# Complex Analysis

## Analyticity and Path-Independence in the Complex Plane

### Baby's First Complex Function Theory

We are interested in functions of the form $f(z)$, where $z = x + iy$ is some complex number. Note that this is specifically different from a function $g\colon \mathbb{R}^2 \to \Omega$ for some domain $\Omega$; in the latter case, we have independent variables $x$ and $y$, while in the former case, we must express $z = x + iy$.

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z)\, dz &= \oint_C w(z)\,(dx + i dy)\\ &= \oint_C w(z)\, dx + i\oint_C w(z)\, dy.\end{aligned}$$

Taking $A_x = w(z)$ and $A_y = iw(z)$, we have

$$= \oint_C \mathbf{A}\cdot\, d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla \times \mathbf{A})\, d\mathbf{a},$$

and when this integral is zero. Note that $(\nabla \times \mathbf{A})\cdot \hat{n} = 0$, so $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$.

Note that we can take

$$w(z) = u(x,y) + iv(x,y),$$

where $z = x + iy$.

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

**Theorem** (Cauchy–Riemann Equations)**:**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that $w$ is analytic,[1] which leads to Cauchy's theorem.

**Theorem** (Cauchy's Theorem)**:** If $C$ is a simple closed curve in a simply connected region, then $w$ is analytic if and only if

$$\oint_C w(z)\, dz = 0. \tag{$\dagger$}$$

**Fact.** The function $w(z)$ is analytic inside the simply connected region $R$ if any of these hold:

- $w$ satisfies the Cauchy–Riemann equations;

[1] Equal to its Taylor series, also holomorphic.

- $w'(z)$ is unique and exists;
- $\frac{\partial w}{\partial \overline{z}} = 0$.
- $w$ can be expanded as $w(z) = \sum_{n\geqslant 0} c_n (z-a)^n$, convergent on some open neighborhood of $a$ for each $a$ on its domain;[II]
- $w(z)$ is path-independent everywhere in R: $\oint_C w(z)\, dz = 0$.

**Example.** Considering $w(z) = z$, we have $u = x$ and $v = y$, so it satisfies the Cauchy–Riemann equations. However, neither $\text{Re}(z)$ nor $\text{Im}(z)$ are analytic, and neither is $\overline{z} = x - iy$.

**Remark:** Whenever we say "analytic at $p$," we mean "analytic in a neighborhood of $p$."

Note that since $\mathbb{C}$ is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of $\mathbb{C}$, by adjoining a point $\{\infty\}$, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. This compactified $\mathbb{C}^*$ is often represented as a unit sphere with the north pole, determined by $(0, 0, 1)$, is the point at infinity. The correspondence between $\mathbb{C}^* \setminus \{\infty\}$ and $\mathbb{C}$ is evaluated via stereographic projection.

We define $\frac{z}{\infty} = 0$ and $\frac{z}{0} = \infty$ for any $z \neq 0, \infty$. The correspondence between $z = x + iy$ in the plane to $Z$ on the Riemann sphere with $\mathbb{R}^3$ coordinates $(\xi_1, \xi_2, \xi_3)$ is

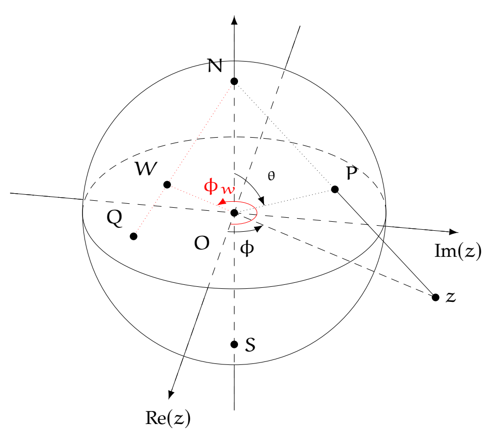
$$\begin{aligned} \xi_1 &= \frac{2\,\text{Re}(z)}{|z|^2 + 1} \\ \xi_2 &= \frac{2\,\text{Im}(z)}{|z|^2 + 1} \\ \xi_3 &= \frac{|z|^2 - 1}{|z|^2 + 1}. \end{aligned}$$

Inverting, we may find

$$\begin{aligned} x &= \frac{\xi_1}{1 - \xi_3} \\ y &= \frac{\xi_2}{1 - \xi_3}, \end{aligned}$$

and with polar coordinates,

$$z = \cot(\theta/2) e^{i\phi}.$$




To determine analyticity at $\infty$, we set $\zeta = \frac{1}{z}$, and analyze the analyticity of $\tilde{w}(\zeta) = w(1/z)$ at 0.

[II] This is technically the real definition of analytic for the case when we're dealing with a function with domain $\mathbb{R}$.

**Cauchy's Integral Formula**

Consider the function $w(z) = c/z$, integrated around a circle of radius $R$. Then, writing $z = Re^{i\varphi}$, we get

$$\begin{aligned}\oint_\Gamma w(z)\,dz &= C\int_0^{2\pi} \frac{e^{-i\varphi}}{R}\underbrace{iRe^{i\varphi}\,d\varphi}_{dz}\\ &= ic\int_0^{2\pi} d\varphi\\ &= 2\pi ic.\end{aligned}$$

If our contour $C$ runs around our singularity at $z = 0$ a total of $n$ times, then we pick up a factor of $n$.

Now, when we consider

$$I = \oint_C \frac{dz}{z^n},$$

this integral actually yields 0 for any $n \neq 1$, despite the fact that 0 is a singularity for $f(z) = \frac{1}{z^n}$. This 0 is not a reflection of Cauchy's integral theorem, but of the fact that

$$z^{-n} = \frac{d}{dz}\left(\frac{z^{-n+1}}{n+1}\right),$$

meaning that $z^{-n}$ is an exact differential, so integrating along a closed curve yields zero change. However, $\frac{1}{z} = \frac{d}{dz}(\ln z)$ may be an exact differential, but for complex $z$, $\ln z = \ln|z| + i\arg(z) = \ln r + i\varphi$. This yields

$$\begin{aligned}\oint_C \frac{c}{z}dz &= c\oint_C d(\ln z)\\ &= c(i(\varphi + 2\pi) - \varphi)\\ &= 2\pi ic.\end{aligned}$$

Ultimately, what this shows is that when we integrate any analytic function $f(\zeta)$ along a closed contour with a singularity at $z$, only the coefficient on $\frac{1}{\zeta - z}$ will remain. This coefficient is known as the residue at 0.

**Theorem** (Cauchy's Integral Formula)**:** If $w$ is analytic in a simply connected region and $C$ is a closed contour winding once around a point $z$ in the region, then

$$w(z) = \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta. \tag{**}$$

Furthermore, this shows that any once-differentiable function is infinitely differentiable, as by differentiating under the integral sign, we get

$$\frac{d^n w}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

**Example** (Deriving Liouville's Theorem)**.** Consider a circle $C$ centered at radius $r$ centered at at $z$, $\zeta - z = Re^{i\varphi}$. We take $d\zeta = iRe^{i\varphi}\,d\varphi$, and taking derivatives, we have

$$w'(z) = \frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi.$$

If $w$ is bounded — i.e., $|w(z)| \leqslant M$ for all $z$ in a given region — then

$$|w'(z)| = \left|\frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi\right|$$

$$\leqslant \frac{1}{2\pi R}\int_0^{2\pi}\left|w\left(z+Re^{i\varphi}\right)\right|\,d\varphi$$
$$\leqslant \frac{M}{R}$$

for all R within the analytic region.

In the case where $w$ is entire (i.e., analytic on $\mathbb{C}$), then this inequality holds for all $R\to\infty$. Thus, $|w'(z)|=0$ for all $z$, meaning that $w$ is constant.

This is known as Liouville's theorem — every bounded entire function is constant. This can be used to prove the fundamental theorem of algebra.

What Liouville's theorem tells us is that any nontrivial behavior will emerge from a function's singularities.

## Singularities and Branches

To understand nontrivial behavior on the complex plane, we need to understand singularities. This will require us to develop understanding of Laurent series.

### Taylor Series

We want to integrate $w(z)$ around some point $a$ in an analytic region of $w(z)$. This yields the form

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)-(z-a)}\,d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)\left(1-\frac{z-a}{\zeta-a}\right)}\,d\zeta. \end{aligned} \tag{‡}$$

Since $\zeta$ is on the contour and $z$ is in the contour, $\left|\frac{z-a}{\zeta-a}\right|<1$, we may expand as a geometric series. Thus, we get

$$= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)}\left(\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n\right)d\zeta.$$

Since the series is uniformly convergent, we are allowed to exchange sum and integral, yielding

$$\begin{aligned} &= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right)}_{=c_n}(z-a)^n \\ &= \sum_{n=0}^{\infty} c_n(z-a)^n, \end{aligned}$$

where

$$c_n = \frac{1}{n!}\left.\frac{d^n w}{dz^n}\right|_{z=a}.$$

If our Taylor series reduces to a known series on the real axis, we find this very desirable. We say this is a type of analytic continuation from the real axis to the complex plane. For example,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is an analytic continuation of $e^x$.

However, more interestingly,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for all $s > 1$. However, we have also shown that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

converges for complex $s$ for all real part greater than 1. Since values of this integral agree with the series representation of $\zeta(s)$ on real axis, we have that this is an analytic continuation of $\zeta(s)$ to the subset of $\mathbb{C}$ defined by $\mathrm{Re}(s) > 1$.

