Problem (Problem 1)**:** Let $f\colon M \to N$ be a smooth map of manifolds. Prove that the graph of f is a smooth submanifold of $M \times N$.

Solution: Let (U, φ) be a chart on M with $\varphi(U) \cong \mathbb{R}^m$, and (V, ψ) a chart on N with $\psi(V) \cong \mathbb{R}^n$ and $f(U) \subseteq V$.

Let $U \times V$ be the corresponding open set in $M \times N$, and let $(p, q) \in U \times V$. We will define a coordinate map on $\rho\colon U \times V \to \mathbb{R}^m \times \mathbb{R}^n$ given by $\rho(p, q) = (\varphi(p), \psi(q) - \psi(f(p)))$. We observe in particular that if $(p, q) = (p, f(p)) \in \Gamma(f) \cap (U \times V)$, then $\rho(p, f(p)) = (\varphi(p), 0)$, meaning that ρ is a smooth chart for $\Gamma(f)$.

Problem (Problem 2)**:** Let $U(n)$ be the set of unitary complex $n \times N$ matrices. Write $SU(n) \leqslant U(n)$ for the kernel of the determinant map.

(a) Show that $U(1)$ is diffeomorphic to the circle, so that $SU(1)$ is a point.

(b) Prove that $U(n)$ is a smooth manifold.

(c) Prove that $SU(2)$ is diffeomorphic to S^3, the three-sphere.

(d) Prove that $U(2)$ is diffeomorphic to $S^1 \times S^3$.

Solution:

(a) Since complex 1×1 matrices are diffeomorphic to $\mathbb{C}$, we see that $x \in U(1)$ if and only if $|x|^2 = 1$, meaning $|x| = 1$, so $x = e^{i\theta}$ for some θ. In particular, this means that the assignment $x \mapsto e^{i\theta}$ gives a diffeomorphism between S^1 and $U(1)$.

(b) Consider the self-map $f\colon \operatorname{Mat}_n(\mathbb{C}) \to \operatorname{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ given by $f(A) = A^*A$. Note that this maps $\operatorname{Mat}_n(\mathbb{C})$ to positive semi-definite (Hermitian) matrices $\operatorname{Mat}_n(\mathbb{C})^+ \subseteq \operatorname{Mat}_n(\mathbb{C})_{\text{s.a.}}$.

Observe that an element of the tangent space to $A \in \operatorname{Mat}_n(\mathbb{C})$ is given by $s_B = A + tB$, where $t \in \mathbb{R}$ and $B \in \operatorname{Mat}_n(\mathbb{C})$. Applying f, we get

$$f(A + tB) = A^*A + t(A^*B + B^*A) + t^2B^*B;$$

meaning that $D_A f$ applied to s_B yields $A^*A + t(A^*B + B^*A)$.

Note that if A is unitary and B is Hermitian, then $(AB)^*(AB) = B^*B$, and

$$A^*A + t(A^*(AB) + (AB)^*A) = I + 2tB,$$

meaning that $D_A f$ is surjective onto the tangent space at the identity when A is unitary (after a scaling), so I is a regular value for f.

(c) We view S^3 as a subset of $\mathbb{C}^2$, so that S^3 consists of all (z_1, z_2) such that

$$|z_1|^2 + |z_2|^2 = 1.$$

We claim that the matrix

$$A_{z_1,z_2} = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}$$

is an element of $SU(2)$. Since it is uniquely determined by z_1 and z_2 in S^3, it follows that $SU(2)$ is diffeomorphic to S^3.

To see this, observe that

$$\det(A) = 1$$

$$\begin{aligned} A^*A &= \begin{pmatrix} \overline{z_1} & -z_2 \\ \overline{z_2} & z_1 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \\ &= \begin{pmatrix} |z_1|^2 + |z_2|^2 & z_2\overline{z_1} - z_2\overline{z_1} \\ z_1\overline{z_2} - z_1\overline{z_2} & |z_1|^2 + |z_2|^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, SU(3) is diffeomorphic to S^3, with the diffeomorphism given by coordinate assignment.

(d) Observe that if $(z_1, z_2) = z \in S^3 \subseteq \mathbb{C}^2$, then if $a \in U(2)$, we have $az \in S^3$. In particular, since unitary matrices are invertible, the operation of $a \in U(2)$ on $z \in S^3$ by multiplication is a group action.

