Introduction: naive set theory

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \dots, \} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \dots, \} \\
\mathbb{Z}_+ &= \{0, 1, 2, \dots, \} \\
\mathbb{Q} &= \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\} \\
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} \\
\mathbb{C}_q &= \{a + bi \mid a, b \in \mathbb{Q}\}
\end{aligned}$$

Recall: given sets X and Y, a relation from X to Y is a subset of $X \times Y$, where $\times$ denotes the cartesian product of X and Y.

A relation $f \subseteq X \times Y$ is a function from X to Y such that $\forall x \in X$, $\exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote f as $f : X \to Y$.

X is the **domain** of f and Y is the **codomain**. The range $\text{ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Examples

$$\text{id}_x : X \to X, \text{id}_X(x) = x$$

This is the identity function.

The Characteristic Function: If $A \subseteq X$

$$\mathbf{1}_A : X \to \mathbb{R},\ \mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Algebra of Functions

Let X be any set, and $(X; \mathbb{R}) = \{f : X \to \mathbb{R}\}$ represent the function space of X with codomain $\mathbb{R}$.

Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$ (scalar multiplication). If $g(x) \neq 0 \forall x \in X$, then $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$.

Finally, we have composition. If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f(x) = g(f(x))$.

Injective, Subjective, and Bijective

A function $f : X \to Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.For example, the shift map $S : \mathbb{N} \to \mathbb{N},\ S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \to \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \to Y$ be a function. f is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \text{id}_X$. f is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \text{id}_Y$.

f is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

Proposition

f is invertible if and only if f is left and right invertible.

Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose g is a left-inverse of f, and h is a right-inverse of f. Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

Theorem

If $f : X \to Y$ is a function:

1. f is injective $\Leftrightarrow$ f is left-invertible.
2. f is surjective $\Leftrightarrow$ f is right-invertible.
3. f is bijective $\Leftrightarrow$ f is invertible.

We will prove the first proposition in the forward direction. Suppose f is injective. Given $y \in \text{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X. We can see that $g \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

Cardinality and Finitude

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets A and B, we can show that A is the same size as B by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

Definition

Sets A and B have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\text{card}(A) = \text{card}(B)$.

Example

Given $a < b$ and $c < d$, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

Example 2

$$\operatorname{card}((0,1)) = \operatorname{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - tan is strictly increasing (and thus injective)
  - $\lim_{x\to\infty} \tan(x) = \infty$ and $\lim_{x\to-\infty} \tan(x) = -\infty$, and by intermediate value theorem, tan is surjective
- $\ell : (0,1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0,1) \to \mathbb{R}$.

Definition

A set F is **finite** if F is empty or $\exists n \in \mathbb{N}$ such that $\operatorname{card}(F) = \operatorname{card}(\{1, 2, \dots, n\})$. A non-finite set is called infinite.

We can *enumerate* F by creating a function $\sigma : \{1, 2, \dots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

Proposition

If $m \neq n$, then $\operatorname{card}\{1, 2, \dots, m\} = \operatorname{card}\{1, 2, \dots, n\}$.

WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is our bijection. This means there are m "pigeons" and n "holes."

One hole, j, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since f is assumed to be injective, this is a contradiction.

Proposition

$\mathbb{N}$ is infinite.

Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m+1\} \to \mathbb{N}$. i is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \to \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

Proposition

If A is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

$\vdots$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of A.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. f is injective as a_n are distinct.

Example

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

f is a bijection as $g : \mathbb{N} \to \mathbb{Z},\ g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of f.

Definition

Given any set X, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of X.

$2^X := \{f \mid f : X \to \{0,1\}\}$.

Proposition

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbf{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$.

Cantor's theorem

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

Fact from calculus: $\forall \sigma \in (0,1)$, σ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \dots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0, 1, \dots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0, 1, \dots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since r is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

Comparing Cardinalities

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

Transitive Property

If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

The composition of two injective functions is injective.

Canonical Set Comparisons

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

Cantor-Schröder-Bernstein

For any set A, $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Let us construct a function: $f : A \to \mathcal{P}(A)$, where $a \mapsto \{a\}$.

f is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

Claim $\nexists g : A \to \mathcal{P}(A)$, g is surjective.

Suppose toward contradiction that such a g exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since g is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. $\perp$

Equivalent Propositions

(i) $\text{card}(A) \leq \text{card}(B)$

(ii) $\exists f : A \hookrightarrow B$

(iii) $\exists g : B \to A$, g surjection.

