

Problem (Problem 1): Let $(a_n)_n$ be a sequence for which $\sum_{n=0}^{\infty} |a_n|^2$ is finite. For each positive N , define $f_N(z) = \sum_{n=0}^{\infty} a_n z^n$, and define $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(a) Show that f is holomorphic on \mathbb{D} .

(b) For each $r \in (0, 1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi} \int_0^{2\pi} |f_N(re^{i\theta})|^2 d\theta.$$

(c) For each $r \in (0, 1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

(d) Determine in terms of $(a_n)_n$ the limit

$$\lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Solution:

(a) Let $0 < r < 1$. Since each f_N is analytic, we can use the Cauchy Integral Formula to compute a_N explicitly, yielding

$$\begin{aligned} |a_N| &= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_N(\xi)}{\xi^{N+1}} d\xi \right| \\ &\leq \frac{1}{r^N} \sup_{|z|=r} |f_N(z)|. \end{aligned}$$

Therefore, if we are able to show that the value

$$\sup_{|z|=r} |f_N(z)|$$

is uniformly bounded by a constant for all N , we will be able to use the Cauchy–Hadamard theorem to show that $\limsup_{N \rightarrow \infty} |a_N|^{1/N} \leq 1$. Toward this end, we use the Cauchy–Schwarz inequality, which yields

$$\begin{aligned} \sup_{|z|=r} |f_N(z)| &= \sup_{|z|=r} \left| \sum_{n=0}^N a_n z^n \right| \\ &\leq \sup_{|z|=r} \left(\sum_{n=0}^N |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^N |z|^{2n} \right)^{1/2} \\ &\leq \sup_{|z|=r} \underbrace{\left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}}_{=:K} \left(\sum_{n=0}^{\infty} |z|^{2n} \right)^{1/2} \\ &= \frac{K}{(1 - |r|^2)^{1/2}}. \end{aligned}$$

Since we have established this uniform bound, we thus find that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence at least 1, so f is analytic on \mathbb{D} , hence holomorphic on \mathbb{D} .

(b) We write out the integral to yield

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f_N(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^N a_n r^n e^{in\theta} \right) \overline{\left(\sum_{m=0}^N a_m r^m e^{im\theta} \right)} d\theta \\ &= \frac{1}{2\pi} \sum_{n=0}^N \sum_{m=0}^N a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\ &= \sum_{n=0}^N |a_n|^2 r^{2n}. \end{aligned}$$

(c) Since f is holomorphic with radius of convergence at least 1, the series expression on $S(0, r)$ converges uniformly, so that we may exchange sum and integral. This yields

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \end{aligned}$$

(d) Since the sequence $(a_n)_n$ is square-summable, the limit is well-defined, and we get

$$\begin{aligned} \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \lim_{r \nearrow 1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \\ &= \sum_{n=0}^{\infty} |a_n|^2. \end{aligned}$$

Problem (Problem 2): Let $\varphi: [0, 1] \rightarrow \mathbb{C}$ be continuous, and define $f: \mathbb{C} \setminus [0, 1] \rightarrow \mathbb{C}$ by

$$f(z) = \int_0^1 \frac{\varphi(t)}{t-z} dt.$$

Show that f is holomorphic and determine the derivative of f in terms of φ .

Problem (Problem 3): Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be entire.

- (a) Suppose there exist $C, R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leq C|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree at most n .
- (b) Suppose that $g: \mathbb{C} \rightarrow \mathbb{C}$ is also entire and $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists some $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ such that $f(z) = \alpha g(z)$ for all $z \in \mathbb{C}$.
- (c) Suppose that there exists some $\theta \in \mathbb{R}$ such that $f(\mathbb{C}) \cap \{re^{i\theta} \mid r > 0\} = \emptyset$. Show that f is constant.

Solution:

(a) Let $r > R$. Then, by the Cauchy estimate, we get that

$$\begin{aligned} |f^{(n+1)}(0)| &\leq \frac{(n+1)!}{r^{n+1}} \sup_{|z|=r} |f(z)| \\ &\leq \frac{(n+1)!}{r^{n+1}} \sup_{|z|=r} (C|z|^n) \\ &= \frac{C(n+1)!}{r}, \end{aligned}$$

so since r is arbitrary and f is entire, we find that $f^{(n+1)}(0) = 0$, so that the power series expansion of f about 0 terminates beyond $n + 1$, meaning that f is a polynomial of degree at most n .

- (b) If g is 0, or f is 0, we are done. Else, assume that g and f are not identically zero. Observe that if g is everywhere non-vanishing, then the function $\frac{f(z)}{g(z)}$ is entire, and satisfies

$$\left| \frac{f(z)}{g(z)} \right| \leq 1,$$

hence $\frac{f(z)}{g(z)} = \alpha$ for some α with $|\alpha| \leq 1$.

If $k(z) = \frac{f(z)}{g(z)}$ is such that $g(z)$ admits zeros, then they must be isolated zeros, or else by the identity theorem, g would be identically zero everywhere. Let a be one of these zeros for g . If ε is small, we observe that for $0 < |z - a| < \varepsilon$, $k(z)$ is bounded. If we let M be this bound, then we observe that the value

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{|\zeta - a| = \varepsilon} \frac{k(\zeta)}{\zeta - a} d\zeta \right| &\leq \frac{1}{2\pi} \int_{|\zeta - a| = \varepsilon} \frac{|k(\zeta)|}{|\zeta - a|} |d\zeta| \\ &\leq \frac{1}{2\pi} \int_{|\zeta - a| = \varepsilon} \frac{M}{|\zeta - a|} |d\zeta| \\ &= \frac{M}{\varepsilon}, \end{aligned}$$

so that the integral is well-defined. In particular, this means that we may define a holomorphic extension of $k(z)$ by

$$h(z) = \begin{cases} k(z) & g(z) \neq 0 \\ \frac{1}{2\pi i} \int_{|\zeta - z| = \varepsilon} \frac{k(\zeta)}{\zeta - z} d\zeta & g(z) = 0. \end{cases}$$

The function $h(z)$ is thus entire, and bounded by 1, so by Liouville's theorem, $h(z) = \alpha$ for some α with $|\alpha| \leq 1$. This means that whenever $g(z) \neq 0$, we have $f(z) = \alpha g(z)$, and clearly when $g(z) = 0$, we have $f(z) = \alpha g(z)$, so that $f(z) = \alpha g(z)$.

