Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a region. Fix $z_0 \in U$. Let

$$\mathcal{F} = \{f \in H(U) \mid \operatorname{im}(f) \subseteq \mathbb{C} \setminus B(0,1), f(z_0) = 2i\}.$$

Show that $\mathcal{F}$ is normal.

Solution: Let $(f_n)_n$ be a sequence in $\mathcal{F}$. We use the conformal map $z \mapsto \frac{1}{z}$ to map $\mathbb{C} \setminus B(0,1)$ to $\mathbb{D}$, giving that the family

$$\mathcal{G} = \left\{ \frac{1}{f} \,\middle|\, f \in \mathcal{F} \right\}$$

is locally bounded (indeed, globally bounded) by 1. Thus, it follows that there is a subsequence

$$\left(\frac{1}{f_{n_k}}\right)_k \to g \colon U \to \mathbb{D}$$

for some holomorphic function $g \colon U \to \mathbb{D}$. Now, since $\frac{1}{f_n}$ has no zeros for each n, it follows from Hurwitz's theorem that either g is uniformly 0 or g also has no zeros. Yet, since $g(z_0) = -\frac{i}{2} \neq 0$, it thus follows that $\frac{1}{g}$ is holomorphic on U, whence

$$(f_{n_k})_k \to \frac{1}{g}.$$

Thus, $\mathcal{F}$ is normal.

Problem (Problem 2)**:**

(a) Using the Schwarz–Pick lemma, show that given $w \in \mathbb{D}$, there exists a holomorphic function $f \colon \mathbb{D} \to \mathbb{D}$ satisfying

$$\begin{aligned} f(w) &= 0 \\ |f'(w)| &= \sup_{\substack{g \in H(\mathbb{D}) \\ g(\mathbb{D}) \subseteq \mathbb{D}}} |g'(w)|. \end{aligned}$$

(b) Show that if $f \colon \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded, then

$$\sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)|f'(z)| \leqslant \sup_{z \in \mathbb{D}} |f(z)|.$$

(c) Let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Show that f either has at most 1 fixed point or f is the identity.

Solution:

(a) We know that the map

$$\psi_w(z) = \frac{w - z}{1 - \overline{w}z}$$

is a conformal map that takes $\psi_w(w) = 0$. Now, we know that

$$|\psi_w'(w)| = \frac{1}{1 - |w|^2}.$$

From the Schwarz–Pick Lemma, we have for all holomorphic functions $f \colon \mathbb{D} \to \mathbb{D}$

$$\frac{|f'(w)|}{1 - |f(w)|^2} \leqslant \frac{1}{1 - |w|^2}.$$

In particular, since $0 \leqslant |f(w)| < 1$, we have

$$|f'(w)| \leqslant \frac{1}{1 - |w|^2},$$

whence $\psi_w(z)$ satisfies

$$\begin{aligned} \psi_w(w) &= 0 \\ |\psi_w'(w)| &= \sup_{\substack{g \in H(\mathbb{D}) \\ g(\mathbb{D}) \subseteq \mathbb{D}}} |g'(w)|. \end{aligned}$$

(b) Let $K = \sup_{z \in \mathbb{D}} |f(z)|$. By the maximum modulus principle, $|f(z)| < K$ for all $z \in \mathbb{D}$, so it follows that $g(z) := \frac{f(z)}{K}$ is a self-map of the unit disk. By the Schwarz–Pick lemma, it then follows that

$$\frac{|g'(z)|}{1 - |g(z)|^2} \leqslant \frac{1}{1 - |z|^2}.$$

Simplifying, we then get

$$\begin{aligned} \left(1 - |z|^2\right)|f'(z)| &\leqslant K\left(1 - \frac{|f(z)|^2}{K^2}\right) \\ &\leqslant K, \end{aligned}$$

so that

$$\sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)|f'(z)| \leqslant \sup_{z \in \mathbb{D}} |f(z)|.$$

(c) The statement is equivalent to showing that if $f \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map with two fixed points, then f is the identity map. Let f be one of these maps, and let $\xi \neq \eta \in \mathbb{D}$ be such that $f(\xi) = \xi$ and $f(\eta) = \eta$.

