Problem (Problem 1)**:** Let R be a Euclidean domain, $n \geq 2$ an integer.

(a) Use the proof of the Smith Normal Form to show that every matrix $A \in \mathrm{GL}_n(R)$ can be written as a product of elementary matrices $E_{ij}(\lambda)$, flip matrices F_{ij}, and a diagonal matrix D.

(b) Now show that the flip matrices can be eliminated from the product in (a), and one can assume that $D = \mathrm{diag}(d, 1, \dots, 1)$. That is, all diagonal entries of D except possibly the $(1, 1)$ entry are equal to 1.

(c) Deduce from (b) that $\mathrm{SL}_n(R)$ is generated by the elementary matrices $E_{ij}(\lambda)$.

Solution:

(a) Observe that a square matrix is in Smith normal form if and only if it is a diagonal matrix of the form $D = \mathrm{diag}(d_1, \dots, d_m, 0, \dots, 0)$ where $d_1|d_2|\cdots|d_m$. By the proof of the Smith normal form, we have that the matrix UAV in Smith normal form is the product of three invertible matrices, so it is invertible, meaning that it is necessarily diagonal with $d_1, \dots, d_n \in R^{\times}$. Since the inverse of any $E_{ij}(\lambda)$ is another matrix of the form $E_{ij}(\lambda)$, and the inverse of F_{ij} is itself, it follows that we may write any $A \in \mathrm{GL}_n(R)$ as

$$A = U^{-1}DV^{-1},$$

where U^{-1} and V^{-1} are collections of flips and $E_{ij}(\lambda)$ and D is a diagonal matrix with $d_1, \dots, d_n \in R^{\times}$.

(b) We observe that the following relation holds between the matrices $E_{ij}(\lambda)$ and F_{ij}:

$$\begin{aligned} E_{ij}(\lambda)F_{jk} &= F_{jk}E_{ik}(\lambda) \\ E_{ij}(\lambda)F_{ik} &= F_{ik}E_{kj}(\lambda) \\ E_{ij}(\lambda)F_{k\ell} &= F_{k\ell}E_{ij}(\lambda) \end{aligned}$$

where in the last case, we have both k, ℓ not equal to either i or j. Since the form of these flip matrices is preserved, upon performing this reduction we may collect all the flip matrices at the front of the expression $A = U^{-1}DV^{-1}$.

Problem (Problem 2)**:** Let R be a Euclidean domain, let $k, n \in \mathbb{N}$, and let $i \leq \min(k, n)$. Given a matrix $A \in \mathrm{Mat}_{k,n}(R)$, define $d_i(A)$ to be the greatest common divisor of all $i \times i$ minors of A. Prove that $d_i(PAQ) = d_i(A)$ for all $P \in \mathrm{GL}_k(R)$ and $Q \in \mathrm{GL}_n(R)$.

Solution: Since P and Q are invertible $k \times k$ and $n \times n$ matrices respectively, it follows from Problem 1 that we may write P and Q as

$$\begin{aligned} P &= \left(\prod_{\alpha=1}^{\alpha_p} E_{i_\alpha j_\alpha}(\lambda_\alpha)\right)(\mathrm{diag}(d_p, 1, \dots, 1)) \\ Q &= (\mathrm{diag}(d_q, 1, \dots, 1))\left(\prod_{\beta=1}^{\beta_q} E_{i_\beta j_\beta}(\lambda_\beta)\right), \end{aligned}$$

where we used the fact that diagonal matrices commute with all other matrices if R is commutative. Furthermore, since P and Q are commutative, d_p and d_q are units. We observe now that

$$PAQ = \left(\prod_{\alpha=1}^{\alpha_p} E_{i_\alpha j_\alpha}(\lambda_\alpha)\right)(\mathrm{diag}(d_p, 1, \dots, 1)A\,\mathrm{diag}(d_q, 1, \dots, 1))\left(\prod_{\beta=1}^{\beta_q} E_{i_\beta j_\beta}(\lambda_\beta)\right).$$

Focusing on the product in the middle, we find that it multiplies the first column of A by d_q and the first row of A by d_p; in particular, it does not affect any of the $i \times i$ minors of A (up to associates). Additionally, since each of the $E_{ij}(\lambda)$ are simply linear combinations of the columns and rows of A respectively, they do not affect the greatest common divisor of any of the $i \times i$ minors of A, meaning that $d_i(A) = d_i(PAQ)$.

