

Problem (Problem 1): Describe the topology of the Grassmanian $\text{Gr}(k, n)$ in a uniform way, so that \mathbb{RP}^n becomes the special case of $\text{Gr}(1, n)$.

Solution: We let elements of $\text{Gr}(k, n)$ be defined as equivalence classes of linearly independent k -tuples of vectors in \mathbb{R}^n , where $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if $\text{span}\{v_1, \dots, v_k\} = \text{span}\{w_1, \dots, w_k\}$.

By extending (v_1, \dots, v_k) and (w_1, \dots, w_k) to ordered bases $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$, we see that these k -tuples are equivalent if and only if there is an invertible linear transformation Q with matrix representation

$$Q = \begin{pmatrix} A & H \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n - k) \times (n - k)$ matrix, so that

$Q[v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n] = [w_1, \dots, w_k, w_{k+1}^*, \dots, w_n]$ The subgroup of all such $Q \subseteq \text{GL}_n(\mathbb{R})$, which we call P , is the stabilizer of $\text{Gr}(k, n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $\text{GL}_n(\mathbb{R})$ acts transitively on all ordered bases of \mathbb{R}^n), we obtain $\text{Gr}(k, n) \cong \text{GL}_n(\mathbb{R})/P$, where the latter coset space is given the quotient topology.

This definition comports with the definition of \mathbb{RP}^n as the space of one-dimensional subspaces, as the invertible 1×1 matrices are precisely the nonzero scalars, so the stabilizers in the case of $\text{Gr}(1, n)$ are the 1×1 invertible block matrices A , or the nonzero scalars.

Problem (Problem 2): Fix an inner product on \mathbb{R}^n . Show that the map $V \mapsto V^\perp$ induces a C^∞ diffeomorphism $\text{Gr}(k, n) \rightarrow \text{Gr}(n - k, n)$.

Solution: Due to the inner product, we make the identification $v \mapsto v^*$ such that $v^*(w) = \langle v, w \rangle$. In particular, we have isomorphisms $V \cong V^*$ and $V^\perp \cong (V^\perp)^*$. Therefore, given an element $T \in \text{Hom}(V, V^\perp)$, dualization gives the transpose map $T^* \in \text{Hom}((V^\perp)^*, V^*)$.

Now, given any chart (U_V, φ_V) in $\text{Gr}(k, n)$, we identify $T \in \text{Hom}(V, V^\perp) \cong U_V$ to $T^* \in \text{Hom}((V^\perp)^*, V^*) \cong U_{V^\perp}$, and identify subspaces $W \in U_V$ with their annihilators

$$W^0 = \{w^* \in (\mathbb{R}^n)^* \mid w^*(v) = 0 \text{ for all } v \in W\},$$

so that $W^0 \cap V^* = 0$. Finally, we define φ_{V^\perp} by

$$\varphi_{V^\perp} = P_{V^*} \circ P_{(V^\perp)^*}|_{W^0}^{-1}.$$

Since every $W \in \text{Gr}(k, n)$ has a unique annihilator subspace $W^0 \in \text{Gr}(n - k, n)$, we have the series of bijective correspondences

$$\begin{aligned} \text{Hom}(V, V^\perp) &\xleftrightarrow{\varphi_V} U_V \\ &\xleftrightarrow{W \mapsto W^0} U_{V^\perp} \\ &\xleftrightarrow{\varphi_{V^\perp}} \text{Hom}((V^\perp)^*, V^*) \\ &\xleftrightarrow{\langle \cdot, \cdot \rangle} \text{Hom}(V^\perp, V), \end{aligned}$$

meaning that this identification is a C^∞ diffeomorphism.

Problem (Problem 3): Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Solution: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^k map that is a C^1 diffeomorphism. In order to show that f is a C^k diffeomorphism, we need to show that $f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ exists and is of class C^k .

First, by the inverse function theorem, since f is a C^1 diffeomorphism, we see that $f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ exists, is continuous, and is such that $D(f^{-1})$ is continuous.

