

## Complex Numbers

A complex number is an ordered pair of real numbers,  $(a, b) = a + bi$ . A vector in  $\mathbb{R}^2$  is also an ordered pair,  $(a, b)$  of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with  $\mathbb{R}^2$ . However, unlike vectors in  $\mathbb{R}^2$ , there is also an operation  $\cdot$ . We desire for  $(0, 1) \cdot (0, 1) = (-1, 0)$ ; essentially,  $i^2 = -1$ . We say that  $i$  is a square foot of  $-1$ ; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di) \\ &:= a(c) + adi + bci + bd(i^2) \\ &:= (ac - bd) + (ad + bc)i \\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus,  $\mathbb{R}^2$  with the operations  $+$  and the above defined complex multiplication is known as  $\mathbb{C}$ . We write as  $a + bi$  instead of  $(a, b)$ .

Given  $z = (a + bi) \in \mathbb{C}$ , we write  $\operatorname{Re}(z) = a$  and  $\operatorname{Im}(z) = b$ . If  $\operatorname{Im}(z) = 0$ , then  $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$ . However, many people say that  $\mathbb{R} \subseteq \mathbb{C}$ , even if  $\mathbb{C}$  isn't defined as such.

## Reciprocals of Complex Numbers

Let  $z \in \mathbb{C}$ , where  $z \neq 0$ . Then,  $\exists w \in \mathbb{C}$  such that  $zw = 1$ .

Let  $w = c + di$ . We want to show that  $zw = 1$ .

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1 \\ ad + bc &= 0.\end{aligned}$$

Thus, let  $w = c + di$ , with  $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2} \\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every  $z \neq 0$ , with  $z = a + bi$ , the *reciprocal* of  $z$  is defined as  $\frac{1}{z} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$ . Then, for  $w \in \mathbb{C}$ , we define

$$\frac{w}{z} := w \left( \frac{1}{z} \right).$$

## Properties of Complex Numbers

Let  $z = a + bi \in \mathbb{C}$ . Then, the (Euclidean) norm (or absolute value) of  $z$  is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of  $z = a + bi$  is  $\bar{z} = a - bi$ .

$$(i) \quad z\bar{z} = |z|^2$$

$$(ii) \quad \overline{(\bar{z})} = z$$

$$(iii) \overline{(z + w)} = \bar{z} + \bar{w}$$

$$(iv) \overline{zw} = \bar{z} \cdot \bar{w}$$

$$(v) z + \bar{z} = 2\operatorname{Re}(z), \text{ so } \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$(vi) z - \bar{z} = 2i\operatorname{Im}(z), \text{ so } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

## Polar Representation

Let  $z = a + bi$  (or  $z = (a, b)$ ). Then,  $|z| = \sqrt{a^2 + b^2}$  is the *radius*, and the *argument* is found by  $\theta = \arctan(b/a)$  for  $a \neq 0$ . Therefore, the full polar representation is as follows:

$$z = |z| (\cos \theta + i \sin \theta). \quad \theta \in [0, 2\pi)$$

If  $z = 0$ , then  $|z| = 0$ , and  $\arg z$  is undefined.

For example, we can find  $\arg i$  in  $[\pi, 3\pi)$  as  $\frac{5\pi}{2}$ .

For  $z_1$  and  $z_2$  in polar form, we have:

$$|z_1 z_2| = |z_1| |z_2| \quad (1)$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi} \quad (2)$$

Proof of (1):

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2) \overline{(z_1 z_2)} \\ &= z_1 z_2 \bar{z}_1 \bar{z}_2 \\ &= z_1 \bar{z}_1 z_2 \bar{z}_2 \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

Since  $|z| \geq 0$ , we get  $|z_1 z_2| = |z_1| |z_2|$ .

Let  $z = 2(\cos \pi/6 + i \sin \pi/6)$ , and let  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined as  $f(w) = zw$ . Then,  $f$  rotates  $w$  by  $\pi/6$  and scales  $w$  by 2.

**Theorem:** For  $n \in \mathbb{N}$ , if  $z = r(\cos \theta + i \sin \theta)$ , then  $z^n = r^n(\cos(n\theta) + i \sin(n\theta))$ .

**Proof:** Induct on  $n$ . For the base case, we know that  $n = 1$  satisfies this property. For  $n > 1$ , we have:

$$\begin{aligned} z^{n+1} &= (z^n)(z) \\ &= (r^n(\cos(n\theta) + i \sin(n\theta))) r(\cos \theta + i \sin \theta) \\ &= (r^n)(r) (\cos(n\theta + \theta) + i \sin(n\theta + \theta)) && \text{Polar Representation Definition} \\ &= r^{n+1}(\cos((n+1)\theta) + i \sin((n+1)\theta)) \end{aligned}$$

We can use this technique to find the “roots of unity.” For example, to find all  $z$  such that  $z^3 = 1$ , we use our

technique:

$$\begin{aligned}
 z^3 &= 1 \\
 |z| &= 1 \\
 \arg z^3 &= 0 \\
 3 \arg z &= 0 \pmod{2\pi} \\
 \arg z &= \frac{k2\pi}{3} \\
 &= 0, \frac{2\pi}{3}, \frac{4\pi}{3} \\
 z_1 &= 1 \\
 z_2 &= (\cos 2\pi/3 + i \sin 2\pi/3) \\
 z_3 &= (\cos 4\pi/3 + i \sin 4\pi/3)
 \end{aligned}$$

We can see that  $z_2^2 = z_3$ .

For the  $n$  case, we find  $z_2 = \cos(2\pi/n) + i \sin(2\pi/n)$ , and  $z_k = z_2^{k-1}$ .

## Exponential, Logarithm, and Trigonometric Functions in $\mathbb{C}$

### Exponential

Let  $z = a + bi$ . We define  $e^{a+bi}$  as follows:

$$e^{a+bi} = e^a (\cos b + i \sin b)$$

Recall that for every nonzero complex number,  $z = |z| (\cos \theta + i \sin \theta)$ , where  $\theta = \arg z$ . Thus,

$$\begin{aligned}
 z &= |z| e^{i\theta} \\
 &= |z| e^{i \arg z}.
 \end{aligned}$$

The function  $e^z$  has some properties similar to the function  $e^x$  in real numbers, and some properties varying with the real numbers.

$$\begin{aligned}
 e^z e^w &= e^{z+w} \\
 e^z &\neq 0
 \end{aligned}$$

However, there are some differences:

$$\begin{aligned}
 |e^{i\theta}| &= 1 \\
 e^{a+bi} &= e^a
 \end{aligned}
 \quad \forall \theta$$

From these properties, we find Euler's equation:

$$e^{i\pi} + 1 = 0$$

Additionally,  $e^z$  is periodic, while  $f(x) = e^x$  is injective:

$$\begin{aligned}
 e^{z+2n\pi} &= e^z (\cos(2n\pi) + i \sin 2n\pi) \\
 &= e^z
 \end{aligned}$$

When examining the function  $f : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ ,  $z \mapsto e^z$ , we find that the following happen:

- $f(\mathbb{R}) = (0, \infty)$  — we apply  $f(x) = e^x$ .
- $f(a + bi) = e^a e^{bi}$  —  $e^a$  is rotated by  $b$ .
- $f(\mathbb{R} + bi)$  is expressed as the line along  $b$  radians through the origin.
- Therefore,  $f(A_0) = \mathbb{C} \setminus \{0\}$ , where  $A_0 = \{a + bi \mid a \in \mathbb{R}, b \in [0, 2\pi)\}$ .

## Logarithm

Recall that for a function  $f : A \rightarrow B$ ,  $f^{-1}$  is a function if  $f$  is injective. However, for any  $f$ , it is the case that  $f^{-1}(b)$  does exist, defined as follows:

$$f^{-1}(b) = \{a \mid f(a) = b\}.$$

For the function  $f(z) = e^z$ ,  $f$  is not one to one, so for  $w = e^z$ ,  $f^{-1}(w) = \{z' \in \mathbb{C} \mid e^{z'} = w\}$ . We can find this as  $f^{-1}(w) = \{z + 2n\pi i \mid n \in \mathbb{Z}\}$ .

We define  $\log(w) := \{z \in \mathbb{C} \mid e^z = w\}$ . For a fixed  $\theta \in \mathbb{R}$ , we define  $\log_{A_\theta}(w) := \{z \mid e^z = w, z \in A_\theta\}$ .

Let  $z = 1 + \frac{5\pi}{2}i$ . Then,

$$\log_{A_{-\pi}} e^z = 1 + \frac{\pi}{2}i$$

Let  $w \in \mathbb{C} \setminus \{0\}$ . To find  $\log w$  (all values), then

$$\begin{aligned} z &\in \log w \\ e^z &= w \\ &= |w|e^{i \arg w} \\ e^{a+bi} &= |w|e^{i \arg w} \\ e^a e^{ib} &= |w|e^{i \arg w}. \end{aligned}$$

Therefore,  $a = \ln |w|$  and  $b = \arg w$ . Additionally, the following hold, for  $z_1, z_2 \in \mathbb{C}$ :

$$\log_{A_\theta}(z_1 z_2) = \log_{A_\theta}(z_1) + \log_{A_\theta}(z_2) + 2n\pi i$$

## Cosine and Sine

$$\begin{aligned} e^{ib} &= \cos b + i \sin b \\ e^{-ib} &= \cos b - i \sin b \\ \cos z &:= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &:= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

## Complex Powers

Recall that for  $s, t \in \mathbb{R}$ ,  $s^t = e^{t \ln s}$ , where  $s > 0$ . For  $z, w \in \mathbb{C}$ ,  $z^w = e^{w \log z}$ , where  $z \neq 0$ .

$$\begin{aligned} (-2)^i &= e^{i \log(-2)} \\ &= e^{i(\ln(2) + i\pi)} \\ &= e^{i \ln 2 - (\pi + 2\pi n)} \\ &= e^{-\pi + 2\pi n + i \ln 2} \end{aligned}$$

This has *infinitely* many values.

