

## The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of  $x$ ).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

## Some Functions and Sets

$$f(x, y) = x^2 - y^2$$

Domain:  $\{(x, y) \mid \exists f(x, y)\}$

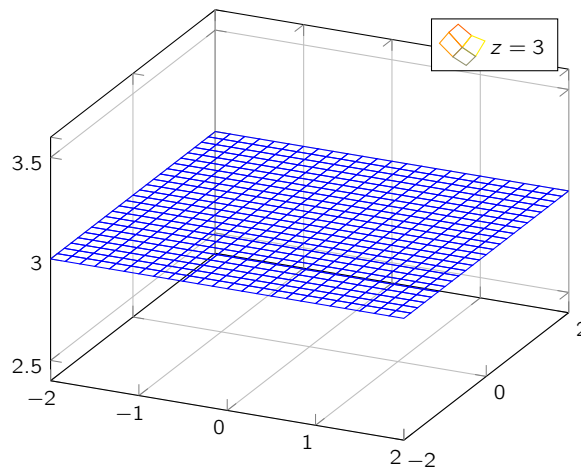
Range:  $\{f(x, y) \mid (x, y) \in \text{Dom}(f)\} = \mathbb{R}$

Graph:  $\text{Graph}(f) = \{x, y, f(x, y) \mid x, y \in \text{Dom}(f)\}$ . For example,  $(1, 3, 4) \notin \text{Graph}(f)$  since  $1^2 - 3^2 \neq 4$ .

## Examples

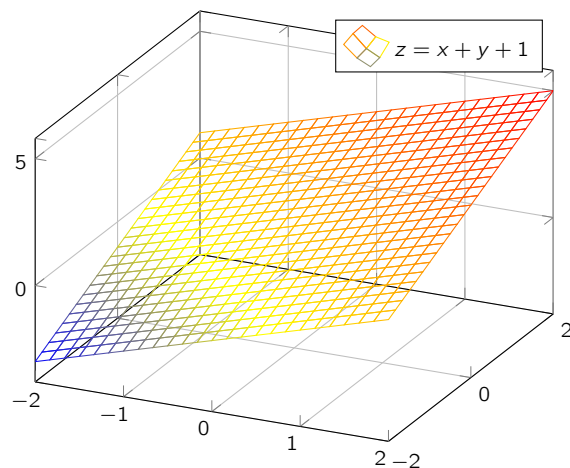
In  $\mathbb{R}^3$ , in  $x, y, z$  coordinates,  $z = 3$  is a plane defined as follows:

- Parallel to the  $xy$  plane.
- Passes through the point  $(0, 0, 3)$ .



Meanwhile,  $y = 0$  would be a “wall” that passes through the origin that contains the line  $y = 0$  in the  $xy$  plane.

Finally,  $z = x + y + 1$  is a plane, as we can see below.

