Problem 1

Recall that a subset $U \subseteq \mathbb{R}$ is **open** if

$$(\forall x \in U)(\exists \varepsilon > 0) \ni V_\varepsilon(x) \subseteq U.$$

Prove that a mapping $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U) \subseteq \mathbb{R}$ is open for every open $U \subseteq \mathbb{R}$.

($\Rightarrow$) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall c \in \mathbb{R}$, $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$. Let U be an open set such that $f(c) \in U$. Then, $\exists \varepsilon_0$ such that $V_{\varepsilon_0}(f(c)) \subseteq U$. So, $\exists \delta_0$ such that $V_{\delta_0}(c) \subseteq f^{-1}(V_{\varepsilon_0}(f(c))) \subseteq f^{-1}(U)$. So, $f^{-1}(U)$ is open.

($\Leftarrow$) Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is open in $\mathbb{R}$.

Since U is open in $\mathbb{R}$, it must be the case that for every $f(c) \in U$, $\exists \varepsilon > 0$ such that $V_\varepsilon(f(c)) \subseteq U$. Since $f^{-1}(U) = \{c \mid f(c) \in U\}$, it must be the case that $\exists \delta > 0$ such that $V_\delta(c) \subseteq f^{-1}(U)$.

Therefore, $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$ for sufficiently small δ. Thus, $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Problem 2

Let $f, g : D \to \mathbb{R}$ be continuous. Show that $f \cdot g$ is continuous.

Since $f : D \to \mathbb{R}$ is continuous, then $\forall (x_n)_n, c \in D$ such that $(x_n)_n \to c$, $(f(x_n))_n \to f(c)$. Similarly, since $g : D \to \mathbb{R}$ is continuous, then $\forall (x_n)_n, c \in D$ such that $(x_n)_n \to c$, $(g(x_n))_n \to g(c)$.

So, $\forall (x_n)_n, c \in D$ such that $(x_n)_n \to c$, $(f(x_n)g(x_n))_n \to f(c)g(c)$ by the properties of sequences. Thus, $f \cdot g$ is continuous.

Problem 3

Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be continuous mappings with $\text{Ran}(f) \subseteq E$. Show that $g \circ f$ is continuous.

Every sequence $(x_n)_n \in D$ with $(x_n)_n \to c \in D$ has $(f(x_n))_n \to f(c)$. Since $(f(x_n))_n \in E$ and $f(c) \in E$, it must be the case that $(g(f(x_n)))_n \to g(f(c))$. So, $g \circ f : D \to \mathbb{R}$ is continuous.

Problem 4

Show that the following functions are Lipschitz:

(i) $f : [-M, M] \to \mathbb{R}$ given by $f(x) = x^2$

(ii) $g : [1, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$

(iii) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sqrt{x^2 + 4}$

(a)

Let $x, y \in [-M, M]$. Then,

$$\begin{aligned}
|f(x) - f(y)| &= |x^2 - y^2| \\
&= |x - y||x + y| \\
&\leq (|x| + |y|)\,|x - y| \\
&\leq 2|M||x - y|
\end{aligned}$$

(b)

Let $x, y \in [1, \infty)$. Then,

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\
&= \frac{1}{xy}|x - y| \\
&\leq |x - y|
\end{aligned}$$

(c)

Let $x, y \in \mathbb{R}$. Then,

$$\begin{aligned}
|f(x) - f(y)| &= |\sqrt{x^2 + 4} - \sqrt{y^2 + 4}| \\
&= \frac{|x^2 - y^2|}{\sqrt{x^2 + 4} + \sqrt{x^2 + 4}} \\
&= \frac{|x + y||x - y|}{\sqrt{x^2 + 4} + \sqrt{x^2 + 4}} \\
&\leq \frac{(|x| + |y|)|x - y|}{\sqrt{x^2 + 4} + \sqrt{y^2 + 4}} \\
&\leq \frac{(|x| + |y|)|x - y|}{\sqrt{x^2} + \sqrt{y^2}} \\
&= \frac{(|x| + |y|)\,|x - y|}{|x| + |y|} \\
&= |x - y|
\end{aligned}$$

Problem 5

Show that the following functions are not Lipschitz:

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

(b) $g : (0, \infty)$ given by $g(x) = \frac{1}{x}$

(a)

Let $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left|x^2 - y^2\right| \\ &= |x-y||x+y| \\ &\leq (|x| + |y|)\,|x-y| \end{aligned}$$

but since $|x| + |y|$ is unbounded, it must be the case that $\nexists c$ such that $|f(x) - f(y)| \leq c|x-y|$.

(b)

Let $x, y \in (0, \infty)$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \frac{|x-y|}{xy} \end{aligned}$$

but since $\frac{1}{xy}$ is unbounded on $(0, \infty)$, it must be the case that $\nexists c$ such that $|f(x) - f(y)| \leq c|x-y|$.

