Contents

First-Order Differential Equations

Introduction to First-Order Differential Equations

Recall that for $y = f(x)$, x is the independent variable and y is the dependent variable.

Definition (Differential Equation)**.** A differential equation is an equation which contains derivatives of a dependent variable with respect to one or more independent variables.

Example (A Basic Differential Equation)**.**

$$\frac{dy}{dx} - 5y - 1 = 0$$

We can classify differential equations by

- type;
- order;
- linearity.

Definition (Classification by Type)**.** There are two types of differential equations:

- ordinary differential equations (ODEs);

- partial differential equations (PDEs).

ODEs are characterized by derivatives of the dependent variable with respect to one independent variable. PDEs are characterized by derivatives of the dependent variable with respect to multiple independent variables.

Example (ODEs and PDEs)**.**

(1) An ODE:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$$

(2) A PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Definition (Classificiation by Order)**.** The order of the highest derivative in a differential equation is the order of the differential equation.

Example (Differential Equations of Varying Orders)**.**

(1)

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) - 4y = x \qquad \text{order 2}$$

(2)

$$2\frac{dy}{dx} + y = 0 \qquad \text{order 1}$$

(3)

$$\sin(x)y''' - (\cos x)\, y' = 2 \qquad \text{order 3}$$

In general, we write a differential equation of order n in the form

$$\underbrace{F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)}_{n+2 \text{ variables}} = 0.$$

Example. Suppose we have the differential equation

$$\frac{d^3y}{dx^3} + y^2 = x.$$

Then, we rewrite as

$$\underbrace{\frac{d^3y}{dx^3} + y^2 - x}_{F\left(x,y,\frac{dx}{dy},\frac{d^2x}{dy^2},\frac{d^3x}{dy^3}\right)} = 0.$$

Alternatively, we can write as

$$\frac{d^3y}{dx^3} = \underbrace{x - y^2}_{f(x,y)}.$$

Definition (Classification by Linearity)**.** We much prefer to analyze linear differential equations over non-linear differential equations.

A differential equation is linear if it has the following form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

(1) The power of each term involving y and all its derivatives is one.

(2) All coefficients are exclusively functions of x.

A differential equation that is not linear is called nonlinear.

Example (Linear Differential Equations (or lack thereof))**.**

(1)

$$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = e^x \qquad \text{Linear}$$

(2)

$$yy'' - 2y' = x \qquad \text{Nonlinear}$$

(3)

$$\frac{d^3y}{dx^3} + y^2 = 0 \qquad \text{Nonlinear}$$

Definition (Autonomous Differential Equations)**.** An autonomous (first-order) differential equation is a differential equation in the following form:

$$\frac{dy}{dx} = f(y).$$

Definition (Solution to an ODE)**.** Consider the general ODE

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0. \qquad (*)$$

A solution of $(*)$ is a function $y = f(x)$ that satisfies the ODE; that is,

$$F\left(x, f(x), f'(x), f''(x), \dots, f^{(n)}(x)\right) = 0$$

for every x in the domain of $f(x)$.

Notice that f is an element of a family of functions that satisfy the differential equation.

Example (Verifying a Solution)**.** We wish to show that $y = xe^x$ is a solution to

$$y'' - 2y' + y = 0$$

on $(-\infty, \infty)$.

In order to do this, we plug the proposed solution into the ODE:

$$\begin{aligned} y'' - 2y' + y &= \frac{d^2}{dx^2}\left(xe^x\right) - 2\frac{d}{dx}\left(xe^x\right) + xe^x \\ &= \left(xe^x + 2e^x\right) - 2\left(xe^x + e^x\right) + xe^x \\ &= 0. \end{aligned}$$

Definition (Equilibrium Solution)**.** An equilibrium solution of a differential equation is a *constant function* that satisfies the differential equation.

Example. We want to find equilibrium solutions for the following equations:

(1) $\frac{dy}{dt} = 2 - y$

(2) $\frac{dy}{dt} = y^2 - 3y - 4$.

In order to find equilibrium solutions, we know that $\frac{dy}{dt} = 0$. Thus, the equilibrium solutions are, respectively,

(1) $y(t) = 2$

(2) $y(t) = -1$ or $y(t) = 4$.

For first-order ODEs of the form

$$\frac{dy}{dt} = f(t, y),$$

equilibrium solutions are found by taking $\frac{dy}{dt} = 0$, and solving for y.

Modeling with Differential Equations

Definition (Initial Value Problem)**.** An initial value problem is a problem with a given ODE and an initial condition.

Example (Initial Value Problems)**.**

(1) We want to find

$$\frac{dy}{dx} = f(x, y)$$

such that $y(x_0) = y_0$.

(2)

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

must satisfy $y(x_0) = y_0$, $y'(x_0) = y_1$.

Modeling primarily occurs via the following feedback loop:

- real-world problem;
- mathematical model;
- solution;
- result/prediction.

As predictions from the model begin to stray from real-world observations, we update the model to reflect these new observations.

Example (Vertical Motion)**.** Consider someone who throws a rock off a building.

We let $y(t)$ denote the height of the ball at time t, with y_0 denoting initial height. The acceleration due to gravity is equal to $a(t) = v'(t) = y''(t) = g$.

Our ODE is

$$y''(t) = -g \qquad \text{for } 0 \leqslant t \leqslant T.$$

We require some initial conditions:

- $y(0) = y_0$ (initial position);
- $y'(0) = v_0$ (initial velocity).

Thus, we have created our second-order initial value problem.

To solve this second-order initial value problem analytically, we start with

$$y'' = -g.$$

Taking our first integral with respect to t, we have

$$y' = -gt + c_1.$$

Now, taking our second integral,

$$y(t) = -\frac{1}{2}gt^2 + c_1t + c_2.$$

This version of $y(t)$ is the general solution.

Applying our initial condition on $y'(t)$, we have $y'(0) = c_1$, meaning $c_1 = v_0$, and applying the initial condition to $y(t)$, we have $y(0) = c_2$, meaning $c_2 = y_0$.

Thus, the solution to this initial value problem is

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0.$$

Example (Population Growth, Exponential and Logistic)**.** Let $P(t)$ be the population of living fish in a lake at time t.

We know that the rate of growth in population is proportional to the population. In other words,

$$\frac{dP}{dt} = kP(t)$$

for some constant $k > 0$.

We can also include an initial condition, $P(0) = P_0$.

We can see (relatively easily) that

$$P(t) = P_0e^{kt}.$$

However, this is not particularly realistic; there is no theoretical upper bound on the model, even though in real life, ecosystems tend to have carrying capacities.

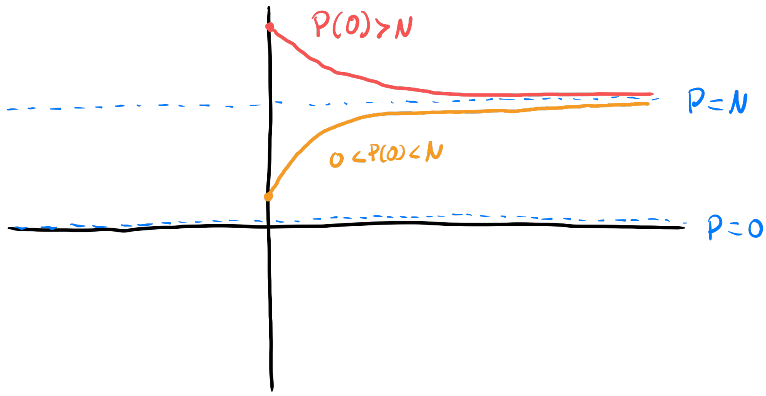
The logistic population model with growth rate k and carrying capacity N is is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right).$$

We can analyze this equation qualitatively first (before finding an analytical solution).

- If $P > N$, we can see that $\frac{dP}{dt} < 0$, which is expected since, if population is greater than carrying capacity, we expect population to approach carrying capacity.
- If $P < N$ (assuming P is positive), we see that $\frac{dP}{dt} > 0$, meaning population increases as it approaches carrying capacity. In particular, as population increases, the growth rate decreases.

- The equilibrium solutions occur at $P = 0$ or $P = N$.



Separable First-Order Differential Equations

Consider the first-order differential equation

$$\frac{dy}{dt} = f(t, y).$$

Definition (Separable Differential Equation)**.** A differential equation of the form

$$\frac{dy}{dt} = g(t)h(y)$$

is called separable.

Note:

$$f(t, y) = g(t)h(y).$$

Example.

(1) We can see that $\frac{dP}{dt} = kP$ is separable; $g(t) = k$, $h(y) = P$.

(2) We can see that $\frac{dy}{dt} = -\frac{t}{y}$ is also separable; $g(t) = -t$, $h(y) = \frac{1}{y}$.

(3) We can see that $\frac{dy}{dt} = y + t$ is not separable.

Method (Separation of Variables)**.** We want to solve $\frac{dy}{dt} = g(t)h(y)$.

(1) We take $\frac{dy}{h(y)} = g(t)\,dt$ by multiplying dt on both sides and dividing by $h(y)$.

(2) Integrate both sides with respect to their corresponding variable, yielding

$$\int \frac{1}{h(y)}\,dy = \int g(t)\,dt.$$

(3) We get

$$H(y) = G(t) + C,$$

where $H(y)$ and $G(t)$ are antiderivatives of $\frac{1}{h(y)}$ and $g(t)$ respectively.

Example (Solving the Exponential Population Growth Model by Separation of Variables)**.** Let $\frac{dP}{dt} = kP$, $P(0) = P_0$.

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= k\,dt \\
\int \frac{1}{P}\,dP &= \int k\,dt \\
\ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
&= e^{kt}e^{C} \\
P &= \left(\pm e^{C}\right)e^{kt} \\
&= Ae^{kt}.
\end{aligned}$$

Our solution is now of the form $P(t) = Ae^{kt}$ (where $A = \pm e^{C}$). This is not the general solution, though, since it lacks our equilibrium solution of $P = 0$. Thus, the general solution is

$$\begin{cases} P(t) = Ae^{kt} \\ P(t) = 0 \end{cases}$$

With the initial condition of $P(0) = P_0$, we have

$$\begin{aligned}
P_0 = P(0) &= Ae^{k\cdot 0} \\
&= A.
\end{aligned}$$

Thus, the particular solution to our initial value problem is $P(t) = P_0e^{kt}$.

Example (Solving a Sample Differential Equation by Separation of Variables)**.** Let $\frac{dy}{dt} = y^2 - 4$. Note that, even though this is not a linear equation, this is a separable equation. We start with the equilibrium solutions, which are at $y(t) = 2$ and $y(t) = -2$.

If $y \neq \pm 2$, we have

$$\begin{aligned}
\frac{dy}{dt} &= y^2 - 4 \\
\frac{1}{y^2 - 4}\,dy &= dt \\
\int \frac{1}{4(y-2)} - \frac{1}{4(y+2)}\,dy &= \int dt \\
\frac{1}{4}\left(\ln\left|\frac{y-2}{y+2}\right|\right) &= t + C_1 \\
\ln\left|\frac{y-2}{y+2}\right| &= 4t + C_2 \\
\left|\frac{y-2}{y+2}\right| &= e^{C_2}e^{4t} \\
&= C_3e^{4t} \\
\frac{y-2}{y+2} &= \pm C_3e^{4t} \\
&= Ce^{4t}
\end{aligned}$$

$$\begin{aligned} y &= 2 + yCe^{4t} + 2Ce^{4t} \\ y &= \frac{2\left(1 + Ce^{4t}\right)}{1 - Ce^{4t}}. \end{aligned}$$

Thus, our general solution is

$$\begin{cases} y(t) = \frac{2\left(1+Ce^{4t}\right)}{1-Ce^{4t}} \\ y(t) = 2 \\ y(t) = -2 \end{cases}$$

Example (Solving the Logistic Population Growth Model)**.** Let $\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$. Our equilibrium solutions are at $P(t) = 0$ and $P(t) = N$. For non-equilibrium solutions, we have

$$\begin{aligned} \frac{dP}{dt} &= kP\left(1 - \frac{P}{N}\right) \\ \frac{1}{P\left(1 - \frac{P}{N}\right)}\,dP &= k\,dt \\ \int \frac{1}{P\left(1 - \frac{P}{N}\right)}\,dP &= \int k\,dt \\ \int \frac{-N}{P\left(P - N\right)}\,dP &= kt + C_1 \\ \int \frac{1}{P} - \frac{1}{P - N}\,dP &= kt + C_1 \\ \ln|P| - \ln|P - N| &= kt + C_1 \\ \ln\left|\frac{P}{P - N}\right| &= kt + C_1 \\ \left|\frac{P}{P - N}\right| &= e^{C_1}e^{kt} \\ \frac{P}{P - N} &= \pm e^{C_1}e^{kt} \\ &= Ce^{kt} \\ P\left(1 - Ce^{kt}\right) &= -NCe^{kt} \\ P &= N\frac{Ce^{kt}}{Ce^{kt} - 1}. \end{aligned}$$

Therefore, our general solution is

$$\begin{cases} P(t) = N\frac{Ce^{kt}}{Ce^{kt}-1} \\ P(t) = 0 \\ P(t) = N \end{cases}.$$

Example (A Non-Separable Linear Differential Equation)**.** Consider the linear differential equation

$$\frac{dy}{dt} + a(t)y = b(t).$$

Notice that

$$\frac{dy}{dt} = -a(t)y + b(t),$$

which is not able to be separated.

In order to solve such an equation, we will need to use an integrating factor.

Slope Fields

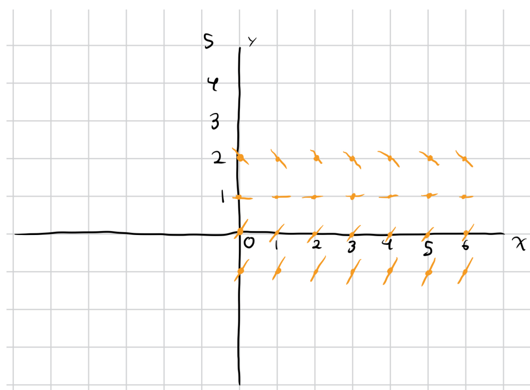
Definition. A slope field is a set of short line segments that indicate slope $\frac{dy}{dx}$ at a set of points (x, y) in the x, y-plane.

It is a graphical method of displaying the general slope and behavior of functions that satisfy $\frac{dy}{dx} = f(x, y)$.

Example. Consider $\frac{dy}{dx} = 1 - y$. We can select some samples of slopes as follows:

Point	$\frac{dy}{dx}$
$(0, 0)$	1
$(1, 0)$	1
$(0, 1)$	0
$(0, 2)$	-1

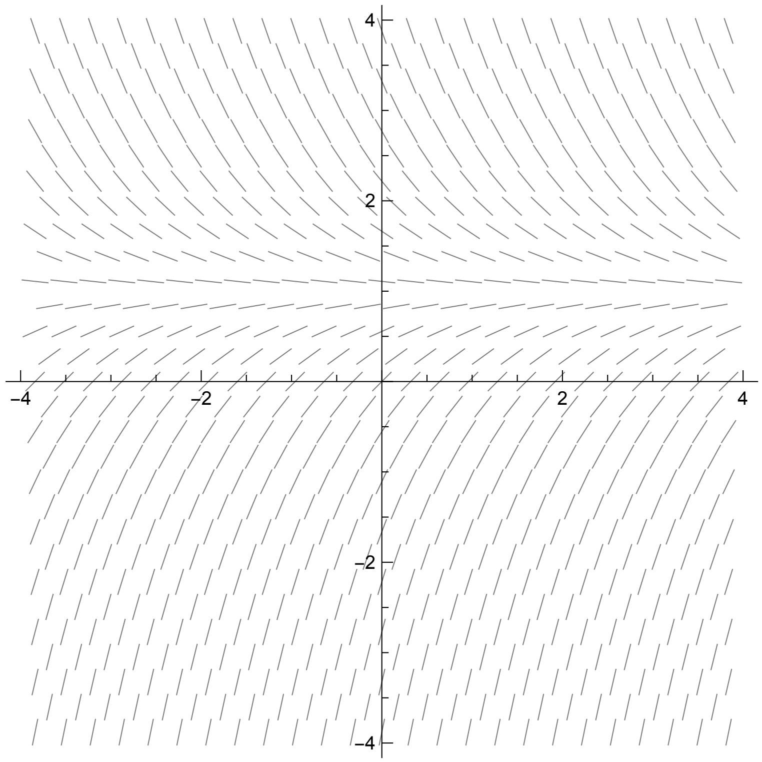
Thus, we can draw the slope field:



Qualitatively, we can see that

- at $y = 1$, all line segments are horizontal;
- for $y < 1$, all line segments have positive slope;
- for $y > 1$, all line segments have negative slope.

