Contents

Cardinality and Countability

Section 1.1: Countable Sets

Definition (Denumerable Set)**.** A set S is denumerable if there exists a function $f : S \to \mathbb{N}$ with f a bijection. We also say S is countably infinite.

Definition (Countable Set)**.** We say S is countable if S is either finite or denumerable.

Theorem (Countability of Unions)**:** If A and B are countable sets, then $A \cup B$ is countable.

Theorem (Countability of Subsets)**:** If $A \subseteq B$, then if B is countable, then A is countable.

Theorem (Union of Finite Sets)**:** If A and B are finite, then $A \cup B$ is finite.

Proof. If A is finite and B has one element, then we show that $A \cup B$ is finite (with two cases).

Afterward, for $|B| > 1$, we use induction on $|B|$. □

Definition (Finite Set)**.** A set A is finite if there exists a bijection $f : S \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N} = \{0, 1, \dots\}$.

We write $|A| = n$.

Theorem (Disjoint Union of Countable Sets)**:** If A is denumerable, B is finite, and $A \cap B = \emptyset$, then $A \cup B$ is denumerable.

Proof. There exists a bijection $f : A \to \mathbb{N}$ (since A is denumerable), and a bijection $g : B \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ (since B is finite).

We create a new bijection $h : A \cup B \to \mathbb{N}$ by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since $A \cap B = \emptyset$, we know that h is well-defined.

Now, we must show that h is a bijection.

Suppose $h(x) = h(y)$.

Case 1: If $x, y \in B$, then $h(x) = g(x) - 1$, and $h(y) = g(y) - 1$, meaning $g(x) - 1 = g(y) - 1$, meaning $g(x) = g(y)$. Since g is a bijection, $x = y$.

Case 2: If $x, y \in A$, a similar argument yields that $x = y$

Case 3: Without loss of generality, let $x \in A$ and $y \in B$. If $x \in A$, then $h(x) = f(x) + n$ and $h(y) = g(y) - 1$. Thus, $f(x) + n = g(y) - 1$. However, since $f(x) + n \geqslant n$ and $0 \leqslant g(y) - 1 \leqslant n - 1$. Thus, we get that $0 \leqslant n \leqslant n - 1$, which is a contradiction.

Thus, we have shown that h is injective. ◻

Theorem (Cartesian Product of Natural Numbers)**:** $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof. We consider $\mathbb{N} \times \mathbb{N}$ as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \cdots,$$

$$\begin{array}{rccccc}
\mathbb{N} \times \{0\} : & (0,0) & (1,0) & (2,0) & (3,0) & \cdots \\
\mathbb{N} \times \{1\} : & (0,1) & (1,1) & (2,1) & (3,1) & \cdots \\
\mathbb{N} \times \{2\} : & (0,2) & (1,2) & (2,2) & (3,2) & \cdots \\
\mathbb{N} \times \{3\} : & (0,3) & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{rccccc}
\mathbb{N} \times \{0\} : & (0,0)^{0} & (1,0)^{2} & (2,0)^{5} & (3,0)^{9} & \cdots \\
\mathbb{N} \times \{1\} : & (0,1)^{1} & (1,1)^{4} & (2,1)^{8} & (3,1) & \cdots \\
\mathbb{N} \times \{2\} : & (0,2)^{3} & (1,2)^{7} & (2,2) & (3,2) & \cdots \\
\mathbb{N} \times \{3\} : & (0,3)^{6} & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can also find a bijection $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, with

$$P(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

A fun challenge is to prove that P is a bijection. ◻

Theorem (Countability of the Rationals)**:** $\mathbb{Q}$ is denumerable.

Theorem (Countability of the Integers)**:** The set $\mathbb{Z}$ is denumerable.

Proof. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 2x & x \geqslant 0 \\ -2x - 1 & x < 0 \end{cases}$$

□

Definition (Cardinality)**.** We say two sets, A and B, have the same cardinality if there exists a bijection $f : A \to B$.

Theorem (Finite Subset Cardinality)**:** If $m, n \in \mathbb{N}$ and $m \neq n$, then $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$ do not have the same cardinality.

Theorem (Infinitude of the Natural Numbers)**:** $\mathbb{N}$ is not finite.

Example. If $A \subsetneq B$ and $|A| = |B|$, then both A and B are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.

