Things You Just Gotta Know

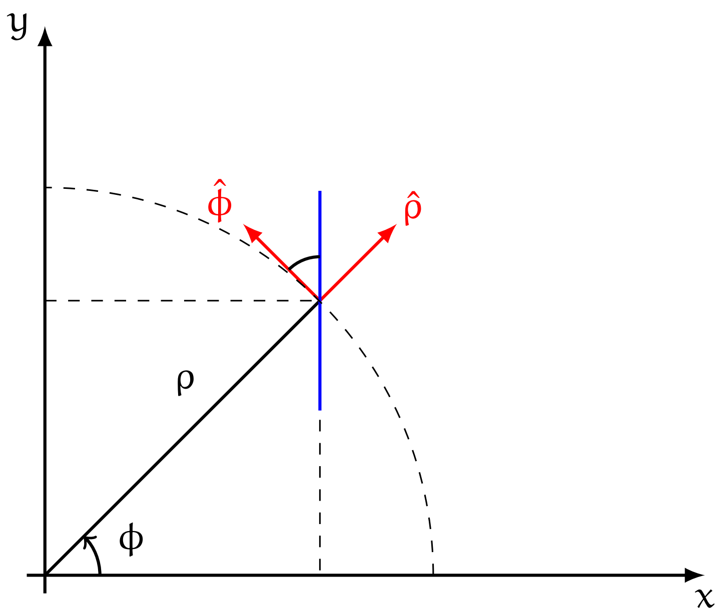
Coordinate Systems

We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x,y)\hat{\imath} + V_y(x,y)\hat{\jmath}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

Polar Coordinates



We can also express the inputs to $\vec{V}$ in polar coordinates, (ρ, ϕ).

$$\vec{V}(\mathbf{r}) = V_\rho\left(\rho,\phi\right)\hat{\imath} + V_\phi\left(\rho,\phi\right)\hat{\jmath}.$$

To extract the input functions, we take

$$\begin{aligned} V_x &= \hat{\imath}\cdot\vec{V} \\ V_y &= \hat{\jmath}\cdot\vec{V}. \end{aligned}$$

Alternatively, we can project $\vec{V}$ onto the $\hat{\rho}, \hat{\phi}$ axis:

$$\vec{V}(\mathbf{r}) = V_\rho\left(\rho,\phi\right)\hat{\rho} + V_\phi\left(\rho,\phi\right)\hat{\phi},$$

and we extract

$$\begin{aligned} V_\rho &= \hat{\rho}\cdot\vec{V} \\ V_\phi &= \hat{\phi}\cdot\vec{V}. \end{aligned}$$

Notice that $\mathbf{r}$ is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned}\mathbf{s} &= x\hat{\imath} + y\hat{\jmath}\\ &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath}\\ &= \rho\hat{\rho}\\ &= \sqrt{x^2+y^2}\hat{\rho}\end{aligned}$$

The main reason we avoided using the $\hat{\rho}$, $\hat{\phi}$ axis up until this point is that ρ and ϕ are *position-dependent*, while the $\hat{\imath}, \hat{\jmath}$ axis is position-independent.

Now, we must figure out the position-dependence of $\hat{\rho}$ and $\hat{\phi}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho}d\rho + \frac{\partial \mathbf{r}}{\partial \phi}d\phi.$$

If we hold ϕ constant, it must be the case that any change in ρ is in the $\hat{\rho}$ direction. Therefore,

$$\begin{aligned}\hat{\rho} &= \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|}\\ &= \frac{\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}}{\left|\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\right|}\\ &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}.\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|}\\ &= \frac{-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}}{\left\|-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}\right\|}\\ &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}.\end{aligned}$$

Thus, we can see that the $\hat{\rho}, \hat{,}\phi$ axis is orthogonal.

$$\begin{aligned}\frac{\partial \hat{\rho}}{\partial \phi} &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\\ &= \hat{\phi},\\ \frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\rho},\\ \frac{\partial \hat{\phi}}{\partial \rho} &= 0,\end{aligned}$$

and

$$\frac{\partial \hat{\rho}}{\partial \rho} = 0$$

Example (Velocity)**.**

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{s}}{dt}\\ &= \frac{d}{dt}\left(x\hat{\imath}\right) + \frac{d}{dt}\left(y\hat{\jmath}\right).\end{aligned}$$

In the case of cartesian coordinates, $\hat{\imath}$ and $\hat{\jmath}$ are constants.

