Amenability: A (Somewhat) Brief Introduction

Avinash Iyer

Occidental College

March 20, 2025

Outline

Contents

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

then we call the pair $(A, \star)$ a *group*.

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

then we call the pair $(A, \star)$ a *group*.

We (usually) abbreviate $a \star b$ as ab.

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

then we call the pair $(A, \star)$ a *group*.

We (usually) abbreviate $a \star b$ as ab. If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.
- If $N \subseteq G$ is a subgroup that satisfies, for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say N is a *normal subgroup*.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.
- If $N \subseteq G$ is a subgroup that satisfies, for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say N is a *normal subgroup*.
- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* G/N.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.
- If $N \subseteq G$ is a subgroup that satisfies, for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say N is a *normal subgroup*.
- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* G/N.
- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

Some Groups

- The integers $\mathbb{Z}$ are a group under addition.

Some Groups

- The integers $\mathbb{Z}$ are a group under addition.
- The group $\mathrm{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.

Some Groups

- The integers $\mathbb{Z}$ are a group under addition.
- The group $\mathrm{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group $\mathrm{E}(3)$ consists of all translations, rotations, and flips in $\mathbb{R}^3$, and is also known as the *isometry group* of $\mathbb{R}^3$.

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X. We write $\rho(g, x) = g \cdot x$.

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X. We write $\rho(g, x) = g \cdot x$. The above lines become $e_G \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$.

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X. We write $\rho(g, x) = g \cdot x$. The above lines become $e_G \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$. If $g \cdot x = x$ only when $g = e_G$, then we say the action is *free*.

Every group is equipped with a family of canonical actions, $\sigma_a\colon G \to G$ for each $a \in G$, given by $x \mapsto ax$, known as *left-multiplication*.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i\in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

If, for any countable collection, $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a σ-*algebra* of subsets.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i\in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

If, for any countable collection, $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a σ*-algebra* of subsets.

The most important σ-algebra, and the one we will be dealing with throughout this talk, is $P(G)$, where G is a group.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say μ is a measure.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say μ is a measure. If $\mu(X) = 1$, then we say μ is a probability measure.

Contents

Motivating Questions

- If G is a group, is it possible to reconstruct G by using some subset of G?

Motivating Questions

- If G is a group, is it possible to reconstruct G by using some subset of G?
- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?

Motivating Questions

- If G is a group, is it possible to reconstruct G by using some subset of G?
- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

- We begin by considering a special group, known as $F(a, b)$ or the *free group on two generators*.

Free Groups

- We begin by considering a special group, known as $F(a,b)$ or the *free group on two generators*.
- We define $F(a,b)$ to be the set of all "words" in the alphabet $\{a,b,a^{-1},b^{-1}\}$, subject to the condition that, for $w,w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

A Curiosity

Let $W(b) \subseteq F(a,b)$ be all the words that start with b. Then, $b^{-1}W(b)$ consists of

A Curiosity

Let $W(b) \subseteq F(a,b)$ be all the words that start with b. Then, $b^{-1}W(b)$ consists of

- all words that start with a;

A Curiosity

Let $W(b) \subseteq F(a, b)$ be all the words that start with b. Then, $b^{-1}W(b)$ consists of

- all words that start with a;
- all words that start with a^{-1};

A Curiosity

Let $W(b) \subseteq F(a,b)$ be all the words that start with b. Then, $b^{-1}W(b)$ consists of

- all words that start with a;
- all words that start with a^{-1};
- all words that start with b — think words that start with b^2 before you multiply b^{-1}.

A Curiosity

Let $W(b) \subseteq F(a,b)$ be all the words that start with b. Then, $b^{-1}W(b)$ consists of

- all words that start with a;
- all words that start with a^{-1};
- all words that start with b — think words that start with b^2 before you multiply b^{-1}.

Thus, all we need to do is add back $W\left(b^{-1}\right)$ to get $F(a,b)$ back.

$$F(a,b) = W\left(b^{-1}\right) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

We're able to take part of the group $F(a,b)$, take some translations, and, miraculously, obtain the entire group back.

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$;

such that

$$\begin{aligned} G &= \bigcup_{i=1}^{n} g_i A_i \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$;

such that

$$\begin{aligned} G &= \bigcup_{i=1}^{n} g_i A_i \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is *G-paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$

such that

$$A = \bigcup_{i=1}^{n} g_i \cdot A_i = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is *G-paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$

such that

$$A = \bigsqcup_{i=1}^{n} g_i \cdot A_i = \bigsqcup_{j=1}^{m} h_j \cdot B_j.$$

A paradoxical group is a paradoxical set under the action of left-multiplication.

Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is *G-paradoxical* if there exist

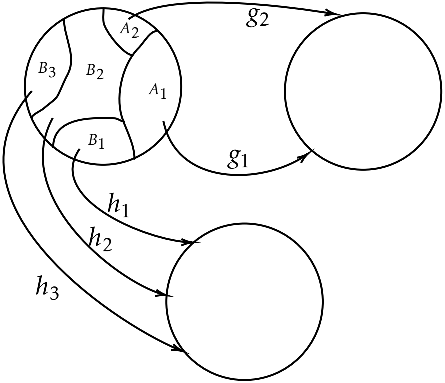
- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$

such that

$$A = \bigcup_{i=1}^{n} g_i \cdot A_i = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

A paradoxical group is a paradoxical set under the action of left-multiplication. If G is paradoxical and acts freely on $A \subseteq X$, then A is G-paradoxical.

Depiction



Some Paradoxical Groups

- The free group $F(a, b)$ is paradoxical.

Some Paradoxical Groups

- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.

Some Paradoxical Groups

- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a,b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

Thus, SO(3) is paradoxical

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

Thus, SO(3) is paradoxical — can we use it to find a paradoxical decomposition?

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

- In other words, not all subsets of $\mathbb{R}^3$ have a definite "volume" invariant under isometry.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

- In other words, not all subsets of $\mathbb{R}^3$ have a definite "volume" invariant under isometry.

We need to introduce a few more concepts before we can show the proof of the paradox.

Equidecomposability

Let G be a group that acts on a set X, and let $A, B \subseteq X$.

Equidecomposability

Let G be a group that acts on a set X, and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \ldots, A_n \subseteq A$, $B_1, \ldots, B_n \subseteq B$
- group elements $g_1, \ldots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say A and B are *G-equidecomposable*.

Equidecomposability

Let G be a group that acts on a set X, and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \ldots, A_n \subseteq A$, $B_1, \ldots, B_n \subseteq B$
- group elements $g_1, \ldots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say A and B are *G-equidecomposable*.

Effectively, A and B are "equal" to each other up to the group action.

Equidecomposability

Let G be a group that acts on a set X, and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \dots, A_n \subseteq A$, $B_1, \dots, B_n \subseteq B$
- group elements $g_1, \dots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say A and B are *G-equidecomposable*.

Effectively, A and B are "equal" to each other up to the group action.

If A is G-paradoxical, then so too is B.

The Banach–Tarski Paradox: Proof Outline

- Use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

  to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a,b)$.

The Banach–Tarski Paradox: Proof Outline

- Use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

  to generate a subgroup of SO(3) isomorphic to $F(a, b)$.

- Copy $F(a, b)$ over by using the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right). \end{aligned}$$

The Banach–Tarski Paradox: Proof Outline

- Use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

  to generate a subgroup of $SO(3)$ isomorphic to $F(a,b)$.
- Copy $F(a,b)$ over by using the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right). \end{aligned}$$

  Removing fixed points yields a decomposition of $S^2 \setminus D$. This is known as the *Hausdorff paradox*.

The Banach–Tarski Paradox: Proof Outline, cont'd

- We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable.

The Banach–Tarski Paradox: Proof Outline, cont'd

- We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable. There is thus a paradoxical decomposition of S^2.

The Banach–Tarski Paradox: Proof Outline, cont'd

- We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable. There is thus a paradoxical decomposition of S^2.
- Use scaling and an off-center rotation to show that $B(0,1) \subseteq \mathbb{R}^3$ is paradoxical under the isometry group E(3).

The Banach–Tarski Paradox: Proof Outline, cont'd

- We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable. There is thus a paradoxical decomposition of S^2.
- Use scaling and an off-center rotation to show that $B(0,1) \subseteq \mathbb{R}^3$ is paradoxical under the isometry group E(3).

Thus, we have shown the *weak* Banach–Tarski paradox.

The Banach–Tarski Paradox: Proof Outline, cont'd

- We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable. There is thus a paradoxical decomposition of S^2.
- Use scaling and an off-center rotation to show that $B(0,1) \subseteq \mathbb{R}^3$ is paradoxical under the isometry group E(3).

Thus, we have shown the *weak* Banach–Tarski paradox. For the full paradox, we need one more thing.