Now, we observe that the association $y \mapsto D_y(f^{-1})$ can be written as

$$y \mapsto f^{-1}(y) \mapsto D_y f(f^{-1}(y)) \mapsto (D_y f(f^{-1}(y)))^{-1} = D_y(f^{-1}),$$

where we observe that f^{-1} is of class C^1 , the derivative D_f is of class C^{k-1} , and matrix inversion is C^∞ ; since $D(f^{-1})$ is a composition of C^1 functions, $D(f^{-1})$ is C^1 , so f^{-1} is C^2 . Inductively, we see that f^{-1} is also of class C^k , so f is a C^k diffeomorphism.

Problem (Problem 4): Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either \mathbb{R} or S^1 , depending on whether it is compact or not.

Solution: Let M be a connected, paracompact manifold with dimension 1, and let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas for M , where φ_i are homeomorphisms between U_i and \mathbb{R} .

Let $\{V_j\}_{j \in J}$ be a locally finite refinement of $\{U_i\}_{i \in I}$, where the restrictions $\psi_j := \varphi_i|_{V_j}$ are homeomorphisms to $O_j \subseteq \mathbb{R}$. We see that for any $p \in M$, since the family of V_j with $p \in V_j$, which we call $\mathcal{V}_p = \{V_j \mid p \in V_j\}$, is finite, the intersection $\bigcap \mathcal{V}_p$ is open; similarly, the intersection $\bigcap \mathcal{O}_p \subseteq \mathbb{R}$ is open, where $\mathcal{O}_p = \{\varphi|_{V_j}(V_j) \subseteq \mathbb{R} \mid V_j \in \mathcal{V}_p\}$.

We see that $M = \bigcup_{p \in M} \bigcap \mathcal{V}_p$. Note that for any distinguished point p_1 , the corresponding sets $\bigcap \mathcal{V}_{p_1}$ and $\bigcup_{p \neq p_1} \bigcap \mathcal{V}_p$ must have nonempty (open) intersection, by the assumption that M is connected. Thus, the corresponding union $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ is an open and connected subset of \mathbb{R} . We may similarly map $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ into S^1 by composing with the quotient map.

Now, if M is compact, then $\bigcup_{p \in M} \bigcap \mathcal{V}_p$ covers M , so there is a finite subcover $M = \bigcup_{i=1}^n \bigcap \mathcal{V}_{p_i}$, so that $\bigcup_{i=1}^n \bigcap \mathcal{O}_{p_i}$ fully covers the corresponding range, meaning that, composing with the quotient map $\bigcup_{i=1}^n \bigcap \mathcal{O}_{p_i}$, we have that $M \cong S^1$. Similarly, if M is non-compact, then $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ is an open and connected subset of \mathbb{R} that does not admit any finite subcover, hence it is homeomorphic to \mathbb{R} .

Problem (Problem 5): In this problem, we prove a weak version of the Whitney Embedding Theorem.

- Find a C^∞ function λ on \mathbb{R}^n with values in $[0, 1]$ such that λ takes the value 1 on the closed ball $B(0, 1)$, and vanishes outside the closed ball $B(0, 2)$.
- Suppose M is a compact C^k manifold of dimension n . Find a C^k atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that $\varphi_i(U_i)$ contains $B(0, 2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0, 1))^\circ$.
- Let λ_i be defined by $\lambda \circ \varphi_i$ on U_i , and 0 outside U_i . Let $f_i: M \rightarrow \mathbb{R}^n$ be defined by $\lambda_i \circ \varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Solution:

- Consider the function $H: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$H(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0, \end{cases}$$

which is a C^∞ function on \mathbb{R} , as $e^{-1/t}$ is C^∞ for all $t > 0$, and the derivative is well-defined at $t = 0$. Next, we see that the function

$$G(t) = \frac{H(4 - t^2)}{H(4 - t^2) + H(t^2 - 1)}$$

takes on the value 1 whenever $-1 \leq t \leq 1$ and is supported on $[-2, 2]$. Furthermore, it is a C^∞ function, as it is a rational function of C^∞ functions where the denominator never takes the value

0. Therefore, if we replace t with $|x|$, when $x \in \mathbb{R}^n$, since the norm is a C^∞ function, we obtain a C^∞ function that is supported on $B(0, 2)$ and is equal to 1 on $B(0, 1)$, given by

$$\lambda(x) = \frac{H(4 - |x|^2)}{H(4 - |x|^2) + H(|x|^2 - 1)}.$$

(b) Let M be a compact C^k manifold, and let $\{(V_i, \psi_i)\}_{i \in I}$ be a C^k atlas for M , where $\{V_i\}_{i \in I}$ is an open cover, the $\psi_i: V_i \rightarrow \mathbb{R}^n$ are homeomorphisms, and the $\psi_j \circ \psi_i^{-1}$ are C^k diffeomorphisms.

