

### The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of  $x$ ).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

### Some Functions and Sets

$$f(x, y) = x^2 - y^2$$

DOMAIN:  $\{(x, y) \mid \exists f(x, y)\}$

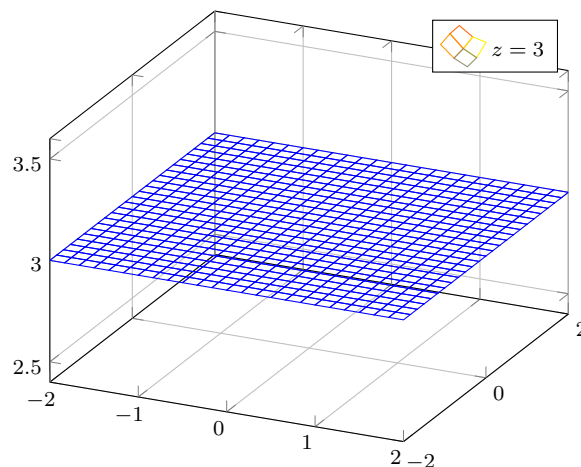
RANGE:  $\{f(x, y) \mid (x, y) \in \text{Dom}(f)\} = \mathbb{R}$

GRAPH:  $\text{Graph}(f) = \{x, y, f(x, y) \mid x, y \in \text{Dom}(f)\}$ . For example,  $(1, 3, 4) \notin \text{Graph}(f)$  since  $1^2 - 3^2 \neq 4$ .

### Examples

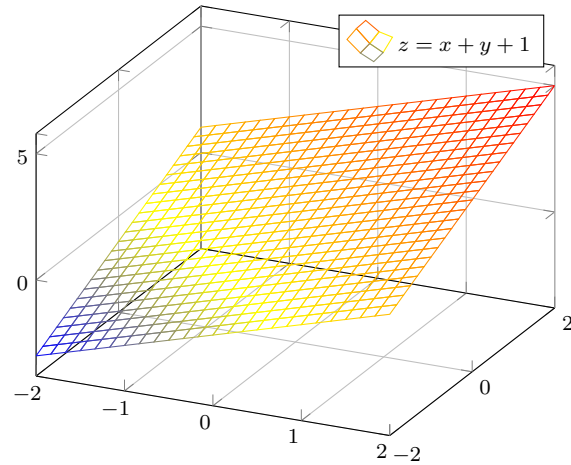
In  $\mathbb{R}^3$ , in  $x, y, z$  coordinates,  $z = 3$  is a plane defined as follows:

- Parallel to the  $xy$  plane.
- Passes through the point  $(0, 0, 3)$ .



Meanwhile,  $y = 0$  would be a “wall” that passes through the origin that contains the line  $y = 0$  in the  $xy$  plane.

Finally,  $z = x + y + 1$  is a plane, as we can see below.