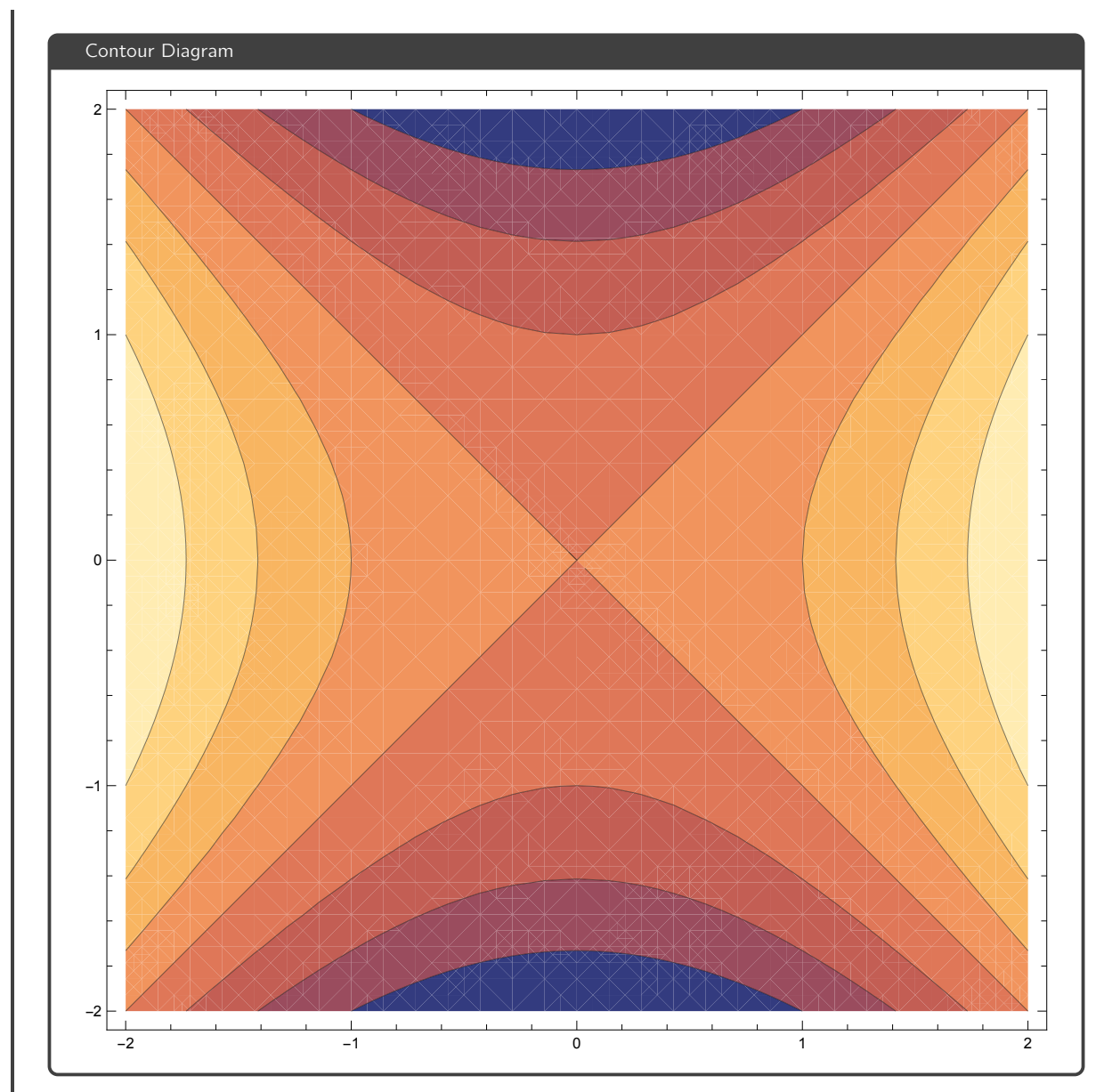


#### Visualizing a function of multiple variables

Consider the function  $f(x, y) = x^2 - y^2$ . We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$
- $f(0, y) = -y^2$
- $f(2, y) = 4 - y^2$
- $f(x, -2) = x^2 + 4$
- $f(x, 0) = x^2$
- $f(x, 2) = x^2 + 4$

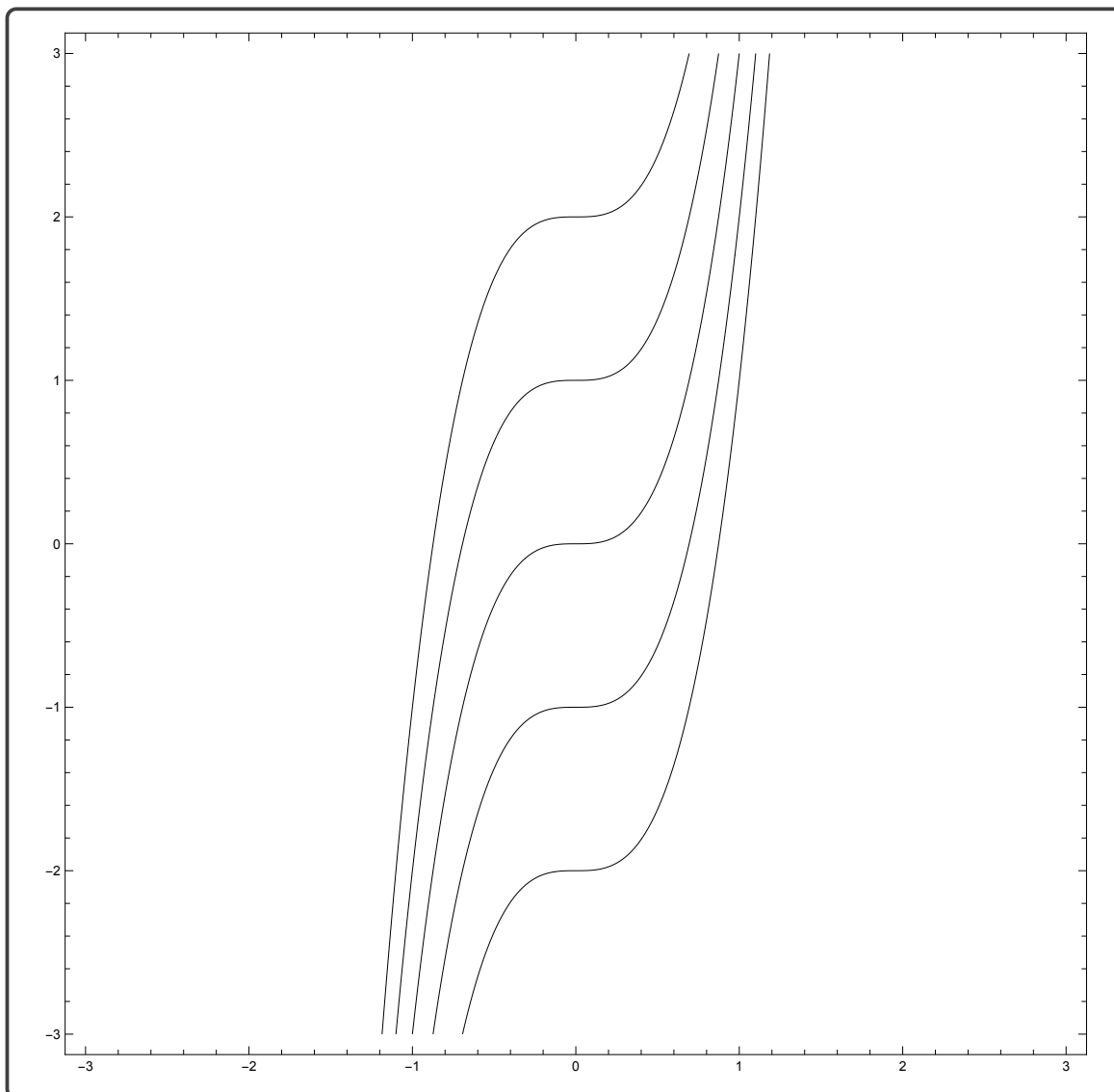
Alternatively, we can visualize via contour diagrams (i.e., everywhere that  $z$  is a certain value), as seen in mathematica as follows:



#### Contour Example

Consider the function  $f(x, y) = y - 3x^2$ . We want to find the contours.

