This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

Useful Results and Proofs

Analytic Functions

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A function $f \colon U \to \mathbb{C}$ is called *analytic* if, for any $z_0 \in U$, there is $r > 0$ and $(a_k)_k \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in U(z_0, r)$.

Analytic functions form a $\mathbb{C}$-algebra.

Theorem (Identity Theorem): Let $f, g \colon U \to \mathbb{C}$ be analytic functions defined a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in U, then $f = g$ on U.

Proof. To begin, we show that if $f \colon U \to \mathbb{C}$ is an analytic function that is not uniformly zero, then for any $z_0 \in U$, there is $\rho > 0$ such that f is nonzero on $\dot{U}(z_0, \rho) \subseteq U$. Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all $z \in U(z_0, r)$, some $r > 0$, and since f is not uniformly zero, there is some minimal ℓ such that $a_\ell \neq 0$. This yields

$$f(z) = (z - z_0)^{\ell} \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function $h \colon U(z_0, r) \to \mathbb{C}$ given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as f and is not zero at z_0, so that g is not zero on some $U(z_0, \rho)$ as g is continuous.

Now, we let V_1 be the set of accumulation points of A in U, and let $V_2 = U \setminus V_1$.

If $z \in V_2$, then there is some $r_1 > 0$ such that $\dot{U}(z_0, r_1) \cap A = \emptyset$, or that $\dot{U}(z_0, r_1) \subseteq A^c$. Meanwhile, since U is open, there is some $r_2 > 0$ such that $U(z_0, r_2) \subseteq U$, meaning that if $r = \min\{r_1, r_2\}$, then $U(z_0, r) \subseteq U \setminus A$. Thus, V_2 is open.

Meanwhile, if $z \in V_1$, then since $V_1 \subseteq U$, it follows that there is $r > 0$ such that $U(z, r)$ and $(a_k)_k$ such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all $w \in U(z, r)$. We claim that $f(w) - g(w)$ is uniformly zero on $U(z, r)$. Else, if there were $w_0 \in U(z, r)$ such that $f(w_0) \neq g(w_0)$, then it would follow that there is $0 < s \leq r$ such that $f(w) \neq g(w)$ for all $w \in \dot{U}(w_0, s)$. Yet, this would contradict the assumption that z is an accumulation point, meaning that V_1 is open.

Since V_1 and V_2 are disjoint open sets whose union is equal to U, it follows that either $V_1 = U$ or $V_2 = U$. If $A \neq \emptyset$, then the identity theorem follows. $\square$

Differentiability

Definition: If $U \subseteq \mathbb{C}$ is an open set, then we say f is differentiable at $z_0 \in U$ if

$$\lim_{w \to z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of f at z_0, and usually write $f'(z_0)$.

If f is differentiable at every $z_0 \in U$, we say f is differentiable on U.

If f is continuous and admits a continuous derivative, then we say f is *holomorphic*.

Note that the limit must be independent of direction. That is, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(w) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Now, given $U \subseteq \mathbb{C}$, write $z = x + iy$ and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$. Observe then that if f is differentiable at $x_0 + iy_0 \in U$, then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x + h, y) + iv(x + h, y)) - (u(x, y) + iv(x, y))}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x, y + h) + iv(x, y + h)) - (u(x, y) + iv(x, y))}{ih} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

Definition: The system of partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if f is differentiable, then the u and v in the definition of f satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly f is differentiable or holomorphic.

> **Proposition:** If $f = u + iv$ is a holomorphic function such that u, v are in $C^2(U)$, then u and v are harmonic. That is, u and v satisfy Laplace's equation:
> $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We call u and v *harmonic conjugates* for each other. That is, if $u \colon U \to \mathbb{R}$ is a harmonic function, then $v \in C^1(U)$ is called a harmonic conjugate if the Cauchy–Riemann equations hold for u and v.

> **Theorem:** Let $U \subseteq \mathbb{R}^2$ be a ball or all of $\mathbb{R}^2$. Then, every harmonic function on U has a harmonic conjugate. If $u \in C^3(U)$, then this conjugate is itself harmonic.

> **Lemma:** Let $g \colon U((x_0, y_0), R) \to \mathbb{R}$ be such that g and $\frac{\partial g}{\partial x}$ are continuous. Then, $G \colon U((x_0, y_0), R) \to \mathbb{R}$, given by
> $$G(x, y) = \int_{y_0}^{y} g(x, t)\, dt$$
> satisfies
> $$\frac{\partial G}{\partial x} = \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\, dt.$$

Proof of Lemma. Write
$$\frac{G(x+h, y) - G(x, y)}{h} - \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\, dt = \int_{y_0}^{y} \left(\frac{g(x+h, t) - g(x, t)}{h} - \frac{\partial g}{\partial x}(x, t) \right) dt.$$

By mean value theorem, the first term is equal to $\frac{\partial g}{\partial x}(x_1, t)$ for some x_1 between x and $x + h$. As $h \to 0$, $x_1 \to x$, as $\frac{\partial g}{\partial x}$ is uniformly continuous on a compact subset that contains x and $x + h$. We may exchange limit and integral to obtain the desired result. $\qquad\square$

Proof of Theorem. We prove for the case of $U = U((x_0, y_0), R)$. Define
$$v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt + \phi(x),$$

with $\phi(x)$ to be determined later. By the fundamental theorem of calculus, we have
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

while by differentiating under the integral sign, and using the fact that u is harmonic, we have
$$\begin{aligned}
\frac{\partial v}{\partial x} &= \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x, t)\, dt + \frac{d\phi}{dx} \\
&= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x, t)\, dt + \frac{d\phi}{dx} \\
&= -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \frac{d\phi}{dx}.
\end{aligned}$$

Defining $\phi \colon \mathbb{R} \to \mathbb{R}$ by
$$\phi(x) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)\, ds,$$

we see that v thus satisfies all the necessary requirements to be a harmonic conjugate.

Now, if u is C^3, then we defined v via the derivative of u, so that v is C^2, and thus v is harmonic.   □

Cauchy's Integral Formula

Proposition: Fix $z_0 \in \mathbb{C}$, $R > 0$, and $f \colon U(z_0, R) \to \mathbb{C}$ holomorphic. For all $z \in U(z_0, R)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof. It suffices to show that

$$\frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = 0.$$

By using the chain rule and fundamental theorem of calculus, we find

$$\frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{\int_0^1 f'((1 - t)z + t\zeta)(\zeta - z) \, dt}{\zeta - z} \, d\zeta$$

$$= \frac{1}{2\pi i} \int_{S(z_0, R)} \int_0^1 f'((1 - t)z + t\zeta) \, dt \, d\zeta$$

$$= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{d}{d\zeta}\left(\frac{1}{t} f((1 - t)z + t\zeta) \right) \, d\zeta$$

$$= 0.$$

□

Proposition: Let $f \colon U \to \mathbb{C}$ be a holomorphic function. The following all hold:

(i) f is analytic;

(ii) f is smooth with $f^{(n)}$ holomorphic;

(iii) for all $z_0 \in U$, if we let $R = \sup\{r > 0 \mid U(z_0, r) \subseteq U\}$, then there is $(a_n)_n \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the power series has radius of convergence R.

Proof.

(i) There exists $r < s$ with $U(z_0, s) \subseteq U$ and $r < r_1 < s$ such that $S(z_0, r_1) \subseteq U$. By Cauchy's Integral Formula, and a power series expansion of $\frac{1}{\xi - z}$ about z_0, this gives

$$f(z) = \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{\xi - z} \, d\xi$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right)}_{=:a_n}$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(ii) Analytic functions are automatically smooth, hence complex-differentiable with continuous

derivative.

(iii) If $r < r_1 < R$, then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi \right),$$

and since the series converges uniformly, we have

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi.$$

Since r was arbitrary, this holds for any $0 < r_1 < R$, whence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for all $z \in U(z_0, R)$.

$\square$

Corollary: Let $U \subseteq \mathbb{C}$ be open, let $z_0 \in U$, and $r > 0$ with $B(z_0, r) \subseteq U$. The following hold:

(i) for all $z \in U(z_0, r)$,

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi;$$

(ii) for all $n > 0$,

$$\left| f^{(n)}(z_0) \right| \leqslant \frac{n!}{r^n} \sup_{\zeta \in S(z_0, r)} |f(\zeta)|.$$

This particular result is known as the *Cauchy Estimate*.

Theorem (Liouville's Theorem)**:** If $f \colon \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded in modulus, then f is constant.

Liouville's Theorem follows from applying Cauchy's estimate to f and using the fact that f is bounded to find that all higher derivatives of f vanish.

Theorem (Fundamental Theorem of Algebra)**:** If $p(z) = a_n z^n + \cdots + a_1 z + a_0$ has $n \geqslant 1$ and $a_n \neq 0$, then there is at least one z_0 such that $p(z_0) = 0$.

Proof. Suppose $p(z)$ were never zero. It would follow then that $\frac{1}{p(z)}$ is also an entire function.

Since $\lim_{|z| \to \infty} |p(z)| = \infty$, it follows that $\lim_{|z| \to \infty} \frac{1}{|p(z)|} = 0$, whence $\left| \frac{1}{p(z)} \right|$ is an entire function that is bounded (as all functions that vanish at infinity are bounded). This means that $\frac{1}{p(z)}$ is constant, so $p(z)$ is constant. $\square$

Corollary: Let $f \colon \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Then, $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Proof. Suppose there were $w \in \mathbb{C}$ and $r > 0$ such that $U(w, r) \cap f(\mathbb{C}) = \emptyset$. Then, $|f(z) - w| \geqslant r$ for all $z \in \mathbb{C}$, meaning that

$$g(z) = \frac{1}{f(z) - w}$$

is bounded and entire (the entirety following from the fact that $f(z) - w$ is nonvanishing). $\square$

Cycles, Winding Numbers, and Homology

Now, we may generalize some of these results related to Cauchy's Integral Formula.

Proposition: Let $\gamma\colon [a, b] \to \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \mathrm{im}(\gamma)$, we have

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{\xi - z} \, d\xi \in \mathbb{Z}.$$

Proof. Let $\phi\colon [a, b] \to \mathbb{C}$ be defined by

$$\phi(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \, ds.$$

Then, we observe

$$\phi(b) = \oint_\gamma \frac{1}{\xi - z} \, d\xi.$$

Then, define $\psi\colon [a, b] \to \mathbb{C}$ by

$$\psi(t) = \frac{e^{\phi(t)}}{\gamma(t) - z}.$$

By the fundamental theorem of calculus, we have

$$\phi'(t) = \frac{\gamma'(t)}{\gamma(t) - z}$$
$$\psi'(t) = \frac{\phi'(t)e^{\phi(t)}}{\gamma(t) - z} - \frac{e^{\phi(t)}\gamma'(t)}{(\gamma(t) - z)^2}$$
$$= 0,$$

whence $\psi(t)$ is constant, and $\psi(t) = \psi(a)$, so

$$\psi(a) = \frac{1}{\gamma(a) - z}.$$

In particular, $\psi(b) = \psi(a)$, so

$$e^{\phi(b)} = \psi(b)(\gamma(b) - z)$$
$$= \psi(a)(\gamma(a) - z)$$
$$= 1,$$

so $\phi(b) = 2\pi i k$ for some $k \in \mathbb{Z}$. $\qquad\square$

Definition: Let $\gamma\colon [a, b] \to \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \mathrm{im}(\gamma)$, define

$$n(\gamma; z) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\xi - z} \, d\xi$$

to be the *winding number* of γ about z.

Definition: A piecewise C^1 *cycle* is a formal sum

$$\Gamma = \gamma_1 + \cdots + \gamma_n,$$

where the $\gamma_j\colon [a_j, b_j] \to \mathbb{C}$ are piecewise C^1 loops. The *length* of Γ is the sum of the lengths of the respective γ_j.

Given a piecewise C^1 cycle Γ, define

$$\oint_\Gamma f(z)\, dz = \sum_{j=1}^n \oint_{\gamma_j} f(z)\, dz,$$

and

$$n(\Gamma; z) = \sum_{j=1}^n n(\gamma_j; z).$$

Proposition: The following hold for the winding number $n(\gamma; z)$:

(i) the function $n(\Gamma; \cdot) \colon \mathbb{C} \setminus \mathrm{im}(\gamma) \to \mathbb{Z}$ is continuous;

(ii) $n(\Gamma; z)$ is constant on each connected component of $\mathbb{C} \setminus \mathrm{im}(\Gamma)$;

(iii) there exists a unique unbounded connected component with $n(\Gamma; z) = 0$ for all z in this unbounded connected component.

Proof.

(i) Since $\mathrm{im}(\Gamma)$ is compact, any $z \notin \mathrm{im}(\Gamma)$ admits a strictly positive

$$\mathrm{dist}_{\mathrm{im}(\Gamma)}(z) = \inf_{w \in \mathrm{im}(\Gamma)} |w - z|.$$

Let $w \in \mathbb{C}$ be such that

$$|w - z| < \frac{1}{2} \mathrm{dist}_{\mathrm{im}(\Gamma)}(z),$$

so that $w \in \mathbb{C} \setminus \mathrm{im}(\Gamma)$. Observe then that

$$
\begin{aligned}
|n(\Gamma; z) - n(\Gamma; w)| &= \left| \frac{1}{2\pi i} \oint_\Gamma \frac{1}{\xi - z} - \frac{1}{\xi - w}\, d\xi \right| \\
&\leqslant \frac{1}{2\pi} \sum_{j=1}^n \oint_{\gamma_j} \left| \frac{1}{\xi - z} - \frac{1}{\xi - w} \right| |d\xi| \\
&= \frac{1}{2\pi} \sum_{j=1}^n \oint_{\gamma_j} \left| \frac{z - w}{(\xi - z)(\xi - w)} \right| |d\xi| \\
&\leqslant \frac{1}{2\pi} \left(\frac{2}{\mathrm{dist}_{\mathrm{im}(\Gamma)}(z)} \right)^2 \ell(\Gamma) |z - w|,
\end{aligned}
$$

whence $|n(\Gamma; z) - n(\Gamma; w)|$ is sufficiently small whenever $|z - w|$ is sufficiently small.

(ii) If C is a connected component of $\mathbb{C} \setminus \mathrm{im}(\Gamma)$, and $n(\Gamma; \cdot) \colon C \to \mathbb{Z}$ is continuous, then since $\mathbb{Z}$ is discrete, $n(\Gamma; \cdot)$ is constant on C.

(iii) For uniqueness, if there are unbounded connected components C_1 and C_2 of $\mathbb{C} \setminus \mathrm{im}(\Gamma)$, then there exists $M > \sup_{z \in \mathrm{im}(\Gamma)} |z|$ and $w_1 \in C_1, w_2 \in C_2$ such that $|w_1| > 2M$ and $|w_2| > 2M$. Since $\mathbb{C} \setminus U(0, 2M)$ is path connected, there exists $\gamma \colon [0, 1] \to \mathbb{C}$ with $|\gamma(t)| \geqslant 2M$ and $\gamma(0) = w_1$, $\gamma(1) = w_2$. Therefore, w_1 and w_2 are in the same connected component.

Existence then follows from $\mathrm{im}(\Gamma)$ being compact.

Finally, let $(z_n)_n \subseteq C$, where C is the unbounded connected component, be such that $\lim_{n\to\infty}|z_n| = \infty$. For $M > \sup_{z\in\text{im}(\gamma)}|z|$, there exists $m \in \mathbb{N}$ such that $|z_m| > M$. Then, we have

$$
\begin{aligned}
|n(\Gamma; z_m)| &= \left| \frac{1}{2\pi i} \oint_\Gamma \frac{1}{\xi - z} \, d\xi \right| \\
&\leqslant \frac{1}{2\pi} \sum_{j=1}^{k} \oint_{\gamma_j} \frac{1}{|\xi - z|} \, |d\xi| \\
&\leqslant \frac{1}{2\pi} \sum_{j=1}^{k} \oint_{\gamma_j} \frac{1}{|z_m| - M} \, |d\xi| \\
&= \frac{\ell(\Gamma)}{2\pi(|z_m| - M)},
\end{aligned}
$$

whence $\lim_{m\to\infty} n(\Gamma; z_m) = 0$, meaning that there exists N such that $|n(\Gamma; z_m)| < 1$ for all $m \geqslant N$, meaning $n(\Gamma; z_m) = 0$ for all sufficiently large m. Since C is connected, it thus follows that $n(\Gamma; z) = 0$ for all $z \in C$.

