

Problem (Problem 2): Let R be a PID. For an R -module M , denote by $d(M)$ the minimal number of generators of M .

- (a) Prove that if M is a finitely generated R -module, and N is a submodule of M , then $d(N) \leq d(M)$.
- (b) Let $a \in R$ be a nonzero non-unit. Find (with proof) the number of submodules of R/aR in terms of the prime decomposition of a .

Solution:

- (a) Let $\{v_1, \dots, v_n\}$ be a minimal generating set for M . Via the surjection $R^n \rightarrow M$ taking $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n r_i v_i$, we observe that $M \cong R^n/G$ for some submodule G of R^n . Since N is a submodule of M , it follows from the fourth isomorphism theorem that N corresponds to a submodule of R^n containing G , which we will call N' ; since N' is a submodule of a free module, it is free with rank $m \leq n$, and N' surjects onto N so that $d(N) \leq d(N') \leq n = d(M)$.
- (b) Without loss of generality, let $a = p_1^{d_1} \cdots p_t^{d_t}$ be the prime decomposition for a , where $d_i \in \mathbb{N}$. From the Chinese Remainder Theorem, we have

$$R/(a) \cong R/\left(p_1^{d_1}\right) \oplus \cdots \oplus R/\left(p_t^{d_t}\right).$$

Observe that any submodule of $R/(a)$ is in correspondence with an ideal containing a (by the fourth isomorphism theorem). These ideals are precisely the ideals of products of prime powers that are less than or equal to a . Since, given $p_i^{d_i}$, there are $0, \dots, d_i$ potential options for the power of p_i , so that there are $(d_1 + 1) \cdots (d_t + 1)$ submodules in $R/(a)$.

Problem (Problem 3):

- (a) Let R be a PID, M a finitely generated R -module, and

$$M = \left(\bigoplus_{i=1}^{\ell} R/(a_i) \right) \oplus R^s$$

its invariant factor decomposition. Prove that $d(M) = m + s$.

- (b) Again let R be a PID. Let F be a free R -module of rank n with basis e_1, \dots, e_n , N the submodule of F generated by some elements $v_1, \dots, v_n \in F$, and let $A \in \text{Mat}_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Find a simple condition on the entries of A which holds if and only if $d(F/N) = n$.

Solution:

- (a) Let p be a prime dividing a_1 . Now, we observe that

$$\begin{aligned} M/pM &\cong \frac{(\bigoplus_{i=1}^m R/(a_i)) \oplus R^s}{(\bigoplus_{i=1}^m p(R/(a_i))) \oplus (pR)^s} \\ &\cong (R/(p))^{m+s}. \end{aligned}$$

Now, since R is a PID, any prime ideal is maximal, meaning that $d(M') = m + s$ is the dimension of $M' = (R/(p))^{m+s}$. Since M' is a quotient of M , it follows that $d(M) \geq m + s$.

Yet, since the set $\{e_i \mid 1 \leq i \leq m + s\}$ generates M as an R -module, it follows that $d(M) \leq m + s$, so that $d(M) = m + s$.

Problem (Problem 6):

- (i) Let V and W be R -modules over a commutative ring. Show that there is a natural homomorphism $\varphi: V^* \otimes W \rightarrow \text{Hom}_R(V, W)$ such that $(\varphi(f \otimes w))(v) = f(v)w$.
- (ii) Assume that W is a finitely generated free R -module. Prove that φ is an isomorphism.
- (iii) Give examples showing that φ need not be surjective if W is either not free or W is free but not finitely generated.