**Solution** (11.2, Problem 2)**:** We evaluate

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)f(x)\,dx\\
&= \begin{cases} 1 & n = 0\\ 0 & \text{else}\end{cases}\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)f(x)\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx\\
&= \frac{1}{\pi}\left(\frac{1}{n}\cos(nx)\bigg|_{0}^{\pi}\right)\\
&= \frac{1}{n\pi}\left((-1)^n - 1\right).
\end{aligned}$$

Therefore, our Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi}\sin(nx).$$

**Solution** (11.2, Problem 8)**:** We evaluate

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)(3-2x)\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} 3\cos(nx) - 2x\cos(nx)\,dx\\
&= \begin{cases} 3 & n = 0\\ 0 & \text{else.}\end{cases}\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)(3-2x)\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} 3\sin(nx) - 2x\sin(nx)\,dx\\
&= \frac{1}{\pi}\left(\frac{3}{n}\cos(nx)\bigg|_{-\pi}^{\pi} - 2\left(\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right)\bigg|_{-\pi}^{\pi}\right)\\
&= \frac{4(-1)^n}{n}.
\end{aligned}$$

Thus, our Fourier series is

$$f(x) = 3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nx).$$

**Solution** (11.2, Problem 10)**:** Using integration by parts, we evaluate

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_{0}^{\pi/2} \cos\left(\frac{n}{2}x\right)\cos(x)\,dx\\
&= \begin{cases} \frac{8}{\pi(4-n^2)}\cos\left(\frac{n\pi}{4}\right) & n \neq 2\\ \frac{1}{2} & n = 2\end{cases}.\\
b_n &= \frac{2}{\pi}\int_{0}^{\pi/2} \sin\left(\frac{n}{2}x\right)\cos(x)\,dx\\
&= \begin{cases} \frac{8}{\pi(4-n^2)}\left(\sin\left(\frac{n\pi}{4}\right) - \frac{n}{2}\cos\left(\frac{n\pi}{4}\right)\right). & n \neq 2\\ \frac{1}{\pi} & n = 2\end{cases}.
\end{aligned}$$

Thus, with $a_0 = \frac{2}{\pi}$, we have the Fourier series

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{2}x\right) + b_n \sin\left(\frac{n}{2}x\right).$$

**Solution** (11.2, Problem 17)**:** We first start by finding the series expansion. Evaluating, we have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} x^2 \cos(nx)\, dx \\
&= \begin{cases} \frac{2(-1)^n}{n^2} & n > 0 \\ \frac{\pi^2}{3} & n = 0 \end{cases} \\
b_n &= \frac{1}{\pi}\int_0^{\pi} x^2 \sin(nx)\, dx \\
&= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right).
\end{aligned}$$

Thus, we have the Fourier series

$$x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos(nx) + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right)\sin(nx).$$

Using the input $x = 0$, we get

$$\begin{aligned}
0 &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos(nx) \\
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}.
\end{aligned}$$

Meanwhile, using the input of $-\pi$, we have $f(-\pi) = 0$, and

$$\begin{aligned}
0 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}$$

**Solution** (11.2, Problem 18)**:** Adding, we get

$$\frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}.$$

**Solution** (11.3, Problem 6)**:** The function

$$f(x) = e^x - e^{-x}$$

is odd.

**Solution** (11.3, Problem 10)**:** The function

$$f(x) = \left|x^5\right|$$

is even.

**Solution** (11.3, Problem 12)**:** This function is even, so we expand in the cosine series. This gives

$$a_n = \int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)\, dx$$

$$= \int_1^2 \cos\left(\frac{n\pi}{2}x\right)\,dx$$
$$= \begin{cases} \frac{2}{n\pi} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

**Solution** (11.3, Problem 18)**:** This function is odd, so we expand in the sine series. This gives

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\,dx \\ &= \frac{(-1)^{n+1}\pi^3}{n} + \frac{6\pi(-1)^n}{n^3}. \end{aligned}$$

**Solution** (11.3, Problem 20)**:** This function is odd, so we expand in a sine series.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} (x+1)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin(nx) + \sin(nx)\,dx \\ &= \frac{2\left(1 + (-1)^{n+1} + (-1)^{n+1}\pi\right)}{n\pi}. \end{aligned}$$

**Solution** (11.3, Problem 34)**:** Since $f(0) = 0$, we expand in a sine series. This gives

$$b_n = \int_0^2 x(2-x)\sin\left(\frac{n\pi}{2}x\right)\,dx \qquad = \frac{16}{n^3\pi^3}\left(1 + (-1)^{n+1}\right).$$

**Solution** (11.4, Problem 2)**:** There are three cases — one with $\lambda = -k^2$, one with $\lambda = 0$, and one with $\lambda = k^2$. In these cases, we have solutions of

$$\begin{aligned} y &= c_1 + c_2 x && \lambda = 0 \\ y &= c_1\cosh(kx) + c_2\sinh(kx) && \lambda = k^2 \\ y &= c_1\cos(kx) + c_2\sin(kx). && \lambda = -k^2 \end{aligned}$$

Now, noting that $y(1) = 0$, we have the cases of

$$\begin{aligned} 0 &= c_1 + c_2 && \lambda = 0 \\ 0 &= c_1\cosh(k) && \lambda = k^2 \\ 0 &= c_1\cos(k) + c_2\sin(k). && \lambda = -k^2 \end{aligned}$$

If it were the case that $c_1 + c_2 = 0$, then by inputting the other boundary value of $y'(0) + y(0) = 0$, we would have $c_2 = 0$, which would be the degenerate case (i.e., useless). Thus, we disregard the case of $\lambda = 0$.

**Solution** (11.4, Problem 4)**:**

**Solution** (11.4, Problem 8)**:**

**Solution** (11.4, Problem 10)**:**

**Solution** (12.3, Problem 2)**:**

**Solution** (12.3, Problem 4)**:**