Solution (40.7)**:** We have

$$\begin{aligned}\langle \psi \mid \mathcal{L}\phi\rangle &= \int_a^b \overline{\psi(x)}\left(\alpha(x)\frac{d^2\phi}{dx^2} + \beta(x)\frac{d\phi}{dx} + \gamma(x)\phi(x)\right)\,dx\\ &= \int_a^b \overline{\psi(x)}\alpha(x)\frac{d^2\phi}{dx^2}\,dx + \int_a^b \overline{\psi(x)}\beta(x)\frac{d\phi}{dx}\,dx + \int_a^b \overline{\psi(x)}\gamma(x)\phi(x)\,dx.\end{aligned}$$

We evaluate these integrals separately. Assuming that α,β,γ are real-valued, we have

$$\begin{aligned}\int_a^b \overline{\psi(x)}\alpha(x)\frac{d^2\phi}{dx^2}\,dx &= \frac{d\phi}{dx}\overline{\psi(x)}\alpha(x)\bigg|_a^b - \int_a^b\left(\frac{d\alpha}{dx}\overline{\psi(x)} + \overline{\frac{d\psi}{dx}}\alpha(x)\right)\frac{d\phi}{dx}\,dx\\ &= \underbrace{\left(\frac{d\phi}{dx}\alpha(x)\overline{\psi(x)} - \phi(x)\left(\frac{d\alpha}{dx}\overline{\psi(x)} + \overline{\frac{d\psi}{dx}}\alpha(x)\right)\right)\bigg|_a^b}_{S_1}\\ &\quad + \int_a^b \overline{\left(\alpha(x)\frac{d^2}{dx^2} + 2\frac{d\alpha}{dx}\frac{d}{dx} + \frac{d^2\alpha}{dx^2}\right)\psi(x)}\phi(x)\,dx.\\ \int_a^b \overline{\psi(x)}\beta(x)\frac{d\phi}{dx}\,dx &= \underbrace{\left(\phi(x)\beta(x)\overline{\psi(x)}\right)\bigg|_a^b}_{S_2} - \int_a^b \phi(x)\left(\frac{d\beta}{dx}\overline{\psi(x)} + \overline{\frac{d\psi}{dx}}\beta(x)\right)\,dx.\end{aligned}$$

Thus, we have

$$\int_a^b \overline{\psi(x)}(\mathcal{L}\phi)(x)\,dx = S_1 + S_2 + \int_a^b \overline{\left(\alpha(x)\frac{d^2}{dx^2} + \left(2\frac{d\alpha}{dx} - \beta(x)\right)\frac{d}{dx} + \left(\frac{d^2\alpha}{dx^2} - \frac{d\beta}{dx} + \gamma(x)\right)\right)\psi(x)}\phi(x)\,dx.$$

Solution (40.23)**:**

(a) We have $p(x) = 1$, and

$$\begin{aligned}\int_0^a \overline{\sin(n\pi x/a)}\sin(m\pi x/a)\,dx &= \frac{a}{m\pi - n\pi}\left(n\pi\cos(n\pi x/)\overline{\sin(m\pi x/a)} - m\pi\cos(m\pi x/a)\overline{\sin(n\pi x/a)}\right)\bigg|_0^a\\ &= 0.\end{aligned}$$

(b) With the eigenfunctions $J_0(\alpha_i r/a)$, we have

$$\int_0^a r\overline{J_0\left(\frac{\alpha_m}{a}r\right)}J_0\left(\frac{\alpha_n}{a}r\right)\,dx = \frac{r\left(\frac{\alpha_n}{a}J_0'\left(\frac{\alpha_n}{a}r\right)\right)\Big|_0^a}{\frac{\alpha_m}{a} - \frac{\alpha_n}{a}}.$$

We use the identity that

$$J_0' = -J_1$$

to use $J_1(0) = 0$ and $J_0\left(\frac{\alpha_i}{a}(a)\right) = 0$, so we recover the orthogonality relation.