**Laurent Series**

Now, what happens if, at (‡), we have $\left|\frac{z-a}{\zeta-a}\right| > 1$. The series as constructed would not converge, but what if we have a series that converges everywhere *outside* $C$? This would entail an expansion in reciprocal integer powers of $z - a$. This yields

$$\begin{aligned}
w(z) &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(z-a)\left(1 - \frac{\zeta - a}{z - a}\right)}\, d\zeta \\
&= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{z-a} \left(\sum_{n=0}^{\infty} \left(\frac{\zeta - a}{z - a}\right)^n\right) d\zeta \\
&= -\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C w(\zeta - a)^n\, d\zeta\right) \frac{1}{(z-a)^{n+1}} \\
&= \sum_{n=1}^{\infty} \underbrace{\left(-\frac{1}{2\pi i} \oint_C w(\zeta - a)^{n-1}\, d\zeta\right)}_{=c_{-n}} \frac{1}{(z-a)^n} \\
&= \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}
\end{aligned}$$

Note that this series has a singularity at $z = a$, but since our series is only defined outside a particular region, that doesn't matter. We call a series in reciprocal powers a Laurent series. More specifically, Laurent series may include expansions in negative powers as well as positive powers.

**Example** (Annuli)**.** If we have a point $a$, we want to surround $a$ by a special contour to apply Cauchy's integral formula.

In particular, for any $z$ in the annulus, we get

$$w(z) = \frac{1}{2\pi i} \oint_{c_1 - c_2} \frac{w(\zeta)}{\zeta - z}\, d\zeta$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\oint_{c_1}\frac{w(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{c_2}\frac{w(\zeta)}{\zeta - z}\,d\zeta\\
&= \sum_{n=-\infty}^{\infty} c_n(z-a)^n\\
&= c_0 + \sum_{n=1}^{\infty}\left(c_{-n}(z-a)^n + c_n(z-a)^n\right).
\end{aligned}
$$

**Example.** Consider the function

$$
\begin{aligned}
w(z) &= \frac{1}{z^2+z-2}\\
&= \frac{1}{(z-1)(z+2)}\\
&= \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).
\end{aligned}
$$

Now, we have three regions to expand $w$ in.

- If $|z| < 1$, then our series is in both $z^n$ and $z^n$.
- If $1 < |z| < 2$, then one of our series is going to in $\frac{1}{z^n}$ and one is in $z^n$.
- If $|z| > 2$, then both of our series are in the form of $\frac{1}{z^n}$ and $\frac{1}{z^n}$

Via tedious, heavily error-prone calculations, we find that

$$
\begin{aligned}
w_1(z) &= -\frac{1}{3}\sum_{n=0}^{\infty}\left(1 + (-1)^n\left(\frac{1}{2}\right)^{n+1}\right)z^n\\
w_2(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}} + \left(-\frac{1}{2}\right)^{n+1} z^n\right)\\
w_3(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(1-(-2)^n\right)\frac{1}{z^{n+1}}.
\end{aligned}
$$

Sewing all of $w_1, w_2, w_3$ together, then we get a full series representation of $w(z)$.

**Definition.** If $w(z)$ is a function that can be written as $w(z) = (z-a)^n g(a)$, where $g(a) \neq 0$, then we say $w$ has an $n$-th order zero at $z = a$. If $n = 1$, then we say $w$ has a simple zero at $a$.

Similarly, if we can write

$$
w(z) = \frac{g(a)}{(z-a)^n}
$$

with $g(a) \neq 0$, then we say $w$ has a pole of order $n$ at $a$. If $n = 1$, then we say $w$ has a simple pole at $a$.

There are three types of isolated singularities (i.e., isolated points where $w(z)$ is not defined).

**Definition.** Let $w$ be an analytic function with isolated singularity at $a$.

- If $w$ remains bounded in any neighborhood of $a$, then it must be the case that $c_{-n} = 0$ for all $n > 1$, so the Laurent series is a pure Taylor expansion. We say $z = a$ is a removable singularity.

For instance, the function

$$\frac{\sin(z-a)}{z-a} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-a)^{2n}}{(2n+1)!}$$

has a removable singularity at $z = a$.

- If not all the $c_{-n}$ are equal to zero, but there is a largest $n > 0$ such that $c_{-n}$ is in the Laurent series expansion, then we say $a$ is an $n$-th order pole. If $n = 1$, we say $a$ is a simple pole.
- If there is no largest value of $n$ such that $c_{-n}$ is in the Laurent series — i.e., that $c_{-n} \neq 0$ for all $n$ — then we say that $a$ is an essential singularity.

  One of the most important facts about an essential singularity is that the behavior is path dependent. For instance,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$$

  has an essential singularity at $z = 0$. We see that $e^{1/z}$ diverges as $z \to 0$ along the positive real axis, but if $z \to 0$ along the negative real axis, we get $e^{1/z} \to 0$.

Singularities can also occur at $\infty$, which occurs when $w(1/z)$ has a singularity at 0.

**Multivalued Function**

Consider the function

$$\begin{aligned} w(z) &= z^2 \\ &= \underbrace{\left(x^2 - y^2\right)}_{u(x,y)} + i \underbrace{(2xy)}_{v(x,y)} \\ &= r^2 e^{2i\varphi}. \end{aligned}$$

Note that if we take a path around the origin going around by an angle of $2\pi$, then the resulting path goes around twice. Note that this means the lines $\varphi$ and $\varphi + \pi$ map to the same point in the $w$ plane.

This isn't such a big deal in and of itself, but if we take $w(z) = z^{1/2}$, we get an issue. Instead of $w$ being a two-to-one function, we now have $w$ is a one-to-two function. This is an implicit problem in $\mathbb{R}$ with the function $w(x) = \sqrt{x}$, which we resolve by taking the "positive" square root. This is known as choosing a branch.

We have to do something similar in the complex plane. Note that if we go around by an angle of $2\pi$ in the $z$ plane, then we only go around by an angle of $\pi$ in the $w$-plane. As we keep going around the plane, we jump from branch to branch, which brings issues of continuity.

To resolve this, we create a "branch cut" that contours are not allowed to cross.

**Example.** The most common branch cut is to start from the branch point at $z = 0$, in the case of $w(z) = z^{1/2}$ or $w(z) = \ln(z)$, and extend along the real axis, meaning our branch cut is $(-\infty, 0]$.

This principal branch restricts *output* values of $\varphi$ to $-\pi < \varphi \leqslant \pi$.

For instance, above the cut, we have $\varphi = \pi$, and below the branch cut, we have $\varphi = -\pi$, meaning we have

$$\sqrt{z} = \sqrt{r} e^{i\pi/2} \qquad \varphi \to \pi$$

$$
\begin{aligned}
&= i\sqrt{r} \\
\sqrt{z} &= \sqrt{r}e^{-i\pi/2} && \varphi \to -\pi \\
&= -i\sqrt{r}.
\end{aligned}
$$

This is why the branch cut "causes" a discontinuity across the branch, but in $\mathbb{C} \setminus (-\infty, 0]$.

Now, if we have

$$
\begin{aligned}
\sqrt{z_1}\sqrt{z_2} &= \left(r_1 e^{i\varphi_1}\right)^{1/2}\left(r_2 e^{i\varphi_2}\right)^{1/2} \\
&= \sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2}.
\end{aligned}
$$

However, if we want to calculate $\sqrt{z_1 z_2}$, and if $|\varphi_1 + \varphi_2| > \pi <$ then our product $z_1 z_2$ crosses the branch cut, and our discontinuity requires $\varphi_1 + \varphi_2$ to be converted to $\varphi_1 + \varphi_2 \pm 2\pi$ so as to bring the angle sum back into the principal branch. This means we have

$$
\begin{aligned}
\sqrt{z_1 z_2} &= \left(r_1 r_2 e^{i(\varphi_1+\varphi_2)/2}\right) \\
&= \begin{cases} \sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| \leqslant \pi \\ -\sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| > \pi \end{cases}.
\end{aligned}
$$

**Example.** Now, if we have $z_1 = 2e^{i(3\pi/4)}$ and $z_2 = e^{i(\pi/2)}$, then we have

$$
\begin{aligned}
\sqrt{z_1} &= \sqrt{2}e^{i3(\pi/8)} \\
\sqrt{z_2} &= e^{i(\pi/4)}.
\end{aligned}
$$

Note that if we take $\sqrt{z_1 z_2}$, then the argument of $z_1 z_2$ is $5\pi/4$, so we have to change our argument to $-3\pi/4$ to return to the principal branch before we may calculate the square root. This gives

$$
\begin{aligned}
\sqrt{z_1 z_2} &= \sqrt{2e^{-i(3\pi/4)}} \\
&= \sqrt{2}e^{-i\pi+i(5\pi/8)} \\
&= -\sqrt{2}e^{i(5\pi/8)} \\
&= -\sqrt{z_1}\sqrt{z_2}.
\end{aligned}
$$

Now, it is possible to have a branch point at $\infty$, by determining if $w\left(\frac{1}{z}\right)$ has a branch point at zero. For instance, if $w = z^{1/2}$, this gives

$$
\begin{aligned}
w\left(\frac{1}{z}\right) &= \frac{1}{\zeta^{1/2}} \\
&= \frac{1}{\sqrt{r}}e^{-i\varphi/2},
\end{aligned}
$$

which has the multivalued behavior around the origin. Thus, $z = \infty$ is a branch point for $z$, and we consider the $(-\infty, 0]$ branch cut that connects the branch points at 0 and $\infty$.

