Problem (Problem 1)**:** Let F be a field. Use JCF to prove that any square matrix $A \in \operatorname{Mat}_n(F)$ is similar to its transpose.

Solution: It suffices to prove this for one Jordan block, as we may then apply the same process for every other separate Jordan block to obtain our desired transformation in the general case. For this, we observe that if $J_d(\lambda)$ is a Jordan block, then it is the expression of a linear transformation $T\colon V \to V$ in terms of the ordered basis

$$\beta_1 = \left\{\overline{(x-\lambda)}^{d-1}, \dots, \overline{(x-\lambda)}, \overline{1}\right\},$$

when V is given the structure of an $F[x]$ module with T acting via $x.v = Tv$. Reversing the order of this basis gives

$$\beta_2 = \left\{\overline{1}, \overline{(x-\lambda)}, \dots, \overline{(x-\lambda)}^{d-1}\right\},$$

and we observe that acting via T gives a matrix representation with λ along the diagonal and 1 along the sub-diagonal, which is exactly $J_d(\lambda)^T$. Thus, via the change of basis matrix $P = [\mathrm{id}]_{\beta_1}^{\beta_2}$, we find that $J_d(\lambda) = P^{-1}J_d(\lambda)^T P$, so $J_d(\lambda)$ is similar to its transpose.

Problem (Problem 2)**:** Let G be a group, F a field, and V a vector space over F. Prove that there is a natural bijection between linear representations of G of the form (ρ, V) and $F[G]$-module structures on V that extend the given F-vector space structure on V.

Solution: Let (ρ, V) be a linear representation of G. We will determine an $F[G]$-module structure on V extending the given F-vector space structure on V by letting the basis $\{\delta_g\}_{g\in G} \subseteq F[G]$ act on vectors in V via

$$\delta_g \cdot v = \rho(g)v,$$

and extending linearly via the universal property of the free module $F[G]$. That this is a module over $F[G]$ follows from the fact that ρ is a homomorphism between G and $\mathrm{GL}(V)$, so for any $g, h \in G$ and $v, v_1, v_2 \in V$, we have

$$\begin{aligned}
\delta_g \cdot (v_1 + v_2) &= \rho(g)(v_1 + v_2)\\
&= \rho(g)v_1 + \rho(g)v_2\\
&= \delta_g \cdot v_1 + \delta_g \cdot v_2\\
(\delta_g + \delta_h) \cdot v &= (\rho(g) + \rho(h))v\\
&= \rho(g)v + \rho(h)v\\
&= \delta_g \cdot v + \delta_h \cdot v\\
(\delta_g\delta_h) \cdot v &= \delta_{gh} \cdot v\\
&= \rho(gh)v\\
&= \rho(g)\rho(h)v\\
&= \rho(g)(\rho(h)v)\\
&= \delta_g(\delta_h \cdot v).
\end{aligned}$$

Now, if we have an $F[G]$-module structure on V extending the F-vector space structure on V, then we claim that by defining $\rho\colon G \to \mathrm{GL}(V)$ by

$$\rho(g)v = \delta_g \cdot v,$$

then this defines a representation of G. We observe that $\delta_e \cdot v = v$ by the definition of the $F[G]$-module structure extending the F-vector space structure, that

$$\begin{aligned} v &= \delta_e \cdot v \\ &= \delta_g \delta_{g^{-1}} \cdot v \\ &= \delta_g \cdot \left(\delta_{g^{-1}} \cdot v\right), \end{aligned}$$

meaning that this is in fact a map into $\mathrm{GL}(V)$. Finally, we observe that if $g, h \in G$, then

$$\begin{aligned} \rho(gh)v &= \delta_{gh} \cdot v \\ &= \delta_g \delta_h \cdot v \\ &= \delta_g \cdot (\delta_h \cdot v) \\ &= \delta_g \cdot (\rho(h)v) \\ &= \rho(g)\rho(h)v, \end{aligned}$$

and since this holds for all $v \in V$, it follows that $\rho(gh) = \rho(g)\rho(h)$, giving that this is a homomorphism.

Problem (Problem 3)**:** This problem collects several related results, each of which may be referred to as Schur's Lemma.

