Introduction: naive set theory

$$\begin{aligned} \mathbb{N} &= \{1,2,3,\dots,\} \\ \mathbb{Z} &= \{0,\pm 1,\pm 2,\dots,\} \\ \mathbb{Z}_+ &= \{0,1,2,\dots,\} \\ \mathbb{Q} &= \left\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\right\} \\ \mathbb{C} &= \{a+bi \mid a,b\in\mathbb{R}\} \\ \mathbb{C}_q &= \{a+bi \mid a,b\in\mathbb{Q}\} \end{aligned}$$

Recall: given sets X and Y, a relation from X to Y is a subset of $X\times Y$, where $\times$ denotes the cartesian product of X and Y.

A relation $f\subseteq X\times Y$ is a function from X to Y such that $\forall x\in X$, $\exists! y\in Y$ such that $(x,y)\in f$. We write $f(x)=y$, and denote f as $f: X\to Y$.

X is the **domain** of f and Y is the **codomain**. The range $\operatorname{ran}(f)=\{f(x)\mid x\in X\}\subseteq Y$.

The graph of a function $\operatorname{Graph}(f)=\{(x,f(x))\mid x\in X\}\subseteq X\times Y$.

Examples

$$\mathrm{id}_x : X\to X, \mathrm{id}_X(x)=x$$

This is the identity function.

The Characteristic Function: If $A\subseteq X$

$$\mathbf{1}_A : X\to\mathbb{R},\ \mathbf{1}_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$$

Algebra of Functions

Let X be any set, and $(X;\mathbb{R})=\{f: X\to\mathbb{R}\}$ represent the function space of X with codomain $\mathbb{R}$.

Let $f,g\in\mathcal{F}(X;\mathbb{R})$. Then, $(f+g)(x)=f(x)+g(x)$, and $(f\cdot g)(x)=f(x)\cdot g(x)$.

If $t\in\mathbb{R}$, then $(tf)(x)=tf(x)$ (scalar multiplication). If $g(x)\neq 0 \forall x\in X$, then $\left(\frac{f}{g}\right)(x):=\frac{f(x)}{g(x)}$.

Finally, we have composition. If $f: X\to Y$ and $g: Y\to Z$ are functions, then $g\circ f(x)=g(f(x))$.

Injective, Subjective, and Bijective

A function $f : X \to Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.For example, the shift map $S : \mathbb{N} \to \mathbb{N},\ S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \to \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \to Y$ be a function. f is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \text{id}_X$. f is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \text{id}_Y$.

f is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

Proposition

f is invertible if and only if f is left and right invertible.

Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose g is a left-inverse of f, and h is a right-inverse of f. Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

Theorem

If $f : X \to Y$ is a function:

1. f is injective $\Leftrightarrow$ f is left-invertible.
2. f is surjective $\Leftrightarrow$ f is right-invertible.
3. f is bijective $\Leftrightarrow$ f is invertible.

We will prove the first proposition in the forward direction. Suppose f is injective. Given $y \in \text{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X. We can see that $g \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

Cardinality and Finitude

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets A and B, we can show that A is the same size as B by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

Definition

Sets A and B have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\text{card}(A) = \text{card}(B)$.

Example

Given $a < b$ and $c < d$, we know that $\text{card}\left([a, b]\right) = \text{card}\left([c, d]\right)$.

We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}\left([a, b]\right) = \text{card}\left([c, d]\right)$.

Example 2

$$\text{card}\left((0, 1)\right) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - tan is strictly increasing (and thus injective)
  - $\lim_{x\to\infty} \tan(x) = \infty$ and $\lim_{x\to-\infty} \tan(x) = -\infty$, and by intermediate value theorem, tan is surjective
- $\ell : (0, 1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0, 1) \to \mathbb{R}$.

Definition

A set F is **finite** if F is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}\left(\{1, 2, \ldots, n\}\right)$. A non-finite set is called infinite.

We can *enumerate* F by creating a function $\sigma : \{1, 2, \ldots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \ldots, x_n\}$.

Proposition

If $m \neq n$, then $\text{card}\{1, 2, \ldots, m\} = \text{card}\{1, 2, \ldots, n\}$.

WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ is our bijection. This means there are m "pigeons" and n "holes."

One hole, j, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \ldots, m\}$). Since f is assumed to be injective, this is a contradiction.

Proposition

$\mathbb{N}$ is infinite.

Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m + 1\} \to \mathbb{N}$. i is injective.

Then, $f \circ i : \{1, 2, \dots, m + 1\} \to \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

Proposition

If A is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

$\vdots$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of A.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. f is injective as a_n are distinct.

Example

$$\operatorname{card}(\mathbb{Z}) = \operatorname{card}(\mathbb{N})$$

$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m + 1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

f is a bijection as $g : \mathbb{N} \to \mathbb{Z},\ g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of f.

Definition

Given any set X, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of X.

$2^X := \{f \mid f : X \to \{0, 1\}\}$.

Proposition

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbf{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$.

Cantor's theorem

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

Fact from calculus: $\forall \sigma \in (0,1)$, σ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \ldots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$, and $\sigma_j(n) \in \{0, 1, \ldots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0, 1, \ldots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\ldots$. Since r is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\ldots\sigma_m(m)\cdots = \tau = 0.\tau(1)\tau(2)\ldots\tau(m)\ldots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.