Textbook Problems

Solution (8.1, Problem 6)**:** We have

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + e^{-t}\begin{pmatrix} \sin(2t) \\ 4\cos(2t) \\ -1 \end{pmatrix}.$$

Solution (8.1, Problem 12)**:** We have

$$\begin{aligned} \frac{dx}{dt} &= 5e^t\cos(t) - 5e^t\sin(t) \\ &= -10e^t\cos(t) + \left(15e^t\cos(t) - 5e^t\sin(t)\right) \\ &= -2x + 5y \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= 3e^t\cos(t) - e^t\sin(t) - 3e^t\sin(t) - e^t\cos(t) \\ &= 2e^t\cos(t) - 4e^t\sin(t) \\ &= \left(-10e^t\cos(t)\right) + 12e^t\cos(t) - 4e^t\sin(t) \\ &= -2x + 4y. \end{aligned}$$

Solution (8.1, Problem 20)**:** Writing our Wronskian, and using the power of Mathematica, we have

$$\begin{aligned} W(t) &= \det\begin{pmatrix} 1 & e^{-4t} & 2e^{3t} \\ 6 & -2e^{-4t} & 3e^{3t} \\ -13 & -e^{-4t} & -2e^{3t} \end{pmatrix} \\ &= -4e^{-t}, \end{aligned}$$

which is zero nowhere along $(-\infty,\infty)$, so the vectors $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$ form a fundamental set of solutions.

Solution (8.1, Problem 26)**:** We start with the homogeneous equation:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{x}.$$

Finding the eigenvalues and eigenvectors, we have

$$\begin{aligned} \det\begin{pmatrix} -1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} &= \lambda^2 - 2 \\ \lambda_1 &= \sqrt{2} \\ \lambda_2 &= -\sqrt{2}, \end{aligned}$$

with associated eigenvectors of

$$\begin{aligned} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \pm\sqrt{2}\begin{pmatrix} a \\ b \end{pmatrix} \\ \mathbf{v}_1 &= \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}. \end{aligned}$$

Therefore, we have the homogeneous solution of

$$\mathbf{x}_h(t) = c_1\begin{pmatrix} -1 \\ -1-\sqrt{2} \end{pmatrix}e^{\sqrt{2}t} + c_2\begin{pmatrix} -1 \\ -1+\sqrt{2} \end{pmatrix}e^{-\sqrt{2}t}.$$

Note that the Wronskian gives

$$\begin{aligned} W(t) &= \det\begin{pmatrix} e^{\sqrt{2}t} & e^{-\sqrt{2}}t \\ \left(-1-\sqrt{2}\right)e^{\sqrt{2}t} & \left(-1+\sqrt{2}\right)\left(e^{-\sqrt{2}t}\right) \end{pmatrix} \\ &= 2\sqrt{2}, \end{aligned}$$

meaning that these solutions are linearly independent. Next, we examine the particular solution.

$$\begin{aligned} \mathbf{x}_p(t) &= \begin{pmatrix} t^2-2t+1 \\ 4t \end{pmatrix} \\ \frac{d\mathbf{x}_p}{dt} &= \begin{pmatrix} 2t-2 \\ 4 \end{pmatrix}. \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} t^2-2t+1 \\ 4t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} &= \begin{pmatrix} -t^2-2t-1+t^2+4t-1 \\ -t^2+2t-1+4t-6t+5 \end{pmatrix} \\ &= \frac{d\mathbf{x}_p}{dt}. \end{aligned}$$

Thus, this is the general solution of our equation on $(-\infty,\infty)$.

Solution (8.2, Problem 12)**:** Computing the characteristic polynomial, we get

$$p(x) = (6-\lambda)(3-\lambda)(-5-\lambda),$$

giving

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -5 \\ \lambda_3 &= 6. \end{aligned}$$

Plugging these into

$$\begin{pmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda_i\begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

we get

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_3 &= \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}e^{-5t} + c_3\begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}e^{6t}.$$

Solution (8.2, Problem 16)**:** Using the power of Mathematica, we find the eigenvalues of $\{5.05452, 4.09561, -2.92362, 2.02882, -0.155338\}$, with corresponding eigenvectors yielding the general solution of

$$
\begin{aligned}
\mathbf{x}(t) &= c_1 e^{5.05452t}\begin{pmatrix} -0.0312209 \\ -0.949058 \\ -0.239535 \\ -0.195825 \\ -0.0508861 \end{pmatrix} \\
&+ c_2 e^{4.09561t}\begin{pmatrix} -0.280232 \\ -0.836611 \\ -0.275304 \\ 0.176045 \\ 0.338775 \end{pmatrix} \\
&+ c_3 e^{-2.92362t}\begin{pmatrix} 0.262219 \\ = 0.162664 \\ -0.826218 \\ -0.346439 \\ 0.31957 \end{pmatrix} \\
&+ c_4 e^{2.02882t}\begin{pmatrix} -0.313235 \\ -0.64181 \\ -0.31754 \\ -0.173787 \\ 0.599018 \end{pmatrix} \\
&+ c_5 e^{-0.155338t}\begin{pmatrix} -0.301294 \\ 0.466599 \\ 0.222136 \\ 0.0534311 \\ -0.799567 \end{pmatrix}.
\end{aligned}
$$

