

## Contents

<b>Introduction</b>	<b>1</b>
<b>Review: Representations, the Reduced Group <math>C^*</math>-Algebra, and the Universal Group <math>C^*</math>-Algebra</b>	<b>1</b>
Left-Regular Representation . . . . .	1
A Bit on Representations and $C^*$ -(Semi)norms . . . . .	2
<b>Using the Left-Regular Representation to Establish Amenability</b>	<b>3</b>
<b>Completely [Property] Maps</b>	<b>10</b>
<b>Positive and Completely Positive Maps</b>	<b>14</b>
Positive Maps . . . . .	14
Completely Positive Maps . . . . .	19
Schur Products and Tensor Products . . . . .	24

## Introduction

Finally, the last part of my notes on  $C^*$ -algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of  $C^*$ -algebras, discussing ideas such as amenability and nuclearity in  $C^*$ -algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of  $C_\lambda^*(G)$  and  $C^*(G)$ .

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's  *$C^*$ -Algebras and Finite-Dimensional Approximations*.

## Review: Representations, the Reduced Group $C^*$ -Algebra, and the Universal Group $C^*$ -Algebra

### Left-Regular Representation

Let  $\Gamma$  be a group. Consider the space  $\ell_2(\Gamma)$ . For every  $s \in \Gamma$ , we define the operator

$$\lambda_s(\xi)(t) = \xi(s^{-1}t).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned} \|\lambda_s(\xi)\|^2 &= \sum_{t \in \Gamma} |\lambda_s(\xi)(t)|^2 \\ &= \sum_{t \in \Gamma} |\xi(s^{-1}t)|^2 \\ &= \sum_{r \in \Gamma} |\xi(r)|^2 \\ &= \|\xi\|^2. \end{aligned}$$

Additionally, each  $\lambda_s$  admits an inverse,  $\lambda_{s^{-1}} = \lambda_s^*$ . Applying to the orthonormal basis  $\{\delta_t\}_{t \in \Gamma}$ , we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus,  $\lambda_s \circ \lambda_r = \lambda_{sr}$ , and we have the unitary representation of  $\Gamma$ ,  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ , where  $\lambda(s) = \lambda_s$ , for  $s \in \Gamma$ . This is the left-regular representation of  $\Gamma$ .

Note that the left regular representation is a faithful representation, hence injective.

Because the  $\lambda$  operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}\lambda_f(\xi)(t) &= \left( \sum_{s \in \Gamma} f(s) \lambda_s(\xi) \right)(t) \\ &= \sum_{s \in \Gamma} f(s) \xi(s^{-1}t)\end{aligned}$$

Note that the space of finitely supported functions on  $\Gamma$ ,  $\mathbb{C}[\Gamma]^1$  is a  $*$ -algebra, where multiplication is given by convolution:

$$\begin{aligned}f * g(t) &= \sum_{s \in \Gamma} f(s) g(s^{-1}t) \\ &= \sum_{r \in \Gamma} f(tr^{-1}) g(r).\end{aligned}$$

Note that we are using  $*$  both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on  $\mathbb{C}[\Gamma]$  is given by

$$f^*(t) = \overline{f(t^{-1})}.$$

## A Bit on Representations and $C^*$ -(Semi)norms

A  $C^*$ -seminorm on a  $*$ -algebra is a seminorm such that defined by

- $\|ab\| \leq \|a\| \|b\|$ ;
- $\|a^*\| = \|a\|$ ;
- $\|a^*a\| = \|a\|^2$ .

If  $A_0$  is a  $*$ -algebra, then a representation of  $A_0$  is a pair  $(\pi_0, \mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space and  $\pi: A_0 \rightarrow \mathcal{B}(\mathcal{H})$  is a  $*$ -homomorphism.

Additionally, if  $A_0$  is a  $*$ -algebra with representation  $\pi_0$ , then we have  $C^*$ -seminorm

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{\text{op}}.$$

If  $\pi_0$  is injective, then  $\|\cdot\|_{\pi_0}$  is a  $C^*$ -norm. If  $\pi_0$  is a  $C^*$ -norm, then the completion of  $A_0$  with respect to  $\|\cdot\|_{\pi_0}$  is a  $C^*$ -algebra.

The universal norm on  $A_0$  is defined as

$$\|a\|_{\text{u}} = \sup_{p \in \mathcal{P}} p(a),$$

where  $\mathcal{P}$  is the collection of all  $C^*$ -seminorms on  $A_0$ . If  $\|a\|_{\text{u}} < \infty$  for all  $a \in A_0$ , then  $\|\cdot\|_{\text{u}}$  is a  $C^*$ -seminorm on  $A_0$ . Note that if one of  $p \in \mathcal{P}$  is a norm, then  $\|\cdot\|_{\text{u}}$  defines a  $C^*$ -norm on  $A_0$ .

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<sup>1</sup>Also known as the free vector space over  $\mathbb{C}$  with basis  $\Gamma$ .

If we have the unitary representation  $u: \mathbb{C}[\Gamma] \rightarrow \mathbb{B}(\mathcal{H})$ , then

$$\pi_u(a) = \sum_{s \in \Gamma} a_s u_s$$

is a representation of  $\mathbb{C}[\Gamma]$ . If  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$  is the left-regular representation, then the left-regular group  $C^*$ -algebra is the group  $*$ -algebra with  $C^*$ -norm defined by  $\|a\| = \|\pi_\lambda(a)\|$ .

The universal group  $C^*$ -algebra is defined as the norm completion of

$$\|a\|_{\max} = \sup \left\{ \|\pi(a)\|_{\text{op}} \mid \pi: \mathbb{C}[\Gamma] \rightarrow \mathbb{B}(\mathcal{H}_\pi) \text{ is a representation} \right\}.$$

Note that

$$\begin{aligned} \|\pi(a)\| &= \left\| \pi \left( \sum_{s \in \Gamma} a_s \delta_s \right) \right\| \\ &= \left\| \sum_{s \in \Gamma} a_s \pi(\delta_s) \right\| \\ &\leq \sum_{s \in \Gamma} \|a_s \pi(\delta_s)\| \\ &= \sum_{s \in \Gamma} |a_s|. \end{aligned}$$

Note that since  $\|\cdot\|_\lambda$  is a norm, we must have  $a = 0$  if and only if  $\|a\|_{\max} = 0$ . The full group  $C^*$ -algebra admits a universal property.

**Proposition:** Let  $\Gamma$  be a discrete group. If  $u: \Gamma \rightarrow \mathbb{B}(\mathcal{H})$ , then there is a contractive  $*$ -homomorphism  $\pi_u: C^*(\Gamma) \rightarrow \mathbb{B}(\mathcal{H})$  that satisfies  $\pi_u(\delta_s) = u(s)$ .

## Using the Left-Regular Representation to Establish Amenability

If  $\pi: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is a unitary representation of  $\mathcal{H}$ , then a vector  $\xi \in \mathcal{H}$  is called invariant for  $\pi$  if  $\pi(g)(\xi) = \xi$  for all  $g \in \Gamma$ .

**Proposition:** The left-regular representation for  $\Gamma$  admits an invariant vector if and only if  $\Gamma$  is finite.

*Proof.* Let  $\Gamma$  be finite. Since  $\Gamma$  is finite, all functions  $a: \Gamma \rightarrow \mathbb{C}$  are square-summable. Thus,  $\xi = \mathbb{1}_\Gamma$  is square-summable, and since  $s\Gamma = \Gamma$  for all  $s \in \Gamma$ , we have  $\mathbb{1}_\Gamma$  is invariant for  $\lambda$ .

Now, let  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$  be the left-regular representation, and suppose there is  $\xi \in \ell_2(\Gamma)$  such that for all  $s \in \Gamma$ , we have

$$\lambda_s(\xi) = \xi.$$

In particular, this means that for any  $t \in \Gamma$ , we have

$$\begin{aligned} \lambda_s(\xi)(t) &= \xi(s^{-1}t) \\ &= \xi(t). \end{aligned}$$

Since this holds for all  $s \in \Gamma$ , we have that  $\xi = c\mathbb{1}_\Gamma$  for some  $c \in \mathbb{C}$ . However, since  $\xi \in \ell_2(\Gamma)$ , we must have that  $\sum_{t \in \Gamma} |c|^2 < \infty$ , which only holds if  $\Gamma$  is finite.  $\square$

An almost-invariant vector for a representation  $\pi: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ , as the name suggests,<sup>II</sup> a sequence (or net) of unit vectors  $(\xi_i)_{i \in I}$  such that

$$\lim_{i \in I} \|\pi(g)(\xi_i) - \xi_i\| = 0.$$

**Theorem:** A group  $\Gamma$  is amenable if and only if the left-regular representation has an almost-invariant vector.

*Proof.* Let  $\Gamma$  be amenable, and let  $F_i$  be a Følner sequence, where  $\frac{|sF_i \Delta F_i|}{|F_i|} \rightarrow 0$  for all  $s \in \Gamma$ .

Define  $\xi_i = \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}$ . Then,

$$\begin{aligned} \|\lambda_s(\xi_i) - \xi_i\|^2 &= \sum_{t \in \Gamma} |\lambda_s(\xi_i)(t) - \xi_i(t)|^2 \\ &= \sum_{t \in \Gamma} \left| \lambda_s \left( \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i} \right)(t) - \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}(t) \right|^2 \\ &= \sum_{t \in \Gamma} \left| \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{sF_i}(t) - \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}(t) \right|^2 \\ &= \frac{|sF_i \Delta F_i|}{|F_i|}. \end{aligned}$$

Thus,  $\lambda$  has an almost-invariant vector.

Suppose there exists an almost-invariant vector  $(\xi_i)_i \in \ell_2(\Gamma)$ . It is sufficient to construct an approximate mean. Since  $\xi_i \in \ell_2(\Gamma)$ , we have that  $\xi_i^2 \in \ell_1(\Gamma)$ . Setting  $\mu_i = \xi_i^2$ , we plug this into the expression for an approximate mean, and obtain

$$\begin{aligned} \|\lambda_s(\mu_i) - \mu_i\|_{\ell_1} &= \sum_{t \in \Gamma} \left| \lambda_s(\xi_i^2)(t) - \xi_i^2(t) \right| \\ &= \sum_{t \in \Gamma} |(\lambda_s(\xi_i)(t) - \xi_i(t))(\lambda_s(\xi_i)(t) + \xi_i(t))| \\ &= \|(\lambda_s(\xi_i) - \xi_i)(\lambda_s(\xi_i) + \xi_i)\|_{\ell_1} \\ &\leq \|\lambda_s(\xi_i) - \xi_i\|_{\ell_2} \|\lambda_s(\xi_i) + \xi_i\| \\ &\leq 2\|\lambda_s(\xi_i) - \xi_i\| \\ &\rightarrow 0. \end{aligned}$$

Thus,  $\mu_i$  is an approximate mean. □

Using the criterion of almost invariant vectors, we may show that a group is amenable if and only if the trivial representation — defined by  $1_\Gamma: \Gamma \rightarrow \mathbb{C}, 1_\Gamma(g) = 1$  is what is known as weakly contained in the left-regular representation.

A representation  $\pi: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is weakly contained in another representation  $\rho: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ , denoted  $\pi < \rho$ , if for every  $\xi \in \mathcal{H}$ , finite  $E \subseteq \Gamma$ , and  $\varepsilon > 0$ , then there are  $\eta_1, \dots, \eta_n \in \mathcal{H}$  such that

$$\left| \langle \pi(g)(\xi), \xi \rangle - \sum_{i=1}^n \langle \rho(g)(\eta_i), \eta_i \rangle \right| < \varepsilon.$$

**Theorem:** A discrete group  $\Gamma$  is amenable if and only if  $1_\Gamma < \lambda$ , where  $\lambda$  is the left-regular representation.

<sup>II</sup>I'm only mostly being facetious here.

