

Abstract

We discuss and prove the three big theorems of real analysis — the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

Integration: An Introduction

In order to discuss integration, we need to start with the building blocks of all functions — simple functions.

Definition. Let X be a measure space, and let $\phi: X \rightarrow [0, \infty]$ be a function. We say ϕ is a *simple function* if it has finite range (and does not take the value $+\infty$).

The *standard form* of a simple function ϕ is

$$\phi = \sum_{k=1}^n c_k \mathbf{1}_{E_k},$$

where $\{c_1, \dots, c_n\} = \text{Ran}(\phi)$, and $E_k = \phi^{-1}(\{c_k\})$.

Recall that a function $f: X \rightarrow \mathbb{R}$, where (X, \mathcal{M}, μ) is a measure space, is called Borel-measurable (or just measurable) if, for every $E \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(E) \in \mathcal{M}$.

Definition. If $\phi: X \rightarrow [0, \infty]$ is a simple, measurable function defined on a measure space (X, \mathcal{M}, μ) , then the *integral* of ϕ is defined to be

$$\int_X \phi \, d\mu = \sum_{k=1}^n c_k \mu(E_k). \quad (\dagger)$$

Proposition: Let $\phi, \psi: X \rightarrow [0, \infty]$ be simple functions with standard forms

$$\begin{aligned} \phi &= \sum_{j=1}^n a_j \mathbf{1}_{E_j} \\ \psi &= \sum_{k=1}^m b_k \mathbf{1}_{F_k}. \end{aligned}$$

Then, the following hold

- (a) for all $c > 0$, $\int_X c\phi \, d\mu = c \int_X \phi \, d\mu$;
- (b) $\int_X \phi + \psi \, d\mu = \int_X \phi \, d\mu + \int_X \psi \, d\mu$;
- (c) if $\phi \leq \psi$ pointwise, then $\int_X \phi \, d\mu \leq \int_X \psi \, d\mu$.

Proof.

- (a) We see that

$$\begin{aligned} \int_X c\phi \, d\mu &= \sum_{j=1}^n (c)(a_j)\mu(E_j) \\ &= c \sum_{j=1}^n a_j \mu(E_j) \\ &= c \int_X \phi \, d\mu. \end{aligned}$$

(b) Note that since

$$\begin{aligned} X &= \bigsqcup_{j=1}^n E_j \\ &= \bigsqcup_{k=1}^m F_k, \end{aligned}$$

we must have

$$\begin{aligned} E_j &= \bigsqcup_{k=1}^m E_j \cap F_k \\ F_k &= \bigsqcup_{j=1}^n F_k \cap E_j \end{aligned}$$

as a disjoint union. Therefore,

$$\begin{aligned} \int_X \phi \, d\mu + \int_X \psi \, d\mu &= \sum_{j=1}^n \sum_{k=1}^m (a_j + b_k) \mu(E_j \cap F_k) \\ &= \int_X \phi + \psi \, d\mu. \end{aligned}$$

(c) If $\phi \leq \psi$, $a_j \leq b_k$ whenever $E_j \cap F_k \neq \emptyset$. Therefore,

$$\begin{aligned} \int_X \phi \, d\mu &= \sum_{k=1}^m \sum_{j=1}^n a_j \mu(E_j \cap F_k) \\ &\leq \sum_{k=1}^m \sum_{j=1}^n b_k \mu(E_j \cap F_k) \\ &= \int_X \psi \, d\mu. \end{aligned}$$

□

Having established integrals for simple functions, we need to establish a convergence property for simple functions for all measurable functions.

Theorem: Let (X, \mathcal{M}, μ) be a measure space, and let $f: X \rightarrow [0, \infty]$ be a measurable function. Then, there is an increasing sequence $(\phi_n)_n$ of simple functions that converges pointwise to f . This sequence converges uniformly to f on any bounded sets.

Proof. For each n , partition the interval $[0, 2^n]$ into subintervals of length 2^{-n} . There are 2^{2n} subintervals, with

$$\begin{aligned} I_{n,0} &= \left[0, \frac{1}{2^n}\right] \\ I_{n,k} &= \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], \end{aligned}$$

where $0 \leq k \leq 2^{2n} - 1$. We define $J_n = (2^n, \infty]$. Define

$$\begin{aligned} E_{n,k} &= f^{-1}(I_{n,k}) \\ F_n &= f^{-1}(J_n). \end{aligned}$$

Then, we may take

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n}.$$

The family ϕ_n are simple, measurable, positive, and increasing.

Fix $x \in X$ such that $f(x) < \infty$, and find N such that $f(x) \leq 2^N$. Then, for a fixed $n \geq N$, there is $0 \leq k \leq 2^{2n} - 1$ such that $x \in E_{n,k}$. Thus,

$$\begin{aligned} |\phi_n(x) - f(x)| &= \left| f(x) - \frac{k}{2^n} \right| \\ &\leq \frac{1}{2^n}. \end{aligned} \tag{*}$$

Thus, this family is pointwise convergent.

