Math 395: Homework 2
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Problem 11

Problem. Let $T \in \text{Hom}_{\mathbb{F}}(P_7(\mathbb{F}), P_7(\mathbb{F}))$ be defined by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the usual derivative of a polynomial $f(x) \in P_7(\mathbb{F})$. For each of the fields below, determine a basis for the image and kernel of T:

(a) $\mathbb{F} = \mathbb{R}$

(b) $\mathbb{F} = \mathbb{F}_3$.

Solution.

(a) For $f(x) \in P_7(\mathbb{R})$, we have

$$f(x) = a_0 + a_1x + \cdots + a_7x^7,$$

where $a_i \in \mathbb{R}$ for each i from 1 through 7. In particular,

$$T(f(x)) = a_1 + 2a_2x + \cdots + 7a_7x^6,$$

and since $a_i \in \mathbb{R}$ for each i, so too is ia_i. For any $p(x) \in P_6(\mathbb{R})$, with $p(x) = p_0 + p_1x + \cdots + p_6x^6$, we can find $q(x) \in P_7(\mathbb{R})$ with

$$q(x) = q_0 + p_0x + \frac{p_1}{2}x^2 + \cdots + \frac{p_5}{6}x^6 + \frac{p_6}{7}x^7,$$

with $q_0 \in \mathbb{R}$ being arbitrary, and

$$T(q(x)) = p_0 + p_1x + \cdots + p_6x^6.$$

Thus, $\text{im}(T) = P_6(\mathbb{R})$. The basis for $\text{im}(T)$ is the basis for $P_6(\mathbb{R})$, which is $\{1, x, x^2, \ldots, x^6\}$.

We know that if $f(x) \in \mathbb{R}$, then $T(f(x)) = 0$, meaning $\ker(T) = \mathbb{R}$. Thus, a basis for $\ker(T)$ is $\{1\}$.

(b) For $f(x) \in P_7(\mathbb{F}_3)$, we have

$$f(x) = a_0 + a_1x + \cdots + a_5x^5 + a_6x^6 + a_7x^7$$

where $a_0, a_1, \ldots, a_6, a_7 \in \mathbb{F}_3$. In particular, we can see that

$$T(f(x)) = a_1 + 2a_2x + 3a_3x^2 + \cdots + 5a_5x^4 + 6a_6x^5 + 7a_7x^6.$$

Since we are working in $\mathbb{F}^3$, in particular, it is the case that $3a_3 \equiv 0a_3 = 0$, and similarly with $6a_6$. Thus, we have

$$T(f(x)) = a_1 + 2a_2x + 4a_4x^3 + 5a_5x^4 + 7a_7x^6.$$

Thus, $\operatorname{im}(T)$ must be of this form, meaning that the set $\left\{1, x, x^3, x^4, x^6\right\}$ is a basis for the image of T.

Similarly, since all polynomials of the form $f(x) = a + bx^3 + cx^6$ with $a, b, c \in \mathbb{F}_3$ are mapped to 0 under T, it is the case that the set $\left\{1, x^3, x^6\right\}$ is a basis for $\ker(T)$.

Problem 12

Problem. Let $T \in \operatorname{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Prove that if $v \in V$ is not in $\ker(T)$, then

$$V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.$$

Solution. Since $T(v) \neq 0$, there exists $(T(v))^{-1} \in \mathbb{F}$. Let $w \in V$. Then,

$$T(w) = \left(T(w)\,(T(v))^{-1}\right) T(v).$$

We let $c = T(w)\,(T(v))^{-1}$. We have

$$\begin{aligned} T(w) &= cT(v) \\ &= T(cv), \end{aligned}$$

meaning

$$T(w - cv) = 0,$$

so $w - cv \in \ker(T)$, or $w \in [cv]_\sim$, where $\sim$ is the equivalence relation defining $V/\ker(T)$.

Thus, we have $w \in \ker(T) + \{cv \mid c \in \mathbb{F}\}$, implying that $V \subseteq \ker(T) + \{cv \mid c \in \mathbb{F}\}$, so $V = \ker(T) + \{cv \mid c \in \mathbb{F}\}$.

For $k \in \ker(T)$, suppose

$$cv + k = 0.$$

Then,

$$\begin{aligned} T(cv + k) &= 0_V \\ cT(v) + T(k) &= 0 \\ cT(v) &= 0. \end{aligned}$$

Since $T(v) \neq 0$ by the definition of v, it must be the case that $c = 0$, meaning $cv = 0_V$. Thus, it is the case that $\ker(T)$ and $\{cv \mid c \in \mathbb{F}\}$ are independent subspaces, meaning

$$V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.$$

Problem 18

Problem. Let V be a $\mathbb{F}$-vector space of dimension n. Let $T \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ such that $T^2 = 0$. Prove that the image of T is contained in the kernel of T, and hence the dimension of the image of T is at most $n/2$.

