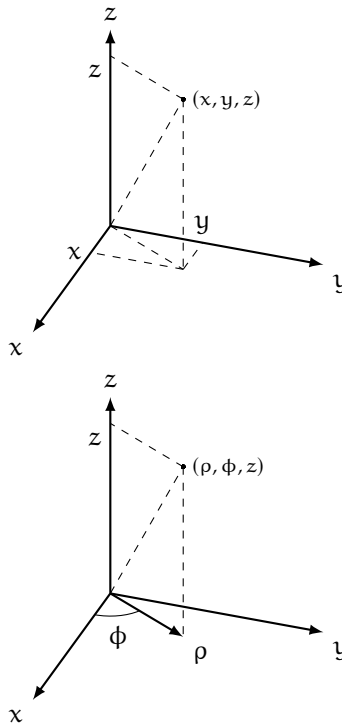


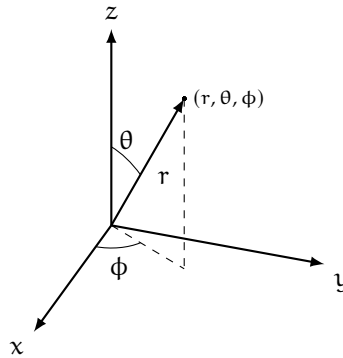
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Things You Just Gotta Know

Coordinate Systems



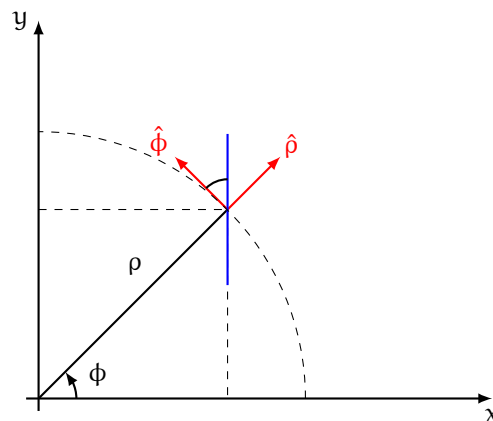


We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x, y)\hat{i} + V_y(x, y)\hat{j}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

Polar Coordinates



We can also express the inputs to \vec{V} in polar coordinates, (ρ, ϕ) .

$$\vec{V}(\mathbf{r}) = V_\rho(\rho, \phi)\hat{i} + V_\phi(\rho, \phi)\hat{j}.$$

To extract the input functions, we take

$$V_x = \hat{i} \cdot \vec{V}$$

$$V_y = \hat{j} \cdot \vec{V}.$$

Alternatively, we can project \vec{V} onto the $\hat{\rho}, \hat{\phi}$ axis:

$$\vec{V}(\mathbf{r}) = V_\rho(\rho, \phi)\hat{\rho} + V_\phi(\rho, \phi)\hat{\phi},$$

and we extract

$$V_\rho = \hat{\rho} \cdot \vec{V}$$

$$V_\phi = \hat{\phi} \cdot \vec{V}.$$

Notice that \mathbf{r} is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned}
 \mathbf{s} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \\
 &= \rho \cos \phi \hat{\mathbf{i}} + \rho \sin \phi \hat{\mathbf{j}} \\
 &= \rho \hat{\rho} \\
 &= \sqrt{x^2 + y^2} \hat{\rho}
 \end{aligned}$$

The main reason we avoided using the $\hat{\rho}, \hat{\phi}$ axis up until this point is that ρ and ϕ are *position-dependent*, while the $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ axis is position-independent.

Now, we must figure out the position-dependence of $\hat{\rho}$ and $\hat{\phi}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi.$$

If we hold ϕ constant, it must be the case that any change in ρ is in the $\hat{\rho}$ direction. Therefore,

$$\begin{aligned}
 \hat{\rho} &= \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|} \\
 &= \frac{\cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}}{\left| \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}} \right|} \\
 &= \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\|} \\
 &= \frac{-\rho \sin \phi \hat{\mathbf{i}} + \rho \cos \phi \hat{\mathbf{j}}}{\left\| -\rho \sin \phi \hat{\mathbf{i}} + \rho \cos \phi \hat{\mathbf{j}} \right\|} \\
 &= -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}.
 \end{aligned}$$

Thus, we can see that the $\hat{\rho}, \hat{\phi}$ axis is orthogonal.

$$\begin{aligned}
 \frac{\partial \hat{\rho}}{\partial \phi} &= -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} \\
 &= \hat{\phi}, \\
 \frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\rho}, \\
 \frac{\partial \hat{\phi}}{\partial \rho} &= 0,
 \end{aligned}$$

and

$$\frac{\partial \hat{\rho}}{\partial \rho} = 0$$

Example (Velocity).

$$\mathbf{v} = \frac{d\mathbf{s}}{dt}$$

$$= \frac{d}{dt} (x\hat{i}) + \frac{d}{dt} (y\hat{j}).$$

In the case of cartesian coordinates, \hat{i} and \hat{j} are constants.