For any  $c$ , we have that  $c = y - 3x^2$ , or  $y = 3x^2 + c$ . Therefore, every contour "looks like"  $3x^2 + c$  for values of  $c$ . For example, in the following, we have  $c = \{-2, -1, 0, 1, 2\}$

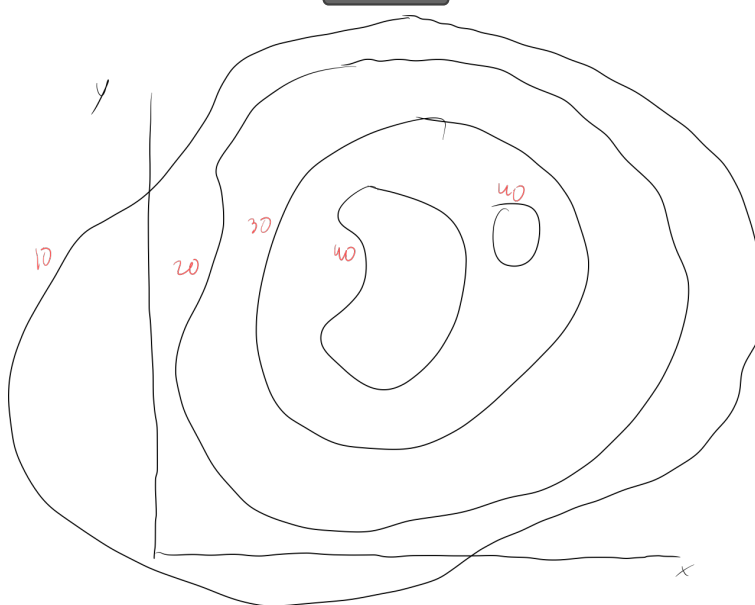


## Distance

In  $\mathbb{R}^5$ , let  $p = (3, 1, 4, 1, 5)$ , and  $q = (1, 0, -2, 0, 2)$ . Using the Euclidean metric, we can find the distance between  $p$  and  $q$  is  $d(p, q) = ((3-1)^2 + (1-0)^2 + (4-(-2))^2 + (1-0)^2 + (5-2)^2)^{1/2} = (4 + 1 + 36 + 1 + 9)^{1/2} = \sqrt{51} = 7.14$ . We can also call this the 2-norm.

$$d(p, q) = \left( \sum_{k=1}^n (p_k - q_k)^2 \right)^{1/2}$$

## Derivatives



To denote a derivative, we can't talk about one value, we must use a *partial* derivative,  $\frac{\partial f}{\partial x}$ , or  $\frac{\partial f}{\partial y}$ . The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

$x \backslash y$	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A "linear" approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where  $(x_0, y_0, z_0) \in \mathbb{R}^3$ , and is an output in  $z = f(x, y)$ , and  $m, n \in \mathbb{R}$ .

For example, with the above table, we can see that the function is linear in  $x$  and  $y$  (i.e., the slope holding the other variable constant is constant).

## Limits in Multivariable Functions

Consider the following:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow  $y = mx$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 - y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2} \\ &= \frac{1 + m^2}{1 - m^2} \end{aligned}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of  $m$

$m$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours “approach” a particular point, we can see that the limit is defined.

### Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have  $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$ . These vectors are equivalent because they are components of  $\mathbb{R}^3$ .

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

### Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

### Properties of Vectors

Let  $\vec{u}, \vec{v} \in \mathbb{R}^n$ . Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $c\vec{u} = \langle cu_1, cu_2, \dots, cu_k \rangle$

Additionally, we define  $\vec{u} \cdot \vec{v}$  as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^n u_k v_k = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

### Partial Derivatives

Consider  $f(x, y) = x^2y + xe^y$ .

$$f_x := \frac{\partial f}{\partial x}$$

$$f_x(a, b) = \left. \frac{\partial f}{\partial x} \right|_{(a,b)}$$

We know that  $f \in C^\infty(\mathbb{R} \times \mathbb{R})$ , meaning  $f$  is endlessly differentiable.

### Functions and Approximations

Let  $f(x, y) = x^2 - y^2$ ,  $g(x, y) = 2xy$

- $f_{xx} + f_{yy} = 0$
- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For  $f(x, y)$  at  $(a, b, f(a, b))$ , we have the following:

$$\begin{aligned}\ell(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ q(x, y) &= \ell(x, y) + \frac{1}{2} (f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2)\end{aligned}$$

In order to get a sense of the “derivative,” we can use the following:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

#### Directional Derivative and Gradient

Given  $f(x, y)$  and  $(a, b)$ , where  $f \in C^2(\mathbb{R}^2)$ . Then, the quadratic approximation is:

$$\begin{aligned}f(x, y) &\approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{1}{2} (f_{xx}(a, b)(x - a)^2 + f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b))\end{aligned}$$

$$df = f_x(a, b)dx + f_y(a, b)dy$$

a differential

$$\Delta f = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

Evaluating  $f(x, y) = xe^y$  at  $(a, b) = (-1, 0)$

$$f_x = e^y$$

$$f_y = xe^y$$

$$f_x(-1, 0) = 1$$

$$f_y(-1, 0) = -1$$

$$\Delta f = \Delta x - \Delta y$$

On a given contour map, let  $\vec{u} = \langle u_1, u_2 \rangle$  denote a *unit* vector in a direction that we want to find the derivative of  $f$  in.

$$f_{\vec{u}}(x, y) = \nabla f(a, b) \cdot \vec{u}$$

Where

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

The directional derivative for all vectors  $\vec{v}$  is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

#### Chain Rule

Let  $f(x, y)$  be a function where  $x = x(t)$  and  $y = y(t)$ . We want to find

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

The chain rule works in higher dimensions too. Consider  $k(x_1(t), x_2(t), \dots, x_{152}(t))$ . Then,

$$\frac{dk}{dt} = \sum_{i=1}^{152} \frac{\partial k}{\partial x_i} \frac{dx_i}{dt}$$

We can also view this as a vector. Let  $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{152}(t) \end{pmatrix}$ . Then, we can write  $\frac{dk}{dt}$  more succinctly as follows:

$$\frac{dk}{dt} = \nabla k \cdot \frac{d\vec{x}}{dt}$$

For example, let  $f(x, y, z) = 3x^2y + zx + 2$ , where  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$

$$\begin{aligned}\frac{df}{dt} &= \begin{pmatrix} 6xy + z \\ 3x^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} \\ &= (6xy + z)x'(t) + 3x^2y'(t) + xz'(t)\end{aligned}$$