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \preceq B$ if A is G-equidecomposable with *a subset of* B.

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \preceq B$ if A is G-equidecomposable with *a subset of* B.

This relation is reflexive (since all sets are subsets of themselves, take the identity action),

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \preceq B$ if A is G-equidecomposable with *a subset of* B.

This relation is reflexive (since all sets are subsets of themselves, take the identity action), transitive (compose the group action),

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \preceq B$ if A is G-equidecomposable with *a subset of* B.

This relation is reflexive (since all sets are subsets of themselves, take the identity action), transitive (compose the group action), and antisymmetric (this one takes a bit more work).

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \preceq B$ if A is G-equidecomposable with *a subset of* B.

This relation is reflexive (since all sets are subsets of themselves, take the identity action), transitive (compose the group action), and antisymmetric (this one takes a bit more work).

Thus, to show that A and B are G-equidecomposable, it suffices to show that $A \preceq B$.

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \preceq B$ if A is G-equidecomposable with *a subset of* B.

This relation is reflexive (since all sets are subsets of themselves, take the identity action), transitive (compose the group action), and antisymmetric (this one takes a bit more work).

Thus, to show that A and B are G-equidecomposable, it suffices to show that $A \preceq B$. We do this, and we are done.

Contents

Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped “create” the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.

Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped “create” the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu\colon F(a,b) \to [0,1]$ be a probability measure

Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped "create" the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu\colon F(a,b) \to [0,1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$1 = \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right)$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \color{blue}{\nu\left(a^{-1}W(a)\right)} + \nu\left(W\left(a^{-1}\right)\right) + \color{blue}{\nu\left(b^{-1}W(b)\right)} + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\ &= \nu(F(a,b)) + \nu(F(a,b)) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned}
1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\
&= \nu(F(a,b)) + \nu(F(a,b)) \\
&= 2.
\end{aligned}$$

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu : P(G) \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu \colon P(G) \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

If G admits a mean, we say G is *amenable*.

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu : P(G) \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

If G admits a mean, we say G is *amenable*.

- In other words, G is sufficiently "well-behaved."

Inheritance Properties of Amenability

- If G is amenable, then any subgroup of G is amenable.

Inheritance Properties of Amenability

- If G is amenable, then any subgroup of G is amenable.
- If G is amenable, then quotient groups, G/N, are amenable.

Inheritance Properties of Amenability

- If G is amenable, then any subgroup of G is amenable.
- If G is amenable, then quotient groups, G/N, are amenable.
- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.

Inheritance Properties of Amenability

- If G is amenable, then any subgroup of G is amenable.
- If G is amenable, then quotient groups, G/N, are amenable.
- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.

Inheritance Properties of Amenability

- If G is amenable, then any subgroup of G is amenable.
- If G is amenable, then quotient groups, G/N, are amenable.
- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If G and H are amenable, then so is $G \times H$.

Inheritance Properties of Amenability

- If G is amenable, then any subgroup of G is amenable.
- If G is amenable, then quotient groups, G/N, are amenable.
- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If G and H are amenable, then so is $G \times H$.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

  Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

  is a mean.
- Abelian groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

Paradoxical Groups and Amenability

Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a,b)$.

Paradoxical Groups and Amenability

Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a,b)$.

More surprisingly, though, every *non*-paradoxical group is amenable.

Paradoxical Groups and Amenability

Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a,b)$.

More surprisingly, though, every *non*-paradoxical group is amenable.

Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

Paradoxical Groups and Amenability

Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a,b)$.

More surprisingly, though, every *non*-paradoxical group is amenable.

Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

Contents

Why Find Alternative Characterizations?

On first glance, it may seem like we're finished, but we're really not.

Why Find Alternative Characterizations?

On first glance, it may seem like we're finished, but we're really not.

Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

Why Find Alternative Characterizations?

On first glance, it may seem like we're finished, but we're really not.

Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|\colon V \to [0, \infty)$ satisfying:

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|\colon V \to [0,\infty)$ satisfying:

- definiteness: $\|v\| \geq 0$, with equality if and only if $v = 0$;

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|: V \to [0, \infty)$ satisfying:

- definiteness: $\|v\| \geq 0$, with equality if and only if $v = 0$;
- homogeneity: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$;

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|\colon V \to [0,\infty)$ satisfying:

- definiteness: $\|v\| \geq 0$, with equality if and only if $v = 0$;
- homogeneity: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$;
- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

A Normed Vector Space

The best example is that of $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sup_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t \in \Gamma} |f(t)|^2 < \infty \right\}.$$

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sup_{t\in\Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sum_{t\in\Gamma} |f(t)| < \infty \right\};$$

$$\ell_2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sum_{t\in\Gamma} |f(t)|^2 < \infty \right\}.$$

They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_1} := \sum_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_2} := \left(\sum_{t\in\Gamma} |f(t)|^2\right)^{1/2}$.