(c) We have

$$\begin{aligned}\int_0^\infty \operatorname{Ai}(\kappa x + \alpha_n)\operatorname{Ai}(\kappa x + \alpha_m)\,dx &= \frac{\kappa x(\operatorname{Ai}'(\kappa x + \alpha_n)\operatorname{Ai}(\kappa x + \alpha_m) - \operatorname{Ai}'(\kappa x + \alpha_m)\operatorname{Ai}'(\kappa x + \alpha_n))\big|_0^\infty}{\kappa^2(\alpha_n - \alpha_m)}\\ &= 0.\end{aligned}$$

Solution (40.27)**:**

(a) We may express the Rayleigh quotient as

$$\rho(v) = \frac{\langle v \mid Av\rangle}{\langle v \mid v\rangle}.$$

(b) We note that if $\mathcal{L}\phi = -\lambda w(x)\phi$, then by multiplying by $\overline{\phi}$, integrating, and dividing we get

$$\begin{aligned}\lambda &= \frac{\int_a^b \overline{\phi(x)}\left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right)\phi(x)\,dx}{\int_a^b |\phi(x)|^2 w(x)\,dx}\\ &= \frac{1}{k_n}\int_a^b \overline{\phi(x)}\left(p(x)\frac{d^2\phi}{dx^2} + \frac{dp}{dx}\frac{d\phi}{dx} + q(x)\phi(x)\right)dx\end{aligned}$$

(c) Splitting things up, we get

$$\lambda = \frac{1}{k_n}\left(\int_a^b \overline{\phi(x)}p(x)\frac{d^2\phi}{dx^2}\,dx + \int_a^b \frac{dp}{dx}\frac{d\phi}{dx}\overline{\phi(x)}\,dx + \int_a^b q(x)|\phi(x)|^2\,dx\right).$$

In the "best case" scenario, we may assume that $\frac{dp}{dx}$ vanishes everywhere, so we are left with

$$\lambda \geqslant \frac{1}{k_n}\left(\int_a^b \overline{\phi(x)}p(x)\frac{d^2\phi}{dx^2}\,dx + \int_a^b q(x)|\phi(x)|^2\,dx\right).$$

Integrating the first term by parts, we may implement the condition that

$$p(x)\left(\left(\frac{d\phi}{dx}\right)\phi(x) - \overline{\phi(x)}\frac{d\phi}{dx}\right)\Bigg|_a^b = 0$$

to simplify down to

$$\lambda \geqslant \frac{1}{k_n}\left(-p(x)\overline{\frac{d\phi}{dx}}\phi(x)\Bigg|_a^b + \int_a^b q(x)|\phi(x)|^2\,dx\right).$$

Solution (41.8)**:** Using the Laplacian in spherical coordinates, we have

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right),$$

which separates

$$\psi(\mathbf{r}) = R(r)\Theta(\theta)\Phi(\phi)$$

into

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{1}{\sin^2(\theta)}\frac{d^2\Phi}{d\phi^2} + \frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = -k^2r^2.$$

The latter two terms are functions of θ, ϕ exclusively, so we have

$$\frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{d^2\Phi}{d\phi^2} = -\lambda,$$

and multiplying out by $\sin^2(\theta)$, we have

$$\frac{1}{\Theta}\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\lambda\sin^2(\theta).$$

Therefore, we recover

$$\begin{aligned}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} &= -m^2\\ \frac{1}{\Theta}\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) &= -\lambda\sin^2(\theta) + m^2\\ \frac{d^2\Phi}{d\phi^2} &= -m^2\Phi(\phi)\end{aligned}$$

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta) = 0.$$

Examining the term in r, we get

$$\begin{aligned}\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) &= -k^2r^2 + \lambda\\ \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2r^2 - \lambda\right)R(r) &= 0.\end{aligned}$$

Using $\lambda = \ell(\ell+1)$, we get

$$\begin{aligned}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2r^2 - \ell(\ell+1)\right)R(r) &= 0\\ \frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta) &= 0\\ \frac{d^2\Phi}{d\phi^2} &= -m^2\Phi.\end{aligned}$$

Using $x = \cos(\theta)$ and $X(x) = \Theta(\theta)$, we have

$$\begin{aligned}\frac{dX}{dx} &= \frac{d\Theta}{d(\cos(\theta))}\\ &= -\frac{1}{\sin(\theta)}\frac{d\Theta}{d\theta},\end{aligned}$$

and

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dX}{dx}\right) = \frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right).$$

Therefore, we have

$$\begin{aligned}R(r) &= a_1 j_\ell(kr) + a_2 n_\ell(kr)\\ \Theta(\theta) &= b_1 P_{\ell,m}(\cos(\theta)) + b_2 Q_{\ell,m}(\cos(\theta))\\ \Phi(\phi) &= c_1 e^{im\phi} + c_2 e^{-im\phi}.\end{aligned}$$