$\square$

Definition: Let $U \subseteq \mathbb{C}$ be open. A cycle Γ is *homologous to zero in* U if $\text{im}(\Gamma) \subseteq U$ and for all $z \in \mathbb{C} \setminus U$, $n(\Gamma; z) = 0$.

Theorem (Cauchy's Integral Formula, General Case): Let $\Gamma = \gamma_1 + \cdots + \gamma_k$ be a piecewise C^1 cycle homologous to zero in U, and $f\colon U \to \mathbb{C}$ holomorphic. Then, for all $z \in U \setminus \text{im}(\Gamma)$,

$$
n(\Gamma; z)f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z} \, d\xi
$$

Theorem (Cauchy's Integral Theorem): Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic, and Γ homologous to zero in U. Then,

$$
\oint_\Gamma f(z) \, dz = 0.
$$

Definition: A region $U \subseteq \mathbb{C}$ is called *simply connected* if its complement in the extended complex plane is connected.

Theorem: If $U \subseteq \mathbb{C}$ is simply connected, then every loop in U is homologous to zero.

Proof. Extend the function $n(\gamma; \cdot)$ to the extended complex plane by defining $n(\gamma; \infty) = 0$. This extended function is continuous on $\hat{\mathbb{C}} \setminus U$, as $n(\gamma; \cdot)$ is zero on the unique unbounded connected component of $\mathbb{C} \setminus \text{im}(\gamma)$. It follows that $n(\gamma; z)$ is equal to zero on $\hat{\mathbb{C}} \setminus U$, whence γ is homologous to zero in U. $\square$

Proposition: Let $U \subseteq \mathbb{C}$ be a region, $f\colon U \to \mathbb{C}$ holomorphic. The following are equivalent:

(i) there exists a holomorphic function $F\colon U \to \mathbb{C}$ such that $F'(z) = f(z)$;

(ii) for every piecewise C^1 loop γ with $\text{im}(\gamma) \subseteq U$, we have

$$
\oint_\gamma f(z) \, dz = 0.
$$

Proof. The direction (i) $\Rightarrow$ (ii) follows immediately from the fundamental theorem of calculus. In the reverse direction, we define $F\colon U \to \mathbb{C}$ by

$$
f(z) = \int_{\sigma(z_0, z)} f(\xi) \, d\xi,
$$

where $\sigma(z_0, z)\colon [0, 1] \to U$ is a piecewise C^1 curve with $\sigma(0) = z_0$ and $\sigma(1) = z$. Such a curve always

exists as U is open and connected (hence path-connected). The integral is well-defined, since if γ_1 and γ_2 are any two such paths, then $\Gamma = \gamma_1 \setminus \gamma_2$ is a piecewise C^1 loop. Additionally, F is continuous.

Now, we evaluate the derivative of F. Let $z \in U$, $r > 0$ such that $U(z, r) \subseteq U$, and $h \in \mathbb{C}$ be such that $z + h \in U(z, r)$. Then,

$$\frac{F(z + h) - F(z)}{h} = \frac{1}{h} \int_{\sigma(z_0, z_0+h)} f(\xi) \, d\xi - \frac{1}{h} \int_{\sigma(z_0, z)} f(\xi) \, d\xi,$$

$$= \frac{1}{h} \int_{\sigma(z, z+h)} f(\xi) \, d\xi.$$

We may assume that $\sigma(z, z + h)$ is a straight line, so that

$$\int_{\sigma(z, z+h)} f(\xi) \, d\xi = hf(z),$$

meaning that

$$\left| \frac{F(z + h) - F(z)}{h} - f(z) \right| = \frac{1}{|h|} \left| \int_{\sigma(z, z+h)} f(\xi) \, d\xi - f(z) \right|$$

$$\leqslant \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)|.$$

Since f is continuous, it follows that the right hand side goes to zero as $|h|$ becomes small. Thus, F' is continuous, so f is holomorphic. $\square$

Observe that $\mathbb{C} \setminus \{0\}$ is not simply connected, meaning that, for instance, the function

$$f(z) = \frac{1}{z}$$

does not have a holomorphic antiderivative defined on the entirety $\mathbb{C} \setminus \{0\}$, as

$$\int_{S^1} f(z) \, dz = 2\pi i.$$

Yet, if we restrict $f(z)$ to a simply connected subset of $\mathbb{C}$, there *is* a holomorphic antiderivative. Choosing such a simply connected subset of $\mathbb{C}$ is known as choosing a *branch* of the logarithm. In fact, there is more that we can say.

Corollary: Let $U \subseteq \mathbb{C}$ be simply connected, and let $f \colon U \to \mathbb{C} \setminus \{0\}$ be a nonvanishing holomorphic function. For each fixed pair $z_0 \in U$ and $w_0 \in \mathbb{C}$ for which $e^{w_0} = f(z_0)$, there exists a unique holomorphic function $g \colon U \to \mathbb{C}$ for which $g(z_0) = w_0$ and $e^{g(z)} = f(z)$.

We call g the logarithm of f, written $g(z) = \log(f(z))$.

Proof. Since f is nonvanishing and U is simply connected, it follows that $\frac{f'}{f}$ is holomorphic on U, meaning there is $\widetilde{g} \colon U \to \mathbb{C}$ such that $\widetilde{g}'(z) = \frac{f'(z)}{f(z)}$. Thus, there is some constant K such that

$$f(z) = Ke^{\widetilde{g}(z)}.$$

Define

$$g(z) = \log(K) + \widetilde{g}(z).$$

$\square$

Theorem (Morera's Theorem): Let $U \subseteq \mathbb{C}$ be open, $f \colon U \to \mathbb{C}$ continuous. Suppose

$$\oint_T f(z) \, dz = 0$$

for all triangles $T \subseteq U$ homologous to zero. Then, f is holomorphic.

Proof. Since U is open, if $z_0 \in U$, there is r such that $U(z_0, r) \subseteq U$. Define $F \colon U(z_0, r) \to \mathbb{C}$ by

$$F(z) = \int_{\sigma(z_0, z)} f(\xi) \, d\xi,$$

where σ is the straight line from z_0 to z. For $0 < |h| < r - |z - z_0|$, we construct the straight lines $\sigma(z, z + h)$ and $\sigma(z_0, z + h)$, such that

$$T = \sigma(z_0, z) + \sigma(z, z + h) - \sigma(z_0, z + h),$$

and

$$\oint_T f(z) \, dz = 0$$

$$= \int_{\sigma(z_0,z)} f(\xi) \, d\xi + \int_{\sigma(z,z+h)} f(\xi) \, d\xi - \int_{\sigma(z_0,z+h)} f(\xi) \, d\xi$$

$$= F(z) - F(z + h) + \int_{\sigma(z,z+h)} f(\xi) \, d\xi,$$

meaning

$$F(z + h) - F(z) = \int_{\sigma(z,z+h)} f(\xi) \, d\xi,$$

$$\frac{F(z + h) - F(z)}{h} = \frac{1}{h} \int_{\sigma(z,z+h)} f(\xi) \, d\xi,$$

$$\left| \frac{F(z + h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{\sigma(z,z+h)} (f(\xi) - f(z)) \, d\xi \right|$$

$$\leqslant \frac{1}{|h|} |h| \sup_{w \in \text{im}(\sigma(z,z+h))} |f(w) - f(z)|$$

$$= \sup_{w \in \text{im}(\sigma(z,z+h))} |f(w) - f(z)|.$$

Since f is continuous, it follows that for sufficiently small $|h|$, the right-hand-side goes to zero, whence $F'(z) = f(z)$, meaning F is holomorphic, so F is analytic, meaning f is analytic, so f is holomorphic. $\qquad\square$

Corollary: Let $U \subseteq \mathbb{C}$ be open, $\gamma \colon [a, b] \to U$ a piecewise C^1 curve, and $g \colon U \times \text{im}(\gamma) \to \mathbb{C}$ continuous. Suppose that for each $w \in \text{im}(\gamma)$, the function $g(\cdot, w)$ is holomorphic. Then,

$$f(z) := \int_\gamma g(z, w) \, dw$$

is holomorphic.

Proof. Let T be a triangle in U homologous to zero. Then, by Fubini's Theorem,

$$\oint_T f(z) \, dz = \oint_T \int_\gamma g(z, w) \, dw \, dz$$

$$= \int_\gamma \oint_T g(z, w) \, dz \, dw.$$

The interior integral vanishes for every w as $g(\cdot, w)$ is holomorphic. Thus, f is holomorphic.   □

Definition: Let $U \subseteq \mathbb{C}$ be open, γ_1, γ_2 piecewise C^1 loops in U. We say γ_1 and γ_2 are homotopic in U if there is a continuous function

$$H \colon [a, b] \times [0, 1] \to U$$

such that

$$H(s, 0) = \gamma_1(s)$$
$$H(s, 1) = \gamma_2(s)$$
$$H(a, t) = H(b, t).$$

For each t, $H(\cdot, t)$ is a continuous loop. We call H a homotopy between γ_0 and γ_1.

Theorem: If γ_0 and γ_1 are homotopic in U, then $\Gamma = \gamma_1 - \gamma_0$ is homologous to zero in U.

Theorem: If $K \subseteq U$ is compact and U is connected, then there is some cycle Γ homologous to zero in U such that $n(\Gamma; z) = 1$ for all $z \in K$.

Corollary: Let U be a region. The following are equivalent:

(i) U is simply connected;

(ii) for every nonvanishing holomorphic function $f \colon U \to \mathbb{C} \setminus \{0\}$, there is a holomorphic function $g \colon U \to \mathbb{C}$ such that $f(z) = e^{g(z)}$;

(iii) for all cycles Γ with $\operatorname{im}(\Gamma) \subseteq U$, Γ is homologous to zero in U.

Maximum Modulus Principle

Theorem (Mean Value Property)**:** Let $U \subseteq \mathbb{C}$ be open, $f \colon U \to \mathbb{C}$ holomorphic, with $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\!\left(z_0 + re^{i\theta}\right) d\theta.$$

Proof. By the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Parametrizing $\gamma(\theta) = z_0 + re^{i\theta}$, we get

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\!\left(z_0 + re^{i\theta}\right)}{re^{i\theta}} ire^{i\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f\!\left(z_0 + re^{i\theta}\right) d\theta.$$

□

Corollary: If $u \colon \mathbb{R}^2 \supseteq U \to \mathbb{R}$ is harmonic, $(x_0, y_0) \in U$, and $r > 0$ is such that $B((x_0, y_0), r) \subseteq U$, then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) \, d\theta.$$

Proof. Take real parts of the mean value property for holomorphic $f = u + iv$.   □

Observe then that the triangle inequality implies that

$$|u(x_0, y_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))| \, d\theta.$$

Functions that satisfy this weaker criterion are known as *subharmonic*. It is subharmonic functions for which the most general case of the *maximum modulus principle* hold.

Theorem (Maximum Modulus Principle): Let $U \subseteq \mathbb{R}^2$ be open and connected, and let $u \colon U \to \mathbb{R}$ be subharmonic. Suppose there exists $(x_0, y_0) \in U$ such that $u(x_0, y_0) \geqslant u(x, y)$ for all $x, y \in U$. Then, u is constant.

Proof. Let $\lambda = u(x_0, y_0)$, and let $E = \{(x, y) \mid u(x, y) = \lambda\} = u^{-1}(\{\lambda\})$. We see immediately that E is closed; we claim that E is also open.

Fix $(x_1, y_1) \in E$. Then, $u(x_1, y_1) = \lambda$. Take $r > 0$ such that $U((x_1, y_1), r) \subseteq U$. Then, for all $0 < s < r$, we have $S((x_1, y_1), s) \subseteq U$, meaning that

$$\begin{aligned}
\lambda &= u(x_1, y_1) \\
&\leqslant \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + s\cos(\theta), y_1 + s\sin(\theta)) \, d\theta \\
&\leqslant \lambda,
\end{aligned}$$

with the latter inequality following from the fact that λ is a local maximum. Therefore, $u(x_1 + s\cos(\theta), y_1 + s\sin(\theta)) = \lambda$ for all $0 < s < r$, whence $U((x_1, y_1), r) \subseteq E$. Thus, E is open, so since U is connected, it follows that E is all of U, meaning u is constant. $\square$

Corollary: If $U \subseteq \mathbb{R}^2$ is bounded and $u \colon \overline{U} \to \mathbb{R}$ is continuous with $u|_U$ subharmonic, then there exists $(x_0, y_0) \in \partial U$ such that $u(x_0, y_0) = \sup_{(x,y) \in U} u(x, y)$.

Corollary: If $U \subseteq \mathbb{C}$ is open and connected, with $f \colon U \to \mathbb{C}$ holomorphic, then if $|f| \colon U \to \mathbb{R}$ has a local maximum at $z_0 \in U$, then f is constant.

Proof. Let $r > 0$ be such that $U(z_0, r) \subseteq U$. Then, restricting $|f|$ to $U(z_0, r)$, we see that $|f|$ restricted to $U(z_0, r)$ is subharmonic viewed as a function on $U(z_0, r)$, hence $|f|$ is constant on $U(z_0, r)$.

Now, by the mean value property and triangle inequality, it follows that for all $0 < s < r$, we have

$$\begin{aligned}
|f(z_0)| &\leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + se^{i\theta})| \, d\theta \\
&= |f(z_0)|,
\end{aligned}$$

meaning that these are equalities. In particular, there exists some θ_s such that $e^{i\theta_s} f(z_0 + se^{i\theta}) \geqslant 0$, meaning that for this value of s, we have

$$\begin{aligned}
|f(z_0)| &= e^{i\theta_s} \int_0^{2\pi} f(z_0 + se^{i\theta}) \, d\theta \\
&= e^{i\theta_s} f(z_0),
\end{aligned}$$

with the latter equality following from the mean value property. Since this holds for any s, it follows that θ_s is independent of s, meaning that $f(z)e^{i\theta_s} \geqslant 0$ for all $z \in U(z_0, r)$, meaning that $\mathrm{Im}(e^{i\theta_s} f(z)) = 0$ on $U(z_0, r)$, whence $f(z)e^{i\theta_s}$ is constant, meaning f is constant on $U(z_0, r)$.

Finally, by the identity theorem, it follows that f is constant on U. $\square$

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. We say a sequence $U \supseteq ((x_n, y_n))_n \to \partial U$ if, for every compact $K \subseteq U$, the set $\{n \in \mathbb{N} \mid (x_n, y_n) \in K\}$ is finite.

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. Given a function $u \colon U \to \mathbb{R}$, define

$$\limsup_{(x,y) \to \partial U} u(x,y) := \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{(x,y) \in U \setminus K} u(x,y).$$

These definitions allow us to extend the maximum modulus principle for subharmonic functions even further.

Theorem: Let $U \subseteq \mathbb{C}$ be a region, $u \colon U \to \mathbb{R}$ a nonconstant subharmonic function. If $((x_n, y_n))_n \subseteq U$ is such that $u(x_n, y_n) \to \sup_{x,y \in U} u(x,y)$, then $((x_n, y_n))_n \to \partial U$. Moreover, $\limsup_{(x,y) \to \partial U} u(x,y) = \sup_{(x,y) \in U} u(x,y)$.

Proof. Suppose toward contradiction that $((x_n, y_n))_n \not\to \partial U$, so there exists a compact subset $K \subseteq U$ and a subset $((x_{n_k}, y_{n_k}))_k$ wholly contained in K. Since K is compact, there is a subsequence of $((x_{n_k}, y_{n_k}))_k$ converging to $(x_0, y_0) \in U$. Therefore, $u(x_0, y_0) = \sup_{(x,y) \in U} u(x,y)$, so u is constant by the maximum modulus principle, which is a contradiction.

Finally, $\limsup_{(x,y) \to \partial U} u(x,y) \leqslant \sup_{(x,y) \in U} u(x,y)$, while if $((x_n, y_n))_n \to \partial U$ is such that $u(x_n, y_n)$ converges to $\sup_{(x,y) \in U} u(x,y)$, then $\sup_{(x,y) \in U} u(x,y) = \lim_{n \to \infty} u(x_n, y_n) \leqslant \limsup_{(x,y) \to \partial U} u(x,y)$. $\square$

Theorem (Open Mapping Principle)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f \colon U \to \mathbb{C}$ be a nonconstant holomorphic function. Then, $f(V) \subseteq \mathbb{C}$ is open.

