Solution (12.4, Problem 6)**:** Upon separation of variables, we get

$$\frac{1}{a^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

$$\begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} A e^{akt} & k^2 \\ At + B & 0 \\ A\cos(akt) + B\sin(akt) & -k^2 \end{cases}$$

$$X(x) = \begin{cases} C e^{kx} & k^2 \\ Cx + D & 0 \\ C\cos(kx) + D\sin(kx) & -k^2 \end{cases}.$$

By plugging in the boundary conditions of $u(0,t) = u(1,t) = 0$, we quickly remove the former two cases, we are of the form

$$T(t) = A\cos(akt) + B\sin(akt)$$
$$X(x) = C\cos(kx) + D\sin(kx).$$

Since $X(0) = 0$, we must have $C = 0$, and since $X(1) = 0$, we have $k = n\pi$, $n \in \mathbb{Z}$. Thus, we have functions of the form

$$u_n(x,t) = (A_n \cos(n\pi a t) + B_n \sin(n\pi a t)) \sin(n\pi x),$$

and the general solution of

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi a t) + B_n \sin(n\pi a t)) \sin(n\pi x).$$

Plugging in the initial condition, we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$
$$= \frac{1}{100} \sin(3\pi x),$$

so that $A_n = \frac{1}{100}$ at $x = 3$ and 0 elsewhere. Writing our amended solution, we have

$$u(x,0) = \left(\frac{1}{100} \cos(3\pi a t) + B_3 \sin(3\pi a t) \right) \sin(3\pi a x).$$

Taking derivatives, we have

$$\left. \frac{\partial u}{\partial t} \right|_{(x,0)} = B_3 \sin(3\pi a x)$$
$$= 0,$$

so $B_3 = 0$, and we arrive at the solution

$$u(x,t) = \frac{1}{100} \cos(3\pi a t) \sin(3\pi x).$$

Solution (12.4, Problem 8)**:** Upon separation of variables, we get

$$\frac{1}{a^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

$$\begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} A e^{akt} & k^2 \\ At + B & 0 \\ A\cos(akt) + B\sin(akt) & -k^2 \end{cases}$$

$$X(x) = \begin{cases} C e^{kx} & k^2 \\ Cx + D & 0 \\ C\cos(kx) + D\sin(kx) & -k^2 \end{cases}.$$

We plug in the boundary conditions of $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$ to obtain

$$X_n(x) = \begin{cases} C_n \cos\left(\frac{n\pi}{L} x\right) & -k^2 \\ Cx + D & 0 \end{cases}$$

$$T_n(t) = \begin{cases} B_n \cos\left(\frac{n\pi a}{L} t\right) & -k^2 \\ At + B & 0 \end{cases}$$

We may evaluate the solution

$$u(x, t) = X_0(x) T_0(t) + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right).$$

To do this, we start with the initial condition, giving $T_0(t) = 1$ and $X_0(x) = x$. Taking the partial derivative with respect to t, we get

$$\frac{\partial u}{\partial t} = X_0(x) \frac{dT_0}{dt} - \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi}{L} x\right)\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi a}{L} t\right).$$

Therefore,

$$u(x, t) = x$$

Solution (12.5, Problem 2)**:** Separating variables, we have

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2}$$

$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

Thus, we have

$$X_n = A_n \cos(\lambda x) + B_n \sin(\lambda x).$$

Using the boundary conditions of $X_n(a) = X_n(0) = 0$, we simplify to

$$X_n = B_n \sin\left(\frac{n\pi}{a} x\right).$$

Similarly, we have

$$Y_n(y) = C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right).$$

Applying the boundary condition of $\left.\frac{\partial u}{\partial y}\right|_{(x,0)} = 0$, we have $D_n = 0$, and

$$u(x,y) = \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right).$$

We have

$$f(x) = u(x,b)$$
$$= \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi}{a}x\right).$$

Using the expansion of Fourier coefficients, we have

$$K_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)\,dx.$$

Solution (12.5, Problem 4)**:** Separating variables, we get

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$
$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

This evaluates to

$$X = A\cos(\lambda x) + B\sin(\lambda x)$$
$$Y = C\cosh(\lambda y) + B\sinh(\lambda y).$$

Using the Neumann boundary condition, we get

$$X_n = A_n\cos\left(\frac{n\pi}{a}x\right)$$
$$Y_n = C_n\cosh\left(\frac{n\pi}{a}y\right) + D_n\sinh\left(\frac{n\pi}{a}y\right).$$

Therefore, $u(x,y)$ is of the form

$$u(x,y) = \sum_{n=0}^{\infty} B_n\cos\left(\frac{n\pi}{a}x\right)\cosh\left(\frac{n\pi}{a}y\right) + \sum_{n=1}^{\infty} D_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right).$$

