Problem 1

Recall that a subset $U \subseteq \mathbb{R}$ is **open** if

$$(\forall x \in U)(\exists \varepsilon > 0) \ni V_\varepsilon(x) \subseteq U.$$

Prove that a mapping $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U) \subseteq \mathbb{R}$ is open for every open $U \subseteq \mathbb{R}$.

($\Rightarrow$) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall c \in \mathbb{R}$, $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$. Let U be an open set such that $f(c) \in U$. Then, $\exists \varepsilon_0$ such that $V_{\varepsilon_0}(f(c)) \subseteq U$. So, $\exists \delta_0$ such that $V_{\delta_0}(c) \subseteq f^{-1}(V_{\varepsilon_0}(f(c))) \subseteq f^{-1}(U)$. So, $f^{-1}(U)$ is open.

($\Leftarrow$) Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is open in $\mathbb{R}$.

Since U is open in $\mathbb{R}$, it must be the case that for every $f(c) \in U$, $\exists \varepsilon > 0$ such that $V_\varepsilon(f(c)) \subseteq U$. Since $f^{-1}(U) = \{c \mid f(c) \in U\}$, it must be the case that $\exists \delta > 0$ such that $V_\delta(c) \subseteq f^{-1}(U)$.

Therefore, $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$ for sufficiently small δ. Thus, $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Problem 2

Let $f, g : D \to \mathbb{R}$ be continuous. Show that $f \cdot g$ is continuous.

Since $f : D \to \mathbb{R}$ is continuous, then $\forall (x_n)_n, c \in D$ such that $(x_n)_n \to c$, $(f(x_n))_n \to f(c)$. Similarly, since $g : D \to \mathbb{R}$ is continuous, then $\forall (x_n)_n, c \in D$ such that $(x_n)_n \to c$, $(g(x_n))_n \to g(c)$.

So, $\forall (x_n)_n, c \in D$ such that $(x_n)_n \to c$, $(f(x_n)g(x_n))_n \to f(c)g(c)$ by the properties of sequences. Thus, $f \cdot g$ is continuous.

Problem 3

Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be continuous mappings with $\operatorname{Ran}(f) \subseteq E$. Show that $g \circ f$ is continuous.

Every sequence $(x_n)_n \in D$ with $(x_n)_n \to c \in D$ has $(f(x_n))_n \to f(c)$. Since $(f(x_n))_n \in E$ and $f(c) \in E$, it must be the case that $(g(f(x_n)))_n \to g(f(c))$. So, $g \circ f : D \to \mathbb{R}$ is continuous.

Problem 4

Show that the following functions are Lipschitz:

(i) $f : [-M, M] \to \mathbb{R}$ given by $f(x) = x^2$

(ii) $g : [1, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$

(iii) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sqrt{x^2 + 4}$