

Problem (Problem 1): Let (X, x_0) be a pointed CW complex. Prove that $\Sigma X = X \wedge S^1$.

Solution: The reduced suspension yields the identifications

$$\begin{aligned}\Sigma X &= X \times [0, 1] / (X \times \{0\} \sim \{x_0\}, X \times \{1\} \sim \{x_0\}, \{x_0\} \times [0, 1] \sim \{x_0\}) \\ &= X \times [0, 1] / (X \times \{1\} \sim \{x_0\} \times [0, 1], X \times \{1\} \sim \{x_0\} \times [0, 1]).\end{aligned}$$

Now, if we consider $\{0\} = \{1\}$ to be the basepoint of S^1 , this yields our desired smash product

$$\begin{aligned}X \wedge S^1 &= X \times S^1 / (X \times \{0\} \sim \{x_0\} \times S^1) \\ &= \Sigma X.\end{aligned}$$

Problem (Problem 2): Prove that, if $f: X \rightarrow Y$ is a map, then Y is a deformation retract of the mapping cylinder M_f .

Solution: Recalling the definition of the mapping cylinder, we have

$$M_f = X \times [0, 1] \sqcup Y / ((x, 1) \sim f(x)).$$

We let q be the quotient map from $X \times [0, 1] \sqcup Y$ to M_f . By the universal property of the quotient map, it follows that if we are able to define a deformation retract from $X \times [0, 1] \sqcup Y$ to $X \times \{1\} \sqcup Y$, then we may compose with the quotient map to find a deformation retract from M_f to Y .

Define the homotopy

$$H: (X \times [0, 1]) \times [0, 1] \rightarrow X \times [0, 1]$$

by taking

$$H((p, s), t) = (p, \max(s, t)).$$

Since the maximum function is continuous, it follows that this is a deformation retract to $X \times \{1\}$, so by extending H to $X \times [0, 1] \sqcup Y$, and composing with the quotient map, we obtain the desired deformation retraction onto Y .

Problem (Problem 3): Prove that, if $f: X \hookrightarrow Y$ is an inclusion map, then the mapping cone C_f is homotopy equivalent to the quotient $Y/f(X)$.

Solution: First, we observe that f is an inclusion map, so $X \subseteq Y$ and $f(x) = x$ for all $x \in X$. Additionally, the mapping cylinder M_f is given by

$$M_f = X \times [0, 1] \sqcup Y / ((x, 1) \sim x),$$

and that Y is homotopy equivalent to M_f from Problem 2. We observe then that the mapping cone C_f is the quotient $M_f/X \times \{0\}$, while we see that

$$M_f/X \times \{1\} = (X \times [0, 1] \sqcup Y/X) / ((x, 1) \sim x),$$

which we can view as an “inverted” mapping cone, of sorts. In particular, this “inverted mapping cone” deformation retracts via straight line homotopy to $[x] \in Y/X$ for each $x \in X$, which is a single point. Therefore, we may define a homotopy (with corresponding homotopy inverse)

$$H: C_f \times [0, 1] \rightarrow Y/f(X)$$

given by

$$H_t(p) = [p],$$

where the equivalence class for p is considered in $M_f/(X \times \{t\})$. This map is continuous by the definition of the quotient topology, so this is our desired homotopy equivalence.