Math 395

Homework 8
Due: 4/30/2024

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Problem 1

Let K/F be a Galois extension with $\text{Gal}(K/F)$ Abelian of order 10. We will compute the intermediate fields between F and K, and their dimensions over F.

Since $\text{Gal}(K/F)$ is Abelian and of order 10, $\text{Gal}(K/F) \cong \mathbb{Z}/10\mathbb{Z}$. (OEIS A000001)

The subgroups of $\text{Gal}(K/F)$ are isomorphic to the subgroups of $\mathbb{Z}/10\mathbb{Z}$; since $10 = 2 \cdot 5$, it must be the case that $\langle 2 \rangle$, with order 5 and $\langle 5 \rangle$, with order 2, are the two proper subgroups of $\mathbb{Z}/10\mathbb{Z}$ (by Lagrange's Theorem). We will let $H_1 \leq \text{Gal}(K/F)$ be isomorphic to $\langle 2 \rangle$, and $H_2 \leq \text{Gal}(K/F)$ be isomorphic to $\langle 5 \rangle$.

Let $A = K^{H_1}$. Then, since $[\mathbb{Z}/10\mathbb{Z} : \langle 2 \rangle] = 2$, it is the case that $[A : F] = 2$. Similarly, for $B = K^{H_2}$, it is the case that $[\mathbb{Z}/10\mathbb{Z} : \langle 5 \rangle] = 5$, so $[B : F] = 5$.

Problem 3

We will find $\text{Gal}(x^4 - 5x^2 + 6)$ over $\mathbb{Q}$.

To start, factoring $x^4 - 5x^2 + 6$, we find it is equal to $(x^2 - 3)(x^2 - 2) = (x - \sqrt{3})(x + \sqrt{3})(x - \sqrt{2})(x + \sqrt{2})$ in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Since $x^4 - 5x^2 + 6$ is separable in $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \text{Spl}(x^4 - 5x^2 + 6)$, it must be the case that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension.

We know that the basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$, meaning that for $\sigma \in \text{Gal}(K/F)$, we have $\sigma(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) + a + b\sigma(\sqrt{2}) + c\sigma(\sqrt{3}) + d\sigma(\sqrt{2})\sigma(\sqrt{6})$. Thus, the possible elements of $\text{Gal}(K/F)$ are

$$\begin{aligned}
\sigma_0 &:= \text{id} \\
\sigma_1 &:= \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \end{cases} \\
\sigma_2 &:= \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases} \\
\sigma_3 &:= \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases} .
\end{aligned}$$

Notice that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0$, meaning we have $\text{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Problem 4

(a) To find the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$, we find its roots, which are $\pm\sqrt[4]{2}$, $\pm i\sqrt[4]{2}$. Thus, $K = \text{Spl}_{\mathbb{Q}}(f(x)) = \mathbb{Q}(i, \sqrt[4]{2})$.

(b) To find $[K : \mathbb{Q}]$, we see

$$\begin{aligned}[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}] &= [\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] \\ &= 8.\end{aligned}$$

(c) To see that such a σ exists, we will verify that it maps a basis for $\mathbb{Q}(i, \sqrt[4]{2})$ to a basis for $\mathbb{Q}(i, \sqrt[4]{2})$, and keeps $\mathbb{Q}$ fixed.

$$\sigma : \begin{cases} 1 \mapsto 1 \\ \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ \sqrt[4]{4} \mapsto -\sqrt[4]{4} \\ \sqrt[4]{8} \mapsto -i\sqrt[4]{8} \\ i \mapsto i \\ i\sqrt[4]{2} \mapsto -\sqrt[4]{2} \\ i\sqrt[4]{4} \mapsto -i\sqrt[4]{4} \\ i\sqrt[4]{8} \mapsto \sqrt[4]{8} \end{cases}.$$

Therefore, $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. We see that $\sigma^2(\sqrt[4]{2}) = -\sqrt[4]{2}$, $\sigma^3(\sqrt[4]{2}) = -i\sqrt[4]{2}$, meaning $\sigma^4 = \mathrm{id}$.

(d) Letting τ be the restriction of complex conjugation to K, we will show that $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ and $\mathrm{Gal}(K/\mathbb{Q}) = \{\mathrm{id}, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$.

