Problem (Problem 1)**:** Let I, J, K be ideals of R.

(a) Show that $(IJ)K = I(JK)$.

(b) Show that $(I + J)K = IK + JK$.

Solution:

(a) Let $u \in (IJ)K$. Then, u is of the form

$$u = \sum_{k=1}^{n} u_k z_k,$$

where the $u_k \in IJ$ and the $z_k \in K$. Since each u_k is an element of IJ, we may write

$$u_k = \sum_{i=1}^{m} x_{k_i} y_{k_i},$$

where the $x_{k_i} \in I$ and the $y_{k_i} \in J$. This yields an expression

$$\begin{aligned} u &= \sum_{k=1}^{n} \left(\sum_{i=1}^{m} x_{k_i} y_{k_i} \right) z_k \\ &= \sum_{k=1}^{n} \sum_{i=1}^{m} x_{k_i} y_{k_i} z_k. \end{aligned}$$

We observe that, for a fixed k, $y_{k_i} z_k \in JK$. So, $x_{k_i}(y_{k_i} z_k) \in I(JK)$ for a fixed k, meaning that $u \in I(JK)$. A similar argument holds in the reverse direction.

(b) Elements of $I + J$ are of the form $x_i + y_i$, where $x_i \in I$ and $y_i \in J$. This means that elements of $(I + J)K$ are of the form

$$\begin{aligned} u &= \sum_{k=1}^{n} \sum_{i=1}^{m} (x_i + y_i) z_k \\ &= \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{m} x_i \right)}_{=:x_k} z_k + \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{m} y_i \right)}_{=:y_k} z_k \\ &= \sum_{k=1}^{n} x_k z_k + \sum_{k=1}^{n} y_k z_k. \end{aligned}$$

Thus, we find that u is in $IK + JK$, and vice versa.

Problem (Problem 4)**:** Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and let $\iota_{S_i} : R \to S_i^{-1}R$ be the corresponding localization homomorphisms. Use the universal property of localization to show that there exists a unique ring homomorphism $\iota' \colon S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. Provide an explicit description of this ring homomorphism. Use this to show that if R is an integral domain and S an arbitrary multiplicative subset of R, then $S^{-1}R$ injects into the fraction field $K = \operatorname{frac}(R)$.

Solution: We observe that $\iota_{S_2} \colon R \to S_2^{-1}R$ maps elements of S_1 to units in $S_2^{-1}R$, as the units in $S_2^{-1}R$ are elements of the form $\frac{s}{s'}$ with $s, s' \in S_2$, so by the universal property, there is a unique ring homomorphism $\iota' \colon S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. In particular, this is the map $\left[\frac{r}{1}\right]_{S_1^{-1}R} \mapsto \left[\frac{r}{1}\right]_{S_2^{-1}R}$.

Since any arbitrary multiplicative subset $S \subseteq R$ of an integral domain is contained in $R \setminus \{0\}$, it follows that $S^{-1}R$ injects into $(R \setminus \{0\})^{-1}R =: \operatorname{frac}(R)$.

Problem (Problem 5)**:** Let $R = \mathbb{Q} \times \mathbb{Q}$ and $S = \{(1,1)\} \cup (\mathbb{Q}^\times \times \{0\})$. The goal of this problem is to identify the localization $S^{-1}R$.

(a) Describe explicitly when $\frac{(a_1,a_2)}{(s_1,s_2)}$ is equal to $\frac{(b_1,b_2)}{(t_1,t_2)}$ in $S^{-1}R$.

(b) Use your result from part (a) to show that the localization $S^{-1}R$ is isomorphic to the localization $T^{-1}\mathbb{Q}$, where $T = \mathbb{Q} \setminus \{0\}$, hence is isomorphic to $\mathbb{R}$.

(c) Find the kernel of the localization homomorphism $\iota_S \colon R \to S^{-1}R$.