Let  $\alpha = u + vi$ . Then,

$$\begin{aligned} z^\alpha &= e^{\alpha \log z} \\ &= e^{(u+vi)(\ln |z| + i \arg z)} \\ &= e^{(u \ln |z| - v \arg z)} e^{i(v \ln |z| + u \arg z)} \end{aligned}$$

Since  $\arg z = \theta + 2\pi n$  for some real  $\theta \in [0, 2\pi)$ ,

$$= e^{u \ln z} e^{-v(\theta + 2\pi n)} e^{i v \ln |z|} e^{i u(\theta + 2\pi n)}$$

Therefore, complex exponentiation is single-valued if  $\alpha \in \mathbb{R}$ . If  $\alpha \in \mathbb{Z}$ , then  $z^\alpha$  has only one value; if  $\alpha \in \mathbb{Q}$ , where  $\alpha = \frac{p}{q}$  and  $\gcd(p, q) = 1$ , then  $z^\alpha$  takes  $q$  distinct values, which are the  $q$ th-roots.

## Continuous Functions with Complex Domains

Let  $z \in \mathbb{C}$ , let  $r > 0$ .

- The set  $D(z; r) := \{w \mid w \in \mathbb{C}, |z - w| < r\}$  is the  $r$ -neighborhood of  $z$ .
- A subset  $A \subseteq \mathbb{C}$  is open if  $(\forall z \in A) (\exists r > 0) \ni D(z; r) \subseteq A$ .

For example, if  $A = \{z \mid \operatorname{Re}(z) > 0\}$ , we can find  $r$  equal to half the magnitude of the real component of  $z$  for any  $z \in A$ , meaning  $A$  is open.

Meanwhile, if  $A = \{z \mid \operatorname{Re}(z) \geq 0\}$ , this is not the case. If  $z = 0$ , then  $\nexists r > 0$  such that  $D(z; r) \subseteq A$ , as any open ball of radius  $r$  will have some element in  $\overline{A}$ .

- A subset  $B \subseteq \mathbb{C}$  is closed if  $\overline{B} \subseteq \mathbb{C}$  is open.

For example,  $A = \emptyset$  is open, by vacuous truth, so  $\overline{A} = \mathbb{C}$  is closed. Similarly, since  $\mathbb{C}$  is open,  $\emptyset$  is closed.

Meanwhile,  $A = \{x + iy \mid -1 \leq x < 1\}$  is neither open nor closed.

## Limits

Let  $A \subseteq \mathbb{C}$ ,  $f : A \rightarrow \mathbb{C}$ ,  $z_0 \in \mathbb{C}$ . Then,

$$\lim_{z \rightarrow z_0} f(z) = \ell$$

means both of the following hold:

- for some  $r > 0$ ,  $D(z_0; r) \setminus \{z_0\} \subseteq \operatorname{dom}(f)$ ,
- $\forall \varepsilon > 0, \exists \delta > 0$  such that  $f(D(z_0; \delta) \setminus \{z_0\}) \subseteq D(\ell; \varepsilon)$ .

For example, if

$$f(z) = \begin{cases} z & z \in \mathbb{C} \setminus \mathbb{R} \\ 3i & z \in \mathbb{R} \end{cases}$$

Then,  $\lim_{z \rightarrow 0} f(z)$  does not exist, as there is no  $\ell$  that satisfies both conditions. Specifically, if  $\ell = 3i$ , and we set  $\varepsilon = 1$ , then a disc of any radius around 0 has some  $z \in \mathbb{C} \setminus \mathbb{R}$  that maps to itself. Similarly, if we set  $\ell = 0$ , then there is a real number in a disc of any radius around 0.

**Note:**  $f$  does not have to be defined at  $z_0$  for the limit to be defined at  $z_0$ .

Let  $A \subseteq \mathbb{C}$  be open,  $f : A \rightarrow \mathbb{C}$ , and  $z_0 \in A$ . We say  $f$  is continuous at  $z_0$  if  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ . We say  $f$  is continuous on  $A$  if  $\forall z_0 \in A$ ,  $f$  is continuous at  $z_0$ .

We will show that  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto 3z$  is continuous.

**Scratch Work:** We want  $\delta$  such that  $f(D(z_0; \delta)) \subseteq D(3z_0; \varepsilon)$ . Let  $z \in D(z_0; \delta)$ , meaning  $f(z) = 3z$ . We want  $3z \in D(3z_0; \varepsilon)$ , meaning we want  $|3z - 3z_0| < \varepsilon$ , or  $|z - z_0| < \frac{\varepsilon}{3}$ .

**Proof:** Let  $\varepsilon > 0$ . Set  $\delta = \frac{\varepsilon}{3}$ . We show  $f(D(z_0; \delta)) \subseteq D(f(z_0); \varepsilon)$ . Let  $z \in D(z_0; \delta)$ . Then,  $|z - z_0| < \delta = \varepsilon/3$ , meaning  $3|z - z_0| < \varepsilon$ , meaning  $|3z - 3z_0| < \varepsilon$ , so  $|f(z) - f(z_0)| < \varepsilon$ . Therefore,  $f(z) \in D(f(z_0); \varepsilon)$ . Since  $f$  is continuous at arbitrary  $z_0$ ,  $f$  is continuous on  $\mathbb{C}$ .

## Sequences

A sequence  $z_1, z_2, \dots \in \mathbb{C}$ . A sequence converges to  $z_0 \in \mathbb{C}$  if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M}, |z_n - z_0| < \varepsilon$$

In words, for any radius around  $z_0$ , we can find  $z_n$  arbitrarily close to  $z_0$  for sufficiently large  $n$ . We write  $z_n \rightarrow z_0$  if this is the case.

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ . Then,  $f$  is continuous on  $\mathbb{C}$  if and only if the following equivalent conditions are met:

- (i) the inverse image of every open set is open ( $f^{-1}(B) := \{a \in \mathbb{C} \mid f(a) \in B\}$ );
- (ii) the inverse image of every closed set is closed;
- (iii) for every sequence  $(z_n)_n$  such that  $(z_n)_n \rightarrow z_0$ ,  $f(z_n) \rightarrow f(z_0)$ .

Let

$$f(z) = \begin{cases} 0 & z = 0 \\ 1 & z \neq 0 \end{cases}.$$

This function is not continuous. We will check that (i)–(iii) fail.

- (i) Let  $B = D(0; 1)$ . Then,  $f^{-1}(B) = \{0\}$ , which is not open set.
- (ii) Let  $B = \text{cl}(D(1; 0.5))$ . Then,  $f^{-1}(B) = \mathbb{C} \setminus \{0\}$ , which is not closed.
- (iii) Let  $z_n = \frac{1}{n}$ . Then,  $(z_n)_n \rightarrow 0$ , but  $f(z_n) = 1$  for all  $n$ , meaning  $f(z_n) \rightarrow 1 \neq f(0)$ .

To show limit divergence, recall the definition of limit convergence:

$$\lim_{n \rightarrow \infty} z_n = z_0 \Leftrightarrow (\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M}, |z_n - z_0| < \varepsilon.$$

Let  $z_1, \dots \in \mathbb{C}$  be a sequence. Then,  $\lim_{n \rightarrow \infty} z_n = \infty$  means

$$(\forall M > 0)(\exists N \in \mathbb{N}) \ni \forall n > N, |z_n| > M.$$

In words,  $|z_n|$  is arbitrarily large for sufficiently large  $n$ .

## Connected Sets

Let  $a, b \in \mathbb{C}$ . A path from  $a$  to  $b$  is a continuous function  $p : [0, 1] \rightarrow \mathbb{C}$  such that  $p(0) = a$  and  $p(1) = b$ . Let  $S \subseteq \mathbb{C}$ . If  $p([0, 1]) \subseteq S$ , then  $p$  is a path in  $S$ .

We say  $S$  is *path-connected* if for any  $s, t \in S$ , there is a path in  $S$  from  $s$  to  $t$ .

Every set that is path-connected is connected, but not necessarily the other way around — if  $A$  is open and path connected, then  $A$  is connected.

An open, path-connected subset of  $\mathbb{C}$  is known as a region, or a domain.

Let  $A = \mathbb{R} \times \{0\}$  (or the  $x$  axis in  $\mathbb{C}$ ).  $A$  is not a region, as  $A$  is not an open set, even if  $A$  is path-connected.

$A \subseteq \mathbb{C}$  is bounded if there exists  $r > 0$  such that  $A \subseteq D(0; r)$ .  $A = \mathbb{R} \times \{0\}$  is not bounded.

If  $A \subseteq \mathbb{C}$ , then  $A$  is compact if  $A$  is closed and bounded. There are various properties of compact sets that make them particularly amenable towards analysis.

**Extreme Value Theorem:** Every real-valued continuous function on a compact domain attains its maximum and minimum values.

**Uniform Continuity Theorem:** Elaborated below.

## Uniform Continuity

Recall that if  $f : A \rightarrow \mathbb{C}$ ,  $f$  is continuous if  $\forall a \in A, \lim_{z \rightarrow a} f(z) = f(a)$ .

$$(\forall a \in A)(\forall \epsilon > 0)(\exists \delta_a > 0) \ni f(D(a; \delta_a)) \subseteq D(f(a); \epsilon) \quad \delta \text{ depends on } a$$

When  $f$  is uniformly continuous, there is one value of  $\delta$ , dependent on  $\epsilon$ , that applies for every value of  $a$ .

$$(\forall \epsilon > 0)(\exists \delta_\epsilon > 0) \ni (\forall a \in A), f(D(a; \delta_\epsilon)) \subseteq D(f(a); \epsilon)$$

## Riemann Sphere

Let  $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ . Let  $N = (0, 0, 1)$  denote the north pole. Then, there is a continuous bijection from  $S^2 \setminus \{N\} \rightarrow \mathbb{C}$ .

We can visualize this by picking a random point on the sphere and drawing a line from the north pole through the sphere to this point, and finding the point that intersects the plane.

Consider the sequence  $z_n = n^2 i$  for  $n = 1, 2, \dots$ . We can see that, on the projection from  $z_n$  to the sphere, all the values of  $p$  converge to  $N$ . Therefore, we write  $\lim_{n \rightarrow \infty} z_n = \infty$ , where  $\infty$  corresponds to  $N$  on  $S^2$ .

