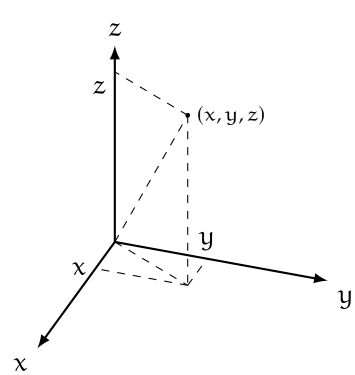
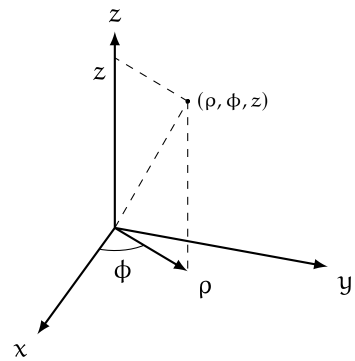
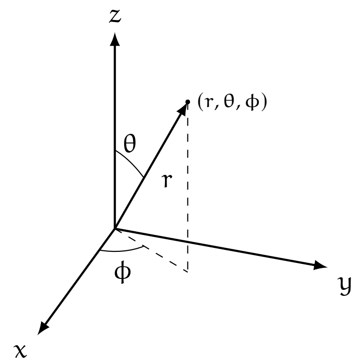
Contents

Things You Just Gotta Know

Coordinate Systems





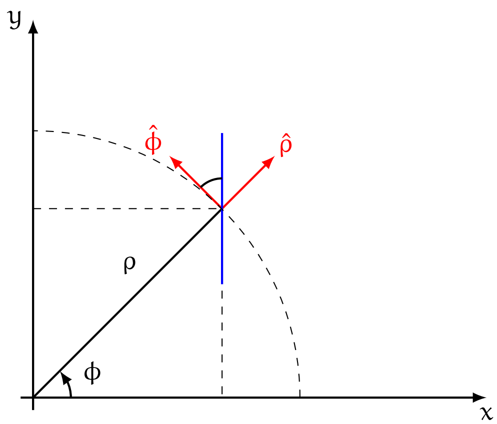


We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x, y)\hat{\imath} + V_y(x, y)\hat{\jmath}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

Polar Coordinates



We can also express the inputs to $\vec{V}$ in polar coordinates, (ρ, ϕ).

$$\vec{V}(\mathbf{r}) = V_\rho\,(\rho, \phi)\,\hat{\imath} + V_\phi\,(\rho, \phi)\,\hat{\jmath}.$$

To extract the input functions, we take

$$\begin{aligned} V_x &= \hat{\imath} \cdot \vec{V} \\ V_y &= \hat{\jmath} \cdot \vec{V}. \end{aligned}$$

Alternatively, we can project $\vec{V}$ onto the $\hat{\rho}, \hat{\phi}$ axis:

$$\vec{V}(\mathbf{r}) = V_\rho(\rho, \phi)\,\hat{\rho} + V_\phi(\rho, \phi)\,\hat{\phi},$$

and we extract

$$\begin{aligned} V_\rho &= \hat{\rho} \cdot \vec{V} \\ V_\phi &= \hat{\phi} \cdot \vec{V}. \end{aligned}$$

Notice that $\mathbf{r}$ is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned} \mathbf{s} &= x\hat{\imath} + y\hat{\jmath} \\ &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} \\ &= \rho\hat{\rho} \\ &= \sqrt{x^2 + y^2}\hat{\rho} \end{aligned}$$

The main reason we avoided using the $\hat{\rho}, \hat{\phi}$ axis up until this point is that ρ and ϕ are *position-dependent*, while the $\hat{\imath}, \hat{\jmath}$ axis is position-independent.

Now, we must figure out the position-dependence of $\hat{\rho}$ and $\hat{\phi}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi.$$

If we hold ϕ constant, it must be the case that any change in ρ is in the $\hat{\rho}$ direction. Therefore,

$$\begin{aligned} \hat{\rho} &= \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|} \\ &= \frac{\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}}{\left|\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\right|} \\ &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}. \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|} \\ &= \frac{-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}}{\left\|-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}\right\|} \\ &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}. \end{aligned}$$

Thus, we can see that the $\hat{\rho}, \hat{}\phi$ axis is orthogonal.

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial \phi} &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\ &= \hat{\phi}, \\ \frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\rho}, \end{aligned}$$

$$\frac{\partial \hat{\phi}}{\partial \rho} = 0,$$

and

$$\frac{\partial \hat{\rho}}{\partial \rho} = 0$$

Example (Velocity)**.**

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{s}}{dt}\\ &= \frac{d}{dt}\left(x\hat{\imath}\right) + \frac{d}{dt}\left(y\hat{\jmath}\right).\end{aligned}$$

In the case of cartesian coordinates, $\hat{\imath}$ and $\hat{\jmath}$ are constants.

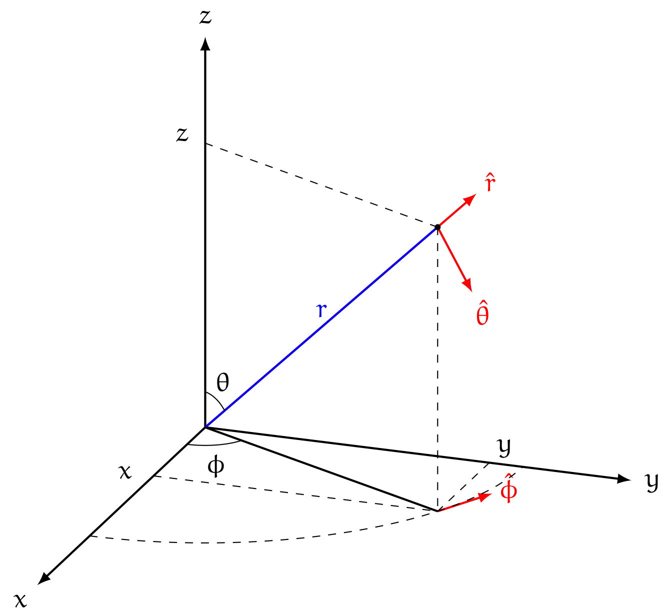
$$= v_x\hat{\imath} + v_y\hat{\jmath}$$

When we examine polar coordinates, since $\hat{\rho}$ and $\hat{\phi}$ are position-dependent, we must use the chain rule.[1]

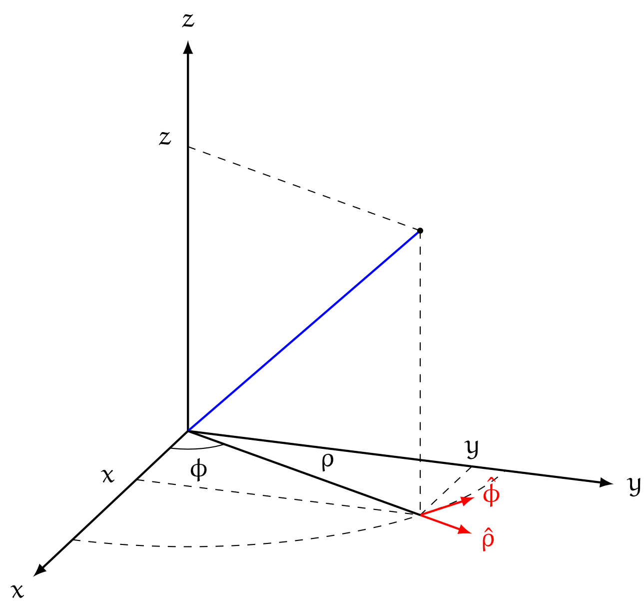
$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{s}}{dt}\\ &= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt}\\ &= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\cancelto{0}{\frac{\partial \hat{\rho}}{\partial \rho}}\frac{d\rho}{dt} + \underbrace{\frac{\partial \hat{\rho}}{\partial \phi}}_{=\hat{\phi}}\frac{d\phi}{dt}\right)\\ &= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi}\\ &= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}.\end{aligned}$$

Notice that $\dot{\rho}$ is the radial velocity and $\dot{\phi} = \omega$ is the angular velocity.

Spherical and Cylindrical Coordinates



[1]Note that $\hat{\rho} = \hat{\rho}\left(\rho, \phi\right)$ and $\hat{\phi} = \hat{\phi}(\rho, \phi)$.



Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho, \phi)$	$\mathbf{s} = s(\rho, \phi, z)$	$\mathbf{s} = s(r, \phi, \theta)$
$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$

Here,[II] ϕ denotes the polar angle and θ denotes the azimuthal angle. Notice that $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

We can see that $\hat{\rho}$, $\hat{\phi}$, and $\hat{\theta}$ in spherical coordinates are also position-dependent.

$$\begin{aligned}
\hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\|\frac{\partial \mathbf{s}}{\partial r}\right\|} \\
&= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k} \\
\hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\|\frac{\partial \mathbf{s}}{\partial \phi}\right\|} \\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
\hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\|\frac{\partial \mathbf{s}}{\partial \theta}\right\|} \\
&= \cos\phi\cos\theta\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}
\end{aligned}$$

Scale Factors and Jacobians

Coordinate System	Line Element	Area Element	Volume Element
Polar	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi$	$d\mathbf{a} = r\ drd\phi$	—
Cylindrical	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz$	—	$d\tau = r\ drd\phi dz$
Spherical	$d\mathbf{s} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta$	$d\mathbf{a} = r^2\sin\theta d\phi d\theta$	$d\tau = r^2\sin\theta\ drd\phi d\theta$

[II]Physicists amirite?

In cylindrical coordinates, we can use the chain rule to find the value of $d\mathbf{r}$:

$$d\mathbf{r} = \hat{\rho} d\rho + \rho\hat{\phi} d\phi + \hat{k} dz.$$

The extra factor of ρ in the expression of $\rho\hat{\phi} d\phi$ is the *scale factor* on ϕ.

Similarly, in spherical coordinates, we have

$$d\mathbf{r} = \hat{r} dr + r\sin\theta\hat{\phi} d\phi + r\hat{\theta} d\theta,$$

with scale factors of $r\sin\theta$ on $\hat{\phi} d\phi$ and r on $\hat{\theta} d\theta$.

When we go from line elements (of the form $d\mathbf{r}$) to area elements (of the form $d\mathbf{a}$), we can see that the area element in polar coordinates is $d\mathbf{a} = \rho d\rho d\phi$ — we need the extra factor of ρ to account for the fact that the magnitude of the area element scales with the radius.

