c \\ f'(c) & x = c \end{cases}$$

It is the case that  $\varphi$  is continuous at  $x = c$  since  $\lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) = \varphi(c)$ .

Clearly,  $f(x) - f(c) = \varphi(x)(x - c)$ .

### Chain Rule

Let  $J \xrightarrow{f} I \xrightarrow{g} \mathbb{R}$ , where  $I$  and  $J$  are intervals. Let  $c \in J$  and  $d = f(c) \in I$ . Assume  $f$  is differentiable at  $x = c$ , and  $g$  is differentiable at  $d = f(c)$ . Then,  $g \circ f$  is differentiable at  $c$  and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

We know that  $\exists \varphi : J \rightarrow \mathbb{R}$  with  $\forall x \in J$ ,  $f(x) - f(c) = \varphi(x)(x - c)$ , with  $\varphi$  continuous at  $x = c$ . Similarly,  $\exists \psi : I \rightarrow \mathbb{R}$  with  $\forall y \in I$ ,  $g(y) - g(d) = \psi(y)(y - d)$ .

In particular,  $\forall x \in J$ ,

$$\begin{aligned} g(f(x)) - g(f(c)) &= \psi(f(x))(f(x) - f(c)) \\ &= \psi(f(x))\varphi(x)(x - c), \end{aligned}$$

so

$$g \circ f(x) - g \circ f(c) = \lambda(x)(x - c) \quad \text{where } \lambda(x) = \psi(f(x))\varphi(x)$$

Note that  $\lambda : J \rightarrow \mathbb{R}$  is continuous at  $x = c$  because

- $\varphi$  is continuous at  $x = c$
- $f$  is differentiable at  $x = c$ , and thus continuous at  $x = c$

- $\psi$  is continuous at  $d = f(c)$

Therefore, by Carathéodory's theorem,  $g \circ f$  is differentiable at  $x = c$ .

Additionally,

$$\begin{aligned}(g \circ f)'(c) &= \lambda(c) \\ &= \psi(f(c))\varphi(c) \\ &= \psi(d)\varphi(c) \\ &= g'(d)f'(c).\end{aligned}$$

#### Inverse Functions

Let  $I$  be an interval,  $f : I \rightarrow \mathbb{R}$  strictly monotone and continuous,  $f(I) = J$ . Let  $g : J \rightarrow I$  be the inverse map.

- $J$  is an interval
- $g$  is continuous and strictly monotone
- If  $f$  is differentiable at  $c \in I$ , and  $f'(c) \neq 0$ , then  $g$  is differentiable at  $y = d = f(c)$ , and

$$g'(d) = \frac{1}{f'(c)}$$

#### Inverse Functions, Example

Let  $T : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ ,  $T(x) = \tan(x)$ . Since  $T$  is strictly monotone, continuous, and  $\lim_{x \rightarrow \pi/2^-} T(x) = +\infty$ , and  $\lim_{x \rightarrow -\pi/2^+} T(x) = -\infty$ ,  $T$  is bijective.

Let  $A : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ .

$$\begin{aligned}A'(d) &= \frac{1}{T'(c)} \\ T(c) &= d \\ A'(d) &= \frac{1}{\sec^2(c)} \\ &= \frac{1}{1 + \tan^2(c)} \\ &= \frac{1}{1 + d^2}\end{aligned}$$

Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f_n(x) = x^n$ , where  $n$  is odd. Since  $f$  is strictly monotone, continuous, and surjective,  $f$  is bijective. Let  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  be the inverse. Then,  $g_n(y) = y^{1/n}$ . Let  $f_n(c) = d$ .

$$\begin{aligned}g'_n(d) &= \frac{1}{f'_n(c)} \\ &= \frac{1}{nc^{n-1}} \\ &= \frac{1}{nd^{1-\frac{1}{n}}} \\ &= \frac{1}{n}d^{\frac{1}{n}-1}\end{aligned}$$

The same idea works when  $n$  is even on  $(0, \infty)$ .

**Exercise:** Let  $\frac{m}{n} \in \mathbb{Q}$ . Show that  $\frac{d}{dx} x^{m/n} = \frac{m}{n} x^{m/n-1}$ .

We can write this as a composition and use the chain rule.

#### Fermat's Theorem

Let  $f : I \rightarrow \mathbb{R}$ ,  $c$  an interior point of  $I$ . Suppose  $f$  has a local maximum or minimum at  $x = c$  ( $\exists \delta \ni x \in V_\delta(c) \Rightarrow f(x) \leq f(c)$  or  $\exists \delta \ni x \in V_\delta(c) \Rightarrow f(x) \geq f(c)$ ). Then,

(1)  $f'(c)$  does not exist.

(2)  $f'(c) = 0$ .

If  $f'(c)$  does not exist, there is nothing to prove. Assume  $f'(c)$  does exist.

Suppose toward contradiction that  $f'(c) \neq 0$ .

**Case 1:**  $f'(c) > 0$ . So,

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} > 0,$$

Meaning  $\exists \delta$  such that  $x \in V_\delta(c)$  implies

$$\frac{f(x) - f(c)}{x - c} > 0.$$

So, if  $x \in (c - \delta, c)$ ,

$$\begin{aligned} f(x) - f(c) &= \frac{f(x) - f(c)}{x - c} (x - c) \\ &< 0 \\ f(x) &< f(c), \end{aligned} \tag{*}$$

and if  $x \in (c, c + \delta)$ ,

$$\begin{aligned} f(x) - f(c) &= \frac{f(x) - f(c)}{x - c} (x - c) \\ &> 0 \\ f(x) &> f(c). \end{aligned} \tag{**}$$

If  $c$  is a local minimum, (\*) violates the assumption, and if  $c$  is a local maximum, (\*\*) violates the assumption.  $\perp$

**Warning:** Fermat's theorem does not run in converse:  $f(x) = x^3$ ,  $f'(0) = 0$  but  $x = 0$  is not a local minimum or maximum. Similarly,  $f(x) = x^{1/3}$ ,  $f'(0) = 0$  but  $x = 0$  is not a local minimum or maximum.

#### Rolle's Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$  with  $f$  continuous on  $[a, b]$  and  $f$  differentiable on  $(a, b)$ . If  $f(a) = f(b)$ ,  $\exists c \in (a, b)$  with  $f'(c) = 0$ .

If  $f$  is a constant function, we are done.

Suppose  $f$  is not a constant function.

**Case 1:**  $\exists x \in (a, b)$  with  $f(x) > f(a)$ . By the extreme value theorem and the hypothesis,  $\exists x_M \in (a, b)$  with  $f(x_M) = \sup_{x \in [a, b]} f(x)$ . By Fermat's Theorem,  $f'(x_M) = 0$ .

**Case 2:**  $\exists x \in (a, b)$  with  $f(x) < f(a)$ . By the extreme value theorem,  $\exists x_m \in (a, b)$  with  $f(x_m) = \inf_{x \in [a, b]} f(x)$ . By Fermat's Theorem,  $f'(x_m) = 0$ .

#### Using Rolle's Theorem

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Suppose  $f(a)f(b) < 0$ , and  $f'(x) \neq 0$ . Show  $f$  has a unique real root in  $[a, b]$ .

Without loss of generality,  $f(a) < 0$  and  $f(b) > 0$ . By the intermediate value theorem,  $\exists z \in (a, b)$  with  $f(z) = 0$ .

Suppose toward contradiction  $\exists z' \in (a, b)$  with  $z' \neq z$ . Use Rolle's theorem on  $[z, z']$  or  $[z', z]$ .