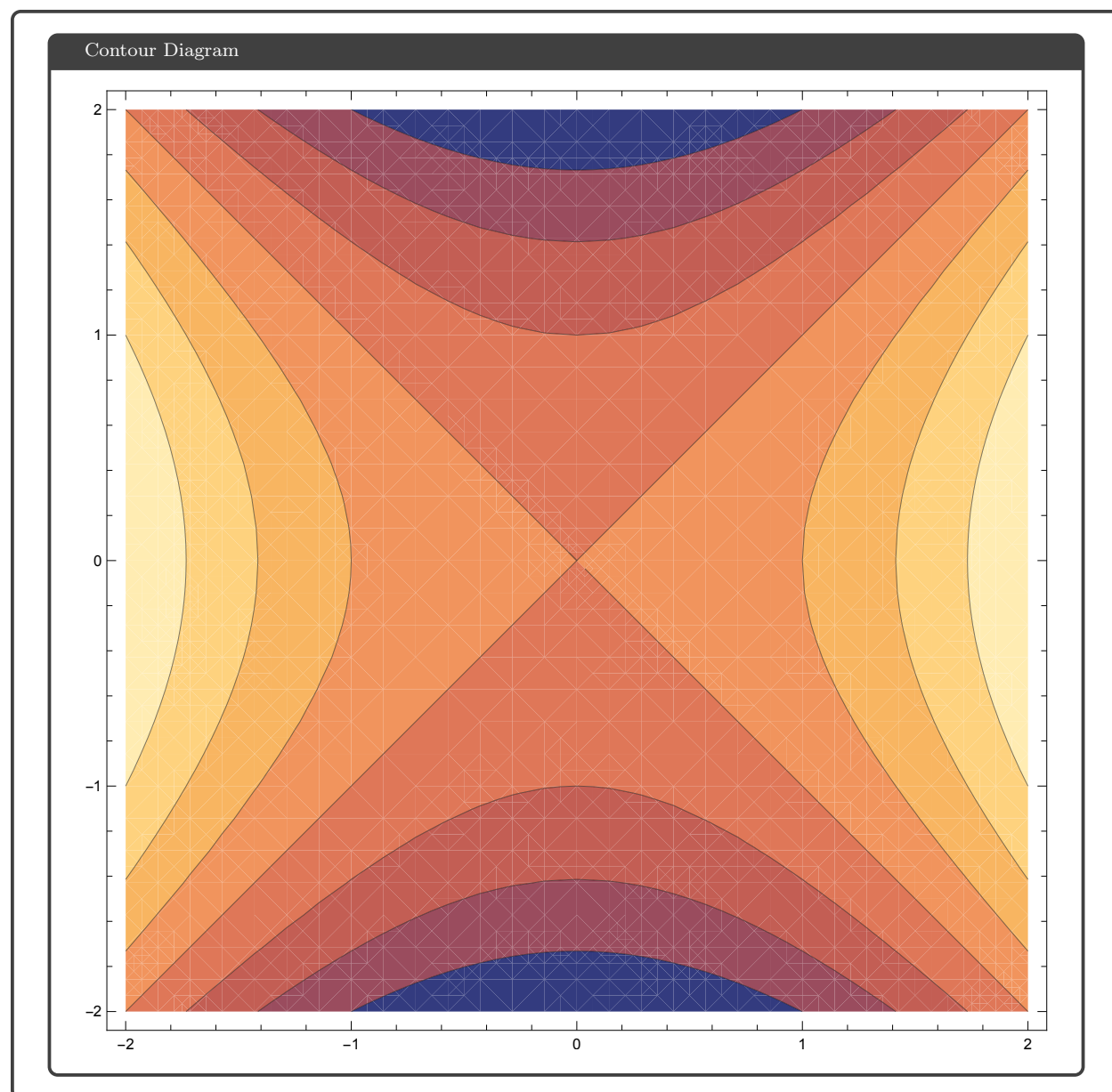


### Visualizing a function of multiple variables

Consider the function  $f(x, y) = x^2 - y^2$ . We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$
- $f(0, y) = -y^2$
- $f(2, y) = 4 - y^2$
- $f(x, -2) = x^2 + 4$
- $f(x, 0) = x^2$
- $f(x, 2) = x^2 + 4$