Section 1.2: Uncountable Sets

Definition (Uncountable Set)**.** A set is uncountable if it is not countable.

Theorem (Uncountability of $\mathbb{R}$)**:** $\mathbb{R}$ is uncountable.

Proof. For all $x \in \mathbb{R}$, and for all $j \in \mathbb{N}$, we define $[x]_j$ to denote the $j + 1$-th digit after the decimal point in the decimal expansion of x.

For example, $[\pi]_0 = 1$, $[\pi]_1 = 4$, etc.

Let $f : \mathbb{N} \to \mathbb{R}$. We will show that f is not surjective.

Let $y \in [0, 1) \subseteq \mathbb{R}$ defined by $\forall j \in \mathbb{N}$,

$$[y]_j = \begin{cases} 0 & [f(j)]_j = 1 \\ 1 & [f(j)]_j \neq 1 \end{cases}.$$

We claim that $y \notin f(\mathbb{N})$. We will show that $\forall j \in \mathbb{N}$, $f(j) \neq y$.

We can see that if $[f(j)]_j = 1$, then $[y]_j = 0$. Similarly, if $[f(j)]_j \neq 1$, then $[y]_j = 1$. Either way, $[f(j)]_j \neq [y]_j$ for all $j \in \mathbb{N}$. □

Remark: The above proof is an example of a diagonalization proof. It can be imagined as

$$\begin{array}{c|c} f(0) & *.\cancel{a_1}\overset{\neq}{a_2}a_3\dots \\ f(1) & *.b_1\cancel{b_2}\overset{\neq}{b_3}\dots \\ f(2) & *.c_1c_2\cancel{c_3}\overset{\neq}{\dots} \\ \vdots & \vdots \end{array}$$

Note: A substantial problem that we might need to deal with is that a real number does not necessarily have a unique decimal representation. For instance, $3.999\cdots = 4.000\dots$.

In order to resolve this issue, we can default to the option with trailing 0 over trailing 9.

Definition (Power Set)**.** The power set of a set S is

$$P(S) = \{A \mid A \subseteq S\}.$$

Theorem (Power Set Surjection)**:** Let $f : S \to P(S)$. Then, f is not surjective.

Proof. Let $T = \{x \in S \mid x \notin f(x)\}$. Then, $T \notin f(S)$.

Let $y \in S$. We want to show that $f(y) \neq T$. Suppose toward contradiction that $f(y) = T$. Then, if $y \in T$, then $y \in f(y)$, which implies that $y \notin T$.

If $y \notin T$, then $y \notin f(y)$, which implies that $y \in T$.

Thus, it cannot be the case that $f(y) = T$. □

Definition (Cardinality Comparison)**.** Let A and B be sets. Then, we write $\text{card}(A) \leqslant \text{card}(B)$ if there exists an injective map $f : A \hookrightarrow B$.

We write $\text{card}(A) < \text{card}(B)$ if there exists an injection $f : A \hookrightarrow B$ but no bijection.

Example (Cardinality of the Power Set)**.** For every set,

$$\text{card}(S) < \text{card}(P(S)).$$

(1) We know that $\text{card}(S) \leqslant \text{card}(P(S))$, defining $f : S \hookrightarrow P(S)$, $f(a) = \{a\}$, since if $f(x) = f(y)$, then $\{x\} = \{y\}$, meaning $x \in \{y\}$, so $x = y$.

    In the case of $f : \varnothing \to \{\varnothing\}$, we define $\varnothing = f \subseteq \varnothing \times \{\varnothing\}$.

(2) Since there exists no bijection $f : S \to P(S)$, it is the case that $\text{card}(S) \neq \text{card}(P(S))$.

Example (Decimal Expansion)**.** We know that for some decimal expansion

$$\begin{aligned} 3.14159\ldots &= 3 + \frac{1}{10} + \frac{4}{100} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{n_i}{10^i}, \end{aligned}$$

with $0 \leqslant n_i \leqslant 9$ for $i \geqslant 1$.

However, we can also write any real number as

$$\sum_{i=0}^{\infty} \frac{n_i}{3^i}$$

with $0 \leqslant n_i \leqslant 2$ for all $i \geqslant 1$.

Example (Finite Strings)**.** Let S be the set of all finite strings of 0 and 1. S is countable.