So, if we let  $x(t) = \sin(t)$ ,  $y(t) = e^t$ , and  $z(t) = t^2 + 1$ . Then, we have

$$\frac{df}{dt} = 6\sin(t)\cos(t)e^t + t^2\cos(t) + \cos(t) + 3e^t\sin^2(t) + 2t\sin(t)$$

Alternatively, consider  $f(x, y, z) = x^2 + yz + e^y$ , where  $x(s, t) = st$ ,  $y = y(s, t) = t + s^2$ ,  $z = z(s, t) = e^t$ . Let

$$\vec{x} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix}$$

Then, we have

$$\begin{aligned}\frac{\partial f}{\partial t} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial t} \\ \frac{\partial f}{\partial s} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial s}\end{aligned}$$

Evaluating the first expression, we have

$$\begin{aligned}\frac{\partial f}{\partial t} &= \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix} \cdot \begin{pmatrix} s \\ 1 \\ e^t \end{pmatrix} \\ &= 2s^2t + 3^t + e^{t+s^2} + (t + s^2)e^t\end{aligned}$$

Consider  $f(x, y(x))$ . Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

This is the technique we use to find implicit differentiation.

We know as a result that  $\nabla f(a, b)$  is orthogonal to the contour curve at  $(a, b)$

#### Recap

In  $\mathbb{R}^3$ , find the plane that contains  $P = (P_1, P_2, P_3)$ ,  $Q$ , and  $R$ . We can find it by the following:

$$\begin{aligned}0 &= \vec{n} \cdot \begin{pmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{pmatrix} \\ 0 &= n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3)\end{aligned}$$

where

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

#### Differentiability

A function  $f(x)$  of one variable is differentiable at  $x = a$  if

$$f(a) = \lim_{h \rightarrow 0} f(a + h)$$

and

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$



exists and is bounded

We can also linearize the function.  $f$  is differentiable if

$$f(x) = f(a) + f'(a)(x - a) + E(x)$$

where  $\lim_{h \rightarrow 0} \frac{E(a+h)}{h} = 0$ .

In the multiple dimensions example, we have  $f(x, y)$  is differentiable if

$$f(x, y) = \ell(x, y) + E(x, y)$$

where  $\lim_{h \rightarrow 0, k \rightarrow 0} \frac{E(a+h, b+k)}{\sqrt{h^2 + k^2}} = 0$

### Local Maxima

Let  $f(x, y) = x^2 + 2y^2$ . We want to find  $(a, b)$  which are local maxima, minima, or other.

$(a, b)$  is a local maximum if  $f(a, b) \geq f(x, y) \forall (x, y) \in V_\epsilon(a, b)$ , where  $\epsilon > 0$ .

(1) Find Critical Points for  $f(x, y) : f_x(x, y), f_y(x, y) = 0$ ,  $f_x(x, y), f_y(x, y)$  are undefined.

$$f_x(x, y) = 2x$$

$$f_y(x, y) = 4y$$

$$f_x(0, 0) = 0$$

$$f_y(0, 0) = 0$$

$$f(0, 0) = 0$$

$$f(x, y) > 0$$

$$\forall (x, y) \neq (0, 0)$$

For all  $x, y$ ,  $f_{xx} = 2$ ,  $f_{yy} = 4$ , and  $f_{xy} = 0$ . Finally,

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) \cdot f_{yy}(x, y) + f_{xy}(x, y)^2 \\ &= 8 \\ &> 0 \end{aligned}$$

Since  $D(x, y) > 0$ , we look at the sign of  $f_{xx}$ . Since it is positive,  $f(0, 0)$  has a local minimum.

### Local Maxima and Minima Approach

Given  $f(x, y)$ , we want

(1) Find critical points:

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

(2) Compute  $f_{xx}, f_{yy}, f_{xy}, D = f_{xx}f_{yy} - (f_{xy})^2$

(3)

$f_{xx}$	$D$	Critical Point
+	+	Local Minimum
-	+	Local Maximum
$\pm$	-	Saddle Point
$\pm$	0	Nothing

Consider the function

$$f(x, y) = \ln(x^2 + y^2 + 1)$$

$$f(0, 0) = 0$$

$$f(x, y) > 0$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

Critical Points:  $(0, 0)$

$$\left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} = \frac{2(x^2 + y^2 + 1) - 4x^2}{(x^2 + y^2 + 1)^2} = 2$$

$$\left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} = \frac{2(x^2 + y^2 + 1) - 4y^2}{(x^2 + y^2 + 1)^2} = 2$$

$$\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = \frac{-4xy}{(x^2 + y^2 + 1)^2} = 0$$

Now, consider the function

$$f(x, y) = x^2 - 2xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x - 2y$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2$$

$$D = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial x^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = 0$$

Therefore, the critical points of this function are indeterminate with the given approach. However, we know that  $f(x, y) = (x - y)^2 = 0$  when  $x = y$ , so the line  $y = x$  is a local minimum trough in 3-space.

Now, consider the function

$$f(x, y) = (x - 1)^2(y + 2)$$

$$\frac{\partial f}{\partial x} = 2(x - 1)(y + 2)$$

$$\frac{\partial f}{\partial y} = (x - 1)^2$$

Critical points:  $\{(1, y) \mid y \in \mathbb{R}\}$

$$\frac{\partial^2 f}{\partial x^2} = 2(y + 2)$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2(x - 1)$$

$$D = 0 - (2(x - 1))^2 = 0$$

Evaluating  $D$  at critical points

## Finding Critical Points

Let  $f(x, y) = (y^2 + 2) \sin(x)$ . on  $[-2, 2] \times [-2, 2]$

$$\frac{\partial f}{\partial x} = (y^2 + 2) \cos(x)$$

$$= 0$$

$$\frac{\partial f}{\partial y} = 2y \sin(x)$$

$$= 0$$

$$(x, y) = \left( \frac{(2n+1)\pi}{2}, 0 \right)$$

$$= \{(\pi/2, 0), (-\pi/2, 0)\}$$

$$\frac{\partial^2 f}{\partial x^2} = -(y^2 + 2) \sin(x)$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \sin(x)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y \cos(x)$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$$

$$= -2(y^2 + 2) \sin^2(x) - 4y^2 \cos^2(x)$$

$$< 0$$

Therefore, the critical points are saddle points. If there is no domain restriction, we have a series of saddle points all along  $y = 0$ .

