

Normed Vector Spaces

Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . A **vector space** over \mathbb{F} is a nonempty set V equipped with two operations: vector addition and scalar multiplication.

$$\begin{aligned} V \times V &\xrightarrow{+} V \\ (v, w) &\mapsto v + w && \text{Vector Addition} \\ F \times V &\rightarrow V \\ (\alpha, v) &\mapsto \alpha v && \text{Scalar Multiplication} \end{aligned}$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

- (i) $u + (v + w) = (u + v) + w$
- (ii) $\exists 0_v \in V$ with $\forall v \in V, 0_v + v = v + 0_v = v$
- (iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$
- (iv) $\forall v, w \in V, v + w = w + v$
- (v) $\alpha(v + w) = \alpha v + \alpha w, (\alpha + \beta)v = \alpha v + \beta v$
- (vi) $\alpha(\beta w) = (\alpha\beta)w$
- (vii) $1 \cdot v = v$

Remarks:

- (a) 0_v is unique and known as the zero vector.
- (b) The vector w in (iii) is unique, and denoted $-v$.
- (c) $0 \cdot v = 0_v$
- (d) $(-1) \cdot v = -v$
- (e) Property (iv) follows from all the other axioms.
- (f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \dots + v}_{n \text{ times}}$

Subspaces

Let V be a vector space over \mathbb{F} . A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

- (i) $w \in W, \alpha \in \mathbb{F} \rightarrow \alpha w \in W$.
- (ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

Remark: 0_v is always a member of any subspace; a subspace is also a vector space.

Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of V , then, $\bigcap W_i$ is a subspace of V .

Proposition: Union of Subspaces

It is not the case that the union of subspaces of V also a subspace. For example, consider \mathbb{R}^2 with the traditional vector space operations:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} x + x' \\ y + y' \end{pmatrix} \\ \alpha \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \end{aligned}$$

If $W_1, W_2 \subseteq V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Generated Subspaces

Let $S \subseteq V$ be any subset of a vector space V . Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^n \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

Remarks:

- $\text{span}(S) \subseteq V$ is a subspace.
- $\text{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\text{span}(S)$ is the “smallest” subspace containing S , or the subspace generated by S .

Proposition: Quotient Group on Vector Space

Let V be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

- (1) \sim_W is an equivalence relation.
- (2) If $[v]_W$ denotes the equivalence class of v , then $[v]_W = v + W = \{v + w \mid w \in W\}$.
- (3) $V/W := \{[v]_W \mid v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

Proof of (1):

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

Proof of (2):

$$\begin{aligned} [v]_W &= \{u \in V \mid u \sim_W v\} \\ &= \{u \in V \mid u - v \in W\} \\ &= \{u \in V \mid u = v + w \text{ some } w \in W\} \\ &= \{v + w \mid w \in W\} \\ &= v + W \end{aligned}$$

Proof of (3): Prove that the operation is well-defined.

Bases

Let V be a vector space and $S \subseteq V$ be a subset.

- (1) S is said to be spanning for V if $\text{span}(S) = V$.
- (2) S is linearly independent if, for $\sum_{j=1}^n \alpha_j v_j = 0_V$ with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.
- (3) S is a basis for V if S is linearly independent and spanning for V .

Proposition: Existence of Basis

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that B is a basis and $B \supseteq B_0$.

Background: A relation on a set X is a subset $R \subseteq X \times X$. If R is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \rightarrow x \sim z$), and antisymmetric ($x \sim y, y \sim x \rightarrow x = y$), then R is an ordering, and we write $x \leq y$.

If \leq is an ordering of X such that $\forall x, y \in X$, $x \leq y$ or $y \leq x$, then \leq is a total (or linear) ordering.

Let \leq be an ordering of X , let $Y \subseteq X$. An upper bound for Y is an element $u \in X$ such that $y \leq u \forall y \in Y$. A maximal element in X is an element $m \in X$ such that $x \in X$, $x \geq m \rightarrow x = m$.

Example: \mathbb{N} under the division ordering defines $a \leq b \Leftrightarrow a \mid b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of A can be divided by 9 and 12). Meanwhile, \mathbb{N} itself has no maximal elements.

This leads us to ask: given an ordered set, (X, \leq) , does X admit maximal elements.

Zorn's Lemma (or Axiom): Let (X, \leq) be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in X . Then, X admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

Proof: Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with D linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that X has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i \in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \dots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \dots, v_n \in D_j$ because Y is totally ordered. However, by definition, D_j is a linearly independent set — therefore, $\alpha_k = 0$. Thus, D is linearly independent.

Since D is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. D is also an upper bound for Y . So, by Zorn's Lemma, X has a maximal element, B .

So, $B_0 \subseteq B \subseteq V$, B is independent, and B is maximal in X . We claim that B is a basis for V . Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and B' is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \dots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.
- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \text{span}(B)$. \perp

Thus, we have a linearly independent set, B' , with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of B . Therefore, $\text{span}(B) = V$, and B is a basis for V .

Examples: Vector Spaces

(1) n -Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$B = \{e_1, \dots, e_n\}$$

where e_i denotes the unit vector at position i .

(2) $m \times n$ Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$\alpha(a_{ij}) = (\alpha a_{ij})$$

$$B = \{e_{ij}\}$$

where e_{ij} denotes a matrix of 0 everywhere except column i and row j .

(3) Functions with domain Ω :

$$\begin{aligned}\mathcal{F}(\Omega, \mathbb{F}) &= \{f \mid f : \Omega \rightarrow \mathbb{F}\} \\ (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

(4) Bounded functions with domain Ω :

$$\begin{aligned}\ell_\infty(\Omega, \mathbb{F}) &= \{f \in \mathcal{F}(\Omega, \mathbb{F}) \mid \|f\|_u \leq \infty\} \\ \|f\|_u &= \sup_{x \in \Omega} |f(x)|\end{aligned}$$

Exercises:

- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha| \|f\|_u$
- Positive Definite: $\|f\|_u = 0 \Rightarrow f = 0$

Proof of Triangle Inequality: Given $x \in \Omega$,

$$\begin{aligned}|(f+g)(x)| &= |f(x) + g(x)| \\ &\leq |f(x)| + |g(x)| \\ &\leq \|f\|_u + \|g\|_u\end{aligned}$$

Therefore,

$$\begin{aligned}\sup |(f+g)(x)| &\leq \|f\|_u + \|g\|_u \\ \|f+g\|_u &\leq \|f\|_u + \|g\|_u\end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a, b], \mathbb{F}) = \{f : [a, b] \rightarrow \mathbb{F} \mid f \text{ continuous}\}$$

Check that $C([a, b], \mathbb{F}) \subseteq \ell_\infty([a, b], \mathbb{F})$ is a subspace.

(6) Let $f : [a, b] \rightarrow \mathbb{R}$ be any function. Let $\mathcal{P} : a = x_0 < x_1 < x_2 < \dots < x_n = b$.

$$\begin{aligned}\text{var}(f; \mathcal{P}) &:= \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \\ \text{var}(f) &= \sup_{\mathcal{P}} \text{var}(f; \mathcal{P}) \\ \text{BV}([a, b]) &= \{f : [a, b] \rightarrow \mathbb{R} \mid \text{var}(f) < \infty\} \\ \|f\|_{\text{BV}} &= |f(a)| + \text{var}(f)\end{aligned}$$

$\text{BV}([a, b])$ is a vector space.

Question: Is $\mathbb{1}_{\mathbb{Q}} \in \text{BV}([0, 1])$?

(7) Suppose $K \subseteq V$ is a *convex* subset of a vector space: $v, w \in K, t \in [0, 1] \Rightarrow (1-t)v + tw \in K$. Let $\text{Aff}(K) = \{f : K \rightarrow \mathbb{R} \mid f \text{ is affine}\}$, where f is affine if $\forall v, w \in K, t \in [0, 1], f((1-t)v + tw) = (1-t)f(v) + tf(w)$.

Exercise: Show that $\text{Aff}(K) \subseteq \mathcal{F}(K, \mathbb{R})$ is a subspace.

(8) Let S be defined as

$$S = \{(a_k)_{k=1}^\infty \mid a_k \in \mathbb{F}\}.$$

Under pointwise operations, S is a vector space.

$$\begin{aligned}(a_k)_k + (b_k)_k &= (a_k + b_k)_k \\ \alpha(a_k)_k &= (\alpha a_k)_k\end{aligned}$$

Note 1: $S = \mathcal{F}(\mathbb{N}, \mathbb{F})$.

Note 2: $c_{00} \subseteq \ell_1 \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq S$.

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k \neq 0\}$
- $c_0 = \{(a_k)_k \mid (a_k)_k \rightarrow 0\}$

- $c = \{(a_k)_k \mid (a_k)_k \rightarrow a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_\infty < \infty\}$
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| = a < \infty\}$

(9) $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_c(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \rightarrow \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ continuous, } \lim_{x \rightarrow \pm\infty} f(x) = 0\}$

(10) Let S be any non-empty set.

$$\mathbb{F}(S) := \{f : S \rightarrow \mathbb{F} \mid f \text{ finitely supported}\}$$

$$\text{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \rightarrow \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t \in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\text{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^n f(t_k) e_{t_k} \in \text{span}(\xi)$. Therefore, ξ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^n \alpha_{t_k} e_{t_k} = 0$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\left(\sum_{k=1}^n \alpha_{t_k} e_{t_k} \right) = 0(t_1)$$

$$\alpha_{t_1} = 0.$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \dots, n$. Therefore, ξ is linearly independent. Since ξ is linearly independent and spanning, ξ forms a basis for $\mathbb{F}(S)$.

Note: The free vector space, $\mathbb{F}(S)$, displays the universal property.

There are functions $\iota : S \rightarrow \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \rightarrow V$ for V a vector space over \mathbb{F} , $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \rightarrow V$ such that $\iota \circ T_\varphi = \varphi$.

$$\begin{array}{ccc} S & \xrightarrow{\iota} & \mathbb{F}(S) \\ & \searrow \varphi & \downarrow T_\varphi \\ & & V \end{array}$$

Proof: Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^n f(t_k) e_{t_k}$, where $\text{supp}(f) = \{t_1, \dots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^n f(t_k) \varphi(t_k)$$

Exercise: Show T_φ is linear and unique.

Exercise 2: Suppose V is a vector space over \mathbb{F} with basis B . Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists T : V \rightarrow W$ such that T is bijective and linear.

Normed Spaces

To every vector $v \in V$, we want to assign a length to v , $\|v\|$.

A **norm** on a vector space V is a map

$$\|\cdot\| : V \rightarrow \mathbb{R}^+$$

$$v \mapsto \|v\| \geq 0$$

such that

- Homogeneity: $\|\alpha v\| = |\alpha| \|v\|$
- Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

(iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = 0_V$.

If $p : V \rightarrow \mathbb{R}^+$ satisfies (i) and (ii), then p is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\begin{aligned}\|v\| &\leq c_1 \|v\|' \\ \|v\|' &\leq c_2 \|v\|\end{aligned}$$

Note: On \mathbb{R}^n , all norms are equivalent.

Exercise: If p is any seminorm on V , then $|p(v) - p(w)| \leq p(v - w)$.

Notation: If V is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.

Examples of Normed Spaces

(1) Given $V = \mathbb{F}^n$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have different norms:

$$\begin{aligned}\|x\|_1 &= \sum_{j=1}^n |x_j| \\ \|x\|_\infty &= \max_{1 \leq j \leq n} |x_j| \\ \|x\|_2 &= \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.\end{aligned}$$

In general, for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p}.$$

Exercise: Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Show that $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$.

We want to show that $\|\cdot\|_p$ defines a norm for $1 \leq p < \infty$. If $1 \leq p < \infty$, its conjugate index $q \in [1, \infty]$ whereby $\frac{1}{p} + \frac{1}{q} = 1$. For example, if $p = 1$, then $q = \infty$, and if $p = \infty$, then $q = 1$.

Lemma 1: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $f : [0, \infty) \rightarrow \mathbb{R}$, $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then, $f(t) \geq 0$ for all $t \geq 0$.

Proof 1: We can see that $f'(t) = t^{p-1} - 1$. Then, $f'(t) = 0$ at $t = 1$; $f'(t) > 0$ for $t > 1$ and $f'(t) < 0$ for $t \in [0, 1)$.

So, since $f(t) \geq f(1)$ for all $t \geq 0$, and $f(1) = 0$, $f(t) \geq 0$ for all $t \geq 0$.

Lemma 2: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $z, y \geq 0$, $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

Proof 2: We know from Lemma 1, $t \leq \frac{1}{p}t^p + \frac{1}{q}$. Multiply by y^q to get

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Set $t = xy^{1-q}$. Then,

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq}y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $p - pq = -q$, so

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

With these two lemmas in mind, we get two important inequalities.

Hölder's Inequality: For $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. Then, for $x, y \in \mathbb{F}^n$,

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \|x\|_p \|y\|_q.$$

Proof of Hölder's Inequality: For $p = 1$, the solution is as follows:

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n |x_j| \|y\|_\infty \\ &= \|x\|_1 \|y\|_\infty, \end{aligned}$$

and similarly for $p = \infty, q = 1$.

For $1 < p < \infty$, assume $\|x\|_p = \|y\|_q = 1$.

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n \left(\frac{1}{p} |x_j|^p + \frac{1}{q} |y_j|^q \right) \\ &= \frac{1}{p} \left(\sum_{j=1}^n |x_j|^p \right) + \frac{1}{q} \left(\sum_{j=1}^n |y_j|^q \right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = 0_{\mathbb{F}}$ or $y = 0_{\mathbb{F}}$, the inequality still holds.

Assume $\|x\|_p \neq 0, \|y\|_q \neq 0$. Set

$$\begin{aligned} x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_q}. \end{aligned}$$

It can be verified that $\|x'\|_p = 1 = \|y'\|_q$. Therefore,

$$\begin{aligned} \left| \sum_{j=1}^n x'_j y'_j \right| &\leq 1 \\ \left| \sum_{j=1}^n \frac{x_j}{\|x\|_p} \frac{y_j}{\|y\|_q} \right| &\leq 1 \\ \left| \sum_{j=1}^n x_j y_j \right| &\leq \|x\|_p \|y\|_q \end{aligned}$$

Minkowski's Inequality: Given $x, y \in \mathbb{F}^n$, $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Proof of Minkowski's Inequality: We can verify for $p = 1, q = \infty$, and vice versa.

Assume $1 < p < \infty$. Then,

$$\begin{aligned}
 \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p \\
 &= \sum_{j=1}^{\infty} |x_j + y_j| |x_j + y_j|^{p-1} \\
 &\leq \sum_{j=1}^{\infty} |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\
 &\leq \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \left(\sum_{j=1}^n |x_j + y_j|^{p(q-1)} \right)^{1/q} + \left(\sum_{j=1}^n |y_j|^p \right)^{1/p} \left(\sum_{j=1}^n |x_j + y_j|^{p(q-1)} \right)^{1/q} \quad \text{Hölder's Inequality} \\
 &= \|x\|_p \|x + y\|_p^{p/q} + \|y\|_p \|x + y\|_p^{p/q} \\
 &= (\|x\|_p + \|y\|_p) \|x + y\|_p^{p-1}
 \end{aligned}$$

Divide by $\|x + y\|_p^{p-1}$ to get desired inequality.

- (2) $\ell_\infty(\Omega, \mathbb{F})$ with $\|\cdot\|_\infty$. This includes subspaces that inherit the norm, such as

$$\begin{aligned}
 C([a, b]) &\subseteq \ell_\infty(\Omega) \\
 \ell_\infty(\mathbb{R}) &\supseteq C_0(\mathbb{R}) \supseteq C_c(\mathbb{R})
 \end{aligned}$$

Exercise: Show that $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ is a subspace.

