Problem (Problem 1)**:** Prove that smooth homotopy and smooth isotopy are equivalence relations.

Solution: If $f\colon M \to N$ is a smooth map, then we can define a smooth homotopy $F\colon M \times [0,1] \to N$ by taking $F(\cdot,t) = f$. If f is a diffeomorphism, then this is a smooth isotopy. Thus, this relation is reflexive.

The relation is symmetric since, if f and g are smoothly homotopic (isotopic), then $F^*\colon M \times [0,1] \to N$, given by $F^*(\cdot,t) = F(\cdot,1-t)$ is a composition of smooth maps, hence smooth.

The relation is transitive since, if $F\colon M \times [0,1] \to N$ is a homotopy (isotopy) from f to g, and $G\colon M \times [0,1] \to N$ is a homotopy (isotopy) from g to h, then we may find a homotopy from f to h by taking

$$H(\cdot,t) = \begin{cases} F(\cdot,2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ G(\cdot,2t-1) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

This is a smooth map since the derivatives of all orders for F and G agree at $t = \frac{1}{2}$.

Problem (Problem 2)**:** Prove that if M is connected, then for all pairs p and q of points on M, there is a diffeomorphism f of M such that $f(p) = q$ and f is isotopic to the identity.

Solution: We know that the diffeomorphism group, $\operatorname{diff}(M)$, is transitive whenever M is connected, so there is a diffeomorphism $f\colon M \to M$ such that $f(p) = q$. Now, if p and q are in the same Euclidean chart, (U,φ), where $\varphi(p) = 0$ and $\varphi(q) = ax_1$, then we may find the desired isotopy to the identity by taking

$$F\colon M \times [0,1] \to M$$

to be given by

$$F(\cdot,t) = f_t,$$

where f_t is a diffeomorphism such that $\varphi \circ f_t(p) = atx_1$.

Now, if p and q are not in the same chart, then since M is connected, there is a finite chain of k intersecting Euclidean charts that we may compose with each other such that we get our diffeomorphism between p and q. Dividing $[0,1]$ into intervals of length $1/k$, we may then find isotopies from the identity to the diffeomorphism mapping p to the ℓ-th intersection point along in this chain as we showed for the case where both p and q are in the same chart. By chaining these isotopies together, we get the isotopy between f and the identity.

Problem (Problem 3)**:** Suppose M is compact and has no boundary, and that M and N have the same dimension. Let f and g be homotopic maps from M to N. Suppose $p \in N$ is a regular value for both f and g. Prove that $\left|f^{-1}(p)\right| = \left|g^{-1}(p)\right|$ modulo 2.

Solution: Let $F\colon M \times [0,1] \to N$ be a smooth homotopy with $F(\cdot,0) = f$ and $F(\cdot,1) = g$. If $p \in N$ is a regular value for F (in addition to one for f and g), it follows that $F^{-1}(p)$ is a 1-manifold subset of $M \times [0,1]$, where $F^{-1}(p) \cap (M \times \{0\}) = f^{-1}(p) \times \{0\}$, and $F^{-1}(p) \cap (M \times \{1\}) = g^{-1}(p) \times \{1\}$. Since the boundary of $M \times [0,1]$ must contain an even number of points (as every 1-submanifold with boundary of $M \times [0,1]$ must have both of its boundary points touch the boundary of $M \times [0,1]$, which are 0 and 1), we must have $\left|f^{-1}(p)\right| + \left|g^{-1}(p)\right| \equiv 0$ modulo 2, so that $\left|f^{-1}(p)\right| = \left|g^{-1}(p)\right|$.

Suppose y is not a regular value for F. Since $M \times [0,1]$ is compact, and F is continuous, it follows that, by Sard's Theorem, y is part of a closed, measure-zero subset of N. In particular, for any neighborhood of y, there is a regular value for F within this neighborhood. Next, we observe that, for a sufficiently small open neighborhood V of y, the number of regular points mapping to y does not change, as the map $x \mapsto \left|F^{-1}(x)\right|$ is continuous and discrete-valued (for the open subset of regular values for F). Thus, on V, we may find $q \in V$ such that $\left|F^{-1}(q)\right|$ is constant, and thus $\left|f^{-1}(y)\right| + \left|g^{-1}(y)\right|$ is even, hence are equal to each other modulo 2.

Problem (Problem 4)**:** Prove that for M, N, f as in the previous exercise, $\left|f^{-1}(p)\right| \equiv \left|f^{-1}(q)\right|$ modulo 2 for all regular values p and q of f, using the previous exercises.

Solution: There is a diffeomorphism $\varphi\colon N \to N$ of N such that $\varphi(p) = q$ and φ is isotopic to the identity, as shown in the solution to Problem 2. In particular, this means that $\varphi \circ f\colon M \to N$ is homotopic to $f\colon M \to N$, meaning that $\left|f^{-1}(p)\right| = \left|(\varphi \circ f)^{-1}(q)\right| = \left|f^{-1}(q)\right|$, with the latter equality following from Problem 3.

Problem (Problem 5)**:** Let M be compact and have no boundary. Let $p \in M$, and $f\colon M \to M$ be defined by $f(x) = p$. Prove that f is not homotopic to the identity map.

Solution: The identity map, id, is a diffeomorphism of M, so $\mathrm{id}^{-1}(q) = \{q\}$ for all $q \in M$. Notice that, for $q \neq p$, $f^{-1}(q) = \emptyset$, meaning that $q \neq p$ are vacuously regular values for f; since these inverse images have cardinality zero, it follows that f and id cannot be homotopic, since we established in Problem 3 that the cardinality of the preimage of a regular value is invariant under homotopy.

