This is a notes document regarding essential problem-solving methods for the Analysis qualifier.

Real Analysis

August 2019 Qualifier

Problem 1

(a) Recall that the Cantor set $\mathcal{C}$ is defined to consist of all $x \in [0,1]$ such that x only contains 0 and 2 in the ternary expansion of x. Writing $a \in [0,2]$ as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where $a_k \in \{0,1,2\}$, we may then find a_k at each ternary expansion slot for k as follows:

- if $a_k = 0$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = c_k = 0$
- if $a_k = 2$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = 2$ and $c_k = 0$ or vice versa.
- if $a_k = 1$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_{k+1} = c_{k+1} = 2$.

Therefore, since every digit of every ternary expansion in $[0,2]$ can be obtained from $\mathcal{C}$, we see that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) We may set B to be the union of all integer translates of $\mathcal{C}$, and set $A = \mathcal{C}$. This yields closed subsets of $\mathbb{R}$ with Lebesgue measure zero that sum to $\mathbb{R}$.

Problem 2

Consider the sequence of functions

$$f_n(x) = n\mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on $[0,1]$. This sequence is pointwise convergent everywhere to zero, as $f_n(0) = 0$ and the Archimedean property give that for any $x \in (0,1]$, there is some n large enough that gives $\frac{1}{n} < x$. Furthermore, we see that

$$\begin{aligned}\int f_n \, d\mu &= n\left(\frac{1}{n} - \frac{1}{n+1}\right)\\ &= \frac{1}{n+1}\\ &\to 0.\end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned}\int \Phi \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1}\\ &\to \infty.\end{aligned}$$

Problem 4

Suppose toward contradiction that both f and $1/f$ are in $L_1(\mathbb{R})$. Then, from Hölder's Inequality, we have

$$\infty = \int 1 \, d\mu$$

$$
\begin{aligned}
&\leqslant \left(\int f\,d\mu\right)^{1/2}\left(\int \frac{1}{f}\,d\mu\right)^{1/2}\\
&< \infty,
\end{aligned}
$$

which is a contradiction.

Problem 5

(a) Let $f \in L_2([-1,1])$. We may find $g \in C([-1,1])$ such that $\|f-g\|_{L_2} < \varepsilon/2$. Similarly, we may find a polynomial p such that $\|g-p\|_u < \varepsilon/4$, meaning that $|p(x)-g(x)| < \varepsilon/4$ for all $x \in [-1,1]$. This yields

$$
\begin{aligned}
\|p-g\|_{L_2} &= \left(\int_{-1}^{1} |p(x)-g(x)|^2\,dx\right)^{1/2}\\
&< \left(\int_{-1}^{1} \left(\frac{\varepsilon}{4}\right)^2 dx\right)^{1/2}\\
&= \left(\frac{\varepsilon^2}{8}\right)^{1/2}\\
&< \frac{\varepsilon}{2},
\end{aligned}
$$

so $\|f-p\|_{L_2} < \varepsilon$, meaning that the closed linear span of the monomials is dense in L_2, and the Legendre polynomials form an orthonormal system.

(b) We see that at every step in evaluating the expression

$$
L_n(x) = c_n \frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{$*$}
$$

the degree of the polynomial increases by 1, so each $L_n(x)$ has degree n. To verify that the polynomials generated from $(*)$ are orthogonal to each other, we let $n > m$ without loss of generality, and use integration by parts to obtain

$$
\begin{aligned}
\langle L_n, L_m\rangle &= \int_{-1}^{1} \left(\frac{d^n}{dx^n}\left(x^2-1\right)^n\right)\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right) dx\\
&= \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n \frac{d^m}{dx^m}\left(x^2-1\right)^m\Big|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n \frac{d^{m+1}}{dx^{m+1}}\left(x^2-1\right)^m dx\\
&\vdots\\
&= (-1)^n \int_{-1}^{1} \frac{d^{m+n}}{dx^{m+n}}\left(x^2-1\right)^m dx\\
&= (-1)^n \int_{-1}^{1} \frac{d^n}{dx^n}\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right) dx\\
&= (-1)^n \int \frac{d^n}{dx^n} L_m(x)\,dx\\
&= 0,
\end{aligned}
$$

seeing as we are taking n derivatives of a degree $m < n$ polynomial.