Problem (Problem 1)**:** Let F be a field, and for $n \geqslant 1$, let $\mathrm{Mat}_n(F)$ be the set of $n \times n$ matrices with entries in F.

(a) Show that $GL_n(F) := \{x \in \mathrm{Mat}_n(F) \mid \det(x) \neq 0\}$ is a group under matrix multiplication.

(b) Show that $SL_n(F) := \{x \in \mathrm{Mat}_n(F) \mid \det(x) = 1\}$ is a normal subgroup of $GL_n(F)$, and identify the quotient $GL_n(F)/SL_n(F)$.

Solution:

(a) We see that if $a, b \in GL_n(F)$, then since $\det(a) \neq 0$, the properties of the determinant yield $0 \neq \det(a)^{-1} = \det\left(a^{-1}\right)$, meaning that $a^{-1} \in GL_n(F)$, and $0 \neq \det(a)\det(b) = \det(ab)$, meaning that $ab \in GL_n(F)$, since fields have no zero-divisors.

(b) If $a \in SL_n(F)$, then for any $x \in GL_n(F)$, we have

$$\begin{aligned} \det\left(xax^{-1}\right) &= \det(x)\det(a)\det\left(x^{-1}\right) \\ &= \det(x)\det(a)\det(x)^{-1} \\ &= \det(a) \\ &= 1, \end{aligned}$$

meaning that $xax^{-1} \in SL_n(F)$ for any $x \in GL_n(F)$. In particular, we note that the map

$$\det\colon GL_n(F) \to F \setminus \{0\},$$

given by $a \mapsto \det(a)$ is a group homomorphism, as has been established by the properties of the determinant, and it is surjective, as the matrix $\mathrm{diag}(a, 1_F, \dots, 1_F)$ has determinant a, for any $a \in F$. Finally, we see that $\det^{-1}(\{1_F\})$ is $SL_n(F)$, meaning that by the First Isomorphism Theorem, $GL_n(F)/SL_n(F) \cong F \setminus \{0\}$.

Problem (Problem 3)**:** Let G be a group, and let $H_1, H_2 \leqslant G$ be subgroups.

(a) Show that if H_1 and H_2 are finite, with $\gcd(|H_1|, |H_2|) = 1$, then $H_1 \cap H_2 = \{e\}$.

(b) Show that if both H_1 and H_2 are normal subgroups, and $H_1 \cap H_2 = \{e\}$, then $h_1h_2 = h_2h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$.

Solution:

(a) Let $g \in H_1 \cap H_2$. Then, we see that $\mathrm{ord}(g) \mid |H_1|$ and $\mathrm{ord}(g) \mid |H_2|$; yet, since $\gcd(|H_1|, |H_2|) = 1$, this means that $\mathrm{ord}(g) = 1$, meaning $g = \{e\}$.

(b) If H_1 and H_2 are normal subgroups, then for $h_1 \in H_1$ and $h_2 \in H_2$, we consider the commutator $c = h_1h_2h_1^{-1}h_2^{-1}$. Notice that by grouping as $\left(h_1h_2h_1^{-1}\right)h_2^{-1}$, since H_2 is a normal subgroup, $c \in H_2$. Similarly, by grouping as $h_1\left(h_2h_1^{-1}h_2^{-1}\right)$, since H_1 is normal, we see that $c \in H_1$. Since $H_1 \cap H_2 = \{e\}$, we see that $h_1h_2h_1^{-1}h_2^{-1} = e$, so $h_1h_2 = h_2h_1$.

Problem (Problem 8)**:** Construct an explicit isomorphism between the group $(\mathbb{R}_{>0}, \cdot)$ of strictly positive real numbers under multiplication and the group $(\mathbb{R}, +)$ of all real numbers under addition.

On the other hand, show that the group $(\mathbb{Q}_{>0}, \cdot)$ of strictly positive rational numbers under multiplication is not isomorphic to the group $(\mathbb{Q}, +)$ of all rational numbers under addition.

Solution: To see an isomorphism between $(\mathbb{R}_{>0}, \cdot)$ and $(\mathbb{R}, +)$, we define the map $r \mapsto \ln(r)$. Notice that by the definition of the logarithm, $\ln(pr) = \ln(p) + \ln(r)$ (so ln preserves their respective group structures), and that ln admits an inverse, exp, so we have an isomorphism between $(\mathbb{R}_{>0}, \cdot)$ and $(\mathbb{R}, +)$.