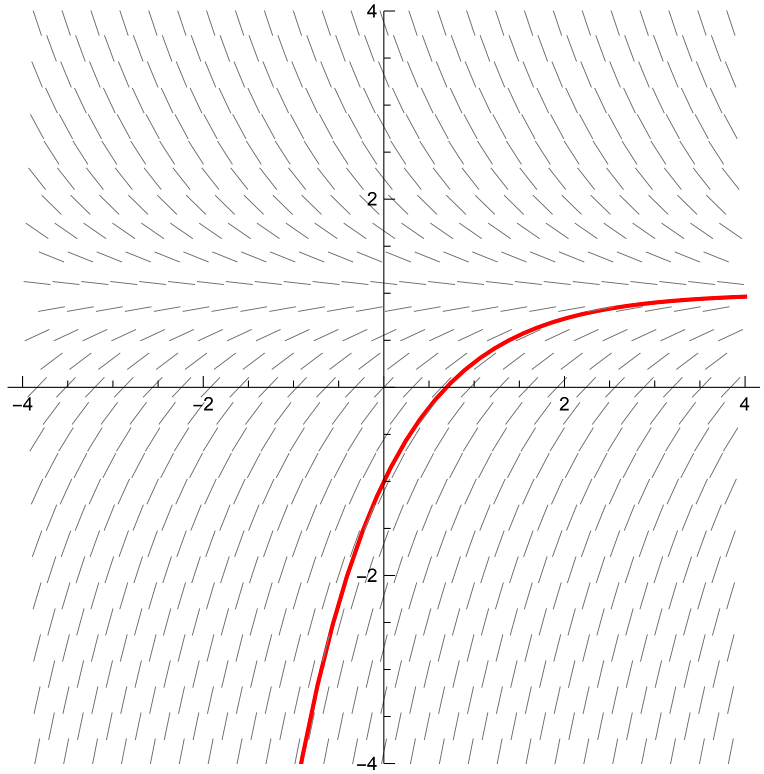
Using a computer, we can generate a better slope field:



Analytically, we solve the equation by separation of variables:

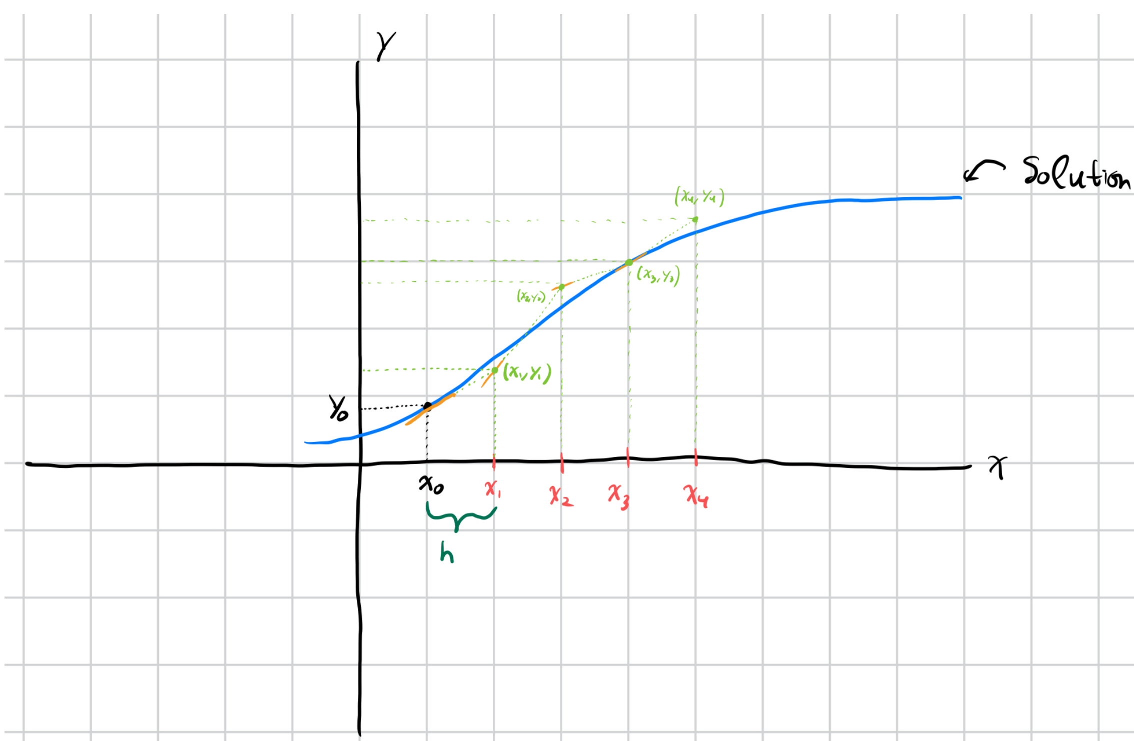
$$\begin{aligned}
\frac{dy}{dx} &= 1 - y \\
\frac{dy}{1-y} &= dx \\
\int \frac{1}{1-y}\,dy &= \int\, dx \\
-\ln|1-y| &= x + C \\
|1-y| &= e^{-x-C_1} \\
1-y &= \pm e^{-C_1}e^{-x} \\
y &= 1 - Ae^{-x}.
\end{aligned}$$

With the initial condition of $y(0) = -1$, we get $y = 1 - 2e^{-x}$.



Euler's Method

We want to approximate solutions to the differential equation $y' = f(x, y)$, $y(x_0) = y_0$. In the diagram, we can see the use of the slope field to calculate the approximate values of (x_i, y_i).



Method (Euler's Method)**.** To approximate the curve at $x_1 = x_0 + h$, we take the point-slope form:

$$\begin{aligned}\frac{y_1 - y_0}{(x_0 + h) - x_0} &= f(x_0, y_0)\\ y_1 &= y_0 + hf(x_0, y_0).\end{aligned}$$

In general, we have

$$y_{k+1} = y_k + hf(x_k, y_k).$$

Example. Consider the differential equation $y' = 2y - 1$. With the step size $\Delta x = h = 0.5$ and $y(0) = 1$, we can approximate $y(1)$ by

k	x_k	y_k	$f(x_k, y_k)$
0	1	1	1
1	1.5	1.5	2
2	2	2.5	—

Thus, using Euler's method with a step size of 0.5, we find that $y(1) \approx 2.5$. The table is read left to right, changing columns after calculating $f(x_k, y_k)$, then using it to calculate y_{k+1}.

Solving the differential equation analytically, we find

$$\begin{aligned}\frac{dy}{dx} &= 2y - 1\\ \int \frac{1}{2y-1}\,dy &= \int\,dx\\ \frac{1}{2}\ln|2y-1| &= x + C_1\\ \ln|2y-1| &= 2x + C_2\\ |2y-1| &= e^{C_2}e^{2x}\\ 2y - 1 &= \pm e^{C_2}e^{2x}\\ y &= \frac{1}{2} + Ae^{2x}.\end{aligned}$$

Plugging in our initial condition, we find $A = \frac{1}{2}$, and the exact value of $y(1)$ is $\frac{1}{2} + \frac{1}{2}e^2$, which is approximately 4.1945.

We can make our approximation via Euler's method better using a shorter step size.

For instance, by using a step size of 0.1, we find:

k	x_k	y_k	$2y_k - 1$
0	0	1	1
1	0.1	1.1	1.2
2	0.2	1.22	1.44
3	0.3	1.364	1.728
4	0.4	1.537	2.07
5	0.5	1.744	2.49
6	0.6	1.993	2.97
7	0.7	2.29	3.58
8	0.8	2.65	4.30
9	0.9	3.08	5.16
10	1	3.596	—

Note that our final approximation of 3.596 is much better than the approximation under $h = 0.5$.

In order to understand if our estimate with Euler's method is an overestimate or underestimate, we use the second derivative test.

$$\begin{aligned} y' &= 2y - 1 \\ y'' &= 2y' \\ &= 2(2y-1) \\ &= 4y - 2. \end{aligned}$$

In particular, our initial condition of $y(0) = 1$ suggests that $y'' = 2 > 0$, meaning Euler's method will return an underestimate (as the tangent lines will lie below the true curve).

Existence and Uniqueness

Given an initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0, \end{aligned}$$

we ask the following two questions.

(1) When does a solution to this initial value problem exist?

(2) If it does exists, is the solution unique?

Example. Consider the polynomial

$$\underbrace{2x^5 - 10x + 5}_{f} = 0,$$

and suppose want to find solutions to this equation.

Notice that f is continuous on $[-1, 1]$, with $f(-1) = 13$ and $f(1) = 3$.

By the Intermediate Value Theorem, this must mean f takes on the value of 0 at at least one value between $x = -1$ and $x = 1$.

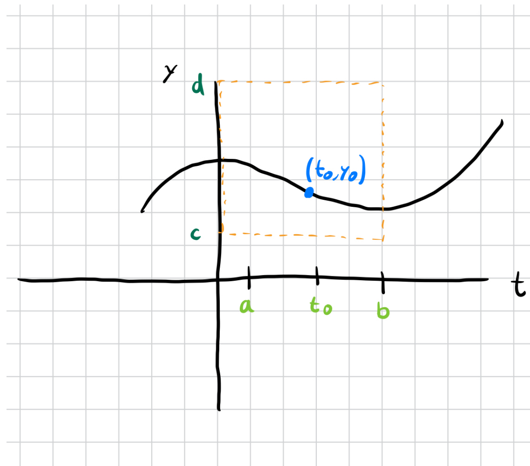
Theorem (Existence and Uniqueness)**:** Let R be a rectangular region in the t, y-plane containing the point (t_0, y_0) in the interior of R. In particular, $R = \{(t, y) \mid a < t < b, c < y < d\}$, and $(t_0, y_0) \in R$.

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists $\varepsilon > 0$ and a unique function $y(t)$ defined on some neighborhood $t_0 - \varepsilon < t < t_0 + \varepsilon$ contained in $a < t < b$ such that $y(t)$ is a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0. \end{aligned}$$

Note the two major conditions here:

- the continuity of f on R; this guarantees existence
- the continuity of $\frac{\partial f}{\partial y}$ on R; this guarantees uniqueness.



If one of the conditions is not satisfied, we may have

- exactly one solution;
- many solutions;
- no solutions.

Example. Consider the differential equation

$$\frac{dy}{dt} = t^2 y^{1/2},$$

and $y(0) = 0$.

We have $f(t, y) = t^2 y^{1/2}$. We have

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{1}{2} t^2 y^{-1/2} \\ &= \frac{t^2}{2\sqrt{y}}. \end{aligned}$$

We can see that f is continuous at $(0, 0)$, but $\frac{\partial f}{\partial y}$ is not continuous at $(0, 0)$.

Additionally, since f is not defined for $y < 0$, we cannot place $(0, 0)$ in the interior of any region on the t, y-plane.

Therefore, we cannot use the existence and uniqueness theorem on f.

Going forward analytically, we start with the equilibrium condition $y(t) = 0$. Using separation of variables, we have

$$\begin{aligned} \frac{dy}{dt} &= t^2 y^{1/2} \\ \int \frac{dy}{y^{1/2}} &= \int t^2 \, dt \\ 2y^{1/2} &= \frac{t^3}{3} + C \\ y &= \left(\frac{t^3}{6} + K\right)^2 \end{aligned}$$

$$
\begin{aligned}
0 &= \left(\frac{(0)^3}{6} + K\right)^2 \\
K &= 0
\end{aligned}
$$

meaning we also have a solution of

$$
\begin{aligned}
y(t) &= \left(\frac{t^3}{6}\right)^2 \\
&= \frac{t^6}{36}.
\end{aligned}
$$

Example. Let $\frac{dy}{dt} = t^2 y^{1/2}$ with the (new) initial condition of $y(2) = 1$.

In particular, we can see that not only is $f(t, y)$ continuous at 0, but so too is $\frac{\partial f}{\partial y}$, and there exists a region about the point $(2, 1)$ such that f and $\frac{\partial f}{\partial y}$ are continuous.

Thus, by the existence and uniqueness theorem, there exists exactly one solution to this initial value problem.

Example. Let $\frac{dy}{dt} = 1 + y^2$, with the initial condition $y(0) = 0$.

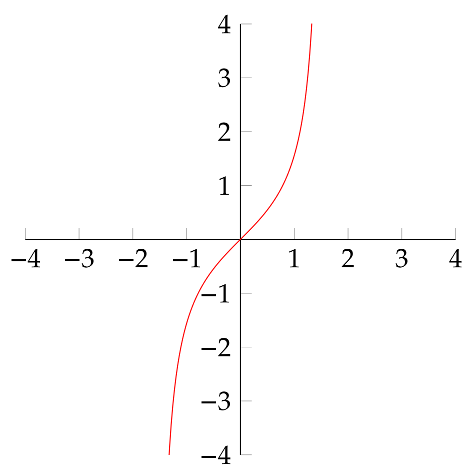
We have $f(t, y) = 1 + y^2$, $\frac{\partial f}{\partial y} = 2y$; both of these functions are continuous on the t, y-plane. Thus, by the existence and uniqueness theorem, there exists a unique solution to this initial value problem.

We solve by separation of variables:

$$
\begin{aligned}
\frac{dy}{dt} &= 1 + y^2 \\
\int \frac{1}{1+y^2}\,dy &= \int dt \\
\arctan(y) &= t + C \\
y &= \tan(t + C) \\
0 &= \tan(0 + C) \\
0 &= \tan(C).
\end{aligned}
$$

We can let $C = 0$, meaning we have the solution

$$
y(t) = \tan t.
$$



Example (Application of the Uniqueness Theorem)**.** Suppose $f(t,y)$ satisfies the conditions for uniqueness.

Assume $y_1(t)$ and $y_2(t)$ are solutions of the differential equation $\frac{dy}{dt} = f(t,y)$.

If these solutions intersect at some $t = t_0$, then $y_1(t)$ and $y_2(t)$ are solutions to this new initial value problem of

$$\begin{aligned}\frac{dy}{dt} &= f(t,y)\\ y(t_0) &= y_1(t_0) = y_2(t_0).\end{aligned}$$

Since f satisfies the uniqueness theorem, it be the case that $y_1(t) = y_2(t)$.

In particular, this means that solutions to differential equations that satisfy the uniqueness conditions cannot intersect. In other words, solutions cannot equal each other at the same "place" at the same "time."

Example. Consider the initial value problem

$$\begin{aligned}\frac{dy}{dt} &= \frac{2y}{t}\\ y(1) &= 1.\end{aligned}$$

We have $f(t,y) = \frac{2y}{t}$, $\frac{\partial f}{\partial y} = \frac{2}{t}$. We can see that f and $\frac{\partial f}{\partial y}$ are continuous at $(1,1)$, as well as a given region with $(1,1)$ in its interior (since continuity is only lost when $t = 0$).

Thus, there exists a unique solution to this initial value problem. We can solve the equation via separation of variables:

$$\begin{aligned}\frac{dy}{dt} &= \frac{2y}{t}\\ \int \frac{1}{2y}\,dy &= \int \frac{1}{t}\,dt\\ \frac{1}{2}\ln|y| &= \ln|t| + C\\ \ln|y| &= 2\ln|t| + 2C\\ |y| &= e^{2\ln|t|}e^{2C}\\ y &= \pm e^{2C}t^2\\ y &= Kt^2\end{aligned}$$

We have the initial condition $y(1) = 1$, meaning we have the solution $y(t) = t^2$.

Notice that as a function, the domain of y is $\mathbb{R}$, but as a solution, we cannot have $t = 0$, meaning that the domain of $y(t) = t^2$ *as a solution* is $t > 0$.

Equilibria and Phase Lines

Given a differential equation

$$\frac{dy}{dt} = f(t,y),$$

we can use slope fields and Euler's method to find an approximate solution, as well as various analytic methods to find a definite solution.

Given an autonomous differential equation, though,

$$\frac{dy}{dt} = f(y),$$

particularly one with f, $\frac{\partial f}{\partial y}$ continuous, we can use some deeper analysis.

In particular, for constant y, the slope at any point (t, y) will be the same as at any other (t, y). This means we can analyze the behavior of *the entire solution* based on one particular line.

Method (Equilibrium Analysis for Autonomous Differential Equations)**.**

(1) Solve $f(y) = 0$ to find the equilibrium solutions.

(2) Analyze the behavior of $f(y)$ around the equilibrium points.

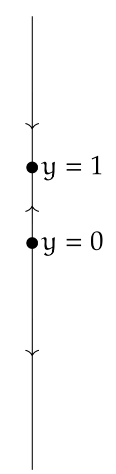
- Draw a particular line, known as a phase line, with positive t.
- Determine if $f(y)$ is positive or negative between equilibrium points.
- If $f(y) > 0$, then the solution is increasing in this region, and if $f(y) < 0$, the solution is decreasing in this region.

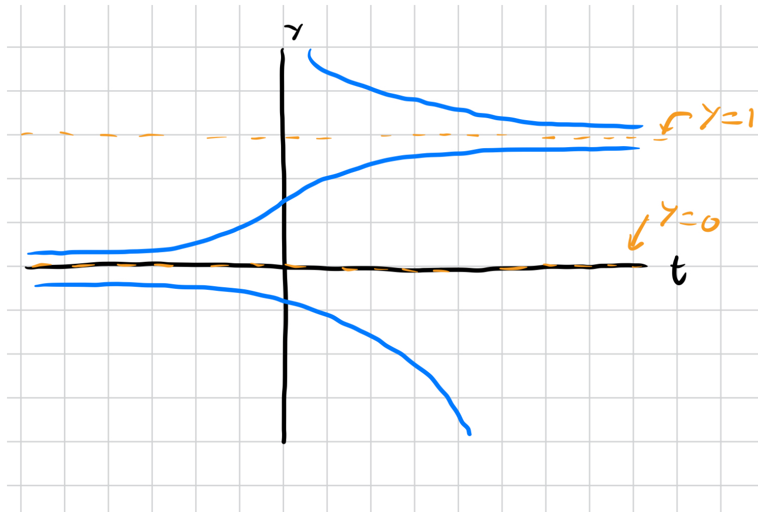
Example. Let

$$\frac{dy}{dt} = (1 - y)y.$$

Thus, we have $f(y) = (1 - y)y$, meaning $y(1 - y) = 0$ for $y = 0$ and $y = 1$.

For $y = 2$, we have $f(2) = -2 < 0$, for $y = 0.5$, $f(y) = 0.25 > 0$, and $f(-1) = -2 < 0$. Thus, our equilibrium analysis looks like the following:





Definition (Classificiation of Equilibrium Points)**.**

(1) An equilibrium point at $y = c$ is called a sink if all the solutions with the initial condition near $y = c$ approach c as t approaches ∞.

- On the phase line, $f(y)$ goes from $+$ to $-$ on the phase line as y goes from $c - \varepsilon$ to $c + \varepsilon$.

(2) An equilibrium point at $y = c$ is called a source if all the solutions with the initial conditions near $y = c$ move away from c as t approaches ∞.

- On the phase line, $f(y)$ goes from $-$ to $+$ on the phase line as y goes from $c - \varepsilon$ to $c + \varepsilon$.

(3) Every equilibrium point that is neither a source nor a sink is called a node.

