Problem (Problem 1)**:** Prove that smooth homotopy and smooth isotopy are equivalence relations.

Solution: If $f\colon M \to N$ is a smooth map, then we can define a smooth homotopy $F\colon M \times [0,1] \to N$ by taking $F(\cdot, t) = f$. If f is a diffeomorphism, then this is a smooth isotopy. Thus, this relation is reflexive.

The relation is symmetric since, if f and g are smoothly homotopic (isotopic), then $F^*\colon M \times [0,1] \to N$, given by $F^*(\cdot, t) = F(\cdot, 1 - t)$ is a composition of smooth maps, hence smooth.

The relation is transitive since, if $F\colon M \times [0,1] \to N$ is a homotopy (isotopy) from f to g, and $G\colon M \times [0,1] \to N$ is a homotopy (isotopy) from g to h, then we may find a homotopy from f to h by taking

$$H(\cdot, t) = \begin{cases} F(\cdot, 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ G(\cdot, 2t - 1) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

This is a smooth map since the derivatives of all orders for F and G agree at $t = \frac{1}{2}$.

Problem (Problem 2)**:** Prove that if M is connected, then for all pairs p and q of points on M, there is a diffeomorphism f of M such that $f(p) = q$ and f is isotopic to the identity.

Solution: We know that the diffeomorphism group, $\operatorname{diff}(M)$, is transitive whenever M is connected, so there is a diffeomorphism $f\colon M \to M$ such that $f(p) = q$. Now, if p and q are in the same Euclidean chart, (U, φ), where $\varphi(p) = 0$ and $\varphi(q) = a x_1$, then we may find the desired isotopy to the identity by taking

$$F\colon M \times [0,1] \to M$$

to be given by

$$F(\cdot, t) = f_t,$$

where f_t is a diffeomorphism such that $\varphi \circ f_t(p) = a t x_1$.

Now, if p and q are not in the same chart, then since M is connected, there is a finite chain of k intersecting Euclidean charts that we may compose with each other such that we get our diffeomorphism between p and q. Dividing $[0,1]$ into intervals of length $1/k$, we may then find isotopies from the identity to the diffeomorphism mapping p to the ℓ-th intersection point along in this chain as we showed for the case where both p and q are in the same chart. By chaining these isotopies together, we get the isotopy between f and the identity.

Problem (Problem 3)**:** Suppose M is compact and has no boundary, and that M and N have the same dimension. Let f and g be homotopic maps from M to N. Suppose $p \in N$ is a regular value for both f and g. Prove that $\left|f^{-1}(p)\right| = \left|g^{-1}(p)\right|$ modulo 2.

Solution: Let $F\colon M \times [0,1] \to N$ be a smooth homotopy with $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. If $p \in N$ is a regular value for F (in addition to one for f and g), it follows that $F^{-1}(p)$ is a 1-manifold subset of $M \times [0,1]$, where $F^{-1}(p) \cap (M \times \{0\}) = f^{-1}(p) \times \{0\}$, and $F^{-1}(p) \cap (M \times \{1\}) = g^{-1}(p) \times \{1\}$. Since the boundary of $M \times [0,1]$ must contain an even number of points (as every 1-submanifold with boundary of $M \times [0,1]$ must have both of its boundary points touch the boundary of $M \times [0,1]$, which are 0 and 1), we must have $\left|f^{-1}(p)\right| + \left|g^{-1}(p)\right| \equiv 0$ modulo 2, so that $\left|f^{-1}(p)\right| = \left|g^{-1}(p)\right|$.

Suppose y is not a regular value for F. Since $M \times [0,1]$ is compact, and F is continuous, it follows that, by Sard's Theorem, y is part of a closed, measure-zero subset of N. In particular, for any neighborhood of y, there is a regular value for F within this neighborhood. Next, we observe that, for a sufficiently small open neighborhood V of y, the number of regular points mapping to y does not change, as the map $x \mapsto \left|F^{-1}(x)\right|$ is continuous and discrete-valued (for the open subset of regular values for F). Thus, on V, we may find $q \in V$ such that $\left|F^{-1}(q)\right|$ is constant, and thus $\left|f^{-1}(y)\right| + \left|g^{-1}(y)\right|$ is even, hence are equal to each other modulo 2.

Problem (Problem 4)**:** Prove that for M, N, f as in the previous exercise, $\left|f^{-1}(p)\right| \equiv \left|f^{-1}(q)\right|$ modulo 2 for all regular values p and q of f, using the previous exercises.

Solution: There is a diffeomorphism $\varphi\colon N \to N$ of N such that $\varphi(p) = q$ and φ is isotopic to the identity, as shown in the solution to Problem 2. In particular, this means that $\varphi \circ f\colon M \to N$ is homotopic to $f\colon M \to N$, meaning that $\left|f^{-1}(p)\right| = \left|(\varphi \circ f)^{-1}(q)\right| = \left|f^{-1}(q)\right|$, with the latter equality following from Problem 3.