Problem:

(a) Show that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$, in which it defines an analytic function, which we denote e^z.

(b) With this as the definition of e^z, prove that $e^z e^w = e^{z+w}$.

(c) Show that for $\theta \in \mathbb{R}$, we have that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, where $\cos(\theta)$ and $\sin(\theta)$ are defined via their usual power series representations.

Solution:

(a) To compute

$$\rho = \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n},$$

we take ordinary natural logarithms and use the fact that logarithms are increasing functions to find that

$$\begin{aligned} \ln(\rho) &= \limsup_{n\to\infty} \left(-n \sum_{k=1}^{n} \ln(k)\right) \\ &= -\infty, \end{aligned}$$

meaning that $\rho = 0$, or that $R = \frac{1}{\rho}$ is infinite.

(b) Computing $e^z e^w$, we get

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{\ell=0}^{\infty} \frac{w^k}{k!}\right) &= \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \frac{1}{(\ell-k)!} \frac{1}{k!} w^k z^{\ell-k} \\ &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k=0}^{\ell} \frac{1}{k!(\ell-k)!} w^k z^{\ell-k} \\ &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (z+w)^{\ell} \\ &= e^{z+w}. \end{aligned}$$

(c) Computing $e^{i\theta}$ by direct substitution, we find that

$$\begin{aligned} e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\ &= \sum_{k \text{ even}} \frac{(-1)^{(k/2)}\theta^k}{k!} + i \sum_{k \text{ odd}} \frac{(-1)^{(k-1)/2}\theta^k}{k!} \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

Problem: Let $U \subseteq \mathbb{C}$ be an open set, $f\colon U \to \mathbb{C}$ an analytic function. Since f is analytic, given $z_0 \in U$, there is $r > 0$ and a sequence $(a_n)_n$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for all $z \in U(z_0, r)$.

Suppose there exists $R > r$ such that $U(z_0, R) \subseteq U$ and $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence at least R. Show that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for all $z \in U(z_0, R)$.

Solution: On the connected open set $V = U(z_0, R)$, define

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Observe that $f|_V$ and g agree on the open subset $U(z_0, r) \subseteq U(z_0, R)$. By the identity theorem, this means that $f = g$ on $U(z_0, R)$.

Problem: Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be an analytic function.

(a) Suppose f is nonconstant, $z_0 \in U$. Show that there exists some $r > 0$ for which $U(z_0, r) \subseteq U$, a positive integer $k \in \mathbb{N}$, an analytic function $g\colon U(z_0, r) \to \mathbb{C}$, and a nonconstant $\lambda \in \mathbb{C} \setminus \{0\}$ such that for $z \in U(z_0, r)$,

$$f(z) = f(z_0) + \lambda(z - z_0)^k + (z - z_0)^{k+1} g(z).$$

(b) Suppose that f is nonconstant, and $z_0 \in U$ is such that $f(z_0) \neq 0$. Show that there exists some $s > 0$ such that $U(z_0, s) \subseteq U$, and $w_1, w_2 \in U(z_0, s)$ such that $|f(w_1)| > |f(z_0)| > |f(w_2)|$.

(c) Show that if $|f|$ is constant, then f is constant.

Solution:

(a) Since f is analytic, we may find $r > 0$ and a sequence $(a_n)_n$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Observe that $f(z_0) = a_0$, so

$$= f(z_0) + \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Next, we find the minimum value of n such that $a_n \neq 0$, which we define to be k. Such a value must exist since f is a nonconstant function. This gives

$$= f(z_0) + a_k (z - z_0)^k + \sum_{n=k+1}^{\infty} a_n (z - z_0)^n.$$

Finally, by reindexing the sum and factoring out $(z - z_0)^{k+1}$, we get

$$= f(z_0) + a_k (z - z_0)^k + (z - z_0)^{k+1} \sum_{n=0}^{\infty} a_{n+k+1} (z - z_0)^n.$$

Define $g(z)$ to be equal to the sum, and define $\lambda = a_k$. Notice that since the radius of convergence of a power series is a limiting case, g and f have the same radius of convergence. This gives

$$= f(z_0) + \lambda(z - z_0)^k + (z - z_0)^{k+1} g(z).$$

(b) Let f be a nonconstant analytic function with $f(z_0) \neq 0$. Since f is nonconstant, we see that λ in the previous problem is nonzero, meaning that $|\lambda|$ is nonzero, in addition to $|f(z_0)|$.

Problem (Problem 6)**:**

(a) For $a \in \mathbb{D}$, define $f_a(z) = \frac{z-a}{1-\overline{a}z}$. Prove that f_a is a bijection from $\mathbb{D}$ to $\mathbb{D}$.

(b) For $a_1, a_2 \in \mathbb{D}$, prove that there is a holomorphic bijection $f\colon \mathbb{D} \to \mathbb{D}$ satisfying $f(a_1) = a_2$.

Solution:

(a) We will show that f_a is a bijection from $\mathbb{D}$ to $\mathbb{D}$ by showing that f_a is defined for all $z \in \mathbb{D}$, that if $z \in \mathbb{D}$, then $f_a(z) \in \mathbb{D}$, then by showing that f_a admits an inverse. First, we observe that f_a is defined so long as $1 - \overline{a}z \neq 0$, meaning that f_a is undefined if

$$\begin{aligned} 1 - \overline{a}z &= 0 \\ z &= \frac{1}{\overline{a}} \\ &= \frac{a}{|a|^2} \\ &= \frac{1}{|a|}\left(\operatorname{sgn}(a)\right), \end{aligned}$$

which necessarily has modulus greater than 1, as $|a| < 1$ and $\operatorname{sgn}(a) = 1$ if $a \neq 0$. Next, we see that $f_a(z)$ is a Möbius transformation that is uniquely determined by

$$\begin{aligned} a &\mapsto 0 \\ 0 &\mapsto -a \\ -a &\mapsto \frac{-2a}{1 + |a|^2}, \end{aligned}$$

all of which stay within the unit disk (for $a \neq 0$ and $a \in \mathbb{D}$). Finally, observe that by taking

$$w = \frac{z - a}{1 - \overline{a}z}$$

and solving for w, we obtain

$$z = \frac{w + a}{1 + \overline{a}w}.$$

This is a left and right inverse, as

$$\begin{aligned} f_a^{-1}(f_a(z)) &= \frac{\frac{z-a}{1-\overline{a}z} + a}{1 + \overline{a}\frac{z-a}{1-\overline{a}z}} \\ &= z, \end{aligned}$$

and

$$\begin{aligned} f_a\left(f_a^{-1}(w)\right) &= \frac{\frac{w+a}{1+\overline{a}w} - a}{1 - \overline{a}\frac{w+a}{1+\overline{a}w}} \\ &= w. \end{aligned}$$

Thus, f is a bijection from $\mathbb{D}$ to $\mathbb{D}$.

(b) Considering the f_a of the previous example, we observe that f_a is holomorphic, as it is Möbius transformation that is undefined at $\frac{1}{|a|}\operatorname{sgn}(a)$, which is outside $\mathbb{D}$. By using the Möbius transformation characterization from earlier, we observe that the composition

$$f = f_{a_2}^{-1} \circ f_{a_1}$$

is holomorphic (as it is a composition of Möbius transformations) and maps a_1 to a_2.