- (c) Let f be such that $f(\mathbb{C}) \cap \{re^{i\theta} \mid r > 0\} = \emptyset$. By adding a sufficient multiple of $2\pi k$, we may assume $\theta > 0$.

Define a branch of the logarithm $\log_\theta(z)$ by taking

$$S_\theta = \{z \in \mathbb{C} \mid \theta < \operatorname{Im}(z) < \theta + 2\pi\}.$$

Then, we observe that $\sqrt{z} = e^{\frac{1}{2} \log_\theta(z)}$ maps S_θ to the set

$$\mathbb{H}_\theta = \{z \mid \theta/2 < \arg(z) < \theta/2 + \pi\} \cup \{0\}.$$

Observe that the map $z \mapsto e^{-i\theta/2} z$ is entire, and the Cayley transform, $\varphi(w) = \frac{w-i}{w+i}$ is holomorphic on $\mathbb{C} \setminus \{-i\}$, maps $0 \mapsto -1$, and maps the open upper half-plane to the unit disk. Therefore, if we fix some $\varepsilon > 0$, we find that the composition

$$\varphi \circ \left(z \mapsto e^{-i\theta/2} z \right) \circ \sqrt{\cdot} \circ f: \mathbb{C} \rightarrow \mathcal{U}(0, 1 + \varepsilon)$$

is an entire function that is bounded in modulus by $1 + \varepsilon$. In particular, since all of $\sqrt{\cdot}$, $z \mapsto e^{-i\theta/2} z$, and φ are nonconstant, this implies that f is constant.

Problem (Problem 4): Let $U = \{z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1\}$. Suppose $f: U \rightarrow \mathbb{C}$ is holomorphic, and there exists $C > 0$ and $\eta \in \mathbb{R}$ such that

$$|f(z)| \leq C(1 + |z|)^\eta$$

for all $z \in U$. Show that for each $n \geq 0$, there exists a constant $C_{n,\eta} \geq 0$ dependent only on n and η such that

$$|f^{(n)}(x)| \leq C_{n,\eta}(1 + |x|)^\eta$$

for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$, $0 < r < 1$, and to start, assume $\eta \geq 0$. Then, from Cauchy's estimate, a bunch of triangle inequalities, and the fact that $\eta \geq 0$ and $r < 1$, we find that

$$\begin{aligned} |f^{(n)}(x)| &\leq \frac{n!}{r^n} \sup_{|w-x|=r} |f(w)| \\ &\leq \frac{n!}{r^n} \sup_{|w-x|=r} (C(1 + |w|)^\eta) \\ &\leq \frac{Cn!}{r^n} \sup_{|w-x|=r} \left(1 + \left|w - \frac{3}{2}x\right| + \frac{3}{2}|x|\right)^\eta \\ &\leq \frac{Cn!}{r^n} \sup_{|w-x|=r} (1 + |w - x| + 2|x|)^\eta \\ &\leq \frac{Cn!}{r^n} \sup_{|w-x|=r} (1 + r + 2|x|)^\eta \\ &\leq \frac{Cn!}{r^n} (2 + 2|x|)^\eta \\ &\leq \frac{C2^\eta n!}{r^n} (1 + |x|)^\eta. \end{aligned}$$

In particular, since this inequality holds for every $0 < r < 1$, it necessarily for $r = 1$, so that $C_{n,\eta} = C2^\eta n!$.

Now, if $\eta < 0$.

Problem (Problem 5): Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$, where $a_0, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$.

- Show that there exist n complex numbers $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ not necessarily distinct such that $P(z) = a_n(z - \alpha_1) \cdots (z - \alpha_n)$.
- Suppose $|\alpha_0| > |\alpha_n|$. Show that there exists some $\alpha \in \mathbb{C}$ for which $|\alpha| > 1$ and $P(\alpha) = 0$.

Solution:

- Dividing out by a_n , we take

$$P(z) = a_n \left(z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_1}{a_n} z + \frac{a_0}{a_n} \right).$$

By the fundamental theorem of algebra, we can find some α_1 such that $P(\alpha_1) = 0$. Therefore, by polynomial division, we have a monic polynomial $q(z)$ with degree $n - 1$ such that

$$P(z) = a_n q(z)(z - \alpha_1).$$

If $q(z)$ is a constant polynomial, it is necessarily equal to 1 and we are done. Else, inductively, we

may find $\alpha_2, \dots, \alpha_n \in \mathbb{C}$ such that $q(z) = (z - \alpha_2) \cdots (z - \alpha_n)$, meaning that

$$P(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

(b) If P is a polynomial, then we may factor

$$P(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

Observe that

$$a_0 = a_n \prod_{i=1}^n \alpha_i,$$

so that

$$\left| \frac{a_0}{a_n} \right| = \prod_{i=1}^n |\alpha_i|.$$

Since $|a_0| > |a_n|$, it follows that

$$\prod_{i=1}^n |\alpha_i| > 1.$$

By the pigeonhole principle, there must be at least one α_i such that $|\alpha_i| > 1$.