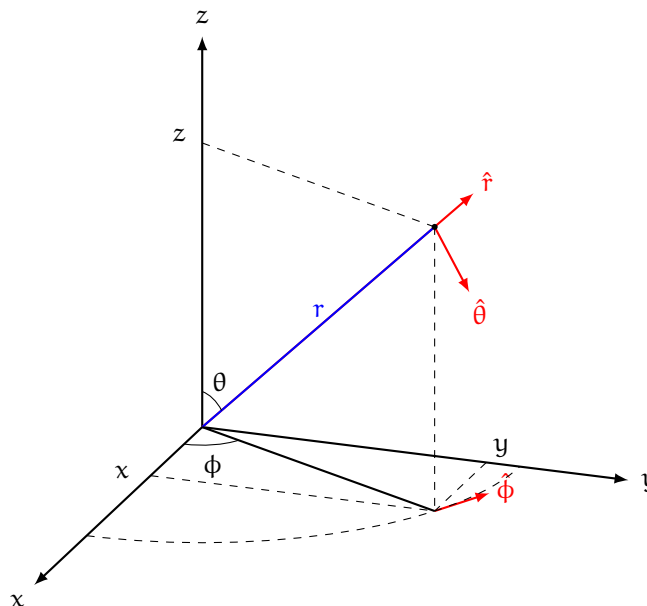
$$= v_x \hat{i} + v_y \hat{j}$$

When we examine polar coordinates, since $\hat{\rho}$ and $\hat{\phi}$ are position-dependent, we must use the chain rule.¹

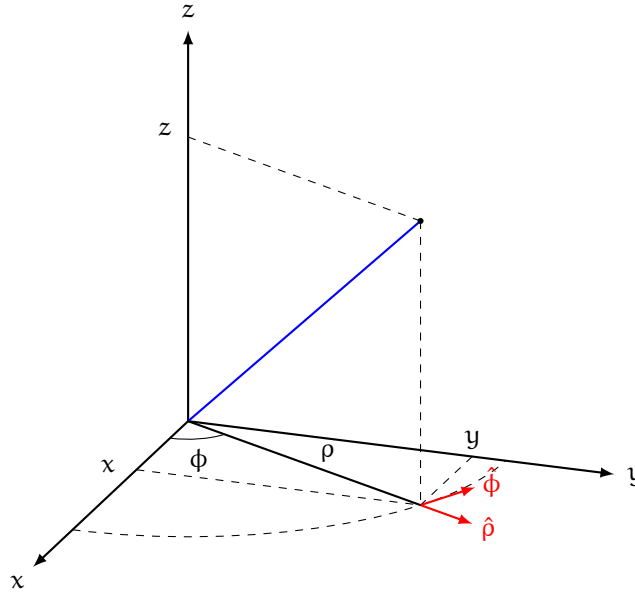
$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{s}}{dt} \\ &= \frac{d\rho}{dt} \hat{\rho} + \rho \frac{d\hat{\rho}}{dt} \\ &= \frac{d\rho}{dt} \hat{\rho} + \rho \left(\frac{\partial \hat{\rho}}{\partial \rho} \frac{d\rho}{dt} + \underbrace{\frac{\partial \hat{\rho}}{\partial \phi} \frac{d\phi}{dt}}_{=\hat{\phi}} \right) \\ &= \frac{d\rho}{dt} \hat{\rho} + \rho \frac{d\phi}{dt} \hat{\phi} \\ &= \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi}. \end{aligned}$$

Notice that $\dot{\rho}$ is the radial velocity and $\dot{\phi} = \omega$ is the angular velocity.

Spherical and Cylindrical Coordinates



¹Note that $\hat{\rho} = \hat{\rho}(\rho, \phi)$ and $\hat{\phi} = \hat{\phi}(\rho, \phi)$.



Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho, \phi)$	$\mathbf{s} = s(\rho, \phi, z)$	$\mathbf{s} = s(r, \phi, \theta)$
$\mathbf{s} = \begin{pmatrix} \rho \cos \phi \\ \rho \sin \phi \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} \rho \cos \phi \\ \rho \sin \phi \\ z \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} r \cos \phi \sin \theta \\ r \sin \phi \sin \theta \\ r \cos \theta \end{pmatrix}$

Here,¹¹ ϕ denotes the polar angle and θ denotes the azimuthal angle. Notice that $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

We can see that \hat{r} , $\hat{\phi}$, and $\hat{\theta}$ in spherical coordinates are also position-dependent.

$$\begin{aligned}
 \hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\| \frac{\partial \mathbf{s}}{\partial r} \right\|} \\
 &= \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \\
 \hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\| \frac{\partial \mathbf{s}}{\partial \phi} \right\|} \\
 &= -\sin \phi \hat{i} + \cos \phi \hat{j} \\
 \hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\| \frac{\partial \mathbf{s}}{\partial \theta} \right\|} \\
 &= \cos \phi \cos \theta \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}
 \end{aligned}$$

Scale Factors and Jacobians

Coordinate System	Line Element	Area Element	Volume Element
Polar	$d\mathbf{s} = \hat{\rho} d\rho + \rho \hat{\phi} d\phi$	$d\mathbf{a} = r dr d\phi$	—
Cylindrical	$d\mathbf{s} = \hat{\rho} d\rho + \rho \hat{\phi} d\phi + \hat{k} dz$	—	$d\tau = r dr d\phi dz$
Spherical	$d\mathbf{s} = \hat{r} dr + r \sin \theta \hat{\phi} d\phi + r \hat{\theta} d\theta$	$d\mathbf{a} = r^2 \sin \theta d\phi d\theta$	$d\tau = r^2 \sin \theta dr d\phi d\theta$

¹¹Physicists amirite?