*Proof.* We show that  $1_\Gamma < \lambda$  is equivalent to the existence of an almost invariant vector for  $\lambda$ . We assume  $\lambda$  admits an almost-invariant vector. It is sufficient to show that for every  $\varepsilon > 0$  and every finite set  $E \subseteq \Gamma$ , there are  $\eta_1, \dots, \eta_n \in \ell_2(\Gamma)$  such that

$$\left| 1 - \sum_{i=1}^n \langle \lambda_t(\eta_i), \eta_i \rangle \right| < \varepsilon$$

for every  $t \in E$ . If we take  $n = 1$  and  $\eta_1 = \xi$ , where  $\xi$  is almost-invariant for all  $g \in E$  — i.e.,  $\|\lambda_g(\xi) - \xi\|_{\ell_2} < \varepsilon$  for all  $g \in E$ . Note that we have

$$\begin{aligned} \|\lambda_g(\xi) - \xi\|^2 &= \langle \lambda_g(\xi) - \xi, \lambda_g(\xi) - \xi \rangle \\ &= \langle \lambda_g(\xi), \lambda_g(\xi) \rangle + \langle \xi, \xi \rangle - 2 \operatorname{Re}(\langle \lambda_g(\xi), \xi \rangle) \\ &= 2 - 2 \operatorname{Re}(\langle \lambda_g(\xi), \xi \rangle) \\ &= 2 \operatorname{Re}(1 - \langle \lambda_g(\xi), \xi \rangle) \\ &\leq 2|1 - \langle \lambda_g(\xi), \xi \rangle|. \end{aligned}$$

Additionally,

$$\begin{aligned} |1 - \langle \lambda_g(\xi), \xi \rangle|^2 &= (1 - \langle \lambda_g(\xi), \xi \rangle)(1 - \overline{\langle \lambda_g(\xi), \xi \rangle}) \\ &= 1 - \overline{\langle \lambda_g(\xi), \xi \rangle} - \langle \lambda_g(\xi), \xi \rangle + |\langle \lambda_g(\xi), \xi \rangle|^2 \\ &\leq 2 - 2 \operatorname{Re}(\langle \lambda_g(\xi), \xi \rangle) \\ &= \|\lambda_g(\xi) - \xi\|^2. \end{aligned}$$

Thus, we have that

$$\begin{aligned} |1 - \langle \lambda_g(\xi), \xi \rangle| &\leq \|\lambda_g(\xi) - \xi\| \\ &< \varepsilon. \end{aligned}$$

We start by showing that  $1_\Gamma < \lambda$  if and only if for every finite  $S \subseteq \Gamma$  and every  $\varepsilon > 0$ , there exists a unit vector  $\xi \in \mathcal{H}$  such that

$$\|\lambda_s(\xi) - \xi\|_{\ell_2} < \varepsilon.$$

In the forward direction, we see that there exists a unit vector  $\xi$  such that  $|1 - \langle \lambda_s(\xi), \xi \rangle| < \varepsilon^2/2$ , meaning  $\|\lambda_s(\xi) - \xi\| < \varepsilon$  by above. Similarly, if  $\|\lambda_s(\xi) - \xi\| < \varepsilon$ , then  $1_\Gamma < \lambda$ .

Now, we assume  $1_\Gamma < \lambda$ . Thus, for a finite  $E \subseteq \Gamma$  and  $\varepsilon > 0$ , then there exists  $f \in \ell_2(\Gamma)$  with  $\|f\|_{\ell_2} = 1$  such that  $\|\lambda_s(f) - f\| < \varepsilon$  for all  $s \in E$ .

Setting  $g = |f|^2$ , we have  $g \in \ell_1(\Gamma)$ . From Hölder's inequality, we have

$$\begin{aligned} \|\lambda_s(g) - g\|_{\ell_1} &\leq \left\| \lambda_s\left(\bar{f}\right) + \bar{f} \right\|_{\ell_2} \|\lambda_s(f) - f\| \\ &\leq 2\|\lambda_s(f) - f\|_{\ell_2} \\ &< 2\varepsilon. \end{aligned}$$

Thus,  $\Gamma$  admits an approximate mean, hence is amenable.  $\square$

Having obtained some more resources on Kesten's criterion, we can now prove that.

**Definition.** Let  $\lambda: \Gamma \rightarrow \mathcal{B}(\ell_2(\Gamma))$  be the left-regular representation. Then, for a finite set  $E \subseteq \Gamma$ , we define the Markov operator  $M(E)$  by

$$M(E) = \sum_{t \in E} \lambda_t.$$

Note that since  $\lambda_t$  is an isometry for each  $t$ , we have

$$\begin{aligned}\|M(E)\|_{\text{op}} &= \left\| \frac{1}{|E|} \sum_{t \in E} \lambda_t \right\|_{\text{op}} \\ &= \frac{1}{|E|} \left\| \sum_{t \in E} \lambda_t \right\|_{\text{op}} \\ &\leq \frac{1}{|E|} \sum_{t \in E} \|\lambda_t\|_{\text{op}} \\ &= 1,\end{aligned}$$

so the Markov operator is a bounded operator (indeed, a contraction).

**Theorem (Kesten's Criterion):** Let  $\Gamma$  contain a finite symmetric generating set  $S$ . Then,  $\Gamma$  is amenable if and only if

$$\|M(S)\|_{\text{op}} = 1.$$

*Proof.* Let  $\Gamma$  be amenable. Then,  $\lambda$  admits an almost-invariant vector,  $(\xi_n)_n \subseteq S_{\ell_2(\Gamma)}$ , such that

$$\|\lambda_s(\xi_n) - \xi_n\|_{\ell_2} \rightarrow 0$$

for all  $s \in \Gamma$ . In particular, we have

$$\begin{aligned}\left| \left\| \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (\xi_n) \right\|_{\ell_2} - \|\xi_n\|_{\ell_2} \right| &\leq \left\| \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (\xi_n) - \xi_n \right\|_{\ell_2} \\ &= \frac{1}{|S|} \left\| \left( \sum_{t \in S} \lambda_t \right) (\xi_n) - |S| \xi_n \right\|_{\ell_2} \\ &\leq \frac{1}{|S|} \sum_{t \in S} \|\lambda_t(\xi_n) - \xi_n\|_{\ell_2} \\ &\rightarrow 0,\end{aligned}$$

meaning that

$$\sup_{\xi \in S_{\ell_2(\Gamma)}} \left\| \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (\xi) \right\| = \|\xi\|,$$

and so the norm of the Markov operator is 1.

Suppose

$$\left\| \frac{1}{|S|} \sum_{t \in S} \lambda_t \right\|_{\text{op}} = 1,$$

or

$$\left\| \sum_{t \in S} \lambda_t \right\|_{\text{op}} = |S|.$$

**Proposition:** If  $T \in \mathcal{B}(\mathcal{H})$  is a self-adjoint operator, then

$$\|T\|_{\text{op}} = \sup_{x \in S_{\mathcal{H}}} |\langle T(x), x \rangle|.$$

*Proof.* We have that

$$\begin{aligned} |\langle T(x), x \rangle| &\leq \|T(x)\| \|x\| \\ &\leq \|T\|_{\text{op}} \|x\|^2 \\ &= \|T\|_{\text{op}}. \end{aligned}$$

Now, we seek to establish the opposite direction. Note that since  $T$  is self-adjoint, we know that  $\langle T(x), x \rangle \in \mathbb{R}$  for any  $x \in \mathcal{H}$ , so by the polarization identity, we have that

$$\langle T(x), y \rangle = \frac{1}{4}(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle).$$

Note that we know that

$$\|T\|_{\text{op}} = \sup_{x, y \in S_{\mathcal{H}}} |\langle T(x), y \rangle|.$$

Now, we set  $\alpha = \sup_{x \in S_{\mathcal{H}}} |\langle T(x), x \rangle|$ . Note that for any nonzero  $x \in \mathcal{H}$ , we have

$$\begin{aligned} \left| \left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|} \right\rangle \right| &\leq \alpha \\ |\langle T(x), x \rangle| &\leq \alpha \|x\|^2. \end{aligned}$$

Now, for any  $x, y \in \mathcal{H}$ , we may assume that  $\langle T(x), y \rangle \in \mathbb{R}$ , as we may multiply  $\langle T(x), y \rangle$  by  $\text{sgn}(\langle T(x), y \rangle)$ . Thus, by the polarization identity and the fact that  $T$  is self-adjoint, we have

$$\begin{aligned} \langle T(x), y \rangle &= \frac{1}{4}(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle) \\ |\langle T(x), y \rangle| &= \left| \frac{1}{4}(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle) \right| \\ &\leq \frac{1}{4}(|\langle T(x+y), x+y \rangle| + |\langle T(x-y), x-y \rangle|) \\ &\leq \frac{\alpha}{4}(\|x+y\|^2 + \|x-y\|^2) \\ &= \frac{\alpha}{4}(2\|x\|^2 + 2\|y\|^2) \\ &= \alpha. \end{aligned}$$

Thus, we have  $\|T\|_{\text{op}} \leq \sup_{x \in S_{\mathcal{H}}} |\langle T(x), x \rangle|$ . □

Now, since  $S$  is symmetric, we have that  $M(S)$  is self-adjoint. Therefore, we know that there is some  $\xi_n \in S_{\mathcal{H}}$  such that

$$\begin{aligned} 1 - \frac{1}{n} &< \left\langle \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (\xi_n), \xi_n \right\rangle \\ &\leq \left\langle \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (|\xi_n|), |\xi_n| \right\rangle. \end{aligned}$$

Thus, rearranging, we have

$$1 - \left\langle \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (|\xi_n|), |\xi_n| \right\rangle < \frac{1}{n}.$$

Since  $M(S)$  is a self-adjoint operator, we have that  $\operatorname{Re} \left\langle \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (\xi_n), \xi_n \right\rangle = \left\langle \left( \frac{1}{|S|} \sum_{t \in S} \lambda_t \right) (\xi_n), \xi_n \right\rangle$ . This gives

$$\begin{aligned} \left\| \left( \frac{1}{S} \sum_{t \in S} \lambda_t \right) (\xi) - \xi \right\| &\leq \frac{1}{|S|} \sum_{t \in S} \|\lambda_t(\xi) - \xi\| \\ &\leq \frac{1}{|S|} \sum_{t \in S} \sqrt{2} |1 - \langle \lambda_t(\xi), \xi \rangle| \\ &= \sqrt{2} \left| 1 - \frac{1}{|S|} \sum_{t \in S} \langle \lambda_t(\xi), \xi \rangle \right| \\ &\rightarrow 0. \end{aligned}$$

Thus,  $\lambda$  admits an almost-invariant vector. □

Next, we turn to Hulanicki's Criterion.

**Definition.** Let  $f \in \ell_1(\Gamma)$ . Then, we define the bounded operator

$$\lambda_{f(t)} = \sum_{t \in \Gamma} f(t) \lambda_t.$$

**Theorem:** If  $\Gamma$  is a discrete group, then  $\Gamma$  is amenable if and only if for every positive finitely-supported  $f: \Gamma \rightarrow \mathbb{C}$ , we have

$$\sum f(t) \leq \|\lambda_{f(t)}\|_{\text{op}}.$$

*Proof.* Suppose  $\Gamma$  is amenable. Let  $f \geq 0$  be a finitely supported function, and let  $(F_n)_n$  be a Følner sequence such that for every  $g \in \text{supp}(f)$ , we have

$$\frac{|g F_n \Delta F_n|}{|F_n|} \leq \frac{1}{n}.$$

Let  $\xi_n = \frac{1}{\sqrt{|F_n|}} \mathbb{1}_{F_n}$ . Note that  $\|\xi_n\|_{\ell_2} = 1$ .