Similarly, the volume element in cylindrical coordinates is $d\tau = r dr d\phi dz$ and the volume element in spherical coordinates is $r^2 \sin\theta dr d\phi d\theta$.

Recall that the definition of an angle ϕ that subtends an arc length s is $\phi\frac{s}{r}$, where r is the radius of a circle. We can imagine a similar concept on a sphere — a *solid angle* measured in steradians is of the form $\Omega = \frac{A}{r^2}$, where A denotes the surface area subtended by the angle Ω. In particular, since $d\Omega = \frac{dA}{r^2}$, we find that $d\Omega = \sin\theta d\phi d\theta$.

When we are dealing with products of scale factors, we need to use the Jacobian to determine the proper scale factor on any given element:

$$\begin{aligned} d\mathbf{a} &= dx dy \\ &= |J| \, du dv, \end{aligned}$$

where $|J|$ denotes the determinant of the Jacobian matrix. We write the Jacobian as follows:

$$\begin{aligned} J &= \frac{\partial (x, y)}{\partial (u, v)} \\ &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}. \end{aligned}$$

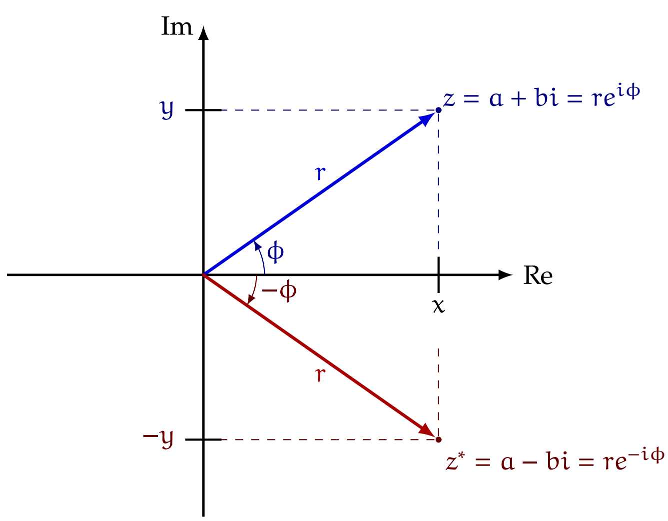
We specifically desire the determinant:

$$|J| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Complex Numbers

Quantity	Expression and/or Criterion
Cartesian form	$z = a + bi$
Polar form	$z = re^{i\phi}$
r	$\sqrt{a^2 + b^2}$
ϕ	$\arg z = \arctan\left(\frac{b}{a}\right)$
Cartesian z^*	$z^* = a - bi$
Polar z^*	$z = re^{-i\phi}$
$\lvert z\rvert$	$\sqrt{zz^*}$
$\mathrm{Re}(z)$	$\mathrm{Re}(z) = \frac{z+z^*}{2}$
$\mathrm{Im}(z)$	$\mathrm{Im}(z) = \frac{z-z^*}{2i}$
$\cos\phi$	$\frac{e^{i\phi}+e^{-i\phi}}{2}$
$\sin\phi$	$\frac{e^{i\phi}-e^{-i\phi}}{2i}$
$e^{i\phi}$	$\cos\phi + i\sin\phi$
$e^{in\phi}$	$\cos(n\phi) + i\sin(n\phi)$

Introduction



A complex number is denoted

$$z = a + bi$$

where $i^2 = -1$ and $a, b \in \mathbb{R}$. This is known as the cartesian representation. However, we can also imagine z as the polar representation:

$$z = re^{i\phi},$$

where $\phi = \arg z$ is known as the argument, and $r = |z|$ is the modulus. We can see the relation between the cartesian and polar representations through Euler's identity:[III]

$$r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

We denote the conjugate of z as z^*[IV], found by $z^* = a - bi = re^{-i\phi}$.

We find $\text{Re}(z)$ and $\text{Im}(z)$, the real and imaginary parts of z, by

$$\begin{aligned}\text{Re}(z) &= \frac{z+z^*}{2}\\ \text{Im}(z) &= \frac{z-z^*}{2i}.\end{aligned}$$

We say that a complex number of the form $e^{i\phi}$ is a *pure phase*, as $\left|e^{i\phi}\right| = 1$.

To find if some complex number z is purely real or purely imaginary, we can use the following criterion:

$$\begin{aligned}z \in \mathbb{R} &\Leftrightarrow z = z^*\\ z \in i\mathbb{R} &\Leftrightarrow z = -z^*.\end{aligned}$$

Example (Real, Imaginary, or Complex?)**.** Consider

$$z_1 = i^i.$$

To find if this is purely real or complex, we take

$$\begin{aligned}z_1^* &= (-i)^{-i}\\ &= \left(\frac{1}{-i}\right)^i\\ &= i^i.\end{aligned}$$

Thus, $z_1 \in \mathbb{R}$. In order to determine the value of i^i, we substitute the polar form:

$$\begin{aligned}z_1 &= \left(e^{i\frac{\pi}{2}}\right)^i\\ &= e^{-\frac{\pi}{2}}.\end{aligned}$$

Some Trigonometry with Complex Exponentials

Consider $z = \cos\phi + i\sin\phi$. We can see that

$$\begin{aligned}\text{Re}(z) &= \cos\phi\\ &= \frac{(\cos\phi + i\sin\phi) + (\cos\phi - i\sin\phi)}{2}\\ &= \frac{e^{i\phi} + e^{-i\phi}}{2}\\ \text{Im}(z) &= \sin\phi\\ &= \frac{(\cos\phi + i\sin\phi) - (\cos\phi - i\sin\phi)}{2i}\\ &= \frac{e^{i\phi} - e^{-i\phi}}{2i}.\end{aligned}$$

We can actually define $\sin\phi$ and $\cos\phi$ with the above derivation.

[III]This can be proven relatively easily through substitution into the Taylor series, which is allowed because e^z is entire.
[IV]Physicists amirite?

Theorem (De Moivre)**:**

$$\begin{aligned} e^{inx} &= \cos(nx) + i\sin(nx) \\ &= \left(e^{ix}\right)^n \\ &= (\cos x + i\sin x)^n. \end{aligned}$$

Example (Finding $\cos(2x)$ and $\sin(2x)$)**.**

$$\begin{aligned} \cos(2x) + i\sin(2x) &= (\cos x + i\sin x)^2 \\ &= \left(\cos^2 x - \sin^2 x\right) + i(2\sin x\cos x). \end{aligned}$$

Since the real parts and imaginary parts have to be equal, this means

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin^2 x &= 2\sin x\cos x. \end{aligned}$$

In particular, we can see that $e^{in\phi} = (-1)^n$ and $e^{in\frac{\pi}{2}} = i^n$.[v]

Additionally, we can see that for $z = re^{i\phi}$,

$$\begin{aligned} z^{1/m} &= \left(re^{i\phi+2\pi n}\right)^{1/m} \\ &= r^{1/m}e^{i\frac{1}{m}(\phi+2\pi n)}, \end{aligned}$$

where $n \in \mathbb{N}$ and m is fixed. For $r = 1$, we call these values the m roots of unity.

Example (Waves and Oscillations)**.** Recall that for a wave with spatial frequency k, angular frequency ω, and amplitude A, the wave is represented by

$$f(x,t) = A\cos(kx - \omega t).$$

The speed of a wave v is equal to $\frac{\omega}{k}$.

Simple harmonic motion is characterized by the solution to the differential equation $\ddot{\mathbf{x}} = -\omega^2\mathbf{x}$, where $\mathbf{x}$ denotes position. In simple harmonic motion, there is no spatial motion, meaning our function is only of time:

$$\begin{aligned} f(t) &= A\cos\omega t \\ &= \operatorname{Re}\left(Ae^{i\omega t}\right). \end{aligned}$$

As a result of the representation of complex numbers in polar form, we can do math entirely in exponentials, then take the real part of our solution to find $f(t)$.

Unfortunately, in the real world, there is friction; as a result, our oscillation is damped by an exponential factor.

Example (Hyperbolic Sine and Hyperbolic Cosine)**.** We wish to calculate $\cos ix$ and $\sin ix$.

$$\begin{aligned} \cos ix &= \frac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right) \\ &= \frac{e^{-x} + e^{x}}{2} \end{aligned}$$

[v] This will be especially useful when we get to Fourier series.

We define $\cosh x = \cos(ix)$. Additionally,

$$\begin{aligned}-i\sin ix &= -i\frac{1}{2i}\left(e^{i(ix)} - e^{-i(ix)}\right)\\ &= i\frac{e^{x} - e^{-x}}{2i}.\\ &= \frac{e^{x} - e^{-x}}{2}.\end{aligned}$$

We define $\sinh x = -i\sin(ix)$.

Similar to how $\cos^2 x + \sin^2 x = 1$, we can find that $\cosh^2 x - \sinh^2 x = 1$.

Index Algebra

We usually denote vectors by either $\vec{A}$, $\mathbf{A}$, or

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is defined by a basis.

If we imagine we are in n-dimensional space, we can let A_i where $i = 1, 2, \dots, n$ denote both

- the ith component of $\vec{A}$;
- the entire vector $\vec{A}$ (since i can be arbitrary).

Contractions and Dummy Indices

Consider $C = AB$, where A, B are $n \times m$ and $m \times p$ matrices respectively.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}.$$

Definition (Matrix Multiplication in Index Notation)**.** For matrices A and B, where A is an $m \times n$ and B is a $n \times p$ matrix, we write

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

We say that k is a dummy index, since k takes values from 1 to n. Note that the value we calculate is C_{ij}; in other words, in the sum $\sum_k A_{ik}B_{kj}$, the indices of the form ij are the "net indices" from the multiplication.

Note that if $C = BA$, then

$$\begin{aligned}C_{ij} &= \sum_{k=1}^{n} B_{ik}A_{kj}\\ &= \sum_{k=1}^{n} A_{kj}B_{ik}\end{aligned}$$

$$\neq \sum_{k=1}^{n} A_{ik}B_{kj}.$$

The corresponding fact is that $AB \neq BA$ necessarily.

Note that the index that is summed over always appears exactly twice.

Definition (Symmetric Matrix)**.** Let C be a matrix. Then, we say C is symmetric if

$$C_{ij} = C_{ji}$$

Definition (Antisymmetric Matrix)**.** Let C be a matrix. We say C is antisymmetric if

$$C_{ij} = -C_{ji}.$$

We can always decompose a random matrix into the sum of a symmetric matrix and an antisymmetric matrix.

