# Math 395

## Homework 1

## Due: 2/1/2024

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**Collaborators:** ______________________________

1. Let $S$ be the subset of $\text{Mat}_2(\mathbf{R})$ be the set consisting of matrices of the form $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$.

   (a) Prove that $S$ is a ring.

   **Proof:** We will show that $S$ is a ring by using the ring axioms.

   - Closure of Addition:

   $$\begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix} = \begin{bmatrix} a+c & a+c \\ b+d & b+d \end{bmatrix}.$$

   - Additive Identity:

   $$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a+0 & a+0 \\ b+0 & b+0 \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} \end{aligned}$$

   - Additive Inverse:

   $$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} -a & -a \\ -b & -b \end{bmatrix} &= \begin{bmatrix} a-a & a-a \\ b-b & b-b \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

   - Associativity of Addition:

   $$\begin{aligned} \left(\begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix}\right) + \begin{bmatrix} e & e \\ f & f \end{bmatrix} &= \begin{bmatrix} (a+c)+e & (a+c)+e \\ (b+d)+f & (b+d)+f \end{bmatrix} \\ &= \begin{bmatrix} a+(c+e) & a+(c+e) \\ b+(d+f) & b+(d+f) \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \left(\begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} e & e \\ f & f \end{bmatrix}\right). \end{aligned}$$

   - Commutativity of Addition:

   $$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix} &= \begin{bmatrix} a+c & a+c \\ b+d & b+d \end{bmatrix} \\ &= \begin{bmatrix} c+a & c+a \\ d+b & d+b \end{bmatrix} \\ &= \begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} a & a \\ b & b \end{bmatrix}. \end{aligned}$$

- Closure of Multiplication:

$$\begin{bmatrix} a & a \\ b & b \end{bmatrix} \cdot \begin{bmatrix} c & c \\ d & d \end{bmatrix} =$$

(b) Show that $J = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a right identity in $S$, i.e., $AJ = A$ for all $A \in \mathrm{Mat}_2(\mathbf{R})$.

(c) Show that $J$ is not a left identity for $S$, i.e., there is an element $B \in S$ so that $JB \neq B$.

2. Show that the subset $S = \{[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}, [15]_{18}\}$ is a subring of $\mathbf{Z}/18\mathbf{Z}$. Does $S$ have an identity?

3. Define a new addition and multiplication on $\mathbf{Z}$ by

$$\begin{aligned} a \oplus b &= a + b - 1 \\ a \odot b &= ab - (a + b) + 2. \end{aligned}$$

Prove that under these operations $\mathbf{Z}$ is an integral domain.

4. Let $R$ be a ring and define $Z(R) = \{a \in R : ar = ra \text{ for every } r \in R\}$. Prove that $Z(R)$ is a subring of $R$. It is referred to as the center of $R$.

5. Let $R$ be a ring and fix an element $x \in R$. Show that the set $\{rx : r \in R\}$ is a subring of $R$.

6. Let $S$ and $T$ be subrings of a ring $R$.

   (a) Is $S \cap T$ a subring of $R$? Justify your answer with a proof or counterexample.

   (b) Is $S \cup T$ a subring of $R$? Justify your answer with a proof or counterexample.

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(F)$ where $F$ is a field.

   (a) Prove that $A$ is invertible if and only if $ad - bc \neq 0_F$.

   (b) Prove that $A$ is a zero divisor if and only if $ad - bc = 0_F$.

   (c) If instead we consider a matrix $A \in \mathrm{Mat}_2(\mathbf{Z})$, do the same conclusions hold? If so, prove them. If not, adjust them to true statements and prove those statements.