We want to find some holomorphic self-map of $\mathbb{D}$ that sends $0 \mapsto 0$. We consider the maps

$$\psi_\xi = \frac{\xi - z}{1 - \overline{\xi} z},$$

which takes $0 \mapsto \xi$ and $\xi \mapsto 0$. Notice that $\psi_\xi \circ \psi_\xi = \mathrm{id}$. Therefore,

$$g = \psi_\xi \circ f \circ \psi_\xi$$

is a holomorphic self-map that sends $0 \mapsto 0$, so by Schwarz's Lemma, we have

$$|g(z)| \leqslant |z|$$

for all $z \in \mathbb{D}$. Yet, we also have

$$\begin{aligned} g(\psi_\xi(\eta)) &= \psi_\xi \circ f \circ \psi_\xi \circ \psi_\xi(\eta) \\ &= \psi_\xi(\eta). \end{aligned}$$

In particular, this means that

$$|g(\psi_\xi(\eta))| = |\psi_\xi(\eta)|,$$

so there exists $\mathbb{D} \ni w := \psi_\xi(\eta)$ such that $|g(w)| = |w|$, so that $g(w) = e^{i\theta} w$. Yet, since the identity relation holds for $\psi_\xi(\eta)$, it follows that $\theta = 0$, so $g(w) = w$. In particular, this means

$$\begin{aligned} \psi_\xi \circ f \circ \psi_\xi(z) &= z \\ f \circ \psi_\xi(z) &= \psi_\xi(z). \end{aligned}$$

Yet, since ψ_ξ is an automorphism, it follows that this relation holds for all $z \in \mathbb{D}$, so that $f(w) = w$ for all $w \in \mathbb{D}$, whence $f = \mathrm{id}$.

Problem (Problem 3)**:** Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$.

(a) Show that $|f(z) + f(-z)| \leqslant 2|z|^2$ for all $z \in \mathbb{D}$.

(b) Show that $|f(z) + f(-z)| = 2|z|^2$ for some $z \in \mathbb{D} \setminus \{0\}$ if and only if $f(z) = e^{i\theta} z^2$.

Solution:

(a) We seek to show that the function

$$k(z) = \frac{f(z) + f(-z)}{2z}$$

maps $\mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$. We may safely assume that $z \neq 0$, as the desired inequality is certainly true for $z = 0$. We observe that since f is a self-map of $\mathbb{D}$ with $f(0) = 0$, Schwarz's Lemma gives

$$|f(z)| \leqslant |z|,$$

or that

$$\frac{|f(z)|}{|z|} \leqslant 1$$

A similar fact holds for $f(-z)$. For all $z \in \mathbb{D}$, we thus have

$$\begin{aligned}\left|\frac{f(z) + f(-z)}{2z}\right| &\leqslant \frac{1}{2}\left(\left|\frac{f(z)}{z}\right| + \left|\frac{f(-z)}{z}\right|\right)\\ &< 1.\end{aligned}$$

Therefore, since k is a self-map of $\mathbb{D}$ with $k(0) = 0$, Schwarz's Lemma gives

$$|f(z) + f(-z)| \leqslant 2|z|^2.$$

(b) Equivalently, we are assuming that

$$\left|\frac{f(z) + f(-z)}{2z}\right| = |z|$$

for some $z \in \mathbb{D} \setminus \{0\}$. From Schwarz's Lemma, we then have that

$$\frac{f(z) + f(-z)}{2z} = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$. This gives

$$\frac{1}{2}(f(z) + f(-z)) = e^{i\theta} z^2.$$

Now, we observe that

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z)).$$

First, we observe that

$$h(z) = \frac{1}{2}(f(z) - f(-z))$$

has $|h(z)| < 1$ for all $z \in \mathbb{D}$, $h(0) = 0$, and

$$|h'(0)| = |f'(0)|,$$

meaning that there is ρ such that $\frac{1}{2}(f(z) - f(-z)) = e^{i\rho} f(z)$, by a corollary to the Riemann Mapping Theorem and Schwarz's Lemma.