Solution (8.2, Problem 24)**:** We start by finding the eigenvalues of the matrix of the equation

$$
\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\mathbf{x}.
$$

Using computational assistance, we find eigenvalues of $\lambda_1 = 8$, $\lambda_2 = -1$, and $\lambda_3 = -1$. By solving the eigenvector equation for $\lambda_1 = 8$, we obtain the eigenvector of

$$
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.
$$

Meanwhile, the eigenvector equation for λ_2, λ_3 gives eigenvectors of

$$
\begin{aligned}
\mathbf{v}_2 &= \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \\
\mathbf{v}_3 &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.
\end{aligned}
$$

Therefore, we get the general solution of

$$
\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{8t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.
$$

Solution (8.2, Problem 28)**:** The matrix in the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}\mathbf{x}$$

is in Jordan canonical form, meaning that we know that its eigenvalues are $4, 4, 4$, and the generalized eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

subject to the chain $\mathbf{v}_1 \to \mathbf{w}_2 \to \mathbf{w}_3$. Thus, our general solution is

$$\mathbf{x}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t}\left(t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + c_3 e^{4t}\left(\frac{t^2}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Solution (8.2, Problem 40)**:** We start by taking the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{pmatrix}\mathbf{x},$$

and finding the eigenvalues. With computational assistance, we find the characteristic polynomial of

$$0 = \lambda^3 + \lambda^2 - \lambda + 15.$$

This has roots of

$$\lambda = 3, 1 \pm 2i.$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$
$$\mathbf{v}_{2,3} = \begin{pmatrix} 1 \\ -9/2 \\ 1 \end{pmatrix} \pm i\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we have

$$\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + e^t\left(c_2 \cos\left(\frac{1}{2}t\right) + c_3 \sin\left(\frac{1}{2}t\right)\right)\begin{pmatrix} 1 \\ -9/2 \\ 1 \end{pmatrix} + e^t\left(c_3 \cos\left(\frac{1}{2}t\right) - c_2 \sin\left(\frac{1}{2}t\right)\right)\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

Solution (8.2, Problem 42)**:** We start by taking the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{pmatrix}\mathbf{x}$$

Solution (8.2, Problem 46)**:** Solving the eigenvectors and eigenvalues for A, we have

$$\det(A - \lambda I) = \lambda^2 - 10\lambda + 29,$$

giving eigenvalues of $5 \pm 2i$ and corresponding eigenvectors of

$$\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Therefore, the general solution is

$$\mathbf{x} = e^{5t} \begin{pmatrix} a\cos(2t) + b\sin(2t) \\ -2a\sin(2t) + 2b\sin(2t) \end{pmatrix}.$$

Solving the initial condition, we have $a = -2$ and $b = 4$. Therefore we have the full solution of

$$\mathbf{x} = e^{5t} \begin{pmatrix} -2\cos(2t) + 4\sin(2t) \\ -4\sin(2t) + 8\cos(2t) \end{pmatrix}.$$

Extra Problems

Solution (Extra Problem 1)**:** To find the linearly independent solutions for $\frac{dx}{dt} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -4 & & \\ 2 & 5 & & \\ & & 5 & 2 \\ & & -4 & 1 \end{pmatrix},$$

we find the eigenvectors and eigenvalues of the two block matrices. Using the power of computational linear algebra, we get

$$\lambda_{1,2} = 3 \pm 2i$$
$$\mathbf{v}_{1,2} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\lambda_{3,4} = 3 \pm 2i$$
$$\mathbf{v}_{3,4} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus, we get the general solution of

$$\mathbf{x} = e^{3t} \begin{pmatrix} -\cos(2t) - \sin(2t) \\ \cos(2t) \\ -\cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix}.$$

Solution (Extra Problem 2)**:**

(i) We have the generalized eigenvector chain $\mathbf{v}_1 \to \mathbf{w}_1$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{w}_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

(ii) We have the generalized eigenvector chain $\mathbf{v}_3 \rightarrow \mathbf{w}_1$, where

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\mathbf{w}_1 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

(iii) We have the generalized eigenvector chain $\mathbf{v}_1 \rightarrow \mathbf{w}_1 \rightarrow \mathbf{w}_2$, where

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\mathbf{w}_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\mathbf{w}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

(iv) We have the generalized eigenvector chain $\mathbf{v}_1 \rightarrow \mathbf{w}_1 \rightarrow \mathbf{w}_2 \rightarrow \mathbf{w}_3$, where

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\mathbf{w}_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\mathbf{w}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\mathbf{w}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

An addition to the chain of generalized eigenvectors adds a 1 to the Jordan block.