Revised Problems

Problem (Homework 3, Problem 3 (b))**:** Prove that S^∞ is contractible.

Solution: We view S^∞ as a topological subspace of $\mathbb{R}^\infty$ (finitely supported real sequences) equipped with the Euclidean norm; i.e., if $(x_n) \in \mathbb{R}^\infty$, then

$$\|(x_n)\| = \left(\sum_{i=0}^{\infty} x_i^2\right)^{1/2},$$

where the sum is finite by definition. We consider $S^\infty \subseteq \mathbb{R}^\infty$ to be the space of all finitely supported sequences (x_n) such that

$$\|(x_n)\| = 1.$$

We consider the continuous 1-parameter family given by

$$f_t(x_n) = (1-t)(x_n) + tV(x_n),$$

where V denote the right unilateral shift mapping $(x_0, x_1, \dots)$ to $(0, x_0, x_1, \dots)$. To show that $f_t(x_n)$ is never zero, we start by considering $(x_0, \dots, x_k, 0)$ viewed in S^{k+1}, and observe then that f_t, restricted to S^{k+1}, yields

$$f_t((x_0, \dots, x_k, 0)) = ((1-t)x_0, (1-t)x_1 + tx_0, \dots, (1-t)x_k + tx_{k-1}, tx_k).$$

Without loss of generality, we may consider x_0 as being nonzero; then, we observe that the second coordinate $1 + t(x_0 - x_1)$ will be nonzero for all $0 \leq t \leq 1$ if $|x_0 - x_1| < 1$, and will be zero at $t = 1$ only when $|x_0 - x_1| = 1$, but that can only happen if either x_0 or x_1 is 1 and every other coordinate is 0; yet, in such a scenario, it is necessarily the case that f_t is nonzero. Therefore, $\|f_t\|$ is nonzero for all $0 \leq t \leq 1$ acting on S^∞.

In particular, when we consider the homotopy $H\colon S^\infty \times [0,1] \to S^\infty$ given by

$$H((x_n), t) = \begin{cases} (1-t)(x_n) + 2tV(x_n) & 0 \leq t \leq 1/2 \\ (2-2t)V(x_n) + (2t-1)(1, 0, \dots) & 1/2 \leq t \leq 1 \end{cases},$$

we observe that H is continuous along each of $S^\infty \times [0, 1/2]$ and $S^\infty \times [1/2, 1]$, and is equal at $t = 1/2$, so by the pasting lemma, H is continuous along $[0,1]$. Since $H(\cdot, t)/\|H(\cdot, t)\|$ is contained in S^∞ (with well-definedness following from the earlier discussion), and is a homotopy between the identity and a constant map, it follows that the identity is null-homotopic, so S^∞ is contractible.

Problem (Homework 6, Problem 2)**:** Prove that, for a path-connected space X, the fundamental group $\pi_1(X)$ is abelian if and only if all the change-of-basepoint isomorphisms β_h depend only on the endpoints of the path h, not on the precise path.

Solution: Let X be path-connected, and suppose $\pi_1(X)$ is abelian. Let x_0, x_1 be distinct points in X with distinct paths h_1 and h_2 connecting x_0 and x_1. We show that $\beta_{\overline{h_2}}\beta_{h_1}$ is identity on $\pi_1(X, x_0)$.

If f is a loop based at x_0, then we have

$$\begin{aligned}\beta_{\overline{h_2}}\beta_{h_1}[f] &= \beta_{\overline{h_2}}\left[\overline{h_1}\cdot f\cdot h_1\right]\\ &= \left[h_2\cdot\overline{h_1}\cdot f\cdot h_1\cdot\overline{h_2}\right]\\ &= [f]\left[h_2\cdot\overline{h_1}\right]\left[h_1\cdot\overline{h_2}\right]\\ &= [f]\left[h_2\cdot\overline{h_1}\cdot h_1\cdot\overline{h_2}\right]\\ &= [f]\left[h_2\cdot c\cdot\overline{h_2}\right]\\ &= [f]\left[h_2\cdot\overline{h_2}\right]\\ &= [f]\left[h_2\cdot\overline{h_2}\right]\\ &= [f][c]\\ &= [f].\end{aligned}$$

Therefore, $\beta_{h_1} = \beta_{h_2}$.

