

These are some notes I have taken from Gerald B. Folland's *A Course in Abstract Harmonic Analysis*, with some other textbooks for various sources.

## Basic Properties of Topological Groups

**Definition:** A *topological group* is a group  $G$  with a topology such that the operation

$$\begin{aligned} m: G \times G &\rightarrow G \\ (x, y) &\mapsto xy \end{aligned}$$

is continuous with respect to the product topology on  $G \times G$  and the operation

$$\begin{aligned} i: G &\rightarrow G \\ x &\mapsto x^{-1} \end{aligned}$$

is continuous with respect to the topology on  $G$ .

For a topological group  $G$ , we denote the unit element as  $1_G$ , and we set

$$\begin{aligned} Ax &= \{yx \mid y \in A\} \\ xA &= \{xy \mid y \in A\} \\ A^{-1} &= \{y^{-1} \mid y \in A\} \\ AB &= \{xy \mid x \in A, y \in B\} \end{aligned}$$

for all subsets  $A, B \subseteq G$  and elements  $x \in G$ .

**Definition:** A subset  $A \subseteq G$  is called *symmetric* if  $A = A^{-1}$ .

**Proposition:** Let  $G$  be a topological group.

- (i) The topology of  $G$  is invariant under translations and inversion; that is, if  $U$  is open, then  $xU, Ux, U^{-1}, AU, UA$  are open for any  $x \in G$  and subset  $A \subseteq G$ .
- (ii) For every neighborhood  $U$  of  $1_G$ , there is a symmetric neighborhood  $V$  of  $1_G$  such that  $VV \subseteq U$ .
- (iii) If  $H$  is a subgroup of  $G$ , so is  $\overline{H}$ .
- (iv) Every open subgroup of  $G$  is closed.
- (v) If  $A$  and  $B$  are compact sets in  $G$ , so is  $AB$ .

*Proof.*

- (i) This is equivalent to the separate continuity of  $(x, y) \mapsto xy$  and  $x \mapsto x^{-1}$ ; furthermore,

$$\begin{aligned} AU &= \bigcup_{x \in A} xU \\ UA &= \bigcup_{x \in A} Ux. \end{aligned}$$

- (ii) Since  $(x, y) \mapsto xy$  is continuous at  $1_G$ , then for every neighborhood  $U$  of  $1_G$ , there are neighborhoods  $W_1, W_2 \subseteq U$ . We may take  $V = W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1}$ .
- (iii) For  $x, y \in \overline{H}$ , there are nets  $(x_\alpha)_\alpha \rightarrow x$  and  $(y_\alpha)_\alpha \rightarrow y$ ; since  $(x_\alpha y_\alpha) \rightarrow xy$  and  $(x_\alpha^{-1})_\alpha \rightarrow x^{-1}$  by continuity of the operations, we have  $xy, x^{-1} \in \overline{H}$ .

- (iv) If  $H$  is open, then so are all the cosets  $xH$ ; since  $G \setminus H$  is the union of all the cosets of  $H$  except for  $H$  itself,  $G \setminus H$  is open, so  $H$  is closed.
- (v) Since  $A \times B$  is compact, and  $AB$  is the continuous image of  $A \times B$  under  $(x, y) \mapsto xy$ , we have  $AB$  is compact.

□

Now, if  $H$  is a subgroup of  $G$ , we let  $G/H$  be the space of left cosets of  $H$ , and  $q: G \rightarrow G/H$  is the canonical quotient map, we may impose the quotient topology on  $G/H$ , meaning that  $U \subseteq G/H$  is open if and only if  $q^{-1}(U)$  is open. Thus,  $q$  maps open sets in  $G$  to open sets in  $G/H$ , as if  $V \subseteq G$  is open,  $q^{-1}(q(V)) = VH$  is also open, so  $q(V)$  is open.

**Proposition:** Let  $H$  be a subgroup of a topological group  $G$ .

- (i) If  $H$  is closed, then  $G/H$  is Hausdorff.
- (ii) If  $G$  is locally compact, so is  $G/H$ .
- (iii) If  $H$  is normal, then  $G/H$  is a topological group.

