The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of x).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

Some Functions and Sets

$$f(x,y) = x^2 - y^2$$

Domain: $\{(x,y) \mid \exists f(x,y)\}$

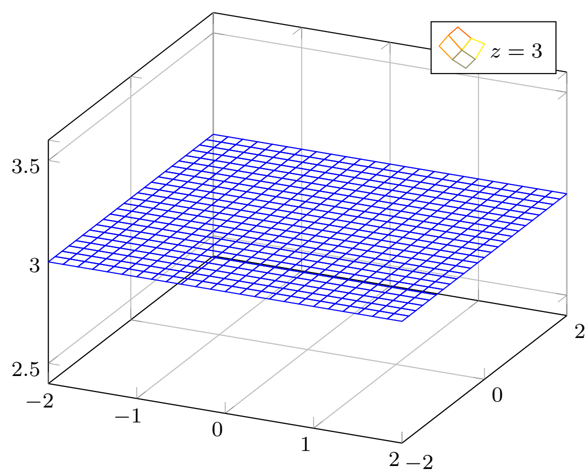
Range: $\{f(x,y) \mid (x,y) \in \text{Dom}(f)\} = \mathbb{R}$

Graph: $\text{Graph}(f) = \{x, y. f(x,y) \mid x, y \in \text{Dom}(f)\}$. For example, $(1,3,4) \notin \text{Graph}(f)$ since $1^2 - 3^2 \neq 4$.

Examples

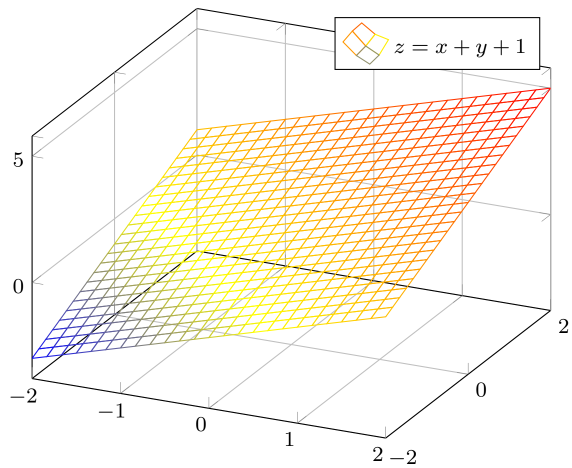
In $\mathbb{R}^3$, in x, y, z coordinates, $z = 3$ is a plane defined as follows:

- Parallel to the xy plane.
- Passes through the point $(0.0, 3)$.



Meanwhile, $y = 0$ would be a "wall" that passes through the origin that contains the line $y = 0$ in the xy plane.

Finally, $z = x + y + 1$ is a plane, as we can see below.