We can define  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  to be the complex plane that includes the “point at infinity” (from the projection on  $S^2$  that corresponds to the north pole).

## Analytic Functions

Let  $f : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$  where  $A$  is open. Let  $z_0 \in A$ . We say  $f$  is differentiable at  $z_0$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

## Rules of Differentiation

- $(f + g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$
- $(f \circ g)' = g'(f' \circ g)$
- For  $n \in \mathbb{Z}$ ,  $(z^n)' = nz^{n-1}$

Let  $f(z) = \bar{z}$ . We will find this value by directly applying the definition of the derivative.

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{\overline{z - z_0}}{z - z_0}. \end{aligned}$$

Let's approach  $z_0$  from the horizontal direction. Suppose  $z = z_0 + t$  for some  $t \in \mathbb{R}$ . Then,

$$\lim_{z \rightarrow z_0} \frac{\overline{z_0 + t} - \bar{z}_0}{z_0 + t - z_0} = 1.$$

Let's approach  $z_0$  from the horizontal direction. Suppose  $z = z_0 + ti$  for some  $t \in \mathbb{R}$ . Then,

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{\overline{z_0 + ti} - \bar{z}_0}{z_0 + ti - z_0} &= \frac{-ti}{ti} \\ &= -1. \end{aligned}$$

Since  $1 \neq -1$ , we find that the limit does not exist.

We see that complex-differentiability is a strong condition.

Suppose that  $f'(z_0) = 2i$ , meaning

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = 2i.$$

If  $z$  is close to  $z_0$ , then  $f(z) - f(z_0) \approx 2i(z - z_0)$ . Pictorially, we can visualize this as, for  $z_0$  sufficiently close to  $z$ , the vector  $z_0 - z$  is akin to a counterclockwise rotation and a scaling by 2. This is applicable for *all*  $z$  in sufficient proximity to  $z_0$ .

Specifically, we can see that the complex differentiable function is *angle-preserving*. The technical name for  $f$  is that  $f$  is *conformal*.

## Analytic Function

Let  $f : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$ . If  $f$  is differentiable at every  $z_0 \in A$ , we say  $f$  is *analytic* on  $A$ .

If  $f$  is analytic on  $A$ , then  $f$  is infinitely differentiable on  $A$ .

If  $f$  is analytic on  $A$  and  $f'(z_0) \neq 0$  for some  $z_0 \in A$ , then  $f$  is conformal at  $z_0 \in A$ .



## Cauchy-Riemann Theorem

Given a function  $f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Recall that we can take partial derivatives,  $\frac{\partial f}{\partial x}$ , and directional derivative  $\frac{\partial f}{\partial u}$  for some unit vector  $u$ .

However, for  $\mathbb{C}$ , there is only one derivative,  $f'(z_0)$ , meaning that regardless of direction,  $f'(z_0)$  exists and has one value. We can contextualize  $f(z) = f(x + yi) = u(x, y) + iv(x, y)$ , where  $u(x, y) \in \mathbb{R}$  and  $v(x, y) \in \mathbb{R}$ . Then,

$$\frac{\partial u}{\partial x} \neq \frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial x} \neq \frac{\partial v}{\partial y}$$

but

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

We can see this by first letting  $z = z_0 + \delta x$ .

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(z_0 + \delta x) - f(z_0)}{z_0 + \delta x - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{u(x_0 + \delta x, y_0) + iv(x_0 + \delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

and in the  $y$  direction,

$$\begin{aligned} f'(z_0) &= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

We set these two values equal to find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned}$$

which are the Cauchy-Riemann equations. The corresponding theorem states that if  $f'(z_0)$  exists, then the Cauchy-Riemann equations must hold.

For example, if  $f(z) = \bar{z}$ , with  $f(x + yi) = x - yi$ , we have  $u(x, y) = x$  and  $v(x, y) = -y$ . Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial v}{\partial x} &= -1, \end{aligned}$$

meaning  $f$  is not complex-differentiable.

If  $f : A \rightarrow \mathbb{C}$  satisfies the Cauchy-Riemann equations at every  $z_0 \in A$ , then  $f$  is analytic on  $A$ .

If  $f : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$  is analytic on  $A$ , then we know  $f'$  and  $f''$  are continuous. From multivariable calculus, we know that  $u_{xy} = u_{yx}$  if both are continuous. So,

$$\begin{aligned} u_{xy} &= \frac{\partial}{\partial y}(u_x) \\ &= \frac{\partial}{\partial y}(v_y) \\ &= v_{yy} \\ u_{yx} &= \frac{\partial}{\partial x}(u_y) \\ &= \frac{\partial}{\partial x}(-v_x) \\ &= -v_{xx} \end{aligned}$$

Therefore,  $v_{xx} + v_{yy} = 0$ . Similarly,  $u_{xx} + u_{yy} = 0$ .

If  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  If  $\varphi_{xx} + \varphi_{yy} = 0$ , then we say  $\varphi$  is a harmonic function. Therefore, if  $f$  is an analytic function, then both the real and imaginary parts of  $f$  are harmonic.

Let  $A \subseteq \mathbb{R}^2$ . If  $u : A \rightarrow \mathbb{R}$  and  $v : A \rightarrow \mathbb{R}$ . Then,  $u$  and  $v$  are harmonic conjugates if  $u + iv$  is an analytic function. Additionally,  $u$  and  $v$  are harmonic conjugates if and only if they satisfy the Cauchy-Riemann equations.

We may ask if there exists an analytic function  $f$  such that  $\operatorname{Re}(f) = x^3 - 3xy^2 + y$ . Then,

$$\begin{aligned} v_y &= u_x = 3x^2 - 3y^2 \\ -v_x &= u_y = 1 - 6xy. \end{aligned}$$

Therefore, we find  $v = -x + 3x^2y - y^3 + c$  through integration. Therefore, we have

$$\begin{aligned} f(z) &= (x^3 - 3xy^2 + y) + i(3x^2y - y^3 - x + c) \\ &= (x - iy)^3 + y - ix + ic \\ &= z^3 + i(-iy + x) + ic \\ &= \bar{z}^3 + i(\bar{z} + c) \end{aligned}$$

Recall from from multivariable calculus that  $\nabla u \perp$  contour lines of  $u$ . Similarly,  $\nabla v \perp$  contour lines of  $v$ . Then, using the Cauchy-Riemann equations, we find

$$\begin{aligned} \nabla u \cdot \nabla v &= (-u_x u_y) + u_x u_y \\ &= 0, \end{aligned}$$

meaning the gradients are orthogonal to each other, meaning the contours of  $u$  are perpendicular to the contours of  $v$ .

## Inverse Functions

Let  $f : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$ . Let  $z_0 \in A$ . If  $f$  is analytic on  $A$  and  $f'(z_0) \neq 0$ , then  $f$  is one to one on some neighborhood of  $z_0$ . Then,  $f^{-1} : f(N) \rightarrow N$  is analytic on  $f(N)$ , and

$$(f^{-1})'(f(z_0)) = \frac{1}{f'(z_0)}.$$

## Derivatives of Elementary Functions

Specifically, we will be working with complex exponentiation, complex trigonometric functions, and complex logarithms.

## Complex Exponential

$$\frac{d}{dz} e^z = e^z,$$

since, letting  $z = x + iy$ ,

$$\begin{aligned} e^z &= e^x e^{iy} \\ &= e^x (\cos(y) + i \sin(y)). \\ \frac{d}{dz} e^z &= \frac{\partial}{\partial x} e^z && \text{treating } y \text{ as constant} \\ &= e^x (\cos(y) + i \sin(y)) \\ &= e^{x+iy} \\ &= e^z. \end{aligned}$$

We know that  $e^z$  is continuous on  $\mathbb{C}$ , but this doesn't imply differentiability at every  $z_0 \in \mathbb{C}$ . We can verify by checking the Cauchy-Riemann equations, where  $u(x, y) = e^x \cos(y)$  and  $v(x, y) = e^x \sin(y)$ . Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos(y) \\ &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial y} &= -e^x \sin(y) \\ &= -\frac{\partial v}{\partial x}. \end{aligned}$$

If a function is analytic on  $\mathbb{C}$ , then  $f$  is known as entire.

## Complex Logarithm

We might ask where  $\log z$  is analytic. Let  $f(z) = e^z$ . Then,  $\log z = f^{-1}(z)$ ; since  $f$  is not one to one, we restrict the domain of  $f$  to  $A_\theta = \{z \mid \operatorname{Im}(z) \in [\theta, \theta + 2\pi)\}$  for any  $\theta$ .

Since  $f|_{A_\theta}$  is one to one, then

$$\left(f|_{A_\theta}\right)^{-1} = \log_{A_\theta}.$$

Fixing  $\theta$ , set  $g = f|_{A_\theta}$ . Then,

$$g^{-1}(g(z)) = z.$$

Because  $g$  is analytic on  $A_\theta$ ,  $g^{-1}$  is analytic on  $A_\theta$ . By chain rule, we have

$$\begin{aligned} \frac{d}{dz} (g^{-1}(g(z))) &= \frac{d}{dz} z \\ g^{-1'}(g(z)) &= \frac{1}{g'(z)} && g'(z) \neq 0 \\ g^{-1}(w) &= \frac{1}{g'(z)} && w = e^z \\ &= \frac{1}{e^z} \\ &= \frac{1}{w}. \end{aligned}$$

Therefore,  $\frac{d}{dw} \log_{A_\theta}(z) = \frac{1}{z}$ . Therefore,  $\text{dom}(\log_{A_\theta}) = \text{ran}(e_{A_\theta}^z) = \mathbb{C} \setminus \{0\}$ . However,  $\log_{A_0}$  (setting  $\theta = 0$ ) is not even continuous on  $\mathbb{C} \setminus \{0\}$ !

Specifically, at  $z = 0$ ,  $e^z = 1$ . Travelling around the unit circle counterclockwise in the image, we see that the preimage of these points travels along the imaginary axis. Approaching 1 “from the bottom,” we find that the preimage of the points approaches  $2\pi$  in the domain. However, they ought to be approaching 0. Therefore, the limit doesn't exist.

