Contents

Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and C*-algebras. This section of notes will be focused on the essential results in functional analysis, starting from normed vector spaces, working our way up through C*-algebras.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

Normed Vector Spaces

Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over $\mathbb{C}$. Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

Definition (Vector Space)**.** A vector space V is a set closed under two operations

$$\begin{aligned} a &: V \times V \rightarrow V,\ (v_1, v_2) \mapsto v_1 + v_2 \\ m &: \mathbb{C} \times V \rightarrow V,\ (\lambda, v) \mapsto \lambda v. \end{aligned}$$

We refer to a as addition, and m as scalar multiplication; $(V, +)$ is an abelian ring.

Definition (Norm)**.** A norm is a function

$$\|\cdot\| : V \rightarrow \mathbb{R}^+,\ x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness: $\|v\| = 0$ if and only if $v = 0_V$.
- Triangle inequality: $\|v + w\| \leqslant \|v\| + \|w\|$.
- Absolute Homogeneity: $\|\lambda v\| = |\lambda|\,\|v\|$, for $\lambda \in \mathbb{C}$.

If a function $p : V \rightarrow \mathbb{R}^+$ satisfies the triangle inequality and absolute homogeneity, we say p is a seminorm.

We say the pair $(V, \|\cdot\|)$ is a normed vector space.

Definition (Balls and Spheres)**.** Let X be a normed vector space, $x \in X$, and $\delta > 0$. Then,

$$\begin{aligned}U(x, \delta) &= \{y \in X \mid d(x, y) < \delta\}\\ B(x, \delta) &= \{y \in X \mid d(x, y) \leqslant \delta\}\\ S(x, \delta) &= \{y \in X \mid d(x, y) = \delta\}.\end{aligned}$$

For a normed vector space, we will use the following conventions for common sets:

$$\begin{aligned}U_X &= U(0, 1)\\ B_X &= B(0, 1)\\ S_X &= S(0, 1)\\ \mathbb{D} &= U_{\mathbb{C}}\\ \mathbb{T} &= S_{\mathbb{C}}.\end{aligned}$$

Definition (Equivalent Norms)**.** Two norms on V, $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there are two constants C_1 and C_2 such that

$$\begin{aligned}\|v\|_a &\leqslant C_1 \|v\|_b\\ \|v\|_b &\leqslant C_2 \|v\|_a\end{aligned}$$

for all $v \in V$. We say $\|\cdot\|_a \sim \|\cdot\|_b$.

Examples

Example (Finite-Dimensional Vector Spaces)**.** The vector space $\mathbb{C}^n$ is with the p-norm is denoted ℓ_p^n, where for $p \in [1, \infty]$, the p-norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

In the case with $p = 2$, this gives the traditional Euclidean norm, and with $p = \infty$, this gives the sup norm:

$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|.$$

Example (A Sequence Space)**.** We let $\ell_p = \left\{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\right\}$ be the collection of sequences in $\mathbb{C}$ with finite p-norm. Here,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

In the case with $p = \infty$, this gives the sequence space ℓ_∞, which has norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Example (A Function Space)**.** We let $\ell^\infty(\Omega)$ denote the set of all bounded functions $f : \Omega \to \mathbb{C}$, equipped with the norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|.$$

If $\Omega = (\Omega, \mathcal{M}, \mu)$ is a measure space, then we let $L^\infty(\Omega)$ be the space of μ-a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

Series Convergence and Completeness

Proposition (Criteria for Banach Spaces)**:** Let X be a normed vector space. The following are equivalent:

(i) X is a Banach space.[1]

(ii) If $(x_k)_k$ is a sequence of vectors such that $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

(iii) If $(x_k)_k$ is a sequence in X such that $\|x_k\| < 2^{-k}$, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. To show (i) implies (ii), for $n > m > N$, we have

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\leqslant \sum_{k=m+1}^{n} \|x_k\| \\ &< \epsilon, \end{aligned}$$

implying that s_n is Cauchy, and thus converges since X is complete.

Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let $(x_n)_n$ be a Cauchy sequence in X. We only need construct a convergent subsequence in order to show that $(x_n)_n$ converges.