- (3) $\Omega = \mathbb{N}$, $\ell_\infty = \ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$. Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \subseteq \ell_\infty.$$

- (4) ℓ_1 with $\|\cdot\|_1$,

$$\|(a_k)_k\|_1 = \sum_{k=1}^n |a_k|.$$

- (5) $C([a, b])$ with

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

- (6) Let $1 \leq p < \infty$.

$$\ell_p = \left\{ (a_k)_{k=1}^\infty \mid \sum_{k=1}^\infty |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left(\sum_{k=1}^\infty |a_k|^p \right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\begin{aligned}
 \left(\sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} &= \left\| \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \right\|_{\ell_p^n} \\
 &= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
 &\leq \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\|_{\ell_p^n} + \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
 &\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p.
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ (by the definition of an infinite series), we find that $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$.

(7) $\mathcal{BV}([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid \text{Var}(f) < \infty\}$ with the norm $\|f\|_{\mathcal{BV}} = |f(a)| + \text{Var}(f)$ is a normed space:

$$\|f\|_{\mathcal{BV}} = 0$$

$$|f(a)| = 0$$

$$\text{Var}(f) = 0$$

given $t \in (a, b]$, look at the partition $a < t \leq b$. Then,

$$\text{Var}(f) \geq |f(t) - f(a)| + |f(b) - f(t)|$$

$$f(t) = 0$$

$$f = 0_f.$$

(8) $\mathcal{M}_{m,n}(\mathbb{F})$ with

$$\|a\|_{\text{op}} = \sup_{\|\xi\|_{\ell_2^m} \leq 1} \|a\xi\|_{\ell_2^n}$$

is a normed vector space. If $\|a\|_{\text{op}} = 0$, then

$$ae_j = 0$$

$$\forall j \in \{1, \dots, n\}.$$

take the dot product with $i \neq j$

$$\begin{aligned} ae_j \cdot e_i &= a_{ij} \\ &= 0 \end{aligned}$$

so $a_{ij} = 0$ for all a_{ij} , so a is the 0 matrix.

(9) Let V, W be vector spaces over \mathbb{F} . Then, $\mathcal{L}(V, W) = \{T \mid T : V \rightarrow W \text{ linear}\}$, where $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

$\mathcal{L}(V, W)$ is a vector space with operations

$$(T + S)(v) = T(v) + S(v)$$

$$(\alpha T)(v) = \alpha T(v).$$

Notation: $\mathcal{L}(V) := \mathcal{L}(V, V)$ is all linear operators on V . $\mathcal{L}(V, \mathbb{F}) = V'$ is all linear functionals.

Suppose V and W are normed vector spaces. If $T : V \rightarrow W$, set

$$\|T\|_{\text{op}} := \sup_{\|v\|_V \leq 1} \|T(v)\|_W,$$

$$\mathcal{B}(V, W) = \{T \in \mathcal{L}(V, W) \mid \|T\|_{\text{op}} \leq \infty\},$$

where $\mathcal{B}(V, W)$ is referred to as the set of all bounded linear maps from V to W . $\mathcal{B}(V, W)$ with $\|\cdot\|_{\text{op}}$ is a normed space.

- Homogeneity:

$$\begin{aligned} \|\alpha T\|_{[\text{op}]} &= \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W \\ &= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W \\ &= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W \\ &= |\alpha| \|T\|_{\text{op}}. \end{aligned}$$

- Triangle Inequality: for $\|v\|_V \leq 1$,

$$\begin{aligned} \|(T + S)(v)\|_W &= \|T(v) + S(v)\|_W \\ &\leq \|T(v)\|_W + \|S(v)\|_W \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

so

$$\begin{aligned} \|T + S\|_{\text{op}} &= \sup_{\|v\| \leq 1} \|T + S(v)\| \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

- Positive Definite: If $\|T\|_{\text{op}} = 0$, then $T(v) = 0$ for all $v \in V$, $\|v\| \leq 1$.

Let $v \in V$, $v \neq 0$. Then, $\frac{v}{\|v\|} \in B_V$.

$$\begin{aligned} T\left(\frac{v}{\|v\|}\right) &= 0 \\ \frac{1}{\|v\|} T(v) &= 0 \\ T(v) &= 0 \end{aligned}$$

Special Cases: $\mathbb{B}(V) = \mathbb{B}(V, V)$, $V^* = \mathbb{B}(V, \mathbb{F})$.

Exercise: $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$.

(10) Inner Product Spaces (expanded upon below).

Inner Product Spaces

An inner product on a vector space V is a pairing

$$V \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}$$

that satisfies

$$(i) \quad \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle, \quad \langle \alpha v, w \rangle = \alpha \langle v, w \rangle.$$

$$(ii) \quad \langle v, w \rangle = \overline{\langle w, v \rangle}$$

$$(iii) \quad \langle v, v \rangle \geq 0.$$

$$(iv) \quad \text{If } \langle v, v \rangle = 0, \text{ then } v = 0.$$

The pair $(V, \langle \cdot, \cdot \rangle)$ is known as an inner product space.

Remarks: $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$, $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$.

If $\langle \cdot, \cdot \rangle$ is an inner product on a linear space V , then set

$$\|v\|_2 := \langle v, v \rangle^{1/2}.$$

Exercise: $\|\alpha v\|_2 = |\alpha| \|v\|_2$, $\|v\|_2 = 0 \Rightarrow v = 0$.

$v, w \in (V, \langle \cdot, \cdot \rangle)$ are *orthogonal* if $\langle v, w \rangle = 0$.

The Pythagorean theorem states that for $v_1, \dots, v_n \in V$ mutually orthogonal, then

$$\left\| \sum_{i=1}^n v_i \right\|^2 = \sum_{j=1}^n \|v_j\|^2.$$

For two vectors $v, w \in V$, $P_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.

Exercise: Check that $\langle P_w(v), v - P_w(v) \rangle$, meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

Cauchy-Schwarz Inequality: In any inner product space,

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|.$$

Proof of Cauchy-Schwarz: From the exercise,

$$\begin{aligned} \|v\| &\geq \|P_w(v)\| \\ \|v\| &\geq \left\| \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\| \\ &= \frac{|\langle v, w \rangle|}{\|w\|^2} \|w\| \end{aligned}$$

therefore,

$$\|v\| \|w\| \geq |\langle v, w \rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

Proof of Triangle Inequality:

$$\begin{aligned}
 \|v + w\|_2^2 &= \langle v + w, v + w \rangle \\
 &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\
 &= \|v\|^2 + \|w\|^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} \\
 &= \|v\|^2 + \|w\|^2 + 2\operatorname{Re} \langle v, w \rangle \\
 &\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w \rangle| \\
 &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\
 &= (\|v\| + \|w\|)^2.
 \end{aligned}$$

Cauchy-Schwarz Inequality

Take square roots on both sides.

(1) $\ell_2^n = \mathbb{F}^n$ with

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left| \sum_{j=1}^n x_j \overline{y_j} \right| \leq \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |y_j|^2 \right)^{1/2}.$$

(2) ℓ_2 with

$$\langle (a_j)_j, (b_j)_j \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

We can see that for any finite n , the Cauchy-Schwarz inequality in ℓ_2^n states

$$\begin{aligned}
 \left| \sum_{j=1}^n a_j \overline{b_j} \right| &\leq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |b_j|^2 \right)^{1/2} \\
 &\leq \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |b_j|^2 \right)^{1/2}.
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we see that $\langle (a_j)_j, (b_j)_j \rangle$ is convergent.

(3) $C([a, b])$ with

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

(4) Let $V = \mathbb{M}_n(\mathbb{C})$.

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let $\operatorname{Tr} : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$, $\operatorname{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$.

- $\operatorname{Tr}(I_n) = n$
- $\operatorname{Tr}(a + \alpha b) = \operatorname{Tr}(a) + \alpha \operatorname{Tr}(b)$
- $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$

Then, if $\text{Tr}(a^*a) = 0$, then $a = 0_{\mathbb{M}_n}$.

$$\begin{aligned} a^*a &= (\overline{a_{ji}})_{i,j} (a_{ij})_{i,j} \\ &= \left(\sum_{k=1}^n \overline{a_{ki}} a_{kj} \right)_{i,j} \\ \text{Tr}(a^*a) &= \sum_{i=1}^n \sum_{k=1}^n \overline{a_{ki}} a_{ki} \\ &= \sum_{i,k=1}^n |a_{ki}|^2 \\ &= \sum_{i,j=1}^n |a_{ij}|^2. \end{aligned}$$

If $\text{Tr}(a^*a) = 0$, then $a_{ij} = 0$ for all i, j .

We define

$$\langle a, b \rangle_{\text{HS}} = \text{Tr}(b^*a).$$

$$(i) \quad (b_1 + b_2)^* = b_1^* + b_2^*$$

$$(ii) \quad (\alpha b)^* = \overline{\alpha} b^*$$

$$(iii) \quad (b_1 b_2)^* = b_2^* b_1^*$$

$$(iv) \quad b^{**} = b$$

The norm is defined as

$$\begin{aligned} \|a\|_{\text{HS}} &= \langle a, a \rangle_{\text{HS}}^{1/2} \\ &= \text{Tr}(a^*a)^{1/2} \\ &= \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2} \end{aligned}$$

Metric Spaces

We looked at normed spaces, where we attach a length $\|v\|$ to every vector v . We can also speak of the distance between two vectors, defined as $d(v, w) = \|v - w\|$.

Notice that the following hold:

- $d(v, w) \geq 0$
-

$$\begin{aligned} d(v, w) &= \|v - w\| \\ &= \|(-1)(w - v)\| \\ &= |-1| \|w - v\| \\ &= \|w - v\| \end{aligned}$$

-

$$\begin{aligned} d(u, w) &= \|u - w\| \\ &= \|u - v + v - w\| \\ &\leq \|u - v\| + \|v - w\| \\ &= d(u, v) + d(v, w). \end{aligned}$$

- $d(v, v) = \|v - v\| = 0$. If $d(v, w) = 0$, then $\|v - w\| = 0$, so $v - w = 0$, so $v = w$.

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely $(\mathbb{R}, |\cdot|)$. We will expand these concepts to all metric spaces.

Definition of a Metric Space

Let X be a non-empty set. A **metric** on X is a map

$$\begin{aligned} d : X \times X &\rightarrow \mathbb{R}^+ \\ (x, y) &\mapsto d(x, y) \geq 0 \end{aligned}$$

such that

- (i) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
- (ii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
- (iii) Zero Distance: $d(x, x) = 0$
- (iv) Definite: $d(x, y) = 0 \Rightarrow x = y$

If d satisfies (i), (ii), and (iii), then d is called a semi-metric. If d satisfies (iv) as well, then d is a metric.

If d is a (semi-)metric on X , the pair (X, d) is called a (semi-)metric space.

Two metrics, d and ρ , on X , are equivalent if $\exists c_1, c_2 \geq 0$ such that $d(x, y) \leq c_1 \rho(x, y)$ and $\rho(x, y) \leq c_2 d(x, y)$ for all x, y .

Examples of Metric Spaces

- (1) Discrete Metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for X any set.

- (2) Hamming distance: between two bit strings of equal length. Let

$$\begin{aligned} X &= \{0, 1\}^n \\ &= \{0, 1\} \underbrace{\times \cdots \times}_{n \text{ times}} \{0, 1\} \\ d_H((x_j)_1^n, (y_j)_1^n) &= |\{j \mid x_j \neq y_j\}|. \end{aligned}$$

- (3) Any normed space $(V, \|\cdot\|)$ is a metric space.

$$d(v, w) = \|v - w\|.$$

Exercise: Show that if two norms are equivalent, their induced metrics are equivalent.

- (4) Subset of Metric Space: If (X, d) is a metric space, and $Y \subseteq X$ is non-empty. Then, (Y, d) is a metric space.

- (5) Paris metric: let (X, ρ) be a metric space. Let $p \in X$ be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

- (6) Bounded metric: Let ρ be a (semi-)metric on X . Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We claim that d is a (semi-)metric. Notice that $0 \leq d(x, y) \leq 1$.

Proof: Clearly, $d(x, y) = d(y, x)$. Additionally, $d(x, x) = 0$. If $d(x, y) = 0$ and ρ is a metric, then $\rho(x, y) = 0$, so $x = y$.

To show the triangle inequality, we examine the function

$$\begin{aligned} f(t) &= \frac{t}{1+t} \\ f'(t) &= \frac{1}{(1+t)^2} > 0. \end{aligned}$$

Since ρ satisfies the triangle inequality, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Apply f on both sides. Then,

$$\begin{aligned} \underbrace{\frac{\rho(x, z)}{1 + \rho(x, z)}}_{d(x, z)} &\leq \frac{\rho(x, y) + \rho(y, z)}{1 + (\rho(x, y) + \rho(y, z))} \\ &= \frac{\rho(x, y)}{1 + \rho(x, y) + \rho(y, z)} + \frac{\rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\ &\leq \underbrace{\frac{\rho(x, y)}{1 + \rho(x, y)}}_{d(x, y)} + \underbrace{\frac{\rho(y, z)}{1 + \rho(y, z)}}_{d(y, z)}. \end{aligned}$$

(7) If d_1, \dots, d_n are metrics on X , $c_1, \dots, c_n \geq 0$. Then,

$$d(x, y) = \sum_{k=1}^n c_k d_k(x, y)$$

is a metric.

(8) Let $\{\rho_k\}_{k=1}^\infty$ be a family of semi-metrics. Assume the family is separating — for all $x \neq y$, there exists k such that $\rho_k(x, y) \neq 0$.

Let d_k be defined as

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Note that $\{d_k\}_{k=1}^\infty$ is also separating.

Then,

$$d(x, y) = \sum_{k=1}^\infty 2^{-k} d_k(x, y)$$

is a metric.

We will now define the Frechet Metric using this method. Let $X = C(\mathbb{R})$. For each $k = 1, 2, 3, \dots$, set $p_k(f) = \sup_{x \in [-k, k]} |f(x)|$.

We can verify that p_k defines a seminorm. We can then check $\rho_k(f, g) = p_k(f - g)$ is a semi-metric.

We claim that $\{\rho_k\}$ is separating: if $f \neq g$, then there exists $x_0 \in \mathbb{R}$ with $f(x_0) \neq g(x_0)$. Since f and g are continuous, there is a neighborhood $[x_0 - \delta, x_0 + \delta]$ such that $f(x) \neq g(x)$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Find k such that $[x_0 - \delta, x_0 + \delta] \subseteq [-k, k]$. Then, $\rho_k(f - g) > 0$.

Construct d_k as above, and then d as follows:

$$d_F = \sum \frac{2^{-k} p_k(f - g)}{1 + p_k(f - g)}$$

(9) Product of metric spaces: let $(X_k, \rho_k)_{k=1}^\infty$ be a countable family of metric spaces. For each k , let

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Remark: If the ρ_k are already uniformly bounded, let $d_k = \rho_k$.

Let

$$\begin{aligned} X &= \prod_{k=1}^\infty X_k \\ &= \{(x_k)_k \mid x_k \in X_k\} \\ &= \left\{ f : \mathbb{N} \rightarrow \bigsqcup_{k=1}^\infty X_k \mid f(k) \in X_k \right\}. \end{aligned}$$

Define $D : X \times X \rightarrow [0, \infty)$ as

$$D(x, y) = \sum_{k=1}^{\infty} 2^{-k} \rho_k(x_k, y_k),$$

$$D(f, g) = \sum_{k=1}^{\infty} 2^{-k} \rho(f(k), g(k)).$$

For example, for each k , let $X_k = \{0, 1\}$ with the discrete metric. Let

$$\begin{aligned} \Delta &= \prod_{k \in \mathbb{N}} \{0, 1\} \\ &= \{(x_k)_k \mid x_k \in \{0, 1\}\} \\ D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k| \end{aligned} \quad (x_k)_k, (y_k)_k \in \Delta.$$

Δ is known as the abstract Cantor set; every compact metric space is a surjective image of the abstract Cantor set.