Proof. Let $z_0 \in U$ and $r > 0$ be such that $B(z_0, r) \subseteq U$. We will show that there exists R such that $U(f(z_0), R) \subseteq f(U(z_0, r)) \subseteq U$, whence $f(U)$ is open.

Since U is a region and f is nonconstant, the zeros of $g(z) := f(z) - f(z_0)$ are isolated, so there exists some $0 < s < r$ such that

$$\delta = \inf_{|z - z_0| = s} |f(z) - f(z_0)|$$

is strictly greater than zero. We claim that $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$. Suppose this were not the case, meaning there would be some $\xi \in U(f(z_0), \delta/2) \setminus f(U(z_0, r))$, and define $h \colon B(z_0, s) \to \mathbb{C}$ by

$$h(z) = \frac{1}{f(z) - \xi}.$$

Since $\xi \notin f(U(z_0, r))$, this is holomorphic, while $\xi \in U(f(z_0), \delta/2)$ implies

$$\sup_{|z - z_0| = s} |h(z)| = \sup_{|z - z_0| = s} \frac{1}{|f(z) - \xi|}$$

$$\leqslant \sup_{|z - z_0| = s} \frac{1}{|f(z) - f(z_0)| - |f(z_0) - \xi|}$$

$$\leqslant \frac{1}{\delta - \delta/2}$$

$$= \frac{2}{\delta}.$$

Yet,

$$|h(z_0)| = \frac{1}{|f(z_0) - \xi|}$$

$$> \frac{2}{\delta},$$

contradicting the maximum modulus principle. Thus, $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$. $\square$

In the proof of the Hadamard Three-Lines Theorem, we used the function $h_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}$ for this purpose.

Classification of Singularities

The classification of singularities seeks to answer two fundamental questions: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, and $f \colon U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic,

- does f have a holomorphic extension to U including z_0;

- and what else can we say about the behavior of f at z_0?

Definition: Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f \colon U \setminus \{z_0\} \to \mathbb{C}$ holomorphic.

- If there exists a holomorphic $g \colon U \to \mathbb{C}$ with $g = f$ on $U \setminus \{z_0\}$, then we say z_0 is a *removable singularity*.

- If $\lim_{z \to z_0} |f(z)| = \infty$, then we say f has a *pole* at z_0.

- Else, we say f has an *essential singularity* at z_0.

Theorem (Riemann's Theorem on Removable Singularities): Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and $f \colon U \setminus \{z_0\} \to \mathbb{C}$ holomorphic. Then, z_0 is a removable singularity if and only if $\lim_{z \to z_0} f(z) = 0$.

Proof. If z_0 is removable, then $g(z)$ is a holomorphic function with $g(z) = f(z)$ on $U \setminus \{z_0\}$, and since g is continuous, it follows that $\lim_{z \to z_0} g(z) = g(z_0)$, whence $\lim_{z \to z_0} (z - z_0)g(z) = \lim_{z \to z_0} (z - z_0)f(z) = 0$.

Now, if $\lim_{z \to z_0} (z - z_0)f(z) = 0$, then there is r such that $B(z_0, r) \subseteq U$, and since f is locally bounded around z_0, it follows that

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

holds for all $z \in \dot{U}(z_0, r)$. Yet, the formula extends to z_0 as it is bounded, whence we may define the holomorphic extension for f by

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{g(\zeta)}{\zeta - z} \, d\zeta & z = z_0 \end{cases}.$$

$\square$

Proposition (Existence of Laurent Series): Suppose $f \colon A(z_0, r, R) \to \mathbb{C}$ is holomorphic, with $0 \leqslant r < R$. Then, there exist holomorphic functions

$$g_1 \colon U(z_0, R) \to \mathbb{C}$$
$$g_2 \colon \mathbb{C} \setminus B(z_0, r) \to \mathbb{C}$$

such that $f = g_1 + g_2$ on $A(z_0, r, R)$. Moreover, there exists $(a_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for all z, and the series converges uniformly on $A(z_0, \rho, s)$ with $r < \rho < s < R$.

Proof. Fix $z \in A(z_0, r, R)$. Then, for $r < \rho_1, \rho_2 < |z - z_0|$, the cycle

$$\Gamma_1 = S(z_0, \rho_1) - S(z_0, \rho_2)$$

is homologous to zero in $A(z_0, r, |z - z_0|)$. By Cauchy's Integral Theorem, it then follows that

$$\oint_{S(z_0, \rho_1)} \frac{f(\xi)}{\xi - z} \, d\xi = \oint_{S(z_0, \rho_2)} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Similarly, for $|z - z_0| < s_1, s_2 < R$, we have

$$\oint_{S(z_0, s_1)} \frac{f(\xi)}{\xi - z} \, d\xi = \oint_{S(z_0, s_2)} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Define $g_1 \colon U(z_0, R) \to \mathbb{C}$ by

$$g_1(z) = \frac{1}{2\pi i} \oint_{S(z_0, s)} \frac{f(\xi)}{\xi - z} \, d\xi,$$

where $|z - z_0| < s < R$. This function is holomorphic by Morera's Theorem. Similarly, we may define $g \colon \mathbb{C} \setminus B(z_0, r) \to \mathbb{C}$ by

$$g_2(z) = -\frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{\xi - z} \, d\xi,$$

where $r < \rho < |z - z_0|$. We claim that $f = g_1 + g_2$ on $A(z_0, r, R)$.

For $z \in A(z_0, r, R)$, we may find, for any $r < \rho < |z - z_0| < s < R$, we let

$$\Gamma = S(z_0, s) - S(z_0, \rho),$$

homologous to zero in $A(z_0, r, R)$, whence

$$f(z) = \frac{1}{2\pi i} \left(\oint_{S(z_0, s)} \frac{f(\xi)}{\xi - z} \, d\xi - \int_{S(z_0, \rho)} \frac{f(\xi)}{\xi - z} \, d\xi \right)$$
$$= g_1(z) + g_2(z).$$

$\square$

Theorem: Let $U \subseteq \mathbb{C}$, $f \colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on $\dot{U}(z_0, R)$ for some R with $U(z_0, R) \subseteq U$. Then,

  (i) f has a removable singularity at z_0 if and only if $a_n = 0$ for all $n < 0$;

 (ii) f has a pole at z_0 if and only if

$$1 \leqslant |\{n < 0 \mid a_n \neq 0\}| < \infty.$$

(iii) f has an essential singularity at z_0 if and only if

$$|\{n < 0 \mid a_n \neq 0\}| = \infty.$$

Proof.

  (i) If $a_n = 0$ for all $n < 0$, then $\lim_{z \to z_0}(z - z_0)f(z) = 0$, so f has a removable singularity at z_0.

Conversely, if f has a removable singularity at z_0, then for $n < 0$, we have

$$a_n = \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi,$$

for any $0 < \rho < R$. Since $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then for any $\varepsilon > 0$, there is sufficiently small ρ such that

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right| \\ &\leqslant \rho^{-1-n} \sup_{|\xi - z_0| = \rho} |(\xi - z_0) f(z)| \\ &\leqslant \varepsilon. \end{aligned}$$

Thus, $|a_n| = 0$ for all $n < 0$.

(ii) If $a_n \neq 0$ for a nonempty finite collection of $n < 0$, we let m be the largest number such that $a_{-m} < 0$, so that $f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$. It follows that $\lim_{z \to z_0} (z - z_0)^m f(z) = a_{-m} \neq 0$. In particular, there is some small δ such that

$$\left| (z - z_0)^m f(z) - a_{-m} \right| < \frac{|a_{-m}|}{2},$$

whence for a sufficiently small ε,

$$\begin{aligned} |f(z)| &> \frac{|a_{-m}| - \left| (z - z_0)^m f(z) - a_{-m} \right|}{|z - z_0|^m} \\ &> \frac{|a_{-m}|}{2|z - z_0|^m} \\ &> \frac{1}{\varepsilon}, \end{aligned}$$

so f has a pole at z_0.

Conversely, if f has a pole at z_0, there is some $r > 0$ such that $|f(z)| \geqslant 1$ whenever $z \in \dot{U}(z_0, r)$. Define $g \colon \dot{U}(z_0, r) \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z)},$$

whence g is holomorphic and bounded on $\dot{U}(z_0, r)$. By the classification of singularities, z_0 is a removable singularity of g, so there exists a holomorphic function $h \colon U(z_0, r) \to \mathbb{C}$ that is equal to $\frac{1}{f(z)}$ for $z \neq z_0$ and $\lim_{z \to z_0} \frac{1}{f(z)}$ if $z = z_0$. We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

where we must have $b_0 = 0$. Let m be the smallest positive integer such that $h(z) = (z - z_0)^m \widetilde{h}(z)$, where $\widetilde{h}(z) = \sum_{n=0}^{\infty} b_{n+m} (z - z_0)^n$ and $b_m \neq 0$. The function $\widetilde{h}$ is holomorphic on $U(z_0, r)$ and nonzero on some $U(z_0, \rho)$ with $0 < \rho \leqslant r$, so that $\widetilde{f}(z) = \frac{1}{\widetilde{h}(z)}$ is holomorphic on $U(z_0, \rho)$, so

$$f(z) = (z - z_0)^{-m} \widetilde{f}(z)$$

on $\dot{U}(z_0, \rho)$. Writing

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

we deduce that

$$f(z) = \sum_{n=-m}^{\infty} c_{n+m} (z - z_0)^n.$$

(iii) Follows from (i) and (ii).

$\square$

Definition: Let $U \subseteq \mathbb{C}$ be open.

- If $f \colon U \setminus \{z_0\} \to \mathbb{C}$ has a pole or a removable singularity at z_0, then the order of the pole or singularity is the smallest $m \geqslant 0$ such that

$$f(z) = (z - z_0)^{-m} g(z)$$

  with $g \colon U \to \mathbb{C}$ is holomorphic and has $g(z_0) \neq 0$.

- If $f \colon U \to \mathbb{C}$ has a zero at $z_0 \in U$, then the order of the zero at z_0 is the smallest $m \geqslant 0$ such that

$$f(z) = (z - z_0)^m g(z)$$

  where $g \colon U \to \mathbb{C}$ is holomorphic and has $g(z_0) \neq 0$.

If $z_0 \in U$ is either a pole or a zero (or a removable singularity that is a zero when f is extended), then the *order* of f is the unique $m \in \mathbb{Z}$ such that

$$f(z) = (z - z_0)^m g(z)$$

with $g(z_0) \neq 0$.

Theorem (Casorati–Weierstrass)**:** Let $U \subseteq \mathbb{C}$ be an open set, and let $f \colon U \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function. If z_0 is an essential singularity of f, then for any $r > 0$ with $U(z_0, r) \subseteq U$, $f\big(\dot{U}(z_0, r)\big) \subseteq \mathbb{C}$ is dense.

Proof. Suppose $f\big(\dot{U}(z_0, r)\big)$ is not dense in $\mathbb{C}$. Then, there exists $\varepsilon > 0$ and $w \in \mathbb{C}$ such that $|f(z) - w| \geqslant \varepsilon$ for all $z \in \dot{U}(z_0, r)$. We will show that z_0 is either removable or a pole.

Define

$$g \colon \dot{U}(z_0, r) \to \mathbb{C}$$

by

$$g(z) = \frac{1}{f(z) - w}.$$

Observe that g is definitionally bounded, so z_0 is a removable singularity of g. Thus, there is a holomorphic function $h \colon U(z_0, r) \to \mathbb{C}$ such that $h(z) = g(z)$ for $z \neq z_0$ and $h(z) = \lim_{z \to z_0} g(z)$. We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

and take

$$h(z) = (z - z_0)^m \widetilde{h}(z),$$

which exists as g is not uniformly zero, and where $\widetilde{h}(z_0) \neq 0$. Consequently, there is $0 < \rho \leqslant r$ such that $\widetilde{f} \colon U(z_0, \rho) \to \mathbb{C}$ given by

$$\widetilde{f}(z) = \frac{1}{\widetilde{h}(z)},$$

which is holomorphic. Thus,

$$f(z) = w + (z - z_0)^{-m}\widetilde{f}(z).$$

Thus, we get

$$f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n,$$

whence z_0 is either removable or a pole. $\hspace{4cm}$ □

The Argument Principle and Rouché's Theorem

Definition: If $U \subseteq \mathbb{C}$ is open, and $V \subseteq U$ is an open subset such that $U \setminus V$ consists solely of isolated points, then a function $f \colon V \to \mathbb{C}$ is *meromorphic* if it is holomorphic on V and every $z_0 \in U \setminus V$ is either a pole or a removable singularity. We say f is meromorphic on U.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U. Let f be meromorphic on U with no poles or zeros on $\mathrm{im}(\Gamma)$.

(i) The set $\left\{z_0 \in U \mid \mathrm{ord}_{z_0}(f) \neq 0, n(\Gamma; z_0) \neq 0\right\}$ is finite.

(ii) We have

$$\frac{1}{2\pi i}\oint_\Gamma \frac{f'(z)}{f(z)}\, dz = \sum_{\substack{z_0 \in U \\ \mathrm{ord}\, z_0(f) \neq 0 \\ n(\Gamma; z_0)}} n(\Gamma; z_0)\, \mathrm{ord}_{z_0}(f).$$

Proof.

(i) Let $K = \{z \in U \mid n(\Gamma; z) \neq 0\} \cup \mathrm{im}(\Gamma)$. We know that for $R > \sup_{w \in \mathrm{im}(\Gamma)} |w|$, we have $n(\Gamma; z) = 0$ for all $z \in \mathbb{C} \setminus B(0, R)$, meaning that K is bounded. Furthermore, if $(z_n)_n \to z \in \mathbb{C}$, then either $z \in \mathrm{im}(\Gamma) \subseteq K$ or $z \in \mathbb{C} \setminus \mathrm{im}(\Gamma)$, so that $n(\Gamma; z) \neq 0$ by the continuity of the map $w \mapsto n(\Gamma; w)$. Thus, $z \in K$ as Γ is homologous to zero in U. It follows thus that K is compact.

Let $E = \left\{z_0 \in K \mid \mathrm{ord}_{z_0}(f) \neq 0\right\}$, meaning that E consists of poles and zeros of f. These points are isolated, meaning E is a closed subset of K, hence compact. Since E is compact and contains isolated points only, it follows that E is finite.

(ii) For each $z \in K$, select $\delta_z > 0$ such that $U(z, \delta_z) \subseteq U$, and $U(z, \delta_z) \cap E \subseteq \{z\}$. Such a δ_z exists since z is isolated in E and U is open. The collection $\{U(z, \delta_z) \mid z \in K\}$ is an open cover of K, so there is a finite subcover $\{U(z_1, \delta_1), \ldots, U(z_m, \delta_m)\}$. Define

$$V = \bigcup_{j=1}^{m} U\big(z_j, \delta_j\big),$$

so that V is open with $K \subseteq V \subseteq U$. Since $\{z \in U \mid n(\Gamma; z) \neq 0\} \subseteq K \subseteq V$, it follows that Γ is homologous to zero in V.

Define

$$g(z) = f(z) \prod_{z \in E} (z - z_0)^{-\operatorname{ord}_{z_0}(f)}.$$

Since $\operatorname{ord}_z(g) = 0$ for all $z \in V$, all the singularities of g are removable, and $\frac{g'}{g}$ is holomorphic on V and satisfies

$$\frac{g'}{g} = \frac{f'}{f} - \sum_{z_0 \in E} \frac{1}{z - z_0} \operatorname{ord}_{z_0}(f).$$

Cauchy's Integral theorem provides the desired result.

$\square$

Theorem (Rouché's Theorem): Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U. Let f and g be meromorphic on U with no poles or zeros on $\operatorname{im}(\Gamma)$. If $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in \operatorname{im}(\Gamma)$, then

$$\frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \oint_\Gamma \frac{g'(z)}{g(z)} \, dz.$$

Proof. Since $|f(z) - g(z)| < |f(z)| + |g(z)|$ on $\operatorname{im}(\Gamma)$, coupled with the fact that $\operatorname{ord}_z(f) = \operatorname{ord}_z(g) = 0$ on $\operatorname{im}(\gamma)$ implies that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1$$

for all $z \in \operatorname{im}(\Gamma)$. This only holds if $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in \operatorname{im}(\Gamma)$. Since $\operatorname{im}(\Gamma)$ is compact, there exists some $\varepsilon > 0$ such that

$$\operatorname{dist}_{(-\infty, 0]} \left| \frac{f}{g}(\operatorname{im}(\Gamma)) \right| \geq \varepsilon.$$

Since $\frac{f}{g}$ is continuous and $\operatorname{im}(\Gamma)$ is compact, there also exists some $\delta > 0$ such that whenever $\operatorname{dist}_{\operatorname{im}(\Gamma)}(z) < \delta$, we have $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$.