We will use the fact that

$$\sup_{x \in S_{\mathcal{H}}} |\langle T(x), x \rangle| \leq \|T\|_{\text{op}}.$$

We see that

$$\begin{aligned} \left| \left\langle \left( \sum_{t \in \Gamma} f(t) \lambda_t \right) (\xi_n), \xi_n \right\rangle \right| &= \left| \sum_{t \in \Gamma} f(t) \langle \lambda_t(\xi_n), \xi_n \rangle \right| \\ &= \left| \sum_{t, s \in \Gamma} f(t) \xi_n(t^{-1}s) \xi_n(s) \right| \\ &\leq \|\lambda_{f(t)}\|, \end{aligned}$$

meaning

$$\lim_{n \rightarrow \infty} \left| \left\langle \left( \sum_{t \in \Gamma} f(t) \lambda_t \right) (\xi_n), \xi_n \right\rangle \right| \leq \|\lambda_{f(t)}\|.$$



Notice that  $\xi_n$  is an almost-invariant vector for  $\lambda$ , meaning that  $\xi_n(t^{-1}s) \rightarrow \xi_n(s)$ . Therefore, this means

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \sum_{t,s \in \Gamma} f(t) \xi_n(t^{-1}s) \xi_n(s) \right| &= \lim_{n \rightarrow \infty} \left| \sum_{t,s \in \Gamma} f(t) |\xi_n(s)|^2 \right| \\ &= \sum_{t \in \Gamma} f(t) \left| \sum_{s \in \Gamma} |\xi_n(s)|^2 \right| \\ &= \sum_{t \in \Gamma} f(t) \\ &\leq \|\lambda_{f(t)}\|_{\text{op}}. \end{aligned}$$

This establishes that there is some  $C > 0$  such that

$$\sum_{t \in \Gamma} f(t) \leq C \|\lambda_{f(t)}\|_{\text{op}}.$$

To show that  $C = 1$ , we note that, by the definition of convolution, we must have

$$\left( \sum_{t \in \Gamma} f(t) \right)^n = \sum_{t \in \Gamma} (f * \dots * f)(t),$$

and

$$\begin{aligned} (\lambda_{f(t)})^n &= \left( \sum_{t \in \Gamma} f(t) \lambda_t \right)^n \\ &= \sum_{t \in \Gamma} (f * \dots * f)(t) \lambda_t \\ &= \lambda_{(f * \dots * f)(t)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left( \sum_{t \in \Gamma} f(t) \right)^n &= \sum_{t \in \Gamma} (f * \dots * f)(t) \\ &\leq C \|\lambda_{(f * \dots * f)(t)}\| \\ &= C \left( \|\lambda_{f(t)}\|_{\text{op}} \right)^n. \end{aligned}$$

This means we have

$$\sum_{t \in \Gamma} f(t) \leq C^{1/n} \|\lambda_{f(t)}\|_{\text{op}}.$$

Since  $n$  is arbitrary, this means  $C = 1$ .

Now, if for all finitely supported  $f$ , we have

$$\sum_{t \in \Gamma} f(t) \leq \|\lambda_{f(t)}\|_{\text{op}}.$$

If  $f = \mathbb{1}_E$  for some finite  $E \subseteq \Gamma$ , we see that

$$\left\| \sum_{t \in E} \lambda_t \right\|_{\text{op}} = |E|,$$

so by Kesten's criterion, we have that  $\Gamma$  is amenable. □

## Completely [Property] Maps

We begin this section with an overview of positive maps, completely positive maps, and extensions. These will be useful for understanding the theorem that a group is amenable if and only if the reduced group  $C^*$ -algebra is nuclear. The ultimate goal here is to prove Arveson's extension theorem (i.e., that  $\mathcal{B}(\mathcal{H})$  is injective with respect to completely positive maps). The primary text for this purpose will be Vern Paulsen's *Completely Bounded Maps and Operator Algebras*.

The idea behind completely positive maps is that they are positive when subjected to a certain amplification process related to the matrix algebras.

**Definition.** An element of a  $C^*$ -algebra is positive if and only if it is self-adjoint and its spectrum is contained in the nonnegative reals. Alternatively,  $b \in A$  is of the form  $b = a^*a$  for some  $a \in A$ .

To introduce a norm such that  $\text{Mat}_n(A)$  becomes a  $C^*$ -algebra, we begin with the most basic  $C^*$ -algebra,  $\mathcal{B}(\mathcal{H})$ , and consider the  $n$ -fold amplification of  $\mathcal{H}$ ,  $\mathcal{H}^{(n)}$ . This is a Hilbert space equipped with inner product

$$\left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \right\rangle = \sum_{j=1}^n \langle h_j, k_j \rangle.$$

Meanwhile, we may consider  $\text{Mat}_n(\mathcal{B}(\mathcal{H}))$  as a linear map on  $\mathcal{H}^{(n)}$ , by taking

$$(T_{ij})_{ij} = \begin{pmatrix} \sum_{j=1}^n T_{1j}(h_j) \\ \vdots \\ \sum_{j=1}^n T_{nj}(h_j) \end{pmatrix}.$$

This yields a  $*$ -isomorphism between  $\text{Mat}_n(\mathcal{B}(\mathcal{H}))$  and  $\mathcal{B}(\mathcal{H}^{(n)})$ .

Given any  $C^*$ -algebra  $A$ , we may theorize  $\text{Mat}_n(A)$  by first isometrically representing  $A$  on some Hilbert space  $\mathcal{H}$ , letting  $A$  be a  $C^*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ , and then identifying  $\text{Mat}_n(A)$  as a  $*$ -subalgebra of  $\text{Mat}_n(\mathcal{B}(\mathcal{H}))$ .

Using a faithful  $*$ -representation of  $A$ , we now have a way to turn  $\text{Mat}_n(A)$  into a  $C^*$ -algebra. However, since the norm is unique on a  $C^*$ -algebra, the norm on  $\text{Mat}_n(A)$  defined in this fashion is independent of the representation of  $A$  that we choose. Furthermore, since  $*$ -isomorphisms are positive maps, the positive elements of  $\text{Mat}_n(A)$  are uniquely determined. This means that every  $C^*$ -algebra carries with it a set of canonically defined norms and orders on each  $\text{Mat}_n(A)$ .

For example, consider  $\text{Mat}_k(\mathbb{C})$ , which can be identified with  $\mathcal{L}(\mathbb{C}^k)$ . We identify  $\text{Mat}_n(\text{Mat}_k(\mathbb{C})) \cong \text{Mat}_{nk}(\mathbb{C})$ . With this identification, the usual multiplication and involution on  $\text{Mat}_n(\text{Mat}_k(\mathbb{C}))$  become multiplication and involution on  $\text{Mat}_{nk}(\mathbb{C})$ .

Now, let  $X$  be a compact Hausdorff space, and let  $C(X)$  be the  $C^*$ -algebra of continuous functions with  $f^*(x) = \overline{f(x)}$ , equipped with the norm  $\|f\| = \sup_{x \in X} |f(x)|$ . Then, an element  $F = (f_{ij})_{ij}$  of  $\text{Mat}_n(C(X))$  can be considered as a continuous  $\text{Mat}_n(\mathbb{C})$ -valued function. Addition, multiplication, and involution in  $\text{Mat}_n(C(X))$  are pointwise. Recalling that the norm on  $\text{Mat}_n(C(X))$  is unique, we may let  $\|F\| = \sup_{x \in X} \|F(x)\|$ , where the latter norm is the canonical matrix norm on  $\text{Mat}_n(C(X))$ . The positive elements of  $\text{Mat}_n(C(X))$  are those  $F$  for which  $F(x)$  is a positive matrix for all  $x$ .

Now, given two  $C^*$ -algebras  $A$  and  $B$  and a map  $\phi: A \rightarrow B$ , there are maps  $\phi_n: \text{Mat}_n(A) \rightarrow \text{Mat}_n(B)$ , given by

$$\phi_n((a_{ij})_{ij}) = (\phi(a_{ij}))_{ij}.$$

In general, when we say that  $\phi$  is completely [property], then we say that all the  $\phi_n$  have that property. For instance, if  $\phi$  is positive, in that it maps positive elements of  $A$  to positive elements of  $B$ , then we say  $\phi$  is completely positive if  $\phi_n$  is a positive map for each  $n$ , where the positive elements of  $\text{Mat}_n(A)$  and  $\text{Mat}_n(B)$  are defined canonically.

Unfortunately, it's not always the case that (e.g.) positive maps are completely positive, or even that  $\|\phi_n\|_{\text{op}} = \|\phi\|_{\text{op}}$  for each  $n$ .

There is an isomorphism between  $\text{Mat}_n(A)$  and the tensor product  $\text{Mat}_n(\mathbb{C}) \otimes A$ . We detail it here. The proof is from Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*.

**Theorem:** Let  $A$  be an algebra, and let  $\text{Mat}_n(A)$  denote the matrix algebra of  $A$ . Then, there is a  $*$ -isomorphism

$$\text{Mat}_n(A) \cong \text{Mat}_n(\mathbb{C}) \otimes A.$$

*Proof.* Define  $\varphi: \text{Mat}_n(A) \rightarrow \text{Mat}_n(\mathbb{C}) \otimes A$  by

$$\varphi\left((a_{ij})_{ij}\right) = \sum_{i,j=1}^n e_{ij} \otimes a_{ij}.$$

Recall that if  $A$  and  $B$  are two algebras, multiplication in  $A \otimes B$  is defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

and if  $A$  and  $B$  are  $*$ -algebras, then the involution is defined by

$$(a \otimes b)^* = a^* \otimes b^*.$$

We start by showing that  $\text{Mat}_n(A) \cong \text{Mat}_n(\mathbb{C}) \otimes A$  as vector spaces. By the definition of the tensor product, the map  $\varphi$  is linear.

Now, suppose

$$\begin{aligned} \varphi\left((a_{ij})_{ij}\right) &= \sum_{i,j=1}^n e_{ij} \otimes a_{ij} \\ &= 0. \end{aligned}$$

Then, since  $\{e_{ij}\}_{ij}$  is linearly independent, we know that  $a_{ij} = 0$  for all  $i, j$ , so  $(a_{ij})_{ij} = 0$ , so  $\varphi$  is injective.

Now, let  $t \in \text{Mat}_n(\mathbb{C}) \otimes A$  be given by

$$t = \sum_k m_k \otimes a_k,$$

where  $m_k \in \text{Mat}_n(\mathbb{C})$  and  $a_k \in A$ . Then, using the matrix units, we write each  $m_k$  as

$$m_k = \sum_{i,j=1}^n m_k(i,j)e_{ij}.$$

This gives

$$\begin{aligned} t &= \sum_k \left( \sum_{i,j=1}^n m_k(i,j)e_{ij} \right) \otimes a_k \\ &= \sum_{i,j=1}^n e_{ij} \otimes \left( \sum_k m_k(i,j)a_k \right). \end{aligned}$$

Defining  $a_{ij} := \sum_k m_k(i, j)a_k$ , we get

$$t = \sum_{i,j=1}^n e_{ij} \otimes a_{ij},$$

meaning that

$$\varphi\left((x_{ij})_{ij}\right) = t.$$

Thus,  $\varphi$  is surjective.

We will show now that  $\varphi$  is multiplicative and  $*$ -preserving. If  $(a_{ij})_{ij}$  and  $(b_{ij})_{ij}$  belong to  $\text{Mat}_n(A)$ .

$$\begin{aligned} \varphi((a_{ik})_{ik})\varphi((b_{lj})_{lj}) &= \left(\sum_{i,k=1}^n e_{ik} \otimes a_{ik}\right)\left(\sum_{l,j=1}^n e_{lj} \otimes b_{lj}\right) \\ &= \sum_{i,j,k,l=1}^n (e_{ik} \otimes a_{ik})(e_{lj} \otimes b_{lj}) \\ &= \sum_{i,j,k,l=1}^n e_{ik}e_{lj} \otimes a_{ik}b_{lj} \\ &= \sum_{i,j,k=1}^n e_{ik}e_{kj} \otimes a_{ik}b_{kj} \\ &= \sum_{ij,k=1}^n e_{ij} \otimes a_{ik}b_{kj} \\ &= \sum_{i,j=1}^n e_{ij} \otimes \left(\sum_{k=1}^n a_{ik}b_{kj}\right) \\ &= \varphi\left(\left(\sum_{k=1}^n a_{ik}b_{kj}\right)_{ij}\right) \\ &= \varphi\left((a_{ij})_{ij}(b_{ij})_{ij}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \varphi\left((a_{ij})_{ij}\right)^* &= \left(\sum_{i,j=1}^n e_{ij} \otimes a_{ij}\right)^* \\ &= \sum_{i,j=1}^n (e_{ij} \otimes a_{ij})^* \\ &= \sum_{i,j=1}^n e_{ij}^* \otimes a_{ij}^* \\ &= \sum_{i,j=1}^n e_{ji} \otimes a_{ij}^* \\ &= \sum_{i,j=1}^n e_{ij} \otimes a_{ji}^* \end{aligned}$$

$$\begin{aligned}
&= \varphi \left( \left( a_{ji}^* \right)_{ij} \right) \\
&= \varphi \left( \left( a_{ij} \right)_{ij}^* \right).
\end{aligned}$$

□

There are lots of useful results using amplification to the matrix algebras.

**Example (Dilating an Isometry).** Let  $V$  be an isometry, and let  $P = I_{\mathcal{H}} - VV^*$  be the projection onto  $\text{Ran}(V)^\perp$ . Define  $U$  on  $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$  by

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix}.$$

We find that

$$\begin{aligned}
U^* &= \begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix} \\
UU^* &= \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} \begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix} \\
&= \begin{pmatrix} VV^* + P & PV \\ V^*P & V^*V \end{pmatrix} \\
&= \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{H}} \end{pmatrix} \\
&= I_{\mathcal{K}} \\
U^*U &= \begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix} \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} \\
&= I_{\mathcal{K}}.
\end{aligned}$$

Thus,  $U$  is a unitary on  $\mathcal{K}$ . We may identify  $\mathcal{H} \cong \mathcal{H} \oplus 0$ , and take

$$V^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$$

for all  $n \geq 0$ . Thus, we are able to realize any isometry  $V$  as the restriction of some unitary to a subspace that respects powers.

