Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $V := \left\{re^{i\theta} \in \mathbb{C} \,\middle|\, -\pi/4 < \theta < \pi/4, r > 0\right\}$. Fix $z_0 \in U$, and let $\mathcal{F} := \{f \in H(U) \mid f(z_0) = 1, \operatorname{im}(f) \subseteq V\}$. Show that $\mathcal{F}$ is normal.

Solution: We observe that a function $f \in H(U)$ if and only if $f(z_0) = 1$ and $\operatorname{im}(f) \subseteq V$, or equivalently, that $e^{i\pi/4} f(z_0) = e^{i\pi/4}$ and $\operatorname{im}(f)$ is a subset of the upper half-plane. Now, by composing with the Cayley Transform, $q(z) = \frac{z-i}{z+i}$, we find that the family

$$\mathcal{G} = \left\{q\left(e^{i\pi/4} f\right) \,\middle|\, f \in \mathcal{F}\right\}$$

is now locally bounded family of holomorphic functions (in fact, it is globally bounded, with every function in $\mathcal{G}$ being bounded above by 1).

Let $(f_n)_n \subseteq \mathcal{F}$. We observe then that $\left(q\left(e^{i\pi/4} f_n\right)\right)_n$ is a sequence in $\mathcal{G}$, meaning that there is a subsequence $\left(q\left(e^{i\pi/4} f_{n_k}\right)\right)_k \to g \colon U \to \mathbb{D}$ for some holomorphic function $g \colon U \to \mathbb{D}$. Since the Cayley Transform has a holomorphic inverse, it follows that $(f_{n_k})_k \to e^{-i\pi/4} q^{-1} \circ g \colon U \to \mathbb{C}$ is a subsequence of $(f_n)_n$ that converges on compact subsets to a holomorphic function, hence $\mathcal{F}$ is normal.

Problem (Problem 2)**:** Let $\mathcal{F} = \{f \in H(\mathbb{D}) \mid \operatorname{im}(f) \subseteq \mathbb{D}\}$. Fix $z_0 \in \mathbb{D}$. Show that there exists a holomorphic function $g \colon \mathbb{D} \to \mathbb{C}$ with $\operatorname{im}(g) \subseteq \mathbb{D}$, $|g'(z_0)| = \max_{f \in \mathcal{F}} |f'(z_0)|$, and $g(z_0) = 0$.

Solution: From Montel's Theorem, we know that the set $\mathcal{F}$ is normal, meaning that $\overline{\mathcal{F}}$ is compact in $H(\mathbb{D})$.

Now, we start by showing that differentiation is a continuous operation. Towards this end, we define the exhaustion

$$K_m = \left\{B\left(0, \frac{m}{m+1}\right) \,\middle|\, m \in \mathbb{N}\right\},$$

and observe that, by the extended maximum modulus principle, for any functions $f_1, f_2 \in \mathbb{D}$,

$$\sup_{z \in K_m} |f_1(z) - f_2(z)| = \sup_{|z| = \frac{m}{m+1}} |f_1(z) - f_2(z)|.$$

Furthermore, we then observe that by the Cauchy estimate,

$$\begin{aligned}
\left|f_1'(z) - f_2'(z)\right| &= \left|\frac{d}{dz}(f_1(z) - f_2(z))\right| \\
&\leqslant \frac{(m+1)}{m} \sup_{|z| = \frac{m}{m+1}} |f_1(z) - f_2(z)| \\
&= \frac{m+1}{m} \|f_1 - f_2\|_{K_m},
\end{aligned}$$

whence

$$\left\|f_1'(z) - f_2'(z)\right\|_{K_m} \leqslant \frac{m+1}{m} \|f_1 - f_2\|_{K_m}.$$

Therefore, we observe that

$$\begin{aligned}
\left\|f_1' - f_2'\right\|_{H(\mathbb{D})} &= \sum_{m=1}^{\infty} 2^{-m} \left\|f_1' - f_2'\right\|_{K_m} \\
&\leqslant \sum_{m=1}^{\infty} 2^{-m} \|f_1 - f_2\|_{K_m} + \sum_{m=1}^{\infty} \frac{1}{m} 2^{-m} \|f_1 - f_2\|_{K_m} \\
&\leqslant 2 \sum_{m=1}^{\infty} 2^{-m} \|f_1 - f_2\|_{K_m}
\end{aligned}$$

$$= 2\|f_1 - f_2\|_{H(\mathbb{D})},$$

meaning that differentiation is 2-Lipschitz, hence continuous. Additionally, since both evaluation and the modulus are continuous, we observe then that the map

$$\begin{aligned} s\colon \overline{\mathcal{F}} &\to \mathbb{R} \\ f &\mapsto |f'(z_0)| \end{aligned}$$

is a continuous map whose domain is a compact space, so there is some $g \in \overline{\mathcal{F}}$ such that $|g'(z_0)| = \sup_{f\in\mathcal{F}}|f'(z_0)|$.