Solution. Suppose $w \in \mathrm{im}(T)$. Then, there exists $v \in V$ such that $T(v) = w$. In particular, this means that

$$\begin{aligned} T(w) &= T(T(v)) \\ &= T^2(v) \\ &= 0, \end{aligned}$$

meaning $T(w) \in \ker(T)$. Thus, $w \in \ker(T)$, implying that $\mathrm{im}(T) \subseteq \ker(T)$. In particular, since $n = \dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(\mathrm{im}(T)) + \dim_{\mathbb{F}}(\ker(T))$, and $\dim_{\mathbb{F}}(\mathrm{im}(T)) \leqslant \dim_{\mathbb{F}}(\ker(T))$, it is the case that $\dim_{\mathbb{F}}(\mathrm{im}(T)) \leqslant n/2$.

Problem 19

Problem. Let W be a subspace of a finite-dimensional vector space V. Let $T \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ be such that $T(W) \subseteq W$. Show that T induces a linear transformation $\overline{T} \in \mathrm{Hom}_{\mathbb{F}}(V/W, V/W)$. Prove that T is nonsingular (i.e., injective) on V if and only if T restricted to W and $\overline{T}$ on V/W are both nonsingular.

Solution. Let $\pi : V \to V/W$ be the projection map, $\pi(v) = v + W$. For $T \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ with $T(W) \subseteq W$, it is the case that $\pi \circ T(W) = 0 + W$. We define $\overline{T} : V/W \to V/W$ by taking

$$\overline{T}(v + W) = T(v) + W.$$

We will show that $\overline{T}$ is well-defined and that $\pi \circ T = \overline{T} \circ \pi$. Suppose $v_1 + W = v_2 + W$. Then, for some $w \in W$, $v_1 = v_2 + w$. Therefore,

$$\begin{aligned} \overline{T}(v_1 + W) &= \overline{T}(v_2 + w + W) \\ &= T(v_2 + w) + W \\ &= T(v_2) + T(w) + W \\ &= T(v_2) + W, \end{aligned}$$

where the property that $T(W) \subseteq W$ was used in the final step.

We will now show that $\overline{T}$ is a linear map. Let $\alpha \in \mathbb{F}$, $v_1 + W, v_2 + W \in V/W$. Then,

$$\begin{aligned} \overline{T}((v_1 + W) + \alpha(v_2 + W)) &= \overline{T}((v_1 + \alpha v_2) + W) \\ &= T(v_1 + \alpha v_2) + W \end{aligned}$$

$$
\begin{aligned}
&= T(v_1) + \alpha T(v_2) + W \\
&= (T(v_1) + W) + \alpha (T(v_2) + W) \\
&= \overline{T}(v_1 + W) + \alpha \overline{T}(v_2 + W).
\end{aligned}
$$

Finally, we can see that for $v \in V$

$$
\begin{aligned}
\pi \circ T(v) &= \pi(T(v)) \\
&= T(v) + W \\
&= \overline{T}(v + W) \\
&= \overline{T}(\pi(v)).
\end{aligned}
$$

Thus, we can see that the following diagram commutes.

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\pi"'] & V \arrow[d, "\pi"] \\
V/W \arrow[r, "\overline{T}"'] & V/W
\end{tikzcd}
$$

Suppose T is injective. Then, by inclusion, $T\big|_W$ is injective. Let $v + W \in \ker\left(\overline{T}\right)$. Then,

$$
\begin{aligned}
\overline{T}(v + W) &= 0 + W \\
&= T(v) + W,
\end{aligned}
$$

Thus, we have $T(v) \in W$. Since V is finite-dimensional, and T is injective, then T is bijective, meaning $T(W) = W$ (as, by assumption, $T(W) \subseteq W$). Thus, $v \in W$, meaning $v + W = 0 + W$, so $\ker\left(\overline{T}\right) = 0 + W$, meaning $\overline{T}$ is injective.

Suppose $\ker\left(\overline{T}\right) = 0 + W$ and $\ker\left(T\big|_W\right) = 0$. Let $v \in \ker(T)$. Then, $T(v) = 0$. Thus,

$$
\begin{aligned}
\pi(T(v)) &= 0 + W \\
&= \overline{T}(\pi(v)),
\end{aligned}
$$

implying that $\pi(v) = 0 + W$, so $v \in W$. So, $T(v) = T\big|_W(v) = 0$, meaning $v = 0$.