In cylindrical coordinates, we can use the chain rule to find the value of $d\mathbf{r}$:

$$d\mathbf{r} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz.$$

The extra factor of ρ in the expression of $\rho\hat{\phi}d\phi$ is the *scale factor* on ϕ .

Similarly, in spherical coordinates, we have

$$d\mathbf{r} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta,$$

with scale factors of $r\sin\theta$ on $\hat{\phi}d\phi$ and r on $\hat{\theta}d\theta$.

When we go from line elements (of the form $d\mathbf{r}$) to area elements (of the form $d\mathbf{a}$), we can see that the area element in polar coordinates is $d\mathbf{a} = \rho d\rho d\phi$ — we need the extra factor of ρ to account for the fact that the magnitude of the area element scales with the radius.

Similarly, the volume element in cylindrical coordinates is $d\tau = r dr d\phi dz$ and the volume element in spherical coordinates is $r^2 \sin\theta dr d\phi d\theta$.

Recall that the definition of an angle ϕ that subtends an arc length s is $\phi = \frac{s}{r}$, where r is the radius of a circle. We can imagine a similar concept on a sphere — a *solid angle* measured in steradians is of the form $\Omega = \frac{A}{r^2}$, where A denotes the surface area subtended by the angle Ω . In particular, since $d\Omega = \frac{dA}{r^2}$, we find that $d\Omega = \sin\theta d\phi d\theta$.

When we are dealing with products of scale factors, we need to use the Jacobian to determine the proper scale factor on any given element:

$$\begin{aligned} d\mathbf{a} &= dx dy \\ &= |J| du dv, \end{aligned}$$

where $|J|$ denotes the determinant of the Jacobian matrix. We write the Jacobian as follows:

$$\begin{aligned} J &= \frac{\partial(x, y)}{\partial(u, v)} \\ &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}. \end{aligned}$$

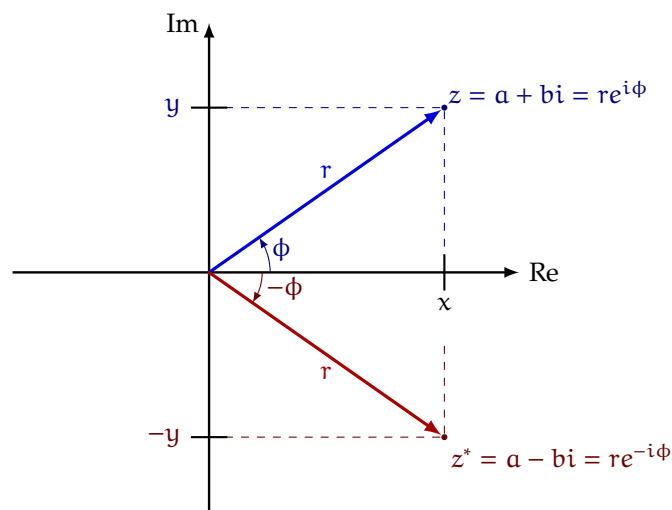
We specifically desire the determinant:

$$|J| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}.$$

Complex Numbers

Quantity	Expression and/or Criterion
Cartesian form	$z = a + bi$
Polar form	$z = re^{i\phi}$
r	$\sqrt{a^2 + b^2}$
ϕ	$\arg z = \arctan\left(\frac{b}{a}\right)$
Cartesian z^*	$z^* = a - bi$
Polar z^*	$z^* = re^{-i\phi}$
$ z $	$\sqrt{zz^*}$
$\operatorname{Re}(z)$	$\operatorname{Re}(z) = \frac{z+z^*}{2}$
$\operatorname{Im}(z)$	$\operatorname{Im}(z) = \frac{z-z^*}{2i}$
$\cos \phi$	$\frac{e^{i\phi} + e^{-i\phi}}{2}$
$\sin \phi$	$\frac{e^{i\phi} - e^{-i\phi}}{2i}$
$e^{i\phi}$	$\cos \phi + i \sin \phi$
$e^{in\phi}$	$\cos(n\phi) + i \sin(n\phi)$

Introduction



A complex number is denoted

$$z = a + bi$$

where $i^2 = -1$ and $a, b \in \mathbb{R}$. This is known as the cartesian representation. However, we can also imagine z as the polar representation:

$$z = re^{i\phi},$$

where $\phi = \arg z$ is known as the argument, and $r = |z|$ is the modulus. We can see the relation between the cartesian and polar representations through Euler's identity:^{III}

$$r(\cos \phi + i \sin \phi) = re^{i\phi}.$$

We denote the conjugate of z as z^* ^{IV}, found by $z^* = a - bi = re^{-i\phi}$.

