Contents

Introduction

This is going to be part of my notes for my Honors Thesis independent study, focused on Amenability and C^*-algebras. This set of notes will be focused on the theory of Hilbert spaces and bounded linear operators on Hilbert spaces. The primary source for this section of notes will be Timothy Rainone's *Functional Analysis: En Route to Operator Algebras*.

I do not claim any of this work to be original.

Hilbert Spaces

In quantum mechanics, the state of a non-relativistic particle is given by a vector in some Hilbert space, which evolves by moving around that space. Specifically, the state of such a particle is determined entirely by the wave function $\xi = \xi(x,t)$, where $x \in \mathbb{R}$ is position and t is time. The wave function is a probability distribution satisfying

$$\int_{\mathbb{R}} |\xi(x,t)|^2 \, d\lambda = 1.$$

In particular, ξ is an element of the space $L_2(\mathbb{R},\lambda)$. The observables on ξ are modeled as operators on $L_2(\mathbb{R},\lambda)$.

Theory of Hilbert Spaces

In undergraduate linear algebra, the dot product of vectors in $\mathbb{R}^n$, $v \cdot w$, is intimately tied to the geometry of $\mathbb{R}^n$ through the equations

$$v \cdot v = \|v\|^2$$
$$v \cdot w = \|v\| \, \|w\| \cos\theta.$$

Inner product spaces help generalize these properties.

Definition. Let X be a vector space over a field $\mathbb{F}$.

(1) An inner product on X is a map

$$\begin{aligned}\langle \cdot, \cdot \rangle : X \times X &\to \mathbb{F}\\ (x, y) &\mapsto \langle x, y \rangle\end{aligned}$$

which satisfies the following conditions for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{F}$.

(i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$;

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

(iii) $\langle x, x \rangle \geqslant 0$;

(iv) $\langle x, x \rangle = 0 \Rightarrow x = 0_X$.

If $\langle \cdot, \cdot \rangle$ satisfies (i)–(iii), but not necessarily (iv), then it is called a semi-inner product.

(2) If $\langle \cdot, \cdot \rangle$ is an inner product on X, the pair $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Remark: A semi inner product also satisfies, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{F}$,

$$\langle x, \lambda y + \mu z \rangle = \overline{\lambda} \langle x, z \rangle + \overline{\mu} \langle y, z \rangle .$$

A semi-inner product is linear in the first variable and conjugate linear in the second variable.

Definition. Let X be a complex vector space. A map

$$F : X \times X \to \mathbb{C}$$

which is linear in the first variable and conjugate linear in the second variable is called a sesquilinear form on X.

A fundamental fact about sesquilinear forms is that for any given sesquilinear form, we are able to pass it into a form that only consists of the same elements in both inputs.

Lemma (Polarization Identity)**:** Let $F : X \times X \to \mathbb{C}$ be a sesquilinear form on X. For all $x, y \in X$, we have

$$\begin{aligned}4F(x, y) &= F(x + y, x + y) + iF(x + iy, x + iy) - F(x - y, x - y) + iF(x - iy, x - iy)\\ &= \sum_{k=0}^{3} i^k F\left(x + i^k y, x + i^k y\right).\end{aligned}$$

Proof. Taking each expression

$$\begin{aligned}F(x + y, x + y) &= F(x, x) + F(x, y) + F(y, x) + F(y, y)\\ iF(x + iy, x + iy) &= iF(x, x) - F(y, x) + F(x, y) + iF(y, y)\\ -F(x - y, x - y) &= -F(x, x) + F(x, y) + F(y, x) - F(y, y)\\ -iF(x - iy, x - iy) &= -iF(x, x) - F(y, x) + F(x, y) - iF(y, y).\end{aligned}$$

Adding these expressions up, we get the polarization identity. $\square$

The following fact follows from the polarization identity.

Fact. If F and G are two sesquilinear forms that agree on the diagonal — i.e., $F(x, x) = G(x, x)$ — then F and G agree everywhere.

Fact. Let X be an inner product space, and suppose $z_1, z_2 \in X$ are such that $\langle x, z_1 \rangle = \langle x, z_2 \rangle$ for all $x \in X$. Then, $z_1 = z_2$.

Proof. We have $\langle x, z_1 \rangle = \langle x, z_2 \rangle$. Then, $\langle x, z_1 - z_2 \rangle = 0$ for all $x \in X$, so $\langle z_1 - z_1, z_1 - z_2 \rangle = 0$, so $z_1 - z_2 = 0$. $\square$

Let's see some inner product spaces.

Example (Finite-Dimensional Space)**.** The finite dimensional space $\mathbb{C}^n$ admits an inner product space given by

$$\langle \xi, \eta \rangle = \sum_{j=1}^{n} \xi_j \overline{\eta_j},$$

where ξ and η are n dimensional vectors over $\mathbb{C}$.

Example (Sequence Space)**.** The space of square-summable sequences,

$$\ell_2 = \left\{ (\lambda_k)_k \;\middle|\; \sum_{n=1}^{\infty} |\lambda_n|^2 := \|\lambda\|^2 < \infty \right\}$$

is an inner product space with the inner product

$$\langle \lambda, \mu \rangle = \sum_{n=1}^{\infty} \lambda_n \overline{\mu_n}.$$

The Cauchy–Schwarz inequality provides for this to be a well-defined inner product.

$$\begin{aligned} \sum_{n=1}^{N} |\lambda_n \overline{\mu_n}| &\leqslant \left(\sum_{n=1}^{N} |\lambda_n|^2 \right)^{1/2} \left(\sum_{n=1}^{N} |\mu_n|^2 \right)^{1/2} \\ &\leqslant \|\lambda\|_2 \, \|\mu\|_2 \\ &< \infty. \end{aligned}$$

Example (Continuous Functions)**.** The space $X = C([0,1])$ admits an inner product given by

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$

Example (Sesquilinear Form on Continuous Function Space)**.** Let Ω be a locally compact Hausdorff space and suppose $\varphi : C_0(\Omega) \to \mathbb{F}$ is a positive linear functional. We know that $\varphi = \varphi_\mu$ for some positive regular finite measure μ on $(\Omega, \mathcal{B}_\Omega)$, and

$$\varphi_\mu(f) = \int_\Omega f \, d\mu.$$

We get a semi inner product on $C_0(\Omega)$ by

$$\begin{aligned} \langle \cdot, \cdot \rangle_\varphi : C_0(\Omega) \times C_0(\Omega) &\to \mathbb{F} \\ (f, g) &\mapsto \int_\Omega f\overline{g}\, d\mu. \end{aligned}$$

We claim that, when μ has full support, $\langle \cdot, \cdot \rangle_\varphi$ is an inner product.

Suppose $g \in C_0(\Omega)$ with $g \geqslant 0$ and $g \neq 0$. Then, there is a nonempty open subset $U \subseteq \Omega$ and $\delta > 0$ such that $g(x) \geqslant \delta$ for all $x \in U$. Since μ has full support, it must be the case that $\mu(U) > 0$, so

$$\begin{aligned} \varphi(g) &= \int_\Omega g \, d\mu \\ &\geqslant \int_\Omega \delta \mathbb{1}_U \, d\mu \\ &= \delta \mu(U) \\ &> 0. \end{aligned}$$

Thus, if $\langle f, f \rangle_\varphi = 0$, then $\varphi\left(|f|^2\right) = 0$, so $f = 0$.

Example (Hilbert–Schmidt Operators)**.** Let $\mathbb{M}_n$ be the $*$-algebra of $n \times n$ matrices over the complex numbers. Let $\text{tr} : \mathbb{M}_n \rightarrow \mathbb{C}$ denote the trace. The trace is a linear, positive, faithful functional satisfying $\text{tr}(a^*) = \overline{\text{tr}(a)}$ for all $a \in \mathbb{M}_n$. The trace induces an inner product

$$\langle a, b \rangle_{\text{HS}} = \text{tr}(b^* a),$$

where the subscript HS stands for Hilbert–Schmidt.

Definition. Let X be an inner product space.

(1) We say two vectors $x, y \in X$ are orthogonal if $\langle x, y \rangle = 0$. We write $x \perp y$.

(2) Let $z \neq 0$ be a fixed vector in X. We define the one dimensional projection

$$P_z(x) = \frac{\langle x, z \rangle}{\langle z, z \rangle} z.$$

Note that P_z is linear and its range is the one-dimensional subspace span(z).

Note: There are a lot of propositions, lemmas, and exercises in this section of my professor's textbook, but I'm not going to be going through all of them since we learn a lot of this in Real Analysis II.

We can turn any semi-inner product space into a seminormed vector space using the semi-inner product. If the semi-inner product is a true inner product, then we can use the inner product to define a norm.

Definition. Let X be a semi-inner product space. For each $x \in X$, we set

$$\|x\| = \langle x, x \rangle^{1/2}.$$

Theorem (Pythagoras)**:** Let X be a semi-inner product space, and suppose $x_1, x_2, \dots, x_n$ are pairwise orthogonal. Then,

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 = \sum_{j=1}^{n} \|x_j\|^2$$

Corollary: Let X be an inner product space, and fix $z \neq 0$ in X. Then, for all $x, y \in X$, we have

(1) $\|x\|^2 = \|x - P_z(x)\|^2 + \|P_z(x)\|^2$;

(2) $\|P_z(x)\| \leqslant \|x\|$;

(3) $|\langle x, z \rangle| \leqslant \|x\| \|y\|$, with equality if and only if x and y are linearly independent (the Cauchy–Schwarz inequality);

(4) $\|x + y\| \leqslant \|x\| + \|y\|$;

(5) $\|\cdot\|$ is a norm on X.

Proposition: If X is an inner product space, then the inner product

$$\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{F}$$

is continuous.

We often start with a semi-inner product, then construct an inner product by quotient out by the null space.

Proposition: Let $\langle \cdot, \cdot \rangle$ be a semi-inner product on X.

(1) The set

$$N = \{x \in X \mid \langle x, x \rangle = 0\}$$

is a subspace of X.

(2) The map
$$\langle x+N, y+N\rangle_{X/N} = \langle x, y\rangle$$
is an inner product on the quotient space X/N.

Proposition (Parallelogram Law)**:** Let X be an inner product space. Then,
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Recall that Banach spaces include ideas regarding isometric isomorphisms — however, we cannot immediately assume this extends to inner product spaces since they include an inherent geometric structure as well. As it turns out, this automatically appears from the definition of an isometry.

Proposition: Let X and Y be inner product spaces. Suppose $V : X \to Y$ is a linear transformation. The following are equivalent.

(i) V is an isometry;

(ii) for each $x, x' \in X$, we have $\langle V(x), V(x')\rangle_Y = \langle x, x'\rangle_X$.

Proof. To show that (ii) implies (i), we see that for $x \in X$,
$$\begin{aligned}\|V(x)\|^2 &= \langle V(x), V(x)\rangle\\ &= \langle x, x\rangle\\ &= \|x\|^2.\end{aligned}$$

We define the sesquilinear forms
$$\begin{aligned}F(x, x') &= \langle V(x), V(x')\rangle_Y\\ G(x, x') &= \langle x, x'\rangle.\end{aligned}$$

Since V is norm-preserving, we have
$$\begin{aligned}F(x, x) &= \|V(x)\|^2\\ &= \|x\|^2\\ &= G(x, x),\end{aligned}$$

so by the polarization identity, F and G agree everywhere. □

Definition. Let X and Y be inner product spaces. A surjective linear isometry $U : X \to Y$ is called a unitary operator.

Equivalently, a unitary operator is a linear isomorphism $U : X \to Y$ that preserves the inner product. We say X and Y are unitarily isomorphic.

Example (A Nonunitary Isometry)**.** Consider the right shift on ℓ_2, defined by
$$R(\xi_1, \xi_2, \dots,) = (0, \xi_1, \xi_2, \dots).$$

Then, R is not onto, but for each $\xi, \eta \in \ell_2$, we have $\langle R(\xi), R(\eta)\rangle = \langle \xi, \eta\rangle$. Thus, R is isometric but not unitary.

Definition (Hilbert Space)**.** A Hilbert space is an inner product space $\mathcal{H}$ over $\mathbb{C}$ such that the norm $\|x\|^2 = \langle x, x\rangle$ is complete.

Example. The space ℓ_2 of all square-summable sequences is a Hilbert space.

Example. If $(\Omega, \mathcal{M}, \mu)$ is any measure space, then $L_2(\Omega, \mu)$ is a Hilbert space with inner product
$$\langle f, g\rangle = \int_\Omega f\overline{g}\, d\mu.$$

Orthogonal Projections

Recall that closed subspaces of Banach spaces may not always admit a topological complement (for instance, $c_0 \subseteq \ell_\infty$). However, in a Hilbert space, a closed subspace always admits an orthogonal projection operator (hence a topological complement).

Theorem (Hilbert Projection Theorem)**:** Let $\mathcal{H}$ be a Hilbert space. Suppose $C \subseteq \mathcal{H}$ is a closed and convex set. Given $x \in \mathcal{H}$, there is a unique $y_x \in C$ such that $\operatorname{dist}_C(x) = d(x, y_x)$. We say y_x is the point in C closest to x

Proof. Set $d = \operatorname{dist}_C(x)$. If $x \in C$, we take $y = x$, so we assume $x \notin C$.

We find a sequence $(y_n)_{n\geqslant 1}$ with $d(x, y_n) \to d$ decreasing. Set $z_n = y_n - x$. We have $\|z_n\| \to d$ decreasing, meaning $\|z_n\|^2 \to d^2$ decreasing. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for $n \geqslant N$,

$$\|z_n\|^2 < d^2 + \varepsilon.$$

We claim that $(y_n)_n$ is a Cauchy sequence in C. If $p, q \in \mathbb{N}$, we see that

$$\begin{aligned} y_p - y_q &= z_p - z_q \\ \left\|\frac{1}{2}(z_p + z_q)\right\| &= \left\|\frac{1}{2}(y_p + y_q) - x\right\| \\ &\geqslant d, \end{aligned}$$

as $\frac{1}{2}(y_p + y_q)$ belongs to C. Thus, for $p, q \geqslant N$, we have

$$\begin{aligned} \|y_p - y_q\|^2 &= \|z_p - z_q\|^2 \\ &= 2\|z_p\|^2 + 2\|z_q\|^2 - \|z_p + z_q\|^2 \\ &= 2\|z_p\|^2 + 2\|z_q\|^2 - 4\left\|\frac{1}{2}(z_p + z_q)\right\|^2 \\ &\leqslant 2d^2 + 2\varepsilon + 2d^2 + 2\varepsilon - 4d^2 \\ &= 4\varepsilon. \end{aligned}$$

Since C is closed, we thus have $d = \lim_{n\to\infty} d(x, y_n) = d(x, y)$ for $(y_n)_n \to y$ for some $y \in C$.

To see uniqueness, suppose $y_1, y_2 \in C$ with $d(x, y_i) = d$. Set $z = y_j - x$ for each j. We have

$$\begin{aligned} 0 &\leqslant \|z_1 - z_2\|^2 \\ &= 2\|z_1\|^2 + 2\|z_2\|^2 - 4\left\|\frac{1}{2}(z_1 + z_2)\right\|^2, \end{aligned}$$

meaning

$$\begin{aligned} 0 &\leqslant \|y_1 - y_2\|^2 \\ &= 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 - 4\left\|\frac{1}{2}(y_1 + y_2) - x\right\|^2 \\ &\leqslant 2d^2 + 2d^2 - 4d^2 \\ &= 0. \end{aligned}$$

Thus, $y_1 = y_2$. ◻

Definition. Let $\mathcal{H}$ be a Hilbert space, and let $M \subseteq \mathcal{H}$ be a closed subspace. We define

$$P_M : \mathcal{H} \to \mathcal{H}$$

by $P_M(x) = y_x$, where y_x is the unique point from the Hilbert projection theorem.

We call P_M the orthogonal projection of $\mathcal{H}$ onto M.

Fact. There are some facts about the orthogonal projection that are useful for us to know.

- $P_M(x) = x \Leftrightarrow x \in M$;
- $\operatorname{Ran}(P_M) = M$;
- $P_M \circ P_M = P_M$ (i.e., that P_M is idempotent).

Definition. Let X be an inner product space, and suppose $S \subseteq X$ is an arbitrary subset. We define the perp of S, $S^\perp$, to be

$$S^\perp = \{x \in X \mid \langle x, y\rangle = 0 \text{ for all } y \in S\}.$$

Exercise: Let $S \subseteq \mathcal{H}$ be an arbitrary subset. Prove the following.

(1) $S^\perp$ is always a closed subspace of $\mathcal{H}$.
(2) $S \subseteq \left(S^\perp\right)^\perp$.
(3) $S \cap S^\perp = \{0\}$.

Solution:

(1) For $x, x' \in S^\perp$ and $\alpha \in \mathbb{C}$, we have for all $y \in S$,

$$\begin{aligned}\left\langle x + \alpha x', y\right\rangle &= \langle x, y\rangle + \alpha\left\langle x', y\right\rangle \\ &= 0,\end{aligned}$$

so $S^\perp$ is a subspace. Additionally, for any sequence $(x_n)_n \subseteq S^\perp$ with $(x_n)_n \to x$ in X, the continuity of the inner product gives

$$\begin{aligned}\langle x_n, y\rangle &\to \langle x, y\rangle \\ &= 0.\end{aligned}$$

(2) For $t \in S$, we have, for all $x \in S^\perp$,

$$\begin{aligned}\langle x, t\rangle &= 0 \\ &= \langle t, x\rangle,\end{aligned}$$

meaning $t \in \left(S^\perp\right)^\perp$.

(3) If $t \in S \cap S^\perp$, then $t \in S$ and $t \in S^\perp$, so

$$\langle t, t\rangle = 0,$$

so $t = 0$.

One of the features of Hilbert spaces is that closed subspaces are always complemented.

Theorem: Let $M \subseteq \mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then, the following are true.

(1) $x - P_M(x) \in M^\perp$ for all $x \in \mathcal{H}$.

(2) $\mathcal{H} = M \oplus M^\perp$.

(3) $(M^\perp)^\perp = M$.

(4) Let P and Q denote the projection operators onto M and $M^\perp$ according to the decomposition $\mathcal{H} = M \oplus M^\perp$. Then, $P = P_M$ and $Q = P_{M^\perp}$.

(5) P_M is linear, $P_M^2 = P_M$, $\operatorname{Ran}(P_M) = M$, $\|P_M\| = 1$, and $\ker(P_M) = M^{\perp}$.

(6) $\mathcal{H}/M \cong M^{\perp}$ are isometrically isomorphic.

Proof.

(1) Let $y = P_M(x)$, and set $z = x - y$. We know that $\|z\| = \operatorname{dist}_M(x) = d$. Let $0 \neq \xi \in M$. Set $\zeta = P_{\xi}(z) = \frac{\langle z,\xi\rangle}{\langle \xi,\xi\rangle}\xi$.

We claim that $\zeta = 0$. Note that

$$\begin{aligned}\|z - \zeta\| &= \|x - y - \zeta\|\\ &= \|x - (y + \zeta)\|\\ &\geqslant d,\end{aligned}$$

as $y + \zeta \in M$.

On the other hand, we have

$$\begin{aligned}\|z - \zeta\|^2 + \|\zeta\|^2 &= \|z\|^2\\ &= d^2.\end{aligned}$$

Thus, $\|z - \zeta\| \leqslant d$. With $\|z - \zeta\| = d$, we have $\|x - y - \zeta\| = d$. Thus, we must have $y + \zeta = y$, so $\zeta = 0$.

(2) If $x \in \mathcal{H}$, we have

$$x = P_M(x) + x - P_M(x),$$

and since $M \cap M^{\perp} = \{0\}$, we have $\mathcal{H} = M \oplus M^{\perp}$.

(3) It is the case that $M \subseteq (M^{\perp})^{\perp}$. Let $x \in (M^{\perp})^{\perp}$. Write $x = y + z$ according to the decomposition $\mathcal{H} = M \oplus M^{\perp}$. Then, $z = x - y \in (M^{\perp})^{\perp} \cap (M^{\perp}) = \{0\}$, so $x = y \in M$, so $M = (M^{\perp})^{\perp}$.

(4) By the way we have defined P and Q, we must have $P(x) = P_M(x)$ for every $x \in \mathcal{H}$. Let $\widetilde{P}$ and $\widetilde{Q}$ be the bounded linear projections according to the decomposition $\mathcal{H} = M^{\perp} \oplus (M^{\perp})^{\perp}$. Since $M = (M^{\perp})^{\perp}$, we have $\widetilde{Q} = P$. Additionally, we must have $\widetilde{P} = P_{M^{\perp}}$. Thus,

$$\begin{aligned}Q &= I - P\\ &= I - \widetilde{Q}\\ &= \widetilde{P}\\ &= P_{M^{\perp}}.\end{aligned}$$

(5) By the Pythagorean theorem, we have

$$\|x\|^2 = \|P_M(x)\|^2 + \|x - P_M(x)\|^2$$

for every $x \in \mathcal{H}$, so $\|P_M(x)\| \leqslant \|x\|$, meaning $\|P_M\| \leqslant 1$. Since $P_M^2 = P_M$, we also have $\|P_M\| \geqslant 1$.

(6) Notice that $P_{M^{\perp}} : \mathcal{H} \to M^{\perp}$ is a 1-quotient map with the kernel $\ker(P_{M^{\perp}}) = M$. Thus, we have $\mathcal{H}/M \cong M^{\perp}$.

□

Corollary: The following are true.

(1) The quotient of a Hilbert space is a Hilbert space.

(2) If $M \subsetneq \mathcal{H}$, then $M^{\perp} \neq \{0\}$. Additionally, if $M^{\perp} = \mathcal{H}$, then $M = \{0\}$.

(3) For any subset $S \subseteq \mathcal{H}$, we have $(S^{\perp})^{\perp} = \overline{\operatorname{span}}(S)$.

Exercise: Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $E \subseteq \mathcal{M}$ be measurable. We look at the set of essentially E-supported square-integrable functions:

$$M_E = \{\xi \in L_2(\Omega, \mu) \mid \xi|_{E^c} = 0 \ \mu\text{-a.e.}\}.$$

(1) Show that M_E is a closed subspace of $L_2(\Omega, \mu)$, and prove that the orthogonal projection onto M_E is given by

$$P_{M_E}(\xi) = \xi \mathbb{1}_E.$$

(2) Note that the restriction $(E, \mathcal{M}|_E, \mu_E)$ is a measure space, where

$$\begin{aligned} \mathcal{M}_E &= \{F \cap E \mid F \in \mathcal{M}\} \\ \mu_E &= \mu|_{\mathcal{M}_E}. \end{aligned}$$

Prove that $L_2(E, \mu_E)$ and M_E are unitarily isomorphic.

Solution:

(1) If ξ and η are two functions that are essentially E-supported, then the sum $\xi + \alpha\eta$, where $\alpha \in \mathbb{C}$, is also essentially E-supported. Similarly, if $(\xi_n)_n \to \xi$ is a sequence of essentially E-supported functions converging in norm to ξ, then we have $(\xi_m - \xi_n)|_{E^c} = 0$ for each ξ_m, ξ_n, so ξ is also essentially E-supported.

To show that P_{M_E} defined by $P_{M_E}(\xi) = \xi\mathbb{1}_E$ is the orthogonal projection onto M_E, we show that P_{M_E} is idempotent and maps all members of M_E to themselves. For $\xi \in L_2(\Omega, \mu)$, we see that

$$\begin{aligned} P_{M_E}^2(\xi) &= P_{M_E}(\xi\mathbb{1}_E) \\ &= \xi(\mathbb{1}_E)(\mathbb{1}_E) \\ &= \xi\mathbb{1}_E \\ &= P_{M_E}(\xi). \end{aligned}$$

Additionally, for any $\xi \in M_E$, we have that $\xi\mathbb{1}_E \equiv \xi$ since $\xi|_{E^c} = 0$ μ-a.e. Thus, P_{M_E} is an idempotent operator that preserves the closed subspace M_E, so by the Hilbert projection theorem, it is necessarily the only (up to μ-a.e. equivalence) orthogonal projection onto M_E.

(2)

Proposition: Let $\mathcal{H}$ be a Hilbert space, and suppose $\{M_i\}_{i=1}^{n}$ is a finite family of mutually orthogonal closed subspaces. Write $M = \sum_{i=1}^{n} M_i$ for the internal sum.

(1) $M \subseteq \mathcal{H}$ is a closed subspace, and $M = \bigoplus_{i=1}^{n} M_i$ is the internal direct sum.

(2) $P_M = \sum_{i=1}^{n} P_{M_i}$.

Proof. To see (1), we know that since $M_i \perp M_j$ for each $i \neq j$, it is the case that $M_i \cap M_j = \{0\}$ for each $i \neq j$, so it is indeed a direct sum.

To see (2), let $x \in \mathcal{H}$, and write $x = y + z$ according to the decomposition $\mathcal{H} = M \oplus M^{\perp}$. Since $M_j \subseteq M$, we have $\ker\left(P_{M_j}\right) \supseteq M^{\perp}$ for each j. Thus, $P_{M_j}(z) = 0$ for every j.

Since $M = \bigoplus_{i=1}^{n} M_i$, we write $y = \sum_{i=1}^{n} y_i$, with $y_i \in M_i$ uniquely. Since M_i are mutually orthogonal, we know that $M_i \subseteq M_j^{\perp} = \ker\left(P_{M_j}\right)$ for each $i \neq j$. We compute

$$\begin{aligned} P_{M_j}(x) &= P_{M_j}(y + z) \\ &= P_{M_j}(y) \\ &= P_{M_j}\left(\sum_{i=1}^{n} y_i\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} P_{M_j}(y_i) \\
&= y_j.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\left(\sum_{i=1}^{n} P_{M_i}\right)(x) &= \sum_{i=1}^{n} P_{M_i}(x) \\
&= \sum_{i=1}^{n} y_i \\
&= y \\
&= P_M(x).
\end{aligned}
$$

□

We can now turn our attention to understanding the continuous dual of Hilbert spaces.

Definition. Let X be an inner product space, and fix $z \in X \setminus \{0\}$. We define $\varphi_z : X \to \mathbb{F}$ by $\varphi_z(x) = \langle x, z\rangle$.

Proposition: Let X be an inner product space. Each $\varphi_z \in X^*$, and the map $X \to X^*$ defined by $z \mapsto \varphi_z$ is a conjugate linear isometry.

Proof. We see that φ_z is linear. We have

$$
\begin{aligned}
|\varphi_z(x)| &= |\langle x, z\rangle| \\
&\leqslant \|x\| \|z\|,
\end{aligned}
$$

with

$$
\begin{aligned}
\varphi_z\left(\frac{z}{\|z\|}\right) &= \frac{1}{\|z\|}\langle z, z\rangle \\
&= \|z\|,
\end{aligned}
$$

so $\|\varphi_z\|_{\text{op}} = \|z\|$. For every $x \in X$, we also have

$$
\begin{aligned}
\varphi_{z_1+\alpha z_2}(x) &= \langle x, z_1 + \alpha z_2\rangle \\
&= \langle x, z_1\rangle + \overline{\alpha}\langle x, z_2\rangle \\
&= (\varphi_{z_1} + \overline{\alpha}\varphi_{z_2})(x).
\end{aligned}
$$

□

If $\mathcal{H}$ is a Hilbert space, then the map $\mathcal{H} \to \mathcal{H}^*$ given by $z \mapsto \varphi_z$ is a bijection. This is known as the Riesz Representation Theorem (not to be confused for the Riesz representation Theorem for measures on $C_c(\Omega)$).