(10) Geodesic Distance: let $\langle \cdot, \cdot \rangle$ be the standard dot product on $\mathbb{R}^3(\mathbb{R}^n)$, then

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$$

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}.$$

To find the geodesic distance, we take $d(x, y) = \arccos(\langle x, y \rangle)$. We claim d is a metric.

- Symmetry: self-evident.
- $d(x, x) = \arccos(1) = 0$. Suppose $d(x, y) = 0$. Then, $\langle x, y \rangle = 1$, meaning $\|x - y\|^2 = 0$, so $x = y$.
- Let $\theta = \arccos(\langle x, y \rangle)$, $\varphi = \arccos(\langle y, z \rangle)$, where $\theta, \varphi \in [0, \pi]$.

$$\begin{aligned} p_x &= \frac{\langle x, y \rangle}{\langle y, y \rangle} y \\ &= \cos(\theta) y \\ x &= \cos(\theta) y + \sin(\theta) u \end{aligned}$$

where

$$u = \frac{x - p_x}{\|x - p_x\|}.$$

Similarly, we can take

$$z = \cos(\varphi) y + \sin(\varphi) v$$

where

$$v = \frac{z - p_z}{\|z - p_z\|}.$$

So,

$$\begin{aligned} \langle x, z \rangle &= \cos(\theta) \cos(\varphi) + \sin(\theta) \sin(\varphi) \langle u, v \rangle \\ &\geq \cos(\theta) \cos(\varphi) - \sin(\theta) \sin(\varphi) \quad \langle u, v \rangle \geq -1 \\ &= \cos(\theta + \varphi). \end{aligned}$$

Since \arccos is decreasing,

$$\begin{aligned} \arccos(\langle x, z \rangle) &\leq \arccos(\cos(\theta + \varphi)) \\ &= \theta + \varphi \\ &= \arccos(\langle x, y \rangle) + \arccos(\langle y, z \rangle). \end{aligned}$$

Therefore, $d(x, y) \leq d(x, y) + d(y, z)$.

- Let $\Gamma = (V, E)$ be a simple connected graph. We define $d : V \times V \rightarrow [0, \infty)$ to be the length of the shortest path between vertices u and v .

Exercise: Show this is a metric.

(11) Let (X, d) be any metric space. If $E \subseteq X$, define $\text{diam}(E) = \sup_{x, y \in E} d(x, y)$. E is bounded if $\text{diam}(E) < \infty$.

Exercise: If $(V, \|\cdot\|)$ is a normed space, $E \subseteq V$ is a subset, show the following are equivalent:

- (i) E is bounded (in the metric sense)
- (ii) $\sup_{v \in E} \|v\| < \infty$
- (iii) $\exists r > 0$ such that $E \subseteq rB_V$.

Let Ω be any set. The function $f : \Omega \rightarrow X$ is bounded if $f(\Omega) \subseteq X$ is bounded. We let.

$$\text{Bd}(\Omega, X) = \{f : \Omega \rightarrow X \mid f \text{ is bounded}\}.$$

Remark: $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty(\Omega, \mathbb{F})$.

(12) $\text{Bd}(\Omega, X)$ with

$$D_u(f, g) = \sup_{x \in \Omega} d(f(x), g(x)).$$

Exercise: Show that D_u defines a metric.

Consider $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty$. Look at the subset

$$E = \{f \in \text{Bd}(\Omega, \mathbb{F}) \mid f(x) \in \{0, 1\}\}.$$

Then,

$$\begin{aligned} D_u(f, g) &= \sup_{x \in \Omega} |f(x) - g(x)|. \\ &= \begin{cases} 1 & f \neq g \\ 0 & f = g \end{cases}. \end{aligned}$$

When we take a particular subset of $D_u(f, g)$, we find that we get the discrete metric.

Taking an overview of the concepts we have learned so far, we see

$$\text{Inner Product Spaces} \subseteq \text{Normed Vector Spaces} \subseteq \text{Metric Spaces}$$

Topology of Metric Spaces

Throughout this section, let (X, d) be a metric space.

(1) Let $x_0 \in X$, $\delta > 0$.

(i) We say

$$U(x_0, \delta) = \{x \in X \mid d(x, x_0) < \delta\}$$

is the open ball centered at x_0 with radius δ .

(ii) We say

$$B(x_0, \delta) = \{x \in X \mid d(x, x_0) \leq \delta\}$$

is the closed ball.

(iii) We say

$$S(x_0, \delta) = \{x \in X \mid d(x, x_0) = \delta\}$$

is the sphere.

(2) $U \subseteq X$ is open if

$$(\forall x \in U)(\exists \delta > 0) \text{ such that } U(x, \delta) \subseteq U.$$

Let

$$\begin{aligned} \tau_X &= \{U \subseteq X \mid U \text{ open}\} \\ &\subseteq \mathcal{P}(X). \end{aligned}$$

(3) $D \subseteq X$ is closed if D^c is open.

- (4) If $x \in U \in \tau_X$, then U is called an open neighborhood of x . If $x \in U \subseteq N$, where $U \in \tau_X$, then N is a neighborhood of x .

$$\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$$

- (5) Let $A \subseteq X$. The interior of A is

$$A^\circ = \bigcup \{V \mid V \subseteq A, V \text{ open}\}.$$

The closure of A is

$$\bar{A} = \bigcap \{D \mid A \subseteq D, D \text{ closed}\}.$$

The boundary of A is

$$\partial A = \bar{A} \setminus A^\circ.$$

Exercise: $\overline{A^c} = (A^\circ)^c$, $(\bar{A})^c = (A^c)^\circ$.

Remarks: A° is the largest open set contained in A . So, if V is open and $V \subseteq A$, then $V \subseteq A^\circ$. Similarly, \bar{D} is the smallest closed set containing D . If C is closed and $D \subseteq C$, then $\bar{D} \subseteq C$.

- For example, $(a, b]^\circ = (a, b)$. This is because (a, b) is open and contained in $(a, b]$, so $(a, b) \subseteq (a, b]^\circ$.
- We will show that $\overline{A^c} \subseteq (A^\circ)^c$.

$$\begin{aligned} A^\circ &\subseteq A \\ (A^\circ)^c &\supseteq A^c \end{aligned}$$

The union of open sets is open, so A° is open, so $(A^\circ)^c$ is closed by definition. Therefore,

$$(A^\circ)^c \supseteq \overline{A^c}.$$

Topology of Open Sets in a Metric Space

The open sets τ_X form a topology:

- (i) $\emptyset, X \in \tau_X$.
- (ii) If $\{V_i\}_{i \in I} \subseteq \tau_X$, then

$$\bigcup_{i \in I} V_i \in \tau_X.$$

- (iii) If $V_1, \dots, V_n \in \tau_X$, then

$$\bigcap_{i=1}^n V_i \in \tau_X.$$

Remark: This is only true of finite intersections. For a counterexample, if $V_n = (-1/n, 1/n) \subseteq \mathbb{R}$ with the Euclidean metric, then the infinite intersection yields $\{0\}$, which is closed in \mathbb{R} with the Euclidean metric.

Proof:

- (1) Clearly, \emptyset (by vacuous truth) and X are open.
- (2) Let $x \in \bigcup_{i \in I} V_i$. Then, $\exists i_0 \in I$ with $x \in V_{i_0}$. Since V_{i_0} is open, $\exists \epsilon > 0$ such that $U(x, \epsilon) \subseteq V_{i_0} \subseteq \bigcup V_i$.
- (3) Let $x \in \bigcap_{i=1}^n V_i$. Then, $x \in V_i$ for all $i \in 1, \dots, n$. Since each V_i is open, $\exists \epsilon_1, \dots, \epsilon_n$ with $U(x, \epsilon_i) \subseteq V_i$ for each $i = 1, \dots, n$. Set $\epsilon = \min\{\epsilon_i\}_{i=1}^n$. Then, $U(x, \epsilon) \subseteq U(x, \epsilon_i) \subseteq V_i$ for all i . Therefore, $U(x, \epsilon) \subseteq \bigcap_{i=1}^n V_i$.

Exercise: Show all open balls are open. In particular, show all open intervals are open.

Exercise: Show the following:

- (1) X, \emptyset are closed.
- (2) If $\{C_i\}_{i \in I}$ is a family of closed sets, then $\bigcap_{i \in I} C_i$ is closed.
- (3) For C_1, \dots, C_n closed, then $\bigcup_{i=1}^n C_i$ is closed.
- (4) Closed balls are closed. Spheres are closed.

Let $x \in X$. Recall that \mathcal{N}_x is the set of all neighborhoods of x .

- (i) $N \in \mathcal{N}_x \Leftrightarrow \exists \delta > 0 : U(x, \delta) \in N$
- (ii) $N \in \mathcal{N}_x, N \subseteq M \Rightarrow M \in \mathcal{N}_x$
- (iii) $N_1, N_2 \in \mathcal{N}_x \Rightarrow N_1 \cap N_2 \in \mathcal{N}_x$

In this sense, \mathcal{N}_x is a directed set with reverse inclusion.

Pointwise Characterization of Subsets

Let $A \subseteq X$.

$$(i) \quad x \in A^\circ \Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A.$$

$$(ii) \quad x \in \bar{A} \Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset.$$

$$(iii) \quad x \in \partial A \Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset \text{ and } U(x, \delta) \cap A^c \neq \emptyset.$$

Proof: Let $A \subseteq X$

(i)

$$\begin{aligned} x \in A^\circ &\Leftrightarrow x \in \bigcup_{\substack{V \in \tau_X \\ V \subseteq A}} V \\ &\Leftrightarrow \exists V \in \tau_X, V \subseteq A, x \in V \\ &\Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A. \end{aligned}$$

(ii)

$$\begin{aligned} x \notin \bar{A} &\Leftrightarrow x \in (\bar{A})^c \\ &\Leftrightarrow x \in (A^c)^\circ \\ &\Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A^c \\ &\Leftrightarrow \exists \delta > 0 : U(x, \delta) \cap A = \emptyset. \end{aligned}$$

We negate both sides.

(iii)

$$\begin{aligned} x \in \partial A &\Leftrightarrow x \in \bar{A} \setminus A^\circ \\ &\Leftrightarrow x \in \bar{A} \cap (A^\circ)^c \\ &\Leftrightarrow x \in \bar{A} \cap \bar{A}^c \\ &\Leftrightarrow x \in \bar{A} \text{ and } x \in \bar{A}^c \\ &\Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset, U(x, \delta) \cap A^c \neq \emptyset \end{aligned}$$

Remark: $\overline{U(v, \delta)} = B(v, \delta)$ in a normed space. $\partial U(v, \delta) = \partial B(v, \delta) = S(v, \delta)$ in a normed space. Also, $B(v, \delta)^\circ = U(v, \delta)$.

Proof: We show that $\overline{U(v, \delta)} = B(v, \delta)$. Since $B(v, \delta)$ is closed, and $U(v, \delta) \subseteq B(v, \delta)$, we know $\overline{U(v, \delta)} \subseteq B(v, \delta)$.

Let $w \in B(v, \delta)$. If $\|w - v\| < \delta$, then $w \in U(v, \delta)$. Assume $\|w - v\| = \delta$. Let $u_t = (1 - t)v + tw$, where $t \in [0, 1]$.

$$\begin{aligned} \|w - u_t\| &= \|w - (1 - t)v - tw\| \\ &= \|(1 - t)(w - v)\| \\ &= (1 - t) \|w - v\| \\ &= (1 - t)\delta. \end{aligned}$$

Let $\epsilon > 0$. Let $t \in (0, 1)$ such that $(1 - t)\delta < \epsilon$. Then, $u_t \in U(w, \epsilon) \cap U(v, \delta)$. Therefore, $w \in \overline{U(v, \delta)}$.

Unions and Intersections of Closure/Interior

Let (X, d) be a metric space.

(i)

$$\left(\bigcup_{i \in I} A_i \right)^\circ \supseteq \bigcup_{i \in I} A_i^\circ$$

may be strict

(ii)

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \bar{A}_i$$

(iii)

$$\bigcap_{k=1}^n A_k^\circ = \left(\bigcap_{k=1}^n A_k \right)^\circ$$

(iv)

$$\overline{\bigcup_{k=1}^n D_k} = \bigcup_{k=1}^n \overline{D_k}$$

Proof:

(i)

$$\begin{aligned} A_i^\circ &\subseteq A_i \\ \bigcup_{i \in I} A_i^\circ &\subseteq \bigcup_{i \in I} A_i \\ \bigcup_{i \in I} A_i^\circ &\subseteq \left(\bigcup_{i \in I} A_i \right)^\circ \end{aligned}$$

Remark: We claim $\overline{\mathbb{Q}} = \mathbb{R}$ under the absolute value metric. We know that $\mathbb{Q} \subseteq \mathbb{R}$, \mathbb{R} is closed, meaning $\overline{\mathbb{Q}} \subseteq \mathbb{R}$. Let $t \in \mathbb{R}$, $\delta > 0$. We know that $(t - \delta, t + \delta) \cap \mathbb{Q} \neq \emptyset$. Therefore, $t \in \overline{\mathbb{Q}}$. Thus, $\overline{\mathbb{Q}} = \mathbb{R}$.

Properties of Boundary

Let $A \subseteq X$.(1) ∂A is closed.(2) $\partial A = \partial A^c$ (3) $\overline{A} = A \cup \partial A$ (4) $A \setminus \partial A = A^\circ$ **Proof:**

(1)

$$\begin{aligned} \partial A &= \overline{A} \setminus A^\circ \\ &= \overline{A} \cap (A^\circ)^c. \end{aligned}$$

(2) Follows from pointwise characterization.

(3) Clearly, $A \cup \partial A \subseteq \overline{A}$. Let $x \in \overline{A}$. If $x \in A$, we're done. Otherwise, $x \in \overline{A} \setminus A \subseteq \overline{A} \setminus A^\circ = \partial A$.

(4)

$$\begin{aligned} A \setminus \partial A &= A \cap (\partial A)^c \\ &= A \cap (\overline{A} \setminus A^\circ)^c \\ &= A \cap (\overline{A} \cap (A^\circ)^c)^c \\ &= A \cap (\overline{A}^c \cup A^\circ) \\ &= (A \cap \overline{A}^c) \cup (A \cap A^\circ) \\ &= A^\circ \end{aligned}$$

Density and Separability

Let (X, d) be a metric space.

- (1) $A \subseteq X$ is d -dense if $\overline{A} = X$.
- (2) $N \subseteq X$ is nowhere dense if $(\overline{N})^\circ = \emptyset$.
- (3) (X, d) is separable if there is a countable dense subset.

Exercise: If $N \subseteq X$ is closed, then N is nowhere dense if and only if N^c is dense.

Exercise: The following are equivalent.

- (1) $A \subseteq X$ is dense.
- (2) $\forall \emptyset \neq U \in \tau_X, U \cap A \neq \emptyset$.
- (3) $\forall x \in X, \forall \varepsilon > 0, U(x, \varepsilon) \cap A \neq \emptyset$.
- (4) $\forall x \in X, \forall \varepsilon > 0, \exists a \in A$ such that $d(x, a) < \varepsilon$.