To start, we will verify that τ maps a basis for $\mathbb{Q}(i, \sqrt[4]{2})$ to a basis for $\mathbb{Q}(i, \sqrt[4]{2})$, keeping $\mathbb{Q}$ fixed.

$$\tau : \begin{cases} 1 \mapsto 1 \\ \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ \sqrt[4]{4} \mapsto \sqrt[4]{4} \\ \sqrt[4]{8} \mapsto \sqrt[4]{8} \\ i \mapsto -i \\ i\sqrt[4]{2} \mapsto -i\sqrt[4]{2} \\ i\sqrt[4]{4} \mapsto -i\sqrt[4]{4} \\ i\sqrt[4]{8} \mapsto -i\sqrt[4]{8} \end{cases}$$

We see that $\tau^2 = \mathrm{id}$, and $\tau \neq \sigma$. Defining $\sigma\tau \cdot x = \sigma(\tau(x))$, we see the elements of $\mathrm{Gal}(K/\mathbb{Q})$ are

$$
\begin{aligned}
e &= \mathrm{id} \\
\sigma &= \begin{cases} \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ i \mapsto i \end{cases} \\
\sigma^2 &= \begin{cases} \sqrt[4]{2} \mapsto -\sqrt[4]{2} \\ i \mapsto i \end{cases} \\
\sigma^3 &= \begin{cases} \sqrt[4]{2} \mapsto -i\sqrt[4]{2} \\ i \mapsto i \end{cases} \\
\sigma^4 &= \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto i \end{cases} \\
&= \mathrm{id} \\
\tau &= \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto -i \end{cases} \\
\tau^2 &= \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto i \end{cases} \\
&= \mathrm{id} \\
\sigma\tau &= \begin{cases} \sqrt[4]{2} \overset{\tau}{\mapsto} \sqrt[4]{2} \overset{\sigma}{\mapsto} i\sqrt[4]{2} \\ i \overset{\tau}{\mapsto} -i \overset{\sigma}{\mapsto} -i \end{cases} \\
\sigma^2\tau &= \begin{cases} \sqrt[4]{2} \overset{\tau}{\mapsto} \sqrt[4]{2} \overset{\sigma^2}{\longmapsto} -\sqrt[4]{2} \\ i \overset{\tau}{\mapsto} -i \overset{\sigma^2}{\longmapsto} -i \end{cases} \\
\sigma^3\tau &= \begin{cases} \sqrt[4]{2} \overset{\tau}{\mapsto} \sqrt[4]{2} \overset{\sigma^3}{\longmapsto} -i\sqrt[4]{2} \\ i \overset{\tau}{\mapsto} -i \overset{\sigma^3}{\longmapsto} -i \end{cases} \\
\tau\sigma &= \begin{cases} \sqrt[4]{2} \overset{\sigma}{\mapsto} i\sqrt[4]{2} \overset{\tau}{\mapsto} -i\sqrt[4]{2} \\ i \overset{\sigma}{\mapsto} i \overset{\tau}{\mapsto} -i \end{cases} \\
&= \sigma^3\tau \\
\tau\sigma^2 &= \begin{cases} \sqrt[4]{2} \overset{\sigma^2}{\longmapsto} -\sqrt[4]{2} \overset{\tau}{\mapsto} -\sqrt[4]{2} \\ i \overset{\sigma^2}{\longmapsto} i \overset{\tau}{\mapsto} -i \end{cases} \\
&= \sigma^2\tau \\
\tau\sigma^3 &= \begin{cases} \sqrt[4]{2} \overset{\sigma^3}{\longmapsto} -i\sqrt[4]{2} \overset{\tau}{\mapsto} i\sqrt[4]{2} \\ i \overset{\sigma^3}{\longmapsto} i \overset{\tau}{\mapsto} -i \end{cases} \\
&= \sigma\tau.
\end{aligned}
$$

Since $|\mathrm{Gal}(K/\mathbb{Q})| = [K : \mathbb{Q}] = 8$, it must be the case that $\{e, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$ are the elements of $\mathrm{Gal}(K/\mathbb{Q})$. This is isomorphic to the dihedral group of order 8, D_4.

(e) We can determine the fixed field of $\langle \sigma^2\tau \rangle$ as follows. We find that for $x = \sum_{j=1}^{8} a_j e_j$, where e_j denotes the jth basis vector of $\mathbb{Q}(i, \sqrt[4]{2})$, we have

$$
\begin{aligned}
\sigma^2\tau(x) &= a_1 - a_2\sqrt[4]{2} + a_3\sqrt[4]{4} - a_4\sqrt[4]{8} - a_5 i + a_6 i\sqrt[4]{2} - a_7 i\sqrt[4]{4} + a_8 i\sqrt[4]{8} \\
\mathrm{id}(x) &= a_1 + a_2\sqrt[4]{2} + a_3\sqrt[4]{4} + a_4\sqrt[4]{8} + a_5 i + a_6 i\sqrt[4]{2} + a_7 i\sqrt[4]{4} + a_8 i\sqrt[4]{8}.
\end{aligned}
$$

