

**Math 395: Homework 2**

**Name: Avinash Iyer**

**Due: 09/12/2024**

**Collaborators: Noah Smith, Gianluca Crescenzo, Carly Venenciano, Timothy Rainone, Clarissa Ly, Ben Langer Weida**

## Problem 11

**Problem.** Let  $T \in \text{Hom}_{\mathbb{F}}(P_7(\mathbb{F}), P_7(\mathbb{F}))$  be defined by  $T(f(x)) = f'(x)$ , where  $f'(x)$  denotes the usual derivative of a polynomial  $f(x) \in P_7(\mathbb{F})$ . For each of the fields below, determine a basis for the image and kernel of  $T$ :

(a)  $\mathbb{F} = \mathbb{R}$

(b)  $\mathbb{F} = \mathbb{F}_3$ .

**Solution.**

(a) For  $f(x) \in P_7(\mathbb{R})$ , we have

$$f(x) = a_0 + a_1x + \cdots + a_7x^7,$$

where  $a_i \in \mathbb{R}$  for each  $i$  from 1 through 7. In particular,

$$T(f(x)) = a_1 + 2a_2x + \cdots + 7a_7x^6,$$

and since  $a_i \in \mathbb{R}$  for each  $i$ , so too is  $ia_i$ . For any  $p(x) \in P_6(\mathbb{R})$ , with  $p(x) = p_0 + p_1x + \cdots + p_6x^6$ , we can find  $q(x) \in P_7(\mathbb{R})$  with

$$q(x) = q_0 + p_0x + \frac{p_1}{2}x^2 + \cdots + \frac{p_5}{6}x^6 + \frac{p_6}{7}x^7,$$

with  $q_0 \in \mathbb{R}$  being arbitrary, and

$$T(q(x)) = p_0 + p_1x + \cdots + p_6x^6.$$

Thus,  $\text{im}(T) = P_6(\mathbb{R})$ . The basis for  $\text{im}(T)$  is the basis for  $P_6(\mathbb{R})$ , which is  $\{1, x, x^2, \dots, x^6\}$ .

We know that if  $f(x) \in \mathbb{R}$ , then  $T(f(x)) = 0$ , meaning  $\ker(T) = \mathbb{R}$ . Thus, a basis for  $\ker(T)$  is  $\{1\}$ .

(b) For  $f(x) \in P_7(\mathbb{F}_3)$ , we have

$$f(x) = a_0 + a_1x + \cdots + a_5x^5 + a_6x^6 + a_7x^7$$

where  $a_0, a_1, \dots, a_6, a_7 \in \mathbb{F}_3$ . In particular, we can see that

$$T(f(x)) = a_1 + 2a_2x + 3a_3x^2 + \cdots + 5a_5x^4 + 6a_6x^5 + 7a_7x^6.$$

Since we are working in  $\mathbb{F}^3$ , in particular, it is the case that  $3a_3 \equiv 0a_3 = 0$ , and similarly with  $6a_6$ . Thus, we have

$$\begin{aligned} T(f(x)) &= a_1 + 2a_2x + 4a_4x^3 + 5a_5x^4 + 7a_7x^6 \\ &\equiv a_1 + 2a_2x + a_4x^3 + 2a_5x^4 + a_7x^6. \end{aligned}$$

Thus,  $\text{im}(T)$  must be of the above form, meaning that the set  $\{1, x, x^3, x^4, x^6\}$  is a basis for the image of  $T$ .

Similarly, since all polynomials of the form  $f(x) = a + bx^3 + cx^6$  with  $a, b, c \in \mathbb{F}_3$  are mapped to 0 under  $T$ , it is the case that the set  $\{1, x^3, x^6\}$  is a basis for  $\ker(T)$ .

## Problem 12

**Problem.** Let  $T \in \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$ . Prove that if  $v \in V$  is not in  $\ker(T)$ , then

$$V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.$$

**Solution.** Since  $T(v) \neq 0$ , there exists  $(T(v))^{-1} \in \mathbb{F}$ . Let  $w \in V$ . Then,

$$T(w) = \left( T(w)(T(v))^{-1} \right) T(v).$$

We let  $c = T(w)(T(v))^{-1}$ . We have

$$\begin{aligned} T(w) &= cT(v) \\ &= T(cv), \end{aligned}$$

meaning

$$T(w - cv) = 0,$$

so  $w - cv \in \ker(T)$ , or  $w \in [cv]_{\sim}$ , where  $\sim$  is the equivalence relation defining  $V/\ker(T)$ .

Thus, we have  $w \in \ker(T) + \{cv \mid c \in \mathbb{F}\}$ , implying that  $V \subseteq \ker(T) + \{cv \mid c \in \mathbb{F}\}$ , so  $V = \ker(T) + \{cv \mid c \in \mathbb{F}\}$ .