Setting $V = \left\{ z \in U \mid \operatorname{dist}_{\operatorname{im}(\Gamma)}(z) < \delta \right\}$, we let $h \colon V \to \mathbb{C}$ be defined by

$$h(z) = \log \left(\frac{f(z)}{g(z)} \right)$$

for the branch of the logarithm that excludes $(-\infty, 0]$, which is well-defined as $\frac{f}{g} \notin (-\infty, 0]$ on V. This satisfies

$$\frac{h'}{h} = \frac{f'}{f} - \frac{g'}{g},$$

whence by Cauchy's Integral Theorem,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_\Gamma \frac{h'(z)}{h(z)} \, dz \\ &= \frac{1}{2\pi i} \left(\oint_\Gamma \frac{f'(z)}{f(z)} \, dz - \oint_\Gamma \frac{g'(z)}{g(z)} \, dz \right). \end{aligned}$$

$\square$

> **Remark:** Most use cases for Rouché's Theorem involve finding $g(z)$ such that $|f(z) - g(z)| < |g(z)|$ on $\text{im}(\Gamma)$, where both $f(z)$ and $g(z)$ have no zeros or poles on $\text{im}(\Gamma)$.

It is possible to use Rouché's Theorem to prove the fundamental theorem of algebra.

> **Corollary:** Let $P \colon \mathbb{C} \to \mathbb{C}$ be a nonconstant polynomial. Then, there exists $z_0 \in \mathbb{C}$ with $P(z_0) = 0$.

Proof. Write $P(z) = a_n z^n + \cdots + a_1 z + a_0$, where $n \geqslant 1$, $a_n \neq 0$, and $a_1, \ldots, a_n \in \mathbb{C}$.

Let $Q(z) = a_n z^n$, R sufficiently large, and $\Gamma = S(0, R)$. Then,

$$
\begin{aligned}
|P(z) - Q(z)| &= |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
&\leqslant |a_n| R^n \left(\frac{|a_{n-1}|}{|a_n| R} + \cdots + \frac{|a_0|}{|a_n| R^n} \right) \\
&< |a_n| R^n \\
&\leqslant |P(z)| + |Q(z)|.
\end{aligned}
$$

Thus, P has the same number of zeros counted with multiplicity as Q. $\qquad\square$

> **Corollary:** Let $U \subseteq \mathbb{C}$ be an open set, $f \colon U \to \mathbb{C}$ be a holomorphic injective function. Then, $f'(z) \neq 0$ for all $z \in U$, whence f admits a holomorphic inverse.

Proof. Suppose there is some $z_0 \in U$ such that $f'(z_0) = 0$. By the identity theorem, there must be some $r > 0$ such that $B(z_0, r) \subseteq U$ and $f'(z) \neq 0$ for all $z \in \dot{U}(z_0, r)$, else f would be equal to a constant on some nonempty open set, hence not injective.

Let $g(z) = f(z) - f(z_0)$. Define $m := \text{ord}_{z_0}(g)$. We observe that $m \geqslant 2$, as $g(z_0) = 0$ and $g'(z_0) = 0$. We may write

$$
g(z) = \lambda (z - z_0)^m + (z - z_0)^{k+1} h(z)
$$

for some holomorphic function $h \colon U \to \mathbb{C}$ and a constant $\lambda \in \mathbb{C} \setminus \{0\}$, meaning

$$
f(z) = f(z_0) + \lambda (z - z_0)^m + (z - z_0)^{k+1} h(z)
$$

for all $z \in U$.

Let $C = \sup_{z \in B(z-0,r)} |h(z)|$, which is finite as $B(z_0, r)$ is compact. Letting $\rho = \min\left(r, \frac{|\lambda|}{2C}\right)$ and $\eta = \frac{|\lambda| \rho^m}{2}$, and fixing $w \in \dot{U}(f(z_0,), \eta)$, we observe that if $z \in S(z - 0, \rho)$,

$$
\begin{aligned}
\left| f(z) - w - \lambda (z - z_0)^m \right| &= \left| f(z_0) - w + (z - z_0)^{m+1} h(z) \right| \\
&\leqslant |f(z_0) - w| + |z - z_0|^{m+1} |h(z)| \\
&< \eta + \rho^{m+1} C \\
&< \rho^m |\lambda| \\
&= \left| (z - z_0)^m \lambda \right|.
\end{aligned}
$$

Therefore, by Rouché's theorem, the number of zeros in $U(z_0, \rho)$ for $f(z) - w$ is equal to the number of zeros counted with multiplicity in $U(z_0, \rho)$ of $(z - z_0)^m \lambda$. Since the latter $m \geqslant 2$, it follows that the former is also $m \geqslant 2$. Since $f'(z) \neq 0$ for all $z \in U(z_0, \rho)$, no zero of $f(z) - w$ can have order at least 2, meaning that there are at least two distinct zeros of $f(z) - w$ in $U(z_0, \rho)$, whence f is not injective. $\qquad\square$

Residues

Definition: Let $U \subseteq \mathbb{C}$, $z_0 \in U$, $r > 0$ such that $U(z_0, r) \subseteq U$, and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

on $\dot{U}(z_0, r)$. The *residue* of f at z_0 is the coefficient a_{-1} of $(z - z_0)^{-1}$. We write $\operatorname{Res}(f; z_0)$.

Proposition: Let $U \subseteq \mathbb{C}$ be an open set, and let f be meromorphic on U with a pole of order $m \geqslant 1$ at $z_0 \in U$. Then,

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z).$$

Proof. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

Differentiating term by term, we find that

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = a_{-1}.$$

$\square$

Proposition: Let $f, g \colon U \to \mathbb{C}$ be holomorphic with z_0 a simple zero for g that is not a zero or pole for f. Then,

$$\operatorname{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. We compute the residue directly, using the fact that $g(z_0) = 0$ to find

$$\lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} = \frac{f(z_0)}{\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}}$$

$$= \frac{f(z_0)}{g'(z_0)}.$$

$\square$

Example:

- If $f(z) = \frac{e^{iz}}{z}$, then

$$\operatorname{Res}(f; 0) = 1.$$

- If $f(z) = \pi \cot(\pi z)$, then

$$\operatorname{Res}(f; n) = \frac{\pi \cos(\pi n)}{\frac{d}{dz}\big|_{z=n} \sin(\pi z)}$$

$$= \frac{\pi \cos(\pi n)}{\pi \cos(\pi n)}$$

$$= 1.$$

- Let $f(z) = \frac{e^{3z}}{(z-2)^2}$. To compute the residue at $z = 2$, we may directly find

$$\text{Res}(f; 2) = \lim_{z \to 2} \frac{d}{dz}(z-2)^2 \frac{e^{3z}}{(z-2)^2}$$
$$= 3e^6.$$

Theorem: Let $f: \dot{U}(z_0, r) \to \mathbb{C}$ be holomorphic. Then, for all $0 < \rho < r$, we have

$$\oint_{S(z_0, \rho)} f(z) \, dz = 2\pi i \, \text{Res}(f; z_0).$$

Proof. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The series converges uniformly on compact sets, so we may exchange order of integration and summation to find

$$\oint_{S(z_0, \rho)} f(z) \, dz = \sum_{n=-\infty}^{\infty} a_n \oint_{S(z_0, \rho)} (z - z_0)^n \, dz$$
$$= 2\pi i a_{-1}.$$

$\square$

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U. Let f be meromorphic on U with no poles on $\text{im}(\Gamma)$. Then,

$$\oint_\Gamma f(z) \, dz = 2\pi i \sum_{z_0 \in E} n(\Gamma; z_0) \, \text{Res}(f; z_0),$$

where

$$E = \left\{ z_0 \in U \mid \text{ord}_{z_0}(f) < 0, n(\Gamma; z_0) \neq 0 \right\}.$$

Proof. As in the proof of the argument principle, we see that E is finite, which we write $\{z_1, \dots, z_m\}$. Select $r > 0$ such that $B(z_j, r)$ are pairwise disjoint and contained in U. Set $\widetilde{\Gamma} = \Gamma - \sum_{j=1}^{n} S(z_j, r)$. Then, $\widetilde{\Gamma}$ is homologous to zero in $U \setminus \{z_1, \dots, z_m\}$, while f is holomorphic on $U \setminus \{z_1, \dots, z_m\}$, whence

$$0 = \oint_{\widetilde{\Gamma}} f(z) \, dz$$
$$= \oint_\Gamma f(z) \, dz - \sum_{j=1}^{m} \oint_{S(z_j, r)} f(z) \, dz$$
$$= \oint_\Gamma f(z) \, dz - 2\pi i \sum_{z_0 \in E} n(\Gamma; z_0) \, \text{Res}(f; z_0).$$

$\square$

Conformal Maps and Spaces of Holomorphic Functions

Given an open subset $U \subseteq \mathbb{C}$, we define $H(U)$ to be the set of all holomorphic functions $f: U \to \mathbb{C}$. Similarly, we write $C(U)$ for the continuous functions $f: U \to \mathbb{C}$.

Definition: A sequence $(f_n)_n \subseteq C(U)$ converges uniformly on compacts to $f \in C(U)$ if, for every compact $K \subseteq U$, the sequence $(f_n|_K)_n \to f|_K$ uniformly.

Proposition: Let $U \subseteq \mathbb{C}$ be open. If a sequence $(f_n)_n \subseteq H(U)$ converges uniformly on compacts to $f \in C(U)$, then $f \in H(U)$. Moreover, the sequence $(f'_n)_n \to f'$.

Proof. Since any $(f_n)_n$ converges on compacts to f, it converges uniformly on any triangle T homologous to zero in U, so that

$$
\begin{aligned}
0 &= \lim_{n \to \infty} \oint_T f_n(z) \, dz \\
&= \oint_T \lim_{n \to \infty} f_n(z) \, dz \\
&= \oint_T f(z) \, dz.
\end{aligned}
$$

To show that $(f'_n)_n \to f'$ converges uniformly on compacts, we use Cauchy's integral formula to find that

$$
f'_n(z) - f'(z) = \frac{1}{2\pi i} \oint_{S(z_0, R)} \frac{f_n(w) - f(w)}{(w - z)^2} \, dw,
$$

where $z \in U(z_0, R)$ and $B(z_0, R) \subseteq U$. Cauchy's estimate then show that this tends to 0 as $n \to \infty$ uniformly for all $z \in U(z_0, R)$. Since compact sets are totally bounded, it thus follows that the convergence is uniform on compact subsets. $\qquad\square$

Definition: Let $U \subseteq \mathbb{C}$ be an open set. An exhaustion of U is a collection of compacts $(K_m)_m$ for which $K_m \subseteq K^\circ_{m+1}$, and

$$
U = \bigcup_{m=1}^{\infty} K_m.
$$

The primary example we will use is

$$
K_m := \left\{ z \in U \,\middle|\, |z| \leqslant m, \mathrm{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{m} \right\}.
$$

Definition: Let $U \subseteq \mathbb{C}$ be an open set, and let $(K_m)_m$ be an Exhaustion of U. For $f, g \in C(U)$, we define

$$
d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f - g\|_{K_m}}{1 + \|f - g\|_{K_m}}.
$$

Now, despite the fact that the metric space $(C(U), d)$ depends on the choice of exhaustion, it can be shown that any two metrics based on exhaustions $(K_m)_m$ and $(K'_m)_m$ are uniformly equivalent.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, $(K_m)_m$ an exhaustion of U, and let

$$
d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f - g\|_{K_m}}{1 + \|f - g\|_{K_m}}
$$

for $f, g \in C(U)$. Then, $(H(U), d)$ is a complete metric space.

Proposition: The topology on $(H(U), d)$ is equal to the topology of uniform convergence on compact subsets.

Proof. Let $(K_m)_m$ be an exhaustion of U. Let $(X_m, d_m) = \left(C(K_m,), \|\cdot - \cdot\|_{K_m} \right)$, and set

$$
X = \prod_{m=1}^{\infty} C(K_m).
$$

The isometry $\iota \colon C(U) \to X$, given by $\iota(f) = (f|_{K_m})_m$ yields that a sequence $(f_n)_n \to f$ if and only if $(f_n|_{K_m})_n \to f|_{K_m}$ for each K_m. In particular, if $(f_n)_n \to f$ uniformly on compact subsets, then it

converges uniformly on each K_m, hence $\lim_{n \to \infty} d(f_n, f) = 0$.

Conversely, if $K \subseteq \mathbb{C}$, $\{K_m^\circ\}_{m=1}^\infty$ is an open cover of K, so there exists a finite subcover. Since $K_m \subseteq K_{m+1}^\circ$, this means there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$. In particular, if $\lim_{n \to \infty} d(f_n, f) = 0$, it then follows that $(f_n)_n \to f$ uniformly on K. $\qquad \square$

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A family $\mathcal{F} \subseteq H(U)$ is called *normal* if its closure $\overline{\mathcal{F}}$ is compact in $H(U)$.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

  (i) $\mathcal{F}$ is normal;

  (ii) for each $K \subseteq U$, the family $\mathcal{F}|_K$ has compact closure in $C(K)$:

  (iii) for each $z \in U$, there exists a bounded open set $W_z \subseteq U$ containing z such that $\mathcal{F}|_{W_z}$ has compact closure in $C\left(\overline{W_z}\right)$.

Proof. We start by showing that (i) implies (iii). For $z \in U$, let $R = R_z > 0$ such that $B(z, R) \subseteq U$, and let $g_n \in \mathcal{F}|_{B(z,R)}$. Choose $f_n \in \mathcal{F}$ such that $f_n|_{B(z,R)} = g_n$. If $\mathcal{F}$ is normal, there exists a subsequence $(f_{n_k})_k \to f \in \overline{\mathcal{F}}$. In particular, $f_{n_k}|_{B(z,R)} \to f|_{B(z,R)}$. Thus, $\mathcal{F}|_{B(z,R)}$ has compact closure in $C(B(z, R))$.

Next, we show that (iii) implies (ii). Let W_z be as above. Given $K \subseteq U$, the collection $\{W_z\}_{z \in U}$ is an open cover of K that has a finite subcover $\{W_1, \ldots, W_\ell\}$. Given $g_n \in \mathcal{F}|_K$, write $g_n = f_n|_K$ for some $f_n \in \mathcal{F}$. There then exists a subsequence $(f_{n_k})_k$ such that $\left(f_{n_k}|_{W_j}\right) \to f_j \in C\left(\overline{W_j}\right)$ for each j. The function $f := \lim_{k \to \infty} f_{n_k}(z)$ is well-defined and satisfies $f|_{\overline{W_j}} = f_j$ for each j. Moreover, since

$$\limsup_{k \to \infty} \|f_{n_k} - h\|_K \leqslant \limsup_{k \to \infty} \max_{1 \leqslant j \leqslant \ell} \|f_{n_k} - f_j\|_{\overline{W_j}}$$
$$= 0,$$

it follows that $\mathcal{F}|_K$ has compact closure in $C(K)$.

Finally, we show that (ii) implies (i). Let $(K_m)_m$ be an exhaustion of U. By Tychonoff's Theorem, $\prod_{m=1}^\infty \overline{\{f \mid_{K_m} | f \in \mathcal{F}\}}$ is compact. Thus, given a sequence $(f_n)_n$ in $\mathcal{F}$, there is a subsequence $(f_{n_k})_k$ such that for each $m \in \mathbb{N}$, there is some $g_m \in C(K_m)$ such that $f_{n_k}|_{K_m}$ converges uniformly to g_m. Thus, the function $f(z) = \lim_{k \to \infty} f_{n_k}(z)$ is well-defined and satisfies $f|_{K_m} = g_m$ for all m. Given a compact $K \subseteq U$, there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$, and consequently, $f_{n_k}|_K \to f|_K$ uniformly, so $\mathcal{F}$ is normal. $\qquad \square$

We will now create a much more workable criterion for normality.

Definition: Let (X, d) be a compact metric space. A family $\mathcal{F} \subseteq C(X)$ is (uniformly) equicontinuous if, for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$.

Recall the Arzelà–Ascoli theorem.

