Contents

Introduction

Finally, the last part of my notes on C*-algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of C*-algebras, discussing ideas such as amenability and nuclearity in C*-algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of $C^*_\lambda(G)$ and $C^*(G)$.

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's C*-*Algebras and Finite-Dimensional Approximations*.

Review: Representations, the Reduced Group C^*-Algebra, and the Universal Group C^*-Algebra

Left-Regular Representation

Let Γ be a group. Consider the space $\ell_2(\Gamma)$. For every $s \in \Gamma$, we define the operator

$$\lambda_s(\xi)(t) = \xi\left(s^{-1}t\right).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned}
\|\lambda_s(\xi)\|^2 &= \sum_{t\in\Gamma}|\lambda_s(\xi)(t)|^2\\
&= \sum_{t\in\Gamma}\left|\xi\left(s^{-1}t\right)\right|^2\\
&= \sum_{r\in\Gamma}|\xi(r)|^2\\
&= \|\xi\|^2.
\end{aligned}$$

Additionally, each λ_s admits an inverse, $\lambda_{s^{-1}} = \lambda_s^*$. Applying to the orthonormal basis $\{\delta_t\}_{t\in\Gamma}$, we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus, $\lambda_s \circ \lambda_r = \lambda_{sr}$, and we have the unitary representation of Γ, $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, where $\lambda(s) = \lambda_s$, for $s \in \Gamma$. This is the left-regular representation of Γ.

Note that the left regular representation is a faithful representation, hence injective.

Because the λ operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}
\lambda_f(\xi)(t) &= \left(\sum_{s\in\Gamma} f(t)\lambda_s(\xi)\right)(t)\\
&= \sum_{s\in\Gamma} f(s)\xi\left(s^{-1}t\right)
\end{aligned}$$

Note that the space of finitely supported functions on Γ, $\mathbb{C}[\Gamma]$,[1] is a $*$-algebra, where multiplication is given by convolution:

$$\begin{aligned} f * g(t) &= \sum_{s\in\Gamma} f(s)g\left(s^{-1}t\right) \\ &= \sum_{r\in\Gamma} f\left(tr^{-1}\right)g(r). \end{aligned}$$

Note that we are using $*$ both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on $\mathbb{C}[\Gamma]$ is given by

$$f^*(t) = \overline{f\left(t^{-1}\right)}.$$

A Bit on Representations and C^*-(Semi)norms

A C^*-seminorm on a $*$-algebra is a seminorm such that defined by

- $\|ab\| \leqslant \|a\|\|b\|$;
- $\|a^*\| = \|a\|$;
- $\|a^*a\| = \|a\|^2$.

If A_0 is a $*$-algebra, then a representation of A_0 is a pair $(\pi_0, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi\colon A_0 \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism.

Additionally, if A_0 is a $*$-algebra with representation π_0, then we have C^*-seminorm

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{op}.$$

If π_0 is injective, then $\|\cdot\|_{\pi_0}$ is a C^*-norm. If π_0 is a C^*-norm, then the completion of A_0 with respect to $\|\cdot\|_{\pi_0}$ is a C^*-algebra.

The universal norm on A_0 is defined as

$$\|a\|_u = \sup_{p\in\mathcal{P}} p(a),$$

where $\mathcal{P}$ is the collection of all C^*-seminorms on A_0. If $\|a\|_u < \infty$ for all $a \in A_0$, then $\|\cdot\|_u$ is a C^*-seminorm on A_0. Note that if one of $p \in \mathcal{P}$ is a norm, then $\|\cdot\|_u$ defines a C^*-norm on A_0.

If we have the unitary representation $u\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H})$, then

$$\pi_u(a) = \sum_{s\in\Gamma} u_s$$

is a representation of $\mathbb{C}[\Gamma]$. If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ is the left-regular representation, then the left-regular group C^*-algebra is the group $*$-algebra with C^*-norm defined by $\|a\| = \|\pi_\lambda(a)\|$.

The universal group C^*-algebra is defined as the norm completion of

$$\|a\|_u = \sup\left\{\|\pi(a)\|_{op} \;\middle|\; \pi\colon \mathbb{C}[\Gamma] \to \mathbb{B}(\mathcal{H}_\pi)\right\}.$$

[1] Also known as the free vector space over $\mathbb{C}$ with basis Γ.