Problem (Problem 3)**:** Let R be a commutative ring with 1.

(a) Let C be an R-algebra, and $A, B \subseteq C$ R-subalgebras that commute with each other; that is, $ab = ba$ for any $a \in A$ and $b \in B$. Prove that there is an R-algebra homomorphism $\varphi\colon A \otimes B \to C$ such that $\varphi(a \otimes b) = ab$ for each $a \in A$ and $b \in B$.

(b) Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as rings.

(c) Now assume that R is a field, and let A be a finite-dimensional R-algebra. Prove that $A \otimes A$ cannot be a field unless $\dim(A) = 1$.

Solution:

(a) Let $\phi\colon A \times B \to C$ be defined by $(a, b) \mapsto ab$. Then, ϕ is an R-bilinear map, so it induces a unique linear map on the tensor product $\varphi\colon A \otimes B \to C$. We claim that this map is compatible with the R-algebra structure of $A \otimes B$.

To see this, observe that if $a_1, a_2 \in A$ and $b_1, b_2 \in B$, then

$$\begin{aligned}\varphi((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \varphi(a_1 a_2 \otimes b_1 b_2)\\ &= a_1 a_2 b_1 b_2\\ &= a_1 b_1 a_2 b_2\\ &= \varphi(a_1 \otimes b_1)\varphi(a_2 \otimes b_2).\end{aligned}$$

This gives our desired R-algebra homomorphism.

(b) We observe that both $\mathbb{R}$ and $\mathbb{Z}[i]$ are $\mathbb{Z}$-subalgebras of $\mathbb{C}$. Therefore, from above, we have a $\mathbb{Z}$-algebra homomorphism

$$\begin{aligned}\varphi\colon \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] &\to \mathbb{C}\\ t \otimes (a + bi) &\mapsto ta + tbi.\end{aligned}$$

To see that this map is injective, observe that $ta + tbi = 0$ if and only if $ta = 0$ and $tb = 0$, meaning either that $t = 0$ or $a, b = 0$; in either case, the corresponding element of the tensor product is the zero tensor. As for surjectivity, if we have $x + yi \in \mathbb{C}$, then we may find the element $x \otimes 1 + y \otimes i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ that maps to $x + yi$. Since this is a bijective $\mathbb{Z}$-algebra homomorphism, it follows that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as $\mathbb{Z}$-algebras, hence as rings.

(c) Suppose A is an R-algebra such that $A \otimes_R A$ is a field. Then, $A \otimes_R A$ is generated by $1 \otimes 1$. Now, consider the subalgebra $N = \{\lambda 1 \mid \lambda \in R\}$. Then, we see that $N \otimes_R A$ is also generated by $1 \otimes 1$, so it has the same dimension as A, and N commutes with A since it consists of scalar multiples of 1. This means that $N \otimes_R A$ admits a homomorphism of R-algebras

$$\begin{aligned}\varphi\colon N \otimes_R A &\to A\\ \lambda 1 \otimes a &\mapsto \lambda a.\end{aligned}$$

This homomorphism is surjective, though, meaning that $\dim_R(A) \leq 1$, so $\dim_R(A) = 1$.

Problem (Problem 5)**:** Let V and W be finite-dimensional vector spaces over F, with $\{v_1, \dots, v_n\}$ a basis for V and $\{w_1, \dots, w_m\}$ a basis for W. Let $\varphi\colon V \otimes W \to \mathrm{Mat}_{n,m}(F)$ be given by $\varphi(v_i \otimes w_j) = e_{ij}$, where e_{ij} is the matrix unit whose (i, j) entry is 1 and all other entries are 0.

