Amenability in Discrete Groups

Conditions and Applications

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December 19, 2024

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Chapter 0

Prelude

This thesis will be an introduction and broad overview of the theory of amenable groups, which admit a rich set of structures and characterizations. While we will not fully track the development of the theory to its research nowadays, we will discuss the origin of the theory of amenability — namely, the Banach–Tarski paradox — and develop different characterizations for amenability that will draw from group theory, measure theory, general topology, and functional analysis. Most of the necessary material will be covered either in the chapters themselves or in the appendix.

Amenability is a truly fascinating topic that, while its core is in functional analysis, allows significant insights into group theory nonetheless. This project may seem like an utterly daunting undertaking, but amenability is a very natural follow-on to the upper division mathematics curriculum (along with measure theory).

Chapter 1

Introduction: Paradoxical Decompositions

The primary goal of this section will be to introduce the idea of a paradoxical decomposition (and its effects on the analytic properties of $\mathbb{R}^3$) through the Banach–Tarski Paradox. The ultimate goal is to prove the following statement.

Proposition 1.0.1 (General Banach–Tarski Paradox). *If A and B are bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such a sequence of isometries applied to these subsets yields B.*

The existence of the Banach–Tarski paradox throws a wrench into a major idea that we may have about subsets of $\mathbb{R}^3$ — namely, that they always have some "volume" to them that is invariant under isometry, similar to how "area" in $\mathbb{R}^2$ is invariant under isometry.

1.1 Essentials of Group Actions

We begin by discussing some of the basic properties of group actions.

Definition 1.1.1 (Group Action). Let G be a group, and A be a set. A left group action of G onto A is a map $\alpha: G \times A \to A$ that satisfies

- $\alpha(g_1, (g_2, a)) = \alpha(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$;
- $\alpha(e_G, a) = a$ for all $a \in A$.

For the sake of brevity, we write $\alpha(g, a) = g \cdot a$.

Every group action can be represented by a permutation on A.

Definition 1.1.2 (Permutation Representation). For each g, the map $\sigma_g: A \to A$ defined by $\sigma_g(a) = g \cdot a$ is a permutation of A. There is a homomorphism associated to these actions, $\varphi: G \to \mathrm{Sym}(A)$, where $\mathrm{Sym}(A)$ is the symmetric group on the elements of A.

The permutation representation can run in the opposite direction in the following sense — given a nonempty set A and a homomorphism $\psi: G \to \mathrm{Sym}(A)$, we can take $g \cdot a = \psi(g)(a)$, where $\psi(g) = \sigma_g \in \mathrm{Sym}(A)$ is a permutation.

Just as we can pass group actions into permutation representations, and discuss ideas like the kernel of homomorphisms, we can also discuss the kernel of an action.

Definition 1.1.3 (Kernel). The kernel of the action of G on A is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}.$$

The kernel of the group action is the kernel of the permutation representation $\varphi\colon G \to \mathrm{Sym}(A)$.

Definition 1.1.4 (Stabilizer). For each $a \in A$, we define the stabilizer of a under G to be the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

Remark 1. *The kernel of the group action is the intersection of the stabilizers of every element of A.*

For each $a \in A$, G_a is a subgroup of G.

Definition 1.1.5 (Faithful Action). An action is faithful if the kernel of the action is the identity, e_G. Equivalently, the permutation representation $\varphi\colon G \to \mathrm{Sym}(A)$ is injective.

The following definition will be useful in the future as we dig deeper into the idea of paradoxical groups.

Definition 1.1.6 (Free Action). For a set X with G acting on X, the action of G on X is free if, for every $x \in X$, $g \cdot x = x$ if and only if $g = e_G$.

The most important theorem relating to group actions is the orbit-stabilizer theorem. As we prove the following theorem, we will reveal the definition of an orbit as a type of equivalence class.

Theorem 1.1.1 (Orbit-Stabilizer Theorem). *Let G be a group that acts on a nonempty set A. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking the index of the stabilizer of a in G, $|G : G_a|$.*

Proof. We start by seeing that $a \sim a$, as $e_G \cdot a = a$. Similarly, if $a \sim b$, then there exists $g \in G$ such that $a = g \cdot b$. Thus,

$$\begin{aligned} g^{-1} \cdot a &= g^{-1} \cdot (g \cdot b) \\ &= g^{-1} g \cdot b \\ &= e \cdot b \\ &= b, \end{aligned}$$

meaning that $b \sim a$. Finally, if we have $a \sim b$ and $b \sim c$, we have $a = g \cdot b$ and $b = h \cdot c$ for some $g, h \in G$. Therefore,

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$. Thus, the relation $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

We claim there is a bijection between the left cosets of G_a and the elements of $[a]_\sim$.

Define $C_a = \{g \cdot a \mid g \in G\}$, which is the set of elements in the equivalence class of a. Define the map $g \cdot a \mapsto gG_a$. Since $g \cdot a$ is always an element of C_a, this map is surjective. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1} g\right) \cdot a = a$, we have $h^{-1} g \in G_a$, which is only true if $gG_a = hG_a$. Thus, the map is injective.

Since there is a one to one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we know that the number of equivalence classes of a under the action of G is $|G : G_a|$. □

Definition 1.1.7 (Orbit). Let G act on A, and let $a \in A$. The orbit of a under G is the set

$$G \cdot a = \{g \cdot a \in A \mid g \in G\}$$

1.2 Free Groups, Free Products, and the Ping Pong Lemma

A fundamental fact that we will use to show the general Banach–Tarski is the fact that the group of isometries on $\mathbb{R}^3$ (also known as the Euclidean group E(3)) contains a copy of the free group on two generators. We will need to understand some basic facts about free groups to deepen our understanding. This understanding will evolve into the fundamental result of this section — the Ping Pong Lemma.

1.2.1 Essentials of Free Groups

Definition 1.2.1. Let G be a group, and $S \subseteq G$ be a subset. We define the subgroup generated by S to be

$$\langle S \rangle_G = \bigcap \{H \mid S \subseteq H,\ H \text{ a subgroup}\}.$$

We say S generates G if $\langle S \rangle_G = G$.

Generated subgroups can be broadly characterized as follows:

$$\langle S \rangle_G = \left\{ s_1^{a_1} s_2^{a_2} \cdots s_n^{a_n} \;\middle|\; n \in \mathbb{N},\ s_1, \ldots, s_n \in S,\ a_1, \ldots, a_n \in \{-1, 1\} \right\}.$$

We say $\langle S \rangle_G$ is finitely generated if $\operatorname{card}(S) < \infty$.

There are two major ways we can conceive of the free group. One way (used in [Löh17]) characterizes the free group directly through the universal property — the traditional definition of the free group is then shown to satisfy the universal property.

We will not be using this method, though, instead we will be using the construction in [Har00], which constructs a more general object (the free product) on a collection of groups by taking a quotient on the coproduct (or disjoint union), of which the free group is a special case. For the sake of completeness, we will state the universal property characterization.

Definition 1.2.2. Let S be a set. A group F containing S is said to be freely generated if, for every group G, and every map $\phi \colon S \to G$, there is a unique group homomorphism $\varphi \colon F \to G$ that extends φ. The following diagram, where ι denotes the inclusion of S into F, commutes:

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, hook, "\iota"'] & G \\ F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We say F is the free group generated by S.

Definition 1.2.3 (Free Monoid). A monoid is a set with multiplication that is associative and contains a neutral element.

Given a set A, the free monoid on A is the set $W(A)$ of all finite sequences of elements of A. We write an element $W(A)$ as $w = a_1 a_2 \cdots a_n$, with $a_j \in A$ For each j. We identify A with the subset $W(A)$ of words with length 1.

The operation on $W(A)$ is concatenation.

Definition 1.2.4. Let $\{\Gamma_i\}_{i \in I}$ be a family of groups. Set $A = \coprod_{i \in I} \Gamma_i = \{(g_i, i) \mid g_i \in \Gamma_i,\ i \in I\}$ to be the coproduct of the family $\{\Gamma_i\}_{i \in I}$.

Let $\sim$ be the equivalence relation generated by

$$\begin{gathered} w e_i w' \sim w w' \\ w a b w' \sim w c w', \end{gathered}$$

for all $w, w' \in W(A)$, where $e_i \in \Gamma_i$ is the neutral element, and $a, b, c \in \Gamma_i$ with $ab = c$ for some $i \in I$.

The quotient $W(A)/\sim$ with the operation of concatenation is a group, known as the free product of the groups $\{\Gamma_i\}_{i\in I}$. We write it as

$$\bigstar_{i\in I}\Gamma_i.$$

The inverse of the equivalence class for $w = a_1 a_2 \cdots a_n$ is $w^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$. The neutral element is ϵ, denoting the empty word.

A word $w = a_1 a_2 \cdots a_n \in W(A)$ with $a_j \in \Gamma_{i_j}$ is said to be reduced if $i_{j+1} \neq i_j$ and $a_j \neq e_{i_j}$ for each j.

It is the case that free products exist.

Proposition 1.2.1. *Let $\{\Gamma_i\}_{i\in I}$ be a family of groups, with $A = \coprod_{i\in I}\Gamma_i$, and $\bigstar_{i\in I}\Gamma_i = W(A)/\sim$ as in Definition 1.2.4.*

Then, any element in $\bigstar_{i\in I}\Gamma_i$ is represented by a unique reduced word in $W(A)$.

Proof. We start by showing existence. Consider an integer $n \geq 0$ and a reduced word $w = a_1 a_2 \cdots a_n$ in $W(A)$, an element $a \in A$, and the word $aw \in W(A)$.

If k is the index for which $a_1 \in \Gamma_k$, we set

$$\mathcal{R}(aw) = \begin{cases} w & a = e_i \\ aa_1 a_2 \cdots a_n & a \in \Gamma_i,\ a \neq e_i,\ i \neq k \\ ba_2 \cdots a_n & a \in \Gamma_k, aa_1 = b \neq e_k \\ a_2 \cdots a_n & a \in \Gamma_k, a = a_1^{-1} \in \Gamma_k. \end{cases}$$

Then, $\mathcal{R}(aw)$ is another reduced word, and $\mathcal{R}(aw) \sim aw$ by our construction, meaning that any word $w \in W(A)$ is equivalent to some reduced word by inducting on the length of w.

Now, we show uniqueness. For each $a \in A$, let $T(a)$ be the map $w \xmapsto{T(a)} \mathcal{R}(aw)$, which is a self-map on the set of reduced words. For each $w = b_1 b_2 \cdots b_n$, we set $T(w) = T(b_1)\,T(b_2)\cdots T(b_n)$. For $a,b,c \in \Gamma_i$ with $ab = c$, we have $T(a)\,T(b) = T(c)$, and $T(e_i) = \epsilon$ for all $i \in I$.

If w is a reduced word, notice that $T(w)(\epsilon) = w$.

Let w be some word in $W(A)$ with w_1, w_2 being reduced words equivalent to w under the equivalence relation $\sim$. Since $w_1 \sim w_2$, we have $T(w_1) = T(w_2)$, and

$$\begin{aligned} w_1 &= T(w_1)(\epsilon) \\ &= T(w_2)(\epsilon) \\ &= w_2. \end{aligned}$$

Since $w_1 = w_2$, it is the case that the reduced representations of any word $w \in W(A)$ are unique. $\square$

Corollary 1.2.1. *If $\{\Gamma_i\}_{i\in I}$ and $\Gamma = \bigstar_{i\in I}\Gamma_i$ are as in Definition 1.2.4, then for each $i_0 \in I$, the canonical inclusion*

$$\iota\colon \Gamma_{i_0} \hookrightarrow \Gamma,$$

where an element $a \in \Gamma_{i_0}$ maps to its one-letter reduced word representation in Γ, is injective.

Proof. For any $a \in \Gamma_{i_0} \setminus \left\{e_{i_0}\right\}$, the one-letter reduced word representation $\iota(a)$ is unique, and not equivalent to the empty word, meaning $\ker(\iota) = \left\{e_{i_0}\right\}$, so ι is injective. $\square$

We can now define the free group as a special type of free product.

Definition 1.2.5. Let X be a set. The free group over X is the free product of the infinite cyclic groups generated by elements $a \in X$:

$$F(X) = \bigstar_{a\in X} \langle a \rangle.$$

As the free product is in one-to-one correspondence with the collection of reduced words over a collection $\{\Gamma_i\}_{i\in I}$, it is also the case that $F(X)$ is the collection of reduced words with the "alphabet" in $X \sqcup X^{-1}$.

The cardinality of X is the rank of $F(X)$.

We can now state and prove a universal property for free products of groups, which we will then apply to the case of the free group.

Theorem 1.2.1 (Universal Property for Free Products). *Let Γ be a group, and let $\{\Gamma_i\}_{i\in I}$ be a family of groups. Let $\{h_i \colon \Gamma_i \to \Gamma\}_{i\in I}$ be a family of homomorphisms.*

Then, there exists a unique homomorphism $h \colon \bigstar_{i\in I}\Gamma_i \to \Gamma$ that extends each h_{i_0}. The following diagram commutes:

$$\begin{tikzcd} \Gamma_{i_0} \arrow[r, "h_{i_0}"] \arrow[d, "\iota"'] & \Gamma \\ \bigstar_{i\in I}\Gamma_i \arrow[ur, "h"'] & \end{tikzcd}$$

In particular, if Γ is a group, and $\phi \colon X \to \Gamma$ is a set map, there exists a unique homomorphism $\Phi \colon F(X) \to \Gamma$ such that $\Phi(x) = \phi(x)$ for each $x \in X$.

Proof. For a reduced word $a_1 a_2 \cdots a_n \in \bigstar_{i\in I}\Gamma_i$, where $a_j \in \Gamma_{i_j} \setminus \left\{e_{i_j}\right\}$, and $i_{j+1} \neq i_j$ for each j, we set

$$h(w) = h_{i_1}(a_1)\, h_{i_2}(a_2) \cdots h_{i_n}(a_n),$$

which defines h uniquely in terms of the homomorphisms h_i. □

1.2.2 The Ping Pong Lemma and Applications

If we are given an arbitrary group, we may be curious as to whether or not the group (or a subgroup of it) is freely generated. The Ping Pong Lemma allows us to ascertain various sufficient conditions that yield a free group.

Theorem 1.2.2 (Ping Pong Lemma). *Let G be a group that acts on a set X, and let Γ_1, Γ_2 be subgroups of G. Let $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$. Assume Γ_1 contains at least 3 elements, and Γ_2 contains at least 2 elements.*

Suppose there exist nonempty subsets $X_1, X_2 \subseteq X$ with $X_1 \triangle X_2 \neq \varnothing$ such that for all $\gamma \in \Gamma_1$ with $\gamma \neq e_G$,

$$\gamma(X_2) \subseteq X_1,$$

and for all $\gamma \in \Gamma_2$ with $\gamma \neq e_G$,

$$\gamma(X_1) \subseteq X_2.$$

Then, Γ is isomorphic to the free product $\Gamma_1 \star \Gamma_2$.

Proof. Let w be a nonempty reduced word with letters in the disjoint union of $\Gamma_1 \setminus \{e_G\}$ and $\Gamma_2 \setminus \{e_G\}$. We must show that the element of Γ defined by w is not the identity.

If $w = a_1 b_1 a_2 b_2 \cdots a_k$ with $a_1, \dots, a_k \in \Gamma_1 \setminus \{e_G\}$ and $b_1, \dots, b_{k-1} \in \Gamma_2 \setminus \{e_G\}$, then,

$$w(X_2) = a_1 b_1 \cdots a_{k-1} b_{k-1} a_k (X_2)$$

$$
\begin{aligned}
&\subseteq a_1 b_1 \cdots a_{k-1} b_{k-1} (X_1) \\
&\subseteq a_1 b_1 \cdots a_{k-1} (X_2) \\
&\vdots \\
&\subseteq a_1 (X_2) \\
&\subseteq X_1.
\end{aligned}
$$

Seeing as $X_2 \not\subseteq X_1$ (by the definition of symmetric difference), it is the case that $w \neq e_G$.

If $w = b_1 a_2 b_2 a_2 \cdots b_k$, we select $a \in \Gamma_1 \setminus \{e_G\}$, and we find that $awa^{-1} \neq e_G$, meaning $w \neq e_G$. Similarly, if $w = a_1 b_1 \cdots a_k b_k$, we select $a \in \Gamma_1 \setminus \left\{e_G, a_1^{-1}\right\}$, similarly finding that $awa^{-1} \neq e_G$. If $w = b_1 a_2 b_2 \cdots a_k$, then we select $a \in \Gamma_1 \setminus \{1, a_k\}$, and find $awa^{-1} \neq e_G$. □

We can refine Theorem 1.2.2 to the case of "doubles" wherein we find a different (yet more readily applicable) sufficient condition for a group that contains a copy of the free group on two generators.

Corollary 1.2.2 (Ping Pong Lemma for "Doubles"). *Let G act on X, and let A_+, A_-, B_+, B_- be disjoint subsets of X whose union is not equal to X. Then, if*

$$
\begin{aligned}
a \cdot (X \setminus A_-) &\subseteq A_+ \\
a^{-1} \cdot (X \setminus A_+) &\subseteq A_- \\
b \cdot (X \setminus B_-) &\subseteq B_+ \\
b^{-1} \cdot (X \setminus B_+) &\subseteq B_-,
\end{aligned}
$$

then it is the case that $\langle a, b\rangle$ is isomorphic to the free group on two generators.

Proof. We let $A = A_+ \sqcup A_-$, $B = B_+ \sqcup B_-$, $\Gamma_1 = \langle a\rangle$, and $\Gamma_2 = \langle b\rangle$. Then, A, B, Γ_1, Γ_2 satisfy the conditions for Theorem 1.2.2. □

Remark 2. *Instead of typing out "the free group on two generators," we will henceforth use $F(a,b)$ to refer to the free group on two generators.*

We can apply Theorem 1.2.2 to show the existence of a set of isometries of $\mathbb{R}^n$ that is isomorphic to the free group on two generators.

Definition 1.2.6 (Special Orthogonal Group). For $n \in \mathbb{N}$, we define $\mathrm{SO}(n)$ to be the group of all real $n \times n$ matrices A such that $A^T = A^{-1}$ and $\det(A) = 1$.

In terms of an isometry of $\mathbb{R}^3$, the group SO(3) denotes the set of all rotations about any line through the origin.

Theorem 1.2.3. *There are elements $a, b \in SO(3)$ such that $\langle a, b\rangle_{SO(3)} \cong F(a,b)$.*

Proof. We let

$$
\begin{aligned}
a &= \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
a^{-1} &= \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
b &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}
\end{aligned}
$$

$$b^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}.$$

We specify

$$X = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$A_+ = \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, k \in \mathbb{Z}, x \equiv 3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\}$$

$$A_- = \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, k \in \mathbb{Z}, x \equiv -3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\}$$

$$B_+ = \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, k \in \mathbb{Z}, z \equiv 3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}$$

$$B_- = \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, k \in \mathbb{Z}, z \equiv -3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}.$$

To verify that the conditions of Theorem 1.2.2 hold, we calculate

$$\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 3x + 4y \\ -4x + 3y \\ 5z \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 3x - 4y \\ 4x + 3y \\ 5z \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \left(\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 5x \\ 3y - 4z \\ 4y + 3z \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix} \left(\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \frac{1}{5^{k+1}} \begin{pmatrix} 5x \\ 3y + 4z \\ -4y + 3z \end{pmatrix}. \tag{4}$$

We verify that the conditions for Corollary 1.2.2 hold for each of these four conditions.

(1) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_-,$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x + 4y \equiv 3(-4x + 3y)$ modulo 5, and that $z' = 5z \equiv 0$ modulo 5.

(2) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_+,$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x - 4y \equiv -3(4x + 3y)$ modulo 5, and $z' = 5z \equiv 0$ modulo 5.

(3) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_-,$$

we see that $k+1 \in \mathbb{Z}$, $z' = 4y + 3z \equiv 3(3y - 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

(4) For any vector

$$\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_+,$$

we see that $k+1 \in \mathbb{Z}$, $z' = -4y + 3z \equiv -3(3y + 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

Thus, by Theorem 1.2.2 and Corollary 1.2.2, it is the case that $\langle a, b\rangle \cong F(a,b)$. $\square$

1.3 Paradoxical Decompositions in $\mathbb{R}^3$

With the essential facts about free groups and group actions in mind, we can turn our attention to "paradoxical" actions that seem to recreate a set by using some of its disjoint proper subsets.

Definition 1.3.1 (Paradoxical Decompositions and Paradoxical Groups)**.** Let G be a group that acts on a set X, with $E \subseteq X$. We say E is G-paradoxical if there exist pairwise disjoint proper subsets $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ of E and group elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

If G acts on itself by left-multiplication, and G satisfies these conditions, we say G is a paradoxical group.

Example 1.3.1. The free group on two generators, $F(a,b)$, is a paradoxical group.

To see that $F(a,b)$ is a paradoxical group, we let $W(x) = \{w \in F(a,b) \mid w \text{ starts with } x\}$. For instance, $ba^2ba^{-1} \in W(b)$.

Since every word in F is either the empty word, or starts with one of a, b, a^{-1}, b^{-1}, we see that

$$F(a,b) = \left\{e_{F(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For $w \in F(a,b) \setminus W(a)$, it is the case that $a^{-1}w \in W\left(a^{-1}\right)$, so $w \in aW\left(a^{-1}\right)$. Thus, for any $t \in F(a,b)$, $t \in W(a)$ or $t \in F(a,b) \setminus W(a) = aW\left(a^{-1}\right)$, so $F(a,b) = W(a) \sqcup aW\left(a^{-1}\right)$.
Similarly, for any $w \in F(a,b) \setminus W(b)$, it is the case that $b^{-1}w \in W\left(b^{-1}\right)$, so $w \in bW\left(b^{-1}\right)$. Thus, for any $t \in F(a,b)$, $t \in W(b)$ or $t \in F(a,b) \setminus W(b) = bW\left(b^{-1}\right)$. Thus, $F(a,b) = W(b) \sqcup bW\left(b^{-1}\right)$.

We have thus constructed

$$\begin{aligned} F(a,b) &= W(a) \sqcup aW\left(a^{-1}\right) \\ &= W(b) \sqcup bW\left(b^{-1}\right), \end{aligned}$$

a paradoxical decomposition of $F(a,b)$ with the action of left-multiplication.

Now that we understand a little more about paradoxical groups, we now want to understand the actions of paradoxical groups on sets.

Proposition 1.3.1. *Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.*

Proof. Let $A_1,\dots,A_n,B_1,\dots,B_m \subset G$ be pairwise disjoint, and let $g_1,\dots,g_n,h_1,\dots,h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

Let $M \subseteq X$ contain exactly one element from every orbit in X.

Claim. The set $\{g\cdot M \mid g \in G\}$ is a partition of X.