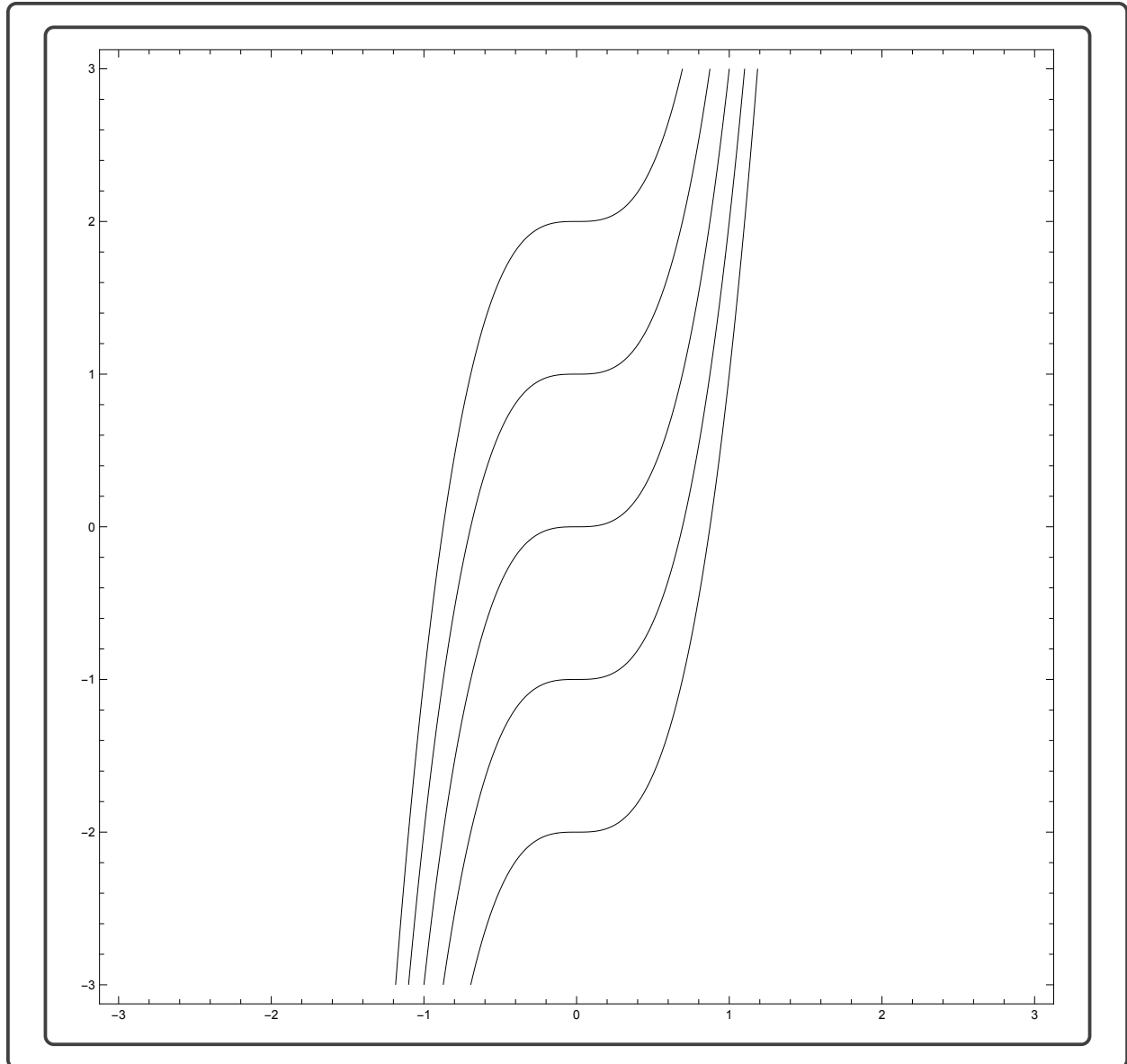
Alternatively, we can visualize via contour diagrams (i.e., everywhere that  $z$  is a certain value), as seen in mathematica as follows:



### Contour Example

Consider the function  $f(x, y) = y - 3x^2$ . We want to find the contours.

For any  $c$ , we have that  $c = y - 3x^2$ , or  $y = 3x^2 + c$ . Therefore, every contour “looks like”  $3x^2 + c$  for values of  $c$ . For example, in the following, we have  $c = \{-2, -1, 0, 1, 2\}$