Theorem (Riesz Representation Theorem)**:** Let $\mathcal{H}$ be a Hilbert space. If $\varphi \in H^*$, then there exists a unique $z \in \mathcal{H}$ such that $\varphi = \varphi_z$.

Proof. We assume $\varphi \neq 0$. We have $M = \ker(\varphi) \subseteq \mathcal{H}$ is a proper closed subspace, so we can choose $w \in M^\perp$ such that $w \neq 0$.

We see that $\ker(\varphi) \subseteq \ker(\varphi_w)$, meaning that $\varphi = \lambda\varphi_w$ for some $\lambda \in \mathbb{F}$. We compute

$$
\begin{aligned}
\varphi(x) &= \lambda\varphi_w(x) \\
&= \lambda\langle x, w\rangle
\end{aligned}
$$

$$= \left\langle x, \overline{\lambda} w \right\rangle.$$

Set $z = \overline{\lambda} w$.

To show uniqueness, if $\varphi = \varphi_{z_1} = \varphi_{z_2}$, then $\langle x, z_1 - z_2 \rangle = 0$ for all $x \in \mathcal{H}$, so $z_1 - z_2 \in \mathcal{H}^\perp = \{0\}$, so $z_1 = z_2$. □

Theorem: Every Hilbert space is reflexive.

Proof. Let $\iota : \mathcal{H} \to \mathcal{H}^{**}$ be the canonical embedding. Let $f \in \mathcal{H}^{**}$, and define $\psi : \mathcal{H} \to \mathbb{C}$ by $\psi(x) = \overline{f(\varphi_x)}$. For all $x_1, x_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\psi(x_1 + \lambda x_2) &= \overline{f(\varphi_{x_1 + \lambda x_2})} \\
&= \overline{f\left(\varphi_{x_1} + \overline{\lambda}\varphi_{x_2}\right)} \\
&= \overline{f(\varphi_{x_1}) + \overline{\lambda} f(\varphi_{x_2})} \\
&= \overline{f(\varphi_{x_1})} + \lambda \overline{f(\varphi_{x_2})} \\
&= \psi(x_1) + \lambda \psi(x_2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\psi(x)| &= \left| \overline{f(\varphi_x)} \right| \\
&= |f(\varphi_x)| \\
&\leqslant \|f\| \|\varphi_x\| \\
&= \|f\| \|x\|.
\end{aligned}$$

Thus, $\psi \in \mathcal{H}^*$, so we know that $\psi = \varphi_z$ for some $z \in H$. Thus,

$$\begin{aligned}
\overline{f(\varphi_x)} &= \psi(x) \\
&= \varphi_z(x) \\
&= \langle x, z \rangle \\
&= \overline{\langle z, x \rangle},
\end{aligned}$$

so $f(\varphi_x) = \langle z, x \rangle = \varphi_x(z) = \hat{z}(\varphi_x)$, so $f = \hat{z}$, so ι is surjective. □

Orthonormal Sets and Orthonormal Bases

Definition. Let X be an inner product space, and let A be an indexing set.

(1) A subset $\{x_\alpha\}_{\alpha \in A}$ is called orthogonal if $\left\langle x_\alpha, x_\beta \right\rangle = 0$ for $\alpha \neq \beta$.

(2) An orthonormal set is an orthogonal set consisting of unit vectors. The set $\{e_\alpha\}_{\alpha \in A}$ is orthonormal if

$$\left\langle e_\alpha, e_\beta \right\rangle = \begin{cases} 1 & \alpha = \beta \\ 0 = \alpha \neq \beta & \end{cases}.$$

Exercise: Show every orthogonal set is linearly independent.

Solution: Let $\{x_\alpha\}_{\alpha \in A}$ be an orthogonal set. Then, for

$$\sum_{i=1}^{n} a_i x_{\alpha_i} = 0,$$

we take

$$\begin{aligned}\left\langle x_{\alpha_j}, \sum_{i=1}^{n} a_i x_{\alpha_i} \right\rangle &= a_j \left\|x_{\alpha_j}\right\|^2\\ &= 0,\end{aligned}$$

so $a_i = 0$ for all i.

Remark: Given an inner product space X and a finite linearly independent subset $F = \{x_1, \dots, x_n\}$, we can always use the Gram–Schmidt process to generate an orthonormal subset $G = \{u_1, \dots, u_n\} \subseteq X$ with $\operatorname{span}(G) = \operatorname{span}(F)$. Inductively, we take

$$\begin{aligned}v_1 &= x_1\\ v_k &= x_k - \sum_{j=1}^{k-1} \frac{\langle x_k, v_j\rangle}{\langle v_j, v_j\rangle} v_j\\ u_k &= \frac{1}{\|v_k\|} v_k.\end{aligned}$$

Exercise: Let A be an arbitrary set, and consider the Hilbert space $\ell_2(A)$. Show that $\{e_\alpha\}_{\alpha\in A} \subseteq \ell_2(A)$ is an orthonormal set.

Example. The family of continuous functions $(e_n : \mathbb{T} \to \mathbb{C})_{n\in\mathbb{Z}}$ is an orthonormal basis for the arc length measure space $(\mathbb{T}, \mathcal{L}_{\mathbb{T}}, \nu)$.

$$\begin{aligned}\langle e_n, e_m\rangle &= \int_{\mathbb{T}} e_n \overline{e_m}\, d\nu\\ &= \int_{\mathbb{T}} e_n e_{-m}\, d\nu\\ &= \int_{\mathbb{T}} e_{n-m}\, d\nu\\ &= \delta_{mn}.\end{aligned}$$

Theorem: Let $\mathcal{H}$ be a Hilbert space, and suppose $(e_\alpha)_{\alpha\in A}$ is an orthonormal family in $\mathcal{H}$.

(1) If $(c_\alpha)_{\alpha\in A} \in \ell_2(A)$, then $\sum_{\alpha\in A} c_\alpha e_\alpha$ is summable in $\mathcal{H}$, and

$$\left\|\sum_{a\in A} c_\alpha e_\alpha\right\| = \|(c_\alpha)_\alpha\|.$$

(2) The map $T : \ell_2(A) \to \mathcal{H}$ defined by $T(\xi) = \sum_{\alpha\in A} \xi(\alpha) e_\alpha$ is a linear isometry.

(3) If $x \in \mathcal{H}$, then $\sum_{\alpha\in A} |\langle x, e_\alpha\rangle|^2 \leqslant \|x\|^2$. This is known as Bessel's inequality.

(4) If $M = \overline{\operatorname{span}}(\{e_\alpha\}_{\alpha\in A})$, then $P_M(x) = \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha$ is the orthogonal projection onto M.

Proof.

(1) We let $\mathcal{F}$ be the collection of finite subsets of A directed by inclusion. For $F \in \mathcal{F}$, we define

$$\begin{aligned}s_F &= \sum_{\alpha\in F} c_\alpha e_\alpha\\ c_F &= \sum_{\alpha\in F} |c_\alpha|^2.\end{aligned}$$

By the Pythagorean theorem, we have

$$\|s_F\|^2 = \left\|\sum_{\alpha\in F} c_\alpha e_\alpha\right\|^2$$

$$
\begin{aligned}
&= \sum_{\alpha\in F} \|c_\alpha e_\alpha\|^2 \\
&= \sum_{\alpha\in F} |c_\alpha|^2 \\
&= c_F.
\end{aligned}
$$

We claim the net $(s_F)_{F\in\mathcal{F}}$ is Cauchy in $\mathcal{H}$. For F and G in $\mathcal{F}$, we set

$$
d_\alpha = \begin{cases} c_\alpha & \alpha \in F \\ -c_\alpha & \alpha \in G \end{cases}.
$$

Then,

$$
\begin{aligned}
\|s_F - s_G\|^2 &= \left\| \sum_{\alpha\in F} c_\alpha e_\alpha - \sum_{\alpha\in G} c_\alpha e_\alpha \right\|^2 \\
&= \left\| \sum_{\alpha\in F\triangle G} d_\alpha e_\alpha \right\| \\
&= \sum_{\alpha\in F\triangle G} |d_\alpha|^2 \\
&= c_{F\triangle G}.
\end{aligned}
$$

Let $\varepsilon > 0$. Since $\sum_{\alpha\in A} |c_\alpha|^2$ is summable, there is a finite $F_0 \subseteq A$ such that for all $F \in \mathcal{F}$ with $F \cap F_0 = \emptyset$, we have $c_F \leqslant \varepsilon^2$.

If F and G are finite subsets of A with $F \supseteq F_0$ and $G \supseteq F_0$, then $F_0 \subseteq F \cap G$, so $(F\triangle G) \cap F_0 = \emptyset$, so

$$
\begin{aligned}
\|s_F - s_G\|^2 &= c_{F\triangle G} \\
&< \varepsilon^2.
\end{aligned}
$$

We define $s = \sum_{\alpha\in A} c_\alpha e_\alpha$. This limit exists since $\mathcal{H}$ is complete and Cauchy nets converge. The norm of s is computed as

$$
\|s\|^2 = \sum_{\alpha\in A} |c_\alpha|^2.
$$

(2) This follows directly from (1).

(3) Let $F \subseteq A$ be finite, and set $M_F = \operatorname{span}\{e_\alpha \mid \alpha \in F\}$. Since M_F is finite-dimensional, M_F is closed. For $x \in \mathcal{H}$ and $\beta \in A \setminus F$, the orthogonality of $(e_\alpha)_{\alpha\in A}$ provides

$$
\left\langle x - \sum_{\alpha\in F} \langle x, e_\alpha\rangle e_\alpha, e_\beta \right\rangle = 0.
$$

Thus, we write

$$
x = x - \sum_{\alpha\in F} \langle x, e_\alpha\rangle e_\alpha + \sum_{\alpha\in F} \langle x, e_\alpha\rangle e_\alpha,
$$

which gives $P_{M_F} = \sum_{\alpha\in F} \langle x, e_\alpha\rangle e_\alpha$. Using the Pythagorean theorem, we get

$$
\sum_{\alpha\in F} |\langle x, e_\alpha\rangle|^2 = \|P_{M_F}(x)\|^2
$$

$$\leqslant \|x\|^2.$$

The inequality follows by taking the supremum,

$$\begin{aligned}\sum_{\alpha\in A} |\langle x, e_\alpha\rangle|^2 &= \sup_{F\subseteq A}\left(\sum_{\alpha\in F} |\langle x, e_\alpha\rangle|^2\right)\\ &\leqslant \|x\|^2.\end{aligned}$$

(4) Fix $x \in \mathcal{H}$, and for each $\alpha \in A$, we set $c_\alpha = \langle x, e_\alpha\rangle$. We have $(c_\alpha)_{\alpha\in A} \in \ell_2(A)$, so $\sum_{\alpha\in A} c_\alpha e_\alpha$ is norm-summable in $\mathcal{H}$. Continuity of the inner product yields

$$\left\langle x - \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha, e_\beta\right\rangle = 0,$$

so $x - \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha \in M^\perp$, meaning $P_M(x) = \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha$.

□

Corollary: If $\mathcal{H}$ is a Hilbert space, and $\{x_\alpha\}_{\alpha\in A}$ is an orthogonal set such that $\sum_{\alpha\in A} \|x_\alpha\|^2$ is summable, then $\sum_{\alpha\in A} x_\alpha$ is summable and

$$\left\|\sum_{\alpha\in A} x_\alpha\right\|^2 = \sum_{\alpha\in A} \|x_\alpha\|^2.$$

Proof. Set $e_\alpha = \frac{1}{\|x_\alpha\|}x_\alpha$, and $c_\alpha = \|x_\alpha\|$ in the proof of the theorem above. □

Example. If $(e_n)_{n\geqslant 1}$ is the set of standard coordinate vectors in ℓ_2, then $\sum_{n\in\mathbb{N}} \frac{1}{n}e_n$ is summable, but the series does not converge absolutely.

Definition. Let $\mathcal{H}$ be a Hilbert space. An orthonormal basis in $\mathcal{H}$ is a maximal orthonormal set E. That is, if $E \subsetneq E'$, then E' is not an orthonormal basis.

Lemma: Let $\mathcal{H}$ be a Hilbert space. Every orthonormal set in $\mathcal{H}$ is contained in an orthonormal basis.

Proof. Let $F \subseteq \mathcal{H}$ be an orthonormal set. Let

$$\mathcal{E} = \{E \subseteq \mathcal{H} \mid F \subseteq E,\ E \text{ orthonormal}\},$$

and order $\mathcal{E}$ by inclusion. For any chain $\mathcal{C}$ in $\mathcal{E}$, then $U = \bigcup_{C\in\mathcal{C}} C$ is an upper bound for $\mathcal{C}$, as for any two vectors $e_\alpha, e_\beta \in U$, both e_α and e_β are contained in some $C \in \mathcal{C}$, so $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$. Applying Zorn's lemma, we get the desired result. □

Orthonormal bases, like Schauder bases, have dense linear span in a Hilbert space.

Theorem: Let $\mathcal{H}$ be a Hilbert space, and $E = (e_\alpha)_{\alpha\in A}$ be an orthonormal set. Let $M = \overline{\operatorname{span}}\{e_\alpha \mid \alpha \in A\}$. The following are equivalent:

(i) E is an orthonormal basis;

(ii) $M^\perp = \{0\}$;

(iii) $M = \mathcal{H}$;

(iv) for each $x \in \mathcal{H}$, we have $x = \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha$;

(v) for each $x \in \mathcal{H}$, we have $\|x\|^2 = \sum_{\alpha\in A} |\langle x, e_\alpha\rangle|^2$ (known as Parseval's identity);

(vi) for each $x, y \in \mathcal{H}$, we have $\langle x, y\rangle = \sum_{\alpha\in A} \langle x, e_\alpha\rangle \overline{\langle y, e_\alpha\rangle}$.

Proof. To see (i) implies (ii), we suppose there is $v \in M^{\perp}$ with $\|v\| = 1$. Then, $\{v\} \cup E$ is an orthonormal set containing E, which contradicts the maximality of E.

The equivalence of (ii) and (iii) follows from the fact that $\mathcal{H}^{\perp} = \{0\}$ and $\{0\}^{\perp} = \mathcal{H}$.

To see that (iii) implies (i), we suppose there is $v \in \mathcal{H}$ such that $v \notin E$ and $\{v\} \cup E$ is an orthonormal set. THen, for each $\alpha \in A$, we have $\langle v, e_\alpha \rangle = 0$, so $\langle v, x \rangle = 0$ for each $x \in \operatorname{span}\{e_\alpha \mid \alpha \in A\}$. Since the inner product is continuous, we have $\langle v, x \rangle = 0$ for each $x \in \overline{\operatorname{span}}\{e_\alpha \mid \alpha \in A\} = M = \mathcal{H}$, implying that $\|v\| = 0$.

To see that (iii) implies (iv), recall that we proved that $P_M(x) = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$, but since $M = \mathcal{H}$, we have $P_M(x) = x$.

We see that (v) follows from (iv) by the previous theorem.

To see that (v) implies (i), if $\langle v, e_\alpha \rangle = 0$ for each $\alpha \in A$, we must have $\|v\| = 0$, so E is a maximal orthonormal set.

To see that (vi) implies (v), we let $x = y$ in the hypothesis of (vi).

To see that (iv) implies (vi), we let $x, y \in \mathcal{H}$. We let $x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$, and $y = \sum_{\beta \in A} \left\langle y, e_\beta \right\rangle e_\beta$. By the continuity of the inner product and the orthonormality of E, we have

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha, \sum_{\beta \in A} \left\langle y, e_\beta \right\rangle e_\beta \right\rangle \\
&= \sum_{\alpha, \beta \in A} \langle x, e_\alpha \rangle \overline{\left\langle y, e_\beta \right\rangle} \left\langle e_\alpha, e_\beta \right\rangle \\
&= \sum_{\alpha \in A} \langle x, e_\alpha \rangle \overline{\left\langle y, e_\beta \right\rangle}.
\end{aligned}$$

□

For an orthonormal basis $\{e_\alpha\}_{\alpha \in A}$ and a given $x \in \mathcal{H}$, we often refer to the terms $\langle x, e_\alpha \rangle$ as the Fourier coefficients of x with respect to the basis $\{e_\alpha\}_{\alpha \in A}$.

Recall that any two vector spaces X and Y are isomorphic if and only if $\dim(X) = \dim(Y)$. A similar idea holds for Hilbert spaces.

Proposition: Let $\mathcal{H}$ be a Hilbert space. Any two orthonormal bases for $\mathcal{H}$ have the same cardinality.

Proof. Let $E = \{e_\alpha\}_{\alpha \in A}$ and $F = \left\{f_\beta\right\}_{\beta \in B}$ be two orthonormal bases for $\mathcal{H}$. If E is finite, then it must be a Hamel basis as orthogonal sets are independent and finite orthonormal bases are spanning by Parseval's identity. Thus, $\dim(\mathcal{H}) < \infty$, and since F is independent, F is finite, so it must be a Hamel basis, with $\operatorname{card}(E) = \operatorname{card}(F)$.

Suppose A and B are both infinite. For each $\beta \in B$, consider

$$A_\beta := \left\{\alpha \,\middle|\, \left\langle f_\beta, e_\alpha \right\rangle \neq 0\right\}.$$

Since

$$\begin{aligned}
\left\|f_\beta\right\|^2 &= \sum_{\alpha \in A} \left|\left\langle f_\beta, e_\alpha \right\rangle\right|^2 \\
&= 1
\end{aligned}$$

is summable, A_β must be countable. Additionally, $A \subseteq \bigcup_{\beta \in B} A_\beta$, since

$$\|e_\alpha\|^2 = \sum_{\beta \in B} \left|\left\langle e_\alpha, f_\beta \right\rangle\right|^2.$$

Since card $\left(A_\beta\right) \leqslant \aleph_0 \leqslant \text{card}(B)$, we get

$$\text{card}(A) \leqslant \text{card}\left(\bigcup_{\beta \in B} A_\beta\right) \leqslant \text{card}(B).$$

Similarly, card(B) ⩽ card(A), so card(A) = card(B) by Cantor–Schröder–Bernstein. □

Definition. Let $\mathcal{H}$ be a Hilbert space. The Hilbert dimension of $\mathcal{H}$, written hdim($\mathcal{H}$), is the cardinality of E for any orthonormal basis E of $\mathcal{H}$.

We can characterize all Hilbert spaces with countable Hilbert dimension.

Proposition: Let $\mathcal{H}$ be a Hamel basis with $\dim(H) = n < \infty$. Then, $\mathcal{H} \cong \ell_2^n$ are unitarily isomorphic.

Proof. Let $\{v_1, \dots, v_n\}$ be a Hamel basis for $\mathcal{H}$. Applying the Gram–Schmidt process, we obtain an orthonormal set $\{u_1, \dots, u_n\}$ with the same span as $\{v_1, \dots, v_n\}$. The map $T\colon \ell_2^n \to \mathcal{H}$ given by $T\left(e_j\right) = u_j$ is a surjective isometry. □

Proposition: Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. The following are equivalent.

(i) $\mathcal{H}$ is separable;

(ii) $\text{hdim}\left(\mathcal{H}\right) = \aleph_0$;

(iii) $\mathcal{H} \cong \ell_2$.

Proof. Let $\{x_k\}_{k=1}^\infty \subseteq \mathcal{H}$ be norm-dense, and let $(e_\alpha)_\alpha$ be an orthonormal basis. Note that for $\alpha \neq \beta$, we have

$$\begin{aligned} \left\|e_\alpha - e_\beta\right\|^2 &= \left\langle e_\alpha - e_\beta, e_\alpha - e_\beta \right\rangle \\ &= 2, \end{aligned}$$

so $\left\|e_\alpha - e_\beta\right\| = \sqrt{2}$. For each $\alpha \in A$, the density of $\{x_k\}_{k=1}^\infty$ allows us to find $J\left(\alpha\right) \in \mathbb{N}$ such that

$$\|e_\alpha - x_{J_\alpha}\| < \frac{1}{2}.$$

We have a map $J : A \to \mathbb{N}$. We claim that J is injective. If not, then there are $\alpha, \beta \in A$ with $\alpha \neq \beta$, $J\left(\alpha\right) = J\left(\beta\right)$. We then have

$$\begin{aligned} \sqrt{2} &= \left\|e_\alpha - e_\beta\right\| \\ &\leqslant \left\|e_\alpha - x_{J(\alpha)}\right\| + \left\|x_{J(\alpha)} - e_\beta\right\| \\ &= \left\|e_\alpha - x_{J(\alpha)}\right\| + \left\|x_{J(\beta)} - e_\beta\right\| \\ &< 1. \end{aligned}$$

Thus, J is injective, so A is countable.

If $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then we have $\mathcal{H} \cong \ell_2\left(\mathbb{N}\right) = \ell_2$.

If $\left(e_n\right)_{n \geqslant 1}$ is the canonical orthonormal basis for ℓ_2, then we know that span$\left(E\right)$ is dense in ℓ_2, so E is a countable total subset of ℓ_2, so $\mathcal{H}$ is separable. □

Tensor Products and Direct Sums of Hilbert Spaces

We have shown that closed subspaces and quotient spaces of Hilbert spaces are Hilbert spaces. Now, we turn our attention to external direct sums and tensor products.

Direct Sums

In linear algebra, we learn that, for a normal $n \times n$ matrix, we can decompose ℓ_2^n into orthogonal pieces that the matrix acts on by scalar multiplication. In order to understand the spectral theorem for normal operators on Hilbert spaces, we need to understand such a decomposition.

Proposition: Let $\{\mathcal{H}_i\}_{i\in I}$ be a family of Hilbert spaces. The set

$$\bigoplus_{i\in I}\mathcal{H}_i = \left\{ (x_i)_{i\in I} \,\middle|\, x_i \in \mathcal{H}_i \text{ and } \sum_{i\in I} \|x_i\|^2 \text{ is summable} \right\}$$

equipped with pointwise operations is a vector space, with inner product

$$\langle x, y\rangle := \sum_{i\in I} \langle x_i, y_i\rangle$$

for $(x_i)_{i\in I}, (y_i)_{i\in I} \in \bigoplus_{i\in I}\mathcal{H}_i$ that induces the complete norm

$$\|(x_i)_i\| := \left(\sum_{i\in I} \|x_i\|^2\right)^{1/2}.$$

The Hilbert space $\bigoplus_{i\in I}\mathcal{H}_i$ is known as the external direct sum of the family $\{\mathcal{H}_i\}_{i\in I}$.

Example. If I is a set, and for each $i \in I$, we have $\mathcal{H}_i = \mathbb{C}$, then $\bigoplus_{i\in I}\mathcal{H}_i = \ell_2(I)$.

Example. If we fix a Hilbert space $\mathcal{H}$, the external direct sum $\bigoplus_{n\geq 1}\mathcal{H}$ is often denoted by $\mathcal{H}^\infty$ or $\ell_2(\mathcal{H})$.

Example. Let $\{(\Omega_n, \mathcal{M}_n, \mu_n)\}_n$ be a countable family of measure spaces, and let $(\Sigma, \mathcal{M}, \mu)$ be the coproduct of these spaces, defined by

$$\begin{aligned} \Sigma &:= \coprod_{n=1}^{\infty} \Omega_n \\ \mathcal{M} &:= \left\{ E \subseteq \Sigma \,\middle|\, \iota_n^{-1}(E) \in \mathcal{M}_n \text{ for all } n \right\} \\ \mu(E) &= \sum_{n=1}^{\infty} \mu_n\left(\iota_n^{-1}(E)\right). \end{aligned}$$

Then, the map

$$V : L_2(\Sigma, \mu) \to \bigoplus_{n\geq 1} L_2(\Omega_n, \mu_n),$$

defined by

$$V(\xi) = (\xi \circ \iota_n)_n$$

is a well-defined unitary isomorphism.

Let $\xi \in L_2(\Sigma, \mu)$. Since each $\iota_n : \Omega_n \to \Sigma$ is measurable, $\xi \circ \iota_n : \Omega_n \to \mathbb{C}$ is also measurable. Additionally, if ξ is 0 μ-a.e., then so is $\xi \circ \iota_n$. Moreover, we have

$$\|V(\xi)\|^2 = \|(\xi \circ \iota_n)_n\|^2$$

$$
\begin{aligned}
&= \sum_{n\geqslant 1} \|\xi \circ \iota_n\|^2 \\
&= \sum_{n\geqslant 1} \int_{\Omega_n} |\xi \circ \iota_n(x)|^2 \, d\mu_n \\
&= \sum_{n\geqslant 1} \int_{\Omega_n} |\xi|^2 \circ \iota_n(x) \, d\mu_n \\
&= \int_{\Sigma} |\xi|^2 \, d\mu \\
&= \|\xi\|^2 .
\end{aligned}
$$

This shows V is a well-defined linear map. Our calculation shows that V is an isometry.

We only need to write an inverse, for which which we define

$$W : \bigoplus_{n\geqslant 2} L_2(\Omega_n, \mu_n) \to L_2(\Sigma, \mu),$$

defined by

$$W((\xi_n)_n) = \xi,$$

where

$$
\begin{aligned}
\xi &\coloneqq \coprod_{n\geqslant 1} (\xi_n : \Omega \to \mathbb{C}) \\
\xi(x, n) &= \xi_n(x).
\end{aligned}
$$

Lemma: Let $\{\mathcal{H}_i\}_{i\in I}$ be a family of Hilbert spaces. For each $k \in I$, the maps

$$
\begin{aligned}
\pi_k &: \bigoplus_{i\in I} \mathcal{H}_i \to \mathcal{H}_k \\
\iota_k &: \bigoplus_{i\in I} \mathcal{H}_i,
\end{aligned}
$$

given by

$$
\begin{aligned}
\pi_k((x_i)_i) &= x_k \\
\iota_k(x) &= (x_i)_i \text{ where } x_{i\neq k} = 0, x_k = x
\end{aligned}
$$

are bounded linear operators, with

$$\langle \pi_k((x_i)_i), y\rangle = \langle (x_i)_i, \iota_k(y)\rangle .$$

Breaking apart an operator into smaller parts is often useful, such as in the spectral theorem.

Proposition: Let $\mathcal{H}$ be a Hilbert space, and let $\{M_i\}_{i\in I}$ be a collection of pairwise orthogonal closed subspaces of $\mathcal{H}$. The following are equivalent

(i) The internal sum $\sum_{i\in I} M_i$ is dense in $\mathcal{H}$.

(ii) Given $x \in \mathcal{H}$, there are unique $x_i \in M_i$ such that

$$x = \sum_{i\in I} x_i$$

is a norm convergent sum. We have $x_i = P_{M_i}(x)$.

Proof. We see that (ii) implies (i) by the definition of the internal sum and summability.

To see that (i) implies (ii), let $x \in \mathcal{H}$, and we set $x_i = P_{M_i}(x)$. Since M_i are mutually orthogonal, it follows that $(x_i)_i$ are mutually orthogonal vectors.

Let $F \subseteq I$ be finite, and set $P = \sum_{i \in F} P_{M_i}$. We know that P is the orthogonal projection onto $\sum_{i \in F} M_i$. By the Pythagorean theorem, we have that

$$\begin{aligned} \sum_{i \in F} \|x_i\|^2 &= \left\| \sum_{i \in F} x_i \right\|^2 \\ &= \left\| \sum_{i \in F} P_{M_i}(x) \right\|^2 \\ &= \|P(x)\|^2 \\ &\leqslant \|x\|^2. \end{aligned}$$

Taking the supremum across finite subsets, we have $\sum_{i \in I} \|x_i\|^2 \leqslant \|x\|^2$. Thus, $\sum_{i \in I} x_i$ is summable in $\mathcal{H}$ to z, and that $\|z\|^2 = \sum_{i \in I} \|x_i\|^2$.