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted $\|T\|_{\mathrm{op}}$.

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted $\|T\|_{\text{op}}$. A linear map is bounded if and only if it is continuous.

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

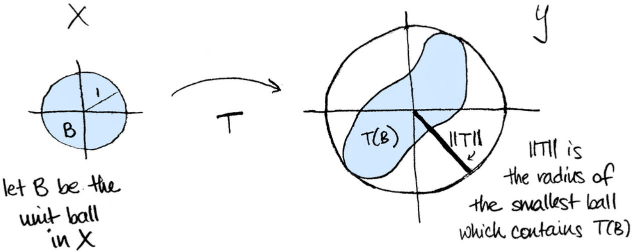
$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted $\|T\|_{\mathrm{op}}$. A linear map is bounded if and only if it is continuous.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Operator Norm Pictorial Depiction

Courtesy of Tai-Danae Bradley.



Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}}$.

Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}}$. All positive linear functionals are automatically continuous.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

for all $t \in \Gamma$ and fixed $s \in \Gamma$.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s \colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

for all $t \in \Gamma$ and fixed $s \in \Gamma$.

If $\varphi \colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

Invariant States and Means

Invariant states and means are interchangeable.

If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

Approximations and Amenability

There is actually one way that working with sets makes life easier.

Approximations and Amenability

There is actually one way that working with sets makes life easier.

Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Approximations and Amenability

There is actually one way that working with sets makes life easier.

Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was “significantly” “bigger” than $W\left(a^{-1}\right)$;

Approximations and Amenability

There is actually one way that working with sets makes life easier.

Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was “significantly” “bigger” than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

Approximations and Amenability

There is actually one way that working with sets makes life easier.

Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was “significantly” “bigger” than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

But what does “bigger” actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n\to\infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n\to\infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

If $\Gamma = \langle S \rangle$ for some finite generating set S, we only need to check for all $s \in S$.

$\mathbb{Z}$ is Amenable

Consider $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ under addition.

$\mathbb{Z}$ is Amenable

Consider $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ under addition.

We can find a Følner sequence by defining

$$F_n = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}.$$

Then, since $\mathbb{Z}$ is generated by 1, we verify

$$\begin{aligned} \frac{|(F_n + 1) \cap F_n|}{|F_n|} &= \frac{2n-1}{2n+1} \\ &\to 1. \end{aligned}$$

$\mathbb{Z}$ is Amenable

Consider $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ under addition.

We can find a Følner sequence by defining

$$F_n = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}.$$

Then, since $\mathbb{Z}$ is generated by 1, we verify

$$\begin{aligned} \frac{|(F_n + 1) \cap F_n|}{|F_n|} &= \frac{2n-1}{2n+1} \\ &\to 1. \end{aligned}$$

Thus, $\mathbb{Z}$ is amenable.

Approximate Means

The Følner condition allows us to find an “approximate” version of a mean.

Approximate Means

The Følner condition allows us to find an "approximate" version of a mean.

Keeping $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|} \mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a "sufficient" finitely supported function g,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

then use a "layer cake" decomposition to find our Følner sets:

$$g = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

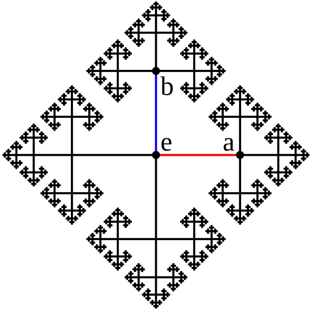
where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S, we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by "walking" along the generators.

Graphs and Amenability

Given a group Γ with generating set S, we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by "walking" along the generators.



Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

$$\inf\left\{ \frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), |S| \text{ finite} \right\} = 0.$$

Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

$$\inf\left\{ \frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), |S| \text{ finite} \right\} = 0.$$

- Essentially, the Cayley graph doesn't "get too big" "too fast."

Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

$$\inf\left\{ \frac{|N(S)|}{|S|} \,\middle|\, S \subseteq V(G), |S| \text{ finite} \right\} = 0.$$

- Essentially, the Cayley graph doesn't "get too big" "too fast."
- This is proven with the Følner condition.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies

- $\langle x, x \rangle \geq 0$, with equality only when $x = 0$;

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies

- $\langle x, x \rangle \geq 0$, with equality only when $x = 0$;
- $\langle x_1 + \alpha x_2, y \rangle = \langle x_1, y \rangle + \alpha \langle x_2, y \rangle$;

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ that satisfies

- $\langle x,x\rangle \geq 0$, with equality only when $x=0$;
- $\langle x_1+\alpha x_2, y\rangle = \langle x_1,y\rangle + \alpha\langle x_2,y\rangle$;
- $\langle x, y_1+\alpha y_2\rangle = \langle x,y_1\rangle + \overline{\alpha}\langle x,y_2\rangle$.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ that satisfies

- $\langle x,x\rangle \geq 0$, with equality only when $x = 0$;
- $\langle x_1+\alpha x_2, y\rangle = \langle x_1,y\rangle + \alpha\langle x_2,y\rangle$;
- $\langle x, y_1+\alpha y_2\rangle = \langle x,y_1\rangle + \overline{\alpha}\langle x,y_2\rangle$.

The inner product induces a norm $\|x\|^2 = \langle x,x\rangle$.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ that satisfies

- $\langle x,x\rangle \geq 0$, with equality only when $x=0$;
- $\langle x_1+\alpha x_2, y\rangle = \langle x_1,y\rangle + \alpha\langle x_2,y\rangle$;
- $\langle x, y_1+\alpha y_2\rangle = \langle x,y_1\rangle + \overline{\alpha}\langle x,y_2\rangle$.

The inner product induces a norm $\|x\|^2 = \langle x,x\rangle$.

If $\mathcal{H}$ is complete with respect to this norm, we call $\mathcal{H}$ a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that "plays nicely" with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that “plays nicely” with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

If $U : \mathcal{H} \to \mathcal{H}$ is such that

$$\begin{aligned} U^*U &= I \\ UU^* &= I, \end{aligned}$$

then we call U a *unitary operator*.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that "plays nicely" with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

If $U : \mathcal{H} \to \mathcal{H}$ is such that

$$U^*U = I$$
$$UU^* = I,$$

then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$
$$\lambda\left(s^{-1}\right) = \lambda(s)^*$$

is called a *unitary representation* of Γ.

All discrete groups are able to be unitarily represented

Representations

A map $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$
$$\lambda\left(s^{-1}\right) = \lambda(s)^*$$

is called a *unitary representation* of Γ.

All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma\colon \Gamma \to \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

As it turns out, the map $\lambda_s(f)(t) = f\left(s^{-1}t\right)$ is a unitary operator on $\ell_2(\Gamma)$, where $\lambda_s^* = \lambda_{s^{-1}}$.

The Left-Regular Representation

As it turns out, the map $\lambda_s(f)(t) = f\left(s^{-1}t\right)$ is a unitary operator on $\ell_2(\Gamma)$, where $\lambda_s^* = \lambda_{s^{-1}}$.

It can also be shown that $\lambda_s\lambda_t = \lambda_{st}$, meaning that the map $s \mapsto \lambda_s$ is a unitary representation.

The Left-Regular Representation

As it turns out, the map $\lambda_s(f)(t) = f\left(s^{-1}t\right)$ is a unitary operator on $\ell_2(\Gamma)$, where $\lambda_s^* = \lambda_{s^{-1}}$.

It can also be shown that $\lambda_s\lambda_t = \lambda_{st}$, meaning that the map $s \mapsto \lambda_s$ is a unitary representation.

The map $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$ is a very special representation, known as the *left-regular representation*.

The Left-Regular Representation

As it turns out, the map $\lambda_s(f)(t) = f\left(s^{-1}t\right)$ is a unitary operator on $\ell_2(\Gamma)$, where $\lambda_s^* = \lambda_{s^{-1}}$.

It can also be shown that $\lambda_s\lambda_t = \lambda_{st}$, meaning that the map $s \mapsto \lambda_s$ is a unitary representation.

The map $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$ is a very special representation, known as the *left-regular representation*.

This is because it "encodes" the group's left-multiplication action, in the sense that $\lambda_s(\delta_t) = \delta_{st}$, where δ_t is the point mass at $t \in \Gamma$.

