Prelude

My REU mentor recently bought me the book *Banach Algebra Techniques in Operator Theory*, so I'm going to be reading through it here. Astute readers may already know that I am also reading through the book *Quantum Theory for Mathematicians*, and may be wondering if this is going to crowd out that book. The answer is yes — but I don't really care that much. If I come out of the summer knowing more things than I knew entering, then I will have succeeded.

Prerequisite Notes

Since Douglas's book is very advanced, I'm going to end up going back and reading other important material in order to contextualize the parts of the book I don't fully understand.

Tychonoff's Theorem

I'm drawing information for this section from Volker Runde's book *A Taste of Topology*, specifically from Chapter 3.

Definition (Product Topology)**.** Let $\{(X_i, \tau_i)\}_i$ be a family of topological spaces, and $X = \prod_{i\in I} X_i$.

The product topology on X is the coarsest topology τ on X such that

$$\prod_i : X \to X_i; \quad f \mapsto f(i)$$

is continuous.

The product topology's open sets are of the form

$$\bigcap_{j=1}^{n} \pi_{i_j}(U_j),$$

where $i_j \in I$. The product topology is the topology of coordinatewise convergence.

Theorem (Tychonoff's)**.** *Let $\{(K_i, \tau_i)\}_{i\in I}$ be a nonempty family of compact topological spaces. Then, the product space $K = \prod_{i\in I} K_i$ is compact in the product topology.*

Proof. Let $\{f_\alpha\}_{\alpha\in A}$ be a net[i] in K. Let $J \subseteq I$ be nonempty, and let $f \in K$.

We call (J, f) a partial accumulation point of $\{f_\alpha\}_{\alpha\in A}$ if $f|_J$ is a accumulation point of $\{f_\alpha|_J\}_{\alpha\in A}$ in $\prod_{j\in J} K_j$. A partial accumulation point of $\{f_\alpha\}_{\alpha\in A}$ is a accumulation point of

[i]See future definition of nets.

$\{f_\alpha\}_{\alpha\in A}$ if and only if $J = I$.

Let $\mathcal{P}$ be the set of partial accumulation points of $\{f_\alpha\}_{\alpha\in A}$ For any two $(J_f, f), (J_g, g) \in \mathcal{P}$, define the order $(J_f, f) \leq (J_g, g)$ if and only if $J_f \subseteq J_g$ and $g|_{J_f} = f$.

Since K_i is compact for each $i \in I$, the net $\{f_\alpha\}_\alpha$ has partial accumulation points $(\{i\}, f_i)$ for each $i \in I$ (since each K_i is compact, the net analogue to sequential compactness holds); in particular, $\mathcal{P}$ is nonempty.

Let $\mathcal{Q}$ be a totally ordered subset of $\mathcal{P}$, and $J_g = \bigcup \{J_f \mid (J_f, f) \in \mathbb{Q}\}$. Define g by letting $g(j) = f(j)$ for each $j \in J_f$ with $(J_f, f) \in \mathcal{Q}$, and arbitrarily on $I \setminus J_g$.

Since $\mathcal{Q}$ is totally ordered, g is well-defined. We claim that (J_g, g) is a partial accumulation point of $\{f_\alpha\}_\alpha$.

Let $N \subseteq \prod_{j\in J_g} K_j$ be a neighborhood of $g|_{J_g}$. We may suppose that

$$N = \pi_{j_1})^{-1}(U_{j_1}) \cap \cdots \cap \pi_{j_n}(U_{j_n}),$$

where $j_1, \dots, j_n \in J_g$, and $U_{j_i} \subseteq K_{j_i}$ are open.

Let $(J_h, h) \in \mathcal{Q}$ be such that $\{j_1, \dots, j_n\} \subseteq J_h$, which is possible since $\mathcal{Q}$ is totally ordered. Since (J_h, h) is a partial accumulation point of $\{f_\alpha\}_\alpha$, there is an index α and a $\beta \geq \alpha$, where

$$f_\beta(j_k) = \pi_{j_k}(f_\beta)\, U_{j_k},$$

so $f_\beta \in N$. Thus, (J_g, g) is a partial accumulation point of $\{f_\alpha\}_\alpha$, and is an element of $\mathcal{P}$.

By Zorn's lemma,[ii] $\mathcal{P}$ has a maximal element, $(J_{\max}, f_{\max})$.

Suppose toward contradiction that $J_{\max} \subset I$, meaning there is an $i_0 \in I \setminus J_{\max}$. Since $(J_{\max}, f_{\max})$ is a partial accumulation point of $\{f_\alpha\}_\alpha$, there is a subnet $\{f_{\alpha_\beta}\}_\beta$ such that $\pi_j(f_{\alpha_\beta}) \to \pi_j(f_{\max})$ for each $j \in J_{\max}$.

Since K_{i_0} is compact, we find a subnet $\left\{f_{\alpha_{\beta_\gamma}}\right\}_\gamma$ such that $\pi_{i_0}\left(f_{\alpha_{\beta_\gamma}}\right)_\gamma$ converges to x_{i_0} in K_{i_0}.

Define $\tilde{f} \in K$ by setting $\tilde{f}|_{J_{\max}} = f_{\max}$, and $\tilde{f}(i_0) = x_{i_0}$. Thus, $\left(J_{\max} \cup \{i_0\}, \tilde{f}\right)$ is a partial accumulation point, which contradicts the maximality of $(J_{\max}, f_{\max})$. $\square$

[ii] In a partially ordered set, if every totally ordered subset contains an upper bound, the set contains a maximal element

Banach Spaces

Let X be a compact Hausdorff space, and let $C(X)$ denote the set of continuous functions $f : X \to \mathbb{C}$. For $f_1, f_2 \in C(X)$ and $\lambda \in \mathbb{C}$, we define

(1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

(2) $(\lambda f_1)(x) = \lambda f_1(x)$

(3) $(f_1 f_2)(x) = f_1(x) f_2(x)$

With these operations, $C(X)$ is a commutative algebra[iii] with identity over the field $\mathbb{C}$.