Let X be a metric space.

- (1) A base for τ_X is a family of open subsets \mathcal{B} such that:

$$(\forall U \in \tau_X) (\forall x \in U) \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U.$$

Equivalently,

$$\forall U \in \tau_X, U = \bigcup_{i \in I} B_i. \quad B_i \in \mathcal{B}$$

- (2) We say that (X, d) is second countable if τ_X admits a countable base.

- For any (X, d) a metric space, $\mathcal{B} = \{U(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a base. Indeed, given any $x \in U \subseteq \tau_X$, by definition, $\exists \varepsilon > 0$ such that $U(x, \varepsilon) \subseteq U$. Alternatively, $\mathcal{B}' = \{U(x, 1/n) \mid x \in X, n \geq 1\}$ is a topological base.
- Let $X = \mathbb{R}^d$ with the Euclidean metric. Then, for $\mathcal{B} = \{U(q, 1/n) \mid n \geq 1, q \in \mathbb{Q}^d\}$, we claim this is a base.

Let $V \subseteq \mathbb{R}^d$ be open, $r \in V$. Since V is open, $\exists \delta > 0$ with $U(r, \delta) \subseteq V$. Find n large such that $1/n < \delta$. Find $q \in \mathbb{Q}^d$ with $\|r - q\| < 1/2n$. This is always possible as \mathbb{Q}^d is dense in \mathbb{R}^d .

Consider $U(q, 1/2n)$. Then, $r \in U(q, 1/2n) \subseteq U(r, \delta) \subseteq V$ because $\|r - q\| < 1/2n$, and if $t \in U(q, 1/2n)$, then

$$\begin{aligned} \|t - r\| &\leq \|t - q\| + \|q - r\| \\ &< 1/2n + 1/2n \\ &= 1/n \\ &< \delta. \end{aligned}$$

Separable, Non-Separable, Dense, and Non-Dense Sets

- (1) $(\mathbb{R}^d, \|\cdot\|_p)$ is separable for any $p \in [1, \infty]$. Indeed, $\mathbb{Q}^d \subseteq \mathbb{R}^d$ is the countable dense subset of \mathbb{R}^d .

Let $r = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \in \mathbb{R}^d$. Find $q = \begin{bmatrix} q_1 \\ \vdots \\ q_d \end{bmatrix} \in \mathbb{Q}^d$ with $|r_j - q_j| < \varepsilon/d$. Then,

$$\begin{aligned} \|r - q\|_1 &= \sum_{j=1}^d |r_j - q_j| \\ &< d. \end{aligned}$$

We know that for any vector $r \in \mathbb{R}^d$, we can find a vector q such that

$$\|q - r\|_p \leq c \|q - r\|_1,$$

so for arbitrary p , find q such that $\|q - r\|_1 < \varepsilon/c$.

- (2) Similarly, $\mathbb{C}_{\mathbb{Q}} = \{a + bi \mid a, b \in \mathbb{Q}\}$ is also countable, meaning $\mathbb{C}_{\mathbb{Q}}^d \subseteq \mathbb{C}^d$ is dense and \mathbb{C}^d is dense.

Proposition: Separable Subsets

If (X, d) is separable, and $Y \subseteq X$, then (Y, d) is also separable.

Let $\{a_k\}$ be a countable dense subset in X . Let $N = \{(m, n) \mid U(a_m, 1/n) \cap Y \neq \emptyset\}$. Clearly, N is nonempty. For each $(m, n) \in N$, choose $b_{(m,n)} \in Y \cap U(a_m, 1/n)$. We claim $\{b_{(m,n)} \mid m, n \geq 1\}$ is dense in Y .

Let $y \in Y$, $\varepsilon > 0$. Find N large so that $\frac{1}{n} < \varepsilon/2$. Since $A \subseteq X$ is dense, find $U(y, 1/n) \cap A \neq \emptyset$. Suppose $d(a_m, y) < 1/n$. Then,

$$\begin{aligned} d(b_{(m,n)}, y) &\leq d(b_{(m,n)}, a_m) + d(a_m, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n} \\ &< \varepsilon. \end{aligned}$$

(1) ℓ_p^n is separable.

(2) $c_{00} = \{(a_k)_{k=1}^n \mid \text{finitely many } a_k \neq 0\}$ with $\|\cdot\|_u$ is separable.

Recall that $e_k = (0, 0, \dots, 1, 0, 0, \dots)$ where 1 is at position k . Consider $E = \mathbb{Q}\text{-span}\{e_k \mid k \geq 1\}$,

$$E = \left\{ \sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q}, n \geq 1 \right\}.$$

The set E is countable. If we fix $n \geq 1$, we have

$$E_n = \left\{ \sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\}.$$

Then, $E = \bigcup E_n$. Note

$$\begin{aligned} \underbrace{\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}}_n &\rightarrow E_n \\ (\alpha_1, \dots, \alpha_n) &\mapsto \sum_{k=1}^n \alpha_k e_k. \end{aligned}$$

Thus, E_n is countable, and E is a countable union of countable sets.

We claim that E is dense. Given $z \in c_{00}$, $\varepsilon > 0$, we know that $z = \sum_{k=1}^n a_k e_k$ for some n and $a_k \in \mathbb{R}$. Find $\alpha_k \in \mathbb{Q}$ such that $|\alpha_k - a_k| < \varepsilon$. Set $w = \sum_{k=1}^n \alpha_k e_k$. Then, $\|z - w\|_u = \sup |\alpha_k - a_k| < \varepsilon$.

(3) c_0 with $\|\cdot\|_u$ is separable.

(4) ℓ_∞ is not separable.

Suppose ℓ_∞ were separable. Consider $E = \{(a_k)_k \in \ell_\infty \mid a_k \in \{0, 1\}\}$. Then, E is separable. Recall that $(E, \|\cdot\|_u)$ has the discrete metric.

In the discrete metric, every subset is open, meaning every subset is closed. Therefore, if X is separable and discrete, then X is countable.

However, E is not countable by Cantor's theorem. $\text{card}(E) = 2^{\aleph_0}$.

Alternatively, we can show that

$$(a_k)_k \mapsto \sum_{k=1}^{\infty} 2^{-k} a_k$$

is onto.

Exercise: ℓ_p is separable for $1 \leq p < \infty$.

(5) We will show that

$$\mathbb{P}[0, 1] \left\{ \sum_{k=1}^n a_k x^k \mid a_k \in \mathbb{R}, n \geq 1 \right\}$$

is $\|\cdot\|_u$ -dense in $C([0, 1])$. Using this, we can show that $(C([0, 1]), \|\cdot\|_u)$ is separable.

The Cantor Set

$$C_0 = [0, 1]$$

$$C_1 = [0, 1/3] \cup [2/3, 1]$$

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$

$$C_3 = [0, 1/27] \cup [2/27, 1/9] \cup \dots \cup [26/27, 1]$$

\vdots

In each step, we delete the middle third of each interval. This process repeated ad infinitum yields the Cantor set.

$$C = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right).$$

(i) C is closed as it is the intersection of closed sets.

(ii) $\text{length}(C) = 0$. Look at the total length of the removed intervals,

$$\begin{aligned} I &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots \\ &= \sum_{k=1}^{\infty} \left(\frac{2^{k-1}}{3^k} \right) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^k \\ &= 1. \end{aligned}$$

Thus, $\text{length}(C) = 0$.

(iii) C is nowhere dense — $(\bar{C})^\circ = \emptyset$. Since C is closed, $C^\circ = \emptyset$.

Suppose $C^\circ \neq \emptyset$. Then, $\exists x \in C, \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq C$. So, $(x - \epsilon, x + \epsilon) \subseteq C_n$ for all n .

Note C_n is the disjoint union of 2^n subintervals, each with length $1/3^n$. Find m so large such that $3^{-m} < \epsilon$. We know that $(x - \epsilon, x + \epsilon) \subseteq C_m$.

However, $(x - \epsilon, x + \epsilon)$ has length $2\epsilon > \frac{2}{3^m}$. Each subinterval in C_m has length $1/3^m$. This implies C_m contains an interval of length greater than $\frac{2}{3^m}$. \perp

(iv) $\text{card}(C) = \text{card}(\mathbb{R})$

Claim 1: Given $n \geq 1$,

$$E_n = \left\{ \sum_{k=1}^n \frac{w_k}{3^k} \mid w_k \in \{0, 2\} \right\}$$

is precisely the set of *left* endpoints of the subintervals of C_n .

For $n = 1$, if $w_1 = 0$, then we get 0, and $w_1 = 2$ yields $2/3$. Meanwhile, if $n = 2$, then we have

$$w_1 = 0, w_2 = 0 \mapsto 0$$

$$w_1 = 0, w_2 = 2 \mapsto 2/9$$

$$w_1 = 2, w_2 = 0 \mapsto 2/3$$

$$w_1 = 2, w_2 = 2 \mapsto 8/9.$$

By induction, we have shown for $n = 1, 2$. Assume this is true for n .

$$\sum_{k=1}^{n+1} w_k 3^{-k} = \underbrace{\sum_{k=1}^n w_k 3^{-k}}_{(1)} + \underbrace{w_{n+1} 3^{-(n+1)}}_{(2)}$$

Part (1) denotes one of the left endpoints of C_n , called $C_{n,k}$ for some $1 \leq k \leq 2^n$. Then, if $w_{n+1} = 0$, we get the left endpoint of $C_{n+1,2k-1}$, and if $w_n = 2$, we get the left endpoint of $C_{n+1,2k}$.

Claim 2:

$$C = \left\{ \sum_{k=1}^{\infty} w_k 3^{-k} \mid w_k \in \{0, 2\} \right\}$$

is precisely the Cantor set.

Let $x = \sum_{k=1}^{\infty} w_k 3^{-k}$. We will show that $x \in C_n$ for all n . Fix $n \geq 1$. Then,

$$x = \underbrace{\sum_{k=1}^n w_k 3^{-k}}_y + \underbrace{\sum_{k>n} w_k 3^{-k}}_z.$$

From our previous claim, y is the left endpoint of some subinterval of C_n . Additionally,

$$\begin{aligned} z &= \sum_{k>n} w_k 3^{-k} \\ &\leq 2 \sum_{k>n} 3^{-k} \\ &= \frac{2}{3^{n+1}} \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) \\ &= \frac{1}{3^n}. \end{aligned}$$

Since the length of a subinterval in C_n is exactly 3^{-n} , it is the case that $x = y + z$ remains an element of $C_{n,k}$.

Let $x \in C$. Then, $x \in C_n$ for all n . Then, $x \in C_1$, so let x_1 be the left endpoint of the interval $C_{1,j}$ that contains x . Then, $|x - x_1| < \frac{1}{3}$, and $x_1 = w_1 3^{-1}$ for some $w_1 \in \{0, 2\}$.

Let x_2 be the left endpoint of the subinterval $C_{2,j}$ that contains x . Then, $|x - x_2| < \frac{1}{3^2}$. Therefore,

$$\begin{aligned} x_2 &= x_1 + w_2 3^{-2} \\ &= w_1 3^{-1} + w_2 3^{-2}. \end{aligned}$$

Iterating, we have x_n , the left endpoint of the subinterval $C_{n,j}$ that contains x .

$$x_n = \sum_{k=1}^n w_k 3^{-k}.$$

We have that $|x - x_n| < 3^{-n}$.

Therefore, $(x_n)_n \rightarrow x$. Also,

$$\begin{aligned} x_n &= \sum_{k=1}^n w_k 3^{-k} \\ &\rightarrow \sum_{k=1}^{\infty} w_k 3^{-k}. \end{aligned}$$

Thus,

$$x = \sum_{k=1}^{\infty} w_k 3^{-k}.$$

To prove $\text{card}(C) = \text{card}(\mathbb{R})$, we will show that $\text{card}(\{0, 1\}^{\mathbb{N}}) = \text{card}(C)$.

$$(a_k)_k \mapsto \sum_{k=1}^{\infty} 2a_k 3^{-k}.$$

Relative (or Subspace) Topology

We know that if (X, d) is a metric space, and $Y \subseteq X$ is any subset, then (Y, d) is a metric space. The question now is: what are the open sets of Y ?

For example, let $X = \mathbb{R}$, $Y = [0, 1]$. Consider $U = [0, 1/2]$. U is not open in \mathbb{R} , as if $x = 0$, then there is no open ball completely contained in U . However, in Y , U is open.

Let (X, d) be a metric space, $Y \subseteq X$ any subset. $V \subseteq Y$ is open if and only if $\exists U \subseteq X$ open such that $V = U \cap Y$. That is, $\tau_Y = \{U \cap Y \mid U \in \tau_X\}$.

Let V be open in Y . Then, $\forall x \in V$, $\exists \delta_x > 0$ such that $U_Y(x, \delta_x) \subseteq V$. We have $U_Y(x, \delta_x) = \{y \in Y \mid d(y, x) < \delta_x\}$. Let

$$\begin{aligned} U &= \bigcup_{x \in V} U_X(x, \delta_x) \\ U \cap Y &= \left(\bigcup_{x \in V} U_X(x, \delta_x) \right) \cap Y \\ &= \bigcup_{x \in V} U_X(x, \delta_x) \cap Y \\ &= \bigcup_{x \in V} U_Y(x, \delta_x). \end{aligned}$$

Let U be open in X . Then, for $x \in U \cap Y$, $\exists \delta_x$ such that $U(x, \delta_x) \subseteq U$.

- (1) ℓ_∞ is not a discrete metric space. However, $E = \{(a_k)_k \mid a_k \in \{0, 1\}\}$ with the induced metric. Then, E is a discrete metric space.

Convergent Sequences

Fix a metric space (X, d) . A sequence in X is a map $x : \mathbb{N} \rightarrow X$, $n \mapsto x(n) = x_n$.

A natural sequence $(n_k)_k$ is a sequence in \mathbb{N} with $n_k \geq k$ for all k . A subsequence of $(x_n)_n$ is a sequence $(x_{n_k})_k$, where $(n_k)_k$ is a natural sequence.

A sequence $(x_n)_n$ converges to $x \in X$ if $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that $n \geq N_\epsilon$ implies $d(x_n, x) < \epsilon$. We write $(x_n)_n \xrightarrow{d} x$.

Exercise: A sequence can have at most one limit, as metric spaces are Hausdorff.

Proposition: Equivalent Definitions of Convergence

Given $(x_n)_n \in X$, $x \in X$, the following are equivalent.

- (i) $(x_n)_n \rightarrow x$ in X
- (ii) $(d(x_n, x))_n \rightarrow 0$ in \mathbb{R}
- (iii) $\forall V \in \mathcal{N}_x$, $\exists N \in \mathbb{N}$ with $n \geq N \Rightarrow x_n \in V$.

Exercise: Let (X, ρ) be a metric space, let $d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$. A sequence $(x_n)_n \xrightarrow{d} x$ if and only if $(x_n)_n \xrightarrow{\rho} x$.

Proposition: Convergent Sequences are Bounded

Let $(x_n)_n \rightarrow x$ in (X, d) . Let $\epsilon = 1$. Then, $\exists N \in \mathbb{N}$ large such that for $n \geq N$, $d(x_n, x) < 1$.

If $m, n \geq N$, then $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < 2$. Let $c = \max_{1 \leq n, m \leq N} d(x_n, x_m)$. Then,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\ &\leq 1 + c. \end{aligned}$$

Let $k = \max\{1 + c, 2\}$. Then, $\text{diam}(\{x_n\}) \leq k$.

