

Problem 1

Let $X = \{0, 1\}^n$. Show that the Hamming distance:

$$d_H : X \times X \rightarrow [0, \infty)$$

$$d_H \left((x_j)_{j=1}^n, (y_j)_{j=1}^n \right) = |\{j \mid x_j \neq y_j\}|$$

defines a metric on X .

Proof:

- Symmetry:

$$\begin{aligned} d_H \left((x_j)_{j=1}^n, (y_j)_{j=1}^n \right) &= |\{j \mid x_j \neq y_j\}| \\ &= |\{j \mid y_j \neq x_j\}| \\ &= d_H \left((y_j)_{j=1}^n, (x_j)_{j=1}^n \right) \end{aligned}$$

- Definiteness: it is only the case that $d_H(x_j, y_j) = 0$ if $x_j = y_j$ for all j , by the definition of the distance.
- Similarly, since $x_j = x_j$ for all j , $d_H(x_j, x_j) = 0$.
- Let $(x_j)_j$, $(y_j)_j$, and $(z_j)_j$ be sequences of bits. The set $\{j \mid x_j \neq z_j\}$ is formed by taking all the values $\{j \mid x_j \neq y_j\}$ along with $\{j \mid y_j \neq z_j\}$, net of particular indices where $x_j = z_j$, but $x_j \neq y_j$. Therefore,

$$d(x, z) \leq d(x, y) + d(y, z).$$

Problem 2

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V , show that the induced metrics d and d' are equivalent.

Proof: Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms. Then, $\exists c_1, c_2 \in \mathbb{R}$ such that $\|v - w\|' \leq c_1 \|v - w\|$ and $\|v - w\| \leq c_2 \|v - w\|'$. However, this is the exact same statement as $d(v, w) \leq c_1 d'(v, w)$ and $d'(v, w) \leq c_2 d(v, w)$. Thus, d and d' are equivalent metrics.

Problem 3

Let $\{X_k, d_k\}$ be a sequence of metric spaces with uniformly bounded metrics. Let

$$X := \prod_{k \geq 1} X_k$$

denote the product.

- (a) Show that

$$D : X \times X \rightarrow [0, \infty)$$

$$D(x, y) := \sum_{k \geq 1} 2^{-k} d_k(x_k, y_k)$$

defines a metric on X .

- (b) Consider the case where $\{X_k\} = \{0, 2\}$ and $d_k(a, b) = |a - b|$ for every $k \geq 1$. We get the abstract Cantor set

$$\Delta := \prod_{k \geq 1} \{0, 2\};$$

$$D(x, y) := \sum_{k=1}^{\infty} 3^{-k} |x_k - y_k|.$$

Prove that $D(x, z) = D(y, z)$ implies $x = y$.

Proof:

(i) Let D be defined as above. Then, $D((x_k)_k, (x_k)_k)$ is a sum of $d_k(x_k, x_k)$, all uniformly zero, meaning $D((x_k)_k, (x_k)_k) = 0$.

Similarly, $D((x_k)_k, (y_k)_k) = 0$ implies that $d_k(x_k, y_k) = 0$ for all x_k, y_k . Since d_k is a metric, this means $x_k = y_k$ for all k , implying that $(x_k)_k = (y_k)_k$.

Additionally, $d_k(x_k, y_k) = d_k(y_k, x_k)$, it is the case that $D((x_k)_k, (y_k)_k) = D((y_k)_k, (x_k)_k)$.

Finally, we must show the triangle inequality:

$$\begin{aligned} D((x_k)_k, (z_k)_k) &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, z_k) \\ &\leq \sum_{k=1}^{\infty} 2^{-k} (d_k(x_k, y_k) + d_k(y_k, z_k)) \\ &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, y_k) + \sum_{k=1}^{\infty} 2^{-k} d_k(y_k, z_k) \\ &= D((x_k)_k, (y_k)_k) + D((y_k)_k, (z_k)_k). \end{aligned}$$

Problem 4

Let $(V, \|\cdot\|)$ be a normed space, and suppose $E \subseteq V$. Show that the following are equivalent:

- (1) E is bounded — $\text{diam}(E) < \infty$;
- (2) $\sup_{v \in E} \|v\| < \infty$;
- (3) there is an $r > 0$ such that $E \subseteq B(0, r)$.

Proof: We will start by showing (i) implies (ii). Let E be a bounded subset of V . Thus, for all $v, w \in E$, $\|v - w\| \leq c$ for some $c \in \mathbb{R}^+$.

Problem 5

Let (X, d) be a metric space and suppose $A \subseteq X$. Show:

- (i) $\overline{A^c} = (A^\circ)^c$
- (ii) $(\overline{A})^c = (A^c)^\circ$

Proof:

(i) We have previously established that $\overline{A^c} \subseteq (A^\circ)^c$. Let $x \in (A^\circ)^c$. Then, $x \notin A^\circ$, meaning $\forall \delta > 0$, $U(x, \delta) \cap A^c \neq \emptyset$. Thus, $x \in \overline{A^c}$.

(ii) Let $x \in \overline{A^c}$. Then, $x \notin \overline{A}$, meaning $\exists \delta > 0$ such that $U(x, \delta) \cap A = \emptyset$. Thus, $U(x, \delta) \subseteq A^c$, meaning $x \in (A^c)^\circ$.

