

These are some definitions and ideas I will be using regularly throughout this presentation.

**Definition** (Groups). Let  $A$  be a set, and let  $\star: A \times A \rightarrow A$  be such that

- $a \star (b \star c) = (a \star b) \star c$ ;
- there exists  $e_A \in A$  such that  $e_A \star a = a \star e_A$  for all  $a \in A$ ;
- for each  $a \in A$ , there is  $a^{-1}$  such that  $a \star a^{-1} = a^{-1} \star a = e_A$ .

Then, we say the pair  $(A, \star)$  is a *group*. We abbreviate  $a \star b = ab$ .

**Definition** (Subgroups and Quotient Groups). Let  $G$  be a group.

- If  $H \subseteq G$  is a subset such that for all  $a, b \in H$ ,  $ab^{-1} \in H$ , then we say  $H$  is a *subgroup*.
- If  $N \subseteq G$  is a subgroup such that for all  $g \in G$  and  $h \in N$ ,  $ghg^{-1} \in N$ , then we say  $N$  is a *normal subgroup*.
- If  $N$  is a normal subgroup, we may define the equivalence relation  $g \sim_N g'$  if  $g^{-1}g' \in N$ ; the equivalence classes  $gN := [g]_{\sim_N}$  form the *quotient group*,  $G/N$ .
- If  $H \subseteq G$  is a subgroup, then the *index* of  $H$  is the number of cosets,  $gH := \{gh \mid h \in H\}$ , written  $[G : H]$ .

**Definition** (Group Actions). If  $G$  is a group, and  $X$  is a set, then  $\rho: G \times X \rightarrow X$  is called an *action* of  $G$  onto  $X$  if  $\rho$  satisfies

- $\rho(e_G, x) = x$ ;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$ .

We write  $\rho(g, x) = g \cdot x$ .

Every group is equipped with a family of canonical actions,  $\sigma: G \times G \rightarrow G$ , given by  $(a, x) \mapsto ax$ , known as *left-multiplication*.

**Definition** (Algebras,  $\sigma$ -Algebras of Subsets). If  $X$  is a set, then a collection  $\mathcal{A} = \{A_i\}_{i \in I} \subseteq P(X)$  is known as an *algebra* of subsets of  $X$  if

- (1)  $\emptyset, X \in \mathcal{A}$ ;
- (2) for all  $A_i \in \mathcal{A}$ ,  $A_i^c \in \mathcal{A}$ ;
- (3) for all  $A_i, A_j \in \mathcal{A}$ ,  $A_i \cup A_j \in \mathcal{A}$ .

If condition (3) holds for any countable subcollection  $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$ , then we say  $\mathcal{A}$  is a  $\sigma$ -*algebra* of subsets.

The primary  $\sigma$ -algebra we are concerned with is  $P(G)$ , where  $G$  is a group.

**Definition** (Measures). If  $X$  is a set and  $\mathcal{A}$  is a  $\sigma$ -algebra, then a map  $\mu: \mathcal{A} \rightarrow [0, \infty]$  that satisfies

- $\mu(\emptyset) = 0$ ;
- for disjoint  $A, B \in \mathcal{A}$ ,  $\mu(A \sqcup B) = \mu(A) + \mu(B)$ ,

then we say  $\mu$  is a *finitely additive measure*. If, for any countable collection of disjoint sets,  $\{A_n\}_{n \geq 1}$ , we have

$$\mu\left(\bigsqcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n),$$

then we say  $\mu$  is a *measure*. If  $\mu(X) = 1$ , then we say  $\mu$  is a *probability measure*.