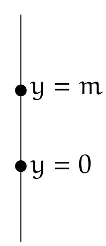
- On the phase space, $f(y)$ maintains the same sign as y goes from $c - \varepsilon$ to $c + \varepsilon$.

Example (Logistic Population Model)**.** Recall that the logistic population model is

$$\frac{dy}{dt} = \underbrace{k\left(1 - \frac{y}{m}\right)y}_{f(y)},$$

with the conditions of $k > 0$, $m > 0$. We want to draw a phase line for this model.

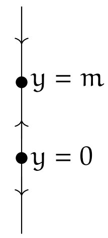
The equilibrium solutions are at $y = 0$ and $y = m$. We draw our phase line with its equilibrium points as



To find out what happens in between, we choose

- $y = -1$; $f(y) < 0$.
- $y = m/2$; $f(y) > 0$.
- $y = 2m$; $f(y) < 0$.

Therefore, our phase line is



Theorem (Linearization)**:** Suppose y_0 is an equilibrium point of the differential equation $\frac{dy}{dt} = f(y)$, where $f(y)$ is a continuously differentiable function. Then,

- if $f'(y_0) < 0$, then y_0 is a sink;
- if $f'(y_0) > 0$, then y_0 is a source;
- if $f'(y_0) = 0$, then we do not have enough information to determine the type of y_0.

Remark: In order to do an equilibrium analysis given a graph of $f(y)$, we identify all our equilibrium points by finding the roots of f, then examining the slope of f at each of the identified equilibrium points.

Example. Suppose

$$\frac{dy}{dt} = (y-1)\left(y^5 - 7y^4 + 3y^3 + 8y^2 - 11\right).$$

We want to classify the equilibrium point $y = 1$. In order to do this, we find its derivative:

$$\frac{\partial f}{\partial y} = \left(y^5 - 7y^4 + 3y^3 + 8y^2 - 11\right) + (y-1)\left(5y^4 - 28y^3 + 9y^2 + 16y\right).$$

Evaluating

$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = -6,$$

meaning our equilibrium point at $y = 1$ is a sink.

Bifurcations

Consider a first-order differential equation

$$y' = f(y, c).$$

The value $c \in \mathbb{R}$ means this differential equation is actually a family of differential equations. We say c is a parameter for the differential equation.

Example (A Parametrized in a Differential Equation)**.** Consider the differential equation modelling the population of fish in a pond, denoted

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - h.$$

Here, k is a constant, while h is a parameter denoting the harvesting rate.

In general, as c varies in our parametrized differential equation, an equilibrium solution may split into two equilibrium solutions disappear entirely.

Example. Consider the family

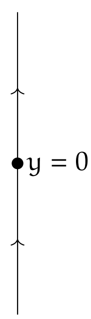
$$\frac{dy}{dt} = y^2 - c.$$

The equilibrium solutions occur when $y^2 - c = 0$, meaning that for $c > 0$, our equilibrium solutions are at $y = \pm\sqrt{c}$. If $c = 0$, then $y = 0$ is an equilibrium solution, and if $c < 0$, there are no equilibrium solutions.

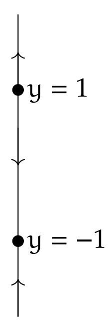
We can see that $c = 0$ is a bifurcation point.

For $c = -1$, our differential equation is $\frac{dy}{dt} = y^2 + 1$.

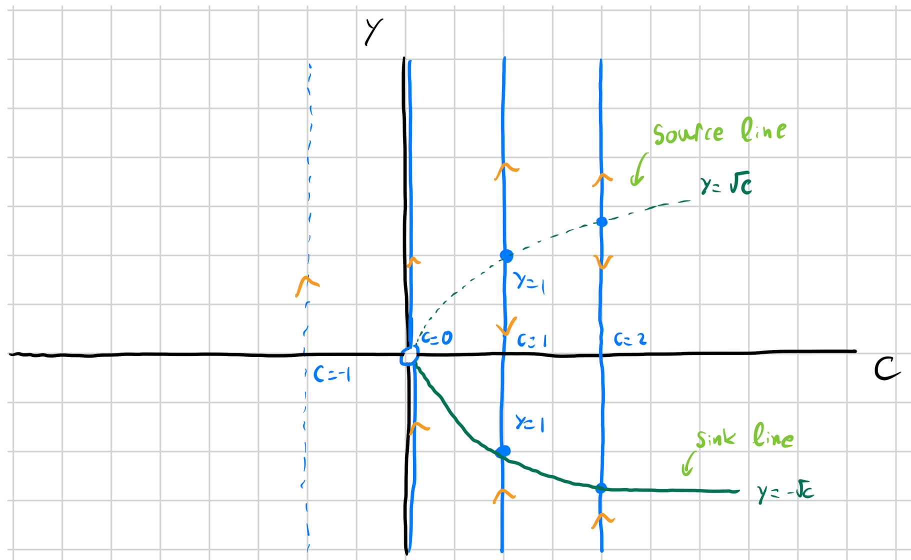
For $c = 0$, our differential equation is $\frac{dy}{dt} = y^2$.



For $c = 1$, our differential equation is $\frac{dy}{dt} = y^2 - 1$.



The bifurcation diagram is as follows.



Notice that the bifurcation point occurs when the source line and sink line meet.

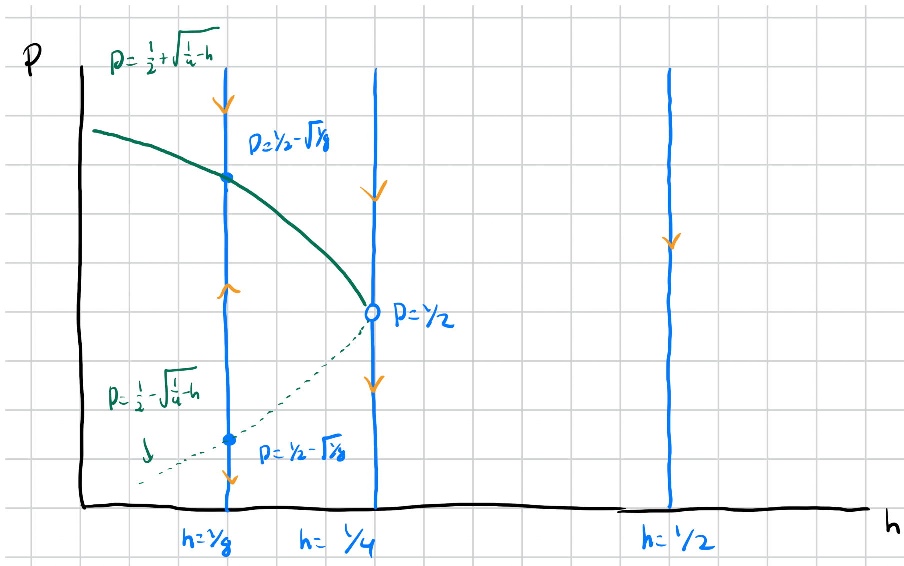
Example (Solving a Population Model)**.** Consider

$$\frac{dP}{dt} = \underbrace{P(1-P) - h}_{=f(P,h)}$$

Equilibrium solutions occur at

$$\begin{aligned} P(1-P) - h &= 0 \\ P^2 - P + h &= 0 \\ P^2 - P + \frac{1}{4} &= \frac{1}{4} - h \\ \left(P - \frac{1}{2}\right)^2 &= \frac{1}{4} - h \\ P &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - h}. \end{aligned}$$

Based on the values of h, we have no equilibrium solutions for $h > \frac{1}{4}$, one equilibrium solution for $h = \frac{1}{4}$, and two equilibrium solutions for $h < \frac{1}{4}$.



- If $h > 1/4$, the fish population will die out;
- If $h = 1/4$, the fish population will stabilize at $P = 1/2$;
- If $0 < h < 1/4$, the fish population will stabilize at a new, lower population.

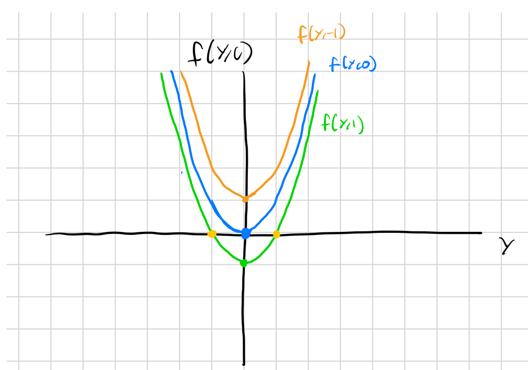
Recall: For

$$\frac{dy}{dt} = y^2 - c,$$

there are three cases.

- $c > 0$: two equilibrium solutions.
- $c = 0$: one equilibrium solution.
- $c < 0$: no equilibrium solutions.

When we graph f against y, we see the following.



In particular, we can see that the bifurcation value occurs at $c = 0$, when the root of f is tangent to the y axis. This can be formalized in the following theorem.

Theorem: Let y_0 be an equilibrium solution. The point c_0 is a bifurcation value for the autonomous differential equation

$$\frac{dy}{dt} = f(y, c)$$

if and only if $f(y_0, c_0) = 0$ and $\left.\frac{\partial f}{\partial y}\right|_{(y_0, c_0)} = 0$.

First Order Linear Differential Equations

Definition. A first-order differential equation of the form

$$\frac{dy}{dt} = a(t)y + b(t)$$
$$\frac{dy}{dt} - a(t)y = b(t)$$

with $a(t), b(t)$ arbitrary functions of t is known as a (first order) linear differential equation.

Example.

(1) $\frac{dy}{dt} + 2y = e^{-t}$ is linear.

Definition (Homogeneous Linear Differential Equations)**.** For

$$\frac{dy}{dt} - a(t)y = b(t),$$

if $b(t) = 0$ for all t, the linear differential equation is called homogeneous (or unforced). If $b(t) \neq 0$, we call this a non-homogeneous differential equation.

Definition (Linearity Principle for Homogeneous Equations)**.** Consider the homogeneous differential equation

$$\frac{dy}{dt} - a(t)y = 0.$$

(1) If $y_h(t)$ is a solution of $\frac{dy}{dt} - a(t)y = 0$, then $cy_h(t)$ is a solution for $\frac{dy}{dt} - a(t)y = 0$, for any $c \in \mathbb{R}$.

(2) If $y_1(t)$ and $y_2(t)$ are solutions of $\frac{dy}{dt} - a(t)y = 0$, then $(y_1 + y_2)(t)$ is a solution to $\frac{dy}{dt} - a(t)y = 0$.

The proof of the linearity principle follows from the linearity of the derivative.

Example. Let

$$\frac{dy}{dt} = a(t)y + b(t).$$

Note that the associated homogeneous differential equation is

$$\frac{dy}{dt} = a(t)y.$$

We can solve the associated homogeneous differential equation using separation of variables:

$$\begin{aligned} \frac{dy}{dt} &= a(t)y \\ \int \frac{1}{y}\, dy &= \int a(t)\, dt \\ \ln(y) &= \int a(t)dt + C \\ y &= Ke^{\int a(t)\, dt}. && C \in \mathbb{R} \end{aligned}$$

Note that the linearity principle does *not* (necessarily) work for non-homogeneous differential equations.

Example (Failure of the Linearity Principle)**.** Consider

$$\frac{dy}{dt} = -y + 2. \tag{$*$}$$

Note that $b(t) = 2 \neq 0$.

One of the solutions to this equation is

$$y(t) = 2 - e^{-t}.$$

We will show that $2y(t)$ is not a solution of $(*)$.

$$\begin{aligned}\frac{d}{dt}\left(2\left(2 - e^{-t}\right)\right) &= 2e^{-t}\\ -y(t) + 2 &= 2e^{-t} - 2.\end{aligned}$$

Theorem (Extended Linearity Principle)**:** Let

$$\frac{dy}{dt} = a(t)y + b(t). \tag{$**$}$$

(1) If $y_h(t)$ is any solution of the homogeneous differential equation

$$\frac{dy}{dt} = a(t)y,$$

and $y_p(t)$ is any solution of $(**)$. Then, $y_h(t) + y_p(t)$ is a solution to $(**)$.

(2) If $y_1(t)$ and $y_2(t)$ are solutions to $(**)$, $y_1(t) - y_2(t)$ provides a solution to the homogeneous differential equation

$$\frac{dy}{dt} = a(t)y.$$

Proof.

(1)

$$\begin{aligned}\frac{d}{dt}\left(y_h(t) + y_p(t)\right) &= \frac{d}{dt}\left(y_h(t)\right) + \frac{d}{dt}\left(y_p(t)\right)\\ &= a(t)y_h(t) + a(t)y_p(t) + b(t)\\ &= a(t)\left(y_h(t) + y_p(t)\right) + b(t).\end{aligned}$$

(2)

$$\begin{aligned}\frac{d}{dt}\left(y_1(t) - y_2(t)\right) &= \frac{d}{dt}\left(y_1(t)\right) - \frac{d}{dt}\left(y_2(t)\right)\\ &= \left(a(t)y_1(t) + b(t)\right) - \left(a(t)y_2(t) - b(t)\right)\\ &= a(t)\left(y_1(t) - y_2(t)\right).\end{aligned}$$

□

Note that, as a result of the extended linearity principle, all solutions to a non-homogeneous first-order linear equation are of the form $y(t) = cy_h(t) + y_p(t)$, where $y_p(t)$ is *any* solution to the equation.

Example. Let

$$\frac{dy}{dt} = -2y + e^t.$$

We can see that $b(t) = e^t$, and the general solution to $\frac{dy}{dt} = -2y$ is Ke^{-2t} for $K \in \mathbb{R}$.

Now, we look at

$$\frac{dy}{dt} + 2y = e^t$$

We make a guess that

$$y_p(t) = \alpha e^t.$$

Then,

$$\begin{aligned}\frac{dy}{dt} + 2y &= \alpha e^t + 2\alpha e^t \\ &= e^t.\end{aligned}$$

Thus, $\alpha = \frac{1}{3}$, implying that $y_p(t) = \frac{1}{3}e^t$.

Thus, the general solution is

$$y(t) = \frac{1}{3}e^t + Ke^{-2t}.$$

Example. Let's try to find the general solution to

$$\frac{dy}{dt} = -2y + \cos(3t).$$

We are aware of the general solution of the homogeneous equation $\frac{dy}{dt} = -2y$, which is $y_h(t) = Ke^{-2t}$.

Now, we look at

$$\frac{dy}{dt} + 2y = \cos(3t)$$

to find a particular solution. We take a guess of $y_p(t) = A\cos(3t) + B\sin(3t)$. Then,

$$\frac{dy}{dt} + 2y = (-3A + 2B)\sin(3t) + (2A + 3B)\cos(3t).$$

Thus, $3A = 2B$ and $2A + 3B = 1$, yielding $B = \frac{3}{13}$ and $A = \frac{2}{13}$.

Therefore, the general solution is

$$y(t) = \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t) + Ke^{-2t}.$$

$b(t)$	Guess for $y_p(t)$
$ae^{\alpha t}$	$Ae^{\alpha t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\cos(\beta t)$
$a\sin(\beta t)$	$A\cos(\beta t) + B\cos(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\cos(\beta t)$
n-th degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$

Integrating Factors

We can formalize the aforementioned "lucky guess" method by creating an integrating factor.

Derivation (Integrating Factor)**.** Consider the equation Consider a factor $\mu(t)$.

$$\begin{aligned}
\frac{dy}{dt} &= a(t)y + b(t)\\
\frac{dy}{dt} - a(t)y &= b(t)\\
\frac{dy}{dt} + g(t)y &= b(t),
\end{aligned}$$

where we define $g(t) = -a(t)$. We multiply each side of this differential equation by $\mu(t)$, yielding

$$\begin{aligned}
\mu(t)\left(\frac{dy}{dt} + g(t)y\right) &= \mu(t)b(t)\\
\mu(t)\frac{dy}{dt} + \mu(t)g(t)y &= \mu(t)b(t).
\end{aligned}$$

Examining the left-hand side, it would be very convenient if $\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + y(t)\frac{d\mu}{dt}$. Then, we would have

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)b(t),$$

where $\frac{d\mu}{dt} = \mu(t)g(t)$. Then,

$$\begin{aligned}
\mu(t)y(t) &= \int \mu(t)b(t)\,dt\\
y(t) &= \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt.
\end{aligned}$$

Orienting our focus to the condition of $\frac{d\mu}{dt} = \mu(t)g(t)$, we solve for μ by separation of variables.

$$\begin{aligned}
\frac{d\mu}{dt} &= \mu g(t)\\
\int \frac{1}{\mu}\,d\mu &= \int g(t)\,dt\\
\ln|\mu| &= \int g(t)\,dt\\
|\mu| &= e^{C+\int g(t)\,dt}\\
\mu &= Ke^{\int g(t)\,dt}.
\end{aligned}$$

Definition (Integrating Factor)**.** For a non-homogeneous linear differential equation,

$$\frac{dy}{dt} + g(t)y = b(t)$$

the integrating factor is a family of functions such that

$$\mu(t) = Ke^{\int g(t)\,dt}.$$

In particular, we can let $K = 1$, yielding the factor

$$\mu(t) = e^{\int g(t)\,dt}.$$

Example. We wish to solve

$$x\frac{dy}{dx} - 4y = x^6e^x.$$

First, we divide out x, yielding

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x.$$

Now, we find the integrating factor,

$$\begin{aligned}
\mu(x) &= e^{\int -\frac{4}{x}\,dx}\\
&= e^{-4\ln|x|}\\
&= e^{\ln\left|x^{-4}\right|}\\
&= e^{\ln\left|x^{-4}\right|} && = \frac{1}{x^4}.
\end{aligned}$$

Multiplying through, we get

$$\begin{aligned}
x^{-4}\frac{dy}{dx} - 4x^{-5}y &= xe^x\\
\frac{d}{dx}\left(x^{-4}y\right) &= xe^x\\
x^{-4}y &= \int xe^x\,dx\\
y &= x^4\left(xe^x - e^x + C\right)\\
&= \underbrace{x^5e^x - x^4e^x}_{y_p(x)} + \underbrace{Cx^4}_{y_h(x)}.
\end{aligned}$$

Example. Consider the equation

$$\frac{dy}{dt} + y = 17\sin(4t).$$

The coefficient in front of y is 1, so our integrating factor is

$$\begin{aligned}
\mu(t) &= e^{\int\,dt}\\
&= e^t.
\end{aligned}$$

Multiplying through, we get

$$\begin{aligned}
\frac{d}{dt}\left(e^ty\right) &= 17e^t\sin(4t)\\
e^ty &= \int 17e^t\sin(4t)\,dt\\
&= 17\int e^t\sin(4t)\,dt\\
&= e^t\sin(4t) - 4e^t\cos(4t) + C && \text{Integration by Parts.}\\
y &= \underbrace{\sin(4t) - 4\cos(4t)}_{y_p(t)} + \underbrace{Ce^{-4t}}_{y_h(t)}.
\end{aligned}$$

Alternatively, we can solve this equation using a "lucky guess" method. Starting with the homogeneous equation, we find

$$\begin{aligned}\frac{dy}{dt} + y &= 0\\ \int \frac{1}{y}\,dy &= \int -1\,dt\\ \ln|y| &= -t + C\\ y_h &= Ce^{-t}.\end{aligned}$$

Now, looking at the non-homogeneous equation, we look at

$$\frac{dy}{dt} + y = 17\sin(4t).$$

The lucky guess we take is $y_p(t) = A\sin(4t) + B\cos(4t)$, yielding

$$\begin{aligned}\frac{dy_p}{dt} + y &= 17\sin(4t)\\ 4A\cos(4t) - 4B\sin(4t) + (A\sin(4t) + B\cos(4t)) &= 17\sin(4t)\\ (4A + B)\cos(4t) + (A - 4B)\sin(4t) &= 17\sin(4t).\end{aligned}$$

The system of equations yields $4A + B = 0$ and $A - 4B = 17$, meaning $A = 1$ and $B = -4$. Thus, $y_p(t)$ is

$$y_p(t) = \cos(4t) - 4\sin(4t).$$

Our general solution is, finally,

$$y(t) = \cos(4t) - 4\sin(4t) + Ce^{-t}.$$

Systems of First-Order Differential Equations

Modeling with Systems

Consider a second-order differential equation

$$y'' + 2y' + 3y + 5 = 0.$$

In order to solve these, we need to start by passing our second order differential equation into a system of first-order differential equations.

