Solution (29.5)**:**

(a) We have

$$\left(\vec{w}\cdot\overleftrightarrow{T}\right)_k = \sum_{i,j,k} w_i T_{jk}\delta_{ij}$$
$$= \sum_{i,k} w_i T_{ik},$$

which is a first-rank tensor.

(b) Since $\vec{w}\cdot\overleftrightarrow{T}$ is a first-rank tensor, and we are taking the dot product of two first rank tensors the expression $\vec{w}\cdot\overleftrightarrow{T}\cdot\vec{v}$ is a scalar (or rank zero tensor).

(c) We have

$$\overleftrightarrow{T}\cdot\overleftrightarrow{U} = \left(\sum_{i,j} T_{ij}e_i\otimes e_j\right)\cdot\left(\sum_{k,\ell} U_{k\ell}e_k\otimes e_\ell\right)$$
$$= \sum_{i,j,k,\ell} T_{ij}U_{k\ell}(e_k\cdot e_i)(e_j\cdot e_\ell),$$

which is a scalar.

(d) The expression $\overleftrightarrow{T}\vec{v}$ expresses the operation of

$$\overleftrightarrow{T} = \sum_{i,j} T_{ij}e_i\otimes e_j$$

on

$$\vec{v} = \sum_i v_i e_i,$$

meaning that $\overleftrightarrow{T}\vec{v}$ is a vector.

(e) The expression $\overleftrightarrow{T}\overleftrightarrow{U}$ is a composition of two linear maps on $V\otimes V$, so it is a rank 2 tensor (or another linear map on $V\otimes V$).

Solution (29.7)**:**

Solution (29.10)**:**

Solution (29.11)**:**

Solution (29.12)**:**

Solution (29.14)**:**

Solution (29.23)**:**

Solution (29.24)**:**

Solution (29.25)**:**