Solution (41.13)**:**

(a) Separating variables, we have

$$0 = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}.$$

We assume that

$$\begin{aligned}\frac{d^2X}{dx^2} &= -\alpha^2 X\\ \frac{d^2Y}{dy^2} &= -\beta^2 Y\\ \frac{d^2Z}{dz^2} &= -\gamma^2 Z,\end{aligned}$$

subject to the condition that

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

We have the boundary conditions of

$$V_0 = V(0, y, z)$$

$$\begin{aligned}0 &= V(a, y, z)\\ &= V(x, 0, z)\\ &= V(x, a, z)\\ &= V(x, y, 0)\\ &= V(x, y, a)\end{aligned}$$

Due to the Neumann boundary conditions in fixed x, we know that our eigenfunctions in y and z are of the form $\sin\left(\frac{n\pi}{a}y\right)$ and $\sin\left(\frac{m\pi}{a}z\right)$. This gives

$$\begin{aligned}Y(y)Z(z) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}z\right)\\ a_{m,n} &= \frac{4V_0}{a^2}\int_0^a\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}z\right)\,dz\,dy\\ &= \int_0^a \frac{2\sqrt{V_0}}{a}\sin\left(\frac{m\pi}{a}\right)dz\int_0^a \frac{2\sqrt{V_0}}{a}\sin\left(\frac{n\pi}{a}\right)dy\\ &= \frac{4V_0}{\pi^2 mn},\end{aligned}$$

and

$$V = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4V_0}{\pi^2 mn}e^{-\frac{\pi}{a}x\sqrt{m^2+n^2}}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}z\right).$$

Remark: I do not know where I lost the $V(a, y, z) = 0$ condition, but I did.

(b) Via linearity, we may consider the cube as being a sum of cubes with faces at $x = 0$ and $z = a$ held at V_0, then add together.

Solving for this case by using the Dirichlet conditions in x and y, we get

$$\begin{aligned}X_mY_n(x, y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\\ a_{n,m} &= \frac{4V_0}{a^2}\int_0^a\int_0^a \sin\left(\frac{m\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)dy\,dx\\ &= \frac{4V_0}{\pi^2 mn},\end{aligned}$$

and

$$Z_{m,n}(z) = a_1\cosh\left(\frac{\pi}{a}\sqrt{m^2+n^2}z\right) + a_2\sinh\left(\frac{\pi}{a}\sqrt{m^2+n^2}z\right).$$

Evaluating the condition that $Z_{m,n}(a) = 1$, we get

$$Z_{m,n}(z) = a_1\cosh\left(\pi\sqrt{m^2+n^2}\right) + a_2\sinh\left(\pi\sqrt{m^2+n^2}\right).$$

Using the power of safe assumptions, we will assume $a_2 = 0$ for all such a_2, giving

$$Z_{m,n}(z) = \tanh\left(\pi\sqrt{m^2+n^2}\right)\cosh\left(\frac{\pi}{a}\sqrt{m^2+n^2}z\right).$$

Thus, we get the solution in the case of *only* $z = a$ at V_0 of

$$V = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4V_0}{\pi^2 mn}\tanh\left(\pi\sqrt{m^2+n^2}\right)\cosh\left(\frac{\pi}{a}\sqrt{m^2+n^2}z\right)\sin\left(\frac{m\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right).$$

and the solution to the full cube of

$$\begin{aligned}V = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&\frac{4V_0}{\pi^2 mn}\tanh\left(\pi\sqrt{m^2+n^2}\right)\cosh\left(\frac{\pi}{a}\sqrt{m^2+n^2}z\right)\sin\left(\frac{m\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)\\ &+ \frac{4V_0}{\pi^2 mn}e^{-\frac{\pi}{a}x\sqrt{m^2+n^2}}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}z\right)\end{aligned}$$

Solution (41.14)**:** We know that solutions of Laplace's equation in cylindrical coordinates are of the form

$$\begin{aligned}R(r) &= a_1 J_m(\beta r) + a_2 N_m(\beta r)\\ \Phi(\phi) &= b_1\cos(m\phi) + b_2\sin(m\phi)\\ Z(z) &= c_1 e^{\beta z} + c_2 e^{-\beta z}.\end{aligned}$$