We claim that $x = z$. Let $y \in M_i$ be arbitrary, and note that $\langle z, y \rangle = \langle x_i, y \rangle$. Thus, we have

$$\begin{aligned} \langle x - z, y \rangle &= \langle x, y \rangle - \langle z, y \rangle \\ &= \langle x, P_{M_i}(y) \rangle - \langle x_i, y \rangle \\ &= \langle P_{M_i}(x), y \rangle - \langle x_i, y \rangle \\ &= \langle x_i, y \rangle - \langle x_i, y \rangle \\ &= 0. \end{aligned}$$

Since y is arbitrary, we have $x - z \in M_i^{\perp}$. Since i is arbitrary, we see that $x - z$ is orthogonal to $\sum_{i \in I} M_i$. Since the latter is dense in $\mathcal{H}$, continuity of the inner product shows that $x - z \in \mathcal{H}^{\perp} = \{0\}$.

For uniqueness, suppose $x = \sum_{i \in I} x_i'$ is an norm-convergent sum with $x_i' \in M_i$ for all $i \in I$. For $j \in I$, we have

$$\begin{aligned} x_j' &= P_j(x) \\ &= P_j\left(\sum_{i \in I} x_i' \right) \\ &= \sum_{i \in I} P_j(x_i') \\ &= x_j' \end{aligned}$$

□

Definition. Let $\mathcal{H}$ be a Hilbert space, and suppose $\mathcal{M} = \{M_i\}_{i \in I}$ is a collection of mutually orthogonal closed subspaces that satisfy the conditions above. Then, we say $\mathcal{H}$ is the internal direct sum of the family $\mathcal{M}$, and write

$$\mathcal{H} = \bigoplus_{i \in I} M_i.$$

Exercise: Let $\mathcal{H} = \bigoplus_{i \in I} M_i$ be an internal direct sum. Viewing each M_i as a Hilbert space, prove that

$$U \colon \bigoplus_{i \in I} M_i \to H$$

$$(x_i)_{i\in I} \overset{U}{\longmapsto} \sum_{i\in I} x_i$$

is unitary. In other words, the external direct sum of the spaces M_i is unitarily isomorphic to $\mathcal{H}$.

Solution: To see that U is an isometry, we can see that for $x = \sum_{i\in I} x_i$,

$$\begin{aligned}
\|U((x_i)_i)\|^2 &= \left\|\sum_{i\in I} x_i\right\|^2 \\
&= \sum_{i\in I} \|x_i\|^2 \\
&= \|(x_i)_i\|^2 .
\end{aligned}$$

Additionally, for any $x \in \mathcal{H}$, we can find an indexed family $(x_i)_i$ such that $\sum_{i\in I} x_i = x$. Thus, we can select the indexed family $(x_i)_i$ in $\bigoplus_{i\in I} M_i$ such that $U((x_i)_i) = x$.

Thus, U is a surjective isometry, so U is unitary.

Exercise: Let $\mathcal{H} = \bigoplus_{i\in I} M_i$ be an internal direct sum, and let $j \in I$ be fixed, $x \in M_j^{\perp}$. Prove that x has the form

$$x = \sum_{\substack{i\in I\\ i\neq j}} x_i,$$

a norm-convergent sum, where $x_i \in M_i$.

Solution: Note that we have proved $x_i = P_{M_i}(x)$. Thus, $x_j = P_{M_j}(x) = 0$, so we have

$$\begin{aligned}
x &= \sum_{i\in I} P_{M_i}(x) \\
&= \sum_{\substack{i\in I\\ i\neq j}} P_{M_i}(x) \\
&= \sum_{\substack{i\in I\\ i\neq j}} x_i.
\end{aligned}$$

Corollary: Suppose $\mathcal{H} = \bigoplus_{i\in I} M_i$ is an internal direct sum, and let $P_i = P_{M_i}$ be the orthogonal projections onto each M_i. Then,

$$I_{\mathcal{H}} = \sum_{i\in I} P_i$$

in the strong operator topology. That is,

$$\begin{aligned}
x &= \sum_{i\in I} P_i(x) \\
\|x\|^2 &= \sum_{i\in I} \|P_i(x)\|^2 .
\end{aligned}$$

Exercise: Let $(\Omega, \mathcal{L}, \mu)$ be a measure space, and let $\bigsqcup_{n\geqslant 1} E_n = \Omega$ be a measurable partition. Show that there is an internal sum

$$\begin{aligned}
L_2(\Omega, \mu) &= \bigoplus_{n\geqslant 1} M_{E_n} \\
&\cong \bigoplus_{n\geqslant 1} L_2\left(E_n, \mu_{E_n}\right).
\end{aligned}$$

Bounded Operators on Hilbert Spaces

Hilbert spaces are, according to John Conway, anyway, "boring," so we are interested in understanding the effects of operators on Hilbert spaces.

In the case of quantum mechanics, a particle with wave function ξ moving along the x axis has position equivalent to its expected value,

$$\int_{\mathbb{R}} x\,|\xi(x)|^2 \; d\lambda = \langle \mathrm{id}_{\mathbb{R}}\, \xi, \xi\rangle\,,$$

where the x coordinate is now an observable of an operator $\xi \overset{Q}{\mapsto} \mathrm{id}_{\mathbb{R}}\, \xi$, which is known as position. This operator is only defined on its domain, as it is not bounded.

Similarly, linear momentum is the map $\xi \overset{P}{\mapsto} \xi'$ (on the domain that it is defined), yielding

$$\langle P(\xi), \xi\rangle = \int_{\mathbb{R}} \frac{d\xi}{dx}\overline{\xi(x)}\; d\lambda,$$

from which we get the uncertainty principle

$$PQ(\xi) = I(\xi) + QP(\xi)\,.$$

Structure of $\mathcal{B}(\mathcal{H})$

If X is a Banach space, then $\mathcal{B}(X)$, the space of bounded linear operators on X, is a unital Banach algebra. We will study the structure of $\mathcal{B}(\mathcal{H})$, which is the space of bounded linear operators on a Hilbert space.

Algebraic-Analytic Structure

Fact. Let $T, S\colon \mathcal{H} \to \mathcal{K}$ be linear maps between Hilbert spaces.

(1) We have $T = S$ if and only if $\langle T(x), y\rangle = \langle S(x), y\rangle$ for all $x \in \mathcal{H}, y \in \mathcal{K}$.

(2) If $\mathcal{H} = \mathcal{K}$, then $T = S$ if and only if $\langle T(x), x\rangle = \langle S(x), x\rangle$ for all $x \in \mathcal{H}$.

Proof.

(1) This follows from the fact that $\langle x, z_1\rangle = \langle x, z_2\rangle$ for all x if and only if $z_1 = z_2$.

(2) We define the sesquilinear forms $F\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$, $G\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ by

$$\begin{aligned} F(x,y) &= \langle T(x), y\rangle \\ G(x,y) &= \langle S(x), y\rangle\,. \end{aligned}$$

We see that $T = S$ if and only if $F = G$, if and only if F and G agree on the diagonal, meaning $\langle T(x), x\rangle = \langle S(x), x\rangle$ for all $x \in \mathcal{H}$.

□

Fact. If $T\colon \mathcal{H} \to \mathcal{K}$ is a linear map, then

$$\|T\|_{op} = \sup\left\{ |\langle T(x), y\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\}.$$

Proof. By the Riesz Representation theorem, we have that $B_{K^*} = \{\langle\cdot, y\rangle \mid y \in B_{\mathcal{K}}\}$, meaning we have

$$\|T(x)\| = \sup_{y\in B_{\mathcal{K}}} |\langle T(x), y\rangle|\,.$$

Taking the supremum over $x \in B_{\mathcal{Y}}$ yields

$$\begin{aligned}\|T\|_{\text{op}} &= \sup_{x\in B_{\mathcal{H}}} \|T(x)\| \\ &= \sup\left\{|\langle T(x), y\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\}.\end{aligned}$$

□

Definition. Let $F : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ be a sesquilinear form. We define the norm

$$\|F\| := \sup\left\{|F(x, y)| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\}.$$

We say F is bounded if $\|F\| < \infty$.

Proposition: If $F : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a bounded sesquilinear form, then there exists a unique $S \in B(\mathcal{K}, \mathcal{H})$ such that

$$F(x, y) = \langle x, S(y)\rangle.$$

Proof. Fix $y \in \mathcal{K}$, and consider the linear functional $\varphi : \mathcal{H} \to \mathbb{C}$ given by $\varphi(x) = F(x, y)$. Since φ is linear, we have

$$\begin{aligned}|\varphi(x)| &= |F(x, y)| \\ &\leqslant \|F\| \|y\|\end{aligned}$$

for all $x \in B_{\mathcal{H}}$, meaning $\varphi \in \mathcal{H}^*$. Thus, there is a unique $z \in \mathcal{H}$ such that $\varphi = \varphi_z$. We define $S(y) := z$. Doing this for each $y \in \mathcal{K}$, we get a map $S : \mathcal{K} \to \mathcal{H}$ such that

$$F(x, y) = \langle x, S(y)\rangle.$$

We show that S is linear and bounded. Let $y_1, y_2 \in \mathcal{K}$ and $\alpha \in \mathcal{C}$. For all $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle x, S(y_1 + \alpha y_2)\rangle &= F(x, y_1 + \alpha y_2) \\ &= F(x, y_1) + \overline{\alpha} F(x, y_2) \\ &= \langle x, S(y_1)\rangle + \overline{\alpha}\langle x, S(y_2)\rangle \\ &= \langle x, S(y_1) + \alpha S(y_2)\rangle.\end{aligned}$$

Thus, S is linear. We also have

$$\begin{aligned}\|S\|_{\text{op}} &= \sup\left\{|\langle x, S(y)\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\} \\ &= \sup\left\{|F(x, y)| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\} \\ &= \|F\|.\end{aligned}$$

For uniqueness, we see that if $\langle x, S_1(y)\rangle = F(x, y) = \langle x, S_2(y)\rangle$, then $S_1 = S_2$ necessarily. □

Theorem: Let $\mathcal{H}, \mathcal{K}, \mathcal{L}$ be Hilbert spaces. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then there is a unique bounded operator $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We call T^* the Hilbert space adjoint of T. Moreover, the following are true for $T, S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $R \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, and $\lambda \in \mathbb{C}$:

(1) $(T + \lambda S)^* = T^* + \overline{\lambda} S^*$;

(2) $T^{**} = T$;

(3) $(R \circ T)^* = T^* \circ R^*$;

(4) if T is invertible, then $\left(T^{-1}\right)^* = (T^*)^{-1}$;

(5) $\|T^*\| = \|T\|$;

(6) $\|T^*T\| = \|T\|^2$ (known as the C^*-property).

Proof. We define $F : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ by $F(x, y) = \langle T(x), y\rangle$. We have F is a sesquilinear form, and

$$\begin{aligned} \|F\| &= \sup\left\{|F(x,y)| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\} \\ &= \sup\left\{|\langle T(x), y\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\} \\ &= \|T\|_{op}. \end{aligned}$$

Thus, there is a unique operator $S_T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\langle T(x), y\rangle = \langle x, S_T(y)\rangle$, with $\|S_T\| = \|T\|$. We define $T^* = S_T$.

We will show (6).

$$\begin{aligned} \|T^*T\| &= \sup_{\substack{x\in B_{\mathcal{H}} \\ y \in B_{\mathcal{K}}}} |\langle T^*T(x), y\rangle| \\ &\geqslant \sup_{x \in B_{\mathcal{H}}} |\langle T^*T(x), x\rangle| \\ &= \sup_{x \in B_{\mathcal{H}}} |\langle T(x), T(x)\rangle| \\ &= \sup_{x \in B_{\mathcal{H}}} \|T(x)\|^2 \\ &= \left(\sup_{x \in B_{\mathcal{H}}} \|T(x)\|\right)^2 \\ &= \|T\|^2 \\ &= \|T\|\,\|T\| \\ &= \|T^*\|\,\|T\| \\ &\geqslant \|T^*T\|. \end{aligned}$$

□

Exercise: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Write $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ to be the (Hilbert space) adjoint, and $T^\dagger : K^* \to H^*$ to be the Banach space adjoint. Let $\rho_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^*$ be the conjugate linear isometry $x \mapsto \varphi_x$, and let $\rho_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^*$ to be the conjugate linear isometry $y \mapsto \varphi_y$. Show that the following diagram commutes.

$$\begin{tikzcd} \mathcal{K}^* \arrow[r, "T^\dagger"] & \mathcal{H}^* \\ \mathcal{K} \arrow[u, "\rho_{\mathcal{K}}"] \arrow[r, "T^*"'] & \mathcal{H} \arrow[u, "\rho_{\mathcal{H}}"'] \end{tikzcd}$$

Proof. Let $x \in \mathcal{H}$, $y \in \mathcal{K}$. By the Riesz representation theorem, we have $\varphi_x = \langle \cdot, x\rangle$ and $\varphi_y = \langle \cdot, y\rangle$. Thus, we have

$$\begin{aligned} T^\dagger\left(\varphi_y\right)((x)) &= \varphi_y\left(T(x)\right) \\ &= \langle T(x), y\rangle \\ &= \langle x, T^*(y)\rangle \\ &= \varphi_x\left(T^*(y)\right). \end{aligned}$$

□

Corollary: The adjoint map $* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $T \mapsto T^*$ is an involution, meaning $\mathcal{B}(\mathcal{H})$ is a unital $*$-algebra. If $\dim(\mathcal{H}) > 1$, then $\mathcal{B}(\mathcal{H})$ is noncommutative.

Definition. A Banach $*$-algebra is a Banach algebra A with an involution satisfying

$$\|a^*\| = \|a\|$$

for all $a \in A$. If A is a Banach $*$-algebra that satisfies the C^*-property, then A is called a C^*-algebra.

We can now look at some examples of operators and adjoints.

Example. Let $a = (a_{ij})_{i,j} \in \text{Mat}_{m,n}(\mathbb{C})$, with the linear operator

$$T_a \colon \ell_2^n \to \ell_2^m$$

defined by $T_a(\xi) = a\xi$. Since ℓ_2^n is finite-dimensional, T_a is bounded. The conjugate transpose $a^* = (\overline{a_{ji}})_{i,j}$ is an $n \times m$ matrix satisfying

$$\begin{aligned}\langle T_a(\xi), \eta\rangle &= \langle a\eta, \xi\rangle \\ &= (a\xi)^* \eta \\ &= \xi^* a^* \eta \\ &= \langle \xi, a^*\eta\rangle \\ &= \langle \xi, T_{a^*}(\eta)\rangle,\end{aligned}$$

meaning $T_a^* = T_{a^*}$.

Example. The canonical projection

$$\pi_k \colon \bigoplus_{i \in I} \mathcal{H}_i \to \mathcal{H}_k$$

and canonical injection

$$\iota_k \colon \mathcal{H}_k \to \bigoplus_{i \in I} \mathcal{H}_i,$$

defined by

$$\begin{aligned}\pi_k((x_i)_i) &= x_k \\ \iota_k(x_k) &= \begin{cases} x_k & i = k \\ 0 & i \neq k \end{cases}\end{aligned}$$

are adjoints.

Example. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. For every pair of nonzero vectors $x \in \mathcal{H}$ and $y \in \mathcal{K}$, we define the rank-one bounded operator $\theta_{x,y} : \mathcal{K} \to \mathcal{H}$ by $\theta_{x,y}(z) = \langle z, y\rangle x$.

In physics, $\theta_{x,y} = |x\rangle \langle y|$.

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left\|\theta_{x,y}(z)\right\| &= \|\langle z, y\rangle x\| \\ &= |\langle z, y\rangle| \|x\| \\ &\leqslant \|z\| \|y\| \|x\|,\end{aligned}$$

meaning $\left\|\theta_{x,y}\right\|_{op} \leqslant \|x\| \|y\|$. Alternatively, we also know that

$$\theta_{x,y}\left(\frac{y}{\|y\|}\right) = \|x\| \|y\|,$$

meaning that $\left\|\theta_{x,y}\right\|_{op} = \|x\| \|y\|$.

The adjoint of $\theta_{x,y}$ is $\theta_{y,x}$. We can see this by taking, for $z \in \mathcal{K}$ and $u \in \mathcal{H}$,

$$\begin{aligned}\left\langle \theta_{x,y}(z), u\right\rangle &= \langle \langle z, y\rangle x, u\rangle \\ &= \langle z, y\rangle \langle x, u\rangle \\ &= \langle z, y\rangle \overline{\langle u, x\rangle} \\ &= \langle z, \langle u, x\rangle y\rangle \\ &= \left\langle z, \theta_{y,x}(u)\right\rangle.\end{aligned}$$

Topologies on $\mathcal{B}(\mathcal{H})$

Given a Banach space X, we can introduce two locally convex topologies on $\mathcal{B}(X)$ — namely, the weak operator topology and the strong operator topology, both of which are weaker than the norm topology.

Lemma: Let $\mathcal{H}$ be a Hilbert space, and let $(T_\alpha)_\alpha$ be a net in $\mathcal{B}(\mathcal{H})$. The following are equivalent:

(i) $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$;

(ii) for all $\xi, \eta \in \mathcal{H}$, $\langle T_\alpha(\xi), \eta\rangle \to \langle T(\xi), \eta\rangle$;

(iii) for all $\xi, \eta \in B_{\mathcal{H}}$, $\langle T_\alpha(\xi), \eta\rangle \to \langle T(\xi), \eta\rangle$;

(iv) for all $\xi \in \mathcal{H}$, $\langle T_\alpha(\xi), \xi\rangle \to \langle T(\xi), \xi\rangle$;

(v) for all $\xi \in B_{\mathcal{H}}$, $\langle T_\alpha(\xi), \xi\rangle \to \langle T(\xi), \xi\rangle$.

Proof. We only need to prove the equivalence between (i) and (ii). The rest follow from scaling or the polarization identity.

We know that $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$ in $\mathcal{B}(\mathcal{H})$ if and only if $\varphi(T_\alpha(\xi)) \to \varphi(T(\xi))$ for each $\xi \in \mathcal{H}$ and $\varphi \in \mathcal{H}^*$. By the Riesz representation theorem, each $\varphi \in \mathcal{H}^*$ is of the form $\varphi(\cdot) = \langle \cdot, \eta\rangle$. □

Example. Let $(\Omega, \mathcal{M}, \mu)$ be a semi-finite[1] measure space. We consider the multiplication operators $M_f \in \mathcal{B}(L_2(\Omega, \mu))$, where $f \in L_\infty(\Omega, \mu)$. If $f \in L_\infty(\Omega, \mu)$ and $(f_\alpha)_\alpha$ is a net in $L_\infty(\Omega, \mu)$, we claim that $(f_\alpha)_\alpha \xrightarrow{w^*} f$ if and only if $M_{f_\alpha} \xrightarrow{\text{WOT}} M_f$. Here, the w^* topology on $L_\infty(\Omega, \mu)$ is given by the duality $(L_1(\Omega, \mu))^* \cong L_\infty(\Omega, \mu)$.

If $(f_\alpha)_\alpha \xrightarrow{w^*} f$, then $\int_\Omega f_\alpha g \, d\mu \to \int_\Omega fg \, d\mu$ for every $g \in L_1(\Omega, \mu)$. For $\xi \in L_2(\Omega, \mu)$, we have $|\xi|^2 \in L_1(\Omega, \mu)$, so

$$\begin{aligned}\langle M_{f_\alpha}(\xi), \xi\rangle &= \int_\Omega f_\alpha \overline{\xi}\xi \, d\mu \\ &= \int_\Omega f_\alpha |\xi|^2 \, d\mu \\ &\to \int_\Omega f |\xi|^2 \, d\mu\end{aligned}$$

[1] A measure space where, for any $E \in \mathcal{M}$ with $\mu(E) = \infty$, there is $F \subseteq E$ and $0 < \mu(F) < \infty$.

$$= \langle M_f(\xi), \xi \rangle.$$

Thus, $M_{f_\alpha} \xrightarrow{\text{WOT}} M_f$.

Suppose $M_{f_\alpha} \xrightarrow{\text{WOT}} M_f$. Given $g \in L_1(\Omega, \mu)$, we can write $g = \xi\overline{\eta}$ for $\xi, \eta \in L_2(\Omega, \mu)$. Thus, we have

$$\begin{aligned}
\int_\Omega f_\alpha g \, d\mu &= \int_\Omega f_\alpha \xi\overline{\eta} \, d\mu \\
&= \langle M_{f_\alpha}\xi, \eta \rangle \\
&\to \langle M_f \xi, \eta \rangle \\
&= \int_\Omega f\xi\overline{\eta} \, d\mu \\
&= \int_\Omega fg \, d\mu,
\end{aligned}$$

so $(f_\alpha)_\alpha \xrightarrow{w^*} f$ in $L_\infty(\Omega, \mu)$.

We can define

$$\mathcal{L}_\infty(\Omega, \mu) = \{M_f \mid f \in L_\infty(\Omega, \mu)\},$$

and we see that $M : L_\infty(\Omega, \mu) \to \mathcal{L}_\infty(\Omega, \mu)$ is an isometric $*$-isomorphism and a w^*-WOT-homeomorphism.

Proposition: Let $\mathcal{H}$ be a Hilbert space.

(1) Left and right multiplication defined by a fixed operator is SOT-continuous. If $S \in \mathcal{B}(\mathcal{H})$, the maps $L_S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and $R_S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\begin{aligned}
L_S(T) &= ST \\
R_S(T) &= TS
\end{aligned}$$

are SOT-SOT-continuous.

(2) The maps L_S and R_S are WOT-WOT-continuous.

(3) The adjoint map $*: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, defined by $T \mapsto T^*$, is WOT-WOT-continuous. If $\dim(\mathcal{H}) = \infty$, the adjoint map is not SOT-SOT-continuous.

(4) If $\dim(\mathcal{H}) = \infty$, joint multiplication, $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, defined by $(T, S) \mapsto TS$, is neither WOT-WOT-continuous nor SOT-SOT-continuous.

Proof.

(1) Let $(T_\alpha)_\alpha \xrightarrow{\text{SOT}} T$ in $\mathcal{B}(\mathcal{H})$. For $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
\|ST_\alpha(\xi) - ST(\xi)\| &= \|S(T_\alpha(\xi) - T(\xi))\| \\
&\leqslant \|S\|_{op} \|T_\alpha(\xi) - T(\xi)\| \\
&\to 0,
\end{aligned}$$

meaning $(ST_\alpha)_\alpha \xrightarrow{\text{SOT}} ST$, meaning L_S is SOT-SOT-continuous.

Similarly, we have

$$\begin{aligned}
\|T_\alpha S(\xi) - TS(\xi)\| &= \|T_\alpha(\xi) - T(S(\xi))\| \\
&\to 0,
\end{aligned}$$

so $(T_\alpha S)_\alpha \xrightarrow{\text{SOT}} TS$, meaning R_S is SOT-SOT-continuous.

(2) Let $(T_{\alpha})_{\alpha} \xrightarrow{\text{WOT}} T$. For $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}\langle ST_{\alpha}(\xi), \eta\rangle &= \langle T_{\alpha}(\xi), S^{*}(\eta)\rangle\\ &\to \langle T(\xi), S^{*}(\eta)\rangle\\ &= \langle ST(\xi), \eta\rangle,\end{aligned}$$

meaning $(ST_{\alpha})_{\alpha} \xrightarrow{\text{WOT}} ST$, so L_S is WOT-WOT-continuous. Additionally, by definition, we must have $\langle T_{\alpha}(S(\xi)), \eta\rangle \to \langle T(S(\xi)), \eta\rangle$, so $(T_{\alpha}S)_{\alpha} \xrightarrow{\text{WOT}} TS$, so R_S is also WOT-WOT-continuous.

(3) If $(T_{\alpha})_{\alpha} \xrightarrow{\text{WOT}} T$, then we have

$$\begin{aligned}\langle T_{\alpha}^{*}(\xi), \eta\rangle &= \langle \xi, T_{\alpha}(\eta)\rangle\\ &\to \langle \xi, T(\eta)\rangle\\ &= \langle T^{*}(\xi), \eta\rangle,\end{aligned}$$

so $(T_{\alpha}^{*})_{\alpha} \xrightarrow{\text{WOT}} T^{*}$.

To see that the adjoint map is not SOT-SOT-continuous, we consider ℓ_2 with the orthonormal basis of canonical coordinate vectors, $(e_n)_{n\geqslant 1}$. Then, the outer product operators

$$\begin{aligned}T_n &= \theta_{e_1, e_n}\\ T_n(\xi) &= \langle \xi, e_n\rangle e_1\end{aligned}$$

give

$$\begin{aligned}\|T_n(\xi)\| &= \|\langle \xi, e_n\rangle e_1\|\\ &= |\langle \xi, e_n\rangle|\\ &\to 0,\end{aligned}$$

since, by Bessel's inequality, the sum of the squares of $|\langle \xi, e_n\rangle|$ converges. Thus, $(T_n)_n \xrightarrow{\text{SOT}} 0$. However, since $T_n^{*} = \theta_{e_n, e_1}$, we have

$$\begin{aligned}\|T_n^{*}(e_1)\| &= \|\langle e_1, e_1\rangle e_n\|\\ &= |\langle e_1, e_1\rangle|\\ &= 1.\end{aligned}$$

Thus, $(T_n^{*})_n \nrightarrow 0$ in SOT.

(4) Consider $(T_n)_n$ as above. Since $(T_n)_n \xrightarrow{\text{SOT}} 0$, we know that $(T_n)_n \xrightarrow{\text{WOT}} T$. However, though $(T_n^{*})_n \nrightarrow 0$ in SOT, we do have $(T_n^{*})_n \xrightarrow{\text{WOT}} 0$. We have

$$\begin{aligned}T_n T_n^{*} &= \theta_{e_1, e_n} \circ \theta_{e_n, e_1}\\ &= \theta_{e_1, e_1},\end{aligned}$$

which does not converge in WOT to 0, since $\langle \theta_{e_1, e_1}(e_1), e_1\rangle = 1$.

□

Exercise: Let X be a Banach space, and suppose $(T_n)_n \xrightarrow{\text{SOT}} T$ and $(S_n)_n \xrightarrow{\text{SOT}} S$ in $\mathcal{B}(\mathcal{X})$. Show that $(T_n S_n)_n \xrightarrow{\text{SOT}} TS$.

Solution: Let $x \in X$. We have that $\|T_n(x) - T(x)\| \xrightarrow{n\to\infty} 0$, so we see that the $(T_n)_n$ are a pointwise bounded family. Similarly, we see that $\|S_n(x) - S(x)\| \xrightarrow{n\to\infty} 0$, so the $(S_n)_n$ are a pointwise bounded family.

Thus, we see that

$$\begin{aligned}\|T_nS_n(x) - TS(x)\| &\leqslant \|T_nS_n(x) - T_nS(x)\| + \|T_nS(x) - TS(x)\| \\ &\leqslant \sup_{n\in\mathbb{N}} \|T_n\|_{op} \|S_n(x) - S(x)\| + \|T_n(S(x)) - T(S(x))\| \\ &\to 0.\end{aligned}$$

Theorem: Let B be the closed unit ball of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Then, B is compact in WOT.

Proof. Let K be defined by

$$K = \prod_{\xi,\eta\in B_{\mathcal{H}}} \overline{\mathbb{D}},$$

where $\overline{\mathbb{D}}$ is the closed unit disk in the complex plane. Then, K is compact in the product topology.

