Section 4.1

Solution (Problem 4)**:** Evaluating with the initial conditions, we get

$$\begin{aligned} c_1 - c_2 &= 0 \\ -c_3 &= 2 \\ c_2 &= -1. \end{aligned}$$

We see that $c_1 = -1$, $c_2 = -1$, and $c_3 = -2$. This yields the particular solution of

$$y = -1 - \cos x - 2\sin x.$$

Solution (Problem 10)**:** The interval $(-\pi, \pi)$ contains a unique solution to the initial value problem.

Solution (Problem 14)**:**

(a) We have

$$\begin{aligned} c_1 + c_2 + 3 &= 0 \\ c_1 + c_2 + 3 &= 4, \end{aligned}$$

which is not possible.

(b) We have

$$\begin{aligned} 3 &= 0 \\ c_1 + c_2 + 3 &= 2, \end{aligned}$$

which is yet again not possible.

(c) We have

$$\begin{aligned} 3 &= 3 \\ c_1 + c_2 + 3 &= 0, \end{aligned}$$

meaning that the solution set is all pairs (c_1, c_2) such that $c_1 + c_2 = -3$.

(d) We have

$$\begin{aligned} c_1 + c_2 + 3 &= 3 \\ 4c_1 + 16c_2 + 3 &= 15, \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 &= 0 \\ 4c_1 + 16c_2 &= 12 \end{aligned}$$

meaning

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1. \end{aligned}$$

Solution (Problem 22)**:** Since

$$\sinh(x) = \frac{1}{2}(e^x + e^{-x}),$$

the functions are not linearly independent anywhere on $(-\infty, \infty)$.

Solution (Problem 28)**:** First, we verify that both solutions work.

$$\begin{aligned} x^2 \frac{d^2}{dx^2}(\cos(\ln(x))) + x\frac{d}{dx}(\cos(\ln(x))) + \cos(\ln(x)) &= x^2\left(-\frac{\cos(\ln(x))}{x^2} + \frac{\sin(\ln(x))}{x^2}\right) + x\left(-\frac{\sin(\ln(x))}{x}\right) + \cos(\ln(x)) \\ &= 0 \end{aligned}$$

$$x^2\frac{d^2}{dx^2}(\sin(\ln(x))) + x\frac{d}{dx}(\sin(\ln(x))) + \sin(\ln(x)) = x^2\left(-\frac{\cos(\ln(x))}{x^2} - \frac{\sin(\ln(x))}{x^2}\right) + x\left(\frac{\cos(\ln(x))}{x}\right) + \sin(\ln(x))$$
$$= 0.$$

Additionally, we find that

$$\det\begin{pmatrix}\cos(\ln(x)) & \sin(\ln(x)) \\ \frac{-\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x}\end{pmatrix} = \frac{1}{x}$$
$$\neq 0,$$

so the solutions are linearly independent. Since the differential equation $x^2y'' + xy' + y' = 0$ is a second order equation, there are no other linearly independent solutions. Thus, we have the general solution of

$$y = \alpha\cos(\ln(x)) + \beta\sin(\ln(x)).$$

Solution (Problem 30)**:** I'm not checking the Wronskian on this one, they're clearly linearly independent. However, I will be doing the derivatives.

$$\frac{d^4}{dx^4}(1) + \frac{d^2}{dx^2}(1) = 0$$
$$\frac{d^4}{dx^4}(x) + \frac{d^2}{dx^2}(x) = 0$$
$$\frac{d^4}{dx^4}(\cos(x)) + \frac{d^2}{dx^2}(\cos(x)) = \cos(x) - \cos(x)$$
$$= 0$$
$$\frac{d^4}{dx^4}(\sin(x)) + \frac{d^2}{dx^2}(\sin(x)) = \sin(x) - \sin(x)$$
$$= 0.$$

Thus, since the solutions are linearly independent and have dimension 4, they form a basis for the general solution of $y^{(4)} + y'' = 0$. The general solution is

$$y(x) = c_1 + c_2x + c_3\cos(x) + c_4\sin(x).$$

Solution (Problem 36)**:**

(a) We have $y = 5$ is a particular solution to $y'' + 2y = 10$.

(b) We have $y = -2x$ is a particular solution to $y'' + 2y = 10$.

(c) Using linearity, we get that $y = -2x + 5$ is a particular solution to $y'' + 2y = -4x + 10$.

(d) Using a similar process, we have a particular solution of $y = 4x + \frac{5}{2}$.

(e) Neither of these linear combinations are general solutions of the differential equation, as the linearity principle only applies to solutions of the corresponding homogeneous equation.

Section 4.2

Solution (Problem 2)**:** Using the power of inspection, we find that our other solution is $y_2 = e^{-2x}$.

Solution (Problem 8)**:**

$$y_2(x) = e^{x/3}\int\frac{e^{-x/6}}{e^{2x/3}}\,dx$$
$$= -\frac{6}{5}e^{-x/2}$$

Solution (Problem 16)**:**

$$y_2(x) = \int e^{-\int\frac{2x}{1-x^2}\,dx}\,dx$$

$$= \frac{1}{3}x^2 - x.$$

Solution (Problem 20)**:** Using the power of inspection, the other homogeneous solution is $y_2(x) = e^{3x}$. Letting $y_p(x) = v(x)e^x$, we get

$$\begin{aligned}\left(v'' - 2v'\right)e^x &= x\\ v'' - 2v' &= xe^{-x}\\ \frac{d}{dx}\left(v'(x)\right) - 2v'(x) &= xe^{-x}.\end{aligned}$$

Using the integrating factor e^{-2x}, we have

$$\begin{aligned}\frac{d}{dx}\left(e^{-2x}v'(x)\right) &= xe^{-x}\\ e^{-2x}v'(x) &= \int xe^{-x}\,dx\\ &= -xe^{-x} - e^{-x}\\ v(x) &= -xe^x.\end{aligned}$$

Thus, we have the general solution of

$$y(x) = c_1e^x + c_2e^{3x} + c_3xe^{2x}.$$

Solution (Problem 22)**:** We know that

$$x\frac{d^2}{dx^2}(x) - x\frac{d}{dx}(x) + x = 0.$$

Let $y_2(x) = v(x)x$. Then,

Section 4.3

Solution (Problem 4)**:** Using the power of inspection, we have

$$y = c_1e^x + c_2e^{2x}.$$

Solution (Problem 6)**:** Using the power of inspection, we have

$$y = c_1e^{5x} + c_2xe^{5x}.$$

Solution (Problem 12)**:** Using the power of inspection+ε, we have

$$y(x) = e^{-\frac{1}{2}x}\left(c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right)\right).$$

Solution (Problem 16)**:** Using the power of inspection+ε, we have

$$y(x) = c_1e^x + c_2e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Solution (Problem 22)**:** Using the power of inspection+2ε, we have

$$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}.$$

Solution (Problem 36)**:** Using the power of inspection+ε, we have

$$y(x) = c_1e^{-x} + c_2e^{-3x} + c_3e^{2x}.$$

Evaluating the initial conditions, we have

$$1 = c_1 + c_2 + c_3$$

$$
\begin{aligned}
1 &= -c_1 - 3c_2 + 2c_3\\
1 &= c_1 + 9c_2 + 4c_3,
\end{aligned}
$$

and using the power of inspection+ε, we get

$$
\begin{aligned}
c_1 &= \frac{2}{3}\\
c_2 &= -\frac{1}{5}\\
c_3 &= \frac{8}{15}.
\end{aligned}
$$

Solution (Problem 38)**:**

Solution (Problem 50)**:**