**Example (Dilating a Contraction).** Similarly, we may define the isometric dilation of a contraction. Let  $T$  be an operator on  $\mathcal{H}$  with  $\|T\| \leq 1$ , and define  $D_T = (I - T^*T)^{1/2}$ . We see that

$$\begin{aligned}
\|T(h)\|^2 + \|D_T(h)\|^2 &= \langle T^*T(h), h \rangle + \langle D_T^2(h), h \rangle \\
&= \|h\|^2.
\end{aligned}$$

We consider now the sequence space

$$\ell_2(\mathcal{H}) = \left\{ (h_n)_{n \in \mathbb{N}} \mid h_n \in \mathcal{H}, \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \right\}.$$

We have the norm

$$\|(h_n)_n\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2$$

and the inner product

$$\langle (h_n)_n, (k_n)_n \rangle = \sum_{n=1}^{\infty} \langle h_n, k_n \rangle.$$

We define the operator  $V: \ell_2(\mathcal{H}) \rightarrow \ell_2(\mathcal{H})$  by

$$V((h_n)_n) = (T(h_1), D_T(h_1), h_2, \dots).$$

It then follows that  $V$  is an isometry on  $\ell_2(\mathcal{H})$ , and that if we identify  $\mathcal{H} \cong \mathcal{H} \oplus 0 \oplus \dots$ , then  $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$ .

**Theorem** (Sz.-Nagy's Dilation Theorem): Let  $T$  be a contraction operator on  $\mathcal{H}$ . There is a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$  as a subspace, and a unitary operator  $U$  on  $\mathcal{K}$  such that  $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ .

*Proof.* Take  $\mathcal{K} = \ell_2(\mathcal{H}) \oplus \ell_2(\mathcal{H})$ , and identify  $\mathcal{H}$  as  $(\mathcal{H} \oplus 0 \oplus \dots) \oplus 0$ . Let  $V$  be the isometric dilation of  $T$  on  $\ell_2(\mathcal{H})$ , and let  $U$  be the unitary dilation of  $V$  on  $\ell_2(\mathcal{H}) \oplus \ell_2(\mathcal{H})$ . Then, since  $\mathcal{H} \subseteq \ell_2(\mathcal{H}) \oplus 0$ , we have that  $P_{\mathcal{H}} U^n|_{\mathcal{H}} = P_{\mathcal{H}} V^n|_{\mathcal{H}} = T^n$  for all  $n \geq 0$ .  $\square$

Whenever  $Y$  is an operator on  $\mathcal{K}$ ,  $\mathcal{H}$  a (closed) subspace of  $\mathcal{K}$ , and  $X = P_{\mathcal{H}} Y|_{\mathcal{H}}$ , then we say  $X$  is a compression of  $Y$ .

**Corollary** (Von Neumann's Inequality): Let  $T$  be a contraction on a Hilbert space. Then, for any polynomial  $p$ ,

$$\|p(T)\| \leq \sup_{|z| \leq 1} |p(z)|.$$

*Proof.* Let  $U$  be a unitary dilation of  $T$ . Since  $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ , linearity means we have  $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$ . Since  $U$  is defined on a larger space than  $T$ , then  $\|p(T)\| \leq \|p(U)\|$ . Furthermore, since unitaries are normal, we have

$$\|p(U)\| = \sup_{\lambda \in \sigma(U)} |p(\lambda)|,$$

where  $\sigma(U)$  is the spectrum of  $U$ . Since  $U$  is unitary,  $\sigma(U) \subseteq \mathbb{T}$ , so von Neumann's inequality follows.  $\square$

## Positive and Completely Positive Maps

### Positive Maps

There are certain results on positive maps that are useful in the study of completely positive maps. We introduce them here.

**Definition.** If  $S$  is a subset of a  $C^*$ -algebra  $A$ , we say  $S$  is an operator system if  $A$  is unital and  $S$  is a self-adjoint subspace of  $A$  with  $1_A \in S$ .

Note that if  $S$  is an operator system and  $h \in S$  is self-adjoint, then though the values  $h_+$  and  $h_-$ , defined by the continuous functional calculus with

$$\begin{aligned} f^+(x) &= \max\{0, x\} \\ f^-(x) &= \min_{0, -x} \end{aligned}$$

may not belong to  $S$ , we can write  $h$  as the difference of two positive elements in  $S$  by

$$h = \frac{1}{2}(\|h\|1_A + h) - \frac{1}{2}(\|h\|1_A - h).$$

**Definition.** If  $S$  is an operator system,  $B$  is a  $C^*$ -algebra, and  $\phi: S \rightarrow B$  is a linear map, then  $\phi$  is called positive if it maps positive elements of  $S$  to positive elements of  $B$ .

**Theorem:** If  $\phi$  is a positive linear functional on an operator system  $S$ , then  $\|\phi\| = \phi(1_A)$ .

When the range of  $\phi$  is not  $\mathbb{C}$ , but rather a  $C^*$ -algebra, then the situation is a bit different.

**Proposition:** Let  $S$  be an operator system, and let  $B$  be a  $C^*$ -algebra. If  $\phi: S \rightarrow B$  is a positive map, then  $\phi$  is bounded, with

$$\|\phi\| \leq 2\|\phi(1_A)\|.$$

*Proof.* Note that if  $p$  is positive, then  $0 \leq p \leq \|p\|1_A$ , so  $0 \leq \phi(p) \leq \|p\|\phi(1_A)$  since positive functions are order-preserving. Thus, we get  $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$  when  $p \geq 0$ .

Note that when  $p_1$  and  $p_2$  are positive, then  $\|p_1 - p_2\| \leq \max\{\|p_1\|, \|p_2\|\}$ . If  $h$  is self-adjoint, then we have

$$\|\phi(h)\| = \frac{1}{2}\phi(\|h\|1_A + h) - \frac{1}{2}\phi(\|h\|1_A - h),$$

which is the difference of two positive elements in  $B$ . Thus, we have

$$\begin{aligned} \|\phi(h)\| &\leq \frac{1}{2} \max\{\|\phi(\|h\|1_A + h)\|, \|\phi(\|h\|1_A - h)\|\} \\ &\leq \|h\|\|\phi(1)\|. \end{aligned}$$

Finally, if  $a$  is arbitrary then write  $a = h + ik$  via the Cartesian decomposition, where  $\|h\|, \|k\| \leq \|a\|$ , and  $h, k$  are self-adjoint. Thus, we have

$$\begin{aligned} \|\phi(a)\| &\leq \|\phi(h)\| + \|\phi(k)\| \\ &\leq 2\|a\|\|\phi(1_A)\|. \end{aligned}$$

□

As it turns out, 2 is the best constant.

**Example.** Let  $\mathbb{T}$  be the unit circle in  $\mathbb{C}$ , and  $C(\mathbb{T})$  be the continuous functions on  $z$ . Let  $z$  be the coordinate function, and let  $S \subseteq C(\mathbb{T})$  be the subspace spanned by  $1, z, \bar{z}$ . Defining

$$\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix},$$

An element of  $S$  is positive if and only if  $c = \bar{b}$  and  $a \geq 2|b|$ , and an element of  $\text{Mat}_2(\mathbb{C})$  is positive if and only if its diagonal entries and determinant are nonnegative real numbers. Thus, it is the case that  $\phi$  is a positive map, but also

$$\begin{aligned} 2\|\phi(1)\| &= 2 \\ &= \|\phi(z)\| \\ &\leq \|\phi\|, \end{aligned}$$

meaning  $\|\phi\| = 2\|\phi(1)\|$ .

We are interested in seeing when unital, positive maps are contractive.

**Lemma:** Let  $A$  be a  $C^*$ -algebra, and let  $p_i$  be positive elements of  $A$  such that

$$\sum_{i=1}^n p_i \leq 1.$$

If  $\lambda_i$  are scalars with  $|\lambda_i| \leq 1$ , then

$$\left\| \sum_{i=1}^n \lambda_i p_i \right\| \leq 1.$$

*Proof.* Note that

$$\begin{pmatrix} \sum_{i=1}^n \lambda_i p_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} p_1^{1/2} & \cdots & p_n^{1/2} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \text{diag}(\lambda_1, \dots, \lambda_n) \begin{pmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_n^{1/2} & 0 & \cdots & 0 \end{pmatrix}.$$

The norm on the matrix on the left is  $\|\sum_{i=1}^n \lambda_i p_i\|$ , while the three matrices on the right have norm less than 1, using the fact that  $\|a^* a\| = \|a\|^2$ .  $\square$

**Theorem:** Let  $B$  be a  $C^*$ -algebra,  $X$  a compact Hausdorff space, and  $C(X)$  the continuous functions on  $X$ . Let  $\phi: C(X) \rightarrow B$  be a positive map. Then,  $\|\phi\| = \|\phi(1)\|$ .

*Proof.* We may assume  $\phi(1) \leq 1$ . Let  $f \in C(X)$  with  $\|f\| \leq 1$ , and let  $\varepsilon > 0$ . Now, we may choose a finite open cover  $\{U_i\}_{i=1}^n$  of  $X$  such that  $|f(x) - f(x_i)| < \varepsilon$  for all  $x \in U_i$ , and let  $\{p_i\}_{i=1}^n$  be a partition of unity subordinate to the cover. That is,  $\{p_i\}_{i=1}^n$  are nonnegative continuous functions satisfying  $\sum_{i=1}^n p_i = 1$  and  $p_i(x) = 0$  for  $x \notin U_i$ .

Set  $\lambda_i = f(x_i)$ , and note that if  $p_i(x) \neq 0$  for some  $i$ , then  $x \in U_i$  and  $|f(x) - \lambda_i| < \varepsilon$ . Hence, for any  $x$ , we have

$$\begin{aligned} \left| f(x) - \sum_{i=1}^n \lambda_i p_i(x) \right| &= \left| \sum_{i=1}^n (f(x) - \lambda_i) p_i(x) \right| \\ &\leq \sum_{i=1}^n |f(x) - \lambda_i| p_i(x) \\ &< \sum_{i=1}^n \varepsilon p_i(x) \\ &= \varepsilon. \end{aligned}$$

By above, we know that  $\|\sum_{i=1}^n \lambda_i p_i\| \leq 1$ , we have

$$\begin{aligned} \|\phi(f)\| &\leq \left\| \phi\left(f - \sum_{i=1}^n \lambda_i p_i\right) \right\| + \left\| \sum_{i=1}^n \phi(p_i) \right\| \\ &< 1 + \varepsilon \|\phi\|. \end{aligned}$$

Since  $\varepsilon$  was arbitrary, we have  $\|\phi\| \leq 1$ .  $\square$

**Lemma (Riesz–Fejér Theorem):** Let  $\tau(e^{i\theta}) = \sum_{n=-N}^N a_n e^{in\theta}$  be a strictly positive function on  $\mathbb{T}$ . Then, there is a polynomial  $p(z) = \sum_{n=0}^N p_n z^n$  such that

$$\tau(e^{i\theta}) = |p(e^{i\theta})|^2.$$

*Proof.* Note that  $\tau$  is real-valued, so  $a_{-n} = \overline{a_n}$ , and  $a_0$  is real. Assuming  $a_{-N} \neq 0$ , we take  $g(z) = \sum_{n=-N}^N a_n z^{n+N}$ , so that  $g$  is a polynomial of degree  $2N$ ,  $g(0) \neq 0$ .

We have  $g(e^{i\theta}) = \tau(e^{i\theta}) e^{iN\theta} \neq 0$ , and that  $\overline{g(1/\bar{z})} = z^{-2N} g(z)$ .

We write the  $2N$  zeros of  $g$  as  $z_1, \dots, z_N, 1/\bar{z}_1, \dots, 1/\bar{z}_N$ .