*Proof.*

- (i) If  $\bar{x} = q(x)$  and  $\bar{y} = q(y)$  are distinct points in  $G/H$ , and since  $H$  is closed,  $xHy^{-1}$  is a closed set that does not contain  $1_G$ . There is a symmetric neighborhood  $U$  of  $1_G$  such that  $UU \cap xHy^{-1} = \emptyset$ ; since  $U = U^{-1}$  and  $H = HH$  ( $H$  is a subgroup), we have  $1_G \notin UxH(Uy)^{-1} = (UxH)(UyH)^{-1}$ , so  $UxH \cap UyH = \emptyset$ . Therefore,  $q(Ux)$  and  $q(Uy)$  are disjoint neighborhoods of  $\bar{x}$  and  $\bar{y}$ .
- (ii) If  $U$  is a compact neighborhood of  $1_G$ ,  $q(Ux)$  is a compact neighborhood of  $q(x)$  in  $G/H$ .
- (iii) If  $x, y \in G$ , and  $U$  is a neighborhood of  $G/H$ , continuity of multiplication in  $G$  implies that there are neighborhoods  $V$  of  $x$  and  $W$  of  $y$  such that  $VW \subseteq q^{-1}(U)$ . We see that  $q(V)$  and  $q(W)$  are neighborhoods of  $q(x)$  and  $q(y)$  such that  $q(V)q(W) \subseteq U$ , meaning multiplication is continuous in  $G/H$ . Similarly, inversion is continuous.

□

**Corollary:** If  $G$  is T1, then  $G$  is Hausdorff, and if  $G$  is not T1, then  $\overline{\{1_G\}}$  is a closed normal subgroup, and  $G/\overline{\{1_G\}}$  is a Hausdorff topological group.

*Proof.* Since singletons are closed in any T1 space, the first assertion follows from part (i) in the previous proposition by taking  $H = \{1_G\}$ .

To see the second assertion, we note that  $\overline{\{1_G\}}$  is a subgroup, and it is the smallest closed subgroup of  $G$ ; it is normal, as otherwise we would obtain a smaller closed subgroup by intersection with one of the conjugates, meaning the result follows from parts (i) and (iii) in the previous proposition by taking  $H = \overline{\{1_G\}}$ .

□

Thus, without loss of generality, we may assume that a topological group is Hausdorff (else take  $G/\overline{\{1_G\}}$ ), and when we talk about locally compact groups, we are talking about topological groups that are locally compact and Hausdorff.

**Proposition:** Every locally compact group  $G$  has a subgroup  $G_0$  that is open, closed, and  $\sigma$ -compact.

*Proof.* Let  $U$  be a symmetric compact neighborhood of  $1_G$ , let  $U_n = \prod_{i=1}^n U$ , and let

$$G_0 = \bigcup_{n=1}^{\infty} U_n.$$

Then,  $G_0$  is the group generated by  $U$ , so it is a subgroup;  $G_0$  is open since  $U_{n+1}$  is a neighborhood of  $U_n$  for all  $n$ , and so  $G_0$  is closed as all open subgroups are closed. Finally, since each  $U_n$  is a finite product of compact subsets of  $G$ ,  $G_0$  is  $\sigma$ -compact.

□

We thus see that  $G_0$  is the disjoint union of cosets of  $G_0$ , meaning  $G$  is a disjoint union of  $\sigma$ -compact spaces. In particular, if  $G$  is connected, then  $G$  is necessarily  $\sigma$ -compact.

**Definition:** Let  $f: G \rightarrow \mathbb{C}$  be a function. The *translates* of  $f$  via  $y \in G$  are defined by

$$\begin{aligned} L_y f(x) &= f(y^{-1}x) \\ R_y f(x) &= f(xy). \end{aligned}$$

Note that the maps  $y \mapsto L_y$  and  $y \mapsto R_y$  are group homomorphisms.

The function  $f$  is called left/right uniformly continuous if

$$\begin{aligned} \|L_y f - f\|_u &\rightarrow 0 \\ \|R_y f - f\|_u &\rightarrow 0 \end{aligned}$$

as  $y \rightarrow 1_G$  respectively.

**Proposition:** If  $f \in C_c(G)$ , then  $f$  is left and right uniformly continuous.

*Proof.* We will prove this for  $R_y f$ .

If  $f \in C_c(G)$ , and  $\varepsilon > 0$ , then for every  $x \in K = \text{supp}(f)$ , there is a neighborhood  $U_x$  of  $1_G$  such that

$$|f(xy) - f(x)| < \frac{1}{2}\varepsilon$$

for any  $y \in U_x$ . Similarly, there is a symmetric neighborhood  $V_x$  of  $1_G$  such that  $V_x V_x \subseteq U_x$ ; the sets  $xV_x$  cover  $K$ , so there exist  $x_1, \dots, x_n \in K$  such that  $K \subseteq \bigcup_{j=1}^n x_j V_{x_j}$ .

