Problem 1

Problem: An ordinal A is a successor ordinal if $A = B \cup \{B\}$ for some ordinal B. An element $m \in A$ is maximal if $\forall x \in A\, (x \in \vee x = m)$. Show that an ordinal is a successor ordinal if and only if it contains a maximal element.

Solution: Let y be an ordinal that contains a maximal element m. Then, for all $x \in y$, $x \in m$ or $x = m$. Thus, for $z = y \setminus \{m\}$, $\forall t \in z$, $t \in m$. Thus, $z \subseteq m$.

We claim that $m \subseteq z$. Since $m \in y$, $m \subset y$. Thus, $m \subset y \cup \{m\}$. However, since $m \notin m$, we have $m \subseteq z$. Thus, $z = m$, and $y = m \cup \{m\}$.

If y is a successor ordinal, then $y = m \cup \{m\}$ for some ordinal m, meaning that for all $x \in y$, $x \in m$ or $x \in \{m\}$, meaning $x \in m$ or $x = m$.

Problem 2

Problem: A limit ordinal is a nonzero ordinal that is not a successor ordinal. Prove that an ordinal A is a limit ordinal if and only if $A = \bigcup A$.

Solution: Let $A = \bigcup A$ and $A \neq 0$. Let $x \in A$. Then, $x \in \bigcup A = \bigcup_{y \in A} y$. Thus, $x \in y$ for some $y \in A$. Thus, $x < y$, so x is not maximal, meaning A has no maximal element. Thus, by problem 2, A is not a successor ordinal.

Problem 3

Problem: Prove that the following two versions of the Axiom of Choice are equivalent.

AC 1: Let T be a set such that every element of T is nonempty. Then, there exists a function f with domain T such that $\forall x \in T, f(x) \in x$.

AC 2: Let T be a set and F a function with domain T such that $\forall x \in T$, $F(x)$ is nonempty. Then, there exists a function f with domain T such that $\forall x \in T$, $f(x) \in F(x)$.

Problem 4

Problem: Let $(S, \leqslant)$ be a partially ordered set where every chain in S has an upper bound in S. Prove that there is a maximal element in S.

Solution: Let Γ be an ordinal. We define $g : \Gamma \to H$ recursively by

$$g(\alpha) = \begin{cases} g(\beta) \cup f\left(g\left(\beta\right)\right) & \alpha = \beta \cup \{\beta\} \\ \bigcup_{\beta \in \alpha} g(\alpha) & \text{else} \end{cases}$$

We wish to prove that for all $\alpha \in \Gamma$, $g\left(\alpha\right)$ is a chain in S.

If $\alpha = 0$, then $g(\alpha) = \varnothing$, which is a chain (by vacuous truth).

Suppose toward contradiction there is some α such that $g\left(\alpha\right)$ is not a chain. Set $B = \{\alpha \in \Gamma \mid g\left(\alpha\right) \text{ is not a chain}\}$. Thus, $B \neq \varnothing$ by our assumption. Let α_0 be the least element of B (which exists because $B \subseteq \Gamma$ and Γ is well-ordered). Additionally, we know that $\alpha_0 \neq \varnothing$ since $g\left(\varnothing\right) = \varnothing$ is a chain.

(i) If $\alpha_0 = \alpha' \cup \{\alpha'\}$. Then $g\left(\alpha_0\right) = g\left(\alpha'\right) \cup \{f\left(g\left(\alpha'\right)\right)\}$.

We know that $g\left(\alpha'\right)$ is a chain, since $\alpha' < \alpha_0$ and α_0 is the least element of B. Since $m = f\left(g\left(\alpha'\right)\right)$ is a strict upper bound for $g\left(\alpha'\right)$, $\forall x \in g\left(\alpha'\right)$, $x < m$.

Let $a, b \in g\left(\alpha_0\right) = g\left(\alpha'\right) \cup \{m\}$. Then, either $a, b \in g\left(\alpha'\right)$, $a, b \in \{m\}$, or (without loss of generality), $a \in g\left(\alpha'\right)$ and $b \in \{m\}$. If $a, b \in g\left(\alpha'\right)$, then $a < b$, $b < a$, or $a = b$, since $g\left(\alpha'\right)$ is totally ordered. If $a, b \in \{m\}$, then $a = b$. Else, if $a \in g\left(\alpha'\right)$ and $b \in \{m\}$, then $a < b$.

Thus, $g(\alpha_0)$ is totally ordered. $\perp$

(ii) Let α_0 be a limit ordinal. Then,

$$g(\alpha_0) = \bigcup_{\beta \in \alpha_0} g(\beta).$$

Let $a, b \in g(\alpha_0)$.

Let $g : \Gamma \to H$. We have shown that $g(\alpha) \in H$ for any $\alpha \in H$.

Let $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. Without loss of generality, let $\alpha \subset \beta$. We will show that $g(\alpha) \subset g(\beta)$.

Suppose toward contradiction that $\alpha \subset \beta$ does not imply $g(\alpha) \subset g(\beta)$. Let β_0 be the smallest element of Γ such that there exists $\alpha_0 \subset \beta_0$ with $g(\alpha_0) \not\subset g(\beta_0)$. We know that $\beta_0 \neq 0$ because 0 has no proper subsets.

(i) If $\beta_0 = \beta' \cup \{\beta'\}$ for some $\beta' \in \Gamma$, then $g(\beta) = g(\beta') \cup \{f(g(\beta'))\}$.

Since $\alpha_0 \subset \beta_0$, $\alpha_0 \in \beta_0$, then $\alpha_0 \in \beta'$ or $\alpha_0 = \beta'$, meaning $\alpha_0 \in \beta'$ or $\alpha_0 = \beta'$.

If $\alpha_0 = \beta'$, then $g(\alpha_0) = g(\beta') \subset g(\beta_0)$ since $\{f(g(\beta'))\}$ is a strict upper bound.

If $\alpha_0 \subset \beta'$, then $g(\alpha_0) \subset g(\beta')$ since β' satisfies the assumption. Thus, $g(\alpha_0) \subset g(\beta_0)$.

Problem 5

Problem: Show that there exists an uncountable set T of countable subsets of $\mathbb{R}$.

Solution: Let S be the set of all countable subsets of $\mathbb{R}$, partially ordered by inclusion.

Let C be a chain in S. Since $C \subseteq S$, C consists of countable sets.

Suppose toward contradiction that there exists no chain with uncountable length. Then, C is countable, so

$$C' = \bigcup_{A \in C} A$$

is a countable union of countable sets, so C' is countable, and C' is an upper bound for C. It is thus the case that S has a maximal element M, which is a countable set.

However, since $\mathbb{R}$ is uncountable, there is some $q \in \mathbb{R} \setminus M$, meaning $M \subsetneq M \cup \{q\}$, contradicting the maximality of M.

Thus, there must be at least one uncountable chain, T, in S.