(a) Let R be a ring, M and N irreducible (left) R-modules, and $f\colon M \to N$ a homomorphism of R-modules. Prove that f is either an isomorphism or the zero map.

(b) Let R be a ring and M an irreducible R-module. Prove that $\mathrm{end}_R(M)$ is a division ring.

(c) Let G be a group, $g \in Z(G)$, and (ρ, V) a linear representation over some field. Prove that for any $\lambda \in F$, the map $\rho(g) - \lambda I$ lies in $\mathrm{end}_{F[G]}(V)$.

(d) In the setting of (c), assume that F is algebraically closed and V is finite-dimensional and irreducible. Use (b) and (c) to prove that $\rho(g) = \lambda I$ for some $\lambda \in F$. In other words, if we are given a finite-dimensional irreducible representation of an algebraically closed field, then any central element must act as a scalar operator.

Solution:

(a) Suppose $f\colon M \to N$ is a nonzero homomorphism of irreducible R-modules M and N. Then, $\ker(f) \leq M$ is a submodule, and since $\ker(f)$ is not equal to all of M, it follows that $\ker(f) = \{0\}$, so f is injective. Similarly, $\mathrm{im}(f) \leq N$ is a nonzero submodule, so $\mathrm{im}(f) = N$, so f is an isomorphism.

(b) Observe that if $f \in \mathrm{end}_R(M)$, then we know from (a) that f is either the zero map or an isomorphism. In particular, every nonzero element of $\mathrm{end}_R(M)$ is invertible, so $\mathrm{end}_R(M)$ is a division ring.

(c) Suppose δ_h is any basis element of $F[G]$. We will show that $(\rho(g) - \lambda I)(\delta_h \cdot v) = \delta_h \cdot ((\rho(g) - \lambda I)v)$. We have

$$\begin{aligned} (\rho(g) - \lambda I)(\delta_h \cdot v) &= (\rho(g) - \lambda I)(\rho(h)v) \\ &= (\rho(gh) - \lambda \rho(h))v \\ &= (\rho(hg) - \lambda \rho(h))v \\ &= \rho(h)(\rho(g) - \lambda I)v \end{aligned}$$

$$= \delta_h \cdot ((\rho(g) - \lambda I)v).$$

Thus, this is an $F[G]$-module endomorphism.

(d) If (ρ, V) is an irreducible representation of G over an algebraically closed field F, then if $g \in Z(G)$, for all $\lambda \in F$, we have $\rho(g) - \lambda I$ is in $\text{end}_{F[G]}(V)$. Since $\text{end}_{F[G]}(V)$ is a division ring by (b), it follows that for all $\lambda \in F$, either $\rho(g) - \lambda I$ is zero or it is invertible. From linear algebra, we know that every linear map on a finite-dimensional vector space over an algebraically closed field admits an eigenvalue, so there is $\lambda \in F$ such that $\rho(g) - \lambda I = 0$, so $\rho(g) = \lambda I$.

Problem (Problem 4)**:** Let R be a ring, and M an R-module. We will say that M has the complement property if, for every submodule N of M, there exists a submodule P such that $M = N \oplus P$. From a theorem in class, M has the complement property if and only if M is completely reducible.

(a) Suppose $M = P \oplus Q$ for some submodules P and Q. Prove that if N is any submodule containing P, then $N = P \oplus (N \cap Q)$.

(b) Deduce from (a) that if M has the complement property, then so does any submodule of M.

(c) Now prove Lemma 11.2 from class, which asserts that if M has the complement property, then any nonzero submodule L of M contains an irreducible submodule.

Solution:

(a) Since $M = P \oplus Q$, we have that $M/P \cong Q$ via the short exact sequence $0 \to P \to M \to Q \to 0$. Furthermore, since N contains P, the fourth isomorphism theorem gives that N/P is a submodule of M/P with $N/P \cong N \cap Q$. Therefore, via the short exact sequence $0 \to P \to N \to N \cap Q \to 0$, we observe that $N \cong P \oplus N \cap Q$.