For each $f \in C(X)$, f is bounded (since X is compact and f is continuous); thus, $\sup |f| < \infty$. We call this the norm of f, and denote it

$$\|f\|_\infty = \sup \{|f(x)| \mid x \in X\}.$$

Proposition (Properties of the Norm on $C(X)$)**.**

(1) Positive Definiteness: $\|f\|_\infty = 0 \Leftrightarrow f = 0$

(2) Absolute Homogeneity: $\|\lambda f\|_\infty = |\lambda| \, \|f\|_\infty$

(3) Subadditivity (Triangle Inequality): $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

(4) Submultiplicativity: $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$

We define a metric ρ on $C(X)$ by $\rho(f, g) = \|f - g\|_\infty$.

Proposition (Properties of the Induced Metric on $C(X)$)**.**

(1) $\rho(f, g) = 0 \Leftrightarrow f = g$

(2) $\rho(f, g) = \rho(g, f)$

(3) $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$

Proposition (Completeness of $C(X)$)**.** *If X is a compact Hausdorff space, then $C(X)$ is a complete metric space.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be Cauchy. Then,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq \|f_n - f_m\|_\infty \\ &= \rho(f_n, f_m) \end{aligned}$$

for each $x \in X$. Thus, $\{f_n(x)\}_{n=1}^\infty$ is Cauchy for each $x \in X$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We will need to show that this implies $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

[iii] A vector space with multiplication.

Let $\varepsilon > 0$; choose N such that $n, m \geq N$ implies $\|f_n - f_m\|_\infty < \varepsilon$. For $x_0 \in X$, there exists a neighborhood U such that $|f_N(x_0) - f_N(x)| < \varepsilon$ for $x \in U$.[iv] Thus,

$$\begin{aligned}|f(x_0) - f(x)| &= |f_n(x_0) - f_N(x_0) + f_N(x_0) - f_N(x) + f_N(x) - f_n(x)|\\ &\leq |f_n(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f_n(x)|\\ &\leq 3\varepsilon.\end{aligned}$$

Thus, f is continuous. Additionally, for $n \geq N$ and $x \in X$, we have

$$\begin{aligned}|f_n(x) - f(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)|\\ &\leq \lim_{m\to\infty} \|f_n - f_m\|_\infty\\ &\leq \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, meaning $C(X)$ is complete. □

Definition (Banach Space)**.** A Banach space is a vector space over $\mathbb{C}$ with a norm $\|\cdot\|$ is complete with respect to the induced metric.

Proposition (Properties of the Banach Space Operations)**.** *Let $\mathcal{X}$ be a Banach space. The functions*

- $a : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$; $a(f, g) = f + g$,
- $s : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$; $s(\lambda, f) = \lambda f$,
- $n : \mathcal{X} \to \mathbb{R}^+$; $n(f) = \|f\|$

are continuous.

Definition (Directed Sets and Nets)**.** Let A be a partially ordered set with ordering $\leq$. We say A is directed if for each $\alpha, \beta \in A$, there exists a γ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

A net is a map $\alpha \mapsto \lambda_\alpha$, where $\alpha \in A$ for some directed set A.

Definition (Convergence of Nets)**.** Let $\{\lambda_\alpha\}$ be a net in X. We say the net converges to $\lambda \in X$ if for every neighborhood U of λ, there exists α_U such that for $\alpha \geq \alpha_U$, every λ_α is contained in U.[v]

Definition (Cauchy Nets in Banach Spaces)**.** A net $\{f_\alpha\}_\alpha$ in a Banach space $\mathcal{X}$ is said to be a Cauchy net if for every $\varepsilon > 0$, there exists α_0 in A such that $\alpha_1, \alpha_2 \geq \alpha_0$ implies $\|f_{\alpha_1} - f_{\alpha_2}\| < \varepsilon$.

Proposition (Convergence of Cauchy Nets in Banach Spaces)**.** *In a Banach space, every Cauchy net is convergent.*

[iv] This is by the continuity of $\{f_n\}_n$.

[v] The net convergence generalizes sequence convergence in a metric space to the case where X does not have a metric.

Proof. Let $\{f_\alpha\}_\alpha$ be a Cauchy net in $\mathcal{X}$. Choose α_1 such that $\alpha \geq \alpha_1$ implies $\|f_\alpha - f_{\alpha_1}\| < 1$.

We iterate this process by choosing $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies $\left\|f_\alpha - f_{\alpha_{n+1}}\right\| < \frac{1}{n+1}$.

The sequence $\{f_{\alpha_n}\}_{n=1}^{\infty}$ is Cauchy, and since $\mathcal{X}$ is complete, there exists $f \in \mathcal{X}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.

We must now prove that $\lim_{\alpha \in A} f_\alpha = f$. Let $\varepsilon > 0$. Choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$, and $\|f_{\alpha_n} - f_\alpha\| < \frac{\varepsilon}{2}$. Then, for $\alpha \geq \alpha_n$, we have

$$\begin{aligned}\|f_\alpha - f\| &\leq \|f_\alpha - f_{\alpha_n}\| + \|f_{\alpha_n} - f\|\\ &< \frac{1}{n} + \frac{\varepsilon}{2}\\ &< \varepsilon.\end{aligned}$$

$\square$

Definition (Convergence of Infinite Series)**.** Let $\{f_\alpha\}_\alpha$ be a set of vectors in $\mathcal{X}$. Let $\mathcal{F} = \{F \subseteq A \mid F \text{ finite}\}$.