Problem (Problem 6)**:** Let $f\colon M \to N$ be smooth and oriented, with M compact and boundaryless and M and N of the same dimension. Show that if $M = \partial W$ for some smooth manifold W, and f extends smoothly to W, then for all $p \in N$ a regular value, we have $\deg(f, p) = 0$.

Solution: Let $\hat{f}$ be the smooth extension of f to W. Since p is a regular value for $\hat{f}$, there are points q_1 and q_2 on M such that $\hat{f}^{-1}(p)$ contains a path γ starting at q_1 and ending at q_2; this follows from the regular value theorem and the fact that W is a manifold of dimension $n+1$ when M is a manifold of dimension N. In particular, we may cover γ by finitely many charts that connect q_1 to q_2.

Since W is oriented, we may select orientations such that all the interior points of γ remain the same orientation in W; yet, if ∂_{n+1} denotes the tangent vector at q_1 that allows for positive orientation at q_1, then upon following this path, the sign of the image of ∂_{n+1} under the family of composed differential maps flips, as we go from an "inward" orientation at q_1 to an "outward" orientation right as γ approaches q_2. This gives that the degree of $\hat{f}$ when it comes to the pair (q_1, q_2) is zero. This holds for all such pairs (q_i, q_{i+1}) that land on M, meaning that $\deg(f, p) = 0$.

Problem (Problem 7)**:** Let M and N be as in the previous exercise. Prove that if f and g are homotopic, and $p \in N$ is a common regular value for both, then $\deg(f, p) = \deg(g, p)$.

Solution: If $F\colon M \times [0,1] \to N$ is a homotopy from f to g, then we see that for any regular values p for F, $F^{-1}(p)$ is a 1-manifold with two boundary points, so that these 1-manifolds intersect $M \times \{0\}$ or $M \times \{1\}$. Observe that the orientation at $M \times \{1\}$ is negative to that at $M \times \{0\}$, meaning that $\deg(F, p) = 0 = \deg(f, p) - \deg(g, p)$.

Problem (Problem 8)**:** Show that $\deg(f, p)$ is independent of the choice of regular value p, so that the *degree*, $\deg(f)$, can be defined. Show that homotopic maps have equal degrees.

Solution: We have shown in Problem 7 that, if p is a common regular value for homotopic maps f and g, then $\deg(f, p) = \deg(g, p)$. Additionally, we have shown that, if p and q are regular values for f, then there is a diffeomorphism φ of N that maps p to q that is isotopic to the identity; we may then compose this isotopy with f such that we get a homotopy between f and $\varphi \circ f$; this means that $\deg(f, p) = \deg(\varphi \circ f, q)$, so that the degree of a map f is independent of the regular value.

If f and g are homotopic to each other, it follows that for each regular value in common, f and g have the same degree with respect to this regular value; since any regular value in common for f and g admits a diffeomorphism isotopic to the identity, it follows that both f and g have the same degree for all regular values in common. Since any regular value for one function is arbitrarily close to a regular value for both functions, it follows that $\deg(f) = \deg(g)$.

Problem (Problem 9)**:** Prove that for a sphere S^n viewed as the unit sphere in $\mathbb{R}^{n+1}$, the antipodal map $x \mapsto -x$ has degree -1 if and only if n is even.

Solution: Let $x \in S^n \subseteq \mathbb{R}^{n+1}$. We view S^n as the boundary of the unit ball $B(0,1) \subseteq \mathbb{R}^{n+1}$; a point

$p \in S^n$ then has an orientation defined by

$$\left(-\frac{\partial}{\partial x_{n+1}}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right).$$

The map $x \mapsto -x$ in $\mathbb{R}^{n+1}$ then flips signs for all the tangent vectors on this orientation, giving

$$\left(\frac{\partial}{\partial x_{n+1}}, -\frac{\partial}{\partial x_1}, \dots, -\frac{\partial}{\partial x_n}\right) = (-1)^{n+1}\left(-\frac{\partial}{\partial x_{n+1}}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right).$$

The degree of $x \mapsto -x$ is thus negative if and only if $n+1$ is odd, meaning it holds if and only if n is even.

Problem (Problem 10)**:** Prove that the sphere S^n admits a nowhere-vanishing vector field if and only if n is odd.

Solution: In order to construct a smooth homotopy $F\colon S^n \times [0,1] \to S^n$ between the identity and antipodal map, we want $F(p,0) = p$, $F(p,1) = -p$, and $\|F(p,t)\|^2 = 1$ for all p and all t. We start by taking

$$F(p,t) = p\cos(\pi t) + y(p)\sin(\pi t);$$

and taking the dot product of F with itself to yield

$$\begin{aligned}
\|F(p,t)\|^2 &= \langle p\cos(\pi t) + y(p)\sin(\pi t), p\cos(\pi t) + y(p)\sin(\pi t)\rangle\\
&= \cos^2(\pi t) + \|y(p)\|^2\sin^2(\pi t) + \langle p, y(p)\rangle\sin(2\pi t)\\
&= 1.
\end{aligned}$$

Ideally, we desire $\langle p, y(p)\rangle = 0$, and $\|y(p)\|^2 = 1$ for all $p \in S^n$. This entails the existence of a smooth map between S^n and S^n that maps p to a vector orthogonal to p. Defining a smooth vector field on S^n in this fashion, we find that such a vector field is necessarily nowhere-vanishing. This implies that the existence of a nowhere-vanishing vector field implies that the antipodal map is smoothly homotopic to the identity.

Meanwhile, we have shown in Problem 9 that, when n is even, the degree of the antipodal map is -1, while the degree of the identity map is always 1, meaning that when n is even, the identity map cannot be smoothly homotopic to the identity, meaning there does not exist a nowhere-vanishing vector field when n is even.