Abstract

Measures are just set functions that follow some particular basic properties, but we can expand them beyond the positive real numbers towards complex numbers; to conceptualize these signed and complex measures, we need to make use of results like the Lebesgue–Radon–Nikodym Theorem and the Hahn Decomposition Theorem that allow us to understand their structural properties.

Signed Measures and the Hahn Decomposition

We know that a measure is a set function $\mu \colon \mathcal{M} \to [0, \infty]$ on a σ-algebra such that

- $\mu(\varnothing) = 0$;
- for a family of disjoint sets $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$,

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

We may ask what happens if we change the codomain from $[0, \infty]$ to $\mathbb{R}$ or $\mathbb{C}$. This is where *signed measures* come in.

Definition. A *signed measure* μ is a real-valued countably additive set function such that $\mu(\varnothing) = 0$ and μ takes on at most one of $-\infty$ or ∞.

We begin by establishing some basic properties of signed measures (akin to the basic properties of measures).

Theorem: Let μ be a signed measure.

(a) If E and F are measurable sets with $E \subseteq F$ and $|\mu(F)| < \infty$, then $|\mu(E)| < \infty$.

(b) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ is a disjoint sequence of measurable subsets such that $\left|\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right)\right| < \infty$, then the series $\sum_{j=1}^{\infty} \mu(E_j)$ is absolutely convergent.

(c) If $\{E_j\}_{j=1}^{\infty}$ is a monotone sequence of measurable sets — and if decreasing, $|\mu(E_n)| < \infty$ for at least one such n — then

$$\mu\left(\lim_{j\to\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Proof.

(a) We see that $\mu(F) = \mu(F \setminus E) + \mu(E)$. If exactly one of the summands is infinite, then so is $\mu(F)$. If both are infinite, then since μ takes on at most one of $-\infty$ or ∞, they are equal and then $\mu(F)$ is infinite. Therefore, both summands must be finite.

(b) We set

$$E_j^+ = \begin{cases} E_j & \mu(E_j) \geq 0 \\ \varnothing & \mu(E_j) < 0 \end{cases}$$

$$E_j^- = \begin{cases} E_j & \mu(E_j) \leq 0 \\ \varnothing & \mu(E_j) > 0 \end{cases}.$$

Then,

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j^+\right) = \sum_{j=1}^{\infty} \mu(E_j^+)$$

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j^-\right) = \sum_{j=1}^{\infty} \mu\left(E_j^-\right).$$

Since the terms of both series have constant sign, and μ takes on at most one of $\pm\infty$, it follows that at least one of these series is convergent, and since $\sum_{j=1}^{\infty} \mu(E_j)$ is convergent, both series converge; therefore, the series is absolutely convergent.

(c) If $\{E_n\}_{n=1}^{\infty}$ is increasing, then we take

$$\begin{aligned}
\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigsqcup_{j=2}^{\infty} (E_j \setminus E_{j-1})\right) \\
&= \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\
&= \lim_{n\to\infty} \sum_{j=2}^{n} \mu(E_j \setminus E_{j-1}) \\
&= \lim_{n\to\infty} \mu\left(\bigsqcup_{j=2}^{n} (E_j \setminus E_{j-1})\right) \\
&= \lim_{j\to\infty} \mu(E_j),
\end{aligned}$$

and similarly for a decreasing sequence, using part (a) to ensure finiteness.

□

Now, we discuss the structure of positive-valued and negative-valued measurable sets.

Definition. Let μ be a signed measure on $(X, \mathcal{M})$. We call a set $E \in \mathcal{M}$ *positive* if, for every measurable $F \subseteq E$, $\mu(F) \geq 0$; similarly, we call $E \in \mathcal{M}$ *negative* if, for every measurable $F \subseteq E$, $\mu(F) \leq 0$.

Theorem (Hahn Decomposition Theorem)**:** If μ is a signed measure, then there exist two disjoint sets A and B such that $A \sqcup B = X$, A is positive with respect to μ, and B is negative with respect to μ. This decomposition unique up to μ-null symmetric difference.