Introduction

Oh hey, it's another one of those textbook notes that I never complete. I've decided to try something different in order to develop my understanding of measure theory. One of the primary for understanding measure theory is Gerald B. Folland's *Real Analysis and Applications* — and one of the benefits it has over a lot of other texts is that it has a significant number of exercises. I'm going to try to do them all — I'll start with Chapters 1–3, and if that goes well enough, continue up through whatever chapter I end up having to tap out at. Interspersed, I will include various notes. I figure that in order to make a subject like measure theory really stick, I need to deal with it consistently.

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Chapter 1

Section 1.2

Definition (σ-Algebra)**.** An algebra of sets on X is a nonempty collection $\mathcal{A}$ of X that is closed under finite unions and complements.

A σ-algebra is an algebra that is closed under countable unions.

Exercise (Exercise 1)**:** A family of sets $\mathcal{R} \subseteq P(X)$ is called a ring if it is closed under finite unions and differences. A ring that is closed under countable unions is called a σ-ring.

(a) Rings (σ-rings) are closed under finite (countable) intersections.
(b) If $\mathcal{R}$ is a ring (σ-ring), then $\mathcal{R}$ is an algebra (σ-algebra) if and only if $X \in \mathcal{R}$.
(c) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
(d) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Solution:

(a) Note that for any $E, F \in \mathcal{R}$, that $E \cap F = E \cup F \setminus ((E \setminus F) \cup (F \setminus E))$.
(b) Let $\mathcal{R}$ be a σ-ring. Then, $\mathcal{R}$ is a σ-algebra if for some $E \in \mathcal{R}$, $E^c \in \mathcal{R}$. Since $E^c = X \setminus E \in \mathcal{R}$, we have $X \setminus E \cup E \in \mathcal{R}$ as $\mathcal{R}$ is closed under (countable) unions. Hence, $X \in \mathcal{R}$.

    If $X \in \mathcal{R}$, then for any $E \in \mathcal{R}$, $E^c = X \setminus E \in \mathcal{R}$. Thus, $\mathcal{R}$ is closed under intersections.
(c) Since $\mathcal{R}$ is a σ-ring, we only need show that the set $\mathcal{A} = \{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is closed under complements. We see that for any $E \in \mathcal{A}$, it is the case that either $E \in \mathcal{R}$ or $E^c \in \mathcal{R}$, so $E^c \in \mathcal{A}$ if and only if $E^c \in \mathcal{R}$ or $E \in \mathcal{R}$, so $\mathcal{A}$ is closed under complements.
(d) Let $\mathcal{R}$ be a σ-ring, and let $\mathcal{A} = \{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. We will show that $\mathcal{A}$ is closed under unions and complements.

    Let $E, F \in \mathcal{A}$. Then, for all $S \in \mathcal{R}$, we have $E \cap S \in \mathcal{R}$ and $F \cap S \in \mathcal{R}$. Since $\mathcal{R}$ is closed under unions, we must have $(E \cup F) \cap S = (E \cap S) \cup (F \cap S) \in \mathcal{R}$, so $E \cup F \in \mathcal{A}$.

    Let $E \in \mathcal{A}$.

Proposition (Proposition 1.2)**:** The Borel σ-algebra, $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following:

(a) the open intervals, $\mathcal{E}_1 = \{(a, b) \mid a < b\}$;

(b) the closed intervals, $\mathcal{E}_2 = \{[a,b] \mid a < b\}$;

(c) the half-open intervals, $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a,b) \mid a < b\}$;

(d) the open rays, $\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) \mid a \in \mathbb{R}\}$;

(e) the closed rays, $\mathcal{E}_7 = \{[a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] \mid a \in \mathbb{R}\}$.

Proof. The elements for $\mathcal{E}_j$ for $j \neq 3,4$ are open or closed, and the elements of $\mathcal{E}_3$, $\mathcal{E}_4$ are G_δ sets — for instance,

$$(a,b] = \bigcap_{n=1}^{\infty}\left(a, b+\frac{1}{n}\right).$$

Thus, $\sigma(\mathcal{E}_j) \subseteq \mathcal{B}_{\mathbb{R}}$ for each j. On the other hand, every open set in $\mathbb{R}$ is a countable union of open intervals, so $\mathcal{B}_{\mathbb{R}} \subseteq \sigma(\mathcal{E}_1)$. Thus, $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{E}_1)$. □

Section 1.3

Theorem (Theorem 1.9)**:** Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$, and let $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$. Then, $\mathcal{M}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, so is $\overline{\mathcal{M}}$. If $E \cup F \in \overline{\mathcal{M}}$, with $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{N}$, we may assume $E \cap N = \varnothing$ — else, we replace F with $F \setminus E$ and N with $N \setminus E$. Then, $E \cup F = (E \cup N) \cap (N^c \cup F)$, so $(E \cup F)^c = (E \cup N)^c \cup (N \setminus F)$. Since $(E \cup N)^c \in \mathcal{M}$ and $N \setminus F \subseteq N$, we have $(E \cup F)^c \in \overline{\mathcal{M}}$, so $\overline{M}$ is a σ-algebra.