Two Special Tensors

Name	Notation	Definition
Kronecker Delta	δ_{ij}	$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
Levi–Civita Symbol	ϵ_{ijk}	$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3) \text{ cyclically} \\ -1 & (i,j,k) = (2,1,3) \text{ cyclically} \\ 0 & \text{else} \end{cases}$

Order of (i,j,k)	Value of ϵ_{ijk}
$1,2,3$	1
$3,1,2$	1
$2,3,1$	1
$1,3,2$	-1
$2,1,3$	-1
$3,2,1$	-1
else	0

Value	Index Notation
$\mathbf{A} \times \mathbf{B}$	$\sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k$
$(\mathbf{A} \times \mathbf{B})_\ell$	$\sum_{i,j} \epsilon_{ij\ell} A_i B_j$
$\left(\hat{e}_i \times \hat{e}_j\right) \cdot \hat{e}_k$	ϵ_{ijk}
B_i	$\sum_{\alpha} B_\alpha \delta_{\alpha i}$
$\mathbf{A} \cdot \mathbf{B}$	$\sum_{i,j} A_i B_j \delta_{ij}$
$\sum_{j,k} \epsilon_{mjk}\epsilon_{njk}$	$2\delta_{mn}$
$\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ij\ell}$	$\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$

Definition (Kronecker Delta)**.** The Kronecker Delta, δ_{ij}, is the tensor that denotes the identity matrix.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example (Extracting an Index)**.** Consider A as vector. Then,

$$\sum_i A_i \delta_{ij} = A_j.$$

In other words, the Kronecker Delta collapses the sum to the jth index.

Example (Orthonormal Basis from Kronecker Delta)**.** Let $\{\hat{e}_i\}_{i=1}^n$ be a basis for some vector space V. If

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

for every i, j, then $\{\hat{e}_i\}_{i=1}^n$ is an orthonormal basis for V.

Definition (Levi–Civita Symbol)**.** In two dimensions, as a matrix, we write

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

meaning

$$\epsilon_{ij} = \begin{cases} 1 & i = 1, j = 2 \\ -1 & i = 2, j = 1 \\ 0 & \text{else} \end{cases}.$$

The Levi–Civita Symbol is antisymmetric, just as the Kronecker Delta is symmetric.

In three dimensions, we define

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = (1, 2, 3) \text{ cyclically} \\ -1 & (i, j, k) = (2, 1, 3) \text{ cyclically} \\ 0 & \text{else} \end{cases}.$$

In other words, $\epsilon_{ijk} = -\epsilon_{jik}$.

Exercise (Relations between δ_{ij} and ϵ_{ijk})**:**

$$\sum_{j,k} \epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn}$$

$$\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ijl} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$$

Definition (Dot Product)**.** Let $\{\hat{e}_i\}_{i=1}^n$ be an orthonormal basis for V. Let $\mathbf{A} = \sum_i A_i \hat{e}_i$ and $\mathbf{B} = \sum_i B_i \hat{e}_i$. Then,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \sum_{i,j} (A_i \hat{e}_i) \cdot (B_j \hat{e}_j) \\ &= \sum_{i,j} A_i B_j (\hat{e}_i \cdot \hat{e}_j) \\ &= \sum_{i,j} A_i B_j \delta_{ij} \\ &= \sum_i A_i B_i \end{aligned}$$

Definition (Cross Product). Let $\{\hat{e}_i\}_{i=1}^3$ be the standard basis over $\mathbb{R}^3$. Let $\mathbf{A} = \sum_i A_i\hat{e}_i$ and $\mathbf{B} = \sum_i B_i\hat{e}_i$. Then,

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \sum_{i,j}(A_i\hat{e}_i)\times\left(B_j\hat{e}_j\right)\\ &= \sum_{i,j}A_iB_j\left(\hat{e}_i\times\hat{e}_j\right)\\ &= \sum_{i,j,k}A_iB_j\left(\epsilon_{ijk}\hat{e}_k\right).\end{aligned}$$

Instead of asking about $\mathbf{A}\times\mathbf{B}$, we ask about $(\mathbf{A}\times\mathbf{B})_\ell$, yielding

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})_\ell &= (\mathbf{A}\times\mathbf{B})\cdot\hat{e}_\ell\\ &= \left(\sum_{i,j,k}A_iB_j\left(\epsilon_{ijk}\hat{e}_k\right)\right)\cdot\hat{e}_\ell\\ &= \sum_{i,j}\epsilon_{ij\ell}A_iB_j.\end{aligned}$$

Remark: This notation for $\mathbf{A}\times\mathbf{B}$ automatically shows us that

$$\begin{aligned}(\mathbf{B}\times\mathbf{A})_\ell &= \sum_{i,j}\epsilon_{ij\ell}B_iA_j\\ &= -\sum_{i,j}\epsilon_{ji\ell}B_iA_j\\ &= -\sum_{i,j}\epsilon_{ji\ell}A_jB_i\\ &= -\sum_{i,j}\epsilon_{ij\ell}A_iB_j && i,j \text{ are dummy indices}\\ &= -(\mathbf{A}\times\mathbf{B})_\ell.\end{aligned}$$

Example (Central Force and Angular Momentum). A central force is defined by

$$\mathbf{F} = f(r)\hat{r},$$

where $\hat{r}$ is a radial vector.

Angular momentum is defined by

$$\mathbf{L} = \mathbf{r}\times\mathbf{p},$$

where $\mathbf{r}$ denotes position and $\mathbf{p}$ denotes momentum. Then,

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{r}\times\mathbf{p})\\ &= \left(\frac{d}{dt}\mathbf{r}\times\mathbf{p}\right)+\mathbf{r}\times\left(\frac{d\mathbf{p}}{dt}\right)\\ &= m\left(\frac{d}{dt}\mathbf{r}\times\frac{d}{dt}\mathbf{r}\right)+\mathbf{r}\times(f(r)\hat{r})\\ &= f(r)\left(\mathbf{r}\times\hat{r}\right).\end{aligned}$$

This implies that $\frac{d\mathbf{L}}{dt} = 0$ under a central force.

Example (Determinant)**.** Let $\mathbf{M} = M_{ij}$ be square. We denote $\mathbf{M}_i$ to be the vector denoting the ith-row. Then,

$$\begin{aligned} m &= |\mathbf{M}| \\ &= \mathbf{M}_1 \cdot (\mathbf{M}_2 \times \mathbf{M}_3) \\ &= \mathbf{M}_3 \cdot (\mathbf{M}_1 \times \mathbf{M}_2) \\ &= \mathbf{M}_2 \cdot (\mathbf{M}_3 \times \mathbf{M}_1). \end{aligned}$$

Example (Trace)**.** Let $\mathbf{M} = M_{ij}$ be a square matrix. We define $\operatorname{tr}(\mathbf{M}) = \sum_i M_{ii}$. Equivalently,

$$\begin{aligned} \operatorname{tr}(\mathbf{M}) &= \sum_{ij} M_{ij}\delta_{ij} \\ &= \sum_i M_{ii}. \end{aligned}$$

Note that

$$\begin{aligned} \operatorname{tr}(I_n) &= \sum_i \delta_{ii} \\ &= n. \end{aligned}$$

When we upgrade to 3 matrices, we take

$$\begin{aligned} \operatorname{tr}(ABC) &= \sum_{i,j}\left(\sum_{k,\ell} A_{ik}B_{k\ell}C_{\ell j}\right)\delta_{ij} \\ &= \sum_{i,k,\ell} A_{ik}B_{k\ell}C_{\ell i} \\ &= \sum_{i,k,\ell} C_{\ell i}A_{ik}B_{k\ell} \\ &= \operatorname{tr}(CAB). \end{aligned}$$

In other words, the trace is invariant under cyclic permutations.

Example (Moment of Inertia Tensor)**.** Recall that

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p}, \\ &= I\boldsymbol{\omega}. \end{aligned}$$

where $\mathbf{p} = m\dot{\mathbf{x}}$, and I denotes the moment of inertia. Note that $I \sim mr^2$. On a more fundamental level, it is the case that the first equation, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is the "true" definition of $\mathbf{L}$.
Consider a small portion m_α about some axis at radius $\mathbf{r}_\alpha$ and momentum $\mathbf{p}_\alpha$. Then, we have

$$\begin{aligned} \mathbf{L}_\alpha &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\ &= \sum_\alpha m_\alpha (\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)). \end{aligned}$$

In the infinitesimal case (i.e., as $\alpha \to 0$), we get

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, \rho \, d\tau,$$

where ρ denotes volume density. Applying the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, we find

$$\mathbf{L} = \int (\boldsymbol{\omega}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}))\, \rho \, d\tau.$$

Switching to index notation, we have

$$\begin{aligned}
L_i &= \int \left(\omega_i r^2 - r_i \sum_j r_j \omega_j\right) \rho\, d\tau \\
&= \sum_j \int \omega_j \left(\delta_{ij} r^2 - r_i r_j\right) \rho\, d\tau \\
&= \sum_j \omega_j \underbrace{\left(\int \left(\delta_{ij} r^2 - r_i r_j\right) \rho\, d\tau\right)}_{\text{moment of inertia tensor}} \\
&= \sum_j I_{ij} \omega_j.
\end{aligned}$$

Binomial Theorem

The binomial theorem allows us to calculate the expansion

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m.$$

In the case of $(x+y)^2 = x^2y^0 + 2x^1y^1 + x^0y^2 = x^2 + 2xy + y^2$. Recall that

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}.$$

Recall that $0! = 1$.

Infinite Series

Let

$$S = \sum_{k=0}^{\infty} a_k$$

be an infinite series. We are often curious as to the convergence of this sum (for a variety of reasons). Formally, we have to invoke partial sums

$$S_N = \sum_{k=0}^{N} a_k,$$

and see if the sequence of partial sums is convergent. However, we will prefer to use series convergence tests.

Example (Geometric Series)**.** Let

$$\begin{aligned}
S &= \sum_{k=0}^{\infty} r^k \\
&= 1 + r + r^2 + \cdots
\end{aligned}$$

Then, we have

$$S_N = \sum_{k=0}^{N} r^k$$

$$rS_N = \sum_{k=0}^{N} r^k.$$

Subtracting, we get

$$\begin{aligned}(1-r)S_N &= 1 - r^{N+1}\\ S_N &= \frac{1-r^{N+1}}{1-r}.\end{aligned}$$

In the limit, we expect that if $r \to \infty$, and $r < 1$, then $r^{N+1} \to 0$. In the infinite case, we have

$$\begin{aligned}S &= \sum_{k=0}^{\infty} r^k\\ &= \frac{1}{1-r},\end{aligned}$$

if $r < 1$.