Instead of R, we will use S to denote the radius of the cylinder. Given the boundary conditions of

$$\begin{aligned}V(r,\phi,0) &= 0\\ V(S,\phi,z) &= 0\\ V(r,\phi,L) &= V_0,\end{aligned}$$

we know that a_2 must be zero, as the N_m blow up at the origin. We let $\alpha_{m,n}$ denote the nth zero of J_m, and since $Z(0)=0$, we must have $Z = d\sinh\left(\frac{\alpha_{m,n}}{S}z\right)$. We are left with the expression

$$V = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sinh\left(\frac{\alpha_{m,n}}{S}z\right)J_m\left(\frac{\alpha_{m,n}}{S}r\right)(A_{m,n}\cos(m\phi) + B_{m,n}\sin(m\phi)).$$

Since we have polar symmetry, we may disregard m for $m \neq 0$. Renaming $\alpha_{0,n} =: \alpha_n$ and $A_{0,n} =: A_n$, we have

$$V(r,\phi,z) = \sum_{n=1}^{\infty} A_n\sinh\left(\frac{\alpha_n}{S}z\right)J_0\left(\frac{\alpha_n}{S}r\right).$$

Evaluating

$$\begin{aligned}V(r,\phi,L) &= V_0\\ &= \sum_{n=1}^{\infty} A_n\sinh\left(\frac{\alpha_n}{S}L\right)J_0\left(\frac{\alpha_n}{S}r\right),\end{aligned}$$

so

$$A_n = \frac{2V_0}{\sinh\left(\frac{\alpha_n L}{S}\right)S^2 J_1(\alpha_n)^2}\int_0^S J_0\left(\frac{\alpha_n}{S}r\right)r\,dr.$$

Solution (41.16)**:** Since we have oscillation in z, our separated solutions to Laplace's equations are of the form

$$\begin{aligned}R(r) &= a_1 I_m(kr) + a_2 K_m(kr)\\ \Phi(\phi) &= b_1\cos(m\phi) + b_2\sin(m\phi)\\ Z(z) &= c_1\cos(kz) + c_2\sin(kz).\end{aligned}$$

We may disregard the term in K_m as the function blows up towards the origin. We may also disregard the term in $\sin(kz)$ as we have periodic Neumann conditions rather than Dirichlet conditions. Thus, we get

$$V(r,\phi,z) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{4V_0\sin\left(\frac{k\pi}{2}\right)}{k\pi}\cos\left(\frac{k\pi}{L}z\right)I_m(kr)\big(A_{k,m}\cos(m\phi) + B_{k,m}\sin(m\phi)\big).$$

As we have polar symmetry, we may disregard all but the $m=0$ term, giving

$$V(r,\phi,z) = \sum_{k=1}^{\infty} A_k\frac{4V_0\sin\left(\frac{k\pi}{2}\right)}{k\pi}\cos\left(\frac{k\pi}{L}z\right)I_0(kr).$$

Finally, we must have

$$A_k = \frac{1}{I_0(kR)},$$

so we have a full solution of

$$V(r,\phi,z) = \sum_{k=1}^{\infty} \frac{4V_0 \sin\left(\frac{k\pi}{2}\right)}{k\pi I_0(kS)} \cos\left(\frac{k\pi}{L}z\right) I_0(kr).$$

Solution (41.25)**:**

(a) The only J_m such that there is displacement at the center is J_0, so only the modes $\alpha_{0,n}$ are excited. Evaluating the ratios, they are not particularly harmonic. The only ratios that appear are $\frac{\alpha_{0,5}}{\alpha_{0,4}}$ approximating a major third and $\frac{\alpha_{0,4}}{\alpha_{0,3}}$ a perfect fourth.

(b) Exciting the preferred modes of J_1 suppresses the J_0 modes because J_m for all $m \geqslant 1$ have no displacement at the center.

(c) The respective frequencies are

$$\begin{aligned} \frac{\alpha_{1,2}}{\alpha_{1,1}} &\approx 1.83 \\ \frac{\alpha_{1,3}}{\alpha_{1,2}} &\approx 1.45 \\ \frac{\alpha_{1,4}}{\alpha_{1,3}} &\approx 1.31 \\ \frac{\alpha_{1,5}}{\alpha_{1,4}} &\approx 1.24 \\ \frac{\alpha_{1,6}}{\alpha_{1,5}} &\approx 1.19 \end{aligned}$$

These are quite close to their respective harmonies of equal temperament.

Solution (41.28)**:**

(a) We have

$$\begin{aligned} 0 &= a_1 J_n(ka) + a_2 N_n(ka) \\ 0 &= a_1 J_n(kb) + a_2 N_n(kb). \end{aligned}$$

Therefore,

$$a_1 = -a_2 \frac{N_n(kb)}{J_n(kb)}.$$

Here, k is the parameter of eigenmodes, with units of inverse radius.

