Math 395

Homework 1
Due: 2/1/2024

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Collaborators: ____________________

Problem 1

Let S be the subset of $\mathrm{Mat}_2(\mathbf{R})$ be the set consisting of matrices of the form $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$.

(a) To show that S is a ring, we will show that S is a subring of the ring $\mathrm{Mat}_2(\mathbf{R})$, by showing that S is not empty, S is closed under subtraction, and S is closed under multiplication.

To show non-emptiness, we can see that the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an element of S by its definition.

To show S is closed under subtraction, let $a, b, c, d \in \mathbf{R}$, and let $e = a - c$ and $f = b - d$. Then,

$$\begin{aligned}
\begin{bmatrix} a & a \\ b & b \end{bmatrix} - \begin{bmatrix} c & c \\ d & d \end{bmatrix} &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} -c & -c \\ -d & -d \end{bmatrix} \\
&= \begin{bmatrix} a + (-c) & a + (-c) \\ b + (-d) & b + (-d) \end{bmatrix} \\
&= \begin{bmatrix} a - c & a - c \\ b - d & b - d \end{bmatrix} \\
&= \begin{bmatrix} e & e \\ f & f \end{bmatrix},
\end{aligned}$$

which is an element of S. Thus, S is closed under subtraction.

Next, we need to show that S is closed under multiplication. Letting $a, b, c, d \in \mathbf{R}$ as before, let $g = ac + ad$ and $h = bc + bd$. Then,

$$\begin{aligned}
\begin{bmatrix} a & a \\ b & b \end{bmatrix} \cdot \begin{bmatrix} c & c \\ d & d \end{bmatrix} &= \begin{bmatrix} ac + ad & ac + ad \\ bc + bd & bc + bd \end{bmatrix} \\
&= \begin{bmatrix} g & g \\ h & h \end{bmatrix},
\end{aligned}$$

which is an element of S. Thus, S is closed under multiplication.

Since S is non-empty, closed under subtraction, and closed under multiplication, S is a subring of $\mathrm{Mat}_2(\mathbf{R})$, and so is a ring.

(b)