Convergence in Different Metric Spaces

Convergence for Bounded Functions: Recall that for (Y, d) a metric space is

$$\begin{aligned} \text{Bd}(\Omega, Y) &= \{f : \Omega \rightarrow Y \mid f \text{ bounded}\} \\ D_u(f, g) &= \sup_{x \in \Omega} d(f(x), g(x)). \end{aligned}$$

Then, $(f_n)_n \rightarrow f$ in $\text{Bd}(\Omega, Y)$ if and only if $D_u(f_n, f) \rightarrow 0$ in \mathbb{R} .

$$\begin{aligned} (\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N &\Rightarrow D_u(f_n, f) < \varepsilon \\ \Leftrightarrow \\ (\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N &\Rightarrow \sup_{x \in \Omega} d(f_n(x), f(x)) < \varepsilon \\ \Leftrightarrow \\ (\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N &\Rightarrow \forall x, d(f_n(x), f(x)) < \varepsilon. \end{aligned}$$

This is exactly the definition of uniform convergence.

Since $\ell_\infty(\Omega) = \text{Bd}(\Omega, \mathbb{F})$, convergence in $\ell_\infty(\Omega)$ is uniform convergence. This is also the case for subspaces, such as c , c_0 , and c_{00} .

Convergence in the Frechet Metric: Consider a separating family of semimetrics ρ_k on a set X . Set $d_k = \frac{\rho_k}{1+\rho_k}$. We saw that

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric on X .

We claim that $(x_n)_n \rightarrow x$ in (X, d) if and only if for all $k \geq 1$, $\rho_k(x_n, x) \rightarrow 0$.

In the forward direction, we know that $(x_n)_n \rightarrow x$ with respect to d if and only if $d(x_n, x) \rightarrow 0$ in \mathbb{R} . Since $0 \leq 2^{-k} d_k(x_n, x) \leq d(x_n, x)$ for fixed k , we have that

$$0 \leq d_k(x_n, x) \leq 2^k d(x_n, x),$$

and as $n \rightarrow \infty$, $d(x_n, x) \rightarrow 0$, meaning $d_k(x_n, x) \rightarrow 0$. Therefore, $\rho_k(x_n, x) \rightarrow 0$.

In the reverse direction, suppose $\rho_k(x_n, x) \rightarrow 0$ in \mathbb{R} as $n \rightarrow \infty$ for all $k \geq 1$. Thus, $d_k(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $k \geq 1$.

Let $\varepsilon > 0$. Let K be so large such that

$$\sum_{k \geq K} 2^{-k} < \varepsilon/2.$$

Therefore, $d_k(x_n, x) \rightarrow 0$ for all $k = 1, \dots, K$. Therefore, $\exists N_1, \dots, N_K$ such that for $n \geq N_k$,

$$d_k(x_n, x) < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, \dots, N_K\}$. Therefore, for $n \geq N$,

$$d_k(x_n, x) < \frac{\varepsilon}{2}$$

for all $k = 1, \dots, K$.

Thus, for all $n \geq N$,

$$\begin{aligned} d(x_n, x) &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_n, x) \\ &= \sum_{k=1}^K 2^{-k} d_k(x_n, x) + \sum_{k=K+1}^{\infty} 2^{-k} d_k(x_n, x) \\ &\leq \frac{\varepsilon}{2} \sum_{k=1}^K 2^{-k} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Therefore, $(x_n)_n \rightarrow x$.

Recall that, for the Frechet metric, our set was $X = C(\mathbb{R})$. For $k = 1, 2, 3, \dots$, we had

$$\rho_k(f) = \sup_{[-k, k]} |f(x)|$$

as our seminorm, and our semimetric was

$$\rho_k(f, g) = \rho_k(f - g).$$

We also showed that the ρ_k family is separating. We make $d_k(f, g) = \frac{\rho_k(f, g)}{1 + \rho_k(f, g)}$ as the bounded family of separating metrics, and

$$d_F(f, g) = \sum_{k=1}^{\infty} \frac{2^{-k} \rho_k(f - g)}{1 + \rho_k(f - g)}.$$

In $(C(\mathbb{R}), d_F)$, $(f_n)_n \rightarrow f$ if and only if $\rho_k(f_n, f) \rightarrow 0$ for all k , meaning $(f_n)_n \rightarrow f$ uniformly on $[-k, k]$ for all k .

This is known as convergence on compact subsets.

Convergence in a Product Space: Let (X, d) and (Y, ρ) be metric spaces. Then,

$$\begin{aligned} X \times Y &= \{(x, y) \mid x \in X, y \in Y\}, \\ D_1((x, y), (x', y')) &= d(x, x') + \rho(y, y') \\ D_{\infty}((x, y), (x', y')) &= \max\{d(x, x'), \rho(y, y')\}. \end{aligned}$$

Both D_1 and D_{∞} are equivalent metrics.

Exercise: $((x_n, y_n))_n \rightarrow (x, y)$ if and only if $(x_n)_n \xrightarrow{d} x$ and $(y_n)_n \xrightarrow{\rho} y$.

Series in a Normed Vector Space

Let $(V, \|\cdot\|)$ be a normed vector space. Consider a sequence $(v_k)_k$ of vectors.

$$\begin{aligned} s_1 &= v_1 \\ s_2 &= v_1 + v_2 \\ &\vdots \\ s_n &= \sum_{k=1}^n v_k. \end{aligned}$$

If $s_n \rightarrow s$ in $(V, \|\cdot\|)$, meaning $\|s_n - s\| \rightarrow 0$, then we say the series $\sum_{k=1}^{\infty} v_k$ converges to s . We write

$$\sum_{k=1}^{\infty} v_k = s.$$

The series converges absolutely if

$$\sum_{k=1}^{\infty} \|v_k\|$$

converges in \mathbb{R} .

Proposition: Sequential Characterization of Closure

Let (X, d) be a metric space with $A \subseteq X$. $x \in \bar{A}$ if and only if $\exists (a_n)_n$ in A with $(a_n)_n \rightarrow x$.

In the forward direction, recall that $x \in \bar{A}$ if and only if $\forall \delta > 0$, $U(x, \delta) \cap A \neq \emptyset$. If $x \in \bar{A}$, then set $\varepsilon_n = 1/n$, and since $U(x, 1/n) \cap A \neq \emptyset$. Let $a_n \in U(x, 1/n) \cap A$. Then, $d(a_n, x) < 1/n \rightarrow 0$, meaning $a_n \rightarrow x$ and $a_n \in A$.

In the reverse direction, if $(a_n)_n \rightarrow x$ and $\varepsilon > 0$, $\exists N$ with $n \geq N \Rightarrow a_n \in U(x, \varepsilon) \cap A$. Thus, $x \in \bar{A}$.

Proposition: Sequential Characterization of Closed Sets

If (X, d) is a metric space, $A \subseteq X$, then the following are equivalent:

- (i) A is closed.
- (ii) Whenever $(a_n)_n$ in A with $(a_n)_n \xrightarrow{d} x$ in X , then $x \in A$.

Continuous Bounded Functions: $C([a, b]) \subseteq \ell_\infty([a, b])$ is closed under $\|\cdot\|_\infty$, since if $(f_n)_n \rightarrow f$ uniformly, and f_n is continuous, then f is continuous.

Sequence Closure: $c_0 \subseteq \ell_\infty$ is closed under $\|\cdot\|_\infty$. Let $(f_n)_n$ be a sequence

$$\begin{aligned} f_1 &= (f_1(1), f_1(2), \dots) \\ f_2 &= (f_2(1), f_2(2), \dots) \\ \lim_{k \rightarrow \infty} f_n(k) &= 0 \end{aligned} \quad \forall n$$

Suppose $(f_n)_n \xrightarrow{\|\cdot\|_\infty} f \in \ell_\infty$.

Let $\varepsilon > 0$. Then, $\exists n \in \mathbb{N}$ such that for $n \geq N$, $\|f - f_n\|_\infty < \varepsilon/2$. Also, $\lim_{k \rightarrow \infty} f_n(k) = 0$. Then, $\exists K \in \mathbb{N}$ such that for $k \geq K$, $|f_n(k)| < \varepsilon/2$. Thus, for $k \geq K$,

$$\begin{aligned} |f(k)| &= |f(k) - f_n(k) + f_n(k)| \\ &\leq |f(k) - f_n(k)| + |f_n(k)| \\ &\leq \|f - f_n\|_\infty + |f_n(k)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, $f \in c_0$.

Distance to a Set

Let (X, d) be a metric space, $A \subseteq X$. Then, $\text{dist}_A : X \rightarrow [0, \infty)$ is defined as

$$\text{dist}_A(x) = \inf_{a \in A} d(x, a).$$

- (1) $\overline{A} = \{x \mid \text{dist}_A(x) = 0\}$
- (2) $\text{dist}_A(\cdot) = \text{dist}_{\overline{A}}(\cdot)$
- (3) $|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y)$

Proof of (1): Let $x \in \overline{A}$. Then, $\exists (a_n)_n$ such that $(a_n)_n \rightarrow x$. Then, $d(a_n, x) \rightarrow 0$. Since $0 \leq \text{dist}_A(x) \leq d(x, a_n)$, $\text{dist}_A(x) = 0$.

Let x be such that $\text{dist}_A(x) = 0$. By the definition of \inf , we construct a_n by finding $a_n \in U(x, 1/n) \cap A$. Thus, $d(a_n, x) \rightarrow 0$, meaning $(a_n)_n \rightarrow x$, so $x \in \overline{A}$.

Proof of (2): Exercise; use (1).

Proof of (3): For all $a \in A$,

$$\begin{aligned} \text{dist}_A(x) &\leq d(x, a) \\ &\leq d(x, y) + d(y, a). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{dist}_A(x) - d(x, y) &\leq d(y, a) \\ \text{dist}_A(x) - d(x, y) &\leq \inf_{a \in A} d(y, a) \\ &= \text{dist}_A(y) \\ \text{dist}_A(x) - \text{dist}_A(y) &\leq d(x, y). \end{aligned}$$

Similarly,

$$\text{dist}_A(y) - \text{dist}_A(x) \leq d(y, x) = d(x, y)$$

meaning

$$|\text{dist}_A(y) - \text{dist}_A(x)| \leq d(x, y).$$

Continuity

Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \rightarrow Y$

(1) is continuous at $x_0 \in X$ if

$$\begin{aligned} & (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon \\ & (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in U_X(x_0, \delta) \Rightarrow f(x) \in U_Y(f(x_0), \varepsilon) \\ & (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } f(U_X(x_0, \delta)) \subseteq U_Y(f(x_0), \varepsilon). \end{aligned}$$

(2) is continuous if f is continuous at every $x_0 \in X$.

Proposition: Equivalent Continuity Criteria

Let $f : (X, d) \rightarrow (Y, \rho)$, $x_0 \in X$. The following are equivalent:

(1) f is continuous at x_0 ;

(2) $(\forall V \in \mathcal{N}_{f(x_0)})(\exists U \in \mathcal{N}_{x_0})$ such that $f(U) \subseteq V$.

(3) $\forall (x_n)_n \rightarrow x_0, (f(x_n))_n \rightarrow f(x_0)$.

(1) \Leftrightarrow (2): Clearly follows from definitions.

(1) \Rightarrow (3): Let $(x_n)_n \rightarrow x_0$. Let $\varepsilon > 0$. Then, $\exists \delta > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \varepsilon$.

Thus, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x_0) < \delta$. So, if $n \geq N$, $d(x_n, x_0) < \delta$, implying $\rho(f(x_n), f(x_0)) < \varepsilon$. So, $(f(x_n))_n \rightarrow f(x_0)$.

(3) \Rightarrow (1): Suppose toward contradiction that $\exists \varepsilon_0 > 0$ such that for $\delta = 1/n$ where $n \in \mathbb{N}$, $\exists (x_n)_n : d(x_n, x_0) < \delta$ and $\rho(f(x_n), f(x_0)) \geq \varepsilon_0$. Then, $(x_n)_n \rightarrow x_0$, but $f(x_n)_n \not\rightarrow f(x_0)$. \perp

Proposition: Topological Criterion for Continuity

Let $f : (X, d) \rightarrow (Y, \rho)$. The following are equivalent:

(1) f is continuous.

(2) $\forall V \in \tau_Y, f^{-1}(V) \in \tau_X$.

(3) $\forall x \in X, \forall (x_n)_n \rightarrow x$, we have $(f(x_n))_n \rightarrow f(x)$.

Proof: Exercise.

Proposition: Composition of Functions

Let $(X, d) \xrightarrow{f} (Y, \rho) \xrightarrow{g} (Z, p)$. If f and g are continuous, then $g \circ f$ is continuous.

Proof: Exercise.

Uniform Continuity

Let $f : (X, d) \rightarrow (Y, \rho)$.

(1) f is uniformly continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } \forall x, x' \in X, d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon$$

(2) f is Lipschitz if $\exists c > 0$ with

$$\rho(f(x), f(x')) \leq c d(x, x')$$

for all $x, x' \in X$.

(3) If $\rho(f(x), f(x')) = d(x, x')$, then f is an isometry. Isometries are always injective.

Exercise:

Isometry \Rightarrow Lipschitz \Rightarrow Uniformly Continuous \Rightarrow Continuous.

For example, $f(x) = x^2$ on $[0, \infty)$ is continuous but not uniformly continuous, and \sqrt{x} on $[0, 1]$ is uniformly continuous but not Lipschitz.

If $(V, \|\cdot\|)$ is a normed space, we might want to care that the following operations are continuous:

- $a : V \times V \rightarrow V$, $a(v, w) = v + w$:

$$\begin{aligned} \|a(v, w) - a(v', w')\| &= \|v + w - (v' + w')\| \\ &= \|v - v' + w - w'\| \\ &\leq \|v - v'\| + \|w - w'\| \\ &= d(v, v') + d(w, w') \\ &= d_1((v, w), (v', w')), \end{aligned}$$

meaning a is Lipschitz.

- $m : \mathbb{F} \times V \rightarrow V$, $m(\alpha, v) = \alpha v$:

$$\begin{aligned} \|m(\alpha, v) - m(\beta, w)\| &= \|\alpha v - \beta w\| \\ &= \|\alpha v - \alpha w + \alpha w - \beta w\| \\ &\leq |\alpha| \|v - w\| + |\alpha - \beta| \|w\| \end{aligned}$$

If $(\alpha_n)_n \rightarrow \beta$ and $(v_n)_n \rightarrow w$, then

$$\begin{aligned} \|\alpha_n v_n - \beta w\| &\leq |\alpha_n| \|v_n - w\| + |\alpha_n - \beta| \|w\| \\ &\rightarrow 0. \end{aligned}$$

- $\|\cdot\| : V \rightarrow \mathbb{F}$:

$$\| \|v\| - \|w\| \| \leq \|v - w\|,$$

meaning $\|\cdot\|$ is Lipschitz.

Let (X, d) be a metric space. Then, $\text{dist}_A : X \rightarrow [0, \infty)$, $\text{dist}_A(x) = \inf_{a \in A} d(x, a)$ is continuous. We had shown

$$|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y),$$

meaning dist_A is Lipschitz.

Proposition: Normal Property of Metric Spaces

Given $A, B \subseteq X$ with $A \cap B = \emptyset$, then $\exists U, V \in \tau_X$ with $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Proof: Set

$$f(x) = \frac{\text{dist}_A(x)}{\text{dist}_A(x) + \text{dist}_B(x)}.$$

Note that $\text{dist}_A(x) + \text{dist}_B(x) = 0$ if and only if $x \in \overline{A} = A$ and $x \in \overline{B} = B$. Therefore, the denominator in $f(x)$ is always positive.

