Complex Numbers

A complex number is an ordered pair of real numbers, $(a, b) = a + bi$. A vector in $\mathbb{R}^2$ is also an ordered pair, (a, b) of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with $\mathbb{R}^2$. However, unlike vectors in $\mathbb{R}^2$, there is also an operation $\cdot$. We desire for $(0, 1) \cdot (0, 1) = (-1, 0)$; essentially, $i^2 = -1$. We say that i is a square foot of -1; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di) \\ &:= a(c) + adi + bci + bd(i^2) \\ &:= (ac - bd) + (ad + bc)i \\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus, $\mathbb{R}^2$ with the operations $+$ and the above defined complex multiplication is known as $\mathbb{C}$. We write as $a+bi$ instead of (a, b).

Given $z = (a + bi) \in \mathbb{C}$, we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$. If $\text{Im}(z) = 0$, then $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$. However, many people say that $\mathbb{R} \subseteq \mathbb{C}$, even if $\mathbb{C}$ isn't defined as such.

Reciprocals of Complex Numbers

Let $z \in \mathbb{C}$, where $z \neq 0$. Then, $\exists w \in \mathbb{C}$ such that $zw = 1$.

Let $w = c + di$. We want to show that $zw = 1$.

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1 \\ ad + bc &= 0.\end{aligned}$$

Thus, let $w = c + di$, with $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2} \\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every $z \neq 0$, with $z = a + bi$, the *reciprocal* of z is defined as $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Then, for $w \in \mathbb{C}$, we define

$$\frac{w}{z} := w\left(\frac{1}{z}\right).$$

Properties of Complex Numbers

Let $z = a + bi \in C$. Then, the (Euclidean) norm (or absolute value) of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

(i) $z\overline{z} = |z|^2$

(ii) $\overline{(\overline{z})} = z$

(iii) $\overline{(z+w)} = \overline{z} + \overline{w}$

(iv) $\overline{zw} = \overline{z} \cdot \overline{w}$

(v) $z + \overline{z} = 2\text{Re}(z)$, so $\text{Re}(z) = \frac{z+\overline{z}}{2}$

(vi) $z - \overline{z} = 2\text{Im}(z)i$, so $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

Polar Representation

Let $z = a+bi$ (or $z = (a,b)$). Then, $|z| = \sqrt{a^2+b^2}$ is the *radius*, and the *argument* is found by $\theta = \arctan(b/a)$ for $a \neq 0$. Therefore, the full polar representation is as follows:

$$z = |z|\left(\cos\theta + i\sin\theta\right). \qquad \theta \in [0, 2\pi)$$

If $z = 0$, then $|z| = 0$, and $\arg z$ is undefined.

For example, we can find $\arg i$ in $[\pi, 3\pi)$ as $\frac{5\pi}{2}$.

For z_1 and z_2 in polar form, we have:

$$\begin{align}
|z_1 z_2| &= |z_1||z_2| \tag{1}\\
\arg(z_1 z_2) &= \arg z_1 + \arg z_2 \mod 2\pi \tag{2}
\end{align}$$

Proof of (1):

$$\begin{align*}
|z_1 z_2|^2 &= (z_1 z_2)\overline{(z_1 z_2)}\\
&= z_1 z_2 \overline{z_1}\,\overline{z_2}\\
&= z_1 \overline{z_1} z_2 \overline{z_2}\\
&= |z_1|^2 |z_2|^2
\end{align*}$$

Since $|z| \geq 0$, we get $|z_1 z_2| = |z_1||z_2|$.

Let $z = 2(\cos\pi/6 + i\sin\pi/6)$, and let $f : \mathbb{C} \to \mathbb{C}$ defined as $f(w) = zw$. Then, f rotates w by $\pi/6$ and scales w by 2.

