8.1

(a)

$$\begin{aligned}\int_0^1 2^x \, dx &= \int_0^1 e^{x(\ln 2)}\, dx \\ &= \frac{1}{\ln 2}\left(e^{x(\ln 2)}\Big|_0^1\right) && u = x(\ln 2) \\ &= \frac{1}{\ln 2}\left(2^x\Big|_0^1\right) \\ &= \frac{1}{\ln 2}(2-1) \\ &= \frac{1}{\ln 2}.\end{aligned}$$

(b)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^x \, dx &= \int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2}+x-\frac{1}{2}\right)+\frac{1}{2}}\, dx && \text{Completing the square.} \\ &= e^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2}\, dx \\ &= \sqrt{2\pi e} && \text{Gaussian Integral}\end{aligned}$$

(c)

(d)

$$\int_{-a}^{a} \sin x e^{-\alpha x^2}\, dx = 0 \qquad \text{Even/odd.}$$

(e)

$$\begin{aligned}\int_0^1 e^{\sqrt{x}}\, dx &= x e^{\sqrt{x}}\Big|_0^1 - \frac{1}{2}\int_0^1 x e^{\sqrt{x}}\, dx && \text{Integration by Parts} \\ &= e - \int_0^1 u^3 e^u \, du && u = \sqrt{x} \\ &= e - \left(u^3 e^u\Big|_0^1 - 3u^2 e^u\Big|_0^1 + 6u\Big|_0^1 - 6e^u\Big|_0^1\right) && \text{Repeated integration by parts.} \\ &= 3e - 6.\end{aligned}$$

To evaluate $\int_0^1 u^3 e^u \, du$, we used tabular integration as follows:

Sign	Differentiate	Integrate
+	u^3	e^u
-	$3u^2$	e^u
+	$6u$	e^u
-	6	e^u
+	0	e^u

Taking the boundary integrals, we obtain

$$u^3 e^u\Big|_0^1 - 3u^2 e^u\Big|_0^1 + 6ue^u\Big|_0^1 - 6e^u\Big|_0^1 = 6 - 2e$$

(f)

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\cosh(u)}\cosh(u)\,du && x=\sinh(u)\\ &= u + C\\ &= \sinh^{-1}(x) + C.\end{aligned}$$

(g)

$$\begin{aligned}\int \tanh x\,dx &= \int \frac{\sinh x}{\cosh x}\,dx\\ &= \int \frac{1}{u}\,du && u=\cosh x\\ &= \ln|u| + C\\ &= \ln|\cosh x| + C.\end{aligned}$$

(h)

$$\begin{aligned}\int \tan^{-1} x\,dx &= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx && \text{integration by parts}\\ &= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C. && \text{u-substitution implicit}\end{aligned}$$

(i)

$$\begin{aligned}\int_S z^2\,d\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2} \cos^2\theta\sin\theta\,d\phi d\theta\\ &= \frac{\pi}{2}\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta\\ &= -\frac{\pi}{2}\int_0^{-1} t^2\,dt && t=\cos\theta\\ &= \frac{\pi}{2}\left(\frac{t^3}{3}\Big|_{-1}^{0}\right)\\ &= \frac{\pi}{6}\end{aligned}$$

8.8

(a)

$$\begin{aligned}\int_0^\infty \frac{x}{e^x-1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx\\ &= \int_0^\infty xe^{-x}\left(\sum_{k=0}^\infty e^{-kx}\right)dx\\ &= \sum_{k=0}^\infty \int_0^\infty xe^{-(k+1)x}\,dx\\ &= \sum_{k=0}^\infty \frac{1}{(k+1)^2}\int_0^\infty ue^{-u}\,du && u=(k+1)x\\ &= \frac{\pi^2}{6}. && \text{Basel Problem}\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty \frac{x}{e^x+1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx \\
&= \int_0^\infty xe^{-x}\sum_{k=0}^{\infty}(-1)^k e^{-kx}\,dx \\
&= \sum_{k=0}^{\infty}(-1)^k\int_0^\infty xe^{-(k+1)x}\,dx \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\int_0^\infty ue^{-u}\,dx && u=(k+1)x \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}.
\end{aligned}$$

To resolve

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2} = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots$$

we take

$$= \left(1+\frac{1}{9}+\frac{1}{25}+\cdots\right)-\frac{1}{4}\underbrace{\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots\right)}_{\frac{\pi^2}{6}},$$

meaning

$$\int_0^\infty \frac{x}{e^x+1}\,dx = \frac{\pi^2}{12}.$$