**Example.** Consider

$$
w(z) = \sqrt{(z-a)(z-b)}.
$$

where $a, b \in \mathbb{R}$ with $a < b$. We expect the only finite branch points to be $a$ and $b$. Introducing polar coordinates, we have

$$
r_1 e^{i\varphi_1} = z - a
$$

$$r_2 e^{i\varphi_2} = z - b,$$

giving

$$w(z) = \sqrt{r_1 r_2} e^{i\varphi_1} e^{i\varphi_2}.$$

Closed contours around *either* $a$ or $b$ are double-valued. However, if our closed contour goes around *both* $a$ and $b$, then both $\varphi_1$ and $\varphi_2$ add up to $2\pi$, meaning we don't have the multivalued behavior.

Now, to select our branch cut, we need to find out if the point at infinity is a branch point. We take $\zeta = \frac{1}{z}$, and we have

$$w(\zeta) = \frac{1}{\zeta}\sqrt{(1 - a\zeta)(1 - b\zeta)},$$

which blows up at $\infty$, but only takes a singular value.[III]

In general, $z^{1/m}$ for integral $m$ will require $m$ branch cuts.

**Example.** Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{xe^{ikx}}{\sqrt{x^2 + a^2}}\, dx.$$

This is a hard integral to evaluate. To resolve this, we extend the integrand to the complex plane, and invoke Cauchy's theorem to deform the contour.

Note that $\sqrt{x^2 + a^2}$ is multivalued, with branch points at $x = \pm ia$. We choose the branch cut such that our integration contour does not cross the branch cut — i.e., from $-ia$ to $\infty$ to $ia$.

Now, we may deform the contour so as to closely wrap around the branch cut from $ia$ to $\infty$. Remembering the sign discontinuity over the branch cut, this gives the integral

$$\begin{aligned}
\int_{i\infty}^{i\infty} \frac{ze^{ikz}}{\sqrt{x^2 + a^2}}\, dz &= \int_{i\infty}^{ia} \frac{ze^{ikz}}{-i\sqrt{x^2 + a^2}}\, dz + \int_{-a}^{\infty} \frac{ze^{ikz}}{i\sqrt{z^2 + a^2}}\, dz \\
&= 2\int_{ia}^{i\infty} \frac{ze^{ikz}}{i\sqrt{z^2 + a^2}}\, dz \\
&= 2\int_{a}^{\infty} \frac{ye^{-ky}}{\sqrt{y^2 - a^2}}\, dy && z = iy \\
&= 2aK_1(ka) \\
&\sim e^{-ka}
\end{aligned}$$

Here, $K_1$ refers to the modified Bessel function.

### Logarithms

In the complex plane, we say

$$\begin{aligned}
\ln z &= \ln\left(re^{i\varphi}\right) \\
&= \ln r + i\varphi \\
&= \ln|z| + i\arg(z).
\end{aligned}$$

[III] Alternatively, we may see that a positively-oriented contour that surrounds both $a$ and $b$ is a negatively-oriented contour around $\infty$. Since such a contour is valid, $\infty$ is not a branch point.

Unfortunately, this $\ln z$ is a multivalued function — a very multivalued one indeed. This yields many branch points, including 0 and $\infty$:

$$\ln(1/\zeta) = -\ln(\zeta).$$

However, we choose the principal branch, $\pi < \varphi \leqslant \pi$, giving

$$\operatorname{Ln} z = \operatorname{Ln}|z| + i \operatorname{Arg}(z).$$

**Example.** Consider $\ln(z_1 z_1)$ and $\operatorname{Ln}(z_1 z_2)$. If we have

$$\begin{aligned} z_1 &= 1 + i \\ z_2 &= i, \end{aligned}$$

then

$$\begin{aligned} \arg(z_1) &= \pi/4 \\ \arg(z_2) &= \pi/2, \end{aligned}$$

so

$$\begin{aligned} \arg(z_1 z_2) &= 3\pi/4 \\ &= \arg(z_1) + \arg(z_2) \\ &= \operatorname{Arg}(z_1 z_2). \end{aligned}$$

However, if $z_1 = z_2 = -1$, then

$$\begin{aligned} \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\ &= 2\pi \\ \operatorname{Arg}(z_1 z_2) &= \operatorname{Arg}(1) \\ &= 0. \end{aligned}$$

Thus, we get that $\operatorname{Ln}(z_1 z_2) \neq \operatorname{Ln}(z_1) + \operatorname{Ln}(z_2)$.

**Example** (Logarithms vs Inverse Trig)**.** Here, we will derive $\arctan(z)$ in terms of the complex logarithm.

Recall that

$$\begin{aligned} \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \end{aligned}$$

so we have

$$\begin{aligned} z &= \tan(w) \\ &= -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}, \end{aligned}$$

which after much tedious, error-prone symbolic manipulation, gives

$$e^{2iw} = \frac{i - z}{i + z}.$$

Thus, we have

$$w = \arctan(z)$$

$$= \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right).$$

Note that since ln has branch points at 0 and $\infty$, $\ln\left(\frac{i-z}{i+z}\right)$ has branch points when $z = \pm i$.

Now, we must choose a branch cut. Specifically, we want our branch cut to continue the real $\arctan(x)$. We dub this $\text{Arctan}(x)$. Along the real axis, we have

$$\begin{aligned}\text{Arctan}(x) &= \frac{1}{2i}\,\text{Ln}\left(\frac{i-x}{i+x}\right)\\ &= \frac{1}{2i}\left(\text{Ln}\left|\frac{i-x}{i+x}\right| + i\,\text{Arg}\left(\frac{i-x}{i+x}\right)\right)\\ &= \frac{1}{2}\,\text{Arg}\left(\frac{i-x}{i+x}\right).\end{aligned}$$

The principal values are from $-\pi$ to $\pi$, so the output of $\text{Arctan}(x)$ ranges from $-\pi/2$ to $\pi/2$.

## Conformal Maps

A conformal map is a special type of map $w\colon \mathbb{C} \to \mathbb{C}$ that "preserves angles." If, in $z$, we map curves whose intersections are at some angle $\varphi$, then the image of those curves also intersect at the angle $\varphi$.

**Example** (Our First Conformal Map)**.** Consider the map

$$\begin{aligned}w(z) &= z^2\\ &= \left(x^2 - y^2\right) + i(2xy)\\ &= u(x,y) + iv(x,y).\end{aligned}$$

Examining the line elements in the $z$ and $w$ planes, we have

$$\begin{aligned}ds^2 &= du^2 + dv^2\\ &= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2\\ &= \left(\frac{\partial u}{\partial x}dx - \frac{\partial v}{\partial x}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy\right)^2\\ &= \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)\left(dx^2 + dy^2\right)\\ &= \left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)\left(dx^2 + dy^2\right)\\ &= 4\left(x^2 + y^2\right)\left(dx^2 + dy^2\right)\end{aligned}$$

Note that $dx^2$ and $dy^2$ have identical scale factors. Since angles are determined by the ratio of $dx$ and $dy$, it is the case that *all* angles are preserved. This is what is meant by a conformal map.

**Example** (Analyticity and Conformality)**.** Consider an analytic function $w(z)$, with its Taylor expansion about $z_0$.

$$w(z) = w(z_0) + w'(z_0)(z - z_0) + \cdots.$$

For a very small $\xi = z - z_0$, we may truncate it into first order, and place into polar form

$$w(z) - w(z_0) = w'(z_0)\xi$$

$$= |w'(z_0)|e^{i\alpha_0}\xi.$$

Moving from $z$ to $w$, we get a magnification (or shrinkage) by $|w'(z_0)|$ and a rotation by $\alpha_0$.

Since, close to $z_0$, $\xi_1 = z_1 - z_0$ and $\xi_2 = z_2 - z_0$ are magnified by (effectively) the same amount, and rotated by (effectively) the same amount, conformality is established.

**Definition.** A conformal map is an analytic function $w(z)$ defined on a domain $\Omega$ such that $w'(z_0) \neq 0$ for all $z_0 \in \Omega$.

**Example** (Möbius Transformations)**.** A Möbius transformation is a fractional linear transformation of the form

$$w(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. We can calculate $w'(z)$ to be

$$w'(z) = \frac{ad - bc}{(cz + d)^2}.$$

Since $w(z)$ is conformal, it is invertible, so

$$\begin{aligned} w^{-1}(z) &= z(w) \\ &= \frac{dw - b}{-cw + a}. \end{aligned}$$

The Möbius transformations include $\infty$, as we have $w(\infty) = \frac{a}{c}$, meaning that it is an automorphism of the Riemann sphere. Note that because of the constraint, we only need three numbers to specify a Möbius transformation.

Consider the Möbius transformation

$$w(z) = \frac{z - i}{z + i}.$$

We let $z_1 = -1$, $z_2 = 1$, and $z_3 = \infty$. Then, we have

$$\begin{aligned} w(z_2) &= \frac{-1 - i}{-1 + i} \\ &= \frac{2i}{2} \\ &= i. \end{aligned}$$

Similarly, this gives $w(z_3) = 1$. After a bit more playing, we can find that this is a map of the (closed) upper half-plane to the (closed) unit disk, $\mathbb{D}$.

