2.1

Problem: Recall that an ordered pair (a, b) can be defined as the set $\{\{a\}, \{a, b\}\}$. Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Solution. Let $L = \{\{a\}, \{a, b\}\}$ and $R = \{c, \{c, d\}\}$. Suppose $L = R$. Since $\{a\} \in L$, we have $\{a\} \in R$. Thus, $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$.

Case 1: If $\{a\} = \{c\}$, then $a \in \{c\}$, meaning $a = c$.

Case 2: If $\{a\} = \{c, d\}$, then $c \in \{a\}$, meaning $c = a$.

Since $\{a, b\} \in L$, we have $\{a, b\} \in R$, meaning $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$.

Case 3: If $\{a, b\} = \{c\}$, then it must be the case that $\{a\} = \{c, d\}$, meaning $a = b = c = d$, so $b = d$.

Case 4: If $\{a, b\} = \{c, d\}$, then it must be the case that $\{a\} = \{c\}$, meaning $a = c$, and thus $b = d$.

2.2

Problem: Define the ordered triple (a, b, c) to be the ordered pair $((a, b), c)$, where the ordered pair is defined as usual. Show that

$$(a_1, b_1, c_1) = (a_2, b_2, c_2)$$

if and only if $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$.

Solution. Since

$$(a_1, b_1, c_1) = (a_2, b_2, c_2)$$

implies

$$((a_1, b_1), c_1) = ((a_2, b_2), c_2),$$

this is true if and only if $(a_1, b_1) = (a_2, b_2)$ and $c_1 = c_2$, which is true if and only if $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$.

2.3

Problem: Show that the replacement schema implies the comprehension schema.

Solution. Let $\psi(u, v) = \phi(v) \wedge u = v$. Then, the replacement schema becomes

$$\begin{aligned} \forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow \exists u\, (u \in a \wedge \psi(u, v))) \\ \forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow \exists u\, (u \in a \wedge \forall u\, (\phi(v) \wedge u = v))) \\ \forall a\, \exists b\, \forall v\, (v \in b \Leftrightarrow v \in a \wedge \phi(v)) \end{aligned}$$

2.4

Problem: In this question, we show how the pairing axiom follows from the replacement schema. Let sets a and b be given.

(a) We originally used the pairing axiom to construct the set $\{\varnothing, \{\varnothing\}\}$. Instead, us the power set axiom.

(b) Let $\psi(u, v)$ be the formula

$$(u = \varnothing \wedge v = a) \vee (u \neq \varnothing \wedge v = b).$$

Show that this is a function-like formula.

(c) Use the replacement schema on the set $\{\emptyset, \{\emptyset\}\}$ and the function-like formula $\psi(u, v)$ to show the existence of the set with elements a and b.

Solution.

(a) Consider $\{\emptyset\}$. By the power set axiom, there exists a set c such that c consists of all subsets of $\{\emptyset\}$. Thus, $c = \{\emptyset, \{\emptyset\}\}$.

(b) Let $\psi(u, v) = (u = \emptyset \wedge v = a) \vee (u \neq \emptyset \wedge v = b)$. Then, if $\psi(u, v) = \psi(u, w) = \text{true}$,

$$(u = \emptyset \wedge v = a) \vee (u \neq \emptyset \wedge v = b)$$

and

$$(u = \emptyset \wedge w = a) \vee (u \neq \emptyset \wedge w = b)$$

If $v = b$, then $u \neq \emptyset$, implying $w = b$, and similarly, if $v = a$, then $w = a$. Thus, $u = w$.

(c) Using the replacement schema on $\{\emptyset, \{\emptyset\}\}$, we see there is a set b such that for $\emptyset \in \{\emptyset, \{\emptyset\}\}$, $\psi(u, v)$ maps $\emptyset$ to a, and for $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$, $\psi(u, v)$ maps $\{\emptyset\}$ to b.

Extra Problem 1

Problem:

(a) Explain what would go wrong if we defined $(a, b) = \{a, \{b\}\}$.

(b) Can you figure out why the book defines $(a, b) = \{\{a\}, \{a, b\}\}$ instead of $\{a, \{a, b\}\}$.

Solution.

(a)

(b) If we consider $(a, b) = (a, b)$, we must then have $\{a, \{a, b\}\} = \{a, \{a, b\}\}$, meaning our cases would yield $a \in \{a, \{a, b\}\}$, or $a = \{a, b\}$, implying $a \in a$ or $a \in b$. In particular, for $a \in a$, we get a descending membership chain, which ends up requiring the regularity axiom.

Extra Problem 2

Problem: Let s be a set. Use mathematical symbols exclusively to express t, the set of all singleton subsets of s.

Solution.

$$\forall s\, \exists t\, \forall x\, (x \in t \Leftrightarrow x \in s \wedge \forall a\, \forall b\, (a \in x \wedge b \in x \Rightarrow a = b))$$

Extra Problem 4

Problem: Show that if A and B are nonempty sets, then $\bigcap (A \cup B) = \bigcap A \cup \bigcap B$.

Solution.

$$\begin{aligned}\bigcap (A \cup B) &= \forall A \forall B \exists C\, \forall x\, (x \in C \wedge (x \in A \vee x \in B)) \\ &= \forall A \forall B \exists C\, \forall x\, ((x \in C \wedge x \in A) \vee (x \in C \wedge x \in B)) \\ &= \bigcap A \cup \bigcap B.\end{aligned}$$

Extra Problem 5

Problem: Show there exists a set s such that $x \in s$ if and only if x is a natural number.

Solution.

$$\exists s\, \forall x \left(\underbrace{(x \in s \land x \cup \{x\} \in s)}_{\text{Axiom of Infinity}} \land \forall y\, (y \in s \Rightarrow \exists z\, (y = z \cup \{z\})) \right).$$