Let $x \in (A^c)^\circ$. Then, $\exists \delta > 0$ such that $U(x, \delta) \subseteq A^c$. Therefore, $U(x, \delta) \cap A = \emptyset$, meaning $x \notin \overline{A}$, so $x \in \overline{A^c}$.

Problem 6

In any metric space, show that open balls are open, closed balls are closed, and spheres are closed. Moreover, in a normed space, show that $\partial U(v, r) = \partial B(v, r) = S(v, r)$.

Proof:

- (i) Let $\delta > 0$, and $A = U(x, \delta)$ for some $x \in X$. Then, for any $y \in A$, set $\varepsilon = \min\{d(x, y), \delta - d(x, y)\}$. Then, $U(y, \varepsilon) \subseteq A$.
- (ii) Let $M = B(x, \delta)$. Let $y \in M^c$. Set $\varepsilon = d(x, y) - \delta$. Then, $U(y, \varepsilon) \subseteq M^c$, meaning M^c is open, and M is thus closed.
- (iii) Let $A = S(x, \delta)$ for some $\delta > 0$. Then, $A^c = U(x, \delta) \cup (B(x, \delta))^c$, meaning A^c is a union of open sets, which is open. Thus, A is closed.

(iv) We have previously established that, in a normed space, $\overline{U(v, r)} = B(v, r)$. Therefore,

$$\begin{aligned}\partial U(v, r) &= \overline{U(v, r)} \setminus U(v, r)^\circ \\ &= \{x \mid d(x, v) \leq r\} \setminus \{x \mid d(x, v) < r\} \\ &= \{x \mid d(x, v) = r\} \\ &= S(v, r).\end{aligned}$$

Similarly, in a normed vector space, $B(v, r)^\circ = U(v, r)$. Therefore,

$$\begin{aligned}\partial B(v, r) &= \overline{B(v, r)} \setminus B(v, r)^\circ \\ &= \{x \mid d(x, v) \leq r\} \setminus \{x \mid d(x, v) < r\} \\ &= \{x \mid d(x, v) = r\} \\ &= S(v, r).\end{aligned}$$

Problem 7

Let (X, d) be a metric space, and suppose $A \subseteq X$. Show that the following are equivalent:

- (i) A is dense in X ;
- (ii) For all $U \in \tau_X$, $U \cap A \neq \emptyset$;
- (iii) For all $x \in X$ and for all $\varepsilon > 0$, $U(x, \varepsilon) \cap A \neq \emptyset$;
- (iv) For all $x \in X$ and for all $\varepsilon > 0$, there is an $a \in A$ with $d(x, a) < \varepsilon$.

Proof:

- (i) \Rightarrow (iii): $\overline{A} = X \Leftrightarrow \forall x \in \overline{A}, \forall \varepsilon > 0, U(x, \varepsilon) \cap A \neq \emptyset \Leftrightarrow \forall x \in X, \forall \varepsilon > 0, U(x, \varepsilon) \cap A \neq \emptyset$.
- (iii) \Leftrightarrow (iv): $\forall x \in X, \forall \varepsilon > 0, U(x, \varepsilon) \cap A \neq \emptyset \Leftrightarrow \forall x \in X, \exists a \in A \ni a \in U(x, \varepsilon) \Leftrightarrow \forall x \in X, \exists a \in A \ni d(x, a) < \varepsilon$.
- (iii) \Rightarrow (ii): Suppose $\forall x \in X, U(x, \varepsilon) \cap A \neq \emptyset$. Then, since $U(x, \varepsilon) \subseteq U$ for some $U \in \tau_X$, it is the case that $U \cap A \neq \emptyset$.
- (ii) \Rightarrow (iii): Since $U(x, \varepsilon) \in \tau_X$, it is the case that for any $x \in X$ and any $\varepsilon > 0$, $U(x, \varepsilon) \cap A \neq \emptyset$.

Problem 9

Show that c_0 with $\|\cdot\|_u$ is separable.

Proof: Let $z \in c_0$. Set $\varepsilon_1 > 0$, then finding N_1 large such that for all $n > N_1$, $z_n < \varepsilon_1$. Set $z' \in c_{00}$ to be equal to z on $1, \dots, N_1$ and equal to 0 for all $n > N_1$.

Recall that for

$$\begin{aligned}E_n &= \left\{ \sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\}, \\ E &= \bigcup E_n,\end{aligned}$$

E is dense in c_{00} , meaning that there exists some $w \in c_{00}$ such that $\|z' - w\| < \varepsilon$ for any $\varepsilon > 0$. However, since $z' = z$ for all n from $1, \dots, N_1$, and the index of $\|z\|_u$ is contained in $1, \dots, N_1$, this means $\|z - w\| < \varepsilon$, meaning E is dense in c_0 .

Since E is countable, this means c_0 is countable.

Problem 10

Let \mathcal{C} denote the Cantor set. Show that \mathcal{C} is nowhere dense.

Proof: We know that \mathcal{C} is closed, meaning all we need show is that $\mathcal{C}^\circ = \emptyset$.

Suppose toward contradiction that \mathcal{C}° is not empty. Then, $\exists x \in \mathcal{C}$ and $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}$.

Find m so large such that $3^{-m} < \varepsilon$. Then, $(x - \varepsilon, x + \varepsilon)$ must be contained in a subinterval with length $\frac{1}{3^m}$. However, $2\varepsilon > \frac{1}{3^m}$, and every subinterval in the element \mathcal{C}_m has length $\frac{1}{3^m}$.