These are some notes I'm taking on the Fourier transform for functions with domain on a particular type of subset of $\mathbb{C}$.

Definition: Let $0 < a < \infty$. The class $\mathcal{F}_a$ is the family of holomorphic functions

$$f\colon \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < a\} \to \mathbb{C}$$

for which

$$A_{f,\varepsilon} := \sup_{|\mathrm{Im}(z)|<a} (1 + |\mathrm{Re}(z)|)^{1+\varepsilon}|f(z)|$$

is finite for some $\varepsilon > 0$.

We define

$$\mathcal{F} = \bigcup_{0<a<\infty} \mathcal{F}_a$$

Definition: If $f \in \mathcal{F}$, we define the *Fourier transform* of f by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx.$$

Lemma: If $f \in \mathcal{F}_a$, then for any $0 \leqslant b < a$, we have

$$\hat{f}(\xi) = \begin{cases} \int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i(x-ib)\xi}\,dx & \xi \geqslant 0 \\ \int_{-\infty}^{\infty} f(x + ib)e^{-2\pi i(x+ib)\xi}\,dx & \xi \leqslant 0. \end{cases}$$

Proof. For $0 < b < a$, define $g(z) = f(z)e^{-2\pi i z\xi}$. Let γ_R be the rectangle with corners $-R, R, R - ib, -R - ib$ traversed *clockwise*. As $R \to \infty$, the integral on the top horizontal line tends to $\hat{f}(\xi)$.

The integral over the left vertical line is estimated by

$$\begin{aligned}\left|i\int_{-b}^{0} f(-R + iy)e^{-2\pi i(-R+iy)\xi}\,dy\right| &\leqslant \frac{A_{f,\varepsilon}}{R^{1+\varepsilon}}\int_{-B}^{0} e^{-2\pi y\xi}\,dy \\ &\leqslant \frac{A_{f,\varepsilon}}{R^{1+\varepsilon}},\end{aligned}$$

which tends to zero as $R \to \infty$. A similar bound holds for the integral over the right horizontal line. Since f is holomorphic, Cauchy's Integral theorem implies that

$$0 = \hat{f}(\xi) - \int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i(x-ib)\xi}\,dx,$$

whence

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i(x-ib)\xi}\,dx.$$

An analogous argument holds for $\xi \leqslant 0$ when the rectangle is taken over the upper half-plane. □

Corollary: If $f \in \mathcal{F}_a$, then $\hat{f}(\xi)$ has exponential decay, in the sense that there is some $B \geqslant 0$ such that

$$\left|\hat{f}(\xi)\right| \leqslant Be^{-2\pi b|\xi|}$$

for all $0 \leqslant b < a$.

Proof. For $0 < b < a$, and $\xi \geqslant 0$, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i(x-ib)\xi}\,dx.$$

Therefore, we have

$$\left|\hat{f}(\xi)\right| \leqslant A_{f,\varepsilon}e^{-2\pi b\xi}\int_{-\infty}^{\infty}\frac{1}{(1+|x|)^{1+\varepsilon}}\,dx.$$

Therefore, we get the desired bound for $\xi \geqslant 0$. A similar argument yields the desired bound for $\xi \leqslant 0$. □

Theorem (Fourier Inversion Formula)**:** If $f \in \mathcal{F}$, we have

$$f(x) = \int_{-\infty}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi.$$

Proof. Write

$$\int_{-\infty}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi = \int_{-\infty}^{0}\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi + \int_{0}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi.$$

We consider the second integral. Since $f \in \mathcal{F}_a$ for some a, we let $0 < b < a$. We use the lemma to write

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(u - ib)e^{-2\pi i(u-ib)\xi}\,du.$$

Using the convergence of integration over ξ and Fubini's Theorem,

$$\begin{aligned}
\int_{0}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi &= \int_{0}^{\infty}\int_{-\infty}^{\infty} f(u-ib)e^{-2\pi i(u-ib)\xi}e^{2\pi i x\xi}\,du\,d\xi\\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-2\pi i(u-ib-x)\xi}\,d\xi\,du\\
&= \int_{-\infty}^{\infty} f(u-ib)\frac{1}{2\pi b + 2\pi(u-x)}\,du\\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(u-ib)}{u-ib-x}\,du\\
&= \frac{1}{2\pi i}\int_{L_1}\frac{f(\zeta)}{\zeta - x}\,d\zeta,
\end{aligned}$$

where L_1 is the line $u - ib$ traversed from left to right. Similarly, we may write

$$\int_{-\infty}^{0}\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi = -\frac{1}{2\pi i}\int_{L_2}\frac{f(\zeta)}{\zeta - x}\,d\zeta,$$

where L_2 is the line $u + ib$ traversed from right to left. If we close up this contour into a rectangle, then we find that we get

$$\begin{aligned}
f(x) &= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R}\frac{f(\zeta)}{\zeta - x}\,d\zeta\\
&= \frac{1}{2\pi i}\int_{L_1}\frac{f(\zeta)}{\zeta - x}\,d\zeta - \frac{1}{2\pi i}\int_{L_2}\frac{f(\zeta)}{\zeta - x}\,d\zeta.
\end{aligned}$$

□

The Fourier inversion formula gives access to a converse that provides structural properties of functions.

Lemma: Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous integrable function. Suppose its Fourier transform is such that there exists some $B \geqslant 0$ and $a > 0$ such that $\left|\hat{f}(\xi)\right| \leqslant Be^{-2\pi a|\xi|}$. Then, there exists a holomorphic function $g \in \mathcal{F}_a$ such that $g(x) = f(x)$.

Proof. Define the entire function

$$f_n(z) = \int_{-n}^{n} \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi.$$

For $z \in \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < b\}$, the function

$$g(z) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i z\xi}\,d\xi$$

is well-defined, since the decay condition on $\hat{f}$ ensure the absolute convergence of the integral. By Fourier inversion, $g(x) = f(x)$ for all $x \in \mathbb{R}$. Finally, g is holomorphic, as the sequence converges uniformly to g, as

$$\begin{aligned} |f_n(z) - g(z)| &\leqslant B\left(\int_{-\infty}^{-n} e^{-2\pi(a-b)\xi}\,d\xi + \int_{n}^{\infty} e^{-2\pi(a-b)\xi}\,d\xi\right) \\ &= \frac{2Be^{-2\pi(a-b)n}}{2\pi(a-b)}, \end{aligned}$$

which converges to 0 uniformly as n tends to infinity. ◻

Theorem (Paley–Wiener Theorem)**:** Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, integrable, and have integrable Fourier transform. Then, $\hat{f}$ is supported on an interval of the form $[-M, M]$ for some $M > 0$ if and only if there exists an entire function $g\colon \mathbb{C} \to \mathbb{C}$ for which there exists some $A \geqslant 0$ such that $|g(z)| \leqslant Ae^{2\pi M|z|}$ and $g(x) = f(x)$ for all $x \in \mathbb{R}$.