Set  $q(z) = (z - z_1) \cdots (z - z_N)$  and  $h(z) = (z - 1/\bar{z}_1) \cdots (z - 1/\bar{z}_N)$ . We have that

$$g(z) = a_N q(z) h(z),$$



where

$$\overline{h(z)} = \frac{(-1)^N \bar{z}^N q(1/\bar{z})}{z_1 \cdots z_N}.$$

Thus, we have

$$\begin{aligned} \tau(e^{i\theta}) &= e^{-iN\theta} g(e^{i\theta}) \\ &= |g(e^{i\theta})| \\ &= |a_N q(e^{i\theta}) \bar{h}(e^{i\theta})| \\ &= \frac{a_N}{z_1 \cdots z_N} |q(e^{i\theta})|^2. \end{aligned}$$

□

**Theorem:** Let  $T$  be an operator on  $\mathcal{H}$  with  $\|T\| \leq 1$ , and let  $S \subseteq C(T)$  be the operator system defined by

$$S = \left\{ p(e^{i\theta}) + \overline{q(e^{i\theta})} \mid p, q \text{ are polynomials} \right\}.$$

Then,  $\phi: S \rightarrow \mathbb{B}(\mathcal{H})$ , given by  $\phi(p + \bar{q}) = p(T) + q(T)^*$  is positive.

*Proof.* It is enough to prove that  $\phi(\tau)$  is positive for every *strictly* positive  $\tau$ .

Let  $\tau(e^{i\theta})$  be strictly positive in  $S$ , meaning  $\tau(e^{i\theta}) = \sum_{\ell, k=0}^n \alpha_\ell \bar{\alpha}_k e^{i(\ell-k)\theta}$ . We must prove that

$$\phi(\tau) = \sum_{\ell, k=0}^n \alpha_\ell \bar{\alpha}_k T(\ell - k),$$

where

$$T(j) = \begin{cases} T^j & j \geq 0 \\ (T^*)^{-j} & j < 0. \end{cases}$$

Fix  $x \in \mathcal{H}$ . Note that

$$\langle \phi(\tau)(x), x \rangle = \left\langle \begin{pmatrix} I & T^* & \cdots & (T^*)^n \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^n & \cdots & T & I \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1 x \\ \bar{\alpha}_2 x \\ \vdots \\ \bar{\alpha}_n x \end{pmatrix}, \begin{pmatrix} \bar{\alpha}_1 x \\ \bar{\alpha}_2 x \\ \vdots \\ \bar{\alpha}_n x \end{pmatrix} \right\rangle, \quad (*)$$

where our matrix operator acts on  $\mathcal{H}^{(n)}$ . Thus, we only need to show that this matrix operator is positive.

To that end, define the  $n \times n$  matrix

$$R = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ T & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T & 0 \end{pmatrix},$$

and note that  $R^{n+1} = 0$ , with  $\|R\|_{\text{op}} \leq 1$  (as  $T$  is a contraction).

We let  $I$  denote the identity operator on  $\mathcal{H}^{(n)}$ . The matrix operator  $(*)$  can be written as

$$I + R + R^2 + \cdots + R^n + R^* + \cdots + (R^*)^n = (I - R)^{-1} + (I - R^*)^{-1} - I,$$

where we used the fact that  $R^{n+1} = 0$  in the geometric series for  $(I - R)^{-1}$  and  $(I - R^*)^{-1}$ . To see that this operator is positive, we let  $h \in \mathcal{H}^{(n)}$ , and let  $h = (I - R)y$  for some  $y \in \mathcal{H}^{(n)}$ . Then,

$$\begin{aligned} \left\langle \left( (I - R)^{-1} + (I - R^*)^{-1} - I \right) (h), h \right\rangle &= \langle y, (I - R)y \rangle + \langle (I - R^*)(y), y \rangle - \langle (I - R)(y), (I - R)(y) \rangle \\ &= \|y\|^2 - \|R(y)\|^2 \\ &\geq 0, \end{aligned}$$

since  $R$  is a contraction. □

Now, we may prove von Neumann's inequality in a different way.

**Theorem** (von Neumann's Inequality): Let  $T$  be an operator on a Hilbert space with  $\|T\|_{\text{op}} \leq 1$ . Then, for any polynomial  $p$ , we have

$$\|p(T)\|_{\text{op}} \leq \|p\|,$$

where  $\|p\| = \sup_{\theta} |p(e^{i\theta})|$ .

*Proof.* The operator system defined by

$$S = \left\{ p(e^{i\theta}) + \overline{q(e^{i\theta})} \mid p, q \text{ polynomials} \right\}$$

is a  $*$ -algebra that separates points, so by the Stone–Weierstrass theorem,  $S$  is dense in  $C(\mathbb{T})$ . We know that  $\phi$  is bounded, so it extends  $C(\mathbb{T})$ . The extension to  $\bar{S} = C(\mathbb{T})$  is also positive, so  $\phi$  is contractive. □

Note that if  $A(\mathbb{D})$  denotes the functions analytic on  $\mathbb{D}$  and continuous on  $\bar{\mathbb{D}}$ , we know that by the maximum modulus principle that the supremum of any function in  $A(\mathbb{D})$  occurs on  $\mathbb{T}$ . We may thus consider  $A(\mathbb{D})$  as a closed subalgebra of  $C(\mathbb{T})$ .

Furthermore, polynomials are dense in  $A(\mathbb{D})$ . Thus, the homomorphism  $p \mapsto p(T)$  extends to a homomorphism  $f \mapsto f(T)$  that satisfies  $\|f(T)\|_{\text{op}} \leq \|f\|$  for all  $f \in A(\mathbb{D})$ .

Another consequence is that if  $a$  is an element of some unital  $C^*$ -algebra  $A$  with  $\|a\| \leq 1$ , then there is a unital, positive map  $\phi: C(\mathbb{T}) \rightarrow A$  such that  $\phi(p) = p(a)$ .

**Corollary:** Let  $B$  and  $C$  be unital  $C^*$ -algebras. Let  $A$  be a unital subalgebra of  $B$ , and let  $S = A + A^*$  be an operator space. If  $\phi: S \rightarrow C$  is positive, then  $\|\phi(a)\| \leq \|\phi(1)\| \|a\|$ .

*Proof.* Let  $a \in A$  with  $\|a\| \leq 1$ . We may extend  $\phi$  to a positive map on  $\bar{S}$ . There is also a positive map  $\psi: C(\mathbb{T}) \rightarrow B$  with  $\psi(p) = p(a)$ . Since  $A$  is an algebra, we must have  $\text{Ran}(\psi) \subseteq \bar{S}$ .

The composition of positive maps is positive, so we have

$$\begin{aligned} \|\phi(a)\| &= \left\| \phi \circ \psi(e^{i\theta}) \right\| \\ &\leq \|\phi \circ \psi(1)\| \|e^{i\theta}\| \\ &= \|\phi(1)\|. \end{aligned}$$

□

If  $\phi(1) = 1$ , then  $\phi$  is a contraction on  $A$ , though  $\phi$  may not be a contraction on all of  $S$ .

**Corollary:** Let  $A$  and  $B$  be unital  $C^*$ -algebras with  $\phi: A \rightarrow B$  a positive map. Then,  $\|\phi\|_{\text{op}} = \|\phi(1)\|$ .

**Lemma:** Let  $A$  be a  $C^*$ -algebra,  $S \subseteq A$  an operator system, and  $f: S \rightarrow \mathbb{C}$  a linear functional with  $f(1) = 1 = \|f\|$ . If  $a$  is a normal element of  $A$ , and  $a \in S$ , then  $f(a) \in \overline{\text{conv}}(\sigma(a))$ .

*Proof.* Suppose not.

The convex hull of a compact set is the intersection of all closed disks containing the set. Then, there exists  $\lambda$  and  $r > 0$  such that  $|f(a) - \lambda| > r$ , where

$$\sigma(a) \subseteq \{z \mid |z - \lambda| \leq r\}.$$

Then,  $\sigma(a - \lambda 1) \subseteq \{z \mid |z| \leq r\}$ . Since norm and spectral radius agree for normal elements, we have  $\|a - \lambda 1\| \leq r$ , while  $|f(a - \lambda 1)| > r$ . This contradicts the fact that  $\|f\| \leq 1$ .  $\square$

**Proposition:** Let  $S$  be an operator system,  $B$  a unital  $C^*$ -algebra, and let  $\phi: S \rightarrow B$  be a unital contraction. Then,  $\phi$  is positive.

*Proof.* Since we can represent  $B$  on  $\mathbb{B}(\mathcal{H})$ , we assume  $B = \mathbb{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . Fix  $x \in \mathcal{H}$  with  $\|x\| = 1$ .

Setting  $f(a) = \langle \phi(a)(x), x \rangle$ , we have  $f(1) = 1$  and  $\|f\| \leq \|\phi\|$ . If  $a$  is positive, then  $f(a)$  is positive by the previous lemma, so since  $x$  was arbitrary,  $\phi(a)$  is also positive.  $\square$

**Proposition:** Let  $A$  be a unital  $C^*$ -algebra, and let  $M$  be a unital subspace of  $A$ . If  $B$  is a unital  $C^*$ -algebra, and  $\phi: M \rightarrow B$  is a unital contraction, then the map  $\tilde{\phi}: M + M^* \rightarrow B$ , given by

$$\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$$

is well-defined and the unique positive extension of  $\phi$  to  $M + M^*$ .

*Proof.* To prove that  $\tilde{\phi}$  is well-defined, it is enough to prove that if  $a$  and  $a^*$  belong to  $M$ , then  $\phi(a)^* = \phi(a^*)$ . Set

$$S_1 = \{a \mid a \in M \text{ and } a^* \in M\}.$$

Then,  $S_1$  is an operator system, and  $\phi$  is a unital, contractive map on  $S_1$ , hence positive by the previous proposition. Since  $\phi$  is positive,  $\phi$  is self-adjoint, so  $\phi(a^*) = \phi(a)^*$ , meaning  $\tilde{\phi}$  is well-defined.

To see that  $\tilde{\phi}$  is positive, we may assume  $B = \mathbb{B}(\mathcal{H})$ . Fix  $x \in S_{\mathcal{H}}$ , and set  $\tilde{\rho}(a) = \langle \tilde{\phi}(a)(x), x \rangle$ . We will show that  $\tilde{\rho}$  is positive.

Let  $\rho: M \rightarrow \mathbb{C}$  be defined by  $\rho(a) = \langle \phi(a)(x), x \rangle$ . Then,  $\|\rho\| = 1$ , and so by the Hahn–Banach theorem,  $\rho$  extends to  $\rho_1: M + M^* \rightarrow \mathbb{C}$  with  $\|\rho_1\| = 1$ . Since  $\rho_1$  is positive,  $\rho_1(a + b^*) = \rho(a) + \overline{\rho(b)} = \tilde{\rho}(a + b^*)$ . Thus  $\tilde{\rho}$  is positive.  $\square$

## Completely Positive Maps

**Definition.** If  $A$  is a  $C^*$  algebra and  $M \subseteq A$  is a linear subspace, then we call  $M$  an operator space.

We may regard  $\text{Mat}_n(M)$  as a subspace of  $\text{Mat}_n(A)$ , with the norm structure inherited from the unique norm structure on  $\text{Mat}_n(A)$ . The primary distinguishing feature of an operator space is the fact that  $\text{Mat}_n(M)$  has a unique norm for all  $n \geq 1$ .

Similarly, if  $S \subseteq A$  is an operator system, then we endow  $\text{Mat}_n(S)$  with the norm and order it inherits from  $\text{Mat}_n(A)$ .

**Definition.** If a matrix  $S \in \text{Mat}_n(\mathbb{C})$  is positive definite and Hermitian, then  $S$  is positive.

*Proof.* If  $S$  is Hermitian, then we know that all the eigenvalues of  $S$  are real and that  $S$  is diagonalizable with orthonormal vectors  $\{v_1, \dots, v_n\}$ . Therefore, if

$$\langle S(x), x \rangle \geq 0$$

for all  $x \in \mathbb{C}^n$ , then so too does this hold for  $v_j$  and corresponding  $\lambda_j$ . Thus,  $\lambda_j \geq 0$  for all  $j$ , so  $S$  is positive.  $\square$

**Lemma** (Ordering of  $\text{Mat}_n(\mathbb{B}(\mathcal{H}))$ ): We have that  $(T_{ij})_{ij} \in \text{Mat}_n(\mathbb{B}(\mathcal{H}))_+$  if and only if, for all  $x_1, \dots, x_n \in \mathcal{H}$ , we have  $(\langle T_{ij}(x_j), x_i \rangle)_{ij} \in \text{Mat}_n(\mathbb{C})_+$ .