Definition (System of Differential Equations)**.** A pair of differential equations of the form

$$\begin{aligned}\frac{dx}{dt} &= f_2(x, y, 2)\\ \frac{dy}{dt} &= f_1(x, y, t)\end{aligned}$$

where f_1 and f_2 are functions of x, y, t defined on a common set S is called a system of two first-order differential equations.

Definition (Linear System of Differential Equations)**.** The system of two first order differential equations is linear if $f_1(x, y, t)$ and $f_2(x, y, t)$ are linear in x and y.

Example. If

$$\begin{aligned}\frac{dx}{dt} &= a_1(t)x + a_2(t)y + a_3(t)\\ \frac{dy}{dt} &= b_1(t)x + b_2(t)y + b_3(t)\end{aligned}$$

Example (Passing Second-Order Equation into System of First-Order Equations)**.** Let

$$2y'' - 5y' + y = 0.$$

To solve this equation, we define

$$\begin{aligned} x_1 &= y \\ x_2 &= y'. \end{aligned}$$

Taking derivatives, we get

$$\begin{aligned} x_1' &= y' \\ x_2' &= y''. \end{aligned}$$

Substituting, we get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= \frac{5}{2}x_2 - \frac{1}{2}x_1. \end{aligned}$$

Example (Passing Third-Order Differential Equation into System of First-Order Equations)**.** Let

$$y''' + 3y'' + 2y' - 5y = \sin(2t)$$

Let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y''. \end{aligned}$$

In general, when creating our system, we stop one derivative short of the order of the equation. Taking derivatives, we get

$$\begin{aligned} x_1' &= y' \\ x_2' &= y'' \\ x_3' &= y''', \end{aligned}$$

and substituting, we get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -3x_3'' - 2x_2' + 5x_1 + \sin(2t). \end{aligned}$$

Definition (Essentials for a System)**.** Let

$$\begin{aligned} \frac{dR}{dt} &= \underbrace{aR - bRF}_{f_1(R,F)} \\ \frac{dF}{dt} &= \underbrace{cRF - dF}_{f_2(R,F)}, \end{aligned}$$

where $a, b, c, d > 0$ are positive constants.[1]

- The equilibrium solutions are a pair of constant functions $R(t)$ and $F(t)$ that solve the system, or for which $f_1(R(t), F(t))$ and $f_2(R(t), F(t))$ are simultaneously equal to zero.

[1]This is known as the Lotka–Volterra model.

- A general solution to the system is a pair of functions $R(t)$ and $F(t)$ that, taken together, satisfy the system of equations.
- As t varies, the pair $(R(t), F(t))$ traces out the *solution curve* in the RF-plane. Its size and shape are determined by the initial condition, $(R(0), F(0))$.
- The phase plane for the system of differential equations is the RF-plane in which the vector
$$\begin{aligned} \mathbf{V} &= R'(t)\hat{\imath} + F'(t)\hat{\jmath} \\ &= f_1\left(R, F\right)\hat{\imath} + f_2\left(R, F\right)\hat{\jmath} \end{aligned}$$
is drawn at a grid of points (R_i, F_i).
- A phase portrait is the phase plane with enough solution curves to show how the solutions behave in every part of the plane.

Example (Solutions for a System of Differential Equations)**.** Consider the following Lotka–Volterra model.

$$\begin{aligned} \frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= -F + 0.9RF. \end{aligned}$$

We start with the equilibrium points.

$$\begin{aligned} \frac{dR}{dt} &= 0 \\ 0 &= R\left(2 - 1.2F\right), \\ R &= 0F & &= \frac{2}{1.2} \\ \frac{dF}{dt} &= 0 \\ 0 &= -F + 0.9RF \end{aligned}$$

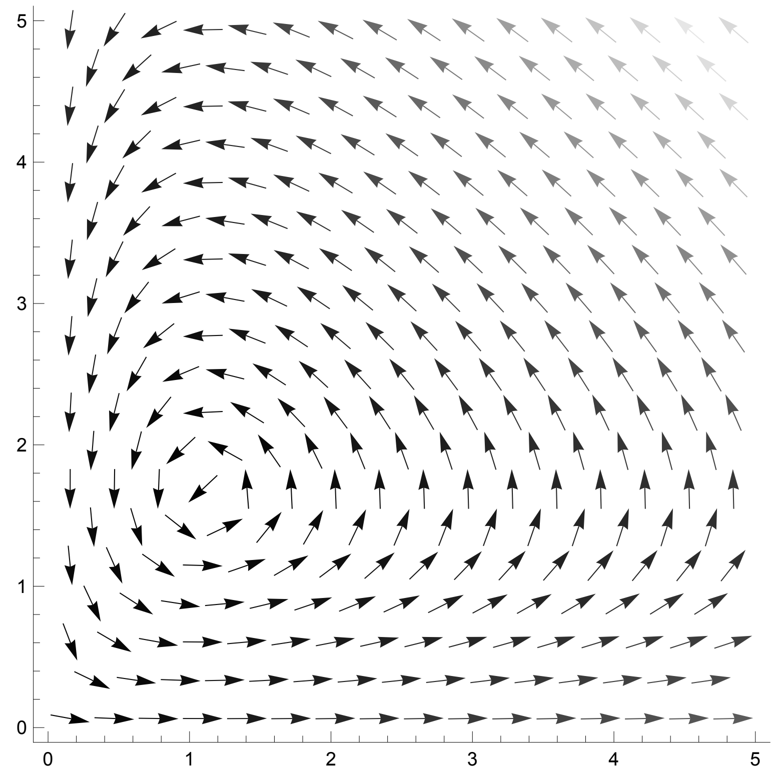
When $R = 0$

$$\begin{aligned} -F &= 0 \\ F &= 0. \end{aligned}$$

When $F = \frac{2}{1.2}$

$$\begin{aligned} 0 &= F\left(0.9R - 1\right) \\ R &= \frac{1}{0.9}. \end{aligned}$$

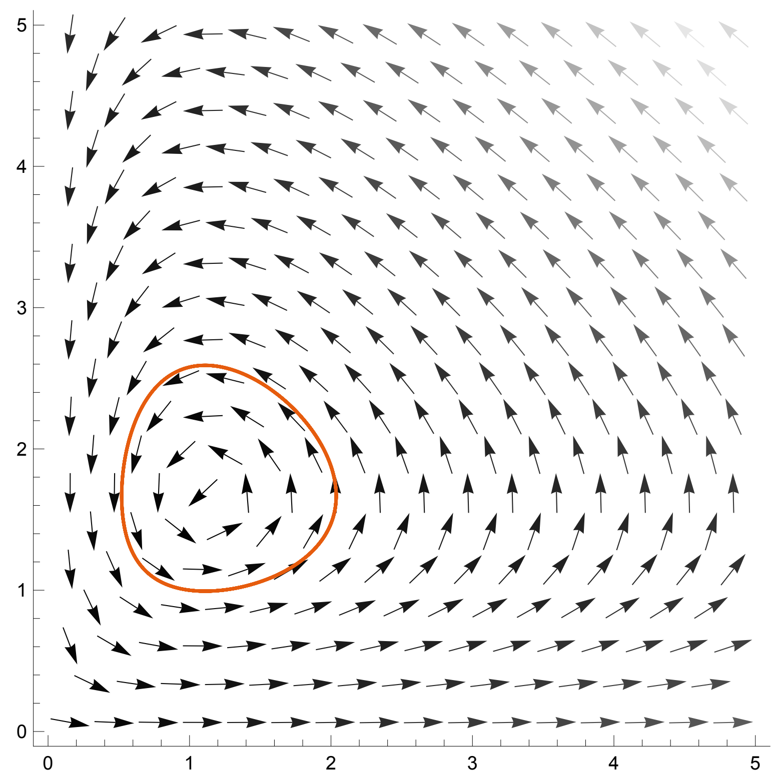
Thus, the equilibrium solutions are at $\left(\frac{1}{0.9}, \frac{2}{1.2}\right)$ and at $(0, 0)$. We can see that the phase portrait seems to approach a fixed point towards the point $(1.11, 1.67)$.



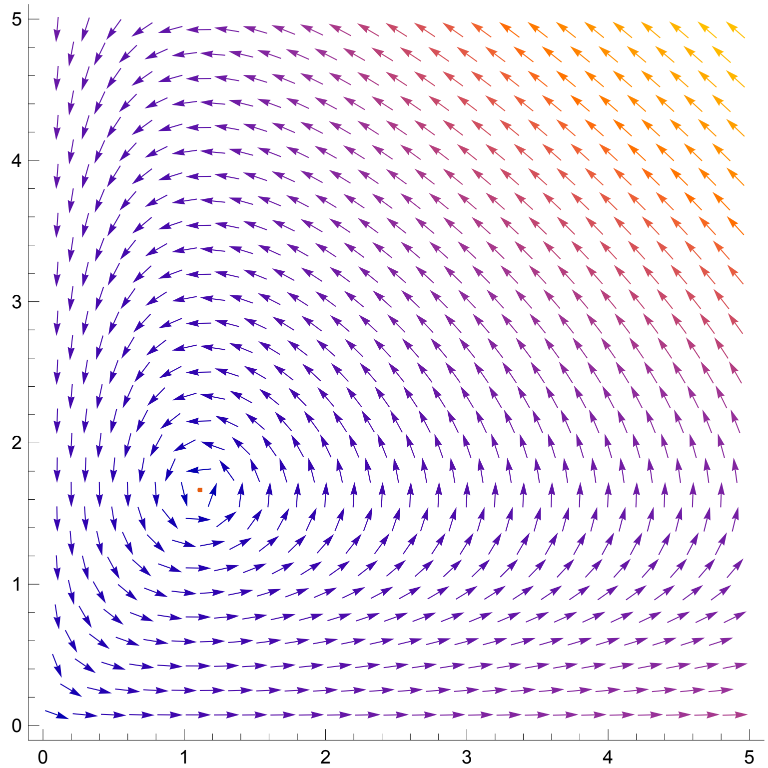
If we take the initial condition of $R(0) = 1$ and $F(0) = 1$, and send F to 0, we see that R tends toward infinity.

Meanwhile, if we send R to 0, we see that the number of tends to 0.

Using a computer, we can plot the solution to these initial conditions as seen below.



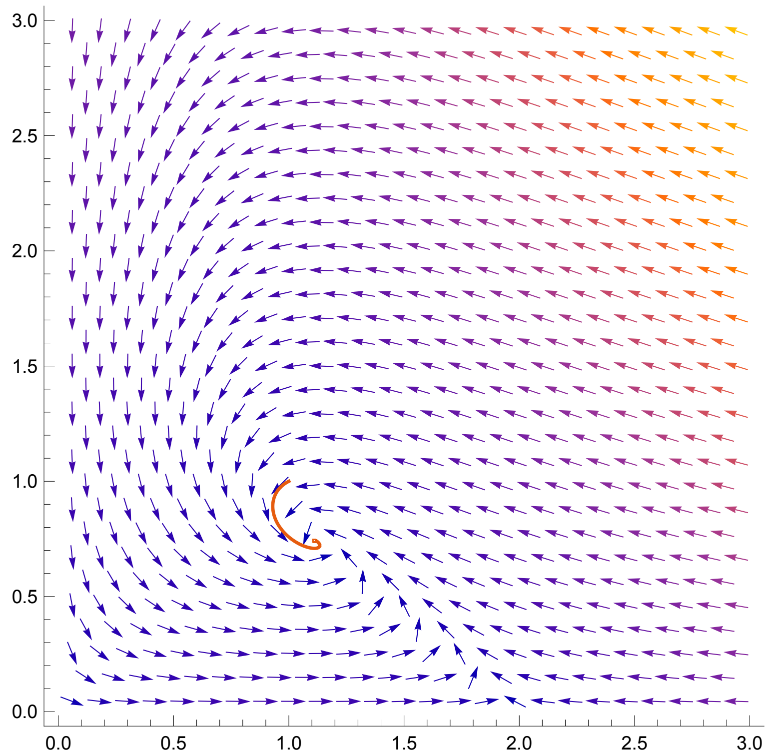
As we send the initial conditions closer to $(1.11, 1.67)$, we see that the solution curves seem to approach an equilibrium value.



Example (An Alternative Lotka–Volterra Model)**.** Consider the following altered model.

$$\frac{dR}{dt} = 2R\left(1 - \frac{R}{2}\right) - 1.2RF$$
$$\frac{dF}{dt} = -F + 0.9RF.$$

Note that, with this new initial condition, we see that the solution approaches an equilibrium point.



Solving for the equilibrium solutions, we get $R = F = 0$, $R = 2, F = 0$, and $R = \frac{1}{0.9}, F = 0.74$.

In particular, with this modified equation, we can see that as F tends to 0, the number of rabbits approaches a stable equilibrium of $R = 2$ as $t \to \infty$. Meanwhile, if R tends to 0, then F also tends to 0.

In general, this modified Lotka–Volterra model approaches some equilibrium point regardless of the starting condition, which differs greatly to the periodic solutions that were found in the original model.

Representing Systems of Equations with Vector Fields

Consider the system of differential equations,

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y). \end{aligned}$$

Note that this system is autonomous. We let

$$\begin{aligned} \vec{Y}(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ \vec{F}(x, y) &= \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}. \end{aligned}$$

We can see that

$$\begin{aligned} \frac{d\vec{Y}}{dt} &= \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} \\ &= \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \\ &= \vec{F}(x, y). \end{aligned}$$

Definition. An equilibrium point $\vec{Y}_0$ of the system $\frac{d\vec{Y}}{dt} = \vec{F}\left(\vec{Y}\right)$ is a point a which $\vec{F}\left(\vec{Y}\right) = \vec{0}$.

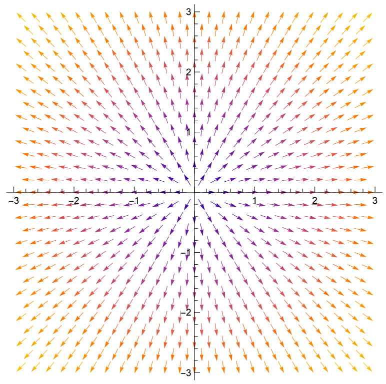
Example. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= x\\ \frac{dy}{dt} &= y.\end{aligned}$$

The corresponding vector field $\vec{Y}$ is

$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix},$$

which has the following vector field.

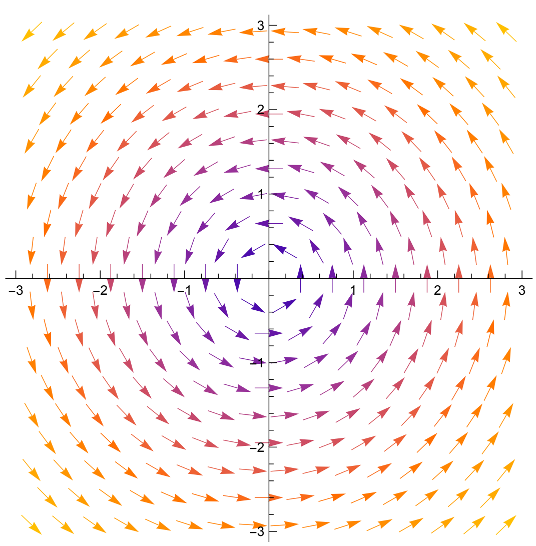


The direction field is the depiction of the vector field with where all vectors are of unit length.

Meanwhile, if we have the system

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x,\end{aligned}$$

the direction field looks as follows.



We can verify that

$$\vec{y}(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

is a solution by plugging it into the expression for $\frac{d\vec{y}}{dt}$.