## Why Finding Critical Points Works

We create the Taylor series of  $f(x, y)$  at  $(x_0, y_0)$ :

$$f(x, y) \approx f(x_0, y_0) + \frac{1}{2} (f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2)$$

$$= f(x_0, y_0) + \underbrace{\nabla f(x, y) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}_{=0 \text{ at critical points}} + \frac{1}{2} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^T \underbrace{\begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}}_{\text{Hessian}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

If the Hessian is positive definite, then  $\lambda_1, \lambda_2 > 0$  and the critical point is a local min. If the Hessian is negative definite, then  $\lambda_1, \lambda_2 < 0$  and the critical point is a local max.

In any given  $2 \times 2$  matrix, the eigenvalues  $\lambda_1, \lambda_2$  are such that  $\lambda_1 + \lambda_2 = \text{Tr}(A)$  and  $\lambda_1 \lambda_2 = \text{Det}(A)$ .

## Optimization

Let  $f(x, y) = 2x - y$ . We want to optimize  $f$  with respect to  $g(x, y) = x^2 - y^2 - 4 = 0$ .

Define  $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$ . Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , then  $L : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ .

Then, we take

$$\nabla L = \nabla f - \lambda \nabla g$$

$$= 0$$

critical points of  $L$

We find  $x, y, \lambda$  for each critical point.

$$\nabla f = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$2 = 2\lambda x$$

$$-3 = -2\lambda y$$

$$x^2 - y^2 = 4$$

$$\lambda = \frac{1}{x}$$

$$\lambda = \frac{3}{2y}$$

$$x = \frac{2y}{3}$$

$$\frac{4y^2}{9} - y^2 = 4$$

$$-\frac{5}{9}y^2 = 4$$

No Solution

However, if  $g(x, y) = x^2 + y^2 - 4 = 0$ , we have

$$\nabla f = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$2 = 2\lambda x$$

$$-3 = 2\lambda y$$

$$x^2 + y^2 = 4$$

$$\lambda = \frac{1}{x}$$

$$\lambda = \frac{-3}{2y}$$

$$x = \frac{-2y}{3}$$

$$\frac{4y^2}{9} + y^2 = 4$$

$$-\frac{13}{9}y^2 = 4$$

$$y = \pm \frac{6}{\sqrt{13}}$$

$$x = \mp \frac{4}{\sqrt{13}}$$

$$f_{\max} = 2\sqrt{13}$$

$$f_{\min} = -2\sqrt{13}$$

This system of Lagrange multipliers applies in the  $n$  dimensional case.

Let  $f(x, y, z) = x + 2y + z^2$  subject to the constraint  $g(x, y, z) = x^2 + y^2 + z^2 = 1$ .

$$\nabla f = \lambda \nabla g$$

$$\begin{pmatrix} 1 \\ 2 \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$2\lambda x = 1$$

$$2\lambda y = 2$$

$$2\lambda z = 2z$$

(\*)

$$x^2 + y^2 + z^2 = 1$$

Consider (\*):

$$\lambda = 1$$

$$x = 1/2$$

$$y = 1$$

$$\frac{1}{4} + 1 + z^2 = 1$$

no solution

$$z = 0$$

$$x^2 + y^2 = 1$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 1$$

$$\frac{5}{4\lambda^2} = 1$$

$$\lambda = \pm \frac{\sqrt{5}}{2}$$

Case 1:

$$\lambda = \frac{\sqrt{5}}{2}$$

$$x = \frac{1}{\sqrt{5}}$$

$$y = \frac{2}{\sqrt{5}}$$

Case 2:

$$\lambda = -\frac{\sqrt{5}}{2}$$

$$x = -\frac{1}{\sqrt{5}}$$

$$y = -\frac{2}{\sqrt{5}}$$

Evaluating  $f$ :

$x$	$y$	$z$	$\lambda$	$f(x, y, z)$
$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	$\frac{\sqrt{5}}{2}$	$\sqrt{5}$
$-\frac{1}{\sqrt{5}}$	$-\frac{2}{\sqrt{5}}$	0	$-\frac{\sqrt{5}}{2}$	$-\sqrt{5}$

If we want to optimize  $f$  with respect to multiple constraint functions  $g_1, g_2, g_3, \dots, g_k$ , we would do:

$$\nabla f = \sum_{i=1}^k \lambda_i \nabla g_i$$

### Integration

Consider  $f(x, y)$ . We want to integrate along the rectangle  $D = [0, 3] \times [0, 2]$ . We can find this as follows:

$$\begin{aligned} \int_D f(x, y) &= \int_0^3 \int_0^2 f(x, y) dy dx \\ &= \int_0^3 dx \int_0^2 dy f(x, y) \end{aligned}$$

For any two regions  $D_1$  and  $D_2$ , we have:

$$\begin{aligned} \int_{D_1} f(x, y) + \int_{D_2} f(x, y) &= \int_{D_1 \cup D_2} f(x, y) \\ &= \int_{D_1 \cup D_2 \setminus D_1 \cap D_2} f(x, y) \end{aligned}$$

### Multidimensional Integral Approximation

We want to find

$$\int_0^2 dy \int_0^3 dx f(x, y)$$

where  $f(x, y)$  is expressed as below.

$y \backslash x$	0	1	2	3
0	1	2	5	4
1	2	1	2	0
2	1	-1	1	-2

Just as we can use the left/right endpoint method for evaluating integrals in one dimension, we can use left/right and top/bottom endpoints to approximate the integral.

