2.6

Problem: Suppose $X = \{\alpha \mid \alpha \text{ is an ordinal}\}$ were a set. Show that it would follow that X is transitive and well-ordered by $\in$.

Solution. Let $s \in t$ and $t \in X$. Then, $t = \alpha$ for some ordinal α, meaning $s \in \alpha$. By the definition of an ordinal, this implies s is an ordinal, implying $s \in X$.

It is necessarily the case that X is (strictly) totally ordered, since for any two distinct ordinals α and β, either $\alpha \in \beta$ or $\beta \in \alpha$. Let $A \subseteq X$ be nonempty. Suppose toward contradiction that A did not have a least element. Since A is a subset of a totally ordered set, A is totally ordered, meaning that for any $\alpha \in A$, there is a β such that $\beta \in \alpha$, and so on. In particular, if it were not the case that A had a least element, there would be an infinite descending membership chain, which is a violation of the axiom of regularity.

2.7

Problem: Suppose α is an ordinal. Show that $\alpha \cup \{\alpha\}$ is an ordinal.

Solution. We will start by showing that $\alpha \cup \{\alpha\}$ is transitive with respect to $\in$. Let $s \in t$ and $t \in \alpha \cup \alpha$. Since $\alpha \cap \{\alpha\} = \emptyset$, it is the case that $t \in \alpha$ or $t \in \{\alpha\}$.

If $t \in \alpha$, then t is also an ordinal, and since ordinals are transitive with respect to $\in$, $s \in \alpha$, so $s \in \alpha \cup \{\alpha\}$. If $t \in \{\alpha\}$, then $t = \alpha$, meaning $s \in \alpha$, so $s \in \alpha \cup \{\alpha\}$.

Let $A \subseteq \alpha \cup \{\alpha\}$ be nonempty. If $A \cap \alpha = \emptyset$, then $A \subseteq \{\alpha\}$, meaning $A = \alpha$, and A has the least element of α. Else, we define the least element of A by taking the least element of $A \cap \alpha$; since $A \cap \alpha$ is a nonempty subset of α, $A \cap \alpha$ has a least element since ordinals are well-ordered.

2.8

Problem: Let α and β be ordinals, and let $S = \{(0, x) \mid x \in \alpha\}$ and $T = \{(1, x) \mid x \in \beta\}$. Define an ordering on $S \cup T$ by taking $(m, n) < (n, y)$ if $m < n$ or if $m = n$ and $x < y$. Show that this is a well-ordering of $S \cup T$.

Solution. Let $A \subseteq S \cup T$ be nonempty. If $A \cap S = \emptyset$, then $A \subseteq T$ is nonempty, meaning any element of A is of the form $(1, t)$, where $t \in \beta$. By the definition of an ordinal, it is the case that the $A_T = \{t \in \beta \mid (1, t) \in T\}$ contains a least element, which means A has a least element.

If $A \cap S \neq \emptyset$, then the ordering gives the least element of A to be the least element in $A \cap S$, which exists since $A_S = \{t \in \alpha \mid (0, t) \in A\}$ is a nonempty set of ordinals.

Thus, this is a well-ordering.

2.9

Problem: Let α and β be ordinals. We define an ordering on $\alpha \times \beta = \{(x, y) \mid x \in \alpha, y \in \beta\}$ by taking $(x, y) < (t, u)$ if $y < u$ or if $y = u$ and $x < t$. Show this is a well-ordering on $\alpha \times \beta$.

Solution. I don't know how to do this problem.

Extra Problem 3

Problem:

(a) If T is $\in$-transitive, then $\bigcup T \subseteq T$.

(b) If $\bigcup T \subseteq T$, then T is $\in$-transitive.

Solution.

(a) I don't know how to do this problem.

(b) Let $\bigcup T \subseteq T$. Let $s \in T$, $x \in s$. Then, $s \subseteq \bigcup T$, so $x \in \bigcup T$, so $x \in T$.

Extra Problem 4

Problem: Prove that ordinal addition is associative.

Solution. Let α, β, γ be ordinals.

$$\alpha + \beta \cong \{0\} \times \alpha \cup \{1\} \times \beta$$

under the lexicographical order.

Additionally,

$$(\alpha + \beta) + \gamma \cong \underbrace{\{0\} \times (\alpha + \beta) \cup \{1\} \times \gamma}_{S},$$

ordered lexicographically.

Finally,

$$\alpha + (\beta + \gamma) \cong \underbrace{\{0\} \times \alpha \cup \{1\} \times (\beta + \gamma)}_{T}$$

ordered lexicographically.

It is enough to show that S is order isomorphic to T, since ordinals are unique up to order isomorphism.

Let $f : S \to T$. Then, for $x \in S$, we have $x \in \{0\} \times (\alpha + \beta)$ or $x \in \{1\} \times \gamma$.

$$f(x) = \begin{cases} (0, a) & x = (0, a);\ \text{for some } a \in \alpha \\ (1, a) & x = (0, a);\ \text{for some } a \in (\alpha + \beta) \setminus \alpha \\ (1, \beta + c) & x = (1, c);\ c \in \gamma \end{cases}$$

We need to show that f is well-defined and order-preserving.