

This is a collection of old real analysis qualifier exam solutions.

## August 2019

### Problem 1

- (a) Recall that the Cantor set  $\mathcal{C}$  is defined to consist of all  $x \in [0, 1]$  such that  $x$  only contains 0 and 2 in the ternary expansion of  $x$ . Writing  $a \in [0, 2]$  as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where  $a_k \in \{0, 1, 2\}$ , we may then find  $a_k$  at each ternary expansion slot for  $k$  as follows:

- if  $a_k = 0$ , we may find  $b_k, c_k \in \mathcal{C}$  such that  $b_k = c_k = 0$
- if  $a_k = 2$ , we may find  $b_k, c_k \in \mathcal{C}$  such that  $b_k = 2$  and  $c_k = 0$  or vice versa.
- if  $a_k = 1$ , we may find  $b_k, c_k \in \mathcal{C}$  such that  $b_{k+1} = c_{k+1} = 2$ .

Therefore, since every digit of every ternary expansion in  $[0, 2]$  can be obtained from  $\mathcal{C}$ , we see that  $\mathcal{C} + \mathcal{C} = [0, 2]$ .

- (b) We may set  $B$  to be the union of all integer translates of  $\mathcal{C}$ , and set  $A = \mathcal{C}$ . This yields closed subsets of  $\mathbb{R}$  with Lebesgue measure zero that sum to  $\mathbb{R}$ .

### Problem 2

Consider the sequence of functions

$$f_n(x) = n \mathbb{1}_{[\frac{1}{n+1}, \frac{1}{n}]},$$

defined on  $[0, 1]$ . This sequence is pointwise convergent everywhere to zero, as  $f_n(0) = 0$  and the Archimedean property give that for any  $x \in (0, 1]$ , there is some  $n$  large enough that gives  $\frac{1}{n} < x$ . Furthermore, we see that

$$\begin{aligned} \int f_n \, d\mu &= n \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{n+1} \\ &\rightarrow 0. \end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned} \int \Phi \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1} \\ &\rightarrow \infty. \end{aligned}$$

### Problem 4

Suppose toward contradiction that both  $f$  and  $1/f$  are in  $L_1(\mathbb{R})$ . Then, from Hölder's Inequality, we have

$$\begin{aligned} \infty &= \int 1 \, d\mu \\ &\leq \left( \int f \, d\mu \right)^{1/2} \left( \int \frac{1}{f} \, d\mu \right)^{1/2} \\ &< \infty, \end{aligned}$$

which is a contradiction.

**Problem 5**

- (a) Let  $f \in L_2([-1, 1])$ . We may find  $g \in C([-1, 1])$  such that  $\|f - g\|_{L_2} < \varepsilon/2$ . Similarly, we may find a polynomial  $p$  such that  $\|g - p\|_{\infty} < \varepsilon/4$ , meaning that  $|p(x) - g(x)| < \varepsilon/4$  for all  $x \in [-1, 1]$ . This yields

$$\begin{aligned}\|p - g\|_{L_2} &= \left( \int_{-1}^1 |p(x) - g(x)|^2 dx \right)^{1/2} \\ &< \left( \int_{-1}^1 \left( \frac{\varepsilon}{4} \right)^2 dx \right)^{1/2} \\ &= \left( \frac{\varepsilon^2}{8} \right)^{1/2} \\ &< \frac{\varepsilon}{2},\end{aligned}$$

so  $\|f - p\|_{L_2} < \varepsilon$ , meaning that the closed linear span of the monomials is dense in  $L_2$ , and the Legendre polynomials form an orthonormal system.