**Definition.** If  $B$  is a  $C^*$ -algebra, and  $\phi: S \rightarrow B$  is a linear map, then  $\phi_n: \text{Mat}_n(S) \rightarrow \text{Mat}_n(B)$  is defined by  $\phi_n((a_{ij})_{ij}) = (\phi(a_{ij}))_{ij}$ . We call  $\phi$   $n$ -positive if  $\phi_n$  is positive, and  $\phi$  is called completely positive if it is  $n$ -positive for all  $n$ .

We call  $\phi$  completely bounded if  $\sup_n \|\phi_n\|$  is finite. We set

$$\|\phi\|_{cb} = \sup_n \|\phi_n\|.$$

We say  $\phi$  is completely isometric or completely contractive if each  $\phi_n$  is isometric and that  $\|\phi\|_{cb} \leq 1$  respectively.

We investigate some of the properties of classes of completely positive maps such that we may prove when they are automatically completely positive.

**Lemma:** Let  $A$  be a  $C^*$ -algebra, and let  $a, b \in A$ . Then, the following hold.

(i) We have  $\|a\| \leq 1$  if and only if

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$$

is positive in  $\text{Mat}_2(A)$ .

(ii) We have

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$$

is positive in  $\text{Mat}_2(A)$  if and only if  $a^*a \leq b$ .

*Proof.* Let  $A$  be represented by  $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ , and set  $T = \pi(a)$ . If  $\|T\| \leq 1$ , then for any  $x, y \in \mathcal{H}$ , we have

$$\begin{aligned} \left\langle \begin{pmatrix} I & T \\ T^* & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle T(y), x \rangle + \langle x, T(y) \rangle + \langle y, y \rangle \\ &\geq \|x\|^2 - 2\|T\|_{op}\|y\|\|x\| + \|y\|^2 \\ &\geq 0. \end{aligned}$$

Conversely, if  $\|T\|_{op} > 1$ , then there exist unit vectors  $x$  and  $y$  such that  $\langle T(y), x \rangle < -1$ , and the above inner product is negative.  $\square$

**Exercise** (Exercise 3.2): Let  $P, Q, A$  be operators on a Hilbert space  $\mathcal{H}$ , with  $P, Q$  positive.

(i) Show that

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0$$

if and only if

$$|\langle Ax, y \rangle|^2 \leq \langle Px, y \rangle \langle Qy, x \rangle.$$

(ii) Show that

$$\begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix} \geq 0$$

if and only if  $B \geq A^*A$ .

(iii) Show that if

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0,$$

then for any  $x \in \mathcal{H}$ , we have

$$\begin{aligned} 0 &\leq \langle (P + A + A^* + Q)x, x \rangle \\ &\leq \left( \sqrt{\langle Px, x \rangle} + \sqrt{\langle Qx, x \rangle} \right)^2, \end{aligned}$$

hence

$$\|P + AA^* + Q\| \leq \left( \|P\|^{1/2} + \|Q\|^{1/2} \right)^2.$$

(iv) Show that if

$$\begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0,$$

then  $A^*A \leq \|P\|P$ , implying  $\|A\| \leq \|P\|$ .

**Solution:**

(i) We see that

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0$$

if and only if, for any  $x, y \in \mathcal{H}$ , we have

$$\begin{pmatrix} \langle Px, x \rangle & \langle Ay, x \rangle \\ \langle A^*x, y \rangle & \langle Qy, y \rangle \end{pmatrix} \geq 0.$$

Thus, we have

$$\begin{aligned} \det \begin{pmatrix} \langle Px, x \rangle & \langle Ay, x \rangle \\ \langle A^*x, y \rangle & \langle Qy, y \rangle \end{pmatrix} &= \langle Px, x \rangle \langle Qy, y \rangle - |\langle Ay, x \rangle|^2 \\ &\geq 0, \end{aligned}$$

so that

$$|\langle Ay, x \rangle|^2 \leq \langle Px, x \rangle \langle Qy, y \rangle.$$

Suppose that

$$|\langle Ay, x \rangle|^2 \leq \langle Px, x \rangle \langle Qy, y \rangle.$$

Now, for any  $x, y \in \mathcal{H}$ , we have

$$\begin{aligned}
 \left\langle \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle Px, x \rangle + \langle Ay, x \rangle + \langle A^*x, y \rangle + \langle Qy, y \rangle \\
 &= \langle Px, x \rangle + 2 \operatorname{Re}(\langle Ay, x \rangle) + \langle Qy, y \rangle \\
 &\geq \langle Px, x \rangle - 2|\langle Ay, x \rangle| + \langle Qy, y \rangle \\
 &\geq \langle Px, x \rangle - 2\langle Px, x \rangle^{1/2} \langle Qy, y \rangle^{1/2} + \langle Qy, y \rangle \\
 &= \left( \langle Px, x \rangle^{1/2} - \langle Qy, y \rangle^{1/2} \right)^2 \\
 &\geq 0.
 \end{aligned}$$

(ii) We begin by assuming that  $B \geq A^*A$ . Since  $B \geq A^*A$ , we have

$$\langle (B - A^*A)y, y \rangle \geq 0,$$

so that

$$\langle By, y \rangle \geq \|Ay\|^2.$$

Thus, in the  $2 \times 2$  case, we have, for any  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}^{(2)}$ ,

$$\begin{aligned}
 \left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle Ay, x \rangle + \langle A^*x, y \rangle + \langle By, y \rangle \\
 &\geq \langle x, x \rangle + \langle Ay, x \rangle + \langle A^*x, y \rangle + \langle Ay, Ay \rangle \\
 &= \langle x, x \rangle + \langle Ay, Ay \rangle + 2 \operatorname{Re}(\langle Ay, x \rangle) \\
 &\geq \langle x, x \rangle + \langle Ay, Ay \rangle - 2\|Ay\|\|x\| \\
 &= \|x\|^2 + \|Ay\|^2 - 2\|Ay\|\|x\| \\
 &\geq 0.
 \end{aligned}$$

Thus, the matrix is positive.

For the converse direction, we suppose  $B \not\geq A^*A$ . Then, there is some  $y \in \mathcal{H}$  such that  $\langle (B - A^*A)y, y \rangle < 0$ . This gives  $\langle By, y \rangle < \|Ay\|^2$ . We may select  $y$  such that  $\|Ay\|^2 = 1$ . Setting  $x = -Ay$ , we have

$$\begin{aligned}
 \left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle Ay, x \rangle + \langle A^*x, y \rangle + \langle By, y \rangle \\
 &= \langle x, x \rangle + \langle Ay, x \rangle + \langle x, Ay \rangle + \langle By, y \rangle \\
 &= \langle -Ay, -Ay \rangle + \langle Ay, -Ay \rangle + \langle -Ay, Ay \rangle + \langle By, y \rangle \\
 &= \|Ay\|^2 - 2\|Ay\|^2 + \langle By, y \rangle \\
 &= -1 + \langle By, y \rangle \\
 &< -1 + \|Ay\|^2 \\
 &= 0.
 \end{aligned}$$

Thus, the matrix is negative.

(iii) We apply the result in (i) to the vector  $\begin{pmatrix} x \\ x \end{pmatrix}$ . This gives

$$\begin{aligned}
 \left\langle \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} x \\ x \end{pmatrix} \right\rangle &= \langle Px, x \rangle + \langle Ax, x \rangle + \langle A^*x, x \rangle + \langle Qx, x \rangle \\
 &= \langle Px, x \rangle + 2 \operatorname{Re}(\langle Ax, x \rangle) + \langle Qx, x \rangle \\
 &\leq \langle Px, x \rangle + 2|\langle Ax, x \rangle| + \langle Qx, x \rangle \\
 &\leq \langle Px, x \rangle + 2\langle Px, x \rangle^{1/2} \langle Qx, x \rangle^{1/2} + \langle Qx, x \rangle \\
 &= \left( \langle Px, x \rangle^{1/2} + \langle Qx, x \rangle^{1/2} \right)^2.
 \end{aligned}$$

(iv) Setting  $Q = P$  in the result from (i), we have

$$|\langle Ay, x \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle,$$

which holds for all  $x, y \in \mathcal{H}$ . In particular, setting  $x = Ay$ , we have

$$\begin{aligned} |\langle Ay, Ay \rangle| &\leq \langle PAy, Ay \rangle \langle Py, y \rangle \\ &\leq \|PAy\| \|Ay\| \langle Py, y \rangle \\ &\leq \|P\| \|Ay\|^2 \langle Py, y \rangle. \end{aligned}$$

This gives

$$\begin{aligned} \|Ay\|^4 &\leq \|P\| \|Ay\|^2 \langle Py, y \rangle \\ \|Ay\|^2 &\leq \|P\| \langle Py, y \rangle \\ \langle A^*Ay, y \rangle &\leq \|P\| \langle Py, y \rangle, \end{aligned}$$

or that  $A^*A \leq \|P\|P$ .

**Proposition:** Let  $S$  be an operator system,  $B$  a unital  $C^*$ -algebra, and  $\phi: S \rightarrow B$  a unital 2-positive map. Then,  $\phi$  is contractive.

*Proof.* Let  $a \in S$  with  $\|a\| \leq 1$ . Then,

$$\phi_2 \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & \phi(a) \\ \phi(a)^* & 1 \end{pmatrix}$$

is positive, hence  $\|\phi(a)\| \leq 1$ . □

**Proposition** (Cauchy–Schwarz for 2-positive Maps): Let  $A, B$  be unital  $C^*$ -algebras, and let  $\phi: A \rightarrow B$  be a unital 2-positive map. Then,

$$\phi(a)^* \phi(a) \leq \phi(a^*a)$$

for all  $a \in A$ .

*Proof.* We have that

$$\begin{aligned} \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & \phi(a) \\ \phi(a)^* & \phi(a^*a) \end{pmatrix} \\ &\geq 0, \end{aligned}$$

meaning that  $\phi(a)^* \phi(a) \leq \phi(a^*a)$  by above. □

**Proposition:** Let  $A$  and  $B$  be unital  $C^*$ -algebras, and let  $M$  be a unital subspace of  $M$ , with  $S = M + M^*$ . If  $\phi: M \rightarrow B$  is unital and 2-contractive, then  $\tilde{\phi}: S \rightarrow B$  given by  $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$  is 2-positive and contractive.

*Proof.* Since  $\phi$  is contractive, we know from above that  $\tilde{\phi}$  is well-defined. Furthermore, note that

$$\text{Mat}_2(S) = \text{Mat}_2(M) + \text{Mat}_2(M)^*,$$

and

$$\left( \tilde{\phi} \right)_2 = \left( \tilde{\phi}_2 \right).$$

Now, since  $\phi_2$  is contractive, we have that  $\tilde{\phi}_2$  is positive, so  $\tilde{\phi}$  is contractive. □

**Proposition:** Let  $A$  and  $B$  be unital  $C^*$ -algebras, let  $M$  be a unital subspace, and let  $S = M + M^*$ . If  $\phi: M \rightarrow B$  is unital and completely contractive, then  $\tilde{\phi}: S \rightarrow B$  is completely positive and completely contractive.

*Proof.* Since  $\phi_n$  is unital and contractive,  $\tilde{\phi}_n$  is positive. Additionally, since  $(\tilde{\phi}_n)_2$  is positive,  $\tilde{\phi}_n$  is contractive.  $\square$

Note that since  $\text{Mat}_2(\text{Mat}_n(A)) \cong \text{Mat}_{2n}(A)$  are  $*$ -isomorphic, the norm on  $\text{Mat}_2(\text{Mat}_n(A))$  is equal to the norm on  $\text{Mat}_{2n}(A)$ .

Now, we may see some examples that belong to these categories.

**Example.** If  $A$  and  $B$  are  $C^*$ -algebras, and  $\pi: A \rightarrow B$  is a  $*$ -homomorphism, then  $\pi$  is completely positive and completely contractive, since each  $\pi_n: \text{Mat}_n(A) \rightarrow \text{Mat}_n(B)$  is a  $*$ -homomorphism, and  $*$ -homomorphisms are both positive and contractive.

**Example.** Fixing  $x, y \in A$ , we may define  $\phi: A \rightarrow A$  by  $\phi(a) = xay$ . Note that if  $(a_{ij})_{ij} \in \text{Mat}_n(A)$ , then

$$\begin{aligned} \left\| \phi_n \left( (a_{ij})_{ij} \right) \right\| &= \left\| (xa_{ij}y)_{ij} \right\| \\ &= \left\| (xI_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} (yI_n) \right\| \\ &\leq \|x\| \left\| (a_{ij})_{ij} \right\| \|y\|. \end{aligned}$$

This means  $\phi$  is completely bounded with  $\|\phi\|_{cb} \leq \|x\| \|y\|$ . Similarly, if  $x = y^*$ , then  $\phi_n$  is positive.

