Part 1

2.6, Problem 2

(a) Using Mathematica and effective guessing, we land upon an initial condition of $\vec{Y}(0) = \begin{pmatrix} 0 \\ 2.13 \end{pmatrix}$.

(b) All solutions with initial conditions in this curve will have the same periodic solution.

2.6, Problem 3

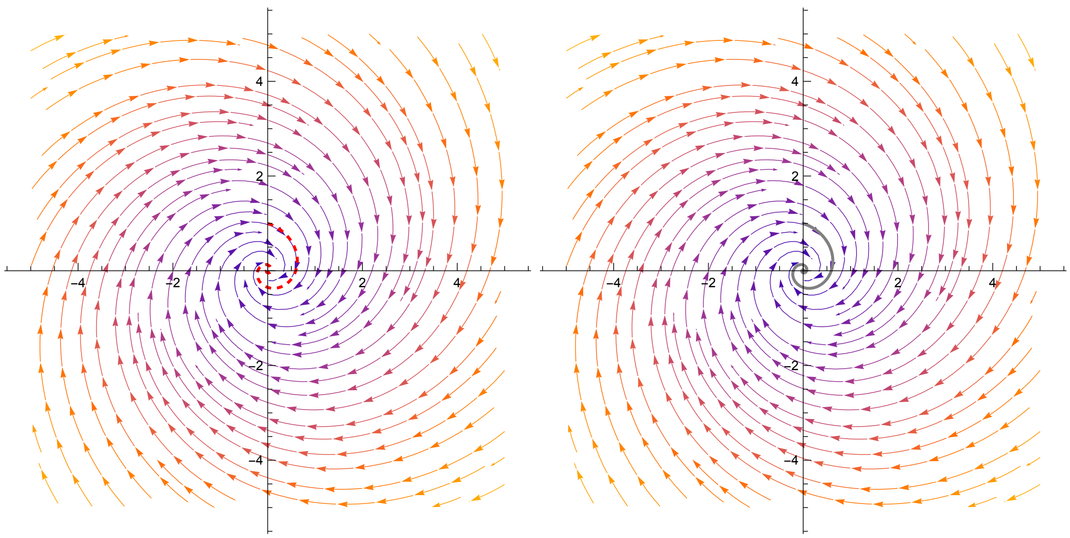
$$\begin{aligned}\frac{d\vec{Y}_1}{dt} &= \frac{d}{dt}\begin{pmatrix} e^{-t}\sin(3t) \\ e^{-t}\cos(3t)\end{pmatrix} \\ &= \begin{pmatrix} -e^{-t}\sin(3t) + 3e^{-t}\cos(3t) \\ -e^{-t}\cos(3t) - 3e^{-t}\sin(3t)\end{pmatrix} \\ &= \begin{pmatrix} -x + 3y \\ -3x - y\end{pmatrix}.\end{aligned}$$

2.6, Problem 4

Since $\vec{Y}_2(t) = \vec{Y}_1(t-1)$ and $\vec{Y}_1(t)$ is a solution, so too is $\vec{Y}_2(t)$.

2.6, Problem 5

Plotting, we see the following.



This does not violate the uniqueness theorem since if $t_0 = 0$ for $\vec{Y}_1$ and $t_0 = 1$ for $\vec{Y}_1$, then the solutions are exactly the same.

2.6, Problem 9

We must have $\vec{Y}_1$ is a phase shift of $\vec{Y}_2$. Specifically, $\vec{Y}_1(t) = \vec{Y}_2(t-1)$.

Chapter 2 Review, Problem 2

Solving $\frac{dx}{dt}$, we must have $y = 0$, which yields $\frac{dy}{dt} = x^2 + 1$. Therefore, there are no equilibrium solutions for this equation.