Theorem: For $n \in \mathbb{N}$, if $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Proof: Induct on n. For the base case, we know that $n = 1$ satisfies this property. For $n > 1$, we have:

$$\begin{align*}
z^{n+1} &= (z^n)(z)\\
&= \left(r^n(\cos(n\theta) + i\sin(n\theta))\right) r(\cos\theta + i\sin\theta)\\
&= (r^n)(r)\left(\cos(n\theta+\theta) + i\sin(n\theta+\theta)\right) && \text{Polar Representation Definition}\\
&= r^{n+1}\left(\cos\left((n+1)\theta\right) + i\sin((n+1)\theta)\right)
\end{align*}$$

We can use this technique to find the “roots of unity.” For example, to find all z such that $z^3 = 1$, we use our technique:

$$\begin{aligned}
z^3 &= 1\\
|z| &= 1\\
\arg z^3 &= 0\\
3\arg z &= 0 \mod 2\pi\\
\arg z &= \frac{k2\pi}{3}\\
&= 0, \frac{2\pi}{3}, \frac{4\pi}{3}\\
z_1 &= 1\\
z_2 &= (\cos 2\pi/3 + i\sin 2\pi/3)\\
z_3 &= (\cos 4\pi/3 + i\sin 4\pi/3)
\end{aligned}$$

We can see that $z_2^2 = z_3$.

For the n case, we find $z_2 = \cos(2\pi/n) + i\sin(2\pi/n)$, and $z_k = z_2^{k-1}$.

Exponential, Logarithm, and Trigonometric Functions in $\mathbb{C}$

Exponential

Let $z = a + bi$. We define e^{a+bi} as follows:

$$e^{a+bi} = e^a\left(\cos b + i\sin b\right)$$

Recall that for every nonzero complex number, $z = |z|\left(\cos\theta + i\sin\theta\right)$, where $\theta = \arg z$. Thus,

$$\begin{aligned}
z &= |z|e^{i\theta}\\
&= |z|e^{i\arg z}.
\end{aligned}$$

The function e^z has some properties similar to the function e^x in real numbers, and some properties varying with the real numbers.

$$\begin{aligned}
e^z e^w &= e^{z+w}\\
e^z &\neq 0
\end{aligned}$$

However, there are some differences:

$$\begin{aligned}
|e^{i\theta}| &= 1 && \forall\theta\\
e^{a+bi} &= e^a
\end{aligned}$$

From these properties, we find Euler's equation:

$$e^{i\pi} + 1 = 0$$

Additionally, e^z is periodic, while $f(x) = e^x$ is injective:

$$\begin{aligned}
e^{z+2n\pi} &= e^z\left(\cos(2n\pi) + i\sin 2n\pi\right)\\
&= e^z
\end{aligned}$$

When examining the function $f : \mathbb{C} \to \mathbb{C}\setminus\{0\}$, $z \mapsto e^z$, we find that the following happen:

- $f(\mathbb{R}) = (0, \infty)$ — we apply $f(x) = e^x$.
- $f(a + bi) = e^a e^{bi}$ — e^a is rotated by b.
- $f(\mathbb{R} + bi)$ is expressed as the line along b radians through the origin.
- Therefore, $f(A_0) = \mathbb{C} \setminus \{0\}$, where $A_0 = \{a + bi \mid a \in \mathbb{R}, b \in [0, 2\pi)\}$.

Logarithm

Recall that for a function $f : A \to B$, f^{-1} is a function if f is injective. However, for any f, it is the case that $f^{-1}(b)$ does exist, defined as follows:

$$f^{-1}(b) = \{a \mid f(a) = b\}.$$

For the function $f(z) = e^z$, f is not one to one, so for $w = e^z$, $f^{-1}(w) = \{z' \in \mathbb{C} \mid e^{z'} = w\}$. We can find this as $f^{-1}(w) = \{z + 2n\pi i \mid n \in \mathbb{Z}\}$.

We define $\log(w) := \{z \in \mathbb{C} \mid e^z = w\}$. For a fixed $\theta \in \mathbb{R}$, we define $\log_{A_\theta}(w) := \{z \mid e^z = w, z \in A_\theta\}$.

Let $z = 1 + \frac{5\pi}{2} i$. Then,

$$\log_{A_{-\pi}} e^z = 1 + \frac{\pi}{2} i$$

Let $w \in \mathbb{C} \setminus \{0\}$. To find $\log w$ (all values), then

$$\begin{aligned}
z &\in \log w \\
e^z &= w \\
&= |w| e^{i \arg w} \\
e^{a+bi} &= |w| e^{i \arg w} \\
e^a e^{ib} &= |w| e^{i \arg w}.
\end{aligned}$$

