Solution (12.4, Problem 14)**:**

(a) We may write

$$\begin{aligned}\xi &= x + at\\ \eta &= x - at,\end{aligned}$$

so

$$\begin{aligned}\frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x} + a\frac{\partial}{\partial t}\\ \frac{\partial}{\partial \eta} &= \frac{\partial}{\partial x} - a\frac{\partial}{\partial t}.\end{aligned}$$

Thus,

$$\left(\frac{\partial}{\partial x} - a\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial t}\right)u = \frac{\partial^2 u}{\partial \xi \partial \eta}.$$

(b) Integrating, we have

$$\begin{aligned}\int \frac{\partial^2 u}{\partial \eta \partial \xi}\, d\eta &= c_1(\xi)\\ \int \frac{\partial u}{\partial \xi}\, d\xi &= c_2(\eta),\end{aligned}$$

so

$$\int\int \frac{\partial^2 u}{\partial \eta \partial \xi}\, d\xi\, d\eta = c_1(x + at) + c_2(x - at).$$

Using this solution, we know that if

$$u(x,t) = c_1(x + at) + c_2(x - at),$$

then

$$\begin{aligned}c_1(x) + c_2(x) &= f(x)\\ a\frac{d}{dx}(c_1(x) - c_2(x)) &= g(x).\end{aligned}$$

Using the fundamental theorem of calculus, we get that

$$\begin{aligned}c_1(x) + c_2(x) &= f(x)\\ c_1(x) - c_2(x) &= \frac{1}{a}\int_{x_0}^{x} g(s)\, ds + k\end{aligned}$$

which gives the solutions

$$\begin{aligned}c_1(x) &= \frac{1}{2}f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\, ds + k/2\\ c_2(x) &= \frac{1}{2}f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\, ds - k/2.\end{aligned}$$

(c) Summing and substituting, we get

$$\begin{aligned}u(x,t) &= c_1(x + at) + c_2(x - at)\\ &= \frac{1}{2}(f(x + at) + f(x - at)) + \frac{1}{2a}\int_{x_0}^{x+at} g(s)\, ds - \frac{1}{2a}\int_{x_0}^{x-at} g(s)\, ds\\ &= \frac{1}{2}(f(x + at) + f(x - at)) + \frac{1}{2a}\left(\int_{x_0}^{x+at} g(s)\, ds + \int_{x-at}^{x_0} g(s)\, ds\right)\\ &= \frac{1}{2}(f(x + at) + f(x - at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds.\end{aligned}$$

Solution (12.4, Problem 16)**:**

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\sin(x+at) + \sin(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at}\cos(s)\,ds \\ &= \frac{1}{2}(\sin(x+at) + \sin(x-at)) + \frac{1}{2a}(\sin(x+at) - \sin(x-at)) \end{aligned}$$

Solution (12.4, Problem 18)**:**

$$u(x,t) = \frac{1}{2}\left(e^{-(x+at)^2} + e^{-(x-at)^2}\right).$$

Solution (Method of Characteristics Problems)**:**

(i) We have

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 1,$$

giving the vector identity

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ u^2 \\ -1 \end{pmatrix} = 0.$$

Writing the Lagrange–Charpit equations, we get

$$\begin{aligned} \frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= u^2 \\ \frac{du}{ds} &= 1. \end{aligned}$$

We have the parametrization of $x(s) = su^2 + x_0$, so $x_0 = x - tu^2$, substituting $t = s$. Finally, we have $u(s) = s + u_0$, so $u_0 = u - t$. Therefore, we have the implicitly defined function

$$\begin{aligned} u(x,t) &= u_0(x_0) \\ &= e^{x-tu^2} - t. \end{aligned}$$

(ii) Putting into standard form, we have the equation

$$\frac{\partial u}{\partial t} + te^t\frac{\partial u}{\partial x} = e^t u.$$

There are two equations here to solve. We start with the equation in x, which gives

$$\frac{dx}{dt} = te^t,$$

so

$$x = te^t - e^t + x_0,$$

and

$$x_0 = x + e^t - te^t.$$

Solving the equation in u, we have

$$\begin{aligned} \frac{du}{dt} &= e^t u \\ u &= u_0 e^{e^t}. \end{aligned}$$

Therefore,

$$\begin{aligned}u(x,t) &= u_0(x_0)e^{e^t}\\ &= \cos\left(\left(x + e^t - te^t\right)^2\right)e^{e^t}.\end{aligned}$$