Additionally, $f : X \rightarrow [0, 1]$ is continuous. Note that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Let $U = f^{-1}((-1/2, 1/2)) = f^{-1}([0, 1/2))$, and $V = f^{-1}((1/2, 3/2)) = f^{-1}((1/2, 1])$. Obviously, $U \subseteq A$ and $V \subseteq B$, and $U \cap V = \emptyset$.

Proposition: Quotient Space

Let $(V, \|\cdot\|)$ be a normed space, and let $W \subseteq V$ be a closed subspace. Then, V/W is a normed space with

$$\begin{aligned} \|v + W\| &= \text{dist}_W(v) \\ &= \inf_{w \in W} \|v - w\|. \end{aligned}$$

Proposition: Uniform Continuity of Linear Transformations

Let $T : V \rightarrow W$ be a linear transformation between two normed spaces. The following are equivalent:

- (1) T is continuous at 0_V .
- (2) T is continuous.
- (3) T is uniformly continuous.
- (4) T is Lipschitz.
- (5) $\exists c \geq 0$ such that $\|T(v)\| \leq c \|v\|$ for all $v \in V$.
- (6) $\|T\|_{\text{op}} = \sup_{\|v\| \leq 1} \|T(v)\| < \infty$. In other words, T is bounded linear.

Proof:

(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1): Obvious.

(6) \Rightarrow (5) Let $v \in V$. If $v = 0_V$, then $T(v) = 0_W$. Suppose $v \neq 0_V$. We know

$$\begin{aligned} \left\| T \left(\frac{v}{\|v\|} \right) \right\| &\leq \|T\|_{\text{op}} \\ \frac{1}{\|v\|} \|T(v)\| &\leq \|T\|_{\text{op}} \\ \|T(v)\| &\leq \|T\|_{\text{op}} \|v\|. \end{aligned}$$

Therefore, $c = \|T\|_{\text{op}}$.

(5) \Rightarrow (6): We will have $\|T(v)\| \leq c$ for all $v \in B_V$. Thus, $\|T\|_{\text{op}} \leq c$ for such c .

(5) \Rightarrow (4): Let $v, w \in V$. Then,

$$\begin{aligned} \|T(v) - T(w)\| &= \|T(v - w)\| \\ &\leq c \|v - w\|, \end{aligned}$$

meaning T is Lipschitz.

(1) \Rightarrow (5): Let $\varepsilon = 1$. Then, $\exists \delta$ such that

$$T(U_V(0, \delta)) \subseteq U_W(T(0), 1).$$

Since T is linear,

$$T(U_V(0, \delta)) \subseteq U_W(0, 1).$$

Let $v \in V \neq 0_V$. We know $\frac{\delta v}{2\|v\|} \in U_V(0, \delta)$. Then,

$$\begin{aligned} \left\| T \left(\frac{\delta v}{2\|v\|} \right) \right\| &\leq 1, \\ \frac{\delta}{2\|v\|} \|T(v)\| &\leq 1 \\ \|T(v)\| &\leq \frac{2}{\delta} \|v\|. \end{aligned}$$

Set $c = \frac{2}{\delta}$. Clearly, $\|T(0)\| \leq \frac{2}{\delta} \|0\|$.

A corollary to this is that any linear map $T : \ell_p^n \rightarrow W$ for W a normed space is uniformly continuous.

Proposition: Continuous Functions on Dense Sets

Let (X, d) , (Y, ρ) be metric spaces, and $A \subseteq X$ dense. If $f, g : X \rightarrow Y$ and $f(A) = g(A)$, then $f(X) = g(X)$.

Proof: Given $x \in X$, there exists $(a_n)_n \rightarrow x$. We know that $(g(a_n))_n \rightarrow g(x)$ and $(f(a_n))_n \rightarrow f(x)$. Since $f(a_n) = g(a_n)$ for all a_n , it is the case that $f(x) = g(x)$.

Morphisms in the Category of Metric Spaces

Let (X, d) and (Y, ρ) be metric spaces, $f : X \rightarrow Y$ a map.

- (1) f is a homeomorphism if f is bijective, continuous, and has a continuous inverse. We write $X \cong Y$ are homeomorphic.
- (2) f is a uniformism if f is bijective, uniformly continuous, and has a uniformly continuous inverse. We write $X \cong Y$ are uniformly isomorphic.
- (3) f is a metric isomorphism if f is bijective, Lipschitz, and has a Lipschitz inverse. We write $X \cong Y$ are metrically isomorphic.
- (4) f is an isometric isomorphism if f is bijective and isometric. We write $X \cong Y$ are isometrically isomorphic.

For example, $\mathbb{R} \cong (-\pi/2, \pi/2)$ are homeomorphic (using $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$). However, \mathbb{R} is not uniformly isomorphic to $(-\pi/2, \pi/2)$.

Suppose $f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ is a uniformism. Let $(x_n)_n = \pi/2 - 1/n$. Then, $(x_n)_n$ is Cauchy. Therefore, $(f(x_n))_n$ is Cauchy. Since \mathbb{R} is complete, $(f(x_n))_n \rightarrow y$ for some $y \in \mathbb{R}$. Then, $f^{-1}(f(x_n))_n \rightarrow f^{-1}(y)$, meaning $(x_n)_n \rightarrow f^{-1}(y) \in (-\pi/2, \pi/2)$. However, $(x_n)_n \rightarrow \pi/2 \notin (-\pi/2, \pi/2)$.

Completeness

Proposition: Weierstrass M-Test

Let V be a Banach space (complete normed vector space). Suppose $(v_k)_k$ is such that $\sum \|v_k\|$ is convergent. Then, $(s_n)_n = \sum_{k=1}^n v_k$ converges in V . Additionally,

$$\left\| \sum_{k=1}^{\infty} v_k \right\| \leq \sum_{k=1}^{\infty} \|v_k\|.$$

Proof: Let $s_n = \sum_{k=1}^n v_k$, and $t_n = \sum_{k=1}^n \|v_k\|$. Let $n > m$. Then,

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^n v_k \right\| \\ &\leq \sum_{k=m+1}^n \|v_k\| \\ &= |t_n - t_m|. \end{aligned}$$

Since $(t_n)_n$ converges, it is Cauchy, and thus s_n is Cauchy. Since V is complete, $(s_n)_n$ converges.

$$\begin{aligned} \|s_n\| &= \left\| \sum_{k=1}^n v_k \right\| \\ &\leq \sum_{k=1}^n \|v_k\| \\ &\leq \sum_{k=1}^{\infty} \|v_k\|. \end{aligned}$$

Let $n \rightarrow \infty$. Using the continuity of the norm, we get

$$\left\| \sum_{k=1}^{\infty} v_k \right\| \leq \sum_{k=1}^{\infty} \|v_k\|.$$

Proposition: Convergence in Hilbert Space

Let H be a Hilbert space (inner product space with a complete norm). Let $(e_n)_n$ be an orthonormal sequence in H . Let $(t_k)_k$ be a sequence in ℓ_2 . Then, $\sum_{k=1}^{\infty} t_k e_k$ converges in H , but not absolutely.

Proof: Let $s_n = \sum_{k=1}^n t_k e_k$. For $n > m$,

$$\begin{aligned}\|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^n t_k e_k \right\|^2 \\ &= \sum_{k=m+1}^n \|t_k e_k\|^2 && \text{Pythagorean Theorem} \\ &= \sum_{k=m+1}^n |t_k|^2\end{aligned}$$

Since $(t_k)_k \in \ell_2$, we know that $(t_k)_k$ is convergent and thus Cauchy. Thus, $(s_n)_n$ is Cauchy.

Note that for $t_k = \frac{1}{k}$, $(t_k)_k$ is square-summable, but not summable in absolute value.

Exercise: Show that

$$\left\| \sum_{k=1}^{\infty} t_k e_k \right\|^2 = \sum_{k=1}^{\infty} |t_k|^2.$$

This result is known as Parseval's Theorem.

Extensions of Continuous Functions

Lemma: Cauchy Sequences in Uniformly Continuous Functions

Let $f : (X, d) \rightarrow (Y, \rho)$ be uniformly continuous. If $(x_n)_n$ is Cauchy, then $(f(x_n))_n$ is Cauchy.

Proof: Let $\epsilon > 0$. Then, $\exists \delta > 0$ such that

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon.$$

Similarly, there exists $N \in \mathbb{N}$ such that for $p, q \geq N$, $d(x_p, x_q) < \delta$. So, for $p, q \geq N$, $d(f(x_p), f(x_q)) < \epsilon$.

Remark: This is not true for continuous functions. For example, if $f(t) = 1/t$ on $(0, 1)$, $x_n = 1/n$ is Cauchy but not convergent.

Theorem: Extension on a Dense Subset

Let (X, d) be a metric space with $A \subseteq X$ dense. Suppose $f : A \rightarrow Y$ is uniformly continuous with (Y, ρ) complete. Then, $\exists!$ uniformly continuous extension, $\tilde{f} : X \rightarrow Y$ that agrees with f on A .

Proof: Let $x \in X$. Then, $\exists (a_n)_n \in A$ with $(a_n)_n \rightarrow x$. Therefore, $(a_n)_n$ is Cauchy, and since f is uniformly continuous, we know that $(f(a_n))_n$ is Cauchy. Thus, $\lim_{n \rightarrow \infty} (f(a_n))_n = \tilde{f}(x)$ exists.

To show \tilde{f} is well-defined, suppose $(b_n)_n$ is another sequence in A with $(b_n)_n \rightarrow x$. Consider $(c_n)_n = (a_1, b_1, a_2, b_2, \dots)$. It must be the case that $(c_n)_n \rightarrow x$. Thus, $(f(c_n))_n$ converges to $y \in Y$. The subsequence of $(f(a_n))_n \rightarrow y$ and $(f(b_n))_n \rightarrow y$. So, we must have $\lim f(a_n) = \lim f(b_n)$.

Note that $\tilde{f}(a) = f(a)$ for all $a \in A$, by choosing the sequence (a, a, a, \dots) .

We claim that \tilde{f} is uniformly continuous. Let $\epsilon > 0$. We know $\exists \delta > 0$ such that for any $a, b \in A$, with $d(a, b) < \delta$, then $\rho(f(a), f(b)) < \epsilon/2$. Now, let $x, x' \in X$ with $d(x, x') < \delta/4$. Find sequences $(a_n)_n \rightarrow x$ and $(b_n)_n \rightarrow x'$ with $(a_n)_n, (b_n)_n \in A$. Find N large such that $n \geq N$ implies $d(a_n, x) < \delta/4$ and $d(b_n, x') < \delta/4$. For $n \geq N$, we have

$$\begin{aligned}d(a_n, b_n) &\leq d(a_n, x) + d(x, x') + d(x', b_n) \\ &< \frac{3\delta}{4} \\ &< \delta\end{aligned}$$

Thus, for $n \geq N$, $\rho(f(a_n), f(b_n)) < \epsilon/2$. By continuity of ρ , taking $n \rightarrow \infty$, we get $\rho(\tilde{f}(x), \tilde{f}(x')) < \epsilon/2$. Therefore, we have $d(x, x') < \delta/4 \Rightarrow d(\tilde{f}(x), \tilde{f}(x')) < \epsilon$. Therefore, \tilde{f} is uniformly continuous.

Suppose $g : X \rightarrow Y$ is another continuous extension of f . Therefore, $g(a) = \tilde{f}(a)$ for all $a \in A$. However, A is dense. Therefore, $g = \tilde{f}$.

Completion of a Metric Space

Let (X, d) be a fixed metric space. A completion of X is a pair $((Z, \rho), i)$.

(i) (Z, ρ) is a complete metric space.

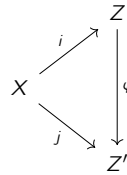
(ii) $i : X \rightarrow Z$ is an isometry.

(iii) $\overline{i(X)}^\rho = Z$.

For example, the completion of $(0, 1)$ is $([0, 1], |\cdot|), i(t) = t$.

Isometric Isomorphism of Completions

Given $((Z, \rho), i)$ and $((Z', \rho'), j)$ completions of X , then there exists a unique isometric isomorphism $\varphi : Z \rightarrow Z'$ such that the following diagram commutes.



Corollary: Isometric Map and Completion of Metric Space

If (X, d) is a metric space, and $i : (X, d) \rightarrow (Y, \rho)$ is an isometry into a complete metric space, then $((\overline{i(X)}), \rho, i)$ is the completion of X .

Theorem: Every Metric Space has a Completion

Consider the Banach space $(C_b(X), \|\cdot\|_u)$. We embed $X \hookrightarrow C_b(X)$ as follows. Fix $x_0 \in X$. Given $x \in X$, $i(x) = X \rightarrow \mathbb{F}$ where $i(x)(t) = d(t, x) - d(t, x_0)$.

Clearly, $i(x)$ is continuous for all x as the distance function is continuous. Also,

$$\begin{aligned} |i(x)(t)| &= |d(t, x) - d(t, x_0)| \\ &\leq d(x, x_0) \\ \|i(x)\|_u &\leq d(x, x_0). \end{aligned}$$

We need only show that $i(x)$ is an isometry.

$$\begin{aligned} \|i(x) - i(y)\|_u &= \sup_{t \in X} |i(x)(t) - i(y)(t)| \\ &= \sup_{t \in X} |d(t, x) - d(t, y)| \\ &= d(x, y). \end{aligned}$$

Nowhere Dense Sets

Let (X, d) be a metric space. Recall that a subset A if $(\overline{A})^\circ = \emptyset$. For example, $G = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ is nowhere dense.

Proposition: Equivalent Conditions for Nowhere Dense Sets

For a $A \subseteq X$, the following are equivalent:

- (i) A is nowhere dense.
- (ii) $\exists F \subseteq X$ closed with $F^\circ = \emptyset$, $A \subseteq F$.
- (iii) $\exists U \subseteq X$ open and dense with $U \subseteq A^c$.

Proof:

- (i) \Rightarrow (ii): Take $F = \overline{A}$.
- (ii) \Rightarrow (i): $\overline{A} \subseteq \overline{F}$, so $\overline{A}^\circ \subseteq \overline{F}^\circ = \emptyset$
- (ii) \Rightarrow (iii): Take $U = F^c$. Note that $U = F^c \subseteq A^c$. Then, $\overline{U} = \overline{F^c} = (F^\circ)^c = X$. Therefore, U is dense and open, and U is contained in A^c .
- (iii) \Rightarrow (ii): Take $F = U^c$.

A point $x \in X$ is isolated if $\exists \varepsilon > 0$ such that $U(x, \varepsilon) = \{x\}$.

Proposition: Extension of Nowhere Dense Sets

Let (X, d) be a metric space.

- (i) If $A \subseteq X$ is nowhere dense and $B \subseteq A$, then B is nowhere dense.
- (ii) If $A \subseteq X$ is nowhere dense, then \overline{A} is nowhere dense.
- (iii) Let A_1, \dots, A_n be nowhere dense. Then, $\bigcup A_i$ is nowhere dense.
- (iv) If X has no isolated points, then every finite set is nowhere dense.

Proof:

- (i) $B \subseteq A$ implies $\overline{B} \subseteq \overline{A}$, so $\overline{B}^\circ = \emptyset$, so B is nowhere dense.
- (ii) If A is nowhere dense, then $\overline{\overline{A}^\circ} = \overline{A}^\circ = \emptyset$.
- (iii) Let A_1 and A_2 be nowhere dense. By the alternate characterization, $U_1 \subseteq A_1^c$, where U_1 is open and dense. Similarly, $U_2 \subseteq A_2^c$, where U_2 is open and dense.

$$\begin{aligned} (A_1 \cup A_2)^c &= A_1^c \cap A_2^c \\ &\supseteq U_1 \cap U_2 \end{aligned}$$

We know $U_1 \cap U_2$ is open. We claim that $U_1 \cap U_2$ is dense.