Solution:

(a) By the definition of the equivalence relation, we must have an element $(r_1, r_2) \in S$ such that

$$(r_1(a_1t_1 - b_1s_1), r_2(a_2t_2 - b_2s_2)) = (0,0).$$

In particular, since $r_1 \in \mathbb{Q}^\times$, and we may always select $r_2 = 0$, it follows that

$$r_1(a_1t_1 - b_1s_1) = 0,$$

so that $a_1t_1 - b_1s_1 = 0$ (as $\mathbb{Q}$ is an integral domain).

(b) We consider the map $\pi_1 \colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, which maps $(a_1, a_2) \mapsto a_1$. Observe then that $S^{-1}R$ satisfies the universal property for localization, as we may write $S = (\mathbb{Q}^\times \times \{0\}) \cup (\mathbb{Q}^\times \times \{1\})$, which maps to $\mathbb{Q}^\times \subseteq \mathbb{Q}$ under this projection map.

In particular, we see that the induced map $\widetilde{\pi_1} \colon S^{-1}R \to \mathbb{Q}$ is given by

$$\widetilde{\pi_1}\left(\frac{(a_1, a_2)}{(s_1, s_2)}\right) = a_1 s_1^{-1}$$

for $s_1 \in \mathbb{Q}^\times$ and $a_1 \in \mathbb{Q}$.

Now, we observe that the map $\mathrm{id} \circ \pi_1 = \pi_1$, and that $T^{-1}\mathbb{Q}$ satisfies the universal property for localization with respect to id, inducing the homomorphism $\widetilde{\mathrm{id}}$ that takes

$$\widetilde{\mathrm{id}}\left(\frac{a}{s}\right) = as^{-1}$$

for $s \in \mathbb{Q}^\times$. Yet, we also observe that, if we set $\iota_T' = \iota_T \circ \widetilde{\pi_1} \circ \iota_S$, that

$$\begin{aligned}
\widetilde{\mathrm{id}} \circ \iota_T'(a_1, a_2) &= \widetilde{\mathrm{id}} \circ \iota_T \circ \widetilde{\pi_1} \circ \iota_S(a_1, a_2) \\
&= \widetilde{\mathrm{id}} \circ \iota_T \circ \widetilde{\pi_1}\left(\frac{(a_1, a_2)}{(1,1)}\right) \\
&= \widetilde{\mathrm{id}} \circ \iota_T(a_1) \\
&= \widetilde{\mathrm{id}}\left(\frac{a_1}{1}\right) \\
&= a_1 \\
&= \pi_1(a_1, a_2).
\end{aligned}$$

Thus, $T^{-1}\mathbb{Q}$ also satisfies the universal property for localization, implying that $T^{-1}\mathbb{Q}$ and $S^{-1}R$ are isomorphic.

$$\begin{tikzcd}
& S^{-1}R \arrow[dr, "\widetilde{\pi_1}"] & \\
\mathbb{Q}\times\mathbb{Q} \arrow[ur, "\iota_S"] \arrow[rr, "\pi_1"'] \arrow[drr, "\iota'_T"'] & & \mathbb{Q} \arrow[loop right, "\mathrm{id}"] \arrow[d, bend right, "\iota_T"'] \\
& & T^{-1}\mathbb{Q} \arrow[u, bend right, "\widetilde{\mathrm{id}}"']
\end{tikzcd}$$

(c) We see that an element (a, b) in $S^{-1}R$ is equivalent to $(0, 0)$ in $S^{-1}R$ if and only if there is $(r_1, r_2) \in (\mathbb{Q}^{\times} \times \{0\}) \cup (\mathbb{Q}^{\times} \times \{1\})$ such that

$$(r_1 a, r_2 b) = 0.$$

Since we may select $r_2 = 0$ for all $b \in \mathbb{Q}$, it follows that we must have $a = 0$, so that the kernel of ι_S is $\{0\} \times \mathbb{Q}$.