There are a few prerequisites for series convergence:

- there exists some K for which for all $k \geqslant K$, $a_{k+1} \leqslant a_k$;
- $\lim_{k\to\infty} < \infty$;
- we need the series to reduce "quickly" enough.

Example (Ratio Test)**.** A series $S = \sum_k a_k$ converges if the ratio of consecutive terms is (eventually) less than 1:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1.$$

Example (Applying the Ratio Test)**.** Consider $S = \sum_k \frac{1}{k!}$. Then,

$$\begin{aligned}r &= \lim_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}\\ &= \lim_{k\to\infty} \frac{1}{k+1}\\ &= 0 < 1\end{aligned}$$

Example (Riemann Zeta Function)**.** We write

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

In order to evaluate the convergence of the Riemann zeta function. We have

$$\begin{aligned}r &= \lim_{k\to\infty} \frac{\frac{1}{(k+1)^s}}{\frac{1}{k^s}}\\ &= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^s\\ &= 1.\end{aligned}$$

Unfortunately, this means the ratio test is inconclusive.

For examples of evaluations of the zeta function, we have

$$\begin{aligned}\zeta(1) &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ \zeta(2) &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots \\ &= \frac{\pi^2}{6}.\end{aligned}$$

Example (Absolute Convergence)**.** In our original ratio test, we had assumed that a_k are real and positive. However, if the $a_k \in \mathbb{C}$, we have to look at the convergence in modulus:

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|.$$

If $\sum_k |a_k|$ converges, this is known as absolute convergence.

Example (Alternating Series Test)**.** If the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

has the following conditions:

- $a_{k+1} < a_k$ for $k > K$;
- $\lim_{k\to\infty} a_k = 0$;

then $\sum_k (-1)^k a_k$ converges.

For instance, the alternating harmonic series converges

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2.$$

Power Series

Consider the function

$$S(x) = \sum_{k=0}^{\infty} a_k x^k.$$

This is a series both in a_k and in x. In order to determine convergence, we use the ratio test as follows:

$$\begin{aligned}\lim_{k\to\infty} \left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| &= |x| \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \\ &\equiv |x|\, r.\end{aligned}$$

In particular, for convergence , it must be the case that

$$|x|\, r < 1.$$

We define

$$R = \begin{cases} \frac{1}{r} & 0 < r < \infty \\ 0 & r = \infty \\ \infty & r = 0 \end{cases}.$$

In particular, this means

$$|x| < R.$$

Definition (Radius of Convergence)**.** For a power series $\sum_k a_k x^k$, the series converges for $|x| < R$,[VI] where

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$
$$R = \begin{cases} \frac{1}{r} & 0 < r < \infty \\ 0 & r = \infty \\ \infty & r = 0 \end{cases}.$$

Note that convergence for $|x| < R$ does not provide information regarding convergence at the boundary.

Example (Geometric Series)**.** We have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

has $R = 1$, meaning the power series converges for $|x| < 1$.

Example (Exponential Function)**.** We have

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

with $R = \infty$.

Example (Natural Log)**.** We have

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

In particular, since $R = 1$, we know that the radius of convergence is $|x| < 1$. However, the series does converge on the boundary when $x = 2$, but not when $x = 0$ (for obvious reasons).

Example (Why Radius of Convergence?)**.** Consider two series

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$$
$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

We can see that the first series converges for $|x| < 1$. However, even though $\frac{1}{1+x^2}$ has a domain across the entire real numbers, it is still the case that the *series* converges for $|x| < 1$.

The primary reason that the radius of convergence is defined as such is because, over the complex numbers, it is the case that $x^2 + 1 = 0$ at $x = \pm i$, meaning $\frac{1}{1+z^2}$ has singularities at those values of z.

The main reason power series are useful is that, when truncated, they are simply polynomials. In particular, with power series, we can reverse the order of sum and derivative.

[VI] The definition is not the true radius of convergence; it is actually that $r = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$. It just happens to be the case that the ratio test and root test return the same value when they're regular limits (rather than limits superior).

Taylor Series

Function	Taylor Series
$f(x)$	$\sum_{k=0}^{\infty} \frac{(x-x_0)^n}{n!} \left(\frac{d^n f}{dx^n} \bigg\vert_{x=x_0} \right)$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$
$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$ [VII]

Definition. The Taylor series of a function $f(x)$ about x_0 is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left(\frac{d^n f}{dx^n} \bigg\vert_{x=x_0} \right).$$

Remark: The reason we write $\frac{d^n f}{dx^n}$ is because $\frac{d^n}{dx^n}$ is an operator in and of itself.

Example (The Most Important Taylor Series)**.**

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \end{aligned}$$

Example (Equilibrium Points)**.** Let $U(x)$ denote a potential over x. Then, $\mathbf{F} = -\nabla U$. We have

$$U(x) = U(x_0) + (x-x_0)\, U'(x_0) + \frac{1}{2!}(x-x_0)^2\, U''(x_0) + \frac{1}{3!}(x-x_0)^3\, U'''(x_0) + \cdots$$

When we analyze an equilibrium point, we disregard the $U(x_0)$ term, and see that the derivative of U is zero; thus, we can truncate our series at the second derivative close to $x = x_0$:

$$\begin{aligned} U(x) &\approx \frac{1}{2} U''(x_0)(x-x_0)^2 \\ &= \frac{1}{2} m\omega^2 (x-x_0)^2. \end{aligned}$$

In other words, when we are very close to equilibrium, we have simple harmonic motion.

Example (Faster Taylor Series)**.** Consider the function

$$\exp\left(\frac{x}{1-x}\right).$$

In order to create its Taylor series, we can create this Taylor series piecewise:

$$\exp\left(\frac{x}{1-x}\right) = 1 + \left(\frac{x}{1-x}\right) + \frac{1}{2!}\left(\frac{x}{1-x}\right)^2 + \frac{1}{3!}\left(\frac{x}{1-x}\right)^3.$$

[VII] We define $(\alpha)_n = \prod_{k=0}^{n-1} (\alpha - k)$

Now, we expand the denominators as geometric series:

$$= 1 + x\left(\sum_{k=0}^{\infty} x^k\right) + \frac{x^2}{2!}\left(\sum_{k=0}^{\infty} x^k\right)^2 + \frac{x^3}{3!}\left(\sum_{k=0}^{\infty} x^k\right) + \cdots$$

If we want to expand through x^3, we have to expand by keeping track of *every* term:

$$= 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + O\left(x^4\right).$$

We say we have expanded the series through the third order; the lowest order correction, denoted $O(x^n)$, is the fourth order (in this case).

Example (Exponentiated Operator). Consider a (square) matrix M. Then, we define

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!},$$

where $M^k = \prod_{i=1}^{k} M$; we define $M^0 = I$. Similarly,

$$e^{\frac{d}{dx}} = \sum_{k=0}^{\infty} \frac{d^k}{dx^k}\frac{1}{k!}.$$

In particular, $e^{\frac{d}{dx}}$ is the Taylor series operator.

Remark: In quantum mechanics, the momentum operator is

$$P = -i\hbar\frac{d}{dx}.$$

Example (Binomial Expansion). For any $\alpha \in \mathbb{C}$ and $|x| < 1$, we have

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Note that if $\alpha \in \mathbb{Z}^+$, then the series truncates (and we recover the binomial theorem again).

The main use of the binomial expansion is with very small quantities. For instance,

$$\begin{aligned} E &\sim \frac{1}{(x^2+a^2)^{3/2}} \\ &= \frac{1}{x^3\left(1+\frac{a^2}{x^2}\right)^{3/2}} \\ &\approx \frac{1}{x^3}\left(1 - \frac{3}{2}\frac{a^2}{x^2}\right) && \text{For } x \gg a \end{aligned}$$

Remark: The binomial expansion only applies to the form $(1+x)^\alpha$. If we are dealing with an expression of the form $(a+x)^\alpha$, we need to factor out a, making the expression $a^\alpha(1+x/a)^\alpha$.

Example (Special Relativity with the Binomial Expansion). In the theory of special relativity, Einstein came up with the equations

$$\begin{aligned} E &= \gamma mc^2 \\ \gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned}$$

We can use the binomial expansion to find more information about γ.

$$\begin{aligned}
E &= \left(1-\frac{v^2}{c^2}\right)^{-1/2}mc^2\\
&= \left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{v^2}{c^2}\right)^2+\cdots\right)mc^2\\
&= mc^2+\underbrace{\frac{1}{2}mv^2\left(1+\frac{3}{4}\frac{v^2}{c^2}+\frac{5}{8}\left(\frac{v^2}{c^2}\right)^2+\cdots\right)}_{\text{Kinetic Energy}}
\end{aligned}$$

As we take $v \ll c$, we only need to keep the first order term in the expansion, meaning we have $E = mc^2 + \frac{1}{2}mv^2$.

Thus, we can find kinetic energy as $KE = (\gamma - 1)\,mc^2$. Notice that this means that *most* energy is internal energy emergent as mass.

Ten Integration Techniques

While Mathematica may exist,[VIII] it is still valuable to know how to take various integrals. More importantly, knowing how to take integrals provides valuable insights into *what* exactly integrals are.

Integration by Parts

Definition (Integration by Parts)**.** Using the product rule, we have

$$\begin{aligned}
\int \frac{d}{dx}(uv)\;dx &= \int \frac{du}{dx}v - \frac{dv}{dx}u\;dx\\
&= \int \frac{du}{dx}v\;dx - \int \frac{dv}{dx}u\;dx.
\end{aligned}$$

Thus, we get

$$\int u\;dv = uv - \int v\;du.$$

In the case where our integrals are definite, we have

$$\int_a^b u\;dv = uv\Big|_a^b - \int_a^b v\;du.$$

We say $uv\big|_a^b$ is the boundary term (or surface term).[IX]

Example.

$$\begin{aligned}
\int xe^{ax}\;dx &= \frac{1}{a}xe^{ax} - \int \frac{1}{a}e^{ax}\;dx && u = x,\ dv = e^{ax}dx\\
&= \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}\\
&= \frac{1}{a^2}e^{ax}(ax-1).
\end{aligned}$$

The $+C$ is implicit.

[VIII] Citation needed.

[IX] We can also use integration by parts to define the (weak) derivative, assuming the boundary term is zero.

