I am using $\overline{z}$ to denote the conjugate of a complex number and T^* to denote the adjoint of an operator.

Chapter 25 Problems

Problem 1

(a)

$$\begin{aligned}
\||1\rangle\|^2 &= \langle 1 \,|\, 1\rangle \\
&= (1)\,\overline{(1)} + (i)\,\overline{(i)} \\
&= 2 \\
\||2\rangle\|^2 &= \langle 2 \,|\, 2\rangle \\
&= (-i)\,\overline{(-i)} + (2i)\,\overline{(2i)} \\
&= 5 \\
\||3\rangle\|^2 &= \langle 3 \,|\, 3\rangle \\
&= \left(e^{i\phi}\right)\overline{(e^{i\phi})} + (-1)\,\overline{(-1)} \\
&= 2 \\
\||4\rangle\|^2 &= \langle 4 \,|\, 4\rangle \\
&= (1)\,\overline{(1)} + (-2i)\,\overline{(-2i)} + (1)\,\overline{(1)} \\
&= 6 \\
\||5\rangle\|^2 &= \langle 5 \,|\, 5\rangle \\
&= (i)\,\overline{(i)} + (1)\,\overline{(1)} + (i)\,\overline{(i)} \\
&= 3.
\end{aligned}$$

(b)

$$\begin{aligned}
\langle 2 \,|\, 1\rangle &= (1)\,\overline{(-i)} + (i)\,\overline{(2i)} \\
&= \overline{-i\overline{(1)} + 2i\overline{(i)}} \\
&= 2 + i \\
&= \overline{\langle 1 \,|\, 2\rangle} \\
\langle 3 \,|\, 1\rangle &= (1)\,\overline{(e^{i\phi})} + (i)\,\overline{(-1)} \\
&= \overline{e^{i\phi}\overline{(1)} + (-1)\,\overline{(i)}} \\
&= e^{-i\phi} - i \\
&= \overline{\langle 1 \,|\, 3\rangle} \\
\langle 3 \,|\, 2\rangle &= (-i)\,\overline{(e^{i\phi})} + (2i)\,\overline{(-1)} \\
&= \overline{(e^{i\phi})\,\overline{(-i)} + (-1)\,\overline{(2i)}} \\
&= -ie^{-i\phi} - 2i \\
&= \overline{\langle 2 \,|\, 3\rangle}. \\
\langle 5 \,|\, 4\rangle &= (1)\,\overline{(i)} + (-2i)\,\overline{(1)} + (1)\,\overline{(i)} \\
&= \overline{i\overline{(1)} + (1)\,\overline{(-2i)} + (i)\,\overline{(1)}} \\
&= -4i
\end{aligned}$$

$$= \overline{\langle 4 \,|\, 5\rangle}$$

Problem 4

(a)

$$\begin{aligned}
|u\rangle^* &= (M\,|v\rangle)^* \\
&= \left(\begin{pmatrix} 1 & i \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right)^* \\
&= \left(\begin{pmatrix} 2 \\ 2-i \end{pmatrix}\right)^* \\
&= \begin{pmatrix} 2 & 2+i \end{pmatrix} \\
&= \begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -i & 1 \end{pmatrix} \\
&= \langle u| \,.
\end{aligned}$$

(b)

$$\begin{aligned}
\langle w \,|\, v\rangle &= \langle w \,|\, Mv\rangle \\
&= \langle w \,|\, u\rangle \\
&= \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2-i \end{pmatrix} \\
&= -i \\
&= \overline{\langle u \,|\, w\rangle} \\
&= \overline{\begin{pmatrix} 2 & 2+i \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \\
&= \overline{(i)} \\
&= -i.
\end{aligned}$$

Problem 5

$$\begin{aligned}
\langle v \,|\, Lw\rangle &= \langle v|\, L\,|w\rangle \\
&= \langle L^* v \,|\, w\rangle \\
&= \overline{\langle w \,|\, L^* v\rangle} \\
&= \overline{\langle w|\, L^*\,|v\rangle}.
\end{aligned}$$

Problem 6

(a)

$$\begin{aligned}
\overline{\overline{\langle v|\, T\,|w\rangle}} &= \overline{\langle w|\, T^*\,|v\rangle} \\
&= \langle v|\, T^{**}\,|w\rangle \,.
\end{aligned}$$