We observe now that the action of U(2) on $S^3 \subseteq \mathbb{C}^2$ by matrix multiplication is transitive, since for any element $(w_1, w_2) \in S^3$, the matrix

$$\begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

and

$$\begin{pmatrix} \overline{w_1} & \overline{w_2} \\ -w_2 & w_1 \end{pmatrix}\begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Additionally, we observe that for any θ,

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

meaning

$$S^3 \cong U(2)/P,$$

where P consists of all matrices of the form

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

We observe that P is diffeomorphic to S^1 via a coordinate assignment, so $U(2) \cong S^3 \times S^1$.

Problem (Problem 3)**:** In this exercise, we will prove the Frobenius theorem.

Let M be a smooth manifold of dimension n, and let D be an r-dimensional distribution on M, where $r \leqslant n$. That is, D picks out an r-dimensional D_p of T_pM for each $p \in M$. In other words, at every point, there are r distinguished, linearly independent vector fields defined in a neighborhood of the point.

A submanifold $N \subseteq M$ is called an *integral submanifold* for D if $T_pN = D_p$ for each $p \in M$. We say D is *completely integrable* if an integral submanifold exists through every point. Integral curves of a vector field are integral submanifolds of a vector field.

We call a distribution that is closed under taking Lie brackets involutive. That is, for any vector fields $X, Y \in D$ (i.e., local 1-distributions that lie in D), then $[X, Y] \in D$.

The Frobenius Theorem says that a distribution D on M is completely integrable if and only if it is involutive.

(a) Show that if D is a completely integrable distribution, then D is involutive.

(b) We say vector fields X and Y commute if $[X, Y] = 0$. For fixed vector fields X and Y, write φ_t and ψ_t for the corresponding flows. Show that the following are equivalent:

(i) X and Y commute;

(ii) Y is invariant under φ_t;

(iii) the flows φ_t and ψ_t commute, so that $\psi_s \circ \varphi_t = \varphi_t \circ \psi_s$ for all t and s where defined.

(c) Assume D is r-dimensional. Choose local coordinates $\{x_1, \dots, x_n\}$ near a point p and r-linearly independent vector fields $Y_1, \dots, Y_r$ near p. Write Y_i as

$$\sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j},$$

and show that there is some permutation of the coordinates such that the $r \times r$ matrix $\left(a_{ij}\right)_{1 \leqslant i,j \leqslant r}$ is invertible near p.

(d) Let $\left(b_{ij}\right)_{1 \leqslant i,j \leqslant r}$ be the inverse of the smoothly varying family of matrices $\left(a_{ij}\right)_{1 \leqslant i,j \leqslant r}$ from the previous part, and let $X_i = \sum_j b_{ij} Y_j$. Show that

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j > r} c_{ij} \frac{\partial}{\partial x_j}$$

for some suitable smooth functions. Show that $X_1, \dots, X_r$ form a basis for D at every point.

(e) Show that $\left[X_i, X_j\right] = 0$ for $1 \leqslant i, j \leqslant r$.

(f) Use the flows generated by $\{X_1, \dots, X_n\}$ to define a smooth map $\phi \colon V \to U$ where V is a neighborhood of $0 \in \mathbb{R}^r$ and U is a neighborhood of $p \in M$.

(g) Choose coordinates $\{t_1, \dots, t_r\}$ on $\mathbb{R}^r$ such that $\phi_*\left(\frac{\partial}{\partial t_i}\right) = X_i$. Argue by shrinking V and U if necessarily that V is a submanifold of U. Use the fact that the flows generated by $X_1, \dots, X_r$ commute to prove that at an arbitrary point $q \in \phi(V)$, we have $D_q = T_q \phi(V)$. Conclude that $\phi(V)$ locally defines an integral submanifold N of the distribution D.

Solution:

(a) Let $(U; x_1, \dots, x_r)$ be a chart about $p \in N$ that extends to coordinates $x_1, \dots, x_n$. Let $\pi \colon M \to N$ is the projection onto N that takes coordinates $x_1, \dots, x_n$ and maps the first r to $x_1, \dots, x_r$, and the rest to 0. Then if $p \in U$, we have that $\pi(p) = p$, meaning that $D_p(\pi) = D_p(\mathrm{id}) = \mathrm{id}$.

