

## Problem 1

Show that  $C_0(\mathbb{R})$  is a Banach space.

**Proof:** Let  $(f_n)_n$  be a Cauchy sequence in  $C_0(\mathbb{R})$ . Since each  $f_k \in C_0(\mathbb{R})$ , it must be the case that each  $f_k$  is uniformly continuous. For each  $x \in \mathbb{R}$ , it is thus the case that  $(f_n(x))_n$  is Cauchy in  $\mathbb{R}$ . Since  $\mathbb{R}$  is complete,  $(f_n(x))_n \rightarrow f(x)$  for each  $x \in \mathbb{R}$ , and since each  $f_k$  is uniformly continuous, it must be the case that  $f(x)$  is continuous.

For  $\varepsilon > 0$ , there must be  $N$  large such that for  $m, n \geq N$  and  $m \geq n$ , it must be the case that  $|f_m(x) - f_n(x)| < \varepsilon$  for all  $x \in \mathbb{R}$ . Letting  $m \rightarrow \infty$ , we have  $|f_n(x) - f(x)| < \varepsilon$ , meaning  $(f_n)_n \rightarrow f$ . Thus,  $f \in C_0(\mathbb{R})$ .

## Problem 2

Show that  $\ell_2$  is a Hilbert space.

**Proof:** Let  $\|x\|_2 = \langle x, x \rangle^{1/2}$  for  $x \in \ell_2$ . Let  $\varepsilon > 0$ . Let  $(x_n)_n$  be a Cauchy sequence in  $\ell_2$ .