Now, if we look at the "ribbon" between the real axis and the line $\operatorname{Im}(z) = i$, we see that it maps to the region

$$S = \mathbb{D} \setminus \left\{ z \;\middle|\; \left| z - \frac{1}{2} \right| \leqslant \frac{1}{2} \right\}.$$

**Example.** Consider the map $w(z) = e^z$. This gives

$$\begin{aligned} w(z) &= e^x e^{iy} \\ &= \rho e^{i\beta}. \end{aligned}$$

This sends curves of constant $y$ to curves of constant argument, and maps curves of constant $x$ to circles of constant radius.

**Complex Potentials**

Consider the analytic function

$$\Omega(z) = \Phi(x,y) + i\Psi(x,y).$$

We know that

$$\begin{aligned}\frac{\partial \Phi}{\partial x} &= \frac{\partial \Psi}{\partial y}\\ \frac{\partial \Phi}{\partial y} &= -\frac{\partial \Psi}{\partial x}.\end{aligned}$$

Thus, we separate to get

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial y}\\ &= \frac{\partial}{\partial y}\frac{\partial \Psi}{\partial x}\\ &= -\frac{\partial^2 \Phi}{\partial y^2},\end{aligned}$$

so

$$\begin{aligned}\nabla^2\Phi &= 0\\ \nabla^2\Psi &= 0.\end{aligned}$$

The converse is also true — if there is some real harmonic function $\Phi(x,y)$, there is a conjugate harmonic function $\Psi(x,y)$ such that $\Omega(z) = \Phi(x,y) + i\Psi(x,y)$ is analytic.

If $\Omega$ is analytic, then $\Phi$ and $\Psi$ must satisfy the Cauchy–Riemann equations, meaning that

$$\begin{aligned}\Psi(x,y) &= \int \frac{\partial \Psi}{\partial y}\,dy + \frac{\partial \Psi}{\partial x}\,dx\\ &= \int \frac{\partial \Phi}{\partial x}\,dy - \frac{\partial \Phi}{\partial x}\,dx.\end{aligned}$$

For $\Psi$ to be a proper single-valued real function, the integral must be path-independent. Using Green's theorem, we may close the path in a simply connected region, and consider it as a surface integral. This gives

$$\begin{aligned}\oint_C \frac{\partial \Phi}{\partial x}\,dy - \frac{\partial \Phi}{\partial y}\,dx &= \int_S \left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \Phi}{\partial y}\right)\right)dx\,dy\\ &= \int_x \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)dx\,dy\\ &= 0.\end{aligned}$$

We call $\Omega(z) = \Phi(x,y) + i\Psi(x,y)$ the complex potential.

This gives

$$\begin{aligned}\frac{d\Omega}{dz} &= \frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial x}\\ &= \frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y}\end{aligned}$$

$$= \frac{\partial \Psi}{\partial y} + i\frac{\partial \Psi}{\partial x}.$$
$$= \overline{\mathcal{E}},$$

where $\mathcal{E}$ is the complex representation of the electric field, **E**. We have

$$\begin{aligned}\mathcal{E} &= \overline{\frac{\partial \Omega}{\partial z}}\\ &= \frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y},\end{aligned}$$

with

$$E = \left|\frac{d\Omega}{dz}\right|.$$

The physics of electric fields is then determined entirely by the complex potential.

What makes harmonic functions useful is that, if there are complicated boundary conditions, we may apply a conformal map and the functions remain harmonic.

**Example** (Cylindrical Capacitor)**.** Consider a cylindrical capacitor with nonconcentric plates meeting at insulated point $u = 1$ and $v = 0$. The larger cylinder with radius 1 is grounded, and the smaller cylinder with radius $1/2$ is held at voltage $V_0$. We want to find the electric field.

We want to find $\widetilde{\Phi}(w)$ such that

$$\nabla^2\widetilde{\Phi}(u, v) = 0.$$

This domain is kind of difficult, so we will solve the problem on a simpler domain and use a conformal map. Note that from Figure 20.4 in the book, we may use the Möbius transformation

$$w(z) = \frac{z - i}{z + i}$$

to transform *to* our cylindrical capacitor *from* a two-plate infinite capacitor with one plate at $\operatorname{Im}(z) = 1$ and one plate at $\operatorname{Im}(z) = 0$. From physics, we know that $\Phi(x, y) = \frac{V_0 y}{d}$, where $d = 1$. Thus, the harmonic conjugate, $\Psi = -V_0 x$, gives us a complex potential of $\Phi = -iV_0 z$.

Solving

$$\frac{z - i}{z + i} = u(x, y) + iv(x, y),$$

we find

$$\begin{aligned}x(u, v) &= -\frac{2v}{(1 - u)^2 + v^2}\\ y(u, v) &= \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}.\end{aligned}$$

Now, this gives

$$\begin{aligned}\widetilde{\Phi}(u, v) &= \Phi(x(u, v), y(u, v))\\ &= V_0\frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}.\end{aligned}$$

**Example** (Fluid Flow)**.** Consider fluid flow around a rock with disk of radius $a$; far away from the rock, we have uniform flow speed of $\alpha$.

Symmetry allows us to focus only on the upper half-plane. Now, there is a conformal map in Table 20.1 of the textbook, which is the map $w(z) = z + \frac{a^2}{z} = u(x,y) + iv(x,y)$ that maps the upper half-plane to the upper half-plane. Furthermore, this map sends the boundary hugging the rock into the $u$-axis.

After applying the conformal map, we get the stream lines $\widetilde{\Psi}(u,v) = \beta v$, as they are streamlines of uniform horizontal flow.

Building the complex potential, we have

$$\begin{aligned}\widetilde{\Omega}(w) &= \Phi(u,v) + i\Psi(u,v)\\ \widetilde{\Omega}(w) &= \beta w,\end{aligned}$$

as we must have $\frac{d\Phi}{du} = \frac{d\Psi}{dv} = \beta$.

Mapping back into the $z$-plane, we have

$$\Omega(z) = \beta\left(z + \frac{a^2}{z}\right).$$

Note that as $z$ becomes very big, the term $\frac{a^2}{z}$ goes to 0, so we must have $\beta = \alpha$.

Now, we may find the streamlines and potentials. Note that we have

$$\begin{aligned}\Phi &= \operatorname{Re}(\Omega)\\ \Psi &= \operatorname{Im}(\Omega).\end{aligned}$$

Now, we have

$$\begin{aligned}\Omega(z) &= \alpha r\left(e^{i\varphi} + \frac{a^2}{r^2}e^{-i\varphi}\right)\\ &= \alpha r\left(\cos(\varphi) + i\sin(\varphi) + \frac{a^2}{r^2}(\cos(\varphi) - i\sin(\varphi))\right).\end{aligned}$$

Taking real and imaginary parts, we have

$$\begin{aligned}\Phi &= \alpha r\left(1 + \frac{a^2}{r^2}\right)\cos(\varphi)\\ \Psi &= \alpha r\left(1 - \frac{a^2}{r^2}\right)\sin(\varphi).\end{aligned}$$

**Example.** Considering our conformal map

$$w(z) = z + \frac{a^2}{z}$$

again, we see that if $|z| = a$, then $|u| \leqslant 2a$. Meanwhile, if $r > a$, then

$$\begin{aligned}w(z) &= z + \frac{a^2}{z}\\ &= re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\end{aligned}$$

$$
\begin{aligned}
&= \left(r + \frac{a^2}{r}\right)\cos(\varphi) + i\left(r - \frac{a^2}{r}\right)\sin(\varphi) \\
&= u + iv.
\end{aligned}
$$

This gives

$$
\frac{u^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{v^2}{\left(r - \frac{a^2}{r}\right)} = 1.
$$

Note that $w$ fails to be conformal when $\frac{dw}{dz} = 0$, meaning that it fails to be conformal at $z = \pm a$.

This is occasionally used in the real world[IV] to design airfoils.

## Residues

Consider a function $f(z)$ with an $n$th order pole. Then, $f$ can be written as

$$
f(z) = \frac{g(z)}{(z-a)^n},
$$

where $g(z)$ is analytic and $g(a) \neq 0$. Recalling Cauchy's integral formula, we see that this expression for $f$ is tantalizingly close to our desired state.

We may expand $g$ in a Taylor series:

$$
g(z) = \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!}(z-a)^m.
$$

Letting $C$ be a positively oriented contour in the analytic domain of $f$ that encircles the singularity, we get

$$
\oint_C f(z)\,dz = \sum_{m=0}^{\infty} \frac{g^{(m)(a)}}{m!} \oint_C (z-a)^{m-n}\,dz.
$$

Note that if $m - n \neq -1$, then the integral on the right vanishes, so we only obtain a nonzero contribution at $m = n-1$. Thus, we get

$$
\oint_C f(z)\,dz = 2\pi i \frac{g^{(n-1)(a)}}{(n-1)!}.
$$

**Definition.** Let $f(z)$ be an analytic function with a pole at $z = a$ with order $n$. We define the residue of $f$ at $a$ as

$$
\operatorname{Res}[f(z), a] := \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right).
$$

This gives an alternative statement of Cauchy's integral formula, giving

$$
\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f(z), a].
$$

However, when we have lots of poles for $f$, and $C$ is a contour that surrounds all the poles, we may deform $C$ such that it surrounds each pole. This gives the residue theorem.