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* for $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* for $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T : \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\text{op}}$, is a very special vector space. The adjoint map satisfies:

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T : \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\text{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T : \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\text{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Furthermore, the operator norm "plays well" with operator composition and the adjoint, in the sense that:

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T+\alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Furthermore, the operator norm "plays well" with operator composition and the adjoint, in the sense that:

- $\|TS\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$;

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Furthermore, the operator norm "plays well" with operator composition and the adjoint, in the sense that:

- $\|TS\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$;
- $\|T^*\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$;

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Furthermore, the operator norm "plays well" with operator composition and the adjoint, in the sense that:

- $\|TS\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$;
- $\|T^*\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$;
- $\|T^* T\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}^2$.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Furthermore, the operator norm "plays well" with operator composition and the adjoint, in the sense that:

- $\|TS\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$;
- $\|T^*\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$;
- $\|T^* T\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}^2$.

These make $\mathbb{B}(\mathcal{H})$ a C^*-*algebra*.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

Furthermore, the operator norm "plays well" with operator composition and the adjoint, in the sense that:

- $\|TS\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$;
- $\|T^*\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$;
- $\|T^* T\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}^2$.

These make $\mathbb{B}(\mathcal{H})$ a C^*-*algebra*. However, there are other C^*-algebras.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t\in\Gamma} x(t)\delta_t,$$

where δ_t is the point mass at $t \in \Gamma$.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t)\delta_t,$$

where δ_t is the point mass at $t \in \Gamma$.

This becomes a $*$-algebra when endowed with multiplication (by convolution) and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g\left(s^{-1} t\right)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^*-Algebra, cont'd

If we represent $\pi_\lambda : \mathbb{C}[\Gamma] \to \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\mathrm{op}},$$

we get the *reduced group C^*-algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

The $n \times n$ matrices, $\mathrm{Mat}_n(\mathbb{C})$, are also C^*-algebras.

Finite-Dimensional Approximations

The $n \times n$ matrices, $\mathrm{Mat}_n(\mathbb{C})$, are also C^*-algebras. In fact, they're a very special kind of C^*-algebra — we care a lot about whether other C^*-algebras can be "sufficiently" approximated by matrices.

Finite-Dimensional Approximations

The $n \times n$ matrices, $\mathrm{Mat}_n(\mathbb{C})$, are also C^*-algebras. In fact, they're a very special kind of C^*-algebra — we care a lot about whether other C^*-algebras can be "sufficiently" approximated by matrices.

We can use these sufficient approximations to establish amenability.

Nuclearity

A C^*-algebra, A, is called *nuclear* if there exist two sequences of maps, $\varphi_n : A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n : \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$, such that

$$\left\| a - \psi_n \circ \varphi_n(a) \right\| \xrightarrow{n\to\infty} 0.$$

Nuclearity

A C^*-algebra, A, is called *nuclear* if there exist two sequences of maps, $\varphi_n\colon A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n\colon \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$, such that

$$\left\| a - \psi_n \circ \varphi_n(a) \right\| \xrightarrow{n\to\infty} 0.$$

- Essentially, any $a \in A$ is "close enough" to a certain family of finite-dimensional analogues.

Nuclearity and Amenability

A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

Nuclearity and Amenability

A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

This is also proven using the Følner condition.

Nuclearity and Amenability

A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

This is also proven using the Følner condition.

Specifically, by showing that the approximation of $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ corresponds to the existence of maps $\varphi_n\colon C^*_\lambda(\Gamma) \to \mathrm{Mat}_{|F_n|}(\mathbb{C})$ and $\psi_n\colon \mathrm{Mat}_{|F_n|}(\mathbb{C}) \to C^*_\lambda(\Gamma)$ that satisfy

$$\left\| x - \psi_n \circ \varphi_n(x) \right\| \xrightarrow{n\to\infty} 0.$$

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);
- there is a sequence $(f_k)_k \subseteq \ell_1(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_1} \to 0$ (Approximate Means);

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);
- there is a sequence $(f_k)_k \subseteq \ell_1(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_1} \to 0$ (Approximate Means);
- the Cayley graph of Γ satisfies $\inf\left\{ \frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), S \text{ finite} \right\} = 0$ (graph amenability);

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);
- there is a sequence $(f_k)_k \subseteq \ell_1(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_1} \to 0$ (Approximate Means);
- the Cayley graph of Γ satisfies $\inf\left\{\frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), S \text{ finite}\right\} = 0$ (graph amenability);
- there is a sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_2} \to 0$ (almost-invariant vectors);

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's theorem);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);
- there is a sequence $(f_k)_k \subseteq \ell_1(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_1} \to 0$ (Approximate Means);
- the Cayley graph of Γ satisfies $\inf\left\{\frac{|N(S)|}{|S|} \,\middle|\, S \subseteq V(G), S \text{ finite}\right\} = 0$ (graph amenability);
- there is a sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_2} \to 0$ (almost-invariant vectors);
- the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear (nuclearity).

