Chapter 5 Theorems

Definitions: A **walk** in a graph is a list of vertices such that any two consecutive vertices are adjacent to each other.

A **trail** is a walk that does not repeat edges, but can repeat vertices.

A **path** is a trail that does not repeat vertices.

A **closed walk** is a walk that ends at the same vertex that it started at. A walk is open if it is not closed. A **circuit** is a closed trail, and a **cycle** is a closed path.

An **Eulerian circuit** is a circuit that traverses all the edges of a graph. A graph is Eulerian if it contains an Eulerian circuit. An **Eulerian trail** traverses all the edges of a graph, and does not return to the same vertex it started from.

Theorem 5.1: A connected graph G is Eulerian if and only if every vertex of G has even degree.

Corollary 5.2: A connected graph G contains an (open) Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, every Eulerian trail of G begins at one of these odd vertices and ends at the other.

Chapter 6 Theorems

Definitions: A **Hamiltonian cycle** is a cycle that contains all the vertices of a graph. A graph is Hamiltonian if it contains a Hamiltonian cycle.

Theorem 6.2 (Dirac's Theorem): If G is a graph of order $n \geq 3$ such that $d(v) \geq n/2$ for all vertices of G, then G is Hamiltonian.

Theorem 6.3 (Ore's Theorem): If G is a graph of order $n \geq 3$ such that $d(u) + d(v) \geq n$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

Theorem 6.5: For any graph G, if there is a positive integer k such that deleting k vertices results in a graph with more than k components, then G is not Hamiltonian.

Chapter 7 Theorems

Definitions: A **matching** is a set of pairwise disjoint edges. A **perfect matching** is a matching that is incident on every vertex.

The subgraph whose edges are a perfect matching is a **1-factor**.

If G contains 1-factors $F_1, F_2, \ldots, F_k$ such that $E(G)$ is partitioned by $E(F_1), E(F_2), \ldots, E(F_k)$, then $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is a **1-factorization** of G, and G is **1-factorable**.

A **bridge** is an edge upon whose deletion the number of components in a connected graph increases.

If a graph can be decomposed into edge-disjoint Hamiltonian cycles, then the graph is **Hamiltonian-factorable**.

Theorem 7.1 (Hall's Theorem): A sequence $(C_1, \ldots, C_n)$ of n nonempty finite sets has a system of distinct representatives $(s_1, \ldots, s_i)$ where $s_i \in C_i$ if and only if for each subsequence Y, the union of the sets in Y has at least as many elements as Y.

Alternatively, if G is a bipartite graph on vertices $C \sqcup S$, where $C = \{c_1, \ldots, c_n\}$ and $S = \{s_1, \ldots, s_m\}$, then G has a C-perfect matching (a matching that contains every vertex in C) if and only if $\forall r$ where $1 \leq r \leq n$, any r vertices in C are adjacent to at least r vertices in S.

Theorem 7.7 (Petersen's Theorem): Every bridgeless 3-regular graph contains a perfect matching.

Theorem 7.10: For every even integer $n \geq 2$, K_n is 1-factorable.

Theorem 7.13: For every odd integer $n \geq 3$, K_n is Hamiltonian-factorable.