I don't know how to do the other two parts of the problems.

Solution (42.1)**:**

(a) We have the Green's Function of

$$G(x,t) = \frac{1}{L} \begin{cases} x(t-L) & x < t \\ t(x-L) & x > t \end{cases}$$

(b) With $\psi(0) = 0$ and $\psi'(L) = 0$, we have

$$G(x,t) = \frac{1}{L} \begin{cases} ax & x < t \\ k & x > t \end{cases}.$$

We must have $at = k$ and $-a = 1$. Thus,

$$G(x,t) = \begin{cases} -x & x < t \\ -t & x > t \end{cases}$$

(c) With $\psi'(0) = 0$ and $\psi(L) = 0$, we have

$$G(x,t) = \begin{cases} k & x < t \\ b(x-L) & x > t \end{cases}.$$

We must have $b(t-L) = k$ and $b = 1$. Therefore,

$$G(x,t) = \begin{cases} t-L & x < t \\ x-L & x > t \end{cases}$$

Solution (42.2)**:** We have

$$\begin{aligned}
\psi_{p,1}(x) &= \int_0^L G(x,t)t^2\,dt \\
&= \frac{1}{L}\int_0^x t(x-L)t^2\,dt + \frac{1}{L}\int_x^L x(t-L)t^2\,dt \\
&= \frac{1}{12}x\left(x^3 - L^3\right). \\
\psi_{p,2} &= \int_0^L G(x,t)t^2\,dt \\
&= \int_0^x -t^3\,dt + \int_x^L -xt^2\,dx \\
&= \frac{1}{12}x\left(x^3 - 4L^3\right) \\
\psi_{p,3} &= \int_0^L G(x,t)t^2\,dt \\
&= \int_0^x t^2(x-L)\,dt + \int_x^L t^2(t-L)\,dt \\
&= \frac{1}{12}\left(x^4 - L^4\right).
\end{aligned}$$

- We see that $\psi_{p,1}(0) = 0$ and $\psi_{p,1}(L) = 0$.
- We see that $\psi_{p,2} = 0$ and $\frac{d\psi_{p,2}}{dx}\big|_{x=L} = 4L^3 - 4L^3 = 0$.
- We see that $\frac{d\psi_{p,3}}{dx}\big|_{x=0} = 0$ and $\psi_{p,3}(L) = 0$.

Solution (42.11)**:**

(a) Since $\delta(x-t) = 0$ whenever $x \neq t$, we may implement the Neumann conditions to take

$$G(x,t) = \begin{cases} a\cos(3x) & x < t \\ b\cos(3(x-L)) & x > t, \end{cases}$$

subject to the conditions that

$$\begin{aligned}
a\cos(3t) &= b\cos(3(t-L)) \\
-3b\sin(3(t-L)) + 3a\sin(3t) &= 1 \\
3a\sin(3t) &= 1 + 3b\sin(3(t-L)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
b &= a\frac{\cos(3t)}{\cos(3(t-L))} \\
3a\sin(3t) &= 1 + 3a\sin(3(t-L))\frac{\cos(3t)}{\cos(3(t-L))} \\
3a\sin(3t) &= 1 + 3a\tan(3(t-L))\cos(3t)
\end{aligned}$$

$$a = \frac{1}{3(\sin(3t) - \tan(3(t-L)))\cos(3t)}$$
$$b = \frac{\cos(3t)}{3(\cos(3(t-L))\sin(3t) - \sin(3(t-L))\cos(3t))}.$$

(b) The eigenfunctions of the Sturm–Liouville operator $\frac{d^2}{dx^2} + 9$ subject to the Neumann boundary conditions are $3\cos\left(\frac{n\pi}{L}t\right)$. Therefore,

$$G(x,t) = \frac{9L}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}t\right)}{n}.$$

(c) We will use the eigenfunction expansion for this purpose. This gives

$$\begin{aligned} y_p(x) &= \int_0^L \frac{9L}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}t\right)}{n}t^2\,dt \\ &= \frac{9L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{L}x\right)\int_0^L t^2\cos\left(\frac{n\pi}{L}t\right)dt \\ &= \frac{9L}{\pi}\sum_{n=1}^{\infty}\frac{2L^3}{n^3\pi^2}\cos\left(\frac{n\pi}{L}x\right). \end{aligned}$$