Problem (Problem 1)**:** Let I, J, K be ideals of R.

(a) Show that $(IJ)K = I(JK)$.

(b) Show that $(I + J)K = IK + JK$.

Problem (Problem 4)**:** Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and let $\iota_{S_i} : R \to S_i^{-1}R$ be the corresponding localization homomorphisms. Use the universal property of localization to show that there exists a unique ring homomorphism $\iota' \colon S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. Provide an explicit description of this ring homomorphism. Use this to show that if R is an integral domain and S an arbitrary multiplicative subset of R, then $S^{-1}R$ injects into the fraction field $K = \operatorname{frac}(R)$.

Solution: We observe that $\iota_{S_2} \colon R \to S_2^{-1}R$ maps elements of S_1 to units in $S_2^{-1}R$, as the units in $S_2^{-1}R$ are elements of the form $\frac{s}{s'}$ with $s, s' \in S_2$, so by the universal property, there is a unique ring homomorphism $\iota' \colon S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. In particular, this is the map $\left[\frac{r}{1}\right]_{S_1^{-1}R} \mapsto \left[\frac{r}{1}\right]_{S_2^{-1}R}$.

Since any arbitrary multiplicative subset $S \subseteq R$ of an integral domain is contained in $R \setminus \{0\}$, it follows that $S^{-1}R$ injects into $(R \setminus \{0\})^{-1}R =: \operatorname{frac}(R)$.