Let $x \in X$, $\varepsilon > 0$. We want to show that $U(x, \varepsilon) \cap (U_1 \cap U_2) \neq \emptyset$. Since U_1 is dense, we know $U_1 \cap U(x, \varepsilon) \neq \emptyset$. Let $z \in U_1 \cap U(x, \varepsilon)$. Therefore, $\exists \delta > 0$ such that $U(z, \delta) \subseteq U_1 \cap U(x, \varepsilon)$. Since U_2 is dense, $U(z, \delta) \cap U_2 \neq \emptyset$. Therefore, $\emptyset \neq U(z, \delta) \cap U_2 \subseteq U(x, \varepsilon) \cap (U_1 \cap U_2)$.

By induction, assuming $A_1 \cup \dots \cup A_{n-1}$ are nowhere dense, then $(A_1 \cup \dots \cup A_{n-1}) \cup A_n$ is nowhere dense.

- (iv) Since X has no isolated points, $\{x\}$ is closed but not open. Therefore, $(\{x\})^\circ = \emptyset$. Use (iii).

Remark: Note that \mathbb{Q} is not nowhere dense, but \mathbb{Q} is the countable union of nowhere dense sets.

Meager Sets

Let (X, d) be a metric space.

- (i) $A \subseteq X$ is meager if A is the countable union of nowhere dense sets. Or, A is of the first category.
- (ii) $B \subseteq X$ is called residual if B^c is meager.

Examples: $\mathbb{Q} \subseteq \mathbb{R}$ is meager, so $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$ is residual. $\mathbb{Z} \subseteq \mathbb{R}$ is meager, but $\mathbb{Z} \subseteq \mathbb{Z}$ is not meager.

Proposition: Extension of Meager Sets

- (i) If A is meager, and $B \subseteq A$, then B is meager.
- (ii) If A_k is meager for $k = 1, \dots$, then A_k is meager.
- (iii) If X has no isolated points, then every countable set is meager.

Proof:

- (i) $A = \bigcup A_k$, with A_k nowhere dense. Then, $B = B \cap A = \bigcup B \cap A_k$.
- (ii) Each A_k is meager, meaning $A_k = \bigcup A_{k_j}$ with A_{k_j} nowhere dense. Thus, $A = \bigcup A_k$ is the countable union of A_{k_j} . Thus, A is meager.
- (iii) Since singleton sets are nowhere dense, we write the countable set as the union of singleton sets.

Proposition: Cantor's Intersection Theorem

Let (X, d) be a complete metric space, and $F_1 \supseteq F_2 \supseteq \dots$ be a sequence of closed, nonempty sets with $(\text{diam}(F_n))_n \rightarrow 0$. Then, $\bigcap F_n = \{x\}$ for some $x \in X$.

Proof: Let $x_n \in F_n$ for $n \geq 1$. Note that $(x_n)_n$ is Cauchy. For $\varepsilon > 0$, let N be large such that $n \geq N \Rightarrow \text{diam}(F_n) < \varepsilon$. For $m, n \geq N$, $d(x_n, x_m) < \varepsilon$ because $x_n, x_m \in F_N$. Therefore, $(x_n)_n \rightarrow x$ for $x \in X$.

We claim that $\{x\} = \bigcap F_n$. To see this, fix $m \in \mathbb{N}$, and consider $(x_{m+k})_k \in F_m$. The tail sequence $(x_{m+k})_k \rightarrow x$. Since F_m is closed, we know $x \in F_m$. Therefore, since m is arbitrary, $x \in \bigcap F_n$.

Now, suppose $\exists x, x' \in \bigcap F_n$ distinct. Then, $d(x, x') > 0$. However, $\exists N \in \mathbb{N}$ large with $\text{diam}(F_N) < d(x, x')$. However, $x, x' \in F_N$, which is a contradiction. Therefore, $\bigcap F_n = \{x\}$.

Baire's Theorem

Let (X, d) be a complete metric space.

- (i) If $\{V_k\}_{k \geq 1}$ is a countable family of open and dense subsets, then $\bigcap V_k$ is dense.
- (ii) X is not meager.

Proof:

- (i) Let U_0 be any open ball. Since V_1 is open and dense, $U_0 \cap V_1$ is open and nonempty. So, $\exists U_1$ with $B_1 = \overline{U_1} \subseteq U_0 \cap V_1$. We can assure that $\text{diam}(B_1) < 1$.

Consider $U_1 \cap V_2$. Since V_2 is dense and open, $U_1 \cap V_2$ is open and nonempty. Therefore, there must be $B_2 = \overline{U_2} \subseteq U_1 \cap V_2$. We can insure that $\text{diam}(B_2) < 1/2$.

Now, with $U_2 \cap V_3$, we have $B_3 = \overline{U_3} \subseteq U_2 \cap V_3$, with $\text{diam}(B_3) < 1/3$.

Inductively, we have U_1, \dots, U_{n-1} and B_1, \dots, B_{n-1} , we see that $U_{n-1} \cap V_n$ is open and nonempty, so we have U_n with $B_n = \overline{U_n} \subseteq U_{n-1} \cap V_n$, with $\text{diam}(B_n) < 1/n$.

Observe that we have $B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \dots$. In particular, $\{B_n\}_{n \geq 1}$ is a nested sequence of closed sets with $\text{diam}(B_n) \rightarrow 0$. Therefore, $\bigcap B_n = \{x\}$.

We claim that $x \in U_0 \cap (\bigcap V_k)$. Note that $B_n \subseteq U_{n-1} \cap V_n \subseteq V_n$. Therefore, $x \in \bigcap B_n$ implies $x \in \bigcap V_n$. Also, $x \in B_1 = \overline{U_1} \subseteq U_0 \cap V_1 \subseteq U_0$. Therefore, $\bigcap V_k$ is dense.

- (ii) Suppose $X = \bigcup A_k$ for A_k nowhere dense. Therefore, $\exists V_k$ open and dense with $V_k \subseteq A_k^c$. Then,

$$\begin{aligned} \emptyset &= X^c \\ &= \left(\bigcup A_k \right)^c \\ &= \bigcap A_k^c \\ &\supseteq \bigcap V_k. \end{aligned}$$

Therefore, by the previous result, $\bigcap V_k$ is open and dense, which is a contradiction. Therefore, X is not meager.

Question: Is $\mathbb{Q} \subseteq \mathbb{R}$ meager? Yes, \mathbb{Q} is the countable union of singleton sets. Is $\mathbb{R} \setminus \mathbb{Q}$ meager? The answer is no — otherwise, we would write $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be a union of meager sets, but \mathbb{R} is complete.

Applying Baire's Theorem

Let (X, d) be a metric space.

- (i) $G \subseteq X$ is a G_δ -set if

$$G = \bigcap_{k \geq 1} V_k$$

with V_k open.

- (ii) $F \subseteq X$ is a F_σ -set if

$$F = \bigcap_{k \geq 1} C_k$$

with C_k closed.

For example, $\mathbb{Q} \subseteq \mathbb{R}$ is F_σ , since \mathbb{Q} is the countable union of singleton sets (which are closed in \mathbb{R}). It can be shown that A is F_σ if and only if A^c is G_δ .

We claim that \mathbb{Q} is not G_δ .

Proof: If \mathbb{Q} is G_δ , then $\mathbb{R} \setminus \mathbb{Q}$ is F_σ , so

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup F_k$$

for F_k closed. Thus,

$$\begin{aligned} \mathbb{R} &= \mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q} \\ &= \bigcup \{q_k\} \cup \bigcup F_k. \end{aligned}$$

Therefore, \mathbb{R} is the countable union of closed sets. Since \mathbb{R} is complete, by Baire's Theorem, we must have $\{q_k\}^\circ \neq \emptyset$, or that $F_k^\circ \neq \emptyset$ for some k . However, $\{q_k\}^\circ = \emptyset$, and $F_k^\circ = \emptyset$ since $F_k \subseteq \mathbb{R} \setminus \mathbb{Q}$, and $\mathbb{R} \setminus \mathbb{Q}$ cannot contain an interval. Therefore, \mathbb{Q} is not G_δ .

Let (X, d) be a metric space. If A is closed, then A is G_δ .

Proof: Recall $\text{dist}_A : X \rightarrow \mathbb{R}$ is continuous. Therefore, $\text{dist}_A^{-1}((-1/n, 1/n)) = \{x \mid \text{dist}_A(x) < 1/n\}$ is open. Recall that $x \in A$ if and only if $\text{dist}_A(x) = 0$.

Therefore, we can write

$$A = \bigcap_{n \geq 1} \{x \mid \text{dist}_A(x) < 1/n\}.$$

Therefore, A is G_δ .

It follows that if A is open, then A is F_σ .

Theorem: Set of Continuities

Let $f : (X, d) \rightarrow (Y, \rho)$ be a map. Then, $C_f := \{x \in X \mid f \text{ is continuous at } x\}$ is a G_δ set.

Oscillation of a Function

Let $f : (X, d) \rightarrow (Y, \rho)$. Fix $x_0 \in X$. The oscillation $\omega_f(x_0) = \inf_{\delta > 0} \text{diam}(f(U(x, \delta)))$, or

$$\omega_f(x_0) = \inf_{\delta > 0} \left(\sup_{x, x' \in U(x, \delta)} \rho(f(x), f(x')) \right).$$

Note that $\omega_f(x_0) \in [0, \infty]$.

(i) f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

(ii) Given $c > 0$, $\{x \mid \omega_f(x_0) < c\} \subseteq X$ is open.

Proof:

(i) Suppose f is continuous at x_0 . Let $\varepsilon > 0$. Then, $\exists \delta > 0$ such that $d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon/2$. Therefore,

$$\text{diam}(f(U(x_0, \delta))) \leq \varepsilon,$$

since for $x, x' \in U(x_0, \delta)$, we have

$$\begin{aligned} \rho(f(x), f(x')) &\leq \rho(f(x), f(x_0)) + \rho(f(x_0), f(x')) \\ &< \varepsilon. \end{aligned}$$

In particular, $\omega(f(x_0)) \leq \varepsilon$. Since ε was arbitrary, we have $\omega_f(x_0) = 0$.

Suppose $\omega_f(x_0) = 0$. Let $\varepsilon > 0$. By the property of infimum, then $\exists \delta > 0$ such that

$$\text{diam}(f(U(x_0, \delta))) < \varepsilon.$$

In particular, if $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \varepsilon$. Thus, f is continuous at x_0 .

(ii) Let $V = \{x \mid \omega_f(x_0) < c\}$. Let $x_0 \in V$. Since $x_0 \in V$, $\omega_f(x_0) < c$. By the property of infimum, $\exists \delta > 0$ such that $\text{diam}(f(U(x_0, \delta))) < c$. Let $\varepsilon = \delta/2$. We claim that $U(x_0, \varepsilon) \subseteq V$.

Let $z \in U(x_0, \varepsilon)$. Note that $U(z, \delta/2) \subseteq U(x_0, \delta)$. Therefore, $f(U(z, \delta/2)) \subseteq f(U(x_0, \delta))$. Thus, $\text{diam}(f(U(z, \delta/2))) \leq \text{diam}(f(U(x_0, \delta))) < c$.

By property of oscillation, $\omega_f(z) < c$. So, $U(x_0, \varepsilon) \subseteq V$.

Proof of Theorem:

$$C_f = \{x \mid f \text{ is continuous at } x\}$$

$$= \bigcap_{n \geq 1} \underbrace{\{x \mid \omega_f(x) < 1/n\}}_{\text{open sets}}$$

meaning $x \in C_f \leftrightarrow \omega_f(x) = 0 \leftrightarrow \omega_f(x) < 1/n$ for all n .

Applying Set of Continuities

There does exist a function continuous at every irrational point and discontinuous at every rational point. Recall from Real Analysis that such f is

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

However, there does not exist $f : \mathbb{R} \rightarrow \mathbb{R}$ with $C_f = \mathbb{Q}$, since the set of continuities is always a G_δ set.

Nowhere Differentiable Functions

Does there exist a function $f : [0, 1] \rightarrow \mathbb{R}$ such that f is continuous on $[0, 1]$ but differentiable nowhere? The answer is yes.

$$f(x) = \sum_{n \geq 1} a^n \cos(b^n x),$$

where $0 < a < 1$ and $ab > 1$ is such a function. This is known as the Weierstrass function.

Such functions are not rare at all. In the complete normed vector space $X = (C[0, 1], \|\cdot\|_\infty)$, $\{f \in X \mid f \text{ differentiable nowhere}\}$ is the complement of a meager set (meaning it is topologically "big").

Compactness

Compactness can best be analogized to finite dimensionality in a metric space.

Let (X, d) be a metric space, and let $K \subseteq X$.

- (1) A cover for K is a family of subsets $\mathcal{U} = \{U_i\}_{i \in I} \subseteq \mathcal{P}(X)$ with $K \subseteq \bigcup U_i$.

The cover \mathcal{U} is called an open cover if each $U_i \subseteq X$ is open. The cover \mathcal{U} is called finite if I is finite. If \mathcal{U} is a cover for K , a subcover of \mathcal{U} is a subfamily $\mathcal{V} = \{U_i\}_{i \in J}$, with $J \subseteq I$, and $K \subseteq \bigcup_{i \in J} U_i$.

- (2) K is called compact if every open cover of K admits a finite subcover. If $\{U_i\}_{i \in I}$ is any family that covers K , then there exists a finite $F \subseteq I$ such that $\{U_i\}_{i \in F}$ covers K .

For example, the set $(0, 1] \subseteq \mathbb{R}$ is not compact, because

$$(0, 1] \subseteq \bigcup_{n \in \mathbb{N}} (1/n, 3/2)$$

does not admit a finite subcover.

Any finite set is compact.

A discrete metric space X is compact if and only if X is finite.

Let (X, d) be a metric space, and $Y \subseteq X$. Let $K \subseteq Y$; K is compact in X if and only if K is compact in Y . This can be shown by taking the relative topology of Y on every open cover of K in X .

Proposition: Properties of Compactness

Let (X, d) be a metric space.

- (1) If $K \subseteq X$ is compact, then K is closed and bounded.
- (2) If X is a compact metric space, and $K \subseteq X$ is closed, then K is compact.

Proof of (2): Let $K \subseteq \bigcup U_i$, with $U_i \subseteq X$ open. Then, $X = (X \setminus K) \cup (\bigcup_{i \in I} U_i)$. This is an open cover for X , meaning it admits a finite subcover $F \subseteq I$ such that $X = (X \setminus K) \cup \bigcup_{i \in F} U_i$. Clearly, $K \subseteq \bigcup_{i \in F} U_i$. Thus, K is compact.

Proof of (1): Let $K \subseteq X$ be compact. Then,

$$K \subseteq \bigcup_{x \in K} U(x, 1).$$

Since K is compact, there exist $\{x_1, \dots, x_n\}$ with $K \subseteq \bigcup_{j=1}^n U(x_j, 1)$. Let $c = \max d(x_i, x_j)$. If $x, y \in K$, then $x \in U(x_i, 1)$ and $y \in U(x_j, 1)$ for some x_i, x_j . Then,

$$\begin{aligned} d(x, y) &\leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \\ &< 1 + c + 1 = 2 + c. \end{aligned}$$

Thus, $\text{diam}(K) < \infty$.