Consider the map $\phi\colon B \to K$ defined by

$$\phi(T) = (\langle T(\xi), \eta\rangle)_{\xi,\eta\in B_{\mathcal{H}}}.$$

Since, by Cauchy–Schwarz, we have that

$$\begin{aligned}|\langle T(\xi), \eta\rangle| &\leqslant \|T\|_{op} \|\xi\| \|\eta\| \\ &\leqslant 1,\end{aligned}$$

for all $\xi, \eta \in B_{\mathcal{H}}$, it is the case that ϕ is well-defined. Additionally, ϕ is injective since $T = S$ if and only if $\langle T(\xi), \eta\rangle = \langle S(\xi), \eta\rangle$ for all $\xi, \eta \in B_{\mathcal{H}}$, and ϕ is an embedding by the definition of the weak operator topology.

We wish to show that the range of ϕ is closed. Let

$$(\langle T_\alpha(\xi), \eta\rangle)_{\xi,\eta} \to (z_{\xi,\eta})_{\xi,\eta} \in K$$

be a net in K. Scaling, we see that $(\langle T_\alpha(\xi), \eta\rangle)_{\xi,\eta}$ converges in $\mathbb{C}$ for all $\xi, \eta \in \mathcal{H}$, so we have a bounded sesquilinear form defined by

$$F(\xi, \eta) = \lim_{\alpha\in A} \langle T_\alpha(\xi), \eta\rangle,$$

with $\|F\| \leqslant 1$. Thus, there is $T \in \mathcal{B}(\mathcal{H})$ with $\|T\|_{op} = \|F\| \leqslant 1$ and $\langle T(\xi), \eta\rangle = F(\xi, \eta)$, meaning $\phi(T) = (z_{\xi,\eta})_{\xi,\eta}$. □

Proposition: Let $\mathcal{H}$ be a Hilbert space, and let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional. The following are equivalent.

(i) There exist $\xi_1, \dots, \xi_n$ and $\eta_1, \dots, \eta_n$ in $\mathcal{H}$ such that

$$\varphi(T) = \sum_{j=1}^{n} \left\langle T\left(\xi_j\right), \eta_j\right\rangle.$$

(ii) φ is WOT-continuous.

(iii) φ is SOT-continuous.

Proof. The implication (i) to (ii) follows from the definition of φ as a finite sum of inner products. The implication (ii) to (iii) follows from the fact that left and right multiplication is WOT-continuous.

We only need to show that (iii) implies (i). Let φ be SOT continuous. There exists $K > 0$ and $\xi_1, \dots, \xi_n \in \mathcal{H}$ with

$$\begin{aligned} |\varphi(T)| &\leqslant K \max_{j=1}^{n} \left\| T\left(\xi_j\right) \right\| \\ &\leqslant K \left(\sum_{j=1}^{n} \left\| T\left(\xi_j\right) \right\| \right)^{1/2} \end{aligned}$$

Consider the subspace $\mathcal{K}_0 \subseteq \bigoplus_{j=1}^{n} \mathcal{H}$ given by

$$\mathcal{K}_0 = \left\{ \left(T\left(\xi_1\right), \dots, T\left(\xi_n\right)\right) \mid T \in \mathcal{B}\left(\mathcal{H}\right) \right\}.$$

Let $\mathcal{K} = \overline{\mathcal{K}_0}$. By the above inequality, the linear functional

$$\psi_0\left(\left(T\left(\xi_1\right), \dots, T\left(\xi_n\right)\right)\right) := \varphi(T)$$

is continuous, meaning it extends continuously to a bounded linear functional on $\mathcal{K}$. There exists a vector $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{K}$ such that $\psi\left(\xi\right) = \langle \xi, \eta \rangle$ for all $\xi \in \mathcal{K}$. Thus, we get

$$\begin{aligned} \varphi\left(T\right) &= \psi\left(\left(T\left(\xi_1\right), \dots, T\left(\xi_n\right)\right)\right) \\ &= \left\langle \left(T\left(\xi_1\right), \dots, T\left(\xi_n\right)\right), \left(\eta_1, \dots, \eta_n\right) \right\rangle \\ &= \sum_{j=1}^{n} \left\langle T\left(\xi_j\right), \eta_j \right\rangle. \end{aligned}$$

□

Remark: The above result implies that $\left(\mathcal{B}\left(\mathcal{H}\right), \text{WOT}\right)^* = \left(\mathcal{B}\left(\mathcal{H}\right), \text{SOT}\right)^*$.

Corollary: Let $\mathcal{H}$ be a Hilbert space, and suppose $C \subseteq \mathcal{B}\left(\mathcal{H}\right)$ is a convex subset. Then, $\overline{C}^{\text{WOT}} = \overline{C}^{\text{SOT}}$.

Proof. Since WOT is weaker than SOT, and we have shown $\left(\mathcal{B}\left(\mathcal{H}\right), \text{WOT}\right)^* = \left(\mathcal{B}\left(\mathcal{H}\right), \text{SOT}\right)^*$, the result follows from the analogous result on locally convex topological vector spaces. □

Particular WOT-closed self-adjoint subalgebras of $\mathcal{B}\left(\mathcal{H}\right)$ are one of the central subjects of the study of operator algebras.

Definition. Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra acting on $\mathcal{H}$ is a WOT-closed unital $*$-subalgebra $N \subseteq \mathcal{B}\left(\mathcal{H}\right)$.

Exercise: Let $\{N_i\}_{i \in I}$ be a family of von Neumann algebras acting on $\mathcal{H}$. Prove that $\bigcap_{i \in I} N_i$ is a von Neumann algebra.

Solution: Since $\{N_i\}_{i \in I}$ is a family of WOT-closed unital $*$-subalgebras, each N_i is WOT-closed, so the intersection $N := \bigcap_{i \in I} N_i$ is WOT-closed. Additionally, since the identity map is in each N_i, it is also in N, and for any $A \in N$, we must have $A \in N_i$ for all $i \in I$, so $A^* \in N_i$ for all N_i, so $A^* \in N$. Finally, since the intersection of subrings is a subring, it is the case that N is a WOT-closed unital $*$-subalgebra.

Example. Let $\mathcal{S} \subseteq \mathcal{B}\left(\mathcal{H}\right)$ be a family of bounded operators. We define the von Neumann algebra generated by $\mathcal{S}$ to be the smallest von Neumann algebra containing $\mathcal{S}$.

$$W^*\left(\mathcal{S}\right) = \bigcap \left\{ N \subseteq \mathcal{B}\left(\mathcal{H}\right) \;\middle|\; N \text{ is a von Neumann algebra and } N \supseteq \mathcal{S} \right\}.$$

Definition. Let $\mathcal{H}$ be a Hilbert space, and suppose $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is a nonempty collection of bounded operators. The commutant of $\mathcal{S}$ is

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) \mid TS = ST \text{ for all } S \in \mathcal{S}\}.$$

The double commutant of $\mathcal{S}$ is $\mathcal{S}'' = (\mathcal{S}')'$.

Exercise: Let $\mathcal{H}$ be a Hilbert space, and suppose $\emptyset \neq \mathcal{S}, \mathcal{T} \subseteq \mathcal{B}(\mathcal{H})$ are subsets.

(1) The commutant $\mathcal{S}'$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$.
(2) If $\mathcal{S}$ is self-adjoint, then $\mathcal{S}'$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.
(3) If $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{T}' \subseteq \mathcal{S}'$.
(4) $\mathcal{S} \subseteq \mathcal{S}''$.
(5) $\mathcal{S}''' = \mathcal{S}'$.

Solution:

(1) If $Q, S \in \mathcal{S}$ and $\alpha \in \mathbb{C}$, we have, for $T \in \mathcal{S}'$,

$$\begin{aligned}(Q + \alpha S)T &= QT + \alpha ST\\ &= TQ + \alpha TS\\ &= T(Q + \alpha S),\end{aligned}$$

and

$$\begin{aligned}(SQ)T &= S(QT)\\ &= S(TQ)\\ &= (ST)Q\\ &= (TS)Q\\ &= T(SQ).\end{aligned}$$

Finally, since $TI = IT$ for $I = \text{id}$, $\mathcal{S}$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$.

(2) If $\mathcal{S}$ is $*$-closed, we have, for $T \in \mathcal{S}$ and $S \in \mathcal{S}'$, we have

$$\begin{aligned}S^*T^* &= (TS)^*\\ &= (ST)^*\\ &= T^*S^*,\end{aligned}$$

so $S^* \in \mathcal{S}$.

(3) Let $\mathcal{S} \subseteq \mathcal{T}$. Then, for $Q \in \mathcal{T}'$, we must have, for all $T \in \mathcal{S}$, $TQ = QT$, so $Q \in \mathcal{S}'$.
(4) Let $T \in \mathcal{S}$. Then, for any $Q \in \mathcal{S}'$, we have $TQ = QT$, so $T \in (\mathcal{S}')' = \mathcal{S}''$.
(5) Consult Rainone

Lemma: Let $\mathcal{H}$ be a Hilbert space, and suppose $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is a subset. Then, $\mathcal{S}'$ is WOT-closed, so it is SOT-closed.

Proof. Let $(T_\alpha)_\alpha$ be a net in $\mathcal{S}'$ converging in WOT to an operator $T \in \mathcal{B}(\mathcal{H})$. For $S \in \mathcal{S}$, we have

$$\begin{aligned}TS &= \lim_\alpha T_\alpha S\\ &= \lim_\alpha ST_\alpha\\ &= ST,\end{aligned}$$

where the limits are taken in the weak operator topology. Thus, $T \in \mathcal{S}'$. □

Corollary: If $\mathcal{H}$ is a Hilbert space, and $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint subset of operators, then $\mathcal{S}'$ is a von Neumann algebra acting on $\mathcal{H}$. Also, $\mathcal{S}''$ is a von Neumann algebra containing $\mathcal{S}$.

There are also various locally convex topologies on $\mathcal{B}(\mathcal{H})$.

Definition. Let $\mathcal{H}$ be a Hilbert space. The locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the family of seminorms $\{q_\xi \mid \xi \in \mathcal{H}\}$, defined by

$$q_\xi(T) = \|T(\xi)\| + \|T^*(\xi)\|,$$

is known as the strong* operator topology, or S*OT.

Proposition: Let $\mathcal{H}$ be a Hilbert space. The identity maps

$$\left(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\text{op}}\right) \xrightarrow{\text{id}} (\mathcal{B}(\mathcal{H}), \text{S*OT}) \xrightarrow{\text{id}} (\mathcal{B}(\mathcal{H}), \text{SOT}) \xrightarrow{\text{id}} (\mathcal{B}(\mathcal{H}), \text{WOT})$$

are continuous. If $\dim(\mathcal{H}) = \infty$, then the inverses are not continuous.

Proof. Consult Rainone. □

Order on $\mathcal{B}(\mathcal{H})$

Since $\mathcal{B}(\mathcal{H})$ is a $*$-algebra, we can talk about self-adjoint elements. We write

$$\mathcal{B}(\mathcal{H})_{\text{s.a.}} = \{T \in \mathcal{B}(\mathcal{H}) \mid T = T^*\}$$

to denote the collection of self-adjoint operators.

Example. Given $\lambda = (\lambda_n)_n$, the multiplication operator $D_\lambda \in \mathcal{B}(\ell_2)$ is self-adjoint if and only if $\lambda_n \in \mathbb{R}$ for all n.

Note that since adjoint operators are unique, we can say that $T = T^*$ if and only if $\langle T(\xi), \eta\rangle = \langle \xi, T(\eta)\rangle$. We can use the polarization identity to show another criterion for self-adjoint elements.

Lemma: An operator $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint if and only if $\langle T(\xi), \xi\rangle \in \mathbb{R}$ for all $\xi \in \mathcal{H}$.

Proof. We have

$$\begin{aligned}\langle T(\xi), \xi\rangle &= \langle \xi, T^*(\xi)\rangle\\ &= \langle \xi, T(\xi)\rangle\\ &= \overline{\langle T(\xi), \xi\rangle},\end{aligned}$$

meaning $\langle T(\xi), \xi\rangle \in \mathbb{R}$.

In the reverse direction, we define $F(x, y) = \langle T(x), y\rangle$ and $G(x, y) = \langle x, T(y)\rangle$. Along the diagonal, we have

$$\begin{aligned}F(x, x) &= \langle T(x), x\rangle\\ &= \overline{\langle T(x), x\rangle}\\ &= \langle x, T(x)\rangle\\ &= G(x, x),\end{aligned}$$

so $F = G$, meaning $\langle T(x), x\rangle = \langle x, T(x)\rangle$, so $T = T^*$. □

Proposition: Let A be a Banach $*$-algebra. The self-adjoint elements $A_{\text{s.a.}}$ form a Banach space over $\mathbb{R}$.

Proof. The self-adjoint elements $A_{\text{s.a.}}$ form a vector space over $\mathbb{R}$. To see this, let $x, y \in A_{\text{s.a.}}$, $\alpha \in \mathbb{R}$. Then,

$$\begin{aligned}(x + \alpha y)^* &= x^* + \overline{\alpha} y^*\\ &= x^* + \alpha y^*\end{aligned}$$

$$= x + \alpha y,$$

so $x + \alpha y$ is self-adjoint.

We show that $A_{\text{s.a.}}$ is norm-closed. Let $(a_n)_n$ be a sequence in $A_{\text{s.a.}}$ that converges in norm to an element $a \in A$. Then, $(a_n)_n = (a_n^*)_n \to a^*$, since the adjoint operator is continuous. This implies $a^* = a$. $\square$

If $A = \mathcal{B}(\mathcal{H})$, then we can say more about the qualities of $A_{\text{s.a.}}$.

Proposition: The self-adjoint operators $\mathcal{B}(\mathcal{H})_{\text{s.a.}} \subseteq \mathcal{B}(\mathcal{H})$ are closed in WOT, meaning they are closed in SOT.

Proof. Let $(T_\alpha)_\alpha$ be a net in $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$ converging in WOT to $T \in \mathcal{B}(\mathcal{H})$. We have

$$\langle T(\xi), \xi\rangle = \lim_{\alpha} \langle T_\alpha(\xi), \xi\rangle .$$

Since each T_α is self-adjoint, we know that $\langle T_\alpha(\xi), \xi\rangle \in \mathbb{R}$ for every $\xi \in \mathcal{H}$, meaning $\langle T(\xi), \xi\rangle \in \mathbb{R}$ for every $\xi \in \mathcal{H}$, so T is self-adjoint. $\square$

Definition. For $T \in \mathcal{B}(\mathcal{H})$, we define the numerical range of T to be

$$W(T) = \{\langle T(\xi), \xi\rangle \mid \xi \in B_{\mathcal{H}}\}$$

and the numerical radius of T to be

$$\nu(T) = \sup_{\xi \in B_{\mathcal{H}}} |\langle T(\xi), \xi\rangle| .$$

We see that for all ξ,

$$|\langle T(\xi), \xi\rangle| \leqslant \nu(T) \|\xi\|^2 ,$$

and by Cauchy–Schwarz,

$$\nu(T) \leqslant \|T\|_{\text{op}} .$$

When T is self-adjoint, we can say more.

Proposition: If $T = T^*$, then $\nu(T) = \|T\|_{\text{op}}$.

Proof. Using the fact that T is self-adjoint, we get

$$4\operatorname{Re}(\langle T(x), y\rangle) = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle$$

for $x, y \in \mathcal{H}$. Using the parallelogram law, we get

$$\begin{aligned} |4\operatorname{Re}(T(x,y))| &\leqslant |\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| \\ &\leqslant \nu\left(\|x+y\|^2 + \|x-y\|^2\right) \\ &= 2\nu(T)\left(\|x\|^2 + \|y\|^2\right), \end{aligned}$$

giving

$$|\operatorname{Re}(\langle T(x), y\rangle)| \leqslant \frac{1}{2}\nu(T)\left(\|x\|^2 + \|y\|^2\right).$$

Let $x, y \in B_{\mathcal{H}}$, and set $\lambda = \operatorname{sgn}(\langle T(x), y\rangle)$. Then, $\left|\overline{\lambda}\right| = 1$, yielding

$$|\langle T(x), x\rangle| = \lambda \langle T(x), y\rangle$$

$$
\begin{aligned}
&= \left\langle T(x), \overline{\lambda} y \right\rangle \\
&= \left| \mathrm{Re}\left(\left\langle T(x), \overline{\lambda} y \right\rangle \right) \right| \\
&\leqslant \frac{1}{2}\nu(T)\left(\|x\| + \left\|\overline{\lambda} y\right\|^2 \right) \\
&= \frac{1}{2}\nu(T)\left(\|x\|^2 + \|y\|^2 \right) \\
&\leqslant \frac{1}{2}\nu(T)(2) \\
&= \nu(T).
\end{aligned}
$$

Thus, taking the supremum, we get

$$\|T\|_{\mathrm{op}} \leqslant \nu(T).$$

□

We can now establish an ordering of $\mathcal{B}(\mathcal{H})_{\mathrm{s.a.}}$ by constructing a cone of positive elements.

Definition. Let $\mathcal{H}$ be a Hilbert space, and suppose $T \in \mathcal{B}(\mathcal{H})$. We say T is a positive operator if $W(T) \subseteq [0, \infty)$. In other words, for all $\xi \in \mathcal{H}$,

$$\langle T(\xi), \xi \rangle \geqslant 0.$$

We write $\mathcal{B}(\mathcal{H})_+$ for all the positive operators in $\mathcal{B}(\mathcal{H})$.

Exercise: Suppose $T \in \mathcal{B}(\mathcal{H})_+$. Prove that the sesquilinear form $(\xi, \eta) \mapsto \langle T(\xi), \eta \rangle$ is a semi-inner product. Using the Cauchy–Schwarz inequality, show that

$$\|T(\xi)\|^2 \leqslant \langle T(\xi), \xi \rangle^{1/2} \left\langle T^3(\xi), \xi \right\rangle^{1/2}.$$

Solution: Let $F : \mathcal{H} \to \mathcal{H}$ be defined by $(x, y) \to \langle T(x), y \rangle$. We have

$$
\begin{aligned}
F(\lambda x_1 + \mu x_2, y) &= \langle T(\lambda x_1 + \mu x_2), y \rangle \\
&= \langle T(\lambda x_1) + T(\mu x_2), y \rangle \\
&= \lambda \langle T(x_1), y \rangle + \mu \langle T(x_2), y \rangle \\
&= \lambda F(x_1, y) + \mu F(x_2, y)
\end{aligned}
$$

$$
\begin{aligned}
F(x, y) &= \langle T(x), y \rangle \\
&= \overline{\langle y, T(x) \rangle} \\
&= \overline{\langle T^*(y), x \rangle} \\
&= \overline{\langle T(y), x \rangle} \\
&= \overline{F(y, x)}
\end{aligned}
$$

$$
\begin{aligned}
F(x, x) &= \langle T(x), x \rangle \\
&\geqslant 0.
\end{aligned}
$$

Thus, F is a semi-inner product.

We use the Cauchy–Schwarz inequality on F to find

$$
\begin{aligned}
F(T(\xi), \xi) &\leqslant F(\xi, \xi)^{1/2} F(T(\xi), T(\xi))^{1/2} \\
\left\langle T^2(\xi), \xi \right\rangle &\leqslant \langle T(\xi), \xi \rangle^{1/2} \langle T(T(\xi)), T(\xi) \rangle \\
\|T(\xi)\|^2 &\leqslant \langle T(\xi), \xi \rangle^{1/2} \left\langle T^3(\xi), \xi \right\rangle^{1/2}.
\end{aligned}
$$

Example. Given $\lambda = (\lambda_n)_n \in \ell_\infty$, the multiplication operator $D_\lambda \in \mathcal{B}(\ell_2)$ is positive if and only if $\lambda_n \geqslant 0$ for all n. For $\xi = (\xi_n)_n$, we have

$$\langle D_\lambda(\xi), \xi\rangle = \sum_{n=1}^{\infty} \lambda_n |\xi_n|^2,$$

which is greater than or equal to zero for $\lambda_n \geqslant 0$. Conversely, if $D_\lambda \in \mathcal{B}(\ell_2)_+$, then $\langle D_\lambda(e_n), e_n\rangle = \lambda_n \geqslant 0$ for all $n \geqslant 1$.

Similarly, if $f \in L_\infty(\Omega, \mu)$, then the multiplication operator $M_f \in \mathcal{B}(L_2(\Omega, \mu))$ is positive if and only if $f(x) \geqslant 0$ μ-a.e.

Example. If $M \subseteq \mathcal{H}$ is a closed subspace, and $P_M \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection, then $P_M \geqslant 0$. This is because

$$\begin{aligned}\langle P_M(x), x\rangle &= \langle P_M(x), P_M(x)\rangle \\ &= \|P_M(x)\|^2 \\ &\geqslant 0.\end{aligned}$$

Proposition: The positive operators $\mathcal{B}(\mathcal{H})_+$ form a WOT-closed cone in $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$. Thus, $\mathcal{B}(\mathcal{H})_+$ is SOT-closed and norm-closed.[II]

Proof. Note that $\mathcal{B}(\mathcal{H})_+ \subseteq \mathcal{B}(\mathcal{H})_{\text{s.a.}}$.

Let $(T_\alpha)_\alpha$ be a net in $\mathcal{B}(\mathcal{H})_+$ converging in WOT to $T \in \mathcal{B}(\mathcal{H})$. Thus, we have

$$\begin{aligned}\langle T(\xi), \xi\rangle &= \lim_{\alpha} \langle T_\alpha(\xi), \xi\rangle \\ &\geqslant 0,\end{aligned}$$

so T is positive.

To see that $\mathcal{B}(\mathcal{H})_+$ is a cone, let $T, S \in \mathcal{B}(\mathcal{H})$, and let $t \in [0, \infty)$. We see that for any $\xi \in \mathcal{H}$,

$$\begin{aligned}\langle (T+S)(\xi), \xi\rangle &= \langle T(\xi), \xi\rangle + \langle S(\xi), \xi\rangle \\ &\geqslant 0\end{aligned}$$

$$\begin{aligned}\langle tT(\xi), \xi\rangle &= t\langle T(\xi), \xi\rangle \\ &\geqslant 0.\end{aligned}$$

Thus, $T + S$ and tT belong to $\mathcal{B}(\mathcal{H})_+$.

Suppose T and $-T$ are in $\mathcal{B}(\mathcal{H})_+$. Then, we have $\langle T(\xi), \xi\rangle \geqslant 0$ and $-\langle T(\xi), \xi\rangle \geqslant 0$, so $\langle T(\xi), \xi\rangle = 0$ for every $\xi \in \mathcal{H}$, meaning $\nu(T) = 0$. Since T is self-adjoint, we have $\|T\|_{op} = 0$. □

We can now define an ordering on the space of self-adjoint operators.

Definition. Let $\mathcal{H}$ be a Hilbert space, and suppose $S, T \in \mathcal{B}(\mathcal{H})_{\text{s.a.}}$. We have $T \leqslant S$ if and only if $S - T \in \mathcal{B}(\mathcal{H})$, which is true if and only if $\langle T(\xi), \xi\rangle \leqslant \langle S(\xi), \xi\rangle$ for all $\xi \in \mathcal{H}$.

Remark: This ordering is generally not well-behaved. For instance, the product of positive operators is not necessarily positive, and it is not necessarily the case that $S \leqslant T$ implies $S^2 \leqslant T^2$.

Proposition: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.

(1) If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $T^*T \geqslant 0$.

[II]The positive operators, as well as self-adjoint operators, are convex.

(2) If $T \geqslant 0$, then $W^*TW \geqslant 0$ For all $W \in \mathcal{B}(\mathcal{H})$.

(3) If $S \leqslant T$ in $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$, then $W^*SW \leqslant W^*TW$ for all $W \in \mathcal{B}(\mathcal{H})$.

(4) If $0 \leqslant S \leqslant T$, then $\|S\|_{\text{op}} \leqslant \|T\|_{\text{op}}$.

(5) If T is positive, then $T \leqslant I$ if and only if $\|T\|_{\text{op}} \leqslant 1$.

(6) If T is self-adjoint and $r \geqslant 0$, then $-rI \leqslant T \leqslant rI$ if and only if $\|T\|_{\text{op}} \leqslant r$.

(7) For T self-adjoint, we have $-\|T\|_{\text{op}} I \leqslant T \leqslant \|T\|_{\text{op}} I$, meaning I is an order unit for $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$, and $\|T\|_{\text{op}} = \inf\{r \mid -rI \leqslant T \leqslant rI\}$.

Proof.

(1) Given $\xi \in \mathcal{H}$, we have $\langle T^*T(\xi), \xi\rangle = \langle T(\xi), T(\xi)\rangle = \|T(\xi)\|^2 \geqslant 0$.

(2) For $\xi \in \mathcal{H}$ and $W \in \mathcal{B}(\mathcal{H})$, we have $\langle W^*TW(\xi), \xi\rangle = \langle T(W(\xi)), \xi\rangle \geqslant 0$.

(3) If $S \leqslant T$, then $T - S \geqslant 0$, so $W^*TW - W^*SW = W^*(S - T)W \geqslant 0$.

(4) Let $0 \leqslant S \leqslant T$. Then,

$$\begin{aligned}
\|S\|_{\text{op}} &= \sup_{\xi \in B_{\mathcal{H}}} \langle S(\xi), \xi\rangle \\
&\leqslant \sup_{\xi \in B_{\mathcal{H}}} \langle T(\xi), \xi\rangle \\
&= \|T\|_{\text{op}}.
\end{aligned}$$

(5) If $0 \leqslant T \leqslant I$, then $\|T\|_{\text{op}} \leqslant \|I\|_{\text{op}}$. Conversely, if $T \geqslant 0$ and $\|T\|_{\text{op}} \leqslant 1$, then for $\xi \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
\langle I(\xi), \xi\rangle &= \langle \xi, \xi\rangle \\
&\leqslant \|\xi\|^2 \\
&= 1 \\
&\geqslant \|T\|_{\text{op}} \\
&= \sup_{\xi \in \mathcal{B}(\mathcal{H})} \langle T(\xi), \xi\rangle.
\end{aligned}$$

(6) We have

$$\begin{aligned}
-rI \leqslant T \leqslant rI &\Leftrightarrow \langle -r\xi, \xi\rangle \leqslant \langle T(\xi), \xi\rangle \leqslant \langle r\xi, \xi\rangle \\
&\Leftrightarrow -r \leqslant \langle T(\xi), \xi\rangle \leqslant r \\
&\Leftrightarrow |\langle T(\xi), \xi\rangle| \leqslant r \\
&\Leftrightarrow \nu(T) \leqslant r \\
&\Leftrightarrow \|T\|_{\text{op}} \leqslant r.
\end{aligned}$$

(7) Follows directly from (6).

□

Exercise: If $S, T \in \mathcal{B}(\mathcal{H})$ are such that $S^*S \leqslant T^*T$, show that $\ker(T) \subseteq \ker(S)$.

Solution: We are aware that $S^*S \leqslant T^*T$ if and only if for all ξ, $\langle S^*S(\xi), \xi\rangle \leqslant \langle T^*T(\xi), \xi\rangle$, meaning $\|S(\xi)\|^2 \leqslant \|T(\xi)\|^2$.

Thus, if $\xi \in \ker(T)$, then $\|S(\xi)\|^2 = 0$, so $S(\xi) = 0$, so $\xi \in \ker(S)$.

Proposition (Douglas Factorization)**:** Let $S, T \in \mathcal{B}(\mathcal{H})$ be such that $S^*S \leqslant T^*T$. Then, there is a contraction $R \in \mathcal{B}(\mathcal{H})$ such that $RT = S$.