If $f(x) = +\infty$, then $\phi_n(x) = 2^n$ for all n , meaning $\phi_n(x)$ also converges to $f(x)$.

If $f(x)$ is bounded, then for a sufficiently large n , $F_n = \emptyset$, and the construction in (*) is valid for all $x \in X$, meaning $\|\phi_n - f\|_u \leq \frac{1}{2^n}$, and $\sup_n \|\phi_n\|_u \leq \|f\|_u$. \square

Remark: By decomposing any complex-valued function f using the Cartesian decomposition to yield $f = (f_+ - f_-) + i(g_+ - g_-)$, the above theorem can be extended to all complex-valued functions. There, the modulus of the simple functions, $(|\phi_n|)_n$ can be taken to be pointwise increasing and bounded above by $|f|$, with uniform convergence on sets where f is bounded in modulus.

The Monotone Convergence Theorem

Since any measurable function $f: X \rightarrow [0, \infty]$ is a pointwise limit of simple functions, we may define the integral of a function as follows.

Definition. Let (X, \mathcal{M}, μ) be a measure space, and let $f: X \rightarrow [0, \infty]$ be a measurable function. The *integral* of f is defined to be

$$\int_X f \, d\mu = \sup \left\{ \int_X \phi_n \, d\mu \mid \phi \text{ simple, } 0 \leq \phi \leq f \right\}.$$

This definition of the integral agrees with the definition in (†) whenever f is simple. Furthermore, it follows that, for all $c \in [0, \infty)$,

$$\int_X cf \, d\mu = c \int_X f \, d\mu,$$

and whenever $f \leq g$,

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

Yet, the issue is that our family of simple functions is uncountable. In order to (more easily) establish this integral, we need to be able to extract a sequence.

Theorem (Monotone Convergence Theorem): Let $(f_n)_n$ be a family of $[0, \infty]$ -valued measurable functions on X such that $f_j \leq f_{j+1}$ for all j . Define

$$\begin{aligned} f &= \lim_{n \rightarrow \infty} f_n \\ &= \sup_{n \in \mathbb{N}} f_n. \end{aligned}$$

Then,

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Proof. The sequence $(\int_X f_n \, d\mu)$ is an increasing sequence of real numbers, so it has a limit (which may be equal to ∞). Furthermore, $\int_X f_n \, d\mu \leq \int_X f \, d\mu$ for all n , meaning $\sup(\int_X f_n \, d\mu) \leq \int_X f \, d\mu$.

To establish the reverse inequality, let $\alpha \in (0, 1)$, $0 \leq \phi \leq f$ a simple function, and let

$$E_n = \{x \mid f_n(x) \geq \alpha \phi(x)\}.$$

The family $\{E_n\}_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets whose union is X .¹ We have

$$\begin{aligned} \int_X f_n \, d\mu &\geq \int_{E_n} f_n \, d\mu \\ &\geq \alpha \int_{E_n} \phi \, d\mu. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \int_{E_n} \phi \, d\mu = \int_X \phi \, d\mu,$$

we have

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \geq \alpha \int_X \phi \, d\mu.$$

We may take the supremum over all $\alpha \in (0, 1)$, meaning

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \geq \int_X \phi \, d\mu.$$

Taking the supremum over all simple $0 \leq \phi \leq f$, we obtain

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu.$$

□

There are a variety of applications of the Monotone Convergence Theorem when it comes to establishing properties of sequences and series of functions.

Theorem: Let $(f_n)_n$ be a sequence of $[0, \infty]$ -valued measurable functions. Then,

$$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof. We start with functions $f_1, f_2: X \rightarrow [0, \infty]$. Let $(\phi_j)_j$ and $(\psi_j)_j$ be sequences of simple functions increasing to f_1 and f_2 respectively. Then,

$$\begin{aligned} \int_X f_1 + f_2 \, d\mu &= \lim_{n \rightarrow \infty} \int_X \phi_j + \psi_j \, d\mu \\ &= \lim_{n \rightarrow \infty} \int_X \phi_j \, d\mu + \lim_{n \rightarrow \infty} \int_X \psi_j \, d\mu \end{aligned} \tag{*}$$

¹To see that their union is equal to X , recall that f is the pointwise limit of f_n .

$$= \int_X f_1 d\mu + \int_X f_2 d\mu, \quad (**)$$

where in (*), we used the linearity of integration for simple functions, and in (**), we used the monotone convergence theorem.