Theorem (Arzelà–Ascoli)**:** Let $K \subseteq \mathbb{C}$ be compact, and let $\mathcal{F} \subseteq C(K)$ be a family of continuous functions. The following are equivalent:

  (i) $\overline{\mathcal{F}}$ is compact;

  (ii) $\mathcal{F}$ is bounded and, for all $z \in K$, $\overline{\{f(z) \mid f \in \mathcal{F}\}} \subseteq \mathbb{C}$ is compact.

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A family $\mathcal{F} \subseteq H(U)$ is called locally (uniformly) bounded on U if, for all $z_0 \in U$, there is some $\delta > 0$ such that $U(z_0, \delta) \subseteq U$ and there is some $C \geqslant 0$ such that $|f(z)| \leqslant C$ for all $z \in U(z_0, \delta)$ and all $f \in \mathcal{F}$.

Proposition: Let $U \subseteq \mathbb{C}$ be open, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

(i) $\mathcal{F}$ is locally bounded;

(ii) for every compact $K \subseteq U$, $\sup_{f \in \mathcal{F}} \|f\|_K$ is finite.

Proof. We start by showing that (i) implies (ii). For each $z \in U$, there is $\delta_z > 0$ and $C_z > 0$ such that $U(z, \delta_z) \subseteq U$ and $|f(w)| \leqslant C_z$ for all $w \in U(z, \delta_z)$. Let $K \subseteq U$ be compact. There is a finite subcover $\{U(z_1, \delta_1), \ldots, U(z_k, \delta_k)\}$, with corresponding bounds $C_1, \ldots, C_k$, so by defining $C = \max\{C_1, \ldots, C_k\}$, it follows that $\sup_{f \in \mathcal{F}} \|f\|_K \leqslant C < \infty$.

Now, we show that (ii) implies (i). Given $z_0 \in U$, there is $r > 0$ such that $B(z_0, r) \subseteq U$. By taking $K = B(z_0, r)$, we are done. $\qquad \square$

Theorem (Montel's Theorem)**:** Let $U \subseteq \mathbb{C}$ be open, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

(i) $\mathcal{F}$ is normal;

(ii) $\mathcal{F}$ is locally bounded.

Proof. We start by showing that (i) implies (ii). Given $K \subseteq U$, the map $\|\cdot\|_K$ is continuous, so $\sup_{f \in \mathcal{F}} \|f\|_K \leqslant \sup_{f \in \overline{\mathcal{F}}} \|f\|_K$, which is finite as $\overline{\mathcal{F}}$ is compact.

Now, we show (ii) implies (i). Fix $z_0 \in U$, and choose $R_0 > 0$ such that $B(z_0, R_0) \subseteq U$. Let $W_0 = U(z_0, R_0/2)$. It suffices to show that $\mathcal{F}|_{\overline{W_0}}$ has compact closure in $C\left(\overline{W_0}\right)$, which by the Arzelà–Ascoli theorem, is equivalent to showing that $\mathcal{F}|_{\overline{W_0}}$ is equicontinuous. Since $\mathcal{F}$ is locally bounded, there is some $C_0 \geqslant 0$ such that $\sup_{f \in \mathcal{F}} \|f\|_{B(z_0, R_0)} \leqslant C_0$. By Cauchy's Integral Formula, we then have for all $z \in \overline{W_0}$ and all $f \in \mathcal{F}$,

$$
\begin{aligned}
|f'(z)| &= \left| \frac{1}{2\pi i} \oint_{S(z_0, R_0)} \frac{f(\xi)}{(\xi - z)^2} \, d\xi \right| \\
&\leqslant \frac{C_0}{2\pi} \oint_{S(z_0, R_0)} \frac{1}{|\xi - z|^2} \, |d\xi| \\
&\leqslant \frac{4 C_0}{R_0} \\
&=: A_0.
\end{aligned}
$$

We have thus shown that $|f'(z)| \leqslant A_0$ for all $z \in \overline{W_0}$, meaning that each $f|_{\overline{W_0}} \in \mathcal{F}|_{\overline{W_0}}$ is A_0-Lipschitz, hence $\mathcal{F}|_{\overline{W_0}}$ is equicontinuous, hence normal. $\qquad \square$

Worked Examples and Problem-Solving Methods

Example: Suppose U is a region in $\mathbb{C}$ that contains 0, and suppose $f \colon U \to \mathbb{C}$ is a holomorphic function satisfying

$$
\left| f\left(\frac{1}{n} \right) \right| < e^n
$$

for sufficiently large n. We will show that this means f is 0 everywhere.

Toward this end, since U is open, there is some $r > 0$ such that $U(0, r) \subseteq U$. Since f is holomorphic, on

$U(0, r)$, we may write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for some sequence $(a_n)_n \subseteq \mathbb{C}$. Now, we also observe that

$$\begin{aligned} |f(0)| &= \lim_{n \to \infty} \left| f\left(\frac{1}{n}\right) \right| \\ &\leqslant \lim_{n \to \infty} e^{-n} \\ &= 0. \end{aligned}$$

Suppose toward contradiction that f were nonconstant. Then, there would be some minimal positive value ℓ such that

$$f(z) = z^{\ell} \sum_{n=0}^{\infty} a_{n+\ell} z^n$$

has $a_\ell \neq 0$. Thus, defining

$$g(z) = \sum_{n=0}^{\infty} a_{n+\ell} z^n,$$

we observe that $g(0) \neq 0$, meaning that on some sufficiently small ball $U(0, \delta) \subseteq U(0, r)$, we have $|g(z)| > \left| \frac{a_\ell}{2} \right|$ for all $z \in U(0, \delta)$. In particular, this means that for n with $\frac{1}{n} < \delta$,

$$\begin{aligned} e^{-n} &\geqslant \left| f\left(\frac{1}{n}\right) \right| \\ &= n^{-\ell} \left| g\left(\frac{1}{n}\right) \right| \\ &\geqslant \frac{|a_\ell|}{2n^\ell}, \end{aligned}$$

whence

$$|a_\ell| \leqslant \frac{2n^\ell}{e^n}.$$

Yet, since n is arbitrary and ℓ is constant, this implies that $|a_\ell| = 0$, contradicting the assumption that there were such a g. Thus, in particular, we have that $f(z) = 0$ on $U(0, r)$, whence f is zero everywhere by the identity theorem.

Cauchy Estimate Problems

Example: Suppose f is an entire function, and suppose there exists a constant C such that for all $z \in \mathbb{C}$,

$$|f(z)| \leqslant C(1 + |z|)^{1/2}.$$

We will show that f is then constant. Toward this end, we will be able to use the Cauchy estimate by taking

$$\left| f^{(n)}(z) \right| \leqslant \frac{n!}{R^n} \sup_{|z|=R} |f(z)|$$

$$\leqslant \frac{Cn!}{R^n} \sup_{|z|=R} (1 + |z|)^{1/2}$$

$$= \frac{Cn!}{R^n} (1 + R)^{1/2},$$

whence for all $n \geqslant 1$, since R is arbitrary, we have $\left|f^{(n)}(z)\right| = 0$, so f is constant.

Maximum Modulus Principle Problems

Example: We show that if $f \colon U \to \mathbb{C}$ is holomorphic on a connected open set, and

$$u(x, y) = |f(x + iy)|$$

is harmonic on U, then f is constant.

Toward this end, we let $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. For any $0 < s < r$, the mean value property gives

$$|f(z_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left|f\!\left(z_0 + se^{i\theta}\right)\right| d\theta$$

$$= |f(z_0)|.$$

In particular, for any $0 < s < r$, we have the equality

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} \left|f\!\left(z_0 + se^{i\theta}\right)\right| d\theta.$$

Since f is continuous, there is some θ_s such that $\left|f\!\left(z_0 + se^{i\theta}\right)\right| = e^{i\theta_s} f\!\left(z_0 + se^{i\theta}\right)$, whence

$$|f(z_0)| = e^{i\theta_s} \frac{1}{2\pi} \int_0^{2\pi} f\!\left(z_0 + se^{i\theta}\right) d\theta$$

$$= e^{i\theta_s} f(z_0),$$

meaning that $\theta_s =: \theta_0$ is independent of s. Yet, this means that $e^{i\theta_0} f(z)$ is holomorphic on $U(z_0, r)$ and has $\operatorname{Im}\!\left(e^{i\theta} f(z)\right) = 0$, meaning that by the open mapping principle, $f(z)$ is constant on $U(z_0, r)$, and so f is constant on U by the identity theorem.

The Phragmén–Lindelöf Method

The maximum modulus principle is primarily useful in the case where f is continuous on the closure of a bounded open set U and holomorphic on the interior. Yet, this fails to be true if U is unbounded.

For instance, if

$$U = \left\{ z \in \mathbb{C} \ \middle|\ -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \right\},$$

and $f(z) = e^{e^z}$, then

$$f\!\left(x \pm \frac{\pi}{2}i\right) = e^{\pm ie^x},$$

whence $|f(z)| = 1$ for $z \in \partial U$. Yet, $f(z) \to \infty$ very rapidly along the positive real axis, which is contained in U.

Yet, all hope is not lost in the case that U is unbounded. If U is unbounded and there is $g \colon U \to \mathbb{C}$ such that $|f| < |g|$, and $g \to \infty$ "slowly" (so to speak) as $z \to \infty$, then it turns out that f is actually bounded in U, and we can use the maximum modulus principle to obtain other conclusions about f.

Finding such a g is part of the *Phragmén–Lindelöf* method, which we expand upon here.

Example: From the Cauchy estimates, we know that if f is entire and

$$|f(z)| \leqslant C\left(1 + |z|^{1/2}\right),$$

then f is constant.

Theorem (Hadamard Three-Lines Theorem)**:** Let $a, b \in \mathbb{R}$ be fixed with $a < b$. Let $U = \{z \mid a < \operatorname{Re}(z) < b\}$. Suppose $|f(z)| < B$ for all $z \in U$ and some fixed $B < \infty$. Define

$$M(x) = \sup\left\{|f(z)| \,\Big|\, z \in \overline{U}\right\}.$$

Then,

$$M(x)^{b-a} \leqslant M(a)^{b-x} M(b)^{x-a}.$$

Proof. Suppose $M(a) = M(b) = 1$. Our task now is to show that $|f(z)| \leqslant 1$ for all $z \in U$. Toward this end, define

$$h_\varepsilon(z) = \frac{1}{1 + \varepsilon(z - a)}$$

for $z \in \overline{U}$. We have $|h_\varepsilon| \leqslant 1$ in $\overline{U}$, so that

$$|f(z) h_\varepsilon(z)| \leqslant 1$$

for all $z \in \partial U$. Furthermore, since $|1 + \varepsilon(z - a)| \geqslant \varepsilon |\operatorname{Im}(z)|$, we have

$$|f(z) h_\varepsilon(z)| \leqslant \frac{B}{\varepsilon |\operatorname{Im}(z)|}$$

for all $z \in \overline{U}$. Cut out a (closed) rectangle R from $\overline{U}$ via the lines $\operatorname{Im}(z) = \pm\frac{B}{\varepsilon}$. Thus, along ∂R, we have $|f(z) h_\varepsilon(z)| \leqslant 1$, so that $|f(z) h_\varepsilon(z)| \leqslant 1$ on R by the maximum modulus principle.

Yet, since $|f(z) h_\varepsilon(z)| \leqslant \frac{B}{\varepsilon |\operatorname{Im}(z)|}$ on the entirety of $\overline{U}$, and $\frac{B}{\varepsilon |\operatorname{Im}(z)|} < 1$ outside R, it follows that $|f h_\varepsilon| \leqslant 1$ on $\overline{U}$, so $|f(z) h_\varepsilon(z)| \leqslant 1$ for all $z \in U$ and all $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0$, we obtain the desired result, that $|f(z)| \leqslant 1$.

In the general case, we define

$$g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)},$$

where for all $M > 0$ and complex w, we have $M^w = e^{w \ln(M)}$. Then, g is entire, g is always nonzero, $\frac{1}{g}$ is bounded on $\overline{U}$, and has

$$|g(a + iy)| = M(a)$$
$$|g(b + iy)| = M(b),$$

meaning that $\frac{f}{g}$ satisfies the previous assumptions, so that $|f/g| \leqslant 1$ in U. $\qquad\square$

In the Phragmén–Lindelöf method, we seek to find a particular ε-dependent function $h_\varepsilon \colon U \to \mathbb{C}$ such

that the following hold:

- $|fh_\varepsilon(z)| \leqslant M$ for all $z \in \partial U$;

- $\lim_{\varepsilon \to 0} h_\varepsilon(z) = 1$;

- there exists a *bounded* $V \subsetneq U$ such that $|fh_\varepsilon| \leqslant M$ on ∂V and on $U \setminus \overline{V}$.

Rouché's Theorem Problems

Example: We show that if f and g are holomorphic on a neighborhood of $B(0,1)$, and $f(z)$ has a simple zero at $z = 0$ and no other zero in $B(0,1)$, then $f_\varepsilon(z) = f(z) + \varepsilon g(z)$ has exactly one zero in $\mathbb{D}$ for sufficiently small ε.

To show this, we start by showing that the conditions of Rouché's Theorem are satisfied for both $f_\varepsilon(z)$ and $f(z)$. It is clear from the fact that f has no other zeros in $B(0,1)$ that f has no zeros on $S(0,1)$, while we may find ε small enough such that

$$|f_\varepsilon(z)| \geqslant |f(z)| - \varepsilon|g(z)|$$
$$> 0$$

by selecting ε such that $\varepsilon \inf_{z \in S(0,1)}|g(z)| < \inf_{z \in S(0,1)}|f(z)|$. Thus, if we set $m_1 = \inf_{z \in S(0,1)}|f(z)|$ and $m_2 = \sup_{z \in S(0,1)}|g(z)|$, we have for $\varepsilon < \frac{m_1}{m_2}$ and all $z \in S(0,1)$,

$$|f_\varepsilon(z) - f(z)| \leqslant |\varepsilon g(z)|$$
$$\leqslant \varepsilon m_2$$
$$< m_1$$
$$< |f(z)|,$$

whence $f(z)$ and $f_\varepsilon(z)$ have the same number of zeros in $\mathbb{D}$ counted with multiplicity.

Residue Integrals

Example: We will evaluate

$$I = \int_0^\infty \frac{1}{1 + x^n}\,dx$$

via contour integration. Toward this end, let $f(z) = \frac{1}{1+z^n}$. We observe that $z^n + 1$ has roots for $e^{i\theta}$ at values $\theta = \frac{\pi + 2\pi k}{n}$ for $0 \leqslant k < n$.

