The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of x).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

Some Functions and Sets

$$f(x,y) = x^2 - y^2$$

Domain: $\{(x,y) \mid \exists f(x,y)\}$

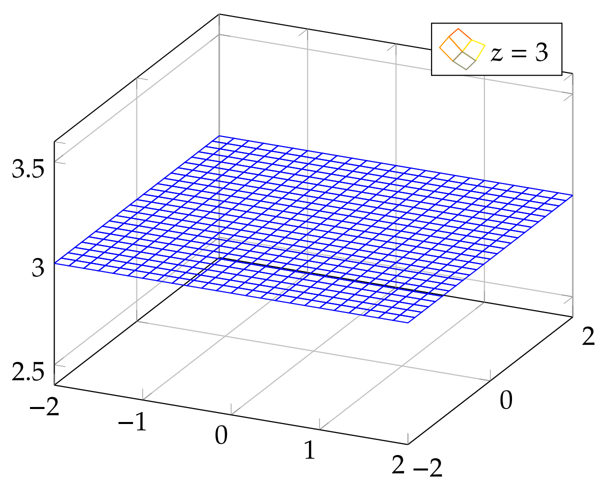
Range: $\{f(x,y) \mid (x,y) \in \text{Dom}(f)\} = \mathbb{R}$

Graph: $\text{Graph}(f) = \{x, y.f(x,y) \mid x,y \in \text{Dom}(f)\}$. For example, $(1,3,4) \notin \text{Graph}(f)$ since $1^2 - 3^2 \neq 4$.

Examples

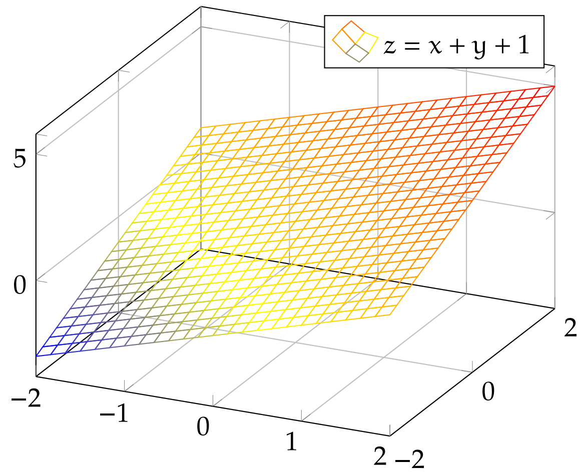
In $\mathbb{R}^3$, in x, y, z coordinates, $z = 3$ is a plane defined as follows:

- Parallel to the xy plane.
- Passes through the point $(0.0, 3)$.



Meanwhile, $y = 0$ would be a "wall" that passes through the origin that contains the line $y = 0$ in the xy plane.

Finally, $z = x + y + 1$ is a plane, as we can see below.



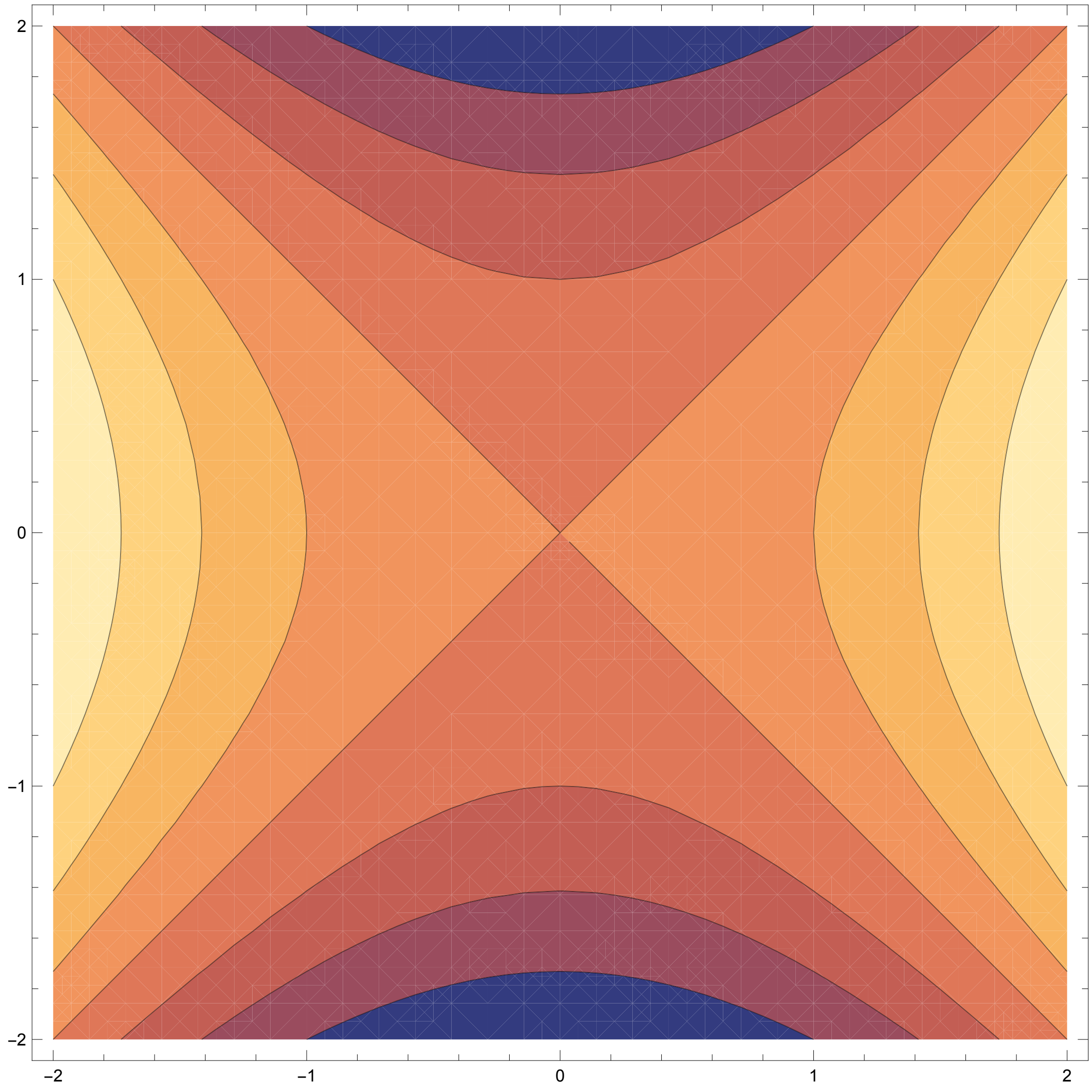
Visualizing a function of multiple variables