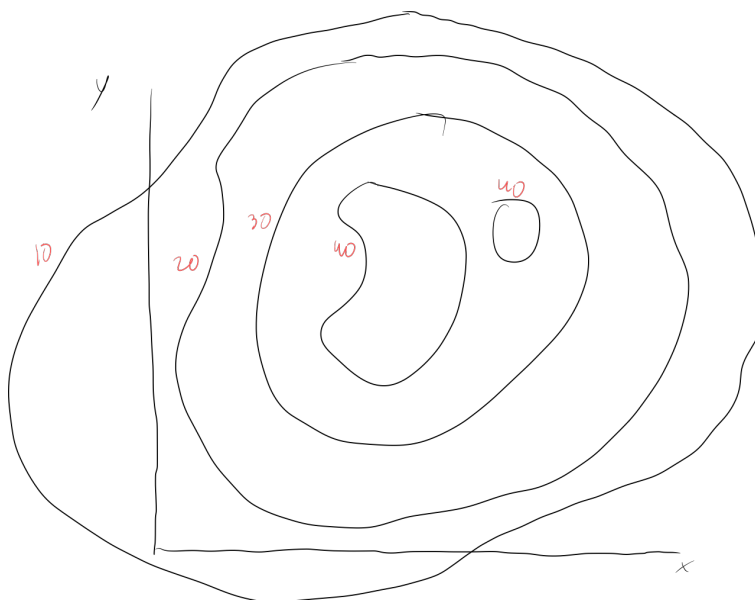


### Distance

In  $\mathbb{R}^5$ , let  $p = (3, 1, 4, 1, 5)$ , and  $q = (1, 0, -2, 0, 2)$ . Using the Euclidean metric, we can find the distance between  $p$  and  $q$  is  $d(p, q) = ((3 - 1)^2 + (1 - 0)^2 + (4 - (-2))^2 + (1 - 0)^2 + (5 - 2)^2)^{1/2} = (4 + 1 + 36 + 1 + 9)^{1/2} = \sqrt{51} = 7.14$ . We can also call this the 2-norm.

$$d(p, q) = \left( \sum_{k=1}^n (p_k - q_k)^2 \right)^{1/2}$$

## Derivatives



To denote a derivative, we can't talk about one value, we must use a *partial* derivative,  $\frac{\partial f}{\partial x}$ , or  $\frac{\partial f}{\partial y}$ . The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

$x \backslash y$	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A “linear” approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where  $(x_0, y_0, z_0) \in \mathbb{R}^3$ , and is an output in  $z = f(x, y)$ , and  $m, n \in \mathbb{R}$ .

For example, with the above table, we can see that the function is linear in  $x$  and  $y$  (i.e., the slope holding the other variable constant is constant).

## Limits in Multivariable Functions

Consider the following:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow  $y = mx$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 - y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2} \\ &= \frac{1 + m^2}{1 - m^2} \end{aligned}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of  $m$

$m$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours “approach” a particular point, we can see that the limit is defined.

### Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have  $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$ . These vectors are equivalent because they are components of  $\mathbb{R}^3$ .

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

#### Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

#### Properties of Vectors

Let  $\vec{u}, \vec{v} \in \mathbb{R}^n$ . Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $c\vec{u} = \langle cu_1, cu_2, \dots, cu_k \rangle$

Additionally, we define  $\vec{u} \cdot \vec{v}$  as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^n u_k v_k = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

### Partial Derivatives

Consider  $f(x, y) = x^2 y + x e^y$ .

$$f_x := \frac{\partial f}{\partial x}$$

$$f_x(a, b) = \left. \frac{\partial f}{\partial x} \right|_{(a, b)}$$

We know that  $f \in C^\infty(\mathbb{R} \times \mathbb{R})$ , meaning  $f$  is endlessly differentiable.

### Functions and Approximations

Let  $f(x, y) = x^2 - y^2$ ,  $g(x, y) = 2xy$

- $f_{xx} + f_{yy} = 0$
- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For  $f(x, y)$  at  $(a, b, f(a, b))$ , we have the following:

$$\begin{aligned}\ell(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(y - b) \\ q(x, y) &= \ell(x, y) + \frac{1}{2} (f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2)\end{aligned}$$

In order to get a sense of the “derivative,” we can use the following:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

### Directional Derivative and Gradient

Given  $f(x, y)$  and  $(a, b)$ , where  $f \in C^2(\mathbb{R}^2)$ . Then, the quadratic approximation is:

$$\begin{aligned}f(x, y) &\approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{1}{2} (f_{xx}(a, b)(x - a)^2 + f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b))\end{aligned}$$

$$df = f_x(a, b)dx + f_y(a, b)dy$$

a differential

$$\Delta f = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

Evaluating  $f(x, y) = xe^y$  at  $(a, b) = (-1, 0)$

$$f_x = e^y$$

$$f_y = xe^y$$

$$f_x(-1, 0) = 1$$

$$f_y(-1, 0) = -1$$

$$\Delta f = \Delta x - \Delta y$$

On a given contour map, let  $\vec{u} = \langle u_1, u_2 \rangle$  denote a *unit* vector in a direction that we want to find the derivative of  $f$  in.

$$f_{\vec{u}}(x, y) = \nabla f(a, b) \cdot \vec{u}$$

Where

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

The directional derivative for all vectors  $\vec{v}$  is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

### Chain Rule

Let  $f(x, y)$  be a function where  $x = x(t)$  and  $y = y(t)$ . We want to find