Proof. We see the map $R_0 : \operatorname{Ran}(T) \to \operatorname{Ran}(S)$ defined by $R_0(T(x)) = S(x)$ is well-defined and linear. Additionally,

$$\begin{aligned}\|R_0(T(x))\|^2 &= \|S(x)\|^2\\ &= \langle S(x), S(x)\rangle\\ &= \langle S^*S(x), x\rangle\\ &\leqslant \langle T^*T(x), x\rangle\\ &= \langle T(x), T(x)\rangle\\ &= \|T(x)\|^2,\end{aligned}$$

meaning R_0 is contractive.

We can extend R_0 continuously to $R_1 : \overline{\operatorname{Ran}}(T) \to \mathcal{H}$. Define $R : \mathcal{H} \to \mathcal{H}$ by the direct sum decomposition $R(x+y) = R_1(x)$, where $x \in \overline{\operatorname{Ran}}(T)$ and $y \in \overline{\operatorname{Ran}}(T)^{\perp}$.[III] ◻

Proposition: Let $\mathcal{H}$ be a Hilbert space, and let $(T_\alpha)_\alpha$ be a norm-bounded increasing net in $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$. Then, there is $T \in \mathcal{B}(\mathcal{H})_{\text{s.a.}}$ such that $T = \lim_\alpha T_\alpha$ in SOT, and $\sup_\alpha T_\alpha = T$.

Proof. Set $C = \sup_\alpha \|T_\alpha\|_{op}$.

For $\xi \in \mathcal{H}$, we see that $(\langle T_\alpha(\xi), \xi\rangle)_\alpha$ is a bounded increasing net in $\mathbb{R}$, since

$$\begin{aligned}|\langle T_\alpha(\xi), \xi\rangle| &\leqslant \|T_\alpha\|_{op}\|\xi\|^2\\ &\leqslant C\|\xi\|^2,\end{aligned}$$

and is increasing by the definition of the ordering on $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$. Thus, this net is convergent in $\mathbb{R}$.

Since $(\xi, \eta) \mapsto \langle T_\alpha(\xi), \eta\rangle$ is a sesquilinear form, we may use the polarization identity to write

$$F(\xi,\eta) = \lim_\alpha\left(\sum_{k=0}^{3} i^k \left\langle T_\alpha\left(\xi + i^k\eta\right), \xi + i^k\eta\right\rangle\right),$$

which defines a sesquilinear form with

$$\begin{aligned}|F(\xi,\eta)| &= \left|\lim_\alpha \langle T_\alpha(\xi), \eta\rangle\right|\\ &\leqslant C\|\xi\|\|\eta\|.\end{aligned}$$

Thus, there is a unique $T \in \mathcal{B}(\mathcal{H})$ with $F(\xi,\eta) = \langle T(\xi), \eta\rangle$, and $\|T\|_{op} = \|F\|_{op} \leqslant C$. By our definition of F, we have $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$, and $T^* = T$.

Since $\sup_\alpha \langle T_\alpha(\xi), \xi\rangle = \langle T(\xi), \xi\rangle$ for each $\xi \in \mathcal{H}$, we see that $T_\alpha \leqslant T$ for every α, and no lesser $S \in \mathcal{B}(\mathcal{H})_{\text{s.a.}}$ has this property. Thus, $\sup_\alpha T_\alpha = T$.

Finally, we must show that $(T_\alpha)_\alpha \xrightarrow{\text{SOT}} T$. If α_0 is fixed in the indexing set, and $\alpha \geqslant \alpha_0$, then we have $0 \leqslant T - T_\alpha \leqslant T - T_{\alpha_0}$, so $\|T - T_\alpha\|_{op} \leqslant \|T - T_{\alpha_0}\|_{op}$. Thus, we have

$$\|(T - T_\alpha)(\xi)\|^2 \leqslant \langle (T - T_\alpha)(\xi), \xi\rangle^{1/2}\left\langle (T - T_\alpha)^3(\xi), \xi\right\rangle^{1/2}$$

[III] It is the case that $\ker(T^*) = \overline{\operatorname{Ran}}(T)$.

$$\begin{aligned}
&\leqslant \langle (T - T_\alpha)\,\xi, \xi\rangle^{1/2} \|T - T_\alpha\|_{op}^{3/2} \|\xi\| \\
&\leqslant \langle (T - T_\alpha)(\xi), \xi\rangle \|T - T_{\alpha_0}\|_{op}^{3/2} \|\xi\| \\
&\to 0.
\end{aligned}$$

Thus, $(T_\alpha)_\alpha \xrightarrow{\text{SOT}} T$. □

Special Classes of Operators

Lemma: Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. For any bounded operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the following are true.

(1) $\ker(T^*) = \operatorname{Ran}(T)^\perp$

(2) $\ker(T) = \operatorname{Ran}(T^*)^\perp$

(3) $\ker(T)^\perp = \overline{\operatorname{Ran}}(T^*)$

(4) $\ker(T^*)^\perp = \overline{\operatorname{Ran}}(T)$

Proof.

(1) For $y \in \mathcal{K}$, we have, for all $x \in \mathcal{H}$,

$$\begin{aligned}
T^*(y) = - &\Leftrightarrow \langle x, T^*(y)\rangle = 0 \\
&\Leftrightarrow \langle T(x), y\rangle = 0 \\
&\Leftrightarrow y \in \operatorname{Ran}(T)^\perp.
\end{aligned}$$

(2) Apply (1) to T^*.

(3) Taking orthogonal complements, we get

$$\begin{aligned}
\ker(T)^\perp &= \left(\operatorname{Ran}(T^*)^\perp\right)^\perp \\
&= \overline{\operatorname{Ran}}(T^*).
\end{aligned}$$

(4) Apply (3) to T^*.

□

Normal Operators

Recall from linear algebra that a square matrix $a \in \operatorname{Mat}_n(\mathbb{C})$ is called normal if $a^*a = aa^*$. These matrices are unitarily diagonalizable, which we studied in linear algebra. However, we can define normal operators in any $*$-algebras.

Definition. Let $\mathcal{H}$ be a Hilbert space. A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is called normal if $TT^* = T^*T$.

Example.

(1) All self-adjoint operators are normal, as are all positive operators and orthogonal projections.

(2) The right shift operator $R : \ell_2 \to \ell_2$ is not normal, since $RR^* \neq I$ and $R^*R = I$. Similarly, the left shift operator is not normal.

(3) The multiplication operator $D_\lambda : \ell_2 \to \ell_2$, where $\lambda \in \ell_\infty$, is normal. This is because all multiplication operators commute. Similarly, the multiplication operators $M_f \in \mathcal{B}(L_2(\Omega, \mu))$ with $f \in L_\infty(\Omega, \mu)$, are also normal.

Lemma: A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is normal if and only if, for all $x \in \mathcal{H}$,

$$\|T(x)\| = \|T^*(x)\|.$$

Proof. For all $x \in \mathcal{H}$, we have

$$\begin{aligned}
\|T(x)\| = \|T^*(x)\| &\Leftrightarrow \|T(x)\|^2 = \|T^*(x)\|^2 \\
&\Leftrightarrow \langle T(x), T(x)\rangle = \langle T^*(x), T^*(x)\rangle \\
&\Leftrightarrow \langle T^*T(x), x\rangle = \langle TT^*(x), x\rangle \\
&\Leftrightarrow T^*T = TT^*.
\end{aligned}$$

□

Exercise: Prove that the adjoint map is SOT-continuous on the set of normal operators.

Solution: Let $(T_\alpha)_\alpha \xrightarrow{\text{SOT}} T$. Then,

$$\begin{aligned}
\|(T_\alpha^* - T^*)(x)\| &= \|(T_\alpha - T)^*(x)\| \\
&\|(T_\alpha - T)(x)\| \\
&\to 0.
\end{aligned}$$

We know that the numerical radius coincides with the operator norm for self-adjoint operators. The same holds for normal operators.

Exercise: If $T \in \mathcal{B}(\mathcal{H})$ is normal, prove that $\nu(T) = \|T\|_{\text{op}}$.

Solution:

(i) From the same property on the sesquilinear form $F(x,y) = \langle T(x), y\rangle$, we can see that

$$|\langle T(x), x\rangle| \leqslant \|T\| \|x\|^2,$$

so $\nu(T) \leqslant \|T\|$.

(ii) We have, for every $x \in B_{\mathcal{H}}$, that

$$|\langle T(x), x\rangle| \leqslant \nu(T).$$

For $y = \|y\| x$, we scale both sides to get

$$|\langle T(y), y\rangle| \leqslant \|y\|^2 \nu(T).$$

(iii) Letting $F(x,y) = \langle T(x), y\rangle$, we have

$$\begin{aligned}
4F(x,y) + 4F(y,x) &= (F(x+y, x+y) + iF(x+iy, x+iy) - F(x-y, x-y) - iF(x-iy, x-iy)) \\
&\quad + (F(y+x) + iF(y+ix, y+ix) - F(y-x, y-x) - iF(y-ix, y-ix)) \\
&= 2(F(x+y) - F(x-y)),
\end{aligned}$$

yielding

$$\frac{1}{2}\operatorname{Re}(F(x,y) + F(y,x)) = \frac{1}{4}\operatorname{Re}(F(x+y) - F(x-y)).$$

(iv)

$$\begin{aligned}
\frac{1}{4}\operatorname{Re}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle) &\leqslant \frac{1}{4}|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle| \\
&\leqslant \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|) \\
&\leqslant \frac{1}{4}\left|\nu(T)\|x+y\|^2 + \nu(T)\|x-y\|^2\right| \\
&\leqslant \frac{1}{4}\left(\nu(T)\left(2\|x\|^2 + 2\|y\|^2\right)\right)
\end{aligned}$$

$$\leqslant \nu(T).$$

Setting $y = \frac{T(x)}{\|T(x)\|}$, we get

$$\begin{aligned}\frac{1}{2}\operatorname{Re}\left(\left\langle T(x), \frac{T(x)}{\|T(x)\|}\right\rangle + \left\langle T\left(\frac{T(x)}{\|T(x)\|}\right), x\right\rangle\right) &= \frac{1}{2}\operatorname{Re}\left(\|T(x)\| + \frac{1}{\|T(x)\|}\left\langle T^2(x), x\right\rangle\right)\\ &= \frac{1}{2}\|T(x)\| + \frac{1}{2}\left(\frac{1}{\|T(x)\|}\operatorname{Re}\left(\left\langle T^2(x), x\right\rangle\right)\right)\\ &\leqslant \nu(T)\end{aligned}$$

Thus, we recover

$$\|T(x)\| + \frac{1}{\|T(x)\|}\operatorname{Re}\left(\left\langle T^2(x), x\right\rangle\right) \leqslant 2\nu(T).$$

(v) Let ω be such that $\omega^2 = \operatorname{sgn}\left(\left\langle T^2(x), x\right\rangle\right)$. Then, we have

$$\begin{aligned}\|\omega T(x)\| + \frac{1}{\|\omega T(x)\|}\operatorname{Re}\left(\left\langle (\omega T)^2(x), x\right\rangle\right) &= \|T(x)\| + \frac{1}{\|T(x)\|}\operatorname{Re}\left(\omega^2\langle T(x), x\rangle\right)\\ &= \|T(x)\| + \frac{1}{\|T(x)\|}\operatorname{Re}\left(|\langle T(x), x\rangle|\right)\\ &= \|T(x)\| + \frac{1}{\|T(x)\|}|\langle T(x), x\rangle|\\ &\leqslant 2\nu(\omega T)\\ &= 2\nu(T).\end{aligned}$$

(vi) We have

$$\begin{aligned}\|T(x)\| &\leqslant \|T(x)\| + \frac{1}{\|T(x)\|}\left|\left\langle T^2(x), x\right\rangle\right|\\ &\leqslant 2\nu(T)\end{aligned}$$

for all $x \in B_{\mathcal{H}}$, meaning

$$\|T\| \leqslant 2\nu(T).$$

Additionally, for $x \neq 0$, we have

$$\|T(x)\|^2 + \left|\left\langle T^2(x), x\right\rangle\right| \leqslant 2\nu(T)\|T(x)\|,$$

so taking the supremum on both sides, we get

$$\|T\|^2 + \nu\left(T^2\right) \leqslant 2\nu(T)\|T\|.$$

(vii) We have

$$\begin{aligned}(\|T\| - \nu(T))^2 &\geqslant 0\\ \|T\|^2 + \nu(T)^2 &\geqslant 2\nu(T)\\ &\geqslant \|T\|^2 + \nu\left(T^2\right),\end{aligned}$$

so

$$\nu\left(T^2\right) \leqslant \nu(T)^2.$$

Inductively, we have

$$\nu\left(T^{2^n}\right) \leqslant \nu\left(T^{2^{n-1}}\right)^2,$$

giving

$$\nu\left(T^{2^n}\right) \leqslant \nu(T)^{2^n}.$$

(viii) Let T be normal. Then, $A = T^*T$ is self-adjoint, so $\|A\|^2 = \|A^*A\| = \|A^2\|$. Additionally, since $T^*T = TT^*$, we have $(T^*T)^{2^n} = (T^*)^{2^n} T^{2^n}$. Combining, this gives

$$\begin{aligned}\|T\|^{2^n} &= \left(\|T^*T\|^{2^n}\right)^{1/2}\\ &= \left\|(T^*T)^{2^n}\right\|^{1/2}\\ &= \left\|(T^*)^{2^n} T^{2^n}\right\|^{1/2}\\ &= \left\|\left(T^{2^n}\right)^* T^{2^n}\right\|^{1/2}\\ &= \left\|T^{2^n}\right\|.\end{aligned}$$

Thus, we see that

$$\begin{aligned}(\nu(T))^{2^n} &\leqslant \|T\|^{2^n}\\ &= \left\|T^{2^n}\right\|\\ &\leqslant 2\nu\left(T^{2^n}\right)\\ &\leqslant 2\nu(T)^{2^n}.\end{aligned}$$

Taking roots and sending $n \to \infty$, we see that

$$\|T\| \leqslant \nu(T),$$

establishing $\nu(T) = \|T\|$.

Proposition: Let $\mathcal{H}$ be a Hilbert space, and suppose $T \in \mathcal{B}(\mathcal{H})$ is normal. The following are true.

(1) $\ker(T) = \ker(T^*) = (\operatorname{Ran}(T))^{\perp}$.

(2) T is injective if and only if $\operatorname{Ran}(T) \subseteq \mathcal{H}$ is dense.

(3) T is invertible if and only if T is bounded below.

Proof.

(1) Since $\|T\| = \|T^*\|$ and $\ker(T^*) = (\operatorname{Ran}(T))^{\perp}$, we know that $\ker(T) = \ker(T^*) = (\operatorname{Ran}(T))^{\perp}$.

(2) We see that T is injective if and only if $\ker(T) = \{0\}$, which is true if and only if $\ker(T)^{\perp} = \mathcal{H}$, so $\left(\operatorname{Ran}(T)^{\perp}\right)^{\perp} = \mathcal{H}$, so $\overline{\operatorname{Ran}}(T) = \mathcal{H}$.

(3) We know that an invertible operator on a Banach space is always bounded below. If T is bounded below, then T is injective and has closed range. Since T is assumed to be normal, its range is dense as well, so $\operatorname{Ran}(T) = \overline{\operatorname{Ran}}(T) = \mathcal{H}$.

□

A linear operator fails to be invertible if it has a non-trivial kernel — however, the converse is false. However, if T is normal, then T is non-invertible if it admits an approximate kernel.

Corollary: Let $\mathcal{H}$ be a Hilbert space. A normal operator $T \in \mathcal{B}(\mathcal{H})$ is noninvertible if and only if there is a sequence $(\xi_n)_n$ in $\mathcal{H}$ with $\|\xi_n\| = 1$ and $\lim_{n\to\infty} \|T(\xi_n)\| = 0$.

Proof. Such a sequence exists if and only if T is not bounded below. □

Isometric and Unitary Operators

Lemma: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and suppose $U, V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

(1) V is an isometry if and only if $V^*V = I_{\mathcal{H}}$.

(2) U is a unitary if and only if $U^*U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{K}}$.

Proof. For $\xi \in \mathcal{H}$, we have

$$\begin{aligned}\|V(\xi)\|^2 &= \langle V(\xi), V(\xi)\rangle \\ &= \langle V^*V(\xi), \xi\rangle.\end{aligned}$$

If $V^*V = I_{\mathcal{H}}$, then

$$\begin{aligned}\|V(\xi)\| &= \langle V^*V(\xi), \xi\rangle \\ &= \langle \xi, \xi\rangle \\ &= \|\xi\|^2,\end{aligned}$$

meaning V is an isometry. Similarly, if V is isometric, then $\|\xi\|^2 = \|V(\xi)\|^2 = \langle V^*V(\xi), \xi\rangle$, so $V^*V = I_{\mathcal{H}}$.

A similar proof shows that U is unitary if and only if $U^*U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{K}}$. □

Example. The left bilateral shift on $\ell_2(\mathbb{Z})$ is defined by $U(\xi)(n) = \xi(n+1)$ for $n \in \mathbb{Z}$. This is a well-defined operator, since $U(\xi) \in \ell_2(\mathbb{Z})$ for $\xi \in \ell_2(\mathbb{Z})$, as

$$\begin{aligned}\|U(\xi)\|^2 &= \sum_{n\in\mathbb{Z}} |U(\xi)(n)|^2 \\ &= \sum_{n\in\mathbb{Z}} |\xi(n+1)|^2 \\ &= \sum_{m\in\mathbb{Z}} |\xi(m)|^2 \\ &= \|\xi\|^2.\end{aligned}$$

We also see that U is an isometry, as $U^*U = I$. We see that $U(e_k) = e_{k-1}$ for all $k \in \mathbb{Z}$, where $(e_k)_{k\in\mathbb{Z}}$ is the canonical orthonormal basis for $\ell_2(\mathbb{Z})$.

We may also define the right bilateral shift operator by $V(\xi)(n) = \xi(n-1)$. We have $V(e_k) = e_{k+1}$. We have $UV = I$ and $VU = I$, so

$$\begin{aligned}V &= IV \\ &= (U^*U)V \\ &= U^*(UV) \\ &= U^*(I) \\ &= U^*,\end{aligned}$$

so $U^*U = UU^* = V^*V = VV^* = I$, so the bilateral shifts are unitary.

Example. The right and left unilateral shifts $R, L : \ell_2 \to \ell_2$ are nonunitary isometries, since $R^*R = LL^* = I$, but $R^*R = L^*L \neq I$.

Exercise: Show that the multiplication operator D_λ associated to $(\lambda_n)_n \in \ell_\infty$ is unitary if and only if $|\lambda_n| = 1$ for all n.

Show that the multiplication operator M_f on $L_2(\Omega, \mu)$ associated to $f \in L_\infty(\Omega, \mu)$ is unitary if and only if $|f| = 1$ μ-a.e.

Solution: Let D_λ be unitary. Then, $D_\lambda^* D_\lambda = D_\lambda D_\lambda^* = I$. Thus, we have

$$\begin{aligned}\left\langle D_\lambda D_\lambda^* (\xi), \xi\right\rangle &= \sum_{n\in\mathbb{N}} |\lambda_n \xi_n|^2\\ &= \sum_{n\in\mathbb{N}} |\lambda_n|^2 |\xi_n|\\ &= \sum_{n\in\mathbb{N}} |\xi_n|^2,\end{aligned}$$

which holds if and only if $|\lambda_n|^2 = 1$, or $|\lambda_n| = 1$ for each n.

Similarly, if $|\lambda_n| = 1$ for each n, then we have

$$\begin{aligned}\left\langle D_\lambda^* D_\lambda (\xi), \xi\right\rangle &= \sum_{n\in\mathbb{N}} |\lambda_n|^2 |\xi_n|^2\\ &= \sum_{n\in\mathbb{N}} |\xi_n|^2\\ &= \left\langle D_\lambda^* D_\lambda (\xi), \xi\right\rangle,\end{aligned}$$

meaning $D_\lambda^* D_\lambda = D_\lambda D_\lambda^* = I$.

Similarly, if M_f is unitary, then

$$\begin{aligned}\left\langle M_f^* M_f (\xi), \xi\right\rangle &= \left\langle M_f M_f^* (\xi), \xi\right\rangle\\ &= \int_\Omega |f|^2 |\xi|^2 \, d\mu\\ &= \int_\Omega |\xi|^2 \, d\mu,\end{aligned}$$

which holds if and only if $|f|^2 = 1$ μ-a.e., meaning $|f| = 1$ μ-a.e.

If $|f| = 1$ μ-a.e., then $|f|^2 = 1$ μ-a.e. We have

$$\begin{aligned}\int_\Omega |f|^2 |\xi|^2 \, d\mu &= \left\langle M_f^* M_f (\xi), \xi\right\rangle\\ &= \left\langle M_f M_f^* (\xi), \xi\right\rangle,\end{aligned}$$

meaning $M_f M_f^* = M_f^* M_f = I$.

Exercise: Let $\omega \in \mathbb{T}$. Consider the sequence $(\omega^n)_n \in \ell_\infty$. We know that the multiplication operator $d_\omega = D_{(\omega^n)_n}$ is unitary. If R denotes the right shift operator, show that

$$d_\omega R = \omega R d_\omega.$$

Similarly, if $(\omega^n)_{n\in\mathbb{Z}}$ is a multiplier in $\ell_\infty(\mathbb{Z})$, and V is the right bilateral shift, show that

$$d_\omega V = \omega V d_\omega.$$

Definition. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.

(1) We write $\mathcal{U}(\mathcal{H}, \mathcal{K})$ to be the set of all unitary operators between $\mathcal{H}$ and $\mathcal{K}$. We say $\mathcal{U}(\mathcal{H}) = \mathcal{U}(\mathcal{H}, \mathcal{H})$.

(2) Two operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are said to be unitarily equivalent if there exists $U \in \mathcal{U}(\mathcal{H}, \mathcal{K})$ such that

$$UTU^* = S.$$

Exercise: Let $T \in \mathcal{B}(\mathcal{H})$, $S \in \mathcal{B}(\mathcal{K})$, and $U \in \mathcal{U}(\mathcal{H}, \mathcal{K})$. Prove that $\|UT\|_{op} = \|T\|_{op}$ and $\|SU\|_{op} = \|S\|_{op}$. Conclude that unitarily equivalent operators have the same norm.

Solution: We have, for any $x \in B_{\mathcal{H}}$, that

$$\|UT(x)\|^2 = \|T(x)\|^2,$$

so

$$\|UT\|_{op} = \|T\|_{op}.$$

Similarly, we must have $\|SU\|_{op} = \|S\|_{op}$. Thus, we must have

$$\begin{aligned}\|S\|_{op} &= \|UTU^*\|_{op}\\ &= \|(UT)U^*\|_{op}\\ &= \|UT\|_{op}\\ &= \|T\|_{op}\end{aligned}$$

Exercise: Suppose $U \in \mathcal{H}(\mathcal{K})$ is a unitary isomorphism of Hilbert spaces. Prove that conjugation by U, $\mathrm{Ad}_U(T) = UTU^*$ is an isometric $*$-isomorphism of C^*-algebras.

Fact (Unitary Group)**.** If $\mathcal{H}$ is a Hilbert space, then $\mathcal{U}(\mathcal{H})$ is a group under composition.

Proof. If $U, V \in \mathcal{U}(\mathcal{H})$, then

$$\begin{aligned}(UV)^*(UV) &= V^*U^*UV\\ &= V^*V\\ &= I,\end{aligned}$$

and

$$\begin{aligned}(UV)(UV)^* &= UVV^*U^*\\ &= UU^*\\ &= I,\end{aligned}$$

meaning $UV \in \mathcal{H}$. We have I is the neutral element of $\mathcal{U}(\mathcal{H})$, and $U^* = U^{-1}$ is an element of $\mathcal{U}(\mathcal{H})$ for $U \in \mathcal{U}(\mathcal{H})$. □

Definition (Unitary Representation)**.** Let Γ be a group. A unitary representation of Γ is a pair $(U, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and $U : \Gamma \to \mathcal{U}(\mathcal{H})$ is a group homomorphism.

Exercise: Let $\mathbb{T}$ be the circle group, and let d_ω be the corresponding multiplication operator on $\ell_2(\mathbb{Z})$. Show that the map $\omega \overset{d}{\mapsto}_{\omega}$ is a unitary representation.

Solution: We are aware that d_ω is a unitary operator if $\omega \in \mathbb{T}$. We must now show that the map $\omega \mapsto d_\omega$ is a homomorphism.

To do this, we show that for $\omega, \tau \in \mathbb{T}$, that $\omega\tau^{-1} \mapsto d_\omega d_\tau^{-1} = d_\omega d_\tau^*$. We have

$$\begin{aligned}\left(\omega\tau^{-1}\right) &\mapsto (\omega^n\tau^{-n})_n\\ &= (\omega^n)_n(\tau^{-n})_n\\ &= (\omega^n)_n\left(\overline{\tau}^n\right)_n\\ &= d_\omega d_{\overline{\tau}}\\ &= d_\omega d_\tau^*.\end{aligned}$$

Thus, this is a unitary representation.

Fact. Let $U : \Gamma \to \mathcal{U}(\mathcal{H})$, $s \mapsto U_s$ be a unitary representation of Γ. Then,

(1) if e is the neutral element of Γ, then $U_e = I$;

(2) for all $s \in \Gamma$, $U_{s^{-1}} = U_s^{-1} = U_s^*$.

Proof. A group homomorphism necessarily preserves identity, which shows (1).

To see (2), for $s \in \Gamma$, we have

$$\begin{aligned} I &= U_e \\ &= U_{ss^{-1}} \\ &= U_s U_s^{-1} \end{aligned}$$

$$\begin{aligned} I &= U_e \\ &= U_{s^{-1}s} \\ &= U_{s^{-1}} U_s, \end{aligned}$$

so by the uniqueness of inverses, we have $U_{s^{-1}} = U_s^{-1} = U_s^*$. □

Example (Left Regular Representation)**.** Let Γ be a group with neutral element e, and consider the Hilbert space $\ell_2(\Gamma)$. For every $s \in \Gamma$, we define $\lambda_s : \ell_2(\Gamma) \to \ell_2(\Gamma)$ by

$$\lambda_s(\xi)(t) = \xi\left(s^{-1}t\right)$$

for each $t \in \Gamma$. This map is linear and well-defined, as

$$\begin{aligned} |\lambda_s(\xi)|^2 &= \sum_{t\in\Gamma} |\lambda_s(\xi)(t)|^2 \\ &= \sum_{t\in\Gamma} \left|\xi\left(s^{-1}t\right)\right|^2 \\ &= \sum_{r\in\Gamma} |\xi(r)|^2 \\ &= \|\xi\|^2. \end{aligned}$$

This also shows that λ_s is an isometry. Moreover, each λ_s has an inverse $\lambda_{s^{-1}}$, so each λ_s is unitary with $\lambda_s^* = \lambda_{s^{-1}}$.

For the vectors $\{\delta_t\}_{t\in\Gamma}$ that form the orthonormal basis for $\ell_2(\Gamma)$, we have

$$\begin{aligned} \lambda_s(\delta_t)(r) &= \delta_t\left(s^{-1}r\right) \\ &= \begin{cases} 1 & s^{-1}r = t \\ 0 & s^{-1}r \neq t \end{cases} \\ &= \begin{cases} 1 & r = st \\ 0 & r \neq st \end{cases} \\ &= \delta_{st}(r), \end{aligned}$$

meaning $\lambda_s(\delta_t) = \delta_{st}$.