Chose $n_1 \in \mathbb{N}$ such that for $n, m \geqslant n_1$, $\|x_m - x_n\| < \frac{1}{2^2}$, and inductively define $n_j > n_{j-1}$ such that $n, m \geqslant n_j$ implies $\|x_m - x_n\| < \frac{1}{2^{j+1}}$.

Let $y_1 = x_{n_1}$, $y_j = x_{n_j} - x_{n_{j-1}}$. Then,

$$\begin{aligned} \|y_j\| &= \left\|x_{n_j} - x_{n_{j-1}}\right\| \\ &< \frac{1}{2^j}, \end{aligned}$$

so $\sum_{j=1}^{\infty} y_j$ converges by our assumption. By telescoping, we see that $\sum_{j=1}^{k} y_j = x_{n_k}$, so $(x_{n_k})_k$ converges. □

Quotient Spaces

Let X be a normed vector space. Then, for $E \subseteq X$ a subspace, there is a quotient space X/E with the projection map $\pi : X \to X/E$, $x \mapsto x + E$. We want to make X/E into a normed space — in order to do this, we use the distance function:

$$\operatorname{dist}_E(x) = \inf_{y \in E} d(x, y),$$

which is uniformly continuous. For E closed, then $\operatorname{dist}_E(x) = 0$ if and only if $x \in E$.

Proposition (Quotient Space Norm)**:** Let X be a normed vector space, and $E \subseteq X$ a subspace. Set

$$\|x + E\|_{X/E} = \operatorname{dist}_E(x).$$

Then,

(1) $\|\cdot\|_{X/E}$ is a well-defined seminorm on X/E.

[1] Complete normed vector space.

(2) If E is closed, then $\|\cdot\|_{X/E}$ is a norm on X/E.

(3) $\|x + E\|_{X/E} \leqslant \|x\|$ for all $x \in X$.

(4) If E is closed, then $\pi : X \to X/E$ is Lipschitz.

(5) If X is a Banach space and E is closed, then X/E is also a Banach space.

Proof.

(1) We will show that $\|\cdot\|_{X/E}$ is well-defined. If $x + E = x' + E$, $x' - x \in E$, so for every $y \in E$, $x' - x + y \in E$. Thus,

$$\begin{aligned} \|x - y\| &= \|x' - (x' - x + y)\| \\ &\geqslant \inf_{z \in E} \|x' - z\| \\ &= \|x' + E\|_{X/E}. \end{aligned}$$

Thus, $\|x + E\|_{X/E} \geqslant \|x' + E\|_{X/E}$, and vice versa.

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and $x \in X$. Then,

$$\begin{aligned} \|\lambda(x + E)\|_{X/E} &= \|\lambda x + E\|_{X/E} \\ &= \inf_{y \in E} \|\lambda x - y\| \\ &= |\lambda| \inf_{y \in E} \left\|x - \lambda^{-1} y\right\| \\ &= |\lambda| \inf_{y' \in E} \|x - y\| \\ &= |\lambda| \|x + E\|_{X/E} \end{aligned}$$

Given $x, x' \in X$ and a fixed $\varepsilon > 0$, we have

$$\|x + E\| + \frac{\varepsilon}{2} > \|x - y\|$$

for some $y \in E$, and

$$\|x' + E\| + \frac{\varepsilon}{2} > \|x' - y'\|$$

for some $y' \in E$. Thus,

$$\begin{aligned} \|(x + x') - (y + y')\| &\leqslant \|x - y\| + \|x' - y'\| \\ &< \varepsilon + \|x + E\| + \|x' + E\|. \end{aligned}$$

Since $y + y' \in E$, we have

$$\begin{aligned} \|(x + E) + (x' + E)\|_{X/E} &= \|x + x' + E\|_{X/E} \\ &\leqslant \|(x + x') - (y + y')\| \\ &< \varepsilon + \|x + E\|_{X/E} + \|x' + E\|_{X/E}, \end{aligned}$$

meaning

$$\|(x + E) + (x' + E)\| \leqslant \|x + E\| + \|x' + E\|.$$

(2) If E is closed, and $\|x + E\| = 0$, then $x \in E$ so $x + E = 0_{X/E}$.