Definition (Decoupled System)**.** A system of linear equations of the form

$$\frac{dx}{dt} = f(x)$$
$$\frac{dy}{dt} = f(y)$$

is known as a completely decoupled system.

Example. Considering the system

$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = -y,$$

we can solve to get

$$x(t) = k_1 e^{-2t}$$
$$y(t) = k_2 e^{-t}.$$

Thus, the general solution to the system is of the form

$$\vec{F}(t) = \begin{pmatrix} k_1 e^{-2t} \\ k_2 e^{-t} \end{pmatrix}.$$

Consider the IVP with

$$\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, we get

$$k_1 = 1$$
$$k_2 = 1,$$

meaning the solution to the particular IVP is

$$\vec{F}(t) = \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix}.$$

Note that we can write a closed form function, $x = y^2$, which allows us to draw a solution curve on the phase plane.

Example (A Partially Decoupled System)**.** Consider the system

$$\frac{dx}{dt} = 3x + 2y$$
$$\frac{dy}{dt} = -y.$$

Since $\frac{dx}{dt}$ depends on both x and y, but $\frac{dy}{dt}$ depends only on y, we can solve for y independent of x, and substitute.

$$y(t) = k_2 e^{-t}$$

Substituting, we have

$$\frac{dx}{dt} - 3x = 2k_2 e^{-t}.$$

Using the integrating factor $\mu(t) = e^{-3t}$, we have

$$\begin{aligned}
e^{-3t}\frac{dx}{dt} - 3xe^{-3t} &= 2k_2 e^{-4t} \\
\frac{d}{dt}\left(xe^{-3t}\right) &= 2k_2 e^{-4t} \\
xe^{-3t} &= -\frac{1}{2}k_2 e^{-4t} + C \\
x &= -\frac{1}{2}k_2 e^{-t} + k_1 e^{3t}.
\end{aligned}$$

Thus, we have

$$\vec{F}(t) = \begin{pmatrix} -\frac{1}{2}k_2 e^{-t} + k_1 e^{3t} \\ k_2 e^{-t} \end{pmatrix}.$$

Consider the initial condition

$$\vec{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then, substituting into our equation, we get $k_2 = 0$, $k_1 = 1$, so our solution is

$$\vec{F}(t) = \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}.$$

Euler's Method for Systems of Differential Equations

Consider a first order autonomous system

$$\begin{aligned}
\frac{dx}{dt} &= f(x,y) \\
\frac{dy}{dt} &= g(x,y),
\end{aligned}$$

with $x(0) = x_0$ and $y(0) = y_0$.

To approximate this solution, we can use the phase diagram. Converting to vector notation, we have

$$\begin{aligned}
\vec{F}(x,y) &= \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \\
\frac{d\vec{Y}}{dt} &= \vec{F}\left(\vec{Y}\right).
\end{aligned}$$

Starting from (x_0, y_0) in the phase diagram, we get the first step, $(x_1, y_1) = (x_0, y_0) + (\delta t)\,\vec{F}\,(x_0, y_0)$. Similarly, we inductively find

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) + \delta t \vec{F}\,(x_k, y_k).$$

Componentwise, we have

$$x_{k+1} = x_k + (\delta t)\, f(x_k, y_k)$$
$$y_{k+1} = y_k + (\delta t)\, g(x_k, y_k)\,.$$

Example. Consider the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - y$$

with the initial condition

$$\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and step size $\delta t = 0.25$. We hope to approximate the solution at $t = 0.5$.

k	t_k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$
0	0	1	1	−1	0
1	0.25	0.75	1	−1	−0.25
2	0.5	0.5	0.9375	—	—

Example. Consider the Lotka–Volterra model

$$\frac{dR}{dt} = 2R - 1.2RF$$
$$\frac{dF}{dt} = -F + 0.9RF$$

with initial condition $R(0) = 1$ and $F(0) = 1$. Using Euler's method with $\delta t = 1$, we can approximate the population of rabbits and foxes after 3 time steps.

k	t_k	R_k	F_k	$f(R_k, F_k)$	$g(R_k, F_k)$
0	0	1	1	0.8	−0.1
1	1	1.8	0.9	1.656	0.558
2	2	3.456	1.458	0.865	3.077
3	3	4.321	4.535	—	—

Existence and Uniqueness of Solutions

Just as with single equations, systems of differential equations have a corresponding existence and uniqueness theorem.

Theorem (Existence and Uniqueness for Systems of Ordinary Differential Equations)**:** Let

$$\frac{d\vec{Y}}{dt} = \vec{F}\left(t, \vec{Y}\right),$$

Suppose t_0 is an initial time and $\vec{Y}_0$ is an initial value.

If $\vec{F}$ is continuously differentiable,[II] then there is $\varepsilon > 0$ and a function $\vec{Y}(t)$ defined on $t_0 - \varepsilon < t < t_0 + \varepsilon$ such that $\vec{Y}(t)$ satisfies the initial value problem,

$$\frac{d\vec{Y}}{dt} = \vec{F}\left(t, \vec{Y}\right),\ \vec{Y}(t_0) = \vec{Y}_0.$$

This solution is unique for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

[II] We say $\vec{F}$ is continuously differentiable if all the partial derivatives of f and g, $f_t, f_x, f_y, g_t, g_x, g_y$ exist and are continuous on some open subset of txy-space, where (t_0, x_0, y_0) is contained in this open subset.

Example. Consider the following system of equations.

$$\begin{aligned}\frac{dx}{dt} &= x^2 + 1\\ \frac{dy}{dt} &= 1,\end{aligned}$$

under the initial condition $(x_0, y_0) = (0, 0)$.

To verify the existence and uniqueness of a solution, we take the partial derivatives

$$\begin{aligned}\frac{\partial f}{\partial t} &= 0\\ \frac{\partial f}{\partial x} &= 2x\\ \frac{\partial f}{\partial y} &= 0\\ \frac{\partial g}{\partial t} &= 0\\ \frac{\partial g}{\partial x} &= 0\\ \frac{\partial g}{\partial y} &= 0.\end{aligned}$$

Thus, since all partial derivatives are continuous in a neighborhood around $(0, 0, 0)$ in txy-space, there is a unique solution for this equation.

Solving the equation, we get

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \,dt\\ \arctan(x) &= t + C\\ x &= \tan\left(t + C\right)\\ C &= 0\\ x(t) &= \tan t\\ y(t) &= t,\end{aligned}$$

thus, we get the solution.

$$\vec{Y}(t) = \begin{pmatrix}\tan(t)\\ t\end{pmatrix}.$$

Example. Consider an autonomous two-dimensional system

$$\frac{d\vec{Y}}{dt} = \vec{F}\left(\vec{Y}\right),$$

of the form

$$\begin{aligned}\frac{dx}{dt} &= f\left(x, y\right)\\ \frac{dy}{dt} &= g\left(x, y\right).\end{aligned}$$

We can draw two conclusions from this case.

(1) Unless a solution is a simple closed curve on the phase plane, solutions cannot intersect themselves at any point.

(2) If two solution curves intersect at any point on the xy-plane, they are the same solution.

Lemma: Suppose $\vec{Y}_1$ is a solution to the autonomous system

$$\frac{d\vec{Y}}{dt} = \vec{F}\left(\vec{Y}\right).$$

Then, $\vec{Y}_1\left(t - t_0\right)$ is a solution to the system for any constant $t - t_0$.

Proof. Given $\vec{Y}_1(t)$ is a solution, so

$$\frac{d}{dt}\left(\vec{Y}_1\left(t\right)\right) = \vec{F}\left(\vec{Y}_1(t)\right).$$

We want to show that

$$\frac{d}{dt}\left(\vec{Y}_1\left(t - t_0\right)\right) = \vec{F}\left(\vec{Y}_1\left(t - t_0\right)\right).$$

Let $\tau = t - t_0$. Then,

$$\begin{aligned}\frac{d}{dt}\left(\vec{Y}_1(\tau)\right) &= \frac{d}{d\tau}\left(\vec{Y}_1(\tau)\right)\frac{d\tau}{dt}\\ &= \frac{d}{d\tau}\left(\vec{Y}_1\left(\tau\right)\right)\\ &= \vec{F}\left(\vec{Y}\left(\tau\right)\right).\end{aligned}$$

□

Theorem: Let $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ be solutions to

$$\frac{d\vec{Y}}{dt} = \vec{F}\left(\vec{Y}\right).$$

If $\vec{Y}_1\left(t_1\right) = \vec{Y}_2\left(t_2\right) = \vec{Y}_0$ for some $t_1 \neq t_2$, Then, $\vec{Y}_2\left(t\right) = \vec{Y}_1\left(t - c\right)$ for some $c = t_2 - t_1$.

Proof. Since $\vec{Y}_1(t)$ is a solution, $\vec{Y}_1(t - c)$ is a solution. Specifically,

$$\vec{Y}_1\left(t - c\right) = \vec{Y}_1\left(t - \left(t_2 - t_1\right)\right).$$

We can see that

$$\begin{aligned}\vec{Y}_1\left(t_2 - c\right) &= \vec{Y}_1\left(t_1\right)\\ &= Y_0\\ &= \vec{Y}_2\left(t_2\right).\end{aligned}$$

Thus, $Y_1\left(t - c\right) = Y_2\left(t\right)$ for all t (by the uniqueness theorem). □

For an autonomous system of equations with uniqueness, there are thus three types of solution curves on the phase plane.

- Simple closed curves;
- simple open curves;
- singular points (equilibrium solutions).

Here, "simple" refers to the fact that the curves do not intersect themselves.

Solving Systems of Linear Differential Equations

Consider a system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy.$$

This is what is known as a linear system with constant coefficients, where the coefficients are a, b, c, d. Note that this system is autonomous, so solution curves do not intersect. We can also express this system as follows.

$$\underbrace{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}}_{\frac{d\vec{Y}}{dt}} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{Y}}.$$

Thus, this system is of the form

$$\frac{d\vec{Y}}{dt} = A\vec{Y}.$$

Theorem: If A is a nonsingular matrix, then the only equilibrium point for the linear system

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

is the origin.

Proof. Note if $\vec{Y}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an equilibrium solution, then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

meaning

$$ax_0 + by_0 = 0$$
$$cx_0 + dy_0 = 0.$$

In the first equation, solving for x_0, we have

$$x_0 = -\frac{b}{a} y_0,$$

and substituting,

$$c\left(-\frac{b}{a}\right) y_0 + dy_0 = 0$$
$$(ad - bc)\, y_0 = 0.$$

Since $\det(A) \neq 0$, $ad - bc \neq 0$. Thus, $y_0 = x_0 = 0$. $\square$

There is also a linearity principle for systems of equations.

Theorem (Linearity Principle)**:** Let $\frac{d\vec{Y}}{dt} = A\vec{Y}$ be a linear system of equations.

(1) If $\vec{Y}(t)$ is a solution, and k is any constant, then $k\vec{Y}(t)$ is a solution.

(2) If $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions, then $\vec{Y}_1(t) + \vec{Y}_2(t)$ is also a solution.

Example.

(1)

$$\begin{aligned} \frac{d}{dt}\left(k\vec{Y}(t)\right) &= k\frac{d\vec{Y}}{dt} \\ &= k\left(A\vec{Y}\right) \\ &= A\left(k\vec{Y}\right). \end{aligned}$$

(2)

$$\begin{aligned} \frac{d}{dt}\left(\vec{Y}_1(t) + \vec{Y}_2(t)\right) &= \frac{d\vec{Y}_1}{dt} + \frac{d\vec{Y}_2}{dt} \\ &= A\vec{Y}_1 + A\vec{Y}_2 \\ &= A\left(\vec{Y}_1 + \vec{Y}_2\right). \end{aligned}$$

Essentially, all linear combinations of solutions to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ are solutions.

Example. Consider

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\vec{Y},$$

with $\vec{Y}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Suppose we are given two solutions,

$$\begin{aligned} \vec{Y}_1(t) &= \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \\ \vec{Y}_2(t) &= \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}. \end{aligned}$$

Then, we can see that any linear combination of $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$. In particular, we need to find the particular linear combination such that $\left(a\vec{Y}_1 + b\vec{Y}_2\right)(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Recall: Recall that two vectors,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

are linearly independent in $\mathbb{R}^2$ if they do not lie on the same line through the origin. Alternatively, it is not the case that

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

for some $a \neq 0$, or that

$$s_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + s_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if $s_1 = s_2 = 0$.

Definition (Wronskian)**.** In general, the Wronskian of $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ is

$$W\left(\vec{Y}_1, \vec{Y}_2\right) = \det\begin{pmatrix} Y_{1x}(t) & Y_{2x}(t) \\ Y_{1y}(t) & Y_{2y}(t)\end{pmatrix}$$

Our Wronskian is

$$\begin{aligned}\det\begin{pmatrix} e^{2t} & -e^{-4t} \\ 0 & 2e^{-4t}\end{pmatrix} &= 2e^{-2t} \\ &\neq 0\end{aligned}$$

Thus, we can see that $\vec{Y}_1$ and $\vec{Y}_2$ are linearly independent.

In general, if $W\left(\vec{Y}_1, \vec{Y}_2\right)(t_0) \neq 0$ for some t_0, then $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are linearly independent.

Theorem (General Solution of Systems of Linear Equations)**:** Let $\vec{Y}_1(t)$, $\vec{Y}_2(t)$ be solutions of the linear system

$$\frac{d\vec{Y}}{dt} = A\vec{Y}.$$

If $\vec{Y}_1(0)$ and $\vec{Y}_2(0)$ are linearly independent, then for any initial condition $\vec{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, we can find constants k_1 and k_2 such that $k_1\vec{Y}_1(t) + k_2\vec{Y}(t)$ is the solution to the initial value problem.

The general solution is of the form $k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t)$.

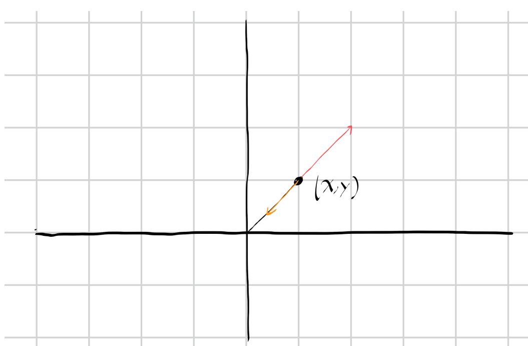
Straight Line Solutions

Let

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\vec{Y}.$$

We begin by examining the corresponding vector field for this system, where we have $\vec{F}\left(\vec{Y}\right) = A\vec{Y}$. We are specifically looking for straight line solutions in the phase plane.

At a point (x, y), the direction field for $\frac{d\vec{Y}}{dt}$ must be in the same (or opposite) direction to the vector pointing towards (x, y).



Mathematically, we want to find a nonzero vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and a number λ such that

$$A\vec{v} = \lambda\vec{v}.$$

- If $\lambda > 0$, then at $\vec{v}$, the vector field points away from the origin.
- If $\lambda < 0$, then at $\vec{v}$, the vector field points towards the origin.

The number λ is the eigenvalue of A and the vector $\vec{v}$ is the eigenvector corresponding to λ.

Note:

(1) Given A, if $\vec{v}$ is an eigenvector for the eigenvalue λ, then so too is any $k\vec{v}$ for $k \in \mathbb{R}$.

(2) Given an eigenvector $\vec{v}$ for λ, the entire line through the origin span $(\vec{v})$ are all eigenvectors for λ.

Example. Consider the differential equation

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \vec{Y}.$$

To find the eigenvalue of A, we take

$$\begin{aligned} (A - \lambda I)\vec{v} &= 0 \\ \begin{pmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0, \end{aligned}$$

meaning we must have

$$\det(A - \lambda I) = 0.$$

to find non-trivial solutions. Calculating, we have

$$\begin{aligned} \det \begin{pmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{pmatrix} &= (4-\lambda)(-1-\lambda) + 6 \\ &= (\lambda - 4)(\lambda + 1) + 6 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 2)(\lambda - 1). \end{aligned}$$

We have eigenvalues of $\lambda_2 = 2$ and $\lambda_1 = 1$.

To find the corresponding eigenvectors, we start with $\lambda_1 = 1$.

$$\begin{aligned} \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 3x - 2y &= 0 \\ 3x - 2y &= 0, \end{aligned}$$

meaning we have $y = \frac{3}{2}x$, or $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Similarly, we have

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

meaning we have $2x = 2y$, so $x = y$. Thus, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Definition (Straight-Line Solution)**.** If λ is an eigenvalue associated with eigenvector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then,

$$\vec{Y}_\lambda(t) = e^{\lambda t}\vec{v}$$

is a solution of $\frac{d\vec{Y}}{dt} = A\vec{Y}$.

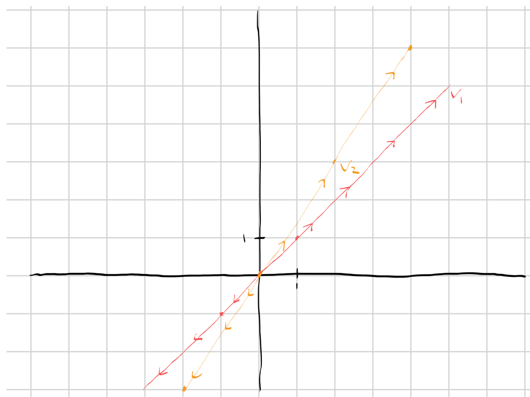
If A has two distinct eigenvalues, λ_1 and λ_2, then the corresponding eigenvectors are linearly independent, meaning there are two straight line solutions on the phase plane.

Exercise: Verify that the straight line solutions of the previous example are indeed straight line solutions.

Thus, we have $\vec{Y}_1(t) = e^{t}\begin{pmatrix}2\\3\end{pmatrix}$ and $\vec{Y}_2(t) = e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$. Thus, the general solution is

$$\begin{aligned}\vec{Y}(t) &= k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t)\\ &= \begin{pmatrix}2k_1e^{t} + k_2e^{2t}\\ 3k_1e^{t} + k_2e^{2t}\end{pmatrix}.\end{aligned}$$

Examining the phase plane, we see the following.