Proof of Claim: Since M contains exactly one element from every orbit in X, it is the case that $G\cdot M = X$, so

$$\bigcup_{g\in G} g\cdot M = X$$

Additionally, for $x,y \in M$, if $g\cdot x = h\cdot y$, then $\left(h^{-1}g\right)\cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Since G acts freely on X, we must have $h^{-1}g = e_G$.

Thus, we can see that $g_1\cdot M \neq g_2\cdot M$, implying $\{g\cdot M \mid g \in G\}$ is a partition of X. □

We define

$$A_j^* = \bigcup_{g\in A_j} g\cdot M,$$

and similarly define

$$B_j^* = \bigcup_{h\in B_j} h\cdot M.$$

As a useful shorthand, we can also write $A_j^* = A_j\cdot M$, and similarly, $B_j^* = B_j\cdot M$, to denote the union of the elements of A_j and B_j respectively acting on M.

Since $\{g\cdot M \mid g\in G\}$ is a partition of X, and $A_1,\dots,A_n,B_1,\dots,B_m \subset G$ are pairwise disjoint, it must be the case that $A_1^*,\dots,A_n^*,B_1^*,\dots,B_m^* \subset X$ are also pairwise disjoint.

For the original $g_1,\dots,g_n,h_1,\dots,h_m$ that defined the paradoxical decomposition of G, we thus have

$$\begin{aligned} \bigcup_{j=1}^{n} g_j\cdot A_j^* &= \bigcup_{j=1}^{n}\left(g_jA_j\right)\cdot M \\ &= G\cdot M \\ &= X, \end{aligned}$$

and

$$\bigcup_{j=1}^{m} h_j\cdot B_j^* = \bigcup_{j=1}^{m}\left(h_jB_j\right)\cdot M$$

$$
\begin{aligned}
&= G \cdot M \\
&= X.
\end{aligned}
$$

Thus, X is G-paradoxical. □

Remark 3. *This proof requires the axiom of choice, as we invoked it to define M to contain exactly one element from every orbit in X.*

Now that we have established $F(a,b)$ as being a paradoxical group, we wish to use it to construct paradoxical decompositions of the unit sphere $S^2 \subseteq \mathbb{R}^3$.

Fact 1.3.1. If H is a paradoxical group, and $H \leq G$, then G is a paradoxical group.

With this fact in mind, we will show that SO(3) is a paradoxical group.

Theorem 1.3.1. *There are rotations A and B that about lines through the origin in $\mathbb{R}^3$ that generate a subgroup of $SO(3)$ isomorphic to $F(a,b)$*

Proof. We take a and b as in the proof of Theorem 1.2.3. □

Remark 4. *Since $SO(n)$ contains a subgroup isomorphic to $SO(3)$ for all $n \geq 3$, it is the case that $SO(n)$ also contains a subgroup isomorphic to $F(a,b)$.*

Since we have shown that SO(3) is paradoxical, as it contains a paradoxical subgroup, we can now begin to examine the action of SO(3) on subsets of $\mathbb{R}^3$.

Theorem 1.3.2 (Hausdorff Paradox). *There is a countable subset D of S^2 such that $S^2 \setminus D$ is $SO(3)$-paradoxical.*

Proof. Let A and B be the rotations in SO(3) that serve as the generators of the subgroup isomorphic to $F(a,b)$.

Since A and B are rotations, so too is any reduced word over $\left\{A, A^{-1}, B, B^{-1}\right\}$. Thus, any such non-empty word contains two fixed points.

We let

$$
F = \left\{x \in S^2 \mid s \text{ is a fixed point for some word } w\right\}.
$$

Since $F(A,B)$ is countably infinite, so too is F. Thus, the union of all these fixed points under the action of all such words w is countable.

$$
D = \bigcup_{w \in F(A,B)} w \cdot F.
$$

Therefore, $F(A,B)$ acts freely on $S^2 \setminus D$, so $S^2 \setminus D$ is SO(3)-paradoxical. □

Unfortunately, the Hausdorff paradox is not enough for us to be able to prove the Banach–Tarski paradox. In order to do this, we need to be able to show that two sets are "similar" under the action of a group.

Definition 1.3.2 (Equidecomposable Sets). Let G act on X, and let $A, B \subseteq X$. We say A and B are G-equidecomposable if there are partitions $\left\{A_j\right\}_{j=1}^n$ of A and $\left\{B_j\right\}_{j=1}^n$ of B, and elements $g_1, \ldots, g_n \in G$, such that for all j,

$$
B_j = g_j \cdot A_j.
$$

We write $A \sim_G B$ if A and B are G-equidecomposable.

Fact 1.3.2. The relation $\sim_G$ is an equivalence relation.

Proof. Let A, B, and C be sets.

To show reflexivity, we can select $g_1 = g_2 = \cdots = g_n = e_G$. Thus, $A \sim_G A$.

To show symmetry, let $A \sim_G B$. Set $\left\{A_j\right\}_{j=1}^{n}$ to be the partition of A, and set $\left\{B_j\right\}_{j=1}^{n}$ to be the partition of B, such that there exist $g_1, \ldots, g_n \in G$ with $g_j \cdot A_j = B_j$. Then,

$$\begin{aligned} g_j^{-1} \cdot \left(g_j \cdot A_j\right) &= g_j^{-1} \cdot B_j \\ A_j &= g_j^{-1} \cdot B_j, \end{aligned}$$

so $B_j \sim_G A_j$.

To show transitivity, let $A \sim_G B$ and $B \sim_G C$. Let $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ be the partitions of A and B respectively and $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$. Let $\left\{B_j\right\}_{j=1}^{m}$ and $\left\{C_j\right\}_{j=1}^{m}$ be partitions of B and C, and $h_1, \ldots, h_m \in G$, such that $h_j \cdot B_j = C_j$.

We refine the partition of A to A_{ij} by taking $A_{ij} = g_i^{-1}\left(B_i \cap B_j\right)$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then, $\left(h_j g_i\right) \cdot A_{ij}$ maps the refined partition of A to C, so A and C are G-equidecomposable. $\square$

Fact 1.3.3. For $A \sim_G B$, there is a bijection $\phi \colon A \to B$ by taking $C_i = C \cap A_i$, and mapping $\phi(C_i) = g_i \cdot C_i$.

In particular, this means that for any subset $C \subseteq A$, it is the case that $C \sim \phi(C)$.

We can now use this equidecomposability to glean information about the existence of paradoxical decompositions.

Proposition 1.3.2. *Let G act on X, with $E, E' \subseteq X$ such that $E \sim_G E'$. Then, if E is G-paradoxical, then so too is E'.*

Proof. Let $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$ be pairwise disjoint, with $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$\begin{aligned} E &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

We let

$$A = \bigsqcup_{i=1}^{n} A_i$$
$$B = \bigsqcup_{j=1}^{m} B_j.$$

It follows that $A \sim_G E$ and $B \sim_G E$, since we can take the partition of A to be $A_1, \ldots, A_n$, and partition E by taking $g_i \cdot A_i$ for $i = 1, \ldots, n$, and similarly for B.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$. Thus, there is a paradoxical decomposition of E' in $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. $\square$

We will now show that S^2 is SO(3) paradoxical.

Proposition 1.3.3. *Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \setminus D$ are $SO(3)$-equidecomposable.*

Proof. Let L be a line in $\mathbb{R}^3$ such that $L \cap D = \varnothing$. Such an L must exist since S^2 is uncountable.

Define $\rho_\theta \in \mathrm{SO}(3)$ to be a rotation about L by an angle of θ. For a fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{ x \in D \mid \rho_\theta^n \cdot x \in D \right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

We define $W_n = \{\theta \mid R_{n,\theta} \neq \varnothing\}$. Since the map $\theta \mapsto \rho_\theta^n \cdot x$ into D is injective, it is the case that W_n is countable. Therefore,

$$W = \bigcup_{n \in \mathbb{N}} W_n$$

is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. We define ρ_ω to be a rotation about L by ω. Then, for every $n, m \in \mathbb{N}$, we have

$$\rho_\omega^n \cdot D \cap \rho_\omega^m \cdot D = \varnothing.$$

We define $\widetilde{D} = \bigsqcup_{n=0}^{\infty} \rho_\omega^n D$. Note that

$$\begin{aligned} \rho_\omega \cdot \widetilde{D} &= \rho_\omega \cdot \bigsqcup_{n=0}^{\infty} \rho_\omega^n \cdot D \\ &= \bigsqcup_{n=1}^{\infty} \rho_\omega^n \cdot D \\ &= \widetilde{D} \setminus D, \end{aligned}$$

meaning $\widetilde{D}$ and D are SO(3)-equidecomposable.

Thus, we have

$$\begin{aligned} S^2 &= \widetilde{D} \sqcup \left(S^2 \setminus \widetilde{D} \right) \\ &\sim_{\mathrm{SO}(3)} \left(\rho_\omega \cdot \widetilde{D} \right) \sqcup \left(S^2 \setminus \widetilde{D} \right) \\ &= \left(\widetilde{D} \setminus D \right) \sqcup \left(S^2 \setminus \widetilde{D} \right) \\ &= S^2 \setminus D, \end{aligned}$$

establishing S^2 and $S^2 \setminus D$ as SO(3)-equidecomposable.

In particular, this means S^2 is also SO(3)-paradoxical. □

To prove the Banach–Tarski paradox, we need a slightly larger group than SO(3) — one that includes translations in addition to the traditional rotations.

Definition 1.3.3 (Euclidean Group)**.** The Euclidean group, E(n), consists of all isometries of a Euclidean space. An isometry of a Euclidean space consists of translations, rotations, and reflections.

Corollary 1.3.1 (Weak Banach–Tarski Paradox)**.** *Every closed ball in* $\mathbb{R}^3$ *is* $E(3)$*-paradoxical.*

Proof. We only need to show that $B(0,1)$ is E(3)-paradoxical. To do this, we start by showing that $B(0,1) \setminus \{0\}$ is SO(3)-paradoxical.

Since S^2 is SO(3)-paradoxical, there exists pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset S^2$ and elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in \mathrm{SO}(3)$ such that

$$S^2 = \bigcup_{i=1}^{n} g_i \cdot A_i$$

$$= \bigcup_{j=1}^{m} h_j \cdot B_j.$$

Define

$$A_i^* = \{tx \mid t \in (0,1], x \in A_i\}$$
$$B_j^* = \left\{ty \mid t \in (0,1], y \in B_j\right\}.$$

Then, $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^* \subset B(0,1) \setminus \{0\}$ are pairwise disjoint, and

$$\begin{aligned} B(0,1) \setminus \{0\} &= \bigcup_{i=1}^{n} g_i \cdot A_i^* \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j^*. \end{aligned}$$

Thus, we have established that $B(0,1) \setminus \{0\}$ is E(3)-paradoxical.

Now, we want to show that $B(0,1) \setminus \{0\}$ and $B(0,1)$ are E(3)-equidecomposable. Let $x \in B(0,1) \setminus \{0\}$, and let ρ be a rotation through x by a line not through the origin such that $\rho^n \cdot 0 \neq \rho^m \cdot 0$ when $n \neq m$.

Let $D = \{\rho^n \cdot 0 \mid n \in \mathbb{N}\}$. We can see that $\rho \cdot D = D \setminus \{0\}$, and that D and $\rho \cdot D$ are E(3)-equidecomposable. Thus,

$$\begin{aligned} B(0,1) &= D \sqcup (B(0,1) \setminus D) \\ &\sim_{\mathrm{E}(3)} (\rho \cdot D) \sqcup (B(0,1) \setminus D) \\ &= (D \setminus \{0\}) \sqcup (B(0,1) \setminus D) \\ &= B(0,1) \setminus \{0\}. \end{aligned}$$

Therefore, $B(0,1)$ is E(3)-equidecomposable. □

In order to prove the general case of the Banach–Tarski paradox, we need one more piece of mathematical machinery.

Our relation of $A \sim_G B$ is useful, but in order to show the general case, we want to refine the relation slightly.

Definition 1.3.4. Let G act on a set X with $A, B \subseteq X$. We write $A \preceq_G B$ if A is equidecomposable with a subset of B.

Fact 1.3.4. The relation $\preceq_G$ is a reflexive and transitive relation.

Proof. To see reflexivity, we can see that since $A \sim_G A$, and $A \subseteq A$, $A \preceq_G A$.

To see transitivity, let $A \preceq_G B$ and $B \preceq_G C$. Then, there exist $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_{\alpha,i}$ for each i, where $A \sim_G B_\alpha \subseteq B$. Similarly, there exist $h_1, \ldots, h_m \in G$ such that $h_j \cdot B_j = C_{\beta,j}$ for each j, where $B \sim_G C_\beta \subseteq C$.

We take a refinement of B by taking intersections $B_{\alpha,ij} = B_{\alpha,i} \cap B_j$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We define $C_{\beta,\alpha,ij} = h_j \cdot B_{\alpha,ij}$ for each $j = 1, \ldots, m$. Then, $h_j g_i \cdot A_i = C_{\beta,\alpha,ij}$, meaning $A \sim_G C_{\beta,\alpha,ij} \subseteq C_\beta \subseteq C$, so $A \preceq_G C$. □

We know from Fact 1.3.3 that $A \preceq_G B$ implies the existence of a bijection $\phi\colon A \to B' \subseteq B$, meaning $\phi\colon A \hookrightarrow B$ is an injection. Similarly, if $B \preceq_G A$, then Fact 1.3.3 implies the existence of an injection $\psi\colon B \hookrightarrow A$.

One may ask if an analogue of the Cantor–Schröder–Bernstein theorem exists in the case of the relation $\preceq_G$, implying that the preorder established in Fact 1.3.4 is indeed a partial order. The following theorem establishes this result.

Theorem 1.3.3. *Let G act on X, and let $A, B \subseteq X$. If $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$.*

Proof. Let $B' \subseteq B$ with $A \sim_G B'$, and let $A' \subseteq A$ with $B \sim_G A'$. Then, we know from Fact 1.3.3 that there exist bijections $\phi\colon A \to B'$ and $\psi\colon B \to A'$.

Define $C_0 = A \setminus A'$, and $C_{n+1} = \psi(\phi(C_n))$. We set

$$C = \bigcup_{n \geq 0} C_n.$$

Since $\psi^{-1}(\psi(\phi(C_n))) = \phi(C_n)$, we have

$$\psi^{-1}(A \setminus C) = B \setminus \phi(C).$$

Having established in Fact 1.3.3 that for any subset of $C \subseteq A$, $C \sim_G \phi(C)$, we also see that $A \setminus C \sim_G B \setminus \phi(C)$.

Thus, we can see that

$$\begin{aligned} A &= (A \setminus C) \sqcup C \\ &\sim_G (B \setminus \phi(C)) \sqcup \phi(C) \\ &= B. \end{aligned}$$

□

Finally, we are able to prove Proposition 1.0.1.

Proof of Proposition 1.0.1: By symmetry, it is enough to show that $A \preceq_{\mathrm{E}(3)} B$.

Since A is bounded, there exists $r > 0$ such that $A \subseteq B(0, r)$.

Let $x_0 \in B^\circ$. Then, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq B$.

Since $B(0, r)$ is compact (hence totally bounded), there are translations $g_1, \ldots, g_n$ such that

$$B(0, r) \subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup g_n \cdot B(x_0, \varepsilon).$$

We select translations $h_1, \ldots, h_n$ such that $h_j \cdot B(x_0, \varepsilon) \cap h_k \cdot B(x_0, \varepsilon) = \varnothing$ for $j \neq k$. We set

$$S = \bigcup_{j=1}^{n} h_j \cdot B(x_0, \varepsilon).$$

Each $h_j \cdot B(x_0, \varepsilon) \subseteq S$ is E(3)-equidecomposable with any arbitrary closed ball subset of $B(x_0, \varepsilon)$, it is the case that $S \preceq B(x_0, \varepsilon)$.

Thus, we have

$$\begin{aligned} A &\subseteq B(0, r) \\ &\subseteq g_1 \cdot B(x_0, \varepsilon) \cup \cdots \cup b_n \cdot B(x_0, \varepsilon) \\ &\preceq S \\ &\preceq B(x_0, \varepsilon) \\ &\preceq B. \end{aligned}$$

□

Chapter 2

Characterizations through Paradoxicality (or lack thereof): Tarski's Theorem

Ultimately, the reason the Banach–Tarski paradox "works" is because the paradoxical group $F(a,b)$, lacks a property known as amenability — specifically, that a group admitting a paradoxical decomposition is not amenable. Before we go further into the characterizations of amenability that will be discussed in Chapter 3, we will start by proving Tarski's theorem, which will establish the equivalence between amenability and non-paradoxicality.

Theorem 2.0.1 (Tarski's Theorem). *Let G be a group that acts on a set X, and let $E \subsetneq X$ be nonempty. There is a finitely additive translation-invariant measure $\mu\colon P(X) \to [0,\infty]$ with $\mu(E) \in (0,\infty)$ if and only if E is not G-paradoxical.*

We can even prove one of the directions of Tarski's theorem now.

Proof of the Forward Direction of Theorem 2.0.1: Let E be G-paradoxical. Suppose toward contradiction that such a translation-invariant finitely additive ν existed with $\nu(E) \in (0,\infty)$.

Let $A_1,\ldots,A_n,B_1,\ldots,B_m \subseteq E$ be pairwise disjoint, and let $t_1,\ldots,t_n,s_1,\ldots,f_m \in G$ such that

$$\begin{aligned} E &= \bigsqcup_{i=1}^{n} t_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} s_j \cdot B_j. \end{aligned}$$

Then, it would be the case that

$$\begin{aligned} \nu(E) &= \nu\left(\bigsqcup_{i=1}^{n} t_i \cdot A_i\right) \\ &= \sum_{i=1}^{n} \nu\left(t_i \cdot A_i\right) \\ &= \sum_{i=1}^{n} \nu\left(A_i\right), \end{aligned}$$

and

$$\nu(E) = \sum_{j=1}^{m} \nu\left(B_j\right).$$

However, this also yields

$$\begin{aligned}
\nu(E) &= \nu\left(\left(\bigsqcup_{i=1}^{n} A_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} B_j\right)\right) \\
&= \sum_{i=1}^{n} \nu(A_i) + \sum_{j=1}^{m} \nu\left(B_j\right) \\
&= \sum_{i=1}^{n} \nu(t_i \cdot A_i) + \sum_{j=1}^{m} \nu\left(x_j \cdot B_j\right) \\
&= \nu(E) + \nu(E) \\
&= 2\nu(E).
\end{aligned}$$

implying that $\nu(E) = 0$ or $\nu(E) = \infty$. □

The opposite direction, unfortunately, will be significantly harder to prove. We will need to know some results from graph theory, understand the properties of the type semigroup of an action, and use some results on commutative semigroups to show the existence of a mean.

2.1 A Little Bit of Graph Theory

To prove the reverse direction of Tarski's theorem, we need to develop some machinery from graph theory that will allow us to prove that a certain semigroup we will construct in the next section satisfies the cancellation identity.

We start by defining graphs and paths, before proving a special case of Hall's theorem, ultimately extending to the infinite case with König's theorem.

Definition 2.1.1 (Graphs and Paths). A graph is a triple (V, E, ϕ), with V, E nonempty sets and $\phi \colon E \to P_2(V)$ a map from E to the set of all unordered subset pairs of V.

For $e \in E$, if $\phi(e) = \{v, w\}$, then we say v and w are the endpoints of e, and e is incident on v and w.

A path in (V, E, ϕ) is a finite sequence $(e_1, \ldots, e_n)$ of edges, with a finite sequence of vertices $(v_0, \ldots, v_n)$, such that $\phi(e_k) = \{v_{k-1}, v_k\}$.

The degree of a vertex, $\deg(v)$, is the number of edges incident on v.

We define the neighbors of $S \subseteq V$ to be the set of all vertices $v \in V \setminus S$ such that v is an endpoint to an edge incident on S. We denote this set $N(S)$.

Definition 2.1.2 (Bipartite Graphs and k-Regularity). Let (V, E, ϕ) be a graph, with $k \in \mathbb{N}$.

(i) If $\deg(v) = k$ for each $v \in V$, we say (V, E, ϕ) is k-regular.

(ii) If $V = X \sqcup Y$, with each edge in E having one endpoint in X and one endpoint in Y, then we say V is bipartite, and write (X, Y, E, ϕ).

Definition 2.1.3 (Perfect Matching). Let (X, Y, E, ϕ) be a bipartite graph. Let $A \subseteq X$ and $B \subseteq Y$. A perfect matching of A and B is a subset $F \subseteq E$ with

(i) each element of $A \cup B$ is an endpoint of exactly one $f \in F$;

(ii) all endpoints of edges in F are in $A \cup B$.

Definition 2.1.4 (Hall Condition)**.** We say a bipartite graph (X, Y, E, ϕ) satisfies the Hall Condition on X if, for all $S \subseteq X$, $|N(S)| \geq |S|$.

Equivalently, we say a (finite) collection of not necessarily distinct finite sets $\mathcal{X} = \{X_i\}_{i=1}^{n}$ satisfies the marriage condition if and only if for all subcollections $\mathcal{Y}_k = \left\{X_{i_k}\right\}_{k=1}^{m}$,

$$|\mathcal{Y}_k| \leq \left|\bigcup_{k=1}^{m} X_{i_k}\right|.$$

Remark 5. *These two formulations of the Hall condition are equivalent regarding an X-perfect matching.*

Theorem 2.1.1 (Hall's Theorem for Finite k-Regular Bipartite Graphs)**.** *Let (X, Y, E, ϕ) be a k-regular bipartite graph for some $k \in \mathbb{N}$, and let $V = X \sqcup E$ be finite. Then, there is a perfect matching of X and Y.*

Proof. Note that since $|E| = k|K| = k|Y|$, it is the case that $|X| = |Y|$.

Let $M \subseteq V$ be any subset. We will show that $|N(M)| \geq |M|$ — that is, (X, Y, E, ϕ) satisfies the Hall condition.

Let $M_X = M \cap X$ and $M_Y = M \cap Y$, where $M = M_X \sqcup M_Y$. Let $[M_X, N(M_X)]$ be the set of edges with endpoints in M_X and $N(M_X)$, and $[M_Y, N(M_Y)]$ be the set of edges with endpoints in M_Y and $N(M_Y)$. We also let $[X, N(M_X)]$ denote the set of edges with endpoints in X and $N(M_X)$, and similarly, $[Y, N(M_Y)]$ is the set of edges with endpoints in Y and $N(M_Y)$.

We can see that $[M_X, N(M_X)] \subseteq [X, N(M_X)]$, and similarly, $[M_Y, N(M_Y)] \subseteq [Y, N(M_Y)]$.