Proof 1: We define $f : S \to \mathbb{N}$ by, for a string $x \in S$, x starts with n_1 zeroes, then has n_2 ones, then n_3 zeroes, etc. We define $f(x) := 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times 7^{n_4} \times 11^{n_5} \cdots$, or

$$f(x) = \prod_{i}^{\infty} p_i^{n_i},$$

where p_i denotes the ith prime number. We can see that f is an injection.

Since S is infinite (proof omitted), we can see that $f(S)$ is also infinite.[1] Since $f(S)$ is an infinite subset of $\mathbb{N}$, $f(S)$ is denumerable, meaning there exists a bijection $q : f(S) \to \mathbb{N}$. Therefore, we have $q \circ f : S \to \mathbb{N}$ is a bijection, meaning S is denumerable.

[1] If $f(S)$ is finite, then there exists a bijection $g : f(S) \to \{1, \ldots, n\}$. Composing g and f, we find S is finite as $g \circ f|_S$ is a bijection.

Proof 2: List the elements of S by length and lexicographic order: short strings come before long strings, and 0s come before 1s.

Rank	String
0	0
1	1
2	00
3	01
4	10
5	11
$\vdots$	$\vdots$

This pattern yields a systematic way to map S to the natural numbers.

Proof 3: We can see that

$$S = \bigcup_{i=1}^{\infty} S_i,$$

where S_i is the set of all strings of length i, each of which contains 2^i elements.

Since each S_i is finite, and $S_i \cap S_j = \emptyset$ (by definition). Thus, S is a countable union of pairwise disjoint countable sets, so S is countable.

Example (All Possible Writings)**.** Let W be the set of all possible writings in English. We let W_n denote the writing with n characters. Then,

$$W = \bigcup_{n=1}^{\infty} W_n,$$

which is a countable union of disjoint finite sets, which is countable.

Similarly, we can list all the writings by length and lexicographic order.

This result implies that "almost all" real numbers, in a sense, are unable to be described.

Section 1.3: Cantor–Schröder–Bernstein Theorem

Example. If we have $|A| \leqslant |B|$ and $|B| \leqslant |A|$, it does not necessarily imply $|A| = |B|$.

This is because the $\leqslant$ in the cardinality comparison implies there exist injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, not that the cardinalities are necessarily "less than or equal to" each other.

However, at the same time, this fact is true — this is what is known as the Cantor–Schröder–Bernstein Theorem.

Theorem (Cantor–Schröder–Bernstein)**:** Let $f : C \hookrightarrow D$ and $g : D \hookrightarrow C$ be injective maps. Then, $|C| = |D|$.

An Informal Proof Sketch. Consider C to be a set of cats and D to be a set of dogs. Every cat chases a dog, and every dog chases a cat, with different cats chasing different dogs and vice versa.

There are four potential arrangements:

(1) A set of cats and dogs are chasing each other in a circle.

(2) A chain of dogs chasing cats that starts with a dog.

(3) A chain of cats chasing dogs that starts with a cat.

(4) An endless chain of cats chasing dogs with no discernible start or end point.

These four cases create a bijection from C to D:

(1) Pair each cat with the dog that it is chasing.

(2) Pair each cat with the dog that it is chasing.

(3) Pair each cat with the dog that *is chasing it*.

(4) Pair each cat with the dog that it is chasing.

$\square$

A More Formal Proof Sketch. For $C = \{c_i\}_{i\in I}$ and $D = \{d_i\}_i$, we have four types of sequences.

(i) Circular sequence: for some $m \in \mathbb{N}$, there exist $c_1, \dots, c_m$ and $d_1, \dots, d_m$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$, where $c_{m+1} = c_1$.

(ii) Cat sequence: there is $c_1, c_2, \dots$ and $d_1, d_2, \dots$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

(iii) Dog sequence: there is $c_1, c_2, \dots$ and $d_1, d_2, \dots$ such that $f(c_i) = d_{i+1}$ and $g(d_i) = c_i$.

(iv) Bi-infinite sequence: $\{c_i\}_{i\in\mathbb{Z}}$ and $\{d_i\}_{i\in\mathbb{Z}}$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

Claim 1: For every $c \in C$, c is in exactly one sequence that is either a circular sequence, a cat sequence, a dog sequence, or a bi-infinite sequence.

We define our bijection $h : C \to D$ by

$$h(c) = \begin{cases} g^{-1}(c) & c \text{ in a dog sequence} \\ f(c) & \text{else} \end{cases}.$$

Claim 2: h is well-defined.