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

The chain rule works in higher dimensions too. Consider  $k(x_1(t), x_2(t), \dots, x_{152}(t))$ . Then,

$$\frac{dk}{dt} = \sum_{i=1}^{152} \frac{\partial k}{\partial x_i} \frac{dx_i}{dt}$$

We can also view this as a vector. Let  $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{152}(t) \end{pmatrix}$ . Then, we can write  $\frac{dk}{dt}$  more succinctly as follows:

$$\frac{dk}{dt} = \nabla k \cdot \frac{d\vec{x}}{dt}$$

For example, let  $f(x, y, z) = 3x^2y + zx + 2$ , where  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$

$$\begin{aligned} \frac{df}{dt} &= \begin{pmatrix} 6xy + z \\ 3x^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} \\ &= (6xy + z)x'(t) + 3x^2y'(t) + xz'(t) \end{aligned}$$

So, if we let  $x(t) = \sin(t)$ ,  $y(t) = e^t$ , and  $z(t) = t^2 + 1$ . Then, we have

$$\frac{df}{dt} = 6\sin(t)\cos(t)e^t + t^2\cos(t) + \cos(t) + 3e^t\sin^2(t) + 2t\sin(t)$$

Alternatively, consider  $f(x, y, z) = x^2 + yz + e^y$ , where  $x(s, t) = st$ ,  $y = y(s, t) = t + s^2$ ,  $z = z(s, t) = e^t$ . Let

$$\vec{x} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix}$$

Then, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial t} \\ \frac{\partial f}{\partial s} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial s} \end{aligned}$$

Evaluating the first expression, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix} \cdot \begin{pmatrix} s \\ 1 \\ e^t \end{pmatrix} \\ &= 2s^2t + 3^t + e^{t+s^2} + (t + s^2)e^t \end{aligned}$$

Consider  $f(x, y(x))$ . Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

This is the technique we use to find implicit differentiation.

We know as a result that  $\nabla f(a, b)$  is orthogonal to the contour curve at  $(a, b)$

### Recap

In  $\mathbb{R}^3$ , find the plane that contains  $P = (P_1, P_2, P_3)$ ,  $Q$ , and  $R$ . We can find it by the following:

$$0 = \vec{n} \cdot \begin{pmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{pmatrix}$$

$$0 = n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3)$$

where

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$



## Differentiability

A function  $f(x)$  of one variable is differentiable at  $x = a$  if

$$f(a) = \lim_{h \rightarrow 0} f(a + h)$$

and

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

exists and is bounded

We can also linearize the function.  $f$  is differentiable if

$$f(x) = f(a) + f'(a)(x - a) + E(x)$$

where  $\lim_{h \rightarrow 0} \frac{E(a+h)}{h} = 0$ .

In the multiple dimensions example, we have  $f(x, y)$  is differentiable if

$$f(x, y) = \ell(x, y) + E(x, y)$$

where  $\lim_{h \rightarrow 0, k \rightarrow 0} \frac{E(a+h, b+k)}{\sqrt{h^2 + k^2}} = 0$

## Local Maxima

Let  $f(x, y) = x^2 + 2y^2$ . We want to find  $(a, b)$  which are local maxima, minima, or other.

$(a, b)$  is a local maximum if  $f(a, b) \geq f(x, y) \forall (x, y) \in V_\varepsilon(a, b)$ , where  $\varepsilon > 0$ .

(1) Find Critical Points for  $f(x, y) : f_x(x, y), f_y(x, y) = 0$ ,  $f_x(x, y), f_y(x, y)$  are undefined.

$$f_x(x, y) = 2x$$

$$f_y(x, y) = 4y$$

$$f_x(0, 0) = 0$$

$$f_y(0, 0) = 0$$

$$f(0, 0) = 0$$

$$f(x, y) > 0$$

$$\forall (x, y) \neq (0, 0)$$

For all  $x, y$ ,  $f_{xx} = 2$ ,  $f_{yy} = 4$ , and  $f_{xy} = 0$ . Finally,

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) \cdot f_{yy}(x, y) + f_{xy}(x, y)^2 \\ &= 8 \\ &> 0 \end{aligned}$$

Since  $D(x, y) > 0$ , we look at the sign of  $f_{xx}$ . Since it is positive,  $f(0, 0)$  has a local minimum.