Example.

$$\begin{aligned}\int \ln x\ dx &= x\ln x - \int x\left(\frac{1}{x}\right)\ dx && u = \ln x,\ dv = dx\\ &= x\ln x - x.\end{aligned}$$

Change of Variables

Definition (u-Substitution)**.** Let $x = x(u)$, meaning $dx = \frac{dx}{du}du$. Thus, we get

$$\int_{x_1}^{x_2} f(x)\ du = \int_{u(x_1)}^{u(x_2)} f\left(x(u)\right)\frac{dx}{du}\ du.$$

Example.

$$\begin{aligned}I_1 &= \int_0^\infty xe^{-ax^2}\ dx\\ &= \frac{1}{2}\int_0^\infty e^{-au}\ du && u = x^2\\ &= \frac{1}{2a}\end{aligned}$$

Example.

$$\begin{aligned}\int_0^\pi \sin\theta\ d\theta &= \int_{-1}^1\ du && u = \cos\theta\\ &= 2.\end{aligned}$$

More generally, we have, for $f(\theta) = f(\cos\theta)$,

$$\int_0^\pi f(\theta)\sin\theta\ d\theta = \int_{-1}^1 f(u)\ du.$$

Example (Trig Substitution)**.**

$$\begin{aligned}\int_0^a \frac{x}{x^2+a^2}\ dx &= \int_0^{\pi/4}\frac{a^2\tan\theta\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\ d\theta && x = a\tan\theta\\ &= \int_0^{\pi/4}\tan\theta\ d\theta\\ &= -\ln\left(\cos\theta\right)\Big|_0^{\pi/4}\\ &= \ln\left(\sqrt{2}\right)\\ &= \frac{1}{2}\ln(2).\end{aligned}$$

Example (Trig Substitution 2.0)**.** For rational functions of $\sin\theta$ and $\cos\theta$, we can use the half-angle trig substitution $u = \tan(\theta/2)$.[x] This yields

$$\begin{aligned}d\theta &= \frac{2du}{1+u^2}\\ \sin\theta &= \frac{2u}{1+u^2}\end{aligned}$$

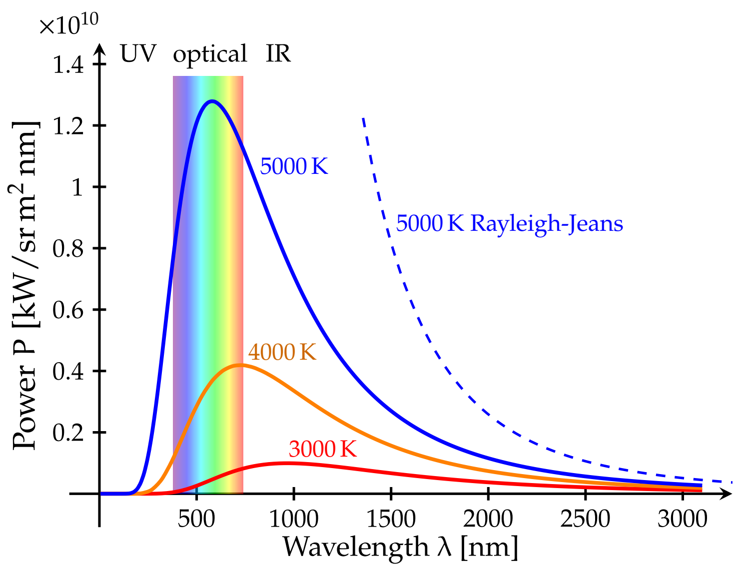
[x] $\tan(\theta/2) = \frac{\sin\theta}{1+\cos\theta}$

$$\cos\theta = \frac{1-u^2}{1+u^2}.$$

For instance,

$$\begin{aligned}
\int \frac{1}{1+\cos\theta}\,d\theta &= \int \frac{1}{1+\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\,du\\
&= \int\,du\\
&= \tan(\theta/2)\\
&= \frac{\sin\theta}{1+\cos\theta}.
\end{aligned}$$

Example (Dimensionless Integrals)**.**



Anything that has a nonzero absolute temperature radiates some energy. In particular, we want to know how this radiation is distributed among various wavelengths.

For a box of photons in equilibrium at temperature T, the energy per volume per wavelength λ[xi] is

$$u(\lambda) = \frac{8\pi hc}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}.$$

Here, h denotes Planck's constant, c is the speed of light, and k is Boltzmann's constant.

In order to find the total energy density, we have to integrate $u(\lambda)$ over all possible values of λ:

$$\begin{aligned}
U &= \int_0^\infty u(\lambda)\,d\lambda\\
&= 8\pi hc\int_0^\infty \frac{1}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}\,d\lambda
\end{aligned}$$

[xi] read as (energy per volume) per wavelength

This integral is, for lack of a better word, hard. However, if we remove the dimensions of λ by substituting $x = \frac{hc}{\lambda kt}$, we can verify that the value of U now becomes

$$U = 8\pi hc \left(\frac{kT}{hc}\right)^4 \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\,dx}_{\text{scalar}}.$$

Thus, all the physics[XII] is captured as a coefficient on the integral; namely, this integral captures the Stefan–Boltzmann law, which has that energy density scales by T^4.

Using some fancy techniques we will learn later, we can evaluate

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}.$$

Even/Odd

Definition (Even and Odd Functions)**.** A function $f(x)$ is

- even if $f(-x) = f(x)$;
- odd if $f(-x) = -f(x)$.

Just as a matrix can be decomposed into a sum of a symmetric and antisymmetric matrix, we can decompose a function into a sum of an even function and an odd function.

Integrals over symmetric intervals on functions with definite parity are very simple:

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & f \text{ odd} \\ 0 & f \text{ even} \end{cases}.$$

For the case of a function $g(x) = g(|x|)$, we have

$$\int_{-a}^{b} g(|x|)\,dx = \int_{-a}^{0} g(-x)\,dx + \int_0^b g(x)\,dx.$$

Products and Powers of Sines and Cosines

Value	Expression
$\sin(\alpha \pm \beta)$	$\sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
$\cos(\alpha \pm \beta)$	$\cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin\alpha\cos\beta$	$\frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$
$\cos\alpha\cos\beta$	$\frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$
$\sin\alpha\sin\beta$	$\frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$

Example. If we have an integral

$$\begin{aligned} \int \sin(3x)\cos(2x)\,dx &= \frac{1}{2}\int \sin(5x) + \sin(x)\,dx \\ &= \frac{1}{2}\left(-\frac{1}{5}\cos(5x) - \cos(x)\right). \end{aligned}$$

[XII] Who cares about that stuff?

Integral	Shortcut
$\int \sin^m(x)\cos^{2k+1}(x)\ dx$	$\int u^m\left(1-u^2\right)^k\ du$
$\int \sin^{2k+1}(x)\cos^n(x)\ dx$	$-\int\left(1-u^2\right)^k u^n\ du$
$\int \sin^2(x)\ dx$	$\frac{x}{2}-\frac{1}{4}\sin(2x)$
$\int \cos^2(x)\ dx$	$\frac{x}{2}+\frac{1}{4}\sin(2x)$

Example. To evaluate

$$\int \sin^2(x)\ dx,$$
$$\int \cos^2(x)\ dx$$

we use the identity

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}\left(1-\cos(2x)\right)\\ \cos^2(x) &= \frac{1}{2}\left(1+\cos(2x)\right),\end{aligned}$$

and take

$$\begin{aligned}\int \sin^2(x)\ dx &= \frac{1}{2}\int \left(1-\cos(2x)\right)\ dx\\ &= \frac{x}{2}-\frac{1}{4}\sin(2x)\\ \int \cos^2(x)\ dx &= \frac{1}{2}\int \left(1+\cos(2x)\right)\ dx\\ &= \frac{x}{2}+\frac{1}{4}\sin(2x).\end{aligned}$$

Thus, we can see that

$$\int_0^\pi \sin^2(x)\ dx = \frac{\pi}{2}$$
$$\int_0^\pi \cos^2(x)\ dx = \frac{\pi}{2}$$

Axial and Spherical Symmetry

Consider a function of the form $f(x,y) = x^2+y^2$. If we were to integrate with respect to $dxdy$, we would need a two dimensional integral. With polar coordinates, though, we would have $dxdy = rdrd\phi$. Since f is axially symmetric, we would have our $dxdy = 2\pi rdr$, which is a one-dimensional integral.

If we have something with spherical symmetry, then there is no dependence on either θ or ϕ, yielding a function $f(\mathbf{r}) = f(r)$, meaning

$$\begin{aligned}\int f(\mathbf{r})\ d\tau &= \int f(r)r^2\sin\theta\ drd\theta d\phi\\ &= 4\pi\int f(r)r^2\ dr.\end{aligned}$$

Note that $\int \sin\theta\ d\theta d\phi$ over the sphere is 4π.

Example. Consider a surface S with charge density $\sigma(\mathbf{r})$. Finding the total charge requires evaluating

$$Q = \int_S \sigma(\mathbf{r})\ dA.$$

If S is hemispherical with $z > 0$ with radius R, and $\sigma = k\frac{x^2+y^2}{R^2}$, the integrand is axially symmetric.

Using spherical coordinates, we evaluate

$$\begin{aligned}
Q &= \int_S \sigma(\mathbf{r})\ dA \\
&= \frac{k}{R^2}\int x^2 + y^2\ dA \\
&= \frac{k}{R^2}\int \left(R^2\sin^2\theta\cos^2\phi + R^2\sin^2\theta\sin^2\phi\right) R^2\sin\theta\ d\theta d\phi \\
&= kR^2\int_S \sin^3\theta\ d\theta d\phi \\
&= 2\pi kR^2\int_0^{\pi/2}\sin^3\theta\ d\theta \\
&= \frac{4\pi kR^2}{3}.
\end{aligned}$$

Example. Let

$$\Phi(\mathbf{r}) = \int \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^3\|\mathbf{k}\|^2}\ d^3k$$

where k-space is an abstract 3-dimensional Euclidean space. In Cartesian coordinates, $d^3k = dk_x\,dk_y\,dk_z$, which yields the integral

$$\Phi(\mathbf{r}) = \int \frac{e^{-ik_xx}e^{-ik_yy}e^{-ik_zz}}{(2\pi)^3\left(k_x^2+k_y^2+k_z^2\right)}\ dk_x\,dk_y\,dk_z.$$

This integral is very hard to evaluate (over Cartesian coordinates, anyway),[XIII] so we need to use some other methods.