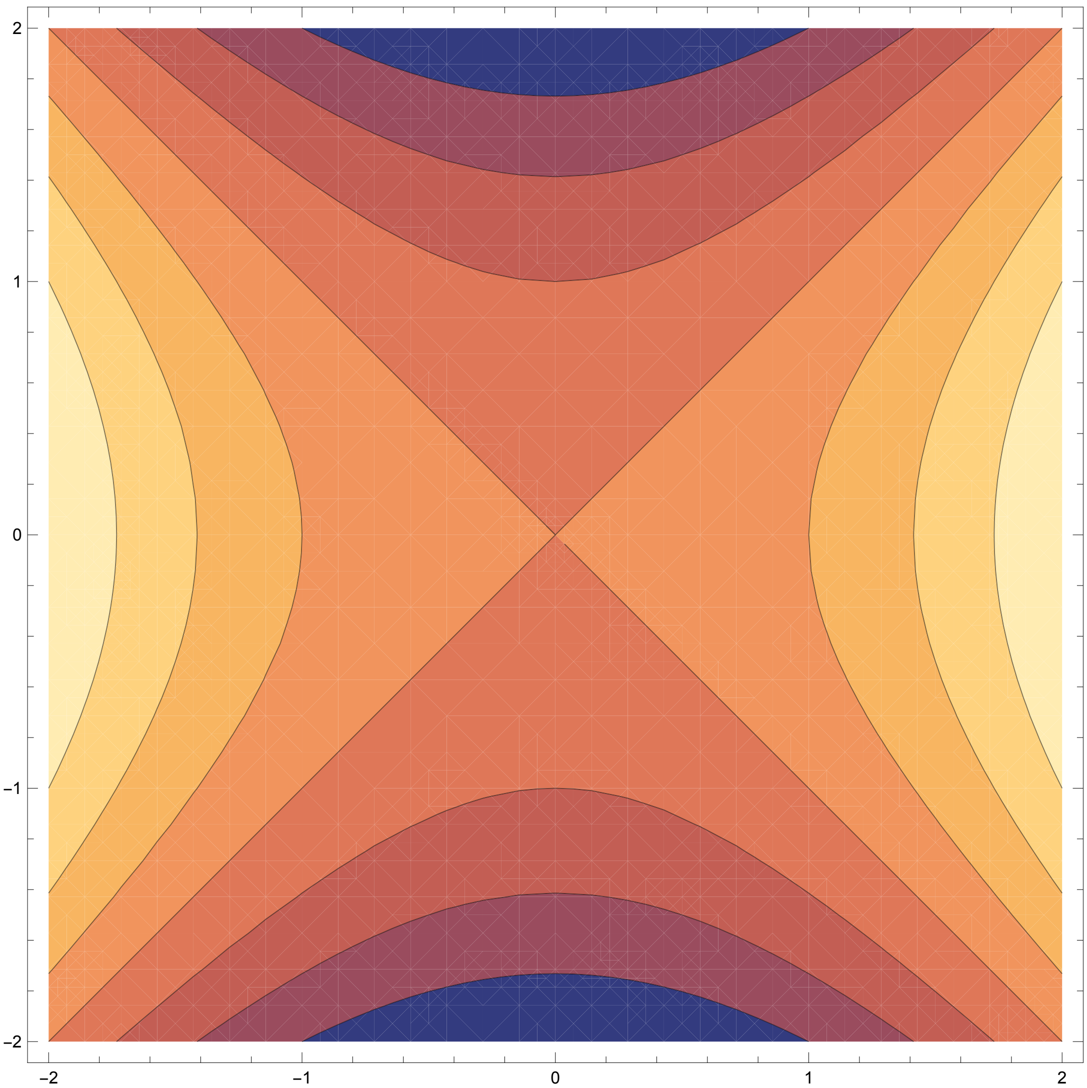
Visualizing a function of multiple variables

Consider the function $f(x, y) = x^2 - y^2$. We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$
- $f(0, y) = -y^2$
- $f(2, y) = 4 - y^2$
- $f(x, -2) = x^2 + 4$
- $f(x, 0) = x^2$
- $f(x, 2) = x^2 + 4$

Alternatively, we can visualize via contour diagrams (i.e., everywhere that z is a certain value), as seen in mathematica as follows:

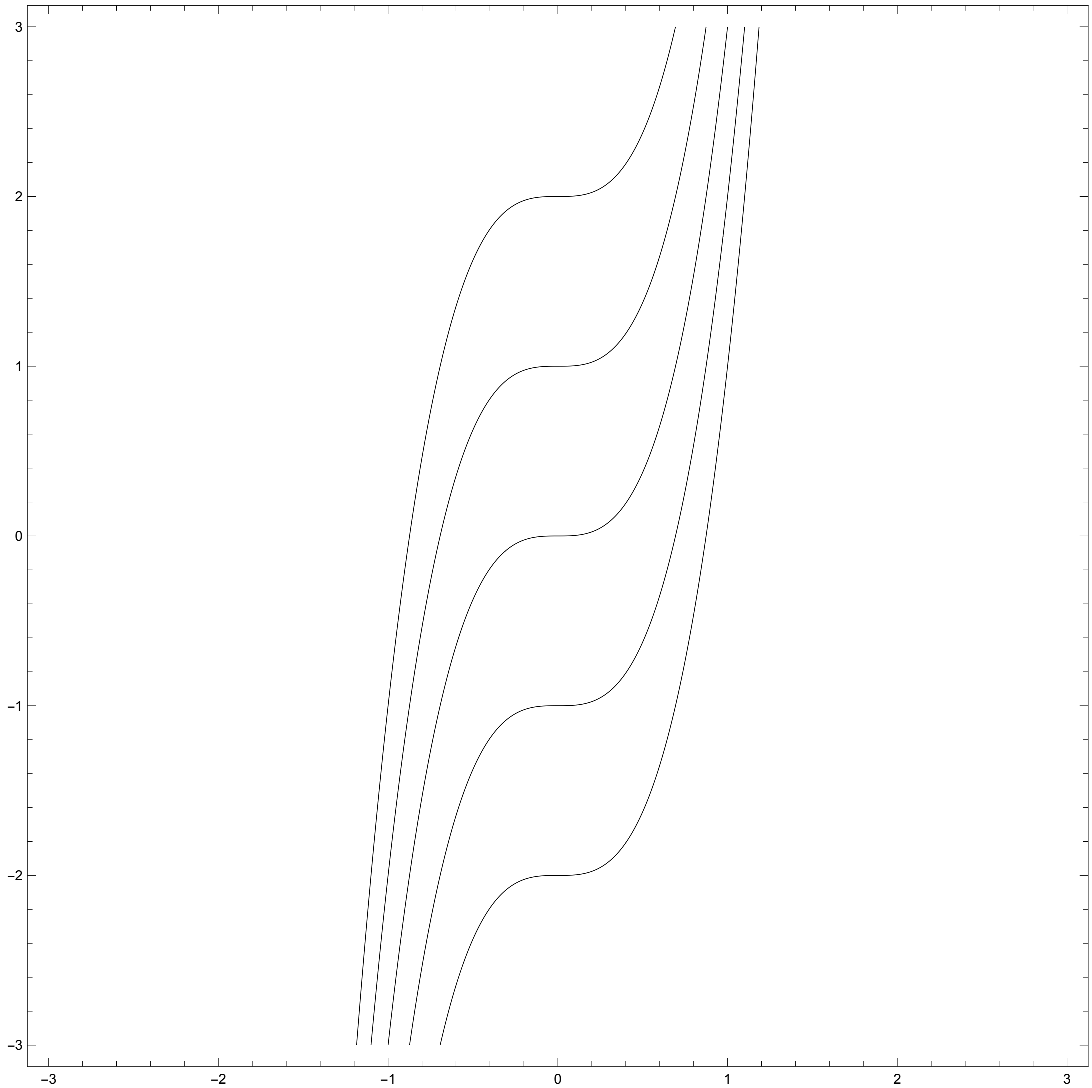
Contour Diagram



Contour Example

Consider the function $f(x, y) = y - 3x^2$. We want to find the contours.

For any c, we have that $c = y - 3x^3$, or $y = 3x^3 + c$. Therefore, every contour "looks like" $3x^3 + c$ for values of c. For example, in the following, we have $c = \{-2, -1, 0, 1, 2\}$