Now, suppose all change of basepoint isomorphisms are independent of the path. This necessarily holds for loops, so if $[\gamma], [f] \in \pi_1(X, x_0)$, since β_γ does not change the basepoint, we must have that $[f] = \beta_\gamma[f]$, or that

$$\begin{aligned}[f] &= [\overline{\gamma}\cdot f\cdot\gamma]\\ &= [\gamma]^{-1}[f][\gamma],\end{aligned}$$

giving that $[\gamma][f] = [f][\gamma]$.

Current Problems

Problem (Problem 1)**:** Let A be a path-connected subspace of a topological space X, and let $i\colon A \to X$ be inclusion. Show that for any $x_0 \in A$, the induced map $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective if and only if every path in X with endpoints in A is homotopic to a path in A.

Solution: Suppose $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. That is, for any homotopy class $[f] \in \pi_1(X, x_0)$ of a loop, there is some homotopy class $[g] \in \pi_1(A, x_0)$ such that $i_*[g] = [f]$. In particular, this means that

$$\begin{aligned}i_*[g] &= [i\circ g]\\ &= [g]\\ &= [f],\end{aligned}$$

so that any loop based at x_0 in X is homotopic to a loop based at x_0 in A.

Now, let γ be a path in X with endpoints x_0 and x_1, and let ζ be a path from x_1 to x_0 in A. Then, the concatenation $\gamma\cdot\zeta$ is a loop in X based at x_0, so it is homotopic to a loop γ' in A based at x_0. Let $x_2 = \gamma'(1/2)$; then, since A is path-connected, it follows that the path $\eta(t) = \gamma'(t/2)$ is homotopic (as a map) to a path in A that has an endpoint at x_1, which we will call η'. Similarly, the path $\xi(t) = \gamma'((t+1)/2)$ is homotopic (as a map) to ζ, meaning that we have the chain of homotopies

$$\begin{aligned}\gamma \cdot \zeta &\simeq \gamma' \\ &= \eta \cdot \xi \\ &\simeq \eta' \cdot \xi \\ &\simeq \eta' \cdot \zeta,\end{aligned}$$

so when we concatenate $\overline{\zeta}$ on both sides (which is kosher as we have fixed endpoints x_0 and x_1), we have that $\gamma \cdot c \simeq \eta' \cdot c$, or that $\gamma \simeq \eta$. For the general case of any two endpoints, y_0 and y_1, we create the desired loop, use the change-of-basepoint isomorphism, then use the inverse change-of-basepoint isomorphism.

In the reverse direction, we observe that since any loop in X with an endpoint in A is necessarily homotopic to a loop in A, meaning that any homotopy class of loops in X based at x_0 includes a representative that is a loop in A, meaning that the induced homomorphism is surjective.

Problem (Problem 2)**:** Show that there is no retraction $r\colon S^1 \times D^2 \to S^1 \times S^1$.

Solution: Suppose such a retraction existed. We see that the sequence $S^1 \times S^1 \xrightarrow{i} S^1 \times D^2 \xrightarrow{r} S^1 \times S^1$ induces homomorphisms $\mathbb{Z}^2 \xrightarrow{i_*} \mathbb{Z} \xrightarrow{r_*} \mathbb{Z}^2$ such that $r_* \circ i_* = \mathrm{id}$. The map r_* is then a surjective homomorphism between $\mathbb{Z}$ and $\mathbb{Z}^2$, meaning that we have $\mathbb{Z}/\ker(r_*) \cong \mathbb{Z}^2$, but since $\mathbb{Z}$ is a principal ideal domain, we have that $\ker(r_*) = (v)$ for some $v \in \mathbb{Z}$; since $\mathbb{Z}^2$ is infinite, and $\mathbb{Z}/(v)$ is finite whenever $v \neq 0$, it follows that we have that v must be equal to 0, so r_* is thus injective. Yet, this implies the existence of an isomorphism between $\mathbb{Z}$ and $\mathbb{Z}^2$, which violates the structure of finitely generated abelian groups.