Problem 1

Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Show that the following are equivalent:

(i) c is a limit point of D.

(ii) There is a sequence $(x_n)_n$ in $D \setminus \{c\}$ with $(x_n)_n \to c$.

($\Rightarrow$) Let c be a limit point of D. Then, taking $\delta_n = 1/n$, let $x_n \in \dot{V}_{\delta_n}(c)$. Then, $(x_n)_n \to c$.

($\Leftarrow$) Let $(x_n)_n$ be a sequence in $D \setminus \{c\}$ with $(x_n)_n \to c$.

Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ with, $\forall n \geq N$, $|x_n - c| < \varepsilon$. Thus, $\forall \varepsilon > 0$, $\exists x_n$ such that $x_n \in \dot{V}_\varepsilon(c)$. Thus, c is a limit point.

Problem 2

Show that f can have at most one limit at c.

Suppose toward contradiction that $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, where $L_1 \neq L_2$. Then, $\exists \varepsilon_0 > 0$ such that $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$.

Let δ_1 be such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon_0$, and δ_2 be such that $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon_0$. Set $\delta = \min(\delta_1, \delta_2)$.

Then, $|x - c| < \delta \Rightarrow |f(x) - L_1| < \varepsilon_0$ and $|x - c| < \delta \Rightarrow |f(x) - L_2| < \varepsilon_0$. So, $\exists k$ such that $f(k) \in V_\varepsilon(L_1)$ and $f(k) \in V_\varepsilon(L_2)$. $\perp$

Problem 3

Show that the following are equivalent:

(i) $\lim_{x\to c} f(x) = L$

(ii) For every sequence $(x_n)_n$ in $D \setminus \{c\}$ such that $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

($\Rightarrow$) Let $\lim_{x\to c} f(x) = L$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

So, $\forall \varepsilon > 0$, $\exists f(x_k) \in V_\varepsilon(L)$, such that $x_k \in \dot{V}_\delta(c)$. So, we have a sequence $(x_n)_n \to c$ defined by $\delta(\varepsilon, c)$, where $(f(x_n))_n \to L$.

($\Leftarrow$) Assume toward contradiction that $\lim_{x\to c} f(x) \neq L$. Then, $\exists \varepsilon_0$ such that $\forall \delta > 0$, $\exists x \in \dot{V}_\delta(c) \cap D$ such that $|f(x) - L| > \varepsilon_0$.

Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in \dot{V}_{1/n}(c) \cap D$ with $|f(x_n) - L| > \varepsilon_0$.

Since $0 < |x - c| < 1/n$, $(x_n)_n \in D \setminus \{c\}$ and $(x_n)_n \to c$, meaning $(f(x_n))_n \to L$. However, $|f(x_n) - L| > \varepsilon_0$. $\perp$

Problem 4

If $\lim_{x\to c} f = L$ exists, show that there is a $\delta > 0$ such that

$$\sup_{x\in \dot{V}_\delta(c)} |f(x)| < \infty$$

Let $\varepsilon = 1$. Then, $\exists \delta > 0$ such that $\forall x \in \dot{V}_\delta(c)$, $|f(x) - L| < 1$. Therefore,

$$\begin{aligned}
|f(x)| &= |f(x) - L + L| \\
&\leq |f(x) - L| + |L| && \text{Triangle Inequality} \\
&< 1 + |L|
\end{aligned}$$

So,

$$\sup_{x\in \dot{V}_\delta(c)} |f(x)| \leq 1 + |L|$$

Problem 5

Establish the following limits:

(a)

$$\lim_{x\to 1} \frac{3x}{1+x} = \frac{3}{2}$$

Preliminary Work: Let $\varepsilon > 0$.

$$\left|\frac{3x}{1+x} - \frac{3}{2}\right| = \frac{3|x-1|}{2|x+1|}$$

If $x \in (0,2)$, or $|x-1| < 1$, then

$$\begin{aligned}
\frac{3|x-1|}{2|x+1|} &< \frac{3}{2}|x-1| \\
&< \varepsilon
\end{aligned}$$

Proof: Given $\varepsilon > 0$, let $\delta = \frac{1}{2}\min\left(1, \frac{2}{3}\varepsilon\right)$. Then,

$$\begin{aligned}
0 < |x - c| &< \delta \\
\left|\frac{3x}{1+x} - \frac{3}{2}\right| &< \frac{3}{2}|x-1| \\
&< \frac{3}{2}\frac{2}{3}\varepsilon \\
&= \varepsilon
\end{aligned}$$

(b)

$$\lim_{x\to 6}\frac{x^2-3x}{x+3}=2$$

Preliminary Work: Let $\varepsilon > 0$.