Since $|[M_X, N(M_X)]| = k|M_X|$ and $|[X, N(M_X)]| = k|N(M_X)|$, we have

$$|M_X| \leq |N(M_X)|,$$

and similarly,

$$|M_Y| \leq |N(M_Y)|.$$

Thus, $|M| \leq |N(M)|$.

We will now show that there is an X-perfect matching. Suppose toward contradiction that F is a maximal perfect matching on $A \subseteq X$ and $B \subseteq Y$ with $X \setminus A \neq \emptyset$.

Then, there is $x \in X \setminus A$. Consider $Z \subseteq V$ consisting of all vertices z such that there exists a F-alternating path $(e_1, \dots, e_n)$ between $z \in Z$ and x.

It cannot be the case that $Z \cap Y$ is empty, since the number of neighbors of x is greater than or equal to 1 by the Hall condition — if it were the case that $Z \cap Y$ were empty, we could add an edge to F consisting of x and one element of $N(\{x\})$, which would contradict the maximality of F.

Consider a path traversing along Z, $(e_1, \dots, e_n)$. It must be the case that $e_n \in F$, or else we would be able to "flip" the matching F by exchanging e_i with e_{i+1} for $e_i \in F$, which would contradict the maximality of F yet again. Thus, every element of $Z \cap Y$ is satisfied by F, so $Z \cap Y \subseteq B$.

Since each element in $Z \cap Y$ is paired with exactly one element of $Z \cap X$ (with one left over), it is the case that $|Z \cap X| = |Z \cap Y| + 1$.

Suppose toward contradiction that there exists $y \in N(Z \cap X)$ with $y \notin Z \cap Y$. Then, there exists $v \in Z \cap X$ and $e \in E$ such that $\phi(e) = \{v, y\}$. However, this means v is connected via a path to x, meaning $y \in Z$, so $y \in Z \cap Y$. Thus, we must have $N(Z \cap X) = Z \cap Y$.

Therefore,

$$\begin{aligned} |Z \cap X| &= |Z \cap Y| + 1 \\ &= |N(Z \cap X)| + 1, \end{aligned}$$

which contradicts the fact that (X, Y, E, ϕ) satisfies the Hall condition. Therefore, $A = X$.

By symmetry, there is a perfect matching of X and Y in (X, Y, E, ϕ). ☐

Remark 6. *An equivalent formulation to Hall's theorem states that there is a system of distinct representatives on $\mathcal{X}$, which is a set $\{x_k\}_{k=1}^n$ such that $x_k \in X_k$ and $x_i \neq x_j$ for $i \neq j$.*

This implies the existence of an injection $f : \mathcal{X} \hookrightarrow \bigcup_{k=1}^n X_k$, such that $f(X_k) \in X_k$.

Theorem 2.1.2 (Infinite Hall's Theorem). *Let $\mathcal{G} = \{X_i\}_{i \in I}$ be a collection of (not necessarily distinct) finite sets. If, for every finite subcollection $\mathcal{Y} = \left\{X_{i_k}\right\}_{k=1}^n$,*

$$n \leq \left| \bigcup_{k=1}^n X_{i_k} \right|,$$

then there is a choice function on G.

Proof. We endow each $X_i \in \{X_i\}_{i \in I}$ with the discrete topology. Since each X_i is finite, each X_i is compact.

Thus, by Tychonoff's theorem, it is the case that $\prod_{i \in I} X_i$ is compact.

For every finite subset $Y \subseteq \mathcal{G}$, we define

$$S_Y = \left\{ f \in \prod_{i \in I} X_i \,\middle|\, f|_Y \text{ is injective} \right\}.$$

The injectivity of $f|_Y$ is equivalent to the existence of a system of distinct representatives on Y. Since Y satisfies the Hall condition, each S_Y is nonempty. Additionally, for any net of functions $f_\alpha \in S_Y$ with $\lim_\alpha f_\alpha = f$, it is the case that $f_\alpha|_Y$ is injective, so $f|_Y$ is injective, meaning S_Y is closed.

We define $F = \{S_Y \mid Y \subseteq \mathcal{G} \text{ finite}\}$. For finite $Y_1, Y_2 \subseteq \mathcal{G}$, every system of distinct representatives in $Y_1 \cup Y_2$ is necessarily a system of distinct representatives on Y_1 and a system of distinct representatives on Y_2, meaning $S_{Y_1 \cup Y_2} \subseteq S_{Y_1} \cap S_{Y_2}$. Thus, F has the finite intersection property.

Since $\prod_{i \in I} X_i$ is compact, $\bigcap F$ is nonempty, where the intersection is taken over all finite subsets of $\mathcal{G}$. For any $f \in \bigcap F$, f is necessarily a choice function. ☐

Remark 7. *This is equivalent to the existence of an injection $f : \mathcal{G} \hookrightarrow \bigcup_{i \in I} X_i$.*

We will use this infinite case of Hall's theorem to prove König's theorem.

Theorem 2.1.3 (König's Theorem). *Let (X, Y, E, ϕ) be a k-regular bipartite graph (not necessarily finite). Then, there is a perfect matching of X and Y.*

Proof. If $k = 1$, it is clear that there is a perfect matching in (X, Y, E, ϕ) consisting of the edges in (X, Y, E, ϕ).

Let $k \geq 2$. Since any finite subset of X satisfies the Hall condition, as displayed in the proof of Theorem 2.1.1, there is some X-perfect matching in (X, Y, E, ϕ). We call this X-perfect matching F. There is an injection $f: X \hookrightarrow Y$ following the edges in F.

Similarly, since any finite subset of Y satisfies the Hall condition, there is some Y-perfect matching in (X, Y, E, ϕ). We call this Y-perfect matching G. There is an injection $g: Y \hookrightarrow X$ following the edges of G.

Consider the subgraph $(X, Y, F \cup G, \phi|_{F \cup G})$. The injections f and g still hold in this graph. By the Cantor–Schröder–Bernstein theorem, there is a bijection $h: X \to Y$ in $(X, Y, F \cup G, \phi|_{F \cup G})$. □

2.2 Type Semigroups

Definition 2.2.1. Let G be a group that acts on a set X.

(i) We define $X^* = X \times \mathbb{N}_0$, and

$$G^* = \{(g, \pi) \mid g \in G, \pi \in \mathrm{Sym}(\mathbb{N}_0)\}.$$

(ii) If $A \subseteq X^*$, the values of n for which there is an element of A whose second coordinate is n are called the levels of A.

Fact 2.2.1. If $E_1, E_2 \subseteq X$, then $E_1 \sim_G E_2$ if and only if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$ for all $m, n \in \mathbb{N}_0$.

Proof of Fact 2.2.1: Let $E_1 \sim_G E_2$. Then, there exist pairwise disjoint $A_1, \ldots, A_n \subset E_1$, pairwise disjoint $B_1, \ldots, B_n \subset E_2$, and elements $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$. We select the permutation $\pi_i \in \mathrm{Sym}(\mathbb{N}_0)$ such that $\pi_i(n) = m$ and $\pi_i(m) = n$ for each i. Then,

$$(g_i, \pi_i) \cdot (A_i \cdot \{n\}) = B_i \cdot \{m\}.$$

Similarly, if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$, then of the pairwise disjoint subsets

$$A_1 \times \{n\}, \ldots, A_n \times \{n\} \subset E_1 \times \{n\}$$

and

$$B_1 \times \{m\}, \ldots, B_n \times \{m\} \subset E_2 \times \{m\},$$

we set $A_1, \ldots, A_n \subset E_1$ and $B_1, \ldots, B_n \subset E_2$. Similarly, for

$$(g_1, \pi_1), \ldots, (g_n, \pi_n) \in G^*$$

such that

$$(g_i, \pi_i) \cdot A_i \times \{n\} = B_i \times \{m\},$$

we select $g_1, \ldots, g_n \in G$. Then, by definition,

$$g_i \cdot A_i = B_i$$

for each i. Thus, $E_1 \sim_G E_2$. □

Definition 2.2.2. Let G be a group that acts on X, and let G^*, X^* be defined as in 2.2.1.

(i) A set $A \subseteq X^*$ is said to be bounded if it has finitely many levels.

(ii) If $A \subseteq X^*$ is bounded, the equivalence class of A with respect to G^*-equidecomposability is called the type of A, which is denoted $[A]$.

(iii) If $E \subseteq X$, we write $[E] = [E \times \{0\}]$.

(iv) Let $A, B \subseteq X^*$ be bounded with $k \in \mathbb{N}_0$ such that for

$$B' = \{(b, n+k) \mid (b, n) \in B\},$$

we have $B' \cap A = \varnothing$. Then, $[A] + [B] = [A \sqcup B']$. Note that $[B'] = [B]$.

(v) We define

$$\mathcal{S} = \{[A] \mid A \subseteq X^* \text{ bounded}\}$$

under the addition defined in (iv) to be the type semigroup of the action of G on X.

Fact 2.2.2. Addition is well-defined in $(\mathcal{S}, +)$, and $(\mathcal{S}, +)$ is a well-defined commutative semigroup with identity $[\varnothing]$.

Proof of Fact 2.2.2: To show that addition is well-defined, we let $[A_1] = [A_2]$, and $[B_1] = [B_2]$. Without loss of generality, $A_1 \cap B_1 = \varnothing$ and $A_2 \cap B_2 = \varnothing$.

By the definition of the type, $A_1 \sim_{G^*} A_2$ and $B_1 \sim_{G^*} B_2$, meaning

$$A_1 \sqcup B_1 \sim_{G^*} A_2 \sqcup B_2,$$

so

$$\begin{aligned}[A_1] + [B_1] &= [A_1 \sqcup B_1] \\ &= [A_2 \sqcup B_2] \\ &= [A_2] + [A_2],\end{aligned}$$

meaning addition is well-defined.

Since addition is well-defined, and $A \sqcup B = B \sqcup A$, we can see that addition is also commutative. We also have

$$\begin{aligned}[A] + [\varnothing] &= [A \sqcup \varnothing] \\ &= [A],\end{aligned}$$

so $[\varnothing]$ is the identity on $\mathcal{S}$.

Finally, since for any $[A], [B] \in \mathcal{S}$, A and B have finitely many levels, it is the case that $A \cup B$ has finitely many levels for any A and B, so $[A] + [B] \in \mathcal{S}$. $\square$

Definition 2.2.3. For any commutative semigroup $\mathcal{S}$ with $\alpha \in S$ and $n \in \mathbb{N}$, we define

$$n\alpha = \underbrace{\alpha + \cdots + \alpha}_{n \text{ times}}$$

Definition 2.2.4. For $\alpha, \beta \in \mathcal{S}$, if there exists $\gamma \in \mathcal{S}$ such that $\alpha + \gamma = \beta$, we write $\alpha \leq \beta$.

Fact 2.2.3. If G is a group acting on X with corresponding type semigroup $\mathcal{S}$, then the following are true.

(i) If $\alpha, \beta \in \mathcal{S}$ with $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$.

(ii) A set $E \subseteq X$ is G-paradoxical if and only if $[E] = 2[E]$.

Proof of Fact 2.2.3: Let G act on X, and let $\mathcal{S}$ be the corresponding type semigroup.

(i) If $[A] \leq [B]$, then there exists $C_1 \in \mathcal{S}$ such that $[A] + [C_1] = [B]$. Without loss of generality, $C_1 \cap A = \varnothing$, meaning $[B] = [A \sqcup C_1]$. Thus, $A \sqcup C_1 \sim_{G^*} B$, meaning $B \preceq_{G^*} A$.

Similarly, if $[B] \leq [A]$, then $B \preceq_{G^*} A$. By Theorem 1.3.3, it is thus the case that $A \sim_{G^*} B$.

(ii) Let E be G-paradoxical. Then, $E \sim_G \bigsqcup_{i=1}^{n} A_i$ and $E \sim_G \bigsqcup_{j=1}^{m} B_j$ for some disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$. Thus, we have

$$\begin{aligned} [E] &= \left[\left(\bigsqcup_{i=1}^{n} A_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} B_j\right)\right] \\ &= \left[\bigsqcup_{i=1}^{n} A_i\right] + \left[\bigsqcup_{j=1}^{m} B_j\right] \\ &= 2\,[E]. \end{aligned}$$

Similarly, if $[E] = 2\,[E]$, then there exist A and B such that

$$\begin{aligned} [E] &= [A] + [B] \\ &= [A \sqcup B], \end{aligned}$$

meaning A and B are each G-equidecomposable with E, so E is G-paradoxical.

□

We can now prove the cancellation identity, which we will be useful as we construct our desired finitely additive measure.

Theorem 2.2.1 (Cancellation Identity on $\mathcal{S}$). *Let $\mathcal{S}$ be the type semigroup for some group action, and let $\alpha, \beta \in \mathcal{S}$, $n \in \mathbb{N}$ such that $n\alpha = n\beta$. Then, $\alpha = \beta$.*

Proof. Let $n\alpha = n\beta$. Then, there are two disjoint bounded subsets $E, E' \subseteq X^*$ with $E \sim_{G^*} E'$, and pairwise disjoint subsets $A_1, \ldots, A_n \subseteq E$, $B_1, \ldots, B_n \subseteq E'$ such that

- $E = A_1 \cup \cdots \cup A_n$, $E' = B_1 \cup \cdots \cup B_n$
- $\left[A_j\right] = \alpha$ and $\left[B_j\right] = \beta$ for each $j = 1, \ldots, n$.

Let $\chi \colon E \to E'$ be a bijection as in Fact 1.3.3, with $\phi_j \colon A_1 \to A_j$, $\psi_j \colon B_1 \to B_j$ also being bijections as in Fact 1.3.3; here we define ϕ_1 and ψ_1 to be the identity map.

For each $a \in A_1$ and $b \in B_1$, we define

$$\begin{aligned} \overline{a} &= \{a, \phi_2(a), \ldots, \phi_n(a)\} \\ \overline{b} &= \{b, \psi_2(b), \ldots, \psi_n(b)\}. \end{aligned}$$

We construct a graph by letting $X = \{\overline{a} \mid a \in A_1\}$ and $Y = \left\{\overline{b} \;\middle|\; b \in B_1\right\}$, and, for each j, define edges $\left\{\overline{a}, \overline{b}\right\}$ if $\chi\left(\phi_j(a)\right) \in \overline{b}$.

Since χ is a bijection, for each $j = 1, \ldots, n$, $\chi\left(\phi_j(a)\right)$ must be an element of B_k for some k, and since $\{B_k\}_{k=1}^{n}$ are disjoint, $\chi\left(\phi_j(a)\right)$ is an element of exactly one B_k. Thus, the graph is n-regular.

By Theorem 2.1.3, this graph has a perfect matching F. As a result, for each $\overline{a} \in X$, there is a unique $\overline{b} \in Y$ and a unique edge $\left\{\overline{a}, \overline{b}\right\} \in F$ such that $\chi\left(\phi_j(a)\right) = \psi_k(b)$ for some $j, k \in \{1, \ldots, n\}$.

We define

$$C_{j,k} = \left\{a \in A_1 \,\middle|\, \left\{\overline{a}, \overline{b}\right\} \in F,\ \chi\left(\phi_j(a)\right) = \psi_k(b)\right\}$$
$$D_{j,k} = \left\{b \in B_1 \,\middle|\, \left\{\overline{a}, \overline{b}\right\} \in F,\ \chi\left(\phi_j(a)\right) = \psi_k(b)\right\}.$$

Therefore, we must have $\psi_k^{-1} \circ \chi \circ \phi_j$ is a bijection from $C_{j,k}$ to $D_{j,k}$, so $C_{j,k} \sim_{G^*} D_{j,k}$.

Since $C_{j,k}$ and $D_{j,k}$ are partitions of A_1 and B_1 respectively, it follows that $A_1 \sim_{G^*} B_1$, so $\alpha = \beta$. □

Corollary 2.2.1. *Let $\mathcal{S}$ be the type semigroup of some group action, and let $\alpha \in \mathcal{S}$ and $n \in \mathbb{N}$ such that $(n+1)\alpha \leq n\alpha$. Then, $\alpha = 2\alpha$.*

Proof. We have

$$\begin{aligned} 2\alpha + n\alpha &= (n+1)\alpha + \alpha \\ &\leq n\alpha + \alpha \\ &= (n+1)\alpha \\ &\leq n\alpha. \end{aligned}$$

Inductively repeating this argument, we get $n\alpha \geq 2n\alpha$; since $n\alpha \leq 2n\alpha$ by definition, we must have $n\alpha = 2n\alpha$, so $\alpha = 2\alpha$. □

Remark 8. *We will call such an α a paradoxical element.*

2.3 Two Results on Commutative Semigroups

Now that we are aware of paradoxical elements and the relationship between G-paradoxicality and the properties of the particular elements of the type semigroup (Fact 2.2.3), we will now relate these properties to finitely additive measures of sets by using the following lemma and theorem.

Lemma 2.3.1. *Let $\mathcal{S}$ be a commutative semigroup, with $\mathcal{S}_0 \subseteq \mathcal{S}$ finite, and $\epsilon \in \mathcal{S}_0$ satisfying the following assumptions:*

(a) $(n+1)\epsilon \nleq n\epsilon$ for all $n \in \mathbb{N}$ (i.e., that ϵ is non-paradoxical);

(b) for each $\alpha \in \mathcal{S}$, there is $n \in \mathbb{N}$ such that $\alpha \leq n\epsilon$.

Then, there is a set function $\nu \colon \mathcal{S}_0 \to [0, \infty]$ that satisfies the following conditions:

(i) $\nu(\epsilon) = 1$;

(ii) for $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathcal{S}_0$ with $\alpha_1 + \cdots + \alpha_n \leq \beta_1 + \cdots \beta_m$,

$$\sum_{j=1}^{n} \nu\left(\alpha_j\right) \leq \sum_{j=1}^{m} \nu\left(\beta_j\right).$$

Proof. We will prove this result by inducting on the cardinality of $\mathcal{S}_0$.

We start with $|\mathcal{S}_0| = 1$. In that case, we define $\nu(\epsilon) = 1$, satisfying condition (i). To satisfy condition (ii), we see that for $n, m \in \mathbb{N}$ with $n\epsilon \leq m\epsilon$, if $n \geq m+1$, then $(m+1)\epsilon \leq n\epsilon \leq m\epsilon$, implying that $\epsilon = 2\epsilon$, which contradicts assumption (a).

Let $\alpha_0 \in \mathcal{S}_0 \setminus \{\epsilon\}$. The induction hypothesis says there is a set function satisfying conditions (i) and (ii), $\nu \colon \mathcal{S}_0 \setminus \{\alpha_0\} \to [0, \infty]$.

For $r \in \mathbb{N}$, there are $\gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q \in \mathcal{S} \setminus \{\alpha_0\}$ such that

$$\delta_1 + \cdots + \delta_q + r\alpha_0 \leq \gamma_1 + \cdots + \gamma_p. \tag{†}$$

Consider the set N defined as follows:

$$N = \left\{ \frac{1}{r}\left(\sum_{j=1}^{p} \nu\left(\gamma_j\right) - \sum_{j=1}^{q} \nu\left(\delta_j\right) \right) \middle| \gamma_j, \delta_j \text{ satisfy } (†) \right\}. \tag{‡}$$

We define the extension of ν as follows:

$$\nu(\alpha_0) = \inf N.$$

This infimum is well-defined since, by assumption (b), there is some $n \in \mathbb{N}$ such that $\alpha_0 \leq n\epsilon$, and $\nu(\epsilon)$ is defined.

Now, we must show that this extension of ν satisfies condition (ii).

Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathcal{S}_0 \setminus \{\alpha_0\}$ and $s, t \in \mathbb{N}_0$ such that

$$\alpha_1 + \cdots + \alpha_n + s\alpha_0 \leq \beta_1 + \cdots + \beta_m + t\alpha_0. \tag{$*$}$$

We will verify condition (ii) in the three following cases.

Case 0: If $s = t = 0$, then the induction hypothesis states that $(*)$ satisfies condition (ii).

Case 1: Let $s = 0$ and $t > 0$. We want to show that

$$\sum_{j=1}^{n} \nu\left(\alpha_j\right) \leq t\nu(\alpha_0) + \sum_{j=1}^{m} \nu\left(\beta_j\right),$$

which implies that

$$\nu(\alpha_0) \geq \frac{1}{t}\left(\sum_{j=1}^{n} \nu\left(\alpha_j\right) - \sum_{j=1}^{m} \nu\left(\beta_j\right) \right).$$

By the definition of infimum, it suffices to show that for $r \in \mathbb{N}$ and $\delta_1, \dots, \delta_q, \gamma_1, \dots, \gamma_p \in \mathcal{S} \setminus \{\alpha_0\}$ satisfying (†), it is the case that

$$\frac{1}{r}\left(\sum_{j=1}^{p} \nu\left(\gamma_j\right) - \sum_{j=1}^{q} \nu\left(\delta_j\right) \right) \geq \frac{1}{t}\left(\sum_{j=1}^{n} \nu\left(\alpha_j\right) - \sum_{j=1}^{m} \nu\left(\beta_j\right) \right).$$

Multiplying $(*)$ by r on both sides, and adding $t\delta_1 + \cdots + t\delta_q$ to both sides, we have

$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q \leq r\beta_1 + \cdots + r\beta_m + t(r\alpha_0) + t\delta_1 + \cdots + t\delta_q.$$

Substituting (†), we find

$$r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q \leq r\beta_1 + \cdots + r\beta_m + t\gamma_1 + \cdots + t\gamma_p.$$

Applying the induction hypothesis, we have

$$r\sum_{j=1}^{n} \nu\left(\alpha_j\right) + t\sum_{j=1}^{q} \nu\left(\delta_j\right) \leq r\sum_{j=1}^{m} \nu\left(\beta_j\right) + t\sum_{j=1}^{p} \nu\left(\gamma_j\right),$$

yielding

$$\frac{1}{r}\left(\sum_{j=1}^{p} \nu\left(\gamma_j\right) - \sum_{j=1}^{q} \nu\left(\delta_j\right) \right) \geq \frac{1}{t}\left(\sum_{j=1}^{n} \nu\left(\alpha_j\right) - \sum_{j=1}^{m} \nu\left(\beta_j\right) \right).$$

Case 2: Let $s > 0$. For $z_1, \ldots, z_t \in N$ (‡), we need to show that

$$s\nu(\alpha_0) + \sum_{j=1}^{n} \nu\left(\alpha_j\right) \leq z_1 + \cdots + z_t + \sum_{j=1}^{n} \nu\left(\beta_j\right).$$

Without loss of generality, we can set $z_1, \ldots, z_n = z$, as for each $z \in N$, $z \geq \nu(\alpha_0)$.