Consider the function $f(x, y) = x^2 - y^2$. We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$
- $f(0, y) = -y^2$
- $f(2, y) = 4 - y^2$
- $f(x, -2) = x^2 + 4$
- $f(x, 0) = x^2$
- $f(x, 2) = x^2 + 4$

Alternatively, we can visualize via contour diagrams (i.e., everywhere that z is a certain value), as seen in mathematica as follows:

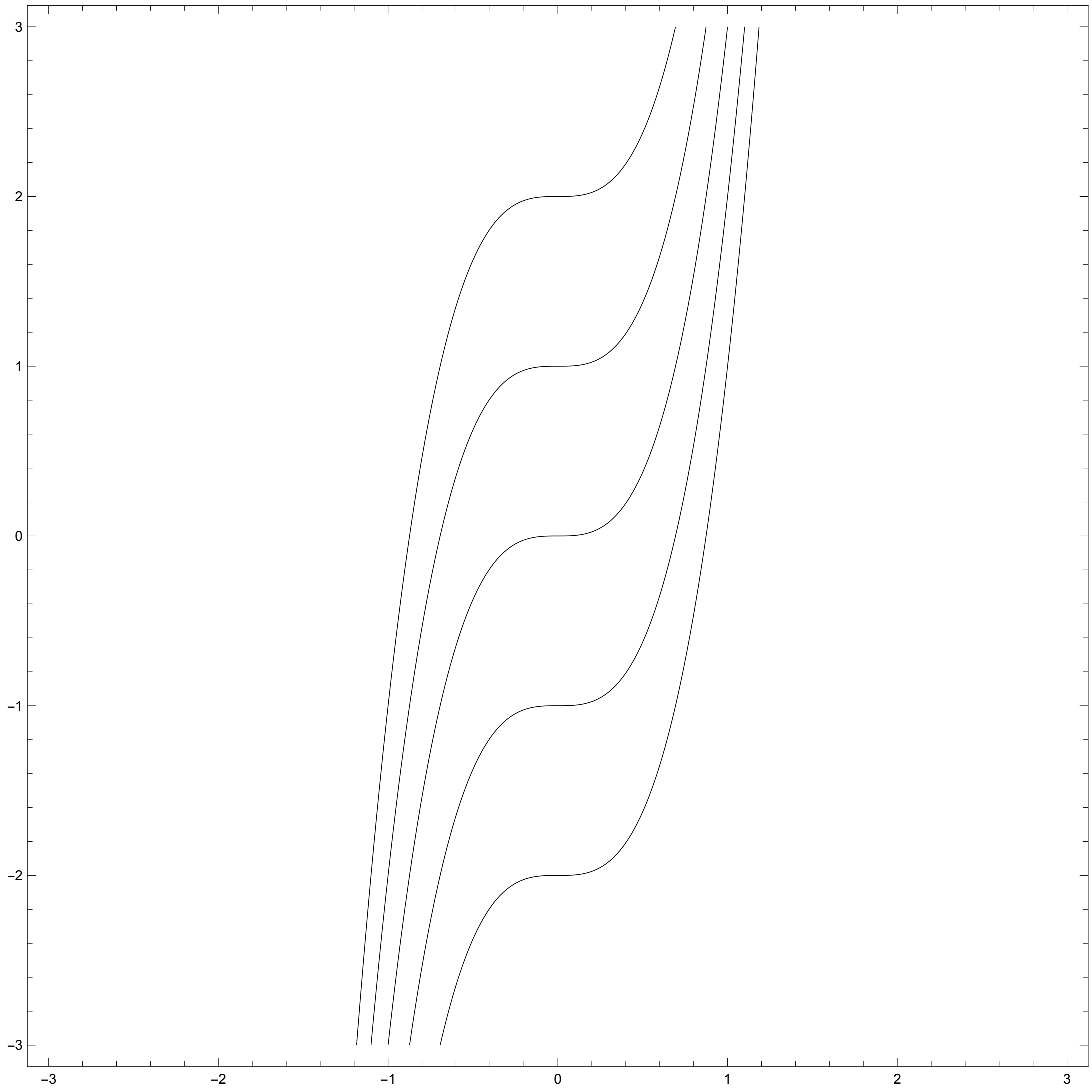
Contour Diagram



Contour Example

Consider the function $f(x, y) = y - 3x^2$. We want to find the contours.

For any c, we have that $c = y - 3x^3$, or $y = 3x^3 + c$. Therefore, every contour "looks like" $3x^3 + c$ for values of c. For example, in the following, we have $c = \{-2, -1, 0, 1, 2\}$



3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3