Claim 3: h is a bijection.

$\square$

Theorem: For every set A, B, either $|A| \leqslant |B|$ or $|B| \leqslant |A|$.

In order to prove this, we need the axiom of choice.

Example (Cardinality of the Reals)**.** Recall that $|\mathbb{N}| < |P(\mathbb{N})|$ and $|\mathbb{N}| < |\mathbb{R}|$. According to the previous theorem, it is the case that either $|P(\mathbb{N})| \leqslant |\mathbb{R}|$ or $|\mathbb{R}| \leqslant |P(\mathbb{N})|$.

In particular, $|P(\mathbb{N})| = |\mathbb{R}|$.

An Informal Proof. Let S be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. We will show that $|S| = |P(\mathbb{N})|$ and $|S| = |\mathbb{R}|$. This will show that $|P(\mathbb{N})| = |\mathbb{R}|$ (by composing bijections).

To show that $|S| = |P(\mathbb{N})|$, define a subset of $\mathbb{N}$ by the support[II] of some element of S. This is a bijection between $P(\mathbb{N})$ and S.

To show $|S| = |\mathbb{R}|$, we place a decimal point in front of the string, and consider it as a real number in base 2, which yields a bijection between S and $[0, 1]$.

Next, we show that $|[0, 1]| = |(0, 1)|$.

Finally, we show that $|(0, 1)| = \mathbb{R}$. Take $f : (0, 1) \to \mathbb{R}$ to be $\cot(\pi x)$ — or $\tan(\pi x - \pi/2)$. These are bijections from $(0, 1)$ to $\mathbb{R}$. $\square$

[II] The elements that f does not map to 0 for some $f \in S$.

Definition (Continuum Hypothesis)**.** We are aware that

$$|\mathbb{N}| < |\mathbb{R}| = |P(\mathbb{N})|.$$

The continuum hypothesis states that there exists no set S such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

The continuum hypothesis is independent of the ZFC axioms.[III]

Exercise (Challenge Problem)**:** Let $T = \left\{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{N};\ \text{finitely many nonzero } a_i\right\}$. Is T countable? We also write

$$T = \bigoplus_{i=0}^{\infty} \mathbb{N}.$$

Axiomatic Set Theory

Question: Is there a set A such that $A \in A$?

Answer. Yes! There is the set $\{\cdots\{\}\cdots\}$, which contains infinitely many sets in itself. Additionally, there is the set $A = \{x \mid x \text{ is a set}\}$.

Example (Russell's Paradox)**.** Consider the set

$$R = \{x \mid x \notin x\}.$$

The question is if $R \in R$. However, this cannot be true, because if $R \in R$, then $R \notin R$ and vice versa.

Axioms of Set Theory

We cannot just say

$$S = \{x \mid x \text{ is blah}\},$$

as evidenced by Russell's paradox. We need to carefully construct rules to create a rigorous description of formal set theory.

Axiom (Existence)**:** The existence axiom states that there exists a set:

$$\exists a\,(a = a).$$

Axiom (Empty Set)**:** The empty set axiom states that there exists a set with no elements:

$$\exists a\,\forall x\,(x \notin a).$$

Axiom (Pairing)**:** The pairing axiom states that, given any sets a and b, there is a set c such that the only elements of c are a and b:

$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \Leftrightarrow x = a \vee x = b)$$

Axiom (Extensionality)**:** The axiom of extensionality states that if two sets have the same elements, they are the same sets:

$$\forall a\,\forall b\,(\forall x\,(x \in a \Leftrightarrow x \in b) \Rightarrow a = b)$$

Question: What is a set?

[III] Zermelo–Fraenkel Axioms with the Axiom of Choice.

Answer. The unsatisfying answer is that "set" and "element" have no meaning *per se*. The main reason we define these axioms is to define relationships between objects (rather than objects themselves).

Example. We want to prove that for every set b, there exists a set $\{b\}$.

Symbolically, we want to show

$$\forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = b).$$

In particular, we can see that, in the pairing axiom, there is no requirement that a and b be distinct. Therefore, we can use the pairing axiom of $a = b$ and $b = b$. Therefore, the pairing axiom becomes

$$\forall b\, \forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = b \vee x = b),$$

which reduces to

$$\forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = b).$$

In particular, if $b = \{\}$ in the previous example, then the pairing axiom implies the uniqueness of the empty set. We will denote $\{\} = \emptyset$. We can create a tower

$$\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \dots,$$

entirely consisting of the empty set.