In spherical coordinates, we have $d^3k = k^2\,dk\,d\Omega$, yielding

$$\Phi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int k^2\frac{e^{-ikr\cos\theta}}{k^2}\ dk\,d(\cos\theta)\,d\phi.$$

Since we are summing away all our k-dependence, we can orient r along the k_z axis. Thus, we can evaluate the integral as

$$\begin{aligned}
\Phi(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int k^2\frac{e^{-ikr\cos\theta}}{k^2}\ dk\,d(\cos\theta)\,d\phi \\
&= \frac{1}{(2\pi)^2}\int_{-1}^{1}\int_0^\infty e^{-ikr\cos\theta}\ dk\,d(\cos\theta) \\
&= \frac{1}{(2\pi)^2}\int \frac{1}{(-ikr)}\left(e^{-ikr} - e^{ikr}\right)\ dk
\end{aligned}$$

[XIII] Citation needed.

$$\begin{aligned} &= \frac{1}{(2\pi)^2}\int_0^\infty \frac{2\sin(kr)}{kr}\,dk \\ &= \frac{1}{2\pi^2}\underbrace{\int_0^\infty \frac{\sin(kr)}{kr}\,dk}_{\text{sinc integral}}. \end{aligned}$$

In order to evaluate the sinc integral, we have to use some different techniques.

Differentiation with Respect to a Parameter

Example. We can evaluate

$$\begin{aligned} \int xe^{ax}\,dx &= \frac{\partial}{\partial a}\left(\int e^{ax}\,dx\right) \\ &= \frac{\partial}{\partial a}\left(\frac{1}{a}e^{ax}\right) \\ &= -\frac{1}{a^2}e^{ax} + \frac{1}{a}xe^{ax} \\ &= \frac{1}{a^2}e^{ax}\left(ax-1\right) \end{aligned}$$

When differentiating with respect to a parameter, it is important to remember that we are often differentiating *with respect to the parameter*, not with respect to our main variable.

Example (Introducing a Parameter)**.** We wish to solve the sinc integral,

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

In order to do this, we will introduce a parameter such that differentiation will cancel out the x in the denominator:

$$J(\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx. \qquad \alpha > 0$$

In particular, $\alpha > 0$. We calculate

$$\begin{aligned} \frac{dJ}{d\alpha} &= -\int_0^\infty e^{-\alpha x}\sin x\,dx \\ &= -\frac{1}{1+\alpha^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} J(\alpha) &= -\int \frac{1}{\alpha^2}\,d\alpha \\ &= -\arctan(\alpha) + C. \end{aligned}$$

In order to determine the value of C, we need to make sure $J(\infty) = 0$. Therefore, $C = \frac{\pi}{2}$. Therefore, we have

$$J(0) = \frac{\pi}{2}.$$

Gaussian Integral

We cannot evaluate $I_0 = \int_0^\infty e^{-ax^2}\,dx$ using elementary methods, because e^{-ax^2} is not an elementary function. The reason we care a lot about e^{-ax^2} is because it is very important in quantum mechanics and statistics.[XIV]

It is clear that I_0 converges. We can see that the dimension of a is x^{-2}, and since we are integrating with respect to dx, we can see that our integral is related to $\frac{1}{\sqrt{a}}$.

Example. We will not solve for I_0, but for I_0^2. Thus, we have

$$\begin{aligned}
I_0^2 &= \left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right)\\
&= \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a\left(x^2+y^2\right)}\,dx\,dy\\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\infty} re^{-ar^2}\,dr\,d\phi\\
&= \frac{\pi}{2}\int_0^{\infty} re^{-ar^2}\,dr\\
&= \frac{\pi}{2}\left(\frac{1}{2}\int_0^{\infty} e^{-au}\,du\right)\\
&= \frac{\pi}{4a}.
\end{aligned}$$

Therefore, $I_0 = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

Definition (Family of Gaussian Integrals)**.**

$$I_n = \int_0^\infty x^n e^{-ax^2}\,dx.$$

Expression	Value
I_0	$\frac{1}{2}\sqrt{\frac{\pi}{a}}$
I_1	$\frac{1}{2a}$
I_{2n}	$(-1)^n\frac{d^n}{da^n}I_0$
I_{2n+1}	$(-1)^n\frac{d^n}{da^n}I_1$

It is important to note that there are different expressions for the Gaussian integral:

$$\begin{gathered}
\int e^{-ax^2}\,dx\\
\int e^{-a^2x^2}\,dx\\
\int e^{-a^2x^2/2}\,dx\\
\int e^{-x^2/a}\,dx\\
\int e^{-x^2/a^2}\,dx,
\end{gathered}$$

meaning we have to be careful when evaluating these integrals.

[XIV]Who cares about that?

Example (Error Function)**.** Consider the integral

$$\int_0^{53} e^{-ax^2}\,dx.$$

Unfortunately, there is no way to do this integral analytically. It is only able to be calculated numerically.

We define

$$\operatorname{erf}(u) = \int_0^u e^{-ax^2}\,dx$$

Completing the Square

Example. Consider the integral

$$\int_{-\infty}^{\infty} e^{-ax^2-bx}\,dx.$$

This integral is Gaussian-esque, but it isn't fully Gaussian, yet.

To do this, we will complete the square:

$$\begin{aligned} ax^2+bx &= a\left(x^2+\frac{b}{a}x\right) \\ &= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right) \\ &= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}. \end{aligned}$$

In particular, this turns the integral into

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ax^2-bx}\,dx &= \int_{-\infty}^{\infty} e^{-a(x+b/2a)^2+b^2/4a}\,dx \\ &= e^{b^2/4a}\int_{-\infty}^{\infty} e^{-a(x+b/2a)}\,dx \\ &= e^{b^2/4a}\left(\sqrt{\frac{\pi}{a}}\right) \\ &= e^{b^2/4a}\sqrt{\frac{\pi}{a}}. \end{aligned}$$

Series Expansion

Function	Expression
$\Gamma(s)$	$\int_0^{\infty} x^{s-1}e^{-x}\,dx$
$\zeta(s)$	$\sum_{k=1}^{\infty}\frac{1}{k^s}$
$\Gamma(s+1)$	$s\Gamma(s)$

Consider the integral

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx.$$

This is a very nasty integral,[xv] but we will need to know this value because it is useful in statistical mechanics.[xvi] We want to ensure this converges.

Notice that for large x, the integrand looks like $e^{-x}x^{s-1}$.

Example. To resolve the integral we take

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \int_0^\infty \frac{e^{-x}x^{s-1}}{1 - e^{-x}}\, dx$$

We will use the geometric series expansion for the denominator:

$$\begin{aligned} &= \int_0^\infty e^{-x}x^{s-1}\sum_{k=0}^\infty e^{-kx}\, dx \\ &= \sum_{k=0}^\infty \int_0^\infty x^{s-1}e^{-(k+1)x}\, dx. \end{aligned}$$

We make the change of variables $u = (n+1)x$.

$$\begin{aligned} &= \sum_{n=0}^\infty \frac{1}{(n+1)^s}\int_0^\infty u^{s-1}e^{-u}\, du \\ &= \underbrace{\sum_{n=1}^\infty \frac{1}{n^s}}_{\zeta(s)} \underbrace{\int_0^\infty u^{s-1}e^{-u}\, du}_{\Gamma(s)}. \end{aligned}$$

Thus, our integral resolves to

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \Gamma(s)\zeta(s).$$

Partial Fractions

Example (A Partial Fraction Decomposition)**.**

$$\begin{aligned} \frac{1}{1-x^2} &= \frac{\alpha}{1-x} + \frac{\beta}{1+x} \\ &= \frac{1/2}{1-x} + \frac{1/2}{1+x}. \end{aligned}$$

Example (Integrating using Partial Fractions)**.** To evaluate

$$\int \frac{4-2x}{(x^2+1)(x-1)^2}\, dx,$$

we do the partial fraction decomposition to find

$$\int \frac{4-2x}{(x^2+1)(x-1)^2}\, dx = \int \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}\, dx.$$

[xv] Citation needed.

[xvi] Okay actually I do kinda care about this.

Example (Mean Value Theorem)**.** If we have a function f defined on $[a, b]$, then there is a point $c \in (a, b)$ such that

$$f(c)(b-a) = \int_a^b f(x)\, dx.$$

More generally, the mean value theorem says there exists $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\, dx = f(c)\int_a^b g(x)\, dx$$

Delta Distribution

Consider a "function" $\delta(x)$ such that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a).$$

This idea seems absurd on its face — after all, singletons have measure zero, so the idea of an integral collapsing into a single point doesn't sound normal.

The structure of the delta distribution is

$$\delta(x-a) = \begin{cases} +\infty & x = a \\ 0 & \text{else} \end{cases}.$$

In particular, we also have to define

$$\int_{-\infty}^{\infty} \delta(x-a)\ dx = 1.$$

This is known as the Dirac delta function (or rather, distribution). The delta distribution "weights" f to infinity at $x = a$ and zero everywhere else.

Example (Delta Distribution as a Limit)**.** Imagine a Gaussian function with area under the curve 1. In particular,

$$f_n(x) = \frac{1}{\sqrt{\pi}} n e^{-n^2x^2}.$$

In particular, we have

$$\delta(x) = \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} n e^{-n^2x^2}$$

Example (A Physical Example)**.** Imagine a ball is kicked. The force is dependent on time, $F(t)$.

There isn't an easy to find the force, but by Newton's second law, we have

$$\delta p = \int F(t)\, dt,$$

where

$$I \equiv \int F(t)\, dt$$

is the impulse.

If we want to model $F(t)$, where we don't care about a nonzero time over which the force is occurring, we can simply state $F(t)$ as

$$F(t) = \Delta p \delta(t - t_0).$$

Taking this integral yields I.

Example (Fourier Integral Representation of Delta Distribution)**.** A different representation of $\delta(x)$ is

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk.$$

We are superimposing all the waves e^{ikx} — in particular, for all values of $k \neq 0$, both e^{ikx} and e^{-ikx} are "added" together, yielding absolute destructive interference.

The factor of $\frac{1}{2\pi}$ is necessary to normalize the integral.

Properties of the Delta Distribution

Normalization:

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\int_{x_1}^{x_2} \delta(x - a)\, dx = \begin{cases} 1 & x_1 < a < x_2 \\ 0 & \text{else} \end{cases}.$$

Sieve:

$$\int_{x_1}^{x_2} f(x)\delta(x - a)\, dx = \begin{cases} f(a) & x_1 < a < x_2 \\ 0 & \text{else} \end{cases}.$$

Example (Delta Distribution as a Limit of Rectangles)**.** We define the family of functions

$$\phi_k(x) = \begin{cases} k/2 & |x| < 1/k \\ 0 & |x| > 1/k \end{cases}.$$

We can see that integrating ϕ_k over $\mathbb{R}$ yields 1 for each k.