However, notice that the domain is not open! To fix this, we will let  $B_\theta = \{z \in \mathbb{C} \mid \text{Im}(z) \in (\theta, \theta + 2\pi)\}$ .

Our log function *is* when  $e^z$  is restricted to  $B_\theta$ . Then,  $\log_{B_\theta}$  is analytic on  $\mathbb{C} \setminus \{re^{i\theta} \mid r \geq 0\}$ . When  $\theta = -\pi$ , then  $\log_{B_{-\pi}}$  is the principle branch of  $\log z$ .

Then, the domain is  $\mathbb{C} \setminus \{z \mid z = x + 0i, x < 0\}$  and the range is  $B_{-\pi}$ .

## Powers

Let  $\alpha \in \mathbb{C}$ . We might ask

$$\frac{d}{dz} \alpha^z$$

$$\frac{d}{dz} z^\alpha.$$

Recall that  $a^b = e^{b \log a}$ . Specifically,  $a^b = e^{b(\ln|a| + i \arg a)}$ .

$$\frac{d}{dz} \alpha^z = \frac{d}{dz} e^{z \log \alpha}$$

Fix  $\theta$ . Then,

$$\begin{aligned} &= \frac{d}{dz} e^{z \log_{A_\theta} \alpha} \\ &= \log_{A_\theta} \alpha e^{z \log_{A_\theta} \alpha} && \text{assuming analytic domain} \\ &= \alpha^z \log_{A_\theta} \alpha. \end{aligned}$$

Specifically, as long as  $\alpha \notin \{re^{i\theta} \mid r \geq 0\}$ ,  $z \log_{A_\theta} \alpha$  is analytic, meaning  $e^{z \log_{A_\theta} \alpha}$  is analytic (composition of analytic functions).

$$\begin{aligned} z^\alpha &= e^{\alpha \log z} \\ &= e^{\alpha \log_{B_\theta} z} \\ &= e^{\alpha \log_{B_\theta} z} \frac{\alpha}{z} \\ &= \alpha z^{\alpha-1}. \end{aligned}$$

Specifically, this holds for  $z \notin \{re^{i\theta} \mid r \geq 0\}$ .

We know that  $\frac{d}{dz} \log_{B_{-\pi}}(z) = \frac{1}{z}$ . The domain of  $\log_{B_{-\pi}}$  is  $\mathbb{C} \setminus (-\infty, 0]$ .

## Contour Integrals

Recall from multivariable that  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is called a curve.

For example,  $\gamma : [0, \pi] \rightarrow \mathbb{R}^2$ , defined as  $\gamma(\theta) = (\cos \theta, \sin \theta)$ . The image of the given curve is a half circle.

We want to have  $\gamma$  be continuous and differentiable. Then,

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$$

is continuous/differentiable if and only if every  $\gamma_i$  is continuous/differentiable.

$$\gamma'(t) = (\gamma'_1(t), \dots, \gamma'_n(t))$$

If  $\gamma'$  is continuous, we say  $\gamma$  is smooth. For us,  $\gamma \in C^1$  is enough,  $\gamma \in C^\infty$  is not necessary.

For  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define

$$\int_{\gamma} f := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

as the line integral of  $f$  over  $\gamma$ .

Let  $f : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$  for  $A$  open, where  $\gamma : [a, b] \rightarrow A$ . Then,

$$\begin{aligned} \int_{\gamma} f &:= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(z_k) \Delta z \end{aligned}$$

Rather than the dot product, we use complex multiplication.

To define  $\gamma'(t)$ , we can imagine it as

$$\begin{aligned} \gamma(t) &= \gamma_1(t) + i\gamma_2(t) \\ \gamma'(t_0) &= \lim_{t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \\ &= \gamma'_1(t_0) + i\gamma'_2(t_0). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\gamma} f &= \int_{\gamma} \underbrace{f(\gamma(t)) \gamma'(t)}_{u(t) + iv(t)} dt \\ &= \int_a^b u(t) dt + i \int_a^b v(t) dt \end{aligned}$$

Let  $\gamma$  be the line from  $i$  to  $2$ , and  $f$  as  $\text{Im}(z)$ . Find  $\int_{\gamma} f$ .

To solve, we need a formula for  $\gamma : [0, 1] \rightarrow \mathbb{C}$ . We can consider  $\gamma(t) = i(1-t) + 2t$ . For any straight line, we can define  $\gamma : [0, 1] \rightarrow \mathbb{C}$  as  $\gamma(t) = p(1-t) + qt$ , or  $p + t(q-p)$ .

So,

$$\begin{aligned}
 \int_{\gamma} f &= \int_0^1 f(\gamma(t))\gamma'(t)dt \\
 &= \int_0^1 \operatorname{Im}(2t + i(1-t))(2-i)dt \\
 &= (2-i) \int_0^1 (1-t)dt \\
 &= (2-i) \left( t - \frac{t^2}{2} \right) \Big|_0^1 \\
 &= \frac{1}{2} (2-i)
 \end{aligned}$$

We could also have  $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{C}$ ,  $\tilde{\gamma}(t) = 2t^2 + i(1-t^2)$ . The image of  $\tilde{\gamma}$  is the same as the image of  $\gamma$ , and (not coincidentally), so is its line integral.

### Theorem: Reparametrization

Let  $f : A \rightarrow \mathbb{C}$  be analytic,  $\gamma : [a, b] \rightarrow A$  and  $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow A$  smooth curves such that  $\tilde{\gamma}$  is a reparametrization of  $\gamma$ . Then,

$$\int_{\gamma} f = \int_{\tilde{\gamma}} f.$$

If  $\gamma : [a, b] \rightarrow A$ , then  $\tilde{\gamma}[\tilde{a}, \tilde{b}] \rightarrow A$  is a reparametrization if  $\exists r : [a, b] \rightarrow [\tilde{a}, \tilde{b}]$  such that  $r(a) = \tilde{a}$  and  $r(b) = \tilde{b}$ , and  $\tilde{\gamma} \circ r = \gamma$ .

For a quick proof, we look at

$$\begin{aligned}
 \int_{\gamma} f &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
 &= \int_a^b f(\tilde{\gamma} \circ r(t))(\tilde{\gamma} \circ r)'(t)dt \\
 &= \int_a^b f(\tilde{\gamma} \circ r(t))\tilde{\gamma}'(r(t))r'(t)dt
 \end{aligned}$$

$$u = r(t), du = r'(t)dt$$

$$\begin{aligned}
 &= \int_{r(a)}^{r(b)} f(\tilde{\gamma}(u))\tilde{\gamma}'(u)du \\
 &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(u))\tilde{\gamma}'(u)du
 \end{aligned}$$

### Cauchy's Theorem: A Generalization

**Note:** I was out of class the previous week so we jumped to this location

So far, we know that if  $\gamma$  is a simple closed curve and  $f$  is analytic on and inside  $\gamma$ , then  $\int_{\gamma} f = 0$ . However, the theorem is much stronger.

If  $\gamma$  is a closed curve, and  $f$  is analytic on  $A \subseteq \mathbb{C}$ , with  $\gamma$  contained in  $A$ , and  $\gamma$  is homotopic to a point in  $A$ , then  $\int_{\gamma} f = 0$ .

Let  $A \subseteq \mathbb{C}$ , with  $j = 0, 1$ , and  $\gamma_j : [0, 1] \rightarrow A$  closed curves. We say  $\gamma_0$  is homotopic in  $A$  to  $\gamma_1$  if there exists continuous  $H : [0, 1] \times [0, 1] \rightarrow A$  such that

- $H_t : [0, 1] \rightarrow A$  defined by  $x \mapsto H(x, t)$  is a closed curve
- $H_0 = \gamma_0$  and  $H_1 = \gamma_1$ .

If such  $H$  exists, we write  $\gamma_0 \sim \gamma_1$ .

For example, if  $\gamma_0(\theta) = e^{2\pi i\theta}$  and  $\gamma_3(\theta) = 3e^{2\pi i\theta}$ , we can show they are homotopic by using a linear homotopy:

$$H_t(\theta) = (1 - t)e^{2\pi i\theta} + t(3e^{2\pi i\theta}),$$

which is both continuous and satisfies the given requirements.

In general, for two arbitrary closed curves  $\gamma_0$  and  $\gamma_1$ , we can't go wrong by trying the linear homotopy  $H_t(\theta) := (1 - t)\gamma_0 + t\gamma_1$ .

If a closed curve  $\gamma$  is homotopic in  $A$  to a point in  $A$  (i.e., the curve is homotopic to a constant map), we say  $\gamma$  is null-homotopic.

A set in  $\mathbb{C}$  is simply connected if it is path-connected and every closed curve in the set is null-homotopic in the set. A set  $A \subseteq \mathbb{C}$  is convex if  $\forall z_0, z_1 \in A, t \in [0, 1], tz_1 + (1 - t)z_0 \in A$ .

Let  $f : A \rightarrow \mathbb{C}$ , where  $f$  is analytic on  $A$ . If  $\gamma_0$  and  $\gamma_1$  are curves in  $A$  such that  $\gamma_0 \sim \gamma_1$  in  $A$ , then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Consider  $\rho$ , a path connecting some point in  $\gamma_0$  to some point in  $\gamma_1$  (if they are closed), which exists by the homotopy. Then,  $\Gamma := \gamma_0 + \rho - \gamma_1 - \rho$  (where we traverse along  $\gamma_0$ , then  $\rho$ , then  $\gamma_1$ , then reverse  $\rho$ .) is null-homotopic. So, Cauchy's Theorem implies that

$$\begin{aligned} \int_{\Gamma} f &= \int_{\gamma_0} f + \int_{\rho} f - \int_{\gamma_1} f - \int_{\rho} f \\ &= 0 \\ \int_{\gamma_0} f &= \int_{\gamma_1} f. \end{aligned}$$

## Cauchy's Integral Formula

We know that

$$\int_{\gamma} f(z) dz = 0$$

occurs if one of these conditions is satisfied.

- If  $\gamma$  is a simple closed curve and  $f$  is analytic on and inside  $\gamma$ .
- If  $\gamma$  is homotopic in a region  $R$  to a point, where  $f$  is analytic on  $R$ .
- If  $f$  has an antiderivative in the region, and  $\gamma$  is a closed curve.
- If  $\gamma$  is closed and contained in a simply connected region  $R$  that  $f$  is analytic on.