(b)

$$\begin{aligned}
\langle v|\,(ST)^*\,|w\rangle &= \overline{\langle w|\, S\,(T\,|v\rangle)} \\
&= \overline{\langle w|\, S\,|u\rangle} && |u\rangle = T\,|v\rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle u|\, S^* \,|w\rangle\\
&= \langle Tv|\, S^* \,|w\rangle\\
&= \langle v|\, T^*S^* \,|w\rangle .
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\langle v|\,(ST)^*\,|w\rangle &= \langle (ST)\,v \mid w\rangle\\
&= \langle Tv|\, S^* \,|w\rangle\\
&= \langle v|\, T^*S^* \,|w\rangle .
\end{aligned}$$

Problem 8

(a) It is clear that

$$\begin{aligned}
\||1\rangle\| &= 1\\
\||2\rangle\| &= 1,
\end{aligned}$$

and

$$\begin{aligned}
\langle 1 \mid 2\rangle &= \overline{\begin{pmatrix} i & 0\end{pmatrix}}\begin{pmatrix}1\\0\end{pmatrix}\\
&= 0.
\end{aligned}$$

(b) It is clear that

$$\begin{aligned}
\||+\rangle\| &= 1\\
\||-\rangle\| &= 1,
\end{aligned}$$

and

$$\begin{aligned}
\langle + \mid -\rangle &= \frac{1}{2}\overline{\begin{pmatrix} 1 & i\end{pmatrix}}\begin{pmatrix}1\\-i\end{pmatrix}\\
&= 0.
\end{aligned}$$

(c) Similarly, it is clear that

$$\begin{aligned}
\||\uparrow\rangle\| &= 1\\
\||\downarrow\rangle\| &= 1.
\end{aligned}$$

Taking inner products, we have

$$\begin{aligned}
\langle \uparrow \mid \downarrow\rangle &= \frac{1}{2}\overline{\begin{pmatrix} 1 & 1\end{pmatrix}}\begin{pmatrix}1\\-1\end{pmatrix}\\
&= 0.
\end{aligned}$$

(d)

$$\begin{aligned}
\||I\rangle\|^2 &= \frac{1}{9}\left(\left|i-\sqrt{3}\right|^2 + |1+2i|^2\right)\\
&= \frac{1}{9}(9)\\
&= 1
\end{aligned}$$

$$\begin{aligned}
\| |\mathrm{II}\rangle \|^2 &= \frac{1}{9}\left(|1-2i|^2 + \left|i+\sqrt{3}\right|^2\right)\\
&= \frac{1}{9}(9)\\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{I} \,|\, \mathrm{II}\rangle &= \frac{1}{9}\overline{\begin{pmatrix} i-\sqrt{3} & 1+2i\end{pmatrix}}\begin{pmatrix} 1-2i\\ i+\sqrt{3}\end{pmatrix}\\
&= 0.
\end{aligned}$$

Problem 9

(a)

$$\begin{aligned}
a_1 &= \langle 1 \,|\, A\rangle\\
&= 1-2i\\
a_2 &= \langle 2 \,|\, A\rangle\\
&= 2+2i\\
\| |A\rangle \|^2 &= |a_1|^2 + |a_2|^2\\
&= 10
\end{aligned}$$

$$\begin{aligned}
b_1 &= \langle 1 \,|\, B\rangle\\
&= 1+i\\
b_2 &= \langle 2 \,|\, B\rangle\\
&= 2i\\
\| |B\rangle \|^2 &= |b_1|^2 + |b_2|^2\\
&= 6
\end{aligned}$$

(b)

$$\begin{aligned}
a_+ &= \langle + \,|\, A\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix} 1 & i\end{pmatrix}}\begin{pmatrix} 1-i\\ 2+2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(3-3i)\\
a_- &= \langle - \,|\, A\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix} 1 & -i\end{pmatrix}}\begin{pmatrix} 1-i\\ 2+2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(-1+i)\\
\| |A\rangle \|^2 &= |a_+|^2 + |a_-|^2\\
&= 10
\end{aligned}$$

$$\begin{aligned}
b_+ &= \langle + \,|\, B\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix} 1 & i\end{pmatrix}}\begin{pmatrix} 1+i\\ 2i\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}(3+i)\\
b_- &= \langle - \mid B\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix}1 & -i\end{pmatrix}}\begin{pmatrix}1+i\\ 2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(2i)\\
\|\lvert B\rangle\|^2 &= |b_+|^2 + |b_-|^2\\
&= 6
\end{aligned}
$$