If $E \cup F \in \overline{\mathcal{M}}$ as above, we set $\overline{\mu}(E \cup F) = \mu(E)$. This is well-defined, since if $E_1 \cup F_1 = E_2 \cup F_2$, with $F_j \subseteq N_j \in \mathcal{N}$, then $E_1 \subseteq E_2 \cup N_2$, so $\mu(E_1) \leqslant \mu(E_2) + \mu(N_2) = \mu(E_2)$. Similarly, $\mu(E_2) \subseteq \mu(E_1)$. □

Exercise (Exercise 6)**:** Complete the proof of Theorem 1.9.

Solution: We now wish to show that every subset of a null set in $\mathcal{M}$ is an element of $\overline{\mathcal{M}}$. This can be seen by the fact that for some $F \subseteq N \in \mathcal{N}$, we have $F = \varnothing \cup F \in \overline{\mathcal{M}}$.

To show uniqueness, we suppose there is some other measure $\nu\colon \overline{\mathcal{M}} \to [0,\infty)$ such that ν agrees with μ on $\mathcal{M}$. For some $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{N}$, we have

$$\begin{aligned}\nu(E \cup F) &= \mu(E)\\ &= \overline{\mu}(E \cup F).\end{aligned}$$

Exercise (Exercise 7)**:** If $\mu_1,\dots,\mu_n$ are measures on $(X,\mathcal{M})$, and $a_1,\dots,a_n \in [0,\infty)$, then $\mu = \sum_{j=1}^{n} a_j\mu_j$ is a measure on $(X,\mathcal{M})$.

Solution: It is clear that $\mu(\varnothing) = \varnothing$. If we have a sequence of disjoint sets $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, then

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \sum_{j=1}^{n} a_j\mu_j\left(\bigcup_{i=1}^{\infty} E_i\right)\\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{\infty} \mu_j(E_i)\\ &= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} a_j\mu_j\right)(E_i)\\ &= \sum_{i=1}^{\infty} \mu(E_i).\end{aligned}$$

Exercise (Exercise 9)**:** If $(X, \mathcal{M}, \mu)$ is a measure space, and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Solution: We have

$$\begin{aligned} \mu(E) &= \mu(((E \cup F) \setminus F) \sqcup E \cap F) \\ \mu(E) &= \mu(E \cup F) - \mu(F) + \mu(E \cap F) \\ \mu(E) + \mu(F) &= \mu(E \cup F) + \mu(E \cap F). \end{aligned}$$

Exercise (Exercise 12)**:** Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ with $\mu(E \triangle M) = 0$, then $\mu(E) = \mu(F)$.
(b) Let $E \sim F$ if $\mu(E \triangle F) = 0$. Then, $\sim$ is an equivalence relation on $\mathcal{M}$.
(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then, $\rho(E, G) \leqslant \rho(E, F) + \rho(F, G)$, hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Solution:

(a) Note that $E = (E \setminus F) \sqcup (E \cap F)$, and $F = (F \setminus E) \sqcup (F \cap E)$. We also have $\mu(E \triangle F) = \mu(E \setminus F) + \mu(F \setminus E) = 0$, so $\mu(F \setminus E) = \mu(E \setminus F) = 0$. Thus,

$$\begin{aligned} \mu(F) &= \mu(F \cap E) \\ &= \mu(E \cap F) \\ &= \mu(E). \end{aligned}$$

Exercise (Exercise 14)**:** If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subseteq E$ such that $C < \mu(F) < \infty$.

Solution: By the definition of a semifinite measure, there exists $F_1 \subseteq E$ such that $0 < \mu(F_1) < \infty$. We let $\delta_1 = \mu(F_1)$.

Now, it must be the case that $\mu(E \setminus F_1) = \infty$, else $\infty = \mu(E) = \mu(E \setminus F_1) + \mu(F_1) < \infty$, a contradiction.

Hence, there exists $F_2 \subseteq E \setminus F_1$ with $0 < \mu(F_2) < \infty$. We let $\delta_2 = \mu(F_2)$. Similarly, we find $\delta_n = \mu(F_n)$, where $F_n \subseteq E \setminus (F_1 \cup \cdots \cup F_{n-1})$.

Now, consider the series $\sum_{n \geqslant 1} \delta_n = \sum_{n \geqslant 1} \mu(F_n) = \mu\left(\bigsqcup_{n \geqslant 1} F_n\right)$. This series must diverge, as otherwise we would have $\mu(E) = \mu\left(\bigsqcup_{n \geqslant 1} F_n\right) < \infty$, which is yet again a contradiction.

Thus, for a given $C > 0$, we find N so large such that $\sum_{n=1}^{N} \delta_n > C$. Then, $F = \bigsqcup_{n=1}^{N} F_n$ is our desired set.