By definition, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) If $f : A \hookrightarrow B$, f is left-invertible, and thus $\exists g : B \to A$ with $g \circ f = id_A$. So, g is right-invertible, so g is surjective.

(iii) $\Rightarrow$ (ii) If $g : B \to A$ is surjective, then g is right-invertible, so $\exists f : A \to B$ such that $g \circ f = id_B$. So, f is left-invertible, so f is injective.

Corollary

If $f : A \to B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$.

Consider $g : A \to f(A)$, where $g(a) = f(a)$. So, g is onto, so $\exists$ an injection $f(A) \hookrightarrow A$.

More Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, q(m,n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $H(m,n) = (h(m), n)$.

Claim H is a bijection.

Proof of Injection If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since h is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

Proof of Surjection Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m,n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m,n) = (k, \ell)$. Set $n = \ell$, and since h is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m,n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, φ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned}
\text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\
&\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\
&= \text{card}(\mathbb{N} \times \mathbb{N}) \\
&\leq \text{card}(\mathbb{N})
\end{aligned}$$

Cardinality Rules

(i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)

(ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)

(iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)

(iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

Proof of (iii) We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

Claim We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, g and f are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1$. $\perp$

We let $A_n = g \circ f(A_{n-1})$ for arbitrary n, and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \to B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where y_x is the unique element in B with $g(y_x) = x$.

Claim We claim h is the desired bijection.

Proof of Injection Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of H, $f(x_1) = f(x_2)$, f is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y(x_2)) = x_2 \notin A_\infty$. This case is not possible.

Thus, h is injective.

Proof of Surjection Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where y_x is the unique element in B with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since g is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Therefore, we have $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$.

Countability

A set X is *countable* if $\exists f : x \hookrightarrow \mathbb{N}$ ($\text{card}(X) \leq \text{card}(\mathbb{N})$). $\text{card}(\mathbb{N}) = \aleph_0$. If X is countable and infinite, X is *denumerable*.

Corollary to Cantor-Schröder-Bernstein

If X is denumerable, then $\text{card}(X) = \aleph_0$.

Since X is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since X is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

(as shown earlier)

Countability under Union

The countable union of countable sets is countable. If I is a countable indexing set and for each $i \in I$, A_i is countable, then $\bigcup\limits_{i\in I} A_i$ is countable.

Since each A_i is countable, $\exists \pi_i : \mathbb{N} \twoheadrightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \to \bigcup_{i\in I} A_i$$

defined as $\pi(i, j) = \pi_i(j)$.

Claim 1 π is a surjection.

Proof 1 Let $x \in \bigcup_{i\in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since π_{i_0} is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, π is surjective.

Claim 2 $I \times \mathbb{N}$ is countable.

Proof 2 We know $\exists f : I \hookrightarrow \mathbb{N}$ since I is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i, n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m, n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since π is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i\in I} A_i$ is countable.

Continuum Hypothesis

We saw that $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(2^{\mathbb{N}})$, where $2^{\mathbb{N}}\{f \mid f : \mathbb{N} \to \{0, 1\}\}$.

Theorem $\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}})$, where I is any non-degenerate interval.

Lemma 1 $\text{card}([0, 1]) \leq \text{card}(2^{\mathbb{N}})$.

Proof 1 Every $t \in [0, 1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0, 1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\varphi} [0, 1]$, defined as $\phi(f) = \sum\limits_{k=1}^{\infty} \dfrac{f(k)}{2^k}$. Set $f : \mathbb{N} \to \{0, 1\}$, $f(k) = \sigma_k$.

Therefore, φ is surjective, so $\exists\{0, 1\} \hookrightarrow 2^{\mathbb{N}}$, so $\text{card}([0, 1]) \leq 2^{\mathbb{N}}$

Lemma 2 $\text{card}([0, 1]) = \text{card}(\mathbb{R})$.

Proof 2 We have $[0, 1] \overset{i}{\hookrightarrow} \mathbb{R}$ via inclusion, so $\text{card}([0, 1]) \leq \text{card}(\mathbb{R})$.

Also, $\text{card}(\mathbb{R}) = \text{card}((0, 1)) \leq \text{card}([0, 1])$, so by Cantor-Schröder-Bernstein, $\text{card}(\mathbb{R}) = \text{card}([0, 1])$.