For $s, r \in \Gamma$, we have

$$\begin{aligned} \lambda_s \circ \lambda_r(\xi)(t) &= \lambda_r(\xi)\left(s^{-1}t\right) \\ &= \xi\left(r^{-1}s^{-1}t\right) \end{aligned}$$

$$\begin{aligned} &= \xi\left((sr)^{-1} t\right) \\ &= \lambda_{sr}(\xi)(t). \end{aligned}$$

Thus, $\lambda_s \circ \lambda_r = \lambda_{sr}$.

We thus have a unitary representation $\lambda : \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, defined by $\lambda(s) = \lambda_s$ for $s \in \Gamma$. This is known as the left regular representation of Γ.

Example. If $T \in \mathcal{B}(\mathcal{H})_{\text{s.a.}}$, then $U = \exp(iT)$ is unitary. We have

$$\begin{aligned} U^{-1} &= \exp(iT)^{-1} \\ &= \exp(-iT) \\ &= \exp((iT)^*) \\ &= \exp(iT)^*. \end{aligned}$$

Theorem (Fuglede's Theorem)**:** If $S, T \in \mathcal{B}(\mathcal{H})$ are such that T is normal and $ST = TS$, then $ST^* = T^*S$.

Proof. Fix $z \in \mathbb{C}$, and consider the operator $\exp(i\overline{z}T)$. Since S commutes with T, it follows that $ST^n = T^nS$ for all $n \geqslant 0$, and by linearity and continuity, we have $S\exp(i\overline{z}T) = \exp(i\overline{z}T)S$. Equivalently, we have

$$\begin{aligned} S &= \exp(i\overline{z}T)\, S \exp(i\overline{z}T)^{-1} \\ &= \exp(i\overline{z}T)\, S \exp(-i\overline{z}T). \end{aligned}$$

Consider the entire function $f : \mathbb{C} \to \mathcal{B}(\mathcal{H})$ defined by $f(z) = \exp(izT^*)\, S \exp(-izT^*)$.

We substitute the expression for S, and recall that T commutes with T^* to get

$$\begin{aligned} f(z) &= \exp(izT^*) \exp(i\overline{z}T)\, S \exp(-i\overline{z}T) \exp(-izT^*) \\ &= \exp(i(zT^* + \overline{z}T))\, S \exp(-i(\overline{z}T + zT^*)). \end{aligned}$$

Note that $zT^* + \overline{z}T$ and $\overline{z}T + zT^*$ are self-adjoint operators, so both $\exp(i(zT^* + \overline{z}T))$ and $\exp(-i(\overline{z}T + zT^*))$ are both unitary. Thus, $\|f(z)\|_{op} \leqslant \|S\|_{op}$.

By Liouville's theorem, we must have f is constant, so

$$\begin{aligned} 0 &= f'(z) \\ &= \exp(izT^*)\, iT^*S \exp(-izT^*) + \exp(izT^*)\, s \exp(-izT^*)(-iT^*). \end{aligned}$$

Setting $z = 0$, we have $0 = iT^*S - iST^*$, meaning $ST^* = T^*S$. □

Projections and Partial Isometries

An operator $P \in \mathcal{B}(\mathcal{H})$ is called a projection of $P^2 = P^* = P$. We set $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ to be the set of all projections.

Fact. Every projection P is positive.

Proof. For $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle P(x), x \rangle &= \langle P(P(x)), x \rangle \\ &= \langle P(x), P(x) \rangle \\ &= \|P(x)\|^2 \\ &\geqslant 0. \end{aligned}$$

□

We have studied orthogonal projections already, where we constructed $P_M \in \mathcal{B}(\mathcal{H})$ that projects onto the closed subspace M. For each $x \in \mathcal{H}$, $P_M(x)$ is the point in M closest to x. We have seen the following

(a) $P_M^2 = P_M^* = P_M$, $P_M \geqslant 0$;

(b) $\operatorname{Ran}(P_M) = M$;

(c) $\|P_M\| = 1$;

(d) $\ker(P_M) = M^\perp$.

We know that, given an idempotent operator $E^2 = E \in \mathcal{B}(\mathcal{H})$, we can decompose $\mathcal{H}$ as the topological direct sum $\mathcal{H} = \operatorname{Ran}(E) \oplus \ker(E)$. However, it may not be the case that $\operatorname{Ran}(E)$ and $\ker(E)$ are not orthogonal, or E is not self-adjoint.

However, we can prove that if E is an idempotent, then E is self-adjoint and equal to the projection onto its range.

Proposition: Let $\mathcal{H}$ be a Hilbert space, and suppose $E \in \mathcal{B}(\mathcal{H})$ is an idempotent. Setting $M = \operatorname{Ran}(E)$, the following are equivalent:

(i) $M = \ker(E)^\perp$;

(ii) $E = P_M$;

(iii) $E \in \mathcal{B}(\mathcal{H})_+$;

(iv) $E^* = E$;

(v) E is normal;

(vi) $\|E\|_{op} \leqslant 1$.

Proof. To see that (i) implies (ii), we remember that $\mathcal{H} = \operatorname{Ran}(E) \oplus \ker(E)$. By the assumption, $M^\perp = \left(\ker(E)^\perp\right)^\perp = \ker(E)$, meaning $H = M \oplus M^\perp$. Since E and P_M are the projections onto M, we must have $E = P_M$.

To see that (ii) implies (iii), we write $x = y + z$, where $y \in M$ and $z \in M^\perp$. Then, $P_M(x) = y$, so we have

$$\begin{aligned}\langle P_M(x), x\rangle &= \langle y, y + z\rangle \\ &= \langle y, y\rangle \\ &= \|y\|^2 \\ &= \|P_M(x)\|^2 \\ &\geqslant 0.\end{aligned}$$

If $E \geqslant 0$, then E is self-adjoint, and E is normal, which proves the chain (iii) through (v).

To see that (v) implies (i), if E is normal, then $\ker(E) = \ker(E^*) = \operatorname{Ran}(E)^\perp$. Moreover, since E is bounded and idempotent, $\operatorname{Ran}(E)$ is closed, so

$$\begin{aligned}M &= \left(M^\perp\right)^\perp \\ &= \left(\operatorname{Ran}(E)^\perp\right)^\perp \\ &= \ker(E)^\perp.\end{aligned}$$

To see that (iv) implies (vi), we see

$$\|E\|^2 = \|E^*E\|$$

$$\begin{aligned} &= \left\|E^2\right\| \\ &= \|E\|, \end{aligned}$$

meaning $\|E\| = 0$ or $\|E\| = 1$.

To see that (vi) implies (iv), if E is a contraction, then so is E^*, as $\|E\| = \|E^*\|$. If $y \in \mathcal{H}$, then

$$\begin{aligned} \|E(y) - E^*(y)\|^2 &= \langle E(y) - E^*(y), E(y) - E^*(y)\rangle \\ &= \|E(y)\|^2 - 2\operatorname{Re}(\langle E(y), E^*(y)\rangle) + \|E^*(y)\| \\ &\leqslant \|E(y)\|^2 - 2\operatorname{Re}\left(\left\langle E^2(y), y\right\rangle\right) + \|y\|^2 \\ &= \|E(y)\|^2 - 2\operatorname{Re}(\langle E(y), y\rangle) + \|y\|^2 \\ &= \langle E(y) - y, E(y) - y\rangle \\ &= \|E(y) - y\|. \end{aligned}$$

If $x \in \mathcal{H}$, we set $y = E(x)$ such that $E(y) = E^2(x) = E(x) = y$. We find $\|y - E^*y\| = 0$, so $E(x) = E^*E(x)$. Since $x \in \mathcal{H}$ was arbitrary, we have $E = E^*E$, meaning

$$\begin{aligned} E^* &= (E^*E)^* \\ &= E^*E \\ &= E, \end{aligned}$$

so E is self-adjoint. $\square$

Exercise: Let $E_\lambda \in \mathcal{B}(\ell_2)$ be the multiplication operator associated with $(\lambda_n)_n \in \ell_\infty$. Prove that D_λ is a projection if and only if $\lambda_n \in \{0, 1\}$ for all n.

Solution: Let D_λ be a projection. Then, $D_\lambda = D_\lambda^*$, implying that $\lambda_n = \overline{\lambda_n}$ for each n, so $\lambda_n \in \mathbb{R}$. Additionally, since $D_\lambda \geqslant 0$, we must have $\lambda_n \geqslant 0$ for each n. Finally, since $D_\lambda^2 = D_\lambda$, we have, for every $(\xi_n)_n \in \ell_2$, that

$$\begin{aligned} \lambda_n^2 \xi_n &= \lambda_n \xi_n \\ \xi_n\left(\lambda_n^2 - \lambda_n\right) &= 0 \\ \xi_n((\lambda_n)(\lambda_n - 1)) &= 0. \end{aligned}$$

Since there exists a sequence $(\xi_n)_n \in \ell_2$ with all $\xi_n \neq 0$, it must be the case that $\lambda_n = 0$ or $\lambda_n = 1$ for each n.

If $\lambda_n \in \{0, 1\}$ for each n, then $D_\lambda^2 = D_\lambda$, as

$$\lambda_n^2 \xi_n = \lambda_n \xi_n$$

for each n. Thus, D_λ is a projection.

Example. Let $(\Omega, \mathcal{L}, \mu)$ be a measure space, and suppose $E \in \mathcal{L}$ is measurable. We may identify $L_2(E, \mu_E)$ with the closed subspace of $L_2(\Omega, \mu)$ consisting of all essentially E-supported functions. The orthogonal projection is the multiplier $M_{\mathbb{1}_E}$.

Exercise: Let Q_n, Q be projections in $\mathcal{B}(\mathcal{H})$ for each n. Prove that

$$(Q_n)_n \xrightarrow{\text{SOT}} Q \Leftrightarrow (Q_n)_n \xrightarrow{\text{WOT}} Q$$

Solution: We have

$$\begin{aligned} Q_n \xrightarrow{\text{WOT}} Q &\Leftrightarrow \langle Q_n(x), x\rangle \to \langle Q(x), x\rangle \\ &\Leftrightarrow \langle (Q_n - Q)(x), x\rangle \to 0 \\ &\Leftrightarrow \|(Q_n - Q)(x)\| \to 0 \\ &\Leftrightarrow Q_n \xrightarrow{\text{SOT}} Q. \end{aligned}$$

Lemma: If P and Q are projections in $\mathcal{B}(\mathcal{H})$, then $PQ = 0$ if and only $QP = 0$ if and only if $\operatorname{Ran}(P) \perp \operatorname{Ran}(Q)$.

Proof. If $PQ = 0$, then

$$\begin{aligned} 0 &= 0^* \\ &= (PQ)^* \\ &= Q^*P^* \\ &= QP. \end{aligned}$$

If $PQ = 0$, then

$$\begin{aligned} \operatorname{Ran}(Q) &\subseteq \ker(P) \\ &= \operatorname{Ran}(P^*)^{\perp} \\ &= \operatorname{Ran}(P)^{\perp}, \end{aligned}$$

meaning $\operatorname{Ran}(Q) \perp \operatorname{Ran}(P)$. Conversely, if $\operatorname{Ran}(Q) \perp \operatorname{Ran}(P)$, then

$$\begin{aligned} \ker(P) &= \operatorname{Ran}(P^*)^{\perp} \\ &= \operatorname{Ran}(P)^{\perp} \\ &\supseteq \operatorname{Ran}(Q), \end{aligned}$$

so $PQ = 0$. ◻

Definition. A family of projections $\{P_i\}_{i\in I}$ in $\mathcal{B}(\mathcal{H})$ is said to be pairwise orthogonal if $P_iP_j = 0$ for $i \neq j$.

Projections are pairwise orthogonal if their ranges are pairwise orthogonal subspaces of $\mathcal{H}$.

Lemma: Let $P_1, \dots, P_n$ be pairwise orthogonal nonzero projections in $\mathcal{B}(\mathcal{H})$. Let $\lambda_1, \dots, \lambda_n$ be scalars. Then,

$$\left\|\sum_{j=1}^{n} \lambda_j P_j\right\|_{op} = \max_{j=1}^{n} |\lambda_j|.$$

Proof. Using the Pythagorean theorem, we get

$$\begin{aligned} \left\|\left(\sum_{j=1}^{n} \lambda_j P_j\right)(\xi)\right\|^2 &= \left\|\sum_{j=1}^{n} \lambda_j P_j(\xi)\right\|^2 \\ &= \sum_{j=1}^{n} \|\lambda_j P_j(\xi)\|^2 \\ &= \sum_{j=1}^{n} |\lambda_j|^2 \|P_j(\xi)\|^2 \\ &\leqslant \max_{j=1}^{n}\left(|\lambda_j|^2\right) \sum_{j=1}^{n} \|P_j(\xi)\|^2 \\ &= \left(\max_{j=1}^{n} |\lambda_j|\right)^2 \left\|\sum_{j=1}^{n} P_j(\xi)\right\|^2 \\ &\leqslant \left(\max_{j=1}^{n} |\lambda_j|\right)^2 \|\xi\|^2, \end{aligned}$$

meaning

$$\left\|\sum_{j=1}^{n}\lambda_j P_j\right\|_{op} \leqslant \max_{j=1}^{n}\left|\lambda_j\right|.$$

In the other direction, if e_i is a unit vector in $\operatorname{Ran}(P_i)$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\lambda_j P_j\right\|_{op} &\geqslant \left\|\left(\sum_{j=1}^{n}\lambda_j P_j\right)(e_i)\right\| \\ &= \|\lambda_i e_i\| \\ &= |\lambda_i|,\end{aligned}$$

meaning

$$\max_{j=1}^{n}\left|\lambda_j\right| \leqslant \left\|\sum_{j=1}^{n}\lambda_j P_j\right\|_{op}.$$

□

We know that $\mathcal{P}(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{B}(\mathcal{H})_+ \subseteq \mathcal{B}(\mathcal{H})_{\text{s.a.}}$, so we are now interested in understanding the ordering on $\mathcal{P}(\mathcal{B}(\mathcal{H}))$.

Proposition: Let $\mathcal{H}$ be a Hilbert space, and let $P, Q \in \mathcal{P}(\mathcal{B}(\mathcal{H}))$. The following are equivalent.

(i) $P \leqslant Q$;

(ii) $PQ = P$;

(iii) $QP = P$;

(iv) $\operatorname{Ran}(P) \subseteq \operatorname{Ran}(Q)$.

(v) $\|P(x)\| \leqslant \|Q(x)\|$ for all $x \in \mathcal{H}$.

Proof. To show that (ii) is true if and only if (iii) is true, we see that

$$\begin{aligned}P &= P^* \\ &= (PQ)^* \\ &= Q^*P^* \\ &= QP.\end{aligned}$$

We see (iii) is true if and only if (iv) is true follows from the definition of $QP = P$.

To see that (iv) implies (i), suppose $\operatorname{Ran}(P) \subseteq \operatorname{Ran}(Q)$, and set $M = \operatorname{Ran}(P)$. If $x \in \mathcal{H}$, we write $x = y + z$ via the decomposition $\mathcal{H} = M \oplus M^{\perp}$. We know that $\langle P(x), x\rangle = \langle y, y\rangle = \|y\|^2$, so

$$\begin{aligned}\langle Q(x), x\rangle &= \langle Q(y) + Q(z), y + z\rangle \\ &= \langle y + Q(z), y + z\rangle \\ &= \|y\|^2 + \langle Q(z), y\rangle + \langle Q(z), z\rangle \\ &= \|y\|^2 + \langle z, Q(y)\rangle + \langle Q(z), z\rangle \\ &= \|y\|^2 + \langle Q(z), z\rangle \\ &\geqslant \langle P(x), x\rangle,\end{aligned}$$

where we use the fact that $Q(y) = y$ and $\langle z, y\rangle = 0$. Thus, we see that $0 \leqslant P \leqslant Q$.

To see that (ii) implies (i), we know that since $0 \leqslant P \leqslant I$, and multiplying by Q preserves order, we have $0 \leqslant QPQ \leqslant QIQ$, so $0 \leqslant P \leqslant Q$.

To see that (i) implies (iv), if $Q(x) = -$, then $\langle Q(x), x\rangle = 0$, so $0 \leqslant \langle P(x), x\rangle \leqslant \langle Q(x), x\rangle = 0$, so $\|P(x)\|^2 = \langle P(x), x\rangle = 0$, so $P(x) = 0$. Thus, we have $\ker(Q) \subseteq \ker(P)$, so $\operatorname{Ran}(P) \subseteq \operatorname{Ran}(Q)$.

To see that (i) is true if and only if (v) is true, we recall that $\|P(x)\|^2 = \langle P(x), x\rangle$ and $\|Q(x)\|^2 = \langle Q(x), x\rangle$. □

Given a Hilbert space, the collection of all closed subspaces $\mathcal{M} = \left\{M \mid M \subseteq H \text{ is a closed subspace}\right\}$ is a complete lattice under inclusion. For $\{M_i\}_{i\in I}$ a collection of closed subspaces, we have an infimum of $\bigcap_{i\in I} M_i$ and a supremum of $\overline{\sum_{i\in I} M_i}$.

We can extend this lattice to the projections in $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$.

Proposition: Let $\mathcal{H}$ be a Hilbert space. The set of projections $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ with the induced ordering from $\mathcal{B}(\mathcal{H})_{\text{s.a.}}$ forms a complete lattice. The map

$$\mathcal{M} \to \mathcal{P}(\mathcal{B}(\mathcal{H})),$$

defined by $M \mapsto P_M$, is a lattice isomorphism.

Proof. We know that every $P \in \mathcal{P}(\mathcal{B}(\mathcal{H}))$ is of the form P_M, where $M = \operatorname{Ran}(P)$ is a closed subspace. It follows that $P \mapsto \operatorname{Ran}(P)$ gives the inverse of the map $M \mapsto P_M$.

If $M \subseteq N$, then we have proven that $P_M \leqslant P_N$ previously.

For M and N closed subspaces, we will show that $P_{M\wedge N} = P_M \wedge P_N$ and $P_{M\vee N} = P_M \vee P_N$, where

$$\begin{aligned} M \wedge N &= M \cap N \\ M \vee N &= \overline{M + N}. \end{aligned}$$

Since $M \cap N \subseteq M, N$, it follows that $P_{M\cap N} \leqslant P_M, P_N$. If $Q \leqslant P_M, P_N$, it follows that $\operatorname{Ran}(Q) \subseteq M, N$, so $\operatorname{Ran}(Q) \subseteq M \cap N$, so $Q \leqslant P_{M\cap N}$. Thus, $P_{M\wedge N} = P_M \wedge P_N$.

Similarly, since $M, N \subseteq \overline{M + N}$, it follows that $P_M, P_N \subseteq P_{\overline{M+N}}$. For Q with $P_M, P_N \leqslant Q$, we must have $M + N \subseteq \operatorname{Ran}(Q)$. Since $\operatorname{Ran}(Q)$ is closed, we must also have $\overline{M + N} \subseteq \operatorname{Ran}(Q)$, so $P_{\overline{M+N}} \leqslant Q$, meaning $P_{M\vee N} = P_M \vee P_N$. □

An operator can often be decomposed into smaller components acting on orthogonal pieces of the Hilbert space.

Definition. Let $\mathcal{H}$ be a Hilbert space, and suppose $T \in \mathcal{B}(\mathcal{H})$. A closed subspace $M \subseteq \mathcal{H}$ is said to be

(1) invariant for T if $T(M) \subseteq M$;

(2) reducing for T if $T(M) \subseteq M$ and $T(M^{\perp}) \subseteq M^{\perp}$.

We can prove some simple properties of invariant and reducing subspaces.

Lemma: Let $M \subseteq \mathcal{H}$ be a closed subspace of a Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$.

(1) M is invariant for T if and only if $M^{\perp}$ is invariant for T^*.

(2) M is invariant for T^* if and only if $M^{\perp}$ is invariant for T.

(3) M is reducing for T if and only if M is invariant for T and T^*.

(4) M reduces T if and only if M reduces T^*.

Proof. To see (1), if $T(M) \subseteq M$, then for $x \in M$ and $y \in M^{\perp}$, we have $\langle T(x), y\rangle = \langle x, T^*(y)\rangle = 0$, so $T^*(M^{\perp}) \subseteq M^{\perp}$. If $M^{\perp}$ is invariant for T^*, then for $y \in M^{\perp}$ and $x \in M$, we have $\langle x, T^*(y)\rangle = \langle T(x), y\rangle = 0$, so T is invariant for $(M^{\perp})^{\perp} = M$.

To see (2), we replace T with T^* in the proof of (1).

It is the case that (3) and (4) follow from (1) and (2). □

Definition. Let $M \subseteq \mathcal{H}$ reduce a bounded operator $T \in \mathcal{B}(\mathcal{H})$. We write

$$\begin{aligned} T_M &= T|_M \\ T_{M^{\perp}} &= T|_{M^{\perp}} \end{aligned}$$

to be the restrictions of T to M and $M^{\perp}$.

Exercise: Let $M \subseteq \mathcal{H}$ be a closed subspace.

(a) If M reduces $T, S \in \mathcal{B}(\mathcal{H})$, show that M also reduces T^*, TS, and $T + \alpha S$ for any $\alpha \in \mathbb{C}$. Moreover, show that

$$\begin{aligned} (T+\alpha S)_M &= T_M + \alpha S_M \\ (TS)_M &= T_M S_M \\ (T^*)_M &= (T_M)^*. \end{aligned}$$

(b) Suppose M reduces a net of bounded operators $(T_\alpha)_\alpha$. If $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$, show that M reduces T as well.

(c) Conclude that

$$R_M = \{T \in \mathcal{B}(\mathcal{H}) \mid M \text{ reduces } T\}$$

is a von Neumann algebra.

Solution:

(a) Since M reduces T, M is invariant for T and T^*, so $T^*\left(M^{\perp}\right) \subseteq M^{\perp}$, meaning $T^*(M) \subseteq M$.

Similarly, since M reduces T and M reduces S, we have

$$\begin{aligned} TS(M) &\subseteq T(M) \\ &\subseteq M. \end{aligned}$$

Similarly, $TS\left(M^{\perp}\right) \subseteq M^{\perp}$.

(b) Let M reduce a net of bounded operators $(T_\alpha)_\alpha$ with $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$. Then, we have, for all $\eta \in M^{\perp}$ and $\xi \in M$,

$$\begin{aligned} 0 &= \langle T_\alpha(\xi), \eta\rangle \\ &\to \langle T(\xi), \eta\rangle, \end{aligned}$$

so $T(\xi) \in M$. Thus, $T(M) \subseteq M$, so M is invariant for T. Similarly, we also find $M^{\perp}$ is invariant for T, so M is reducing for T.

(c) We see that the collection R_M is a WOT-closed subalgebra of $\mathcal{B}(\mathcal{H})$, so R_M is a von Neumann algebra.

Definition. Let $B \subseteq \mathcal{B}(\mathcal{H})$ be a collection of bounded operators on a Hilbert space $\mathcal{H}$. A closed subspace $M \subseteq \mathcal{H}$ is said to be invariant/reducing for B if M is invariant/reducing for every $b \in B$.

Definition. Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$ a set of vectors, and let $B \subseteq \mathcal{B}(\mathcal{H})$ be a collection of bounded operators. Set

$$[BS] = \overline{\operatorname{span}}\{b(\eta) \mid b \in B, \eta \in S\}.$$

If $S = \{\xi\}$, then we write $[B\xi]$. If $B \subseteq \mathcal{B}(\mathcal{H})$ is a subspace, and $\xi \in \mathcal{H}$, then

$$[B\xi] = \overline{\{b(\xi) \mid b \in B\}}.$$

Exercise: If $B \subseteq \mathcal{B}(\mathcal{H})$ is a subalgebra of bounded operators on a Hilbert space $\mathcal{H}$ and $S \subseteq \mathcal{H}$ is a collection of vectors, prove that $[BS]$ is invariant for B. If B is a $*$-subalgebra, prove that $[BS]$ is reducing for B.

Solution: Let B be a subalgebra. Then, for any $\xi \in [BS]$, we see that $b(\xi) \in [BS]$ for every $b \in B$, meaning $b([BS]) \subseteq [BS]$ for each $b \in B$, so $[BS]$ is invariant.

Similarly, for a $*$-subalgebra B, since $b([BS]) \subseteq [BS]$, and $b^*([BS]) \subseteq [BS]$ for each $b \in B$, it is the case that $[BS]$ is reducing for B.

We can understand invariant and reducing subspaces through purely algebraic expressions as well.

Lemma: Let $\mathcal{H}$ be a Hilbert space, and suppose $M \subseteq \mathcal{H}$ is a closed subspace with orthogonal projection $P = P_M$.

(1) M is invariant for $T \in \mathcal{B}(H)$ if and only if $PTP = TP$.

(2) M is reducing for $T \in \mathcal{B}(H)$ if and only if $PT = TP$.

(3) M is reducing for the family $B \subseteq \mathcal{B}(\mathcal{H})$ if and only if $P \in B'$, where B' is the commutant of B.

Proof.

(1) If $PTP = TP$, then $T(M) = TP(M) = PTP(M) \subseteq \operatorname{Ran}(P) = M$. Conversely, if M is invariant for T, then $TP(x) \in M$ for any $x \in \mathcal{H}$, so $PTP(x) = x$.

(2) We know that M and $M^{\perp}$ are invariant for T, so we have $PTP = TP$ and $(I-P)T(I-P) = T(I-P)$, meaning $TP = PT$.

(3) Follows from (2).

□

We can now understand the characterization of partial isometries, which are (effectively) a combination of a projection and isometry.

Proposition: Let $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator between Hilbert spaces, and let $L = \ker(T)^{\perp}$, $M = \operatorname{Ran}(T)$. The following are equivalent:

(i) T^*T is the orthogonal projection onto L;

(ii) TT^* is the orthogonal projection onto M, which is a closed subspace;

(iii) $T|_L : L \to K$ is an isometry;

(iv) $TT^*T = T$;

(v) $T^*TT^* = T^*$.

All of these cases imply that $T|_L : L \to M$ is an isometric isomorphism of Hilbert spaces.

Proof. The equivalence between (iv) and (v) follows from adjoints.

To show that (v) implies (i), we see that T^*T is self-adjoint, and that

$$\begin{aligned}(T^*T)(T^*T) &= (T^*TT^*)T \\ &= T^*T,\end{aligned}$$

so T^*T is an idempotent. Thus, T^*T is the orthogonal projection onto $\operatorname{Ran}(T^*T)$, which is close since T^*T is idempotent. We also see that

$$\begin{aligned}\operatorname{Ran}(T^*T) &\subseteq \operatorname{Ran}(T^*) \\ &\subseteq \overline{\operatorname{Ran}(T^*)}\end{aligned}$$

$$\begin{aligned} &= \ker(T)^{\perp} \\ &= L. \end{aligned}$$

Since $T^*TT^* = T^*$, we get $\operatorname{Ran}(T^*) \subseteq \operatorname{Ran}(T^*T)$. Taking closures, we get

$$\begin{aligned} L &= \ker(T)^{\perp} \\ &= \overline{\operatorname{Ran}(T^*)} \\ &\subseteq \operatorname{Ran}(T^*T), \end{aligned}$$

meaning T^*T is the orthogonal projection onto L.

To see the implication (ii) to (v), we see that if $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfies $S^*S = 0$, then $S = 0$ by the C^* identity. We will show that $T^*TT^* - T^* = 0$, by seeing

$$\begin{aligned} (T^*TT^* - T^*)^* (T^*TT^* - T^*) &= (TT^*T - T)(T^*TT^* - T^*) \\ &= TT^*TT^*TT^* - TT^*TT^* - TT^*TT^* + TT^* \\ &= TT^*TT^* - TT^* - TT^* + TT^* \\ &= 0. \end{aligned}$$

Similarly, the implication (i) to (iv) follows from replacing TT^* with T^*T.

