Problem 1

(i) Let $f\colon X \to Y$ be a function, and let $A \subseteq X$. Then, if $x \in A$, we must have $f(x) \in f(A)$, so that $x \in f^{-1}(f(A))$, meaning $A \subseteq f^{-1}(f(A))$. If f is injective, then for any $z \in f^{-1}(f(X))$, there is a unique $y \in f(X)$ such that $f(z) = y$ (by injectivity), meaning that f is left-invertible; therefore, $f^{-1}(f(A)) = A$ for all such $A \subseteq X$.

(ii) If $B \subseteq Y$, then for any $f(x) \in f\left(f^{-1}(B)\right)$, $x \in f^{-1}(B)$, or that $f(x) \in B$, meaning that $f\left(f^{-1}(B)\right) \subseteq B$. If f is surjective, then for any $y \in Y$, there is some $z \in X$ such that $f(z) = y$, or that $z \in f^{-1}(\{y\})$; therefore, we may select a right-inverse for f, meaning that $f\left(f^{-1}(X)\right) = Y$, or $f\left(f^{-1}(B)\right) = B$ for all subsets $B \subseteq Y$.

Problem 6

Let $X \neq \emptyset$ and $\mathcal{C} \subseteq \mathcal{P}(X)$. We define the topology $\tau_{\mathcal{C}}$ on X by

$$\tau_{\mathcal{C}} = \bigcap\left\{\tau \;\middle|\; \tau \subseteq \mathcal{P}(X) \text{ is a topology}, \mathcal{C} \subseteq \tau\right\}.$$

The intersection is indeed well-defined, as the discrete topology includes $\mathcal{C}$, and it is the smallest such topology as any other topology on X that contains $\mathcal{C}$ is included in the intersection.

Problem 9

Let τ_1, τ_2 be topologies on X with respective bases $\mathcal{B}_1, \mathcal{B}_2$.

Let $\tau_1 \subseteq \tau_2$, $x \in X$, and $B_1 \in \mathcal{B}_1$ with $x \in B_1$. Since $B_1 \in \mathcal{B}_1$, $B_1 \in \tau_1$, so $B_1 \in \tau_2$, meaning that $B_1 = \bigcup_{i \in I} U_i$ for some family $\{U_i\}_{i \in I} \subseteq \mathcal{B}_2$. There is at least one such $U_i \in \mathcal{B}_2$ with $x \in U_i$; setting $B_2 := U_i$, we have $x \in B_2 \subseteq B_1$.

Suppose now that for any $x \in X$, $B_1 \in \mathcal{B}_1$ with $x \in B_1$, there is $B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$. Let $U \in \tau_1$. Then, there is some family $\{B_i\}_{i \in I} \subseteq \mathcal{B}_1$ such that $\bigcup_{i \in I} B_i = U$. For each $x \in B_i$, we find $V_{x,i} \in \mathcal{B}_2$ such that $x \in V_{x,i} \subseteq B_i$. Then, we get $\bigcup\{V_{x,i} \mid x \in U, i \in I\} = U$, whence $U \in \tau_2$.

Problem 14

Let $f\colon X \to Y$ be a map of topological spaces.

(i) Let f be open. Then, for any $A \subseteq X$, we see that $f(A^\circ)$ is open in Y. Since $A^\circ \subseteq A$, we see that $f(A^\circ) \subseteq f(A)$, so since $f(A^\circ)$ is open, it is contained in $f(A)^\circ$, meaning that $f(A^\circ) \subseteq f(A)^\circ$.

Now, let f be such that $f(A^\circ) \subseteq f(A)^\circ$ for all $A \subseteq X$. Let $U \subseteq X$ be open, so $U^\circ = U$. Then, since $f(U^\circ) \subseteq f(U)^\circ$, we see that $f(U) \subseteq f(U)^\circ$, and since $f(U)^\circ \subseteq f(U)$ necessarily, we have $f(U) = f(U)^\circ$, or that f is an open map.

(ii) Let f be closed. Then, for any $A \subseteq X$, we have that $f\left(\overline{A}\right)$ is closed in X. Since $A \subseteq \overline{A}$, we have that $f(A) \subseteq f\left(\overline{A}\right)$, and since $f\left(\overline{A}\right)$ is closed, we also have $\overline{f(A)} \subseteq f\left(\overline{A}\right)$.

Now, let f be such that $\overline{f(A)} \subseteq f\left(\overline{A}\right)$, and let $C \subseteq X$ be closed, meaning $\overline{C} = C$. Then, as assumed, we have $\overline{f(C)} \subseteq f\left(\overline{C}\right)$, but since $C = \overline{C}$, we have $f\left(\overline{C}\right) = f(C) \subseteq \overline{f(C)}$, meaning $\overline{f(C)} = f(C)$, or that $f(C)$ is closed.

Problem 18

(i) Let

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{i \in I} U_i \;\middle|\; U_i = B_i \in \mathcal{B}_i \text{ for finitely many } i \right\}.$$

We show that $\mathcal{B}_{\text{prod}}$ is a basis. First, for each $\mathcal{B}_i$, $\bigcup \mathcal{B}_i = X_i$, so by taking products we find that $\bigcup \mathcal{B}_{\text{prod}} = \prod_{i \in I} X_i$. Then, for any two elements U and V of $\mathcal{B}_{\text{prod}}$, there are finitely many $\{U_{i_k}\}_{k=1}^{n}$ and $\left\{V_{i_j}\right\}_{j=1}^{n}$ in $\mathcal{B}_{i_k}$ and $\mathcal{B}_{i_j}$ such that U and V are the respective products with X_i elsewhere. We have three cases.

- If $U_{i_j} \neq X_{i_j}$ and $V_{i_j} = X_{i_j}$ at index i_j, then the intersection $U \cap V$ has U_{i_j} at index i_j, meaning that their intersection is yet again a product of basis elements with all but finitely many equal to X_i.
- Similarly, if $U_{i_j} = X_{i_j}$ and $V_{i_j} \neq X_{i_j}$, the intersection $U \cap V$ yields yet another product of basis with elements with all but finitely many equal to X_i.
- If $U_{i_j}, V_{i_j} \neq X_{i_j}$, then there is some $W_{i_j} \in \mathcal{B}_{i_j}$ such that $W_{i_j} \subseteq U_{i_j} \cap V_{i_j}$, meaning that the intersection $U \cap V$ contains a product of basis elements with all but finitely many equal to X_i.

(ii) Let

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{i \in I} B_i \;\middle|\; B_i \in \mathcal{B}_i \right\}.$$

We can see immediately that $\mathcal{B}_{\text{box}}$ is a basis, as $\bigcup \mathcal{B}_i = X_i$ for each i, so by taking products, we get that $\bigcup \mathcal{B}_{\text{box}} = \prod_{i \in I} X_i$. Similarly, since the finite intersection of basis elements always contains a basis element, taking $U, V \in \mathcal{B}_{\text{box}}$, we have $\left(\prod_{i \in I} U_i\right) \cap \left(\prod_{i \in I} V_i\right) = \prod_{i \in I}(U_i \cap V_i)$, which contains some box $\prod_{i \in I} W_i$ where $W_i \in \mathcal{B}_i$.

(iii) If the indexing set I is finite, we see that $\tau_{\text{box}} = \tau_{\text{prod}}$. Else, if I is infinite, we have that $\tau_{\text{prod}} \subsetneq \tau_{\text{box}}$.