Define the ordering $F_1 \leq F_2 \Leftrightarrow F_1 \subseteq F_2$.[vi] For each F, define

$$g_F = \sum_{\alpha \in F} f_\alpha.$$

If $\{g_F\}_{F \in \mathcal{F}}$ converges to some $g \in \mathcal{X}$, then

$$\sum_{\alpha \in A} f_\alpha$$

converges, and we write

$$g = \sum_{\alpha \in A} f_\alpha.$$

Proposition (Absolute Convergence of Series in Banach Space)**.** *Let $\{f_\alpha\}_\alpha$ be a set of vectors in the Banach space $\mathcal{X}$. Suppose $\sum_{\alpha \in A} \|f_\alpha\|$ converges in $\mathbb{R}$. Then, $\sum_{\alpha \in A} f_\alpha$ converges in $\mathcal{X}$.*

Proof. All we need show is $\{g_F\}_{F \in \mathcal{F}}$ is Cauchy. Since $\sum_{\alpha \in A} \|f_\alpha\|$ converges, there exists $F_0 \in \mathcal{F}$ such that $F \geq F_0$ implies

$$\sum_{\alpha \in F} \|f_\alpha\| - \sum_{\alpha \in F_0} \|f_\alpha\| < \varepsilon.$$

[vi] the inclusion ordering

Thus, for $F_1, F_2 \geq F_0$, we have

$$\begin{aligned}
\|g_{F_1} - g_{F_2}\| &= \left\| \sum_{\alpha \in F_1} f_\alpha - \sum_{\alpha \in F_2} f_\alpha \right\| \\
&= \left\| \sum_{\alpha \in F_1 \setminus F_2} f_\alpha - \sum_{\alpha \in F_2 \setminus F_1} \right\| \\
&\leq \sum_{\alpha \in F_1 \setminus F_2} \|f_\alpha\| + \sum_{\alpha \in F_2 \setminus F_1} \|f_\alpha\| \\
&\leq \sum_{\alpha \in F_1 \cup F_2} \|f_\alpha\| - \sum_{\alpha \in F_0} \|f_\alpha\| \\
&< \varepsilon.
\end{aligned}$$

Thus, $\{g_F\}_{F \in \mathcal{F}}$ is Cauchy, and thus the series is convergent. □

Theorem (Absolute Convergence Criterion for Banach Spaces)**.** *Let $\mathcal{X}$ be a normed vector space. Then, $\mathcal{X}$ is a Banach space if and only if for every sequence $\{f_n\}_{n=1}^\infty$ of vectors in $\mathcal{X}$,*

$$\sum_{n=1}^\infty \|f_n\| < \infty \Rightarrow \sum_{n=1}^\infty f_n \textit{ convergent.}$$

Proof. The forward direction follows from the previous proposition.

Let $\{g_n\}_{n=1}^\infty$ be a Cauchy sequence in a normed vector space where

$$\sum_{n=1}^\infty \|f_n\| < \infty \Rightarrow \sum_{n=1}^\infty f_n \text{ convergent.}$$

We select a subsequence $\{g_{n_k}\}_{k=1}^\infty$ as follows. Choose n_1 such that $i, j \geq n_1$ implies $\|g_i - g_j\| < 1$; recursively, we select n_{N+1} such that $\|g_{N+1} - g_N\| < 2^{-N}$. Then,

$$\sum_{k=1}^\infty \|g_{k+1} - g_k\| < \infty.$$

Set $f_k = g_{n_k} - g_{n_{k-1}}$ for $k > 1$, with $f_1 = g_{n_1}$. Then,

$$\sum_{k=1}^\infty \|f_k\| < \infty,$$

meaning $\displaystyle\sum_{k=1}^\infty f_k$ converges. Thus, $\{g_{n_k}\}_{k=1}^\infty$ converges, meaning $\{g_n\}_{n=1}^\infty$ converges in $\mathcal{X}$. □

Definition (Bounded Linear Functional)**.** Let $\mathcal{X}$ be a Banach space. A function $\varphi : \mathcal{X} \to \mathbb{C}$ is known as a bounded linear functional if

(1) $\varphi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1\varphi(f_1) + \lambda_2\varphi(f_2)$ for each $\lambda_1, \lambda_2 \in \mathbb{C}$ and $f_1, f_2 \in \mathcal{X}$.

(2) There exists M such that $|\varphi(f)| \leq M\|f\|$ for each $f \in \mathcal{X}$.

Proposition (Equivalent Criteria for Bounded Linear Functionals)**.** *Let φ be a linear functional on $\mathcal{X}$. Then, the following conditions are equivalent:*

(1) φ is bounded;

(2) φ is continuous;

(3) φ is continuous at 0.

Proof. (1) $\Rightarrow$ (2): If $\{f_\alpha\}_{\alpha\in A}$ is a net in $\mathcal{X}$ converging to f, then $\lim_{\alpha\in A}\|f_\alpha - f\| = 0$. Thus,

$$\begin{aligned}\lim_{\alpha\in A}|\varphi(f_\alpha) - \varphi(f)| &= \lim_{\alpha\in A}|\varphi(f_\alpha - f)|\\ &\leq \lim_{\alpha\in F} M\|f_\alpha - f\|\\ &= 0\end{aligned}$$

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): If φ is continuous at 0, then there exists $\delta > 0$ such that $\|f\| < \delta \Rightarrow |\varphi(f)| < 1$. Thus, for any $g \in X$ nonzero, we have

$$\begin{aligned}|\varphi(g)| &= \frac{2\|g\|}{\delta}\left|\varphi\left(\frac{\delta}{2\|g\|}g\right)\right|\\ &< \frac{2}{\delta}\|g\|,\end{aligned}$$

meaning φ is bounded.

□

Definition (Dual Space)**.** Let $\mathcal{X}^*$ be the set of bounded linear functionals on $\mathcal{X}$. For each $\varphi \in \mathcal{X}^*$, define

$$\|\varphi\| = \sup_{\|f\|=1}|\varphi(f)|.$$

We say $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Proposition (Completeness of the Dual Space)**.** *For $\mathcal{X}$ a Banach space, $\mathcal{X}^*$ is a Banach space.*

Proof. Both positive definiteness and absolute homogeneity are apparent from the definition of the norm. We will now show the triangle inequality as follows. Let $\varphi_1, \varphi_2 \in \mathcal{X}^*$. Then,

$$\begin{aligned}\|\varphi_1 + \varphi_2\| &= \sup_{\|f\|=1} |\varphi_1(f) + \varphi_2(f)| \\ &\leq \sup_{\|f\|=1} |\varphi_1(f)| + \sup_{\|f\|=1} |\varphi_2(f)| \\ &= \|\varphi_1\| + \|\varphi_2\|.\end{aligned}$$

