

Problem (Problem 1): Let $U \subseteq \mathbb{C}$ be a region. Fix $z_0 \in U$. Let

$$\mathcal{F} = \{f \in H(U) \mid \text{im}(f) \subseteq \mathbb{C} \setminus B(0, 1), f(z_0) = 2i\}.$$

Show that \mathcal{F} is normal.

Solution: Let $(f_n)_n$ be a sequence in \mathcal{F} . We use the conformal map $z \mapsto \frac{1}{z}$ to map $\mathbb{C} \setminus B(0, 1)$ to \mathbb{D} , giving that the family

$$\mathcal{G} = \left\{ \frac{1}{f} \mid f \in \mathcal{F} \right\}$$

is locally bounded (indeed, globally bounded) by 1. Thus, it follows that there is a subsequence

$$\left(\frac{1}{f_{n_k}} \right)_k \rightarrow g: U \rightarrow \mathbb{D}$$

for some holomorphic function $g: U \rightarrow \mathbb{D}$. Now, since $\frac{1}{f_n}$ has no zeros for each n , it follows from Hurwitz's theorem that either g is uniformly 0 or g also has no zeros. Yet, since $g(z_0) = -\frac{i}{2} \neq 0$, it thus follows that $\frac{1}{g}$ is holomorphic on U , whence

$$(f_{n_k})_k \rightarrow \frac{1}{g}.$$

Thus, \mathcal{F} is normal.

Problem (Problem 2):

- (a) Using the Schwarz–Pick lemma, show that given $w \in \mathbb{D}$, there exists a holomorphic function $f: \mathbb{D} \rightarrow \mathbb{D}$ satisfying

$$\begin{aligned} f(w) &= 0 \\ |f'(w)| &= \sup_{\substack{g \in H(\mathbb{D}) \\ g(\mathbb{D}) \subseteq \mathbb{D}}} |g'(w)|. \end{aligned}$$

- (b) Show that if $f: \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and bounded, then

$$\sup_{z \in \mathbb{D}} \left(1 - |z|^2 \right) |f'(z)| \leq \sup_{z \in \mathbb{D}} |f(z)|.$$

- (c) Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic function. Show that f either has at most 1 fixed point or f is the identity.

Solution:

- (a) We know that the map

$$\psi_w(z) = \frac{w - z}{1 - \bar{w}z}$$

is a conformal map that takes $\psi_w(w) = 0$. Now, we know that

$$|\psi'_w(w)| = \frac{1}{1 - |w|^2}.$$

From the Schwarz–Pick Lemma, we have for all holomorphic functions $f: \mathbb{D} \rightarrow \mathbb{D}$

$$\frac{|f'(w)|}{1 - |f(w)|^2} \leq \frac{1}{1 - |w|^2}.$$

In particular, since $0 \leq |f(w)| < 1$, we have

$$|f'(w)| \leq \frac{1}{1 - |w|^2},$$

whence $\psi_w(z)$ satisfies

$$\begin{aligned} \psi_w(w) &= 0 \\ |\psi'_w(w)| &= \sup_{\substack{g \in H(\mathbb{D}) \\ g(\mathbb{D}) \subseteq \mathbb{D}}} |g'(w)|. \end{aligned}$$

- (b) Let $K = \sup_{z \in \mathbb{D}} |f(z)|$. By the maximum modulus principle, $|f(z)| < K$ for all $z \in \mathbb{D}$, so it follows that $g(z) := \frac{f(z)}{K}$ is a self-map of the unit disk. By the Schwarz–Pick lemma, it then follows that

$$\frac{|g'(z)|}{1 - |g(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Simplifying, we then get

$$\begin{aligned} (1 - |z|^2) |f'(z)| &\leq K \left(1 - \frac{|f(z)|^2}{K^2} \right) \\ &\leq K, \end{aligned}$$

so that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \leq \sup_{z \in \mathbb{D}} |f(z)|.$$

- (c) The statement is equivalent to showing that if $f: \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic self-map with two fixed points, then f is the identity map. Let f be one of these maps, and let $\xi \neq \eta \in \mathbb{D}$ be such that $f(\xi) = \xi$ and $f(\eta) = \eta$.

We want to find some holomorphic self-map of \mathbb{D} that sends $0 \mapsto 0$. We consider the maps

$$\psi_\xi = \frac{\xi - z}{1 - \bar{\xi}z},$$

which takes $0 \mapsto \xi$ and $\xi \mapsto 0$. Notice that $\psi_\xi \circ \psi_\xi = \text{id}$. Therefore,

$$g = \psi_\xi \circ f \circ \psi_\xi$$

is a holomorphic self-map that sends $0 \mapsto 0$, so by Schwarz's Lemma, we have

$$|g(z)| \leq |z|$$

for all $z \in \mathbb{D}$. Yet, we also have

$$\begin{aligned} g(\psi_\xi(\eta)) &= \psi_\xi \circ f \circ \psi_\xi \circ \psi_\xi(\eta) \\ &= \psi_\xi(\eta). \end{aligned}$$