Therefore, $a = \ln |w|$ and $b = \arg w$. Additionally, the following hold, for $z_1, z_2 \in \mathbb{C}$:

$$\log_{A_\theta}(z_1 z_2) = \log_{A_\theta}(z_1) + \log_{A_\theta}(z_2) + 2n\pi i$$

Cosine and Sine

$$\begin{aligned}
e^{ib} &= \cos b + i \sin b \\
e^{-ib} &= \cos b - i \sin b \\
\cos z &:= \frac{e^{iz} + e^{-iz}}{2} \\
\sin z &:= \frac{e^{iz} - e^{-iz}}{2i}
\end{aligned}$$

Complex Powers

Recall that for $s, t \in \mathbb{R}$, $s^t = e^{t \ln s}$, where $s > 0$. For $z, w \in \mathbb{C}$, $z^w = e^{w \log z}$., where $z \neq 0$.

$$\begin{aligned}
(-2)^i &= e^{i \log(-2)} \\
&= e^{i(\ln(2) + i\pi)} \\
&= e^{i \ln 2 - (\pi + 2\pi n)} \\
&= e^{-\pi + 2\pi n + i \ln 2}
\end{aligned}$$

This has *infinitely* many values.

Let $\alpha = u + vi$. Then,

$$\begin{aligned} z^{\alpha} &= e^{\alpha \log z} \\ &= e^{(u+vi)(\ln|z| + i \arg z)} \\ &= e^{(u \ln|z| - v \arg z)} e^{i(v \ln|z| + u \arg z)} \end{aligned}$$

Since $\arg z = \theta + 2\pi n$ for some real $\theta \in [0, 2\pi)$,

$$= e^{u \ln z} e^{-v(\theta + 2\pi n)} e^{iv \ln|z|} e^{iu(\theta + 2\pi n)}$$

Therefore, complex exponentiation is single-valued if $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Z}$, then z^{α} has only one value; if $\alpha \in \mathbb{Q}$, where $\alpha = \frac{p}{q}$ and $\gcd(p, q) = 1$, then z^{α} takes q distinct values, which are the qth-roots.

Continuous Functions with Complex Domains

Let $z \in \mathbb{C}$, let $r > 0$.

- The set $D(z; r) := \{w \mid w \in \mathbb{C}, |z - w| < r\}$ is the r-neighborhood of z.
- A subset $A \subseteq \mathbb{C}$ is open if $(\forall z \in A)\,(\exists r > 0) \ni D(z; r) \subseteq A$.

For example, if $A = \{z \mid \text{Re}(z) > 0\}$, we can find r equal to half the magnitude of the real component of z for any $z \in A$, meaning A is open.

Meanwhile, if $A = \{z \mid \text{Re}(z) \geq 0\}$, this is not the case. If $z = 0$, then $\nexists r > 0$ such that $D(z; r) \subseteq A$, as any open ball of radius r will have some element in $\overline{A}$.

- A subset $B \subseteq \mathbb{C}$ is closed if $\overline{B} \subseteq \mathbb{C}$ is open.

For example, $A = \emptyset$ is open, by vacuous truth, so $\overline{A} = \mathbb{C}$ is closed. Similarly, since $\mathbb{C}$ is open, $\emptyset$ is closed.

Meanwhile, $A = \{x + iy \mid -1 \leq x < 1\}$ is neither open nor closed.

Limits

Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, $z_0 \in \mathbb{C}$. Then,

$$\lim_{z \to z_0} f(z) = \ell$$

means both of the following hold:

(i) for some $r > 0$, $D(z_0; r) \setminus \{z_0\} \subseteq \text{dom}(f)$,

(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $f(D(z_0; \delta) \setminus \{z_0\}) \subseteq D(\ell; \varepsilon)$.

For example, if

$$f(z) = \begin{cases} z & z \in \mathbb{C} \setminus \mathbb{R} \\ 3i & z \in \mathbb{R} \end{cases}$$

Then, $\lim_{z \to 0} f(z)$ does not exist, as there is no ℓ that satisfies both conditions. Specifically, if $\ell = 3i$, and we set $\varepsilon = 1$, then a disc of any radius around 0 has some $z \in \mathbb{C} \setminus \mathbb{R}$ that maps to itself. Similarly, if we set $\ell = 0$, then there is a real number in a disc of any radius around 0.

Note: f does not have to be defined at z_0 for the limit to be defined at z_0.

Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$, and $z_0 \in A$. We say f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$. We say f is continuous on A if $\forall z_0 \in A$, f is continuous at z_0.

We will show that $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto 3z$ is continuous.

Scratch Work: We want δ such that $f(D(z_0;\delta)) \subseteq D(3z_0;\varepsilon)$. Let $z \in D(z_0;\delta)$, meaning $f(z) = 3z$. We want $3z \in D(3z_0;\varepsilon)$, meaning we want $|3z - 3z_0| < \varepsilon$, or $|z - z_0| < \frac{\varepsilon}{3}$.

Proof: Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{3}$. We show $f(D(z_0;\delta)) \subseteq D(f(z_0);\varepsilon)$. Let $z \in D(z_0;\delta)$. Then, $|z - z_0| < \delta = \varepsilon/3$, meaning $3|z - z_0| < \varepsilon$, meaning $|3z - 3z_0| < \varepsilon$, so $|f(z) - f(z_0)| < \varepsilon$. Therefore, $f(z) \in D(f(z_0);\varepsilon)$. Since f is continuous at arbitrary z_0, f is continuous on $\mathbb{C}$.

Sequences

A sequence $z_1, z_2, \cdots \in \mathbb{C}$. A sequence converges to $z_0 \in \mathbb{C}$ if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M},\ |z_n - z_0| < \varepsilon$$

In words, for any radius around z_0, we can find z_n arbitrarily close to z_0 for sufficiently large n. We write $z_n \to z_0$ if this is the case.

Let $f : \mathbb{C} \to \mathbb{C}$. Then, f is continuous on $\mathbb{C}$ if and only if the following equivalent conditions are met:

(i) the inverse image of every open set is open $(f^{-1}(B) := \{a \in \mathbb{C} \mid f(a) \in B\})$;

(ii) the inverse image of every closed set is closed;

(iii) for every sequence $(z_n)_n$ such that $(z_n)_n \to z_0$, $f(z_n) \to f(z_0)$.

Let

$$f(z) = \begin{cases} 0 & z = 0 \\ 1 & z \neq 0 \end{cases}.$$

This function is not continuous. We will check that (i)–(iii) fail.

(i) Let $B = D(0;1)$. Then, $f^{-1}(B) = \{0\}$, which is not open set.

(ii) Let $B = \text{cl}(D(1;0.5))$. Then, $f^{-1}(B) = \mathbb{C} \setminus \{0\}$, which is not closed.

(iii) Let $z_n = \frac{1}{n}$. Then, $(z_n)_n \to 0$, but $f(z_n) = 1$ for all n, meaning $f(z_n) \to 1 \neq f(0)$.

To show limit divergence, recall the definition of limit convergence:

$$\lim_{n\to\infty} z_n = z_0 \Leftrightarrow (\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M},\ |z_n - z_0| < \varepsilon.$$

Let $z_1, \ldots, \in \mathbb{C}$ be a sequence. Then, $\lim_{n\to\infty} = \infty$ means

$$(\forall M > 0)(\exists N \in \mathbb{N}) \ni \forall n > N,\ |z_n| > M.$$

In words, $|z_n|$ is arbitrarily large for sufficiently large n.

Connected Sets

Let $a, b \in \mathbb{C}$. A path from a to b is a continuous function $p : [0,1] \to \mathbb{C}$ such that $p(0) = a$ and $p(1) = b$. Let $S \subseteq \mathbb{C}$. If $p([0,1]) \subseteq S$, then p is a path in S.

We say S is *path-connected* if for any $s, t \in S$, there is a path in S from s to t.

Every set that is path-connected is connected, but not necessarily the other way around — if A is open and path connected, then A is connected.

An open, path-connected subset of $\mathbb{C}$ is known as a region, or a domain.

Let $A = \mathbb{R} \times \{0\}$ (or the x axis in $\mathbb{C}$). A is not a region, as A is not an open set, even if A is path-connected.

$A \subseteq \mathbb{C}$ is bounded if there exists $r > 0$ such that $A \subseteq D(0; r)$. $A = \mathbb{R} \times \{0\}$ is not bounded.

If $A \subseteq \mathbb{C}$, then A is compact if A is closed and bounded. There are various properties of compact sets that make them particularly amenable towards analysis.

Extreme Value Theorem: Every real-valued continuous function on a compact domain attains its maximum and minimum values.