(b) Suppose M has the complement property. If N is any submodule of M, then either N contains no nonzero submodules, so it has the complement property via $N = N \oplus 0$, or N contains a nonzero submodule, which we will call P. Since P is complemented in M, there is Q such that $M = P \oplus Q$. Yet, this means that $N = P \oplus (N \cap Q)$, so N has the complement property.

(c) Suppose M has the complement property, and let $N \subseteq M$ be a submodule, with $x \in N$. Then, the submodule generated by x, $L = \langle x \rangle$ also has the complement property by (b). Since $Rx \cong R$ by forgetting the x, so any submodule of Rx can be viewed as any submodule of R viewed as an R-module.

Since the submodules of R are the ideals, then R has a maximal ideal P; the corresponding submodule is $K = Px$, and does not contain x as P does not contain 1. Since L has the complement property, we have $L = K \oplus B$, where B is another submodule of B. Thus, we have $L/K = B$, and we observe that $L/K = Rx/Px \cong R/P$. Since R/P is a field, it follows that R/P has no R-submodules(/ideals) other than 0 and the whole space. Thus, B has no submodules; since $x \notin K$, it follows that $x \in B$, so B is an irreducible submodule of L, which is a submodule of N. Thus, every submodule of M has an irreducible submodule.

Problem (Problem 5)**:** Let $n \in \mathbb{N}$, and let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Let S_n be the symmetric group over $[n]$. Let F be any field, (ρ, V) the permutation representation of S_n over F corresponding to the defining action of S_n on $[n]$. If we think of $V = F^n$, then $\rho \colon S_n \to \text{GL}_n(F)$ is given by $\rho(\sigma)(e_i) = e_{\sigma(i)}$.

(a) Let

$$Z = F(e_1 + \cdots + e_n)$$
$$W = \left\{ (x_1, \dots, x_n) \in F^n \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\}.$$

Prove that Z and W are both subrepresentations of V.

(b) Prove that $V = W \oplus Z$ if and only if $\operatorname{char}(F)$ does not divide n.

(c) Assume that $\operatorname{char}(F)$ does not divide $n!$. Prove that W is an irreducible representation of S_n.

Solution:

(a) If σ is any permutation, then we observe that

$$e_1 + \cdots + e_n = e_{\sigma(1)} + \cdots + e_{\sigma(n)},$$

so that Z is invariant under the representation of ρ. Similarly, we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_{\sigma(i)},$$

so W is invariant under the representation of ρ. Furthermore, both Z and W are subspaces, the former by definition and the latter since for any $\lambda \in F$, we have

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n),$$

so if $(x_1, \dots, x_n) \in W$, so too is $(\lambda x_1, \dots, \lambda x_n)$, and we can split sums to find that W is a subspace.

(b) Consider the (linear) map

$$\varphi\colon V \to F$$

given by

$$\varphi((x_1, \dots, x_n)) = x_1 + \cdots + x_n.$$

If the characteristic of F does not divide n, then we have that $\ker(\varphi) = W$, and $\operatorname{im}(\varphi) = F \cong Z$ (since the basis vector for Z does not map to 0 under φ), so that $V = W \oplus Z$.

Now, suppose the characteristic of F divides n. In that case, we observe that $e_1 + e_2 + \cdots + e_n \mapsto 0$ under φ, meaning that $Z \subseteq \ker(\varphi)$, or that $Z \subseteq W$, meaning that the sum $V + W$ cannot be direct.

(c) Let $\beta(w)$, the weight of w, denote the number of nonzero coordinates of w. Observe that if $\beta(w) > 2$, then there is a transposition $\tau \in S_n$ that leaves at least one coordinate invariant; therefore, we would necessarily have $\beta(\rho(\tau)w - w)$ has weight smaller than w.

Now, suppose $0 \neq w \in X \leq W$ is a nonzero S_n-invariant subspace. Either the weight of w is 2 or is greater than 2, since if there is any nonzero entry in w, then there has to be at least one more nonzero entry in w such that the sum of the weights cancels out.