We take the closed contour γ_R given by

$$\oint_{\gamma_R} f(z)\,dz = \int_0^R f(x)\,dx + \int_0^{\frac{2\pi}{n}} f(Re^{i\theta})\,d(Re^{i\theta}) + \int_R^0 f(xe^{i(2\pi/n)})\,d(xe^{i(2\pi/n)}).$$

The left-hand side encloses the residue of f at $e^{i\pi/n}$, so by the Residue Theorem, since f has a simple pole at $e^{i\pi/n}$,

$$\oint_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}\left(f; e^{i\pi/n}\right)$$

$$= 2\pi i \frac{1}{n\left(e^{i\pi/n}\right)^{n-1}}$$

$$= 2\pi i \frac{1}{n\left(e^{i\frac{(n-1)\pi}{n}}\right)}$$

$$= \frac{2\pi}{n} \frac{e^{i\pi/2}}{e^{i\pi \frac{n-1}{n}}}$$

$$= \frac{2\pi}{n} e^{i\frac{\pi}{n} - \frac{\pi}{2}}.$$

We now observe that the original integral expression equals

$$\oint_{\gamma_R} f(z) \, dz = \int_0^R \frac{1}{1 + x^n} \, dx + \int_0^{\frac{2\pi}{n}} \frac{1}{1 + R^n e^{in\theta}} iRe^{i\theta} \, d\theta + e^{i(2\pi/n)} \int_R^0 \frac{1}{1 + x^n} \, dx$$

$$= \left(1 - e^{i(2\pi/n)}\right) I + \int_0^{2\pi/n} \frac{1}{1 + R^n e^{in\theta}} iRe^{i\theta} \, d\theta.$$

Estimating the second integral, we get for $R > 1$,

$$\left| \int_0^R \frac{iRe^{i\theta}}{1 + R^n e^{in\theta}} \, d\theta \right| \leq \frac{\frac{2\pi}{n} R}{R^n - 1},$$

so that the integral tends to 0 as $R \to \infty$. Thus,

$$2\pi i \operatorname{Res}\left(f; e^{i\pi/n}\right) = \lim_{R \to \infty} \oint_{\gamma_R} f(z) \, dz$$

$$= \left(1 - e^{i(2\pi/n)}\right) I,$$

whence

$$I = \frac{2\pi}{n} \frac{e^{i\left(\frac{\pi}{n} - \frac{\pi}{2}\right)}}{1 - e^{i(2\pi/n)}}$$

$$= \frac{2\pi}{n} \frac{e^{-i\frac{\pi}{2}}}{e^{-i\pi/n} - e^{i\pi/n}}$$

$$= \frac{2\pi}{n} \frac{-i}{-2i \sin\left(\frac{\pi}{n}\right)}$$

$$= \frac{\pi}{n \sin\left(\frac{\pi}{n}\right)}.$$

Old Exams

August 2019

Problem (Problem 1)**:** Let ξ be a nonnegative real number. Compute

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x^2 + 1} \, dx.$$

Solution: We consider

$$f(z) = \frac{e^{i\xi z}}{z^2 + 1}$$

$$\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2 + 1} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{i\xi x}}{x^2 + 1} \, dx.$$

We consider the contour γ_R given closing with the semicircle of radius R in the upper half-plane,

parametrized by $\left\{ Re^{i\theta} \mid 0 \leqslant \theta \leqslant \pi \right\}$. Then,

$$\oint_{\gamma_R} f(z) \, dz = 2\pi i \, \text{Res}(f; i)$$

$$= \int_{-R}^{R} \frac{e^{i\xi x}}{x^2 + 1} \, dx + \int_0^\pi \frac{e^{i\xi Re^{i\theta}}}{R^2 e^{2i\theta} + 1} i Re^{i\theta} \, d\theta.$$

On the circular integral, we observe that for $R > 1$,

$$\left| \int_0^\pi \frac{iRe^{i\theta} e^{i\xi R(\cos(\theta) + i \sin(\theta))}}{R^2 e^{2i\theta} + 1} \, dz \right| \leqslant \pi \frac{Re^{-R\xi \sin(\theta)}}{R^2 - 1}$$

$$\to 0$$

as $R \to \infty$. Therefore, since the pole at i is simple, we get

$$2\pi i \, \text{Res}(f; i) = 2\pi i \left(\lim_{z \to i} \frac{(z - i)e^{i\xi x}}{(z - i)(z + i)} \right)$$

$$= \pi e^{-\xi},$$

whence

$$\pi e^{-\xi} = \lim_{R \to \infty} \oint_{\gamma_R} f(z) \, dz$$

$$= \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{i\xi x}}{x^2 + 1} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2 + 1} \, dx.$$

Problem (Problem 2)**:** Let f be an entire function, and suppose there is some $\alpha \in (0, \infty)$ such that

$$|f(z)| \leqslant C|z|^\alpha$$

for all $z \geqslant 1$. Show that f is a polynomial.

Solution: From the Archimedean property, we know that there is some natural number N such that $N > \alpha$. We observe then that, from Cauchy's estimates,

$$\left| f^{(N)}(z) \right| \leqslant \frac{N!}{r^N} \sup_{|z| = r} |f(z)|$$

$$\leqslant \frac{N!}{r^N} \sup_{|z| = r} C|z|^\alpha$$

$$= \frac{CN!}{r^{N-\alpha}}$$

$$\to 0$$

as $r \to \infty$, whence the Taylor expansion for f about 0 terminates at some N. In particular, this means that f is a polynomial.

Problem (Problem 3)**:** Let f be an entire function. Suppose that $\lim_{z \to \infty} f(z) = \infty$. Show that f is a polynomial.

Solution: Consider the transformation $z \mapsto 1/z$, giving

$$\lim_{z \to 0} f(1/z) = \lim_{z \to \infty} f(z)$$

$$= \infty.$$

In particular, from the classification of singularities, this means that $f(1/z)$ has a pole at 0. This gives some n such that

$$f(1/z) = \sum_{k=0}^{n} a_k z^{-k},$$

whence

$$f(z) = \sum_{k=0}^{n} a_k z^k,$$

so f is a polynomial.

Problem (Problem 4): Let $\Omega = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. Suppose $f \colon \overline{\Omega} \to \mathbb{C}$ be continuous with $f|_\Omega$ holomorphic. Suppose $|f(iy)| \leqslant 1$ for all $y \in \mathbb{R}$ and $|f(z)| \leqslant 2$ for all $z \in \Omega$. Show that in fact $|f(z)| \leqslant 1$ for all $z \in \Omega$.

Solution: Consider the function

$$f_\varepsilon(z) = \frac{f(z)}{1 + \varepsilon z}.$$

We observe that

$$
\begin{aligned}
|f_\varepsilon(z)| &= \frac{|f(z)|}{|1 + \varepsilon z|} \\
&\leqslant \frac{2}{|1 + \varepsilon z|} \\
&\leqslant \frac{2}{\varepsilon |\operatorname{Im}(z)|}.
\end{aligned}
$$

Now, we observe that for z in the rectangle with corners $i2/\varepsilon$, $-i2/\varepsilon$, $1/\varepsilon + i2/\varepsilon$, and $1/\varepsilon - i2/\varepsilon$, that

$$|f_\varepsilon(z)| \leqslant 1$$

for all z on this rectangle, so by the maximum modulus principle, the inequality holds on the interior of the rectangle, and

$$|f_\varepsilon(z)| \leqslant 1$$

for all z in Ω outside this this rectangle, so that

$$
\begin{aligned}
|f(z)| &= \lim_{\varepsilon \to 0} |f(z)| \\
&\leqslant 1
\end{aligned}
$$

for all $z \in \Omega$.

Problem (Problem 5): Let $\mathbb{D} = \{z \mid |z| < 1\}$. Let $\mathcal{F}$ be a family of holomorphic functions on $\mathbb{D}$, and that $\sup_{f \in \mathcal{F}} |f(0)| < \infty$. Show that $\mathcal{F}$ is normal if and only if $\{f' \mid f \in \mathcal{F}\}$ is normal.

Solution: Call the family $\mathcal{G} = \{f' \mid f \in \mathcal{F}\}$. First, we observe that if $(f_n)_n \subseteq \mathcal{F}$ is a sequence with convergent subsequence $(f_{n_k})_k \to f \colon \mathbb{D} \to \mathbb{C}$ uniformly on compact sets, then it has been well-established that $(f'_{n_k})_k \to f'$ uniformly on compact sets, whence $(f'_n)_n \subseteq \mathcal{G}$ admits a convergent subsequence.

Now, let $(f_n)_n \subseteq \mathcal{F}$, so that $(f'_n)_n \subseteq \mathcal{G}$. Then, $(f'_n)_n$ admits a subsequence $(f'_{n_k})_k \to g \colon \mathbb{D} \to \mathbb{C}$.

First, we observe that since $\mathbb{D}$ is simply connected, g admits an antiderivative $f\colon \mathbb{D} \to \mathbb{C}$. We will show that $(f_{n_k})_k \to f$ uniformly on compacts.

Let $K \subseteq \mathbb{D}$ be compact, and let $z \in K$. Let $(K_m)_m$ be an exhaustion of $\mathbb{D}$ by closed balls of radius $\frac{m}{m+1}$. Then, there is some M such that $K \subseteq K_M^\circ$. We observe that the path $\gamma\colon [0,1] \to \mathbb{D}$ given by $\gamma(t) = tz$ is then contained wholly in K_M. Furthermore, we have

$$
\begin{aligned}
|f_{n_k}(z) - f(z)| &\leqslant \left| \int_0^1 z\big(f'_{n_k}(tz) - g(tz)\big)\,dt \right| \\
&\leqslant |z| \sup_{t \in [0,1]} \left| f'_{n_k}(tz) - g(tz) \right| \\
&\leqslant \sup_{z \in K_M} \left| f'_{n_k}(z) - g(z) \right| \\
&\to 0
\end{aligned}
$$

whence

$$
\begin{aligned}
\sup_{z \in K} |f_{n_k}(z) - f(z)| &\leqslant \sup_{z \in K_M} |f_{n_k}(z) - f(z)| \\
&\leqslant \sup_{z \in K_M} \left| f'_{n_k}(z) - g(z) \right| \\
&\to 0
\end{aligned}
$$

so that $(f_{n_k})_k \to f$ uniformly on K. Thus, $\mathcal{F}$ is normal.

January 2020

Problem (Problem 1)**:** Compute

$$
\lim_{R \to \infty} \int_{-R}^R \frac{\sin(x)}{x}\,dx
$$

using residue theory.

Solution: We start by taking $f(z) = \frac{e^{iz}}{z}$, and let γ_R be the contour defined by a semicircle below the real axis with radius δ centered at the origin and an outer semicircle with radius R, alongside rays from $-R$ to $-\delta$ and δ to R. We observe then that

$$
\begin{aligned}
\oint_{\gamma_R} f(z)\,dz &= \int_{-R}^{-\delta} \frac{e^{ix}}{x}\,dx + \int_\delta^R \frac{e^{ix}}{x}\,dx \\
&\quad + i\int_\pi^{2\pi} \frac{e^{i\delta e^{i\theta}}}{\delta e^{i\theta}} \delta e^{i\theta}\,d\theta \\
&\quad + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}\,d\theta.
\end{aligned}
$$

The left-hand side evaluates to

$$
\begin{aligned}
2\pi i \operatorname{Res}(f; 0) &= 2\pi i \lim_{z \to 0} \left(\frac{ze^{iz}}{z} \right) \\
&= 2\pi i.
\end{aligned}
$$

Now, estimating the integral in R, we have

$$
\left| \frac{iRe^{i\theta} e^{iRe^{i\theta}}}{Re^{i\theta}} \right| = \left| e^{iR(\cos(\theta) + i\sin(\theta))} \right|
$$

$$\leqslant e^{-R},$$

whence the integral over R goes to zero. Finally, we observe that

$$\left|e^{i\delta e^{i\theta}}\right| = \left|e^{i\delta(\cos(\theta) + i \sin(\theta))}\right|$$

$$= e^{-\delta \sin(\theta)}$$

For a fixed θ, we observe that $e^{-\delta \sin(\theta)} \to 0$ uniformly as $\delta \to 0$, whence upon taking these various limits, we get

$$2\pi i = i \lim_{R \to \infty} \int_{-R}^{R} \frac{\sin(x)}{x} \, dx + i\pi,$$

so that

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\sin(x)}{x} \, dx = \pi.$$

Problem (Problem 2): Let f be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Solution: Suppose toward contradiction that there were some $w \in \mathbb{C}$ such that there exists $r > 0$ with $U(w, r) \notin f(\mathbb{C})$. Then, we observe that the function

$$g(z) = \frac{1}{f(z) - w}$$

is entire since $w \notin f(\mathbb{C})$ and has

$$|g(z)| = \frac{1}{|f(z) - w|}$$

$$\leqslant \frac{1}{r},$$

since $|f(z) - w| \geqslant r$. Thus, g is constant by Liouville's Theorem, so f is constant, which contradicts the assumption that f is nonconstant.

Problem (Problem 3): Let f be an entire function. Assume there is a sequence $(r_n)_n \subseteq (0, \infty)$ with $r_n \to \infty$ and constants $C, \alpha \in (0, \infty)$ such that

$$\sup_{|z| = r_n} |f(z)| \leqslant C r_n^{\alpha}.$$

Show that f is a polynomial.

Solution: Let $N \in \mathbb{N}$ be such that $\alpha < N$. Then, by Cauchy's estimate, we have

$$\left|f^{(N)}(0)\right| \leqslant \frac{N!}{r_n^N} \sup_{|z| = r_n} |f(z)|$$

$$\leqslant \frac{CN!}{r_n^{N-\alpha}},$$

and since $r_n \to \infty$, so too does $r_n^{N-\alpha}$, whence $\left|f^{(N)}(0)\right| = 0$. In particular, this means that the power series expansion of f about 0, which is equal to f on $\mathbb{C}$ as f is entire, terminates past N, meaning that f is a polynomial.

Problem (Problem 4): Let $(f_n)_n$ be a sequence of holomorphic functions, and suppose $(f_n)_n \to f$ uniformly on compact subsets of $\mathbb{D}$ to a function f. Suppose f has no zeros on $\left\{z \in \mathbb{C} \,\middle|\, |z| = \frac{1}{2}\right\}$. Show that there exists an $N \in \mathbb{N}$ such that if $n, m \geqslant N$, then f_m and f_n have the same number of zeros in

$\left\{ z \in \mathbb{C} \;\middle|\; |z| < \frac{1}{2} \right\}$, counted with multiplicity.

Solution: Since $S(0, 1/2)$ is compact, it follows that $(f_n)_n \to f$ uniformly on $S(0, 1/2)$. Let $\varepsilon_0 > 0$ be such that $\inf_{z \in S(0,1/2)} |f(z)| \geqslant \varepsilon_0 > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$\sup_{|z|=1/2} |f_n(z) - f(z)| < \frac{\varepsilon_0}{2},$$

whence

$$\begin{aligned}
|f_n(z)| &= |f_n(z) - f(z) + f(z)| \\
&\geqslant |f(z)| - |f_n(z) - f(z)| \\
&\geqslant \frac{\varepsilon_0}{2} \\
&> 0.
\end{aligned}$$

In particular, for every $n \geqslant N$, we have

$$\begin{aligned}
|f_n(z) - f(z)| &< \frac{\varepsilon_0}{2} \\
&< \varepsilon_0 \\
&\leqslant |f(z)| \\
&\leqslant |f(z)| + |f_n(z)|,
\end{aligned}$$

so by Rouché's Theorem, it follows that for every $n \geqslant N$, f_n has the same number of zeros in $U(0, 1/2)$ as f, whence for all $m, n \geqslant N$, f_m and f_n have the same number of zeros on $U(0, 1/2)$.

Problem (Problem 5): Let $U \subseteq \mathbb{C}$ be open, and let $f_n \colon U \to \mathbb{C}$ be a sequence of analytic functions. Suppose that f_n converge pointwise to a function $f \colon U \to \mathbb{C}$. If $|f_n| \leqslant 1$ for all n, show that f is analytic.

Solution: By Montel's Theorem, the collection $\mathcal{F} = \{ f_n \mid n \in \mathbb{N} \}$ is normal as it is locally (globally, even) bounded by 1.

Our goal is to show that under this condition, $(f_n)_n \to f$ uniformly on compact subsets. The completeness of the space $H(U)$ under the topology of uniform convergence on compact subsets would then give us our desired result.

Since $\mathcal{F}$ is normal, it follows that for any subsequence $(f_{n_k})_k$, there is a sub-subsequence $\left(f_{n_{k_j}} \right)_j \to h_k$ uniformly on compact subsets, where h_k is necessarily holomorphic. Yet, since every subsequence of $(f_n)_n$ converges pointwise to f, it follows that $h_k = f$ for all k. In particular, this means that every subsequence $(f_n)_k$ admits a sub-subsequence $\left(f_{n_{k_j}} \right)_j \to f$ uniformly on compact subsets. We show that this implies that $(f_n)_n \to f$ uniformly on compact subsets.

Suppose this were not the case. Then, there would be some compact $K \subseteq U$, some $\varepsilon_0 > 0$, and some subsequence $(f_{n_k})_k$ such that $\| f_{n_k} - f \|_K \geqslant \varepsilon_0$ for all k. Yet, since $(f_{n_k})_k$ admits a convergent subsequence $\left(f_{n_{k_j}} \right)_j$, it follows that for some J, we have $\left\| f_{n_{k_j}} - f \right\|_K < \varepsilon_0$ for all $j \geqslant J$, which is a contradiction. In particular, this means that $(f_n)_n \to f$ uniformly on compact subsets, so f is holomorphic.

August 2020

Problem (Problem 1): Let f be an entire function, and assume that

$$\sup_{|z|=r}|f(z)| \leqslant 10\ln(r)$$

for all $r \geqslant 100$. Show that f is constant.

Solution: If $R > 100$, then by Cauchy's estimates, we have

$$
\begin{aligned}
|f'(0)| &\leqslant \frac{1}{R}\sup_{|z|=R}|f(z)| \\
&\leqslant \frac{1}{R}(10\ln(R)) \\
&\to 0
\end{aligned}
$$

as $R \to \infty$. In particular, this means that the power series expansion for $f(z)$ centered at 0 terminates after the constant term, whence f is constant.

Problem (Problem 2): Let $\Omega = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Suppose that

(i) $|f(z)| > 1$ for all $z \in \Omega$;

(ii) there exists $(z_k)_k \subseteq \Omega$ with $(z_k)_k \to 0$ and $|f(z_k)| \leqslant 10$ for all k.