Uniform Continuity Theorem: Elaborated below.

Uniform Continuity

Recall that if $f : A \to \mathbb{C}$, f is continuous if $\forall a \in A$, $\lim_{z \to a} f(z) = f(a)$.

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta_a > 0) \ni f(D(a; \delta_a)) \subseteq D(f(a); \varepsilon) \qquad \delta \text{ depends on } a$$

When f is uniformly continuous, there is one value of δ, dependent on ε, that applies for every value of a.

$$(\forall \varepsilon > 0)(\exists \delta_\varepsilon > 0) \ni (\forall a \in A), f(D(a; \delta_\varepsilon)) \subseteq D(f(a); \varepsilon)$$

Riemann Sphere

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2\}$. Let $N = (0, 0, 1)$ denote the north pole. Then, there is a continuous bijection from $S^2 \setminus \{N\} \to \mathbb{C}$.

We can visualize this by picking a random point on the sphere and drawing a line from the north pole through the sphere to this point, and finding the point that intersects the plane.

Consider the sequence $z_n = n^2 i$ for $n = 1, 2, \dots$. We can see that, on the projection from z_n to the sphere, all the values of p converge to N. Therefore, we write $\lim_{n \to \infty} z_n = \infty$, where ∞ corresponds to N on S^2.

We can define $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to be the complex plane that includes the "point at infinity" (from the projection on S^2 that corresponds to the north pole).

Analytic Functions

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ where A is open. Let $z_0 \in A$. We say f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Rules of Differentiation

- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$
- $(f \circ g)' = g'(f' \circ g)$
- For $n \in \mathbb{Z}$, $(z^n)' = nz^{n-1}$

Let $f(z) = \overline{z}$. We will find this value by directly applying the definition of the derivative.

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\ &= \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}. \end{aligned}$$

Let's approach z_0 from the horizontal direction. Suppose $z = z_0 + t$ for some $t \in \mathbb{R}$. Then,

$$\lim_{z \to z_0} \frac{\overline{z_0 + t} - \overline{z_0}}{z_0 + t - z_0} = 1.$$

Let's approach z_0 from the horizontal direction. Suppose $z = z_0 + ti$ for some $t \in \mathbb{R}$. Then,

$$\begin{aligned} \lim_{z \to z_0} \frac{\overline{z_0 + ti} - \overline{z_0}}{z_0 + ti - z_0} &= \frac{-ti}{ti} \\ &= -1. \end{aligned}$$

Since $1 \neq -1$, we find that the limit does not exist.

We see that complex-differentiability is a strong condition.

Suppose that $f'(z_0) = 2i$, meaning

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 2i.$$

If z is close to z_0, then $f(z) - f(z_0) \approx 2i(z - z_0)$. Pictorially, we can visualize this as, for z_0 sufficiently close to z, the vector $z_0 - z$ is akin to a counterclockwise rotation and a scaling by 2. This is applicable for *all* z in sufficient proximity to z_0.

Specifically, we can see that the complex differentiable function is *angle-preserving*. The technical name for f is that f is *conformal*.

Analytic Function

Let $f : A \subseteq C \to \mathbb{C}$. If f is differentiable at every $z_0 \in A$, we say f is *analytic* on A.

If f is analytic on A, then f is infinitely differentiable on A.

If f is analytic on A and $f'(z_0) \neq 0$ for some $z_0 \in A$, then f is conformal at $z_0 \in A$.

Cauchy-Riemann Theorem

Given a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. Recall that we can take partial derivatives, $\frac{\partial f}{\partial x}$, and directional derivative $\frac{\partial f}{\partial u}$ for some unit vector u.

However, for $\mathbb{C}$, there is only one derivative, $f'(z_0)$, meaning that regardless of direction, $f'(z_0)$ exists and has one value. We can contextualize $f(z) = f(x + yi) = u(x, y) + iv(x, y)$, where $u(x, y) \in \mathbb{R}$ and $v(x, y) \in \mathbb{R}$. Then,

$$\frac{\partial u}{\partial x} \neq \frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial x} \neq \frac{\partial v}{\partial y}$$

but

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

We can see this by first letting $z = z_0 + \delta x$.

