Problem (Problem 1)**:** Let R be a Euclidean domain, $n \geq 2$ an integer.

(a) Use the proof of the Smith Normal Form to show that every matrix $A \in \mathrm{GL}_n(R)$ can be written as a product of elementary matrices $E_{ij}(\lambda)$, flip matrices F_{ij}, and a diagonal matrix D.

(b) Now show that the flip matrices can be eliminated from the product in (a), and one can assume that $D = \mathrm{diag}(d, 1, \dots, 1)$. That is, all diagonal entries of D except possibly the $(1,1)$ entry are equal to 1.

(c) Deduce from (b) that $\mathrm{SL}_n(R)$ is generated by the elementary matrices $E_{ij}(\lambda)$.

Solution:

(a) Observe that a square matrix is in Smith normal form if and only if it is a diagonal matrix of the form $D = \mathrm{diag}(d_1, \dots, d_m, 0, \dots, 0)$ where $d_1 | d_2 | \cdots | d_m$. By the proof of the Smith normal form, we have that the matrix UAV in Smith normal form is the product of three invertible matrices, so it is invertible, meaning that it is necessarily diagonal with $d_1, \dots, d_n \in R^\times$. Since the inverse of any $E_{ij}(\lambda)$ is another matrix of the form $E_{ij}(\lambda)$, and the inverse of F_{ij} is F_{ji}, it follows that we may write any $A \in \mathrm{GL}_n(R)$ as

$$A = U^{-1} D V^{-1},$$

where U^{-1} and V^{-1} are collections of flips and $E_{ij}(\lambda)$ and D is a diagonal matrix with $d_1, \dots, d_n \in R^\times$.

(b)

Problem (Problem 3)**:** Let R be a commutative ring with 1.

(a) Let C be an R-algebra, and $A, B \subseteq C$ R-subalgebras that commute with each other; that is, $ab = ba$ for any $a \in A$ and $b \in B$. Prove that there is an R-algebra homomorphism $\varphi \colon A \otimes B \to C$ such that $\varphi(a \otimes b) = ab$ for each $a \in A$ and $b \in B$.

(b) Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as rings.

(c) Now assume that R is a field, and let A be a finite-dimensional R-algebra. Prove that $A \otimes A$ cannot be a field unless $\dim(A) = 1$.

Solution:

(a) Let $\phi \colon A \times B \to C$ be defined by $(a, b) \mapsto ab$. Then, ϕ is an R-bilinear map, so it induces a unique linear map on the tensor product $\varphi \colon A \otimes B \to C$. We claim that this map is compatible with the R-algebra structure of $A \otimes B$.

To see this, observe that if $a_1, a_2 \in A$ and $b_1, b_2 \in B$, then

$$\begin{aligned} \varphi((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \varphi(a_1 a_2 \otimes b_1 b_2) \\ &= a_1 a_2 b_1 b_2 \\ &= a_1 b_1 a_2 b_2 \\ &= \varphi(a_1 \otimes b_1) \varphi(a_2 \otimes b_2). \end{aligned}$$

This gives our desired R-algebra homomorphism.

(b) We observe that both $\mathbb{R}$ and $\mathbb{Z}[i]$ are $\mathbb{Z}$-subalgebras of $\mathbb{C}$. Therefore, from above, we have a $\mathbb{Z}$-algebra homomorphism

$$\begin{aligned} \varphi \colon \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] &\to \mathbb{C} \\ t \otimes (a + bi) &\mapsto ta + tbi. \end{aligned}$$

To see that this map is injective, observe that $ta + tbi = 0$ if and only if $ta = 0$ and $tb = 0$, meaning either that $t = 0$ or $a, b = 0$; in either case, the corresponding element of the tensor product is the zero tensor. As for surjectivity, if we have $x + yi \in \mathbb{C}$, then we may find the element $x \otimes 1 + y \otimes i \in$

$\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ that maps to $x + yi$. Since this is a bijective $\mathbb{Z}$-algebra homomorphism, it follows that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as $\mathbb{Z}$-algebras, hence as rings.