In particular, this means that

$$|g(\psi_\xi(\eta))| = |\psi_\xi(\eta)|,$$

so there exists $\mathbb{D} \ni w := \psi_\xi(\eta)$ such that $|g(w)| = |w|$, so that $g(w) = e^{i\theta}w$. Yet, since the identity relation holds for $\psi_\xi(\eta)$, it follows that $\theta = 0$, so $g(w) = w$. In particular, this means

$$\begin{aligned} \psi_\xi \circ f \circ \psi_\xi(z) &= z \\ f \circ \psi_\xi(z) &= \psi_\xi(z). \end{aligned}$$

Yet, since ψ_ξ is an automorphism, it follows that this relation holds for all $z \in \mathbb{D}$, so that $f(w) = w$ for all $w \in \mathbb{D}$, whence $f = \text{id}$.

Problem (Problem 3): Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic function with $f(0) = 0$.

- (a) Show that $|f(z) + f(-z)| \leq 2|z|^2$ for all $z \in \mathbb{D}$.
 (b) Show that $|f(z) + f(-z)| = 2|z|^2$ for some $z \in \mathbb{D} \setminus \{0\}$ if and only if $f(z) = e^{i\theta} z^2$.

Solution:

- (a) We seek to show that the function

$$k(z) = \frac{f(z) + f(-z)}{2z}$$

maps $\mathbb{D} \setminus \{0\} \rightarrow \mathbb{D} \setminus \{0\}$. We may safely assume that $z \neq 0$, as the desired inequality is certainly true for $z = 0$. We observe that since f is a self-map of \mathbb{D} with $f(0) = 0$, Schwarz's Lemma gives

$$|f(z)| \leq |z|,$$

or that

$$\frac{|f(z)|}{|z|} \leq 1$$

A similar fact holds for $f(-z)$. For all $z \in \mathbb{D}$, we thus have

$$\left| \frac{f(z) + f(-z)}{2z} \right| \leq \frac{1}{2} \left(\left| \frac{f(z)}{z} \right| + \left| \frac{f(-z)}{z} \right| \right) < 1.$$

Therefore, since k is a self-map of \mathbb{D} with $k(0) = 0$, Schwarz's Lemma gives

$$|f(z) + f(-z)| \leq 2|z|^2.$$

- (b) Equivalently, we are assuming that

$$\left| \frac{f(z) + f(-z)}{2z} \right| = |z|$$

for some $z \in \mathbb{D} \setminus \{0\}$. From Schwarz's Lemma, we then have that

$$\frac{f(z) + f(-z)}{2z} = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$. This gives

$$\frac{1}{2}(f(z) + f(-z)) = e^{i\theta} z^2.$$

Now, we observe that

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z)).$$

First, we observe that

$$h(z) = \frac{1}{2}(f(z) - f(-z))$$

has $|h(z)| < 1$ for all $z \in \mathbb{D}$, $h(0) = 0$, and

$$|h'(0)| = |f'(0)|,$$

meaning that there is ρ such that $\frac{1}{2}(f(z) - f(-z)) = e^{i\rho} f(z)$, by a corollary to the Riemann Mapping Theorem and Schwarz's Lemma.

Problem (Problem 4):

- (a) Show that if $f: \mathbb{H} \rightarrow \mathbb{D}$ is a conformal map, then there exists some $\theta \in \mathbb{R}$ and $\beta \in \mathbb{H}$ such that

$$f(z) = e^{i\theta} \frac{z - \beta}{z - \bar{\beta}}.$$

- (b) Show that if $f: \mathbb{H} \rightarrow \mathbb{H}$ is a conformal map, then there exists some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$$

such that

$$f(z) = \frac{az + b}{cz + d}.$$

Solution:

- (a) Let $f: \mathbb{H} \rightarrow \mathbb{D}$ be a conformal map, and let $\beta = f^{-1}(0)$. It suffices to show that

$$g(z) = \frac{z - \beta}{z - \bar{\beta}}$$

is a conformal map from \mathbb{H} to \mathbb{D} that takes $\beta \mapsto 0$. The essential uniqueness of conformal maps will give us our desired result.

Letting $\beta = a + bi$ with $b > 0$, we observe that the expression of g can be rewritten as

$$\begin{aligned} g(z) &= \frac{\left(\frac{z-a}{b}\right) - i}{\left(\frac{z-a}{b}\right) + i} \\ &= q \circ L(z), \end{aligned}$$

where $L: \mathbb{H} \rightarrow \mathbb{H}$ takes $z \mapsto \frac{z-a}{b}$ (and is a uniquely determined automorphism of \mathbb{H} that takes $a+bi$ to i), while $q: \mathbb{H} \rightarrow \mathbb{D}$ is the Cayley transform. In particular, this is a composition of conformal maps, hence conformal, maps $\beta \mapsto 0$, and has $|g'(\beta)| \neq 0$, meaning that it must be the case that a general conformal map from \mathbb{H} to \mathbb{D} that maps $\beta \mapsto 0$ must be of the form