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z_0 + \delta x) - f(z_0)}{z_0 + \delta x - z_0} \\ &= \lim_{z \to z_0} \frac{u(x_0 + \delta x, y_0) + iv(x_0 + \delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

and in the y direction,

$$\begin{aligned} f'(z_0) &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

We set these two values equal to find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned}$$

which are the Cauchy-Riemann equations. The corresponding theorem states that if $f'(z_0)$ exists, then the Cauchy-Riemann equations must hold.

For example, if $f(z) = \overline{z}$, with $f(x + yi) = x - yi$, we have $u(x, y) = x$ and $v(x, y) = -y$. Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial v}{\partial v} &= -1, \end{aligned}$$

meaning f is not complex-differentiable.

If $f : A \to \mathbb{C}$ satisfies the Cauchy-Riemann equations at every $z_0 \in A$, then f is analytic on A.

If $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is analytic on A, then we know f' and f'' are continuous. From multivariable calculus, we know that $u_{xy} = u_{yx}$ if both are continuous. So,

$$\begin{aligned}
u_{xy} &= \frac{\partial}{\partial y}(u_x)\\
&= \frac{\partial}{\partial y}(v_y)\\
&= v_{yy}\\
u_{yx} &= \frac{\partial}{\partial x}(u_y)\\
&= \frac{\partial}{\partial x}(-v_x)\\
&= -v_{xx}
\end{aligned}$$

Therefore, $v_{xx} + v_{yy} = 0$. Similarly, $u_{xx} + u_{yy} = 0$.

If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ If $\varphi_{xx} + \varphi_{yy} = 0$, then we say φ is a harmonic function. Therefore, if f is an analytic function, then both the real and imaginary parts of f are harmonic.

Let $A \subseteq \mathbb{R}^2$. If $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$. Then, u and v are harmonic conjugates if $u + iv$ is an analytic function. Additionally, u and v are harmonic conjugates if and only if they satisfy the Cauchy-Riemann equations.

We may ask if there exists an analytic function f such that $\text{Re}(f) = x^3 - 3xy^2 + y$. Then,

$$\begin{aligned}
v_y = u_x &= 3x^2 - 3y^2\\
-v_x = u_y &= 1 - 6xy.
\end{aligned}$$

Therefore, we find $v = -x + 3x^2y - y^3 + c$ through integration. Therefore, we have

$$\begin{aligned}
f(z) &= (x^3 - 3xy^2 + y) + i(3x^2y - y^3 - x + c)\\
&= (x - iy)^3 + y - ix + ic\\
&= z^3 + i(-iy + x) + ic\\
&= \overline{z}^3 + i\left(\overline{z} + c\right)
\end{aligned}$$

Recall from from multivariable calculus that $\nabla u \perp$ contour lines of u. Similarly, $\nabla v \perp$ contour lines of v. Then, using the Cauchy-Riemann equations, we find

$$\begin{aligned}
\nabla u \cdot \nabla v &= (-u_x u_y) + u_x u_y\\
&= 0,
\end{aligned}$$

meaning the gradients are orthogonal to each other, meaning the contours of u are perpendicular to the contours of v.

Inverse Functions

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$. Let $z_0 \in A$. If f is analytic on A and $f'(z_0) \neq 0$, then f is one to one on some neighborhood of z_0. Then, $f^{-1} : f(N) \to N$ is analytic on $f(N)$, and

$$(f^{-1})'(f(z_0)) = \frac{1}{f'(z_0)}.$$

Derivatives of Elementary Functions

Specifically, we will be working with complex exponentiation, complex trigonometric functions, and complex logarithms.

Complex Exponential

$$\frac{d}{dz}e^z = e^z,$$

since, letting $z = x + iy$,

$$\begin{aligned} e^z &= e^x e^{iy} \\ &= e^x(\cos(y) + i\sin(y)). \\ \frac{d}{dz}e^z &= \frac{\partial}{\partial x}e^z && \text{treating } y \text{ as constant} \\ &= e^x(\cos(y) + i\sin(y)) \\ &= e^{x+iy} \\ &= e^z. \end{aligned}$$

We know that e^z is continuous on $\mathbb{C}$, but this doesn't imply differentiability at every $z_0 \in \mathbb{C}$. We can verify by checking the Cauchy-Riemann equations, where $u(x, y) = e^x\cos(y)$ and $v(x, y) = e^x\sin(y)$. Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x\cos(y) \\ &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial y} &= -e^x\sin(y) \\ &= -\frac{\partial v}{\partial x}. \end{aligned}$$

If a function is analytic on $\mathbb{C}$, then f is known as entire.