We must now show completeness. Let $\{\varphi_n\}_n$ be a sequence in $\mathcal{X}^*$. Then, for every $f \in \mathcal{X}$, it is the case that

$$|\varphi_n(f) - \varphi_m(f)| \leq \|\varphi_n - \varphi_m\| \|f\|,$$

meaning $\{\varphi_n(f)\}_n$ is Cauchy for each f. Define $\varphi(f) = \lim_{n\to\infty} \varphi_n(f)$. It is clear that $\varphi(f)$ is linear, and for N such that $n, m \geq N \Rightarrow \|\varphi_n - \varphi_m\| < 1$,

$$\begin{aligned}|\varphi(f)| &\leq |\varphi(f) - \varphi_N(f)| + |\varphi_N(f)| \\ &\leq \lim_{n\to\infty} |\varphi_n(f) - \varphi_N(f)| + |\varphi_N(f)| \\ &\leq \left(\lim_{n\to\infty} \|\varphi_n - \varphi_N\| + \|\varphi_N\|\right) \|f\| \\ &\leq (1 + \|\varphi_N\|) \|f\|,\end{aligned}$$

so φ is bounded. Thus, we must show that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$. Let $\varepsilon > 0$. Set N such that $n, m \geq N \Rightarrow \|\varphi_n - \varphi_m\| < \varepsilon$. Then, for $f \in \mathcal{X}$,

$$\begin{aligned}|\varphi(f) - \varphi_n(f)| &\leq |\varphi(f) - \varphi_m(f)| + |\varphi_m(f) - \varphi_n(f)| \\ &\leq |(\varphi - \varphi_m)(f)| + \varepsilon \|f\|.\end{aligned}$$

Since $\lim_{m\to\infty} |(\varphi - \varphi_m)(f)| = 0$, we have $\|\varphi - \varphi_m\| < \varepsilon$. □

Proposition (Banach Spaces and their Duals)**.**

(1) The space ℓ^∞ consists of the set of bounded sequences. For $f \in \ell^\infty$, the norm on f is computed as $\|f\|_\infty = \sup_n |f(n)|$.

(2) The subspace $c_0 \subseteq \ell^\infty$ consists of all sequences that vanish at ∞. The norm on c_0 is inherited from the norm on ℓ_∞.

(3) The space ℓ^1 consists of the set of all absolutely summable sequences. For $f \in \ell^1$, the norm on f is computed as $\|f\| = \sum_{n=1}^{\infty} |f(n)|$.

We claim that these are all Banach spaces.

We also claim that $c_0^* = \ell^1$, and $(\ell^1)^* = \ell^\infty$.

Proofs of Banach Space.

ℓ^∞:

Proof of Normed Vector Space: Let $a, b \in \ell^\infty$, and $\lambda \in \mathbb{C}$. Then,

$$\sup_n |a(n)| = 0$$

if and only if a is the zero sequence. Additionally, we have that

$$\begin{aligned} \|\lambda a\|_\infty &= \sup_n |\lambda a(n)| \\ &= |\lambda| sup_n |a(n)| \\ &= |\lambda| \, \|a\|_\infty , \end{aligned}$$

meaning $\|\cdot\|_\infty$ is absolutely homogeneous. Finally,

$$\begin{aligned} \|a + b\|_\infty &= \sup_n |a(n) + b(n)| \\ &\leq \sup_n |a(n)| + \sup_n |b(n)| \\ &= \|a\|_\infty + \|b\|_\infty . \end{aligned}$$

Proof of Completeness: Let $\{a_n\}_{n=1}^\infty$ be a Cauchy sequence of elements of ℓ^∞. Let $\varepsilon > 0$, and let N be such that $\|a_n - a_m\|_\infty < \varepsilon$ for $n, m \geq N$. Then, for each k,

$$\begin{aligned} |a_n(k) - a_m(k)| &= |(a_n - a_m)(k)| \\ &\leq \|a_n - a_m\| \\ &< \varepsilon, \end{aligned}$$

meaning that $a_n(k)$ is Cauchy in $\mathbb{C}$ for each k.

Set $a(k) = \lim_{n\to\infty} a_n(k)$. We must now show that $\lim_{n\to\infty} \|a - a_n\| = 0$. Let $\varepsilon > 0$, and set N such that for $n, m \geq N$, $\|a_m - a_n\| < \varepsilon$. Then,

$$\begin{aligned} |a(k) - a_n(k)| &\leq |a(k) - a_m(k)| + |a_m(k) - a_n(k)| \\ &\leq |a(k) - a_m(k)| + \|a_m - a_n\| \\ &< |a(k) - a_m(k)| + \varepsilon. \end{aligned}$$

Since $\lim_{m\to\infty} |a(k) - a_m(k)| = 0$, we have $\|a - a_n\| < \varepsilon$.[vii]

c_0:

[vii] The reason we had to go about it like this was that we defined the sequence a pointwise; however, we need to show convergence *in norm*.

Proof of Subspace: Let $a, b \in c_0$, and $\lambda \in \mathbb{C} \setminus \{0\}$. Let $\varepsilon > 0$. Set N_1 such that $|a(n)| < \frac{\varepsilon}{2|\lambda|}$ for all $n \geq N_1$, and set N_2 such that $|b(n)| < \frac{\varepsilon}{2}$ for all $n \geq N_2$.

Then, for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} |\lambda a(n) + b(n)| &\leq |\lambda||a(n)| + |b(n)| \\ &< |\lambda|\frac{\varepsilon}{2|\lambda|} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Proof of Completeness: In order to show completeness, we must show that c_0 is closed in ℓ^∞. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence in c_0, with $a_k \to a$.

We will need to show that $a \in c_0$.[viii] Let $\varepsilon > 0$, and set K such that for all $k \geq K$, $\|a_k - a\| < \varepsilon/2$. For each k, choose N such that $|a_k(n)| < \varepsilon/2$ for all $n \geq N$. Then, for all $n \geq N$,

$$\begin{aligned} |a(n)| &\leq |a(n) - a_k(n)| + |a_k(n)| \\ &< \|a - a_k\| + |a_k(n)| \\ &< \varepsilon. \end{aligned}$$

Since c_0 is closed in ℓ^∞, it is thus complete.