We can also show that

$$\int_{\gamma} \frac{1}{z - z_0} dz = 2\pi i$$

where  $\gamma$  is a simple closed curve and  $z_0$  is contained within the region with boundary  $\gamma$ .

Let  $f$  be analytic on a simply connected open set  $D \subseteq \mathbb{C}$ . Then, for every piecewise smooth closed curve  $\gamma \in D$  and every point  $z_0 \in D \setminus \text{im}(\gamma)$ ,

$$\begin{aligned}\int_{\gamma} \frac{f(z)}{z - z_0} dz &= 2\pi i f(z_0) \\ f(z_0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz\end{aligned}$$

For every  $z_0$  inside  $\Gamma$ ,  $f(z_0)$  is determined by the values of  $f$  on  $\Gamma$ .

For an outline of the proof, consider  $C$ , a circle of radius  $\varepsilon > 0$  centered at  $z_0$ . Since  $\Gamma \sim C$  in  $D \setminus \{z_0\}$ , we know that

$$\int_{\Gamma} \frac{f(z)}{z - z_0} dz = \int_C \frac{f(z)}{z - z_0} dz$$

Therefore, on  $C$ ,  $f(z) \approx f(z_0)$  if  $\varepsilon$  is small. So,

$$\begin{aligned}&\approx f(z_0) \int_C \frac{1}{z - z_0} dz \\ &= 2\pi i f(z_0)\end{aligned}$$

For example, we can find

$$\begin{aligned}\int_{|z|=4} \frac{\cos(z)}{(z - \pi)(z - 5)} dz &= \int_{|z|=4} \left( \frac{\cos z}{z - 5} \right) \frac{1}{z - \pi} dz \\ &= 2\pi i \frac{\cos(\pi)}{\pi - 5} \\ &= \frac{2\pi i}{5 - \pi}\end{aligned}$$

Suppose  $f(z)$  is continuous on a contour  $\Gamma$  (not necessarily closed). Let

$$g(w) = \int_{\Gamma} \frac{f(z)}{z - w} dz.$$

Then,  $g$  is defined for all  $w \notin \text{im}(\Gamma)$ , and  $g$  is differentiable at every  $w \notin \text{im}(\Gamma)$ . In other words,  $g$  is analytic on  $\mathbb{C} \setminus \text{im}(\Gamma)$ . Additionally,  $g'$  is also analytic on  $\mathbb{C} \setminus \text{im}(\Gamma)$ , with

$$\begin{aligned}g'(w) &= \frac{d}{dw} \int_{\Gamma} \frac{f(z)}{z - w} dz \\ &= \int_{\Gamma} \frac{d}{dw} \frac{f(z)}{z - w} dz \\ &= \int_{\Gamma} \frac{f(z)}{(z - w)^2} dz\end{aligned}$$

This is what we use to prove that any complex-differentiable function is infinitely complex-differentiable.

If  $f$  is analytic on  $D$ , then  $f'$  is analytic on  $D$ . Since  $f$  is analytic, then

$$f(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - w} dz$$



where  $\Gamma$  is a circle centered at  $w$ . So,

$$\begin{aligned} f'(w) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-w)^2} dz \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{(f(z)/(z-w))}{z-w} dz \end{aligned}$$

The numerator  $\frac{f(z)}{z-w}$  is continuous on  $\Gamma$  because  $w \notin \Gamma$ , so by the previous theorem, the integral is analytic on  $D \setminus \text{im}(\Gamma)$ . Therefore,  $f'$  is differentiable at  $w$ , so  $f'$  is analytic on  $D$ .

If  $\Gamma$  is a simple closed curve,  $w$  is inside  $\Gamma$ , and  $f$  is analytic on  $D$  with  $\Gamma \subseteq D$ . Then,

$$\begin{aligned} f'(w) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-w)^2} dz \\ f''(w) &= \frac{2}{2\pi i} 2! \int_{\Gamma} \frac{f(z)}{(z-w)^3} dz \\ f^{(n)}(w) &= \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-w)^{n+1}} dz \end{aligned}$$

For example,

$$\int_{|z|=2} \frac{e^{-z}}{(z+1)^3} dz = e^{-2}\pi i$$

If  $f$  is continuous on a domain  $D$  and  $\int_{\Gamma} f = 0$  for every closed  $\Gamma$  in  $D$ , then  $f$  is analytic on  $D$ .

By the path independence theorem,  $f$  has an antiderivative  $F$  on  $D$ . So,  $F$  is analytic on  $D$  as  $F' = f$ . Thus,  $F^{(n)}$  is analytic for all  $n$ , so  $F'$  is analytic, meaning  $f$  is analytic. The converse does not hold.

Recall that  $\varphi(x, y)$  is harmonic on  $D$  if  $\varphi_{xx} + \varphi_{yy} = 0$ . If  $f(z) = u(x, y) + iv(x, y)$ , then  $f' = u_x + iv_x$ , or  $f' = v_y - iu_y$ . If  $f$  is analytic, then both  $u$  and  $v$  are harmonic. Similarly,  $u_x, v_x$  are harmonic, and  $u_y, v_y$  are harmonic (since the analyticity of  $f$  implies that  $f'$  is also analytic).

## Bounds for Analytic Functions and the Fundamental Theorem of Algebra

Liouville's Theorem: every non-constant entire function is unbounded.

Recall that

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-w)^{n+1}} dz.$$

Suppose that  $f$  is analytic on  $C_R(z_0) = \{z \mid |z - z_0| = R\}$  and  $f$  is bounded on  $C_R$ . Then,  $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$ .

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-w)^{n+1}} dz \right| \\ &= \frac{n!}{2\pi} \left| \int_{C_R} \frac{f(z)}{(z-w)^{n+1}} dz \right| \end{aligned}$$

given  $|f(z)| \leq M$ ,

$$\begin{aligned} \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| &= \frac{|f(z)|}{R^{n+1}} \\ &\leq \frac{M}{R^{n+1}} \end{aligned}$$

So

$$\begin{aligned} |f^{(n)}(z_0)| &\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R \\ &= \frac{n!M}{R^n} \end{aligned}$$

To show Liouville's Theorem, by the above result,  $|f'(z_0)| \leq \frac{M}{R}$ . Since  $f$  is entire and  $M$  is fixed, we can make  $R$  arbitrarily large. So,  $|f'(z_0)| = 0$ , with  $z_0$  arbitrary. Thus,  $f$  is constant.

## Fundamental Theorem of Algebra

Every non-constant polynomial has at least one root in the complex plane.

To prove this, suppose  $P(z) = a_n z^n + \cdots + a_1 z + a_0$  has no root. Then,  $\frac{1}{P(z)}$  is also entire. We have that

$$\begin{aligned} \lim_{|z| \rightarrow \infty} \left| \frac{1}{P(z)} \right| &= \lim_{|z| \rightarrow \infty} \left| \frac{1}{z^n(a_0/z^n + \cdots + a_n)} \right| \\ &= \lim_{|z| \rightarrow \infty} \frac{1}{|z^n|} \left| \frac{1}{a_0/z^n + \cdots + a_n} \right| \\ &\rightarrow 0. \end{aligned}$$

Therefore, there exists  $M$  such that for  $|z| > M$ ,  $|P(z)| < 1$ . Examining  $D_M := \{z \mid |z| \leq M\}$ . Since  $\left| \frac{1}{P(z)} \right|$  is a real-valued continuous function, it attains a maximum value  $A$  on  $D_M$  since  $D_M$  is compact. Thus,  $|1/P(z)| \leq \max\{1, A\}$  for all  $z \in \mathbb{C}$ . Thus,  $1/P(z)$  is bounded and entire, meaning  $P(z)$  is constant.  $\perp$

## Extrema of Non-Constant Analytic Functions

Let  $f$  be analytic on  $A \subseteq \mathbb{C}$  open. Then,  $|f|$  admits a local maximum at  $z_0 \in A$  only if  $f$  is constant.

$f$  has a local maximum at  $z_0$  if  $\exists \varepsilon > 0$  such that for all  $z \in D_\varepsilon(z_0) := \{z \mid |z - z_0| < \varepsilon\}$ ,  $|f(z)| \leq |f(z_0)|$ .

Maximum modulus principle: If  $f$  is analytic on a bounded domain  $D$  and continuous on  $\partial D$ , then  $f$  attains its maximum on  $\partial D$ .

For example, if  $f(z) = z^2 - 1$ , then to find the absolute extrema of  $f$  on  $D_2(0)$  (the closed disk of radius 2 about 0), we know that  $f$  attains its absolute extrema on the boundary of  $D_2(0)$ .

$$\begin{aligned} |f(z)| &= |z^2 - 1| \\ &\leq |z^2| + |1| = 5 \\ &\geq |z^2| - |1| = 3 \end{aligned}$$

We have that  $|f(2)| = 3$  and  $|f(2i)| = 5$ .

If  $f$  is a non-constant, non-zero analytic function on a bounded domain  $D$ ,  $f$  has no local minimum.

**Proof:** Let  $g(z) = \frac{1}{f(z)}$ . Since  $f(z)$  is non-zero on  $D$ , and  $f$  is analytic on  $D$ , so too is  $g$ . Therefore,  $|g|$  admits its maximum on  $\partial D$ . Since  $\max |g| = \min |f|$ ,  $|f|$  attains its minimum on  $\partial D$ .

To prove the maximum modulus principle, we use the following lemma:

**Lemma:** If  $f$  is analytic, and  $|f|$  is non-constant on a disk  $|z - z_0| < r$ , then  $|f(z_0)|$  is not maximal on  $D$ .