Therefore, by induction, we get that

$$\int_X \sum_{n=1}^N f_n d\mu = \sum_{n=1}^N \int_X f_n d\mu.$$

Applying the monotone convergence theorem to the sequence of partial sums, we obtain

$$\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

□

Fatou's Lemma

Going deeper into our quest to find out when (pointwise) convergence of functions implies convergence of their integrals, we establish the “next best” option.

Theorem (Fatou's Lemma): Let $(f_n)_n: X \rightarrow [0, \infty]$ be a sequence of measurable functions. Then,

$$\int_X \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

Proof. For each $k \geq 1$ and for all $j \geq k$, we see that $\inf_{n \geq k} f_n \leq f_j$.

Since integration preserves relative order, this means $\int_X \inf_{n \geq k} f_n d\mu \leq \int_X f_j d\mu$ for all $j \geq k$.

By the definition of infimum, we thus get that $\int_X \inf_{n \geq k} f_n d\mu \leq \inf_{j \geq k} \int_X f_j d\mu$. Applying the monotone convergence theorem, we may take the supremum of both sides to obtain

$$\begin{aligned} \int_X \liminf_{n \rightarrow \infty} f_n d\mu &= \sup_{k \geq 1} \int_X \inf_{n \geq k} f_n d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu. \end{aligned}$$

□

Dominated Convergence Theorem

Fatou's Lemma is primarily used to prove the Dominated Convergence Theorem, the latter of which is significantly more powerful (but also requires one more condition).

Definition. Let (X, \mathcal{M}, μ) be a measure space, and let $f: X \rightarrow \mathbb{R}$ be a measurable function. We define the integral of f to be

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu,$$

where

$$f^+(x) = \max\{0, f(x)\}$$

$$f^-(x) = \max\{0, -f(x)\}.$$

We define the integral of a measurable $f: X \rightarrow \mathbb{C}$ to be

$$\int_X f \, d\mu = \int_X \operatorname{Re}(f) \, d\mu + i \int_X \operatorname{Im}(f) \, d\mu.$$

We say f is *integrable*, or a member of L_1 , if

$$\int_X |f| \, d\mu < \infty.$$

Proposition: If $f \in L_1(X, \mu)$, then

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Proof. If f is real-valued, then

$$\begin{aligned} \left| \int_X f \, d\mu \right| &= \left| \int_X f^+ \, d\mu - \int_X f^- \, d\mu \right| \\ &\leq \int_X f^+ \, d\mu + \int_X f^- \, d\mu \\ &= \int_X |f| \, d\mu. \end{aligned}$$

Now, if f is complex-valued with $\int_X f \, d\mu \neq 0$, we define $\alpha = \operatorname{sgn}(\int_X f \, d\mu)$. Then,

$$\begin{aligned} \left| \int_X f \, d\mu \right| &= \alpha \int_X f \, d\mu \\ &= \int_X \alpha f \, d\mu. \end{aligned}$$

Note that $\int_X \alpha f \, d\mu$ is real-valued, so

$$\begin{aligned} \left| \int_X f \, d\mu \right| &= \operatorname{Re} \left(\int_X \alpha f \, d\mu \right) \\ &= \int_X \operatorname{Re}(\alpha f) \, d\mu \\ &\leq \int_X |\operatorname{Re}(\alpha f)| \, d\mu \\ &\leq \int_X |\alpha f| \, d\mu \\ &= \int_X |f| \, d\mu. \end{aligned}$$

□

Now that we have established some of the important properties of L_1 , we may prove the Dominated Convergence Theorem.

Theorem (Dominated Convergence): Let $(f_n)_n$ be a sequence in L_1 such that $f_n \rightarrow f$ almost everywhere. If there exists a nonnegative $g \in L_1$ such that $|f_n| \leq g$ almost everywhere for every n , then $f \in L_1$ and

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Proof. Since f is the pointwise limit of a sequence of measurable functions, f is measurable, and since $|f| \leq g$ almost everywhere, we have $f \in L_1$. It is sufficient to assume that f_n and f are real-valued, meaning $g + f_n \geq 0$ and $g - f_n \geq 0$ almost everywhere.

Applying Fatou's Lemma, we have

$$\begin{aligned} \int_X g \, d\mu + \int_X f \, d\mu &\leq \liminf_{n \rightarrow \infty} \int_X (g + f_n) \, d\mu \\ &= \int_X g \, d\mu + \int_X f_n \, d\mu, \end{aligned}$$

and

$$\begin{aligned} \int_X g \, d\mu - \int_X f \, d\mu &\leq \liminf_{n \rightarrow \infty} \int_X (g - f_n) \, d\mu \\ &= \int_X g \, d\mu - \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu, \end{aligned}$$

meaning

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_X f_n \, d\mu &\geq \int_X f \, d\mu \\ &\geq \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu. \end{aligned}$$

□