Since M is compact, we have a finite subcover $\{V_j\}_{j=1}^n$ and an exhaustion by compact subsets via

$$U_j = \bigcup_{k=1}^j V_k$$

$$M = \bigcup_{j=1}^n U_j,$$

where, without loss of generality, $\overline{U_j} \subseteq U_{j+1}$.

Now, for each $p \in \overline{U_j} \setminus U_{j-1}$ (define $U_0 = U_1 = \emptyset$), we may find i_p with a corresponding C^k chart (V_{i_p}, ψ_{i_p}) , where without loss of generality, $\psi_{i_p}(p) = 0$, and let $W_p = \psi_{i_p}^{-1}(U(0, 1))$.

Clearly, $B(0, 2) \subseteq \psi_{i_p}(V_{i_p})$, and by finitely enumerating the elements p_{j_k} in $\overline{U_j} \setminus U_{j-1}$, we have an open cover $\{W_{p_{j_k}}\}_{k=1}^m = \{\psi_{p_{j_k}}^{-1}(U(0, 1))\}_{k=1}^m$ of M , and $\{(V_{i_{p_k}}, \psi_{p_k})\}_{k=1}^m$ are C^k charts such that $B(0, 2) \subseteq \psi_{p_k}(V_{i_{p_k}})$.

(c)

Problem (Problem 6): Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Solution: Let $\{U_i\}_{i \in I}$ be a locally finite open cover of M , and let $\{(U_i, \varphi_i)\}_{i \in I}$ be the corresponding C^k atlas for M where $B(0, 2) \subseteq \varphi_i(U_i)$, and M is covered by $\varphi_i^{-1}(U(0, 1))$. Then, we may define

$$f_i = \begin{cases} G \circ \varphi_i & \text{on } U_i \\ 0 & \text{on } U_i^c, \end{cases}$$

where

$$G(x) = \frac{e^{\frac{1}{4-|x|^2}}}{e^{\frac{1}{4-|x|^2}} + e^{\frac{1}{|x|^2-1}}}$$

is a C^∞ function supported on $B(0, 2)$ and equal to 1 on $U(0, 1)$. Defining

$$f = \sum_{i \in I} f_i,$$

we see that $f \neq 0$ everywhere, as M is covered by the family $\varphi_i^{-1}(U(0, 1))$, and since $\{U_i\}_{i \in I}$ is locally finite, f is also C^k as each f_i is the composition of a C^k diffeomorphism and a C^∞ function. The functions

$$g_i = \frac{f_i}{f}$$

are thus smooth, $0 \leq g_i \leq 1$, and $\sum_{i \in I} g_i = 1$.

Problem (Problem 7): Let X and Y be topological spaces, and let $C(X, Y)$ be the set of continuous maps from X to Y . Equip $C(X, Y)$ with the compact-open topology, where the basic open sets are

$$U_{K, V} = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space and let X be compact. We note that a neighborhood basis in the topology of uniform convergence on $C(X, Y)$ consists of sets of the form

$$U_{f, \varepsilon} = \left\{ g \mid \sup_{x \in X} d(f(x), g(x)) < \varepsilon \right\}.$$

Similarly, a subbase for the compact open topology consists of sets of the form

$$U_{f, K, \varepsilon} = \left\{ g \mid \sup_{x \in K} d(f(x), g(x)) < \varepsilon \right\};$$

the fact that Y is a metric space allows us to take this refinement of the compact-open topology.

Thus, to prove that the compact-open topology and the topology of uniform convergence are equivalent, we show that any basis element of the topology of uniform convergence is contained in a basis element of the compact-open topology, and vice versa.

First, we see that almost by definition, if $K \subseteq X$ is any compact subset, then

$$U_{f, \varepsilon} \subseteq U_{f, K, \varepsilon},$$

as any function whose supremum distance is less than ε over X must have that supremum distance hold over $K \subseteq X$.