Theorem: Suppose A has a real eigenvalue λ and associated vector $\vec{v}$.

Then, the linear system

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

has the straight line solution

$$\vec{Y}(t) = e^{\lambda t}\vec{v}.$$

Moreover, if λ_1 and λ_2 are distinct real eigenvalues of A with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$, the straight line solutions

$$\begin{aligned}\vec{Y}_1(t) &= e^{\lambda_1 t}\vec{v}_1\\ \vec{Y}_2(t) &= e^{\lambda_2 t}\vec{v}_2,\end{aligned}$$

are linearly independent, so the general solution to the system is

$$\vec{Y}(t) = k_1e^{\lambda_1 t}\vec{v}_1 + k_2e^{\lambda_2 t}\vec{v}_2.$$

Phase Portraits for Linear Systems with Real Eigenvalues

Consider

$$\begin{aligned}\frac{d\vec{Y}}{dt} &= A\vec{Y},\\ A &= \begin{pmatrix}a & b\\ c & d\end{pmatrix}.\end{aligned}$$

Thus, we have the general eigenvalues

$$\begin{aligned}\det\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix} &= 0\\ \lambda^2-(a+d)\lambda+(ad-bc) &= 0\\ \left(\lambda-\frac{a+d}{2}\right)^2 &= \frac{4bc+(a-d)^2}{4}\\ \lambda &= \frac{a+d}{2}\pm\frac{\sqrt{4bc+(a-d)^2}}{2}.\end{aligned}$$

- We have complex eigenvalues if
$$4bc+(a-d)^2<0.$$
- We have real eigenvalues if
$$4bc+(a-d)^2>0.$$
- We have repeated eigenvalues if
$$4bc+(a-d)^2=0$$

Note that for real eigenvalues with the same sign, the result

$$\vec{Y}(t)=k_1e^{\lambda_1 t}\vec{v}_1+k_2e^{\lambda_2 t}\vec{v}_2$$

provides that all solution curves are parallel to the "faster" growing/shrinking eigenvector, and are tangent to the "slower" growing/shrinking eigenvector.

If we have a saddle (that is, one positive and one negative eigenvalue), then all solution curves are tangent to both eigenvectors, going in the direction of the positive/negative eigenvalue depending on its respective eigenvector.

Phase Portraits for Systems with Complex Eigenvalues

We begin turning our attention to the case when

$$4bc+(a-d)^2<0.$$

Example. Consider the system

$$\frac{d\vec{Y}}{dt}=\begin{pmatrix}2 & 2\\ -4 & 6\end{pmatrix}\vec{Y}.$$

Finding the eigenvalues, we take

$$\begin{aligned}\det(A-\lambda I) &= 0\\ (2-\lambda)(6-\lambda)+8 &= 0\\ \lambda^2-8\lambda+20 &= 0\\ (\lambda-4)^2 &= -4\\ \lambda &= 4\pm 2i.\end{aligned}$$

Solving for the eigenvectors, we have

$$\begin{aligned}
\begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= (4+2i)\begin{pmatrix} x \\ y \end{pmatrix}\\
(-2-2i)\,x + 2y &= 0\\
-4x + (2-2i)\,y &= 0\\
y &= (1+i)\,x\\
\vec{v}_1 &= \begin{pmatrix} 1 \\ 1+i \end{pmatrix}\\
\begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= (4-2i)\begin{pmatrix} x \\ y \end{pmatrix}\\
(-2+2i) + 2y &= 0\\
-4x + (2+2i)\,y &= 0\\
y &= (1-i)\,x\\
\vec{v}_2 &= \begin{pmatrix} 1 \\ 1-i \end{pmatrix}.
\end{aligned}$$

Using our system for straight-line solutions, we take

$$\vec{Y}_1(t) = e^{(4+2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

We are dealing with an issue here — namely, our equation exists in the real numbers, but we have a complex solution. To ameliorate this, we use the identity $e^{a+bi} = e^{a}(\cos b + i\sin b)$, to separate the equation to find

$$\begin{aligned}
\vec{Y}_1(t) &= e^{4t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\\
&= e^{4t}\cos(2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix} + ie^{4t}\sin(2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\\
&= e^{4t}\cos(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + ie^{4t}\cos(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + ie^{4t}\sin(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{4t}\sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\\
&= \underbrace{e^{4t}\begin{pmatrix} \cos(2t) \\ \cos(2t)-\sin(2t) \end{pmatrix}}_{\operatorname{Re}\left(\vec{Y}_1(t)\right)} + i\underbrace{\left(e^{4t}\begin{pmatrix} \sin(2t) \\ \cos(2t)+\sin(2t) \end{pmatrix}\right)}_{\operatorname{Im}\left(\vec{Y}_1(t)\right)}
\end{aligned}$$

Note that $\operatorname{Re}\left(\vec{Y}_1(0)\right)$ and $\operatorname{Im}\left(\vec{Y}_1(0)\right)$ are linearly independent, and $\operatorname{Re}\left(\vec{Y}_1(t)\right)$ and $\operatorname{Im}\left(\vec{Y}_1(t)\right)$ are real solutions.

The general solution is, thus

$$\vec{Y}(t) = k_1\operatorname{Re}\left(\vec{Y}_1(t)\right) + k_2\operatorname{Im}\left(\vec{Y}_1(t)\right).$$

Theorem (Complex Eigenvalues and Solutions)**:** Suppose $\vec{Y}(t)$ is a complex-valued solution to the linear system of differential equation

$$\frac{d\vec{Y}}{dt} = A\vec{Y}.$$

Suppose

$$\vec{Y}(t) = \text{Re}\left(\vec{Y}(t)\right) + i\,\text{Im}\left(\vec{Y}(t)\right),$$

where both real and imaginary parts are real-valued functions of t. Then, $\text{Re}\left(\vec{Y}(t)\right)$ and $\text{Im}\left(\vec{Y}(t)\right)$ are both solutions to $\frac{d\vec{Y}}{dt} = A\vec{Y}$. The general solution is of the form

$$\vec{Y}(t) = k_1\,\text{Re}\left(\vec{Y}(t)\right) + k_2\,\text{Im}\left(\vec{Y}(t)\right).$$

Example. Let $\lambda = \alpha \pm i\beta$. The complex-valued solution is

$$\vec{Y}(t) = e^{(\alpha+i\beta)t}\vec{v},$$

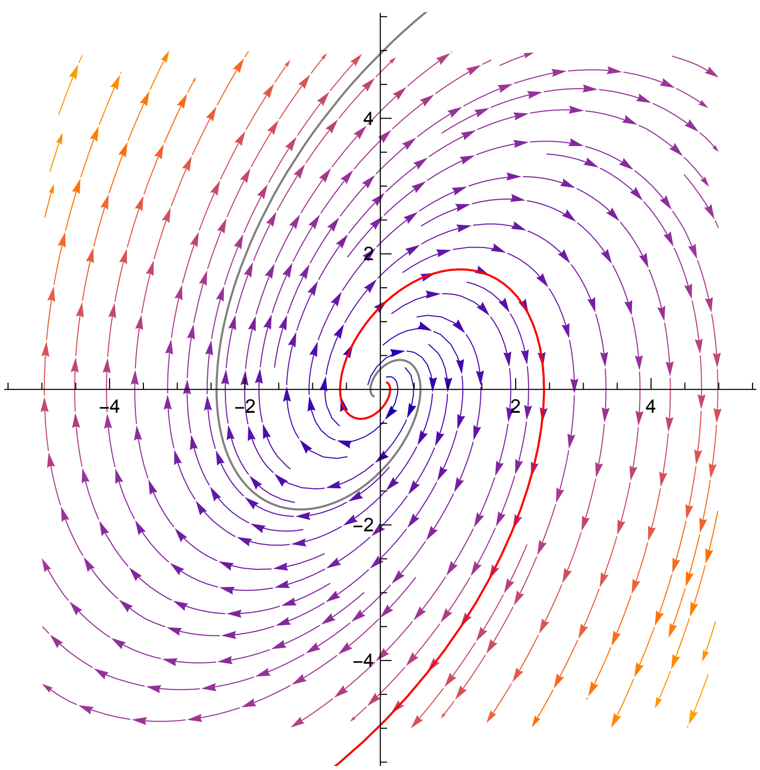
where $\vec{v}$ is the eigenvector corresponding to $\lambda = \alpha + i\beta$.

Splitting up, we get

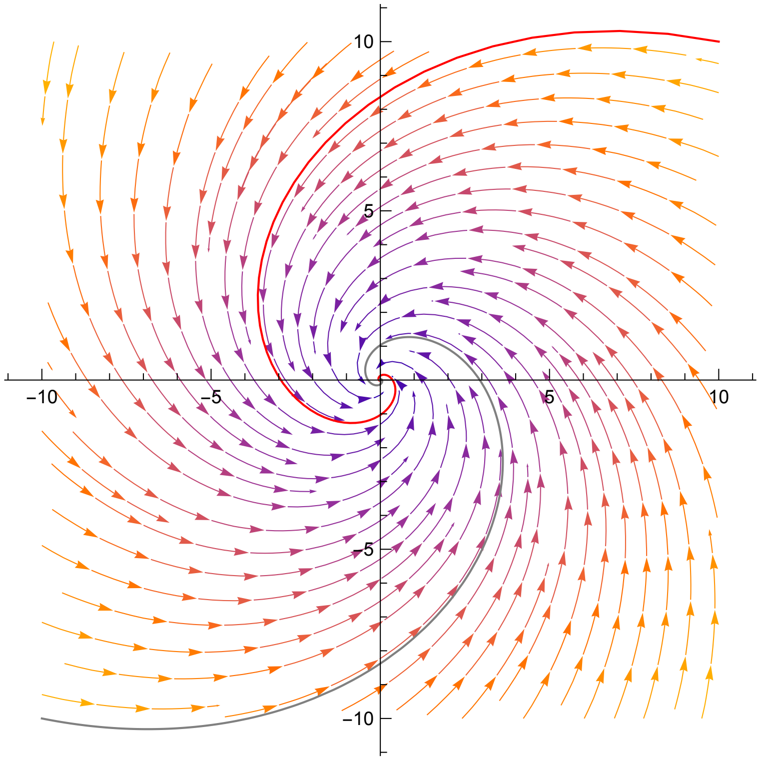
$$\begin{aligned}\vec{Y}(t) &= e^{\alpha t}e^{i\beta t}v\\ &= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)\vec{v}.\end{aligned}$$

Since $\vec{v}$ is constant, the long-term behavior of the solution curves is affected by t.

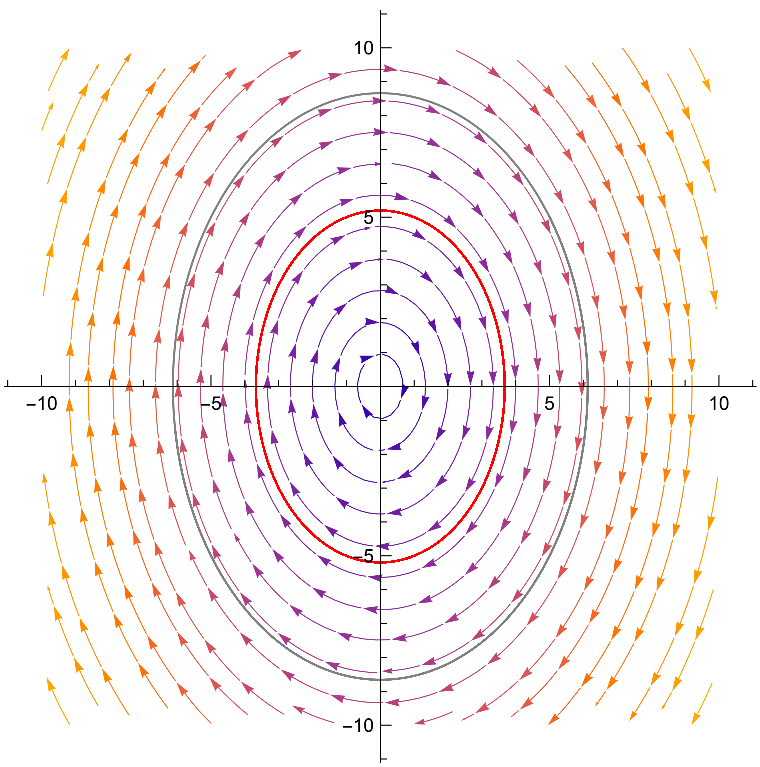
- If $\alpha > 0$, then $e^{\alpha t} \to \infty$ as $t \to \infty$, so our solutions move away from the origin. If $\beta \neq 0$, then they do so in a spiral pattern.



- If $\alpha < 0$, then $e^{\alpha t} \to 0$ as $t \to \infty$, so our solutions move toward the origin. If $\beta \neq 0$, they do so in a spiral pattern.



- If $\alpha = 0$ and $\beta \neq 0$, we have a purely periodic solution.



Recall that when we find the eigenvalues of a matrix A, we see that

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\mathrm{tr}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)} = 0.$$

Solving using the quadratic formula we get

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

We can see that the real part of λ is

$$\mathrm{Re}(\lambda) = \frac{a+d}{2}.$$

This is a quick way to evaluate a whether the origin is a spiral source vs. spiral sink.

Phase Portraits for Repeated Eigenvalues

Consider

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\vec{Y}$$

our linear system. There are repeated eigenvalues of $\lambda = -2$. The corresponding eigenvector is

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We can see the straight line solution of the form

$$\vec{Y}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

To see the general solutions, we start by converting into the system.

$$\begin{aligned} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= -2y. \end{aligned}$$

Since this system is partially decoupled, we are indeed able to solve it.

$$\begin{aligned} y(t) &= k_1 e^{-2t} \\ \frac{dx}{dt} &= -2x + k_1 e^{-2t} \\ \frac{dx}{dt} + 2x &= k_1 e^{-2t} \\ e^{2t}\frac{dx}{dt} + 2xe^{2t} &= k_1 \\ \frac{d}{dt}\left(xe^{2t}\right) &= k_1 \\ xe^{2t} &= k_1 t + k_2 \\ x &= k_1 t e^{-2t} + k_2 e^{-2t}. \end{aligned}$$

Our general solution is, thus

$$\vec{Y}(t) = \begin{pmatrix} k_1 t e^{-2t} + k_2 e^{-2t} \\ k_2 e^{-2t} \end{pmatrix}.$$

Including the initial condition of $x(0) = x_0$, $y(0) = y_0$, we get

$$\begin{aligned}\vec{Y}(t) &= \begin{pmatrix} y_0 t e^{-2t} + x_0 e^{-2t} \\ y_0 e^{-2t} \end{pmatrix} \\ &= te^{-2t} \underbrace{\begin{pmatrix} y_0 \\ 0 \end{pmatrix}}_{\vec{v}} + e^{-2t} \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{\vec{Y}(0)}.\end{aligned}$$

Theorem: Let $\frac{d\vec{Y}}{dt} = A\vec{Y}$ be a linear system where A is a 2×2 matrix with a repeated real eigenvalue λ, but with only one eigenline. Then, the general solution is the following.

$$\vec{Y}(t) = e^{\lambda t}\vec{v}_0 + te^{\lambda t}\vec{v}_1,$$

where $\vec{v}_0$ is an arbitrary initial condition, and $\vec{v}_1$ is determined as follows:

$$\vec{v}_1 = (A - \lambda I)\,\vec{v}_0.$$

If $\vec{v}_1 = 0$, then $\vec{v}_0$ is an eigenvector, and $\vec{Y}(t)$ is a straight line solution. Otherwise, $\vec{v}_1$ is an eigenvector.

Example. Let

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \vec{Y}.$$

The eigenvalues of this matrix are $\lambda = 1$. We let

$$\vec{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and

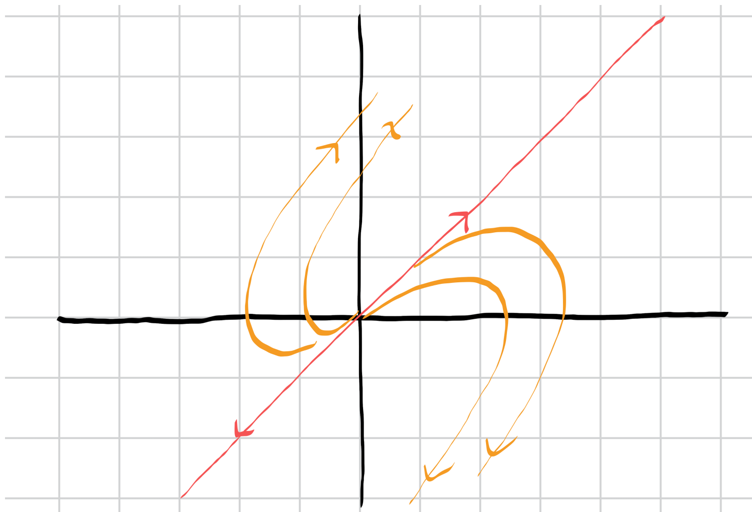
$$\begin{aligned}\vec{v}_1 &= (A - \lambda I)\,\vec{v}_0 \\ &= \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} -2x_0 + 2y_0 \\ -2x_0 + 2y_0 \end{pmatrix}.\end{aligned}$$

Thus, our solution is

$$\vec{Y}(t) = e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^t \begin{pmatrix} -2x_0 + 2y_0 \\ -2x_0 + 2y_0 \end{pmatrix}.$$

To create solutions, we let the initial condition change over time, and find the desired eigenvectors.

For instance, if our initial condition is $\vec{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\vec{v}_1 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, and our solutions look as follows.



In this scenario, we say the origin is a nodal source.

Example (Qualitative Analysis)**.** Note that we can take

$$\begin{aligned}\vec{Y}(t) &= e^{\lambda t}\vec{v}_0 + te^{\lambda t}\vec{v}_1\\ &= e^{\lambda t}\left(\vec{v}_1 + t\vec{v}_1\right).\end{aligned}$$

As $t \to \infty$, the solution asymptotically approaches the direction $\vec{v}_1$. This is more readily apparent if $\lambda > 0$.