(iii) In standard form, the equation becomes

$$\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0,$$

so we have the solution

$$\begin{aligned}u(x,t) &= u_0\left(x - \frac{1}{2}t^2\right)\\ &= \frac{1}{\left(x - \frac{1}{2}t^2\right)^2 + 2}.\end{aligned}$$

(iv) We have

$$\begin{aligned}\frac{dx}{dt} &= 4x\\ x &= x_0e^{4t}\\ x_0 &= xe^{-4t}.\end{aligned}$$

Furthermore, since

$$\frac{du}{dt} = t,$$

we get

$$u(x,t) = \frac{1}{2}t^2 + u_0,$$

so

$$u(x,t) = \frac{1}{2}t^2 + \left(xe^{-4t}\right)^3.$$

(v) We have

$$\begin{aligned}\frac{dy}{dt} &= y\\ y_0 &= ye^{-t}\\ \frac{dx}{dt} &= 1\\ x_0 &= x - t,\end{aligned}$$

so our solution is

$$\begin{aligned}u(x,y,t) &= u_0(x_0,y_0)\\ &= x - t + ye^{-t}.\end{aligned}$$

(vi) We have

$$\begin{aligned}\frac{dy}{dt} &= y\\ y_0 &= ye^{-t}\\ \frac{dx}{dt} &= 1\\ x_0 &= x - t\\ \frac{du}{dt} &= u\\ u &= u_0e^t\\ &= \left(x - t + ye^{-t}\right)e^t.\end{aligned}$$

Solution (D'Alembert's Method Problems)**:**

(i) We have

$$\left(\frac{\partial}{\partial t}+2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u=0,$$

giving characteristic curves of $x+2t$ and $x-t$. Therefore, if

$$u(x,t)=h(x+2t)+k(x-t),$$

we have

$$\begin{aligned} h(x)+k(x)&=u(x,0)\\ 2h(x)-k(x)&=\int_{x_0}^{x} g(s)\,ds, \end{aligned}$$

giving

$$\begin{aligned} h(x)&=\frac{1}{3}u(x,0)+\frac{1}{3}\int_{x_0}^{x} g(s)\,ds\\ k(x)&=\frac{2}{3}u(x,0)-\frac{1}{3}\int_{x_0}^{x} g(s)\,ds, \end{aligned}$$

so that

$$\begin{aligned} u(x,t)&=\frac{1}{3}\sin(x+2t)+\frac{2}{3}\sin(x-t)+\frac{1}{3}\int_{x-t}^{x+2t} e^{s}\,ds\\ &=\frac{1}{3}\sin(x+2t)+\frac{2}{3}\sin(x-t)+\frac{1}{3}\left(e^{x+2t}-e^{x-t}\right). \end{aligned}$$

(ii) We have

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+9\frac{\partial}{\partial x}\right)u=0,$$

so we have characteristic curves of x and $x+9t$. Therefore, if

$$u(x,t)=h(x+9t)+k(x),$$

we have

$$\begin{aligned} h(x)+k(x)&=u(x,0)\\ 9h(x)&=\int_{x_0}^{x} g(s)\,ds, \end{aligned}$$

so

$$\begin{aligned} u(x,t)&=x^2+1+\frac{1}{9}\int_{x}^{x+9t} s\,ds\\ &=x^2+1+\frac{1}{9}\left(\frac{1}{9}(x+9t)^2-\frac{1}{2}x^2\right). \end{aligned}$$

