Math 395: Homework 2
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Problem 1

Problem: Let $V = P_n(\mathbb{F})$. Let $\mathcal{B} = \{1, x, \dots, x^n\}$ be a basis of V. Let $\lambda \in \mathbb{F}$, and set $\mathcal{C} = \left\{1, x-\lambda, \dots, (x-\lambda)^{n-1}, (x-\lambda)^n\right\}$. Define a linear transformation $T \in \operatorname{Hom}_{\mathbb{F}}(V, V)$ by taking $T\left(x^j\right) = (x-\lambda)^j$. Determine the matrix of this linear transformation. Use this to conclude that $\mathcal{C}$ is also a basis of V.

Solution. Considering our basis $\mathcal{B} = \{1, x, \dots, x^n\}$, we evaluate $T\left(x^j\right)$ for each j. In particular, this yields

$$\begin{aligned} T(1) &= 1 \\ T(x) &= x - \lambda \\ &\vdots \\ T\left(x^{n-1}\right) &= (x-\lambda)^{n-1} \\ T(x^n) &= (x-\lambda)^n. \end{aligned}$$

In particular, $T\left(x^j\right) = (1)(x-\lambda)^j$, implying that our matrix is

$$\begin{aligned} [T]_{\mathcal{B}}^{\mathcal{C}} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \\ &= I_n. \end{aligned}$$

In particular, since I_n is an isomorphism, it is the case that T maps one basis of V to another basis of V, meaning $\mathcal{C}$ is a basis of $P_n(\mathbb{F})$.