Problem (Problem 1)**:**

(a) Show that $\mathbb{R}$ is not a free $\mathbb{Z}$-module.

(b) Compute $\hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})$ and $\hom_{\mathbb{Z}}(\mathbb{R},\mathbb{Z})$.

Solution:

(a) Suppose toward contradiction that $\mathbb{R}$ were a free $\mathbb{Z}$-module. Then, there would be some unique $\mathbb{Z}$-linear combination

$$1 = z_1 b_1 + \cdots + z_n b_n,$$

with $b_1, \dots, b_n \in B$, where B is the basis for $\mathbb{R}$. We observe now that for any $k \in \mathbb{Z}_{>0}$,

$$\frac{1}{k} = z_1' b_1' + \cdots + z_m' b_m'$$

for some other basis elements $b_1', \dots, b_m' \in B$ and integers $z_1', \dots, z_m'$. Suppose toward contradiction that there were some b_i' such that $b_i' \notin \{b_1, \dots, b_n\}$. Then, we would have

$$\begin{aligned} 1 &= k\left(z_1' b_1' + \cdots + z_m' b_m'\right) \\ &= k z_1' b_1' + \cdots + k z_m' b_m', \end{aligned}$$

implying that 1 has a non-unique expression of integral linear combinations of basis elements, contradicting the assumption that $\mathbb{R}$ is free over $\mathbb{Z}$.

There is some submodule $Y \supseteq \mathbb{Q}$ of $\mathbb{R}$ defined by $\mathbb{Z}\langle b_1, \dots, b_n\rangle$. The map

$$\begin{aligned} \nu \colon \mathbb{Z}^n &\to Y \\ (z_1, \dots, z_n) &\mapsto z_1 b_1 + \cdots + z_n b_n \end{aligned}$$

is thus an isomorphism, as it is injective by the assumption that B is a basis and surjective by definition. Now, since $\mathbb{Q} \subseteq Y$ is a submodule, we observe that $\nu^{-1}(\mathbb{Q}) \subseteq \mathbb{Z}^n$ is a submodule, as for any $w_1, w_2 \in \nu^{-1}(\mathbb{Q})$, we have $\nu(w_1), \nu(w_2) \in \mathbb{Q}$, whence $\nu(w_1 + w_2) \in \mathbb{Q}$, so that $w_1 + w_2 \in \nu^{-1}(\mathbb{Q})$, and $\nu(z w_1) = z\nu(w_1) \in \mathbb{Q}$ for any $z \in \mathbb{Z}$, whence $z w_1 \in \nu^{-1}(\mathbb{Q})$.

Now, since each $\mathbb{Z}$ is a PID (hence Noetherian), it follows that every $\mathbb{Z}$-submodule(/ideal) of $\mathbb{Z}^n$ is also finitely generated, as it is of the form $I_1 \times \cdots \times I_n$ for ideals $I_1, \dots, I_n \in \mathbb{Z}$. Thus, it follows that $\mathbb{Q} \cong \nu^{-1}(\mathbb{Q})$, whence $\mathbb{Q}$ is then isomorphic to a finitely generated $\mathbb{Z}$-module, which is a contradiction as it has been well-established that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module.

(b) We claim that both $\hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})$ and $\hom_{\mathbb{Z}}(\mathbb{R},\mathbb{Z})$ are zero. Toward this end, observe that

$$\varphi\left(\frac{a}{b}\right) = k\varphi\left(\frac{a}{kb}\right)$$

for all $\frac{a}{b} \in \mathbb{Q}$ with $\frac{a}{b} \neq 0$ and all $k \in \mathbb{Z}_{>0}$. Yet, this can only be the case if $\varphi\left(\frac{a}{b}\right) = 0$, whence $\hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \cong \{0\}$. Similarly, if $r \in \mathbb{R}$ is real with $r \neq 0$, then

$$\varphi(r) = k\varphi\left(\frac{r}{k}\right),$$

for all $k \in \mathbb{Z}_{>0}$, so that $\varphi(r) = 0$, and thus $\hom_{\mathbb{Z}}(\mathbb{R},\mathbb{Z}) \cong \{0\}$.

Problem (Problem 2)**:** Let R be a commutative ring with 1. Suppose there are integers m_1 and m_2 such that $R^{m_1} \cong R^{m_2}$. Prove that $m_1 = m_2$.

