Problem 1

True or false: If H is a minor of G, then H is a contraction of a subgraph of G.

True.

Problem 2

Prove each of the following.

(a) There exists an infinite family F of graphs such that no graph in F is a subgraph of another graph in F.

(b) There exists an infinite family F of graphs such that no graph in F is a contraction of another graph in F.

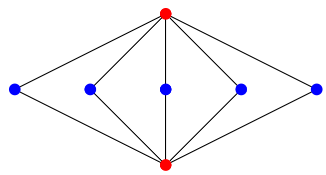
(c) There exists an infinite family F of graphs such that no graph in F is a subgraph or a contraction of another graph in F.

(a)

Cycles; if we delete any vertex or edge of any cycle, then the degree of at least two vertices is reduced by 1, while all vertices in every cycle are of degree 2.

(b)

The family of graphs $K_{2,n}$:



Problem 3

Prove that the set of all planar graphs is minor-closed.

Let G be any planar graph. Then, by Wagner's theorem, it must be the case that neither K_5 nor $K_{3,3}$ are minors of G. Therefore, any minor of G, G', must also not have K_5 nor $K_{3,3}$ as a minor — otherwise, we would take the steps to create G', then the steps to create one of the forbidden minors, and G would have the forbidden minors as a minor.

Thus, since no minor of any planar graph can be non-planar, it must be the case that planarity is minor-closed.

Problem 4

Let P be an arbitrary set of graphs. Let P' be the set of all graphs not in P. By the Graph Minor Theorem, P has a finite subset F of graphs that are minor-minimal in P. Similarly, P' has a finite subset F' of graphs that are minor minimal in P'. Prove that if P is minor-closed, then a graph G is in P' if and only if G has a minor in F'. So, if P is minor-closed, then P and P' are both "characterized" by F'. In fact, if P is minor-closed, then F consists of only one graph, namely the graph with only one vertex. Why?

Suppose P is minor-closed. Then, if $H \in F$, then no minor of H is in P or F.

($\Rightarrow$) Let $G \in P$. Suppose toward contradiction that there is a graph $H \in F'$ that is a minor of G.

Since P is minor-closed, and H is a minor of $G \in P$, $H \in P$.

However, $H \in F' \subseteq P'$. So, $H \in P$ and $H \in P'$. $\perp$

($\Leftarrow$) Suppose that no graph in F' is a minor of G. Suppose toward contradiction that $G \notin P$.

Then, $G \in P'$.

So, by definition of F', G must have a minor in F'.

So, $\exists H \in F'$ such that H is a minor of G. $\perp$

Since any graph that is in F must have a minor that is also in F, every graph in F must reduce to the graph of one vertex.

Problem 5

A graph G is apex if $G - v$ is planar for some vertex v of G. Prove that the set of apex graphs is minor-closed.

Let G be an apex graph. Then, $\exists v \in V(G)$ such that $G - v$ does not contain either of the forbidden minors of planarity — i.e., $G - v$ does not reduce to either K_5 or $K_{3,3}$.

If G' is a minor of G formed via a process that includes deleting v, then G' must be apex (as all planar graphs are apex, since they do not contain any of the forbidden minors) — if G' does not delete v in the process of forming it from G, then $G' - v$ is a minor of G' and of G, and since $G - v$ does not contain any of the forbidden minors , neither must $G' - v$. Therefore, $G' - v$ is apex.

Problem 6

Prove that if G is a connected graph, then for every edge e, $G - e$ has at most two connected components.

Since G is connected, $\exists$ a path $P = (v, v_1, \dots, a, b, \dots, v_n, w)$ in G. It must be the case that $e \in P$, or else $P \in G - e$.

Let $x \in V(G - e)$ such that $x \neq v$ and $x \neq w$. We will show that $\exists$ a path in $G - e$ from x to v or x to w.

Let P_{xa} be a path in G from x to a. If $e \notin P_{xa}$, then $(v, \dots, a) \cup P_{xa}$ is a path from v to x without e.

If $e \in P_{xa}$, then $P_{xa} = (x, \dots, b, a)$. So, $(x, \dots, b) \cup (b, \dots, w)$ is a path from x to w without e.

Problem 7

In this problem, we (try to) prove Robbins' Theorem.

We will use induction on the number of edges to prove. Clearly, the statement holds for $|E(G)| = 3$. Suppose $|E(G)| > 3$ and the statement holds for every graph with fewer than $|E(G)|$ edges.

(a) Let $e \in G$. If $G - e$ is bridgeless, then by the inductive hypothesis, $G - e$ admits an orientation in which every distinct pair of vertices is within a directed circuit. So, if we add e back into the graph, this does not change the fact that every distinct pair of vertices is in a circuit with the given orientation of $G - e$. Therefore, G is circuit-oriented.

(b) If $G - e$ has a bridge, e'. We showed in the previous problem that deleting a bridge yields exactly two components — thus, it must be the case that $G - e - e'$ contains two components.

Since there is only one bridge in $G - e$, each of G_1 and G_2 must be bridgeless, meaning there is a circuit orientation in each component. Additionally, it must be the case that e' has one endpoint in G_1 and one endpoint in G_2 — if, WLOG, both of the endpoints of e' were in G_1, then e' would not be a bridge. Finally, if both endpoints of e were in, WLOG, G_1, then $G - e$ would still contain the bridge e', violating the assumption.

(c) Let $C = (v_1', v_2', w_1, \dots, w_k, v_1')$ be a circuit in G. Let $v_1, v_1' \in G_1$, $v_2, v_2' \in G_2$, where e has endpoints v_1 and v_2, while e' has endpoints v_1' and v_2'. Since C traverses $v_1'v_2'$, in $G - e$ it must be the case that $v_1'v_2'$ is the bridge between G_1 and G_2. So, C traverses through various edges in G_2 — however, since C is a circuit that includes v_1', it must be the case that C traverses an edge between G_2 and G_1 — this must be $e = v_2v_1$. If it were not e, then $G - e$ would not be bridgeless, as there would be another edge x_2x_1 from G_2 to G_2 contained in C, implying that $G - e - x_2x_1$ was still connected.

(d) Case 1: If x, y are in the same G_i, then since the particular G_i is bridgeless, there is a circuit orientation in G_i, meaning x and y are in the same directed circuit.

Case 2: If $x \in G_1$ and $y \in G_2$, where $xy \neq e$, then x and y are distinct vertices in G/e. We trace out the circuit in G_1 from x to v_1', then e', then the circuit in G_2 from v_2' to y to the vertex v from contracting v_2v_1 in G/e. Finally, we trace along the circuit in G_1 from v to x. This is a directed circuit.

I don't know how to show Case 3.

Problem 8

Problem 9

Problem 10

Prove Theorem 10.2 and Corollary 10.3. (I'm sorry, I'm doing this at like 10pm on the day before it was due because of various time constraints, it's very sloppy).

Theorem 10.2: If G is a planar graph of order $n \geq 3$ and size m, then for G to be planar, it must be the case that

$$n - m + r = 2.$$

In the maximal case, every edge is double-counted in finding the number of faces, and each face is a triangle, meaning that $3r \leq 2m$. So,

$$\begin{aligned} 3n - 3m + 2m &\leq 6 \\ m &\leq 3n - 6 \end{aligned}$$

Corollary 10.3: Suppose every vertex in G is of degree 6 or greater. Then, $|V(G)| \geq 7$, meaning

$$\begin{aligned} 2m &= \sum d(v) \\ &\geq 6|V(G)| \\ m &\geq 3|V(G)| \end{aligned}$$