Let  $V = \bigcap_{j=1}^n V_{x_j}$ . If  $x \in K$ , then there is some  $j$  such that  $x_j^{-1}x \in V_{x_j}$ , so  $xy = x_j(x_j^{-1}x)y \in x_j U_{x_j}$ , so

$$\begin{aligned} |f(xy) - f(x)| &\leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon, \end{aligned}$$

for any  $y \in V$ , meaning that  $\|R_y f - f\|_u < \varepsilon$ . Similarly, if  $xy \in K$ , then  $|f(xy) - f(x)| < \varepsilon$ ; meanwhile, if  $x, xy \notin K$ , then  $f(x) = f(xy) = 0$ , so we are done.  $\square$

## Haar Measure

**Definition:** We define a subset of  $C_c(G)$  to be

$$C_c^+(G) = \{f \in C_c(G) \mid f \geq 0, f \neq 0\}.$$

**Definition:** A left/right Haar measure on  $G$  is a nonzero Radon measure  $\mu$  on  $G$  such that  $\mu(xE) = \mu(E)$  for every Borel  $E \subseteq G$  and all  $x \in G$ .

**Proposition:** Let  $\mu$  be a Radon measure on the locally compact group  $G$ , and let  $\tilde{\mu}(E) = \mu(E^{-1})$ . Then, the following hold:

- (a)  $\mu$  is a left Haar measure if and only if  $\tilde{\mu}$  is a right Haar measure.
- (b)  $\mu$  is a left Haar measure if and only if  $\int L_y f \, d\mu = \int f \, d\mu$  for all  $f \in C_c^+(G)$  and every  $y \in G$ .

*Proof.* The result in (a) follows from basic properties of the inverse.

To see (b), note that for any Radon measure  $\mu$ , one has  $\int L_y f \, d\mu = \int f \, d\mu_y$ , where  $\mu_y(E) = \mu(yE)$ , which follows from approximation via simple functions. Thus, if  $\mu$  is a Haar measure, then  $\int L_y f \, d\mu = \int f \, d\mu$  for all  $f \in C_c^+(G)$ , so it holds for all  $f \in C_c(G)$ . The measure  $\mu$  is unique from the [Riesz–Markov–Kakutani Representation Theorem](#).  $\square$

Now, our focus turns to the question of establishing the existence and (essential) uniqueness of the Haar measure.

**Theorem:** Every locally compact group  $G$  possesses a left Haar measure  $\lambda$ .

*Proof.* We will construct  $\lambda$  as a linear functional on  $C_c(G)$ .

Let  $f, \phi \in C_c^+(G)$ . We define  $(f : \phi)$  to be the infimum of all such finite sums  $\sum_{j=1}^n c_j$  such that

$$f \leq \sum_{j=1}^n c_j L_{x_j} \phi$$

for some  $x_1, \dots, x_n \in G$ . Such a value necessarily exists as  $\text{supp}(f)$  can be covered by some finite number of translates of  $\phi^{-1}(1/2\|\phi\|_u, \infty)$ , meaning that  $(f : \phi) \leq 2N\|f\|_u/\|\phi\|_u$ . We see the following:

- (i)  $(f : \phi) = (L_y f : \phi)$ ;
- (ii)  $(f_1 + f_2 : \phi) \leq (f_1 : \phi) + (f_2 : \phi)$ ;
- (iii)  $(cf : \phi) = c(f : \phi)$  for any  $c \geq 0$ ;
- (iv)  $(f_1 : \phi) \leq (f_2 : \phi)$  whenever  $f_1 \leq f_2$ ;
- (v)  $(f : \phi) \geq \|f\|_u/\|\phi\|_u$ ;
- (vi)  $(f : \phi) \leq (f : \psi)(\psi : \phi)$  for any  $\psi \in C_c^+(G)$ .

To see (vi), notice that if  $f \leq \sum_{i=1}^n c_i L_{x_i} \phi$  and  $\psi \leq \sum_{j=1}^m b_j L_{y_j} \phi$ , then  $f \leq \sum_{i=1}^n \sum_{j=1}^m c_i b_j L_{x_i y_j} \phi$ .

We fix a function  $f_0 \in C_c^+(G)$ , and define

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)}.$$

This functional is left-invariant, subadditive, homogeneous, and monotone, and also satisfies

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0).$$

Now,  $I_\phi$  is not necessarily additive, but on a neighborhood it is very close to being so.

**Lemma:** If  $f_1, f_2 \in C_c^+(G)$ , and  $\varepsilon > 0$ , then there is a neighborhood  $V$  of  $1_G$  such that  $I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \varepsilon$  whenever  $\text{supp}(\phi) \subseteq V$ .

*Proof of Lemma.* Fix  $g \in C_c^+(G)$  such that  $g = 1$  on  $\text{supp}(f_1 + f_2)$ , and let  $\delta$  be a (to be determined) positive number. Let  $h = f_1 + f_2 + \delta g$ , and let  $h_i = f_i/h$  for each  $i$ ; note that  $h_i = 0$  whenever  $f_i = 0$ .