For  $k \in \ker(T)$ , suppose

$$cv + k = 0.$$

Then,

$$\begin{aligned} T(cv + k) &= 0_V \\ cT(v) + T(k) &= 0 \\ cT(v) &= 0. \end{aligned}$$

Since  $T(v) \neq 0$  by the definition of  $v$ , it must be the case that  $c = 0$ , meaning  $cv = 0_V$ . Thus, it is the case that  $\ker(T)$  and  $\{cv \mid c \in \mathbb{F}\}$  are independent subspaces, meaning

$$V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.$$

## Problem 18

**Problem.** Let  $V$  be a  $\mathbb{F}$ -vector space of dimension  $n$ . Let  $T \in \text{Hom}_{\mathbb{F}}(V, V)$  such that  $T^2 = 0$ . Prove that the image of  $T$  is contained in the kernel of  $T$ , and hence the dimension of the image of  $T$  is at most  $n/2$ .

**Solution.** Suppose  $w \in \text{im}(T)$ . Then, there exists  $v \in V$  such that  $T(v) = w$ . In particular, this means that

$$\begin{aligned} T(w) &= T(T(v)) \\ &= T^2(v) \\ &= 0, \end{aligned}$$

meaning  $T(w) \in \ker(T)$ . Thus,  $w \in \ker(T)$ , implying that  $\text{im}(T) \subseteq \ker(T)$ . In particular, since  $n = \dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(\text{im}(T)) + \dim_{\mathbb{F}}(\ker(T))$ , and  $\dim_{\mathbb{F}}(\text{im}(T)) \leq \dim_{\mathbb{F}}(\ker(T))$ , it is the case that  $\dim_{\mathbb{F}}(\text{im}(T)) \leq n/2$ .

## Problem 19

**Problem.** Let  $W$  be a subspace of a finite-dimensional vector space  $V$ . Let  $T \in \text{Hom}_{\mathbb{F}}(V, V)$  be such that  $T(W) \subseteq W$ . Show that  $T$  induces a linear transformation  $\bar{T} \in \text{Hom}_{\mathbb{F}}(V/W, V/W)$ . Prove that  $T$  is nonsingular (i.e., injective) on  $V$  if and only if  $T$  restricted to  $W$  and  $\bar{T}$  on  $V/W$  are both nonsingular.

**Solution.** Let  $\pi : V \rightarrow V/W$  be the projection map,  $\pi(v) = v + W$ . For  $T \in \text{Hom}_{\mathbb{F}}(V, V)$  with  $T(W) \subseteq W$ , it is the case that  $\pi \circ T(W) = 0 + W$ . We define  $\bar{T} : V/W \rightarrow V/W$  by taking

$$\bar{T}(v + W) = T(v) + W.$$

We will show that  $\bar{T}$  is well-defined and that  $\pi \circ T = \bar{T} \circ \pi$ . Suppose  $v_1 + W = v_2 + W$ . Then, for some  $w \in W$ ,  $v_1 = v_2 + w$ . Therefore,

$$\begin{aligned} \bar{T}(v_1 + W) &= \bar{T}(v_2 + w + W) \\ &= T(v_2 + w) + W \\ &= T(v_2) + T(w) + W \\ &= T(v_2) + W, \end{aligned}$$

where the property that  $T(W) \subseteq W$  was used in the final step.

We will now show that  $\bar{T}$  is a linear map. Let  $\alpha \in \mathbb{F}$ ,  $v_1 + W, v_2 + W \in V/W$ . Then,

$$\begin{aligned} \bar{T}((v_1 + W) + \alpha(v_2 + W)) &= \bar{T}((v_1 + \alpha v_2) + W) \\ &= T(v_1 + \alpha v_2) + W \end{aligned}$$

$$\begin{aligned}
&= T(v_1) + \alpha T(v_2) + W \\
&= (T(v_1) + W) + \alpha (T(v_2) + W) \\
&= \bar{T}(v_1 + W) + \alpha \bar{T}(v_2 + W).
\end{aligned}$$

Finally, we can see that for  $v \in V$

$$\begin{aligned}
\pi \circ T(v) &= \pi(T(v)) \\
&= T(v) + W \\
&= \bar{T}(v + W) \\
&= \bar{T}(\pi(v)).
\end{aligned}$$

Thus, we can see that the following diagram commutes.

$$\begin{array}{ccc}
V & \xrightarrow{T} & V \\
\pi \downarrow & & \downarrow \pi \\
V/W & \xrightarrow{\bar{T}} & V/W
\end{array}$$

Suppose  $T$  is injective. Then, by inclusion,  $T|_W$  is injective. Let  $v + W \in \ker(\bar{T})$ . Then,

$$\begin{aligned}
\bar{T}(v + W) &= 0 + W \\
&= T(v) + W,
\end{aligned}$$

Thus, we have  $T(v) \in W$ . Since  $V$  is finite-dimensional, and  $T$  is injective, then  $T$  is bijective, meaning  $T(W) = W$  (as, by assumption,  $T(W) \subseteq W$ ). Thus,  $v \in W$ , meaning  $v + W = 0 + W$ , so  $\ker(\bar{T}) = 0 + W$ , meaning  $\bar{T}$  is injective.

Suppose  $\ker(\bar{T}) = 0 + W$  and  $\ker(T|_W) = 0$ . Let  $v \in \ker(T)$ . Then,  $T(v) = 0$ . Thus,

$$\begin{aligned}
\pi(T(v)) &= 0 + W \\
&= \bar{T}(\pi(v)),
\end{aligned}$$

implying that  $\pi(v) = 0 + W$ , so  $v \in W$ . So,  $T(v) = T|_W(v) = 0$ , meaning  $v = 0$ .