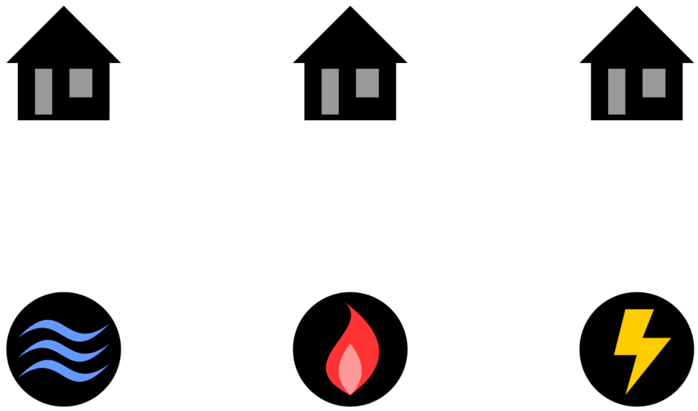
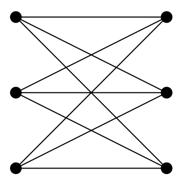
Graphs and the Three Utilities Problem

We can imagine trying to connect three houses below with three utilities without the utility lines crossing.



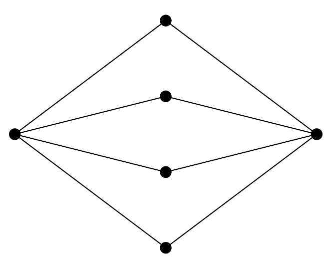
This problem is akin to the graph $K_{3,3}$ (the complete bipartite graph with three vertices in each partite set).



A *graph* is an ordered pair of sets (V, E), where $E \subseteq V \times V$.

For example, if $V = \{a, b, c\}$ and $E = \{(a, b), (a, c)\}$, then (V, E) is a graph. The goal of the three utilities puzzle is to draw $K_{3,3}$ in $\mathbb{R}^2$ without any edges crossing. A graph that can be drawn as such is *planar*.

- $K_{3,3}$ is not planar.
- $K_{2,4}$ is planar.



Euler's Theorem

Let $G \subseteq \mathbb{R}^2$ be a planar graph (i.e., drawn in $\mathbb{R}^2$ without edge crossings). Each disjoint subset of $\mathbb{R}^2 - G$ is a *face* of G.

For every graph G embedded in $\mathbb{R}^2$ (i.e., drawn without edge crossings) with V vertices, E edges, and F faces, the following is true:

$$V - E + F = 2$$

We will use this theorem to show that you cannot connect the three houses to the three utilities as follows:

Outline Proof (of $K_{3,3}$'s non-planarity)

Suppose toward contradiction that $K_{3,3}$ is planar. Then, by Euler's Theorem, we know that $V - E + F = 2$.

We know that $K_{3,3}$ has six vertices and nine edges, so we know that $6 - 9 + F = 2$. Therefore, we know that there must be 5 faces. In order to enclose a face, there must be at least four edges in $K_{3,3}$ (as there is no edge between two members of a partite set). Additionally, each edge encloses two faces. Therefore, $E \geqslant 2F$. However, since $E = 9$, and we assume that $F \geqslant 5$, we have reached a contradiction (as $9 < 10$). Thus, $K_{3,3}$ is not planar.

Four-Color Theorem

Every planar graph can be colored (adjacent vertices do not have the same color) with four colors. The planar graph can be colored by fewer colors.

Polynomial Example

Let $p(a, b, c, d) = ab + ac + ad + bc + bd + cd$. When we factor, we get $p(a, b, c, d) = a(b + c + d) + b(c + d) + cd$. In the first equation, we had to carry out 6 multiplications, while in the second equation we only had to carry out 3 multiplications. We could factor differently:

$$\begin{aligned} p(a, b, c, d) &= ab + ac + ad + bc + bd + cd \\ &= a(b + c + d) + b(c + d) + cd \\ &= (a + b)(c + d) + ab + cd \end{aligned}$$

We have a lower bound of three multiplications to carry out.

In the arbitrary case, we have the following. We want to find the lowest number of multiplications.

$$p(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j$$

The minimum number of multiplications we can do is $n-1$. We can find this via a graph with n vertices $\{x_1, \dots, x_n\}$, and for $x_i x_j$ in p, we have an edge from x_i to x_j. This is the complete graph on n vertices, K_n. Each complete bipartite subgraph represents a multiplication — so our question can be restated as follows:

> Given a complete graph on n vertices, K_n, partition its edges into as few complete graphs as possible.

The answer for this is $n - 1$, with a proof in linear algebra. However, there is no graph theory-specific proof for this question.