**Theorem** (Residue Theorem)**:**

$$
\oint_C f(z)\,dz = 2\pi i \sum_{a\in C} \operatorname{Res}[f(z), a] \tag{††}
$$

| Type | Method |
|---|---|
| $n$-th order pole | $\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right)$ |
| simple pole | $\lim\limits_{z\to a}(z-a)f(z)$ |
| $f=\frac{p}{q}$, $q(a)$ simple zero | $\dfrac{p(a)}{q'(a)}$ |
| pole at infinity | $\lim\limits_{z\to 0}\left(-\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right)\right)$ |
| pole at infinity, $\lim_{\lvert z\rvert\to\infty} f(z)=0$ | $-\lim\limits_{\lvert z\rvert\to\infty}(zf(z))$ |

Table 1: Finding $\text{Res}[f(z), a]$

We can find the residue in a variety of ways.

**Example.** We will find the residue for $\cot(z)$ for each of the residues.

$$\begin{aligned}\text{Res}[\cot(z), n\pi] &= \lim_{z\to n\pi}(z-n\pi)\frac{\cos(z)}{\sin(z)}\\ &= (-1)^n \lim_{z\to n\pi}\frac{z-n\pi}{\sin(z)}\\ &= (-1)^n \lim_{z\to n\pi}\frac{z-n\pi}{(-1)^n\sin(z-n\pi)}\\ &= 1.\end{aligned}$$

**Example.** We may find

$$\begin{aligned}\text{Res}\left[\frac{z}{\sinh(z)}, in\pi\right] &= \left.\frac{z}{\frac{d}{dz}(\sinh(z))}\right|_{z=in\pi}\\ &= \frac{in\pi}{\cosh(in\pi)}\\ &= (-1)^n in\pi\end{aligned}$$

**Example.** Let's evaluate

$$\oint_C \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

Finding the residue at each pole, we get

$$\begin{aligned}\text{Res}[f(z), 0] &= \frac{2}{3}\\ \text{Res}[f(z), -1] &= -\frac{3}{2}\\ \text{Res}[f(z), 3] &= -\frac{1}{6}.\end{aligned}$$

These are evaluated using the cover-up method.

Now, we may find the integral by taking

$$\oint_{|z|=2} f(z)\,dz = -i\frac{5\pi}{3}.$$

[IV] I guess people do things over there.

**Example.** Let

$$\begin{aligned} f(z) &= \frac{1}{z^2\sinh(z)}\\ &= \frac{1}{-iz^2\sin(iz)}. \end{aligned}$$

The simple zeros of $\sinh(z)$ are at $in\pi$, so we have an order 3 pole at $z = 0$

$$\begin{aligned} \operatorname{Res}[f(z), 0] &= \frac{1}{(n-1)!}\frac{d^2}{dz^2}\left[z^3 f(z)\right]\bigg|_{z=0}\\ &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z}{\sinh(z)}\right)\bigg|_{z=0}\\ &= -\frac{1}{6}. \end{aligned}$$

Thus, integrating about the unit circle, we get

$$\oint_{|z|=1} = -\frac{i\pi}{3}.$$

If we were to evaluate via the Laurent series, we would have

$$\begin{aligned} \frac{1}{z^2\sinh(z)} &= \frac{1}{z^2}\left(\frac{1}{z + z^2/3 + z^5/5! + \cdots}\right)\\ &= \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right)\\ &\approx \frac{1}{z^3}\left(1 - \frac{z^2}{3!} + \cdots\right)\\ &= \frac{1}{z^3} - \frac{1}{6z} + \cdots, \end{aligned}$$

giving a residue of $-\frac{1}{6}$.

Instead of using the contour on the unit circle, if we want to use a circle of radius 4, we get the residues at $z = \pm i\pi$. To evaluate this, we take

$$\begin{aligned} \operatorname{Res}[f(z), i\pi] &= \frac{1}{-\pi^2(-1)}\\ &= \frac{1}{\pi^2}\\ \operatorname{Res}[f(z), -i\pi] &= \frac{1}{\pi^2}. \end{aligned}$$

Evaluating the integral, we would get

$$\begin{aligned} \oint_C f(z)\,dz &= 2\pi i\left(-\frac{1}{6} + \frac{2}{\pi^2}\right)\\ &= -\frac{i\pi}{3} + \frac{4i}{\pi}. \end{aligned}$$

**Example.** We will now use the residue theorem to evaluate a real-valued integral. Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx.$$

Since this integral goes to zero, we will evaluate

$$I' = \oint_C \frac{1}{z^2+1}\,dz,$$

where C is a semicircle with radius $r$ along the real axis from $-r$ to $r$ "pointing upward," so to speak.

This gives

$$\oint_C \frac{1}{z^2+1}\,dz = \int_{C_r} f(z)\,dz + \int_{-r}^{r} f(x)\,dx,$$

which, sending $r$ to infinity, is equal to

$$I = \int_{-\infty}^{\infty} f(x)\,dx.$$

However, since our expression $\frac{1}{z^2+1}$ has poles at $i$ and $-i$, our semicircle gives

$$\begin{aligned}\oint_C \frac{1}{z^2+1} &= 2\pi i \operatorname{Res}[f(z), i]\\ &= 2\pi i \lim_{z\to i} \frac{1}{z+i}\\ &= 2\pi i \frac{1}{2i}\\ &= \pi.\end{aligned}$$

If we have a finite number of isolated singularities, we are always able to draw a contour that encloses all of them, which allows us to use the residue theorem.

Now, we know that we can have poles at infinity — and that any positively-oriented contour in the plane is a negatively-oriented contour around $\infty$. Thus, if we have a contour surrounding all our finite singularities, we get

$$\begin{aligned}\sum_i \operatorname{Res}[f(z), a_i] &= -\operatorname{Res}[f(z), \infty]\\ \operatorname{Res}[f(z), \infty] + \sum_i [f(z), a_i] &= 0,\end{aligned}$$

as we're doing the same integral, but in negative orientation about $\infty$ and positive orientation about our singularities.

We have

$$\operatorname{Res}[f(z), \infty] = \operatorname{Res}\left[-\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right].$$

**Example.** Now, recalling

$$f(z) = \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

The residues are

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= 2/3\\ \operatorname{Res}[f(z), -1] &= -3/2\end{aligned}$$

$$\text{Res}[f(z), 3] = -1/6.$$

Now, calculating the residue at infinity, we have

$$\begin{aligned}
\text{Res}[f(z), \infty] &= \text{Res}\left[-\frac{1}{z^2}\frac{(1/z-1)(1/z-2)}{1/z(1/z+1)(3-1/z)}, 0\right]\\
&= -\text{Res}\left[\frac{(z-1)(2z-1)}{z(z+1)(3z-1)}\right]\\
&= 1.
\end{aligned}$$

Now, if $\lim_{|z|\to\infty} f(z) = 0$, then $f$ is pure Laurent series. In that case, if there is a residue, then we find the residue by evaluating

$$\text{Res}[f(z), \infty] = -\lim_{|z|\to\infty} zf(z)$$

**Example.** Consider functions of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where $q$ is a higher-order polynomial than $p$.

If $q$ has first-order zeros $a$ and second-order zeros at $b$, then

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z-a_k} + \frac{B_k}{z-b_k} + +\frac{C_k}{(z-b_k)^2}.$$

Note that the coefficients are actually residues. This gives

$$\begin{aligned}
A_k &= \text{Res}[f(z), a_k]\\
B_k &= \text{Res}[f(z), b_k]\\
C_k &= \text{Res}[(z-b_k)f(z), b_k].
\end{aligned}$$

For instance,

$$\frac{(z-1)(z-2)}{z(z+1)(3-z)} = \frac{2}{3}\frac{1}{z} - \frac{1}{6}\frac{1}{z-3} - \frac{3}{2}\frac{1}{z+1}.$$

Now, we may also have

$$\frac{(z-1)(z-2)}{z(z+1)^2(3-z)} = \frac{2}{3}\frac{1}{z} - \frac{1}{24}\frac{1}{z-3} - \frac{5}{8}\frac{1}{z+1} - \frac{3}{2}\frac{1}{(z+1)^2}.$$

**Integrating around a Circle**

We want to evaluate angular integrals of the form

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi))\, d\varphi.$$

Now, while this is a real integral over a domain, we may reformulate it about the unit circle by using the substitutions

$$\begin{aligned}
z &= e^{i\varphi}\\
d\varphi &= \frac{dz}{iz},
\end{aligned}$$

which yields

$$\sin(n\varphi) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$
$$\cos(m\varphi) = \frac{1}{2}\left(z^m + \frac{1}{z^m}\right).$$

Thus, our integral becomes

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi))\, d\varphi = \oint_{|z|=1} f\left(\frac{1}{2i}\left(z^n - \frac{1}{z^n}\right), \frac{1}{2}\left(z^m + \frac{1}{z^m}\right)\right)\frac{dz}{iz}.$$

**Example.** Consider

$$\begin{aligned}\int_0^{2\pi} \sin^2(\varphi)\, d\varphi &= -\frac{1}{4}\oint_{|z|=1}\left(z - \frac{1}{z}\right)^2 \frac{dz}{iz}\\ &= \frac{i}{4}\oint_{|z|=1}\frac{1}{z^3}\left(z^4 - 2z^1 + 1\right) dz\\ &= -\frac{1}{2}\pi \operatorname{Res}\left[\frac{1}{z^3}\left(z^4 - 2z^1 + 1\right), 0\right].\end{aligned}$$

The residue at $z = 0$ is $-2$ — this can be found by dividing out by $z^3$.

