1.8

Problem. Fix a natural number $b \geqslant 2$. Show that every positive real number in x in $[0,1]$ has a b-adic expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{x_n}{b^n},$$

with each $0 \leqslant x_n \leqslant b-1$.

1.9

Problem. Suppose

$$\sum_{n=1}^{\infty} \frac{x_n}{b^n} = \sum_{n=1}^{\infty} \frac{y_n}{b^n},$$

with $0 \leqslant x_n \leqslant b-1$ and $0 \leqslant y_n \leqslant b-1$ integers. Show that either $x_n = y_n$ for all n, or there is an m such that one of the following two cases occurs:

- $x_m = y_m + 1$ and for $n \geqslant m+1$, $y_n = b-1$ and $x_n = 0$;
- $y_m = x_m + 1$ and for $n \geqslant m+1$, $x_n = b-1$ and $y_n = 0$.

1.10

Problem. Show that a number $x \in [0,1]$ is rational if and only if its decimal expansion is eventually periodic. Deduce that irrational numbers have unique decimal expansions.

Solution. Let x be rational. Then, $x = \frac{p}{q}$, with $p \in \mathbb{Z}_{\geqslant 0}$, $q \in \mathbb{Z}_{>0}$, with $\frac{p}{q}$ in lowest terms, with $q > p$.

We write $10x = x_1 + y_1$, with $x_1 = \lfloor 10x \rfloor$ and $y_1 = 10x - \lfloor 10x \rfloor$. Thus, we have

$$\begin{aligned} y_1 &= \frac{10p}{q} - \frac{qx_1}{q} \\ &= \frac{10p - qx_1}{q} \\ &= \frac{m_1}{q}. \end{aligned}$$

We want to show that $0 \leqslant m_1 < q$.

Now, we take $10y_1 = x_2 + y_2$, with

$$\begin{aligned} y_2 &= \frac{10m_1}{q} - \frac{qx_2}{q} \\ &= \frac{m_2}{q}. \end{aligned}$$

Repeatedly, we get $y_n = \frac{m_n}{q}$.

We have $0 \leqslant x_i < 10$, and $0 \leqslant m_i < q$. Thus, looking at the set of pairs $(x_1, m_1), (x_2, m_2), \ldots$. Since x_i and m_i are limited, there cannot be infinitely many distinct pairs; thus, there will necessarily be a value of n such that $(x_k, m_k) = (x_{k+n}, m_{k+n})$.

1.11

Problem. Show that the collection of polynomials with rational coefficients is a countably infinite set.

Solution. Let $\mathcal{P}_n(\mathbb{Q})$ denote the set of polynomials with degree n with coefficients in $\mathbb{Q}$. We construct a bijection

$$\mathcal{P}_n(\mathbb{Q}) \to \prod_{k=0}^{n} \mathbb{Q},$$

where $\prod$ denotes the Cartesian product, by taking

$$a_0 + a_1x + \cdots + a_nx^n \mapsto (a_0, a_1, \ldots, a_n).$$

Since $\prod_{k=0}^{n} \mathbb{Q}$ is a countable Cartesian product of countable sets, this means $\mathcal{P}_n(\mathbb{Q})$ is countable.

Finally, we have $\mathbb{Q}[x]$, the set of all polynomials with rational coefficients, is

$$\mathbb{Q}[x] = \bigcup_{k=0}^{\infty} \mathcal{P}_k(\mathbb{Q}),$$

meaning $\mathbb{Q}[x]$ is countable.

Since $\mathbb{Q}[x]$ is countable, and for any $p(x) \in \mathbb{Q}[x]$, $p(x)$ has at most $\deg(p(x))$ roots, it must be the case that the algebraic numbers are countable.

1.12

Problem. Show that the collection of infinite sequences made up of the elements 0 and 1 is uncountable.

Solution. Let S denote the set of all infinite sequences consisting of the elements 0 and 1. Suppose toward contradiction S is countable. In particular, S is infinite (as the subset of sequences consisting of 0 everywhere except for 1 at position n is infinite), meaning we are supposing that S is denumerable.

Let $f : \mathbb{N} \to S$ be a bijection from S to $\mathbb{N}$, defining $f(i) = s_i$, where s_i is a sequence. We let $s_{i,j}$ denote the jth position of sequence i.

Define a new sequence a by taking

$$a_j = \begin{cases} 0 & s_{j,j} = 1 \\ 1 & s_{j,j} = 0 \end{cases}.$$

It is then the case that $a \in S$, but a is not in $\operatorname{im}(f)$. Thus, f cannot be a bijection, meaning S is not countable.

1.13

Problem. Show that the number of functions mapping from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

Solution. Since the set of functions $f : \mathbb{N} \to \{0, 1\}$ is a subset of the set of functions $f : \mathbb{N} \to \mathbb{N}$, and we have shown that the set of functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable (as a sequence is a function from $\mathbb{N}$ to some codomain), so too is the set of functions $f : \mathbb{N} \to \mathbb{N}$.

Extra Problem 1

Problem. Prove that every infinite subset of a denumerable set is denumerable.

Solution. Let A be a denumerable set, and let $S \subseteq A$ be infinite. We will create a denumeration of S.

Let $f : \mathbb{N} \to A$ be a bijection, which exists as A is denumerable. We define $a_i = f(i)$ for each $i \in \mathbb{N}$.

It is then the case that $S = \left\{a_{i_j}\right\}$ for some $\left\{i_j\right\}_j \subseteq \mathbb{N}$, with $\left\{i_j\right\}$ infinite. Define s_0 to be a_{i_0}, where i_0 denotes the least element in $\left\{i_j\right\}_j$. It is the case that i_0 exists by the well-ordering principle. We then define $s_1 = a_{i_1}$, where i_1 is the least element in $\left\{i_j\right\}_j \setminus \{i_0\}$. Repeatedly, we define $s_n = a_{i_n}$, where i_n is the least element in $\left\{i_j\right\}_j \setminus \{i_0, \dots, i_{n-1}\}$.

Finally, we have the bijection $g : S \to \mathbb{N}$ defined by $g(s_i) = i$, meaning S is denumerable.

Extra Problem 2

Problem. If $|A| \leqslant |B|$, then $|P(A)| \leqslant |P(B)|$.

Solution. Let $f : A \hookrightarrow B$ be an injection. Given $S \subseteq A$, we have $f(S) \subseteq B$, meaning $S \in P(A)$ implies $f(S) \in P(B)$. We let $g : P(A) \to P(B)$ be induced by f, with

$$\begin{aligned} g(S) &= f(S) \\ &= \{f(x) \mid x \in S\}. \end{aligned}$$

Extra Problem 3

Problem. If $|A| = |B|$, then $|P(A)| = |P(B)|$.

Solution. Let $f : A \to B$ be a bijection. Given $S \subseteq A$, we know that $f(S) \subseteq B$, meaning $S \in P(A)$ and $f(S) \in P(B)$. We define $g : P(A) \to P(B)$ to be induced by f as follows:

$$g(S) = \{f(x) \mid x \in S\}.$$

Then, g is a bijection, as f is a bijection.

Extra Problem 4