Part 1

3.3, Problem 3

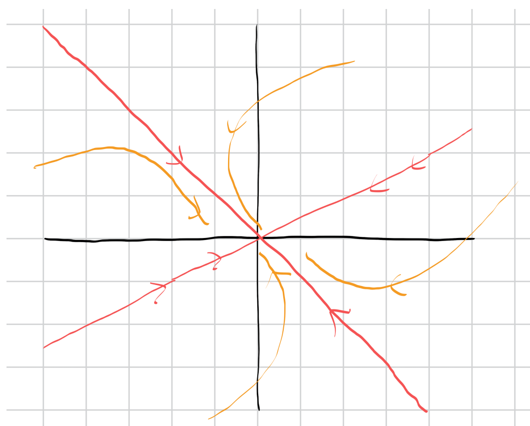
Finding the eigenvalues of the matrix, we have

$$\det\begin{pmatrix}-5-\lambda & -2\\ -1 & -4-\lambda\end{pmatrix} = (5+\lambda)(4+\lambda)-2$$
$$\lambda^2+9\lambda+18=0$$
$$(\lambda+6)(\lambda+3)=0,$$

so the eigenvalues are $\lambda_1 = -6$ and $\lambda_2 = -3$. Their corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix}2\\1\end{pmatrix}$$
$$\vec{v}_2 = \begin{pmatrix}-1\\1\end{pmatrix}.$$

The phase portrait is as follows.



3.3, Problem 4

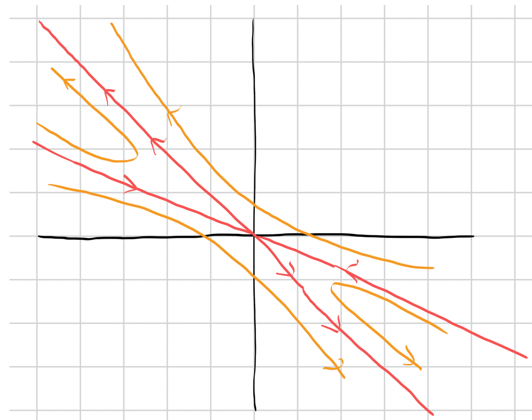
Finding the eigenvalues of the matrix, we have

$$\det\begin{pmatrix}5-\lambda & 4\\ 9 & 0-\lambda\end{pmatrix} = \lambda(\lambda-5)-36$$
$$\lambda^2-5\lambda-36=0$$
$$(\lambda-9)(\lambda+4)=0,$$

so the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = -4$. The corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix}-1\\1\end{pmatrix}$$
$$\vec{v}_2 = \begin{pmatrix}-4/9\\1\end{pmatrix}.$$

The phase portrait is as follows.



3.3, Problem 7

Finding the eigenvalues of the matrix, we have

$$\det\begin{pmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} = (\lambda-2)(\lambda-1)-1$$
$$\lambda^2-3\lambda+1=0,$$

from which we get eigenvalues of $\lambda_{1,2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$. The corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix}1\\ \left(-1+\sqrt{5}\right)/2\end{pmatrix}$$
$$\vec{v}_2 = \begin{pmatrix}1\\ \left(-1-\sqrt{5}\right)/2\end{pmatrix}.$$

The phase portrait is as follows.

3.3, Problem 8

Finding the eigenvalues of the matrix, we have

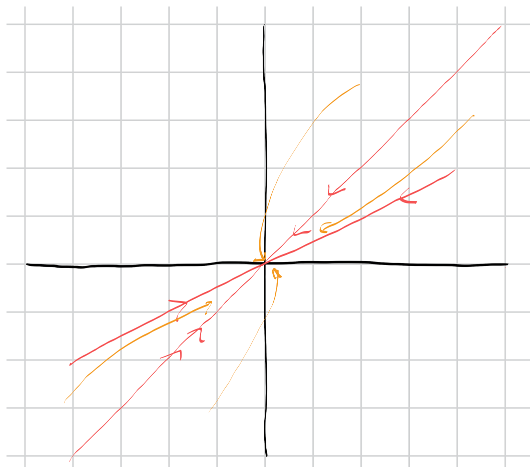
$$\det\begin{pmatrix}-1-\lambda & -2\\ 1 & -4-\lambda\end{pmatrix} = (\lambda+4)(\lambda+1)+2$$

$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda + 3)(\lambda + 2) = 0,$$

giving eigenvalues of $\lambda_1 = -3$ and $\lambda_2 = -2$. The corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The corresponding phase portrait is as follows.



3.3, Problem 20

(a) The equilibrium point at the origin is a saddle.

(b) The straight line solutions are

$$\vec{Y}_1 = e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\vec{Y}_2 = e^{4t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(c) I don't know how to do this problem.

Part 2

3.4, Problem 1

$$\begin{aligned}
\vec{Y}_1 &= e^{t(1+3i)}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^{t}\left(\cos(3t) + i\sin(3t)\right)\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= e^{t}\cos(3t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{t}\sin(3t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ie^{t}\left(\cos(3t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \sin(3t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).
\end{aligned}$$

Thus, the general solution is

$$\vec{Y}(t) = k_1 e^t \cos(3t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^t \sin(3t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^t \left(\cos(3t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \sin(3t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

3.4, Problem 2

$$\begin{aligned} \vec{Y}_2 &= e^{t(-2+5i)} \begin{pmatrix} 1 \\ 4 - 3i \end{pmatrix} \\ &= e^{-2t} \left(\cos(5t) \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 3 \sin(5t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + i e^{-2t} \left(-3 \cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(5t) \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right). \end{aligned}$$

Thus, the general solution is

$$\vec{Y}(t) = k_1 e^{-2t} \left(\cos(5t) \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 3 \sin(5t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + k_2 e^{-2t} \left(-3 \cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(5t) \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right).$$

3.4, Problem 9 (a)

The eigenvalues are at $\pm 2i$. The corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thus, we get

$$\begin{aligned} \vec{Y}_1(t) &= e^{2it} \begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= (\cos(2t) + i \sin(2t)) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &= \cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \left(-\cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right). \end{aligned}$$

The general solution is, thus

$$\vec{Y}(t) = k_1 \left(\cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + k_2 \left(-\cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$