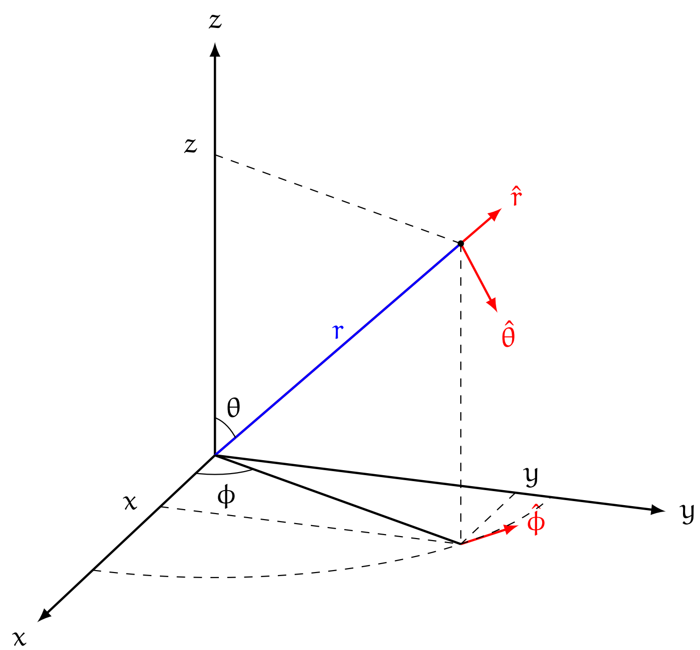
$$= v_x\hat{\imath} + v_y\hat{\jmath}$$

When we examine polar coordinates, since $\hat{\rho}$ and $\hat{\phi}$ are position-dependent, we must use the chain rule.[I]

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{s}}{dt}\\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt}\\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\cancelto{0}{\frac{\partial\hat{\rho}}{\partial\rho}}\frac{d\rho}{dt} + \underbrace{\frac{\partial\hat{\rho}}{\partial\phi}}_{=\hat{\phi}}\frac{d\phi}{dt}\right)\\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi}\\
&= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}.
\end{aligned}$$

Notice that $\dot{\rho}$ is the radial velocity and $\dot{\phi} = \omega$ is the angular velocity.

Spherical Coordinates



Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho,\phi)$	$\mathbf{s} = s(\rho,\phi,z)$	$\mathbf{s} = s(r,\phi,\theta)$
$\mathbf{s} = \begin{pmatrix}\rho\cos\phi\\ \rho\sin\phi\end{pmatrix}$	$\mathbf{s} = \begin{pmatrix}\rho\cos\phi\\ \rho\sin\phi\\ z\end{pmatrix}$	$\mathbf{s} = \begin{pmatrix}r\cos\phi\sin\theta\\ r\sin\phi\sin\theta\\ r\cos\theta\end{pmatrix}$

Here,[II] ϕ denotes the polar angle and θ denotes the azimuthal angle. Notice that $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

[I]Note that $\hat{\rho} = \hat{\rho}(\rho,\phi)$ and $\hat{\phi} = \hat{\phi}(\rho,\phi)$.
[II]Physicists amirite?

We can see that $\hat{\rho}$, $\hat{\phi}$, and $\hat{\theta}$ in spherical coordinates are also position-dependent.

$$\begin{aligned}
\hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\|\frac{\partial \mathbf{s}}{\partial r}\right\|} \\
&= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k} \\
\hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\|\frac{\partial \mathbf{s}}{\partial \phi}\right\|} \\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
\hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\|\frac{\partial \mathbf{s}}{\partial \theta}\right\|} \\
&= \cos\phi\cos\theta\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}
\end{aligned}$$