- (b) We see that at every step in evaluating the expression

$$L_n(x) = c_n \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (*)$$

the degree of the polynomial increases by 1, so each  $L_n(x)$  has degree  $n$ . To verify that the polynomials generated from  $(*)$  are orthogonal to each other, we let  $n > m$  without loss of generality, and use integration by parts to obtain

$$\begin{aligned}\langle L_n, L_m \rangle &= \int_{-1}^1 \left( \frac{d^n}{dx^n} (x^2 - 1)^n \right) \left( \frac{d^m}{dx^m} (x^2 - 1)^m \right) dx \\ &= \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^m}{dx^m} (x^2 - 1)^m \Big|_{-1}^1 - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^{m+1}}{dx^{m+1}} (x^2 - 1)^m dx \\ &\vdots \\ &= (-1)^n \int_{-1}^1 \frac{d^{m+n}}{dx^{m+n}} (x^2 - 1)^m dx \\ &= (-1)^n \int_{-1}^1 \frac{d^n}{dx^n} \left( \frac{d^m}{dx^m} (x^2 - 1)^m \right) dx \\ &= (-1)^n \int_{-1}^1 \frac{d^n}{dx^n} L_m(x) dx \\ &= 0,\end{aligned}$$

seeing as we are taking  $n$  derivatives of a degree  $m < n$  polynomial.

**January 2020****Problem 1**

- (a) This is false. If  $A \subseteq [0, 1]$  is the “fat Cantor set” constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then  $\mu(A) = \frac{1}{2}$ , but  $A$  is nowhere dense, meaning that if  $U \subseteq A$  is open, then  $U = \emptyset$ .

To see that  $A$  is nowhere dense, we see that  $A$  is closed, so if  $x \in A \subseteq [0, 1]$ , and  $\varepsilon > 0$ , we may show that the interval  $(x - \varepsilon, x + \varepsilon)$  is not contained in  $A$ . In the recursive construction of  $A$ , we may see that there is some step  $n_1$  such that  $\frac{1}{4^{n_1}} < 2\varepsilon$ , implying that  $(x - \varepsilon, x + \varepsilon)$  is not contained in the recursive construction at  $n_1$ . Therefore  $A^\circ = \emptyset$ .

- (b) This is true. By the definition of the Lebesgue outer measure, for any  $\varepsilon > 0$ , there are  $\{(a_k, b_k)\}_{k=1}^\infty$  such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^\infty (a_k, b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^\infty (a_k, b_k),$$

we have that  $U$  is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\{U \mid A \subseteq U, U \text{ open}\}.$$

**Remark:** Part (a) can be solved by selecting  $A = \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$ .

### Problem 3

- (a) Consider the algebra of polynomials on  $[0, 1]$  without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and  $f(x) = x$  separates points in  $[0, 1]$ , this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at  $x = 0$ , the uniform closure of the algebra yields all the continuous functions on  $[0, 1]$  with  $f(0) = 0$ .
- (b) In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra  $\mathcal{A}$  to include the constant functions.

### Problem 4

We consider the signed measure on  $\mathcal{F}$  defined by

$$\nu(E) = \int_E f \, d\mu,$$

meaning that  $\nu \ll \mu$ , so the function  $g := \frac{d\nu}{d\mu}$ , where  $\frac{d\nu}{d\mu}$  denotes the Radon–Nikodym derivative of  $\nu$  with respect to  $\mu$  (where we restrict  $\mu$  to  $\mathcal{F}$ ), is  $\mathcal{F}$ -measurable (by Radon–Nikodym) and in  $L_1(\mathbb{R}, \mathcal{F}, \mu)$ . This gives, for all  $E \in \mathcal{F}$ ,

$$\begin{aligned} \int_E g \, d\mu &= \int_E \frac{d\nu}{d\mu} \, d\mu \\ &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f \, d\mu. \end{aligned}$$

**Problem 5**

Let  $M = \mu(X)$ .

Let  $(f_n)_n \rightarrow f$  in measure, and let  $\varepsilon > 0$ . If we let

$$\begin{aligned} A &= \{x \mid |f_n(x) - f(x)| > \varepsilon/2M\} \\ B &= \{x \mid |f_n(x) - f(x)| \leq \varepsilon/2M\}, \end{aligned}$$

we have

$$\begin{aligned} \int_X \min(1, |f_n - f|) \, d\mu &= \int_A \min(1, |f_n - f|) \, d\mu + \int_B \min(1, |f_n - f|) \, d\mu \\ &\leq \mu(A) + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Meanwhile, if

$$\int_X \min(1, |f_n - f|) \, d\mu \rightarrow 0,$$

then by Chebyshev's Inequality, we have, for a fixed  $0 < \varepsilon \leq 1$ ,

$$\begin{aligned} \mu(\{x \mid |f_n - f| \geq \varepsilon\}) &= \mu(\{x \mid \min(1, |f_n - f|) \geq \varepsilon\}) \\ &\leq \frac{1}{\varepsilon} \int_X \min(1, |f_n - f|) \, d\mu \\ &\rightarrow 0, \end{aligned}$$

so  $(f_n)_n \rightarrow f$  in measure.