## Evaluating a Multidimensional Integral

$$\begin{aligned}
 \underbrace{\int_0^1}_{dy} \underbrace{\int_0^1}_{dx} x e^y dx dy &= \left( \int_0^1 e^y dy \right) \left( \int_0^1 x dx \right) \\
 &= \left( e^y \Big|_0^1 \right) \left( \frac{x^2}{2} \Big|_0^1 \right) \\
 &= (e - 1) \left( \frac{1}{2} - 0 \right) \\
 &= \frac{e - 1}{2}
 \end{aligned}$$

This can scale up into multiple dimensions:

$$\begin{aligned}
 \int_0^1 \int_2^4 \int_{-1}^2 x + y + z^2 dx dy dz &= \int_0^1 \int_2^4 \left( \int_{-1}^2 x + y + z^2 dx \right) dy dz \\
 &= \int_0^1 \int_2^4 \left( \frac{x^2}{2} + yx + xz^2 \Big|_{x=-1}^{x=2} \right) dy dz \\
 &= \int_0^1 \int_2^4 \left( \left( 2 - \frac{1}{2} \right) + (2y - (-y)) + (2z^2 - (-z^2)) \right) dy dz \\
 &= \int_0^1 \int_2^4 3z^2 + 3y + \frac{3}{2} dy dz \\
 &= \int_0^1 \left( \frac{3}{2} y^2 + \frac{3}{2} y + 3yz^2 \Big|_{y=2}^{y=4} \right) dz \\
 &= \int_0^1 6z^2 + 21z dz \\
 &= 2z^3 + 21z \Big|_{z=0}^{z=1} \\
 &= 23
 \end{aligned}$$

Consider the integral below:

$$\begin{aligned}
 \iint_D x e^y dx dy &= \int_0^1 \int_y^1 x e^y dx dy \\
 &= \int_0^1 e^y dy \left( \frac{x^2}{2} \Big|_{x=y}^{x=1} \right) \\
 &= \int_0^1 e^y \left( \frac{1}{2} - \frac{y^2}{2} \right) dy \\
 &= \frac{1}{2} \left( \int_0^1 e^y dy - \int_0^1 y^2 e^y dy \right) \\
 &= \frac{1}{2} ((e - 1) - (e - 2)) \\
 &= \frac{1}{2}
 \end{aligned}$$

## Example Integrals

Consider the domain  $D : \{(x, y) \mid 1 \leq x \leq 2, 0 \leq y \leq \ln 2, y = \ln x\}$ . We are going to evaluate  $f(x, y) = 1$ .

$$\begin{aligned}\int_D f(x, y) dD &= \int_0^{\ln 2} \int_{e^y}^2 1 \, dx \, dy \\ &= \int_0^{\ln 2} \left( x \Big|_{x=e^y}^{x=2} \right) dy \\ &= \int_0^{\ln 2} (2 - e^y) dy \\ &= 2 \ln 2 - 1\end{aligned}$$

Consider  $A = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, x^2 \leq (x, y) \leq \sqrt{x}\}$  and evaluating  $f(x, y) = x + 2y$

$$\begin{aligned}\int_A x + 2y dA &= \int_0^1 \int_{y^2}^{\sqrt{y}} x + 2y \, dx \, dy \\ &= \int_0^1 \left( \frac{x^2}{2} + 2xy \Big|_{x=y^2}^{x=\sqrt{y}} \right) dy \\ &= \int_0^1 \frac{y}{2} + 2y^{3/2} - \left( \frac{y^4}{2} + 2y^3 \right) dy \\ &= \frac{y^2}{4} + \frac{4}{5} y^{5/2} - \frac{y^5}{10} - \frac{y^4}{2} \Big|_{y=0}^{y=1} \\ &= \frac{9}{20}\end{aligned}$$

## General Multivariable Integration

In any given area of integration  $A$ , we have the following general form:

$$\begin{aligned}\int_A f(x, y) dA &= \underbrace{\int_{c_1}^{c_2}}_{\text{always constants}} \int_{g(y)}^{h(y)} f(x, y) dx dy \\ &= \int_{d_1}^{d_2} \int_{p(x)}^{q(x)} f(x, y) dy dx\end{aligned}$$

In the three-dimensional case, we have

$$\int_V f(x, y, z) dV = \int_{c_1}^{c_2} \int_{g(z)}^{h(z)} \int_{p(y, z)}^{q(y, z)} f(x, y, z) dx dy dz$$

Consider an integral with domain as follows:

$$D = \{x, y, z \mid -1 \leq x \leq 1, 0 \leq y \leq 10, 0 \leq z \leq 1 - x^2\}$$

For volume, we have that  $f(x, y, z) = 1$

$$\begin{aligned}V &= \int_{-1}^1 \int_0^{1-x^2} \int_0^{10} 1 \, dy \, dz \, dx \\ &= \int_{-1}^1 \int_0^{1-x^2} 10 \, dz \, dx \\ &= 10 \int_{-1}^1 1 \, dx - 10 \int_{-1}^1 x^2 \, dx \\ &= \frac{40}{3}\end{aligned}$$



For another example, consider the pyramid defined by the points  $(0, 0, 0)$ ,  $(0, 2, 0)$ ,  $(1, 0, 0)$ ,  $(0, 0, 3)$ . As established, we do  $f(x, y, z) = 1$  integrated over the domain.

$$V = \int_0^3 \int_0^{g(y)} \int_0^{h(y,z)} f(x, y, z) \, dx \, dy \, dz$$

We find the cross product for the plane

$$\vec{n} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

$$6 = 6x + 3y + 2z$$

$$x = 1 - \frac{1}{2}y - \frac{1}{3}z$$

$$y = 2 - \frac{2}{3}z$$

$$V = \int_0^3 \int_0^{2-\frac{2z}{3}} \int_0^{1-\frac{1}{2}y-\frac{1}{3}z} 1 \, dx \, dy \, dz$$

$$= \int_0^3 \int_0^{2-\frac{2z}{3}} \left(1 - \frac{1}{2}y - \frac{1}{3}z\right) dy \, dz$$

$$= \int_0^3 \left[2 - \frac{2z}{3} - \frac{(2-\frac{2z}{3})^2}{4} - \frac{(2-\frac{2z}{3})z}{3}\right] dz$$