The implication (iv) to (ii) follows from the fact that TT^* is a self-adjoint idempotent with closed range, meaning $\operatorname{Ran}(TT^*) = \operatorname{Ran}(T) = M$.

To see the implication (i) to (iii), we see that if $x \in L$ and $T^*T = P_L$, then

$$\begin{aligned} \|T(x)\|^2 &= \langle T(x), T(x) \rangle \\ &= \langle T^*T(x), x \rangle \\ &= \langle x, x \rangle \\ &= \|x\|^2, \end{aligned}$$

meaning $T|_L : L \to \mathcal{K}$ is an isometry.

If $T|_L : L \to \mathcal{K}$ is an isometry, we know that $\langle T(x), T(z') \rangle = \langle z, z' \rangle$ for all $z, z' \in L$. Let $x \in \mathcal{H}$ be arbitrary. Note that $T^*T(x) \in \operatorname{Ran}(T^*) \subseteq \ker(T)^{\perp}$, so we write $x = y + z$ by the decomposition $\mathcal{H} = \ker(T) \oplus L$. We have $T^*T(x) = T^*T(z)$. For any $z' \in L$, we compute

$$\begin{aligned} \langle x - T^*T(x), z' \rangle &= \langle z - T^*T(z), z' \rangle \\ &= \langle z, z' \rangle - \langle T^*T(z), z' \rangle \\ &= \langle z, z' \rangle - \langle T(z), T(z') \rangle \\ &= 0. \end{aligned}$$

This means $x - T^*T(x) \in L^{\perp}$. The decomposition

$$x = TT^*(x) + (x - TT^*(x))$$

implies $TT^*(x) = P_L(x)$. Since x was arbitrary, we have $TT^* = P_L$. □

Definition. An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying one of the above conditions is known as a partial isometry with initial projection T^*T and final projection TT^*.

Exercise: Let x and y be unit vectors in $\mathcal{H}$. Prove that the rank-one operator $\theta_{x,y}$ is a partial isometry with initial projection $\theta_{y,y}$ and final projection $\theta_{x,x}$.

Partial isometries and unitaries are primarily useful to compare projections, which aid in the classification of von Neumann algebras.

Exercise: Consider the following relations on the set of projections $\mathcal{B}(\mathcal{H})$.

(a) For $P, Q \in \mathcal{P}(\mathcal{B}(\mathcal{H}))$, we define $P \sim Q$ if there exists $V \in \mathcal{B}(\mathcal{H})$ with $V^*V = P$ and $VV^* = Q$. This is known as Murray–von Neumann equivalence. Prove that this defines an equivalence relation on $\mathcal{P}(\mathcal{B}(\mathcal{H}))$.

(b) For $P, Q \in \mathcal{B}(\mathcal{H})$, we set $P \sim_{\mathcal{U}} Q$ if there exists $U \in \mathcal{U}(\mathcal{B}(\mathcal{H}))$ such that $UPU^* = Q$. This is known as unitary equivalence. Prove that $\sim_{\mathcal{U}}$ defines an equivalence relation on $\mathcal{P}(\mathcal{B}(\mathcal{H}))$.

(c) Prove that $P \sim_{\mathcal{U}} Q \Rightarrow P \sim Q$.

Solution: (a) Reflexivity follows from $P^* = P$. Symmetry follows from the fact that $U = V^*$ yields $U^*U = Q$ and $UU^* = P$. Transitivity follows from composition.

(b) Reflexivity follows from $I \in \mathcal{U}(\mathcal{B}(\mathcal{H}))$, symmetry from $U \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \Rightarrow U^* \in \mathcal{U}(\mathcal{B}(\mathcal{H}))$, and transitivity from $U, V \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \Rightarrow UV \in \mathcal{U}(\mathcal{B}(\mathcal{H}))$.

(c) Setting $V = UP$ in the unitary equivalence, we find that

$$\begin{aligned} V^*V &= (UP)^*(UP) \\ &= P^*U^*UP \\ &= P^*P \\ &= P \\ VV^* &= (UP)(UP)^* \\ &= UPP^*U^* \\ &= UP^2U^* \\ &= UPU^* \\ &= Q. \end{aligned}$$

Exercise: Let $\mathcal{H}$ be a Hilbert space. Prove the following.

(a) $P \sim Q$ if and only if $\operatorname{rank}(P) = \operatorname{rank}(Q)$.

(b) $P \sim_{\mathcal{U}} Q$ if and only if $\operatorname{Ran}(P) = \operatorname{rank}(Q)$ and $\operatorname{rank}(I - P) = \operatorname{rank}(I - Q)$.

Solution:

(a) Let $P \sim Q$. Then, there exists $V \in \mathcal{B}(\mathcal{H})$ such that $V^*V = P$ and $VV^* = Q$. Notice that for $x \in \mathcal{H}$, we have

$$\begin{aligned} \|VP(x)\|^2 &= \langle VP(x), VP(x)\rangle \\ &= \langle VV^*V(x), VV^*V(x)\rangle \\ &= \langle V(x), V(x)\rangle \\ &= \langle V^*V(x), x\rangle \\ &= \langle P(x), x\rangle \\ &= \langle P(x), P(x)\rangle \\ &= \|P(x)\|^2. \end{aligned}$$

By definition, we must have V is the partial isometry with initial projection P and final projection Q. In particular, this means $V|_{\operatorname{Ran}(P)}\colon \operatorname{Ran}(P) \to \operatorname{Ran}(Q)$ is an isometric isomorphism, meaning that $\operatorname{hdim}(\operatorname{Ran}(P)) = \operatorname{hdim}(\operatorname{Ran}(Q))$, so $\operatorname{rank}(P) = \operatorname{rank}(Q)$.

If $\operatorname{rank}(P) = \operatorname{rank}(Q)$, then there exists an isometric isomorphism $T\colon \operatorname{Ran}(P) \to \operatorname{Ran}(Q)$. Defining V to be equal to T on $\operatorname{Ran}(P)$ and 0 elsewhere, we then have $V^*V = P$ and $VV^* = Q$.

(b) If $P \sim_{\mathcal{U}} Q$, then P and Q are Murray–von Neumann equivalent, so $\operatorname{rank}(P) = \operatorname{rank}(Q)$. Similarly, we have

$$\begin{aligned} U^*(I - P)U &= I - U^*PU \\ &= I - Q, \end{aligned}$$

so $I - P$ and $I - Q$ are Murray–von Neumann equivalent, so $\operatorname{rank}(I - P) = \operatorname{rank}(I - Q)$.

If $\operatorname{rank}(P) = \operatorname{rank}(Q)$, then there exists $V \in \mathcal{B}(\mathcal{H})$ such that $V^*V = P$ and $VV^* = Q$. Similarly, there exists $T \in \mathcal{B}(\mathcal{H})$ such that $T^*T = I - P$ and $TT^* = I - Q$.

Fact. If P and Q are projections in $\mathcal{B}(\mathcal{H})$ with $\|P-Q\|_{op} < 1$, then P and Q are unitarily equivalent.

Proof. Suppose $0 \neq \xi \in \operatorname{Ran}(P)$. If $Q(\xi) = 0$, then

$$\begin{aligned}
\|\xi\| &= \|P(\xi)\| \\
&= \|P(\xi) - Q(\xi)\| \\
&= \|(P-Q)(\xi)\| \\
&\leqslant \|P-Q\|_{op}\|\xi\| \\
&< \xi.
\end{aligned}$$

Thus, $Q(\xi) \neq 0$, meaning $Q|_{\operatorname{Ran}(P)}\colon \operatorname{Ran}(P) \to \operatorname{Ran}(Q)$ is injective. It follows that $\operatorname{rank}(P) \leqslant \operatorname{rank}(Q)$. Similarly, we must have $\operatorname{rank}(Q) \leqslant \operatorname{rank}(P)$, meaning $\operatorname{rank}(Q) = \operatorname{rank}(P)$.

Note that $\|P^{\perp} - Q^{\perp}\|_{op} = \|Q-P\|_{op} < 1$, meaning $\operatorname{rank}(P^{\perp}) = \operatorname{rank}(Q^{\perp})$. Thus, $P \sim_u Q$. □

Direct Sums and Diagonalizable Operators

To move towards constructing the matrix algebra $\operatorname{Mat}_n(\mathcal{B}(\mathcal{H}))$ and prove the Double Commutant theorem, we begin by constructing the matrix representation of an operator.

Proposition: Let $\{\mathcal{H}_i\}_{i\in I}$ be a family of Hilbert spaces, and suppose for each $i \in I$, we have a bounded operator $T_i \in \mathcal{B}(\mathcal{H}_i)$. If $\sup_{i\in I}\|T_i\|_{op} < \infty$, then $T\colon \bigoplus_{i\in I}\mathcal{H}_i \to \bigoplus_{i\in I}\mathcal{H}_i$, defined by

$$T((\xi_i)_i) = (T_i(\xi_i))_i$$

is a well-defined bounded linear operator with $\|T\|_{op} = \sup_{i\in I}\|T_i\|_{op}$.

Proof. For $\xi = (\xi_i)_i$ in $\bigoplus_{i\in I}\mathcal{H}_i$, we compute

$$\begin{aligned}
\|T(\xi)\|^2 &= \|(T_i(\xi_i))_i\|^2 \\
&= \sum_{i\in I}\|T_i(\xi_i)\|^2 \\
&\leqslant \sum_{i\in I}\|T_i\|_{op}^2\|\xi_i\|^2 \\
&\leqslant \sup_{i\in I}\|T_i\|_{op}^2\sum_{i\in I}\|\xi_i\|^2 \\
&= \left(\sup_{i\in I}\|T_i\|_{op}\right)^2\|\xi\|^2,
\end{aligned}$$

meaning T is well-defined and $\|T\|_{op} \leqslant \sup_{i\in I}\|T_i\|_{op}$. Since T restricts to T_i on each $\mathcal{H}_i$, it follows that $\|T\|_{op} \geqslant \|T_i\|_{op}$ for every $i \in I$. □

Definition. We call T as defined above the external direct sum of the family $\{T_i\}_{i\in I}$. We write $T = \bigoplus_{i\in I} T_i$.

If $\mathcal{H}_i = \mathcal{H}$ and $T_i = T$ for all $i \in I$, we write $\bigoplus_{i\in I} T$ acting on $\bigoplus_{i\in I}\mathcal{H}$ as the I-fold amplification of T.

We write $T^{(\infty)}$ for $\bigoplus_{n\in\mathbb{N}} T$, and $T^{(n)}$ for $\bigoplus_{k=1}^{n} T$.

Exercise: Suppose $\bigoplus_{i\in I} T_i$ and $\bigoplus_{i\in I} S_i$ are external direct sums acting on $\bigoplus_{i\in I}\mathcal{H}_i$, and let α be a scalar. Prove the following algebraic properties.

(a) $\alpha\left(\bigoplus_{i\in I} T_i\right) + \bigoplus_{i\in I} S_i = \bigoplus_{i\in I}(\alpha T_i + S_i)$.

(b) $\left(\bigoplus_{i\in I} T_i\right) \circ \left(\bigoplus_{i\in I} S_i\right) = \bigoplus_{i\in I} (T_i S_i)$.

(c) $\left(\bigoplus_{i\in I} T_i\right)^* = \bigoplus_{i\in I} T_i^*$.

(d) $\left(\bigoplus_{i\in I} T_i\right)^n = \bigoplus_{i\in I} T_i^n$.

Solution: Define

$$\begin{aligned} T &= \bigoplus_{i\in I} T_i \\ S &= \bigoplus_{i\in I} S_i. \end{aligned}$$

(a)

$$\begin{aligned} (\alpha T + S)\left((\xi_i)_i\right) &= \alpha T\left((\xi_i)_i\right) + S\left((\xi_i)_i\right) \\ &= \alpha\left(T_i\left(\xi_i\right)\right)_i + \left(S_i\left(\xi_i\right)\right)_i \\ &= \left((\alpha T_i + S_i)\left(\xi_i\right)\right)_i \\ &= \left(\bigoplus_{i\in I} (\alpha T_i + S_i)\right)(\xi_i)_i \end{aligned}$$

(b)

$$\begin{aligned} TS\left((\xi_i)_i\right) &= T\left((S_i\left(\xi_i\right))_i\right) \\ &= \left(T_i S_i\left(\xi_i\right)\right)_i \\ &= \left(\bigoplus_{i\in I} T_i S_i\right)(\xi_i)_i. \end{aligned}$$

(c)

$$\begin{aligned} T^*\left((\xi_i)_i\right) &= \left(T_i^*\left(\xi_i\right)\right)_i \\ &= \left(\bigoplus_{i\in I} T_i^*\right)(\xi_i)_i. \end{aligned}$$

(d)

$$\begin{aligned} T^n\left((\xi_i)_i\right) &= T^{n-1}\left((T_i\left(\xi_i\right))_i\right) \\ &\vdots \\ &= \left(T_i^n\left(\xi_i\right)\right)_i \\ &= \left(\bigoplus_{i\in I} T_i^n\right)(\xi_i)_i. \end{aligned}$$

We can start by looking at the direct sum of multiplication operators, which will lead us toward a better understanding of block-diagonal and diagonal operators.

Example. Let $\{(\Omega_n, \mathcal{M}_n, \mu_n)\}_n$ be a countable family of measure spaces with disjoint union $(\Sigma, \mathcal{M}, \mu)$. For every $n \geqslant 1$, we let $f_n \in L_\infty\left(\Omega_n, \mu_n\right)$, and suppose $\sup_{n\geqslant 1} \|f_n\|_\infty < \infty$.

We can speak of the external direct sum operator

$$T = \bigoplus_{n\geqslant 1} M_{f_n}$$

acting on $\bigoplus_{n\geqslant 1} L_2(\Omega_n, \mu_n)$.

Let

$$V : L_2(\Sigma, \mu) \to \bigoplus_{n\geqslant 1} L_2(\Omega_n, \mu_n)$$

be defined by

$$V(\xi) = (\xi \circ \iota_n)_n .$$

We claim that the unitarily equivalent V^*TV on $L_2(\Sigma, \mu)$ is equivalent to M_f on $f = \bigsqcup_n f_n$.

We see that for $\xi \in L_2(\Sigma, \mu)$, we have

$$\begin{aligned} TV(\xi) &= T((\xi \circ \iota_n)_n) \\ &= (M_{f_n}(\xi \circ \iota_n))_n \\ &= (f_n(\xi \circ \iota_n))_n , \end{aligned}$$

and

$$\begin{aligned} VM_f(\xi) &= V(f\xi) \\ &= (f\xi \circ \iota_n)_n . \end{aligned}$$

Thus, for any $x \in \Omega_n$, we have

$$\begin{aligned} f\xi \circ \iota_n(x) &= f\xi(x, n) \\ &= f(x, n)\,\xi(x, n) \\ &= f_n(x)(\xi \circ \iota_n)(x) \\ &= f_n(\xi \circ \iota_n)(x), \end{aligned}$$

so

$$f\xi \circ \iota_n = f_n(\xi \circ \iota_n).$$

This holds for every n, so $TV = VM_f$ as claimed.

A Hilbert space with $\operatorname{hdim}(\mathcal{H}) \geqslant 2$ admits infinitely many internal direct sum decompositions. For instance, if $(e_i)_{i\in I}$ is an orthonormal basis, then $\mathcal{H}$ is the internal direct sum of $\left\{\operatorname{span}(e_i)\right\}_{i\in I}$.

In general, if $\mathcal{H}$ is the internal direct sum of a family $\mathcal{M} = \{M_i\}_{i\in I}$, any operator $T \in \mathcal{B}(\mathcal{H})$ can be represented and viewed as a matrix $[T]_{\mathcal{M}}$ acting on the external direct sum $\bigoplus_{i\in I} M_i$.

Recall that if $\mathcal{H} = \bigoplus_{i\in I}$ is an internal direct sum, then $\mathcal{H}$ is unitarily isomorphic to the *external* direct sum $\bigoplus_{i\in I} M_i$ through the operator

$$U((\xi_i)_i) = \sum_{i\in I} \xi_i.$$

Proposition: Let $\mathcal{H}$ be the internal direct sum of the family $\mathcal{M} = \{M_i\}_{i\in I}$. Let P_i be the orthogonal projection onto M_i, and $T \in \mathcal{B}(\mathcal{H})$ be any operator. Set $T_{ij} = P_i T P_j$ for each $i, j \in I$.

(1) The map

$$[T]_{\mathcal{M}} : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} M_i,$$

defined by

$$[T]_{\mathcal{M}}\left((\xi_i)_i\right) = \left(\sum_{j\in I} T_{ij}\left(\xi_j\right)\right)_i$$

defines a bounded operator on the external direct sum $\bigoplus_{i\in I} M_i$ with $\|[T]_{\mathcal{M}}\|_{op} \leqslant \|T\|_{op}$

(2) The operators T and $[T]_{\mathcal{M}}$ are unitarily equivalent via U as defined above. Thus, we have $\|[T]_{\mathcal{M}}\|_{op} = \|T\|_{op}$.

(3) The map $T \mapsto [T]_{\mathcal{M}}$ is an isometric $*$-isomorphism of C^*-algebras called the matrix representation of T with respect to $\mathcal{M}$.

Proof. We will write $[T] := [T]_{\mathcal{M}}$.

(1) If $(\xi_i)_{i\in I}$ belongs to $\bigoplus_{i\in I} M_i$, then $\sum_{j\in I}\left\|\xi_j\right\|^2$ is summable by definition, so $\sum_{j\in I}\xi_j$ is summable in $\mathcal{H}$. If we fix $i \in I$, using continuity of P_jT and the fact that $P_j\left(\xi_j\right) = \xi_j$ for each j, we have

$$\begin{aligned}
P_i\left(\sum_{j\in I}\xi_j\right) &= \sum_{j\in I} P_iT\left(\xi_j\right)\\
&= \sum_{j\in I} P_iTP_j\left(\xi_j\right)\\
&= \sum_{j\in I} T_{ij}\left(\xi_j\right)
\end{aligned}$$

is summable in $\mathcal{H}$. Additionally, $\sum_{i\in I}\|P_i(\eta)\|^2 = \|\eta\|^2$ for every $\eta \in \mathcal{H}$, so we have

$$\begin{aligned}
\sum_{i\in I}\left\|\sum_{j\in I} T_{ij}\left(\xi_j\right)\right\|^2 &= \sum_{i\in I}\left\|\sum_{j\in I} P_iTP_j\left(\xi_j\right)\right\|^2\\
&= \sum_{i\in I}\left\|P_i\left(\sum_{j\in I} T\left(\xi_j\right)\right)\right\|^2\\
&= \left\|\sum_{i\in I} T\left(\xi_j\right)\right\|^2\\
&= \left\|T\left(\sum_{j\in I}\xi_j\right)\right\|^2\\
&\leqslant \|T\|_{op}^2\left\|\sum_{j\in I}\xi_j\right\|^2\\
&= \|T\|_{op}^2\sum_{j\in I}\left\|\xi_j\right\|^2\\
&= \|T\|_{op}^2\left\|(\xi_i)_i\right\|^2,
\end{aligned}$$

showing that $[T]$ is linear and $\|[T]\|_{op} \leqslant \|T\|_{op}$.

(2) We will show that $U[T] = TU$. Let $\xi = (\xi_i)_i \in \bigoplus_{i\in I} M_i$. Then,

$$TU\left(\left(\xi_j\right)_j\right) = T\left(\sum_{j\in I}\xi_j\right)$$

$$\begin{aligned}
&= \sum_{j\in I} TP_j\left(\xi_j\right)\\
&= \sum_{i} P_i\left(\sum_{j\in I} T\left(\xi_j\right)\right)\\
&= \sum_{i\in I}\left(\sum_{j\in I} P_iTP_j\left(\xi_j\right)\right)\\
&= \sum_{i\in I}\left(\sum_{j\in I} T_{ij}\left(\xi_j\right)\right)\\
&= U\left(\left(\sum_{j\in I} T_{ij}\left(\xi_j\right)\right)_i\right)\\
&= U\left[T\right]\left(\left(\xi_j\right)_j\right),
\end{aligned}$$

where we used the fact that $\eta = \sum_{i\in I} P_i\left(\eta\right)$ to move from the second to the third line. Thus, since unitary equivalence preserves norm, $[T]$ and T have the same norm.

(3) We have shown that conjugation is an isometric $*$-isomorphism.

$\square$

Remark: We will write $[T]_{\mathcal{M}}$ as

$$[T]_{\mathcal{M}} = \left(T_{ij}\right)_{ij}$$

to denote the matrix representation of T with respect to the family $\mathcal{M}$.

Proposition: Let $\mathcal{H}$ be an internal direct sum of $\mathcal{M} = \{M_i\}_{i\in I}$. Suppose $T \in \mathcal{B}\left(\mathcal{H}\right)$.

If each M_i is invariant for T, then each M_i reduces T, and the matrix representation of T with respect to $\mathcal{M}$ is the external direct sum

$$[T]_{\mathcal{M}} = \bigoplus_{i\in I} T_{M_i}.$$

Proof. We will start by showing that each M_i reduces T. For a fixed $j \in I$, if $x \in M_j^{\perp}$, then there are $x_i \in M_i$ with

$$x = \sum_{\substack{i\in I\\ i\neq j}} x_i.$$

By continuity, we have

$$T\left(x\right) = \sum_{\substack{i\in I\\ i\neq j}} T\left(x_i\right).$$

Since $T\left(x_i\right) \in M_i$ for each $i \in I$, for any $y \in M_j$, we have

$$\begin{aligned}
\left\langle T\left(x\right), y\right\rangle &= \left\langle \sum_{\substack{i\in I\\ i\neq j}} T\left(x_i\right), y\right\rangle\\
&= \sum_{\substack{i\in I\\ i\neq j}} \left\langle T\left(x_i\right), y\right\rangle
\end{aligned}$$

$$= 0,$$

so $T\left(M_j^{\perp}\right) \subseteq M_j^{\perp}$.

If P_i denotes the orthogonal projection onto M_i, we note that $P_iP_j = 0$ for $i \neq j$ since M_i are mutually orthogonal. Additionally, all the P_i commute with T, so if $i \neq j$, $T_{ij} = P_iTP_j = TP_iP_j = 0$. Thus,

$$\begin{aligned}
[T]_{\mathcal{M}}\left((\xi_i)_i\right) &= \left(\sum_{j\in I} T_{ij}\left(\xi_j\right)\right)_i \\
&= \left(T_{ii}\left(\xi_i\right)\right)_i \\
&= \left(P_iTP_i\left(\xi_i\right)\right)_i \\
&= \left(P_iT\left(\xi_i\right)\right)_i \\
&= \left(TP_i\left(\xi_i\right)\right)_i \\
&= \left(T\left(\xi_i\right)\right)_i \\
&= \left(\sum_{i\in I} T_{M_i}\right)\left((\xi_i)_i\right).
\end{aligned}$$

□

Definition. Let $\mathcal{H}$ be a Hilbert space.

- A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is said to be block-diagonalizable if there exists a family $\mathcal{M} = \{M_i\}_{i\in I}$ of closed subspaces of $\mathcal{H}$ with $\mathcal{H} = \bigoplus_{i\in I} M_i$, with each M_i reducing T. We say T is block-diagonal with respect to $\mathcal{M}$.

  If T is block-diagonal with respect to $\mathcal{M}$, then we write $[T] = \bigoplus_{i\in I} T_{M_i}$. We say T is the internal direct sum and write $T = \bigoplus_{i\in I} T_{M_i}$.
- A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is said to be diagonalizable if it is block-diagonalizable with each M_i having $\dim(M_i) = 1$.

Proposition: Let $\mathcal{H}$ be a Hilbert space, and suppose $M \subseteq \mathcal{H}$ reduces $T \in \mathcal{B}(\mathcal{H})$. Setting $T_M = T|_M$ and $T_{M^\perp} = T|_{M^\perp}$. Then,

$$T = T_M \oplus T_{M^\perp}.$$

Proof. Since M reduces T, both M and $M^\perp$ are invariant for T, so $T = T_M \oplus T_{M^\perp}$. □

Proposition: Let $\mathcal{M} = \{M_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$, with $\mathcal{H} = \sum_{i\in I} M_i$. Write P_i to be the orthogonal projection to be the orthogonal projection onto M_i. For every $T \in \mathcal{B}(\mathcal{H})$, the unconditional sum $\sum_{i\in I} P_iTP_i$ converges in the strong operator topology.

Letting $\mathbb{E}_{\mathcal{M}} \colon \mathcal{B}(\mathcal{H})$ be defined by

$$\mathbb{E}_{\mathcal{M}}(T) = \sum_{i\in I} P_iTP_i,$$

the following are true:

(1) $\mathbb{E}_{\mathcal{M}}$ is linear, positive, contractive, and faithful;

(2)

$$\operatorname{Ran}\left(\mathbb{E}_{\mathcal{M}}\right) = R_{\mathcal{M}} = \left\{T \in \mathcal{B}(\mathcal{H}) \;\middle|\; T \text{ is block diagonal with respect to } \mathcal{M}\right\},$$

and $\mathbb{E}_{\mathcal{M}}(S) = S$ for every $S \in R_{\mathcal{M}}$;

(3) If $T \in \mathcal{B}(\mathcal{H})$ and $S \in R_{\mathcal{M}}$, then

$$\begin{aligned}\mathbb{E}_{\mathcal{M}}(ST) &= S \circ \mathbb{E}_{\mathcal{M}}(T)\\ \mathbb{E}_{\mathcal{M}}(TS) &= \mathbb{E}_{\mathcal{M}}(T) \circ S.\end{aligned}$$

We say $\mathbb{E}_{\mathcal{M}}$ is the conditional expectation onto $R_{\mathcal{M}}$.

Proof.

(1) For a given $\xi \in \mathcal{H}$, we find

$$\begin{aligned}\sum_{i \in I} \|P_i T P_i(\xi)\|^2 &\leqslant \sum_{i \in I} \|T P_i(\xi)\|^2\\ &\leqslant \|T\|_{op}^2 \sum_{i \in I} \|P_i(\xi)\|^2\\ &= \|T\|_{op}^2 \|\xi\|^2.\end{aligned}$$

Thus, $\sum_{i \in I} P_i T P_i(\xi)$ is summable in $\mathcal{H}$, and

$$\begin{aligned}\left\|\sum_{i \in I} P_i T P_i(\xi)\right\|^2 &= \sum_{i \in I} \|P_i T P_i(\xi)\|^2\\ &\leqslant \|T\|_{op}^2 \|\xi\|^2.\end{aligned}$$

The map $S\colon \mathcal{H} \to \mathcal{H}$ defined by

$$S(\xi) = \sum_{i \in I} P_i T P_i(\xi)$$

is well-defined, linear, and bounded with $\|S\|_{op} \leqslant \|T\|_{op}$. It follows that $\mathbb{E}_{\mathcal{M}}$ is contractive, as well as linearity and positivity, as $\mathcal{B}(\mathcal{H})_+$ is SOT-closed.

If $T^*T \geqslant 0$ and $E_{\mathcal{M}}(T^*T) = 0$, then we must have $P_i T^* T P_i = 0$ for every $i \in I$, as if a net of positive operators converges in SOT to 0, every member of the net must be zero. Thus, we have

$$\begin{aligned}\|TP_i\|^2 &= \|(TP_i)^*(TP_i)\|\\ &= \|P_i T^* T P_i\|\\ &= 0,\end{aligned}$$

so $TP_i = 0$ for all $i \in I$. Given $\xi \in \mathcal{H}$, we obtain

$$\begin{aligned}T(\xi) &= T\left(\sum_{i \in I} P_i(\xi)\right)\\ &= \sum_{i \in I} TP_i(\xi)\\ &= 0,\end{aligned}$$

meaning $T = 0$ and $\mathbb{E}_{\mathcal{M}}$ is a faithful map.