Complex Logarithm

We might ask where $\log z$ is analytic. Let $f(z) = e^z$. Then, $\log z = f^{-1}(z)$; since f is not one to one, we restrict the domain of f to $A_\theta = \{z \mid \operatorname{Im}(z) \in [\theta, \theta + 2\pi)\}$ for any θ.

Since $f\big|_{A_\theta}$ is one to one, then

$$\left(f\big|_{A_\theta}\right)^{-1} = \log_{A_\theta}.$$

Fixing θ, set $g = f\big|_{A_\theta}$. Then,

$$g^{-1}(g(z)) = z.$$

Because g is analytic on A_θ, g^{-1} is analytic on A_θ. By chain rule, we have

$$\begin{aligned} \frac{d}{dz}(g^{-1}(g(z))) &= \frac{d}{dz}z \\ g^{-1'}(g(z)) &= \frac{1}{g'(z)} && g'(z) \neq 0 \\ g^{-1}(w) &= \frac{1}{g'(z)} && w = e^z \\ &= \frac{1}{e^z} \\ &= \frac{1}{w}. \end{aligned}$$

Therefore, $\frac{d}{dw}\log_{A_\theta}(z) = \frac{1}{z}$. Therefore, $\text{dom}(\log_{A_\theta}) = \text{ran}(e^z_{A_\theta}) = \mathbb{C} \setminus \{0\}$. However, $\log_{A_0}$ (setting $\theta = 0$) is not even continuous on $\mathbb{C} \setminus \{0\}$!

Specifically, at $z = 0$, $e^z = 1$. Travelling around the unit circle counterclockwise in the image, we see that the preimage of these points travels along the imaginary axis. Approaching 1 "from the bottom," we find that the preimage of the points approaches 2π in the domain. However, they ought to be approaching 0. Therefore, the limit doesn't exist.

However, notice that the domain is not open! To fix this, we will let $B_\theta = \{z \in \mathbb{C} \mid \text{Im}(z) \in (\theta, \theta + 2\pi)\}$.

Our log function *is* when e^z is restricted to B_θ. Then, $\log_{B_\theta}$ is analytic on $\mathbb{C} \setminus \{re^{i\theta} \mid r \geq 0\}$. When $\theta = -\pi$, then $\log_{B_{-\pi}}$ is the principle branch of $\log z$.

Then, the domain is $C \setminus \{z \mid z = x + 0i, x < 0\}$ and the range is $B_{-\pi}$.

Powers

Let $\alpha \in \mathbb{C}$. We might ask

$$\frac{d}{dz}\alpha^z$$
$$\frac{d}{dz}z^\alpha.$$

Recall that $a^b = e^{b \log a}$. Specifically, $a^b = e^{b(\ln|a| + i \arg a)}$.

$$\frac{d}{dz}\alpha^z = \frac{d}{dz}e^{z\log\alpha}$$

Fix θ. Then,

$$\begin{aligned} &= \frac{d}{dz}e^{z\log_{A_\theta}\alpha} \\ &= \log_{A_\theta}\alpha e^{z\log_{A_\theta}\alpha} && = \alpha^z \log_{A_\theta}\alpha. && \text{assuming analytic domain} \end{aligned}$$

Specifically, as long as $\alpha \notin \{re^{i\theta} \mid r \geq 0\}$, $z\log_{A_\theta}\alpha$ is analytic, meaning $e^{z\log_{A_\theta}\alpha}$ is analytic (composition of analytic functions).

$$\begin{aligned} z^\alpha &= e^{\alpha\log z} \\ &= e^{\alpha\log_{B_\theta} z} \\ &= e^{\alpha\log_{B_\theta} z}\frac{\alpha}{z} \\ &= \alpha z^{\alpha-1}. \end{aligned}$$

Specifically, this holds for $z \notin \{re^{i\theta} \mid r \geq 0\}$.

We know that $\frac{d}{dz}\log_{B_{-\pi}(z)} = \frac{1}{z}$. The domain of $\log_{B_{-\pi}}$ is $\mathbb{C} \setminus (-\infty, 0]$.