Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

$$d\mathbf{r} = \rho\cos\phi\,\hat{\imath} + \rho\sin\phi\,\hat{\jmath} + z\hat{k}.$$

We let $\hat{e}_1 = \hat{\rho}$, $\hat{e}_2 = \hat{\phi}$, and $\hat{e}_3 = \hat{z}$, with $u_1 = \rho$, $u_2 = \phi$, and $u_3 = z$. Thus, we get

- Line element:

$$\begin{aligned}
(ds)^2 &= \sum_{i,j} \frac{\partial \mathbf{r}}{\partial u_i}\frac{\partial \mathbf{r}}{\partial u_j}\left(\hat{e}_i\cdot\hat{e}_j\right) du_i\,du_j \\
&= \sum_{i=1}\left(\frac{\partial \mathbf{r}}{\partial u_i}\right)(du_i)^2 && \text{The } \hat{\rho}, \hat{\phi}, \hat{z} \text{ basis is orthogonal} \\
&= (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2.
\end{aligned}$$

- Area element:

$$d\mathbf{a} = \left(\sum_k \epsilon_{ijk}\hat{e}_k\right)\frac{\partial \mathbf{r}}{\partial u_i}\cdot\frac{\partial \mathbf{r}}{\partial u_j}\,du_i\,du_j$$

4.9

Without loss of generality, we have

$$\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ij\ell} = \epsilon_{mn1}\epsilon_{ij1},$$

where $m, n, i, j = 2, 3$. If we have $m = i, n = j$, then $\epsilon_{mn1}\epsilon_{ij1} = 1$; if $m = j, n = i$, then $\epsilon_{mn1}\epsilon_{ij1} = -1$; else, $\epsilon_{mn1}\epsilon_{ij1} = 0$.

4.11

(a)

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &= \sum_{i,j,k}\epsilon_{ijk}A_iB_j\hat{e}_k \\
&= -\sum_{i,j,k}\epsilon_{jik}B_jA_i\hat{e}_k \\
&= -(\mathbf{B}\times\mathbf{A})
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) &= \sum_{i,j,k}\left(\epsilon_{ijk}A_iB_j\hat{e}_k\right)\cdot A_i\hat{e}_i \\
&= \sum_{i,j,k}\delta_{ik}\left(\epsilon_{ijk}A_i^2B_j\right) \\
&= 0.
\end{aligned}$$

(c)

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) =$$

(d)

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_iC_j \\
&= \sum_{i,j,\ell}\left(\epsilon_{\ell ij}A_\ell B_i\right)C_j \\
&= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})
\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_i C_j\\ &= \sum_{i,j,\ell}\left(\epsilon_{j\ell i}C_j A_i\right)B_i\\ &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}).\end{aligned}$$

(e)

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j}\epsilon_{ijk}A_i\left(\sum_{\alpha,\beta}\epsilon_{\alpha\beta j}B_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\epsilon_{ijk}\epsilon_{\alpha\beta j}A_i B_\alpha C_\beta\\ &= -\left(\sum_{i,j,\alpha,\beta}\epsilon_{ikj}\epsilon_{\alpha\beta j}A_i B_\alpha C_\beta\right)\\ &= -\left(\sum_{i,j,\alpha,\beta}\left(\delta_{i\alpha}\delta_{k\beta}-\delta_{i\beta}\delta_{k\alpha}\right)A_i B_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\left(\delta_{k\alpha}\delta_{i\beta}-\delta_{i\alpha}\delta_{k\beta}\right)A_i B_\alpha C_\beta\\ &= \sum_{i,j,\alpha,\beta}\left(B_\alpha\delta_{k\alpha}\right)\left(A_i C_\beta\delta_{i\beta}\right)-\left(C_\beta\delta_{k\beta}\right)\left(A_i B_\alpha\delta_{i\alpha}\right)\\ &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}).\end{aligned}$$

Chapter 5 Problems

5.1

Let $f(x) = x^n$. We use linearity for the general case.

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{x^n + n\left(hx^{n-1}\right)+\cdots+nh^{n-1}x+h^n-x^n}{h}\\ &= \lim_{h\to 0}\left(nx^{n-1}+\cdots+nh^{n-2}x+h^{n-1}\right)\\ &= nx^{n-1}.\end{aligned}$$

5.6

$$\begin{aligned}\cos(N\phi)+i\sin(N\phi) &= (\cos\phi + i\sin\phi)^N\\ &= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k\left(\sin\phi^{N-k}\right)\left(e^{i\frac{\pi}{2}}\right)^{N-k}\\ &= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k(\sin\phi)^{N-k}\left(\cos\left((N-k)\frac{\pi}{2}\right)+i\sin\left((N-k)\frac{\pi}{2}\right)\right)\\ &= \sum_{k=0}^{N}\binom{N}{k}\cos\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}+i\left(\sum_{k=0}^{N}\binom{N}{k}\sin\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\right).\end{aligned}$$

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right| \\ &= 1,\end{aligned}$$

which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \qquad \forall n \geqslant 1.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

(b)
- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\left(\frac{1}{2}\right)\right| \\ &= \frac{1}{2} \\ &< 1,\end{aligned}$$

meaning the series converges by the ratio test.

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$$\frac{1}{n2^n} < \frac{1}{2^n} \qquad \text{for all } n \geqslant 1,$$

and since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty}\frac{1}{n2^n}$ converges.

6.9

$$\begin{aligned}\sum_{n=-N}^{N} e^{inx} &= 1 + \sum_{n=1}^{N} e^{-inx} + \sum_{n=1}^{N} e^{inx} \\ &= \frac{1-e^{-i(N+1)x}}{1-e^{-ix}} + \frac{1-e^{i(N+1)x}}{1-e^{ix}} + 1 \\ &= \frac{e^{-i(N+1)x}-1}{e^{-ix}-1} + \frac{e^{i(N+1)x}-1}{e^{ix}-1}\end{aligned}$$