ℓ^1:

Proof of Normed Vector Space: Let $a, b \in \ell^1$, and $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned} \|\lambda a + b\| &= \sum_{k=1}^{\infty} |\lambda a(k) + b(k)| \\ &\leq \sum_{k=1}^{\infty} |\lambda a(k)| + \sum_{k=1}^{\infty} |b(k)| \\ &= |\lambda| \sum_{k=1}^{\infty} |a(k)| + \sum_{k=1}^{\infty} |b(k)| \\ &= |\lambda| \|a\| + \|b\|. \end{aligned}$$

Thus, $\lambda a + b \in \ell^1$. We have also shown both the triangle inequality and absolute homogeneity. We can also see that, if $\|a\| = 0$,

$$\begin{aligned} \|a\| &= \sum_{k=1}^{\infty} |a(k)| \\ &= 0, \end{aligned}$$

which is only true if $a(k) = 0$ for all k.

[viii] Sequential criterion for closure.

□

Example (Pointwise Convergence and Convergence in Norm)**.** Consider a sequence $\{\varphi_n\}_n$ in $\mathcal{X}^*$. If the sequence converges in norm to φ, then it must also converge pointwise. However, the converse isn't true.

For each k, define $L_k(f) = f(k)$, where $f \in \ell^1$. We can see that $L_k \in (\ell^1)^*$, and $\lim_{k\to\infty} L_k(f) = 0$ for each $f \in \ell^1$. The sequence of L_k thus converges to the zero functional pointwise, but since $\|L_k\| = 1$ always, it isn't the case that L_k converges to the zero functional in norm.

Definition (Weak Topology and w^*-Topology)**.** Let X be a set, Y a topological space, and $\mathcal{F}$ be a family of functions from X to Y. The weak topology on X is the topology for which all functions in $\mathcal{F}$ are continuous.

For each f in $\mathcal{X}$, let $\hat{f} : \mathcal{X}^* \to \mathbb{C}$ be defined by $\hat{f}(\varphi) = \varphi(f)$. The w^*-topology on $\mathcal{X}^*$ is the weak topology on $\mathcal{X}^*$ defined by the family of functions $\left\{\hat{f} \mid f \in \mathcal{X}\right\}$.

If Y is Hausdorff and $\mathcal{F}$ separates the points of X, then the weak topology is Hausdorff.[ix]

Proposition (Hausdorff Property of w^*-Topology)**.** *The w^*-topology on $\mathcal{X}^*$ is Hausdorff.*

Proof. If $\varphi_1 \neq \varphi_2$, then there exists at least one f such that $\varphi_1(f) \neq \varphi_2(f)$, meaning $\left\{\hat{f} \mid f \in \mathcal{X}\right\}$ separates the points of $\mathcal{X}^*$, so the w^*-topology is Hausdorff. □

Proposition (Convergence in the w^*-Topology)**.** *A net $\{\varphi_\alpha\}_\alpha$ converges to $\varphi \in \mathcal{X}^*$ in the w^* topology if and only if $\lim_{\alpha\in A} \varphi_\alpha = \varphi$.*[x]

Proposition (Determination of the w^*-Topology)**.** *Let $\mathcal{M}$ be a dense subset of $\mathcal{X}$, and let $\{\varphi_\alpha\}_{\alpha\in A}$ be a uniformly bounded net in $\mathcal{X}^*$, where $\lim_{\alpha\in A} \varphi_\alpha(f) = \varphi(f)$ for each $f \in \mathcal{M}$. Then, the net $\{\varphi_\alpha\}_{\alpha\in A}$ converges to φ in the w^* topology.*

Proof. Let $M = \sup\limits_{\alpha\in A} \max\{\|\varphi_\alpha\|, \|\varphi\|\}$, and let $\varepsilon > 0$.

Given $g \in \mathcal{X}$, choose $f \in \mathcal{M}$ such that $\|f - g\| < \frac{\varepsilon}{3M}$. Let $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $|\varphi_\alpha(f) - \varphi(f)| < \frac{\varepsilon}{3}$. Then, for all $\alpha \geq \alpha_0$,

$$\begin{aligned} |\varphi_\alpha(g) - \varphi(g)| &\leq |\varphi_\alpha(g) - \varphi_\alpha(f)| + |\varphi_\alpha(f) - \varphi(f)| + |\varphi(f) - \varphi(g)| \\ &\leq \|\varphi_\alpha\| \|f - g\| + \frac{\varepsilon}{3} + \|\varphi\| \|f - g\| \\ &< \varepsilon. \end{aligned}$$

□

Definition (Unit Ball)**.** For $\mathcal{X}$ a Banach space, we denote the unit ball as $B_{\mathcal{X}} = \{f \in \mathcal{X} \mid \|f\| \leq 1\}$.[xi]

[ix] I am trying to find a source to prove this, will include the proof of this implicit proposition hopefully.

[x] In the special case of Hilbert space $\mathcal{H}$, we know from the Riesz Representation Theorem that each $\varphi \in \mathcal{H}^*$ is represented by ψ such that $\varphi(f) = \langle f, \psi\rangle$.

[xi] The book uses a different notation, but I don't like that notation.

Theorem (Banach–Alaoglu)**.** *The set $B_{\mathcal{X}^*}$ is compact in the w^*-topology.*

Proof. Let $f \in B_{\mathcal{X}}$. Let $\overline{\mathbb{D}}^f$ denote the f-labeled copy of the closed unit disc in $\mathbb{C}$. Set

$$P = \prod_{f \in B_{\mathcal{X}}} \overline{\mathbb{D}}^f.$$

Then, P is compact by Tychonoff's theorem.

Define $\Lambda : B_{\mathcal{X}^*} \to P$ by $\Lambda(\varphi) = \varphi|_{B_{\mathcal{X}}}$. Notice that $\Lambda(\varphi_1) = \Lambda(\varphi_2)$ implies that $\varphi_1 = \varphi_2$ on $B_{\mathcal{X}}$, meaning $\varphi_1 = \varphi_2$. Therefore, Λ is injective.

