Part 1

2.6, Problem 2

(a) Using Mathematica and effective guessing, we land upon an initial condition of $\vec{Y}(0) = \begin{pmatrix} 0 \\ 2.13 \end{pmatrix}$.

(b) All solutions with initial conditions in this curve will have the same periodic solution.

2.6, Problem 3

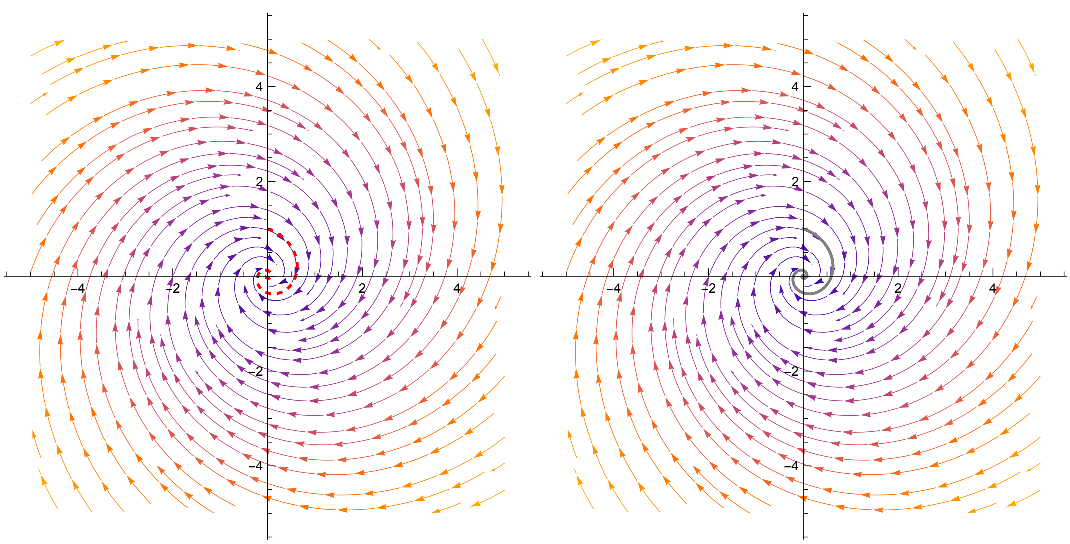
$$\begin{aligned} \frac{d\vec{Y}_1}{dt} &= \frac{d}{dt}\begin{pmatrix} e^{-t}\sin(3t) \\ e^{-t}\cos(3t) \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t}\sin(3t) + 3e^{-t}\cos(3t) \\ -e^{-t}\cos(3t) - 3e^{-t}\sin(3t) \end{pmatrix} \\ &= \begin{pmatrix} -x + 3y \\ -3x - y \end{pmatrix}. \end{aligned}$$

2.6, Problem 4

Since $\vec{Y}_2(t) = \vec{Y}_1(t-1)$ and $\vec{Y}_1(t)$ is a solution, so too is $\vec{Y}_2(t)$.

2.6, Problem 5

Plotting, we see the following.



This does not violate the uniqueness theorem since if $t_0 = 0$ for $\vec{Y}_1$ and $t_0 = 1$ for $\vec{Y}_1$, then the solutions are exactly the same.

2.6, Problem 9

We must have $\vec{Y}_1$ is a phase shift of $\vec{Y}_2$. Specifically, $\vec{Y}_1(t) = \vec{Y}_2(t-1)$.

Chapter 2 Review, Problem 2

Solving $\frac{dx}{dt}$, we must have $y = 0$, which yields $\frac{dy}{dt} = x^2 + 1$. Therefore, there are no equilibrium solutions for this equation.