Contents

Present-Day Programs of Research

Amenability is still a very active field of study, especially when it comes to C^*-algebras.

Present-Day Programs of Research

Amenability is still a very active field of study, especially when it comes to C^*-algebras.

If a C^*-algebra is nuclear, then by calculating a certain invariant, the C^*-algebra can be classified; if the C^*-algebra is not nuclear, then the process of classification is significantly more difficult.

Present-Day Programs of Research

Amenability is still a very active field of study, especially when it comes to C^*-algebras.

If a C^*-algebra is nuclear, then by calculating a certain invariant, the C^*-algebra can be classified; if the C^*-algebra is not nuclear, then the process of classification is significantly more difficult.

Additionally, other questions related to finite-dimensional approximations of C^*-algebras, such as quasidiagonality, were only resolved recently.

Present-Day Programs of Research

Amenability is still a very active field of study, especially when it comes to C^*-algebras.

If a C^*-algebra is nuclear, then by calculating a certain invariant, the C^*-algebra can be classified; if the C^*-algebra is not nuclear, then the process of classification is significantly more difficult.

Additionally, other questions related to finite-dimensional approximations of C^*-algebras, such as quasidiagonality, were only resolved recently. A paper in 2015 proved that if Γ is amenable, then $C^*_\lambda(\Gamma)$ is quasidiagonal (the reverse direction was shown in 1987).

Acknowledgments

A large thank you goes to

- the professors of the math department;
- friends, family, and acquaintances both in the math major and outside;
- everyone in attendance.

References I

[AB06] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis*. Third. A Hitchhiker's Guide. Springer, Berlin, 2006, pp. xxii+703. ISBN: 978-3-540-32696-0.

[Alu09] Paolo Aluffi. *Algebra: Chapter 0*. Vol. 104. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xx+713. ISBN: 978-0-8218-4781-7. DOI: 10.1090/gsm/104. URL: https://doi.org/10.1090/gsm/104.

[BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*. Vol. 11. New Mathematical Monographs. Cambridge University Press, Cambridge, 2008, pp. xiv+472. ISBN: 978-0-521-88720-5. DOI: 10.1017/CBO9780511542749. URL: https://doi.org/10.1017/CBO9780511542749.

References II

[Bla06] B. Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4. DOI: 10.1007/3-540-28517-2. URL: https://doi.org/10.1007/3-540-28517-2.

[BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004, pp. xiv+716. ISBN: 0-521-83378-7. DOI: 10.1017/CBO9780511804441. URL: https://doi.org/10.1017/CBO9780511804441.

References III

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+509. ISBN: 978-0-8218-4381-9. DOI: 10.1090/gsm/088. URL: https://doi.org/10.1090/gsm/088.

[CE78] Man-Duen Choi and Edward G. Effros. "Nuclear C^*-Algebras and the Approximation Property". In: *American Journal of Mathematics* 100.1 (1978), pp. 61–79. ISSN: 00029327. URL: http://www.jstor.org/stable/2373876 (visited on 02/07/2025).

[DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. Third. John Wiley & Sons, Inc., Hoboken, NJ, 2004, pp. xii+932. ISBN: 0-471-43334-9.

References IV

[Enc25] The Editors of Encyclopaedia Britannica. *Ship of Theseus*. Accessed: 2025-02-06. 2025. URL: https://www.britannica.com/topic/Ship-of-Theseus.

[Fol84] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984, pp. xiv+350. ISBN: 0-471-80958-6.

[Hal06] Paul R. Halmos. “How to write mathematics”. In: *Butl. Soc. Catalana Mat.* 21.1 (2006). Translation of Enseignement Math. (2) **16** (1970), 123–152 [MR0277319], pp. 53–79, 158. ISSN: 0214-316X,2013-9829.

