

Background: Asymptotic Freeness and Large Deviations

We start by recalling the basic asymptotic freeness result discussed in class.

Proposition: Let (A_1^N, \dots, A_r^N) be an independent r -tuple of GUE $N \times N$ matrices. Then, the family A_1^N, \dots, A_r^N converge in distribution to r independent semicircular elements, $s_1, \dots, s_r \in B(\mathcal{F}(\mathbb{C}^r))$, in the sense that for all $m \geq 1$ and all $1 \leq i_1, \dots, i_m \leq r$, we have

$$\lim_{N \rightarrow \infty} E[\text{tr}(A_{i_1}^N \cdots A_{i_m}^N)] = \varphi(s_{i_1} \cdots s_{i_m}),$$

where φ is the vacuum state, $\varphi(T) = \langle T\Omega, \Omega \rangle$.

In fact, this collection is *almost surely* asymptotically free, in the following sense. Suppose we have two random matrices A^N and B^N defined on probability spaces (X_N, μ_N) . Define

$$X := \prod_{N \in \mathbb{N}} X_N$$

$$\mu := \prod_{N \in \mathbb{N}} \mu_N,$$

where the latter is the product measure on X . The matrices A^N and B^N are said to be *almost surely* asymptotically free if there exists a noncommutative probability space (A, φ) and $a, b \in A$, and for almost all $x = (x_N)_N \in X$, we have $A^N(x_N), B^N(x_N) \in (\mathbb{M}_N, \text{tr})$ converge in distribution to a, b .

Now, from here, we may ask a seemingly simple question: as N grows large, how likely are we to encounter other distributions? To make this sense more precise, we consider a random $N \times N$ self-adjoint matrix A , and let

$$\mu_A = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$$

be its empirical spectral distribution. This is a random probability measure on \mathbb{R} , and as $N \rightarrow \infty$, the semicircle law gives that μ_A converges weakly to the semicircle distribution; this can be strengthened to almost sure convergence by an application of the argument for asymptotic freeness. The question then becomes, how quickly does the deviation between μ_A and any other probability distribution ν decrease as N increases? This is where the theory of large deviations starts to take shape.

Much of this exposition related to the classical notions of entropy will be centered around results discussed in [MS17, Ch. 7].

Large Deviations

We start with one of the classical examples of convergence of random variables to introduce large deviations. Consider a sequence of independent and identically distributed real-valued random variables $(X_i)_i$ with common distribution μ . Set

$$S_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

$$m = E[X_1]$$

$$v = E[X_1^2] - m^2.$$

Then, we have that if $E[X_1^2] < \infty$, the central limit theorem says that $S_n \approx m + \frac{\sigma}{\sqrt{n}}N(0, 1)$.

If μ is the standard Gaussian distribution, then this gives that S_n is distributed as $N(0, 1/n)$; we then get that

$$P(S_n \in [x, x + dx]) \approx \sqrt{\frac{n}{2\pi}} e^{-nx^2} dx.$$

Asymptotically, this gives that the probability that S_n is near the value $x \in \mathbb{R}$ decays exponentially in n determined by a rate function $I(x) = x^2/2$.

We will now generalize this result. In particular, if we let μ be any distribution discussed above (rather than simply the normal distribution), then we will find a rate function $I(x)$ such that

$$e^{-nI(x)} \sim P(S_n > x)$$

whenever $x > m$, and whenever $x < m$

$$e^{-nI(x)} \sim P(S_n < x).$$

For a given distribution μ , we can compute the rate function by using a family of basic manipulations. If $x > m$, then for all $\lambda \geq 0$, we may use Markov's inequality to obtain

$$\begin{aligned} P(S_n > x) &= P(nS_n > nx) \\ &= P\left(e^{\lambda(nS_n - nx)} \geq 1\right) \\ &\leq E\left[e^{\lambda(nS_n - nx)}\right] \\ &= e^{-\lambda nx} E\left[e^{\lambda(X_1 + \dots + X_n)}\right] \\ &= \left(e^{-\lambda x} E\left[e^{\lambda X}\right]\right)^n, \end{aligned}$$

where X is identically distributed to each of the X_i , and we use the fact that the X_i are independent.

One-Dimensional Free Entropy

Microstates Free Entropy

Applications: Structural Properties of Free Group Factors

References

- [MS17] James A. Mingo and Roland Speicher. *Free Probability and Random Matrices*. Vol. 35. Fields Institute Monographs. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017, pp. xiv+336. ISBN: 978-1-4939-6941-8; 978-1-4939-6942-5. DOI: [10.1007/978-1-4939-6942-5](https://doi.org/10.1007/978-1-4939-6942-5). URL: <https://doi.org/10.1007/978-1-4939-6942-5>.