Therefore, $a_2 = -a_2$, $a_4 = -a_4$, $a_5 = -a_5$, and $a_7 = -a_7$, meaning the coefficients on the respective a_i are identically 0, or

$$\begin{aligned} x &= a_1 + a_3\sqrt[4]{4} + a_6 i\sqrt[4]{2} + a_8 i\sqrt[4]{8} \\ &= a_1 + a_6 i\sqrt[4]{2} - a_3\left(i\sqrt[4]{2}\right)^2 - a_8\left(i\sqrt[4]{2}\right)^3. \end{aligned}$$

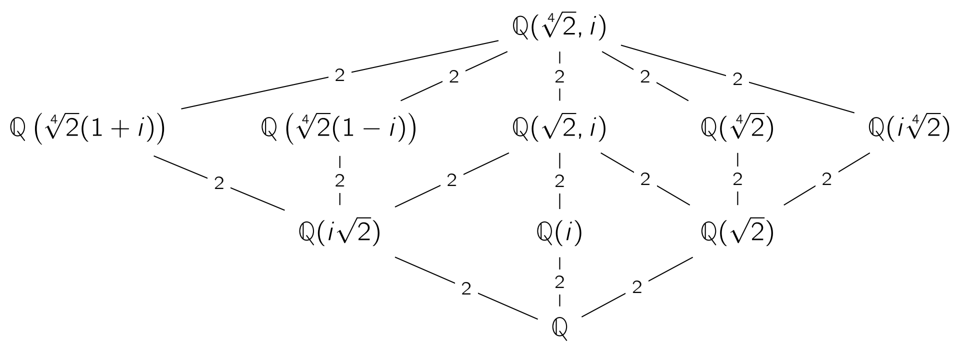
Therefore, $\mathbb{Q}\left(i, \sqrt[4]{2}\right)^{\langle\sigma^2\tau\rangle} = \mathbb{Q}(i\sqrt[4]{2})$. (Answer found with assistance from Adamson (1964), "Introduction to Field Theory.")

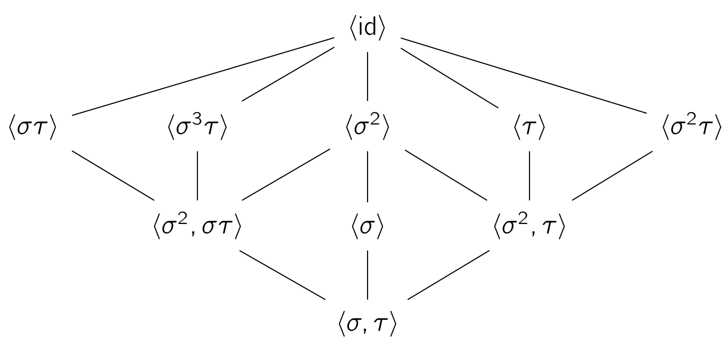
(f) Letting $E = \mathbb{Q}(\sqrt{2}, i)$, we have

$$\begin{aligned} [K : E] &= [\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt{2}, i)] \\ &= 2. \end{aligned}$$

Additionally, since $\mathbb{Q}(\sqrt{2}, i) = \mathrm{Spl}_{\mathbb{Q}}(x^2 + 2)$, it is also Galois over $\mathbb{Q}$, meaning $\mathrm{Gal}(K/E) \trianglelefteq \mathrm{Gal}(K/\mathbb{Q})$ with $|\mathrm{Gal}(K/E)| = 2$. Thus, $\mathrm{Gal}(K/E) = \langle\sigma^2\rangle$.

(g) To find the fixed fields for $\sigma\tau$ and $\sigma^3\tau$, we use the procedure that we used for $\sigma^2\tau$ to find $\mathbb{Q}(\sqrt[4]{2}, i)^{\langle\sigma\tau\rangle} = \mathbb{Q}\left(\sqrt[4]{2}(1+i)\right)$ and $\mathbb{Q}(\sqrt[4]{2}, i)^{\langle\sigma^3\tau\rangle} = \mathbb{Q}\left(\sqrt[4]{2}(1-i)\right)$. Thus, the lattice of subfields and subgroups is as follows.





Problem 6

We will prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of $\mathbb{Q}(\zeta_n)$ for any $n \geq 1$.

It is known that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, which is an Abelian group. Therefore, any subgroup of $\mathrm{Gal}\left(\mathbb{Q}(\zeta_n)/\mathbb{Q}\right)$ is normal, so any subfield $\mathbb{Q} \subseteq E \subseteq \mathbb{Q}(\zeta_n)$ is Galois over $\mathbb{Q}$. However, since $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois extension, it cannot be the case that $\mathbb{Q}(\sqrt[3]{2})$ is a subfield of $\mathbb{Q}(\zeta_n)$. (Answer found using hint from Stack Overflow.)