Solution: Let I be a maximal ideal of R, and let $K = R/I$. We claim that if $M_1 \cong M_2$ are isomorphic R-modules, then $M_1/IM_1 \cong M_2/IM_2$ are isomorphic as R/I-vector spaces. Toward this end, we let

$$\psi\colon M_1 \to M_2/IM_2$$

be a surjective homomorphism of R-modules defined by $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\pi} M_2/IM_2$, whence $\ker(\psi) = IM_1$, as

$$\psi(v_1) = 0 + IM_2$$

if and only if $\varphi(v_1) \in IM_2$, whence $\varphi(v_1) = i\varphi(w_1)$ with $i \in I$, or that $\varphi(iw_1) \in IM_2$, so $iw_1 \in IM_1$. The reverse inclusion follows from the first isomorphism theorem, as $IM_1 \subseteq \ker(\psi)$ by observation. Thus, we have an isomorphism $\overline{\psi}\colon M_1/IM_1 \to M_2/IM_2$.

We claim that the action

$$(r + I) \cdot (m + IM_1) = r \cdot m + IM_1$$

is a well-defined action of R/I on M_1/IM_1. Toward this end, we let $r_1 + I = r_2 + I$, whence $r_1 - r_2 \in I$. For any $m + IM_1 \in M_1/IM_1$, we have (as the quotient module M_1/IM_1 is well-defined)

$$\begin{aligned}(r_1 + I) \cdot (m + IM_1) &= r_1 \cdot m + IM_1\\ &= (r_1 - r_2 + r_2) \cdot m + IM_1\\ &= ((r_1 - r_2) \cdot m + IM_1) + (r_2 \cdot m + IM_1)\\ &= (0 + IM_1) + (r_2 \cdot m + IM_1)\\ &= r_2 \cdot m + IM_1.\end{aligned}$$

The rest of the axioms for the action of R/I on M_1/IM_1 follow from the axioms of R-modules.

Thus, it follows that if $R^{m_1} \cong R^{m_2}$, then we have

$$\begin{aligned}R^{m_1}/IR^{m_1} &\cong R^{m_2}/IR^{m_2}\\ K^{m_1} &\cong K^{m_2},\end{aligned}$$

whence $m_1 = m_2$ by the invariance of dimension for vector spaces.

Problem (Problem 4)**:** Let R be a local ring with maximal ideal I.

(a) Show that if M is a finitely generated module with $I \cdot M = M$, then $M = \{0\}$.

(b) If M is a finitely generated R-module, and $y_1, \dots, y_m \in M$ are such that $\overline{y_1}, \dots, \overline{y_m} \in M/IM$ generate M/IM, then $y_1, \dots, y_m$ generate M.

Solution:

(a) Let $M = \langle x_1, \dots, x_n\rangle$, and suppose $IM = M$. Then, it follows that there are $v_1, \dots, v_n \in I$ such that

$$x_n = v_1 \cdot x_1 + \cdots + v_n \cdot x_n,$$

whence

$$(1 - v_n) \cdot x_n = v_1 \cdot x_1 + \cdots + v_{n-1} \cdot x_{n-1},$$

whence, since I is a local ring,

$$x_n = (1 - v_n)^{-1}(v_1 \cdot x_1 + \cdots + v_{n-1} \cdot x_{n-1}),$$

meaning that $M = \langle x_1, \dots, x_{n-1}\rangle$. Inductively, any generating subset of M can be reduced in this fashion until $M = \{0\}$.

(b) Let $N = \langle y_1, \dots, y_m \rangle$. We wish to show that

$$M = N + IM.$$

Toward this end, let $v \in M$. If $v \in IM$, then we are done. Else, if $v \notin IM$, it follows that $v + IM \neq 0 + IM$, so there are $\alpha_1, \dots, \alpha_m$ such that

$$\begin{aligned} v + IM &= \alpha_1 \cdot (y_1 + IM) + \cdots + \alpha_m \cdot (y_1 + IM) \\ &= (\alpha_1 \cdot y_1 + \cdots + \alpha_m \cdot y_m) + IM. \end{aligned}$$

In particular, this means there is some $q \in IM$ such that

$$v = (\alpha_1 \cdot y_1 + \cdots + \alpha_m \cdot y_m) + q,$$

whence $M = N + IM$.

Consider the subspace $I(M/N)$ of M/N. We seek to show that $I(M/N) = M/N$. Let $v + N \in M/N$. Since $v \in M$, it follows that there are $r_1, \dots, r_n \in I$ and $q \in IM$ such that

$$v = \sum_{i=1}^{n} r_i \cdot y_i + q.$$

In particular, this means that $v + N = q + N$. Since $q + N = ip + N$ for some $p \in M$, we have $i(p + N) = v + N$, whence $I(M/N) = M/N$, meaning that by part (a), we have $M/N \cong \{0\}$, or that $M = N$. Thus, $y_1, \dots, y_n$ generate N.