$u$  substitution:

$$u = 2 - \frac{2z}{3}$$

$$z = \frac{3(u-2)}{-2}$$

$$dz = -\frac{3}{2}du$$

$$= \int_2^0 \left(u - \frac{u^2}{4} - \frac{3u(u-2)}{-6}\right) \left(-\frac{3}{2}du\right)$$

$$= \int_2^0 \frac{u^2}{4}$$

$$= -\frac{3}{2} \int_2^0 \frac{u^2}{4}$$

$$= -\frac{3}{2} \frac{u^3}{12} \Big|_2^0$$

$$= 1$$

#### The Jacobian

Let  $f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)$  be differentiable.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

If we seek to enact a change of variables, we compute  $\det J$ .

For example, let  $x = r \cos \theta = x(r, \theta)$ ,  $y = r \sin \theta = y(r, \theta)$ .

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \\ \det J &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \end{aligned}$$

Consider a circle of radius 5. We will integrate  $f(x, y) = x^2 + y$  over this circle.

(1): Cartesian

$$\begin{aligned} \int_A f(x, y) dA &= \int_{-5}^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} (x^2 + y) dy dx \\ &= \int_{-5}^5 \left( x^2 y + \frac{y^2}{2} \right) \Big|_{y=-\sqrt{25-x^2}}^{y=\sqrt{25-x^2}} dx \\ &= \int_{-5}^5 2x^2 \sqrt{25-x^2} dx \\ &\vdots \end{aligned}$$

(2): Polar

$$\begin{aligned} \int_A f(r, \theta) dr d\theta &= \int_A f(r, \theta) \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} dr d\theta \\ &= \int_0^{2\pi} \int_0^5 (r^2 \cos^2 \theta + r \sin \theta) (r) dr d\theta \\ &= \int_0^{2\pi} \int_0^5 r^3 \cos^2 \theta dr d\theta + \int_0^{2\pi} \int_0^5 r^2 \sin \theta dr d\theta \\ &= \left( \int_0^{2\pi} \cos^2 \theta d\theta \right) \left( \int_0^5 r^3 dr \right) + \left( \int_0^{2\pi} \sin \theta d\theta \right) \left( \int_0^5 r^2 dr \right) \\ &= \frac{625}{4} \left( \int_0^{2\pi} \cos^2 \theta d\theta \right) \\ &= \frac{625\pi}{4} \end{aligned}$$

### Cylindrical Coordinates

We can define a new coordinate base as follows:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned}$$

Thus, any integral over the cylinder has the following form:

$$\begin{aligned} \int_V f(x, y, z) dV &= \int_V f(r, \theta, z) \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} dr d\theta dz \\ &= \int_V f(r, \theta, z) r dr d\theta dz \end{aligned}$$

## Spherical Coordinates

If we have spherical symmetry, we will need to include a second angle  $\varphi$ , relative to the positive  $z$  axis:

$$x = \rho \cos \theta \sin \varphi$$

$$y = \rho \sin \theta \sin \varphi$$

$$z = \rho \cos \varphi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \rho^2 \sin \varphi$$

## Integrating with Spherical and Cylindrical Coordinates

Consider a cylinder defined by  $z \in [0, 4]$  and  $r = 1$ , and a half-sphere defined by  $r = 1$  and  $\varphi \in [0, \pi/2]$ , located on "top" of the cylinder. We want to find

$$\int_V f(x, y, z) dV = \int_{V_1} f(x, y, z) dV_1 + \int_{V_2} f(x, y, z) dV_2$$

where  $f(x, y, z = z)$ . Then,

$$\int_V z dV = \int_{V_1} (z) r dr d\theta dz + \int_{V_2} (z) (\rho^2 \sin \varphi) d\rho d\theta d\varphi$$

$$I_c = \int_0^4 \int_0^{2\pi} \int_0^1 (z)(r) dr d\theta dz$$

$$= \int_0^4 \int_0^{2\pi} \left( \frac{z}{2} r^2 \Big|_0^1 \right) d\theta dz$$

$$= \int_0^4 \int_0^{2\pi} \frac{z}{2} d\theta dz$$

$$= \int_0^4 \pi z dz$$

$$= 8\pi$$

$$I_s = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (4 + \cos \varphi)(r^2 \sin \varphi) d\rho d\theta d\varphi$$

$$= 2\pi \int_0^{\pi/2} \int_0^1 (4 + \cos \varphi)(r^2 \sin \varphi) d\rho d\varphi$$

$$= 2\pi \int_0^1 r^2 dr \int_0^{\pi/2} 4 \sin \varphi + \cos \varphi \sin \varphi d\varphi$$

$$= \frac{2\pi}{3} \frac{9}{2}$$

$$= 3\pi$$

$$I = I_c + I_s$$

$$= 11\pi$$

## Vector Valued Functions

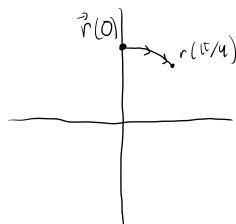
Let

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

$$0 \leq t \leq \frac{\pi}{4}$$

$$\vec{r}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{r}\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

We can expand to  $\mathbb{R}^3$ :

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ 15 \end{pmatrix}$$

Circle at  $z = 15$ 

$$\vec{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ t \sin(t) \end{pmatrix}$$

Helix

$$\vec{r}''(t) = \begin{pmatrix} t \sin(t) \\ t \cos(t) \\ t \end{pmatrix}$$

Conical Helix

We can also add parameters:

$$\vec{r}(t) = \begin{pmatrix} u \cos(v) \\ u \sin(v) \end{pmatrix}$$

Disc

## Differentiation in Vector Valued Functions

For

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

we have

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

For example, in uniform circular motion, we have

$$\begin{aligned}\vec{x}(t) &= r \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} \\ \frac{d\vec{x}}{dt} &= \omega r \begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix} \\ \frac{d^2\vec{x}}{dt^2} &= -\omega^2 r \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} \\ &= -\omega^2 r \vec{x}(t)\end{aligned}$$