**August 2020****Problem 1**

This is false. To see this, let  $\mathcal{C}(x)$  denote the Cantor–Lebesgue function, and let

$$h(x) = \sum_{n=-\infty}^{\infty} \mathcal{C}(x - n) + n.$$

Then, since  $\mathcal{C}(x)$  has derivative zero almost everywhere, the sum of a number of translates of  $\mathcal{C}(x)$  still has derivative zero almost everywhere. Then, setting

$$f(x) = h(x) + x,$$

we get that  $f(x)$  has derivative equal to 1 almost everywhere. However, at the same time,  $f(2) - f(1) = 2$ .

**Problem 2**

We show the inverse problem, which is that every closed set in  $\mathbb{R}^2$  is  $G_\delta$ . To do this, we let  $A \subseteq \mathbb{R}^2$  be closed, nonempty, and proper (if  $A = \emptyset$  or  $A = \mathbb{R}^2$  the answer is trivial).

Then, there is some  $x \in A^c$ , and specifically there is  $x \in A^c$  with rational coordinates (else, select  $y \in \mathbb{Q}^2$  within the ball of radius  $\varepsilon$  that allows  $A^c$  to be open). Furthermore, since  $\mathbb{R}^2$  is a metric space,  $\mathbb{R}^2$  is regular, so there are open  $U_x$  and  $V_x$  such that  $A \subseteq U_x$ ,  $x \in V_x$ , and  $U_x \cap V_x = \emptyset$ .

Therefore, we get

$$A = \bigcap \{U_x \mid x \in \mathbb{Q}^2 \setminus A\},$$

meaning that  $A$  is  $G_\delta$ . Taking complements, we thus get that every open set is  $F_\sigma$ .

### Problem 3

(a) We see that

$$\begin{aligned} \langle Pf_i, f_j \rangle &= \delta_{i+1,j} \\ &= \delta_{i,j-1} \\ &= \langle f_i, f_{j-1} \rangle \\ &= \langle f_i, P^*f_j \rangle, \end{aligned}$$

so that  $Pf_n = f_{n-1}$  if  $n > 1$ . Else, if  $n = 1$ , then  $P^*f_n = 0$ .

(b) We see that, acting on the orthonormal basis  $(f_n)_n$ ,  $P^*P(f_n) = f_n$ , and

$$PP^*(f_n) = \begin{cases} 0 & n = 1 \\ 1 & \text{else,} \end{cases}$$

so that  $P^*P = I$  and  $PP^*$  is as above.

### Problem 4

We see that

$$\mu(\{x \mid f_n(x) > t\}) = \mu(X) - \mu(\{x \mid f_n(x) \leq t\}),$$

so by taking limits, we find that

$$\lim_{n \rightarrow \infty} \mu(\{x \mid f_n(x) > t\}) = \begin{cases} 1 & t < 0 \\ 0 & t \geq 0 \end{cases}.$$

So, if  $\varepsilon > 0$ , then

$$\begin{aligned} \mu(\{x \mid |f_n(x)| > \varepsilon\}) &= \mu(\{x \mid f_n(x) < -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\leq \mu(\{x \mid f_n(x) \leq -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\rightarrow 0. \end{aligned}$$

## August 2022

### Problem 1

We note that

$$\begin{aligned} \left| \frac{n \sin(x/n)}{x(1+x^2)} \right| &\leq \left| \frac{n(x/n)}{x(1+x^2)} \right| \\ &= \frac{1}{1+x^2}, \end{aligned}$$

and since  $\frac{1}{1+x^2}$  is integrable, we may use Dominated Convergence to switch limit and integral, giving

