Problem 1

Prove the following limits:

(i) $\left(\frac{2n}{n+2}\right)_n \to 2$

(ii) $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0$

(iii) $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0$

(iv) $\left(n^k b^n\right)_n \to 0$ where $0 \leq b < 1$ and $k \in \mathbb{N}$

(v) $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \to 3$

(i)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow \left|\frac{2n}{n+2} - 2\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}
\frac{2n}{n+2} &> 2 - \varepsilon \\
2n &> (2n - \varepsilon n) - 2\varepsilon + 4 \\
n &> \frac{4-2\varepsilon}{\varepsilon}
\end{aligned}$$

Proof Let $\varepsilon > 0$, $N = \left\lceil \frac{4-2\varepsilon}{\varepsilon} \right\rceil$. Then,

$$\begin{aligned}
n &> \frac{4-2\varepsilon}{\varepsilon} \\
\varepsilon n &> 4 - 2\varepsilon \\
0 &> 4 - 2\varepsilon - \varepsilon n \\
2n &> 2n + 4 - \varepsilon(n+2) \\
2n &> (2-\varepsilon)(n+2) \\
\frac{2n}{n+2} - 2 &> -\varepsilon \\
\left|\frac{2n}{n+2} - 2\right| &< \varepsilon && \frac{2n}{n+2} < 2 \ \forall n \in \mathbb{N}
\end{aligned}$$

(ii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \to \left|\left(\frac{\sqrt{n}}{n+1}\right)\right| < \varepsilon$$

Preliminary Work We will show that $\left(\frac{1}{\sqrt{n}}\right)_n \to 0$. Let $\varepsilon > 0$ and $N = 1 + \left\lceil \frac{1}{\varepsilon^2} \right\rceil$. Then,

$$\begin{aligned} n &\geq N \\ n &> \frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{n}} &< \varepsilon \\ \left|\frac{1}{\sqrt{n}} - 0\right| &< \varepsilon \end{aligned}$$

Proof We know that $\forall n$, $\frac{\sqrt{n}}{n+1} > 0$ and $\frac{\sqrt{n}}{n+1} < \frac{1}{\sqrt{n}}$. Since we showed earlier that $\frac{1}{\sqrt{n}} \to 0$, it must be the case that $\frac{\sqrt{n}}{n+1} \to 0$.

(iii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow \left|\frac{(-1)^n}{\sqrt{n+7}}\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned} \frac{1}{\sqrt{n+7}} &< \varepsilon \\ \frac{1}{\varepsilon} &< \sqrt{n+7} \\ n &> \frac{1}{\varepsilon^2} - 7 \end{aligned}$$

Proof Let $\varepsilon > 0$, $N = \left\lceil \frac{1}{\varepsilon^2} \right\rceil - 7$. Then,

$$\begin{aligned} n &> \frac{1}{\varepsilon^2} - 7 \\ n + 7 &> \frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{n+7}} &< \varepsilon \\ -\varepsilon &< \frac{-1}{\sqrt{n+7}} \\ \left|\frac{(-1)^n}{\sqrt{n+7}}\right| &< \varepsilon \end{aligned}$$

(iv)

If $b = 0$, then $n^k b^n = 0 \to 0$.

Let $0 < b < 1$. To show that $(n^k b^n)_n \to 0$, we will find what the ratio of consecutive terms tends toward:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^k b^{n+1}}{n^k b^n}\\ &= b\left(\frac{n+1}{n}\right)^k\end{aligned}$$

We claim that $\left(\frac{n+1}{n}\right)^k \to 1$. For this, we need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n \geq N \Rightarrow \left|\left(\frac{n+1}{n}\right)^k - 1\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}\left|\left(1+\frac{1}{n}\right)^k - 1\right| &< \varepsilon\\ \left(1+\frac{1}{n}\right)^k &< \varepsilon + 1\\ 1+\frac{1}{n} &< (\varepsilon+1)^{1/k}\\ n &> \frac{1}{(\varepsilon+1)^{1/k}-1}\end{aligned}$$

Proof Let $\varepsilon > 0$. Let $N = \left\lceil \frac{1}{(\varepsilon+1)^{1/k}-1} \right\rceil + 1$. Then, for $n \geq N$, we have

$$\begin{aligned}n &> \frac{1}{(\varepsilon+1)^{1/k}-1}\\ (\varepsilon+1)^{1/k} &> 1+\frac{1}{n}\\ \left(1+\frac{1}{n}\right)^k - 1 &< \varepsilon\end{aligned}$$

whence $\left|\left(\frac{n+1}{n}\right)^k - 1\right| = \left(1+\frac{1}{n}\right)^k - 1.$

Therefore, since $\left(\frac{n+1}{n}\right)^k \to 1$, the ratio converges to $b < 1$, meaning $n^k b^n \to 0$.