Now, in the reverse direction, we fix f and ε . We wish to show that there is a finite family of subsets U_{K_i, V_i} with $f \in U_{K_i, V_i}$ for each i , and their intersection lies in $U_{f, \varepsilon}$. We see that every point $x \in X$ has a pre-compact open neighborhood U_x such that $f(\overline{U_x}) \subseteq U(f(x), \varepsilon/3)$. The family $\{x \in X\} U_x$ is an open cover for X , so admits a finite subcover $\{U_{x_i}\}_{i=1}^n$. Since each $\overline{U_{x_i}}$ is compact, and $f \in U_{\overline{U_{x_i}}, U(f(x_i), \varepsilon/3)}$ for each i , we see that

$$V = \bigcap_{i=1}^n U_{\overline{U_{x_i}}, U(f(x_i), \varepsilon/3)}$$

is an open subset in the compact-open topology on $C(X, Y)$ that contains f and is contained in $U_{f, \varepsilon}$, so the topologies are thus equal.

Problem (Problem 8): Let $C^k(M, N)$ be the set of C^k maps from M to N . The compact-open topology on $C^k(M, N)$ is defined similarly. Let $f \in C^k(M, N)$, (U, φ) and (V, ψ) charts on M and N , let $K \subseteq U$ be compact such that $f(K) \subseteq V$, and let $\varepsilon > 0$. We obtain a basic neighborhood $N(f, U, \varphi, V, \psi, K, \varepsilon)$ by looking at all the maps $g \in C^k(M, N)$ such that $g(K) \subseteq V$, and

$$\|D^r(\psi f \varphi^{-1})(x) - D^r(\psi g \varphi^{-1})(x)\|_{\text{op}} \leq \varepsilon \quad (*)$$

for all integers $0 \leq r \leq k$.

The Whitney topology is slightly different. Let $\Phi = \{(U_i, \varphi_i)\}_{i \in I}$ be a locally finite atlas on M , let

$K_i \subseteq U_i$ be compact for all i , let Ψ be an atlas on N , and let $\{\varepsilon_i\}_{i \in I}$ be a family of positive numbers. A basic neighborhood of $f \in C^k(M, N)$ in this topology is given by all g such that $g(K_i) \subseteq V_i$ for all i , and

$$\|D^r(\psi_i f \varphi_i^{-1})(x) - D^r(\psi_i g \varphi_i^{-1})(x)\|_{\text{op}} \leq \varepsilon_i \quad (**)$$

for all $x \in \varphi_i(K_i)$ and all integers $0 \leq r \leq k$.

For infinite values of k , we take the compact-open and Whitney topologies on $C^\infty(M, N)$ to be the union of these topologies via the inclusion $C^\infty(M, N) \subseteq C^k(M, N)$. Show the following:

- (a) these basic neighborhoods actually give a basis for a topology in both cases;
- (b) if M is compact, these two topologies coincide;
- (c) if M is compact and has no boundary, then the C^k diffeomorphisms from M to N are open in $C^k(M, N)$ in the Whitney topology.

Solution:

- (a) Clearly, in both the compact open topology and the Whitney topology, the respective neighborhoods cover $C^k(M, N)$, so we only need to verify the condition that if $X_1, X_2 \subseteq C^k(M, N)$ are open subsets such that $f \in X_1 \cap X_2$, then there is $X_3 \subseteq C^k(M, N)$ open such that $X_3 \subseteq X_1 \cap X_2$.

We start with the case of the compact-open topology. Let $f \in X_1 \cap X_2$, where X_1 and X_2 are open in the compact-open topology. Since $f \in X_1$, there is a chart (U_1, φ_1) of M , a chart (V_1, ψ_1) of N , $K_1 \subseteq U_1$ compact such that $f(K_1) \subseteq V_1$, and $\varepsilon_1 > 0$ such that $(*)$ holds and $N(f, U_1, \varphi_1, V_1, \psi_1, \varepsilon_1) \subseteq X_1$. Similarly, since $f \in X_2$, there are charts (U_2, φ_2) and (V_2, ψ_2) of M and N respectively, $K_2 \subseteq U_2$ compact with $f(K_2) \subseteq V_2$, and $\varepsilon_2 > 0$ such that $(*)$ holds, and $N(f, U_2, \varphi_2, V_2, \psi_2, \varepsilon_2) \subseteq X_2$. Note that by the characterization, $(*)$ holds for the supremum over all $x \in \varphi_j(K_j)$ for $j = 1, 2$.