We may plug this into an expression for $u(x,0)$ to get

$$x = \sum_{n=0}^{\infty} B_n\cos\left(\frac{n\pi}{a}x\right),$$

meaning

$$B_0 = a$$
$$B_n = \frac{2\left((-1)^n - 1\right)a}{n^2\pi^2},$$

and plugging in the condition that $u(x,b) = 0$, we have

$$D_n = -\coth\left(\frac{n\pi b}{a}\right)B_n.$$

Solution (12.5, Problem 6)**:** Separating variables, we have

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

We thus have

$$Y = A\cos(\lambda y) + B\sin(\lambda y).$$

Using the Neumann boundary condition in y, we may simplify this to

$$Y_n = A_n\cos(ny).$$

This gives

$$X_n = B_n\cosh(nx) + C_n\sinh(nx).$$

We thus have

$$u(x,y) = \sum_{n=0}^{\infty} A_n\cosh(nx)\cos(ny) + \sum_{n=1}^{\infty} C_n\sinh(nx)\cos(ny).$$

Evaluating the boundary condition at $u(0,y)$, we have

$$g(y) = \sum_{n=0}^{\infty} A_n\cos(ny),$$

so that

$$A_n = \frac{2}{\pi}\int_0^{\pi} g(y)\cos(ny)\,dy.$$

Evaluating the derivative at $x = 1$, we have

$$\left.\frac{\partial u}{\partial x}\right|_{(1,y)} = \sum_{n=1}^{\infty} nA_n\sinh(n)\cos(ny) + \sum_{n=1}^{\infty} nC_n\cosh(n)\cos(ny)$$

$$= 0.$$

Therefore, $C_n = -A_n\tanh(n)$.

Solution (12.5, Problem 8)**:** Separating variables, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

Using the Dirichlet boundary condition on X, we get

$$X_n = A_n\sin(n\pi x)$$
$$Y_n = B_n\cosh(n\pi y) + C_n\sinh(n\pi y).$$

Thus, we have $u(x,y)$ of the form

$$u(x,y) = \sum_{n=1}^{\infty} A_n\sin(n\pi x)\cosh(n\pi y) + \sum_{n=1}^{\infty} B_n\sin(n\pi x)\sinh(n\pi y).$$

Setting

$$\left.\frac{\partial u}{\partial y}\right|_{x=0} = \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x)$$
$$= 0,$$

we have $B_n = 0$, so

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cosh(n\pi y).$$

Solving for the Fourier coefficients, we get

$$A_n = \frac{2}{\cosh(n\pi)} \int_0^1 f(x)\sin(n\pi x)\,dx.$$

Solution (12.6, Problem 2)**:** We may homogenize the boundary condition by letting $u(x, t) = v(x, t) + \psi(x)$, and solving

$$k\frac{d^2\psi}{dx^2} = 0,$$

where $\psi(0) = u_0$ and $\psi(1) = 0$, so $\psi = -\frac{1}{u_0}x + u_0$. This gives the boundary value problem

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = 0$$

with Dirichlet boundary of $v(0) = v(1) = 0$. Separating variables as $v(x, t) = X(x)T(t)$, we then get

$$\frac{1}{T}\frac{dT}{dt} = \frac{k}{X}\frac{d^2X}{dx^2}$$
$$= \begin{cases} -\alpha^2 \\ 0 \\ \alpha^2 \end{cases},$$

so that

$$X_n = A_n \sin\left(n\pi\sqrt{k}x\right)$$
$$T_n = e^{-kn^2\pi^2 t},$$

and

$$v(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi\sqrt{k}x\right)e^{-kn^2\pi^2 t}.$$

We have the initial condition $v(x, 0) = f(x) - \psi(x)$, so

$$A_n = 2\int_0^1 \left(f(x) + \frac{1}{u_0}x - u_0\right)\sin(n\pi x)\,dx,$$

and

$$u(x, t) = v(x, t) - \frac{1}{u_0}x + u_0.$$

Solution (12.6, Problem 4)**:** Homogenizing the boundary conditions, we have

$$u(x, t) = v(x, t) + \psi(x),$$

where

$$k\frac{d^2\psi}{dx^2} = -r.$$

This gives

$$\psi(x) = -\frac{r}{2k}x^2 + Bx + C.$$

Plugging in the boundary conditions, we get

$$\psi(x) = -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0.$$

The homogeneous heat equation is given by

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = 0,$$

where $v(x, 0) = f(x) - \psi(x)$, $v(0, t) = v(1, t) = 0$. By separating variables, we get

$$\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{k}{X}\frac{d^2 X}{dx^2}$$
$$= \begin{cases} -\alpha^2 \\ 0 \\ \alpha^2 \end{cases}$$

Using the Dirichlet boundary conditions, we get

$$X_n = A_n \sin\left(n\pi\sqrt{k}x\right)$$
$$T_n = e^{-kn^2\pi^2 t}.$$

This gives

$$v(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi\sqrt{k}x\right)e^{-kn^2\pi^2 t}.$$

The Fourier coefficients are

$$A_n = 2\int_0^1 (f(x) - \psi(x))\sin(n\pi x)\,dx,$$

and we get the solution of $u(x, t) = v(x, t) + \psi(x)$.

Solution (12.6, Problem 10)**:**

Solution (Extra Problems)**:**