Problem (Problem 1)**:** Let R be a ring in which every element a satisfies $a^2 = a$. Show that

(a) $2a = 0$ for every $a \in R$, so $a = -a$;

(b) R is commutative.

Solution:

(a) Let $a \in R$. We see that, since $a + a \in R$, $(a+a)^2 = a + a$, so that

$$\begin{aligned} a + a &= (a+a)^2 \\ &= (a+a)(a+a) \\ &= a^2 + a^2 + a^2 + a^2 \\ &= a + a + a + a, \end{aligned}$$

and since R is a ring, we see that $a + a = 0$, or that $a = -a$.

(b) Similarly, if $a, b \in R$, then since $(a+b)^2 = a + b$, we have

$$\begin{aligned} a + b &= (a+b)^2 \\ &= (a+b)(a+b) \\ &= a^2 + b^2 + ab + ba \\ &= a + b + ab + ba, \end{aligned}$$

so $ab = -ba$, but since $-ba = ba$ by the previous part, we have $ab = ba$, and so R is commutative.

Problem (Problem 2)**:** Let R be a ring with identity, and let $R^\times$ be the set of invertible elements of R. Show that $R^\times$ is a group under multiplication. What is $\mathbb{Z}[i]^\times$.

Solution: First, $R^\times$ is nonempty, as R contains a multiplicative identity. Next, if $a, b \in R^\times$, we see that ab admits the inverse $b^{-1}a^{-1}$, as

$$\begin{aligned} (ab)\left(b^{-1}a^{-1}\right) &= a\left(bb^{-1}\right)a^{-1} \\ &= aa^{-1} \\ &= 1, \end{aligned}$$

and similarly,

$$\begin{aligned} \left(b^{-1}a^{-1}\right)(ab) &= b^{-1}\left(a^{-1}a\right)b \\ &= b^{-1}b \\ &= 1, \end{aligned}$$

so $R^\times$ is closed under multiplication. Similarly, since $\left(b^{-1}\right)^{-1} = b$ for any $b \in R^\times$, every element of $R^\times$ has a multiplicative inverse, so $R^\times$ is a group.

To understand the picture of $\mathbb{Z}[i]^\times$, we try to understand when, given $a + bi \in \mathbb{Z}[i] \subseteq \mathbb{C}$, $\frac{1}{a+bi} \in \mathbb{Z}[i]$. Doing the hand calculations, we see that

$$\frac{1}{a+bi} = \frac{1}{a^2+b^2}(a - bi).$$

Therefore, we see that this holds if and only if $a = \pm 1$ and $b = 0$, or $b = \pm 1$ and $a = 0$, meaning that $\mathbb{Z}[i]^\times = \{1, i, -1, -i\}$.

Problem (Problem 3)**:** Fix an integer $n > 1$. Recall that for $a, b \in \mathbb{Z}$, we write $a \equiv b$ modulo n if $a - b$ is divisible by n. Show that this relation is an equivalence relation on $\mathbb{Z}$. Furthermore, show that if $a \equiv b$

modulo n, and $c \equiv d$ modulo n, then

$$a + c \equiv b + d \text{ modulo } n, \text{ and } ac \equiv bd \text{ modulo } n.$$

Solution: Since 0 is divisible by n, it is clear that $a \equiv a$ modulo n, so the relation is reflexive.

If $a \equiv b$ modulo n, then since $n|(a-b)$, we must also have $n|(b-a)$, so $b \equiv a$ modulo n, so the relation is symmetric.

Finally, if $a \equiv b$ modulo n and $b \equiv c$ modulo n, then since $n|a-b$ and $n|b-c$, by adding, we see that $n|(a-b)+(b-c)$, so $n|a-c$ and $a \equiv c$ modulo n, so the relation is transitive.

Now, if $a \equiv b$ modulo n, and $c \equiv d$ modulo n, then since $n|(a-b)$ and $n|(c-d)$, by adding, we see that $n|(a+c)-(b+d)$, so $a+c \equiv b+d$ modulo n. To see the last equivalence, we rewrite $a = b + kn$, $c = d + \ell n$, where $k, \ell \in \mathbb{Z}$. Thus, multiplying things out, we see that

$$\begin{aligned} ac &= (b+kn)(d+\ell n) \\ &= bd + nkd + \ell nb + k\ell n^2 \\ &= bd + (kd + \ell b + k\ell n)n, \end{aligned}$$

and since $kd + \ell b + k\ell n \in \mathbb{Z}$, we have $ac \equiv bd$ modulo n.

