Problem (Problem 1)**:** In this exercise, we prove another fundamental result in differential topology, called the tubular neighborhood theorem. Let M be a compact smooth manifold with orientable boundary N. For simplicity, assume that N is connected. The tubular neighborhood theorem asserts that N admits a neighborhood in M which is diffeomorphic to $N \times [0, 1)$.

(a) Choose a Riemannian metric on M, and show that N admits a nonvanishing vector field that is everywhere orthogonal to the tangent space of N. That is, a vector field X such that for all $p \in N$, $g(X_p, T_pN) = 0$.

(b) Use the flow generated by X to find the desired neighborhood.

Solution:

(a) If $p \in N$, then we observe that $T_pN < T_pM$ is a proper subspace with codimension 1. Letting $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for T_pN, then we may extend to a basis for T_pM by taking a representative for a basis for T_pM/T_pN, and observing that such a vector necessarily has

$$g_p(e_n, e_k) = 0$$

for all $k = 1, \dots, n-1$. By smoothly varying over all points $p \in N$, we get our desired everywhere nonvanishing vector field normal to T_pN.

(b) Let φ_t be the one-parameter diffeomorphism group generated by X, where $\varphi_t\colon M \to M$ is such that $\varphi_0(p) = p$ for all $p \in N$. Then, $\varphi\colon (-\varepsilon, \varepsilon) \to \operatorname{diff}(M)$ restricted to $[0, \varepsilon)$ gives our desired neighborhood in M diffeomorphic to $N \times [0, 1)$.

Problem (Problem Set 7, Problem 5)**:** Suppose G is a finite group acting freely on a manifold M by diffeomorphisms.

(a) Show that M/G is a manifold.

(b) Show that the de Rham cohomology of M/G is isomorphic to the G-invariant cohomology of M.

Problem (Problem Set 8, Problem 3)**:** Compute the de Rham cohomology of $\mathbb{RP}^n$.

Solution: We will use the result related to invariant cohomology to compute this.

Problem (Problem Set 8, Problem 5)**:** Use the Mayer–Vietoris sequence to prove the Künneth Formula: if M and N are smooth manifolds, then $H^*_{DR}(M \times N)$ is the tensor product of $H^*_{DR}(M)$ and $H^*_{DR}(N)$. Specifically, in each degree ℓ, we have

$$H^\ell_{DR}(M \times N) = \bigoplus_{i+j=\ell} H^i_{DR}(M) \otimes H^j_{DR}(N).$$

Solution: For the sake of being able to solve this problem, we focus on the case where M and N are closed smooth manifolds.

Let $V = M \times N$ be the product manifold for M and N. If $\pi_1\colon V \to M$ and $\pi_2\colon V \to N$ are the projection maps on M and N respectively, we get the composed maps

$$\mathcal{A}^k(M) \times \mathcal{A}^\ell(N) \to \mathcal{A}^{k+\ell}(V)$$

given by $(\omega, \eta) \mapsto \pi_1^*\omega \wedge \pi_2^*\eta$. If ω and η are closed forms, then we observe that

$$\begin{aligned} d(\pi_1^*\omega \wedge \pi_2^*\eta) &= d\pi_1^*\omega \wedge \pi_2^*\eta + (-1)^k \pi_1^*\omega \wedge d\pi_2^*\eta \\ &= \pi_1^*(d\omega) \wedge \pi_2^*\eta + (-1)^k \pi_1^*\omega \wedge \pi_2^*(d\eta) \\ &= 0. \end{aligned}$$

Furthermore, if we let $\omega' = \omega + d\tau$ and $\eta' = \eta + d\rho$, then we know from earlier work that $\pi_1^*\omega' \wedge \pi_2^*\eta'$ can be expressed as $\pi_1^*\omega \wedge \pi_2^*\eta + d\sigma$ for some form σ by using the fact that d and the pullback commute.

Thus, it follows that the map descends to a map in cohomology, given by

$$\begin{aligned} H^k_{DR}(M) \times H^\ell_{DR}(N) &\to H^{k+\ell}(M \times N) \\ ([\omega],[\eta]) &\mapsto \left[\pi_1^*\omega \wedge \pi_2^*\eta\right], \end{aligned}$$

whence via the universal property of tensor products and direct sums, we get the map

$$\psi\colon H^*_{DR}(M) \otimes H^*_{DR}(N) \to H^*(M \times N).$$

Our goal now is to show that ψ is indeed an isomorphism.

Toward this end, suppose we have two open sets in the good cover for M, given by U_1 and U_2. From the Mayer–Vietoris sequence, this yields the following exact sequence in cohomology for a fixed k, where D_k denote the connecting homomorphisms from $H^k(U_1 \cap V_2)$ to $H^{k+1}(M)$.

$$\cdots \xrightarrow{D_{k-1}} H^k_{DR}(M) \xrightarrow{i} H^k_{DR}(U_1) \oplus H^k_{DR}(U_2) \xrightarrow{j} H^k_{DR}(U_1 \cap U_2) \xrightarrow{D_k} \cdots$$

Since the tensor product preserves exact sequences, we observe that by taking the tensor product with $H^\ell_{DR}(N)$, giving the following.

$$\cdots \xrightarrow{D_{k-1}} H^k_{DR}(M) \otimes H^\ell_{DR}(N) \xrightarrow{i} H^k_{DR}(U_1) \otimes H^\ell_{DR}(N) \oplus H^k_{DR}(U_2) \otimes H^\ell_{DR}(N) \xrightarrow{j} H^k(U_1 \cap U_2) \otimes H^\ell(N) \xrightarrow{D_k} \cdots$$