We now need to evaluate if $\lim_{k\to\infty} \phi_k(x) = \delta(x)$. In order to see this, we take

$$\begin{aligned} \lim_{k\to\infty} \int_{-\infty}^{\infty} f(x)\phi_k(x)\, dx &= \lim_{k\to\infty} \frac{k}{2} \int_{-1/k}^{1/k} f(x)\, dx \\ &= \lim_{k\to\infty} f(c_k) \left(\frac{k}{2} \int_{-1/k}^{1/k} dx \right) \\ &= \lim_{k\to\infty} f(c_k), \end{aligned}$$

where we define c_k from the mean value theorem. In particular, since $c_k \in (-1/k, 1/k)$, it is the case that $c_k \to 0$ as $k \to \infty$, so

$$\begin{aligned} \lim_{k\to\infty} \int_{-\infty}^{\infty} f(k)\phi_k(x)\, dx &= \lim_{k\to\infty} f(c_k) \\ &= f(0). \end{aligned}$$

Thus, $\lim_{k\to\infty} \phi_k(x) = \delta(x)$.

We can imagine the delta distribution to be the density distribution of a single point.

The units of $\delta(x)$ are

$$[\delta(x)] = x^{-1}.$$

Example (Linear Argument for δ)**.** Consider $\delta(ax)$. For instance, we want to evaluate

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\, dx.$$

To do so, we use u substitution with $u = ax$:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(ax)\ dx &= \frac{1}{a}\int_{-\infty}^{\infty} f\left(u/a\right)\delta(u)\, du\\ &= \frac{1}{a}f(0).\end{aligned}$$

It is important to note that the integration variable dx and the argument of $\delta(x)$ must be equal.

In general, we have

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

Example (Function Argument for δ)**.** We now want to evaluate

$$\int_{-\infty}^{\infty} f()\delta(g(x))\, dx.$$

When we take the change of variables, we have

$$\int_{y_1}^{y_2} f(y)\delta(y)\ dy = \int_{x_1}^{x_2} f(g(x))\delta(g(x))\left|\frac{dg}{dx}\right|\ dx.$$

Therefore, we must have $\delta(g(x)) = \frac{1}{|dg/dx|_{g(x)=0}}\delta(x)$.

In the general case, we have

$$\delta(g(x)) = \frac{1}{|dg/dx|_{x_0}}\delta(x - x_0)$$

where $g\left(x_0\right) = 0$.

If, in the region of integration, g takes multiple zeros, we must take a sum:

$$\delta(g(x)) = \sum_i \frac{1}{|dg/dx|_{x_i}}\delta(x_i)\,;$$

where we assume $\left|\frac{dg}{dx}\right|_{x_i} \neq 0$.

Example ($x^2 - a^2$ Argument for δ)**.** Consider the distribution

$$\delta\left(x^2 - a^2\right).$$

The derivative of $g(x)$ is $2x$; the two zeros of g are at $x = \pm a$. Therefore,

$$\begin{aligned} \delta\left(x^2 - a^2\right) &= \frac{1}{|2x|_a}\delta(x-a) + \frac{1}{|2x|_{-a}}\delta(x+a) \\ &= \frac{1}{2a}\left(\delta(x-a) + \delta(x+a)\right). \end{aligned}$$

For example, if we took

$$\begin{aligned} \int_{-\infty}^{\infty} x^3\delta\left(x^2 - a^2\right)\, dx &= \frac{1}{2a}\int_{-\infty}^{\infty} x^3\left(\delta(x-a) + \delta(x+a)\right)\, dx \\ &= \frac{1}{2a}\left(a^3 + (-a)^3\right) \\ &= 0. \end{aligned}$$

Now, evaluating

$$\begin{aligned} \int_{0}^{\infty} x^3\left(\delta\left(x^2 - a^2\right)\right)\, dx &= \frac{1}{2a}\int_{0}^{\infty} x^3\left(\delta(x-a) + \delta(x+a)\right)\, dx \\ &= \frac{1}{2a}\left(a^3\right) \\ &= \frac{1}{2}a^2. \end{aligned}$$

Example ((Weak) Derivative of δ)**.** Obviously we cannot formally take $\delta'(x)$, but we can always place $\delta(x)$ under the integral sign and treat $\delta'(x)$ as the "derivative" via integration by parts:

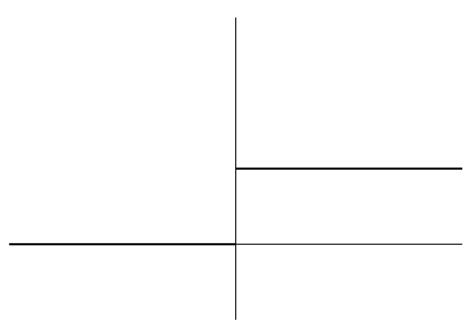
$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta'(x)\, dx &= f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\delta(x)\, dx \\ &= -\frac{df}{dx}\Big|_0 \\ &= -f'(0). \end{aligned}$$

The "identity" for the delta function's derivatives is

$$f(x)\delta'(x) = -f'(x)\delta(x).$$

Example (Heaviside Step Function)**.** The Heaviside step function, $\Theta(x)$, is

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$



Example (Higher Dimension Delta Distributions)**.** In higher dimensions,

$$\int_{\text{all space}} \delta(\mathbf{r})\, d\tau = 1,$$

and

$$\int_V f(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{a})\;d\tau = \begin{cases} f(\mathbf{a}) & \mathbf{a}\in V \\ 0 & \text{otherwise} \end{cases}.$$

One of the common notations for higher dimensional delta functions is $\delta^{(n)}(\mathbf{r})$, where (n) denotes the dimension (not to be confused with nth derivative).

Instead, we can use $\delta(\mathbf{r})$, which lets us know that we are dealing in higher dimensions, and context is evident.

Example (Voltage under a Point Charge)**.** The voltage of a point charge q at a position $\mathbf{a}$ is given by Coulomb's law

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{a}|},$$

with $\Phi = 0$ at ∞.

For a continuous point charge distribution $\rho(\mathbf{r})$, we consider each element of the volume $d\tau$ centered at $\mathbf{r}$ with charge $dq = \rho(\mathbf{r})\,d\tau$.

$$\begin{aligned} d\Phi(\mathbf{r}) &= \frac{dq}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{a}|} \\ &= \frac{1}{4\pi\epsilon_0}\frac{\rho(\mathbf{r})\,d\tau}{|\mathbf{r}-\mathbf{a}|}. \end{aligned}$$

In particular, for some $\mathbf{r}$, we need to add up over $\mathbf{a}$, yielding

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;d\tau'$$

This expression should hold for every physically reasonable volume charge distribution ρ, what $\rho(\mathbf{r})$ denotes a point charge?

In particular, if $\rho(\mathbf{r})$ is a point charge, then $\rho = q\delta(\mathbf{r}-\mathbf{a})$.

Example (Using the Multi-Dimensional Delta Distribution)**.** In Cartesian coordinates, we have

$$\delta(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0).$$

If we want to transform $\delta(\mathbf{r}-\mathbf{r}_0)$ into a different coordinate system such as $d\tau = du\;dv\;dw$, we need the Jacobian. Thus,

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{|J|}\delta(u-u_0)\,\delta(v-v_0)\,\delta(w-w_0).$$

For instance, in spherical coordinates, we have

$$\begin{aligned} \delta(\mathbf{r}-\mathbf{r}_0) &= \frac{1}{r^2\sin\theta}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0) \\ &= \frac{1}{r^2}\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\delta(\phi-\phi_0). \end{aligned}$$

Vector Calculus

Question: What is a vector?

Answer. A vector is an element of a vector space.

Remark: Yes, vectors as defined by "magnitude and direction" also are elements of vector spaces.

For the purposes of this unit, we will focus on vectors in the vector space $\mathbb{R}^n$ over $\mathbb{R}$.

Notation: Vector-valued functions with vector-valued outputs will be denoted

$$\mathbf{F}(\mathbf{r}).$$

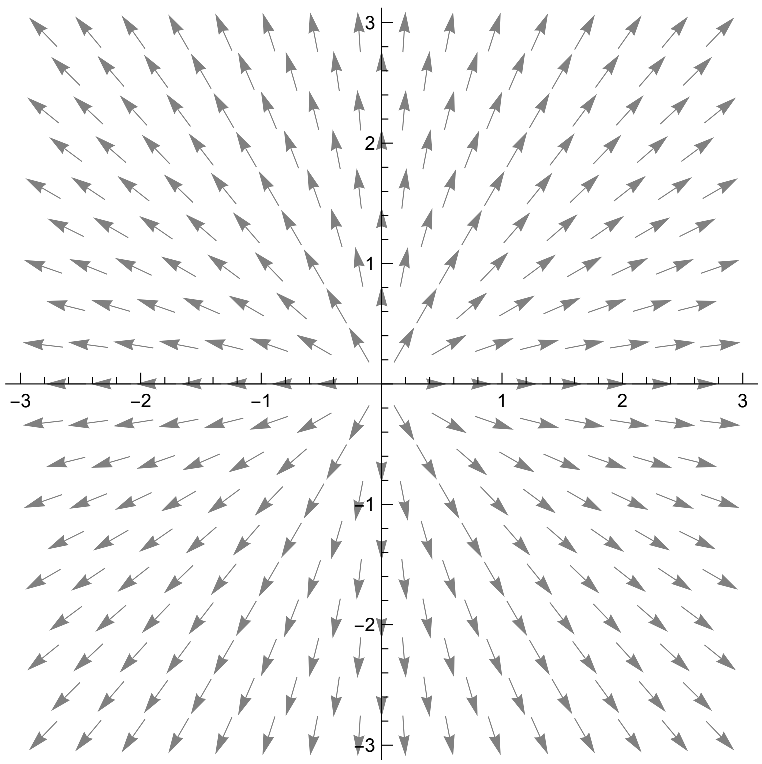
Vector Fields

Definition (Vector Field). A vector-valued function $\mathbf{F}(\mathbf{r})$ with vector-valued outputs is known as a vector field.

Example. The field

$$\mathbf{F}(x, y, z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

can be seen below.

