

### Problem 1

Let  $v_1, \dots, v_n$  be mutually orthogonal vectors in an inner product space  $V$ . Show that

$$\left\| \sum_{k=1}^n v_k \right\|^2 = \sum_{k=1}^n \|v_k\|^2.$$

**Proof:**

$$\begin{aligned} \left\| \sum_{k=1}^n v_k \right\|^2 &= \left\langle \sum_{k=1}^n v_k, \sum_{k=1}^n v_k \right\rangle \\ &= \sum_{i=1}^n \left\langle \sum_{k=1}^n v_k, v_i \right\rangle \\ &= \sum_{i=1}^n \langle v_i, v_i \rangle && \text{since for } i \neq j, \langle v_i, v_j \rangle = 0 \\ &= \sum_{i=1}^n \|v_i\|^2 \end{aligned}$$

### Problem 2

Let  $V$  be an inner product space and fix  $w \neq 0$  in  $V$ . We define the one-dimensional projection

$$P_w : V \rightarrow V; P_w(v) := \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

- (i) Prove that  $v - P_w(v) \perp P_w(v)$ .
- (ii) Show that  $P_w : V \rightarrow V$  is a linear operator with  $\|P_w\|_{\text{op}} = 1$ .
- (iii) Show that  $P_w \circ P_w = P_w$ .

### Problem 3

Let  $V$  be an inner product space. Prove the reverse Cauchy-Schwarz Inequality which states

$$v, w \in V, \text{ and } |\langle v, w \rangle| = \|v\| \|w\| \Rightarrow v = \alpha w.$$

### Problem 4

Let  $V$  be an inner product space. Then, for any  $v, w \in V$ , show that

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

**Proof:**

$$\begin{aligned} \langle v + w, v + w \rangle + \langle v - w, v - w \rangle &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle + \langle v, v \rangle - \langle v, w \rangle - \langle w, v \rangle + \langle w, w \rangle \\ &= \langle v, v \rangle + \langle v, v \rangle + \langle w, w \rangle + \langle w, w \rangle \\ &= 2\|v\|^2 + 2\|w\|^2 \end{aligned}$$

### Problem 5

Let  $\lambda = (\lambda_k)_k$  belong to  $\ell_\infty$ . Show that the map

$$D_\lambda : \ell_2 \rightarrow \ell_2; D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with  $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$

**Proof:**

Well-Defined: Let  $(\zeta_k)_k = 0$  for all  $k \in \mathbb{N}$ . Then,

$$\begin{aligned} D_\lambda((\zeta_k)_k) &= (\lambda_k \zeta_k)_k \\ &= ((\lambda_k)(0))_k \\ &= 0 \end{aligned}$$

Linear:

$$\begin{aligned} D_\lambda((\alpha \xi_k)_k + (\beta \zeta_k)_k) &= D_\lambda((\alpha \xi_k + \beta \zeta_k)_k) \\ &= (\lambda_k(\alpha \xi_k + \beta \zeta_k))_k \\ &= (\alpha \lambda_k \xi_k + \beta \lambda_k \zeta_k)_k \\ &= (\alpha \lambda_k \xi_k)_k + (\beta \lambda_k \zeta_k)_k \\ &= \alpha (\lambda_k \xi_k)_k + \beta (\lambda_k \zeta_k)_k \\ &= \alpha D_\lambda((\xi_k)_k) + \beta D_\lambda((\zeta_k)_k) \end{aligned}$$

Bounded:

$$\begin{aligned} \|D_\lambda\|_{\text{op}} &= \sup_{\|\xi_k\|_k \leq 1} \|D_\lambda((\xi_k)_k)\| \\ \|D_\lambda((\xi_k)_k)\| &= \left( \sum_{k=1}^{\infty} |\lambda_k \xi_k|^2 \right)^{1/2} \\ &\leq \left( \sum_{k=1}^{\infty} \left| \sup_{k \in \mathbb{N}} |\lambda_k| |\xi_k| \right|^2 \right)^{1/2} \\ &= \|\lambda\|_{\infty} \left( \sum_{k=1}^n |\xi_k|^2 \right)^{1/2} \\ &= \|\lambda\|_{\infty} \|\xi_k\| \end{aligned}$$

## Problem 6

Consider the vector space  $C([0, 2\pi])$  equipped with

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt.$$

(i) Show that this pairing defines an inner product on  $C([0, 2\pi])$ .

