Problem (Problem 1)**:** Let $(a_n)_n$ be a sequence for which $\sum_{n=0}^{\infty}|a_n|^2$ is finite. For each positive N, define $f_N(z) = \sum_{n=0}^{\infty} a_n z^n$, and define $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

  (a) Show that f is holomorphic on $\mathbb{D}$.

  (b) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \left|f_N\left(re^{i\theta}\right)\right|^2 d\theta.$$

  (c) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta.$$

  (d) Determine in terms of $(a_n)_n$ the limit

$$\lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta.$$

Solution:

  (a) Let $0 < r < 1$. Since each f_N is analytic, we can use the Cauchy Integral Formula to compute a_N explicitly, yielding

$$|a_N| = \left|\frac{1}{2\pi i} \int_0^{2\pi} \frac{f_N(\xi)}{\xi^{N+1}} d\xi\right|$$

$$\leqslant \frac{1}{r^N} \sup_{|z|=r} |f_N(z)|.$$

Therefore, if we are able to show that the value

$$\sup_{|z|=r} |f_N(z)|$$

is uniformly bounded by a constant for all N, we will be able to use the Cauchy–Hadamard theorem to show that $\limsup_{N\to\infty}|a_N|^{1/N} \leqslant 1$. Toward this end, we use the Cauchy–Schwarz inequality, which yields

$$\sup_{|z|=r} |f_N(z)| = \sup_{|z|=r}\left|\sum_{n=0}^{N} a_n z^n\right|$$

$$\leqslant \sup_{|z|=r}\left(\sum_{n=0}^{N} |a_n|^2\right)^{1/2}\left(\sum_{m=0}^{N}|z|^{2m}\right)^{1/2}$$

$$\leqslant \sup_{|z|=r}\underbrace{\left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}}_{=:K}\left(\sum_{m=0}^{\infty}|z|^{2m}\right)^{1/2}$$

$$= \frac{K}{(1-|r|^2)^{1/2}}.$$

Since we have established this uniform bound, we thus find that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence at least 1, so f is analytic on $\mathbb{D}$, hence holomorphic on $\mathbb{D}$.

(b) We write out the integral to yield

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \left| f_N\left(re^{i\theta}\right) \right|^2 \, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^{N} a_n r^n e^{in\theta} \right) \overline{\left(\sum_{m=0}^{N} a_m r^m e^{im\theta} \right)} \, d\theta \\
&= \frac{1}{2\pi} \sum_{n=0}^{N} \sum_{m=0}^{N} a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \\
&= \sum_{n=0}^{N} |a_n|^2 r^{2n}.
\end{aligned}
$$

Problem (Problem 3): Let $f \colon \mathbb{C} \to \mathbb{C}$ be entire.

(a) Suppose there exist $C, R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leqslant C|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree at most n.

(b) Suppose that $g \colon \mathbb{C} \to \mathbb{C}$ is also entire and $|f(z)| \leqslant |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists some $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$ such that $f(z) = \alpha g(z)$ for all $z \in \mathbb{C}$.

(c) Suppose that there exists some $\theta \in \mathbb{R}$ such that $f(\mathbb{C}) \cap \left\{ re^{i\theta} \mid r > 0 \right\} = \emptyset$. Show that f is constant.