As in Case 1, we multiply (∗) by r, add $t\delta_1 + \cdots + t\delta_q$ to both sides, and substitute with (†), yielding

$$\begin{aligned} r\alpha_1 + \cdots + r\alpha_n + rs\alpha_0 + t\delta_1 + \cdots + t\delta_q &\leq r\beta_1 + \cdots + r\beta_m + t(r\alpha_0) + t\delta_1 + \cdots + t\delta_q \\ r\alpha_1 + \cdots + r\alpha_n + t\delta_1 + \cdots + t\delta_q + rs\alpha_0 &\leq r\beta_1 + \cdots + r\beta_m + t\gamma_1 + \cdots + t\gamma_p. \end{aligned}$$

Defining

$$z = \frac{1}{r}\left(\sum_{j=1}^{p} \nu\left(\gamma_j\right) - \sum_{j=1}^{q} \nu\left(\delta_j\right)\right),$$

we get

$$\begin{aligned} s\nu(\alpha_0) + \sum_{j=1}^{n} \nu\left(\alpha_j\right) &\leq \sum_{j=1}^{n} \nu\left(\alpha_j\right) + \frac{s}{sr}\left(r\sum_{j=1}^{m} \nu\left(\beta_j\right) - r\sum_{j=1}^{n} \nu\left(\alpha_j\right) + t\sum_{j=1}^{p} \nu\left(\gamma_j\right) - t\sum_{j=1}^{q} \nu\left(\delta_j\right)\right) \\ &= tz + \sum_{j=1}^{m} \nu\left(\beta_j\right). \end{aligned}$$

Thus, we have shown that ν extends in a manner that satisfies conditions (i) and (ii). □

We can "upgrade" our finitely additive set function to a semigroup homomorphism as follows.

Theorem 2.3.1. *Let $(\mathcal{S}, +)$ be a commutative semigroup with identity element 0, and let $\epsilon \in \mathcal{S}$. Then, the following are equivalent:*

(i) $(n+1)\epsilon \leq n\epsilon$ *for all* $n \in \mathbb{N}$;

(ii) *there is a semigroup homomorphism* $\nu\colon (\mathcal{S}, +) \to ([0, \infty], +)$ *such that* $\nu(\epsilon) = 1$.

Proof. To show that (ii) implies (i), we let $\nu\colon (\mathcal{S}, +) \to ([0, \infty], +)$ be a semigroup homomorphism with $\nu(\epsilon) = 1$. Then,

$$\begin{aligned} \nu((n+1)\epsilon) &= (n+1)\nu(\epsilon) \\ &= n+1 \\ &> n \\ &= n\nu(\epsilon) \\ &= \nu(n\epsilon), \end{aligned}$$

meaning that $(n+1)\epsilon \nleq n\epsilon$.

To show that (i) implies (ii), we suppose that for each $\alpha \in \mathcal{S}$, there is $n \in \mathbb{N}$ such that $\alpha \leq n\epsilon$ — for any such α for which this is not the case, we define $\nu(\alpha) = \infty$.

For a finite subset $\mathcal{S}_0 \subseteq \mathcal{S}$ with $\epsilon \in \mathcal{S}_0$, we define $M_{\mathcal{S}_0}$ to be the set of all $\kappa\colon \mathcal{S} \to [0, \infty]$ such that

- $\kappa(\epsilon) = 1$;

- $\kappa(\alpha + \beta) = \kappa(\alpha) + \kappa(\beta)$ for $\alpha, \beta, \alpha + \beta \in \mathcal{S}_0$.

Since we assume condition (i), we know that such a κ with $\kappa(\epsilon) = 1$ exists. Additionally, since

$$\alpha + \beta \leq (\alpha + \beta)$$

and

$$(\alpha + \beta) \leq \alpha + \beta,$$

it is the case that

$$\kappa(\alpha + \beta) \leq \kappa(\alpha) + \kappa(\beta) \leq \kappa(\alpha + \beta),$$

meaning $\kappa(\alpha + \beta) = \kappa(\alpha) + \kappa(\beta)$. Thus, $M_{\mathcal{S}_0}$ is nonempty. It is also the case that $M_{\mathcal{S}_0}$ is closed, since any net of functions $\kappa_p \colon \mathcal{S} \to [0, \infty]$ with $\kappa_p(\epsilon) = 1$ and $\kappa_p(\alpha + \beta) = \kappa_p(\alpha) + \kappa_p(\beta)$ will necessarily satisfy these conditions in the limit.

We let $[0, \infty]^{\mathcal{S}} = \{\kappa \mid \kappa : \mathcal{S} \to [0, \infty]\}$ be equipped with the product topology. By Tychonoff's theorem, $[0, \infty]^{\mathcal{S}}$ is compact.

Since, for any $\mathcal{S}_1, \ldots, \mathcal{S}_n$ finite, it is the case that

$$M_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n} \subseteq M_{\mathcal{S}_1} \cap \cdots \cap M_{\mathcal{S}_n},$$

since any such $\kappa \in M_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n}$ must necessarily be in every $M_{\mathcal{S}_i}$. Thus, the family

$$\left\{ M_{\mathcal{S}_0} \,\middle|\, \mathcal{S}_0 \subseteq \mathcal{S} \text{ finite} \right\}$$

has the finite intersection property. Thus, by compactness, there is some ν such that

$$\nu \in \bigcap \left\{ M_{\mathcal{S}_0} \,\middle|\, \mathcal{S}_0 \subseteq \mathcal{S} \text{ finite} \right\},$$

with $\nu(\epsilon) = 1$ and, for all $\alpha, \beta \in \mathcal{S}$, since $\nu \in M_{\{\alpha, \beta, \alpha + \beta\}}$, $\nu(\alpha + \beta) = \nu(\alpha) + \nu(\beta)$. □

2.4 Proof of Tarski's Theorem

Finally, we are able to prove Tarski's Theorem.

Proof of Theorem 2.0.1: Let $\mathcal{S}$ be the type semigroup of the action of G on X.

Suppose E is not G-paradoxical. Then, $[E] \neq 2[E]$, meaning $(n+1)[E] \not\leq n[E]$ for all $n \in \mathbb{N}$.

Thus, there is a map $\nu \colon \mathcal{S} \to [0, \infty]$ with $\nu([E]) = 1$. The map $\mu \colon P(X) \to [0, \infty]$ defined by

$$\mu(A) = \nu([A])$$

is the desired finitely additive measure. □

Chapter 3

Characterizations using Invariant States: the Følner Condition

Having proven Tarski's theorem, we can turn our attention to a more definite understanding of amenability. We will use theorems and techniques from functional analysis to help understand the space $\ell_\infty(G)$, which will open a wide variety of characterizations for amenability, beyond that which was established in Tarski's theorem.

3.1 Means and Invariant States

Definition 3.1.1. Let G be a group, with $P(G)$ denoting its power set.

An invariant mean on G is a set function $m\colon P(G) \to [0,1]$ which satisfies, for all $t \in G$ and $E, F \subseteq G$,

- $m(G) = 1$;
- $m(E \sqcup F) = m(E) + m(F)$;
- $m(tE) = m(E)$.

We say G is amenable if G admits a mean.

The mean m is, in other words, a translation-invariant probability measure on the measurable space $(G, P(G))$.

We have shown in the proof of Theorem 2.0.1 that an equivalent condition for amenability is that the group is not paradoxical.

Using some essential results in group theory, we can establish some preliminary results on subgroups and quotient groups.

Proposition 3.1.1. *Let G be an amenable group with $H \leq G$. Then, the following are true:*

(1) H is amenable;

(2) for $H \trianglelefteq G$, G/H is amenable.

Proof.

(1) Let R be a right transversal for H, wherein we select one element of each right coset of H to make up R.

If m is a mean for G, we set $\lambda\colon P(H) \to [0,1]$ defined by

$$\lambda(E) = m(ER).$$

We have

$$\begin{aligned}\lambda(H) &= m(HR)\\ &= m(G)\\ &= 1.\end{aligned}$$

We claim that if $E \cap F = \varnothing$, then $ER \cap FR = \varnothing$. Suppose toward contradiction this is not the case. Then, $xr_1 = yr_2$ for some $x \in E$, $y \in F$, and $r_1, r_2 \in R$. Then, we must have $r_2 r_1^{-1} = y^{-1}x \in H$, meaning $r_1 = r_2$ as, by definition, R contains exactly one element of each right coset. Thus, $x = y$, so $E \cap F \neq \varnothing$.

We then have

$$\begin{aligned}\lambda(E \sqcup F) &= m((E \sqcup F)R)\\ &= m(ER \sqcup FR)\\ &= m(ER) + m(FR)\\ &= \lambda(E) + \lambda(F),\end{aligned}$$

and

$$\begin{aligned}\lambda(sE) &= m(sER)\\ &= m(ER)\\ &= \lambda(E).\end{aligned}$$

(2) Let $\pi\colon G \to G/H$ be the canonical projection, defined by $\pi(t) = tH$. We define

$$\lambda\colon P(G/H) \to [0,1]$$

by $\lambda(E) = m\left(\pi^{-1}(E)\right)$. We have

$$\begin{aligned}\lambda(G/H) &= m\left(\pi^{-1}(G/H)\right)\\ &= m(G)\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(E \sqcup F) &= m\left(\pi^{-1}(E \sqcup F)\right)\\ &= m\left(\pi^{-1}(E) \sqcup \pi^{-1}(F)\right)\\ &= m\left(\pi^{-1}(E)\right) + m\left(\pi^{-1}(F)\right)\\ &= \lambda(E) + \lambda(F).\end{aligned}$$

To show translation-invariance, we let $sH = \pi(s) \in G/H$, and $E \subseteq G/H$. Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for $r \in s\pi^{-1}(E)$, we have $r = st$ for $t \in \pi(E)$, so $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$.

Additionally, if $r \in \pi^{-1}(\pi(s)E)$, we have $\pi(r) \in \pi(s)E$, so $\pi\left(s^{-1}r\right) \in E$, meaning $s^{-1}r \in \pi^{-1}E$.

Thus,

$$\begin{aligned}\lambda(\pi(s)E) &= m\left(\pi^{-1}(\pi(s)E)\right)\\ &= m\left(s\pi^{-1}(E)\right)\\ &= m\left(\pi^{-1}(E)\right)\\ &= \lambda(E).\end{aligned}$$

$\square$

Now that we understand some useful properties of means in relation to groups and subgroups, we turn our attention toward finding means on groups. In order to do this, we turn our attention towards the space $\ell_\infty(G)$, which allows us to use theories from functional analysis to better understand means on G.

Definition 3.1.2. Let G be a group.

(1) The space $\mathcal{F}(G,\mathbb{R})$ is defined by

$$\mathcal{F}(G,\mathbb{R}) = \{f \mid f : G \to \mathbb{R} \text{ is a function}\}.$$

(2) A function $f \in \mathcal{F}(G,\mathbb{R})$ is called positive if $f(x) \geq 0$ for all $x \in G$.

(3) A function $f \in \mathcal{F}(G,\mathbb{R})$ is called simple if $\mathrm{Ran}(f)$ is finite. We let

$$\Sigma = \{f \in \mathcal{F}(G,\mathbb{R}) \mid f \text{ is simple}\}.$$

Fact 3.1.1. It is the case that $\Sigma \subseteq \mathcal{F}(G,\mathbb{R})$ is a linear subspace.

Definition 3.1.3. For $E \subseteq G$, we define

$$\mathbb{1}_E : G \to \mathbb{R}$$

by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

This is the characteristic function of E.

Fact 3.1.2. We have

$$\mathrm{span}\{\mathbb{1}_E \mid E \subseteq G\} = \Sigma.$$

Proof. We see that $\mathbb{1}_E \in \Sigma$ for any $E \subseteq G$, and that Σ is a subspace.

If $\phi \in \Sigma$ with $\mathrm{Ran}(\phi) = \{t_1, \ldots, t_n\}$, where t_i are distinct, we set

$$E_i = \phi^{-1}(\{t_i\}),$$

yielding

$$\phi = \sum_{i=1}^{n} t_i \mathbb{1}_{E_i}.$$

$\square$

Definition 3.1.4. (1) A function $f \in \mathcal{F}(G,\mathbb{R})$ is bounded if there exists $M > 0$ such that $\mathrm{Ran}(f) \subseteq [-M, M]$.

(2) The space $\ell_\infty(G)$ is defined by

$$\ell_\infty(G) = \{f \in \mathcal{F}(G,\mathbb{R}) \mid f \text{ is bounded}\}.$$

(3) The norm on $\ell_\infty(G)$ is defined by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

Proposition 3.1.2. *The space $\ell_\infty(G)$ is complete. Additionally, $\overline{\Sigma} = \ell_\infty(G)$.*

Proof. Let $(f_n)_n$ be $\|\cdot\|$-Cauchy in $\ell_\infty(G)$. Then, for all $x \in G$, it is the case that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n - f_m)(x)| \\ &\leq \|f_n - f_m\|, \end{aligned}$$

meaning $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We must show that $f \in \ell_\infty(G)$, and $\|f_n - f\| \to 0$.

We have

$$\begin{aligned} |f(x)| &= \left|\lim_{n\to\infty} f_n(x)\right| \\ &= \lim_{n\to\infty} |f_n(x)| \\ &\leq \limsup_{n\to\infty} \|f_n\| \\ &\leq C, \end{aligned}$$

as Cauchy sequences are always bounded. Thus, $\sup_{x\in G} |f(x)| \leq C$.

Given $\varepsilon > 0$, we find N such that for all $m, n \geq N$, $\|f_n - f_m\| \leq \varepsilon$. Thus, for $x \in G$, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq \|f_n - f_m\| \\ &\leq \varepsilon. \end{aligned}$$

Taking $m \to \infty$, we get $|f_n(x) - f(x)| \leq \varepsilon$, for all $n \geq N$, so $\|f_n - f\| \leq \varepsilon$ for all $n \geq N$.

For $f \in \ell_\infty(G)$, let $\operatorname{Ran}(f) \subseteq [-M, M]$ for some $M > 0$. Let $\varepsilon > 0$. Since $[-M, M]$ is compact, it is totally bounded, so we can find intervals $I_1, \dots, I_n$ with $[-M, M] = \bigsqcup_{k=1}^{n} I_k$, with the length of each I_k less than ε.

Set $E_k = f^{-1}(I_k)$. Pick some $t_k \in I_k$. We set

$$\phi = \sum_{i=1}^{n} t_k \mathbb{1}_{E_k}.$$

Then, it is the case that $\left\|\phi - f\right\| < \varepsilon$. □

Corollary 3.1.1. *For any $f \in \ell_\infty(G)$, there is a sequence $(\phi_n)_n$ with $\left\|\phi_n - f\right\| \to 0$. If $f \geq 0$, then we can select $\phi_n \geq 0$.*

Now that we understand how simple functions relate to $\ell_\infty(G)$, we start by defining a translation action on $\ell_\infty(G)$, from which we will be able to convert the idea of means into invariant elements of the state space of the dual of $\ell_\infty(G)$.

Proposition 3.1.3. *Let G be a group. There is an action*

$$\lambda_s \colon G \to \operatorname{Isom}(\ell_\infty(G))$$

defined by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

Proof. We have

$$\lambda_s(f + \alpha g)(t) = (f + \alpha g)\left(s^{-1}t\right)$$

$$\begin{aligned}
&= f\left(s^{-1}t\right)\alpha g\left(s^{-1}t\right)\\
&= \lambda_s(f)(t) + \alpha\lambda_s(g)(t)\\
&= (\lambda_s(f) + \alpha\lambda_s(g))(t).
\end{aligned}$$

Thus, λ_s is linear. Additionally,

$$\begin{aligned}
\|\lambda_s(f)\| &= \sup_{t\in G}|\lambda_s(f)(t)|\\
&= \sup_{t\in G}\left|f\left(s^{-1}t\right)\right|\\
&= \|f\|,
\end{aligned}$$

and

$$\begin{aligned}
\|\lambda_s(f) - \lambda_s(g)\| &= \|\lambda_s(f-g)\|\\
&= \|f-g\|,
\end{aligned}$$

meaning λ_s is an isometry.

We have

$$\begin{aligned}
\lambda_s \circ \lambda_r(f)(t) &= \lambda_r(f)\left(s^{-1}t\right)\\
&= \lambda_r\left(r^{-1}s^{-1}t\right)\\
&= f\left((sr)^{-1}t\right)\\
&= \lambda_{sr}(f)(t),
\end{aligned}$$

establishing that $\lambda_s \circ \lambda_r = \lambda_{sr}$.

By a similar process, we find that $\lambda_s(\mathbb{1}_E) = \mathbb{1}_{sE}$ for any $E \subseteq G$ and $s \in G$. □

Definition 3.1.5. A state on $\ell_\infty(G)$ is a continuous linear functional $\mu \in (\ell_\infty(G))^*$ such that the following are true:

- μ is positive;
- $\mu(\mathbb{1}_G) = 1$.

A state is called left-invariant if

$$\mu(\lambda_s(f)) = \mu(f).$$

Example 3.1.1. The evaluation functional, $\delta_x : \ell_\infty \to \mathbb{R}$, defined by

$$\delta_x(f) = f(x),$$

is a state. However, it is not necessarily invariant, as

$$\begin{aligned}
\delta_x(\lambda_s(f)) &= \lambda_s(f)(x)\\
&= f\left(s^{-1}x\right)\\
&\neq f(x).
\end{aligned}$$

However, we can use the evaluation functional to create an invariant state. If G is finite, we define

$$\mu = \frac{1}{|G|}\sum_{x\in G}\delta_x,$$

which is indeed an invariant state.

We can characterize states slightly differently, which will enable us to show the equivalence between invariant states and means.

Lemma 3.1.1.

(1) If μ is a state on $\ell_\infty(G)$, then

$$\|\mu\|_{\text{op}} = 1.$$

(2) If $\mu \in (\ell_\infty(G))^*$ is such that

$$\begin{aligned}\|\mu\|_{\text{op}} &= \mu(\mathbb{1}_G)\\ &= 1,\end{aligned}$$

then μ is positive and a state.

Proof.

(1) Let μ be a state. Given $f \in \ell_\infty(G)$, we have

$$\begin{aligned}\|f\|\mathbb{1}_G - f &\geq 0\\ \|f\|\mathbb{1}_G + f &\geq 0,\end{aligned}$$

so

$$\begin{aligned}0 &\leq \mu(\|f\|\mathbb{1}_G - f)\\ &= \|f\|\mu(\mathbb{1}_G) - \mu(f)\end{aligned}$$

meaning

$$\mu(f) \leq \|f\|.$$

Additionally,

$$\begin{aligned}0 &\leq \mu(\|f\|\mathbb{1}_G + f)\\ &= \|f\|\mu(\mathbb{1}_G) + \mu(f),\end{aligned}$$

meaning

$$-\mu(f) \leq \|f\|.$$

Thus, we have $|\mu(f)| \leq \|f\|$, so $\|\mu\|_{\text{op}} \leq 1$. However, since $\mu(\mathbb{1}_G) = 1$, we must have $\|\mu\|_{\text{op}} = 1$.

(2) Suppose $\|\mu\|_{\text{op}} = \mu(\mathbb{1}_G) = 1$. Let $f \geq 0$. Set $g = \frac{1}{\|f\|_u} f$.

Then, $\text{Ran}(g) \subseteq [0,1]$, and $\text{Ran}(g - \mathbb{1}_G) \subseteq [-1,1]$. Thus, $\|g - \mathbb{1}_G\|_u \leq 1$.

Since $\|\mu\|_{\text{op}} = 1$, we must have

$$\begin{aligned}|\mu(g - \mathbb{1}_G)| &\leq 1\\ |\mu(g) - 1| &\leq 1,\end{aligned}$$

and since $\mu(\mathbb{1}_G) = 1$, we have $\mu(g) \in [0,2]$. Thus, $\mu(f) = \|f\|\mu(g) \geq 0$.

□

To show the equivalence between means and invariant states, we need to be able to characterize the state space on $(\ell_\infty(G))^*$. To do this, we make use of some results from functional analysis.

If X is a normed vector space, then the topology on X^* induced by X^{**} is known as the weak* topology. The weak* topology is the topology of pointwise convergence in X^* — a net $(\varphi_\alpha)_\alpha$ converges to φ in the weak* topology if and only if, for all $\hat{x} \in X^{**}$, we have

$$(\hat{x}(\varphi_\alpha))_\alpha \to \hat{x}(\varphi),$$

or by the definition of X^{**}, this is equivalent to

$$(\varphi_\alpha(x)) \to \varphi(x)$$

for all $x \in X$.

We state some important results in functional analysis here without proof. The proofs of these results can be found in functional analysis textbooks such as [Rud73].

Theorem 3.1.1 (Hahn–Banach Continuous Extension Theorem). *Let X be a normed vector space, $E \subseteq X$ a subspace, and $\varphi \in E^*$ a bounded linear functional. Then, there exists a continuous $\psi \in X^*$ such that $\|\varphi\|_{\mathrm{op}} = \|\psi\|_{\mathrm{op}}$, and $\psi|_E = \varphi$.*

Theorem 3.1.2 (Hahn–Banach Separation Theorems). *Let X be a normed vector space.*

(1) Given a nonzero $x_0 \in X$, there is a $\varphi \in X^*$ with $\|\varphi\|_{\mathrm{op}} = 1$ and $\varphi(x_0) = \|x\|$. We call φ a norming functional.

(2) Given a proper closed subspace $E \subseteq X$ and $x_0 \in X \setminus E$, there is a $\varphi \in X^*$ such that $\varphi|_E = 0$, $\|\varphi\|_{\mathrm{op}} = 1$, and $\varphi(x) = \mathrm{dist}_E(x)$ for all $x \in X$.

Theorem 3.1.3 (Banach–Alaoglu Theorem). *Let X be a normed vector space.*

(1) The closed unit ball in the dual space, B_{X^*}, is compact in the w^* topology.

(2) A subset $C \subseteq X$ is w^*-compact if and only if C is w^*-closed and norm bounded.

Corollary 3.1.2. *The set of states in $(\ell_\infty(G))^*$ forms a w^*-compact subset of $B_{(\ell_\infty(G))^*}$.*

Proof. From the Banach–Alaoglu Theorem, we only need to show that the set of states, $S(\ell_\infty(G))$, is w^*-closed, as every element of $S(\ell_\infty(G))$ has norm 1.

Let $f \in \ell_\infty(G)$ be positive, and let $(\varphi_i)_i$ be a net in $S(\ell_\infty(G))$ with $(\varphi_i)_i \xrightarrow{w^*} \varphi \in (\ell_\infty(G))^*$. From Lemma 3.1.1, we must show that φ is positive and $\varphi(\mathbb{1}_G) = 1$.

We start by seeing that, since each φ_i is a state, we have $\varphi_i(f) \geq 0$ for each $i \in I$, so we must have $\varphi(f) \geq 0$.