Chapter 2 Review, Problem 3

$$\begin{aligned} x &= \frac{dy}{dt} \\ \frac{dx}{dt} &= 1 \end{aligned}$$

Chapter 2 Review, Problem 7

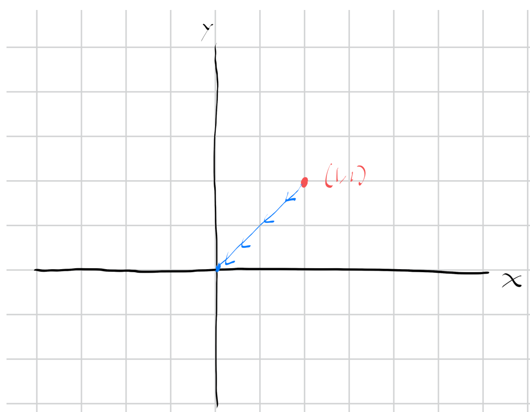
$$\begin{aligned} \frac{dx}{dt} &= -6e^{-6t} \\ &= 2\left(e^{-6t}\right) - 2\left(4e^{-6t}\right) \\ &= 2x - 2y^2 \\ \frac{dy}{dt} &= -6e^{-3t} \\ &= -3y. \end{aligned}$$

Thus, this is a solution to the system of differential equations.

Chapter 2 Review, Problem 12

$$\begin{aligned} \vec{Y}(0.5) &\approx \vec{Y}(0) + 0.5\vec{F}\left(\vec{Y}(0)\right) \\ &= \begin{pmatrix} 3.5 \\ 2 \end{pmatrix}. \end{aligned}$$

Chapter 2 Review, Problem 13



Chapter 2 Review, Problem 15

This is true, as we have shown in the solution to Problem 7.

Chapter 2 Review, Problem 16

This is true, as $y = 0$ means $\frac{dy}{dt} = 0 = -y$, and for $x(t) = 2$, $\frac{dx}{dt} = 0$, meaning this is an equilibrium solution to the differential equation.

Chapter 2 Review, Problem 20

This is true, as phase shifting any solution to a system of differential equations yields another solution to a system of differential equations.

Chapter 2 Review, Problem 30

The phase portrait of the completely decoupled system has all its solution curves as lines.

3.1, Problem 6

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

3.1, Problem 7

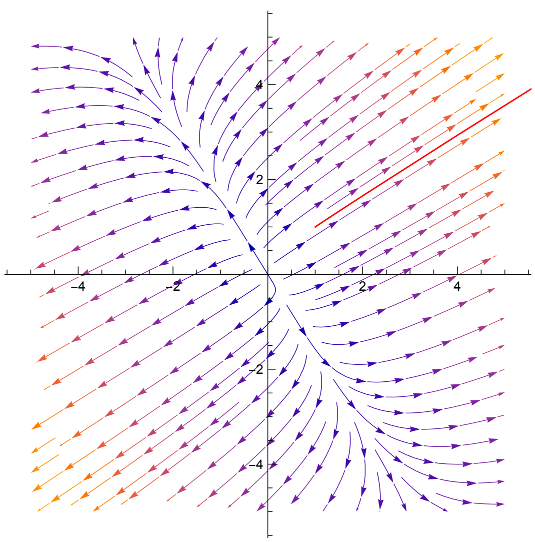
$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 3 & -2 & 7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \\ r(t) \end{pmatrix}$$

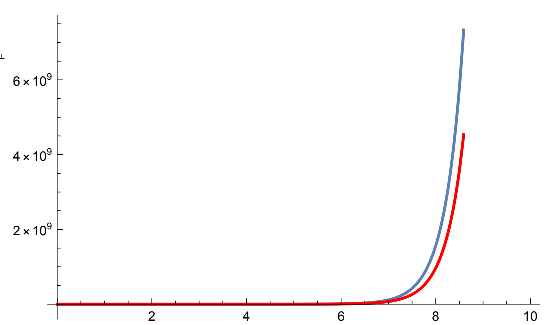
3.1, Problem 8

$$\begin{aligned} \frac{dx}{dt} &= -3x + 2\pi y \\ \frac{dy}{dt} &= 4x - y. \end{aligned}$$