$$f = e^{i\theta} g(z).$$

- (b) We start by showing that all conformal maps $f: \mathbb{H} \rightarrow \mathbb{H}$ that fix i can be expressed by fractional linear transformations from matrices in $\text{SL}_2(\mathbb{R})$. We observe then that $q \circ f: \mathbb{H} \rightarrow \mathbb{D}$, where q is the Cayley Transform, is necessarily of the form

$$q \circ f = e^{i\theta} \frac{z - i}{z + i},$$

following from the uniqueness we showed in part (a). This gives

$$\begin{aligned} f &= -i \frac{z(1 + e^{i\theta}) + i(1 - e^{i\theta})}{z(e^{i\theta} - 1) - i(1 + e^{i\theta})} \\ &= -i \frac{e^{i\theta/2}(z(2 \cos(\theta/2)) + 2 \sin(\theta/2))}{e^{i\theta/2}(z(2i \sin(\theta/2)) - 2i \cos(\theta/2))} \\ &= \frac{z \cos(\theta/2) + \sin(\theta/2)}{-z \sin(\theta/2) + \cos(\theta/2)}. \end{aligned}$$

Since the rotation map

$$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

it follows that any conformal map of \mathbb{H} that fixes i can be expressed in this form.

In the general case, we observe that we can translate an arbitrary element of the form $z = a + bi$ with $b > 0$ to i by taking $L = \frac{z-a}{b}$, which admits a representation as an element of $\mathrm{SL}_2(\mathbb{R})$ via the matrix

$$\begin{pmatrix} 1/\sqrt{b} & -a/\sqrt{b} \\ 0 & \sqrt{b} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

In particular, if $f: \mathbb{H} \rightarrow \mathbb{H}$ is a conformal map, then there is a unique element $a + bi$ that maps to i , meaning that we may necessarily write f as

$$f = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1/\sqrt{b} & -a/\sqrt{b} \\ 0 & \sqrt{b} \end{pmatrix} \cdot z,$$

whence any conformal map from \mathbb{H} to \mathbb{H} can be expressed in this fashion.

Problem (Problem 5): Let f be an entire function satisfying $|f(z)| = 1$ for all $z \in S^1$. Show that there exists some $\theta \in \mathbb{R}$ and a nonnegative integer n such that $f(z) = e^{i\theta} z^n$ for all $z \in \mathbb{C}$.

Problem (Problem 6):

- (a) Show that there does not exist a continuous function $f: \overline{\mathbb{H}} \rightarrow \mathbb{C}$ for which $f|_{\mathbb{H}}$ is holomorphic, $f(\mathbb{R}) \subseteq (-\infty, 0)$, and $f(\mathbb{H}) \subseteq \mathbb{H}$.
- (b) Let $f: \overline{\mathbb{H}} \rightarrow \mathbb{C}$ be a continuous function for which $f|_{\mathbb{H}}$ is holomorphic, $f(\mathbb{R}) \subseteq \mathbb{R}$, and $0 \leq \mathrm{Im}(f(z)) \leq \mathrm{Re}(f(z))$ for all $z \in \overline{\mathbb{H}}$. Show that f is constant.

Solution:

- (a) We extend f to a holomorphic function on all of \mathbb{C} using the Schwarz reflection principle. Observe then that the range of the extension for f (which we call g) is contained in $\mathbb{C} \setminus [0, \infty)$. We may thus define a branch of the square root that maps $\mathbb{C} \setminus [0, \infty)$ to the upper half-plane, so that

$$v(z) = \frac{\sqrt{g(z)} - i}{\sqrt{g(z)} + i}$$

is an entire function whose range is contained within \mathbb{D} . Thus, it follows that g (and thus f) is constant by Liouville's Theorem. Yet, this would lead to a contradiction, since the condition that $f(\mathbb{R}) \subseteq (0, \infty)$ would imply that the constant value for f is some element of \mathbb{R} , while the condition that $f(\mathbb{H}) \subseteq \mathbb{H}$ would imply that the constant value for f is an element of \mathbb{H} , which cannot be in \mathbb{R} .

- (b) Using the Schwarz reflection principle, we extend f to be holomorphic on \mathbb{C} ; we call this extension g . To understand the behavior of g with respect to the alternative condition on f , we observe that on the lower half-plane, we have that $g(z) = \overline{f(\overline{z})}$; in particular, we have

$$\mathrm{Im}(\overline{f(\overline{z})}) \leq 0 \leq \mathrm{Re}(\overline{f(\overline{z})}),$$

so we have that g maps \mathbb{C} to the right half-plane. Thus, we have that $U(-1, -1/2) \not\subseteq g(\mathbb{C})$, meaning that by the converse to a corollary of Liouville's Theorem, we have that g is constant. Since g is a holomorphic extension of f , it follows that f is thus constant.