(a) Prove that for a matrix $A \in \mathrm{Mat}_{n,m}(F)$, the following are equivalent:

  (i) $A = \varphi(v \otimes w)$ for some elements $v \in V$ and $w \in W$;

  (ii) $\mathrm{rk}(A) \leq 1$.

(b) Let $A \in \mathrm{Mat}_{n,m}(F)$. Prove that $\mathrm{rk}(A)$ is the smallest d such that $\varphi^{-1}(A)$ can be written as a sum of d simple tensors.

Solution:

(a) Suppose that $A = \varphi(v \otimes w)$ for some $v \in V$ and $w \in W$. We may write

$$\begin{aligned} v \otimes w &= \sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j e_i \otimes e_j \\ A &= \sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j e_{ij}. \end{aligned}$$

Then, using the identity

$$e_{ij}(e_k) = \delta_{jk} e_i,$$

where δ_{jk} denotes the Kronecker delta, we get that for an arbitrary vector

$$x = \sum_{k=1}^{m} r_k e_k$$

in F^m, we have

$$\begin{aligned} Ax &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j e_{ij}\right)\left(\sum_{k=1}^{m} r_k e_k\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} s_i t_j r_k e_{ij}(e_k) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} s_i t_j r_k \delta_{jk} e_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} t_j r_j s_i e_i \\ &= \sum_{j=1}^{m} t_j r_j \left(\sum_{i=1}^{n} s_i e_i\right) \\ &\in \operatorname{span}\left\{\sum_{i=1}^{n} s_i e_i\right\}. \end{aligned}$$

Therefore, $\operatorname{rk}(A) \leq 1$.

If $\operatorname{rk}(A) = 0$, then $v \otimes w$ is the zero tensor since φ is an isomorphism. Else, we assume $\operatorname{rk}(A) = 1$. Then, there are some coefficients $s_1, \dots, s_n$ such that

$$\operatorname{im}(A) = \operatorname{span}\left\{\sum_{i=1}^{n} s_i e_i\right\}.$$

Now, let

$$x = \sum_{k=1}^{m} r_k e_k.$$

We may then define

$$w = \sum_{j=1}^{m} t_j e_j$$

to be such that

$$\sum_{j=1}^{m} t_j r_j = c,$$

so that

$$\varphi\left(\left(\sum_{i=1}^{n} s_i e_i\right) \otimes \left(\sum_{j=1}^{m} t_j e_j\right)\right)\left(\sum_{k=1}^{m} r_k e_k\right) = c\sum_{i=1}^{n} s_i e_i.$$

Thus, we find $v \otimes w$ such that $A = \varphi(v \otimes w)$.

(b)

Problem (Problem 6)**:** Let R be a ring with 1, and let M be a left R-module, N a submodule. Prove that M is Noetherian if and only if N and M/N are both Noetherian.

Solution: Suppose M is a Noetherian module. Then, any submodule of M is finitely generated, so since any submodule of N is a submodule of M, N is Noetherian. Similarly, since any submodule of M/N corresponds to a submodule of M that contains N by the Fourth Isomorphism Theorem, it follows that M/N is also Noetherian.

Now, suppose M is a module such that M/N and N are Noetherian. Let $P_1 \leq P_2 \leq \cdots$ be an ascending chain of submodules for M. Then, $P_1 \cap N \leq P_2 \cap N \leq \cdots$ is an ascending chain of submodules of N, so there is some index k_1 such that $P_{k_1+i} = P_{k_1}$ for all $i \in \mathbb{N}$. Similarly, the set of submodules $P_1 + N \leq P_2 + N \leq \cdots$ is an ascending chain of submodules that contains N, so the submodules $(P_1 + N)/N \leq (P_2 + N)/N \leq \cdots$ forms an ascending chain of submodules in M/N, so there is some index k_2 such that $P_{k_2+i} = P_{k_2}$ for all $i \in \mathbb{N}$. In particular, this means that for all $i \in \mathbb{N}$, $P_{k+i} = P_k$, where $k = \max(k_1, k_2)$, so M is Noetherian.