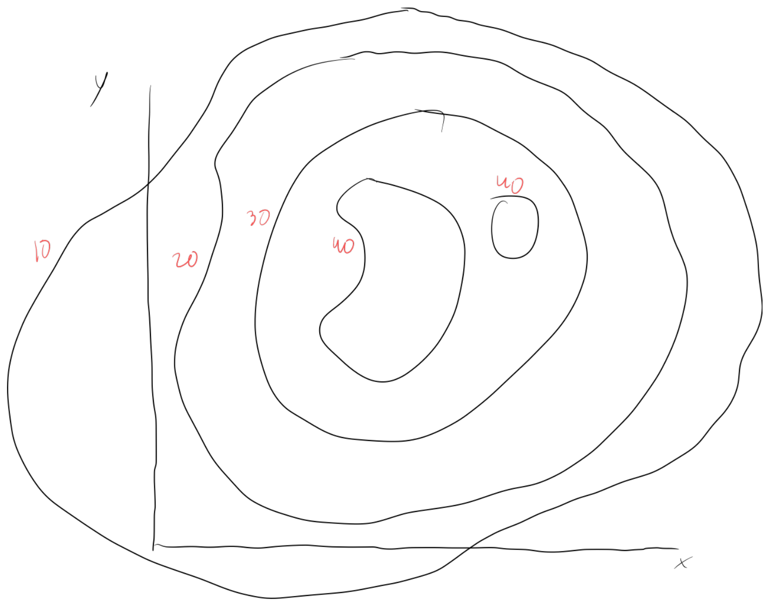


Distance

In $\mathbb{R}^5$, let $p = (3,1,4,1,5)$, and $q = (1,0,-2,0,2)$. Using the Euclidean metric, we can find the distance between p and q is $d(p,q) = ((3-1)^2 + (1-0)^2 + (4-(-2))^2 + (1-0)^2 + (5-2)^2)^{1/2} = (4+1+36+1+9)^{1/2} = \sqrt{51} = 7.14$. We can also call this the 2-norm.

$$d(p,q) = \left(\sum_{k=1}^{n} (p_k - q_k)^2\right)^{1/2}$$

Derivatives



To denote a derivative, we can't talk about one value, we must use a *partial* derivative, $\frac{\partial f}{\partial x}$, or $\frac{\partial f}{\partial y}$. The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

x\y	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A "linear" approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and is an output in $z = f(x, y)$, and $m, n \in \mathbb{R}$.

For example, with the above table, we can see that the function is linear in x and y (i.e., the slope holding the other variable constant is constant).

Limits in Multivariable Functions

Consider the following:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow $y = mx$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2} &= \lim_{(x,y)\to(0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2} \\ &= \frac{1 + m^2}{1 - m^2}\end{aligned}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of m

m	$\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours "approach" a particular point, we can see that the limit is defined.

Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$. These vectors are equivalent because they are components of $\mathbb{R}^3$.

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

Properties of Vectors

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $c\vec{u} = \langle cu_1, cu_2, \dots, cu_k \rangle$

Additionally, we define $\vec{u} \cdot \vec{v}$ as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

Partial Derivatives

Consider $f(x, y) = x^2 y + xe^y$.

$$f_x := \frac{\partial f}{\partial x}$$
$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

We know that $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, meaning f is endlessly differentiable.

Functions and Approximations

Let $f(x, y) = x^2 - y^2$, $g(x, y) = 2xy$

- $f_{xx} + f_{yy} = 0$
- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For $f(x,y)$ at $(a,b,f(a,b))$, we have the following:

$$\begin{aligned}
\ell(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(y-b) \\
q(x,y) &= \ell(x,y) + \frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right)
\end{aligned}$$

In order to get a sense of the "derivative," we can use the following:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

Directional Derivative and Gradient

Given $f(x,y)$ and (a,b), where $f \in C^2(\mathbb{R}^2)$. Then, the quadratic approximation is:

$$\begin{aligned}
f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_x(a,b)(y-b) \\
&\quad + \frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + f_{yy}(a,b)(y-b)^2 + f_{xy}(a,b)(x-a)(y-b)\right) \\
df &= f_x(a,b)dx + f)y(a,b)dy && \text{a differential} \\
\Delta f &= f_x(a,b)\Delta x + f_y(a,b)\Delta y
\end{aligned}$$

Evaluating $f(x,y) = xe^y$ at $(a,b) = (-1,0)$

$$\begin{aligned}
f_x &= e^y \\
f_y &= xe^y \\
f_x(-1,0) &= 1 \\
f_y(-1,0) &= -1 \\
\Delta f &= \Delta x - \Delta u
\end{aligned}$$

On a given contour map, let $\vec{u} = \langle u_1, u_2 \rangle$ denote a *unit* vector in a direction that we want to find the derivative of f in.

$$f_{\vec{u}}(x,y) = \nabla f(a,b) \cdot \vec{u}$$

Where

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

The directional derivative for all vectors $\vec{v}$ is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$