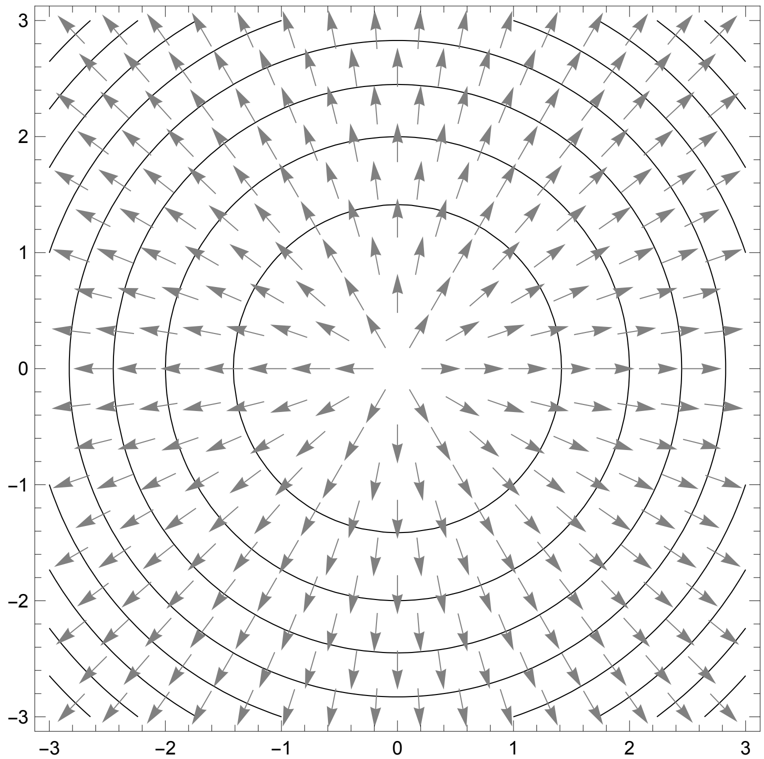


Notice that, in terms of spherical coordinates, $\mathbf{F}(x, y, z) = r\hat{r} = \mathbf{r}$.

Definition (Incompressible Fluid). A fluid is incompressible if its density is constant.

In particular, incompressible fluids cannot have either sources or sinks, since sources imply a local reduction in density, while sinks imply a local increase in density.

Example. Consider a sprinkler with N streams. Since water is incompressible, the density of streamlines σ and the surface area of the spherical shells, A must be inversely proportional to each other.

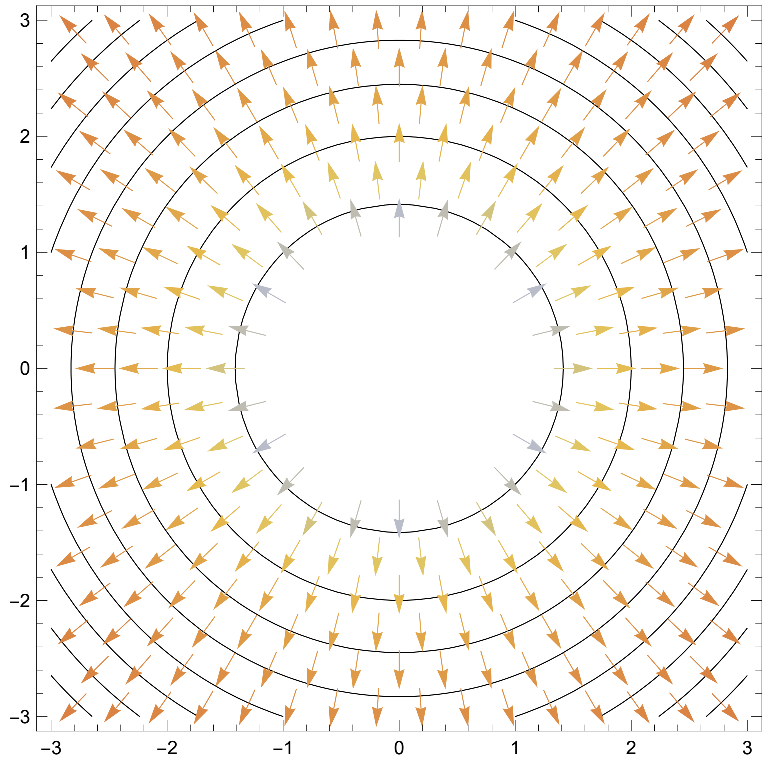


Thus, we have

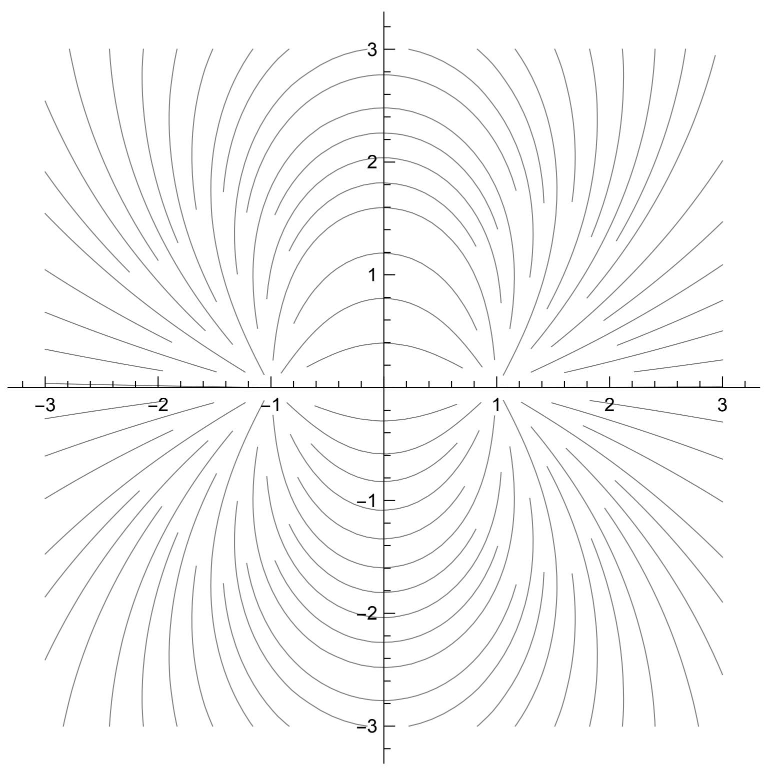
$$N = \sigma A,$$

meaning $\sigma \sim \frac{1}{r^2}$ since $A \sim r^2$.

In particular, the strength of the vector field must diminish with the square of the distance.



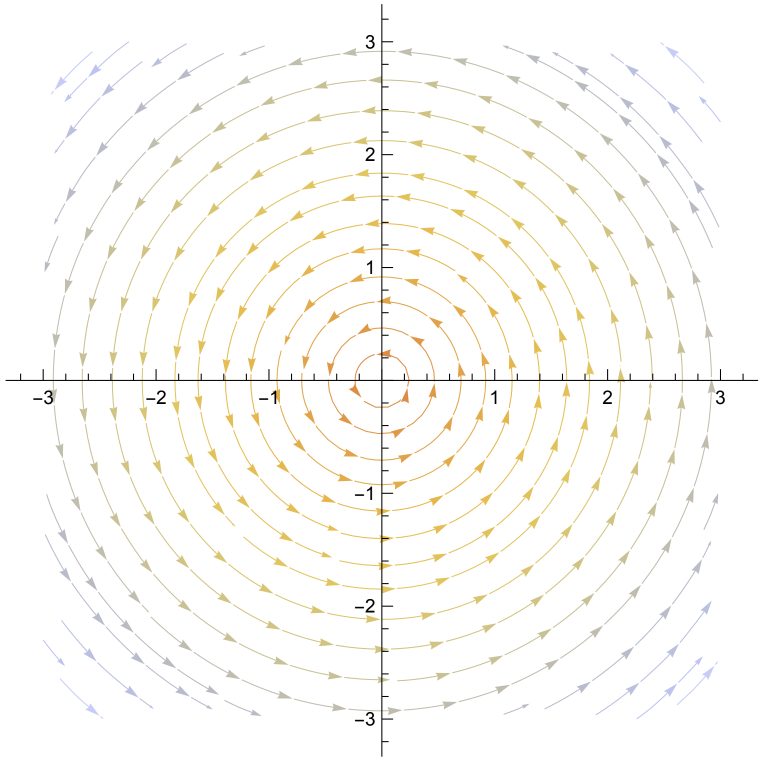
Example. Vector fields can be added.



Example. Consider the field

$$\mathbf{G}(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \left(-y\hat{\imath} + x\hat{\jmath}\right)$$

As depicted, we can see that the vector field looks as follows.



In particular, we can see that $\mathbf{G} = \hat{\phi}$.

Notice that our vector fields are dependent on both the basis and the coordinate system.

In particular, we have reason to prefer a Cartesian basis over the polar or spherical basis, since the Cartesian basis is position-independent.

Grad, Div, and Curl

Consider a scalar function $f(\mathbf{r})$. If we want to imagine how f changes as we move $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$, we use the chain rule.

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz \\ &= \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ &= \nabla f \cdot d\mathbf{r}. \end{aligned}$$

In particular, we define

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}.$$

Notice that, since $dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$ is a vector, and df is a scalar, we know that ∇f *must* be a vector.

Definition (Gradient)**.**

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \\ df &= |\nabla f|\,|d\mathbf{r}|\cos\theta. \end{aligned}$$

If ∇f is in the direction of $d\mathbf{r}$, then $\cos\theta = 1$, meaning df is maximized. In particular, ∇f points in the direction of maximum change in f.

In particular, this means that for every (differentiable) scalar field, there is a natural vector field associated with the direction of largest increase.

Example. The electric field

$$\mathbf{E} = -\nabla V.$$

Similarly, for any given potential U,

$$\mathbf{F} = -\nabla U.$$

Definition (The ∇ Operator)**.**

$$\begin{aligned}\nabla f &= \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}}_{\nabla} (f)\end{aligned}$$

Thus, we get

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}.$$

Example.

(1) For some scalar field $f(\mathbf{r})$, we can take

$$\nabla(f) = \nabla f,$$

which yields the gradient field.

(2) For some vector field $\mathbf{E}$, we can take

$$\nabla \cdot \mathbf{E} = g$$

which yields a scalar field known as the divergence of $\mathbf{E}$.

In particular,

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x}\left(\mathbf{E}\cdot\hat{\imath}\right) + \frac{\partial}{\partial y}\left(\mathbf{E}\cdot\hat{\jmath}\right) + \frac{\partial}{\partial z}\left(\mathbf{E}\cdot\hat{k}\right).$$

(3) For some vector field $\mathbf{B}$, we can take

$$\nabla \times \mathbf{B} = \mathbf{A},$$

which yields a vector field known as the curl of $\mathbf{B}$.

(4)

$$\begin{aligned}\nabla \cdot (\nabla f) &= (\nabla \cdot \nabla) f \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \nabla^2 f \\ &= \Delta f,\end{aligned}$$

which yields an operator known as the Laplacian.