

Problem (Problem 1): A topological group is a group which is also a Hausdorff topological space where the group operations are continuous.

Recall the definition of the concatenation operation on the fundamental group. Now, let G be a path-connected topological group, and let $\pi_1(G, e)$ be the fundamental group of G with base point e . Use the Hilton–Eckmann argument to prove that the concatenation operation on the fundamental group is commutative.

Solution: Define two operations, $*$ and \cdot , on the homotopy-classes of functions $f: S^1 \rightarrow (G, e)$, where $S^1 \cong [0, 1]/(\{0\} \sim \{1\})$ given by

$$f * g = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

$$f \cdot g = f(t)g(t),$$

where the latter is multiplication within the group and the former is concatenation. We see that the identity map

$$\text{id}: S^1 \rightarrow (G, e)$$

$$t \mapsto e$$

is an identity for both $*$ and \cdot . Our task now is to show that the Hilton–Eckmann condition holds. That is, let $a, b, c, d: S^1 \rightarrow (G, e)$ be continuous maps with base point e . Then,

$$\begin{aligned} (a * b) \cdot (c * d) &= (a * b)(t) \cdot (c * d)(t) \\ &= \begin{cases} a(2t)c(2t) & 0 \leq t \leq 1/2 \\ b(2t - 1)d(2t - 1) & 1/2 \leq t \leq 1 \end{cases} \\ &= (a \cdot c) * (b \cdot d), \end{aligned}$$

whence $\cdot = *$ and the concatenation operation is commutative.

Problem (Problems 2–4):

- (2) Let M and N be smooth, orientable, closed manifolds of the same dimension n , and let $f: M \rightarrow N$ be a smooth function. Show that f induces a map $f^*: H_{\text{DR}}^n(N) \rightarrow H_{\text{DR}}^n(M)$ which is multiplication by an integer. This is called the degree of f and is written $\deg(f)$.
- (3) Recall the definition of the degree of f from one of the previous problem sets, counting the sums of signs of determinants of the derivative of f over the preimage of a regular value of f . Prove that the two definitions of the degree agree.
- (4) With the setup of the previous exercises, prove that if ω is an arbitrary n -form on N , then

$$\int_M f^* \omega = \deg(f) \int_N \omega.$$

Solution: Letting $\omega \in H_{\text{DR}}^n(N)$ be a nonvanishing top-dimensional form. By the naturality of the de Rham isomorphism, it follows that there is some $\delta \in \mathbb{R}$ such that

$$\int_M f^* \omega = \delta \int_N \omega$$

Our task now is to show that $\delta \in \mathbb{Z}$. In particular, we will show that $\delta = \deg(f)$, where $\deg(f)$ is defined as before.

Toward this end, let q be a regular value of f . We may use a smooth bump function to restrict ω to a

small open neighborhood V of q . It follows then that $f^{-1}(q) = \{p_1, \dots, p_\ell\}$ for some ℓ , with corresponding disjoint open neighborhoods U_1, \dots, U_ℓ locally diffeomorphic to V , whence the support of $f^*\omega$ is contained in the union of U_1, \dots, U_ℓ . If $f^{-1}(q) = \emptyset$, then

$$\begin{aligned} \int_M f^*\omega &= \int_\emptyset f^*\omega \\ &= \delta \int_N \omega \\ &= 0, \end{aligned}$$

whence $\delta = 0$. If $f^{-1}(q) \neq \emptyset$, then we see that

$$\int_M f^*\omega = \sum_{k=1}^{\ell} \int_{U_k} f^*\omega.$$

Now, since f is a local diffeomorphism on each of the U_k , it follows that

$$\begin{aligned} \int_{U_k} f^*\omega &= \text{sgn}(\det(D_{p_k} f)) \int_V \omega \\ &= \text{sgn}(\det(D_{p_k} f)) \int_N \omega. \end{aligned}$$

Therefore, we find that

$$\begin{aligned} \int_M f^*\omega &= \sum_{k=1}^{\ell} \text{sgn}(\det(D_{p_k} f)) \int_N \omega \\ &= \deg(f) \int_N \omega, \end{aligned}$$

giving that $\deg(f)$ as defined via cohomology and as defined via summation over neighborhoods of preimages of a regular value are equal to each other.

Problem (Problem 5): Prove that for all $d \in \mathbb{Z}$ and all n , there exists a smooth map $S^n \rightarrow S^n$ with degree d .

Solution: If $d = 0$, then we may take a constant map to have degree d .

If $d > 0$, select d points, $p_1, \dots, p_d \in S^n$ and corresponding disjoint open neighborhoods U_1, \dots, U_d . We define $f: S^n \rightarrow S^n$ to be a map that takes p_i to the north pole, a diffeomorphism between U_i and $S^n \setminus \{s\}$ (which follows from the coordinate map for U_i composed with the (inverse) south pole stereographic projection), where s is the south pole, and everything outside of $U_1 \cup \dots \cup U_d$ to the south pole.

Similarly, if $d < 0$, perform the same process, but f should be orientation-reversing on each of the U_i .

Problem (Problem 6): Let K and L be smooth knots in \mathbb{R}^3 , which is to say smooth embeddings of S^1 into \mathbb{R}^3 . Assume that K and L are disjoint subsets of \mathbb{R}^3 . Choosing an orientation on S^1 orients K and L . Write $K = f(s)$ and $L = g(t)$, where $t, s \in S^1$, and let $F(s, t) = g(t) - f(s)$ as a difference of vectors in \mathbb{R}^3 . Notice that F never takes on the value 0 . Renormalizing the vectors in $\mathbb{R}^3 \setminus \{0\}$ gives a map $\pi: \mathbb{R}^3 \setminus \{0\} \rightarrow S^2$. Write $G = \pi \circ F$. Notice that $G: S^1 \times S^1 \rightarrow S^2$ is a smooth map.

Define the linking number $\text{lk}(K, L) \equiv \deg(G)$. Now, let

$$\omega = \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} (x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2),$$

where x_1, x_2, x_3 are (Cartesian) coordinates on \mathbb{R}^3 .

- (a) Show that $d\omega = 0$.
 (b) Let $\eta = \omega|_{S^2}$. Prove that

$$\int_{S^2} \eta = 4\pi.$$

Conclude that

$$\text{lk}(K, L) = \frac{1}{4\pi} \int_{S^1 \times S^1} F^* \omega.$$

- (c) Compute $F^* dx_i$ for $i = 1, 2, 3$.
 (d) Find an explicit expression for $\text{lk}(K, L)$ as an integral over $S^1 \times S^1$, in terms of f and g .

Solution:

- (a) We compute

$$d\omega = \left(\frac{\partial}{\partial x_1} \left(\frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \right) + \frac{\partial}{\partial x_2} \left(\frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \right) + \frac{\partial}{\partial x_3} \left(\frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \right) \right) dx_1 \wedge dx_2 \wedge dx_3.$$

Symmetrically, we observe that

$$\begin{aligned} \frac{\partial}{\partial x_1} \left(\frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \right) &= \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} - \frac{3}{2} \frac{2x_1(x_1)}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} \\ &= \frac{-2x_1^2 + x_2^2 + x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}, \end{aligned}$$

whence their sum is zero.

- (b) We observe that the restriction to S^1 yields $x_1^2 + x_2^2 + x_3^2 = 1$, meaning that the restriction is

$$\iota^* \omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2,$$

where $\iota: S^2 \hookrightarrow \mathbb{R}^3$ is inclusion. In particular, we observe that this is precisely the area form on \mathbb{R}^3 with respect to the vector

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

whence

$$\int_{S^2} \iota^* \omega = 4\pi.$$