Is 0 a removable singularity?

Solution: First, since $|f(z)| > 1$, we observe that the function

$$g(z) = \frac{1}{f(z)}$$

is then holomorphic on Ω, and is bounded above by 1, meaning that by Riemann's Theorem on removable singularities, there is a holomorphic extension $\widetilde{g}$ defined on $\mathbb{D}$.

Observe that $\widetilde{g}$ is continuous at $z = 0$, so that with the aforementioned sequence $(z_k)_k$, we have

$$
\begin{aligned}
|\widetilde{g}(0)| &= \lim_{k\to\infty}|\widetilde{g}(z_k)| \\
&= \lim_{k\to\infty}|g(z_k)| \\
&= \lim_{k\to\infty}\left|\frac{1}{f(z_k)}\right| \\
&\geqslant \frac{1}{10}.
\end{aligned}
$$

In particular, $\widetilde{g}(0) \neq 0$, so that the extension

$$\widetilde{f}(z) = \frac{1}{\widetilde{g}(z)}$$

is a holomorphic extension of f to $\mathbb{D}$, whence 0 is a removable singularity.

Problem (Problem 3): Compute

$$I = \int_0^\infty \frac{\ln(x)}{x^2 + 1}\,dx.$$

Show all estimates.

Solution: Consider the branch of the complex logarithm that excludes the positive real axis, and let $f(z) = \frac{\ln(z)}{z^2+1}$. Then, using the contour hugging the positive real axis with imaginary part ε, a semicircle of

radius R, returning along the negative real axis to $-\varepsilon$, we then get

$$\oint_{\gamma_R} f(z)\,dz = \int_{i\varepsilon}^{R+i\varepsilon} \frac{\ln(x)}{x^2+1}\,dx + \int_{-R}^{\varepsilon} \frac{\ln(x)}{x^2+1}\,dx$$
$$+ \int_{\lambda}^{\pi} \frac{\ln\left(Re^{i\theta}\right)}{R^2 e^{2i\theta}+1} iRe^{i\theta}\,d\theta,$$

so that upon taking $\varepsilon \to 0$ and using a u-substitution on the negative real axis integral, we get

$$\oint_{\gamma_R} f(z)\,dz = 2\int_0^R \frac{\ln(x)}{x^2+1}\,dx + i\pi \int_0^R \frac{1}{x^2+1}\,dx$$
$$+ \int_0^{\pi} \frac{\ln(R)+i\theta}{R^2 e^{2i\theta}+1} iRe^{i\theta}\,d\theta.$$

Estimating the second integral, for $R > 1$, we get

$$\left| \int_0^{\pi} \frac{\ln(R)+i\theta}{R^2 e^{2i\theta}+1} iRe^{i\theta}\,d\theta \right| \leqslant \pi \left(\frac{R\ln(R)}{R^2-1} + \frac{R\theta}{R^2-1} \right)$$
$$\to 0,$$

so that

$$2\pi i \operatorname{Res}(f; i) = 2\pi i \left(\lim_{z \to i} \frac{(z-i)\ln(i)}{2i} \right)$$
$$= \frac{\pi^2}{2} i$$
$$= 2\int_0^{\infty} \frac{\ln(x)}{x^2+1}\,dx + i\pi\left(\frac{\pi}{2}\right),$$

meaning that

$$\int_0^{\pi} \frac{\ln(x)}{x^2+1}\,dx = 0.$$

Problem (Problem 5): Let $\mathcal{F}$ be a family of entire functions with the property that for all circles C there is a constant M_C such that

$$\sup_{f \in \mathcal{F}} \sup_{|z| \in C} |f(z)| \leqslant M_C.$$

(i) Show that there exists a sequence $(f_n)_n \subseteq \mathcal{F}$ converging to an entire function uniformly on compact subsets.

(ii) Suppose that $f_n(z) \neq 7$ for all $n \geqslant 77$ and all $z \in \mathbb{D}$. Can it be the case that $g\left(\frac{1+i}{2}\right) = 7$?

Solution:

(i) We start by showing that the family $\mathcal{F}$ is a normal family. Letting $z \in \mathbb{C}$, take $r > 0$ such that $z \in U(0, r)$, and let $\delta > 0$ be such that $U(z, \delta) \subseteq U(0, r)$. By the assumptions of the problem, we have M_r such that

$$\sup_{|z|=r} |f(z)| \leqslant M_r$$

for all $f \in \mathcal{F}$. By the maximum modulus principle, since every element of $\mathcal{F}$ is entire, and thus has

a continuous extension to $B(0, r)$, we have

$$\sup_{|z| \leqslant r} |f(z)| = \sup_{|z|=r} |f(z)|,$$

whence

$$\sup_{w \in U(z,\delta)} |f(z)| \leqslant \sup_{|z| \leqslant r} |f(z)|$$
$$\leqslant M_r,$$

for all $f \in \mathcal{F}$. Therefore, the family $\mathcal{F}$ is locally bounded, and is thus normal by Montel's Theorem.

Therefore, if $(f_n)_n \subseteq \mathcal{F}$, there is a subsequence $(f_{n_k})_k \to g$ for some $g \in H(\mathbb{C})$ uniformly on compact subsets. We will relabel this subsequence to $(f_n)_n$ for the following question.

(ii) Let $\frac{1}{\sqrt{2}} < r < 1$. Consider the sequence of functions given by $h_n = f_n - 7$. Since 7 is a constant function, it follows that $h_n \to g - 7$ uniformly on compact subsets.

By the assumptions of the problem, $h_n(z) \neq 0$ on $S(0, r)$ and the h_n each have no zeros in $U(0, r)$. Thus, from Hurwitz's Theorem, it follows that either $g - 7$ has no zeros on $U(0, r)$ or is uniformly zero on $U(0, r)$. If it is the latter case, then since $\frac{1+i}{2} \in U(0, r)$, we have that $g\left(\frac{1+i}{2}\right) = 7$.

January 2021

Problem (Problem 1)**:** Compute, for $\xi > 0$,

$$I = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x^2 - 2x + 2} \, dx.$$

Show all estimates.

Solution: We consider the semicircular contour γ_R ranging from $-R$ to R centered at 0. Letting

$$f(z) = \frac{e^{iz\xi}}{z^2 - 2z + 2},$$

we get

$$\oint_{\gamma_R} f(z) \, dz = \int_{-R}^{R} \frac{e^{ix\xi}}{x^2 - 2x + 2} \, dx + \int_0^\pi \frac{e^{i\xi Re^{i\theta}}}{R^2 e^{2i\theta} - Re^{i\theta} + 2} iRe^{i\theta} \, d\theta.$$

Estimating the second integral, we find that for $R > 2$

$$\left| \int_0^\pi \frac{iRe^{i\theta} e^{i\xi Re^{i\theta}}}{R^2 e^{2i\theta} - Re^{i\theta} + 2} \, d\theta \right| \leqslant \frac{\pi R e^{-R}}{R^2 - 2R - 2}$$
$$\to 0$$

as $R \to \infty$. As a result, we get

$$\int_{-R}^{R} \frac{e^{ix\xi}}{x^2 - 2x + 2} \, dx = \lim_{R \to \infty} \int_{\gamma_R} f(z) \, dz$$
$$= 2\pi i \operatorname{Res}(f; 1 + i)$$
$$= \pi e^{i\xi - 1}.$$

Problem: Suppose that $f \colon \mathbb{C} \to \mathbb{C}$ is entire, and

$$\lim_{|z| \to \infty} \frac{|f(z)|}{|z|} = 0.$$

Show that f is constant.

Solution: Since f is entire, and we have

$$\lim_{|z| \to \infty} \frac{|f(z)|}{|z|} = 0,$$

we then have

$$\lim_{|w| \to 0} |wf(1/w)| = 0,$$

where we substitute $w = \frac{1}{z}$. In particular, this means

$$\lim_{w \to 0} wf(1/w) = 0,$$

so by Riemann's theorem on removable singularities, it follows that

$$f(w) = \sum_{n=0}^{\infty} a_n w^{-n}$$

is bounded in a neighborhood of 0. In particular, 0 is not a pole for $f(1/w)$, meaning that the Laurent series expansion for f centered at 0 terminates at a_0. Thus, f is constant.

Problem (Problem 3)**:** Suppose that $f \colon \mathbb{C} \to \mathbb{C}$ is entire, and that there exist constants $C, R > 0$ such that $|f(z)| \geqslant C$ whenever $|z| \geqslant R$. Show that f is a polynomial.

Solution: Consider the function $g \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by

$$g(z) = f(1/z).$$

We then have that $|g(z)| \geqslant C$ whenever $|z| \leqslant \frac{1}{R}$. There are then two cases.

- If $|g|$ is bounded above, then $f(z)$ is bounded on $\mathbb{C} \setminus B(0, R)$, whence f is bounded and entire, hence constant by Liouville's Theorem.

- If $|g|$ is not bounded above, then by the converse to the Casorati–Weierstrass theorem, since

$$U(0, C/2) \not\subseteq g(U(0, 1/R) \setminus \{0\}),$$

  it follows that g has a pole at 0. In particular, this means that the Laurent series expansion for g is of the form

$$g(z) = \sum_{n=0}^{m} a_n z^{-m},$$

  meaning that f is a polynomial.

Problem (Problem 4)**:** Let $U \subseteq \mathbb{C}$ be nonempty and open, and let $(f_n)_n \colon U \to \mathbb{D}$ be a sequence of holomorphic functions converging uniformly on compact sets to $f \colon U \to \mathbb{C}$. Suppose there is $p \in U$ such that $|f(p)| = 1$. Show that f is constant.

Solution: To start, we observe that f is holomorphic since the metric space $H(U)$ is complete in the topology of uniform convergence on compact subsets. Additionally, $(f_n)_n \to f$ pointwise since it

converges uniformly on compact subsets. This means in particular that $f(U) \subseteq \overline{\mathbb{D}}$.

Since there is some $z \in \partial\mathbb{D}$ with $z \in \text{im}(f)$, it then follows that $\text{im}(f)$ is not open, meaning that f is constant by the converse of the open mapping principle.

Problem (Problem 5): Let $f_n : \mathbb{D} \to \mathbb{D}$ be a sequence of holomorphic functions that converges pointwise to 0 on $B(0, 1/2)$.

    (a) Suppose $f : \mathbb{D} \to \mathbb{C}$ is a limit (uniformly on compact subsets) of a subsequence of $(f_n)_n$. Show that $f = 0$.

    (b) Show that $(f_n)_n \to 0$ uniformly on compact subsets.

Solution:

    (a) Since f is the limit of a subsequence of $(f_n)_n$ uniformly on compact subsets, it follows that f is holomorphic by the completeness of $H(\mathbb{D})$ in the topology of uniform convergence on compact subsets. Since $f \equiv 0$ on $B(0, 1/2)$, the identity theorem gives that $f \equiv 0$.

    (b) First, we observe that the family $\{f_n \mid n \in \mathbb{N}\}$ is a normal family by Montel's Theorem since every function in this family is bounded above by 1, meaning that any subsequence of $(f_n)_n$ admits a subsequence converging to some holomorphic function $f : \mathbb{D} \to \overline{\mathbb{D}}$. We have earlier shown that any subsequence of $(f_n)_n$ that converges to some holomorphic function will converge to 0, meaning that every subsequence of $(f_n)_n$ admits a subsequence converging to the same point.

    Therefore, from undergrad real analysis, we observe that if $(f_n)_n$ were not to converge to 0, we would have some $\varepsilon_0 > 0$ and some subsequence $(f_{n_k})_k$ such that

$$d_K(f_{n_k}, 0) \geqslant \varepsilon_0$$

for all k, but this would contradict the fact that $(f_{n_k})_k$ has a subsequence converging to 0. Thus, it follows that $(f_n)_n \to 0$ uniformly on compact subsets.

January 2022

Problem (Problem 1): Compute, for $0 < \alpha < 1$,

$$I = \int_0^\infty \frac{1}{x^\alpha(x+1)} \, dx.$$

Solution: Consider the branch of the logarithm defined on $\mathbb{C} \setminus [0, \infty)$. Define a keyhole contour γ_R with height/inner radius ε and outer radius R hugging the branch cut.

Writing out γ_R as $\varepsilon \to 0$, we get

$$\int_{\gamma_R} f(z) \, dz = \int_0^R \frac{1}{x^\alpha(x+1)} \, dx + \int_0^{2\pi} \frac{1}{R^\alpha e^{i\alpha\theta}(1 + Re^{i\theta})} iRe^{i\theta} \, d\theta + \int_R^0 \frac{1}{(xe^{2\pi i})^\alpha(1+x)} \, dx$$

$$= \left(1 - e^{-2\pi i\alpha}\right) \int_0^R \frac{1}{x^\alpha(1+x)} \, dx + \int_0^{2\pi} \frac{iRe^{i\theta}}{R^\alpha e^{i\alpha\theta}(1 + Re^{i\theta})} \, d\theta.$$

Estimating the second integral, we have, for $R > 1$,

$$\left| \int_0^{2\pi} \frac{iRe^{i\theta}}{R^\alpha e^{i\alpha\theta}(1 + Re^{i\theta})} \, d\theta \right| \leqslant 2\pi \frac{R}{R^\alpha(R-1)}$$

$$\to 0.$$

Thus, we have

$$\left(1 - e^{-2\pi i\alpha}\right) I = 2\pi i \, \text{Res}(f; -1)$$

$$= 2\pi i \frac{1}{e^{i\pi\alpha}},$$

whence

$$I = \frac{2\pi i}{2i \sin(\pi\alpha)}$$
$$= \frac{\pi}{\sin(\pi\alpha)}.$$

Problem (Problem 2)**:** Suppose that f is an entire function such that there are constants $a, b > 0$ with

$$|f(z)| \leqslant a + b|z|$$

for all $z \in \mathbb{C}$. Show that f is a polynomial of degree at most 1.

Solution: Using Cauchy's Estimate, we find that

$$|f''(0)| \leqslant \frac{2}{R^2} \sup_{|z|=R} |f(z)|$$
$$\leqslant \frac{2}{R^2} \sup_{|z|=R} (a + b|z|)$$
$$= \frac{2(a + bR)}{R^2}$$
$$\to 0.$$

Thus, the power series expansion for f about 0 terminates after the expression $a_0 + a_1 z$, meaning f is linear.

Problem (Problem 3)**:** Give an explicit example of an unbounded harmonic function $u \colon \mathbb{D} \to (0, \infty)$ such that

$$\lim_{z \to \zeta} u(z) = 0$$

for all $\zeta \in S^1$ with $\zeta \neq 1$.

Solution: The Cayley Transform, $q \colon \mathbb{H} \to \mathbb{D}$, given by

$$q(z) = \frac{z - i}{z + i}$$

is a conformal map of $\mathbb{H}$ to $\mathbb{D}$. The inverse, then,

$$q^{-1}(z) = -i \frac{z + 1}{z - 1}$$

is then a conformal map of $\mathbb{D}$ to $\mathbb{H}$. Furthermore, $\operatorname{Im}\bigl(q^{-1}(z)\bigr)$ is then harmonic as it is the imaginary part of a holomorphic function, with for any $z = e^{i\theta}$ in $S^1 \setminus \{1\}$

$$q^{-1}\bigl(e^{i\theta}\bigr) = -i \frac{e^{i\theta} + 1}{e^{i\theta} - 1}$$
$$= -i \frac{i \sin(\theta)}{1 - \cos(\theta)}$$
$$= \frac{\sin(\theta)}{1 - \cos(\theta)}$$
$$\in \mathbb{R},$$

whence $\operatorname{Im}\bigl(q^{-1}\bigl(e^{i\theta}\bigr)\bigr) = 0$ for all elements of S^1 that are not equal to 1.

Problem (Problem 4): Let $S = \{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 1\}$. Suppose $f \colon \overline{S} \to \mathbb{C}$ is bounded, $f|_S$ is holomorphic, and

$$\sup_{t \in \mathbb{R}}(\max(|f(it)|, |f(1 + it)|)) \leqslant 1.$$

Show that $|f(z)| \leqslant 1$ for all $z \in S$.