Let $\{\varphi_\alpha\}_{\alpha \in A}$ be a net in $\mathcal{X}^*$ converging to φ in the w^*-topology. Then,

$$\begin{aligned} \lim_{\alpha \in A} \varphi_\alpha(f) &= \varphi(f) \\ \lim_{\alpha \in A} \left(\Lambda\left(\varphi_\alpha\right)\right)(f) &= \lim_{\alpha \in A} \left(\Lambda\left(\varphi\right)\right)(f), \end{aligned}$$

meaning

$$\lim_{\alpha \in A} \Lambda\left(\varphi_\alpha\right) = \Lambda\left(\varphi\right)$$

in P. Since Λ is one-to-one, we can see that $\Lambda : \mathcal{B}_{\mathcal{X}^*} \to \Lambda\left(B_{\mathcal{X}^*}\right) \subseteq P$ is a linear homeomorphism.

Let $\{\Lambda\left(\varphi_\alpha\right)\}_{\alpha \in A}$ be a net in $\Lambda\left(B_{\mathcal{X}^*}\right)$ converging in the product topology to ψ. Let $f, g \in B_{\mathcal{X}^*}$ and $\xi \in \mathbb{C}$ with $f + g \in B_{\mathcal{X}^*}$ and $\xi f \in B_{\mathcal{X}^*}$. Then,

$$\begin{aligned} \psi\left(f + g\right) &= \lim_{\alpha \in A} \left(\Lambda\left(\varphi_\alpha\right)\right)\left(f + g\right) \\ &= \lim_{\alpha \in A} \left(\Lambda\left(\varphi_\alpha\right)\right)(f) + \lim_{\alpha \in A} \left(\Lambda\left(\varphi_\alpha\right)\right)(g) \\ &= \psi(f) + \psi(g) \end{aligned}$$

and

$$\begin{aligned} \psi(\xi f) &= \lim_{\alpha \in A} \left(\Lambda\left(\varphi_\alpha\right)\right)(\xi f) \\ &= \lim_{\alpha \in A} \varphi_\alpha\left(\xi f\right) \\ &= \varphi\left(\xi f\right) \\ &= \xi \varphi\left(f\right) \\ &= \xi\left(\Lambda\left(\varphi\right)\right)(f) \\ &= \xi \psi(f). \end{aligned}$$

Thus, $\psi(f)$ determines $\tilde{\psi}(f) = \frac{1}{\|f\|}\psi\left(f\right)$ in $B_{\mathcal{X}^*}$ for all $f \in \mathcal{X} \setminus \{0\}$. If $f \in B_{\mathcal{X}}$, then $\tilde{\psi} \in \mathcal{B}_{\mathcal{X}^*}$ and $\Lambda(\tilde{\psi}) = \psi$.

Thus, $\Lambda\left(B_{\mathcal{X}^*}\right)$ is closed in P, meaning $B_{\mathcal{X}^*}$ is compact in the w^*-topology. $\square$

We will be able to use the Banach–Alaoglu theorem to prove that every Banach space is isomorphic to a subspace of $C(X)$ for some compact Hausdorff space X. However, we will need some theorems and machinery to prove that

Definition (Sublinear Functionals)**.** Let $\mathcal{E}$ be a real linear space, and let p be a real-valued functional on $\mathcal{E}$. We say p is a sublinear functional if $p(f+g) \leq p(f) + p(g)$ for all $f, g \in \mathcal{E}$, and $p(\lambda f) = \lambda p(f)$.

Theorem (Hahn–Banach Dominated Extension)**.** *Let $\mathcal{E}$ be a real linear space, and p a (real-valued) sublinear functional on $\mathcal{E}$. Let $\mathcal{F} \subseteq \mathcal{E}$ be a subspace, and φ a real linear functional on $\mathcal{F}$ such that $\varphi(f) \leq p(f)$ for all $f \in \mathcal{F}$.*

Then, there exists a real linear functional Φ on $\mathcal{E}$ such that $\Phi(f) = \varphi(f)$ for $f \in \mathcal{F}$, and $\Phi(g) \leq p(g)$ for all $g \in \mathcal{E}$.

Proof. Let $\mathcal{F} \subseteq \mathcal{E}$ be a nonempty subspace, and let $f \notin \mathcal{F}$. Select $\mathcal{G} = \{g + \lambda f \mid g \in \mathcal{F},\ \lambda \in \mathbb{R}\}$.

We will extend φ to $\Phi_{\mathcal{G}}$ by taking $\Phi(g + \lambda f) \leq p(g + \lambda f)$. Dividing by $|\lambda|$, we find that, for all $h \in \mathcal{F}$

$$\Phi(f - h) \leq p(f - h)$$

and

$$-p\left(h - f\right) \leq \Phi\left(h - f\right).$$

Thus, recalling that $\Phi(h) = \varphi(h)$ for $h \in \mathcal{F}$,

$$-p\left(h - f\right) + \varphi(h) \leq \Phi\left(f\right) \leq p(f - h) + \varphi(h).$$

The desired Φ only has this property if

$$\sup_{h \in \mathcal{F}} \left\{\varphi(h) - p(h - f)\right\} \leq \inf_{k \in \mathcal{F}} \left\{\varphi(k) + p(f - k)\right\}.$$

However, we also have

$$\begin{aligned}
\varphi(h) - \varphi(k) &= \varphi(h - k) \\
&\leq p(h - k) \\
&\leq p(f - k) + p(h - f),
\end{aligned}$$

meaning

$$\varphi(h) - p(h - f) \leq \varphi(k) + p(f - k).$$

Therefore, we can thus extend φ on $\mathcal{F}$ to Φ on $\mathcal{G}$, where $\Phi(h) \leq p(h)$. We label this as $\Phi_{\mathcal{G}}$.

Let $\mathcal{P} = \{(\mathcal{G}_\delta, \Phi_{\mathcal{G}_\delta})\}_{\delta\in D}$ denote the class of extensions of φ such that $\Phi_{\mathcal{G}_\delta}(h) \leq p(h)$ for all $h \in \mathcal{G}_\delta$.

An element of $\mathcal{P}$ contains $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{E}$, where $\Phi_{\mathcal{G}}$ extends φ, meaning $\mathcal{P}$ is nonempty.