Similarly, since $\varphi_i(\mathbb{1}_G) = 1$ for each $i \in I$, and $(\varphi_i)_i \xrightarrow{w^*} \varphi$, we have $\varphi(\mathbb{1}_G) = 1$. Thus, by Lemma 3.1.1, we have that $S(\ell_\infty(G))$ is w^*-closed. □

Now, we may show the correspondence between invariant states and means.

Proposition 3.1.4. *If $\mu \in (\ell_\infty(G))^*$ is a state, then $m\colon P(G) \to [0,1]$ defined by $m(E) = \mu(\mathbb{1}_E)$ is a finitely additive probability measure on G.*

Moreover, if μ is invariant, then m is a mean.

Proof. We have

$$\begin{aligned} m(G) &= \mu(\mathbb{1}_G) \\ &= 1 \end{aligned}$$

$$\begin{aligned} m(\varnothing) &= \mu(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} m(E \sqcup F) &= \mu(\mathbb{1}_{E \sqcup F}) \\ &= \mu(\mathbb{1}_E + \mathbb{1}_F) \\ &= \mu(\mathbb{1}_E) + \mu(\mathbb{1}_F) \\ &= m(E) + m(F). \end{aligned}$$

Additionally, since $0 \leq \mathbb{1}_E \leq \mathbb{1}_G$, we have $0 \leq \mu(\mathbb{1}_E) \leq 1$, so $0 \leq m(E) \leq 1$.

If μ is invariant, then

$$\begin{aligned} m(sE) &= \mu(\mathbb{1}_{sE}) \\ &= \mu(\lambda_s(\mathbb{1}_E)) \\ &= \mu(\mathbb{1}_E) \\ &= m(E). \end{aligned}$$

□

Proposition 3.1.5. *If G admits a mean, then $(\ell_\infty(G))^*$ admits an invariant state.*

Proof. Let m be a mean. Define $\mu_0 \colon \Sigma \to \mathbb{R}$ by

$$\mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right) = \sum_{k=1}^{n} t_k m(E_k).$$

Since m is finitely additive, it is the case that μ_0 is well-defined, linear, and positive, with $\mu_0(\mathbb{1}_G) = m(G) = 1$.

Additionally, since m is a mean, then for $f = \sum_{k=1}^{n} t_k \mathbb{1}_{E_k}$, we have

$$\begin{aligned} \mu_0(\lambda_s(f)) &= \mu_0\left(\lambda_s\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right)\right) \\ &= \mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{sE_k}\right) \\ &= \sum_{k=1}^{n} t_k m(sE_k) \\ &= \sum_{k=1}^{n} t_k m(E_k) \\ &= \mu_0(f). \end{aligned}$$

We see that

$$\left|\mu_0(f)\right| = \left|\sum_{k=1}^{n} t_k m(E_k)\right|$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n} |t_k| \, m(E_k) \\
&\leq \sum_{k=1}^{n} \|f\| \sum_{k=1}^{n} m(E_k) \\
&= \|f\| \sum_{k=1}^{n} m(E_k) \\
&\leq \|f\|,
\end{aligned}
$$

meaning μ_0 is continuous, so μ_0 is uniformly continuous.

Since $\overline{\Sigma} = \ell_\infty(G)$, uniform continuity provides that μ_0 extends to a continuous linear functional $\mu \colon \ell_\infty(G) \to \mathbb{R}$ with $\mu(\mathbb{1}_G) = \mu_0(\mathbb{1}_G) = 1$.

For $f \geq 0$, we find a sequence $(\phi_n)_n$ in Σ with $\phi_n \geq 0$ and $\left\|\phi_n - f\right\| \xrightarrow{n\to\infty} 0$. We set

$$
\begin{aligned}
\mu(f) &= \lim_{n\to\infty} \mu(\phi_n) \\
&= \lim_{n\to\infty} \mu_0(\phi_n) \\
&\geq 0,
\end{aligned}
$$

so μ is a state.

If $f \in \ell_\infty(G)$, $s \in G$, and $(\phi_n)_n$ a sequence in Σ with $(\phi_n)_n \to f$, then

$$
\begin{aligned}
\left\|\lambda_s(\phi_n) - \lambda_s(f)\right\| &= \left\|\lambda_s(\phi_n - f)\right\| \\
&= \left\|\phi_n - f\right\| \\
&\to 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mu(\lambda_s(\phi_n)) &= \mu_0(\lambda_s(\phi_n)) \\
&= \mu_0(\phi_n) \\
&= \mu(\phi_n) \\
&\to \mu(f),
\end{aligned}
$$

so $\mu(f) = \mu(\lambda_s(f))$. Thus, $\mu \in (\ell_\infty(G))^*$ is an invariant state. $\square$

3.2 Establishing Amenability using Invariant States

Owing to the correspondence between invariant states and means, we are now able to establish the amenability of large classes of groups.

Proposition 3.2.1. *The group of integers, $\mathbb{Z}$, is amenable.*

Proof. We define the left shift, $\lambda_1 : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$, by

$$\ell_\infty(f)(k) = f(k-1).$$

This is an action as in Proposition 3.1.3.

We set $Y = \operatorname{Ran}(\mathrm{id} - \lambda_1) \subseteq \ell_\infty(\mathbb{Z})$. We claim that $\operatorname{dist}_Y(\mathbb{1}_\mathbb{Z}) \geq 1$.

Suppose toward contradiction that there is $y \in Y$ with $\left\|\mathbb{1}_{\mathbb{Z}} - y\right\|_u = r < 1$. Then, $y = f - \lambda_1 f$ for some $f \in \ell_\infty(\mathbb{Z})$, so

$$\|\mathbb{1}_{\mathbb{Z}} - (f - \lambda_1(f))\| = r.$$

Thus, for all $k \in \mathbb{Z}$, we have

$$|1 - (f(k) - f(k-1))| \leq r,$$

so $|f(k) - f(k-1)| \geq 1 - r > 0$. However, such an f cannot be bounded.

Since $\operatorname{dist}_{\overline{Y}}(\mathbb{1}_{\mathbb{Z}}) = \operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}})$, the Hahn–Banach separation theorems provide $\mu \in (\ell_\infty(\mathbb{Z}))^*$ with $\left\|\mu\right\|_{\text{op}} = 1$, $\mu|_{\overline{Y}} = 0$, and $\mu(\mathbb{1}_{\mathbb{Z}}) = \operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geq 1$.

Since $\left\|\mu\right\|_{\text{op}} = 1$ and $\mu(\mathbb{1}_{\mathbb{Z}}) \geq 1$, we must have $\mu(\mathbb{1}_{\mathbb{Z}}) = 1$.

Additionally, since $\left\|\mu\right\|_{\text{op}} = \mu(\mathbb{1}_{\mathbb{Z}}) = 1$, we have that μ is a state on $\ell_\infty(\mathbb{Z})$, and since $\mu(y) = 0$ for all $y \in Y$, we have

$$\begin{aligned} \mu(f - \lambda_1(f)) &= 0 \\ \mu(f) &= \mu(\lambda_1(f)). \end{aligned}$$

Inductively, this means that $\mu(f) = \mu(\lambda_k(f))$ for all $k \in \mathbb{Z}$, so μ is an invariant state on $\ell_\infty(\mathbb{Z})$. Thus, $\mathbb{Z}$ is amenable. □

Proposition 3.2.2. *If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.*

Proof. Let $\rho \in (\ell_\infty(G/N))^*$ be an invariant state, and let $p\colon P(N) \to [0,1]$ be a mean. For $E \subseteq G$, we define $f_E\colon G/N \to \mathbb{R}$ by

$$f_E(tN) = p\left(N \cap t^{-1}E\right).$$

We start by verifying that f_E is well-defined. For $tN = sN$, we have $s^{-1}t \in N$, so

$$\begin{aligned} p\left(N \cap t^{-1}E\right) &= p\left(s^{-1}t\left(N \cap t^{-1}E\right)\right) \\ &= p\left(s^{-1}tN \cap s^{-1}E\right) \\ &= p\left(N \cap s^{-1}E\right). \end{aligned}$$

Since f_E is defined through p, we can see that f_E is bounded. Additionally,

$$\begin{aligned} f_{E\sqcup F}(tN) &= p\left(N \cap t^{-1}(E \sqcup F)\right) \\ &= p\left(N \cap \left(t^{-1}E \sqcup t^{-1}F\right)\right) \\ &= p\left(\left(N \cap t^{-1}E\right) \sqcup \left(N \cap t^{-1}F\right)\right) \\ &= p\left(N \cap t^{-1}E\right) + p\left(N \cap t^{-1}F\right) \\ &= f_E(tN) + f_F(tN) \\ &= (f_E + f_F)(tN), \end{aligned}$$

and

$$\begin{aligned} f(sE)(tN) &= p\left(N \cap t^{-1}sE\right) \\ &= f_E\left(s^{-1}tN\right) \end{aligned}$$

$$= \lambda_{sN}(f_E)(tN),$$

so $f_{sE} = \lambda_{sN}(f_E)$. Finally,

$$\begin{aligned} f_G(tN) &= p\left(N \cap t^{-1}G\right) \\ &= p(N) \\ &= 1, \end{aligned}$$

meaning $f_G = \mathbb{1}_{G/N}$.

We define $m\colon P(G) \to [0,1]$ by

$$m(E) = \rho(f_E).$$

Then, we have

$$m(E \sqcup F) = m(E) + m(F)$$

$$m(G) = 1$$

$$\begin{aligned} m(sE) &= \rho(f_{sE}) \\ &= \rho(\lambda_{sN}(f_E)) \\ &= \rho(f_E) \\ &= m(E), \end{aligned}$$

so m is a mean. □

Corollary 3.2.1. *The finite direct product of amenable groups is amenable.*

Proof. For H and K amenable groups, we know that $K \cong (H \times K)/H$ and H are amenable, so $H \times K$ is amenable. Induction provides the general case. □

Corollary 3.2.2. *Finitely generated abelian groups are amenable.*

Proof. By the fundamental theorem of finitely generated abelian groups, all finitely generated abelian groups are isomorphic to $\mathbb{Z}^d \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$.

Since $\mathbb{Z}^d$ is a finite direct product of $\mathbb{Z}$, and the torsion subgroup $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$ is finite, we see that a finitely generated abelian group is a direct product of two amenable groups, hence amenable. □

Corollary 3.2.3. *If $\{G_i\}_{i \in I}$ is a directed family of amenable groups, then the direct limit,*

$$G = \bigcup_{i \in I} G_i,$$

is also amenable.

Proof. Let $\mu_i \in (\ell_\infty(G_i))^*$ be invariant states.

Set

$$M_i = \{\mu \in S(\ell_\infty(G)) \,|\, \mu(\lambda_s(f)) = \mu(f) \text{ for all } s \in G_i\}.$$

We set $\mu(f) = \mu_i\left(f|_{G_i}\right)$. Since each G_i is amenable, it is the case that each M_i is nonempty. Similarly, seeing as we have established the state space as w^*-closed in $B_{(\ell_\infty(G))^*}$, it is the case that each M_i is w^*-closed in

$B_{(\ell_\infty(G))^*}$.

For $i_1, \ldots, i_n$, we find $G_j \supseteq G_{i_1}, \ldots, G_{i_n}$, which exists since $\{G_i\}_{i\in I}$ is directed. We have that $M_j \subseteq \bigcap_{k=1}^n M_{i_k}$, so $\{M_i\}_{i\in I}$ has the finite intersection property.

Thus, there is $\mu \in \bigcap_{i\in I} M_i$, which is necessarily invariant on G. □

Corollary 3.2.4. *All abelian groups are amenable.*

Proof. Every abelian group is the direct union of its finitely generated subgroups. □

Corollary 3.2.5. *All solvable groups are amenable.*

Proof. Let $e_G = G_0 \leq G_1 \leq \cdots \leq G_n \leq G$ be such that $G_{j-1} \trianglelefteq G_j$ for $j = 1, \ldots, n$, and G_i/G_j is abelian.

Since G_0 is abelian, it is amenable, as is G_1/G_0, so G_1 is amenable. We see then that G_2 is amenable as G_1 and G_2/G_1 are amenable.

Continuing in this fashion, we see that G is amenable. □

3.3 Følner's Condition and Approximate Means

While showing the existence of an invariant state is necessary and sufficient for showing a group is amenable, as well as showing the group is non-paradoxical, it is often difficult to establish either of these conditions.

However, we can often more easily create a sequence (or net) of finitely supported functions whose limit is an invariant state. This will require the use of the Følner condition.

Definition 3.3.1. A group is said to satisfy the Følner condition if, for every $\varepsilon > 0$ and $E \subseteq G$, there is a nonempty finite subset $F \subseteq G$ such that for all $t \in E$,

$$\frac{|tF \triangle F|}{|F|} \leq \varepsilon.$$

Equivalently, we can also say that the Følner condition is satisfied if and only if

$$\frac{|tF \cap F|}{|F|} \geq 1 - \varepsilon$$

for every $\varepsilon > 0$.

Lemma 3.3.1. *A countable group G satisfies the Følner condition if and only if G admits a sequence $(F_n)_n$ with $F_n \subseteq G$ finite such that*

$$\left(\frac{|tF_n \triangle F_n|}{|F_n|}\right)_n \xrightarrow{n\to\infty} 0$$

for all $t \in G$. Such a sequence is known as a Følner sequence.

Proof. Let G admit a Følner sequence, $(F_n)_n$. Given $\varepsilon > 0$ and $E \subseteq G$ finite, find N such that for all $s \in E$ and $n \geq N$,

$$\frac{|sF_n \triangle F_n|}{|F_n|} \leq \varepsilon.$$

We take $F = F_N$ in the definition of the Følner condition.

Let G satisfy the Følner condition. We write $G = \bigcup_{n \geq 1} E_n$, with $E_1 \subseteq E_2 \subseteq \cdots$, and define F_n such that for all $t \in E_n$,

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leq \frac{1}{n}.$$

Given $t \in G$, then $t \in E_N$ for some N, so $t \in E_n$ For all $n \geq N$, so

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leq \frac{1}{n}$$

for all $n \geq N$. Thus,

$$\left(\frac{|tF_n \triangle F_n|}{|F_n|}\right) \xrightarrow{n \to \infty} 0.$$

□

Lemma 3.3.2. *Let G be a finitely generated group with generating set S. If $(F_n)_n$ is a sequence of finite subsets such that, for all $s \in S$,*

$$\left(\frac{|sF_n \triangle F_n|}{|F_n|}\right)_n \to 0,$$

then $(F_n)_n$ is a Følner sequence for G.

Proof. Note that

- $s(A \triangle B) = sA \triangle sB$;
- $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

We see that for any $s \in S$,

$$\begin{aligned}
\frac{\left|s^{-1} F_n \triangle F_n\right|}{|F_n|} &= \frac{\left|s^{-1}(F_n \triangle sF_n)\right|}{|F_n|} \\
&= \frac{|F_n \triangle sF_n|}{|F_n|} \\
&\to 0.
\end{aligned}$$

Thus, we may assume that S is symmetric — i.e., that $\left\{s^{-1} \;\middle|\; s \in S\right\} = \{s \mid s \in S\}$.

For any $s, t \in S$, we have

$$\begin{aligned}
\frac{|stF_n \triangle F_n|}{|F_n|} &\leq \frac{|stF_n \triangle F_n|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\
&= \frac{|s(tF_n \triangle F_n)|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\
&= \frac{|tF_n \triangle F_n|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|} \\
&\to 0.
\end{aligned}$$

We use induction to find the general case. □

Example 3.3.1. Consider the group $\mathbb{Z}$. Since $\mathbb{Z}$ is generated by the element $\{1\}$, we see that for $F_n = [-n, n]$, that

$$\begin{aligned}
\frac{|(F_n + 1) \triangle F_n|}{|F_n|} &= \frac{2}{2n+1} \\
&\to 0,
\end{aligned}$$

meaning that $\mathbb{Z}$ satisfies the Følner condition.

We have thus far proven that G satisfies the Følner condition if and only if G admits a Følner sequence, and that G is amenable if and only if G admits an invariant state.

We will now begin harmonizing these two characterizations through the use of approximate means, eventually showing that G satisfies the Følner condition if and only if G admits an approximate mean, and that G admits an approximate mean if and only if G is amenable.

Definition 3.3.2. For a group G, we define

$$\mathrm{Prob}(G) = \left\{ f\colon G \to [0,\infty) \,\middle|\, |\mathrm{supp}(f)| < \infty,\ \sum_{t\in G} f(t) = 1 \right\}.$$

Note that $\mathrm{Prob}(G) \subseteq B_{\ell_1(G)}$. For $f \in \mathrm{Prob}(G)$, we set $\varphi_f : \ell_\infty(G) \to \mathbb{C}$ defined by

$$\varphi_f(g) = \sum_{t\in G} g(t) f(t).$$

Fact 3.3.1. For $f \in \mathrm{Prob}(G)$, φ_f is a state on $\ell_\infty(G)$.

Proof. We can see that, by definition, φ_f is positive, linear, and has $\varphi_f(\mathbb{1}_G) = 1$.

We only need to show that $\left\|\varphi_f\right\| = 1$. We see that

$$\begin{aligned}
\left|\varphi_f(g)\right| &= \left|\sum_{t\in G} g(t) f(t)\right| \\
&\leq \sum_{t\in G} |g(t)||f(t)| \\
&\leq \|g\|_\infty \sum_{t\in G} |f(t)| \\
&= \|g\|_\infty .
\end{aligned}$$

□

Proposition 3.3.1. *There is an action* $\lambda\colon G \xrightarrow{\mathrm{Isom}} (\ell_1(G))$ *such that* $\mathrm{Prob}(G)$ *is invariant.*

Proof. Let $\lambda_s(f)(t) = f\left(s^{-1}t\right)$. Then,

$$\begin{aligned}
\|\lambda_s(f)\|_1 &= \sum_{t\in G} |\lambda_s(f)(t)| \\
&= \sum_{t\in G} \left|f\left(s^{-1}t\right)\right| \\
&= \sum_{r\in G} |f(r)| \\
&= \|f\|_1 .
\end{aligned}$$

Just as in Proposition 3.1.3, it is the case that λ_s is linear. Additionally,

$$\begin{aligned}
\lambda_r \circ \lambda_s(f)(t) &= \lambda_s(f)\left(r^{-1}t\right) \\
&= f\left(s^{-1}r^{-1}(t)\right) \\
&= f\left((rs)^{-1}t\right) \\
&= \lambda_{rs}(f)(t).
\end{aligned}$$

We see that if $f \in \mathrm{Prob}(G)$, then for $f \geq 0$, we have $\lambda_s(f) \geq 0$, and

$$\begin{aligned}\sum_{t\in G} \lambda_s(f)(t) &= \sum_{t\in G} f\left(s^{-1}t\right) \\ &= \sum_{r\in G} f(r) \\ &= 1\end{aligned}$$

for any $f \in \mathrm{Prob}(G)$. ☐

Definition 3.3.3. For a countable group G, a sequence $(f_k)_k$ is called an approximate mean if, for all $s \in G$,

$$\|f_k - \lambda_s(f_k)\|_1 \xrightarrow{k\to\infty} 0.$$

To begin the forward direction regarding the equivalence between the Følner condition, approximate means, and means, we begin by showing that the existence of a Følner sequence implies the existence of an approximate mean. Then, we will show that the existence of an approximate mean implies the existence of an invariant state (hence mean).

Proposition 3.3.2. *If G admits a Følner sequence $(F_k)_k$, then G admits an approximate mean.*

Proof. Set $f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k} \in \mathrm{Prob}(G)$. Then,

$$\begin{aligned}\|f_k - \lambda_s(f_k)\|_1 &= \frac{1}{|f_k|}\left\|\mathbb{1}_{F_k} - \lambda_s\left(\mathbb{1}_{F_k}\right)\right\| \\ &= \frac{1}{F_k}\left\|\mathbb{1}_{F_k} - \mathbb{1}_{sF_k}\right\| \\ &= \frac{|F_k \Delta sF_k|}{|F_k|} \\ &\to 0.\end{aligned}$$

☐

Proposition 3.3.3. *If G admits an approximate mean, then G is amenable.*

Proof. Let $(f_k)_k$ be an approximate mean. We define $\varphi_k = \left(\varphi_{f_k}\right)_k$ (as in Definition 3.3.2) to be a sequence of states on $\ell_\infty(G)$.

Since the state space on $\ell_\infty(G)$ is w^*-compact, there is a state μ and a subnet $\left(\varphi_{k_j}\right)_j \xrightarrow{w^*} \mu$.

We only need to show that μ is invariant. Note that

$$\left|\mu(g) - \mu(\lambda_s(g))\right| \leq \left|\mu(g) - \varphi_{k_j}(g)\right| + \left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| + \left|\varphi_{k_j}(\lambda_s(g)) - \mu(\lambda_s(g))\right|$$

for all $g \in \ell_\infty(G)$, $s \in G$, and all j.

Given $\varepsilon > 0$, we find J such that for $j \geq J$,

$$\begin{aligned}\left|\mu(g) - \varphi_{k_j}(g)\right| &< \varepsilon/3 \\ \left|\mu(\lambda_s(g))\varphi_{k_j}(\lambda_s(g))\right| &< \varepsilon/3.\end{aligned}$$

We also see that

$$\left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| = \left|\sum_{t\in G} g(t) f_{k_j}(t) - \sum_{t\in G} g\left(s^{-1}t\right) f_{k_j}(t)\right|$$

$$
\begin{aligned}
&= \left| \sum_{t\in G} g(t) f_{k_j}(t) - \sum_{r\in G} g(r) f_{k_j}(sr) \right| && r = s^{-1}t \\
&= \left| \sum_{t\in G} g(t) \left(f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right) \right| \\
&\leq \|g\|_\infty \sum_{t\in G} \left| f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t) \right| \\
&= \|g\|_\infty \left\| f_{k_j} - \lambda_{s^{-1}}\left(f_{k_j}\right) \right\|_1 \\
&< \varepsilon/3
\end{aligned}
$$

for large j. Thus, we have

$$
\left|\mu(g) - \mu(\lambda_s(g))\right| < \varepsilon,
$$

for all $\varepsilon > 0$, so $\mu(g) = \mu(\lambda_s(g))$. □

We will now commence with the reverse direction, starting by showing that amenability implies the existence of an approximate mean, and then showing that the existence of an approximate mean implies that the Følner condition is satisfied.

Proposition 3.3.4. *If G is amenable, then G admits an approximate mean.*

Proof. Suppose G does not admit an approximate mean. Then, there exists a finite subset $E_0 \subseteq G$ and $\varepsilon_0 > 0$ such that for all $s \in E_0$ and all $f \in \mathrm{Prob}(G)$, we have $\|f - \lambda_s(f) \geq \varepsilon_0\|$.