Lemma 3 Any two non-degenerate intervals I and J have the same cardinality.

Proof 3 We can create injections $I \hookrightarrow J$ and vice-versa.

Lemma 4 $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0, 1])$.

Proof 4 $\psi : 2^{\mathbb{N}} \to [0, 1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

ψ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim ψ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$, $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} + \frac{1}{3^{k_0}} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$\begin{aligned}
|t_f - t_g| &= |\sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k}| \\
&\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\
&\leq \sum_{k>k_0} \frac{1}{3^k} \\
&= \frac{(1/3)^{k_0+1}}{1-(1/3)} \\
&= \frac{1}{2} \cdot \frac{1}{3^{k_0}}
\end{aligned}$$

$\perp$

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0,1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than $\aleph_0$ and strictly less than $2^{\aleph_0}$ is an axiom (i.e., it can be an assumption or not).

Ordering

Let X be a non-empty set. A relation on X is a subset of $X \times X$.

- R is *reflexive* if $\forall x \in X,\ (x,x) \in R$.
- R is *transitive* if $(x,y),(y,z) \in R \to (x,z) \in R$.
- If R is *antisymmetric* $(x,y),(y,x) \in R \to x = y$.

If R is reflexive, transitive, and antisymmetric, then R is an *ordering* of X.

If R is an ordering of X, then we write:

$$(x,y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x\ \forall x \in X$
- $x \leq_R y,\ y \leq_R z \to x \leq_R z$
- $x \leq_R y,\ y \leq_R x \to x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

Algebraic ordering of $\mathbb{N}_0$

$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$ such that $n + k = m$

$\mathbb{N}$ ordered via division

$$n \leq_D m \Leftrightarrow n | m$$

Under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

Inclusion Let S be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_i B \Leftrightarrow A \subseteq B$.

Containment With X defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

For $\mathcal{F}(X,\mathbb{R}) = \{f \mid f : X \to \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x)\ \forall x \in X$.

Types of Orderings

- An ordering $\leq$ of X is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.
- An ordering is *directed* if $\forall x, y \in X\ \exists z \in X$ such that $x \leq z$ and $y \leq z$.

If X is a totally ordered set, X is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D),\ (\mathcal{P}(S), \leq_i),\ (\mathcal{P}(S), \leq_c)$$

Upper/Lower Bounds

(i) Let $(X, \leq)$ be an ordered set, $A \subseteq X$. A is bounded above if $\exists v \in X$ with $a \leq v\ \forall a \in A$. Such a v is an upper bound.

(ii) A is bounded below if $\exists \ell \in X$ such that $a \geq \ell\ \forall a \in A$. Such a w is a lower bound.

(iii) If v is an upper bound of A and $v \in A$, then v is the greatest element of A, or $\max(A) = v$.

(iv) If ℓ is a lower bound for A and $\ell \in A$, then ℓ is the least element of A, or $\min(A) = \ell$.

(v) If u is an upper bound for A, and $u \leq v$ for all other upper bounds v of A, then u is the *least upper bound* of A, or $\sup(A) = u$ (for *supremum*).

(vi) If ℓ is a lower bound for A, and $\ell \leq g$ for all other lower bounds g of A, then ℓ is the *greatest lower bound* of A, or $\inf(A) = \ell$ (for *infimum*).

(vii) If A is bounded above and below, then A is bounded.

Well-Ordering Principle

With $(\mathbb{N}, \leq_a)$, every nonempty $A \subseteq \mathbb{N}$ has a least element.

Examples

Example 1

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \dots, 12\}$, we have the following:

Bounded Above? Yes.

Upper Bounds $12, 13, 14, \dots$

Greatest Element 12

Example 2

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \dots, 10\}$

Bounded Above? Yes.

Upper Bounds 10!

Greatest Element? No.

Supremum $2^3 \cdot 3^2 \cdot 5 \cdot 7$

Bounded Below? Yes.

Lower Bound 1

Least Element? No.

Infimum 1

Example 3

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$.

Supremum $\bigcup_{i \in I} A_i$

Infimum $\bigcap_{i \in I} A_i$

Complete Sets

An ordered set $(X, \leq)$ is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, $\mathbb{Q}$ is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

Ordering of $\mathbb{Z}$

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0,\ n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

Properties of $\mathbb{Z}^+$

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$