(v)

Preliminary Work

$$\begin{aligned}
\left|\frac{2^{n+1}+3^{n+1}}{2^n+3^n}-3\right| &< \varepsilon \\
3-\frac{2^{n+1}+3^{n+1}}{2^n+3^n} &< \varepsilon \\
\frac{3(2^n+3^n)-2^{n+1}-3^{n+1}}{2^n+3^n} &< \varepsilon \\
\frac{2^n}{2^n+3^n} &< \varepsilon \\
2^n &< (2^n+3^n)\varepsilon \\
(1-\varepsilon)2^n &< \varepsilon \cdot 3^n \\
\frac{1-\varepsilon}{\varepsilon} &< \left(\frac{3}{2}\right)^n \\
n &> \frac{\ln(1-\varepsilon)-\ln\varepsilon}{\ln 3-\ln 2}
\end{aligned}$$

Proof Let $\varepsilon > 0$ and $N = \left\lceil \frac{\ln(1-\varepsilon)-\ln\varepsilon}{\ln 3-\ln 2} \right\rceil + 1$. Then, for $n \geq N$, we have

$$\begin{aligned}
n &> \frac{\ln(1-\varepsilon)-\ln\varepsilon}{\ln 3-\ln 2} \\
n\ln\left(\frac{3}{2}\right) &> \ln\left(\frac{1-\varepsilon}{\varepsilon}\right) \\
\frac{3^n}{2^n} &> \frac{1-\varepsilon}{\varepsilon} \\
\varepsilon(3^n+2^n) &> 2^n \\
\frac{2^n}{2^n+3^n} &< \varepsilon
\end{aligned}$$

whence $\left|\frac{2^{n+1}+3^{n+1}}{2^n+3^n}-3\right| = \frac{2^n}{2^n+3^n}$.

Problem 2

Show that the sequence $(\cos(n))_n$ does not converge.

Problem 3

If $(x_n)_n$ is a real sequence converging to x, show that

$$(|x_n|)_n \to |x|$$

Is the converse true?

If $(x_n)_n \to x$, then $|x_n - x| \to 0$. So

$$\begin{aligned}
||x_n|-|x|| &\leq |x_n - x| && \text{Reverse Triangle Inequality} \\
&\to 0
\end{aligned}$$

So, $|x_n| \to |x|$.

The converse is not true. For example, the sequence $\left(|(-1)^n|\right)_n \to 1$, but $\left((-1)^n\right)_n$ does not converge.

Problem 4

If $(x_n)_n$ is a real sequence converging to $x > 0$, show that there is an $N \in \mathbb{N}$ and $c > 0$ such that

$$x_n \geq c \ \forall n \geq N$$

Since $(x_n)_n \to x$, we know that $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})$ such that $n \geq n \to x_n \in V_\varepsilon(x)$.

In particular, let $\varepsilon_0 = \frac{|0-x|}{3}$, $c = \frac{x}{3} < x$, and ε_1 small such that $V_{\varepsilon_1}(c) \cap V_{\varepsilon_0}(x) = \emptyset$.

Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in V_{\varepsilon_0}(x) > c$.

Problem 5

If $(x_n)_n$ is a real sequence of positive terms converging to x, show that $x \geq 0$ and

$$(\sqrt{x_n})_n \to \sqrt{x}$$

$x \geq 0$

Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0-x|}{2}$. Since $x_n \to x$, $\exists N \in \mathbb{N}$ large such that $x_n \in V_\varepsilon(x)$ for $n \geq N$. However, $\forall \ell \in V_\varepsilon(x)$, $\ell < 0$, meaning that $x_n < 0$ for large n. $\perp$

$(\sqrt{x_n})_n \to \sqrt{x}$

Case 1: Suppose $x = 0$. Let $\varepsilon > 0$. Then,

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ x_n &< \varepsilon^2 \\ \sqrt{x_n} &< \varepsilon \\ |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

So, $\sqrt{x_n} \to 0$.

Case 2: Suppose $x > 0$. Let $\varepsilon > 0$. Then,

$$\begin{aligned} \left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \\ &= \frac{1}{\sqrt{x_n} + \sqrt{x}}|x_n - x| \\ &\leq \frac{1}{\sqrt{x}}|x_n - x| \\ &\to 0 \end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \to 0$, so $\sqrt{x_n} \to x$

Problem 6

If $(x_n)_n$ and $(y_n)_n$ are sequences with $(x_n)_n \to 0$ and $(y_n)_n$ bounded. Show that

$$(x_n y_n)_n \to 0$$

Let $y \in \mathbb{R}$ be an upper bound on $(y_n)_n$. Then,

$$\begin{aligned} |x_n y_n| &\leq |x_n||y| \\ &\to 0 \end{aligned}$$

Therefore, $x_n y_n \to 0$.

Problem 7

If $(x_n)_n$ is a sequence of positive terms such that

$$\left(\frac{x_{n+1}}{x_n}\right)_n \to L > 1,$$

show that $(x_n)_n$ is not bounded, and thus not convergent. If $L = 1$, can we make any conclusions?

Suppose toward contradiction that $(x_n)_n$ is bounded. Then, $\exists y$ such that

$$|x_n| < y \ \forall n$$

so,

$$\frac{x_{n+1}}{x_n} < 1 \ \forall n$$

However, this means $L < 1$. $\perp$.

If $L = 1$, we cannot make any conclusions about the nature of the sequence.