Solution: Let $f_\varepsilon(z) = \frac{f(z)}{1 + \varepsilon z}$, and let $K = \sup_{z \in \overline{S}}|f(z)|$. Then, we observe that

$$|f_\varepsilon(z)| \leqslant \frac{|f(z)|}{\varepsilon |\operatorname{Im}(z)|}.$$

In particular, if $|\operatorname{Im}(z)| = K/\varepsilon$, we have $|f_\varepsilon(z)| \leqslant 1$ on the rectangle with vertices at $\pm iK/\varepsilon, 1 \pm iK/\varepsilon$, meaning that by the maximum modulus principle, $|f_\varepsilon(z)| \leqslant 1$ on this rectangle, while $|f_\varepsilon(z)| \leqslant 1$ outside the rectangle by the fact that $|f|$ is bounded by K.

Thus, since $\varepsilon > 0$ is arbitrary, we have $|f(z)| \leqslant 1$ on $\overline{S}$.

Problem (Problem 5): Let $(M_n)_{n=0}^\infty$ be a sequence of positive numbers such that $\sum_{n=0}^\infty M_n z^n$ has a radius of convergence at least 1.

Let $\mathcal{F}$ be the set of holomorphic functions on $\mathbb{D}$ that satisfy

$$\left|\frac{f^{(n)}(0)}{n!}\right| \leqslant M_n.$$

Show that $\mathcal{F}$ is normal.

Solution: Let $z \in \mathbb{D}$, $\delta > 0$ such that $B(z, \delta) \subseteq \mathbb{D}$. For $f \in \mathcal{F}$ and $w \in \mathbb{D}$, uniform convergence and the triangle inequality provide

$$\begin{aligned}
|f(w)| &\leqslant \sum_{n=0}^\infty \left|\frac{f^{(n)}(0)}{n!}\right| |z|^n \\
&\leqslant \sum_{n=0}^\infty \left|\frac{f^{(n)}(0)}{n!}\right| (|z| + \delta)^n \\
&\leqslant \sum_{n=0}^\infty M_n (|z| + \delta)^n \\
&=: C.
\end{aligned}$$

In particular, this means that $\mathcal{F}$ is locally bounded, so $\mathcal{F}$ is normal by Montel's Theorem.

January 2023

Problem (Problem 1): Compute

$$I = \int_{-\infty}^\infty \frac{\sin^2(x)}{x^2 + 1}\, dx.$$

Solution: Take $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Writing the integral out, we have

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\sin^2(x)}{x^2 + 1}\, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{x^2 + 1}\, dx \\
+ \lim_{R \to \infty} \operatorname{Re}\left(\int_{-R}^R \frac{e^{2ix}}{x^2 + 1}\, dx\right).
\end{aligned}$$

To evaluate the second integral, we let γ_R be the semicircle with radius R centered at 0 in the upper half-plane. Then, parametrizing, we get

$$\oint_{\gamma_R} \frac{e^{2iz}}{z^2 + 1}\, dz = \int_{-R}^{R} \frac{e^{2ix}}{x^2 + 1}\, dx + \int_0^{\pi} \frac{e^{2iRe^{i\theta}}}{R^2 e^{2i\theta} + 1} iRe^{i\theta}\, d\theta.$$

Estimating the second integral, we have, for $R > 1$,

$$\left| \int_0^{\pi} \frac{iRe^{i\theta} e^{2iRe^{i\theta}}}{R^2 e^{2i\theta} - 1}\, d\theta \right| \leq \pi \frac{Re^{-2R}}{R^2 - 1},$$

which goes to zero as $R \to \infty$. Thus,

$$\oint_{\gamma_R} \frac{e^{2iz}}{z^2 + 1}\, dz = 2\pi i \operatorname{Res}(f; i)$$

$$= \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{2ix}}{x^2 + 1}\, dx$$

$$= 2\pi i \left(\frac{e^{-2}}{2i} \right)$$

$$= \frac{\pi}{e^2}.$$

Thus, we get

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2 + 1}\, dx = \frac{\pi}{2} - \frac{\pi}{2e^2}.$$

Problem (Problem 2): Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function for which there exist $M, R \geq 0$ such that $|f(z)| \geq M|z|^2$ for all $|z| > R$. Show that f is a polynomial of degree at least 2.

Solution: We observe that an equivalent formulation of the condition is that

$$\left| f\!\left(\frac{1}{z} \right) \right| \geq \frac{M}{|z|^2}$$

for all $0 < |z| < \frac{1}{R}$. In particular, this means that $\lim_{z \to 0} \left| f\!\left(\frac{1}{z} \right) \right| = \infty$, meaning that $f\!\left(\frac{1}{z} \right)$ has a pole at $z = 0$.

Since f is entire, we thus get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and since f has a pole at $z = 0$,

$$f\!\left(\frac{1}{z} \right) = \sum_{n=0}^{m} a_n z^{-n},$$

whence

$$f(z) = \sum_{n=0}^{m} a_n z^n.$$

Writing $f\!\left(\frac{1}{z} \right) = z^{-m} h(z)$ for some holomorphic function $h\colon U(0, 1/R) \to \mathbb{C}$, we get for all $0 < |z| < \frac{1}{R}$,

$$|z^{-m} h(z)| \geq M|z|^{-2}$$

$$\left| z^{-m+2} h(z) \right| \geqslant M,$$

which implies that $m \geqslant 2$, as else we would have that $\lim_{z \to 0} \left| z^{-m+2} h(z) \right| = 0 < M$, implying there would be some $0 < |z| < \frac{1}{R}$ such that $\left| z^{-m+2} h(z) \right| < M$.

Problem (Problem 3)**:** Let $f, g \colon \mathbb{C} \to \mathbb{C}$ be entire functions such that there exists $\lambda \in \mathbb{R}$ with

$$\operatorname{Im}(f(z)) \leqslant \lambda \operatorname{Im}(g(z))$$

for all $z \in \mathbb{C}$. Show that there exist constants $a, b \in \mathbb{C}$ such that $f(z) = ag(z) + b$.

Solution: Since $\operatorname{Im}(f(z)) \leqslant \lambda \operatorname{Im}(g(z))$, it follows that $\operatorname{Im}(f(z) - \lambda g(z)) \leqslant 0$. Now, we have

$$\left| e^{i(\lambda g(z) - f(z))} \right| = e^{\operatorname{Im}(f(z) - \lambda g(z))}$$

$$\leqslant 1.$$

Since $e^{i(\lambda g(z) - f(z))}$ is entire and bounded, it follows from Liouville that there is some $a \in \mathbb{C}$ such that

$$e^{i\lambda g(z) - i f(z)} = e^{a}$$

for all $z \in \mathbb{C}$. We thus have a branch of the logarithm such that

$$i\lambda g(z) - i f(z) = a,$$

whence

$$f(z) = \lambda g(z) + i a.$$

Problem (Problem 4)**:** Let $P(z) = z^9 + z^5 - 8z^3 + 2z + 1$. Determine the number of zeros for P counted with multiplicity in the annulus $A(0, 1, 2)$.

Solution: Let $\gamma = S(0, 2)$. Observe that for all $z \in S(0, 2)$, we have

$$|P(z)| \geqslant |z|^9 - |z|^5 - \left| 8z^3 \right| - 2|z| - 1$$

$$= 512 - 32 - 64 - 4 - 1$$

$$> 0,$$

and that

$$\left| P(z) - z^9 \right| \leqslant |z|^5 + \left| 8z^3 \right| + |2z| + 1$$

$$< 512$$

$$= |z|^9$$

on $S(0, 2)$. By Rouché's Theorem, it follows that all the zeros of P lie in $U(0, 2)$.

Meanwhile, on $S(0, 1)$, we have

$$|P(z)| \geqslant 8|z|^3 - |z|^9 - |z|^5 - 2|z| - 1$$

$$= 8 - 1 - 1 - 2 - 1$$

$$> 0,$$

and

$$\left| P(z) - \left(-8z^3 \right) \right| \leqslant |z|^9 + |z|^5 + 2|z| + 1$$

$$< 8$$

$$= \left| -8z^3 \right|,$$

so there are three zeros inside $U(0, 1)$ by Rouché's Theorem. Thus, there are six zeros in $A(0, 1, 2)$.

Problem (Problem 5): Let $\mathbb{D}$ be the unit disk, and let $\mathcal{F}$ be the family of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying $|f(z)| > 1$ for all $z \in \mathbb{D}$ and $f(0) = 2i$. Show that $\mathcal{F}$ is normal.

Solution: Let $(f_n)_n \subseteq \mathcal{F}$ be a sequence in $\mathcal{F}$. We observe that the family

$$\mathcal{G} = \left\{ \frac{1}{f} \ \middle| \ f \in \mathcal{F} \right\}$$

consists of holomorphic functions from $\mathbb{D}$ to $\mathbb{D}$, which is globally bounded by 1, whence $\mathcal{G}$ is normal. In particular, this means that there is some holomorphic function $g\colon \mathbb{D} \to \overline{\mathbb{D}}$ and a subsequence $\left(\frac{1}{f_{n_k}} \right)_k \to g$ uniformly on compact subsets. Notice that $g(0) = \frac{1}{2i}$, meaning that g is nonconstant; additionally, since each of the $\frac{1}{f_{n_k}}$ do not take on the value 0, and $g(0) \neq 0$, it follows that g has no zeros in $\mathbb{D}$, meaning that $\frac{1}{g}$ is a well-defined holomorphic function. Since $\frac{1}{z}\colon \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus B(0,1)$ is conformal, it follows that $(f_{n_k})_k \to \frac{1}{g}$ uniformly on compact subsets. Thus, $\mathcal{F}$ is normal.

August 2025

Problem (Problem 1): Evaluate

$$\int_0^{2\pi} \frac{1}{5 + 4\cos(\theta)} \, d\theta.$$

Solution: Let $z = e^{i\theta}$. Upon a change of variables, we get

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{5 + 4\cos(\theta)} \, d\theta &= \oint_{S^1} \frac{1}{iz\left(5 + 2\left(z + \frac{1}{z}\right)\right)} \, dz \\
&= \oint_{S^1} \frac{1}{i(2z^2 + 5z + 2)} \, dz \\
&= \oint_{S^1} \frac{1}{i(2z + 1)(z + 2)} \, dz \\
&= 2\pi i \operatorname{Res}\left(f; -\frac{1}{2} \right) \\
&= 2\pi i \left(\frac{1}{2i\left(-\frac{1}{2} + 2\right)} \right) \\
&= \frac{2}{3}\pi
\end{aligned}$$

Problem (Problem 2): Let $f(z)$ be an entire function satisfying $|f(2z)| \leqslant 2|f(z)|$. Show that either $f(z) = f(0)$ or $f(z) = f'(0)z$ for all $z \in \mathbb{C}$.

Solution: From Cauchy's Estimate, we have

$$\begin{aligned}
|f''(0)| &\leqslant \frac{2}{2^{2k}} \sup_{|z| = 2^k} |f(z)| \\
&\leqslant \frac{2}{2^{2k}} \sup_{|z| = 2^k} \left| 2^k f(z 2^{-k}) \right| \\
&= \frac{1}{2^{k-1}} \sup_{|z| = 1} |f(z)|.
\end{aligned}$$

Since k is arbitrary, it follows that $|f''(0)| = 0$, whence the power series expansion for f centered at 0 is given by $f(z) = f(0) + f'(0)z$.

We claim that at least one of these must be zero. If it were the case that both $f(0)$ and $f'(0)$ were nonzero,

then it would follow that there were some isolated zero $z_0 \neq 0$ for f, whence

$$0 < |f(2z_0)|$$
$$\leqslant 2|f(z_0)|$$
$$= 0,$$

which is a contradiction. Thus, if $f(0) = 0$, then $f(z) = f'(0)z$, and if $f'(0) = 0$, then $f(z) = f(0)$.

Problem (Problem 3): Show that the only solution to $e^z = 2z + 1$ inside $\mathbb{D}$ is $z = 0$.

Solution: We let $f(z) = e^z - 2z - 1$ and $g(z) = 2z$. On S^1, both f and g are nonzero. Additionally,

$$|e^z - 1| = \left| \sum_{k=1}^{\infty} \frac{z^k}{k!} \right|$$
$$\leqslant \sum_{k=1}^{\infty} \frac{|z|^k}{k!}$$
$$= e - 1$$
$$< 2$$
$$= |2z|,$$

whence f and g have the same number of zeros in $\mathbb{D}$, namely 1 at $z = 0$.

Problem (Problem 4): Let $U = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1\}$. Let $f \colon U \to \mathbb{C}$ be continuous and bounded with $f|_U$ holomorphic. Suppose there exist constants M_0 and M_1 such that

$$\sup_{\mathrm{Re}(z)=0} |f(z)| \leqslant M_0$$
$$\sup_{\mathrm{Re}(z)=1} |f(z)| \leqslant M_1.$$

Show that for $0 \leqslant r \leqslant 1$,

$$\sup_{\mathrm{Re}(z)=r} |f(z)| \leqslant M_0^{1-r} M_1^r.$$

Solution: Let $\varepsilon > 0$, and define $f_\varepsilon(z) = f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)}$. We will show that $\sup_{z \in \overline{U}} |f_\varepsilon(z)| \leqslant 1$.

Observe that $\mathrm{Re}(z^2 - 1) = (\mathrm{Re}(z)^2 - 1) - \mathrm{Im}(z)^2$; since $0 < \mathrm{Re}(z) < 1$, it follows that this quantity is less than or equal to $-\mathrm{Im}(z)^2$. Thus, we have

$$|f_\varepsilon(z)| \leqslant |f(z)| M_0^{\mathrm{Re}(z)-1} M - 1^{-\mathrm{Re}(z)} e^{-\varepsilon \, \mathrm{Im}(z)^2}.$$

Since f is bounded, it follows that as $\mathrm{Im}(z) \to \infty$, $|f_\varepsilon(z)| \to 0$. Let A be such that $|f_\varepsilon(z)| \leqslant 1$ for all $|\mathrm{Im}(z)| \geqslant A$. Then, on the rectangle with corners at $Ai, 1 + Ai, -Ai, 1 - Ai$ such that $f_\varepsilon(z) \leqslant 1$ on the boundary of this rectangle. From the maximum modulus principle and the fact that $|f_\varepsilon| \leqslant 1$ outside the rectangle, it follows that $|f_\varepsilon(z)| \leqslant 1$ for all $z \in \overline{U}$. Thus, since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{\varepsilon \to 0} |f_\varepsilon(z)| \leqslant 1,$$

and thus

$$|f(z)| \leqslant M_0^{1-\mathrm{Re}(z)} M_1^{\mathrm{Re}(z)}.$$

The desired result follows.

Problem (Problem 5)**:** Let $\mathcal{F}$ be the set of holomorphic functions on $U(0, 2)$ such that

$$\int_0^{2\pi} \left| f(e^{i\theta}) \right| \, d\theta \leqslant 1.$$

Let

$$\mathcal{G} = \{\, g \colon \mathbb{D} \to \mathbb{C} \mid g = f|_{\mathbb{D}} \text{ for some } f \in \mathbb{D} \,\}.$$

Show that $\mathcal{G}$ is normal.

Solution: Let $z \in \mathbb{D}$. We observe then that for every $f \in \mathcal{F}$, we have

$$
\begin{aligned}
|f(z)| &\leqslant \int_0^{2\pi} \frac{\left| f(e^{i\theta}) \right|}{\left| e^{i\theta} - z \right|} \left| i e^{i\theta} \right| \, d\theta \\
&\leqslant \frac{1}{1 - |z|} \int_0^{2\pi} \left| f(e^{i\theta}) \right| \, d\theta \\
&= \frac{1}{1 - |z|}.
\end{aligned}
$$

In particular, if we let δ be such that $U(z, \delta) \subseteq \mathbb{D}$, then for all $w \in U(z, \delta)$ and all $f \in \mathcal{G}$, we have

$$
\begin{aligned}
|f(w)| &\leqslant \frac{1}{1 - |w|} \\
&\leqslant \frac{1}{1 - |z| - \delta} \\
&=: C.
\end{aligned}
$$

Thus, $\mathcal{G}$ is locally bounded, hence normal.

Notation

- $U(z_0, r) = \{\, z \in \mathbb{C} \mid |z - z_0| < r \,\}$
- $B(z_0, r) = \{\, z \in \mathbb{C} \mid |z - z_0| \leqslant r \,\}$
- $S(z_0, r) = \{\, z \in \mathbb{C} \mid |z - z_0| = r \,\}$
- $\dot{U}(z_0, r) = \{\, z \in \mathbb{C} \mid 0 < |z - z_0| < r \,\}$
- $A(z_0, r_1, r_2) = \{\, z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2 \,\}$