Math 395

Homework 1

Due: 2/1/2024

Name: Avinash Iyer

Collaborators: ______________________________

1. Let S be the subset of $\mathrm{Mat}_2(\mathbf{R})$ be the set consisting of matrices of the form $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$.

   (a) Prove that S is a ring.

   Proof: We will show that S is a ring by using the ring axioms.

   - Closure:

   $$\begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix} = \begin{bmatrix} a+c & a+c \\ b+d & b+d \end{bmatrix}$$

   - Additive Identity:

   $$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a+0 & a+0 \\ b+0 & b+0 \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} \end{aligned}$$

   - Additive Inverse:

   $$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} -a & -a \\ -b & -b \end{bmatrix} &= \begin{bmatrix} a-a & a-a \\ b-b & b-b \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

   - Associativity:

   $$\begin{aligned} \left(\begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix}\right) + \begin{bmatrix} e & e \\ f & f \end{bmatrix} &= \begin{bmatrix} (a+c)+e & (a+c)+e \\ (b+d)+f & (b+d)+f \end{bmatrix} \\ &= \begin{bmatrix} a+(c+e) & a+(c+e) \\ b+(d+f) & b+(d+f) \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \left(\begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} e & e \\ f & f \end{bmatrix}\right) \end{aligned}$$

   - Commutativity:

   $$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix} &= \begin{bmatrix} a+c & a+c \\ b+d & b+d \end{bmatrix} \\ &= \begin{bmatrix} c+a & c+a \\ d+b & d+b \end{bmatrix} \\ &= \begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} a & a \\ b & b \end{bmatrix} \end{aligned}$$

(b) Show that $J = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a right identity in S, i.e., $AJ = A$ for all $A \in \mathrm{Mat}_2(\mathbf{R})$.

(c) Show that J is not a left identity for S, i.e., there is an element $B \in S$ so that $JB \neq B$.

2. Show that the subset $S = \{[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}, [15]_{18}\}$ is a subring of $\mathbf{Z}/18\mathbf{Z}$. Does S have an identity?

3. Define a new addition and multiplication on $\mathbf{Z}$ by

$$a \oplus b = a + b - 1$$
$$a \odot b = ab - (a + b) + 2.$$

Prove that under these operations $\mathbf{Z}$ is an integral domain.

4. Let R be a ring and define $Z(R) = \{a \in R : ar = ra \text{ for every } r \in R\}$. Prove that $Z(R)$ is a subring of R. It is referred to as the center of R.

5. Let R be a ring and fix an element $x \in R$. Show that the set $\{rx : r \in R\}$ is a subring of R.

6. Let S and T be subrings of a ring R.

(a) Is $S \cap T$ a subring of R? Justify your answer with a proof or counterexample.

(b) Is $S \cup T$ a subring of R? Justify your answer with a proof or counterexample.

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(F)$ where F is a field.

(a) Prove that A is invertible if and only if $ad - bc \neq 0_F$.

(b) Prove that A is a zero divisor if and only if $ad - bc = 0_F$.

(c) If instead we consider a matrix $A \in \mathrm{Mat}_2(\mathbf{Z})$, do the same conclusions hold? If so, prove them. If not, adjust them to true statements and prove those statements.