Thus, we get the answer of

$$\int_0^{2\pi} \sin^2(\varphi)\, d\varphi = \pi.$$

**Example.** Using residues, we can evaluate a lot of integrals that are quite tricky on their face.

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\varphi)}{5 - 4\sin(\varphi)}\, d\varphi &= \oint_{|z|=1}\frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{5 - \frac{4}{2i}\left(z - \frac{1}{z}\right)}\frac{dz}{iz}\\ &= -\oint_{|z|=1}\frac{z^4 + 1}{2z^2(2z - i)(z - 2i)}\, dz.\end{aligned}$$

Now, we have a simple pole at $i/2$, a simple pole at $2i$, and a pole of order 2 at 0. We only evaluate the residues at 0 and $i/2$. We get

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= -\frac{d}{dz}\left(\frac{z^4 + 1}{2z^2(2z - i)(z - 2i)}\right)\Bigg|_{z=0}\\ &= -\frac{5i}{8}\\ \operatorname{Res}[f(z), i/2] &= \frac{17i}{24}.\end{aligned}$$

Thus, we get the result of

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\varphi)}{5 - 4\sin(\varphi)}\, d\varphi &= 2\pi i\left(-\frac{5i}{8} + \frac{17i}{24}\right)\\ &= -\frac{\pi}{6}.\end{aligned}$$

**Integrating along the Real Axis**

If we want to evaluate integrals along the real axis, such as

$$I = \int_{-\infty}^{\infty} f(x)\,dx,$$

we may be curious as to how we may evaluate this.

To do this, we recall that we created a contour in the upper half-plane of large enough radius $r$, and evaluated the residues inside the contour. We consider the contour to be equal to $C = C_r + l_r$, where $l_r$ is along the real axis and $C_r$ closes our contour. Thus, we get

$$\oint_C f(z)\,dz = \lim_{r\to\infty}\left(\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz\right).$$

Note that the polar coordinate Jacobian gives us the requirement that $\lim_{|z|\to\infty}|zf(z)| = 0$.

When $f(z) = \frac{p(z)}{q(z)}$, this is satisfied when $q$ is of degree at least two more than that of $p$.

**Example.** Consider

$$\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4}\,dx = \oint_C \frac{2z+1}{z^4+5z^2+4}\,dz.$$

Factoring, we get

$$\oint_C \frac{2z+1}{z^4+5z^2+4}\,dz = \oint_C \frac{2z+1}{(z-2i)(z+2i)(z-i)(z+i)}\,dz.$$

We only care about the residues in the upper half-plane. We have residues of

$$\begin{aligned}\operatorname{Res}[f(z), 2i] &= -\frac{1}{3} + \frac{i}{12}\\ \operatorname{Res}[f(z), i] &= \frac{1}{3} - \frac{i}{6}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4}\,dx &= 2\pi i\left(\frac{1}{3} - \frac{i}{6} - \frac{1}{3} + \frac{i}{12}\right)\\ &= \frac{\pi}{6}.\end{aligned}$$

Note that if we chose our contour to be in the lower half-plane, then we would have a *negatively* oriented contour, and evaluate at the residues in the lower half-plane.

**Example.** Consider

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^3-i}\,dx &= \oint_C \frac{1}{z^3-i}\,dz\\ &= \oint_C \frac{1}{(z+i)\left(z-e^{i\pi/6}\right)\left(z-e^{5i\pi/6}\right)}.\end{aligned}$$

Closing $C$ in the lower half-plane, we only need the residue at $-i$. This gives

$$\begin{aligned}\oint_C \frac{1}{z^3-i} &= -2\pi i\left(-\frac{1}{3}\right)\\ &= \frac{2\pi i}{3}.\end{aligned}$$

Consider integrals of the form

$$\int_{-\infty}^{\infty} g(x)e^{ikx}\,dx,$$

where $k$ is real.

Now, we want to know when exactly we are allowed to "close up" the semicircle contour.

We start by assuming $k$ is positive. Closing in the upper half-plane so as to ensure exponential decay, we have

$$\begin{aligned}\left|\int_{C_r} g(z)e^{ikz}\,dz\right| &\leqslant \int_{C_r}\left|g(z)e^{ikz}\right|dz\\ &= \int_0^{\pi}\left|g\left(re^{i\varphi}\right)\right|re^{-kr\sin(\varphi)}\,d\varphi.\end{aligned}$$

Since $\sin(\varphi) \geqslant 0$ on the range of integration, the integral vanishes as $r \to \infty$. Therefore, we are allowed to close up the contour whenever $|g(z)| \to 0$ as $|z| \to \infty$.

**Example.** Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+4}\,dx.$$

This gives

$$\begin{aligned}I &= \mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+4}\,dx\right)\\ &= \mathrm{Re}\left(\oint_C \frac{e^{ikz}}{z^2+4}\,dz\right)\\ &= \oint_C \frac{e^{ikz}}{(z-2i)(z+2i)}\,dz.\end{aligned}$$

We assume $k > 0$. Then, evaluating at $2i$, we have

$$I = \frac{\pi}{2}e^{-2k}.$$

Now, if $k < 0$, we close our contour in the lower half-plane, we get

$$I = \frac{\pi}{2}e^{2k}.$$

Thus, our integral is always

$$I = \frac{\pi}{2}e^{-2|k|}.$$

### Non-Circular Contours

Sometimes, semicircles don't work.

**Example.** Consider

$$\int_{-\infty}^{\infty}\frac{e^{bx}}{e^x+1}\,dx,$$

where $0 < b < 1$. Writing our integral, we have

$$I = \int \frac{e^{bz}}{e^z + 1}\, dz$$

This gives poles at $z = (2n + 1)i\pi$, which means we cannot close this contour with a semicircular arc at $\infty$.

What may work in this case is by drawing a rectangular contour from $-a$ to $a$ such that it encloses exactly one of the poles of our integrand. The vertical segments of this contour go to zero as we send $a \to \infty$. We call the segment of the contour along the line $a + 2\pi i$ to $a - 2\pi i$ as $I'$.

This gives

$$I + I' = \oint_C \frac{e^{bz}}{e^z + 1}\, dz.$$

Now, we constructed $I'$ such that

$$\begin{aligned} I' &= \int_{\infty}^{-\infty} \frac{e^{b(x+2\pi i)}}{e^{x+2\pi i} + 1}\, dx \\ &= -e^{2\pi i b} \int_{-\infty}^{\infty} \frac{e^{bx}}{e^x + 1}\, dx \\ &= -e^{2\pi i b} I. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \oint_C \frac{e^{bz}}{e^z + 1}\, dz &= I\left(1 - e^{2\pi i b}\right) \\ &= 2\pi i \operatorname{Res}\left[\frac{e^{bz}}{e^z + 1}, i\pi\right], \end{aligned}$$

giving

$$I = \frac{\pi}{\sin(\pi b)}.$$

**Example.** We want to evaluate

$$\int_0^\infty \cos\left(x^2\right)\, dx$$
$$\int_0^\infty \sin\left(x^2\right)\, dx.$$

To evaluate this, we draw a slice-shaped contour going along the real axis and returning to 0 along $z = re^{i\pi/4}$. Therefore, we evaluate

$$\begin{aligned} \oint_C e^{iz^2}\, dz &= \int_0^\infty e^{ix^2}\, dx + 0 + \int_\infty^0 e^{i\left(re^{i\pi/4}\right)^2} e^{i\pi/4}\, dr \\ &= \int_0^\infty e^{ix^2}\, dx + 0 + \int_\infty^0 e^{-r^2} e^{i\pi/4}\, dr. \end{aligned}$$

Thus, we get

$$\begin{aligned} \int_0^\infty \cos\left(x^2\right)\, dx &= \int_0^\infty \sin\left(x^2\right)\, dx \\ &= \frac{1}{2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

**Integrating with Branch Cuts**

When we're integrating with residues, branch cuts are a feature rather than a bug.

**Example.** Consider the integral

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^3}\,dx.$$

We need a branch cut to avoid the multivalued behavior. Our poles are at $e^{i\pi/3}, -1, e^{-i\pi/3}$. Since our integral is along the real axis, we take our branch cut along the domain $[0,\infty]$.