Consider the set

$$
X = \bigoplus_{|E_0|} \ell_1(G),
$$

endowed with the norm

$$
\begin{aligned}
\left\|(f_s)_{s\in E_0}\right\| &= \sum_{s\in E_0} \sum_{t\in G} |f_s(t)| \\
&= \sum_{s\in E_0} \|f_s\|_1,
\end{aligned}
$$

and let

$$
C = \left\{ (f - \lambda_s(f))_{s\in E_0} \,\middle|\, f \in \mathrm{Prob}(G) \right\}.
$$

Since $\mathrm{Prob}(G)$ is convex, it is the case that C is convex, and since $|E_0|$ is finite, C is necessarily bounded. Note that $0 \notin \overline{C}$.

By the Hahn–Banach separation theorem for convex sets, there is a real-valued $\varphi \in X^*$ such that $\varphi(C) \geq 1$. Here,

$$
\begin{aligned}
X^* &\cong \bigoplus_{|E_0|} \ell_1(G)^* \\
&\cong \sum_{|E_0|} \ell_\infty(G),
\end{aligned}
$$

endowed with the norm

$$
\left\|(g_s)_{s\in E_0}\right\| = \max_{s\in E_0} \left(\sup_{t\in G} |g_s(t)| \right)
$$

$$= \max_{s\in E_0} \|g_s\|_\infty .$$

We let $\varphi = \left(\varphi_{g_s}\right)_{s\in E_0}$, where $g_s \in \ell_\infty(G)$ is defined by the duality

$$\varphi_{g_s}(f) = \sum_{t\in G} f(t) g_s(t).$$

Thus, for all $f \in \operatorname{Prob}(G)$, we have

$$\begin{aligned}
1 &\leq \varphi\left((f - \lambda_s(f))_{s\in E_0}\right) \\
&= \sum_{s\in E_0} \varphi_{g_s}(f - \lambda - s(f)) \\
&= \sum_{s\in E_0} \sum_{t\in G} (f - \lambda_s(f))(t) g_s(t) \\
&= \sum_{s\in E_0} \left(\sum_{t\in G} f(t) g_s(t) - \sum_{t\in G} f\left(s^{-1}t\right) g_s(t) \right) \\
&= \sum_{s\in E_0} \left(\sum_{t\in G} f(t) g_s(t) - \sum_{r\in G} f(r) g_s(sr) \right) \\
&= \sum_{s\in E_0} \left(\sum_{r\in G} f(r) g_s(r) - \sum_{r\in G} f(r) \lambda_{s^{-1}}(g)(r) \right) \\
&= \sum_{s\in E_0} \sum_{r\in G} f(r)(g_s - \lambda_{s^{-1}}(g))(r).
\end{aligned}$$

Note that this holds for any $f \in \operatorname{Prob}(G)$, including the case of $f = \delta_t$ for a given $t \in G$. We must have

$$\begin{aligned}
&= \sum_{s\in E_0} \sum_{r\in G} \delta_t(r)(g_s(r) - \lambda_{s^{-1}}(g_s))(r) \\
&= \sum_{s\in E_0} (g_s - \lambda_{s^{-1}}(g))(t).
\end{aligned}$$

In particular, we must have

$$\geq \mathbb{1}_G.$$

Since G is amenable, there is a mean $\mu \colon \ell_\infty(G) \to \mathbb{C}$ with $\mu(g_s) = \mu(\lambda_{s^{-1}}(g_s))$, meaning

$$\begin{aligned}
0 &= \mu\left(\sum_{s\in E_0} (g_s - \lambda_{s^{-1}}(g_s))(t) \right) \\
&\geq \mu(\mathbb{1}_G) \\
&= 1,
\end{aligned}$$

which is a contradiction. □

To show that the existence of an approximate mean implies the Følner condition, we require the following lemma.

Lemma 3.3.3. *Let $f \colon S \to \mathbb{R}$ be finitely supported with $\sum_{s\in S} f(s) = 1$. Then, there exist subsets $\{F_i\}_{i=1}^n$, where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$, and constants $\{c_i\}_{i=1}^n$, such that*

$$f = \sum_{i=1}^n c_i \mathbb{1}_{F_i},$$

where

$$\sum_{i=1}^{n} c_i |F_i| = 1.$$

This is known as the layer cake representation for f.

Proof of Lemma 3.3.3: We define $F_1 = \operatorname{supp}(f)$, and take $c_1 = \min(\operatorname{Ran}(f))$. Taking $E_1 = f^{-1}(c_1)$ (as a set-theoretic inverse), we define $F_2 = F_1 \setminus E_1$.

Take $d_1 = \min(f(F_2))$, and define $c_2 = d_1 - c_1$. Then, defining $E_2 = f^{-1}(d_1)$, $F_3 = F_2 \setminus E_2$, and $d_2 = \min(f(F_3))$, we define $c_3 = d_2 - c_2 - c_1$.

Continuing in this pattern, we find $d_{i-1} = \min(f(F_i))$, $E_i = f^{-1}(d_{i-1})$, and $c_i = d_{i-1} - \sum_{j=1}^{i-1} c_i$.

This yields a decomposition $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$, where $\sum_{i=1}^{n} c_i \mathbb{1}_{F_i} = f$ by construction.

We now verify that $\sum_{i=1}^{n} c_i |F_i| = 1$.

$$\begin{aligned} 1 &= \sum_{s \in S} f(s) \\ &= \sum_{s \in S} \sum_{i=1}^{n} c_i \mathbb{1}_{F_i}(s). \end{aligned}$$

By definition, if $s \in F_j$ for some j, we see that c_j is summed for $\left|F_j\right|$ many times. Thus, we obtain

$$= \sum_{i=1}^{n} c_i |F_i|.$$

□

We will use the layer cake decomposition to prove that if G admits an approximate mean, then G satisfies the Følner condition.

Proposition 3.3.5. *Let G admit an approximate mean. Then, G satisfies the Følner condition.*

Proof. Let $(f_k)_k$ be an approximate mean, as in Definition 3.3.3. Fix a finite nonempty set $S \subseteq G$. Then, by the definition of an approximate mean, there must exist some $N \in \mathbb{N}$ such that for all $k \geq N$ and all $s \in G$,

$$\|f_k - \lambda_s(f_k)\|_1 \leq \frac{\varepsilon}{|S|}.$$

In particular, this holds for f_N and for all $s \in S$.

Since $f_N \in \operatorname{Prob}(G)$ is finitely supported and $\sum_{s \in G} f_N(s) = 1$, we may use Lemma 3.3.3 to rewrite f_N as

$$f_N = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$, and $\sum_{i=1}^{n} c_i |F_i| = 1$. □

Thus far, we have shown the following to be equivalent for a discrete group G:

(1) G is non-paradoxical;

(2) G is amenable;

(3) G admits an invariant state;

(4) G admits an approximate mean;

(5) G satisfies the Følner condition.

The equivalence between (1) and (2) follows from Tarski's theorem (Theorem 2.0.1), the equivalence between (2) and (3) follows from Propositions 3.1.4 and 3.1.5, and the equivalence between (3), (4), and (5) follows from

Chapter 4

Characterizations through Fixed Points

Chapter 5

Characterizations using C^*-Algebras

Appendix A

Point-Set Topology

We will need a bit of background in point-set topology in order to satisfactorily understand the functional analysis behind the results in Chapters 3, 4, and 5.

A.1 Axioms of Set Theory

In order to garner sufficient understanding of point-set topology, we need to be able to comprehend some of the essential axioms behind the objects known as "sets." This is where the axioms of set theory come into play.

Definition A.1.1 (Zermelo–Fraenkel Axioms). In Zermelo–Fraenkel set theory, all objects are sets. In order to maintain convention with the way the rest of this section will refer to sets, all sets will be referred to by capital letters, and all elements of sets by lowercase letters.

- Axiom of Existence: $\exists A\,(A = A)$. This axiom guarantees a nonempty universe.
- Axiom of Extensionality: $\forall x\,(x \in A \Leftrightarrow x \in B) \Rightarrow A = B$. This axiom states that if two sets share the same members, then the sets are equal.
- Axiom Schema of Comprehension: $\exists B\,\forall x\,(x \in B \Leftrightarrow x \in A \wedge \varphi(x))$. This axiom states that for any formula $\varphi(x)$, where x is a free variable, there is a set B such that the members of B are the members of A for which φ holds.
- Pairing Axiom: $\forall A\,\forall B\,\exists C\,((A \in Z) \wedge (B \in Z))$. This axiom states that for any sets A and B, there is a set $C = \{A, B\}$ that contains the sets A and B as elements.
- Power Set Axiom: $\forall A\,\exists P(A)\,\forall B\,(B \in P(A) \Leftrightarrow B \subseteq A)$. We use the shorthand $B \subseteq A$ to write the statement $\forall x\,(x \in B \Rightarrow x \in A)$. This axiom states that for any set A there exists a set $P(A)$ such that any element of $P(A)$ is a subset of A, and any subset of A is an element of $P(A)$.
- Union Axiom: $\forall \mathcal{A}\,\exists A\,\forall Y\,\forall x\,((x \in Y \wedge Y \in \mathcal{A}) \Rightarrow x \in A)$. This axiom states that for any collection $\mathcal{A}$, there is a set A, denoted $\bigcup \mathcal{A}$, that contains all the elements of all the sets in the collection $\mathcal{A}$.
- Axiom of Infinity: $\exists A\,(\varnothing \in A \wedge \forall x\,(x \in A \Rightarrow x \cup \{x\} \in A))$. This axiom states that there is a set, A, such that the empty set is in A and, for any element x, if $x \in A$, then so too is the successor, $x \cup \{x\}$.
- Axiom of regularity: $\forall X\,(X \neq \varnothing \Rightarrow \exists Y\,(Y \in X \wedge Y \cap X = \varnothing))$. This axiom states that any nonempty set X contains a set Y such that Y and X are disjoint. As a consequence, any chain of sets descending in membership must terminate.
- Axiom Schema of Replacement: $\forall A\,\exists B\,\forall v\,(v \in B \Rightarrow \exists u\,(u \in A \wedge \psi(u, v)))$. The axiom schema of replacement says that for a function-like formula (a formula such that $\psi(u, v) \wedge \psi(u, w) \Rightarrow v = w$) $\psi(u, v)$, there is a set A consisting of exactly those sets/elements $v \in B$ that correspond to $u \in A$.

The final axiom, the Axiom of Choice, is special, and as a result, we state it separately, for we will be using some of its consequences in the future sections. The following is one way of interpreting the axiom of choice.

Definition A.1.2 (Axiom of Choice)**.** Let $\{S_i\}_{i\in I}$ be an indexed collection of nonempty sets. Then, there exists an indexed set $\{x_i\}_{i\in I}$ such that $x_i \in S_i$ for each I.

Equivalently, if $\{S_i\}_{i\in I}$ is an indexed collection of nonempty sets, then there is some choice function

$$f \in \prod_{i\in I} S_i.$$

On its own, this formulation of the Axiom of Choice is not particularly useful. However, there is a statement of the Axiom of Choice which is just as useful.

Definition A.1.3 (Preorders, Partial Orders, Total Orders, and Well-Orders)**.** Let X be a set, and $\leq$ be a relation on X. We say a relation is a preorder if it is reflexive and transitive:

- $a \leq a$
- $a \leq b \wedge b \leq c \Rightarrow a \leq c.$

We say X is a directed set if, for any $a, b \in X$, there is $c \in X$ such that $a \leq c$ and $b \leq c$.

If $\leq$ is also antisymmetric — that is, $a \leq b \wedge b \leq a \Rightarrow a = b$ — then, we say $\leq$ is a partial order.

We say $m \in X$ is a maximal element if, for any $x \in X$ with $m \leq x$, $m = x$.

If X is partially ordered by $\leq$ and, for any two elements $a, b \in X$, either $a \leq b$ or $b \leq a$, then we say $\leq$ is a total order on X.

If X is a totally ordered set that has the property that, for any nonempty $A \subseteq X$, there is some $x \in A$ such that for any $y \in A$, $x \prec y$ for all $y \in A$ with $y \neq x$, then we say $\leq$ is a well-order on X.

Example A.1.1.

- The set $\mathbb{N}$ with the usual ordering is a well-ordered set.
- If A is a set, then $P(A)$ with the containment ordering, $A \leq B$ if $A \supseteq B$, is a partially ordered set.
- Similarly, if A is a set, then $P(A)$ with the inclusion ordering, $A \leq B$ if $A \subseteq B$, is a partially ordered set.
- A collection of functions $\{\varphi_i : Z_i \to Y\}_{i\in I}$ ordered by $\varphi_i \leq \varphi_j$ if $Z_i \subseteq Z_j$ and $\varphi_j|_{Z_i} = \varphi_i$, is a partially ordered set. This is often known as the extension ordering.

We can state an equivalent formulation of the Axiom of Choice as follows.

Theorem A.1.1 (Zorn's Lemma)**.** *If $(X, \leq)$ is a partially ordered set with the property that for all $C \subseteq X$ with C totally ordered, C has an upper bound, then X has a maximal element.*

There are many proofs of both Zorn's Lemma from the Axiom of Choice and the Axiom of Choice from Zorn's Lemma. However, we will mostly be using it for the purposes of proving other theorems. The following results can be proven using Zorn's Lemma.

Example A.1.2.

- Every $\mathbb{F}$-vector space V has a basis $B \subseteq V$ such that the set of all finite linear combinations of elements of B over $\mathbb{F}$ is V.
- If φ is a continuous linear functional defined on a subspace $W \subseteq V$, there is an extension Φ such that $\Phi|_W = \varphi$. This is one of the Hahn–Banach theorems.
- The arbitrary product of compact spaces is compact. This is known as Tychonoff's Theorem.

A.2 Metric Spaces

Building upon the basics of set theory, we move towards understanding metric spaces.

A.2.1 Basics of Metric Spaces

Definition A.2.1 (Metrics)**.** Let X be a set. A distance metric is a function

$$d\colon X \times X \to [0, \infty)$$

such that the following three properties are satisfied:

- if $x, y \in X$ and $d(x, y) = 0$, then $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A function that satisfies the latter two properties is called a semimetric.

Two metrics d and ρ on X are equivalent if there exist constants $c_1, c_2 \geq 0$ such that

$$d(x, y) \leq c_1 \rho(x, y)$$
$$\rho(x, y) \leq c_2 d(x, y)$$

for all $x, y \in X$.

A metric space is a pair (X, d), where d is a metric.

Example A.2.1 (Some Distance Metrics)**.**

- The discrete metric on any nonempty set is given by

$$d(x, y) \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

- The Euclidean metric between $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is

$$d_2(x, y) = \left(\sum_{j=1}^{n} \left| y_j - x_j \right|^2 \right)^{1/2}.$$

- Other metrics on $\mathbb{R}^n$ include

$$d_1(x, y) = \sum_{j=1}^{n} \left| y_j - x_j \right|$$
$$d_\infty(x, y) = \max_{j=1}^{n} \left| y_j - x_j \right|.$$

  All of d_1, d_2, d_∞ are equivalent metrics.

- The Hamming distance between two strings of bits is

$$d_H\colon \{0,1\}^n \times \{0,1\}^n \to [0, \infty)$$
$$d_H\left(\left(x_j\right)_{j=1}^{n}, \left(y_j\right)_{j=1}^{n} \right) = \left| \left\{ j \mid x_j \neq y_j \right\} \right|.$$

- The set $C([0,1],\mathbb{R})$ consisting of continuous real-valued functions from $[0,1]$ to $\mathbb{R}$ can be equipped with

$$d_u(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|,$$

which is the uniform metric, or

$$d_1(f,g) = \int_0^1 |f(t) - g(t)|\, dt.$$

- All subsets of a metric space X equipped with the same metric is also a metric space.

- If ρ is a metric on X, then we can create a distance metric

$$d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$$

that is bounded on $[0,1]$.

- If $d_1,\ldots,d_n$ are metrics on X and $c_1,\ldots,c_n > 0$ are constants, then

$$d(x,y) = \sum_{k=1}^{n} c_k d_k(x,y)$$

defines a metric on X.

- If $(\rho_k)_k$ is a family of separating semimetrics for X — i.e., for $x,y \in X$ distinct, there is some ρ_j such that $\rho_j(x,y) \neq 0$ — then, we can obtain bounded semimetrics by taking

$$d_k(x,y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}$$

for each k. From each d_k, we define

$$d(x,y) = \sum_{k=1}^{n} 2^{-k} d_k(x,y),$$

which is a metric on X.

- If $(X_k,\rho_k)_{k\geq 1}$ is a sequence of metric spaces, then we can form the product space

$$X = \prod_{k\geq 1} X_k$$

with the metric

$$D(f,g) = \sum_{k\geq 1} d_k(f(k), g(k)).$$

Here, $d_k = \frac{\rho_k}{1+\rho_k}$ is the corresponding bounded metric to ρ_k.

Definition A.2.2 (Open and Closed Sets)**.** Let (X,d) be a metric space.

(1) For $x \in X$ and $\delta > 0$, we define

 (a) the open ball at x with radius $\delta > 0$

$$U(x,\delta) = \{y \in X \mid d(y,x) < \delta\};$$

(b) the closed ball centered at x with radius $\delta > 0$

$$B(x, \delta) = \{y \in X \mid d(y, x) \leq \delta\};$$

(c) the sphere centered at x with radius $\delta > 0$

$$S(x, \delta) = \{y \in X \mid d(y, x) = \delta\}.$$

(2) A set $V \subseteq X$ is open if, for all $x \in V$, there is $\delta > 0$ such that $U(x, \delta) \subseteq V$.

A subset $C \subseteq X$ is closed if C^c is open.

(3) If $x \in V$ and $V \subseteq X$ is open, then we say V is an open neighborhood of x. A neighborhood of x is any subset $N \subseteq X$ such that N contains an open neighborhood of x.

(4) If $A \subseteq X$ is any subset, the interior of A is

$$A^\circ := \bigcup \{V \mid V \text{ is open, } V \subseteq A\},$$

the closure of A is

$$\overline{A} = \bigcap \{C \mid C \text{ is closed, } A \subseteq C\},$$

and the boundary of A is

$$\partial A = \overline{A} \setminus A^\circ.$$

We can now talk about the topology of the metric space.

Fact A.2.1. Let (X, d) be a metric space, and let

$$\mathcal{U} = \{V \mid V \subseteq X \text{ open}\}.$$

Then, the following are true.

- $\varnothing \in \mathcal{U}, X \in \mathcal{U}$.
- If $\{V_i\}_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} V_i \in \mathcal{U}$.
- If $\{V_i\}_{i=1}^n$ is a finite collection of open sets, then $\bigcap_{i=1}^n V_i \in \mathcal{U}$.

Definition A.2.3. Let (X, d) be a metric space. Suppose $A \subseteq X$ is a nonempty subset.

(1) The distance from a point $x \in X$ to the set A is defined by

$$\operatorname{dist}_A(x) = \inf_{a \in A} d(x, a).$$

(2) The diameter of A is defined by

$$\operatorname{diam}(A) = \sup_{x, y \in A} d(x, y).$$

(3) If $\operatorname{diam}(A) < \infty$, then we say A is bounded.

(4) If, for every $\delta > 0$, there is a finite subset $F_\delta \subseteq X$ such that

$$A \subseteq \bigcup_{x \in F_\delta} U(x, \delta).$$

(5) For $A, B \subseteq X$, we define the Hausdorff distance between A and B to be

$$d_H(A,B) = \max\left\{\sup_{x\in A} \operatorname{dist}_B(x), \sup_{y\in B} \operatorname{dist}_A(y)\right\}.$$

Example A.2.2. Let Ω be a nonempty set, and (X,d) be a metric space. A function $f: \Omega \to X$ is said to be bounded if $\operatorname{diam}(\operatorname{Ran}(f)) < \infty$.

The collection $\operatorname{Bd}(\Omega, X)$ denotes all bounded functions with domain Ω and codomain X.

On $\operatorname{Bd}(\Omega, X)$, we define the uniform metric by

$$D_u(f,g) = \sup_{x\in\Omega} d(f(x), g(x)).$$

A.2.2 Convergence and Continuity in Metric Spaces

Definition A.2.4. Let (X,d) be a metric space.

(1) A sequence in X is a map $x: \mathbb{N} \to X$, which we call $(x_n)_n$ or $(x_n)_{n\geq 1}$.

(2) A natural sequence is a strictly increasing sequence of natural numbers $(n_k)_{k\geq 1}$ with $n_k \geq k$ and $n_k < n_{k+1}$.

(3) If $(n_k)_k$ is a natural sequence, the sequence $\left(x_{n_k}\right)_k$ is called a subsequence of $(x_n)_n$.

(4) We say $(x_n)_n \to x$ if $d(x_n, x)_n \xrightarrow{n\to\infty} 0$. We say x is the limit of $(x_n)_n$.

Example A.2.3.

- If Ω is a nonempty set, and (X,d) is a metric space, the sequence of functions $f_n: \Omega \to X$ is said to converge pointwise to $f: \Omega \to X$ if

$$f_n(x) \xrightarrow{n\to\infty} f(x)$$

for each $x \in \Omega$.

- If $(f_n)_n \in \operatorname{Bd}(\Omega, X)$ is a sequence, we say $(f_n)_n \to f$ converges uniformly if

$$D_u(f_n, f) \xrightarrow{n\to\infty} 0,$$

or, equivalently,

$$\sup_{x\in\Omega} d(f_n(x), f(x)) \xrightarrow{n\to\infty} 0.$$

Definition A.2.5 (Sequential Criteria for Closure)**.** If (X,d) is a metric space, and $E \subseteq X$ is nonempty, then E is closed if and only if, for all $(x_n)_n \to x$ with $x_n \in E$, $x \in E$.

If $E \subseteq X$ is any nonempty set, then $\overline{E}$ is precisely the set of all $x \in X$ such that $(x_n)_n \to x$ for some $(x_n)_n \subseteq E$.

Definition A.2.6 (Completeness)**.** Let (X,d) be a metric space.

- If $(x_n)_n$ is a sequence in X such that for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon$, then we say the sequence is called Cauchy.
- If, for any $(x_n)_n$ Cauchy, $(x_n)_n \to x$ in X, then we say X is complete.
- If (X,d) is complete, then for any $A \subseteq X$ closed, A is also complete.
- If $A \subseteq X$ is complete as a metric space, then A is closed.

Example A.2.4. The metric space $\mathbb{Q}$ with the metric inherited from $\mathbb{R}$ is not complete. For instance, there is a sequence of rational numbers $(2, 2.7, 2.71, 2.718, \dots)$ converging to e, but $e \notin \mathbb{Q}$.

The space $\mathrm{Bd}(\Omega, X)$ is complete if X is complete.