This gives us the archetype of a completely bounded map. If  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are Hilbert spaces, and  $v_i: \mathcal{H}_1 \rightarrow \mathcal{H}_2$  are bounded operators for  $i = 1, 2$ , then if  $\pi: A \rightarrow \mathbb{B}(\mathcal{H}_2)$  is a  $*$ -homomorphism, we may define  $\phi: A \rightarrow \mathbb{B}(\mathcal{H}_1)$  by  $\phi(a) = v_2^* \pi(a) v_1$ . This function  $\phi$  is completely bounded with  $\|\phi\|_{cb} \leq \|v_1\| \|v_2\|$ .

In fact, we will show that *every* completely bounded map is of this form.

**Proposition:** Let  $S \subseteq A$  be an operator system,  $B$  a  $C^*$ -algebra, and  $\phi: S \rightarrow B$  completely positive. Then,  $\phi$  is completely bounded, and  $\|\phi(1)\| = \|\phi\| = \|\phi\|_{cb}$ .

*Proof.* We have  $\|\phi(1)\| \leq \|\phi\| \leq \|\phi\|_{cb}$ , so it is sufficient to show that  $\|\phi\|_{cb} \leq \|\phi(1)\|$ . Let  $A = (a_{ij})_{ij}$  be in  $\text{Mat}_n(S)$  with  $\|A\| \leq 1$ , and let  $I_n$  be the unit of  $\text{Mat}_n(A)$ . Then, since

$$T = \begin{pmatrix} I_n & A \\ A^* & I_n \end{pmatrix}$$

is positive, the map

$$\phi_{2n} \left( \begin{pmatrix} I_n & A \\ A^* & I_n \end{pmatrix} \right) = \begin{pmatrix} \phi_n(I_n) & \phi_n(A) \\ \phi_n(A)^* & \phi_n(I_n) \end{pmatrix}$$

is positive, so  $\|\phi_n(A)\| \leq \|\phi_n(I_n)\| = \|\phi(1)\|$ .  $\square$

### Schur Products and Tensor Products

We will apply the previous results on positive and completely positive maps on the Schur product.

**Definition.** If  $A = (a_{ij})_{ij}$  and  $B = (b_{ij})_{ij}$ , then the Schur product is defined by

$$A * B = (a_{ij} b_{ij})_{ij}.$$



Note that for a fixed  $A$ , we get a linear map

$$S_A(B) = A * B.$$

To study the Schur product, we review some results on tensor products.

Let  $A \in \text{Mat}_n(\mathbb{C})$  and  $B \in \text{Mat}_m(\mathbb{C})$ . Then,  $A \otimes B$  is the linear transformation on  $\mathbb{C}^n \otimes \mathbb{C}^m = \mathbb{C}^{nm}$ , defined by  $A \otimes B(x \otimes y) = Ax \otimes By$  with the unique linear extension provided by the tensor product.

Note that we have  $\|A \otimes B\| = \|A\| \|B\|$ , which is shown by writing  $A \otimes B = (A \otimes I)(I \otimes B)$ .

Now, letting  $\{e_1, \dots, e_n\}$  and  $\{f_1, \dots, f_m\}$  be our canonical orthonormal bases for  $\mathbb{C}^n$  and  $\mathbb{C}^m$  respectively, we may order our basis as  $e_1 \otimes f_1$ , then  $e_2 \otimes f_1$ , etc., yielding the block matrices for  $A \otimes B$  is

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

This matrix is known as the Kronecker product of  $A$  and  $B$ . Now, similarly, we may order our basis by  $e_i \otimes f_1$ , then  $e_i \otimes f_2$ , etc., yielding a different block matrix of the form

$$\begin{pmatrix} b_{11}A & \cdots & b_{1m}A \\ \vdots & \ddots & \vdots \\ b_{m1}A & \cdots & b_{mm}A \end{pmatrix},$$

which is the Kronecker product of  $B$  and  $A$ .

Now, since both of these matrices represent the same linear transformation, they are unitarily equivalent, given by the permutation matrix that reorders the basis vectors. One obtains the  $(k, \ell)$  entry of the  $(i, j)$  block of  $b_{ij}A$  by taking the  $(i, j)$  entry of the  $(k, \ell)$  block  $a_{k\ell}B$ . We will call this the *canonical shuffle*.

Now, we let  $A$  and  $B$  be elements of  $\text{Mat}_n(\mathbb{C})$ , and define  $V: \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n$  to be the isometry given by  $V(e_i) = e_i \otimes e_i$ . We will show that  $V^*(A \otimes B)V = A * B$ . Note that

$$\begin{aligned} \langle V^*(A \otimes B)V e_j, e_i \rangle &= \langle (A \otimes B)(e_j \otimes e_j), e_i \otimes e_i \rangle \\ &= \langle A e_j, e_i \rangle \langle B e_j, e_i \rangle \\ &= a_{ij} b_{ij} \\ &= \langle A * B e_j, e_i \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \|S_A(B)\| &\leq \|V^*(A \otimes B)V\| \\ &\leq \|A\| \|B\|, \end{aligned}$$

so that

$$\|S_A\| \leq \|A\|.$$

Now, if  $(B_{ij})_{ij} \in \text{Mat}_k(\text{Mat}_n(\mathbb{C}))$ , then

$$\begin{aligned} (S_A)((B_{ij})_{ij}) &= (V^*(A \otimes B_{ij})V)_{ij} \\ &= \begin{pmatrix} V^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V^* \end{pmatrix} A \otimes \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix} \begin{pmatrix} V & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V \end{pmatrix}, \end{aligned}$$

so that  $\|(S_A)_k\| \leq \|A\|$ . Thus,  $\|S_A\|_{cb} \leq \|A\|$ .

However, this isn't a really good estimate. For instance, if  $A$  is the matrix consisting of all 1s, then the norm of  $A$  is  $n$ , while  $\|S_A\| = 1$ .

We will prove that if  $A$  is positive, then  $S_A$  is completely positive. Thus, for positive matrices, we are able to obtain  $\|S_A\|_{cb}$  by finding

$$\begin{aligned}\|S_A\| &= \|S_A(I)\| \\ &= \|S_A\|_{cb} \\ &= \max\{a_{ii} \mid i = 1, \dots, n\}.\end{aligned}$$

Now, if  $A$  is not positive, then obtaining this norm is a bit more difficult. We can decompose  $A = (P_1 - P_2) + i(P_3 - P_4)$ , and get

$$\|S_A\|_{cb} \leq \|S_{P_1}\|_{cb} + \|S_{P_2}\|_{cb} + \|S_{P_3}\|_{cb} + \|S_{P_4}\|_{cb},$$

but unfortunately this estimate isn't really enough.

Now, we will characterize when the Schur product is completely positive.

**Theorem:** Let  $A = (a_{ij})_{ij} \in \text{Mat}_n(\mathbb{C})$ . The following are equivalent:

- (i)  $A$  is positive;
- (ii)  $S_A : \text{Mat}_n(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C})$  is positive;
- (iii)  $S_A : \text{Mat}_n(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C})$  is completely positive.

*Proof.* We have that (iii) implies (ii), and (ii) implies (i) by choosing  $J$  to be the matrix consisting of 1, which is positive, meaning  $S_A(J) = A$ . Thus, we must prove that (i) implies (iii).

Note that if  $A$  and  $B$  are positive, then  $A \otimes B$  is positive. This follows from the fact that  $A \otimes B = (A^{1/2} \otimes B^{1/2})^2$ .

Now, if  $B \in \text{Mat}_n(\mathbb{C})$  is positive, then

$$\begin{aligned}S_A(B) &= V^*(A \otimes B)V \\ &= \left( (A^{1/2} \otimes B^{1/2})V \right)^* \left( (A^{1/2} \otimes B^{1/2})V \right)\end{aligned}$$

is positive, meaning (i) implies (ii).

Now, to see that (i) implies (iii), we let  $B = (B_{ij})_{ij} \in \text{Mat}_k(\text{Mat}_n(\mathbb{C}))$ , and write  $B = (X_{ij})_{ij}^* (X_{ij})_{ij}$ . We see that

$$\begin{aligned}(S_A)_k(B) &= (V^*(A \otimes B_{ij})V) \\ &= \left( (A^{1/2} \otimes X_{ij})V \right)^* \left( (A^{1/2} \otimes X_{ij})V \right),\end{aligned}$$

meaning  $(S_A)_k$  is positive. □

There is an analogous theory of Schur products in the space  $\mathcal{B}(\ell_2)$ , where we consider the bounded operators as infinite matrices. If we mandate that  $A \in \mathcal{B}(\ell_2)_+$ , then we can use a similar line of argumentation to show that  $S_A$  is completely positive, but this requires a bit more care as the matrix consisting of all 1s regarded as an operator on  $\ell_2$  is not a bounded operator.

Now, we can show a pretty useful result, which is that bounded linear functionals are not only positive, but completely positive.

**Proposition:** Let  $S$  be an operator space, and let  $f: S \rightarrow \mathbb{C}$  be a bounded linear functional. Then,

$$\|f\|_{cb} = \|f\|_{op},$$

and if  $S$  is an operator system with  $f$  positive, then  $f$  is completely positive.

*Proof.* Let  $(a_{ij})_{ij} \in \text{Mat}_n(S)$ , and let  $x, y \in \mathbb{C}^n$  be unit vectors. Then,

$$\begin{aligned} \left| \left\langle f\left((a_{ij})_{ij}\right)(x), y \right\rangle \right| &= \left| \sum_{i,j=1}^n f(a_{ij})(x_j) \overline{y_i} \right| \\ &= \left| f\left(\sum_{i,j=1}^n a_{ij} x_j \overline{y_i}\right) \right| \\ &\leq \|f\|_{op} \left\| \sum_{i,j=1}^n a_{ij} x_j \overline{y_i} \right\|. \end{aligned}$$

Now, all we need to show is that the latter element has norm less than  $\|(a_{ij})_{ij}\|$ . Note that this sum is the entry on the first row and column of the matrix that represents the product

$$\begin{pmatrix} \overline{y_1} & \cdots & \overline{y_n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix}.$$

The outer two factors have norm 1, since  $x$  and  $y$  are chosen to be unit vectors.

To show that  $f$  is completely positive, we only need to show that

$$\left\langle f_n\left((a_{ij})_{ij}\right)(x), x \right\rangle = f\left(\sum_{i,j=1}^n a_{ij} x_j \overline{x_i}\right)$$

is positive whenever  $(a_{ij})_{ij}$  is positive. However, using the above product, we see that the summation is equal to the first row and column entry of a positive matrix, hence positive.  $\square$

Now, we examine the positivity and boundedness of maps with codomain  $C(X)$ , where  $X$  is a compact Hausdorff space.

Note that every element  $F = (f_{ij})_{ij}$  of  $\text{Mat}_n(C(X))$  can be considered as a continuous matrix-valued function, with multiplication and  $*$ -operation as pointwise multiplication and involution of the matrix-valued functions.

To make  $\text{Mat}_n(C(X))$  into a  $C^*$ -algebra is to set  $\|F\| = \sup\{\|F(x)\| \mid x \in X\}$ , and by uniqueness of  $C^*$ -norms, this is the only way to create a  $C^*$ -norm.

**Theorem:** Let  $S$  be an operator space, and let  $\phi: S \rightarrow C(X)$  be a bounded linear map. Then,  $\|\phi\|_{cb} = \|\phi\|_{op}$ . Furthermore, if  $S$  is an operator system and  $\phi$  is positive, then  $\phi$  is completely positive.

*Proof.* Let  $x \in X$ , and define  $\phi^x$  to be pointwise evaluation — i.e.,  $\phi^x(a) = \phi(a)(x)$ . Then,

$$\begin{aligned} \|\phi_n\| &= \sup\{\|\phi_n^x\| \mid x \in X\} \\ &= \sup\{\|\phi^x\| \mid x \in X\} \\ &= \|\phi\|_{op}. \end{aligned}$$

Similarly,  $\phi_n((a_{ij})_{ij})$  is positive if and only if  $\phi_n^x((a_{ij})_{ij})$  is positive for all  $x \in X$ .  $\square$

Thus, when the codomain  $C^*$ -algebra is commutative, boundedness and complete boundedness, as well as positivity and complete positivity, coincide. A commutative domain is enough to show that positive maps are completely positive, but unfortunately a commutative domain is not enough to guarantee that bounded maps are completely bounded.