Problem (Problem 4)**:** Show that a finite commutative ring with 1 and without zero divisors is a field.

Solution: Let $a \in R$, and consider the map $\varphi_a \colon R \setminus \{0\} \to R \setminus \{0\}$ given by $b \mapsto ab$. We see that if $ab = ac$, then $a(b-c) = 0$, and since $a \neq 0$, we see that $b = c$, so φ_a is injective. Since φ_a is an injective self-map of a finite set, φ_a is surjective, so φ_a is bijective, and thus $\varphi_a^{-1}(1)$ is well-defined, so $a\varphi_a^{-1}(1) = 1$, meaning a has a right-inverse. Since R is commutative, we have $\varphi_a^{-1}(1)a = 1$, so R is a field.

Problem (Problem 5)**:** Let $R = \text{Mat}_n(\mathbb{R})$ be the ring of real $n \times n$ matrices. Show that if A satisfies $\det(A) = 0$, then there exist nonzero $B, C \in R$ such that $AB = \mathbf{0}_n$ and $CA = \mathbf{0}_n$.

Solution: If A is the zero matrix, then the problem is trivial, so we assume A is not the zero matrix. By the Cayley–Hamilton theorem, since 0 is an eigenvalue of A, we must have that $c_A(t)$ includes a factor of t, which is irreducible, so that the minimal polynomial $m_A(t)$ has a factor of t. Thus, we may write $m_A(t) = tp(t)$, where $p(t)$ is some other monic polynomial. Since $\deg p(t) < \deg m_A(t)$, this means that $p(A) \neq \mathbf{0}_n$ (else, it would contradict the minimality of p). By setting $B = p(A)$, we find our desired nonzero matrix B such that $AB = BA = \mathbf{0}_n$.

Problem (Problem 6)**:** An element $x \in R$ is called *nilpotent* if there exists $n > 0$ such that $x^n = 0$.

Assume R is a commutative ring with identity. Show that if $x \in R$ is nilpotent, then

(a) rx is nilpotent for any $r \in R$;

(b) $1 + x$ is invertible.

Solution:

(a) We see that, since R is commutative,

$$\begin{aligned} (rx)^n &= (rx)(rx)\cdots(rx) \\ &= r^n x^n \\ &= 0, \end{aligned}$$

so rx is nilpotent.

(b) We see that if a is nilpotent, then

$$1 = 1 - a^n$$

$$= (1 - a)\left(1 + a + \cdots + a^{n-1}\right),$$

meaning that $1 - a$ is invertible. Furthermore, we note that if a is nilpotent, then so is $-a$, as $-a = (-1)a$, allowing us to apply part (a). Thus, $1 + x = 1 - (-x)$ is invertible if x is nilpotent.

Problem (Problem 7)**:** Let $R = \mathrm{Mat}_n(\mathbb{F})$, where $\mathbb{F}$ is a field. Show that if I is a nonzero 2-sided ideal of R, then $I = R$.

Solution: We show that if I is a nonzero two-sided ideal in $\mathrm{Mat}_n(\mathbb{F})$, then $I_n \in I$.

Since I is nonzero, there is some matrix $\left(a_{ij}\right)_{i,j} \in I$ such that at particular indices i_0 and j_0, $a_{i_0j_0} \neq 0$. Since $a_{ij} \in \mathbb{F}$ for all i, j, we have that $a_{i_0j_0}^{-1}$ exists.

Let $e_{k\ell}$ be the matrix unit with a position 1 at index (k, ℓ) and zero elsewhere. Then, via some matrix algebra, we see that

$$a_{i_0j_0}(e_{kk})_{i,j} = \left(e_{ki_0}\right)_{i,j}\left(a_{ij}\right)_{i,j}\left(e_{j_0k}\right)_{i,j}$$

which is necessarily in I, as I is a two-sided ideal. Therefore, since $\mathbb{F}$ is a field, we see that $(e_{kk})_{i,j} \in I$ for each k, so $\sum_{k=1}^{n}(e_{kk})_{i,j} \in I$, so $I_n \in I$, meaning $I = R$.