Problem 6

Suppose $f : \mathbb{R} \to \mathbb{R}$ and for some $C \geq 0$, we have $|f(q)| \leq C$ for all rationals $q \in \mathbb{Q}$. Show that $\|f\|_{\mathbb{R}} \leq C$.

Let $t \in \mathbb{R}$. Then, $\exists (q_n)_n \in \mathbb{Q}$ such that $(q_n)_n \to t$, as the rationals are dense.

Since f is continuous, $(f(q_n))_n \to f(t)$.

Since $|f(q_n)| \leq C$ for all q_n, it must be the case that $f(t) \leq C$.

Problem 7

Suppose $f : \mathbb{R} \to \mathbb{R}$ is an additive map, that is,

$$f(x+y) = f(x) + f(y) \qquad \forall x, y \in \mathbb{R}.$$

If f is continuous at some point, say $x = c$, show that f is continuous everywhere and that $f(x) = ax$ for some $a \in \mathbb{R}$.

Let $t \in \mathbb{R}$. Let $(x_n)_n \in \mathbb{R}$ with $(x_n)_n \to c$. Then, for the sequence $(x_n - c + t)_n \in \mathbb{R}$, with $(x_n - c + t)_n \to t$, we have

$$\begin{aligned} f(x_n - c + t) &= f(x_n) + -f(c) + f(t) \\ &\to f(c) - f(c) + f(t) \\ &= f(t) \end{aligned}$$

so f must be continuous at $x = t$.

Problem 8

Assume $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$g(x+y) = g(x)g(y) \qquad \forall x, y \in \mathbb{R}.$$

If g is continuous at $x = 0$, show that g is continuous everywhere. Then show that there is a $b \geq 0$ with $g(x) = b^x$.

Problem 9

Let p be a polynomial of odd degree. Show that p has a real root.

Let $p(x) = a_{2n+1}x^{2n+1} + \cdots + a_1 x + a_0$. Then, $\lim_{x\to\infty} p(x) = \pm\infty$, and $\lim_{x\to-\infty} p(x) = \mp\infty$. Without loss of generality, suppose $\lim_{x\to\infty} p(x) = +\infty$, and $\lim_{x\to-\infty} = -\infty$.

Then, for any $N > 0$, $\exists x_1 > 0$ such that $p(x) > N$ for all $x > x_1$. So, $p(x_1) > 0$. Similarly, for any $M < 0$, $\exists x_2 < 0$ such that $p(x) < M$ for all $x < x_2$. So, $p(x_2) < 0$.

By the intermediate value theorem on $[x_2, x_1]$, there must be a point where $p(x) = 0$ where $x \in [x_2, x_1]$.

Problem 10

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that vanishes at infinity, that is,

$$\lim_{x\to\pm\infty} f = 0.$$

Show that f is bounded.

Let $\varepsilon > 0$. Then, $\exists N > 0$ and $M < 0$ such that $|f(x)| < \varepsilon$ for all $x > N$ an $x < M$.

So, on $(-\infty, M)$ and (N, ∞), $|f|$ is bounded by ε. Finally, on $[M, N]$, $|f|$ must be bounded by the Extreme Value Theorem.

Therefore, $|f|$ is bounded on $\mathbb{R}$, and thus f is bounded on $\mathbb{R}$.

Problem 11

A function $f : D \to \mathbb{R}$ is said to be lower semicontinuous (LSC) at $x = c$ if

$$(\forall\varepsilon > 0)(\exists\delta > 0) \ni x \in D \cap V_\delta(c) \Rightarrow f(c) - \varepsilon < f(x).$$

A function $f : D \to \mathbb{R}$ is said to be upper semicontinuous (USC) at $x = c$ if

$$(\forall\varepsilon > 0)(\exists\delta > 0) \ni x \in D \cap V_\delta(c) \Rightarrow f(x) < f(c) + \varepsilon$$

(i) Show that f is continuous at c if and only if f is USC and LSC at c.

(ii) Show that f is LSC at c if and only if

$$\liminf_{n\to\infty} f(x_n) \geq f(c),$$

for every sequence $(x_n)_n$ in D that converges to c.

(iii) Show that f is USC at c if and only if

$$\limsup_{n\to\infty} f(x_n) \leq f(c)$$

for every sequence $(x_n)_n$ in D that converges to c.

(iv) Show that a USC function $f : [a, b] \to \mathbb{R}$ admits an absolute maximum on $[a, b]$.

Problem 12

Let $f : [a, b] \to \mathbb{R}$ be a continuous function satisfying the following property:

$$\forall x \in [a, b],\ \exists y \in [a, b] \ni |f(y)| \leq \frac{1}{2}|f(x)|.$$

Show that there is a $c \in [a, b]$ with $f(c) = 0$.