(c)

$$
\begin{aligned}
a_\uparrow &= \langle \uparrow \mid A\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix}1 & 1\end{pmatrix}}\begin{pmatrix}1-i\\ 2+2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(3+i)\\
a_\downarrow &= \langle \downarrow \mid A\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix}1 & -1\end{pmatrix}}\begin{pmatrix}1-i\\ 2+2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(-1-3i)\\
\|\lvert A\rangle\|^2 &= \left|a_\uparrow\right|^2 + \left|a_\downarrow\right|^2\\
&= 10
\end{aligned}
$$

$$
\begin{aligned}
b_\uparrow &= \langle \uparrow \mid B\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix}1 & 1\end{pmatrix}}\begin{pmatrix}1+i\\ 2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(1+3i)\\
b_\downarrow &= \langle \downarrow \mid B\rangle\\
&= \frac{1}{\sqrt{2}}\overline{\begin{pmatrix}1 & -1\end{pmatrix}}\begin{pmatrix}1+i\\ 2i\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}(1-i)\\
\|\lvert B\rangle\|^2 &= \left|b_\uparrow\right|^2 + \left|b_\downarrow\right|^2\\
&= 6.
\end{aligned}
$$

(d)

$$
\begin{aligned}
a_I &= \langle I \mid A\rangle\\
&= \frac{1}{3}\overline{\begin{pmatrix}i-\sqrt{3} & 1+2i\end{pmatrix}}\begin{pmatrix}1-i\\ 2+2i\end{pmatrix}\\
&= \frac{5\sqrt{3}-1}{\sqrt{3}} - \frac{\left(1+\sqrt{3}\right)i}{\sqrt{3}}.
\end{aligned}
$$

$$\begin{aligned}
a_{II} &= \langle II \mid A \rangle \\
&= \frac{1}{3}\overline{\begin{pmatrix} 1-2i & i+\sqrt{3} \end{pmatrix}} \begin{pmatrix} 1-i \\ 2+2i \end{pmatrix} \\
&= \frac{5\sqrt{3}+2}{\sqrt{3}} + \frac{\left(2-\sqrt{3}\right)i}{\sqrt{3}} \\
\| |A\rangle \|^2 &= |a_I|^2 + |a_{II}|^2 \\
&= 10
\end{aligned}$$

$$\begin{aligned}
b_I &= \langle I \mid B \rangle \\
&= \frac{1}{3}\overline{\begin{pmatrix} i-\sqrt{3} & 1+2i \end{pmatrix}} \begin{pmatrix} 1+i \\ 2i \end{pmatrix} \\
&= \left(\frac{5-\sqrt{3}}{3}\right) + \left(\frac{1-\sqrt{3}}{3}\right)i \\
b_{II} &= \langle II \mid B \rangle \\
&= \frac{1}{3}\overline{\begin{pmatrix} 1-2i & i+\sqrt{3} \end{pmatrix}} \begin{pmatrix} 1+i \\ 2i \end{pmatrix} \\
&= \frac{1}{3} + \left(1 + \frac{2}{\sqrt{3}}\right)i \\
\| |B\rangle \|^2 &= |b_I|^2 + |b_{II}|^2 \\
&= 6.
\end{aligned}$$

Problem 13

$$\begin{aligned}
\left|\hat{A}\right\rangle &= \frac{|A\rangle}{\sqrt{\langle A \mid A \rangle}} \\
&= \frac{1}{\sqrt{3}} \left(|\hat{e}_1\rangle + |\hat{e}_2\rangle + |\hat{e}_3\rangle\right).
\end{aligned}$$

Problem 17

$$\begin{aligned}
|e_3\rangle &= |M_3\rangle - \langle \hat{e}_1 \mid M_3 \rangle |\hat{e}_1\rangle - \langle \hat{e}_2 \mid M_3 \rangle |\hat{e}_2\rangle \\
&= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} - \frac{1}{2}\operatorname{tr}\left(\hat{e}_1^* M_3\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\operatorname{tr}\left(\hat{e}_2^* M_3\right) \left(\frac{i}{\sqrt{2}}\right) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} - \frac{1+i}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1+i}{4} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-i}{2} & 0 \\ 0 & -\frac{1-i}{2} \end{pmatrix} \\
|\hat{e}_3\rangle &= \frac{|e_3\rangle}{\| |e_3\rangle \|} \\
&= \frac{1-i}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

To find $|e_4\rangle$, we can see that $\| |M_4\rangle \| = 1$ and $\langle \hat{e}_1 \mid M_4 \rangle = \langle \hat{e}_2 \mid M_4 \rangle = \langle \hat{e}_3 \mid M_4 \rangle = 0$. Thus, $|\hat{e}_4\rangle = |M_4\rangle$.