(iii) Factoring, we have

$$\left(\frac{\partial}{\partial t}-3\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u=0,$$

with the three characteristic curves of $x-3t$, $x-2t$, and $x-t$. We thus have the equations

$$\begin{aligned} c_1+c_2+c_3&=u(x,0)\\ -3c_1-2c_2-c_3&=\int_{x_0}^{x} g(s)\,ds \end{aligned}$$

$$9c_1 + 4c_2 + c_3 = \int_{x_0}^{x} h(s)\, ds,$$

which yields solutions of

$$\begin{aligned}
c_1(x) &= u(x,0) + \frac{3}{2}\int_{x_0}^{x} g(s)\, ds + \frac{1}{2}\int_{x_0}^{x} h(s)\, ds\\
c_2(x) &= -3u(x,0) - 4\int_{x_0}^{x} g(s)\, ds - \int_{x_0}^{x} h(s)\, ds\\
c_3(x) &= 3u(x,0) + \frac{5}{2}\int_{x_0}^{x} g(s)\, ds + \frac{1}{2}\int_{x_0}^{x} h(s)\, ds.
\end{aligned}$$

This gives the solution of

$$\begin{aligned}
u(x,t) &= (x-3t)^2 + 1 - 3(x-2t)^2 - 3 + 3(x-t)^2 + 3\\
&\quad + \frac{1}{2}\left(\frac{1}{2}(x-3t)^2 - \frac{1}{2}(x-2t)^2\right) + \frac{5}{2}\left(\frac{1}{2}(x-t)^2 - \frac{1}{2}(x-2t)^2\right)\\
&\quad + \frac{1}{2}\left(\frac{1}{2}(x-3t)^2 - \frac{1}{2}(x-2t)^2\right) + \frac{1}{2}\left(\frac{1}{2}(x-t)^2 - \frac{1}{2}(x-2t)^2\right)\\
\\
&= (x-3t)^2 - 3(x-2t)^2 + 3(x-t)^2 + \frac{1}{2}(x-3t)^2 - 2(x-2t)^2 + \frac{3}{2}(x-t)^2 + 1\\
&= \frac{3}{2}(x-3t)^2 - 5(x-2t)^2 + \frac{9}{2}(x-t)^2 + 1.
\end{aligned}$$

Solution (Hyperbolic System Problem)**:** We have the hyperbolic system

$$\mathbf{w}(x,t) = \begin{pmatrix} u(x,t)\\ v(x,t)\end{pmatrix}$$

$$\frac{\partial \mathbf{w}}{\partial t} = \begin{pmatrix} -4 & -2\\ -3 & 1\end{pmatrix}\frac{\partial \mathbf{w}}{\partial x}.$$

with

$$\mathbf{w}_0 = \begin{pmatrix} x+2\\ x^2+1\end{pmatrix}.$$

The diagonalization of the matrix A is

$$P = \begin{pmatrix} 1 & 2\\ -3 & 1\end{pmatrix}$$

$$D = \begin{pmatrix} 2 & \\ & -5\end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1/7 & 2/7\\ -3/7 & -1/7\end{pmatrix}.$$

We take the decoupled system

$$\mathbf{z}_0 = \begin{pmatrix} -1/7 & 2/7\\ -3/7 & -1/7\end{pmatrix}\begin{pmatrix} x+2\\ x^2+1\end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} -(x+2) + 2\left(x^2+1\right) \\ -3(x+2) - 1(x^2+1) \end{pmatrix}$$
$$= \frac{1}{7}\begin{pmatrix} 2x^2 - x \\ -x^2 - 3x - 7 \end{pmatrix}.$$

The decoupled system is

$$\begin{aligned} \frac{\partial z_1}{\partial t} &= 2\frac{\partial z_1}{\partial x} \\ \frac{\partial z_2}{\partial t} &= -5\frac{\partial z_1}{\partial x}, \end{aligned}$$

with

$$\begin{aligned} z_1(x,0) &= \frac{1}{7}\left(2x^2 - x\right) \\ z_2(x,0) &= \frac{1}{7}\left(-x^2 - 3x - 7\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} z_1(x,t) &= \frac{1}{7}\left(2(x+2t)^2 - (x+2t)\right) \\ z_2(x,t) &= \frac{1}{7}\left(-(x-5t)^2 - 3(x-5t) - 7\right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{w}(x,t) &= \frac{1}{7}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2(x+2t)^2 - (x+2t) \\ -(x-5t)^2 - 3(x-5t) - 7 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} (x+2t)^2 - (x+2t) - 2(x-5t)^2 - 6(x-5t) - 14 \\ -6(x+2t)^2 + 3(x+2t) - (x-5t)^2 - 3(x-5t) - 7 \end{pmatrix}. \end{aligned}$$