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx &= \int_0^\infty \lim_{n \rightarrow \infty} \frac{n \sin(x/n)}{x(1+x^2)} dx \\ &= \int_0^\infty \lim_{h \rightarrow 0} \frac{\frac{1}{h} \sin(hx)}{x(1+x^2)} dx \\ &= \int_0^\infty \frac{x}{x(1+x^2)} dx \\ &= \frac{\pi}{2}. \end{aligned}$$

## Problem 2

- (a) Let  $f$  be Lipschitz, and let  $M$  denote the Lipschitz constant — i.e.,  $|f(x) - f(y)| \leq |x - y|$  for all  $x, y \in [a, b]$ . Set  $\delta = \frac{\varepsilon}{M}$ . Then, if  $\{(a_j, b_j)\}_{j=1}^k$  is a partition such that  $\sum_{j=1}^k |b_j - a_j| < \delta$ , we have

$$\begin{aligned} \sum_{j=1}^k |f(b_j) - f(a_j)| &\leq M \sum_{j=1}^k |b_j - a_j| \\ &< \varepsilon. \end{aligned}$$

Thus,  $f$  is absolutely continuous. Now, if  $x, x+h \in [a, b]$ , we have that

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq M,$$

meaning that

$$\begin{aligned} |f'(x)| &= \lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x)}{h} \right| \\ &\leq M, \end{aligned}$$

and since  $f'(x)$  exists for a.e.  $x \in [a, b]$ , we have that  $\text{ess sup}_{x \in [a, b]} |f'(x)| \leq M$ , so  $f' \in L_\infty([a, b])$ .

Let  $f$  be absolutely continuous with bounded derivative. Then, if  $M$  is the essential supremum of the  $f'$ , the fundamental theorem of calculus gives

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_x^y f'(t) dt \right| \\ &\leq \int_x^y |f'(t)| dt \\ &\leq \int_x^y M dx \\ &= M|y - x|, \end{aligned}$$

so  $f$  is Lipschitz.

- (b) If  $f$  is such that  $f'(x)$  exists, then for  $x, x+h \in [a, b]$ , we have

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq \|f'\|_{\text{Lip}},$$

so by taking limits, we have

$$|f'(x)| \leq \|f'\|_{\text{Lip}}.$$

Thus, this ordering must respect essential suprema, meaning

$$\|f'\|_{L_\infty} \leq \|f\|_{Lip}.$$

Furthermore, if  $\varepsilon > 0$ , there are  $x, y \in [a, b]$  with  $x < y$  such that

$$\begin{aligned} \|f\|_{Lip} - \varepsilon &< \left| \frac{f(y) - f(x)}{y - x} \right| \\ &= \frac{1}{|y - x|} \left| \int_x^y f'(t) dt \right| \\ &\leq \frac{1}{|y - x|} \int_x^y |f'(t)| dt \\ &\leq \frac{1}{|y - x|} \int_x^y \|f'\|_{L_\infty} dt \\ &= \|f'\|_{L_\infty}, \end{aligned}$$

and since  $\varepsilon$  is arbitrary, we have  $\|f\|_{Lip} \leq \|f'\|_{L_\infty}$ .

### Problem 3

We start by showing that

$$|a - b| = \int_0^\infty |\mathbb{1}_{(t, \infty)}(a) - \mathbb{1}_{(t, \infty)}(b)| dt$$

for all  $a, b \in [0, \infty)$ . Without loss of generality,  $a \leq b$ . To see this, note that there are three cases:

$$|\mathbb{1}_{(t, \infty)}(a) - \mathbb{1}_{(t, \infty)}(b)| = \begin{cases} 0 & t < a, b \\ 1 & a \leq t < b, \\ 0 & a, b \leq t \end{cases}$$

giving

$$\begin{aligned} \int_0^\infty \mathbb{1}_{[a, b)} dt &= \mu([a, b)) \\ &= b - a \\ &= |a - b|. \end{aligned}$$

