2.1.2

Let G be a graph:

(a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.

(b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

(a)

($\Rightarrow$) Let G be a tree. Thus, G is connected (by definition), and acyclic. Since G is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

($\Leftarrow$) Let G be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in G, so G is a connected acyclic graph, or a tree.

(b)

($\Rightarrow$) Let G be a tree, and let e be an edge such that $e \notin E(G)$, and $e = uv$. Then, we create a cycle from the path $uTv + e$ — since there is only one path uTv, this means that $uTv + e$ is a unique cycle.

($\Leftarrow$) Suppose toward contradiction that adding e to the tree G yielded more than one cycle in the graph $G + e$. Then, the graph $G = G + e - e$ would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that G was a tree, we have reached a contradiction, meaning that e added exactly one cycle to the tree G.

2.1.6

Let T be a tree with average degree a. In terms of a, find $n(T)$.

$$\begin{aligned}
d_{\text{avg}} &= \frac{2e(T)}{n(T)} \\
a &= \frac{2(n(T)-1)}{n(T)} \\
an &= 2n - 2 \\
(a-2)n &= -2 \\
n &= \boxed{\frac{2}{2-a}}
\end{aligned}$$

2.1.7

Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

Base Case If $m = 0$, then since this graph has zero edges, it has zero cycles, and since $0 \geq 1 - n$, we have proven the base case.

Inductive Hypothesis For an n-vertex graph with $0 \leq k \leq m$ vertices, then G has at least $k - n + 1$ cycles.

Proof We can consider three cases of edge deletion changing the number of cycles:

Reduction of > 1 cycle If edge deletion reduces the number of cycles by more than one, adding e back into the graph yields a $((m-1)-n+1)+1 = m-n+1$ increase in the lower bound of cycles, while the total number of cycles increases by more than one, meaning that the inductive hypothesis remains valid.

Reduction of exactly 1 cycle If edge deletion reduces the number of cycles by exactly one, then the total number of cycles is reduced by 1 and the lower bound on cycles is reduced by $(m-1)-n+1 = (m-n+1)-1$ cycles, satisfying the inductive hypothesis.

Reduction of 0 cycles If edge deletion does not reduce the number of cycles, then the lower bound on the number of cycles is $(m-1)-n+1 = (m-n+1)-1$, but the total number of cycles remains unchanged, satisfying the inductive hypothesis.

2.1.12

Compute the diameter and radius of $K_{m,n}$.

The diameter of $K_{m,n}$ is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of $K_{m,n}$ is also 2 — the eccentricity of every vertex in $K_{m,n}$ is 2, so the radius must also be 2.

2.1.13

Prove that every graph with diameter d has an independent set with at least $\left\lceil \frac{1+d}{2} \right\rceil$ vertices.

Let G be a graph with diameter d, and let $u \in V(G)$ be a vertex with eccentricity d. Let P be a maximal u, v path of length d. Then, P has $d+1$ vertices. So, P has a maximal independent set containing every other vertex, with total cardinality of $\left\lceil \frac{d+1}{2} \right\rceil$. Therefore, G has an independent set with at least $\left\lceil \frac{d+1}{2} \right\rceil$ vertices.