We find $\text{Re}(z)$ and $\text{Im}(z)$, the real and imaginary parts of z , by

$$\begin{aligned}\text{Re}(z) &= \frac{z + z^*}{2} \\ \text{Im}(z) &= \frac{z - z^*}{2i}.\end{aligned}$$

We say that a complex number of the form $e^{i\phi}$ is a *pure phase*, as $|e^{i\phi}| = 1$.

To find if some complex number z is purely real or purely imaginary, we can use the following criterion:

$$\begin{aligned}z \in \mathbb{R} &\Leftrightarrow z = z^* \\ z \in i\mathbb{R} &\Leftrightarrow z = -z^*.\end{aligned}$$

Example (Real, Imaginary, or Complex?). Consider

$$z_1 = i^i.$$

To find if this is purely real or complex, we take

$$\begin{aligned}z_1^* &= (-i)^{-i} \\ &= \left(\frac{1}{-i}\right)^i \\ &= i^i.\end{aligned}$$

Thus, $z_1 \in \mathbb{R}$. In order to determine the value of i^i , we substitute the polar form:

$$\begin{aligned}z_1 &= \left(e^{i\frac{\pi}{2}}\right)^i \\ &= e^{-\frac{\pi}{2}}.\end{aligned}$$

Some Trigonometry with Complex Exponentials

Consider $z = \cos \phi + i \sin \phi$. We can see that

$$\begin{aligned}\text{Re}(z) &= \cos \phi \\ &= \frac{(\cos \phi + i \sin \phi) + (\cos \phi - i \sin \phi)}{2} \\ &= \frac{e^{i\phi} + e^{-i\phi}}{2} \\ \text{Im}(z) &= \sin \phi \\ &= \frac{(\cos \phi + i \sin \phi) - (\cos \phi - i \sin \phi)}{2i} \\ &= \frac{e^{i\phi} - e^{-i\phi}}{2i}.\end{aligned}$$

We can actually define $\sin \phi$ and $\cos \phi$ with the above derivation.

^{III}This can be proven relatively easily through substitution into the Taylor series, which is allowed because e^z is entire.

^{IV}Physicists amirite?

Theorem (De Moivre).

$$\begin{aligned} e^{inx} &= \cos(nx) + i \sin(nx) \\ &= \left(e^{ix}\right)^n \\ &= (\cos x + i \sin x)^n. \end{aligned}$$

Example (Finding $\cos(2x)$ and $\sin(2x)$).

$$\begin{aligned} \cos(2x) + i \sin(2x) &= (\cos x + i \sin x)^2 \\ &= (\cos^2 x - \sin^2 x) + i(2 \sin x \cos x). \end{aligned}$$

Since the real parts and imaginary parts have to be equal, this means

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin^2 x &= 2 \sin x \cos x. \end{aligned}$$

In particular, we can see that $e^{in\pi} = (-1)^n$ and $e^{in\frac{\pi}{2}} = i^n$.^v

Additionally, we can see that for $z = re^{i\phi}$,

$$\begin{aligned} z^{1/m} &= \left(re^{i\phi+2\pi n}\right)^{1/m} \\ &= r^{1/m} e^{i\frac{1}{m}(\phi+2\pi n)}, \end{aligned}$$

where $n \in \mathbb{N}$ and m is fixed. For $r = 1$, we call these values the m roots of unity.

Example (Waves and Oscillations). Recall that for a wave with spatial frequency k , angular frequency ω , and amplitude A , the wave is represented by

$$f(x, t) = A \cos(kx - \omega t).$$

The speed of a wave v is equal to $\frac{\omega}{k}$.

Simple harmonic motion is characterized by the solution to the differential equation $\ddot{x} = -\omega^2 x$, where x denotes position. In simple harmonic motion, there is no spatial motion, meaning our function is only of time:

$$\begin{aligned} f(t) &= A \cos \omega t \\ &= \operatorname{Re} \left(A e^{i\omega t} \right). \end{aligned}$$

As a result of the representation of complex numbers in polar form, we can do math entirely in exponentials, then take the real part of our solution to find $f(t)$.

Unfortunately, in the real world, there is friction; as a result, our oscillation is damped by an exponential factor.

Example (Hyperbolic Sine and Hyperbolic Cosine). We wish to calculate $\cos ix$ and $\sin ix$.

$$\begin{aligned} \cos ix &= \frac{1}{2} \left(e^{i(ix)} + e^{-i(ix)} \right) \\ &= \frac{e^{-x} + e^x}{2} \end{aligned}$$

^vThis will be especially useful when we get to Fourier series.

We define $\cosh x = \cos(ix)$. Additionally,

$$\begin{aligned} -i \sin ix &= -i \frac{1}{2i} (e^{i(ix)} - e^{-i(ix)}) \\ &= i \frac{e^{ix} - e^{-ix}}{2i} \\ &= \frac{e^x - e^{-x}}{2}. \end{aligned}$$

We define $\sinh x = -i \sin(ix)$.