Now, we have

$$\begin{aligned} \|f - g\|_{L_1} &= \int_X |f(x) - g(x)| d\mu(x) \\ &= \int_X \int_0^\infty |\mathbb{1}_{(t, \infty)}(f(x)) - \mathbb{1}_{(t, \infty)}(g(x))| dt d\mu(x), \end{aligned}$$

and by Tonelli's Theorem, we have

$$\begin{aligned} &= \int_0^\infty \int_X |\mathbb{1}_{f^{-1}((t, \infty))} - \mathbb{1}_{g^{-1}((t, \infty))}| d\mu(x) dt \\ &= \int_0^\infty \int_X \mathbb{1}_{f^{-1}((t, \infty)) \Delta g^{-1}((t, \infty))} d\mu(x) dt \\ &= \int_0^\infty \mu(f^{-1}((t, \infty)) \Delta g^{-1}((t, \infty))) dt. \end{aligned}$$

**Problem 4**

- (a) Since  $|\mu| \perp |\nu|$ , there are  $U, V \subseteq X$  such that  $|\mu|$  is concentrated on  $U$  and  $|\nu|$  is concentrated on  $V$ , with  $U \cap V = \emptyset$ .

Note that by the Jordan decompositions, we have  $|\mu| = \mu_1 + \mu_2 \geq \mu_{1,2}$  so  $\mu_{1,2}$  are concentrated on  $U$ , and similarly  $\nu_{1,2}$  are concentrated on  $V$ , so  $\mu_i \perp \nu_j$ .

- (b) We show that the measures  $\mu_1 + \nu_1$  and  $\mu_2 + \nu_2$  are mutually singular. To see this, note the following:

- $\mu_1 = 0$  on  $N_\mu \cup V$ ;
- $\nu_1 = 0$  on  $N_\nu \cup U$ ;
- $\mu_2 = 0$  on  $P_\mu \cup V$ ;
- $\nu_2 = 0$  on  $P_\nu \cup U$ ,

so  $\mu_1 + \nu_1 = 0$  on  $A = (N_\mu \cup V) \cap (N_\nu \cup U)$ , and  $\mu_2 + \nu_2 = 0$  on  $B = (P_\mu \cup V) \cap (P_\nu \cup U)$ . Therefore, since

$$\begin{aligned} A \cup B &= (N_\mu \cap N_\nu) \cup (N_\mu \cap U) \cup (N_\nu \cap V) \\ &\quad \cup (P_\mu \cap P_\nu) \cup (P_\mu \cap U) \cup (P_\nu \cap V) \\ &= X \end{aligned}$$

$$\begin{aligned} A \cap B &= (N_\mu \cup V) \cap (N_\nu \cup U) \\ &\quad \cap (P_\mu \cup V) \cap (P_\nu \cup U) \\ &= \emptyset, \end{aligned}$$

the measures  $\mu_1 + \nu_1$  and  $\mu_2 + \nu_2$  are mutually singular, so  $A \sqcup B$  forms a Hahn decomposition for  $\mu + \nu$  with corresponding Jordan decomposition of  $(\mu_1 + \nu_1) - (\mu_2 + \nu_2)$ . Thus,

$$\begin{aligned} \|\mu + \nu\|_{TV} &= |\mu + \nu|(X) \\ &= (\mu_1 + \nu_1)(X) + (\mu_2 + \nu_2)(X) \\ &= (\mu_1 + \mu_2)(X) + (\nu_1 + \nu_2)(X) \\ &= |\mu|(X) + |\nu|(X) \\ &= \|\mu\|_{TV} + \|\nu\|_{TV}. \end{aligned}$$

**January 2023****Problem 1**

By using Fatou's Lemma, and assuming WLOG that  $(f_n)_n \rightarrow f$  pointwise everywhere, we get

$$\begin{aligned} \int_X |f|^p \, d\mu &= \int_X \liminf_{n \rightarrow \infty} |f_n|^p \, d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p \, d\mu \\ &\leq 1, \end{aligned}$$

so  $\|f\|_{L_p} \leq 1$ .