Problem 18

(a) Note that

$$\langle \phi_m \mid \phi_m \rangle = k_m$$

so

$$\begin{aligned} 1 &= \frac{1}{k_m} \langle \phi_m \mid \phi_m \rangle \\ &= \left\langle \frac{1}{\sqrt{k_m}} \phi_m \middle| \frac{1}{\sqrt{k_m}} \phi_m \right\rangle \\ &= \left\langle \hat{\phi}_m \middle| \hat{\phi}_m \right\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} |v\rangle &= \sum_m \left\langle \hat{\phi}_m \middle| v \right\rangle \left| \hat{\phi}_m \right\rangle \\ &= \sum_m \left\langle \frac{1}{\sqrt{k_m}} \phi_m \middle| v \right\rangle \left| \frac{1}{\sqrt{k_m}} \phi_m \right\rangle \\ &= \sum_m \frac{1}{k_m} \langle \phi_m \mid v \rangle \left| \phi_m \right\rangle \\ &= \sum_m c_m \left| \phi_m \right\rangle. \end{aligned}$$

Thus, $c_m = \frac{1}{k_m} \langle \phi_m \mid v \rangle$.

(b)

$$\begin{aligned} \mathrm{id}_V &= \sum_m \left| \hat{\phi}_m \right\rangle \left\langle \hat{\phi}_m \right| \\ &= \sum_m \left| \frac{1}{\sqrt{k_m}} \phi_m \right\rangle \left\langle \frac{1}{\sqrt{k_m}} \phi_m \right| \\ &= \sum_m \frac{1}{k_m} \left| \phi_m \right\rangle \left\langle \phi_m \right|. \end{aligned}$$

Problem 19

$$\begin{aligned} \left(M_{ij}\right)^* &= \left(\langle \hat{e}_i | \mathcal{M} \left| \hat{e}_j \right\rangle \right)^* \\ &= \left\langle \hat{e}_j \middle| \mathcal{M}^* \left| \hat{e}_i \right\rangle \right. \\ &= \overline{M_{ji}}. \end{aligned}$$

Problem 26

We can see that $|\phi_1\rangle$ and $|\phi_2\rangle$ are linearly independent, and similarly are $|\phi_3\rangle$ and $|\phi_4\rangle$. Since $|\phi_2\rangle$ and $|\phi_3\rangle$ are necessarily linearly independent, the collection of $|\phi_i\rangle$ are linearly independent.

Therefore,

$$|\hat{e}_1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
|e_2\rangle &= \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{4}\left[\overline{\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\
|\hat{e}_2\rangle &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|e_3\rangle &= |\phi_3\rangle - \langle \hat{e}_1 \mid \phi_3\rangle - \langle \hat{e}_2 \mid \phi_3\rangle \\
&= \frac{i}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\
|\hat{e}_3\rangle &= \frac{i}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|e_4\rangle &= |\phi_4\rangle - \langle \hat{e}_1 \mid \phi_4\rangle |\hat{e}_1\rangle - \langle \hat{e}_2 \mid \phi_4\rangle |\hat{e}_2\rangle - \langle \hat{e}_3 \mid \phi_4\rangle |\hat{e}_3\rangle \\
&= \frac{1-i}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \\
|\hat{e}_4\rangle &= \frac{1-i}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Problem 29

(a)

$$\mathcal{T} = |2\rangle\langle 1| + |1\rangle\langle 2|$$

(b)

$$
\begin{aligned}
\mathcal{T} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}.
\end{aligned}
$$

Problem 30

$$\begin{aligned}
\mathcal{S} &= |\hat{e}_1\rangle\langle\hat{e}_1| + |-\hat{e}_3\rangle\langle\hat{e}_3| \\
\mathcal{T} &= |\hat{e}_3\rangle\langle\hat{e}_1| + |\hat{e}_2\rangle\langle\hat{e}_2| + |\hat{e}_1\rangle\langle\hat{e}_3|.
\end{aligned}$$

In the standard basis, we have

$$\begin{aligned}
\mathsf{S} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\mathsf{T} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$