Definition A.2.7 (Continuity).
- Let (X,d) and (Y,ρ) be metric spaces, and let $f\colon X \to Y$ be a function. We say f is continuous at x if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $z \in U(x,\delta) \Rightarrow \rho(f(x), f(z)) < \varepsilon$.
- If f is continuous at every point in X, then we say f is continuous.
- If f is bijective, continuous, and f^{-1} is continuous, then we say f is a homeomorphism.
- We say f is uniformly continuous on X if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $y, z \in X$, $d(y,z) < \delta \Rightarrow \rho(f(y), f(z)) < \varepsilon$.
- We say f is Lipschitz if there exists $C > 0$ such that $d(x,y) \leq Cd(f(x), f(y))$ for all $x, y \in X$.
- We say f is an isometry if $d(x,y) = d(f(x), f(y))$ for all $x, y \in X$.

Fact A.2.2. Let $f\colon X \to Y$ be a map between metric spaces. The following are equivalent:

(i) f is continuous;

(ii) if $V \subseteq Y$ is open, then $f^{-1}(V) \subseteq X$ is open;

(iii) if $(x_n)_n \to x$ in X, then $(f(x_n))_n \to f(x)$ in Y.

Fact A.2.3. If M and N are metric spaces with N complete, and $A \subseteq M$ is dense, then if $f\colon A \to N$ is uniformly continuous, then there is a unique uniformly continuous map $\tilde{f}\colon M \to N$.

Definition A.2.8. Let (X,d) and (Y,ρ) be metric spaces.

(1) We say X and Y are homeomorphic if there is a homeomorphism $f\colon X \to Y$.

(2) We say X and Y are uniformly isomorphic if there is a uniformly continuous bijection $f\colon X \to Y$ with f^{-1} uniformly continuous. Such an f is called a metric space uniformism.

(3) We say X and Y are isometrically isomorphic if there is a bijective isometry $f\colon M \to N$.

Fact A.2.4. If X and Y are uniformly isomorphic metric spaces with X complete, then so too is Y.

If d and ρ are equivalent metrics on a set X, then the identity map

$$\mathrm{id}_X : (X,\rho) \to (X,d)$$

is a metric space uniformism.

A.3 Topological Spaces

We can now move from metric spaces to the more general setting of topological spaces. This will enable us to understand certain properties (like openness, continuity, etc.) separate from the metric structure (or lack thereof) that a certain set is endowed.

A.3.1 Definitions

Definition A.3.1. Let X be a nonempty set. A topology on X is a family of subsets τ satisfying

(1) $\varnothing \in \tau, X \in \tau$;

(2) if $\{V_i\}_{i\in I} \subseteq \tau$, then $\bigcup_{i\in I} V_i \in \tau$;

(3) if $\{V_i\}_{i=1}^{n} \subseteq \tau$, then $\bigcap_{i=1}^{n} V_i \in \tau$.

If τ is a topology on X, then (X, τ) is called a topological space. We call members of τ open sets.

If $C \subseteq X$ and $C^c \in \tau$, then C is called.

If E is closed and open, it is called clopen.

A countable union of closed sets is called an F_σ set, and a countable intersection of open sets is called a G_δ set.

Definition A.3.2. If X is a nonempty set, then the definition $\tau = P(X)$ is known as the discrete topology.

If X is a nonempty set, and $\tau = \{X, \varnothing\}$, then we call τ the indiscrete topology.

Definition A.3.3. Let X be a nonempty set. Suppose $\tau_1, \tau_2 \subseteq P(X)$ are two topologies on X. If $\tau_1 \subseteq \tau_2$, then we say τ_1 is weaker (or coarser) than τ_2. We say τ_2 is stronger (or finer) than τ_1.

Definition A.3.4. Let X be a nonempty set, and suppose $\mathcal{E} \subseteq P(X)$ is a family of subsets. We define the topology on X generated by $\mathcal{E}$ to be

$$\tau(\mathcal{E}) = \bigcap \{\tau \mid \tau \text{ is a topology on } X, \mathcal{E} \subseteq \tau\}.$$

In other words, $\tau(\mathcal{E})$ is the weakest topology that contains the family $\mathcal{E}$.

Definition A.3.5. Let (X, τ) be a topological space. If $Y \subseteq X$ is a subset, then the subspace topology on Y is defined by

$$\tau_Y = \{V \cap Y \mid V \in \tau\}.$$

Definition A.3.6. Let (X, τ) be a topological space, and let $A \subseteq X$ be a subset.

(1) The interior of A is the open set $A^\circ = \bigcup \{V \mid V \in \tau,\ V \subseteq A\}$.

(2) The closure of A is the closed set $\overline{A} = \bigcap \{C \mid C \text{ closed}, A \subseteq C\}$.

(3) We say A is dense if $\overline{A} = X$.

(4) We say A is nowhere dense if $\left(\overline{A}\right)^\circ = \varnothing$.

If X admits a countable dense subset, then we say X is separable.

If X is the countable union of nowhere dense subsets, then we say X is meager.

Remark 9. *A set A is dense if and only if, for any $U \in \tau$ with $U \neq \varnothing$, it is the case that $A \cap U \neq \varnothing$.*

Fact A.3.1. If (M, d) is a separable metric space, and $E \subseteq M$ is a subset, then E with the subspace topology is also separable.

Definition A.3.7. Let (X, τ) be a topological space.

- An open neighborhood of x_0 is an open set $V \in \tau$ with $x_0 \in V$. We write

$$\mathcal{O}_{x_0} = \{V \mid V \in \tau, x_0 \in V\}$$

to denote the family of all open neighborhoods of x_0.

- If $N \subseteq X$ is a subset with $x_0 \in V \subseteq N$, where $V \in \mathcal{O}_{x_0}$, then we say N is a neighborhood of x_0. We write $\mathcal{N}_{x_0}$ to be the collection of neighborhoods of x_0.
- A neighborhood base for τ at x_0 is a family $\mathcal{O} \subseteq \mathcal{O}_{x_0}$ with such that for all $U \in \mathcal{O}_{x_0}$, there is $V \in \mathcal{O}$ with $V \subseteq U$.
- We say (X, τ) is first countable if every $x \in X$ admits a countable neighborhood base.
- A base for τ is a family $\mathcal{B} \subseteq \tau$ that contains a neighborhood base for τ at x_0 For each $x_0 \in X$.
- We say (X, τ) is second countable if it admits a countable base.

Fact A.3.2. If $\mathcal{B}$ is a base for τ, then every $U \in \tau$ can be written as a union $U = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$.

Fact A.3.3. All metric spaces are first-countable, with a neighborhood base of

$$\mathcal{O}_{x_0} = \{U(x_0, 1/n) \mid n \in \mathbb{N}\}$$

for each $x_0 \in X$.

Fact A.3.4. A metric space (X, d) is second countable if and only if it is separable.

Fact A.3.5. If X is a topological space, and $x_0 \in X$ has a countable neighborhood base, then there is a neighborhood base $(V_n)_{n \geq 1}$ with $V_1 \supseteq V_2 \supseteq \cdots$.

A.3.2 Continuity in Topological Spaces

Definition A.3.8. Let (X, τ) and (Y, σ) be topological spaces, and let $f : X \to Y$ be a map.

(1) We say f is continuous at $x_0 \in X$ if, for every $U \in \mathcal{O}_{f(x_0)}$, there is $V \in \mathcal{O}_x$ with $f(V) \subseteq U$.

(2) We say f is continuous if f is continuous at every point in X.

(3) We say f is a homeomorphism if f is a continuous bijection with a continuous inverse.

(4) We say f is an open map if $U \in \tau$ implies $f(U) \in \sigma$. Similarly, we say f is a closed map if $C \subseteq X$ closed implies $f(C) \subseteq Y$ is closed.

(5) We say f is a quotient map if f is surjective with $V \subseteq Y$ open if and only if $f^{-1}(V) \subseteq X$ open.

(6) We say f is an embedding if $f : X \to \operatorname{Ran}(f)$ is a homeomorphism, where $\operatorname{Ran}(f)$ is endowed with the subspace topology.

(7) We write $C(X, Y)$ to be the continuous functions from X to Y. If $Y = \mathbb{C}$ with the regular topology, then we write $C(X)$.

Fact A.3.6. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U) \subseteq X$ is open for every open $U \subseteq Y$. Equivalently, f is continuous if and only if $f^{-1}(C) \subseteq X$ is closed for every closed $C \subseteq Y$.

Definition A.3.9 (Separation Axioms). Let (X, τ) be a topological space.

- We say X is T1 if $\{x\}$ is closed for every $x \in X$.
- We say X is T2 (or Hausdorff) if, for every $x, y \in X$ with $x \neq y$, there are $U, V \in \tau$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

- We say X is T3 if, for every $x \in X$ and $B \subseteq X$ closed with $x \notin B$, there are $U, V \in \tau$ with $x \in U$, $B \subseteq V$, and $U \cap V = \varnothing$. If X is T1 and T3, we say X is regular.
- We say X is T3.5 if, for every $x_0 \in X$ and closed $B \subseteq X$ with $x_0 \notin B$, there is a continuous function $f\colon X \to [0,1]$ with $f(x_0) = 0$ and $f(B) = 1$. If X is T1 and T3.5, we say X is completely regular.
- We say X is T4 if, for every pair of closed subsets $A, B \subseteq X$ with $A \cap B = \varnothing$, there are $U, V \in \tau$ with $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. If X is T1 and T4, then we say X is normal.

Just as we defined completely regular spaces through the existence of certain continuous functions that act to separate points, we can completely classify normality through a separating family of continuous functions.

Theorem A.3.1 (Urysohn's Lemma). *Let (X, τ) be a topological space. It is the case that X is normal if and only if for every pair of disjoint closed subsets $A, B \subseteq X$, there is a continuous function $f\colon X \to [0,1]$ with $f(A) = 0$ and $f(B) = 1$.*

Remark 10. *Metric spaces are an example of normal spaces.*

A.3.3 Initial and Final Topologies

Definition A.3.10. Let X be a set, and suppose $\{(Y_i, \tau_i)\}_{i \in I}$ is a family of topological spaces. Let $\{f_i\colon X \to Y_i\}$ be a family of maps. Setting

$$\varepsilon = \left\{ f_i^{-1}(V) \;\middle|\; V_i \in \tau_i \right\},$$

and letting $\tau = \tau(\varepsilon)$ be the topology on X generated by ε, we say τ is the initial topology on X induced by the maps $\{f_i\}_{i \in I}$.

Specifically, τ is the weakest topology on X such that each f_i is continuous.

Definition A.3.11 (Product Topology). Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. The topology on the product $\prod_{i \in I} X_i$ is defined to be the initial topology induced by the family of projection maps,

$$\pi_j\colon \prod_{i \in I} X_i \to X_j,$$

defined by $\pi_j((x_i)_{i \in I}) = x_j$.

For each $U \subseteq X_i$ open, we have $\pi_j^{-1}(U) = \prod_{i \in I} U_I$, where $U_i = X_i$ for $i \neq j$, and $U_j = U$. A base for this topology is the collection

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \;\middle|\; U_i = X_i \text{ for all but finitely many open } U_i \subseteq X_i \right\}.$$

If we consider $X_i = X$ for all i, there is a bijection between $X^I :- \{f \mid f\colon I \to X\}$, the set of all functions from I to X, and $\prod_{i \in I} X_i$, with the map $f \mapsto (f(i))_{i \in I}$. The product topology on X^I coincides with the topology of pointwise convergence.

Definition A.3.12 (Final Topology). Let (X, τ) be a topological space, Y a nonempty set, and suppose $q\colon X \to Y$ is a surjection. Then, the collection

$$\tau_q :- \left\{ V \subseteq Y \;\middle|\; q^{-1}(V) \in \tau \right\}$$

is what is known as the final (or quotient) topology on Y produced by q.

A.3.4 Convergence in Topological Spaces

Given a non-first-countable space X and a subset $A \subseteq X$, it is not necessarily the case that $x \in \overline{A}$ is the limit of a sequence $(x_n)_n$. However, we know that the sequential characterization of properties like closure, compactness (which will be covered in an upcoming section), and continuity is useful, so we want to generalize these ideas to non-first-countable spaces. This is where we can use nets.

Definition A.3.13 (Nets). A net is a map $A \to X$, where $\alpha \mapsto x_\alpha$, where A is a directed set. We write nets as $(x_\alpha)_\alpha$.

Example A.3.1 (Some Directed Sets).

(1) The natural numbers, $\mathbb{N}$, or the real numbers, $\mathbb{R}$, equipped with their usual ordering, are examples of directed sets. Every totally ordered set is directed.

(2) If S is any set, the collection $F(S)$ consisting of all finite subsets of S is directed by inclusion.

(3) The collection of finite partitions over a closed and bounded interval, $\mathcal{P}([a,b])$ is by the partition norm. If $P = \left\{x_j\right\}_{j=0}^{n}$ and $Q = \left\{y_j\right\}_{j=0}^{m}$ are partitions, we define

$$\|P\| = \max_{1 \leq j \leq n} \left|x_j - x_{j-1}\right|$$
$$\|Q\| = \max_{1 \leq j \leq m} \left|y_j - y_{j-1}\right|,$$

and the preorder that $P \leq Q$ if and only if $\|P\| \geq \|Q\|$. In other words, we say $P \leq Q$ if Q is finer than P.

For any partitions P and Q, their common refinement is a supremum for both — $P \vee Q \geq P, Q$ for each partition.[I]

(4) Let (X, τ) be a topological space, and for every x, we order the $\mathcal{O}_x$ by containment. That is, for elements $U, V \in \mathcal{O}_x$, we set $U \leq V$ if and only if $U \supseteq V$. This is a directed set by reverse inclusion, as we can always take $U \cap V \subseteq U, V$ (since both U and V contain x).

Similarly, the neighborhood system at x, $\mathcal{N}_x$, is also directed by containment.

(5) If A and B are directed sets, then $A \times B$ with the Cartesian ordering — $(\alpha_1, \beta 1) \leq (\alpha_2, \beta 2)$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ — is also a directed set.

Example A.3.2 (Some Nets).

(1) Any sequence $(x_k)_{k \in \mathbb{N}}$ is a net.

(2) Let $F(\Omega)$ be the set of all finite subsets of Ω directed by inclusion. Let $f : \Omega \to \mathbb{C}$ be a map. Then, we have a net $(s_F)_{F \in F(\Omega)}$ defined by

$$s_F = \sum_{x \in F} f(x).$$

(3) Consider the collection of partitions $\mathcal{P}([a,b])$ directed by the partition norm. For a bounded function $f : [a,b] \to \mathbb{R}$ and a partition $P = \left\{x_j\right\}_{j=0}^{n}$, for each j we set

$$M_j(P) = \sup_{t \in \left[x_j, x_{j-1}\right]} f(t)$$
$$m_j(P) = \inf_{t \in \left[x_j, x_{j-1}\right]} f(t).$$

[I] This is extremely useful in defining the Riemann integral.

We obtain two nets, $U, L \colon \mathcal{P}([a,b])$, defined by

$$U(P) = \sum_{j=1}^{n} M_j(P)\left(x_j - x_{j-1}\right)$$
$$L(P) = \sum_{j=1}^{n} m_j(P)\left(x_j - x_{j-1}\right).$$

These are known as the upper and lower Darboux sums.

Definition A.3.14. Let (X, τ) be a topological space, and let $(x_\alpha)_\alpha$ be a net in X.

(1) For a set $U \subseteq X$, we say $(x_\alpha)_\alpha$ is eventually in U if there is $\alpha_0 \in A$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

(2) We say the net $(x_\alpha)_\alpha$ converges to $x \in X$ if, for every $U \in \mathcal{O}_x$, $(x_\alpha)_\alpha$ is eventually in U. We write $(x_\alpha)_\alpha \xrightarrow{\tau} x$, though if the topology is clear from context the τ is not written.

(3) For a given $U \subseteq X$, we say $(x_\alpha)_\alpha$ is frequently in U if for every $\beta \in A$, there is $\alpha \in A$ with $\alpha \geq \beta$ and $x_\alpha \in U$.

(4) A point $x \in X$ is known as a cluster point of the net $(x_\alpha)_\alpha$ if for every $U \in \mathcal{O}_x$, $(x_\alpha)_\alpha$ is frequently in U. That is, for all $U \in \mathcal{O}_x$ and for all $\beta \in A$, there exists $\alpha \in A$ with $\alpha \geq \beta$ and $x_\alpha \in U$.

Fact A.3.7 (Characterizations Using Nets)**.** Let (X, τ) and (Y, σ) be topological spaces, $E \subseteq X$ a subset, and $f : X \to Y$ a map.

- It is the case that $x \in \overline{E}$ if and only if there is a net $(x_\alpha)_\alpha$ in E with $(x_\alpha)_\alpha \to x$.
- A map f is continuous if and only if for every convergent net $(x_\alpha)_\alpha \xrightarrow{\tau} x$, we have $(f(x_\alpha))_\alpha \xrightarrow{\sigma} f(x)$.
- If X is given by the initial topology induced by the family of maps $\{f_i \colon X \to (Y_i, \tau_i)\}_{i \in I}$, the net $(x_\alpha)_\alpha$ converges to x if and only if $(f_i(x_\alpha))_\alpha \xrightarrow{\tau_i} f_i(x)$ in Y_i for each $i \in I$.
- If $\{(X_i, \tau_i)\}_{i \in I}$ is a family of topological spaces, with $X = \prod_{i \in I} X_i$ equipped with the product topology, then a net $(x_\alpha)_\alpha$ in X converges to $x \in X$ if and only if $(x_\alpha(i))_\alpha \xrightarrow{\tau_i} x(i)$ in X_i for each $i \in I$.

Appendix B

Measure Theory and Integration

In order to properly discuss amenability, we need a strong foundation in measure theory.

B.1 Constructing Measurable Spaces

Fix a set Ω. We let $\mathcal{A} = \{A_i\}_{i \in I}$ be a collection of subsets of Ω.

Definition B.1.1 (Algebra of Subsets). The collection $\mathcal{A} = \{A_i\}_{i \in I}$ is known as an algebra of subsets for Ω if

- $\emptyset, \Omega \in \mathcal{A}$;
- for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

We can refine the concept of an algebra of subsets to consider countable unions rather than finite unions. This is known as a σ-algebra.

Definition B.1.2 (σ-Algebra of Subsets). The collection $\mathcal{A} = \{A_i\}_{i \in I}$ is known as a σ-algebra of subsets for Ω if

- $\emptyset, \Omega \in \mathcal{A}$;
- for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- for any countable collection $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, $\bigcup_{n \geq 1} A_n \in \mathcal{A}$.

Definition B.1.3 (Measurable Space). A pair $(\Omega, \mathcal{A})$, where Ω is a set and $A \subseteq P(\Omega)$ is a σ-algebra, is called a measurable space. Elements in the measurable space are called $\mathcal{A}$-measurable sets.

Definition B.1.4 (Restriction of a σ-Algebra). For a measurable space $(\Omega, \mathcal{A})$, with $E \in \mathcal{A}$, the family

$$\mathcal{A}_E = \{E \cap A \mid A \in \mathcal{A}\}$$

is a σ-algebra on E, known as the restriction of $\mathcal{A}$ to E.

Definition B.1.5 (Produced σ-Algebra). Let $(\Omega, \mathcal{A})$ be a measurable space, and $f : \Omega \to \Lambda$ is a map. The σ-algebra produced by f on Λ is the collection

$$\mathcal{N} = \left\{E \mid E \subseteq \Lambda,\ f^{-1}(E) \in \mathcal{A}\right\}.$$

Definition B.1.6 (Generated σ-Algebra). For a family $\mathcal{E} \subseteq P(\Omega)$, the σ-algebra generated by E is the smallest σ-algebra that contains E.

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{E} \in \mathcal{M}_i \\ \mathcal{M}_i\ \sigma\text{-Algebra}}} \mathcal{M}_i$$

Definition B.1.7 (Borel σ-Algebra)**.** If Ω is a topological space with topology $\tau \subseteq P(\Omega)$, we define

$$\mathcal{B}_\Omega = \sigma(\tau)$$

to be the Borel σ-algebra.

All open, closed, clopen, F_σ, and G_δ subsets of Ω are Borel.

We can now begin examining measurable functions.

Definition B.1.8 (Measurable Functions)**.** Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces.

(1) We say a map $f : \Omega \to \Lambda$ is $\mathcal{M}$-$\mathcal{N}$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

(2) We say a map $f : \Omega \to \mathbb{R}$ is measurable if it is $\mathcal{M}$-$\mathcal{B}_\mathbb{R}$-measurable.

(3) We say a map $f : \Omega \to \mathbb{C}$ is measurable if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.

The set of all measurable functions on $(\Omega, \mathcal{M})$ is denoted $L_0(\Omega, \mathcal{M})$.

The collection of all bounded measurable functions is the set

$$B_\infty(\Omega, \mathcal{M}) = \left\{ f \in L_0(\Omega, \mathcal{M}) \mid \sup_{x \in \Omega} |f(x)| < \infty \right\}.$$

Example B.1.1. If $f : \Omega \to \Lambda$ is a continuous map between topological spaces, then f is $\mathcal{B}_\Omega$-$\mathcal{B}_\Lambda$-measurable, since

$$\mathcal{F} = \left\{ E \subseteq \Lambda \mid f^{-1}(E) \in \mathcal{B}_\Omega \right\}$$

is a σ-algebra containing every open set in Λ, so $\mathcal{F}$ contains $\mathcal{B}_\Lambda$.

Example B.1.2. If $(\Omega, \mathcal{M})$ is a measurable space, and $f : \Omega \to \Lambda$ is a map, the measurable space $(\Lambda, \mathcal{N})$ produced by f is necessarily measurable.

Fact B.1.1. If $(\Omega, \mathcal{M})$, $(\Lambda, \mathcal{N})$, and $(\Sigma, \mathcal{L})$ are measurable spaces, with $f : \Omega \to \Lambda$ and $g : \Lambda \to \Sigma$ measurable, then $g \circ f$ is measurable.

Proposition B.1.1. *Let $(\Omega, \mathcal{M})$ be a measurable space. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Suppose $f, g, h_n : \Omega \to \mathbb{F}$ are measurable for $n \geq 1$.*

(1) If $\alpha \in \mathbb{F}$, then $f + \alpha g$ is measurable.

(2) $\overline{f}$ is measurable.

(3) fg is measurable.

(4) $\frac{f}{g}$ is measurable assuming it is well-defined.

(5) if h_n are $\mathbb{R}$-valued, and $(h_n(x))_n$ is bounded for each $x \in \Omega$, then $\sup h_n$ and $\inf h_n$ are measurable.

(6) If f and g are $\mathbb{R}$ valued, then $\max(f, g)$ and $\min(f, g)$ are measurable. In particular,

$$f_+ = \max(f, 0)$$
$$f_- = \max(0, -f)$$

are measurable.