**Lemma:** Let  $(p_{ij})_{ij}$  be a positive scalar matrix, and let  $q$  be a positive element of some  $C^*$ -algebra  $B$ . Then,  $(p_{ij}q)_{ij}$  is positive in  $\text{Mat}_n(B)$ .

*Proof.* We write  $(p_{ij})_{ij}$  as  $(s_{ij})_{ij}^* (s_{ij})_{ij}$ , and write  $q = n^*n$ . This gives

$$\begin{aligned} (p_{ij}q)_{ij} &= (p_{ij})_{ij} \text{diag}(q, \dots, q) \\ &= (s_{ij})_{ij}^* (s_{ij})_{ij} \text{diag}(n^*n, \dots, n^*n) \\ &= (s_{ij})_{ij}^* (s_{ij})_{ij} \text{diag}(n^*, \dots, n^*) \text{diag}(n, \dots, n) \\ &= (s_{ij})_{ij}^* (s_{ij})_{ij} \text{diag}(n, \dots, n)^* \text{diag}(n, \dots, n) \\ &= \text{diag}(n, \dots, n)^* (s_{ij})_{ij}^* (s_{ij})_{ij} \text{diag}(n, \dots, n) \quad \text{diag}(n, \dots, n) \in \text{Mat}_n(B), (s_{ij})_{ij} \in \text{Mat}_n(\mathbb{C}) \\ &= \left( (s_{ij})_{ij} \text{diag}(n, \dots, n) \right)^* \left( (s_{ij})_{ij} \text{diag}(n, \dots, n) \right). \end{aligned}$$

Thus,  $(p_{ij}q)_{ij}$  is positive in  $\text{Mat}_n(B)$ .  $\square$

**Theorem:** Let  $B$  be a  $C^*$ -algebra, and let  $\phi: C(X) \rightarrow B$  be a positive map. Then,  $\phi$  is completely positive.

*Proof.* Let  $P(x)$  be positive in  $\text{Mat}_n(C(X))$ . We prove that  $\phi_n(P)$  is positive.

Given  $\varepsilon > 0$ , we may find a partition of unity  $\{u_\ell(x)\}_{\ell=1}^m$  and positive matrices  $P_\ell = (p_{ij}^\ell)_{ij}$  such that

$$\left| P - \sum_{\ell=1}^m u_\ell(x) P_\ell \right| < \varepsilon.$$

However, we know that

$$\begin{aligned} \phi_n(u_\ell P_\ell) &= \phi_n \left( (u_\ell p_{ij}^\ell)_{ij} \right) \\ &= \left( \phi(u_\ell) p_{ij}^\ell \right)_{ij}, \end{aligned}$$

which is positive. Therefore,  $\phi_n(P)$  is within  $\varepsilon \|\phi_n\| \|P\|$  of a sum of positive elements. Since  $\text{Mat}_n(B)_+$  is a closed set, we have that  $\phi_n(P)$  is positive.  $\square$

**Corollary:** Let  $T$  be a contractive operator on  $\mathcal{H}$ , and let  $(p_{ij})_{ij}$  be a  $n \times n$  matrix of polynomials. Then,

$$\left\| (p_{ij}(T))_{ij} \right\|_{\text{op}} \leq \sup \left\{ \left\| (p_{ij}(z))_{ij} \right\| \mid |z| = 1 \right\}.$$

*Proof.* The map given by  $\phi(p + \bar{q}) = p(T) + q(T)^*$  extends to a positive map  $\varphi: C(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{H})$ . This map is completely positive as  $C(\mathbb{T})$  is a commutative  $C^*$ -algebra. Thus,  $\|\varphi\|_{\text{cb}} = \|\varphi(1)\| = 1$ . Thus,

$$\begin{aligned} \left\| (p_{ij}(T))_{ij} \right\| &= \left\| \phi_n \left( (p_{ij})_{ij} \right) \right\| \\ &\leq \left\| (p_{ij}(1))_{ij} \right\|. \end{aligned}$$

$\square$

**Lemma:** Let  $A$  be a  $C^*$ -algebra. Then, every positive element of  $\text{Mat}_n(A)$  is a sum of  $n$  positive elements of the form  $(a_i^* a_j)_{ij}$ , where  $\{a_1, \dots, a_n\} \subseteq A$ .

*Proof.* Note that if  $R$  is the element of  $\text{Mat}_n(A)$  whose  $k$ th row is  $a_1, \dots, a_n$  and 0 elsewhere, then  $R^* R = (a_i^* a_j)_{ij}$ , so such an element is positive.

Now, let  $P$  be positive, yielding  $P = B^* B$ . Write  $B = R_1 + \dots + R_n$ , where  $R_k$  is the  $k$ th row of  $B$  and 0 elsewhere.

Then, since  $R_i^* R_j = 0$  whenever  $i \neq j$ , we have that  $P = R_1^* R_1 + \dots + R_n^* R_n$ .  $\square$

Thus, it suffices to check that  $\phi: A \rightarrow B$  is  $n$ -positive by verifying that  $(\phi(a_i^* a_j))_{ij}$  is positive for all  $a_1, \dots, a_n \in A$ .

**Theorem:** Let  $B$  be a  $C^*$ -algebra, let  $\phi: \text{Mat}_n(\mathbb{C}) \rightarrow B$  be a linear map, and let  $\{e_{ij}\}_{i,j=1}^n$  denote the standard matrix units for  $\text{Mat}_n(\mathbb{C})$ . The following are equivalent:

- (i)  $\phi$  is completely positive;
- (ii)  $\phi$  is  $n$ -positive;
- (iii)  $(\phi(e_{ij}))_{ij}$  is positive in  $\text{Mat}_n(B)$ .

*Proof.* It suffices to show that (iii) implies (i), as (i) implies (ii) and  $(e_{ij})_{ij}$  is positive for each  $i, j$ , giving (ii) implies (iii).

It is sufficient to assume that  $B = \mathbb{B}(\mathcal{H})$ . Fix  $k$ , and let  $x_1, \dots, x_k \in \mathcal{H}$ ,  $B_1, \dots, B_k \in \text{Mat}_n(\mathbb{C})$ . It is sufficient to prove that

$$\sum_{i,j}^k \langle \phi(B_i^* B_j) x_j, x_i \rangle \geq 0.$$

Write  $B_\ell = \sum_{r,s=1}^n b_{rs,\ell} e_{rs}$ , such that

$$B_i^* B_j = \sum_{r,s,t=1}^n \overline{b_{rs,i}} b_{rt,j} e_{st}.$$

Set  $y_{t,r} = \sum_{j=1}^k b_{rt,j} x_j$ . Then,

$$\begin{aligned} \sum_{i,j=1}^k \langle \phi(B_i^* B_j) x_j, x_i \rangle &= \sum_{r=1}^n \sum_{s,t=1}^n \left\langle \phi(e_{st}) \left( \sum_{i,j=1}^k b_{rs,i} b_{rt,j} x_j \right), x_i \right\rangle \\ &= \sum_{r=1}^n \sum_{s,t} \langle \phi(e_{st}) y_{t,r}, y_{s,r} \rangle. \end{aligned}$$

However, this latter sum is positive, since  $(\phi(e_{st}))_{st}$  is positive, so we have expressed our original sum as the sum of  $n$  positive quantities.  $\square$

Now, we may obtain some fairly deep results in operator theory via the properties of positive maps.

**Definition.** If  $T \in \mathbb{B}(\mathcal{H})$ , we define the numerical radius of  $T$  by

$$w(T) = \sup_{x \in \mathcal{B}_{\mathcal{H}}} |\langle Tx, x \rangle|.$$

**Exercise:** Let  $S_n$  be the cyclic forward shift on  $\mathbb{C}^n$ . That is,  $S_n e_j = e_{j+1 \bmod n}$ , where  $e_0, \dots, e_{n-1}$  is the canonical basis for  $\mathbb{C}^n$ .

- (i) Show that  $S_n$  is unitarily equivalent to a diagonal matrix whose entries are the  $n$ th roots of unity.
- (ii) Let  $T \in \mathcal{B}(\mathcal{H})$ . Show that  $w(T) = w(T \otimes S_n)$ .
- (iii) Let  $R_n$  be the  $n \times n$  matrix of operators whose subdiagonals are  $T$  and 0 elsewhere. Show that  $w(R_n) \leq w(T \otimes S_n)$ .
- (iv) Show that  $\operatorname{Re}(\langle R_n y, y \rangle) \leq 1$  for all  $\|y\| = 1$  if and only if  $w(R_n) \leq 1$ .

**Solution:**

- (i) By the definition of the cyclic forward shift, defining  $A := S_n$ , we have  $A^n = I_n$ , or  $A^n - I = 0$ . This means that the minimal polynomial for  $S_n$  is  $m_{S_n}(x) = x^n - 1$ , meaning that the  $n$ th roots of unity are eigenvalues for  $S$ . Since  $S_n$  is an operator acting on  $\mathbb{C}^n$ , there are at  $n$  eigenvalues (with multiplicity) for  $S_n$ , meaning that the  $n$ th roots of unity are in fact *the* eigenvalues of  $S_n$ . Thus,  $S_n$  is unitarily equivalent to a diagonal matrix with the  $n$ th roots of unity on the diagonal.
- (ii) The operator  $T \otimes S_n$  acts on  $\mathcal{H} \otimes \mathbb{C}^n$  such that  $(T \otimes S_n)(y \otimes v) = Ty \otimes S_n v$ . Thus, we have

$$\begin{aligned}
 w(T \otimes S_n) &= \sup_{y \otimes v \in \mathcal{B}_{\mathcal{H} \otimes \mathbb{C}^n}} |\langle (T \otimes S_n)(y \otimes v), y \otimes v \rangle| \\
 &= \sup_{y \otimes v \in \mathcal{B}_{\mathcal{H} \otimes \mathbb{C}^n}} |\langle Ty \otimes S_n v, y \otimes v \rangle| \\
 &= \sup_{y \in \mathcal{B}_{\mathcal{H}}} \sup_{v \in \mathcal{B}_{\mathbb{C}^n}} |\langle Ty, y \rangle| |\langle S_n v, v \rangle| \\
 &= w(T)w(S_n) \\
 &= w(T).
 \end{aligned}$$

- (iii) We consider the non-cyclic shift  $S'_n$ , and note that  $w(S'_n) \leq w(S_n)$ , as applying the non-cyclic shift will yield zero in the first entry of the vector  $v$ . We have that  $R_n \cong T \otimes S'_n$ , meaning that  $w(R_n) \leq w(T)$ .
- (iv) If  $w(R_n) \leq 1$  for all  $\|y\| = 1$ , then since  $\operatorname{Re}(\langle R_n y, y \rangle) \leq |\langle R_n y, y \rangle| \leq 1$ , it is clear that  $\operatorname{Re}(\langle R_n y, y \rangle) \leq 1$  for all  $\|y\| = 1$ .

Now, suppose  $\operatorname{Re}(\langle R_n y, y \rangle) \leq 1$  for all  $\|y\| = 1$ .

**Theorem:** Let  $T \in \mathcal{B}(\mathcal{H})$ , let  $S \subseteq C(T)$  be the operator system defined by  $S = \{p + \bar{q} \mid p, q \text{ polynomials}\}$ . The following are equivalent:

- (i)  $w(T) \leq 1$ ;
- (ii) the map  $\phi: S \rightarrow \mathcal{B}(\mathcal{H})$ , defined by

$$\phi(p + \bar{q}) = p(T) + q(T)^* + (p(0) + \overline{q(0)})I$$

is positive.

*Proof.* We start by showing that (i) implies (ii).

Let  $R_n$  be the  $n \times n$  operator matrix with subdiagonal entry  $T$  and remaining entries 0. Note that  $w(R_n) \leq w(T)$ .

Now, we see that  $\phi$  is positive so long as the matrix

$$\begin{pmatrix} 2 & T^* & \cdots & (T^*)^n \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^n & \cdots & T & 2 \end{pmatrix} \quad (*)$$

is positive for all  $n$ .

Note that  $R_n^{n+1} = 0$ , so  $(*)$  can be written as  $(I - R_n)^{-1} + (I - R_n^*)^{-1}$ .

Fix  $x = (I - R_n)y$ , and compute

$$\left\langle \left( (I - R_n)^{-1} + (I - R_n^*)^{-1} \right) x, x \right\rangle = 2\|y\|^2 - 2\operatorname{Re}(\langle R_n y, y \rangle).$$

Thus,  $w(R_n) \leq 1$ .

□