We will show that K^c is open. Let $x_0 \notin K$. For each $x \in K$, there exist $\delta_x > 0$ with $U(x, \delta_x) \cap U(x_0, \delta_x) = \emptyset$. Then,

$$K \subseteq \bigcup_{x \in K} U(x, \delta_x).$$

Since K is compact, there exist $\{x_1, \dots, x_n\}$ with $K \subseteq \bigcup U(x_j, \delta_{x_j})$. Let $\delta = \min\{\delta_{x_j}\} > 0$. Then, $U(x_0, \delta) \subseteq K^c$.

Proposition: Compactness and Intersections of Closed Sets

Let (X, d) be a metric space. The following are equivalent.

- (1) X is compact;
- (2) If $\{C_i\}_{i \in I}$ is a family of closed sets with the finite intersection property (i.e., the intersection of finitely many elements of $\{C_i\}$ is non-empty), then $\bigcap_{i \in I} C_i \neq \emptyset$.

Proposition: Separability of Compact Metric Spaces

Let (X, d) be a compact metric space. Then, (X, d) is separable.

Proof: For fixed $n \geq 1$, consider the cover

$$X = \bigcup U(x, 1/n).$$

By compactness, there exist $\{x_{n,1}, \dots, x_{n,m_n}\}$ with

$$X = \bigcup_{j=1}^{m_n} U(x_{n,j}, 1/n).$$

Let $S = \{x_{n,j} \mid n \in \mathbb{N}, 1 \leq j \leq m_n\}$. Then, S is countable.

Let $x \in X$, $\varepsilon > 0$. Let N be large such that $N^{-1} < \varepsilon$. So,

$$x \in \bigcup_{j=1}^{m_N} U(x_{N,j}, 1/N),$$

so $x \in U(x_{N,j}, 1/N)$ for some j , whence $d(x, x_{N,j}) < 1/N < \varepsilon$, so $x_{N,j} \in U(x, \varepsilon)$. So, $\bar{S} = X$.

Proposition: Sequential Compactness

Let (X, d) be a metric space, $K \subseteq X$. We say K is sequentially compact if every sequence in K admits a convergent subsequence in K .

From Bolzano-Weierstrass, we know that $[a, b] \subseteq \mathbb{R}$ is sequentially compact.

If K is compact, then K is sequentially compact.

Proof: Let $(x_k)_k \in K$. Let $C_0 = \{x_1, x_2, \dots\}$, $C_1 = \overline{\{x_2, x_3, \dots\}}$, etc. such that $C_n = \overline{\{x_{n+1}, x_{n+2}, \dots\}}$.

Observe that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$. Additionally, $\{C_n\}$ has the finite intersection property. Since K is compact, the previous proposition states that $\bigcap C_n \neq \emptyset$. Let $x \in \bigcap C_n$.

$x \in C_1$, meaning $\exists k_1 > 1$ with $d(x, x_{k_1}) < 1$. $x \in C_{k_1}$, meaning $\exists k_2 > k_1$ with $d(x, x_{k_2}) < 1/2$. $x \in C_{k_2}$, meaning $\exists k_3 > k_2$ with $d(x, x_{k_3}) < 1/3$. Continuing, we have $(x_{k_j})_j \in K$ with $d(x, x_{k_j}) < 1/j$. Thus, $(x_{k_j})_j \rightarrow x$.

If (X, d) is sequentially compact, then X is complete.

Lemma: If $(x_n)_n$ is Cauchy, and $(x_n)_n$ admits a convergent subsequence, then $(x_n)_n$ is convergent.

Proof of Lemma: Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for $p, q \geq N$, $d(x_p, x_q) < \varepsilon/2$.

Also, suppose $(x_{n_k})_k \rightarrow x$. Then, $\exists K \in \mathbb{N}$ large such that for $k \geq K$, $d(x_{n_k}, x) < \varepsilon/2$.

Therefore, for $n \geq N$, find $k \geq \max\{N, K\}$, we have

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Proof: If (X, d) is sequentially compact, for $(x_n)_n$ a Cauchy sequence in (X, d) , we have that $(x_n)_n$ admits a convergent subsequence. Then, we use the lemma.

Total Boundedness

Let (X, d) be a metric space. $K \subseteq X$ is totally bounded if $\forall \delta > 0$, $\exists x_1, \dots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n U(x_i, \delta)$.

Exercise: If K is totally bounded, then K is bounded. If $L \subseteq K$, and K is totally bounded, then L is totally bounded.

Sequential Compactness and Total Boundedness

Let (X, d) be a metric space. Let $K \subseteq X$ be sequentially compact. Then, K is totally bounded.

Proof: Suppose K is not totally bounded. Then, $\exists \delta_0 > 0$ such that $K \not\subseteq \bigcup_{x \in F} U(x, \delta_0)$ for any finite F .

Let $x_1 \in K$. Since $K \not\subseteq U(x_1, \delta_0)$, so let $x_2 \in K \setminus U(x_1, \delta_0)$. Since $K \not\subseteq U(x_1, \delta_0) \cup U(x_2, \delta_0)$, let $x_3 \in K \setminus (U(x_1, \delta_0) \cup U(x_2, \delta_0))$. Continuing, we find $x_n \in K \setminus \bigcup_{j=1}^{n-1} U(x_j, \delta_0)$.

Thus, we have a sequence $(x_n)_n$. By sequential compactness, $(x_n)_n$ admits $(x_{n_k})_k \rightarrow x \in K$. Since $(x_{n_k})_k$ is convergent, $(x_{n_k})_k$ is Cauchy. But, $d(x_p, x_q) \geq \delta_0$, since, without loss of generality, for $p > q$, $x_p \notin U(x_q, \delta_0)$. \perp

Corollary: Compact Subsets of Real Numbers

If $K \subseteq \mathbb{R}$ is compact, $\sup K \in K$ and $\inf K \in K$.

Proof: We can always construct sequences $(x_n)_n \rightarrow \sup K$ and $(y_n)_n \rightarrow \inf K$ in K . Since $\sup K < \infty$ and $\inf K < \infty$, since K is compact, and thus bounded.

Since K is also closed, $\sup K \in K$ and $\inf K \in K$.

Theorem: Equivalence of Compactness Definitions

Let (X, d) be a metric space. The following are equivalent.

- (1) X is compact.
- (2) X is sequentially compact.
- (3) X is complete and totally bounded.

Proof: We proved that $(1) \Rightarrow (2) \Rightarrow (3)$. We will now prove $(3) \Rightarrow (1)$.

Suppose \mathcal{V} is an open cover of X that fails to admit a finite subcover. Let $\epsilon = 1$. Since X is totally bounded $X = \bigcup_{j=1}^{m_1} U_{1,j}$, where $U_{1,j}$ are open balls of radius 1.

There must be some open ball among the $U_{1,j}$ not covered by finitely many members of \mathcal{V} . Call this ball $U(x_1, 1)$. Let $\epsilon = 1/2$. By total boundedness, $X = \bigcup_{j=1}^{m_2} U_{2,j}$, where $U_{2,j}$ are open balls of radius $1/2$. Then, $U(x_1, 1) = \bigcup (U(x_1, 1) \cap U_{2,j})$. So, there must be an open ball of radius $1/2$, $U(x_2, 1/2)$, such that $U(x_1, 1) \cap U(x_2, 1/2)$ cannot be covered by finitely many members of \mathcal{V} .

Continuing, we have a sequence $(x_n)_n$, where $F_n = U(x_1, 1) \cap U(x_2, 1/2) \cap \dots \cap U(x_n, 1/n)$ cannot be covered by finitely many members of \mathcal{V} .

Let $C_n = \overline{F_n}$. Notice that $F_1 \supseteq F_2 \supseteq \dots$, meaning $C_1 \supseteq C_2 \supseteq \dots$. We see that $\text{diam}(C_n) = \text{diam}(F_n) \leq 2/n$. Applying Cantor's intersection theorem, we have $\bigcap C_n = \{x\}$.

Since \mathcal{V} is an open cover, locate $V \in \mathcal{V}$ such that $x \in V$. Since V is open, there exists $\epsilon > 0$ such that $U(x, \epsilon) \subseteq V$. Choose N large such that $2/N < \epsilon$. Since $x \in C_N$, $d(z, x) \leq 2/N < \epsilon$ for all $z \in C_N$, meaning $F_N \subseteq C_N \subseteq U(x, \epsilon) \subseteq V$.

Therefore, $\{V\}$ is a cover for F_N . \perp

Proposition: Multi-dimensional Bolzano-Weierstrass Theorem

Let $\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_d, b_d] = \prod_{j=1}^d [a_j, b_j] \subseteq \ell_p^d$. Then, \mathcal{R} is sequentially compact, so \mathcal{R} is compact.

Proof: The proof in \mathbb{R}^d works similarly to the proof in \mathbb{R}^2 . Consider $\pi_x : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\pi_y : \mathbb{R}^2 \rightarrow \mathbb{R}$. We saw that $(v_n)_n \rightarrow v$ in ℓ_p^2 if and only if $(\pi_x(v_n))_n \rightarrow \pi_x(v)$ and $(\pi_y(v_n))_n \rightarrow \pi_y(v)$.

If $(v_n)_n \in \mathcal{R}$, then $(\pi_x(v_n))_n \in [a_1, b_1]$. By Bolzano-Weierstrass, there is a convergent subsequence $(\pi_x(v_{n_k}))_k \rightarrow x \in [a_1, b_1]$.

Now, consider $(\pi_y(v_{n_k}))_k \in [a_2, b_2]$. By Bolzano-Weierstrass, there is a convergent subsequence $(\pi_y(v_{n_{k_j}}))_j \rightarrow y \in [a_2, b_2]$. Thus, $(v_{n_{k_j}})_j \rightarrow (x, y)$ in \mathcal{R} .

Heine-Borel Theorem

Let $K \subseteq \mathbb{R}^d$. The following are equivalent:

- (i) K is compact;
- (ii) K is sequentially compact;
- (iii) K is closed and bounded.

Proof: We have $(i) \Leftrightarrow (ii)$, and $(i) \Rightarrow (iii)$. We will show $(iii) \Rightarrow (ii)$.

If K is bounded, then $K \subseteq \mathcal{R} = \prod_{j=1}^d [a_j, b_j]$. Let $(v_n)_n$ be a sequence in K . By the previous proposition, there exists a subsequence $(v_{n_k})_k \rightarrow v \in \mathcal{R}$. Since K is closed, $v \in K$. Therefore, K is sequentially compact.

There are many examples of closed and bounded sets that are not compact (in infinite-dimensional vector spaces).

For example, in $\ell_1 = \{a = (a_k)_k \mid \sum_{k=1}^{\infty} |a_k| < \infty\}$, we have $e_n = (0, 0, \dots, 0, 1, 0, \dots)$, with 1 at the n th coordinate. For the sequence $(e_n)_n$, $\|e_k\|_1 = 1$ for all e_k , so $(e_n)_n \in B_{\ell_1}$, which is closed and bounded. Observe that $\|e_n - e_m\| = 2$ for all $m \neq n$, so there does not exist a convergent subsequence. Thus, ℓ_1 is not sequentially compact.

Remark: We will show that for a normed space, $(V, \|\cdot\|)$, B_V is compact if and only if $\dim(V) < \infty$.

Proposition: Continuous Image of Compact Sets

If $f : (X, d) \rightarrow (Y, \rho)$ is continuous, and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

Proof: Let $\bigcup_{i \in I} V_i$ be an open cover for $f(K)$, where $V_i \subseteq Y$ open. Taking the preimage, we have

$$\begin{aligned} K &\subseteq f^{-1}(f(K)) \\ &\subseteq f^{-1}\left(\bigcup_{i \in I} V_i\right) \\ &= \bigcup_{i \in I} f^{-1}(V_i) \end{aligned}$$

since f is continuous, $f^{-1}(V_i) \subseteq X$ are open. By compactness, there exists $F \subseteq I$ finite such that

$$K \subseteq \bigcup_{i \in F} f^{-1}(V_i).$$

Taking the image, we have

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i \in F} f^{-1}(V_i)\right) \\ &= \bigcup_{i \in F} f(f^{-1}(V_i)) \\ &= \bigcup_{i \in F} V_i. \end{aligned}$$

Thus, $f(K)$ has a finite subcover.

Corollary: Compactness under Topologically Equivalent Metrics

Let d_1 and d_2 be topologically equivalent ($\text{id}_X : (X, d_1) \rightarrow (X, d_2)$ is a homeomorphism). Then, $K \subseteq X$ is d_1 -compact if and only if K is d_2 -compact.

Corollary: Heine-Borel Theorem Extension

For $K \subseteq \ell_p^n$, K is compact if and only if K is closed and bounded.

Extreme Value Theorem

Let (X, d) be a metric space, $K \subseteq X$ compact, and $f : X \rightarrow \mathbb{R}$ continuous. Then, $\sup_{x \in X} f(x) = f(x_M)$ and $\inf_{x \in X} f(x) = f(x_m)$ for some $x_M, x_m \in K$.

Proof: We know that $f(K) \subseteq \mathbb{R}$ is compact. Then, $\inf f(K)$ and $\sup f(K)$ are elements of $f(K)$.

Proposition: Compactness of Closed Unit Ball

Let V be a finite-dimensional vector space over \mathbb{F} .

- (1) All norms on V are equivalent.
- (2) For any norm, $\|\cdot\|$ on V , $B_{(V, \|\cdot\|)} = \{v \in V \mid \|v\| \leq 1\}$ is compact.

Proof of (1): Let $\{v_1, \dots, v_n\}$ be a linear basis for V . Define

$$\left\| \sum_{j=1}^n t_j v_j \right\|_1 = \sum_{j=1}^n |t_j|.$$

This is a norm on V .

Then, $\varphi : \ell_1^n \rightarrow V$

$$\varphi \left(\sum_{j=1}^n t_j e_j \right) = \sum_{j=1}^n t_j v_j$$

is a linear isometric isomorphism. Since $B_{\ell_1^n}$ is compact, so too is $\varphi(B_{\ell_1^n})$, so $B_{(V, \|\cdot\|)}$ is compact.

Then, $S_1 := \{v \in V \mid \|v\|_1 = 1\}$ is compact since $S_1 \subseteq B_{(V, \|\cdot\|)}$ is closed.

Let $\|\cdot\|$ be any norm on V . We will show that $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. Note that

$$\begin{aligned} \left\| \sum_{j=1}^n t_j v_j \right\| &\leq \sum_{j=1}^n |t_j| \|v_j\| \\ &\leq c \sum_{j=1}^n |t_j| \\ &= c \left\| \sum_{j=1}^n t_j v_j \right\|_1 \end{aligned}$$

where $c = \max \|v_j\|$. Consider $g : (V, \|\cdot\|_1) \rightarrow \mathbb{R}$, with $g(v) = \|v\|$.

$$\begin{aligned} |g(v) - g(w)| &= |\|v\| - \|w\|| \\ &\leq \|v - w\| \\ &\leq c \|v - w\|_1 \end{aligned}$$

so g is Lipschitz, and thus continuous. S_1 is compact in $(V, \|\cdot\|_1)$, so by the extreme value theorem, $\inf_{v \in S_1} g(v) = g(v_0) = \|v_0\|$ for some $v_0 \in S_1$. Note that $D := \|v_0\| > 0$, else $v_0 = 0$. Thus, $g(v) \geq D$ for all $v \in S_1$

$$\|v\| \geq D \quad \forall v \in S_1$$

Let $0 \neq v$. Then,

$$\begin{aligned} \frac{v}{\|v\|_1} &\in S_1 \\ \left\| \frac{v}{\|v\|_1} \right\| &\geq D \end{aligned}$$

so

$$\|v\| \geq D \|v\|_1.$$

Therefore, we have $\|v\|_1 \leq \frac{1}{D} \|v\|$. Thus, any two norms on V are equivalent.

Proof of (2): Exercise.