**Proof of Lemma:** Suppose toward contradiction that  $|f(z_0)|$  is the maximum of  $|f(z)|$ . By the hypothesis, there exists  $z_1 \in D$  with  $|f(z_1)| < |f(z_0)|$ . Let  $\Gamma$  be the circle  $|z - z_0| = |z_1 - z_0|$ . Since  $f$  is analytic,

$$2\pi i f(z_0) = \int_{\Gamma} \frac{f(z)}{z - z_0} dz.$$

On  $\Gamma$ ,  $|z - z_0| = |z_1 - z_0|$ , so

$$\begin{aligned} \left| \int_{\Gamma} \frac{f(z)}{z - z_0} dz \right| &\leq \int_{\Gamma} \left| \frac{f(z)}{z - z_0} \right| dz \\ &= \int_{\Gamma} \frac{|f(z)|}{|z - z_0|} dz \\ &= \frac{1}{|z_1 - z_0|} \int_{\Gamma} |f(z)| dz \\ &< \frac{1}{|z_1 - z_0|} \int_{\Gamma} |f(z_0)| dz \quad (*) \\ &= \frac{\ell(\Gamma) |f(z_0)|}{|z_1 - z_0|} \\ &= \frac{2\pi |z_1 - z_0| |f(z_0)|}{|z_1 - z_0|} \\ &= 2\pi |f(z_0)| \end{aligned}$$

(\*): There must exist  $\varepsilon > 0$  such that for  $|z - z_1| < \varepsilon$ ,  $|f(z)| < |f(z_0)|$ , since  $f$  is continuous and  $|f(z_1)| < |f(z_0)|$ . Let  $\Gamma_1 = \Gamma \cap D(z_1, \varepsilon)$ , and  $\Gamma_2 = \Gamma \setminus \Gamma_1$ . Then,

$$\begin{aligned} \left| \int_{\Gamma} \right| &= \left| \int_{\Gamma_1} + \int_{\Gamma_2} \right| \\ &\leq \left| \int_{\Gamma_1} \right| + \left| \int_{\Gamma_2} \right| \\ &< \left| \int_{\Gamma_1} |f(z_0)| \right| + \left| \int_{\Gamma_2} |f(z_0)| \right|. \end{aligned}$$

However, this means  $|f(z_0)| < |f(z_0)|$ , which is a contradiction.

Alternatively, if  $f'(z_0) \neq 0$ , then  $f$  approximately rotates and stretches or contracts a small disk around  $z_0$ . If we draw a line from 0 to  $f(z_0)$  through the disk, then there is some point in  $\text{im}(f)$  in the disk that has a larger modulus than  $f(z_0)$ .

## Winding Number

Recall the Cauchy Integral Formula: if  $f$  is analytic on a simply connected domain  $D$ , and  $\Gamma$  is a simple closed curve in  $D$ , with  $z_0$  inside  $\Gamma$ , then

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz.$$

There is a generalized version: if  $f$  is analytic on any domain  $D$ , and  $\Gamma$  is any closed curve that is null-homotopic in  $D$ . If  $z_0 \notin \Gamma$ , then

$$f(z_0) I(\Gamma, z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz,$$

where  $I(\Gamma, z_0)$  denotes the winding number of  $\Gamma$  about  $z_0$ .

We define

$$I(\Gamma, z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z - z_0} dz$$

for  $z_0 \notin \Gamma$ . We assert that  $I(\Gamma, z_0)$  is always an integer.

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz &= f(z_0) I(\Gamma, z_0) \\ \int_{\Gamma} \frac{f(z)}{z - z_0} dz &= \int_{\Gamma} \frac{f(z_0)}{z - z_0} dz \end{aligned}$$

## Series and Sequences

A sequence in  $\mathbb{C}$  is a function  $a : \mathbb{N} \rightarrow \mathbb{C}$ . We denote  $a(n) = a_n$ .

A sequence  $(a_n)_n$  converges to  $L \in \mathbb{C}$  if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - L| < \varepsilon$ . In other words,  $(a_n)_n$  converges to  $L$  if  $a_n$  is arbitrarily close to  $L$  for all sufficiently large  $n$ .

A series  $\sum_{n=1}^{\infty} a_n$  converges to some  $S$  if the sequence of partial sums converges to  $S$ , where  $s_n := \sum_{k=1}^n a_k$ .

## Tests for Convergence and Divergence

**Divergence Test:** In real numbers, if  $\lim_{n \rightarrow \infty} x_n \neq 0$ , then  $\sum x_n$  diverges.

Similarly, in complex numbers, if  $\lim_{n \rightarrow \infty} |a_n| \rightarrow 0$ , then  $\sum a_n$  diverges.

**Ratio Test:** Let

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If  $L < 1$ , then  $\sum a_n$  converges, and if  $L > 1$ , then  $\sum a_n$  diverges. If  $L = 1$ , then the test is inconclusive.

**Comparison Test:** Given  $\sum a_n$  and  $\sum b_n$  series. If  $|a_n| \leq |b_n|$  for sufficiently large  $n$ , and  $\sum b_n$  converges, then  $\sum |a_n|$  converges (so  $\sum a_n$  converges).

**Geometric Series:** If  $a_{n+1}/a_n = c$  for all  $n$ , then  $\sum a_n = \sum a_0 c^n$ , and we say  $a_n$  is a geometric series. If  $|c| < 1$ , then  $a_n$  converges, and if  $|c| > 1$ , then  $\sum a_n$  diverges.

The partial sums

$$\begin{aligned} s_n &= a_0 + \cdots + a_0 c^n \\ c s_n &= a_0 c + \cdots + a_0 c^{n+1} \\ s_n(1 - c) &= a_0 - a_0 c^{n+1} \\ s_n &= \frac{a_0 - a_0 c^{n+1}}{1 - c} \\ &= a_0 \frac{1 - c^{n+1}}{1 - c} \\ \lim_{n \rightarrow \infty} s_n &= a_0 \lim_{n \rightarrow \infty} \frac{1 - c^{n+1}}{1 - c} \\ &= a_0 \frac{1}{1 - c} \end{aligned}$$

since  $|c| < 1$

## Convergence of Functions

To find for which  $z \in \mathbb{C}$  does  $\sum \frac{1}{z^n}$  converge, we use the geometric series, meaning  $|\frac{1}{z}| < 1$ , meaning  $|z| > 1$  is necessary for the series to converge. When  $|z| > 1$ , the series converges to  $\frac{1}{1-(1/z)}$ .

Letting  $f_n(z) = s_n(z)$ , we have that  $f_n$  is itself a sequence of functions. Letting  $g(z) = \frac{1}{1-(1/z)}$ . Then, for each fixed  $z$  with  $|z| > 1$ , we see that  $\lim f_n(z) = g(z)$ . So, on the set  $|z| > 1$ , the sequence  $f_n$  converges *pointwise* to  $g$ .

Let  $(f_n)_n$  be a sequence of functions with  $f_n : A \rightarrow \mathbb{C}$ ,  $A \subseteq \mathbb{C}$ . We say  $(f_n)_n$  converges pointwise to  $g$  on  $A$  if  $\forall z \in A, \forall \epsilon > 0, \exists M \in \mathbb{N}$  such that for all  $n \geq M$ ,  $|f_n(z) - g(z)| < \epsilon$ .

We say  $f_n$  converges uniformly to  $g$  on  $A$  if  $\forall \epsilon > 0, \exists M$  such that for all  $z \in A$  and  $\forall n \geq M$ ,  $|f_n(z) - g(z)| < \epsilon$ .

Let  $f_n = \sum_{k=0}^n \frac{1}{z^k}$ . Does  $f_n$  converge to  $g(z) = \frac{1}{1-(1/z)}$  uniformly on  $|z| > 1$ ?

We want to show that for some  $\epsilon_0 > 0$ , there does not exist  $M$  such that  $\forall z \in A, \forall n > M, |f_n(z) - g(z)| < \epsilon_0$ . Let  $\epsilon_0 = 1$ . Fix  $M \in \mathbb{N}$ . We will show  $\exists z$  with  $|z| > 1$  such that for some  $n > M$ ,  $|f_n(z) - g(z)| \geq 1$ . We have

$$\begin{aligned} |f_n(z) - g(z)| &= \left| \frac{1 - \frac{1}{z^{n+1}}}{1 - \frac{1}{z}} - \frac{1}{1 - \frac{1}{z}} \right| \\ &= \frac{1}{z^{n+1} \left(1 - \frac{1}{z}\right)} \end{aligned}$$

Let  $z = 1 + \delta$  for  $\delta > 0$  sufficiently small. Then,

$$\geq 1$$

## Taylor Series

Recall from Calc II that for  $f : \mathbb{R} \rightarrow \mathbb{R}$ , a Taylor series for  $f$  centered at  $x_0$  is

$$T_{x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

If  $f$  is infinitely differentiable at  $x_0$ , we have that  $T_{x_0}(x)$  will converge to  $f$  in an interval of convergence about  $x_0$ . For a finite-degree polynomial, we have that

$$P_k(x) := \sum_{n=0}^k \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

approximates  $f$ . Specifically, we can see that  $P_k^{(j)}(x_0) = f^{(j)}(x_0)$  for  $j \leq k$ .

We say that  $f(z)$  is analytic on  $z_0$  if  $f(z)$  is analytic on  $D(z_0; \delta)$  for some  $\delta > 0$ . If  $f(z)$  is analytic at  $z_0$ , then the Taylor series for  $f(z)$  around  $z_0$  is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

If  $f(z)$  is analytic on an open disk  $D(z_0; r)$ , then the Taylor series for  $f(z)$  around  $z_0$  converges to  $f(z)$  on  $D(z_0; r)$ , and converges uniformly on  $D(z_0; r') \subset D(z_0; r)$ .

For example, if  $f(z) = (c - z)^{-1}$ , we can find a Taylor series for  $f$  about 0, and find the disk of convergence.

$$\begin{aligned} f'(z_0) &= (c - z)^{-2} \\ f''(z_0) &= 2(c - z)^{-3} \\ f^{(3)}(z_0) &= 3!(c - z)^{-4} \\ &\vdots \\ f^{(n)}(z_0) &= n!(c - z)^{-(n+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} T(f, z_0) &= \sum_{n=0}^{\infty} \frac{n!(c - z_0)^{-(n+1)}}{n!} (z - z_0)^n \\ &= \sum_{n=0}^{\infty} c^{-(n+1)} z^n. \end{aligned}$$

To find the radius of convergence, we find that  $f$  is analytic on  $\mathbb{C} \setminus \{c\}$ . Thus,  $T(f, z_0)$  is convergent about  $D(0; |c|)$ .