**Problem 2**

Let

$$f(t) = \mu(E \cap (-\infty, t)),$$

and for any sequence  $(t_n)_n$ , define

$$E_n = E \cap (-\infty, t_n).$$

We will show that  $f$  is left- and right-continuous, hence continuous. To start, if  $(t_n)_n \searrow t$ , then

$$\bigcap_{n \in \mathbb{N}} E_n = E \cap (-\infty, t],$$

so

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n \setminus \{t\}\right) \\ &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) - \mu(\{t\}). \end{aligned}$$

Since  $\mu$  is atomless, we see that  $\mu(\{t\}) = 0$ , so since  $\mu(E) < \infty$ ,

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \lim_{n \rightarrow \infty} f(t_n). \end{aligned}$$

Thus,  $f$  is right-continuous. Similarly, if  $(t_n)_n \nearrow t$ , then

$$\bigcup_{n \in \mathbb{N}} E_n = E \cap (-\infty, t),$$

so by continuity from below,

$$\begin{aligned} f(t) &= \mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \lim_{n \rightarrow \infty} f(t_n). \end{aligned}$$

Therefore,  $f$  is continuous. Since

$$\begin{aligned} \lim_{t \rightarrow -\infty} f(t) &= 0 \\ \lim_{t \rightarrow \infty} f(t) &= \mu(E) \\ &> 0, \end{aligned}$$

the intermediate value theorem gives some  $t_0 \in \mathbb{R}$  such that

$$\begin{aligned} f(t_0) &= \mu(E \cap (-\infty, t_0)) \\ &= \frac{1}{2} \mu(E). \end{aligned}$$

**Problem 4**

We start by showing that  $\|\cdot\|_{W_p}$  is indeed a norm. To see that  $\|\cdot\|_{W_p}$  is positive definite, if

$$\|f\|_{W_p} = 0,$$

then  $|f(0)| = 0$  and  $\|f'\|_{L_p} = 0$ . Since  $\|f'\|_{L_p} = 0$ ,  $f' = 0$  a.e. as  $L_p$  is a Banach space. Note that, by the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t) dt,$$

so  $f(x) = 0$  almost everywhere, hence  $f(x) = 0$  in  $L_p$ .

Next, to see homogeneity, we have for all  $\alpha \in \mathbb{C}$ ,

$$\begin{aligned} \|\alpha f\|_{W_p} &= |\alpha f(0)| + \|(\alpha f)'\|_{L_p} \\ &= |\alpha| \left( |f(0)| + \|f'\|_{L_p} \right) \\ &= |\alpha| \|f\|_{W_p}, \end{aligned}$$

as  $\|\cdot\|_{L_p}$  is a norm. Finally, we have

$$\begin{aligned} \|f + g\|_{W_p} &= |(f + g)(0)| + \|(f + g)'\|_{L_p} \\ &\leq |f(0)| + |g(0)| + \|f'\|_{L_p} + \|g'\|_{L_p} \\ &= \|f\|_{W_p} + \|g\|_{W_p}, \end{aligned}$$

as  $\|\cdot\|_{L_p}$  is a norm, so the triangle inequality holds. Thus,  $\|\cdot\|_{W_p}$  is a norm.

Let  $(f_n)_n$  be Cauchy in  $W_p([0, 1])$ . Then, for all  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,

$$\begin{aligned} \|f_n - f_m\|_{W_p} &= |f_n(0) - f_m(0)| + \|f'_n - f'_m\|_{L_p} \\ &< \varepsilon, \end{aligned}$$

meaning that both

$$\begin{aligned} |f_n(0) - f_m(0)| &< \varepsilon \\ \|f'_n - f'_m\|_{L_p} &< \varepsilon. \end{aligned}$$

Since  $\mathbb{C}$  and  $L_p([0, 1])$  are complete, there is  $c \in \mathbb{C}$  and  $g \in L_p([0, 1])$  such that

$$\begin{aligned} f_n(0) &\rightarrow c \\ f'_n &\rightarrow g. \end{aligned}$$

Define

$$f(x) = c + \int_0^x g(t) dt.$$

Then, we note that by the Fundamental Theorem of Calculus,

$$\begin{aligned} f'(x) &= g(x) \\ &\in L_p([0, 1]), \end{aligned}$$

so  $f \in W_p([0, 1])$ . Finally, we see that

$$\begin{aligned} \|f_n - f\|_{W_p([0, 1])} &= |f_n(0) - f(0)| + \|f'_n - f'\|_{L_p} \\ &= |f_n(0) - c| + \|f'_n - g\|_{L_p} \\ &\rightarrow 0, \end{aligned}$$

so  $(f_n)_n \rightarrow f$  in  $W_p$ , meaning  $W_p$  is complete.