

Problem (Problem 1): In this exercise, we prove another fundamental result in differential topology, called the tubular neighborhood theorem. Let M be a compact smooth manifold with orientable boundary N . For simplicity, assume that N is connected. The tubular neighborhood theorem asserts that N admits a neighborhood in M which is diffeomorphic to $N \times [0, 1]$.

- (a) Choose a Riemannian metric on M , and show that N admits a nonvanishing vector field that is everywhere orthogonal to the tangent space of N . That is, a vector field X such that for all $p \in N$, $g(X_p, T_p N) = 0$.
- (b) Use the flow generated by X to find the desired neighborhood.

Solution:

- (a) If $p \in N$, then we observe that $T_p N < T_p M$ is a proper subspace with codimension 1. Letting $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for $T_p N$, then we may extend to a basis for $T_p M$ by taking a representative for a basis for $T_p M / T_p N$, and observing that such a vector necessarily has

$$g_p(e_n, e_k) = 0$$

for all $k = 1, \dots, n-1$. By smoothly varying over all points $p \in N$, we get our desired everywhere nonvanishing vector field normal to $T_p N$.

- (b) Let φ_t be the one-parameter diffeomorphism group generated by X , where $\varphi_t: M \rightarrow M$ is such that $\varphi_0(p) = p$ for all $p \in N$. Then, $\varphi: (-\varepsilon, \varepsilon) \rightarrow \text{diff}(M)$ restricted to $[0, \varepsilon)$ gives our desired neighborhood in M diffeomorphic to $N \times [0, 1]$.

Problem (Problem Set 7, Problem 5): Suppose G is a finite group acting freely on a manifold M by diffeomorphisms.

- (a) Show that M/G is a manifold.
- (b) Show that the de Rham cohomology of M/G is isomorphic to the G -invariant cohomology of M .

Solution:

- (a) Let $p \in M$, and let U be a neighborhood of p . By shrinking U if necessary, the fact that G acts freely on M implies that for all $g \neq 1$, we have $g \cdot U \cap U = \emptyset$. In particular, this gives that the projection map $q: M \rightarrow M/G$ is a covering map. Thus, to find a chart about $[p] \in M/G$, we consider the image $\bar{U} := q(U) \subseteq M/G$. If φ is the coordinate map for U , define $\bar{\varphi}: \bar{U} \rightarrow \mathbb{R}^n$ by taking $\bar{\varphi}(q(u)) = \varphi \circ q^{-1}(\bar{U})$.

Let $[w] \in \bar{U} \cap \bar{V}$. We have some $w_1 \in U$ and $w_2 \in V$ such that $q(w_1) = q(w_2) = [w]$; in particular, there is $g \in G$ such that $g \cdot w_1 = w_2$. We may define $U' = U \cap g^{-1} \cdot V$ and $V' = V \cap (g \cdot U)$, where $\bar{U}' \cap \bar{V}' \subseteq \bar{U} \cap \bar{V}$ and $[w] \in \bar{U}' \cap \bar{V}'$. Furthermore, we see that $q(U' \cap V') = \bar{U}' \cap \bar{V}'$, as any element in the latter is given by $[x]$, where $x_1 \in U'$ and $x_2 \in V'$ have $q(x_i) = [x]$, meaning that the element $k \cdot x_1 = x_2$ is uniquely determined as the action is free.

We observe now that the transition map $\bar{\psi} \circ \bar{\varphi}^{-1}: \bar{\varphi}(\bar{U}' \cap \bar{V}') \rightarrow \bar{\psi}(\bar{U}' \cap \bar{V}')$ is then given, by the definition of these maps, by the transition map between $\varphi(U \cap (g^{-1} \cdot V))$ to $\psi((g \cdot U) \cap V)$. Therefore, M/G is a manifold.

- (b) Consider a closed form $\omega \in \mathcal{A}^*(M/G)$. The pullback $q^*\omega \in \mathcal{A}^*(M)$ is necessarily G -invariant. This descends to a map in cohomology

$$q^*: H_{\text{DR}}^*(M/G) \rightarrow H_{\text{DR}}^*(M)^G.$$

Our goal is to show that this map is injective and surjective. First, let $[\omega] \in \ker(q^*)$. Then, ω is

exact, meaning that $q^*\omega$ is exact, so there is some η such that $q^*\omega = d\eta$. If we let

$$\xi = \frac{1}{|G|} \sum_{g \in |G|} g^* \eta,$$

then we see that ξ is G -invariant and has $d\xi = \eta$. By the commutativity of pullback and d , there is then some $\bar{\xi} \in H_{DR}^*(M/G)$ such that $q^*\bar{\xi} = \xi$, meaning that $d\xi$ and ω are in the same cohomology class. Thus, $\ker(q^*) = \{0\}$.

Now, to see surjectivity, let $[\omega] \in H_{DR}^*(M)$ be G -invariant. By using the same averaging process, we have a representative of the cohomology class that can be found by pullback of a closed form in $H_{DR}^*(M/G)$, so that $H_{DR}^*(M/G) \cong H_{DR}^*(M)^G$.

Problem (Problem Set 8, Problem 3): Compute the de Rham cohomology of $\mathbb{R}\mathbb{P}^n$.