Similar to how $\cos^2 x + \sin^2 x = 1$, we can find that $\cosh^2 x - \sinh^2 x = 1$.

Index Algebra

We usually denote vectors by either \vec{A} , \mathbf{A} , or

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is defined by a basis.

If we imagine we are in n -dimensional space, we can let A_i where $i = 1, 2, \dots, n$ denote both

- the i th component of \vec{A} ;
- the entire vector \vec{A} (since i can be arbitrary).

Contractions and Dummy Indices

Consider $C = AB$, where A, B are $n \times m$ and $m \times p$ matrices respectively.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}.$$

Definition (Matrix Multiplication in Index Notation). For matrices A and B , where A is an $m \times n$ and B is a $n \times p$ matrix, we write

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

We say that k is a dummy index, since k takes values from 1 to n . Note that the value we calculate is C_{ij} ; in other words, in the sum $\sum_k A_{ik} B_{kj}$, the indices of the form ij are the “net indices” from the multiplication.

Note that if $C = BA$, then

$$\begin{aligned} C_{ij} &= \sum_{k=1}^n B_{ik} A_{kj} \\ &= \sum_{k=1}^n A_{kj} B_{ik} \end{aligned}$$

$$\neq \sum_{k=1}^n A_{ik} B_{kj}.$$

The corresponding fact is that $AB \neq BA$ necessarily.

Note that the index that is summed over always appears exactly twice.

Definition (Symmetric Matrix). Let C be a matrix. Then, we say C is symmetric if

$$C_{ij} = C_{ji}$$

Definition (Antisymmetric Matrix). Let C be a matrix. We say C is antisymmetric if

$$C_{ij} = -C_{ji}.$$

We can always decompose a random matrix into the sum of a symmetric matrix and an antisymmetric matrix.

Two Special Tensors

Name	Notation	Definition
Kronecker Delta	δ_{ij}	$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
Levi-Civita Symbol	ϵ_{ijk}	$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = (1, 2, 3) \text{ cyclically} \\ -1 & (i, j, k) = (2, 1, 3) \text{ cyclically} \\ 0 & \text{else} \end{cases}$

Order of (i, j, k)	Value of ϵ_{ijk}
1, 2, 3	1
3, 1, 2	1
2, 3, 1	1
1, 3, 2	-1
2, 1, 3	-1
3, 2, 1	-1
else	0

Value	Index Notation
$\mathbf{A} \times \mathbf{B}$	$\sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k$
$(\mathbf{A} \times \mathbf{B})_\ell$	$\sum_{i,j} \epsilon_{ij\ell} A_i B_j$
$(\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k$	ϵ_{ijk}
B_i	$\sum_{\alpha} B_{\alpha} \delta_{\alpha i}$
$\mathbf{A} \cdot \mathbf{B}$	$\sum_{i,j} A_i B_j \delta_{ij}$
$\sum_{j,k} \epsilon_{mjk} \epsilon_{njk}$	$2\delta_{mn}$
$\sum_{\ell} \epsilon_{mnl} \epsilon_{ijl}$	$\delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}$

Definition (Kronecker Delta). The Kronecker Delta, δ_{ij} , is the tensor that denotes the identity matrix.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example (Extracting an Index). Consider A as vector. Then,

$$\sum_i A_i \delta_{ij} = A_j.$$

In other words, the Kronecker Delta collapses the sum to the j th index.

Example (Orthonormal Basis from Kronecker Delta). Let $\{\hat{e}_i\}_{i=1}^n$ be a basis for some vector space V . If

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

for every i, j , then $\{\hat{e}_i\}_{i=1}^n$ is an orthonormal basis for V .

Definition (Levi-Civita Symbol). In two dimensions, as a matrix, we write

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

meaning

$$\epsilon_{ij} = \begin{cases} 1 & i = 1, j = 2 \\ -1 & i = 2, j = 1 \\ 0 & \text{else} \end{cases}.$$

The Levi-Civita Symbol is antisymmetric, just as the Kronecker Delta is symmetric.

In three dimensions, we define

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = (1, 2, 3) \text{ cyclically} \\ -1 & (i, j, k) = (2, 1, 3) \text{ cyclically} \\ 0 & \text{else} \end{cases}.$$

In other words, $\epsilon_{ijk} = -\epsilon_{jik}$.

Exercise (Relations between δ_{ij} and ϵ_{ijk}).

$$\sum_{j,k} \epsilon_{mjk} \epsilon_{njk} = 2\delta_{mn}$$

$$\sum_{\ell} \epsilon_{mnl} \epsilon_{ijl} = \delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}$$

Definition (Dot Product). Let $\{\hat{e}_i\}_{i=1}^n$ be an orthonormal basis for V . Let $\mathbf{A} = \sum_i A_i \hat{e}_i$ and $\mathbf{B} = \sum_i B_i \hat{e}_i$. Then,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \sum_{i,j} (A_i \hat{e}_i) \cdot (B_j \hat{e}_j) \\ &= \sum_{i,j} A_i B_j (\hat{e}_i \cdot \hat{e}_j) \\ &= \sum_{i,j} A_i B_j \delta_{ij} \\ &= \sum_i A_i B_i \end{aligned}$$