Then, we see that  $h_i \in C_c^+(G)$ , so there is a neighborhood  $V$  of  $1_G$  such that  $|h_i(x) - h_i(y)| < \delta$  for each  $i$  and all  $y$  such that  $y^{-1}x \in V$ .

Suppose  $\phi \in C_c^+(G)$  and  $\text{supp}(\phi) \subseteq V$ . If  $h \leq \sum_{j=1}^n c_j L_{x_j} \phi$ , then

$$f_i(x) = h(x)h_i(x)$$

$$\begin{aligned}
&\leq \sum_{j=1}^m c_j \phi(x_j^{-1}x) h_i(x) \\
&\leq \sum_{j=1}^m c_j \phi(x_j^{-1}x) (h_i(x_j) + \delta),
\end{aligned}$$

since  $|h_i(x) - h_i(x_j)| < \delta$  whenever  $x_j^{-1}x \in \text{supp}(\phi)$ . Since  $h_1 + h_2 \leq 1$ , we have

$$\begin{aligned}
(f_1 : \phi) + (f_2 : \phi) &\leq \sum_{j=1}^m c_j (h_1(x_j) + \delta) + \sum_{j=1}^m c_j (h_2(x_j) + \delta) \\
&\leq \sum_{j=1}^m c_j (1 + 2\delta).
\end{aligned}$$

Taking the infimum of all such sums, we have

$$\begin{aligned}
I_\phi(f_1) + I_\phi(f_2) &\leq (1 + 2\delta) I_\phi(h) \\
&\leq (1 + 2\delta) (I_\phi(f_1 + f_2) + \delta I_\phi(g))
\end{aligned}$$

Thus, by taking  $\delta$  small enough such that

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \varepsilon,$$

we obtain our desired result.  $\square$

Now, for each  $f \in C_c^+(G)$ , let  $X_f$  be the interval  $[(f_0 : f)^{-1}, (f : f_0)]$ , and let  $X$  be the Cartesian product of all such  $X_f$ . We see that  $X$  is a compact Hausdorff space consisting of functions from  $C_c^+(G)$  into  $(0, \infty)$  with the value at  $f$  equal to  $X_f$ . Thus,  $I_\phi(f) \in X_f$  for each  $\phi \in C_c^+(G)$  via the established bound.

Now, for each  $V \in \mathcal{N}_{1_G}$ , we let  $K(V)$  be the closure in  $X$  of the set  $\{I_\phi \mid \text{supp}(\phi) \subseteq V\}$ . The sets  $K(V)$  have the finite intersection property, as

$$\bigcap_{j=1}^n K(V_j) \supseteq K\left(\bigcap_{j=1}^n V_j\right),$$

so by compactness, there is  $I \in X$  such that  $I$  is in every  $K(V)$ . This means that every neighborhood of  $I$  contains a  $I_\phi$  with  $\text{supp}(\phi)$  arbitrarily small, so for every neighborhood  $V$  of  $1_G$ , any  $\varepsilon > 0$ , and any  $f_1, \dots, f_n \in C_c^+(G)$ , there is  $\phi \in C_c^+(G)$  such that  $\text{supp}(\phi) \subseteq V$ ,  $|I(f_j) - I_\phi(f_j)| < \varepsilon$  for all  $j$ , meaning that  $I$  commutes with left translation, addition, and multiplication by positive scalars.

Any  $f \in C_c(G)$  can be written as  $f = (g_1 - h_1) + i(g_2 - h_2)$  where each of  $g_i, h_i$  are positive. We thus extend  $I(f) = (I(g_1) - I(h_1)) + i(I(g_2) - I(h_2))$ , which gives a nonzero positive linear functional on  $C_c(G)$ , so by the Riesz–Markov–Kakutani Representation Theorem, we have a measure  $\lambda$  associated to  $I$ .  $\square$

**Proposition:** If  $\lambda$  is a left Haar measure, and  $U$  is a nonempty open set, then  $\lambda(U) > 0$ , and  $\int f d\lambda > 0$  for all  $f \in C_c^+(G)$ .

*Proof.* Let  $U$  be nonempty, open, and  $\lambda(U) = 0$ . Then,  $\lambda(xU) = 0$  for all  $x$ , and since any compact set  $K$  can be covered by finitely many such translates,  $\lambda(K) = 0$  for each compact set  $K$ , meaning that  $\lambda(G) = 0$  by inner regularity, contradicting  $\lambda$  being nonzero.

Now, if  $f \in C_c^+(G)$ , then  $U = \{x \mid f(x) > \frac{1}{2}\|f\|_u\}$  is open, and  $\int f d\lambda > \frac{1}{2}\|f\|_u \lambda(U) > 0$ .  $\square$