Chapter 2 Review, Problem 3

$$\begin{aligned} x &= \frac{dy}{dt} \\ \frac{dx}{dt} &= 1 \end{aligned}$$

Chapter 2 Review, Problem 7

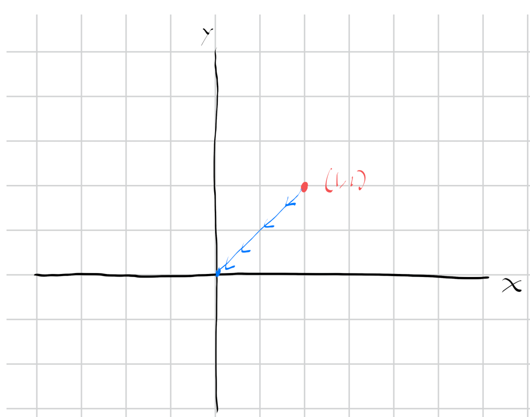
$$\begin{aligned} \frac{dx}{dt} &= -6e^{-6t} \\ &= 2\left(e^{-6t}\right) - 2\left(4e^{-6t}\right) \\ &= 2x - 2y^2 \\ \frac{dy}{dt} &= -6e^{-3t} \\ &= -3y. \end{aligned}$$

Thus, this is a solution to the system of differential equations.

Chapter 2 Review, Problem 12

$$\begin{aligned} \vec{Y}(0.5) &\approx \vec{Y}(0) + 0.5\vec{F}\left(\vec{Y}(0)\right) \\ &= \begin{pmatrix} 3.5 \\ 2 \end{pmatrix}. \end{aligned}$$

Chapter 2 Review, Problem 13



Chapter 2 Review, Problem 15

This is true, as we have shown in the solution to Problem 7.

Chapter 2 Review, Problem 16

This is true, as $y = 0$ means $\frac{dy}{dt} = 0 = -y$, and for $x(t) = 2$, $\frac{dx}{dt} = 0$, meaning this is an equilibrium solution to the differential equation.

Chapter 2 Review, Problem 20

This is true, as phase shifting any solution to a system of differential equations yields another solution to a system of differential equations.

Chapter 2 Review, Problem 30

The phase portrait of the completely decoupled system has all its solution curves as lines.

3.1, Problem 6

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

3.1, Problem 7

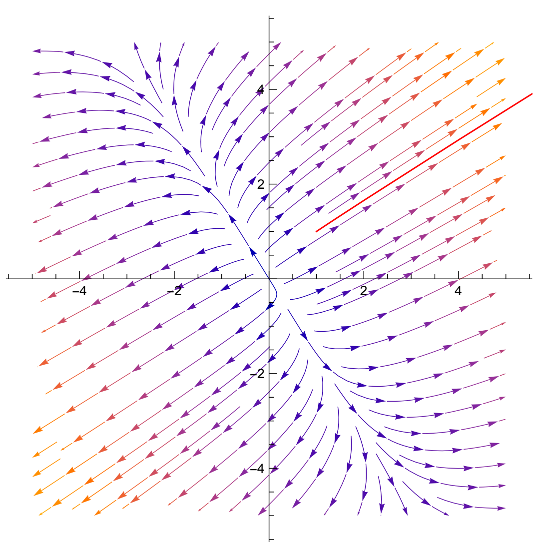
$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 3 & -2 & 7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \\ r(t) \end{pmatrix}$$

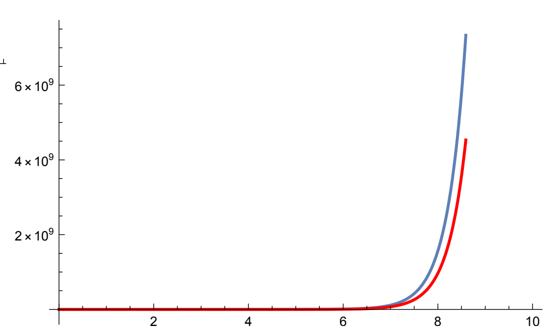
3.1, Problem 8

$$\begin{aligned} \frac{dx}{dt} &= -3x + 2\pi y \\ \frac{dy}{dt} &= 4x - y. \end{aligned}$$