(2) Multiplication by a fixed operator is SOT-continuous, so for all $j \in I$, we see that

$$\begin{aligned}P_j \mathbb{E}_{\mathcal{M}}(T) &= E_{\mathcal{M}}(T) P_j\\ &= P_j T P_j,\end{aligned}$$

which means each M_j reduces $\mathbb{E}_{\mathcal{M}}(T)$, so $E_{\mathcal{M}}(T) \in R_{\mathcal{M}}$. If each $M_i$reduces $S \in \mathcal{B}(\mathcal{H})$, then $SP_i = P_iS = P_iSP_i$< so>

$$\begin{aligned}
\mathbb{E}_{\mathcal{M}}(S) &= \sum_{i\in I} P_iSP_i \\
&= \sum_{i\in I} SP_i \\
&= S\left(\sum_{i\in I} P_i\right) \\
&= SI \\
&= S.
\end{aligned}$$

(3) Since we have $SP_i = P_iS = P_iSP_i$, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{M}}(ST) &= \sum_{i\in I} P_iSTP_i \\
&= \sum_{i\in I} SP_iTP_i \\
&= S\left(\sum_{i\in I} P_iTP_i\right) \\
&= S \circ \mathbb{E}_{\mathcal{M}}(T).
\end{aligned}$$

Similarly, $\mathbb{E}_{\mathcal{M}}(TS) = \mathbb{E}_{\mathcal{M}}(T) \circ S$.

□

We can now study the spectral decomposition of a diagonalizable operator.

Proposition: Let $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent:

(i) T is diagonalizable;

(ii) there is an orthonormal basis $(e_i)_{i\in I}$ of $\mathcal{H}$ and a bounded family $(\mu_i)_{i\in I}$ of scalars with $T(e_i) = \mu_i e_i$ for each $i \in I$;

(iii) there is an internal direct sum decomposition, $\mathcal{H} = \bigoplus_{j\in J} E_j$, with $E_j \neq \{0\}$ for all j, and a family $\{\lambda_j\}_{j\in J}$ of distinct scalars such that

$$T = \bigoplus_{j\in J} \lambda_j \operatorname{id}_{E_j}.$$

Moreover, the following are true.

(1) The collection $\{\mu_i\}_{i\in I}$ is a complete list of eigenvalues for T, and $\{\lambda_j\}_{j\in J}$ is a complete list of *distinct* eigenvalues for T.

(2) If $E_{\lambda_j}(T) = \ker(T - \lambda_j I)$ is the eigenspace for λ_j, then $E_j = E_{\lambda_j}$.

(3) If P_j denotes the orthogonal projection onto E_j, then

$$T = \sum_{j\in J} \lambda_j P_j$$

in SOT.

(4) $\|T\|_{op} = \sup_{j\in J} |\lambda_j|$.

Proof. To see (i) implies (ii), let T be diagonalizable. Then, there is an internal direct sum decomposition $\mathcal{H} = \sum_{i\in I} M_i$, with each M_i reducing T, and $\dim(M_i) = 1$ for each i. We assume $M_i = \operatorname{span}(e_i)$, for $e_i \in \mathcal{H}$ and $\|e_i\| = 1$.

Since M_i reduces T, we must have $T(e_i) = \mu_i e_i$ for scalars $\mu_i \in \mathbb{C}$. We see that $|\mu_i| = \|\mu_i e_i\| \leqslant \|T\|_{op}$, so we find our respective families $(e_i)_{i\in I}$ and $(\mu_i)_{i\in I}$.

To see (ii) implies (i), for each $i \in I$ we set $M_i = \operatorname{span}(e_i)$, so $\mathcal{H} = \bigoplus_{i\in I} M_i$, and each M_i is invariant under T. This yields T as diagonalizable with respect to $\{M_i\}_{i\in I}$.

To see (ii) implies (iii), we partition I into $I = \bigsqcup_{j\in J} I_j$, where $i, i' \in I_j$ implies $\mu_i = \mu_{i'}$, and $i \in I_j, i' \in I_{j'}$ with $j \neq j'$ implies $\mu_i \neq \mu_{i'}$. For each $j \in J$, we set

$$E_j = \overline{\operatorname{span}}\left\{e_i \;\middle|\; i \in I_j\right\},$$

and $\lambda_j = \mu_i$ for any $i \in I_j$. It is the case that $\left\{E_j\right\}_{j\in J}$ are mutually orthogonal, and the sum $\sum_{j\in J} E_j$ is dense in $\mathcal{H}$ by the definition of an orthonormal basis. Thus, $\mathcal{H} = \bigoplus_{j\in J} E_j$.

Since $T(e_i) = \mu_i e_i$ for each $i \in I$, linearity and continuity of T imply that $T_{E_j} = \lambda_j \operatorname{id}_{E_j}$. Thus, each E_j is invariant for T, meaning it is reducing, so we have $T = \bigoplus_{j\in J} \lambda_j \operatorname{id}_{E_j}$.

To see (iii) implies (ii), for each $j \in J$, let $\mathcal{E}_j$ be an orthonormal basis for E_j. Then, $\mathcal{E} = \bigoplus_{j\in J} \mathcal{E}_j$ is an orthonormal basis for $\mathcal{H}$. If $e \in \mathcal{E}$, then $e \in \mathcal{E}_j$ for some j, so $T(e) = \lambda_j e_j$.

We now turn our attention towards evaluating facts (1)–(3).

(1) All the μ_i are eigenvalues for T, just by definition. Suppose λ is an eigenvalue for T. That is, there exists $x \neq 0$ with $T(x) = \lambda x$. By Parseval's theorem, we can write $x = \sum_{i\in I} \langle x, e_i\rangle e_i$. Thus,

$$\begin{aligned} T(x) &= T\left(\sum_{i\in I} \langle x, e_i\rangle e_i\right) \\ &= \sum_{i\in I} \langle x, e_i\rangle T(e_i) \\ &= \sum_{i\in I} \langle x, e_i\rangle \mu_i e_i. \end{aligned}$$

We also have

$$\begin{aligned} T(x) &= \lambda x \\ &= \lambda\left(\sum_{i\in I} \langle x, e_i\rangle e_i\right) \\ &= \sum_{i\in I} \langle x, e_i\rangle \lambda e_i. \end{aligned}$$

For each $k \in I$, we apply the rank-one projection onto $\operatorname{span}(e_k)$ to both expressions, yielding

$$\langle x, e_k\rangle \mu_k e_k = \langle x, e_k\rangle \lambda e_k$$

for each $k \in I$. Thus, $\langle x, e_k\rangle (\mu_k - \lambda) = 0$ for every $k \in I$. If $\lambda \notin \{\mu_i\}_{i\in I}$, then $\langle x, e_k\rangle = 0$ for all $k \in I$, implying $x = 0$, which is a contradiction. Thus, $\lambda = \mu_i$ for some $i \in I$.

(2) By the definition of E_j, it is the case that $E_j \subseteq E_{\lambda_j}(T)$. Suppose $T(x) = \lambda_j x$ for some $x \in \mathcal{H}$. Then, we find, by (3)

$$\begin{aligned}
\sum_{k\in I} \lambda_j P_k(x) &= \lambda_j \left(\sum_{k\in I} P_k(x)\right)\\
&= \lambda_j x\\
&= T(x)\\
&= \sum_{k\in J} \lambda_k P_k(x),
\end{aligned}$$

meaning $\left(\lambda_j - \lambda_k\right) P_k(x) = 0$ for all $k \in I$. If $k \neq j$, we must have $P_k(x) = 0$, so $x = P_j(x) \in E_j$.

(3) Since T is reduced by $\left\{E_j\right\}_{j\in J}$, we find that, for P_j defined as the orthogonal projection onto E_j,

$$\begin{aligned}
T &= \sum_{j\in J} P_j T P_j\\
&= \sum_{j\in J} T P_j\\
&= \sum_{j\in J} \lambda P_j
\end{aligned}$$

in SOT.

(4) By construction, $\sup_{i\in I} |\mu_i| = \sup_{j\in J} \left|\lambda_j\right|$. We must have $\|T\|_{op} \geqslant \|T(e_i)\| = |\mu_i|$, so $\|T\|_{op} \geqslant \sup_{i\in I} |\mu_i|$.

Given $x \in \mathcal{H}$, we use Parseval's theorem to compute

$$\begin{aligned}
\|T(x)\|^2 &= \left\|T\left(\sum_{i\in I} \langle x, e_i\rangle e_i\right)\right\|^2\\
&= \left\|\sum_{i\in I} \langle x, e_i\rangle T(e_i)\right\|^2\\
&= \left\|\sum_{i\in I} \langle x, e_i\rangle \mu_i e_i\right\|^2\\
&= \sum_{i\in I} |\langle x, e_i\rangle|^2 |\mu_i|^2\\
&\leqslant \left(\sup_{i\in I} |\mu_i|^2\right) \sum_{i\in I} |\langle x, e_i\rangle|^2\\
&= \left(\sum_{i\in I} |\mu_i|^2\right) \|x\|^2.
\end{aligned}$$

Thus, $\|T\|_{op} \leqslant \sup_{i\in I} \left|\mu_j\right|$, so $\|T\|_{op} = \sup_{i\in I} |\mu_i| = \sup_{j\in J} \left|\lambda_j\right|$.

□

Corollary: If $T \in \mathcal{B}(\mathcal{H})$ is diagonalizable, then so is T^*. Additionally, T is normal.

Proof. Let $(e_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ be as above. For all $i, j \in I$, we have

$$\begin{aligned}\left\langle e_j, T^*(e_i)\right\rangle &= \left\langle T(e_j), e_i\right\rangle\\ &= \left\langle \mu_j e_j, e_i\right\rangle\\ &= \mu_j\left\langle e_j, e_i\right\rangle\\ &= \mu_i\left\langle e_j, e_i\right\rangle\\ &= \left\langle e_j, \overline{\mu_i} e_i\right\rangle.\end{aligned}$$

Thus, we find $\langle x, T^*(e_i)\rangle = \langle x, \overline{\mu_i} e_i\rangle$ for every $x \in \operatorname{span}\left\{e_j \mid j \in I\right\}$, and by continuity, it follows that $T^*(e_i) = \overline{\mu_i} e_i$. It follows that T^* is diagonalizable.

Note that $T^*T(e_i) = |\mu_i|^2 e_i = TT^*(e_i)$ for all $i \in I$. Thus, $T^*T(x) = TT^*(x)$ for all $x \in \mathcal{H}$, so T is normal. □

Example. If $\lambda = (\lambda_n)_n$ belongs to ℓ_∞, then D_λ is diagonalizable, since $D(e_n) = \lambda_n e_n$, where $(e_n)_n$ is the standard basis on ℓ_2. Thus, $(\lambda_n)_n$ is a complete list of eigenvalues.

The Matrix Algebra and Double Commutant Theorem

Fix a Hilbert space $\mathcal{H}$, and let

$$\operatorname{Mat}_n(\mathcal{B}(\mathcal{H})) = \left\{(T_{ij})_{i,j=1}^n \;\middle|\; T_{ij} \in \mathcal{B}(\mathcal{H})\right\}.$$

This is a $*$-algebra that is $*$-isomorphic to $\operatorname{Mat}_n(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$. We would like to view each matrix $(T_{ij})_{i,j=1}^n$ as an operator on the n-fold direct sum $\mathcal{H}^{(n)}$, similar to the way $n \times n$ matrix acts on a vector in ℓ_2^n.

Proposition: Let $\mathcal{H}$ be a Hilbert space, and suppose $(T_{ij})_{i,j=1}^n \in \operatorname{Mat}_n(\mathcal{B}(\mathcal{H}))$. The map $[T_{ij}] : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$, defined by

$$[T_{ij}]\,(\xi_j)_{j=1}^n \left(\sum_{j=1}^n T_{ij}(\xi_j)\right)_{i=1}^n$$

defines a bounded operator with

$$\left\|T_{ij}\right\|_{op} \leqslant \left(\sum_{i,j=1}^n \left\|T_{ij}\right\|_{op}^2\right)^{1/2}.$$

Moreover, the map $(T_{ij})_{i,j=1}^n \mapsto [T_{ij}]$ is a unital $*$-isomorphism of $*$-algebras.

Proof. Using the Cauchy–Schwarz inequality, we find

$$\begin{aligned}\left\|[T_{ij}]\,(\xi_j)_{j=1}^n\right\|^2 &= \left\|\left(\sum_{j=1}^n T_{ij}(\xi_j)\right)_{i=1}^n\right\|^2\\ &= \sum_{i=1}^n \left\|\sum_{j=1}^n T_{ij}(\xi_j)\right\|^2\\ &\leqslant \sum_{i=1}^n\left(\sum_{j=1}^n \left\|T_{ij}(\xi_j)\right\|\right)^2\\ &\leqslant \sum_{i=1}^n\left(\sum_{j=1}^n \left\|T_{ij}\right\|_{op}\left\|\xi_j\right\|\right)^2\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left\|T_{ij}\right\|_{op}^{2}\right)\left(\sum_{j=1}^{n}\left\|\xi_j\right\|^{2}\right)\\
&= \left(\sum_{i,j=1}^{n}\left\|T_{ij}\right\|_{op}^{2}\right)\left\|\left(\xi_j\right)_{j=1}^{n}\right\|^{2}.
\end{aligned}$$

Thus,

$$\left\|\left[T_{ij}\right]\right\|_{op} \leqslant \left(\sum_{i,j=1}^{n}\left\|T_{ij}\right\|_{op}^{2}\right)^{1/2}.$$

If $\left(T_{ij}\right)_{i,j=1}^{n}$ and $\left(S_{ij}\right)_{i,j=1}^{n}$ are in $\mathrm{Mat}_n\left(\mathcal{B}\left(\mathcal{H}\right)\right)$, and $\alpha \in \mathbb{C}$, we can see that

$$\alpha\left[T_{ij}\right] + \left[S_{ij}\right] = \left[\alpha T_{ij} + S_{ij}\right]$$

as operators on $\mathcal{H}^{(n)}$. To evaluate multiplication, we see

$$\begin{aligned}
[T_ij] \circ \left[S_{ij}\right]\left(\xi_j\right)_j &= \left[T_{ij}\right]\left(\left(\sum_{j=1}^{n} S_{kj}\left(\xi_j\right)\right)_k\right)\\
&= \left(\sum_{k=1}^{n} T_{ik}\left(\sum_{j=1}^{n} S_{kj}\left(\xi_j\right)\right)\right)_i\\
&= \left(\sum_{j,k=1}^{n} T_{ik}S_{kj}\left(\xi_j\right)\right)\\
&= \left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n} T_{ik}S_{kj}\right)\left(\xi_j\right)\right)_i\\
&= \left[\left(\sum_{k=1}^{n} T_{ik}S_{kj}\right)_j\right]\left(\xi_j\right)_j\\
&= \left[\left(T_{ij}\right)\left(S_{ij}\right)\right]\left(\xi_j\right)_j.
\end{aligned}$$

Evaluating the adjoint, we find

$$\begin{aligned}
\left\langle\left[T_{ij}\right]\left(\xi_i\right)_i, \left(\eta_i\right)_i\right\rangle &= \left\langle\left(\sum_{j=1}^{n} T_{ij}\left(\xi_j\right)\right)_i, \left(\eta_i\right)_i\right\rangle\\
&= \sum_{i=1}^{n}\left\langle\sum_{j=1}^{n} T_{ij}\left(\xi_j\right), \eta_i\right\rangle\\
&= \sum_{i,j=1}^{n}\left\langle T_{ij}\left(\xi_j\right), \eta_i\right\rangle\\
&= \sum_{i,j=1}^{n}\left\langle \xi_j, T_{ij}^{*}\left(\eta_i\right)\right\rangle\\
&= \sum_{j=1}^{n}\left\langle \xi_j, \sum_{i=1}^{n} T_{ij}^{*}\left(\eta_i\right)\right\rangle\\
&= \left\langle\left(\xi_j\right)_j, \left(\sum_{i=1}^{n} T_{ij}^{*}\left(\eta_i\right)\right)_j\right\rangle
\end{aligned}$$

$$= \left\langle \left(\xi_j\right)_j, \left[T_{ij}\right]\left(\eta_j\right)_j \right\rangle.$$

Thus, $\left[T_{ij}\right]^* = \left[T_{ij}^*\right]$, meaning the map $\left(T_{ij}\right)_{i,j} \mapsto \left[T_{ij}\right]$ is an injective $*$-homomorphism.

To show surjectivity, let $T \in \mathcal{B}\left(\mathcal{H}^{(n)}\right)$ be arbitrary. Set $T_{ij} = \pi_i T \iota_j$, where $\pi_i \colon \mathcal{H}^{(n)} \to \mathcal{H}$ and $\iota_j \colon \mathcal{H} \to \mathcal{H}^{(n)}$ are the canonical projections and injections respectively. We claim that $\left[T_{ij}\right] = T$ on $\mathcal{H}^{(n)}$.

If $\xi = (\xi_i)_i \in \mathcal{H}^{(n)}$, then

$$\begin{aligned}
\left[T_{ij}\right](\xi) &= \left(\sum_{j=1}^{n} T_{ij}\left(\xi_j\right)\right)_{i=1}^{n}\\
&= \left(\sum_{j=1}^{n} \pi_i T \iota_j\left(\xi_j\right)\right)_{i=1}^{n}\\
&= \left(\pi_i T\left(\sum_{j=1}^{n} \iota_j \xi_j\right)\right)_{i=1}^{n}\\
&= \left(\pi_i T(\xi)\right)_{i=1}^{n}\\
&= T(\xi).
\end{aligned}$$

□

Remark: We will let $\operatorname{diag}(T_1, \dots, T_n)$ denote the operator $\left[T_{ij}\right]$ where $T_{ii} = T_i$ and $T_{ij} = 0$ for $i \neq j$. If $T \in \mathcal{B}(\mathcal{H})$, the n-fold amplification $T^{(n)}$ on $\mathcal{H}^{(n)}$ is precisely $\operatorname{diag}(T, \dots, T)$.

Since $\mathcal{B}\left(\mathcal{H}^{(n)}\right)$ is a C*-algebra, we may identify $\mathrm{Mat}_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}\left(\mathcal{H}^{(n)}\right)$, and use the given complete C*-norm to find one on $\mathrm{Mat}_n(\mathcal{B}(\mathcal{H}))$.

Corollary: If $\mathcal{H}$ is a Hilbert space, and $n \geqslant 1$ is an integer, the $*$-algebra is a C*-algebra when equipped with the norm

$$\left\|\left(T_{ij}\right)_{i,j=1}^{n}\right\| = \left\|\left[T_{ij}\right]\right\|_{op}.$$

We may now use the n-fold amplifications to resolve some properties of the double commutant. Recall that if $A \subseteq \mathcal{B}(\mathcal{H})$ is a $*$-subalgebra of bounded operators, then A'' is a von Neumann algebra containing A. The Double Commutant Theorem actually shows that, under some mild conditions, A'' is actually the von Neumann algebra generated by A.

Definition. Let $\mathcal{H}$ be a Hilbert space, and suppose $A \subseteq \mathcal{B}(\mathcal{H})$ is a subset. We define

$$A^{(n)} = \left\{a^{(n)} \;\middle|\; a \in A\right\}$$

to be the n-fold amplification of A. The kernel of A is defined to be

$$\ker(A) = \{\xi \in \mathcal{H} \mid a(\xi) = 0,\ \forall a \in A\}.$$

We say A acts non-degenerately on $\mathcal{H}$ if $\ker(A) = \{0\}$.

Exercise: Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra of operators. Prove that A acts non-degenerately on $\mathcal{H}$ if and only if $[A\mathcal{H}] = \mathcal{H}$. If A is unital, show that A acts non-degenerately.

Solution: We have that A acts degenerately if and only if $\ker(A) \neq \{0\}$, meaning there exists some $\xi \neq 0$ such that $a(\xi) = 0$ for all $a \in A$, meaning that $\overline{\operatorname{span}}\{a(\xi) \mid a \in A\} = 0$, meaning $\xi \in [A\mathcal{H}]^{\perp}$, so $[A\mathcal{H}]^{\perp} \neq \mathcal{H}$. The reverse direction follows from the fact that if A is a $*$-subalgebra with $[A\mathcal{H}] = \mathcal{H}$, then $\mathcal{H}^{\perp} = \{0\}$, so $a\xi = 0$ for all $a \in A$ if and only if $\xi = 0$.

Lemma: Let $\mathcal{H}$ be a Hilbert space, and suppose $A \subseteq \mathcal{B}(\mathcal{H})$ is a $*$-subalgebra.

(1) The collection $A^{(n)} \subseteq \mathcal{B}\left(\mathcal{H}^{(n)}\right)$ is also a $*$-subalgebra.

(2) If $\ker(A) = \{0\}$, then $\ker\left(A^{(n)}\right) = \{0\}$. Thus, if A acts non-degenerately, then so does $A^{(n)}$.

(3) If $x \in A''$, then $x^{(n)} \in \left(A^{(n)}\right)''$.

Proof.

(1) This result follows from the properties of $A^{(n)} = \bigoplus_{i=1}^{n} A$.

(2) If $\ker(A) = \{0\}$, then $\operatorname{diag}(A, \dots, A)\left(\xi_j\right)_j = 0$ if and only if $\xi_j = 0$ for all j. Thus, $\ker\left(A^{(n)}\right) = \{0\}$.

(3) We identify $\operatorname{Mat}_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}\left(\mathcal{H}^{(n)}\right)$. Let $x \in A''$, and $T = \left(T_{ij}\right)_{ij} \in \left(A^{(n)}\right)'$. Then,

$$\begin{aligned}\left(T_{ij}a\right)_{ij} &= \left(T_{ij}\right)_{ij} a^{(n)}\\ &= a^{(n)}\left(T_{ij}\right)_{ij}\\ &= \left(aT_{ij}\right)_{ij},\end{aligned}$$

meaning $T_{ij}a = aT_{ij}$ for all $a \in A$ and all i, j. Thus, for all i, j, we have $T_{ij} \in A'$, meaning $xT_{ij} = T_{ij}x$ for all i, j. Thus,

$$\begin{aligned}\left(T_{ij}\right)_{ij} x^{(n)} &= \left(T_{ij}x\right)_{ij}\\ &= \left(xT_{ij}\right)_{ij}\\ &= x^{(n)}\left(T_{ij}\right)_{ij}.\end{aligned}$$

Thus, $x^{(n)} \in \left(A^{(n)}\right)''$.

□

Lemma: Let $\mathcal{H}$ be a Hilbert space with $\xi \in \mathcal{H}$, and let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra that acts non-degenerately.

(1) The closed subspace $\mathcal{K} = [A\xi]$ reduces A, and $\xi \in \mathcal{K}$.

(2) If $b \in A''$ and $\varepsilon > 0$, there is $a \in A$ with $\|a(\xi) - b(\xi)\| < \varepsilon$.

Proof.

(1) We have established that $\mathcal{K}$ reduces A earlier. Now, we show that $\xi \in \mathcal{K}$. Let P be the orthogonal projection onto $\mathcal{K}$. Thus, $P \in A'$, so $P^{\perp} = I - P \in A'$. Since $a(\xi) \in \mathcal{K}$ for all $a \in A$, we have

$$\begin{aligned}aP^{\perp}(\xi) &= P^{\perp}a(\xi)\\ &= 0\end{aligned}$$

for all $a \in A$. Since $\ker(A) = \{0\}$, we must have $P^{\perp}(\xi) = 0$, so $P(\xi) = \xi$, so $\xi \in \mathcal{K}$.

(2) With P as the orthogonal projection onto $\mathcal{K}$, we see that $bP = Pb$, since $P \in A'$ Thus, using $P(\xi) = \xi$, we have

$$\begin{aligned}b(\xi) &= bP(\xi)\\ &= Pb(\xi)\\ &\in \mathcal{K},\end{aligned}$$

meaning, by the definition of the closed span, there is an $a \in A$ with $\|a(\xi) - b(\xi)\| < \varepsilon$.

□

Proof. Since $A \subseteq A''$, and A'' is WOT-closed, we see that $\overline{A}^{\text{SOT}} \subseteq \overline{A}^{\text{WOT}} \subseteq A''$. It is thus sufficient to show that $A'' \subseteq \overline{A}^{\text{SOT}}$.

Let $b \in A''$, and consider a basic SOT-open neighborhood of b, which is of the form

$$U = \bigcap_{j=1}^{n} \left\{ T \in \mathcal{B}(\mathcal{H}) \mid \left\| T\left(\xi_j\right) - b\left(\xi_j\right) \right\| < \varepsilon \right\},$$

where $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in \mathcal{H}$. We only need to show that $U \cap A$ is nonempty. Set $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{(n)}$ (as a column vector).

We are aware that $A^{(n)} \subseteq \mathcal{B}\left(\mathcal{H}^{(n)}\right)$ is a $*$-subalgebra with $\ker\left(A^{(n)}\right) = \{0\}$ and $b^{(n)} \subseteq \left(A^{(n)}\right)''$. Thus, we find $a^{(n)}$ with

$$\begin{aligned} \left(\sum_{j=1}^{n} \left\| a\left(\xi_j\right) - b\left(\xi_j\right) \right\|^2\right)^{(1/2)} &= \left\| \begin{pmatrix} a(\xi_1) - b(\xi_1) \\ \vdots \\ a(\xi_n) - b(\xi_n) \end{pmatrix} \right\|_{\mathcal{H}^{(n)}} \\ &= \left\| a^{(n)}(\xi) - b^{(n)}(\xi) \right\|_{\mathcal{H}^{(n)}} \\ &< \varepsilon, \end{aligned}$$

meaning that $\left\| a\left(\xi_j\right) - b\left(\xi_j\right) \right\| < \varepsilon$ for all $j = 1, \dots, n$, so $a \in U$. □

Corollary: Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a unital $*$-subalgebra of operators. Then, A is a von Neumann algebra if and only if $A = A''$.

Definition. Let $\mathcal{H}$ be a Hilbert space. A MASA (maximal abelian self-adjoint algebra) in $\mathcal{B}(\mathcal{H})$ is a maximal commutative $*$-algebra $N \subseteq \mathcal{B}(\mathcal{H})$. A MASA is a maximal element of the set

$$\mathcal{A} = \left\{ A \subseteq \mathcal{B}(\mathcal{H}) \mid A \text{ is a commutative } *\text{-algebra} \right\}$$

ordered by inclusion.

Remark: MASAs do exist — since $\{0\}$ is an abelian self-adjoint subalgebra, and the union of any chain in $\mathcal{A}$ is also an abelian self-adjoint subalgebra, so Zorn's lemma provides a maximal element.

Proposition: Let A be a commutative $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then, A is a MASA if and only if $A = A'$.

Proof. Let A be a MASA. Since A is commutative, we see that $A \subseteq A'$. Let $u \in A'$ be self-adjoint. The $*$-algebra B generated by A and u is commutative and contains A, so by maximality, we must have $B = A$, meaning $u \in A$. Additionally, A' is a $*$-subalgebra, so for a given $x \in A'$, we may write the Cartesian decomposition $x = u + iv$, where $u, v \in A'$ are self-adjoint. Thus, $u, v \in A$, so $x \in A$, so $A = A'$.

Conversely, if $A = A'$, and B is a commutative $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains A, we must have $B \subseteq B' \subseteq A' = A$, so $A = B$, meaning A is a MASA. □