References V

[Hal66] James D. Halpern. "Bases in vector spaces and the axiom of choice". In: *Proc. Amer. Math. Soc.* 17 (1966), pp. 670–673. ISSN: 0002-9939,1088-6826. DOI: 10.2307/2035388. URL: https://doi.org/10.2307/2035388.

[Har00] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000, pp. vi+310. ISBN: 0-226-31719-6.

[Jec03] Thomas Jech. *Set theory*. millennium. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

References VI

[Jus22] Kate Juschenko. *Amenability of discrete groups by examples*. Vol. 266. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2022, pp. xi+165. ISBN: 978-1-4704-7032-6. DOI: 10.1090/surv/266. URL: https://doi.org/10.1090/surv/266.

[Kes59a] Harry Kesten. "Full Banach Mean Values on Countable Groups". In: *Mathematica Scandinavica* 7.1 (1959), pp. 146–156. ISSN: 00255521. URL: http://www.jstor.org/stable/24489015 (visited on 02/05/2025).

[Kes59b] Harry Kesten. "Symmetric Random Walks on Groups". In: *Transactions of the American Mathematical Society* 92.2 (1959), pp. 336–354. ISSN: 00029947. URL: http://www.jstor.org/stable/1993160 (visited on 02/05/2025).

[Knu09] Søren Knudby. "The Banach-Tarski Paradox". 2009.

References VII

[Löh17] Clara Löh. *Geometric group theory*. Universitext. An introduction. Springer, Cham, 2017, pp. xi+389. ISBN: 978-3-319-72253-5. DOI: 10.1007/978-3-319-72254-2. URL: https://doi.org/10.1007/978-3-319-72254-2.

[Mon17] Mehdi Sangani Monfared. "Følner's condition and expansion of Cayley graphs for group actions". In: *New York Journal of Mathematics* 23 (Sept. 2017), pp. 1295–1306.

[Mon13] Nicolas Monod. "Groups of piecewise projective homeomorphisms". In: *Proc. Natl. Acad. Sci. USA* 110.12 (2013), pp. 4524–4527. ISSN: 0027-8424,1091-6490. DOI: 10.1073/pnas.1218426110. URL: https://doi.org/10.1073/pnas.1218426110.

References VIII

[Pau02] Vern Paulsen. *Completely bounded maps and operator algebras*. Vol. 78. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+300. ISBN: 0-521-81669-6.

[Rai23] Timothy Rainone. "Functional Analysis-En Route to Operator Algebras". 2023.

[Rud73] Walter Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, pp. xiii+397.

[Run02] Volker Runde. *Lectures on amenability*. Vol. 1774. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. xiv+296. ISBN: 3-540-42852-6. DOI: 10.1007/b82937. URL: https://doi.org/10.1007/b82937.

References IX

[Run05] Volker Runde. *A taste of topology*. Universitext. Springer, New York, 2005, pp. x+176. ISBN: 978-0387-25790-7.

[Run20] Volker Runde. *Amenable Banach algebras*. Springer Monographs in Mathematics. A panorama. Springer-Verlag, New York, 2020, pp. xvii+462. ISBN: 978-1-0716-0351-2. DOI: 10.1007/978-1-0716-0351-2. URL: https://doi.org/10.1007/978-1-0716-0351-2.

[Tak64] Masamichi Takesaki. "On the cross-norm of the direct product of C^*-algebras". In: *Tohoku Math. J. (2)* 16 (1964), pp. 111–122. ISSN: 0040-8735,2186-585X. DOI: 10.2748/tmj/1178243737. URL: https://doi.org/10.2748/tmj/1178243737.

References X

[Tao09] Terence Tao. *245B, notes 2: Amenability, the ping-pong lemma, and the Banach-Tarski paradox (optional).* https://terrytao.wordpress.com/2009/01/08/245b-notes-2-amenability-the-ping-pong-lemma-and-the-banach-tarski-paradox-optional/. 2009.

[Tit72] J Tits. "Free subgroups in linear groups". In: *Journal of Algebra* 20.2 (1972), pp. 250–270. ISSN: 0021-8693. DOI: https://doi.org/10.1016/0021-8693(72)90058-0. URL: https://www.sciencedirect.com/science/article/pii/0021869372900580.

References XI

[Wei80] Joachim Weidmann. *Linear operators in Hilbert spaces*. Vol. 68. Graduate Texts in Mathematics. Translated from the German by Joseph Szücs. Springer-Verlag, New York-Berlin, 1980, pp. xiii+402. ISBN: 0-387-90427-1.