Definition (Cross Product). Let $\{\hat{e}_i\}_{i=1}^3$ be the standard basis over \mathbb{R}^3 . Let $\mathbf{A} = \sum_i A_i \hat{e}_i$ and $\mathbf{B} = \sum_i B_i \hat{e}_i$. Then,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \sum_{i,j} (A_i \hat{e}_i) \times (B_j \hat{e}_j) \\ &= \sum_{i,j} A_i B_j (\hat{e}_i \times \hat{e}_j) \\ &= \sum_{i,j,k} A_i B_j (\epsilon_{ijk} \hat{e}_k).\end{aligned}$$

Instead of asking about $\mathbf{A} \times \mathbf{B}$, we ask about $(\mathbf{A} \times \mathbf{B})_\ell$, yielding

$$\begin{aligned}(\mathbf{A} \times \mathbf{B})_\ell &= (\mathbf{A} \times \mathbf{B}) \cdot \hat{e}_\ell \\ &= \left(\sum_{i,j,k} A_i B_j (\epsilon_{ijk} \hat{e}_k) \right) \cdot \hat{e}_\ell \\ &= \sum_{i,j} \epsilon_{ij\ell} A_i B_j.\end{aligned}$$

Remark: This notation for $\mathbf{A} \times \mathbf{B}$ automatically shows us that

$$\begin{aligned}(\mathbf{B} \times \mathbf{A})_\ell &= \sum_{i,j} \epsilon_{ij\ell} B_i A_j \\ &= - \sum_{i,j} \epsilon_{ji\ell} B_i A_j \\ &= - \sum_{i,j} \epsilon_{ji\ell} A_j B_i \\ &= - \sum_{i,j} \epsilon_{ij\ell} A_i B_j && i, j \text{ are dummy indices} \\ &= -(\mathbf{A} \times \mathbf{B})_\ell.\end{aligned}$$

Example (Central Force and Angular Momentum). A central force is defined by

$$\mathbf{F} = f(r) \hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a radial vector.

Angular momentum is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where \mathbf{r} denotes position and \mathbf{p} denotes momentum. Then,

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) \\ &= \left(\frac{d}{dt} \mathbf{r} \times \mathbf{p} \right) + \mathbf{r} \times \left(\frac{d\mathbf{p}}{dt} \right) \\ &= m \left(\frac{d}{dt} \mathbf{r} \times \frac{d}{dt} \mathbf{r} \right) + \mathbf{r} \times (f(r) \hat{\mathbf{r}}) \\ &= f(r) (\mathbf{r} \times \hat{\mathbf{r}}).\end{aligned}$$

This implies that $\frac{d\mathbf{L}}{dt} = 0$ under a central force.

Example (Determinant). Let $\mathbf{M} = M_{ij}$ be square. We denote \mathbf{M}_i to be the vector denoting the i th-row. Then,

$$\begin{aligned} m &= |\mathbf{M}| \\ &= \mathbf{M}_1 \cdot (\mathbf{M}_2 \times \mathbf{M}_3) \\ &= \mathbf{M}_3 \cdot (\mathbf{M}_1 \times \mathbf{M}_2) \\ &= \mathbf{M}_2 \cdot (\mathbf{M}_3 \times \mathbf{M}_1). \end{aligned}$$

Example (Trace). Let $\mathbf{M} = M_{ij}$ be a square matrix. We define $\text{tr}(\mathbf{M}) = \sum_i M_{ii}$. Equivalently,

$$\begin{aligned} \text{tr}(\mathbf{M}) &= \sum_{ij} M_{ij} \delta_{ij} \\ &= \sum_i M_{ii}. \end{aligned}$$

Note that

$$\begin{aligned} \text{tr}(\mathbf{I}_n) &= \sum_i \delta_{ii} \\ &= n. \end{aligned}$$

When we upgrade to 3 matrices, we take

$$\begin{aligned} \text{tr}(ABC) &= \sum_{i,j} \left(\sum_{k,\ell} A_{ik} B_{k\ell} C_{\ell j} \right) \delta_{ij} \\ &= \sum_{i,k,\ell} A_{ik} B_{k\ell} C_{\ell i} \\ &= \sum_{i,k,\ell} C_{\ell i} A_{ik} B_{k\ell} \\ &= \text{tr}(CAB). \end{aligned}$$

In other words, the trace is invariant under cyclic permutations.

Example (Moment of Inertia Tensor).

Recall that

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p}, \\ &= I \boldsymbol{\omega}. \end{aligned}$$

where $\mathbf{p} = m\dot{\mathbf{x}}$, and I denotes the moment of inertia. Note that $I \sim m r^2$. On a more fundamental level, it is the case that the first equation, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is the “true” definition of \mathbf{L} .

