Math 395: Homework 4
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Problem 3

Problem: Let

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 12 & 0 & 1 \end{pmatrix}.$$

Calculate $m_T(x)$ and determine the eigenvalues of A.

Solution. We calculate A^2 and A^3[I] to to find

$$A^2 = \begin{pmatrix} -8 & 0 & -3 \\ 2 & 1 & -2 \\ 36 & 0 & 11 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -52 & 0 & 5 \\ -18 & -1 & -4 \\ -60 & 0 & -47 \end{pmatrix}.$$

This yields a minimal polynomial[II] of

$$m_T(x) = x^3 - 2x^2 + 11x + 14.$$

Factoring this polynoimal over $\mathbb{R}$ yields

$$m_T(x) = (x+1)\left(x^3 - 3x + 14\right),$$

which means we need to find the eigenvalues in $\mathbb{C}$. Thus, we get

$$m_T(x) = (x+1)\left(x - \left(\frac{3}{2} + i\frac{\sqrt{47}}{2}\right)\right)\left(x - \left(\frac{3}{2} - i\frac{\sqrt{47}}{2}\right)\right).$$

The eigenvalues are -1, $\frac{3}{2} - i\frac{\sqrt{47}}{2}$, and $\frac{3}{2} + i\frac{\sqrt{47}}{2}$.

Problem 15

Problem: Let $A \in \mathrm{Mat}_n(\mathbb{F})$.

(a) Assume A has eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that $\det(A) = \lambda_1 \cdots \lambda_n$ and $\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n$.
(b) Suppose A does not have n distinct eigenvalues, but $c_A(x)$ splits into linear factors over F. Can you characterize the determinant and trace of A in terms of the eigenvalues?

Solution.

(a) If $A \in \mathrm{Mat}_n(\mathbb{F})$ has distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then there exists $P \in GL_n(\mathbb{F})$ such that

$$A = P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)P^{-1},$$

where $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ denote the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ at entries $a_{11}, \dots, a_{nn}$. In particular, this means

$$\begin{aligned} \det(A) &= \det\left(P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)P^{-1}\right) \\ &= \det\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right) \\ &= \prod_{j=1}^{n} \lambda_j, \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(A) &= \mathrm{tr}\left(P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)P^{-1}\right) \\ &= \mathrm{tr}\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right) \\ &= \sum_{j=1}^{n} \lambda_j. \end{aligned}$$

(b) If $c_A(x)$ splits into linear factors over F, then the Jordan canonical form for A exists, with each of its Jordan blocks consisting of the roots of $c_A(x)$ with multiplicity.[III] Thus, we can characterize $\mathrm{tr}(A)$ to be the sum of the roots of $c_A(X)$ with multiplicity,

[I] with help from Mathematica
[II] with help from Mathematica
[III] Assistance from Wikipedia

and det(A) to be the product of the roots with multiplicity.

Problem 17

Problem: Prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a matrix $A \in \mathrm{Mat}_n(\mathbb{F})$, the $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues for A^k for any $k \geqslant 0$.

Solution. Since A has eigenvalues $\lambda_1, \ldots, \lambda_n$, it is the case that there exists some $P \in \mathrm{GL}_n(\mathbb{F})$ such that

$$A = P\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_k)\right)P^{-1}.$$

For $k = 0$, we have

$$\begin{aligned} A^0 &= \left(P\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)P^{-1}\right)^0 \\ &= I_n \\ &= P\left(\operatorname{diag}\left(\lambda_1^0, \ldots, \lambda_n^0\right)\right)P^{-1}, \end{aligned}$$

meaning $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues for A^k.

For $k > 0$, we have

$$\begin{aligned} A^k &= \underbrace{\left(P\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)P^{-1}\right)\left(P\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)P^{-1}\right)\cdots\left(P\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)P^{-1}\right)}_{k\text{ times}} \\ &= P\underbrace{\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)\cdots\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\right)}_{k\text{ times}}P^{-1} \\ &= P\left(\operatorname{diag}\left(\lambda_1^k, \ldots, \lambda_n^k\right)\right)P^{-1}, \end{aligned}$$

meaning $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues for A^k.

Problem 24

Problem: Prove that any matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ satisfying $A^3 = A$ can be diagonalized. Is this true of any field $\mathbb{F}$? If so, prove it. If not, provide a counterexample.

Solution. We have

$$\begin{aligned} m_A(x) &= x^3 - x \\ &= x(x-1)(x+1), \end{aligned}$$

meaning that it is diagonalizable, with eigenvalues of 0, 1, and -1.

However, it is not the case that this is true for every field. For instance, in $\mathbb{F}_2$, we have $x - 1 = x + 1$, meaning

$$m_A(x) = x(x+1)^2$$

over $\mathbb{F}_2$, which does not yield distinct linear factors.