$$\begin{aligned}\left|\frac{x^2-3x}{x+3}-2\right| &= \left|\frac{x^2-3x-2(x+3)}{x+3}\right| \\ &= \left|\frac{x^2-5x-6}{x+3}\right| \\ &= \frac{|x+1|}{|x-3|}|x-6|\end{aligned}$$

for $|x-6|<1$, we have

$$\begin{aligned}&< 3|x-6| \\ &< \varepsilon\end{aligned}$$

Proof: Let $\varepsilon > 0$, and let $\delta = \frac{1}{2}\min\left(1, \frac{\varepsilon}{3}\right)$. Then,

$$\begin{aligned}0 < |x-6| &< \delta \\ \left|\frac{x^2-3x}{x+3}-2\right| &< 3|x-6| \\ &< 3\frac{\varepsilon}{3} \\ &= \varepsilon\end{aligned}$$

(c)

$$\lim_{x\to 0}\mathbb{1}_{\mathbb{Q}}=0$$

Let $(x_n)_n$ be a sequence defined by $\frac{1}{n}$, and let $(y_n)_n$ be a sequence defined by $\frac{1}{n\sqrt{2}}$. Then,

$$\begin{aligned}(x_n)_n &= (1,1,1,\dots) \\ (y_n)_n &= (0,0,0,\dots) \\ (z_n)_n &:= (x_1,y_1,x_2,y_2,\dots) \\ &= (1,0,1,0,\dots)\end{aligned}$$

Then, $(z_n)_n$ contains two subsequences, namely $(x_n)_n$ and $(y_n)_n$ that converge to two different values (1 and 0 respectively). Therefore $\lim_{x\to 0}\mathbb{1}_{\mathbb{Q}}$ does not exist.

(d)

$$\lim_{x\to 0} \frac{x^2}{|x|} = 0$$

Let $(x_n)_n$ be a sequence such that $(x_n)_n \to 0$ and $x_n \neq 0\ \forall n \in \mathbb{N}$. Then,

$$\begin{aligned} f(x_n) &= \frac{x_n^2}{|x_n|} \\ &= \frac{|x_n|^2}{|x_n|} \\ &= |x_n| \\ &\to 0 \end{aligned}$$

Problem 6

For which values of $k = 0, 1, 2, \dots$ does

$$\lim_{x\to 0} x^k \sin(1/x)$$

exist?

$k = 0$: Suppose $k = 0$. Let $(a_n)_n \in (0,1)$ be a sequence defined by $a_n = \frac{2}{(4n+1)\pi}$, and let $(b_n)_n \in (0,1)$ be a sequence defined by $\frac{1}{\pi n}$. Then,

$$(f(a_n))_n = (1, 1, 1, \dots),$$

and

$$(f(b_n))_n = (0, 0, 0, \dots),$$

meaning that $(f(a_n))_n \to 1$ and $(f(b_n))_n \to 0$. Let $(c_n)_n = (a_1, b_1, a_2, b_2, \dots)$. Then, $(f(c_n))_n$ has a subsequence $(f(a_n))_n \to 1$ and a subsequence $(f(b_n))_n \to 0$. Therefore, $(f(c_n))_n$ is divergent, meaning the limit does not exist.

$k \neq 0$: Suppose $k \neq 0$. Let $(x_n)_n$ be an arbitrary sequence in $D \setminus \{0\}$ such that $(x_n)_n \to 0$. Then,

$$\begin{aligned} |f(x_n)| &= \left| x_n \sin\left(\frac{1}{x_n}\right) \right| \\ &\leq |x_n| \\ &\to 0 \end{aligned}$$

meaning $(f(x_n))_n \to 0$.

Problem 7

Assume $f(x) \geq 0$ for all $x \in D$ and suppose $\lim_{x\to c} f =: L$ exists. Show that $L \geq 0$ and

$$\lim_{x\to c} \sqrt{f} = \sqrt{L}$$

Let $(x_n)_n \in D \setminus \{c\}$ such that $(x_n)_n \to c$. Then, $(f(x_n))_n \to L$, by the sequential definition of limits. Since $f(x_n) \geq 0$ for all x_n, by the properties of sequences, it must be the case that $L \geq 0$.

Similarly, it must be the case that $\left(\sqrt{f(x_n)}\right)_n \to \sqrt{L}$ by the properties of sequences — meaning that $\lim_{x\to c} \sqrt{f} = \sqrt{L}$.

Problem 8

Assume $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $\lim_{x\to 0} f := L$ exists, show that $L = 0$ and show that $\lim_{x\to c} f$ exists for all $c \in \mathbb{R}$.

Part 1: Let $(x_n)_n \in \mathbb{R} - \{0\}$, $(x_n)_n \to 0$. Then, since $f(x+y) = f(x) + f(y)$ and f is defined on $\mathbb{R}$, we have

$$\begin{aligned} f(x_n) &= f(0) + f(x_n) \\ 0 &= f(0), \end{aligned}$$

meaning $(f(x_n))_n \to f(0) = 0$.