We draw our contour of radius $R$ by hugging the branch without crossing it, with a small circle of radius $\varepsilon$ just outside 0. This gives the integral

$$\oint \frac{\sqrt{z}}{1+z^3}\,dz = \int_0^\infty \frac{\sqrt{z}}{1+z^3}\,dx + \int_{C_R} \frac{\sqrt{z}}{1+z^3}\,dz + \int_\infty^0 \frac{\sqrt{z}}{1+z^3}\,dz + \int_{C_\varepsilon} \frac{\sqrt{z}}{1+z^3}\,dz.$$

Note that since $\lim_{|z|\to\infty}|zf(z)| = 0$, and $\lim_{|z|\to 0}|zf(z)| = 0$, our integrals along $C_R$ and $C_\varepsilon$ go to zero, giving the integral

$$\begin{aligned}
I' &= \int_\infty^0 \frac{\sqrt{z}}{1+z^3}\,dz\\
&= \int_\infty^0 \frac{\sqrt{e^{2i\pi}x}}{1+(e^{2i\pi}x)^3}\,dx\\
&= \int_0^\infty \frac{\sqrt{x}}{1+x^3}\,dx\\
&= I.
\end{aligned}$$

Thus,

$$\oint \frac{\sqrt{z}}{1+z^3}\,dz = 2I.$$

Evaluating the residues, we have

$$\begin{aligned}
\operatorname{Res}\left[f(z), e^{i\pi/3}\right] &= \lim_{z\to e^{i\pi/3}} \frac{\sqrt{z}}{3z^2}\\
&= -\frac{i}{3}\\
\operatorname{Res}[f(z), -1] &= \lim_{z\to -1} \frac{\sqrt{z}}{3z^2}\\
&= \frac{i}{3}\\
\operatorname{Res}\left[f(z), e^{5\pi i/3}\right] &= -\frac{i}{3},
\end{aligned}$$

giving the solution of

$$\begin{aligned}
I &= \frac{1}{2}2\pi i\left(-\frac{i}{3}\right)\\
&= \frac{\pi}{3}.
\end{aligned}$$

**Example.** To evaluate

$$I = \int_{0}^{\infty} \frac{1}{1+x^3}\,dx,$$

we start by evaluating

$$\int_{0}^{\infty} \frac{\ln(x)}{1+x^3}\,dx$$

with the branch cut along the real axis. Using the keyhole contour in the previous example, we have that $C_R$ and $C_\varepsilon$ contribute nothing, and ln picks up a phase of $2\pi i$, so that

$$\begin{aligned}\oint_C \frac{\ln(z)}{1+z^3} &= \int_{0}^{\infty} \frac{\ln(x)}{1+x^3}\,dx + \int_{\infty}^{0} \frac{\ln(x)+2\pi i}{1+x^3}\,dx\\ &= -2\pi i I.\end{aligned}$$

Therefore,

$$I = -\sum \operatorname{Res}\left[\frac{\ln(x)}{1+x^3}\right].$$

Thus, we get the solution of

$$\int_{0}^{\infty} \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

**Example.** Consider the integral

$$I = \int_{0}^{1} \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx.$$

The poles are around $\pm ia$.

Our problem is that we have multivalued behavior at $\pm 1$. We may take the cut from $-1$ to $1$ along the real axis, and our contour gives a sign flip across the cut.

We draw a dog-bone style contour hugging the cut in negative orientation to give us $2I$. Thus, we get

$$\begin{aligned}\oint_C \frac{\sqrt{1-z^2}}{z^2+a^2}\,dz &= \int_{-1}^{1} \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx - \int_{1}^{-1} \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx\\ &= 4I,\end{aligned}$$

where the sign flip in the second integral comes from crossing the branch cut.

Now, to evaluate the sum of the residues, we need to evaluate at three poles — $ia$, $-ia$, and the pole at $\infty$. Thus, we get

$$\begin{aligned}\operatorname{Res}[f(z), \pm ia] &= \frac{\sqrt{a^2+1}}{2ia}\\ -\operatorname{Res}[f(z), \infty] &= \lim_{|z|\to\infty} zf(z)\\ &= i.\end{aligned}$$

Therefore,

$$\begin{aligned}4I &= 2\pi i\left(\frac{\sqrt{a^2+1}}{ia} - i\right)\\ &= \frac{\pi}{2a}\left(\sqrt{a^2+1} - a\right).\end{aligned}$$

**Poles on the axis**

If we want to evaluate integrals with the pole on the contour, we need to use principal values.

$$\mathrm{PV}\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\left(\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx\right).$$

Similarly, we want to apply this for the calculus of residues. To do this, we take

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \lim_{\varepsilon\to 0} f(z)\,dz,$$

where $c_\pm$ are small semicircular contour additions of radius $\varepsilon$ to C that hug our pole on the real axis, with $c_-$ excluding the pole and $c_+$ including the pole. Thus, we have

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \lim_{\varepsilon\to 0}\int_{c_\pm} \frac{(z-x_0)f(z)}{z-x_0}\,dz.$$

Introducing $z - x_0 = \varepsilon e^{i\varphi}$, we have $dz = i\varepsilon e^{i\varphi}\,d\varphi$, giving

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \mathrm{Res}[f(z), x_0]\int_{c_\pm} i\,d\varphi.$$

Thus, we have

$$\begin{aligned}\oint_C f(z)\,dz &= \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx \pm i\pi\,\mathrm{Res}[f(z), x_0].\\ &= 2\pi i\sum_{z_i}\mathrm{Res}[f(z) - z_i].\end{aligned}$$

Thus, we have

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = \sum_{z_i \text{ in } C}\mathrm{Res}[f(z), z_i] \mp i\pi\,\mathrm{Res}[f(z), x_0].$$

Note that we only have *half* the residue when the pole is on the contour. Therefore, we have the result of

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\sum_{z_i \text{ in } C}\mathrm{Res}[f(z), z_i] + \frac{1}{2}\sum_{z_i \text{ on } C}\mathrm{Res}[f(z), z_i]\right).$$

**Example.** Consider

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx.$$

We have a simple pole at $x = a$.

We close our contour with a semicircle on the upper half-plane. Since we have no poles inside the contour, we have

$$\begin{aligned}\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx &= \pi i\,\mathrm{Res}\left[\frac{e^{ikx}}{x-a}, a\right]\\ &= \pi i e^{ika}.\end{aligned}$$

Notice that if $k < 0$, we must close the contour in the lower half-plane, giving

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx = \mathrm{sgn}(k)\pi i e^{ika}.$$

Taking real and imaginary components, we get

$$\begin{aligned} \text{PV}\int_{-\infty}^{\infty} \frac{\cos(kx)}{x-a}\,dx &= -\pi\sin(ka)\\ \text{PV}\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\,dx &= \pi\cos(ka). \end{aligned}$$

**Example.** We will evaluate

$$I = \text{PV}\int_{0}^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx.$$

We have a troublesome portion at $x = 0$, so we draw our contour to exclude 0.

We may close the contour with a large semicircle $C_R$. Since $\lim_{|z|\to\infty}|zf(z)| = 0$ and $\lim_{|z|\to 0}|zf(z)| = 0$, we may take these limits to give

$$\begin{aligned} \oint_C \frac{\ln(z)}{z^2+a^2} &= \int_{-\infty}^{0} \frac{\ln\left(e^{i\pi}x\right)}{x^2+a^2}\,dx + \int_{0}^{\infty} \frac{\ln\left(e^{i0}x\right)}{x^2+a^2}\,dx\\ &= \text{PV}\int_{-\infty}^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx + i\pi\int_{0}^{\infty} \frac{1}{x^2+a^2}\,dx\\ &= 2\text{PV}\int_{0}^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx + \frac{i\pi^2}{2a}\\ &= 2\pi i\,\text{Res}[f(z), ia]. \end{aligned}$$

Thus, we get $I = \frac{\pi}{2a}\ln(a)$.

**Example.** Instead of moving our contour up or down by $\varepsilon$ to include (or exclude) a pole, we may move the pole up or down by $\varepsilon$. We consider

$$\int_{-\infty}^{\infty} \frac{g(x)}{x-x_0}\,dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{g(x)}{x-(x_0\pm i\varepsilon)}\,dx.$$

Breaking into real or imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{g(x)}{x-(x_0\pm i\varepsilon)^2}\,dx = \int_{-\infty}^{\infty} g(x)\frac{x-x_0}{(x-x_0)^2+\varepsilon^2}\,dx \pm i\varepsilon\int_{-\infty}^{\infty} \frac{g(x)}{(x-x_0)^2+\varepsilon^2}\,dx.$$

Now, notice that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2} = \begin{cases} 0 & x\neq x_0\\ \infty & x = x_0 \end{cases}.$$

Now, we may take

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}\,dx = \varepsilon\oint_C \frac{dz}{z+\varepsilon^2},$$

where $z = (x-x_0)^2$. This gives

$$\begin{aligned} \varepsilon\oint_C \frac{dz}{z+\varepsilon^2} &= \varepsilon(2\pi i)\left(\frac{1}{2i\varepsilon}\right)\\ &= \pi. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}\,dx = \pi\delta(x-x_0).$$

Therefore, we recover

$$\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{\infty}\frac{g(x)}{(x-x_0)^2+\varepsilon^2}\,dx = \pi g(x_0).$$

This gives the identity under the integral of

$$\frac{1}{z\mp i\varepsilon} = \mathrm{PV}\frac{1}{x}\pm i\pi\delta(x).$$

**Example.** Consider the integral

$$\begin{aligned}\oint_{C_r}\frac{\cos(z)}{z}\,dz &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{\cos(x)}{x\pm i\varepsilon}\,dxj\\ &= \underbrace{\mathrm{PV}\int_{-\infty}^{\infty}\frac{\cos(x)}{x}\,dx}_{=0}\mp i\pi\int_{-\infty}^{\infty}\cos(x)\delta(x)\,dx\\ &= \mp i\pi\end{aligned}$$

**Sommerfeld–Watson Transform and Series Summation**

Thus far, we've been replacing integrals with sums. Now, we're interested in going the other way around.

Consider the sum

$$S = \sum_{-\infty}^{\infty} f(n),$$

given the condition that $f(z)$ is analytic for $z\in\mathbb{R}$ and $\lim_{|z|\to\infty}\left|z^2 f(z)\right| = 0$.

We will introduce the auxiliary function

$$\begin{aligned}g(z) &= \pi\cot(\pi z)\\ &= \pi\frac{\cos(\pi z)}{\sin(\pi z)}.\end{aligned}$$

Note that $g(z)$ has an infinite number of poles at $z = n$ for each $n\in\mathbb{Z}$.