(3) For $x \in X$,

$$\begin{aligned}\|x + E\|_{X/E} &= \inf_{y \in E} \|x - y\| \\ &\leqslant \|x\|.\end{aligned}$$

(4) We have

$$\begin{aligned}\|(x + E) - (x' + E)\|_{X/E} &= \|x - x' + E\|_{X/E} \\ &\leqslant \|x - x'\|.\end{aligned}$$

(5) Let X be complete and $E \subseteq X$ be closed. Let $(x_k + E)_k$ be a sequence in X/E with $\|x_k + E\| < 2^{-k}$. We want to show that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

For each k, since $\|x_k + E\| < 2^{-k}$, there exists $y_k \in E$ such that $\|x_k - y_k\| < 2^{-k}$. Since X is complete, $\sum_{k=1}^{\infty} x_k - y_k$ converges.

Let $\left(\sum_{k=1}^{n} x_k - y_k\right)_n \to x$ in X. Applying the canonical projection map, π, to both sides, we get

$$\begin{aligned}\sum_{k=1}^{n} (x_k + E) &= \sum_{k=1}^{n} \pi(x_k) \\ &= \pi\left(\sum_{k=1}^{n} (x_k - y_k)\right) \\ &\to \pi(x),\end{aligned}$$

implying that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

□

Exercise: Consider ℓ_∞ and its closed subspace c_0. If $\pi : \ell_\infty \to \ell_\infty/c_0$ denotes the canonical quotient map, with $(z_k)_k \in \ell_\infty$, show that

$$\|(z_k)_k + c_0\| = \limsup_{k\to\infty} |z_k|$$

Solution. By the definition of the quotient norm, we have

$$\begin{aligned}\|(z_k)_k + c_0\|_{\ell_\infty/c_0} &= \inf_{(a_k)_k \in c_0} \|(z_k)_k - (a_k)_k\| \\ &= \inf_{(a_k)_k \in c_0} \sup_{k \in \mathbb{N}} |z_k - a_k| \\ &= \limsup_{k\to\infty} |z_k|.\end{aligned}$$

Bounded Linear Operators

Definition (Continuous Functions)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is called Lipschitz if there is a constant $C > 0$ such that

$$d_Y(f(x), f(x')) \leqslant C d_x(x, x')$$

for all $x, x' \in X$.

If $C \leqslant 1$, a Lipschitz map is known as a contraction.

If

$$d_Y(f(x), f(x')) = d_X(x, x')$$

for all $x, x' \in X$, then f is known as an isometry.

Proposition (Categorization of Continuous Linear Maps)**:** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map. The following are equivalent:

(i) T is continuous at 0.

(ii) T is continuous.

(iii) T is uniformly continuous.

(iv) T is Lipschitz.

(v) There exists a constant $C > 0$ such that $\|T(x)\| \leqslant C\|x\|$ for all $x \in X$.

Definition (Bounded Linear Operator)**.** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map.

(1) T is bounded if $T(B_X)$ is bounded in Y. Equivalently, T is bounded if and only if

$$\sup_{x \in B_X} \|T(x)\| < \infty,$$

or that $\exists r > 0$ such that $T(B_X) \subseteq B_Y(0, r)$.

(2) The operator norm of T is the value

$$\|T\|_{op} = \sup_{x \in B_X} \|T(x)\|.$$

Lemma: Let $T : X \to Y$ be a linear map between normed vector spaces. Then,

$$\|T\|_{op} = \sup_{x \in S_X} \|T(x)\|$$

and for all $x \in X$,

$$\|T(x)\| \leqslant \|T\|_{op} \|x\|.$$

Lemma: Let $T : X \to Y$ be a bounded linear map between normed vector spaces. Then, for any $x \in X$ and $r > 0$,

$$r\|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|$$

Proof. Let $C = \sup_{y \in B(x,r)} \|T(y)\|$. If $z \in B(0, r)$, then $z + x, z - x \in B(x, r)$, meaning

$$2T(z) = T(z + x) + T(z - x),$$

so by the triangle inequality, we get

$$\begin{aligned} 2\|T(z)\| &\leqslant \|T(z + x)\| + \|T(z - x)\| \\ &\leqslant 2\max\{\|T(z + x)\|, \|T(z - x)\|\} \\ &\leqslant 2C. \end{aligned}$$

Thus,

$$\|T(z)\| \leqslant \sup_{y \in B(x,r)} \|T(y)\|,$$

meaning

$$r\|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|.$$

□