Problem (Problem 1)**:** Show that every element of order 2 in A_n is the square of an element of order 4 in S_n.

Solution: Let $\alpha \in A_n$ be written as a product of disjoint cycles

$$\alpha = \sigma_1 \cdots \sigma_r,$$

such that $\alpha^2 = e$. Since $\alpha = \alpha^{-1}$, we then have that

$$\alpha = \sigma_1^{-1} \cdots \sigma_r^{-1},$$

whence each of $\sigma_1, \ldots, \sigma_r$ is of order 2. In particular, this means that α is in fact a product of an even number of disjoint transpositions, which we will rewrite as

$$\alpha = \tau_1 \cdots \tau_{2k}.$$

Pairing up these transpositions, we observe that

$$\begin{aligned} \tau_1 \tau_2 &= (a_1, b_1)(a_2, b_2) \\ &= (a_1, a_2, b_1, b_2)^2, \end{aligned}$$

whence we have k 4-cycles $\zeta_1, \ldots, \zeta_k$ given by

$$\zeta_i^2 = \tau_{2i-1} \tau_{2i}$$

Each of these ζ_i are disjoint, of order 4, and we have

$$\gamma = \zeta_1 \cdots \zeta_k$$

is of order 4 in S_n and is such that

$$\gamma^2 = \alpha.$$

Problem (Problem 2)**:** Let $G = \langle x \rangle$ be a cyclic group, H an arbitrary group. Let $\varphi_1, \varphi_2 \colon G \to \mathrm{aut}(H)$ be homomorphisms such that $\mathrm{im}(\varphi_1)$ and $\mathrm{im}(\varphi_2)$ are conjugate. If G is infinite, also assume that φ_1 and φ_2 are injective. Prove that the semidirect products $H \rtimes_{\varphi_1} G$ and $H \rtimes_{\varphi_2} G$ are isomorphic.

Problem (Problem 3)**:**

(a) Construct a nonabelian group of order 75.

(b) Show that up to isomorphism there are three groups of order 75.

Solution:

(a) We observe that $75 = 3 \cdot 5^2$, so by the result on subgroups of the form $p^2 q$, with $q < p$, we have a unique 5-Sylow subgroup. Suppose this 5-Sylow subgroup is of the form $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. Then, this is in fact a 2-dimensional vector space over $\mathbb{Z}/5\mathbb{Z}$, meaning that

$$\mathrm{aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \cong GL_2(\mathbb{Z}/5\mathbb{Z}),$$

which has order $24 \cdot 20 = 480$. In particular, there is some nontrivial automorphism from $\mathbb{Z}/3\mathbb{Z} \to \mathrm{aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$, which we can find by selecting an element of order 3 from $GL_2(\mathbb{Z}/5\mathbb{Z})$, which emerges from the fact that $480 = 3 \cdot 2^5 \cdot 5$ admits a 3-Sylow subgroup. This gives the nonabelian group $(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \rtimes_f \mathbb{Z}/3\mathbb{Z}$.

(b) We observe that there are two abelian groups of order 75, given by

$$\begin{aligned} G_1 &= \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \\ G_2 &= \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}. \end{aligned}$$

The reason G_1 and G_2 are not isomorphic is that there are no elements of order 25 in G_1, while (for example), $(0, 3)$ has order 5^2 in G_2.