Considering  $f(z) = (c - z)^{-1}$  again, we find

$$\begin{aligned} f(z) &= \frac{1}{c} \frac{1}{1 - \frac{z}{c}} \\ &= \frac{1}{c} \sum_{n=0}^{\infty} c^{-n} z^n && \text{true iff } |z/c| < 1 \\ &= \sum_{n=0}^{\infty} c^{-(n+1)} z^n. \end{aligned}$$

To find a Taylor series for  $g(z) = (c - z)^{-2}$ , we have that  $g(z) = f'(z)$ , so we can take the Taylor series for  $f$  and differentiate it. Since  $f$  is analytic on  $|z| < |c|$ , and  $g$  is equal to  $f'$ , we have that  $g$  is convergent on the same disk that  $f$  is convergent on.

If  $f$  is analytic at  $z_0$ , and  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$  on some disk  $D(z_0; r)$ , then  $f'(z) = \sum_{n=1}^{\infty} c_n n(z - z_0)^{n-1}$ , and this series converges on  $D(z_0; r)$ . We can also do integration term-by-term.

For example, to find the Taylor series for  $f(z) = \text{Log}(z)$  around  $z_0 \in \mathbb{C} \setminus (\infty, 0]$ , we take integrals term-by-term on  $g(z) = 1/z$ .

$$\begin{aligned} g(z) &= \frac{1}{z} \\ &= \frac{1}{z_0 - (z_0 - z)} \\ &= \frac{1}{z_0} \frac{1}{1 - \left(1 - \frac{z}{z_0}\right)} \\ &= \frac{1}{z_0} \sum_{n=0}^{\infty} \left(1 - \frac{z}{z_0}\right)^n. \end{aligned}$$

We have that the series converges if  $|1 - z/z_0| < 1$ .

$$\begin{aligned} &= \frac{1}{z_0} \sum_{n=0}^{\infty} \left( \frac{z_0 - z}{z_0} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n \end{aligned}$$

so,

$$\begin{aligned} f(z) &= \int f'(z) dz \\ &= \int \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n dz \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} \int (z - z_0)^n dz \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} \frac{(z - z_0)^{n+1}}{n+1} + C \end{aligned}$$

Specifically,  $C = \text{Log}(z_0)$ . Thus,

$$f(z) = \text{Log}(z_0) + \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} \frac{(z - z_0)^{n+1}}{n+1}.$$

We have that the radius of convergence in  $\mathbb{C}$  is equal to  $\text{dist}_{(-\infty, 0]}(z_0) = |\text{Im}(z_0)|$ .

For  $f$  and  $g$  with respective Taylor series, we can find their sum relatively easily (coefficient-wise addition), but for  $fg$ , we require convolution.

$$\begin{aligned} (a_0 + a_1z + a_2z^2 + \cdots) (b_0 + b_1z + b_2z^2 + \cdots) &= a_0b_0 + (a_0b_1 + a_1b_0)z + \cdots \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) z^n \end{aligned}$$

## Power Series

Recall that the Taylor series for  $f(z)$  about  $z_0$  is

$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j \\ &= \sum_{j=0}^{\infty} c_j (z - z_0)^j \\ c_j &= \frac{f^{(j)}(z_0)}{j!}. \end{aligned}$$

Suppose instead that we start with the sequence  $(c_n)_n$ . For example, let  $c_j = \frac{j^2+1+i}{(2j)^j}$ . We may ask if  $\sum c_j(z-z_0)^j$  is convergent (and thus the Taylor series for some analytic function about  $z_0$ ).

If  $\sum c_j(z - z_0)^j$  converges for some  $z \neq z_0$ , then it indeed is. A series of the form  $f(z) = \sum c_j(z - z_0)^j$  is known as a power series. However, the function that the power series converges to may not be an elementary

function.

For every power series  $\sum c_j(z - z_0)^j$ , there exists a single value  $R \in [0, \infty]$  such that the power series converges on  $|z - z_0| < R$  and diverges on  $|z - z_0| \geq R$ . For every  $r < R$ , the power series converges *uniformly* on  $|z - z_0| < r$ . If  $R$  is finite,  $|z - z_0| = R$  is called the circle of convergence. The power series may converge at some, all, or no points on  $|z - z_0| = R$ .  $R$  is known as the radius of convergence.

Let  $c_k = \frac{k^2+1+i}{(2i)^k}$ , with  $\sum c_k(z - 5i)^k$  the series we must find the radius of convergence for. Using the ratio test, we find

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{\frac{(k+1)^2+1+i}{(2i)^{k+1}}(z - 5i)^{k+1}}{\frac{k^2+1+i}{(2i)^k}(z - 5i)^k} \right| &= \lim_{k \rightarrow \infty} \left| (z - 5i) \frac{(k+1)^2+1+i}{(k^2+1+i)(2i)} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{(z - 5i)}{2i} \right| \\ &= \frac{|z - 5i|}{2}. \end{aligned}$$

If  $\frac{|z-5i|}{2} < 1$ , then the power series converges, so we have  $R = 2$ .

We care about the uniform convergence of the power series since if  $(f_n)_n$  is a sequence of continuous functions that converges uniformly to  $f$  on  $D$ , then  $f$  is continuous on  $D$ . If  $(f_n)_n$  are analytic under the same condition, then  $f$  is analytic.

Notice that  $f_n(z) = \sum_{j=0}^n c_j(z - z_0)^j$  is a polynomial. Since the  $(f_n)_n$  are analytic, if it is the case that the power series converges uniformly on  $D$ , then  $f(z)$  is analytic.

- (i) Every power series with nonzero radius of convergence is an analytic function inside its circle of convergence.
- (ii) The Taylor series for the function  $\sum_{j=0}^{\infty} c_j(z - z_0)^j$  is itself.

For example, let  $g(z) = \sum_{k=0}^{\infty} \frac{k^2+1+i}{(2i)^k}(z - 5i)^k$ . Then,  $g$  is analytic on  $|z - 5i| < 2$ , and  $g^{(9)}(5i) = \left(\frac{82+i}{(2i)^9}\right)(9!)$

To prove (ii), consider  $\sum_{j=0}^{\infty} a_j(z - z_0)^j = \sum_{j=0}^{\infty} b_j(z - z_0)^j$ . We then ask if  $a_j = b_j$  for all  $j$ . The constant term of the  $n$ th-derivative of the left-hand side is  $a_n n!$ , and the constant term of the  $n$ th derivative of the right-hand side is  $b_n n!$ . Plugging in  $z_0$  to the respective  $n$ th derivatives, we find that  $a_n = b_n$ .

To prove that a sequence of continuous functions  $(f_n)_n \rightarrow f$  uniformly to a continuous function  $f$ , we pick  $z_0 \in D$  to show that  $f$  is continuous at  $z_0$ .

Let  $\varepsilon > 0$ . We want to show that there exists  $\delta > 0$  such that  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ .

Since  $(f_n)_n \rightarrow f$  uniformly on  $D$ , there exists  $M$  such that  $\forall n \geq M$  and  $\forall z \in D$ ,  $|f_n(z) - f(z)| < \varepsilon$ . Since  $f_M$  is continuous, we have that  $\exists \delta > 0$  such that  $|f_M(z) - f_M(z_0)| < \varepsilon$  for  $|z - z_0| < \delta$ . Then,

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_M(z) + f_M(z) - f_M(z_0) + f_M(z_0) - f(z_0)| \\ &\leq |f(z) - f_M(z)| + |f_M(z) - f_M(z_0)| + |f_M(z_0) - f(z_0)| \\ &< 3\varepsilon \end{aligned}$$

## Laurent Series

Suppose  $\sum_{j=1}^{\infty} a_j(z - z_0)^j$  converges on  $|z - z_0| < R_1$ . Then,  $\sum_{j=1}^{\infty} a_j w^j$ , where  $w = z - z_0$  converges on  $|w| < R_1$ . Then,  $\sum_{j=1}^{\infty} a_j \left(\frac{1}{z - z_0}\right)^j$  converges where  $\left|\frac{1}{z - z_0}\right| < R_1$ , so it converges with  $|z - z_0| > \frac{1}{R_1}$ .



We write it as  $\sum_{j=1}^{\infty} a_j(z-z_0)^{-j}$ . Let  $c_{-1} = a_1$ ,  $c_{-2} = a_2$ , etc.; then, we write the series as  $\sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j}$ . Suppose also that  $\sum_{j=0}^{\infty} b_j(z-z_0)^j$  converges on  $|z-z_0| < R_2$  such that  $\frac{1}{R_1} < R_2$ . Then, both series converge on the annulus defined by  $\frac{1}{R_1} < |z-z_0| < R_2$ . Let  $c_j = b_j$  for  $j \geq 0$ . Then,

$$\sum_{j=-\infty}^{\infty} c_j(z-z_0)^j = \sum_{j=0}^{\infty} b_j(z-z_0)^j + \sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j}$$

converges on the annulus.

Suppose  $f$  is analytic on the annulus  $r < |z-z_0| < R$ , with  $r, R \in [0, \infty]$ . Then, for all  $z$  in the annulus, the series  $\sum_{j=-\infty}^{\infty} c_j(z-z_0)^j$  converges to  $f(z)$ , where  $c_j$  is given by

$$c_j = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{j+1}} dz,$$

where  $\gamma$  is any counterclockwise simple closed curve in the annulus.

When  $j$  is positive, we have that

$$\begin{aligned} c_j &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{j+1}} dz \\ &= \frac{1}{j!} f^{(j)}(z_0). \end{aligned}$$

To find the Laurent series for  $f(z) = \frac{e^z}{z-i}$  in  $\mathbb{C} \setminus \{i\}$ , we do the following.

We need  $\sum_{j=-\infty}^{\infty} c_j(z-i)^j$ . We can write  $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$  about 0.

$$\begin{aligned} e^z &= e^{z-i+i} \\ &= e^i e^{z-i} \\ &= e^i \sum_{j=0}^{\infty} \frac{(z-i)^j}{j!}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{e^z}{z-i} &= \frac{1}{z-i} e^i \sum_{j=0}^{\infty} \frac{(z-i)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{e^i}{j!} (z-i)^{j-1}, \end{aligned}$$

meaning it converges on  $0 < |z-i| < \infty$ .