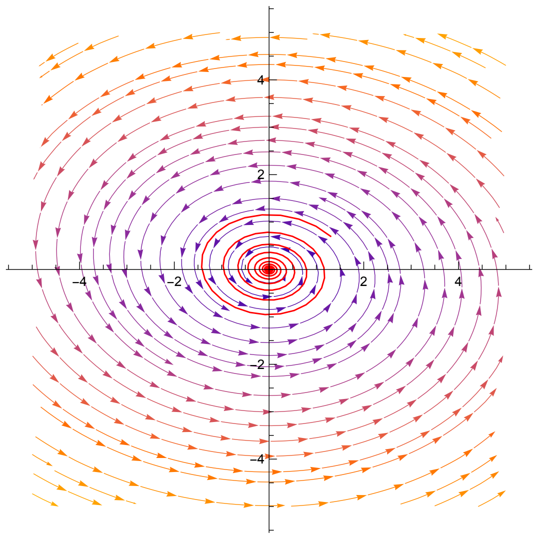
Part 2

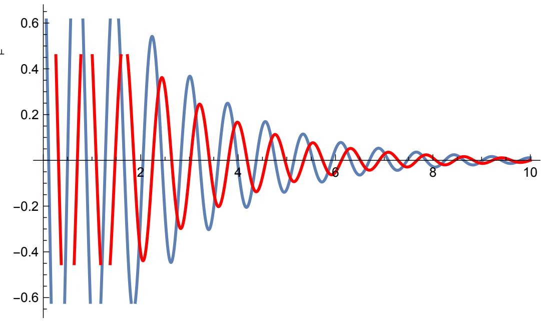
3.1, Problem 10





3.1, Problem 13





3.1, Problem 18

(a) Converting

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = 0,$$

we have

$$v(t) = c$$

for some c.

(b) This means $y(t) = ct + k$ for $k \in \mathbb{R}$.

(c)

3.1, Problem 31

(a) Since

$$3\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 0\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

these vectors are not linearly independent.

(b)

$$-\lambda\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so they are not linearly independent.

(c) If $x_1 \neq 0$, then $y_2 = \frac{x_2 y_1}{x_1}$, meaning $y_2 = \lambda y_1$ and $x_2 = \lambda x_1$, so we use (b). Similarly, if $x_2 \neq 0$, we take $-(x_1 y_2 - x_2 y_1) = x_2 y_1 - x_1 y_2 = 0$ and use (b) again. Finally, if $x_1 = 0$, then we must have y_1 or $x_2 = 0$, both of which yield linear dependence.

3.1, Problem 32

Let

$$x_1y_2 - x_2y_1 \neq 0$$

Suppose toward contradiction that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are not linearly independent. Then, there is λ such that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, meaning we have

$$x_1y_2 - x_2y_1 = \lambda x_2y_2 - x_2\lambda y_2 \qquad = 0.$$

Thus, we must have $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ not linearly independent.

3.1, Problem 35

(a)

$$\frac{dW}{dt} = x_1'(t)y_2(t) + x_1(t)y_2'(t) - \left(x_2'(t)y_1(t) + x_2(t)y_1'(t)\right).$$

(b)

$$\begin{aligned}\frac{dW}{dt} &= x_1'(t)y_2(t) + x_1(t)y_2'(t) - \left(x_2'(t)y_1(t) + x_2(t)y_1'(t)\right) \\ &= \left(ax_1(t) + by_1(t)\right) y_2(t) + x_1(t)\left(cx_2(t) + dy_2(t)\right) - \left(\left(ax_2(t) + by_2(t)\right) y_1(t) + x_2(t)\left(ax_1(t) + by_1(t)\right)\right) \\ &= (a + d)\left(x_1(t)y_2(t) - x_2(t)y_1(t)\right) \\ &= (a + d)\, W(t).\end{aligned}$$

(c)

$$\begin{aligned}\frac{dW}{dt} &= (a + d)\, W(t) \\ W(t) &= e^{(a+d)t}.\end{aligned}$$

(d)

$$\begin{aligned}W(0) &= x_1(0)y_2(0) - x_2(0)y_1(0) \\ &= \det\begin{pmatrix} x_1(0) & x_2(0) \\ y_1(0) & y_2(0) \end{pmatrix} \\ &\neq 0,\end{aligned}$$

meaning

$$\frac{dW}{dt} = (a + d)\, W(t)$$

has a nondegenerate initial condition. Thus, we have

$$W(t) = e^{(a+d)t},$$

which is never zero, meaning $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are always linearly independent.