The partial order on $\mathcal{P}$ can be set by $(\mathcal{G}_1, \Phi_{\mathcal{G}_2}) \leq (\mathcal{G}_2, \Phi_{\mathcal{G}_2})$ if $G_1 \subseteq G_2$ and $\Phi_{\mathcal{G}_1}(f) = \Phi_{\mathcal{G}_2}(f)$ for all $f \in \mathcal{G}_1$.

Consider a chain[xii] $\{(\mathcal{G}_\alpha, \Phi_{\mathcal{G}_\alpha})\}_{\alpha\in A}$. To find an upper bound, consider

$$\mathcal{G} = \bigcup_{\alpha\in A} \mathcal{G}_\alpha,$$

where $\Phi_{\mathcal{G}}(f) = \Phi_{\mathcal{G}_\alpha}(f)$ for every $f \in \mathcal{G}_\alpha$. Then, $\Phi_{\mathcal{G}}$ is a linear functional that satisfies the given properties,[xiii] and $(\mathcal{G}, \Phi_{\mathcal{G}})$ is an upper bound for $\{(\mathcal{G}_\alpha, \Phi_{\mathcal{G}_\alpha})\}$.

Thus, by Zorn's Lemma, there is a maximal element of $\mathcal{P}$, $(\mathcal{G}_{\max}, \Phi_{\mathcal{G}_{\max}})$. If $\mathcal{G}_0 \neq \mathcal{E}$, then we can find a $f \notin \mathcal{G}_0$ and repeat the process performed at the beginning of the proof, which would contradict maximality.

Thus, we have constructed a linear functional Φ such that $\Phi(f) \leq p(f)$ for all $f \in \mathcal{E}$ that extends φ. $\square$

Theorem (Hahn–Banach Continuous Extension)**.** *Let $\mathcal{M}$ be a subspace of the Banach space $\mathcal{X}$. If φ is a bounded linear functional on $\mathcal{M}$, then there exists Φ on $\mathcal{X}^*$ such that $\Phi(f) = \varphi(f)$ for all $f \in \mathcal{M}$ and $\|\Phi\| = \|\varphi\|$.*

Proof. Consider $\tilde{\mathcal{X}}$ as the real linear space on which $\|\cdot\|$ is the sublinear functional. Set $\psi = \operatorname{Re}(\varphi)$ on $\mathcal{M}$.

We can see that, since $\operatorname{Re}(\varphi(f)) \leq |\varphi(f)|$, $\|\psi\| \leq \|\varphi\|$.

Set $p(f) = \|\varphi\|\,\|f\|$. Since $\psi(f) \leq p(f)$ for all $f \in \mathcal{X}$, by the dominated extension theorem, there exists Ψ defined on $\tilde{\mathcal{X}}$ that extends ψ. In particular, we can see that $\Psi(f) \leq \|\varphi\|\,\|f\|$.

Define Φ on $\mathcal{X}$ by $\Phi(f) = \Psi(f) - i\Psi(if)$ for any $f \in \mathcal{X}$. We will show that Φ is a complex bounded linear functional that extends φ and has norm $\|\varphi\|$. We can see that

$$\begin{aligned}\Phi(f+g) &= \Psi(f+g) - i\Psi\left(i\left(f+g\right)\right)\\ &= \Psi(f) - i\Psi(if) + \Psi(g) - i\Psi(ig)\\ &= \Phi(f) + \Phi(g),\end{aligned}$$

[xii] totally ordered subset

[xiii] I am too lazy to prove this.

and for $\lambda_1, \lambda_2 \in \mathbb{R}$,[xiv]

$$\Phi\left(\left(\lambda_1 + i\lambda_2\right) f\right) = \Phi\left(\lambda_1 f\right) + \Phi\left(i\lambda_2 f\right) \qquad = \left(\lambda_1 + i\lambda_2\right)\Phi(f).$$

To verify that $\Phi(f)$ extends $\varphi(f)$, let $f \in \mathcal{M}$, and we can see that

$$\begin{aligned}
\Phi(f) &= \Psi(f) - i\Psi(if) \\
&= \psi(f) - i\psi(if) \\
&= \operatorname{Re}\left(\varphi(f)\right) - i\operatorname{Re}\left(\varphi(if)\right) \\
&= \operatorname{Re}\left(\varphi(f)\right) - i\left(-\operatorname{Im}\left(\varphi(f)\right)\right) \\
&= \varphi(f).
\end{aligned}$$

Finally, to verify that $\|\Phi\| = \|\varphi\|$, all we need show is that $\|\Phi\| \leq \|\Psi\|$. Let $\Phi(f) = re^{i\theta}$. Then,

$$\begin{aligned}
|\Phi(f)| &= r \\
&= e^{-i\theta}\Phi(f) \\
&= \Phi\left(e^{-i\theta} f\right) \\
&= \Psi\left(e^{-i\theta} f\right) \\
&\leq \left|\Psi\left(e^{-i\theta} f\right)\right| \\
&\leq \|\Psi\| \|f\|,
\end{aligned}$$

meaning

$$\|\Phi\| \|f\| \leq \|\Psi\| \|f\|.$$

$\square$

Corollary (Norming Functional). *If $f \in \mathcal{X}$, then there exists $\varphi \in \mathcal{X}^*$ such that $\|\varphi\| = 1$ and $\varphi(f) = \|f\|$.*

Proof. Assume $f \neq 0$. Let $\mathcal{M} = \{\lambda f \mid \lambda \in \mathbb{C}\}$, and define ψ on $\mathcal{M}$ by $\psi(\lambda f) = \lambda \|f\|$. Then, $\|\psi\| = 1$ and an extension of ψ to $\mathcal{X}$ has the desired properties. $\square$

Theorem (Banach). *Let $\mathcal{X}$ be any Banach space. Then, $\mathcal{X}$ is isometrically isomorphic to some closed subspace of $C(X)$ for compact Hausdorff X.*

Proof. Set $X = B_{\mathcal{X}^*}$ in the w^*-topology, which by Banach–Alaoglu, is compact.