Problem (Problem 8)**:** Let $n \in \mathbb{N}$ and consider $\mathbb{Z}^n$ as a ring with component-wise addition and multiplication.

(a) Prove that $\mathrm{aut}_{\text{group}}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$.

(b) Prove that $\mathrm{aut}_{\text{ring}}(\mathbb{Z}^n) \cong \mathrm{Sym}(n)$.

Solution: Before we start, we first notice that every element of $\mathbb{Z}^n$ can be written as

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n,$$

where e_j are the standard basis of $\mathbb{Z}^n$ and $a_j \in \mathbb{Z}$ for each j. Therefore, if φ is any automorphism as either a group or a ring, we may write $\varphi(v)$ as some integer linear combination of $\varphi\left(e_j\right)$, where the e_j are the standard basis vectors for $\mathbb{Z}^n$.

(a) Let $\varphi \in \mathrm{aut}_{\text{group}}(\mathbb{Z}^n)$. If $v \in \mathbb{Z}^n$ is some vector, then

$$\begin{aligned}\varphi(v) &= \varphi(a_1e_1 + a_2e_2 + \cdots + a_ne_n)\\ &= a_1\varphi(e_1) + a_2\varphi(e_2) + \cdots + a_n\varphi(e_n).\end{aligned}$$

Since a linear transformation may be specified uniquely via a basis, we may specify a matrix element $A_\varphi \in \mathrm{Mat}_n(\mathbb{Z})$ by

$$A_\varphi e_j = \varphi\left(e_j\right)$$

for each j. Note that since each φ is invertible, each A_φ may have A_φ^{-1} defined by $A_\varphi^{-1}e_j = \varphi^{-1}\left(e_j\right)$, so each $A_\varphi \in \mathrm{GL}_n(\mathbb{Z})$. Similarly, we see that if $\psi, \varphi \in \mathrm{aut}_{\text{group}}(\mathbb{Z}^n)$, then

$$\begin{aligned}\psi \circ \varphi\left(e_j\right) &= A_\psi\left(\varphi\left(e_j\right)\right)\\ &= A_\psi\left(A_\varphi e_j\right)\\ &= A_\psi A_\varphi e_j.\end{aligned}$$

Therefore, the map $\varphi \mapsto A_\varphi$ is an isomorphism, so $\mathrm{aut}_{\text{group}}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$.

(b) Let $\varphi \in \mathrm{aut}_{\text{ring}}$; notice that $\mathrm{aut}_{\text{ring}} \subseteq \mathrm{aut}_{\text{group}}$ meaning that we know already that φ can be written as some element of $\mathrm{GL}_n(\mathbb{Z})$. Suppose that, for some e_k, we may write

$$\varphi(e_k) = \sum_{k=1}^{n} a_ke_k.$$

Notice then that, since $e_ie_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol, we get

$$\begin{aligned}\varphi\left(e_k^2\right) &= \varphi(e_k)\varphi(e_k)\\ &= \sum_{k=1}^{n} a_k^2 e_k\\ &= \sum_{k=1}^{n} a_k e_k,\end{aligned}$$

meaning in particular that $a_k = 0$ or $a_k = 1$, seeing as the a_k are elements of $\mathbb{Z}$.

Now, given a standard basis vector e_i, we notice that

$$\varphi(e_i) = \sum_{m=1}^{k} e_{i_m}.$$

Yet, since φ is a ring automorphism, so too is φ^{-1}, so that

$$e_i = \sum_{m=1}^{k} \varphi^{-1}(e_{i_m}).$$

Now, this means that the sum $\sum_{m=1}^{k} \varphi^{-1}(e_{i_m})$ yields a 1 in position i and zero everywhere else; yet, since $\varphi^{-1}(e_{i_m})$ yields another sum of basis vectors, we see that this can only hold if $\varphi^{-1}(e_{i_\ell}) = e_i$ for exactly one specific ℓ (else, we get that e_i either has nonzero entries other than at i, or that the entry at i is greater than or equal to 2).

Thus, $\varphi(e_i) = e_k$ for some e_k; since φ is injective on $\mathbb{Z}^n$, φ is necessarily injective on $\{e_1, \dots, e_n\}$ as we just established. Therefore, φ is some permutation of $\{e_1, \dots, e_n\}$, so $\text{aut}_{\text{ring}} \cong \text{Sym}(n)$.