Notice then that if X is a vector field on M and $f \colon N \to \mathbb{R}$ is a C^∞ function, then for any $p \in N$,

$$\begin{aligned} X(f \circ \pi)(p) &= \left(D_p \pi X\right)(f)(p) \\ &= X(f)(p). \end{aligned}$$

Therefore, if $X_1, \dots, X_r$ are vector fields on M about p that define our distribution, and $f \colon N \to \mathbb{R}$ is a C^∞ function, then

$$\begin{aligned} [X_k, X_\ell](f \circ \pi)(p) &= X_k(X_\ell(f \circ \pi))(p) - X_\ell(X_k(f \circ \pi))(p) \\ &= X_k\left(\left(D_p \pi X_\ell\right)(f)\right)(p) - X_\ell\left(\left(D_p \pi X_k\right)(f)\right)(p) \\ &= X_k(X_\ell(f))(p) - X_\ell(X_k(f))(p) \\ &= [X_k, X_\ell](f)(p), \end{aligned}$$

meaning that $[X_k, X_\ell]$ is contained in our distribution.

Problem: Let i, j, k be formal symbols that satisfy the relations $i^2 = j^2 = k^2 = ijk = -1$. The $\mathbb{R}$-vector space over $\{1, i, j, k\}$ together with these multiplication rules is called the quarternion algebra $\mathbb{H}$, which is diffeomorphic to $\mathbb{R}^4$. A typical element is $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. Multiplication is defined by the distributive law, and real scalars commute with everything.

(a) Show that the multiplicative structure on $\mathbb{H}$ is completely determined by the rules above.

(b) The conjugate of $q = a + bi + cj + dk$ is $\overline{q} = a - bi - cj - dk$. A unit quaternion is one where $\overline{q}q = 1$. Show that the unit quaternions are diffeomorphic to S^3.

(c) Find the 2×2 unitary complex matrices representing i, j, k with correct multiplicative structure so that the unit quaternions are explicitly diffeomorphic to SU(2).

(d) Show that the unit quaternions act on $\mathbb{R}^3$, which consists of the vector space spanned by i, j, k.

(e) Writing a vector $v \in \mathbb{R}^3$ as $xi + yj + zk$, show that conjugation by a unit quaternion preserves $x^2 + y^2 + z^2$.

(f) Show that every orthogonal transformation of determinant one, known as SO(3), is realized by quaternionic conjugation. Show that the kernel of the map $SU(2) \to SO(3)$ has order two.

(g) Show that SO(3) is diffeomorphic to $\mathbb{RP}^3$.

Solution:

(a) We must verify that the multiplication table for $1, i, j, k$ is completely determined by the rules shown above. To this end, observe that, if we desire to know the value of $x = ij$, then $xk = ijk = -1$, so that $xk = -1$. Then, multiplying on the right by k, we then get that $xk^2 = -k = x(-1)$, so $x = k$. Similarly, we then find that $jk = i$ and $ki = j$.

With the cyclic multiplication in mind, we may then compute $ji = j(jk) = j^2k = -k$, and similarly we find that the anti-cyclic multiplication table yields $ik = -j$ and $kj = -i$.

(b) Notice that $S^3 \subseteq \mathbb{R}^4$ is given by

$$S^3 = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}.$$

If $q = a + bi + cj + dk$ is a unit quaternion, then by assigning $x_1 = a$, $x_2 = b$, $x_3 = c$, and $x_4 = d$, then we see that

$$\begin{aligned} 1 &= \overline{q}q \\ &= (a - bi - cj - dk)(a + bi + cj + dk) \\ &= a^2 + b^2 + c^2 + d^2, \end{aligned}$$

so that q is uniquely assigned to an element of S^3. Thus, S^3 is diffeomorphic to the unit quaternions.

(c) We start by associating 1 to the identity,

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We then need to find three matrices I, J, K (note here that I does not denote the identity) subject to the constraints of:

- $I^2 = J^2 = K^2 = IJK = -\mathbb{1}$;
- $I^*I = J^*J = K^*K = \mathbb{1}$;
- $\det(I) = \det(J) = \det(K) = 1$;