Axiom (Union)**:** The axiom of union states that if a and b are sets, there exists a set c whose elements are either elements of a or elements of b, and every element of a is in c and every element of b is in c:

$$\forall a\, \forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x \in a \vee x \in b)$$

Definition. The string $a \subseteq b$ is shorthand for

$$\forall x\, (x \in a \Rightarrow x \in b).$$

Axiom (Power Set)**:** The power set axiom states that for all a, there is a set b such that all elements of b are subsets of a and all subsets of a are contained in b:

$$\forall a\, \exists b\, \forall y\, (y \in b \Leftrightarrow y \subseteq a).$$

Definition. We let (a, b) be shorthand for the set

$$\{a, \{a, b\}\}.$$

Exercise: If $\{a, \{a, b\}\} = \{c, \{c, d\}\}$, it is the case that $a = c$ and $b = d$.

Recall that

$$c = \{x \mid x \text{ is blah}\}$$

is a problematic definition of a set. However, if a is a set, we can define

$$c = \{x \mid x \in a \wedge x \text{ is blah}\},$$

which does not cause any contradictions. The following axiom schema formalizes this fact.

Axiom (Comprehension schema)**:** The comprehension schema says that, given any formula $\varphi(x)$, in which x is a free variable, there exists a set c whose elements are those in a that satisfy φ:

$$\forall a\, \exists c\, \forall x\, (x \in c \Leftrightarrow x \in a \wedge \varphi(x)).$$

Remark: There are infinitely many axioms in the comprehension schema, one for each formula φ. This is why it is known as a schema rather than an axiom.

Remark: Since we can specify a formula $\varphi(x) : x \neq x$, the comprehension schema obviates the empty set axiom.

Example (Some Logic)**.** An example of a formula is $\forall p\ \exists q(p \Rightarrow q)$.

In the formula $\exists q\ \ (p \Rightarrow q)$, we say p is a free variable.

The main symbols in logic are $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, ()$ (the symbols that make up propositional logic), as well as $\forall, \exists$ (which form the basis of first-order logic).

In propositional logic, the only two symbols that are needed are $\wedge$ and $\neg$ (or $\vee$ and $\neg$).[IV]

When we get to set theory, the last symbol we need is $\in$.

We can build larger formulae by substituting formulae into other formulae.

Example (Using the Comprehension Schema)**.** Let $\phi(x) : \exists y\,(y \in X)$. This is an axiom:

$$\forall a\ \exists b\ \forall x\,(x \in b \Leftrightarrow x \in a \wedge \exists y\,(y \in x))$$

In particular, this axiom is equivalent to saying

$$\forall a\ \exists b \text{ s.t. } b = \{x \in a \mid x \neq \varnothing\}.$$

Axiom (Union)**:** The union axiom states that for a collection of sets T, there is a union of the sets, $a = \bigcup T$.

$$\forall t\ \exists a\ \forall x\,(x \in a \Leftrightarrow \exists y\,(y \in t \wedge x \in y)).$$

Alternatively, we can say

$$\forall t\ a = \{x \mid x \in \text{some element of } t\}$$

is a set.

Axiom (Infinity)**:** There exists an infinite set.

$$\exists a\,(\varnothing \in a \wedge \forall x\,(x \in a \Rightarrow x \cup \{x\} \in a))$$

Remark: To see that this set, a has an element, $\varnothing$. Thus,

$$a = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \dots\}$$

We define $0 = \varnothing$, $1 = \{\varnothing, \{\varnothing\}\}$, etc. Thus, the axiom of infinity defines the natural numbers.

Axiom (Regularity)**:** There is no infinite chain of the form

$$\cdots \in d \in c \in b \in a.$$

$$\forall s \exists x\,(s = \varnothing \vee s \neq \varnothing \Rightarrow (x \in s \wedge x \cap s = \varnothing))$$

Remark: The existence of this axiom is meant to obviate the case where we imagined a set a with $a \in a$.

Definition (Function-like Formula)**.** Let $\psi(x, y)$ be a formula with x, y free variables such that $\forall x, y, z$, $\psi(x, y) \wedge \psi(x, z) \Rightarrow y = z$.

