Problem 1

Let X be a metric space. Show that X is second countable if and only if X is separable. Conclude that if X is a separable metric space, then every open set is the union of countably many open balls.

Proof:

($\Rightarrow$): Let X be second countable. Then, X contains base $U_1, U_2, \dots \in \mathcal{B}$ such that each U_i is nonempty. Let $x_1 \in U_1, x_2 \in U_2, \dots$.

The set $\{x_i\}_{i\geq 1}$ is countable, as each $x_i \in U_i$. For any $U \in \tau_X$ where $U \neq \emptyset$, $U = \bigcup_{i\in I} U_i$, meaning that $U \cap \{x_i\}_{i\geq 1} \neq \emptyset$. Thus, $\{x_i\}_{i\geq 1}$ is dense in X, meaning X is separable.

($\Leftarrow$): Let X be separable, with countable dense subset $\{x_i\}_{i\geq 1}$. Let

$$\mathcal{B} = \{U(x_i, 1/n) \mid x_i \in \{x_i\}_{i\geq 1}, n \in \mathbb{N}\}.$$

Then, for every $U \in \tau_X$, since $U \cap \{x_i\}_{i\geq 1} \neq \emptyset$, and $\exists n$ such that $U(x_k, 1/n) \subseteq U$, it must be the case that $\mathcal{B}$ is a base for τ_X. Thus, X is second countable.

If X is a separable metric space, then it admits a countable base, and any element of τ_X is a union of the elements of the base, implying that any element of τ_X is a union of countably many open balls.

Problem 2

Let (X, d) be a metric space, $(x_n)_n$ a sequence in x, and $x \in X$. The following are equivalent:

(i) $(x_n)_n \to x$ in X;

(ii) $(d(x_n, x))_n \to 0$ in $\mathbb{R}$;

(iii) For every neighborhood $V \in \mathcal{N}_x$, there is an $N \in \mathbb{N}$ with $n \geq N \Rightarrow x_n \in V$.

Proof: Let $(x_n)_n \to x$ in X. Then, for any $\varepsilon > 0$, $\exists N$ large such that $n \geq N \Rightarrow d(x_n, x) < \varepsilon$. However, this is precisely the same as $|d(x_n, x) - 0| < \varepsilon$, which is true if and only if $(d(x_n, x)) \to 0$.

Problem 3

Let X be a metric space. Show that a sequence $(x_n)_n$ converges in X if and only if every subsequence $\left(x_{n_k}\right)_k$ admits a convergent subsequence $\left(x_{n_{k_j}}\right)_j$.

Proof: I don't know how to do this.

Problem 4

Let $\{(X_k, d_k)\}$ be a family of metric spaces. Assume that for every $k \geq 1$, we have $d_k(x, y) \leq 1$ for all $x, y \in X_k$. Let

$$X := \prod_{k\geq 1} X_k$$

denote the product with metric

$$d(f, g) := \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k)).$$

Show that a sequence $(f_n)_n$ converges to f in X if and only if $(f_n(k))_n \to f(k)$ for every $k \geq 1$.

Proof: Let $(f_n)_n \to f$. Then, $(d(f_n, f))_n \to 0$. Therefore, for $\varepsilon > 0$, there exists an N large such that

$$\sum_{k=1}^{\infty} 2^{-k} d_k(f_n(k), f(k)) < \varepsilon$$

for $n \geq N$.

Problem 5

Let V be a normed space. Show that the operations

$$\begin{aligned} a &: V \times V \to V; \\ a(v, w) &= v + w \end{aligned}$$

and

$$\begin{aligned} \mu &: \mathbb{F} \times V \to V; \\ \mu(\alpha, v) &= \alpha v \end{aligned}$$

are continuous.

Proof:

- $a : V \times V \to V$, $a(v, w) = v + w$:

$$\begin{aligned} \left\|a(v,w) - a(v',w')\right\| &= \left\|v + w - (v' + w')\right\| \\ &= \left\|v - v' + w - w'\right\| \\ &\leq \left\|v - v'\right\| + \left\|w - w'\right\| \\ &= d(v, v') + d(w, w') \\ &= d_1((v,w),(v',w')), \end{aligned}$$

  meaning a is Lipschitz.
- $\mu : \mathbb{F} \times V \to V$, $\mu(\alpha, v) = \alpha v$;

$$\begin{aligned} \|\mu(\alpha, v) - \mu(\beta, w)\| &= \|\alpha v - \beta w\| \\ &= \|\alpha v - \alpha w + \alpha w - \beta w\| \\ &\leq |\alpha| \, \|v - w\| + |\alpha - \beta| \, \|w\| \end{aligned}$$

  If $(\alpha_n)_n \to \beta$ and $(v_n)_n \to w$, then

$$\begin{aligned} \|\alpha_n v_n - \beta w\| &\leq |\alpha_n| \, \|v_n - w\| + |\alpha_n - \beta| \, \|w\| \\ &\to 0. \end{aligned}$$

Problem 6

Let (X, d) be a metric space, $f, g : X \to \mathbb{F}$ continuous maps, and $\alpha \in \mathbb{F}$. Show that $f + g$, fg, and αf are continuous.