Note that

$$\begin{aligned}\|\pi(a)\| &= \left\|\pi\left(\sum_{s\in\Gamma} a_s\delta_s\right)\right\| \\ &= \left\|\sum_{s\in\Gamma} a_s\pi(\delta_s)\right\| \\ &\leqslant \sum_{s\in\Gamma}\|a_s\pi(\delta_s)\| \\ &= \sum_{s\in\Gamma}|a_s|.\end{aligned}$$

Note that since $\|\cdot\|_\lambda$ is a norm, we must have $a = 0$ if and only if $\|a\|_u = 0$. The full group C*-algebra admits a universal property.

Proposition: Let Γ be a discrete group. If $u\colon \Gamma \to \mathcal{U}(\mathcal{H})$, then there is a contractive $*$-homomorphism $\pi_u\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ that satisfies $\pi_u(\delta_s) = u(s)$.

Almost-Invariant Vectors

If $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\mathcal{H}$, then a vector $\xi \in \mathcal{H}$ is called invariant for π if $\pi(g)(\xi) = \xi$ for all $g \in \Gamma$.

Proposition: The left-regular representation for Γ admits an invariant vector if and only if Γ is finite.

Proof. Let Γ be finite. Since Γ is finite, all functions $a\colon \Gamma \to \mathbb{C}$ are square-summable. Thus, $\xi = \mathbb{1}_\Gamma$ is square-summable, and since $s\Gamma = \Gamma$ for all $s \in \Gamma$, we have $\mathbb{1}_\Gamma$ is invariant for λ.

Now, let $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ be the left-regular representation, and suppose there is $\xi \in \ell_2(\Gamma)$ such that for all $s \in \Gamma$, we have

$$\lambda_s(\xi) = \xi.$$

In particular, this means that for any $t \in \Gamma$, we have

$$\begin{aligned}\lambda_s(\xi)(t) &= \xi\left(s^{-1}t\right) \\ &= \xi(t).\end{aligned}$$

Since this holds for all $s \in \Gamma$, we have that $\xi = c\mathbb{1}_\Gamma$ for some $c \in \mathbb{C}$. However, since $\xi \in \ell_2(\Gamma)$, we must have that $\sum_{t\in\Gamma}|c|^2 < \infty$, which only holds if Γ is finite. □

An almost-invariant vector for a representation $\pi\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, as the name suggests,[II] a sequence (or net) of vectors $(\xi_i)_{i\in I}$ such that

$$\lim_{i\in I}\|\pi(g)(\xi_i) - \xi_i\| = 0.$$

Theorem: A group Γ is amenable if and only if the left-regular representation has an almost-invariant vector.

Proof. Let Γ be amenable, and let F_i be a Følner sequence — $\frac{|sF_i \triangle F_i|}{|F_i|} \to 0$ for all $s \in \Gamma$. Define $\xi_i = \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}$. Thus,

$$\|\lambda_s(\xi_i) - \xi_i\|^2 = \sum_{t\in\Gamma}|\lambda_s(\xi_i)(t) - \xi_i(t)|^2$$

[II] I'm only mostly being facetious here.

$$\begin{aligned}
&= \sum_{t\in\Gamma}\left|\lambda_s\left(\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right)(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{F_i}\right|^2\\
&= \sum_{t\in\Gamma}\left|\frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t) - \frac{1}{\sqrt{|F_i|}}\mathbb{1}_{sF_i}(t)\right|^2\\
&= \frac{|sF_i \triangle F_i|}{|F_i|}.
\end{aligned}$$

Thus, λ has an almost-invariant vector.

Suppose there exists an almost-invariant vector $(\xi_i)_i \in \ell_2(\Gamma)$. It is sufficient to construct an approximate mean. Since $\xi_i \in \ell_2(\Gamma)$, we have that $\xi_i^2 \in \ell_1(\Gamma)$. Setting $\mu_i = \xi_i^2$, we plug this into the expression for an approximate mean, and obtain

$$\|\lambda_s(u_i) - u_i\|_{\ell_1} = \sum_{t\in\Gamma}\left|\lambda_s\left(\xi_i^2\right)(t) - \xi_i^2(t)\right|$$

Thus, μ_i is an approximate mean. □