(7) $|f|$ is measurable.

(8) The pointwise limit of measurable functions is measurable — if $\lim_{n\to\infty} h_n(x)$ exists for all $x \in \Omega$, then $h = \lim_{n\to\infty} h_n$ is measurable.

Definition B.1.9 (Simple Functions). A simple function $s\colon \Omega \to \mathbb{F}$ is a function with finite range. In other words, s is of the form

$$s = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k}$$

for $E_k \subseteq \Omega$ and $c_k \in \mathbb{F}$.

A simple function is measurable if and only if $E_k \in \mathcal{M}$ for each k.

B.2 Constructing Measures

A measure assigns a nonnegative "length" or "volume" to measurable sets.

Definition B.2.1 (Basics of Measures). A measure on a measurable space $(\Omega, \mathcal{M})$ is a map $\mu\colon \mathcal{M} \to [0, \infty]$ that satisfies the following.

(i) $\mu(\varnothing) = 0$;

(ii) $\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu\left(E_j\right)$.

The triple $(\Omega, \mathcal{M}, \mu)$ is called a measure space.

A measure μ is finite if $\mu(\Omega) < \infty$

If $\mu(\Omega) = 1$, then μ is called a probability measure.

A measure μ is called finitely additive if $\mu(E \sqcup F) = \mu(E) + \mu(F)$.

A measure μ is called σ-finite if there is a countable family $\{E_n\}_{n\geq 1} \subseteq \mathcal{M}$ such that

$$\Omega = \bigcup_{n\geq 1} E_n$$

and $\mu(E_n) < \infty$.

A measure μ on $(\Omega, \mathcal{M})$ is called semi-finite if, for every $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$.

Lemma B.2.1. *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.*

(1) If $E, F \in \mathcal{M}$ with $F \subseteq E$, then $\mu(F) \subseteq \mu(E)$.

(2) If $(E_n)_n$ is a sequence of measurable sets, then

$$\mu\left(\bigcup_{n\geq 1} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

(3) If $(E_n)_{n\geq 1}$ is an increasing family of measurable sets, then

$$\mu\left(\bigcup_{n\geq 1} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Definition B.2.2 (Pushforward Measure). Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $(\Lambda, \mathcal{N})$ be a measurable space. Let $f : \Omega \to \Lambda$ be measurable. The map

$$f_*\mu \colon \mathcal{N} \to [0, \infty]$$

defined by

$$f_*\mu(E) = \mu\left(f^{-1}(E)\right)$$

defines a measure on $(\Lambda, \mathcal{N})$. This is known as the pushforward measure of μ.

If $\mathcal{N}$ on Λ is produced by f, then the pushforward measure is necessarily defined on $\mathcal{N}$, and that any function $g \colon \Lambda \to \mathbb{F}$ is measurable if and only if $g \circ f$ is measurable.

Definition B.2.3. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

A null set is a measurable set $N \in \mathcal{M}$ with $\mu(N) = 0$.

A property which holds for all $x \in \Omega \setminus N$ for some null set N is said to hold μ-almost everywhere, or μ-a.e.

Definition B.2.4. If $(\Omega, \mathcal{M}, \mu)$ is a measure space, we can define an equivalence relation on the set $L_0(\Omega, \mathcal{M}, \mu)$, by

$$f \sim_\mu g \text{ if and only if } \mu(\{x \mid f(x) \neq g(x)\}) = 0.$$

We define the set of all classes of measurable functions by

$$\begin{aligned} L(\Omega, \mu) &= L_0(\Omega, \mathcal{M}) / \sim_\mu \\ &= \left\{[f]_\mu \mid f \in L_0(\Omega, \mathcal{M})\right\}. \end{aligned}$$

Fact B.2.1. The operations

- $[f]_\mu + [g]_\mu = [f+g]_\mu$;
- $[f]_\mu [g]_\mu = [fg]_\mu$;
- and $\alpha [f]_\mu = [\alpha f]_\mu$

are well-defined.

Definition B.2.5 (Essentially Bounded Functions and Continuous Functions). Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and $f : \Omega \to \mathbb{C}$ be measurable. We say f is μ-essentially bounded if there is $C \geq 0$ such that

$$\mu(\{x \in \Omega \mid |f(x)| \geq C\}) = 0.$$

We say C is an essential bound for f. The infimum of all essential bounds is the essential supremum, which gives

$$\begin{aligned} \|f\|_\infty &= \operatorname{ess\,sup}(f) \\ &= \inf\{C \geq 0 \mid \mu(\{x \in \Omega \mid |f(x)| \geq C\}) = 0\}. \end{aligned}$$

The collection of all μ-essentially bounded functions is denoted

$$L_\infty(\Omega, \mu) = \left\{[f]_\mu \in L(\Omega, \mu) \mid \|f\|_\infty < \infty\right\}.$$

Note that $B_\infty(\Omega, \mu) = L_\infty(\Omega, \mu)$ as sets.

If Ω is a topological space, with $\mathcal{B}_\Omega$ the Borel σ-algebra, we have $C(\Omega) \subseteq L_0(\Omega, \mathcal{B}_\Omega)$.

For μ a measure on $(\Omega, \mathcal{B}_\Omega)$, the μ-equivalence classes of continuous functions are

$$C(\Omega, \mu) = \left\{ [f]_\mu \mid f \in C(\Omega) \right\}.$$

Members of $L(\Omega, \mu)$ and $L_\infty(\Omega, \mu)$ are equivalence classes of functions (rather than functions themselves), but we use the abuse of notation that $[f]_\mu = f$.

Fact B.2.2. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $f, g\colon \Omega \to \mathbb{C}$ be measurable, and $\alpha \in \mathbb{C}$. Then, the following are true:

- $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$;
- $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$;
- if $\|f\|_\infty = 0$, then $f = 0$ μ-a.e.;
- $\|f\|_\infty \le \|f\|_u$;
- if f is essentially bounded, then

$$\mu(\{x \mid |f(x)| \ge \|f\|_\infty\}) = 0.$$

Definition B.2.6 (Complete Measure Space). A measure space $(\Omega, \mathcal{M}, \mu)$ is said to be complete if all subsets of null sets are measurable (and null).

B.3 Integration

Definition B.3.1. If $\phi\colon \Omega \to [0, \infty)$ is a positive, simple, and measurable function,

$$\phi = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k},$$

then the integral of ϕ is defined as

$$\int_\Omega \phi \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k),$$

with the convention that $0 \cdot \infty = 0$.

The value of this integral is not dependent on the representation of ϕ.

Definition B.3.2. If $f : \Omega \to \infty[0, \infty)$ is a positive measurable function, then

$$\int_\Omega f \, d\mu = \sup \left\{ \int_\Omega \phi \, d\mu \mid \phi \text{ measurable and simple, } 0 \le \phi \le f \right\}.$$

If $E \subseteq \Omega$ is measurable, we define

$$\int_E f \, d\mu = \int_\Omega f \mathbb{1}_E \, d\mu.$$

Proposition B.3.1. *Let $(\Omega, \mathcal{M})$ be a measurable space, and let $f : \Omega \to \mathbb{C}$ be measurable. There is a sequence $(\phi_n)_n$ of simple, measurable functions with $(\phi_n(x))_n \xrightarrow{n \to \infty} f(x)$.*

If $f \ge 0$, we can take ϕ_n to be positive and pointwise increasing.

If f is bounded, then this convergence is uniform, and ϕ_n can be uniformly bounded.

Theorem B.3.1 (Monotone Convergence Theorem). *Let $(f_n : \Omega \to [0,\infty))$ be an inreasing sequence of positive, measurable functions converging pointwise to $f : \Omega \to [0,\infty)$. Then, f is measurable, and*

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$

Definition B.3.3. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) A measurable function $f : \Omega \to [0,\infty)$ is integrable if

$$\int_\Omega f \, d\mu < \infty.$$

(2) A measurable function $f : \Omega \to \mathbb{R}$ is integrable if both f_+ and f_- are integrable. We define

$$\int_\Omega f \, d\mu = \int_\Omega f_+ \, d\mu - \int_\Omega f_- \, d\mu.$$

(3) A measurable function $f : \Omega \to \mathbb{C}$ is said to be integrable if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are integrable. We define

$$\int_\Omega f \, d\mu = \int_\Omega \mathrm{Re}(f) \, d\mu + i \int_\Omega \mathrm{Im}(f) \, d\mu.$$

Fact B.3.1. Let $f, g : \Omega \to \mathbb{C}$ be integrable functions, and $\alpha \in \mathbb{C}$. Then,

- $f + \alpha g$ is integrable, and $\displaystyle\int_\Omega (f + \alpha g) \, d\mu = \int_\Omega f \, d\mu + \alpha \int_\Omega g \, d\mu$;
- if f and g are real-valued, and $f \le g$, then $\displaystyle\int_\Omega f \, d\mu \le \int_\Omega g \, d\mu$;
- $\displaystyle\left| \int_\Omega f \, d\mu \right| \le \int_\Omega |f| \, d\mu.$

Fact B.3.2. If $f = g$ μ-a.e., then

$$\int_\Omega f \, d\mu = \int_\Omega g \, d\mu.$$

Fact B.3.3. If $f : \Omega \to \mathbb{C}$ is measurable, then $\int_\Omega |f| \, d\mu = 0$ if and only if $f = 0$ μ-a.e.

Fact B.3.4. A measurable function $f : \Omega \to \mathbb{C}$ is integrable if and only if $|f|$ is integrable.

Definition B.3.4 (Integrable Functions). Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) We define the set of (classes of) integrable functions to be

$$L_1(\Omega, \mu) = \left\{ [f]_\mu \in L(\Omega, \mu) \mid f \text{ is integrable} \right\}.$$

(2) We define the set of (classes of) square-integrable functions to be

$$L_2(\Omega, \mu) = \left\{ [f]_\mu \in L(\Omega, \mu) \mid |f|^2 \text{ is integrable} \right\}.$$

Definition B.3.5. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. If f and $(f_n)_n$ are integrable with $\|f - f_n\|_1 \xrightarrow{n\to\infty} 0$, we say $(f_n)_n$ converges in mean to f.

Fact B.3.5. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) For $f \in L_1(\Omega, \mu)$, the maps

$$[f]_\mu \longmapsto \int_\Omega f \, d\mu$$
$$[f]_\mu \longmapsto \int_\Omega |f| \, d\mu$$

are well-defined.

(2) For $f \in L_1(\Omega, \mu)$, we define

$$\|f\|_1 = \int_\Omega |f| \, d\mu.$$

This is a well-defined norm.

$$\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$$
$$\|\alpha f_1\| = |\alpha| \|f\|_1$$
$$\|f\| = 1 \Leftrightarrow f = 0 \ \mu\text{-a.e.}$$

(3) The form

$$d_1\left([f]_\mu, [g]_\mu\right) = \|f-g\|_1$$

is a metric on $L_1(\Omega, \mu)$.

Theorem B.3.2 (Dominated Convergence Theorem). *Let $(f_n : \Omega \to \mathbb{C})_n$ be a sequence of measurable functions converging pointwise to a measurable function $f : \Omega \to \mathbb{C}$. If there is an integrable $g : \Omega \to [0, \infty)$ with $|f_n| \leq g$ for all n, then*

$$\int_\Omega f_n \, d\mu \xrightarrow{n\to\infty} \int_\Omega f \, d\mu.$$

Corollary B.3.1. *If $f : \Omega \to \mathbb{C}$ is integrable, then there is a sequence of simple integrable functions $(\phi_n)_n$ with $\left\|f - \phi_n\right\|_1 \xrightarrow{n\to\infty} 0$.*

Corollary B.3.2. *If $f : \mathbb{R} \to \mathbb{C}$ is integrable, then there is a sequence $(f_n)_n$ of compactly supported integrable functions such that $\|f - f_n\|_1 \xrightarrow{n\to\infty} 0$.*

Theorem B.3.3. *If $f : \mathbb{R} \to \mathbb{C}$ is integrable, and $\varepsilon > 0$, there is a continuous, compactly supported function g with $\|f-g\|_1 < \varepsilon$.*

Proposition B.3.2. *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $(\Lambda, \mathcal{N})$ be a measurable space with $f : \Omega \to \Lambda$ a measurable map. Let $f_*\mu$ is the pushforward measure on $(\Lambda, \mathcal{N})$. For a measurable function $g : \Lambda \to [0, \infty)$, then*

$$\int_\Lambda g \, d(f_*\mu) = \int_\Omega (g \circ f) \, d\mu.$$

Moreover, if $g : \Lambda \to \mathbb{F}$ is integrable with respect to $f_*\mu$, then so too is $g \circ f$ with respect to μ.

B.4 Complex Measures

Example B.4.1. If $(\Omega, \mathcal{M}, \mu)$ is a measure space, then the map $\mu_f(E) = \int_E f \, d\mu$ is a well-defined measure.

Definition B.4.1. Let $(\Omega, \mathcal{M}, \mu)$ be a measurable space.

(1) A complex number on $(\Omega, \mathcal{M}, \mu)$ is a map $\mu : \mathcal{M} \to \mathbb{C}$ satisfying the following conditions.

- $\mu(\varnothing) = 0$;
- $\mu\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$ for $\{E_k\}_{k\geq 1} \subseteq \mathcal{M}$.

(2) We write $M(\Omega)$ to be the set of all complex measures on $(\Omega, \mathcal{M})$.

(3) If $\mu \in M(\Omega)$, and $\mu(E) \in \mathbb{R}$ for all $E \in \mathcal{M}$, then we say μ is a real measure on $(\Omega, \mathcal{M})$.

(4) If $\mu \in M(\Omega)$ and $\mu(E) \geq 0$ for all $E \in \mathcal{M}$, then we say μ is a positive measure on $(\Omega, \mathcal{M})$.

(5) If μ is a positive measure on $(\Omega, \mathcal{M})$ with $\mu(\Omega) = 1$, we say μ is a probability measure on $(\Omega, \mathcal{M})$. We write $\mathcal{P}(\Omega, \mathcal{M})$ to be the collection of all probability measures on $(\Omega, \mathcal{M})$.

(6) If Ω is a locally compact Hausdorff space, we always let $M(\Omega)$ be the set of all complex Borel measures on Ω.

Definition B.4.2. If $(\Omega, \mathcal{M})$ is a measurable space, and $x \in \Omega$, the Dirac measure at x is defined by

$$\begin{aligned} \delta_x \colon \mathcal{M} &\to [0,1] \\ \delta_x(E) &= \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}. \end{aligned}$$

If $x_1, \dots, x_n$ are distinct points in Ω, and $t_1, \dots, t_n \in [0,1]$ with $\sum_{j=1}^{n} t_j = 1$, then

$$\mu = \sum_{j=1}^{n} t_j \delta_{x_j}$$

is a probability measure on $(\Omega, \mathcal{M})$.

Fact B.4.1. If μ is a complex measure on $(\Omega, \mathcal{M})$, then $\overline{\mu}$, defined by $\overline{\mu}(E) = \overline{\mu(E)}$ for $E \in \mathcal{M}$, is also a complex measure. Additionally, $\operatorname{Re}(\mu)$ and $\operatorname{Im}(\mu)$, defined by

$$\begin{aligned} \operatorname{Re}(\mu)(E) &= \operatorname{Re}(\mu(E)) \\ \operatorname{Im}(\mu)(E) &= \operatorname{Im}(\mu(E)) \end{aligned}$$

are real measures.

Definition B.4.3. If $\mu \in M(\Omega)$, then the total variation of μ is the quantity

$$|\mu| \colon \mathcal{M} \to [0, \infty]$$

with

$$|\mu|(E) = \sup\left\{ \sum_{j=1}^{\infty} \left|\mu\left(E_j\right)\right| \,\middle|\, E = \sqcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{M} \right\}.$$

Fact B.4.2. If $\mu \in M(\Omega)$, then $|\mu|$ is a positive, finite measure. Additionally, if $\mu, \nu \in M(\Omega)$ with $\alpha \in \mathbb{C}$, then

(a) $|\mu(E)| \leq |\mu|(E)$

(b) $|\mu + \nu|(E) \leq |\mu|(E) + |\nu|(E)$

(c) $|\alpha\mu|(E) = |\alpha||\mu|(E)$.

Definition B.4.4 (Absolute Continuity of Measures). Let $(\Omega, \mathcal{M})$ be a measurable space, and let μ and ν be positive measures on this space. If $\mu(A) > 0$ implies $\nu(A) > 0$ for a given $A \in \mathcal{M}$, we say μ is absolutely continuous with respect to ν. We write $\mu \ll \nu$.

Theorem B.4.1 (Radon–Nikodym Theorem). *If $\mu \ll \nu$ on $(\Omega, \mathcal{M})$, then there exists a measurable function $f: \Omega \to [0,\infty]$ such that*

$$\nu(A) = \int_A f \, d\nu$$

for each $A \in \mathcal{M}$.

Remark 11. *The Radon–Nikodym theorem extends to signed and complex measures.*

Fact B.4.3. Let $(\Omega, \mathcal{M}, \lambda)$ be a measure space, and suppose $f \in L_1(\Omega, \lambda)$. Then, $\mu(E) = \int_E f \, d\lambda$ defines a complex measure. We write $f = \frac{d\mu}{d\lambda}$, which is the Radon–Nikodym derivative of μ with respect to λ.

It is also the case that

$$\left|\mu\right|(E) = \int_E |f| \, d\lambda.$$

Fact B.4.4. If $\mu \in M(\Omega)$, there exists a measurable function $f: \Omega \to \mathbb{C}$ such that $|f| = 1$ and $\mu(E) = \int_E f \, d\left|\mu\right|$ for all $E \in \mathcal{M}$.

Definition B.4.5. Let Ω be a locally compact Hausdorff space equipped with the Borel σ-algebra, $\mathcal{B}_\Omega$.

(1) A Borel measure $\mu\colon \mathcal{B}_\Omega \to [0,\infty]$ is called

- inner regular on $E \in \mathcal{B}_\Omega$ if

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\};$$

- outer regular on $E \in \mathcal{B}_\Omega$ if

$$\mu(E) = \inf\{\mu(U) \mid U \supseteq E, U \text{ open}\};$$

- regular on E if it is inner regular and outer regular on E;
- regular if it is regular on all $E \in \mathcal{B}_\Omega$;
- Radon if
  - $\mu(K) < \infty$ for all compact $K \subseteq \Omega$;
  - μ is inner regular on all open sets and outer regular on all Borel sets.

(2) A complex Borel measure $\mu\colon \mathcal{B}_\Omega \to \mathbb{C}$ is regular if $\left|\mu\right|$ is regular; μ is Radon if $\left|\mu\right|$ is Radon.

(3) We write $M_r(\Omega)$ to denote the set of all complex regular measures on $(\Omega, \mathcal{B}_\Omega)$.

Fact B.4.5. Every positive Radon measure is regular. Thus, every complex Borel measure is regular if and only if it is Radon.

Moreover, if Ω is a second countable locally compact Hausdorff space, then every complex Borel measure is regular.

Definition B.4.6. Let (Ω, τ) be a topological space, and suppose $\mu\colon \mathcal{B}_\Omega \to [0,\infty]$ is a Borel measure.

(1) The kernel of μ is the set

$$N_\mu = \bigcup\{U \subseteq \Omega \mid U \in \tau,\ \mu(U) = 0\}.$$

(2) The support of μ is the complement of the kernel, $\operatorname{supp}(\mu) = N_\mu^c$.

Fact B.4.6. If μ is a Radon measure on a locally compact Hausdorff space Ω, then $\mu\left(N_\mu\right) = 0$, meaning $\mu(\Omega) = \mu(\mathrm{supp}(\mu))$.

Theorem B.4.2 (Hahn and Jordan Decomposition). *Let $(\Omega, \mathcal{M})$ be a measurable space, and let $\mu\colon \mathcal{M} \to \mathbb{R}$ be a real measure. Then, there is a measurable partition $\Omega = P \sqcup N$ such that for all $E \subseteq P$, $\mu(E) \geq 0$, and for all $E \subseteq N$, $\mu(E) \leq 0$. This partition is unique up to a μ-null symmetric difference — that is, for any P', N' satisfying the conditions, $\mu(P' \triangle P) = 0$ and $\mu(N' \triangle N) = 0$.*

There is a unique decomposition $\mu = \mu_+ - \mu_-$, with $\mu_\pm$ that are positive such that if $E \subseteq P$, then $\mu_-(E) = 0$, and if $E \subseteq N$, $\mu_+(E) = 0$.

Definition B.4.7. Let $(\Omega, \mathcal{M})$ be a measurable space, and let $f : \Omega \to \mathbb{C}$ be measurable.

(1) If $\mu\colon \mathcal{M} \to \mathbb{R}$ is a real measure with $\mu = \mu_+ - \mu_-$, we say that f is μ-integrable if it is both μ_+ and μ_--integrable. We define

$$\int_\Omega f\, d\mu = \int_\Omega f\, d\mu_+ - \int_\Omega f\, d\mu_-.$$

(2) If $\mu : \mathcal{M} \to \mathbb{C}$ is a complex measure with $\mu_1 = \mathrm{Re}(\mu)$ and $\mu_2 = \mathrm{Im}(\mu)$, we say f is μ-integrable if it is both μ_1 and μ_2-integrable. We define

$$\int_\Omega f\, d\mu = \int_\Omega f\, d\mu_1 + i \int_\Omega f\, d\mu_2.$$

Theorem B.4.3 (Riesz Representation Theorem on $C_c(\Omega)$). *Let Ω be a locally compact Hausdorff space. If $\varphi\colon C_c(\Omega) \to \mathbb{C}$ is a positive linear functional, then there is a unique Radon measure μ such that*

$$\varphi(f) = \int_\Omega f\, d\mu$$

for all $f \in C_c(\Omega)$. Additionally, for every open $U \subseteq \Omega$, we have

$$\mu(U) = \sup\{\varphi(f) \mid f \in C_c(\Omega, [0,1]), \mathrm{supp}(f) \subseteq U\},$$

and for every compact $K \subseteq \Omega$, we have

$$\mu(K) = \inf\{\varphi(f) \mid f \geq \mathbb{1}_K\}.$$

Theorem B.4.4 (Riesz Representation theorem on $C(X)$). *Let X be compact, and let $\varphi \in (C(X))^*$ be a positive linear functional with $\varphi(\mathbb{1}_X) = \|\varphi\| = 1$. Then, for $f \in C(X)$, there is a unique Borel probability measure such that*

$$\varphi(f) = \int_X f\, d\mu.$$

Definition B.4.8. Let Ω be a locally compact Hausdorff space, and let $\tau\colon \Omega \to \Omega$ be a continuous transformation. A regular Borel probability measure $\mu \in \mathcal{P}_r(\Omega)$ is called τ-invariant if $\tau_*\mu = \mu$.

Appendix C

Functional Analysis

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