Consider a small portion m_α about some axis at radius \mathbf{r}_α and momentum \mathbf{p}_α . Then, we have

$$\begin{aligned} \mathbf{L}_\alpha &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\ &= \sum_\alpha m_\alpha (\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)). \end{aligned}$$

In the infinitesimal case (i.e., as $\alpha \rightarrow 0$), we get

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \rho \, d\tau,$$

where ρ denotes volume density. Applying the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, we find

$$\mathbf{L} = \int (\boldsymbol{\omega} (\mathbf{r} \cdot \mathbf{r}) - \mathbf{r} (\mathbf{r} \cdot \boldsymbol{\omega})) \rho \, d\tau.$$

Switching to index notation, we have

$$\begin{aligned}
 L_i &= \int \left(\omega_i r^2 - r_i \sum_j r_j \omega_j \right) \rho \, d\tau \\
 &= \sum_j \int \omega_j \left(\delta_{ij} r^2 - r_i r_j \right) \rho \, d\tau \\
 &= \sum_j \omega_j \underbrace{\left(\int \left(\delta_{ij} r^2 - r_i r_j \right) \rho \, d\tau \right)}_{\text{moment of inertia tensor}} \\
 &= \sum_j I_{ij} \omega_j.
 \end{aligned}$$

Binomial Theorem

The binomial theorem allows us to calculate the expansion

$$(x + y)^n = \sum_{m=0}^n \binom{n}{m} x^{n-m} y^m.$$

In the case of $(x + y)^2 = x^2 y^0 + 2x^1 y^1 + x^0 y^2 = x^2 + 2xy + y^2$. Recall that

$$\binom{n}{m} = \frac{n!}{m! (n - m)!}.$$

Recall that $0! = 1$.

Infinite Series

Let

$$S = \sum_{k=0}^{\infty} a_k$$

be an infinite series. We are often curious as to the convergence of this sum (for a variety of reasons). Formally, we have to invoke partial sums

$$S_N = \sum_{k=0}^N a_k,$$

and see if the sequence of partial sums is convergent. However, we will prefer to use series convergence tests.

Example (Geometric Series). Let

$$\begin{aligned}
 S &= \sum_{k=0}^{\infty} r^k \\
 &= 1 + r + r^2 + \dots
 \end{aligned}$$

Then, we have

$$S_N = \sum_{k=0}^N r^k$$

$$rS_N = \sum_{k=0}^N r^k.$$

Subtracting, we get

$$(1-r)S_N = 1 - r^{N+1}$$

$$S_N = \frac{1 - r^{N+1}}{1 - r}.$$

In the limit, we expect that if $r \rightarrow \infty$, and $r < 1$, then $r^{N+1} \rightarrow 0$. In the infinite case, we have

$$S = \sum_{k=0}^{\infty} r^k$$

$$= \frac{1}{1-r},$$

if $r < 1$.

There are a few prerequisites for series convergence:

- there exists some K for which for all $k \geq K$, $a_{k+1} \leq a_k$;
- $\lim_{k \rightarrow \infty} a_k < \infty$;
- we need the series to reduce “quickly” enough.

Example (Ratio Test). A series $S = \sum_k a_k$ converges if the ratio of consecutive terms is (eventually) less than 1:

$$r = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1.$$

Example (Applying the Ratio Test). Consider $S = \sum_k \frac{1}{k!}$. Then,

$$r = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{k+1}$$

$$= 0 < 1$$

Example (Riemann Zeta Function). We write

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

In order to evaluate the convergence of the Riemann zeta function. We have

$$r = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)^s}}{\frac{1}{k^s}}$$

$$= \lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^s$$

$$= 1.$$

Unfortunately, this means the ratio test is inconclusive.

For examples of evaluations of the zeta function, we have

$$\begin{aligned}\zeta(1) &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ \zeta(2) &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots \\ &= \frac{\pi^2}{6}.\end{aligned}$$

Example (Absolute Convergence). In our original ratio test, we had assumed that a_k are real and positive. However, if the $a_k \in \mathbb{C}$, we have to look at the convergence in modulus:

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

If $\sum_k |a_k|$ converges, this is known as absolute convergence.

Example (Alternating Series Test). If the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

has the following conditions:

- $a_{k+1} < a_k$ for $k > K$;
- $\lim_{k \rightarrow \infty} a_k = 0$;

then $\sum_k (-1)^k a_k$ converges.

For instance, the alternating harmonic series converges

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2.$$

Power Series

Consider the function

$$S(x) = \sum_{k=0}^{\infty} a_k x^k.$$

This is a series both in a_k and in x . In order to determine convergence, we use the ratio test as follows:

$$\begin{aligned}\lim_{k \rightarrow \infty} \left| \frac{a_{k+1} x^{k+1}}{a_k x^k} \right| &= |x| \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| \\ &\equiv |x| r.\end{aligned}$$

In particular, for convergence, it must be the case that

$$|x| r < 1.$$

We define

$$R = \begin{cases} \frac{1}{r} & 0 < r < \infty \\ 0 & r = \infty \\ \infty & r = 0 \end{cases}.$$

In particular, this means

$$|x| < R.$$

Definition (Radius of Convergence). For a power series $\sum_k a_k x^k$, the series converges for $|x| < R$,^{vi} where

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$R = \begin{cases} \frac{1}{r} & 0 < r < \infty \\ 0 & r = \infty \\ \infty & r = 0 \end{cases}.$$

Note that convergence for $|x| < R$ does not provide information regarding convergence at the boundary.