For $\lambda < 0$, we have nodal sinks. To find the initial direction for this case, we plug in the initial condition and evaluate the vector field at the particular point.

Example (Scaled Identity)**.** Consider the equation

$$\frac{d\vec{Y}}{dt} = \underbrace{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}_{A} \vec{Y}.$$

The eigenvalues of A are repeated with value a. Note that since this is a scaled identity matrix, the eigenvectors are every vector in the plane.

Thus, we can select linearly independent eigenvectors. We let

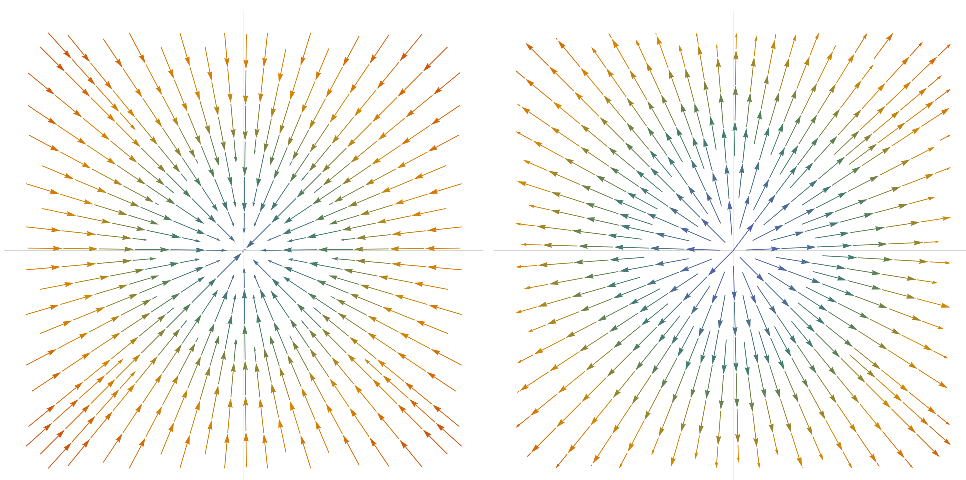
$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Since every vector is an eigenvector, every solution is a straight line solution. The general solution is

$$\begin{aligned} \vec{Y}(t) &= k_1 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{at} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} k_1 e^{at} \\ k_2 e^{at} \end{pmatrix}. \end{aligned}$$

In this case, our phase portrait has its phase lines as follows.



We say these are sink star and source star points respectively.

Example. Consider the equation

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \vec{Y},$$

which has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, with eigenvectors

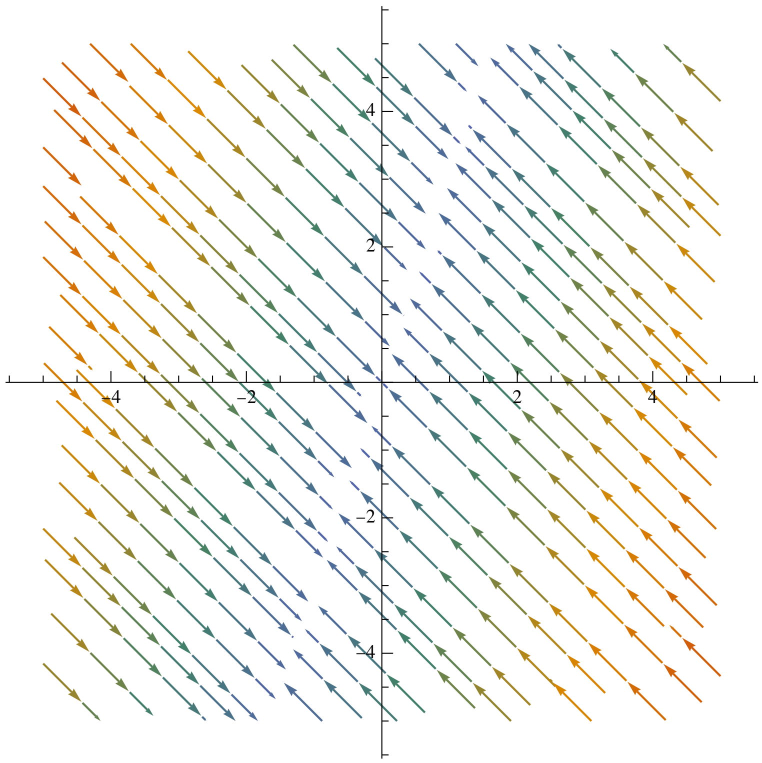
$$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The general solution is of the form

$$\vec{Y}(t) = k_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + k_2 e^{-4t} \begin{pmatrix} -1 \\ 1-1 \\ 1 \end{pmatrix}.$$

We can see that if $k_2 = 0$, then we have equilibrium solutions. Thus, the eigenline for $\vec{v}_1$ consist of all the equilibrium solutions. The phase diagram looks as follows.



Example. Consider the system

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \vec{Y},$$

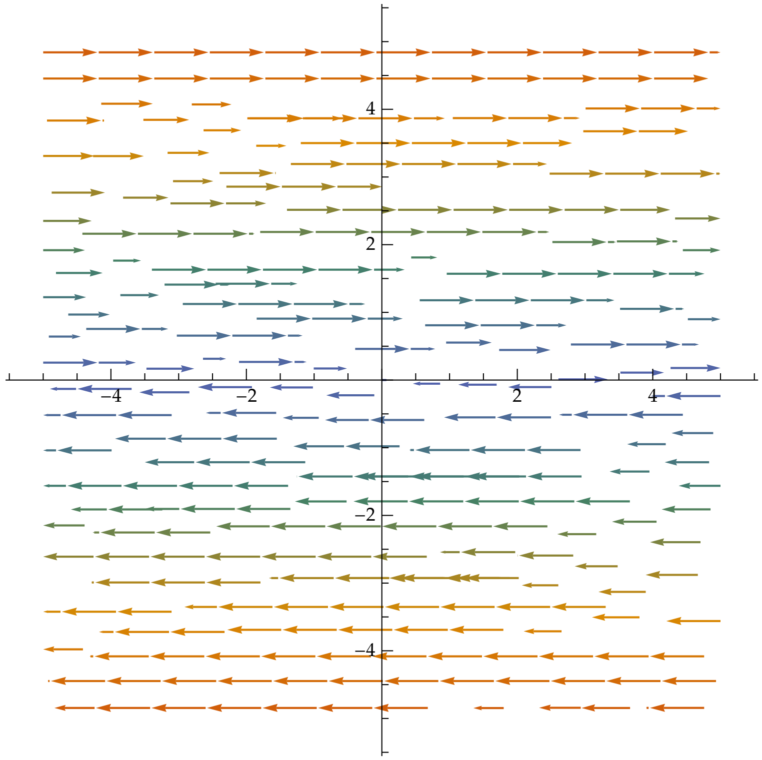
with repeated eigenvalues $\lambda = 0$. We only have one eigenvector

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $\vec{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Then, $v_1 = \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}$. Thus, our general solution is

$$\vec{Y}(t) = k_1 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + 2t \begin{pmatrix} y_0 \\ 0 \end{pmatrix}.$$

The phase diagram looks as follows.



Example. Consider the system

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{Y}.$$

Then, there is one repeated eigenvalue of 0, and every point is an equilibrium solution.

The Trace-Determinant Plane

Recall that in the expression

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{Y},$$

the eigenvalues of A are the solutions to

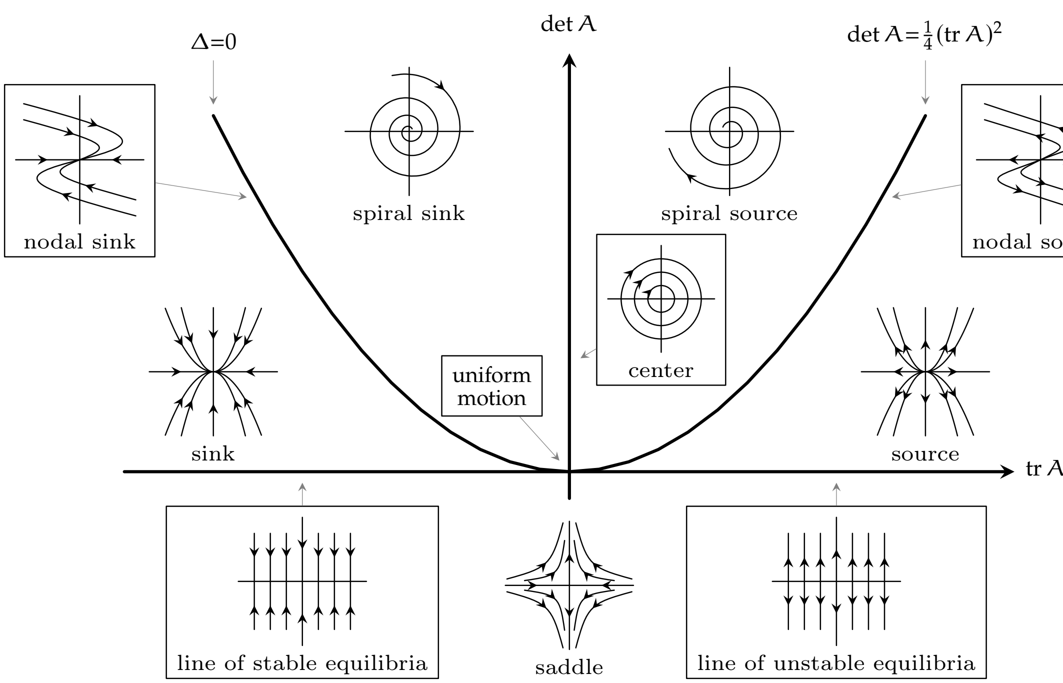
$$\begin{aligned} \lambda^2 - (a+d)\lambda + (ad - bc) &= 0 \\ \lambda^2 - T\lambda + D &= 0 \\ \lambda &= \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}. \end{aligned}$$

- If we have complex eigenvalues with real part $\frac{T}{2}$,
  - for $T < 0$, we have a spiral sink;
  - for $T > 0$, we have a spiral source;
  - for $T = 0$, we have a center.
- If we have repeated eigenvalues, then
  - for $T < 0$, we have a nodal sink (or a sink star point, if every vector is an eigenvector);

  - for $T > 0$, we have a nodal source (or a source star point, if every vector is an eigenvector);
  - for $T = 0$, we have the special cases discussed earlier.

- If we have two real eigenvalues,
  - for $T > 0$ and $D > 0$, we have two positive eigenvalues,[III] so we have a source;
  - for $T > 0$ and $D = 0$, we have one zero eigenvalue and one positive eigenvalue, so we have a line of unstable equilibria;
  - for $T > 0$ and $D < 0$, we have one positive eigenvalue and one negative eigenvalue, so we have a saddle;
  - for $T < 0$ and $D > 0$, we have two negative eigenvalues, so we have a sink;
  - for $T < 0$ and $D = 0$, we have one zero eigenvalue and one negative eigenvalue, so we have a line of stable equilibria;
  - for $T < 0$ and $D < 0$, we have one positive eigenvalue and one negative eigenvalue, so we have a saddle.

The following diagram[IV] visualizes the trace-determinant plane.



Example (Bifurcations for Systems of Equations)**.** Consider the family of systems (where a is a parameter)

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}\vec{Y}.$$

We find that

$$\begin{aligned} \det A &= 2a \\ \operatorname{tr} A &= -2. \end{aligned}$$

[III]Recall that the trace is the sum of the eigenvalues and the determinant is the product of the eigenvalues.
[IV]Taken from LaTeXStack Exchange.

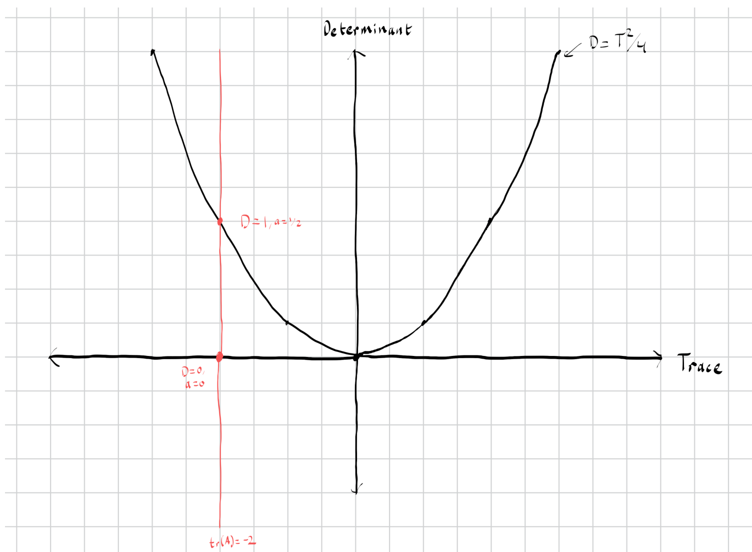
Using the trace-determinant plane, we find the equation

$$\det A = \frac{1}{4}(\operatorname{tr} A)^2$$
$$2a = 1$$

characterizes the bifurcation values.

- If $a > \frac{1}{2}$, the origin is a spiral sink.
- If $0 < a < \frac{1}{2}$, the origin is a sink.
- If $a < 0$, the origin is a saddle.
- If $a = \frac{1}{2}$, the origin is a nodal sink or sink star point.
- If $a = 0$, we have a line of attracting fixed points (as one of our eigenvalues is negative).

The bifurcation values are thus at $a = 0$ and $a = \frac{1}{2}$.



Second-Order Linear Equations

Consider the second order equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

Example. Let

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0.$$

We apply our lucky guess of $y(t) = ke^{\lambda t}$. Plugging it in, we find

$$\lambda^2\left(ke^{\lambda t}\right) + 7\lambda\left(ke^{\lambda t}\right) + 10\left(ke^{\lambda t}\right) = 0$$
$$\left(ke^{\lambda t}\right)\left(\lambda^2 + 7\lambda + 10\right) = 0.$$

Thus, we find $\lambda_1 = -2$ or $\lambda_2 = -5$ satisfy this equation, as well as $k = 0$ (which is the equilibrium solution). Our solutions currently are

$$\begin{aligned} y_1(t) &= k_1 e^{-2t} \\ y_2(t) &= k_2 e^{-5t}. \end{aligned}$$

The linearity principle provides

$$y_3(t) = k_1 e^{-2t} + k_2 e^{-5t}.$$

We may ask if this is the general solution.

To do this, we use the interplay between the second order differential equation and the first order differential equation. We set $x_1 = y$ and $x_2 = \frac{dy}{dt}$. Then,

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -10x_1 - 7x_2, \end{aligned}$$

meaning

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -5$, and eigenvectors

$$\begin{aligned} \vec{v}_1 &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ \vec{v}_2 &= \begin{pmatrix} 1 \\ -5 \end{pmatrix}. \end{aligned}$$

Thus, the general solution for the system is

$$\vec{Y}(t) = k_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{-5t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

Therefore, we only need to take the first component of the general solution for this system for the general solution of the second-order equation.

In general, for a second order differential equation, we find its characteristic polynomial

$$a\lambda^2 + b\lambda + c = 0,$$

and solve for the roots.

Example. Consider

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0.$$

Converting into the system, we get

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -13 & -4 \end{pmatrix} \vec{Y},$$

with eigenvalues

$$\lambda_1 = -2 + 3i$$

$$\lambda_2 = -2 - 3i.$$

We find

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 + 3i \end{pmatrix}.$$

We then find

$$\begin{aligned}
\vec{Y}(t) &= e^{(-2+3i)t} \begin{pmatrix} 1 \\ -2 + 3i \end{pmatrix} \\
&= e^{-2t} \left(\cos(3t) + i\sin(3t)\right) \begin{pmatrix} 1 \\ -2 + 3i \end{pmatrix} \\
&= e^{-2t} \begin{pmatrix} \cos(3t) + i\sin(3t) \\ -2\cos(3t) - 3\sin(3t) + 3i\cos(3t) - 2i\sin(3t) \end{pmatrix} \\
&= e^{-2t} \begin{pmatrix} \cos(3t) \\ -2\cos(3t) - 3\sin(3t) \end{pmatrix} + ie^{-2t} \begin{pmatrix} \sin(3t) \\ 3\cos(3t) - 2\sin(3t) \end{pmatrix}.
\end{aligned}$$

Thus, we have the general solution

$$\vec{Y}_1(t) = \begin{pmatrix} k_1e^{-2t} + k_2e^{-2t}\sin(3t) \\ 2k_1e^{-2t}\cos(3t) - 3k_1e^{-2t}\sin(3t) + 3k_2e^{-2t}\cos(3t) - 2k_2e^{-2t}\sin(3t) \end{pmatrix}.$$

Here, we get the general solution to the second-order equation to be the first component in the general solution to the system.