**Proof:** We will show that  $\langle f, g \rangle$  satisfies the axioms of the inner product.

Addition:

$$\begin{aligned} \langle f_1 + f_2, g \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (f_1(t) + f_2(t)) \overline{g(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (f_1(t) \overline{g(t)} + f_2(t) \overline{g(t)}) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f_1(t) \overline{g(t)} dt + \frac{1}{2\pi} \int_0^{2\pi} f_2(t) \overline{g(t)} dt \\ &= \langle f_1, g \rangle + \langle f_2, g \rangle. \end{aligned}$$

Scalar Multiplication:

$$\begin{aligned} \langle \alpha f, g \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (\alpha f(t)) \overline{g(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \alpha (f(t) \overline{g(t)}) dt \\ &= \alpha \left( \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt \right) \\ &= \alpha \langle f, g \rangle. \end{aligned}$$

Conjugation:

$$\begin{aligned}\overline{\langle g, f \rangle} &= \frac{1}{2\pi} \int_0^{2\pi} \overline{g(t) \overline{f(t)}} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt \\ &= \langle f, g \rangle.\end{aligned}$$

Positive Definition:

$$\begin{aligned}\langle f, f \rangle &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{f(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt \\ &\geq 0.\end{aligned}$$

For  $\langle f, f \rangle = 0$ , we have that the integral equals zero — since  $f$  is continuous, it means that if  $|f(t)|^2 > 0$  for some  $t_0 \in [0, 2\pi]$ , then  $|f(t)|^2 \neq 0$  on some interval  $[t_0 - \delta, t_0 + \delta]$ , meaning the integral can only equal zero if  $f$  is  $\mathbb{0}_f$  on  $[0, 2\pi]$ .

(ii) For  $n \in \mathbb{Z}$ , set  $e_n(t) = \cos(nt) + i \sin(nt)$ . Show that the family  $\{e_n\}_{n \in \mathbb{Z}}$  is orthonormal.

**Proof:** We will show that  $\{e_n\}_{n \in \mathbb{Z}}$  is orthonormal by showing that  $\langle e_n, e_n \rangle = 1$  and  $\langle e_n, e_m \rangle = 0$  for  $m \neq n$ .

$$\begin{aligned}\langle e_n, e_n \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(nt) + i \sin(nt))(\cos(nt) - i \sin(nt)) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\cos^2(nt) + \sin^2(nt)) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} dt \\ &= 1 \\ \langle e_n, e_m \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(nt) + i \sin(nt))(\cos(mt) - i \sin(mt)) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(mt) \cos(nt) + i \sin(nt) \cos(mt) - i \sin(mt) \cos(nt) + \sin(nt) \sin(mt)) dt \\ &= \frac{1}{2\pi} \left( \int_0^{2\pi} (\cos(mt) \cos(nt) + \sin(nt) \sin(mt)) dt + i \int_0^{2\pi} (\sin(nt) \cos(mt) - \sin(mt) \cos(nt)) dt \right) \\ &= 0.\end{aligned}$$

## Problem 7

Let  $V$  be any normed space,  $p \in [1, \infty]$ , and suppose  $T : \ell_p^n \rightarrow V$  is linear. Show that  $T$  is bounded.

## Problem 8

Let  $\mathbb{P}[0, 1] = \{\sum_0^n a_k x^k \mid a_k \in \mathbb{C}\} \subseteq C([0, 1])$  denote the linear subspace of all polynomial functions equipped with the uniform norm  $\|\cdot\|_u$  inherited from  $C([0, 1])$ . We define the map

$$\begin{aligned}D : \mathbb{P}[0, 1] &\rightarrow \mathbb{P}[0, 1] \\ D(p(x)) &= p'(x).\end{aligned}$$

Show that  $D$  is unbounded.

**Proof:** Let  $p(x) = x^n$ . Then, in  $\mathbb{P}[0, 1]$ ,

$$\begin{aligned}\|p\|_u &= 1 \\ \|D(p)\|_u &= n.\end{aligned}$$

For any  $L \in \mathbb{R}$ , we can find a  $n \in \mathbb{N}$  sufficiently large such that  $\|D(p)\|_u = n > L$ , by the Archimedean property. Therefore,  $D$  is unbounded.

**Problem 9**

Let  $V$  be an infinite-dimensional normed space. Show that there is a linear functional  $\varphi : V \rightarrow \mathbb{F}$  that is unbounded.

**Proof:** Let  $B = \{x_n\}$  be the basis for  $V$ . We define  $\varphi : V \rightarrow \mathbb{F}$  as  $\varphi(x) = \sum_n n\alpha_n$  for the  $\alpha_n x_n$  component in  $x$ . Then,  $\varphi$  is linear and unbounded, as the values  $n$  takes are not bounded, seeing as  $V$  is infinite-dimensional.

**Problem 10**

Let  $a, b \in \mathbb{M}_n$ . Show the following properties of the operator norm.

$$(i) \quad \|a\|_{\text{op}} = \sup \left\{ |\langle a\xi, \eta \rangle| \mid \xi, \eta \in B_{\ell_2^n} \right\}$$

$$(ii) \quad \|a^*\|_{\text{op}} = \|a\|_{\text{op}}$$

$$(iii) \quad \|ab\|_{\text{op}} \leq \|a\|_{\text{op}} \|b\|_{\text{op}}$$

$$(iv) \quad \|a^*a\|_{\text{op}} = \|a\|_{\text{op}}^2$$