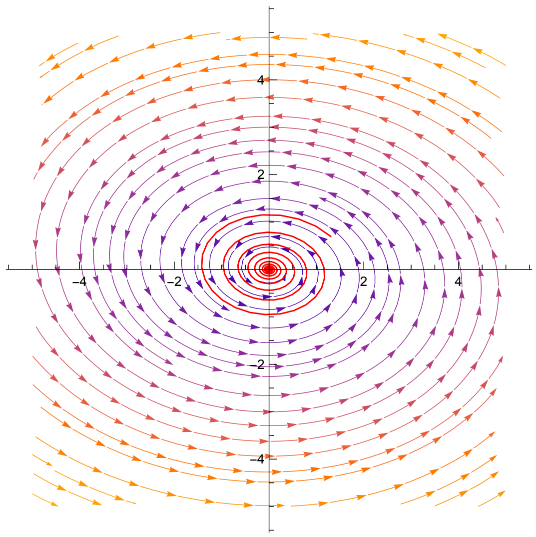
Part 2

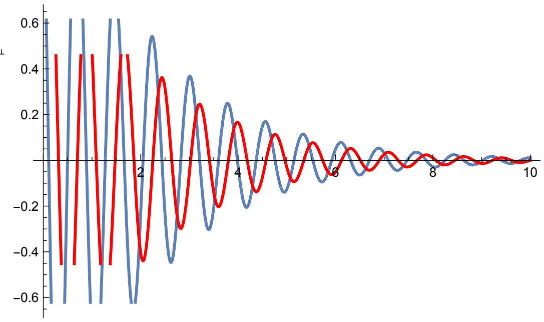
3.1, Problem 10





3.1, Problem 13





3.1, Problem 18

(a) Converting

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = 0,$$

we have

$$v(t) = c$$

for some c.

(b) This means $y(t) = ct + k$ for $k \in \mathbb{R}$.

(c)

3.1, Problem 31

(a) Since

$$3\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 0\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

these vectors are not linearly independent.

(b)

$$-\lambda\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so they are not linearly independent.

(c) If $x_1 \neq 0$, then $y_2 = \frac{x_2 y_1}{x_1}$, meaning $y_2 = \lambda y_1$ and $x_2 = \lambda x_1$, so we use (b). Similarly, if $x_2 \neq 0$, we take $-(x_1 y_2 - x_2 y_1) = x_2 y_1 - x_1 y_2 = 0$ and use (b) again. Finally, if $x_1 = 0$, then we must have y_1 or $x_2 = 0$, both of which yield linear dependence.

3.1, Problem 32

Let

$$x_1y_2 - x_2y_1 \neq 0$$

Suppose toward contradiction that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are not linearly independent. Then, there is λ such that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, meaning we have

$$x_1y_2 - x_2y_1 = \lambda x_2y_2 - x_2\lambda y_2 \qquad = 0.$$

Thus, we must have $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ not linearly independent.

3.1, Problem 35

(a)

$$\frac{dW}{dt} = x_1'(t)y_2(t) + x_1(t)y_2'(t) - \left(x_2'(t)y_1(t) + x_2(t)y_1'(t)\right).$$

(b)

$$\begin{aligned}\frac{dW}{dt} &= x_1'(t)y_2(t) + x_1(t)y_2'(t) - \left(x_2'(t)y_1(t) + x_2(t)y_1'(t)\right)\\ &= \left(ax_1(t) + by_1(t)\right)y_2(t) + x_1(t)\left(cx_2(t) + dy_2(t)\right) - \left(\left(ax_2(t) + by_2(t)\right)y_1(t) + x_2(t)\left(ax_1(t) + by_1(t)\right)\right)\\ &= (a+d)\left(x_1(t)y_2(t) - x_2(t)y_1(t)\right)\\ &= (a+d)\,W(t).\end{aligned}$$

(c)

$$\begin{aligned}\frac{dW}{dt} &= (a+d)\,W(t)\\ W(t) &= e^{(a+d)t}.\end{aligned}$$

(d)

$$\begin{aligned}W(0) &= x_1(0)y_2(0) - x_2(0)y_1(0)\\ &= \det\begin{pmatrix} x_1(0) & x_2(0) \\ y_1(0) & y_2(0) \end{pmatrix}\\ &\neq 0,\end{aligned}$$

meaning

$$\frac{dW}{dt} = (a+d)\,W(t)$$

has a nondegenerate initial condition. Thus, we have

$$W(t) = e^{(a+d)t},$$

which is never zero, meaning $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are always linearly independent.

3.2, Problem 8

3.2, Problem 9

3.2, Problem 12

3.2, Problem 16

3.2, Problem 17

3.2, Problem 18