For the complex roots $\lambda = a \pm bi$ in a second order equation, we get

$$y(t) = k_1e^{at}\cos(bt) + k_2e^{at}\sin(bt),$$

and for repeated roots, the general solution is

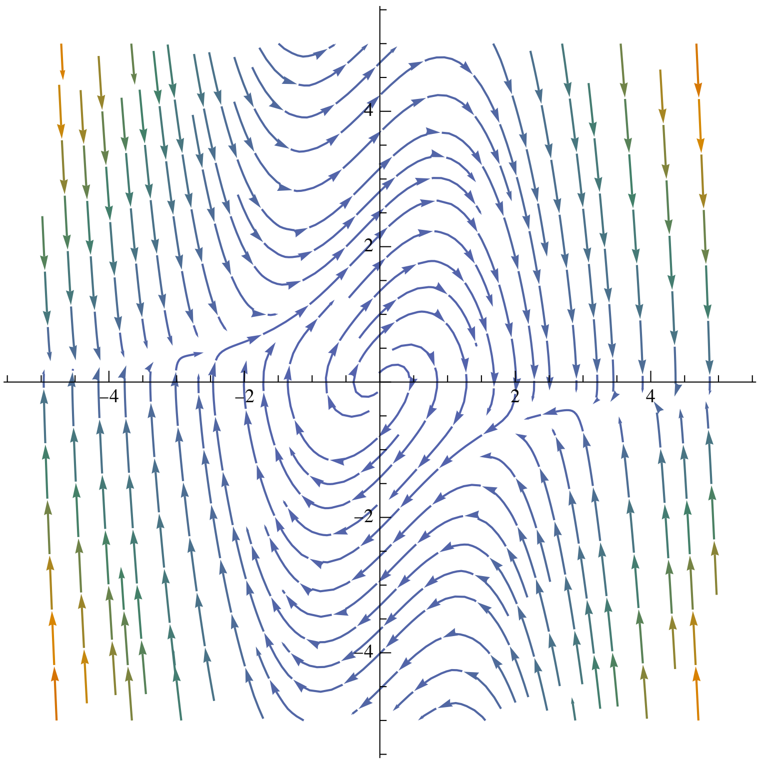
$$y(t) = k_1e^{\lambda t} + k_2te^{\lambda t}.$$

Understanding Nonlinear Systems

Equilibrium Point Analysis

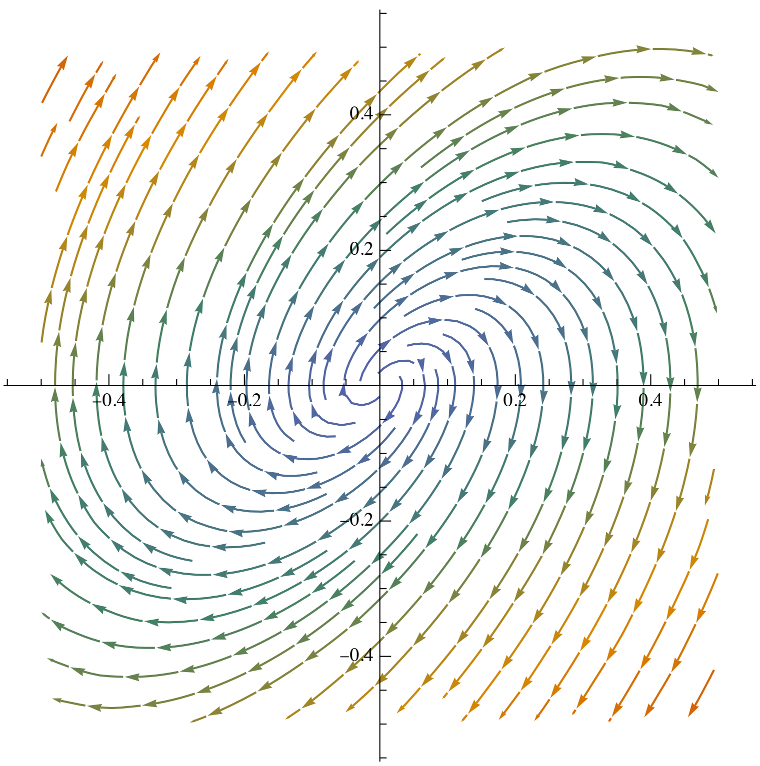
We will start by examining a special system known as the Van der Pol system of equations.

$$\begin{aligned}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x + \left(1 - x^2\right)y.
\end{aligned}$$



We can start by finding the equilibrium solutions. Taking $\frac{dx}{dt} = 0$, we find $y = 0$, from which we place into $\frac{dy}{dt} = -x + \left(1 - x^2\right)(0) = 0$, so $x = 0$.

The only equilibrium point is the origin. Note that we can see the origin is a type of spiral source as we zoom in close to the origin.



We can justify this by approximating the nonlinear system with a linear system.

When we are very close to the origin such as at $x = y = 0.1$, the term x^2y is orders of magnitude smaller than x and y. We can then analyze the linearized system

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + y,\end{aligned}$$

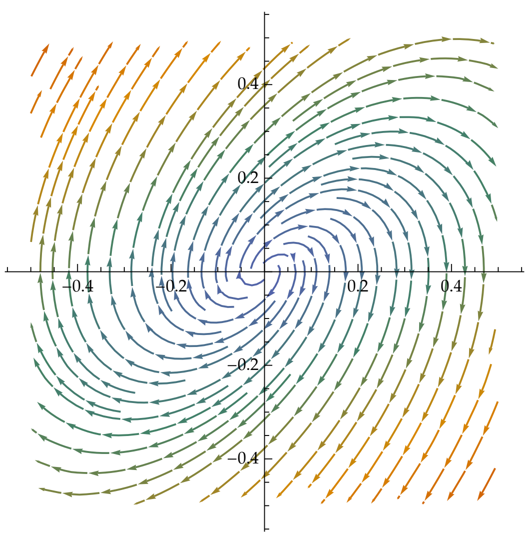
which is

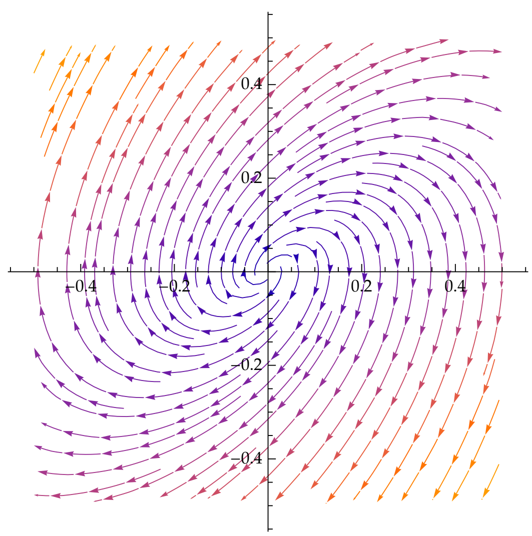
$$\frac{d\vec{Y}}{dt} = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\vec{Y}.$$

Solving this system of equations, we find eigenvalues

$$\begin{aligned}\det\left(A - \lambda I\right) &= \lambda\left(\lambda - 1\right) + 1\\ \lambda^2 - \lambda + 1 &= 0\\ \left(\lambda - \frac{1}{2}\right)^2 &= -\frac{3}{4}\\ \lambda &= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.\end{aligned}$$

We can see that this is a complex solution, and specifically that the origin is a spiral source (as the real part is positive). Note that, near the origin, we can see that the linearization (on the right) and the original system (on the left) are very similar.





Considering an arbitrary nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= f\left(x, y\right)\\ \frac{dy}{dt} &= g\left(x, y\right)\end{aligned}$$

with equilibrium point (a, b), where $f\left(a, b\right) = 0 = g\left(a, b\right)$, we can use techniques from multivariable calculus to find the linear approximation of $f\left(x, y\right)$ near (a, b).

$$f\left(x, y\right) \approx f\left(a, b\right) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}\left(x - a\right) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}\left(y - b\right).$$

Thus, to linearize our system near the equilibrium point, we find

$$\begin{aligned} f(x,y) &\approx \left.\frac{df}{dx}\right|_{(a,b)}(x-a) + \left.\frac{df}{dy}\right|_{(a,b)}(y-b) \\ g(x,y) &\approx \left.\frac{dg}{dx}\right|_{(a,b)}(x-a) + \left.\frac{dg}{dy}\right|_{(a,b)}(y-b). \end{aligned}$$

Note that we can view this as the system

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} \left.\frac{df}{dx}\right|_{(a,b)} & \left.\frac{df}{dy}\right|_{(a,b)} \\ \left.\frac{dg}{dx}\right|_{(a,b)} & \left.\frac{dg}{dy}\right|_{(a,b)} \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

We can now take the change of variables $u = x-a$ and $v = y-b$, to yield

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} \left.\frac{df}{dx}\right|_{(a,b)} & \left.\frac{df}{dy}\right|_{(a,b)} \\ \left.\frac{dg}{dx}\right|_{(a,b)} & \left.\frac{dg}{dy}\right|_{(a,b)} \end{pmatrix}}_{J|_{(a,b)}} \begin{pmatrix} u \\ v \end{pmatrix}.$$

This is our new linearized system,

$$\frac{d\vec{U}}{dt} = J(a,b)\vec{U},$$

where

$$\vec{U} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Example. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= x^2 + y^2 - 2. \end{aligned}$$

We find the equilibrium solutions $(-1,-1), (1,1)$.

To find the linearization, find the Jacobian

$$\begin{aligned} J|_{(x,y)} &= \begin{pmatrix} 1 & -1 \\ 2x & 2y \end{pmatrix} \\ J|_{(1,1)} &= \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \\ J|_{(-1,-1)} &= \begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix}. \end{aligned}$$

- Evaluating the eigenvalues for $J|_{(1,1)}$, we find

$$\begin{aligned} (1-\lambda)(2-\lambda) + 2 &= 0 \\ \lambda^2 - 3\lambda + 4 &= 0 \\ \lambda &= \frac{3}{2} \pm \frac{\sqrt{7}}{2}i, \end{aligned}$$

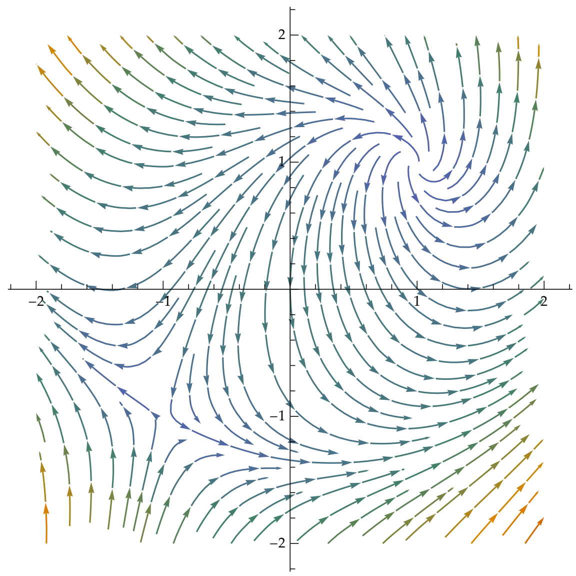
  implying that the equilibrium point $(1,1)$ is a spiral source.

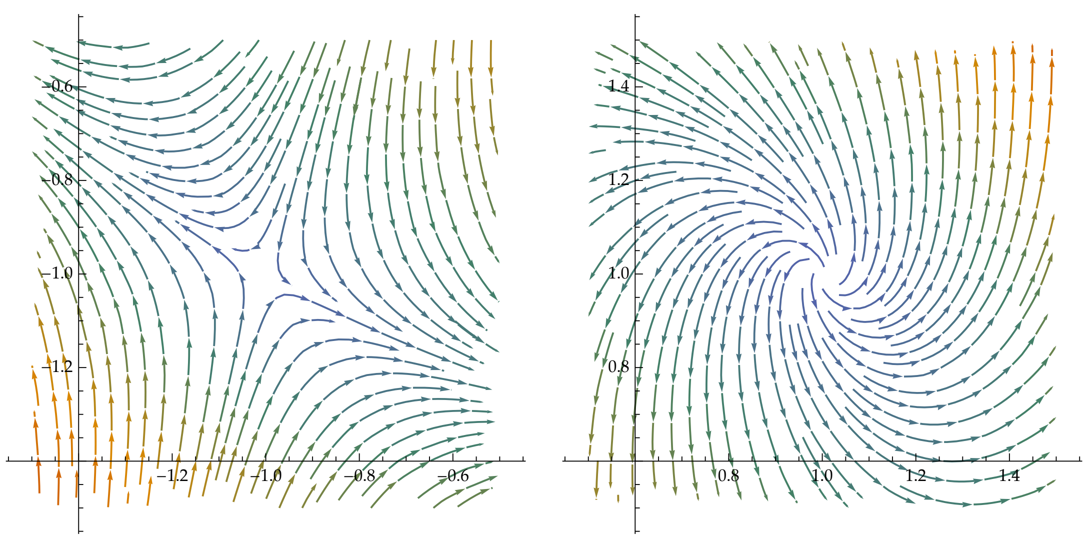
- Evaluating the eigenvalues for $J|_{(-1,-1)}$, we find

$$\begin{aligned}(\lambda - 1)(\lambda + 2) - 2 &= 0\\ \lambda^2 + \lambda - 4 &= 0\\ \left(\lambda + \frac{1}{2}\right)^2 &= \frac{5}{4}\\ \lambda &= \frac{1}{2} \pm \frac{\sqrt{5}}{2}.\end{aligned}$$

  Since there is one positive eigenvalue and one negative eigenvalue, we see that this equilibrium point is a saddle.

We can see in the plot below that near our equilibrium points, our linearization works decently well.





Unfortunately, there are places where linearization may fail.

Example (Linearization's Failure)**.** Consider the linearization

$$\begin{aligned}\frac{dx}{dt} &= y - \left(x^2 + y^2\right)x\\ \frac{dy}{dt} &= -x - \left(x^2 + y^2\right)y.\end{aligned}$$

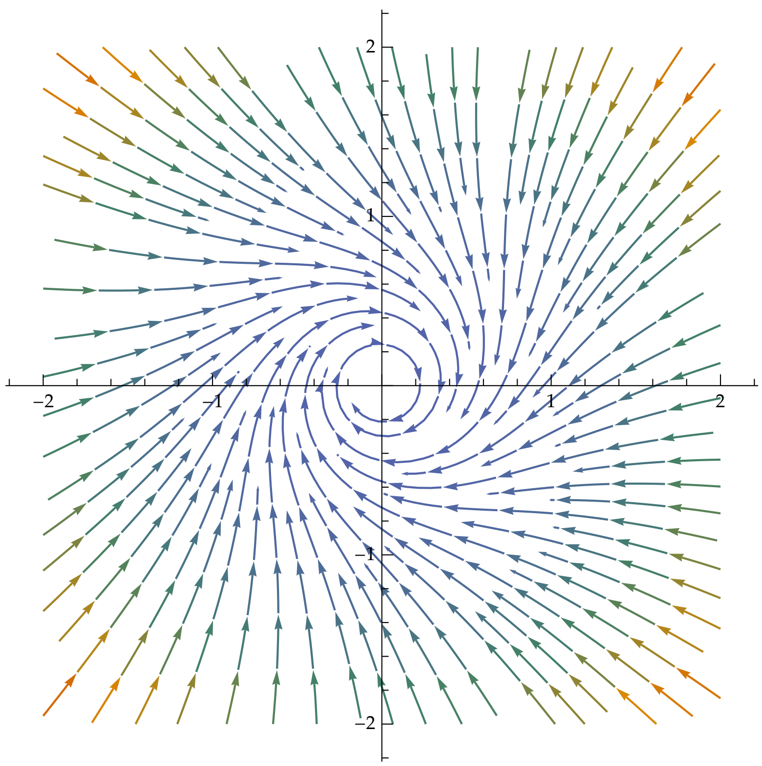
Note that $(0, 0)$ is an equilibrium solution. Solving the Jacobian, we find

$$J|_{(x,y)} = \begin{pmatrix} -3x^2 - y^2 & 1 - 2yx \\ -1 - 2xy & -3y^2 - x^2 \end{pmatrix}$$

Evaluating at the equilibrium solution, we find

$$\frac{d\vec{U}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The eigenvalues are $\lambda = \pm i$. Thus, we find that the origin is a center. Presumably.



However, we see that solutions spiral towards the origin (very slowly).

Thus, we are forced to see that we cannot rely on the linearization method to solve borderline cases.

In particular, the borderline cases on the trace-determinant plane occur at the points where

- $T = 0$ (i.e., centers);
- $D = \frac{1}{4}T^2$ (i.e., repeated eigenvalues);
- $D = 0$ (i.e., attracting/repelling fixed points).

Laplace Transforms

We can use a different method to solve differential equations.

Definition (Laplace Transform)**.** Let $f(t)$ be a function defined on $[0,\infty)$. The Laplace transform of $f(t)$ defined as

$$\begin{aligned} F(s) &= \mathcal{L}\left[f(t)\right]\\ &= \int_0^\infty e^{-st}f(t)\,dt \end{aligned}$$

for all s provided the improper integral converges.

Example. Let $f(t) = 1$. Then,

$$\begin{aligned} \mathcal{L}\left[1\right] &= \int_0^\infty e^{-st}\,dt\\ &= \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_{t=0}^{b}\\ &= \lim_{b\to\infty}\left(\frac{1}{s}e^{-sb} + \frac{1}{s}\right)\\ &= \frac{1}{s}. \end{aligned}$$

This definition is valid for all $s > 0$.

Example. We find

$$\begin{aligned} \mathcal{L}\left[t\right] &= \int_0^\infty te^{-st}\,dt\\ &= \lim_{b\to\infty}\int_0^b te^{-st}\,dt\\ &= \lim_{b\to\infty}\left(-\frac{1}{s}\Big|_0^b - \int_0^b -\frac{1}{s}e^{-st}\,dt\right)\\ &= \lim_{b\to\infty}\left(\left(-\frac{1}{s}be^{-sb} - \frac{1}{s^2}e^{-sb}\right) - \left(-\frac{1}{s} - \frac{1}{s^2}\right)\right)\\ &= \frac{1}{s^2}. \end{aligned}$$

Example. Let $f(t) = e^{at}$. Then, we have

$$\begin{aligned} \mathcal{L}\left[t\right] &= \int_0^\infty e^{at}e^{-st}\,dt\\ &= \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\,dt\\ &= \lim_{b\to\infty}\left(-\frac{1}{s-a}e^{(s-a)t}\Big|_0^b\right)\\ &= \frac{1}{s-a}, \end{aligned}$$

which is defined for all $s > a$.

f	$\mathcal{L}[f(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$

Note that the Laplace transform is a linear operator:

$$\begin{aligned}\mathcal{L}[f(t)+g(t)] &= \mathcal{L}[f(t)]+\mathcal{L}[g(t)]\\ \mathcal{L}[af(t)] &= a\mathcal{L}[f(t)].\end{aligned}$$

The Laplace transform exists for all functions that are bounded above by some exponential function.