To try to find the Laurent series for  $\frac{1}{z^2(z-i)}$ , we may consider on different annuli. For  $0 < |z| < 1$ , we first have to find the Laurent series for  $\frac{1}{z-i}$ .

$$\begin{aligned} \frac{1}{z-i} \cdot \frac{i}{i} &= \frac{i}{1-(iz)} \\ &= \sum_{j=0}^{\infty} i^{j+1} z^j. \end{aligned}$$

This Taylor series converges on  $|z| < 1$ . Thus,

$$\begin{aligned}\frac{1}{z^2(z-i)} &= \frac{1}{z^2} \sum_{j=0}^{\infty} i^{j+1} z^j \\ &= \sum_{j=0}^{\infty} i^{j+1} z^{j-2}.\end{aligned}$$

For  $1 < |z| < \infty$ , we can do

$$\begin{aligned}\frac{1}{z-i} &= \frac{1}{z(1-\frac{i}{z})} \\ &= \frac{1}{z} \sum_{j=0}^{\infty} i^j z^{-j} \\ \frac{1}{z^2(z-i)} &= \sum_{j=0}^{\infty} i^j z^{-j-3}.\end{aligned}$$

To find the Laurent series for  $f(z) = \frac{1}{(z-2)(z-3)}$  on  $|z| < 2$ , we do the following.

$$\begin{aligned}\frac{1}{(z-2)(z-3)} &= \frac{1}{z-3} - \frac{1}{z-2} \\ \frac{1}{z-3} &= \frac{1}{-3(1-(z/3))} \\ &= -\frac{1}{3} \sum_{j=0}^{\infty} \left(\frac{z}{3}\right)^j \\ \frac{1}{z-2} &= -\frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j\end{aligned}$$

Both of these series converge on  $|z| < 2$ , so

$$\begin{aligned}\frac{1}{z-3} - \frac{1}{z-2} &= \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}} - \frac{z^j}{3^{j+1}} \\ &= \sum_{j=0}^{\infty} \frac{3^{j+1} - 2^{j+1}}{6^{j+1}} z^j.\end{aligned}$$

## Cauchy Criterion and Convergence

Let  $(a_n)_n \in \mathbb{C}$  be such that  $\forall \varepsilon > 0, \exists N$  large such that for  $m, n \geq N, |a_m - a_n| < \varepsilon$ . A sequence is convergent if and only if it is Cauchy.

Let  $(a_n)_n \rightarrow \ell \in \mathbb{C}$ . Let  $\varepsilon > 0$ . Then,  $\exists N$  such that for all  $n \geq N, |a_n - L| < \varepsilon$ . Let  $m, n \geq N$ . Then,

$$\begin{aligned}|a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |a_m - L| \\ &< 2\varepsilon\end{aligned}$$

The other direction requires the axiom of choice.

Recall the comparison test: if  $\sum b_j$  converges, and  $|a_j| < b_j$  for all  $j$ , then  $\sum a_j$  converges. To prove this, we require the Monotone Convergence Theorem — every bounded monotone sequence of real numbers converges.

Let  $(a_j)_j$  be nondecreasing and bounded above by  $B$ . Since  $(a_j)_j$  is bounded above, it has a least upper bound  $L$ . Let  $\varepsilon > 0$ . Since  $L$  is the least upper bound,  $L - \varepsilon$  is not an upper bound for  $(a_j)_j$ , meaning  $\exists j$  such that  $a_j > L - \varepsilon$ . Thus,  $L - \varepsilon < a_j \leq a_{j+1} \leq \dots < L$ . So, for all  $k > j$ ,  $|a_k - L| < \varepsilon$ .

To show the comparison test, let  $S_n = \sum_{j=0}^n |a_j|$ . We have that  $S_n$  is monotone increasing. Additionally,  $S_n$  is bounded above since  $S_n \leq \sum_{j=0}^n b_j \leq \sum b_j$ , which converges. Let  $T_n = \sum_{j=0}^n a_j$ . Pick  $m, n$  with  $m < n$ . Given  $\varepsilon > 0$ , we have that for  $m, n \geq N$ ,

$$\begin{aligned} |T_m - T_n| &= \left| \sum_{j=m+1}^n a_j \right| \\ &\leq \sum_{j=m+1}^n |a_j| \\ &= S_n - S_m \\ &< \varepsilon, \end{aligned}$$

so  $T_n$  is Cauchy, and thus convergent.

Let  $A \subseteq \mathbb{R}$ . Then,  $\sup A$  is the least upper bound of  $A$  — if  $A$  is not bounded above, then  $\sup A = \infty$ .

For a sequence  $(a_n)_n \in \mathbb{R}$ , define  $(x_n)_n$  as  $x_n = \sup\{a_j\}_{j \geq n}$ . Then,  $x_n = \sup\{a_n, x_{n+1}\}$ . Thus, we have  $x_n \geq x_{n+1} \geq \dots$ , so  $(x_n)_n$  may converge to some  $L$ , or  $x_n \rightarrow \pm\infty$ . We define  $\lim_{n \rightarrow \infty} x_n$

Let  $(a_n)_n = (-1)^n$ . Then,  $\limsup a_n = 1$ . However,  $\limsup (-2)^n = \infty$ .

## Cauchy-Hadamard Theorem

Given any power series  $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ ,  $\exists R \in [0, \infty]$  such that the series converges uniformly for  $|z - z_0| < R$ , diverges on  $|z - z_0| > R$ , and converges uniformly on  $|z - z_0| < r < R$ .

**Case 1:** Let  $\ell = \limsup \sqrt[n]{|a_n|}$ . Let  $\ell \in (0, \infty)$ . Then,  $R = \frac{1}{\ell}$ .

Let  $z \in \mathbb{C}$  such that  $|z - z_0| < \frac{1}{\ell}$ . Then,  $\exists \ell'$  such that  $|z - z_0| < \frac{1}{\ell'} < \frac{1}{\ell}$ . Then,  $c = \ell'|z - z_0| < 1$ .

We have  $\limsup \sqrt[n]{|a_n|} = \ell$ , let  $x_n = \sup\{\sqrt[n]{|a_n|}, \{\sqrt[n+1]{|a_{n+1}|}, \dots\}\}$ . Then,  $\lim_{n \rightarrow \infty} x_n = \ell$ . Then,  $\exists p$  such that  $\forall n \geq p$ ,  $|x_n - \ell| < \ell' - \ell$ . So,  $\sup\{\sqrt[n]{|a_n|}, \dots\} < \ell'$ , meaning that  $\forall n \geq p$ ,

$$\begin{aligned} \sqrt[n]{|a_n|}|z - z_0| &< \ell'|z - z_0| = c \\ |a_n||z - z_0|^n &< c^n, \end{aligned}$$

so  $\sum a_j(z - z_0)^j$  converges by comparison with  $\sum c^j$ , since  $c < 1$ .

**Case 2:** Let  $\ell = 0$ . We let  $R = \infty$ . Let  $z \in \mathbb{C}$ . Then,  $\limsup \sqrt[n]{|a_n|} = 0$ . If  $z = z_0$ , we have convergence. If not, let  $c = |z - z_0| > 0$ . Then, there exists  $p$  such that for all  $n \geq p$ ,  $|x_n - 0| < \frac{1}{2c}$ , where  $x_n$  denotes the tail sequence.

Then, for all  $j > p$ ,  $\sqrt[j]{|a_j|} < \frac{1}{2c}$ , so  $|a_j| < (\frac{1}{2c})^j$ , so,  $|a_j(z - z_0)^j| < (\frac{1}{2c})^j c^j$ , meaning  $\sum a_j(z - z_0)^j$  converges by comparison with  $\sum \frac{1}{2^j}$ .

**Case 3:** Let  $\ell = \infty$ . We set  $R = 0$ . Let  $z \neq z_0$ . We will show that  $\lim |a_j(z - z_0)^j| \rightarrow 0$ .

With this, we have shown absolute convergence as well. To show uniform convergence, we need the  $M$ -test.

**Weierstrass  $M$ -Test:** Let  $f_j : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$ ,  $j \in \mathbb{Z}_{\geq 0}$ , with  $f_j$  not necessarily continuous. Suppose  $|f_j| \leq M_j$  on  $A$ . If  $\sum M_j$  converges, then  $\sum f_j(z)$  converges absolutely and uniformly on  $A$ .

For each  $z \in A$ , we know  $\sum f_j(z)$  converges by the comparison test with  $\sum M_j$ . Let  $g(z) = \sum f_j(z)$ .

Let  $g_n(z) = \sum_{j=0}^n f_j(z)$ . We want to show  $g_n \rightarrow g$  uniformly. Given  $\varepsilon > 0$ ,  $\exists p$  such that for all  $n \geq p$  and for all  $z \in A$ ,  $|g_n(z) - g(z)| < \varepsilon$ .

$$\begin{aligned} |g(z) - g_n(z)| &= \left| \sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^n f_j(z) \right| \\ &= \left| \sum_{j=n+1}^{\infty} f_j(z) \right| \\ &\leq \sum_{j=n+1}^{\infty} M_j. \end{aligned}$$

So, for sufficiently large  $n$ ,  $|S_n - S_{\infty}| < \varepsilon$  for sufficiently large  $n$ , as  $\sum_{j=0}^{\infty} M_j$  converges.

To show uniform convergence within  $|z - z_0| < r < R$ , let  $z_1 = z_0 + r$ . So,  $|z - z_0| = r < R$ , meaning  $\sum a_j(z_1 - z_0)^j$  converges. So,  $|f_j(z)| \leq |a_j||z - z_0|^j$ . By above, we have  $\sum_{j=0}^{\infty} M_j$  converges, where  $M_j = |a_j(z_1 - z_0)^j|$ .

Then, for all  $z$  with  $|z - z_0| < |z_1 - z_0|$ , we have  $|a_j(z - z_0)^j| \leq M_j$ , meaning  $\sum a_j(z - z_0)^j$  converges uniformly on  $|z - z_0| < r$  by the  $M$ -test.