3.2, Problem 8

(a)

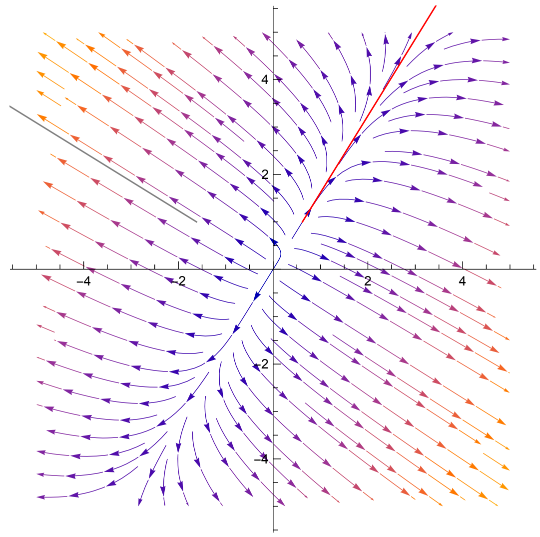
$$\begin{aligned}
\det\begin{pmatrix}2-\lambda & -1\\ -1 & 1-\lambda\end{pmatrix} &= (\lambda-2)(\lambda-1)-1\\
&= \lambda^2-3\lambda-1\\
\frac{5}{4} &= \left(\lambda-\frac{3}{2}\right)^2\\
\lambda_1 &= \frac{3+\sqrt{5}}{2}\\
\lambda_2 &= \frac{3-\sqrt{5}}{2}.
\end{aligned}$$

(b)

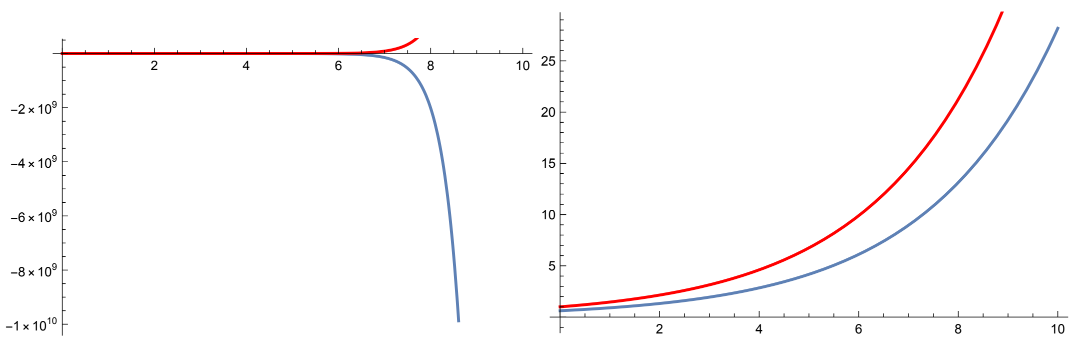
$$\begin{aligned}
\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix} &= \left(\frac{3+\sqrt{5}}{2}\right)\begin{pmatrix}x_1\\ y_1\end{pmatrix}\\
2x_1-y_1 &= \frac{3+\sqrt{5}}{2}x_1\\
-x_1+y_1 &= \frac{3+\sqrt{5}}{2}y_1\\
x_1 &= -\frac{1+\sqrt{5}}{2}y_1\\
\vec{v}_1 &= \begin{pmatrix}-\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix} &= \left(\frac{3-\sqrt{5}}{2}\right)\begin{pmatrix}x_2\\ y_2\end{pmatrix}\\
2x_2-y_2 &= \frac{3-\sqrt{5}}{2}x_2\\
-x_2+y_2 &= \frac{3-\sqrt{5}}{2}y_2\\
x_2 &= -\frac{1-\sqrt{5}}{2}y_2\\
\vec{v}_2 &= \begin{pmatrix}-\frac{1-\sqrt{5}}{2}\\ 1\end{pmatrix}.
\end{aligned}$$

(c) In the following image, the gray line represents $\lambda_1 = -\frac{3+\sqrt{5}}{2}$, while the red line represents $\lambda_2 = \frac{3-\sqrt{5}}{2}$.



(d) Left: $\lambda_1 = -\frac{1+\sqrt{5}}{2}$, Right: $\lambda_2 = \frac{\sqrt{5}-1}{2}$.