Now, we observe that $g(\mathbb{D}) \subseteq \overline{\mathbb{D}}$, and that since the map

$$B_1(z) = \frac{z - z_0}{1 - \overline{z_0}z}$$

is contained in $\mathcal{F}$ (as was established on a previous homework assignment) with $\left|B_1'(z_0)\right| \geqslant 1$, it follows that $|g'(z_0)| \geqslant 1$, meaning that g is a nonconstant holomorphic function, hence $g(\mathbb{D}) \subseteq \mathbb{D}$ by the open mapping principle.

We claim now that $g(z_0) = 0$. Suppose this were not the case, meaning that there is some $0 < r < 1$ such that $|g(z_0)| = r$. We have established already that $g(z_0) \in \mathbb{D}$. The map

$$h(z) = \frac{g(z) - g(z_0)}{1 - \overline{g(z_0)}g(z)},$$

is thus the composition of g with the function

$$B_2(z) = \frac{z - g(z_0)}{1 - \overline{g(z_0)}z},$$

and since both g and B_2 map $\mathbb{D}$ to $\mathbb{D}$, we have $h = B_2 \circ g$ is a holomorphic function that maps maps $\mathbb{D}$ to $\mathbb{D}$, whence $h \in \mathcal{F}$. Yet,

$$\begin{aligned} |h'(z_0)| &= |g'(z_0)|\frac{1}{1 - |g(z_0)|^2} \\ &= |g'(z_0)|\frac{1}{1 - r^2}, \end{aligned}$$

implying that $|h'(z_0)| > |g'(z_0)|$, contradicting the maximality of $|g'(z_0)|$. Thus, it must be the case that $g(z_0) = 0$.

Problem (Problem 3)**:** Let $(a_n)_n$ be a sequence of nonnegative real numbers such that the radius of convergence of

$$\sum_{n=0}^{\infty} a_n z^n$$

is at least 1. Let

$$\mathcal{F} := \bigcap_{n=0}^{\infty}\left\{ f \in H(\mathbb{D}) \;\middle|\; \left|\frac{f^{(n)}(0)}{n!}\right| \leqslant a_n \right\}.$$

Show that $\mathcal{F}$ is a normal family.

Problem (Problem 5)**:** Let $(f_n)_n$ be a sequence of holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$ that is locally bounded, and suppose there exists a holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ such that the set $\{z \in \mathbb{D} \mid \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in $\mathbb{D}$. Show that $(f_n)_n$ converges uniformly on compact sets to f.

Solution: Since $(f_n)_n$ is locally bounded, it follows that the family $\{f_n \mid n \in \mathbb{N}\}$ is a normal family, by Montel's theorem. In particular, this means that for any subsequence $(f_{n_k})_k$, there is a subsequence of $(n_k)_k$, which we call $\left(n_{k_j}\right)_j$ and a holomorphic function $g_k\colon \mathbb{D} \to \mathbb{C}$ such that

$$\left(f_{n_{k_j}}\right)_j \to g_k$$

on compact subsets. Yet, since uniform convergence on compact subsets implies pointwise convergence, we have that $\{z \in \mathbb{D} \mid g_k(z) = f(z)\}$ has an accumulation point in $\mathbb{D}$, whence each of the g_k are equal to f by the identity theorem.

Now, if it were not the case that $(f_n)_n \to f$ uniformly on compacts, then we would be able to find some subsequence $(f_{n_k})_k$ with $\|f_{n_k} - f\| \geqslant \varepsilon_0$ for some $\varepsilon_0 > 0$ and all k. Yet, since this is a subsequence of $(f_n)_n$, it admits a subsequence converging to f, contradicting the assertion that $\|f_{n_k} - f\| \geqslant \varepsilon_0$ for all k.