If the weight of w is greater than 2, then from the earlier discussion, there is a transposition τ leaving at least one nonzero coordinate invariant. Since the characteristic of F does not divide n, it follows that $Z \nsubseteq W$, so there are coordinates with distinct values, meaning that if we select τ to transpose these two coordinates, then $\beta(\rho(\tau)w - w) = 2$. By the assumptions of X being a subspace and X being S_n-invariant, it follows that $\rho(\tau)w - w$ is contained in X.

Now, let w be a weight 2 vector in X. Then, since the sum of all the entries in w is equal to 0, we must have that the nonzero entries at indices i and j are equal to r and $-r$ for some $r \in F$. Scaling, we then find a vector $w' \in X$ with the entries at indices i and j equal to 1 and -1 respectively. Then, by using a transposition, we may assume that $j = 1$ and $i = 2$; by using a series of transpositions, we find a linearly independent subset of vectors $u_2, \dots, u_n$ in X that have -1 in the first entry and 1 in position i for the vector u_i.

Yet, since W has dimension equal to $n - 1$, since the characteristic of F does not divide $n!$, and hence does not divide n, we must have that $\{u_2, \dots, u_n\}$ is a basis for W as dimension is invariant, so $X = W$.

Problem (Problem 6)**:** Let R be a commutative ring with 1, M_1, M_2, N_1, N_2 R-modules. Let $\varphi\colon M_1 \to M_2$ and $\psi\colon N_1 \to N_2$ be homomorphisms of R-modules.

(a) Prove that there exists a unique linear map $\varphi \otimes \psi\colon M_1 \otimes N_1 \to M_2 \otimes N_2$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$ for all $m \in M_1$ and $n \in N_1$.

(b) Now assume R is a field, M_1, M_2, N_1, N_2 are finite-dimensional vector spaces, and choose bases α_i for M_i and β_i for N_i for each $i = 1, 2$. Note that $\gamma_i = \alpha_i \otimes \beta_i$ is a basis for $M_i \otimes N_i$. Consider the matrices $A = [\varphi]_{\alpha_1}^{\alpha_2}$, $[\psi]_{\beta_1}^{\beta_2}$, and $C = [\varphi \otimes \psi]_{\gamma_1}^{\gamma_2}$. Prove that if γ_1 and γ_2 are ordered in a suitable way, then C is the Kronecker product of A and B,

$$C = \begin{pmatrix} a_{11}B & \cdots & a_{1\ell}B \\ \vdots & \ddots & \vdots \\ a_{k1}B & \cdots & a_{k\ell}B \end{pmatrix}.$$

(c) In the setting of (b), assume that $M_1 = M_2$ and $N_1 = N_2$. Prove that $\operatorname{tr}(\varphi \otimes \psi) = \operatorname{tr}(\varphi)\operatorname{tr}(\psi)$.

Solution:

(a) Consider the map $\varphi \times \psi\colon M_1 \times N_1 \to M_2 \otimes N_2$, given by $(m, n) \mapsto \varphi(m) \otimes \psi(n)$. Then, since φ and ψ are homomorphisms, we have, for any $m_1, m_2 \in M$ and $n \in N$,

$$\begin{aligned}(\varphi \times \psi)(m_1 + m_2, n) &= \varphi(m_1 + m_2) \otimes \psi(n) \\ &= (\varphi(m_1) + \varphi(m_2)) \otimes \psi(n) \\ &= \varphi(m_1) \otimes \psi(n) + \varphi(m_2) \otimes \psi(n) \\ &= (\varphi \times \psi)(m_1, n) + (\varphi \times \psi)(m_2, n),\end{aligned}$$

and analogously for the second coordinate. Additionally, if $r \in R$, then

$$\begin{aligned}(\varphi \times \psi)(rm, n) &= \varphi(rm) \otimes \psi(n) \\ &= r\varphi(m) \otimes \psi(n) \\ &= \varphi(m) \otimes r\psi(n) \\ &= \varphi(m) \otimes \psi(rn)\end{aligned}$$

$$= (\varphi \times \psi)(m, rn).$$

Thus, by the universal property, we obtain a unique linear map $\varphi \otimes \psi \colon M_1 \otimes N_1 \to M_2 \otimes N_2$ given by

$$(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n).$$