Set $\beta : \mathcal{X} \to C(X)$ by $\beta(f)(\varphi) = \varphi(f)$. Then, for $\lambda_1, \lambda_2 \in \mathbb{C}$, $f_1, f_2 \in \mathcal{X}$,

$$\begin{aligned}
\beta\left(\lambda_1 f_1 + \lambda_2 f_2\right)(\varphi) &= \varphi\left(\lambda_1 f_1 + \lambda_2 f_2\right) \\
&= \lambda_1 \varphi(f_1) + \lambda_2 \varphi\left(f_2\right) \\
&= \left(\lambda_1 \beta\left(f_1\right) + \lambda_2 \beta\left(f_2\right)\right)(\varphi).
\end{aligned}$$

[xiv] Notice that $\Phi(if) = \Psi(if) - i\Psi(-f) = i\Psi(f) + \Psi(if) = i\Phi(f)$

Let $f \in \mathcal{X}$. Then,

$$\begin{aligned}\|\beta(f)\|_\infty &= \sup_{\varphi \in B_{\mathcal{X}^*}} |\beta(f)(\varphi)| \\ &= \sup_{\varphi \in B_{\mathcal{X}^*}} |\varphi(f)| \\ &\leq \sup_{\varphi \in B_{\mathcal{X}^*}} \|\varphi\| \, \|f\| \\ &\leq \|f\|.\end{aligned}$$

Additionally, since there exists a norming functional in $B_{\mathcal{X}^*}$, we have that $\|\beta(f)\|_\infty = \|f\|$, meaning β is an isometric isomorphism. □

Note: The preceding construction cannot yield an isometric isomorphism to $C(B_{\mathcal{X}^*})$ itself, even if $\mathcal{X} = C(Y)$ for some Y.

It can be shown via topological arguments that if $\mathcal{X}$ is separable, we can take X to be the interval $[0,1]$.

Now, we turn to finding the dual space of $C([0,1])$. In particular, we will soon find out that $C([0,1]) = \text{BV}([0,1])$, which is the space of all functions of bounded variation.

Definition (Bounded Variation)**.** If φ is a complex function with domain $[0,1]$, φ is said to be of bounded variation if for every partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, it is the case that

$$\sum_{i=0}^{n} |\varphi(t_{n+1}) - \varphi(t_n)| \leq M.$$

The infimum of all such values of M is denoted $\|\varphi\|_{\text{BV}}$.[xv] Henceforth, all functions of bounded variation will be referred to as BV functions.

Proposition (Limits of BV Functions)**.** *A BV function possesses a limit from the left and right at each endpoint.*

Proof. Let $\varphi : [0,1] \to \mathbb{C}$ not have a limit from the left at some point $t \in (0,1]$.

Then, for any $\delta > 0$, there exist s_1, s_2 such that $t - \delta < s_1 < s_2 < t$ and $|\varphi(s_2) - \varphi(s_1)| \geq \varepsilon$. Selecting $\delta_2 = t - s_2$, we inductively create a sequence $\{s_n\}_{n=1}^{\infty}$ where $0 < s_1 < s_2 < \cdots < s_n < \cdots < t$.

Consider a partition $t_0 = 0$, and $t_k = s_k$ for $k = 1, 2, \dots, N$, and $t_{N+1} = 1$, we have

$$\begin{aligned}\sum_{k=0}^{N} |\varphi(t_{k+1}) - \varphi(t_k)| &\geq \sum_{k=1}^{N} |\varphi(s_{k+1}) - \varphi(s_k)| \\ &\geq N\varepsilon.\end{aligned}$$

Thus, φ is not a BV function. □

[xv] The book uses $\|\varphi\|_\nu$, but I think that's more confusing than BV.

Corollary (Discontinuities of a BV Function). *Let $\varphi : [0,1] \to \mathbb{C}$ be a BV function. Then, φ has countably many discontinuities.*

Proof. Notice that φ is discontinuous at a point t if and only if $\varphi(t) \neq \varphi(t^+)$ or $\varphi(t) \neq \varphi(t^-)$.

If $t_0, t_1, \dots, t_n$ are distinct points of $[0,1]$, then

$$\sum_{i=0}^{N} \left|\varphi(t) - \varphi(t^+)\right| + \sum_{i=0}^{N} \left|\varphi(t) - \varphi(t^-)\right| \leq \|\varphi\|_{\text{BV}}.$$

Thus, for every $\varepsilon > 0$, there exist at most finitely many t such that $|\varphi(t) - \varphi(t^+)| + |\varphi(t) - \varphi(t^-)| \geq \varepsilon$, meaning there can be at most countably many discontinuities. $\square$

Definition (Riemann–Stieltjes Integral). Let $f \in C([0,1])$, and let $\varphi \in \text{BV}([0,1])$. Then, we denote the Riemann–Stieltjes integral

$$\int_0^1 f \, d\varphi = \sum_{i=0}^{n} f(t_i') \left[\varphi(t_{i+1}) - \varphi(t_i)\right],$$

where $\{t_i\}$ is a partition and $t_i' \in [t_i, t_{i+1}]$.

Proposition (Essential properties of the Riemann–Stieltjes Integral). *If $f \in C([0,1])$ and $\varphi \in BV([0,1])$, then*

(1) $\displaystyle\int_0^1 f \, d\varphi$ *exists;*

(2) $\displaystyle\int_0^1 (\lambda_1 f_1 + \lambda_2 f_2) \, d\varphi = \lambda_1 \int_0^1 f_1 \, d\varphi + \lambda_2 \int_0^1 f_2 \, d\varphi$ *for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $f_1, f_2 \in C([0,1])$;*

(3) $\displaystyle\int_0^1 f \, d(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1 \int_0^1 f_1 \, d\varphi_1 + \lambda_2 \int_0^1 f_2 \, d\varphi_2$ *for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\varphi_1, \varphi_2 \in BV([0,1])$;*

(4) $\displaystyle\left|\int_0^1 f \, d\varphi\right| \leq \|f\|_\infty \|\varphi\|_{BV}$ *for $f \in C([0,1])$ and $\varphi \in BV([0,1])$.*