Part 2: Let $(x_n)_n \to c$. Then, $(x_n - c)_n \to 0$. So,

$$\begin{aligned} f(x_n) &= f(x_n - c + c) \\ &= f(x_n - c) + f(c) \\ &\to f(c) \end{aligned}$$

Problem 9

Let $f : (0,1) \to \mathbb{R}$ be a bounded function such that $\lim_{x\to 0} f$ does not exist. Show that there are two sequences $(x_n)_n$ and $(y_n)_n$ with $(x_n)_n \to 0$, $(y_n)_n \to 0$, and $(f(x_n))_n$ and $(f(y_n))_n$ are both convergent, but with different limits.

Since $\lim_{x\to 0} f$ does not exist, $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_0 \in (0,1)$ such that $|f(x_0) - L| \geq \varepsilon_0$.

Let $\delta_{x,n} = \frac{1}{n}$, and let $\delta_{y,n} = \frac{1}{n^2}$. Select $x_n \in (0, \delta_{x,n})$, and $y_n \in (0, \delta_{y,n})$ for each n, where $x_n \neq y_n$. Set L_x and L_y, where $L_x \neq L_y$ such that $|f(x_n) - L_x| \geq \varepsilon_0$, and $|f(y_n) - L_y|$ $\forall x_n, y_n$.

Since f is bounded, $a \leq f(x_n) \leq b$ and $c \leq f(y_n) \leq d$. Then, $\exists n_j, n_k$ such that $\left(f(x_{n_j})\right) \to a$ and $(f(y_{n_k})) \to d$.

With proper selection of x_n and y_n, we find that $(x_{n_j})_j \to 0$, $(y_{n_k})_k \to 0$, and the image of these sequences converges to different points.

Problem 10

Suppose $f : (0,\infty) \to \mathbb{R}$. Show that the following are equivalent:

(i) $\lim\limits_{x\to\infty} f = L$

(ii) For every sequence $(x_n)_n$ in $(0,\infty)$ with $(x_n)_n \to \infty$, we have $(f(x_n))_n \to L$.

($\Rightarrow$) Let $\lim_{x\to\infty} f = L$. Then, $\forall \varepsilon > 0$, $\forall k > 0$, $\exists x \geq k$ such that $f(x) \in V_\varepsilon(L)$.

Selecting x_n such that $x_n > k$, we have $(x_n)_n \to +\infty$, and $f(x_n) \in V_\varepsilon(L)$.

($\Leftarrow$) Assume $\lim_{x\to\infty} f \neq L$. Then, $(\exists \varepsilon_0)(\forall k > 0)(\exists x > k)$ such that $|f(x) - L| > \varepsilon_0$. Let $k = n$. Then, $\exists x_n > n$ with $|f(x_n) - L| \geq \varepsilon_0$.

Since $(x_n)_n \to +\infty$, it must be the case by (ii) that $(f(x_n))_n \to L$. However, $|f(x_n) - L| \geq \varepsilon_0$. $\perp$

Problem 11

If $f : (a, \infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} xf(x) =: L$ exists, show that

$$\lim_{x\to\infty} f(x) = 0.$$

Let $(x_n)_n \to +\infty$ where $(x_n)_n \in (a, \infty)$. Then, it must be the case that $(x_n f(x_n))_n \to L$. So, for $\varepsilon > 0$,

$$\begin{aligned}
|x_n f(x_n) - L| &< \varepsilon \\
|f(x_n)| &= \frac{|x_n f(x_n)|}{|x_n|} \\
&= \frac{|x_n f(x_n) - L + L|}{|x_n|} \\
&\leq \frac{|x_n f(x_n) - L|}{|x_n|} + \frac{|L|}{x_n} \\
&< \frac{\varepsilon}{N} + \frac{|L|}{N} && \text{for } N \text{ large, by the Archimedean Property} \\
&< \varepsilon,
\end{aligned}$$

meaning $|f(x_n)| \to 0$.

Problem 12

Suppose $f, g : (0, \infty) \to \mathbb{R}$ are such that $\lim_{x\to\infty} f =: L > 0$, and $\lim_{x\to\infty} g = \infty$. Show that $\lim_{x\to\infty} fg = \infty$. Does this hold if $L = 0$?

Let $(x_n)_n \to \infty$. Then, $\forall M > 0$, $\exists N_1$ large such that $n \geq N \Rightarrow g(x_n) > M$, and $\exists N_2$ large such that $n \geq N_2 \Rightarrow |f(x_n) - L| < \varepsilon$. Let $N = \max(N_1, N_2)$.

I don't know how to commence further on the problem.