#### Examining the Helix

Let

$$\begin{aligned}\vec{r}(t) &= \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \\ &= \begin{pmatrix} r \cos \omega t \\ r \sin \omega t \\ ct \end{pmatrix} \\ \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ &= \begin{pmatrix} -\omega r \sin \omega t \\ \omega r \cos \omega t \\ c \end{pmatrix} \\ \vec{a}(t) &= \begin{pmatrix} -\omega^2 r \cos \omega t \\ -\omega^2 r \sin \omega t \\ 0 \end{pmatrix} \\ \|\vec{r}(t)\| &= \sqrt{r^2 + m^2 t^2} \\ \|\vec{v}(t)\| &= \sqrt{\omega^2 r^2 + m^2} \\ \|\vec{a}(t)\| &= \omega^2 r\end{aligned}$$

#### Path Length of Helix

Along  $\vec{r}(t)$ , the total length traveled along  $t \in [a, b]$  is

$$\ell = \int_a^b \|\vec{r}'(t)\| dt$$

So, for the helix, we have

$$\begin{aligned}\ell &= \int_a^b \sqrt{\omega^2 r^2 + m^2} dt \\ &= \frac{2\pi \sqrt{\omega^2 r^2 + m^2}}{\omega}\end{aligned}$$

## Vector Fields

The following are vector fields:

$$\vec{F}(x, y) = \begin{pmatrix} x + y \\ y^2 - x \end{pmatrix}$$

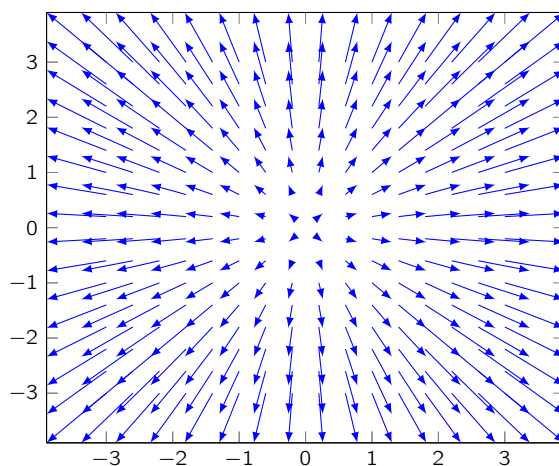
$$\vec{F}(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{F}(x, y, z, w) = \begin{pmatrix} xy \\ yz \\ 0 \\ w \end{pmatrix}$$

For the vector field

$$\vec{F}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix},$$

we select various points and draw their resulting vectors.



## Curves in a Vector Field

Given a field  $\vec{F}(x, y)$ , a flow line  $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$  is such that  $\vec{F}(\vec{r}(t)) = \vec{r}'(t)$ . For example, let

$$\vec{F}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$x'(t) = x(t)$$

$$y'(t) = y(t)$$

$$x(0) = x_0$$

$$y(0) = y_0$$

$$x(t) = x_0 e^t$$

$$y(t) = y_0 e^t.$$

This time, let

$$\vec{F}(x, y) = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$\vec{r}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$x'(t) = x(t)$$

$$y'(t) = 1$$

$$x(t) = x_0 e^t$$

$$y(t) = t + y_0$$

$$t = \ln(x) - \ln(x_0)$$

$$y = \ln(x) + (y_0 - \ln(x_0))$$

$$x \neq 0$$

## Flow Lines in Conservative Vector Fields

Let  $f(x, y) = x^2 + y$ . If

$$\begin{aligned} \vec{F}(x, y) &= \nabla f(x, y) \\ &= \begin{pmatrix} 2x \\ 1 \end{pmatrix}, \end{aligned}$$

then  $\vec{F}(x, y)$  is known as a **conservative** vector field — i.e., it is derived from a gradient.

Let

$$\vec{F}(x, y) = \begin{pmatrix} 2x + y \\ y \end{pmatrix}$$

$$\vec{r}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Solve the system:

$$x'(t) = 2x + y$$

$$y'(t) = y$$

Guess:

$$x(t) = Ae^t + Be^{2t}$$

$$x(0) = x_0$$

$$x_0 = A + B$$

$$x'(t) = Ae^t + 2Be^{2t}$$

$$= 2(Ae^t + Be^{2t}) + y_0e^t$$

$$A = 2A + y_0$$

$$A = -y_0$$

$$B = x_0 + y_0$$

$$x(t) = -y_0e^t + (x_0 + y_0)e^{2t}$$

$$y(t) = y_0e^t$$

$$x(t) = -y + (x_0 + y_0) \left( \frac{y(t)}{y_0} \right)^2$$

Let

$$\vec{F} = \begin{pmatrix} y \\ -x \\ 2 \end{pmatrix}$$

$$x'(t) = y(t)$$

$$y'(t) = -x(t)$$

$$z'(t) = z(t)$$

$$z(t) = 2$$

$$x''(t) = -x(t)$$

$$y''(t) = -y(t)$$

$$x(t) = A \cos(t) + B \sin(t)$$

$$y(t) = P \cos(t) + Q \sin(t)$$

$$z(t) = 2t + z_0$$

#### Conservative Vector Fields and Calculating Work

We can find work using the traditional formula from physics:

$$W = \int \vec{F} \cdot d\vec{r}$$



Let

$$\vec{F} = \begin{pmatrix} x \\ y^2 \end{pmatrix}$$

$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

$$0 \leq t \leq 1$$

$$d\vec{r} = \vec{r}'(t)dt$$

$$W = \int \begin{pmatrix} x(t) \\ y(t)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$

$$= \int_0^1 t + 2t^5 dt$$

$$= \frac{5}{6}$$