Now, what we will do here is integrate the product $f(z)g(z)$ around a long enough symmetric contour hugging the real axis. This gives

$$\begin{aligned}\frac{1}{2\pi i}\oint_C f(z)g(z)\,dz &= \sum_{n=-\infty}^{\infty}\operatorname{Res}[\pi\cot(\pi z)f(z), n]\\ &= \sum_{n=-\infty}^{\infty} f(z)\left.\frac{\pi\cos(\pi z)}{\frac{d}{dz}(\sin(\pi z))}\right|_{z=n}.\\ &= \sum_{n=-\infty}^{\infty} f(n).\end{aligned}$$

Now, this doesn't *seem* that helpful, until we remember that our contour C surrounds all the other poles of f in negative orientation.

$$\frac{1}{2\pi}\oint_C f(z)g(z)\,dz = -\sum_i \operatorname{Res}[\pi\cot(\pi z)f(z), z_i].$$

Thus, we have converted our infinite sum into a finite sum.

Similarly, if we have an alternating sign series

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty}(-1)^n f(n) \\ &= -\sum_i \operatorname{Res}[\pi\csc(\pi z)f(z), z_i] \end{aligned}$$

**Example.** Consider

$$S = \sum_{n=0}^{\infty}\frac{1}{n^2+a^2}.$$

Our analogous function is

$$f(z) = \frac{1}{z^2+a^2}.$$

Then,

$$\begin{aligned} S' &= -\operatorname{Res}\left[\frac{\pi\csc(\pi z)}{z^2+a^2}, \pm ia\right] \\ &= -\frac{\pi}{2a}\coth(\pi a). \end{aligned}$$

Therefore, we have

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a).$$

Now, we write

$$S = \frac{1}{2a^2} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}.$$

Thus, we get the sum

$$\sum_{n=0}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{2a^2}(1+\pi a\coth(\pi a)).$$

**Example.** Now, we may consider

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2} \\ &= -\operatorname{Res}\left[\frac{\pi\csc(\pi z)}{z^2+a^2}, \pm ia\right] \\ &= -\frac{\pi}{2a}\frac{1}{\sinh(\pi a)}, \end{aligned}$$

giving

$$S = \frac{1}{2a^2}\left(1+\frac{\pi a}{\sinh(\pi a)}\right).$$

## Oscillators and Forcing

Consider a damped harmonic oscillator with position $u(t)$. Then, $u$ obeys Newton's second law,

$$\ddot{u} + 2\beta\dot{u} + \omega_0^2 u = 0.$$

Here, $\beta$ is the damping factor, and $\omega_0$ denotes the natural frequency.

The solutions of this equation are

$$\begin{aligned} u(t) &= e^{-\beta t}\left(ae^{i\Omega t} + be^{-i\Omega t}\right) \\ &= e^{-\beta t}(a\cos(\Omega t) + b\sin(\Omega t)), \end{aligned}$$

where $\Omega^2 = \omega_0^2 - \beta^2$. This is known as a transient solution.

There are three types of motion.

- An underdamped system occurs when $\omega_0 > \beta$, so $\Omega$ is real, meaning we get oscillation that is damped out.
- An overdamped system occurs when $\beta > \omega_0$, so $\Omega$ is imaginary, and the damping slows down the return of the wave.
- When $\omega = \beta$, then $\Omega = 0$, and the solution is of the form $u(t) = e^{-\beta t}(at + b)$, and the system returns to equilibrium as quickly as possible. This is known as critical damping.

A forced system occurs when we have the differential equation

$$\ddot{u} + \beta\dot{u} + \omega_0^2 u = f(t). \tag{†}$$

We may consider a forcing function of the form $f(t) = f_0\cos(\omega t)$. We may also write

$$f(t) = f_0\operatorname{Re}\left(e^{i\omega t}\right).$$

We expect to have a complex steady-state solution of the form

$$U_\omega = C(\omega)e^{i\omega t}.$$

We solve for $U$ by sticking it into the differential equation of †. This will give the equation

$$U_\omega = \frac{f_0 e^{i\omega t}}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}.$$

Note that the real solution is $u = \operatorname{Re}(U_\omega)$, or

$$\begin{aligned} u(t) &= \frac{1}{2}(U_\omega + U_{-\omega}) \\ &= \frac{1}{2}\left(U_\omega + \overline{U_\omega}\right). \end{aligned}$$

Now, when we consider a generalized forcing function $f(t)$, where $f$ is a continuum sum of forcing frequencies where the amplitudes are functions of $\omega$, $\hat{f}(\omega)$, we get an integral:

$$u(t) = \int \frac{F(\omega)e^{i\omega t}}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}\,d\omega.$$

Plugging this solution into the differential equation, we get

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega,$$

which is a Fourier transform (see Math Methods 1).

**Impulse Forcing**

Consider a hammer blow forcing function, known as an impulse forcing function.

The impulse forcing is of the form

$$\begin{aligned} f(t) &= f_0\delta(\omega(t-t_0)) \\ &= \frac{\hat{f}_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t_0)}\,d\omega, \end{aligned}$$

where

$$\hat{f}_0 = \frac{f_0}{\omega_0}.$$

We want to find the impulse solution,

$$\begin{aligned} G(t) &\coloneqq u_\delta(t) \\ &= \frac{\hat{f}_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-t_0)}}{(\omega_0-\omega^2)+2i\beta\omega}\,d\omega. \end{aligned}$$

To do this integral, we will make use of residues. Writing our denominator as

$$\left(\omega_0^2-\omega^2\right)+2i\beta\omega = (\omega-\omega_+)(\omega-\omega_-),$$

where

$$\begin{aligned} \omega_\pm &= i\beta \pm \sqrt{\omega_0^2-\beta^2} \\ &= i\beta \pm \Omega. \end{aligned}$$

We close our contour in the upper half-plane so that get a decaying exponential. Evaluating the residues, we get

$$\begin{aligned} G(t) &= \hat{f}_0\frac{e^{-\beta(t-t_0)}}{2i\Omega}\left(e^{i\Omega(t-t_0)}-e^{-i\Omega(t-t_0)}\right) \\ &= \frac{f_0}{\Omega}\sin(\Omega(t-t_0))e^{-\beta(t-t_0)}, \end{aligned}$$

where $t > t_0$.

Now, if $t < t_0$, then we must close our contour in the lower half-plane, and since there are no poles in the lower half-plane, we get $G(t) = 0$ for $t < t_0$. Thus, we must have

$$G(t) = \frac{f_0}{\Omega}\sin(\Omega(t-t_0))e^{-\beta(t-t_0)}\Theta(t-t_0),$$

where $\Theta$ is the Heaviside step function.

- The imaginary and real parts of $\omega_\pm$ give the damping, $\beta$, and parameter, $\Omega$, respectively. Now, we may interpret the different types of damping in this respect.
  - If $\Omega$ is real (i.e., underdamped motion), then $\omega_\pm$ have constant magnitude of $\omega_0$, meaning that varying the damping only moves the poles around in a circle.
  - If $\beta = \omega_0$ (i.e., critically damped motion), then the poles converge at $i\beta$ along the imaginary axis.
  - If $\beta > \omega_0$ (i.e., overdamped mption), then the poles separate along the imaginary axis, giving non-oscillatory motion.
- The poles also encode resonance characteristics, where we have $\omega_{\text{res}}^2 = \omega_0^2 - 2\beta^2$.
- If $\beta \ll \omega_0$, then the damping is mathematically equivalent to the $i\varepsilon$ prescription moving the resonance pole at $\omega_0$ off the real axis and into the upper half-plane.

**Waves on a String**

Whereas an undamped oscillator is harmonic only in time, a wave is harmonic in both space and time.

A wave satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where $c$ is the wave speed.

The general solution is of the form

$$U(x, t) = ce^{i(kx=\omega t)}.$$

A forced wave occurs via

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(x, t).$$

Now, we start with the impulse solution,

$$\begin{aligned} f(x, t) &= f_0\delta(x - x_0)\delta(t - t_0) \\ &= f_0 \int_{-\infty}^{\infty} \frac{e^{ik(x-x_0)}}{2\pi}\,dk \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{2\pi}\,d\omega. \end{aligned}$$

Now, we have

$$G(x, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{ikx}\,dk \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2/c^2 - k^2}\,d\omega.$$

To evaluate this integral, we start with the integral in $\omega$, given by

$$I = \frac{c^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - c^2k^2}\,d\omega.$$

Unfortunately here, our simple poles lie on the real axis at $\pm ck$.

To find the solution, we need boundary and initial conditions to know where we want to make our $i\varepsilon$ adjustment.

If there is no wave before our impulse hits, we need our integral to vanish whenever $t < 0$ which occurs when we close our contour in the lower half plane, so we subtract $i\varepsilon$ from $\pm ck$. Factoring, we have

$$I = \frac{c^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - (ck - i\varepsilon))(\omega + (ck + i\varepsilon))}\,d\omega.$$

Thus, we have

$$I(t > 0) = -\frac{c}{k}\sin(ckt)e^{-\varepsilon t},$$

and in the limit as $\varepsilon \to 0$, we have

$$I(t) = -\frac{c}{k}\sin(ckt).$$

Now, sticking our value of $I$ into the integral in $k$, we have

$$\begin{aligned} G(x, t > 0) &= -\frac{c}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ictk} - e^{-ictk}}{2ik} e^{ikx}\,dk \\ &= -\frac{c}{2}\Theta(ct - |x|). \end{aligned}$$

The expression $-\frac{c}{2}\Theta(ct - |x|)$ gives causality, and the expression $|x|$ comes from symmetry.