(e) The general solution is

$$\vec{Y}(t) = k_1 e^{\frac{3+\sqrt{5}}{2}t}\begin{pmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} + k_2 e^{\frac{\sqrt{5}-1}{2}t}\begin{pmatrix} \frac{3-\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

3.2, Problem 9

(a)

$$\begin{aligned} \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} &= (\lambda-2)(\lambda-1)-1 \\ \frac{5}{4} &= \left(\lambda - \frac{3}{2}\right)^2 \\ \lambda_1 &= \frac{3+\sqrt{5}}{2} \\ \lambda_2 &= \frac{3-\sqrt{5}}{2}. \end{aligned}$$

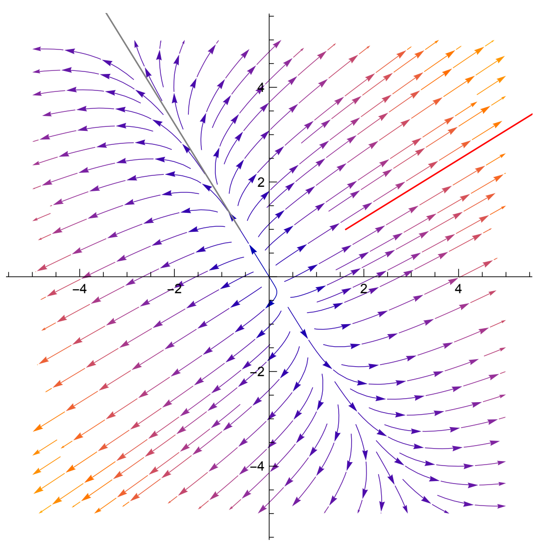
(b)

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \left(\frac{3+\sqrt{5}}{2}\right)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

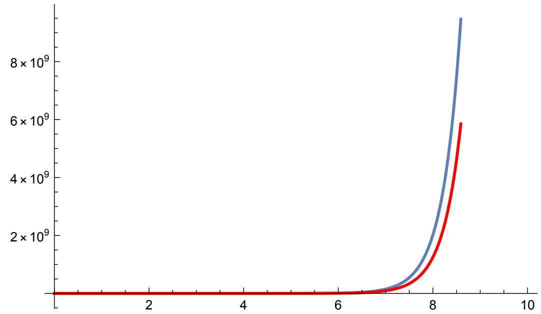
$$
\begin{aligned}
2x_1 + y_1 &= \frac{3+\sqrt{5}}{2}x_1\\
x_1 + y_1 &= \frac{3+\sqrt{5}}{2}y_1\\
x_1 &= \frac{1+\sqrt{5}}{2}y_1\\
\vec{v}_1 &= \begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}
\end{aligned}
$$

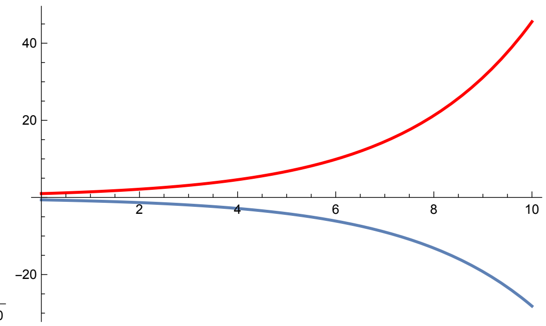
$$
\begin{aligned}
\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix} &= \left(\frac{3-\sqrt{5}}{2}\right)\begin{pmatrix}x_2\\ y_2\end{pmatrix}\\
2x_2 + y_2 &= \frac{3-\sqrt{5}}{2}x_2\\
x_2 + y_2 &= \frac{3-\sqrt{5}}{2}y_2\\
x_2 &= \frac{1-\sqrt{5}}{2}y_2\\
\vec{v}_2 &= \begin{pmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{pmatrix}.
\end{aligned}
$$

(c) In the following image, the red line represents $\lambda_1 = \frac{3+\sqrt{5}}{2}$, while the gray line represents $\lambda_2 = \frac{3-\sqrt{5}}{2}$.