## Local Maxima and Minima Approach

Given  $f(x, y)$ , we want

(1) Find critical points:

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

(2) Compute  $f_{xx}, f_{yy}, f_{xy}, D = f_{xx}f_{yy} - (f_{xy})^2$

(3)

$f_{xx}$	$D$	Critical Point
+	+	Local Minimum
-	+	Local Maximum
$\pm$	-	Saddle Point
$\pm$	0	Nothing

Consider the function

$$f(x, y) = \ln(x^2 + y^2 + 1)$$

$$f(0, 0) = 0$$

$$f(x, y) > 0$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

Critical Points:  $(0, 0)$

$$\begin{aligned} \left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} &= \frac{2(x^2 + y^2 + 1) - 4x^2}{(x^2 + y^2 + 1)^2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} &= \frac{2(x^2 + y^2 + 1) - 4y^2}{(x^2 + y^2 + 1)^2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} &= \frac{-4xy}{(x^2 + y^2 + 1)^2} \\ &= 0 \end{aligned}$$

Now, consider the function

$$f(x, y) = x^2 - 2xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x - 2y$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2$$

$$\begin{aligned} D &= \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial x^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \\ &= 0 \end{aligned}$$

Therefore, the critical points of this function are indeterminate with the given approach. However, we know that  $f(x, y) = (x - y)^2 = 0$  when  $x = y$ , so the line  $y = x$  is a local minimum trough in 3-space.

Now, consider the function

$$f(x, y) = (x - 1)^2(y + 2)$$

$$\frac{\partial f}{\partial x} = 2(x - 1)(y + 2)$$

$$\frac{\partial f}{\partial y} = (x - 1)^2$$

Critical points:  $\{(1, y) \mid y \in \mathbb{R}\}$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= 2(y+2) \\ \frac{\partial^2 f}{\partial y^2} &= 0 \\ \frac{\partial^2 f}{\partial x \partial y} &= 2(x-1) \\ D &= 0 - (2(x-1))^2 \\ &= 0\end{aligned}$$

Evaluating  $D$  at critical points

### Finding Critical Points

Let  $f(x, y) = (y^2 + 2) \sin(x)$ . on  $[-2, 2] \times [-2, 2]$

$$\begin{aligned}\frac{\partial f}{\partial x} &= (y^2 + 2) \cos(x) \\ &= 0 \\ \frac{\partial f}{\partial y} &= 2y \sin(x) \\ &= 0 \\ (x, y) &= \left( \frac{(2n+1)\pi}{2}, 0 \right) \\ &= \{(\pi/2, 0), (-\pi/2, 0)\} \\ \frac{\partial^2 f}{\partial x^2} &= -(y^2 + 2) \sin(x) \\ \frac{\partial^2 f}{\partial y^2} &= 2 \sin(x) \\ \frac{\partial^2 f}{\partial x \partial y} &= 2y \cos(x) \\ D(x, y) &= \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \\ &= -2(y^2 + 2) \sin^2(x) - 4y^2 \cos^2(x) \\ &< 0\end{aligned}$$

Therefore, the critical points are saddle points. If there is no domain restriction, we have a series of saddle points all along  $y = 0$ .

### Why Finding Critical Points Works

We create the Taylor series of  $f(x, y)$  at  $(x_0, y_0)$ :

$$\begin{aligned}f(x, y) &\approx \ell(x_0, y_0) + \frac{1}{2} (f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2) \\ &= f(x_0, y_0) + \underbrace{\nabla f(x, y) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}_{=0 \text{ at critical points}} + \frac{1}{2} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^T \underbrace{\begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}}_{\text{Hessian}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}\end{aligned}$$

If the Hessian is positive definite, then  $\lambda_1, \lambda_2 > 0$  and the critical point is a local min. If the Hessian is negative definite, then  $\lambda_1, \lambda_2 < 0$  and the critical point is a local max.

In any given  $2 \times 2$  matrix, the eigenvalues  $\lambda_1, \lambda_2$  are such that  $\lambda_1 + \lambda_2 = \text{Tr}(A)$  and  $\lambda_1 \lambda_2 = \text{Det}(A)$ .