Proof: Let $(x_n)_n \to x \in X$. Then, we know that $|f(x_n) - f(x)| \to 0$ and $|g(x_n) - g(x)| \to 0$ (where $|\cdot|$ denotes absolute value in $\mathbb{F}$). Let $\varepsilon > 0$. Therefore, for N large, we know that

$$\begin{aligned} |f(x_n) + g(x_n) - (f(x) + g(x))| &\leq |f(x_n) - f(x)| + |g(x_n) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

meaning $|f(x_n) + g(x_n) - (f(x) + g(x))| \to 0$, so $(f(x_n) + g(x_n))_n \to f(x) + g(x)$. Thus, $f + g$ is continuous.

Similarly,

$$\begin{aligned} |f(x_n)g(x_n) - f(x)g(x)| &= |f(x_n)g(x_n) - f(x_n)g(x) + f(x_n)g(x) - f(x)g(x)| \\ &= |f(x_n)(g(x_n) - g(x)) + g(x)(f(x_n) - f(x))| \\ &\leq |f(x_n)||g(x_n) - g(x)| + |g(x)||f(x_n) - f(x)| \\ &\leq c|g(x_n) - g(x)| + g(x)|f(x_n) - f(x)| && \text{convergent sequences are bounded} \\ &< \varepsilon \end{aligned}$$

so $(f(x_n)g(x_n))_n \to f(x)g(x)$.

Problem 8

Let $h : X \to Y$ be a homeomorphism of metric spaces. Show that the map

$$\begin{aligned} T_h : (C(X), \|\cdot\|_u) &\to (C(Y), \|\cdot\|_u) \\ T_h(f) &= f \circ h \end{aligned}$$

is an isometric isomorphism of normed spaces.

Proof: We will show that T is linear, bijective, and isometric.

$$\begin{aligned} T_h(f+g) &= (f+g) \circ h \\ &= f \circ h + g \circ h \\ &= T_h(f) + T_h(g). \end{aligned}$$

Let $T_h(f) = T_h(g)$. Then,

$$\begin{aligned} f \circ h &= g \circ h \\ (f \circ h) \circ h^{-1} &= (g \circ h) \circ h^{-1} \\ f \circ (h \circ h^{-1}) &= g \circ (h \circ h^{-1}) \\ f &= g. \end{aligned}$$

Problem 9

Suppose $T : V \to W$ is a bijective linear map between normed spaces with $\|T\|_{\text{op}} \leq 1$ and $\left\|T^{-1}\right\|_{\text{op}} \leq 1$. Show that T is an isometry.

Proof: Since the operator norm for T is less than or equal to 1, we know that for $v, w \in V$,

$$\|T(v) - T(w)\|_W \leq \|v - w\|_V$$

and

$$\left\|T^{-1}(T(v)) - T^{-1}(T(w))\right\|_V \leq \|T(v) - T(w)\|_W$$

so, since T is bijective,

$$\|v - w\|_V \leq \|T(v) - T(w)\|_W$$

meaning

$$\|T(v) - T(w)\|_W = \|v - w\|_V$$

so T is an isometry.

Problem 10

For each $\lambda = (\lambda_k)_k$ in ℓ_∞, define

$$\begin{aligned} \varphi_\lambda &: \ell_1 \to \mathbb{F}; \\ \varphi_\lambda((a_k)_k) &= \sum_{k=1}^{\infty} \lambda_k a_k. \end{aligned}$$

(i) Show that φ_λ is well-defined and bounded linear.

Proof: We will show that φ_λ is linear, then well-defined, and we will show it is bounded in part (ii).

$$\begin{aligned}
\varphi_\lambda((a_k)_k + (b_k)_k) &= \sum_{k=1}^{\infty} \lambda_k(a_k + b_k) \\
&= \sum_{k=1}^{\infty} (\lambda_k a_k + \lambda_k b_k) \\
&= \sum_{k=1}^{\infty} \lambda_k a_k + \sum_{k=1}^{\infty} \lambda_k b_k \\
&= \varphi_\lambda((a_k)_k) + \varphi_\lambda((b_k)_k) \\
\varphi_\lambda(\alpha(a_k)_k) &= \sum_{k=1}^{\infty} \lambda_k(\alpha a_k) \\
&= \sum_{k=1}^{\infty} \alpha\lambda_k a_k \\
&= \alpha\sum_{k=1}^{\infty} \lambda_k a_k \\
&= \alpha\varphi_\lambda((a_k)_k).
\end{aligned}$$

Additionally, it is the case that $\varphi_\lambda((a_k)_k) = 0$ if and only if $a_k = 0$ for all k, meaning φ_λ is linear.

(ii) Show that $\|\varphi_\lambda\|_{\text{op}} = \|\lambda\|_\infty$.

Proof:

$$\begin{aligned}
\|\varphi_\lambda((a_k)_k)\|_1 &= \sum_{k=1}^{\infty} |\lambda_k a_k| \\
&\leq \sum_{k=1}^{\infty} \|\lambda\|_\infty |a_k| \\
&= \|\lambda\|_\infty \sum_{k=1}^{\infty} |a_k| \\
&= \|\lambda\|_\infty \|(a_k)_k\|_1
\end{aligned}$$

Therefore, $\|\varphi_\lambda\|_{\text{op}} = \|\lambda\|_\infty$.