(d) Left: $\lambda_1 = \frac{3+\sqrt{5}}{2}$, Right: $\lambda_2 = \frac{3-\sqrt{5}}{2}$





(e) The general solution is

$$\vec{Y}(t) = k_1 e^{\frac{3+\sqrt{5}}{2}t}\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix} + k_2 e^{\frac{3-\sqrt{5}}{2}t}\begin{pmatrix}\frac{1-\sqrt{5}}{2}1\end{pmatrix}.$$

3.2, Problem 12

Solving for the eigenvalues, we have

$$\det\begin{pmatrix}3-\lambda & 0\\ 1 & -2-\lambda\end{pmatrix} = (\lambda-3)(\lambda+2),$$

meaning

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -2.\end{aligned}$$

The corresponding eigenvectors are

$$\begin{aligned}\vec{v}_1 &= \begin{pmatrix}5\\ 1\end{pmatrix}\\ \vec{v}_2 &= \begin{pmatrix}0\\ 1\end{pmatrix},\end{aligned}$$

meaning the general solution is

$$\vec{Y}(t) = k_1 e^{3t}\begin{pmatrix}5\\ 1\end{pmatrix} + k_2 e^{-2t}\begin{pmatrix}0\\ 1\end{pmatrix}.$$

(1) With initial condition $(1, 0)$, we have

$$\begin{aligned}5k_1 &= 1\\ k_1 + k_2 &= 0,\end{aligned}$$

so

$$\begin{aligned}k_1 &= \frac{1}{5}\\ k_2 &= -\frac{1}{5},\end{aligned}$$

and our solution is

$$\vec{Y}_1(t) = \frac{1}{5}e^{3t}\begin{pmatrix}5\\ 1\end{pmatrix} - \frac{1}{5}e^{-2t}\begin{pmatrix}0\\ 1\end{pmatrix}.$$

(2) With initial condition $(0, 1)$, we have $k_1 = 0$ necessarily and $k_2 = 1$. Thus, our solution is

$$\vec{Y}_2(t) = e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}.$$

(3) With initial condition $(2, 2)$, we have

$$\begin{aligned}5k_1 &= 2\\ k_1 + k_2 &= 2,\end{aligned}$$

so $k_1 = \frac{2}{5}$ and $k_2 = \frac{8}{5}$. Thus, our solution is

$$\vec{Y}_3 = \frac{2}{5}e^{3t}\begin{pmatrix}5\\1\end{pmatrix} + \frac{8}{5}e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}.$$

3.2, Problem 16

$$\begin{aligned}\det(A - \lambda I) &= \det\begin{pmatrix}a-\lambda & b\\ 0 & d-\lambda\end{pmatrix}\\ &= (a-\lambda)(d-\lambda),\\ \lambda_1 &= a\\ \lambda_2 &= d\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix} &= a\begin{pmatrix}x_1\\y_1\end{pmatrix}\\ ax_1 + by_1 &= ax_1\\ dy_1 &= ay_1\\ x_1 &= 1\\ y_1 &= 0\\ \vec{v}_1 &= \begin{pmatrix}1\\0\end{pmatrix},\end{aligned}$$

and similarly,

$$\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}.$$

3.2, Problem 17

$$\begin{aligned}\det\begin{pmatrix}a-\lambda & b\\ b & d-\lambda\end{pmatrix} &= (\lambda - a)(\lambda - d) - b^2\\ b^2 &= \lambda^2 - (a+d)\lambda + ad\\ b^2 + \frac{(a-d)^2}{4} &= \left(\lambda - \frac{a+d}{2}\right)^2.\end{aligned}$$

Since the left hand side is positive and nonzero, it is the case that there are two distinct real eigenvalues if $b \neq 0$.

3.2, Problem 18

$$\begin{aligned}
\det\begin{pmatrix} a-\lambda & b \\ c & -\lambda \end{pmatrix} &= \lambda(\lambda - a) - bc \\
bc &= \lambda^2 - a\lambda \\
bc + \frac{a^2}{4} &= \left(\lambda - \frac{a}{2}\right)^2 \\
\lambda &= \frac{a}{2} \pm \frac{\sqrt{4bc + a^2}}{2}.
\end{aligned}$$

It may not necessarily be the case that $4bc + a^2$ is positive, meaning that, unlike the case of problem 16, there is no guarantee of real eigenvalues.