Example (Geometric Series). We have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

has $R = 1$, meaning the power series converges for $|x| < 1$.

Example (Exponential Function). We have

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

with $R = \infty$.

Example (Natural Log). We have

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$$

In particular, since $R = 1$, we know that the radius of convergence is $|x| < 1$. However, the series does converge on the boundary when $x = 2$, but not when $x = 0$ (for obvious reasons).

Example (Why Radius of Convergence?). Consider two series

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

We can see that the first series converges for $|x| < 1$. However, even though $\frac{1}{1+x^2}$ has a domain across the entire real numbers, it is still the case that the *series* converges for $|x| < 1$.

The primary reason that the radius of convergence is defined as such is because, over the complex numbers, it is the case that $x^2 + 1 = 0$ at $x = \pm i$, meaning $\frac{1}{1+z^2}$ has singularities at those values of z .

The main reason power series are useful is that, when truncated, they are simply polynomials. In particular, with power series, we can reverse the order of sum and derivative.

^{vi}The definition is not the true radius of convergence; it is actually that $r = \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|}$. It just happens to be the case that the ratio test and root test return the same value when they're regular limits (rather than limits superior).

Taylor Series

Function	Taylor Series
$f(x)$	$\sum_{k=0}^{\infty} \frac{(x-x_0)^n}{n!} \left(\left. \frac{d^n f}{dx^n} \right _{x=x_0} \right)$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$
$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$ ^{vii}

Definition. The Taylor series of a function $f(x)$ about x_0 is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left(\left. \frac{d^n f}{dx^n} \right|_{x=x_0} \right).$$

Remark: The reason we write $\frac{d^n f}{dx^n}$ is because $\frac{d^n}{dx^n}$ is an operator in and of itself.

Example (The Most Important Taylor Series).

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \end{aligned}$$

Example (Equilibrium Points). Let $U(x)$ denote a potential over x . Then, $F = -\nabla U$. We have

$$U(x) = U(x_0) + (x-x_0) U'(x_0) + \frac{1}{2!} (x-x_0)^2 U''(x_0) + \frac{1}{3!} (x-x_0)^3 U'''(x_0) + \dots$$

When we analyze an equilibrium point, we disregard the $U(x_0)$ term, and see that the derivative of U is zero; thus, we can truncate our series at the second derivative close to $x = x_0$:

$$\begin{aligned} U(x) &\approx \frac{1}{2} U''(x_0) (x-x_0)^2 \\ &= \frac{1}{2} m\omega^2 (x-x_0)^2. \end{aligned}$$

In other words, when we are very close to equilibrium, we have simple harmonic motion.

Example (Faster Taylor Series). Consider the function

$$\exp\left(\frac{x}{1-x}\right).$$

In order to create its Taylor series, we can create this Taylor series piecewise:

$$\exp\left(\frac{x}{1-x}\right) = 1 + \left(\frac{x}{1-x}\right) + \frac{1}{2!} \left(\frac{x}{1-x}\right)^2 + \frac{1}{3!} \left(\frac{x}{1-x}\right)^3 + \dots$$

^{vii}We define $(\alpha)_n = \prod_{k=0}^{n-1} (\alpha - k)$

Now, we expand the denominators as geometric series:

$$= 1 + x \left(\sum_{k=0}^{\infty} x^k \right) + \frac{x^2}{2!} \left(\sum_{k=0}^{\infty} x^k \right)^2 + \frac{x^3}{3!} \left(\sum_{k=0}^{\infty} x^k \right) + \dots$$

If we want to expand through x^3 , we have to expand by keeping track of *every* term:

$$= 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + O(x^4).$$

We say we have expanded the series through the third order; the lowest order correction, denoted $O(x^n)$, is the fourth order (in this case).

Example (Exponentiated Operator). Consider a (square) matrix M . Then, we define

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!},$$

where $M^k = \prod_{i=1}^k M$; we define $M^0 = I$. Similarly,

$$e^{\frac{d}{dx}} = \sum_{k=0}^{\infty} \frac{d^k}{dx^k} \frac{1}{k!}.$$

In particular, $e^{\frac{d}{dx}}$ is the Taylor series operator.

Remark: In quantum mechanics, the momentum operator is

$$P = -i\hbar \frac{d}{dx}.$$

Example (Binomial Expansion). For any $\alpha \in \mathbb{R}^+$ and $|x| < 1$, we have

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

Note that if $\alpha \in \mathbb{Z}^+$, then the series truncates (and we recover the binomial theorem again).

The main use of the binomial expansion is with very small quantities. For instance,

$$\begin{aligned} E &\sim \frac{1}{(x^2 + a^2)^{3/2}} \\ &= \frac{1}{x^3 \left(1 + \frac{a^2}{x^2}\right)^{3/2}} \\ &\approx \frac{1}{x^3} \left(1 - \frac{3}{2} \frac{a^2}{x^2}\right) \end{aligned}$$

For $x \gg a$