Problem (Problem 7)**:** Let $S \subseteq R$ be a multiplicative subset, and let $\iota_S \colon R \to S^{-1}R$ be the corresponding localization homomorphism. Consider the map

$$\begin{aligned} \alpha \colon \left\{P' \mid P' \text{ is a prime ideal of } S^{-1}R\right\} &\to \left\{P \mid P \text{ is a prime ideal of } R \text{ such that } S \cap P = \emptyset\right\} \\ P' &\mapsto \iota_S^{-1}(P'). \end{aligned}$$

(a) Verify that α is well-defined.

(b) Define an inverse map β by $\beta(P) = P \cdot S^{-1}R$. Show that β is well-defined. That is, $\beta(P)$ is a prime ideal of $S^{-1}R$.

(c) Show that α and β are mutual inverses.

Solution:

(a) We observe that ι_S takes 1_R to $\frac{1}{1} \equiv 1_{S^{-1}R}$, the latter equality coming from the fact that $\frac{a}{1} \cdot \frac{1}{1} = \frac{a}{1}$, so that if P' is a prime ideal in $S^{-1}R$, then $\iota_S^{-1}(P')$ is a prime ideal in $S^{-1}R$. Additionally, we observe that $\iota_S^{-1}(P')$ does not contain any element of S, as otherwise P' would contain an invertible element in $S^{-1}R$, and thus P' would not be prime.

(b) Let P be a prime ideal in R such that $P \cap S = \emptyset$. Elements of $P \cdot S^{-1}R$ are of the form $q \cdot \frac{r}{t}$, where $q \in P$, $r \in R$, and $t \in S$. Equivalently, we may write this element as $(qr) \cdot \frac{1}{t}$, where $qr \in P$ and $\frac{1}{t} \in S^{-1}R$. We observe that if $\frac{a}{s} \cdot \frac{b}{t} \in P \cdot S^{-1}R$, then $ab \in P$ and $\frac{1}{st} \in S^{-1}R$, so that either $a \in P$ or $b \in P$, as P is prime. Thus, since $P \cdot S^{-1}R$ is an ideal, we have $\frac{a}{s} \in P \cdot S^{-1}R$ or $\frac{b}{t} \in P \cdot S^{-1}R$.

(c) We will show that if P' is a prime ideal in $S^{-1}R$, then $\iota_S^{-1}(P') \cdot S^{-1}R = P'$. Let $a \cdot \frac{b}{s} \in \iota_S^{-1}(P') \cdot S^{-1}R$, where $a \in \iota_S^{-1}(P')$ and $\frac{b}{s} \in S^{-1}R$. We may write $(ab)\frac{1}{s} \in \iota_S^{-1}(P') \cdot S^{-1}R$, meaning that $ab \in \iota_S^{-1}(P')$, so that $\frac{ab}{1} \in P'$, meaning that $\frac{ab}{s} \in P'$, giving one direction of inclusion. The other direction of inclusion follows from the fact that if $\frac{a}{s} \in P'$, then $\frac{a}{1} \in P'$, meaning $a \in \iota_S^{-1}(P')$, and thus $\frac{a}{s} \in \iota_S^{-1}(P') \cdot S^{-1}R$. This gives that $\beta \circ \alpha$ is identity on the set of prime ideals of $S^{-1}R$.

If P is a prime ideal of $S^{-1}R$ such that $P \cap S = \emptyset$, and if $a \in P$, then $a \cdot \frac{b}{s} \in P \cdot S^{-1}R$ for any $\frac{b}{s} \in S^{-1}R$. In particular, this holds for $b = s = 1$, meaning that $\frac{a}{1} \in P \cdot S^{-1}R$, so that $a \in \iota_S^{-1}\left(P \cdot S^{-1}R\right)$, so one inclusion holds. The other inclusion holds by the fact that if $a \in \iota_S^{-1}\left(P \cdot S^{-1}R\right)$, then $\frac{a}{1} \in P \cdot S^{-1}R$, so that $a \cdot \frac{1}{1} \in P \cdot S^{-1}R$, meaning that $a \in P$. Thus, α and β are mutual inverses.