Axiom (Replacement Schema)**:**

$$\forall a\ \exists b\ \forall x\,(x \in b \Leftrightarrow \exists y\,(y \in a \wedge \psi(x, y)))$$

Remark: It is possible to prove the comprehension schema from the replacement schema.

[IV] In computers, the only gate that is necessary is the NAND gate.

The axioms that we have discussed so far are known as the Zermelo–Fraenkel axioms.

Question: If A and B are nonempty, is it the case that $A \times B \neq \emptyset$

Answer. This is true. There exists $a \in A$ and $b \in B$ such that $(a, b) \in A \times B$. This can be proven using the ZF axioms.

Question: If $A_1, A_2, \ldots, \neq \emptyset$, then is $A_1 \times A_2 \times \cdots \neq \emptyset$?

Answer. This requires the axiom of choice.

Axiom (Choice)**:** If T is a collection of sets, $\exists b$ such that $\forall a \in T$, $a \cap b \neq \emptyset$.

$$\forall t\, \exists b\, (\forall a\, (a \in t \Rightarrow \exists x\, (x \in a \wedge x \in b))) .$$

Remark: We define $x \in (a \cap b)$ as shorthand for $x \in a \wedge x \in b$.

Remark: The axiom of choice is controversial.

Remark: The axiom of choice entails certain counterintuitive results, such as the Banach–Tarski paradox[v] and the existence of non-measurable sets.

The Banach–Tarski paradox states that for any two bounded subsets of $\mathbb{R}^3$ with nonempty interior, one of the sets can be partitioned into finitely many subsets, with certain isometries applied to said partition, and reconstituted into the second set.

Recall:

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

Definition. For any sets A and B, each subset of $A \times B$ is a relation from A to B.

Definition. A relation $R \subseteq A \times B$ is a function if

$$\forall x \forall y \forall z\, ((x, y) \in R \wedge (x, z) \in R \Rightarrow y = z) .$$

Definition. A function $F \subseteq A \times B$ is injective if

$$\forall x \forall x' \forall y\, ((x, y) \in F \wedge (x', y) \in F \Rightarrow x = x')$$

Notation: For some statement φ,

$$\forall x \in A\, (\varphi)$$

is shorthand for

$$\forall x\, (x \in A \Rightarrow \varphi)$$

Notation: If $F \subseteq A \times B$ and $\forall x \in A, (x, y) \in F$, then we write $F : A \to B$.

Also, $\forall (x, y) \in F$, we write $F(x) = y$.

Definition. A function F is onto B if

$$\forall y \in B\, \exists x\, (x, y) \in F.$$

Remark: Do not say "onto" without mentioning B. It is okay to say $F : A \to B$ is onto (or surjective).

Example. We wish to show that if $f : A \xrightarrow{\text{onto}} B$, then there exists a function $g : B \to A$ such that g is an injection.

Since f is onto B, for every $b \in B$, there exists $a \in A$ such that $f(a) = b$. We define $g(b)$ to be a particular choice function on the set of all a such that $f(a) = b$.

[v]Hey, one of the topics for my Honors thesis is on this.

Remark: The above statement (that every surjective function has a right-inverse, which is necessarily injective) is an equivalent statement to the axiom of choice.

Example (Natural Numbers)**.** Since the empty set exists, we can define $\emptyset = \{\} = 0$. We set $1 = \{0\}$, $2 = \{0, 1\}$, etc. We have $n = \{0, \dots, n-1\}$.

If we take $n \cup \{n\}$, we have

$$\begin{aligned}\{0, \dots, n-1\} \cup \{n\} &= \{0, \dots, n\}\\ &= n + 1.\end{aligned}$$

In other words, we define addition by taking $n \cup \{n\}$.

Question: Is $n \in n + 1$? Is $n \subseteq n + 1$?

Answer. Yes and yes.

Definition. We say $m < n$ if $m \in n$, or $m \subseteq n$.

Example. We will use the ZF axioms to show that there exists a set whose elements are all the natural numbers.

Defining using the axiom of infinity, we get

$$\exists s\ (\emptyset \in s \wedge \forall x\,(x \in s \Rightarrow x \cup \{x\} \in s) \wedge \forall y\,(y \in s \Rightarrow y = \emptyset \vee \exists x\,(x \cup \{x\} = y)))$$