A smooth map between manifolds $f\colon M \to N$ includes a certain family of local information; for instance, the derivative $D_p f\colon T_p M \to T_{f(p)} N$, which is a linear map between tangent spaces at p and q, is defined on a coordinate chart $U \subseteq M$ for p and a corresponding coordinate chart $V \subseteq N$ for $f(p)$. Yet, the properties of this linear map can give us information about the underlying map f.

To understand this, we need to dive into the world of regular and critical values.

Much of this document is based on the book *Topology from the Differentiable Viewpoint* and assorted notes from my Differential Topology class.

Sard's Theorem and the Regular Value Theorem

Definition: Let $f\colon M \to N$ be a smooth map, and let $p \in M$. We say p is a *critical point* for f if $D_p f$ does not have the same rank as the dimension of $T_{f(p)} N$.

If $D_p f$ has the same rank as the dimension of $T_{f(p)} N$, then we say that p is a *regular point* of f.

We say $q \in N$ is a *critical value* for f if $f^{-1}(\{q\})$ contains a critical point for f. Else,we say that q is a *regular value.*

We start with the case of Sard's Theorem on $\mathbb{R}^n$. Then, we will expand this to the case of any arbitrary manifold by means of a technical lemma.

Theorem (Sard's Theorem)**:** Let $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be a smooth map. Then, if C is the set of critical points for f, we have $f(C) \subseteq \mathbb{R}^m$ has measure zero.

The proof of Sard's Theorem is very technical, so we will not be showing the full proof. A proof can be found at this link.

A useful result used in conjunction with Sard's Theorem is the Regular Value Theorem. We will show some important results using these two theorems.

Theorem (Regular Value Theorem)**:** Let $f\colon M \to N$ be a smooth map of manifolds with dimensions $m \geq n$. If $q \subseteq N$ is a regular value, then $f^{-1}(\{q\}) \subseteq M$ is a submanifold of dimension $m - n$.

Proof. Let $p \in f^{-1}(\{q\})$, and let (U, φ) be a chart about p where $\varphi(U) \cong \mathbb{R}^m \cong T_p M$ are identified together. Since $D_p f$ is full rank, we have that $K = \ker(D_p f)$ is of codimension n, meaning that $K \cong \mathbb{R}^{m-n}$.

Let $L\colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ be a projection, and define $F\colon U \to N \times \mathbb{R}^{m-n}$ by $x \mapsto (f(x), L(x))$. Then, since L is a linear map and the matrix representation for $D_p F$ is block-diagonal, we have that $D_p F = (D_p f, L)$. In particular, $D_p F\colon \mathbb{R}^n \to \mathbb{R}^n$ is full rank, so by the inverse function theorem, F is invertible on a neighborhood $V \times W \subseteq N \times \mathbb{R}^{m-n}$, where W is a neighborhood of 0. We may thus identify $U \cong V \times W$.

By composing with the projection $\pi\colon N \times \mathbb{R}^{m-n} \to N$ given by $(q, W) \mapsto q$, we have that $f = \pi \circ F$, meaning $f^{-1}(\{q\}) = F^{-1}\big(\pi^{-1}(\{q\})\big)$, so that $f^{-1}(\{q\}) \cong \mathbb{R}^{m-n}$. $\square$

Remark: If M is compact and N has the same dimension as M, $f^{-1}(\{q\})$ is discrete. Additionally, the cardinality $\left|f^{-1}(\{q\})\right|$ is a locally constant function of q.

To see this, let $p_1, \dots, p_k$ be the elements of $f^{-1}(\{q\})$ with corresponding disjoint open neighborhoods $U_1, \dots, U_k$. These neighborhoods are necessarily mapped diffeomorphically onto neighborhoods $V_1, \dots, V_k$ in N. If we let

$$V = (V_1 \cap \dots \cap V_k) \setminus f(M \setminus (U_1 \cup \dots \cup U_k)),$$

then for any $w \in V$, we have $\left|f^{-1}(\{w\})\right|$ is equal to $\left|f^{-1}(\{q\})\right|$.

The No-Retraction Theorem

One of the primary applications of the Regular Value Theorem is the No-Retraction Theorem, which is essentially a generalization of the smooth version of the Brouwer Fixed-Point Theorem.

Theorem: Let M be a compact smooth n-dimensional manifold with boundary $N = \partial M$. There does not exist any smooth surjective function $r\colon M \to N$ such that $r|_N = \mathrm{id}_N$.

Proof. Suppose toward contradiction that there were a retraction. Let X be the set of critical points for r in M; by Sard's Theorem, $r(X) \subseteq N$ has measure zero, so there exists a regular value $y \in N$.

By the Regular Value Theorem, $r^{-1}(\{y\})$ is a smooth 1-dimensional manifold, so $r^{-1}(\{y\})$ is either S^1 or an open interval. If $r^{-1}(\{y\})$ is S^1, then $r^{-1}(\{y\})$ is necessarily contained in the interior of M, which would contradict the fact that $y \in \partial M$ and $y \in r^{-1}(\{y\})$. Therefore, $r^{-1}(\{y\})$ is an interval, and specifically it is one that has both of its endpoints on N. This follows from the fact that on the interior of M, such an interval must be identified to a 1-dimensional topological subspace of M. Therefore, there is some $z \neq y \in N$ such that $z \in r^{-1}(\{y\})$, implying that $r(z) = y \neq z$, which means r is not a retraction. □

The usefulness of results like the No-Retraction Theorem is often amplified in homology theory, where one can find an analogous result through cohomology groups.

Mapping Degree Theory

The best use case for the Regular Value Theorem is in proving the equivalence between two ways to describe the "wrapping" of a map from one manifold about another.

Definition: Let $f\colon M \to N$ be a smooth map between manifolds of the same dimension. Then, the degree of f at $q \in N$, is given by

$$\deg_q(f) = \sum_{p \in f^{-1}(\{q\})} \mathrm{sgn}(D_p f),$$

where $\mathrm{sgn}(D_p f)$ denotes the sign of the determinant of $D_p f$.