Solution: We observe that the antipodal map, $x \mapsto -x$, is a finite free action on the manifold S^n , and is such that the orbit space is $\mathbb{R}\mathbb{P}^n$, given by $\mathbb{Z}/2\mathbb{Z}$.

We know that the cohomology for S^n is given by \mathbb{R} at H^0 , and \mathbb{R} at H^n , with 0 everywhere else. The antipodal map is of degree 1 if and only if n is odd, which means that the antipodal map is thus a sign-preserving local diffeomorphism. In particular, this means that for an n -form $\omega \in \mathcal{A}^n(\mathbb{R}\mathbb{P}^n)$, we have

$$\int_{S^n} q^* \omega = \int_{\mathbb{R}\mathbb{P}^n} \omega,$$

so that the simplicial cohomology is identical. Thus, if n is odd, then $\mathbb{R}\mathbb{P}^n$ and S^n have the same cohomology.

Now, if n is even, we know that the degree of the antipodal map is -1 . Yet, this means that there are no invariant closed n -forms, meaning that the top-dimensional cohomology of $\mathbb{R}\mathbb{P}^n$ is 0.

Problem (Problem Set 8, Problem 5): Use the Mayer–Vietoris sequence to prove the Künneth Formula: if M and N are smooth manifolds, then $H_{DR}^*(M \times N)$ is the tensor product of $H_{DR}^*(M)$ and $H_{DR}^*(N)$. Specifically, in each degree ℓ , we have

$$H_{DR}^\ell(M \times N) = \bigoplus_{i+j=\ell} H_{DR}^i(M) \otimes H_{DR}^j(N).$$

Solution: For the sake of being able to solve this problem, we focus on the case where M and N are closed smooth manifolds.

Let $V = M \times N$ be the product manifold for M and N . If $\pi_1: V \rightarrow M$ and $\pi_2: V \rightarrow N$ are the projection maps on M and N respectively, we get the composed maps

$$\mathcal{A}^k(M) \times \mathcal{A}^\ell(N) \rightarrow \mathcal{A}^{k+\ell}(V)$$

given by $(\omega, \eta) \mapsto \pi_1^* \omega \wedge \pi_2^* \eta$. If ω and η are closed forms, then we observe that

$$\begin{aligned} d(\pi_1^* \omega \wedge \pi_2^* \eta) &= d\pi_1^* \omega \wedge \pi_2^* \eta + (-1)^k \pi_1^* \omega \wedge d\pi_2^* \eta \\ &= \pi_1^*(d\omega) \wedge \pi_2^* \eta + (-1)^k \pi_1^* \omega \wedge \pi_2^*(d\eta) \\ &= 0. \end{aligned}$$

Furthermore, if we let $\omega' = \omega + d\tau$ and $\eta' = \eta + d\rho$, then we know from earlier work that $\pi_1^* \omega' \wedge \pi_2^* \eta'$ can be expressed as $\pi_1^* \omega \wedge \pi_2^* \eta + d\sigma$ for some form σ by using the fact that d and the pullback commute. Thus, it follows that the map descends to a map in cohomology, given by

$$H_{DR}^k(M) \times H_{DR}^\ell(N) \rightarrow H^{k+\ell}(M \times N)$$

$$([\omega], [\eta]) \mapsto [\pi_1^* \omega \wedge \pi_2^* \eta],$$

whence via the universal property of tensor products and direct sums, we get the map

$$\psi: H_{\text{DR}}^*(M) \otimes H_{\text{DR}}^*(N) \rightarrow H^*(M \times N).$$

Our goal now is to show that ψ is indeed an isomorphism.

Toward this end, suppose we have two open sets in the good cover for M , given by U_1 and U_2 . From the Mayer–Vietoris sequence, this yields the following exact sequence in cohomology for a fixed k , where D_k denote the connecting homomorphisms from $H^k(U_1 \cap U_2)$ to $H^{k+1}(M)$.

$$\dots \xrightarrow{D_{k-1}} H_{\text{DR}}^k(M) \xrightarrow{i} H_{\text{DR}}^k(U_1) \oplus H_{\text{DR}}^k(U_2) \xrightarrow{j} H_{\text{DR}}^k(U_1 \cap U_2) \xrightarrow{D_k} \dots$$

Since the tensor product preserves exact sequences, we observe that by taking the tensor product with $H_{\text{DR}}^\ell(N)$, giving the following.

$$\dots \xrightarrow{D_{k-1}} H_{\text{DR}}^k(M) \otimes H_{\text{DR}}^\ell(N) \xrightarrow{i} H_{\text{DR}}^k(U_1) \otimes H_{\text{DR}}^\ell(N) \oplus H_{\text{DR}}^k(U_2) \otimes H_{\text{DR}}^\ell(N) \xrightarrow{j} H^k(U_1 \cap U_2) \otimes H^\ell(N) \xrightarrow{D_k} \dots$$

Taking direct sums with the same dimension, we obtain the following diagram.

$$\begin{array}{ccc} & \vdots & \vdots \\ & \downarrow & \downarrow \\ \bigoplus_{k+\ell=i-1} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^{i-1}((U_1 \cap U_2) \times N) \\ D_{i-1} \downarrow & & \downarrow D_{i-1} \\ \bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i(M \times N) \\ i \downarrow & & \downarrow i \\ \bigoplus_{k+\ell=i} (H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) & \xrightarrow{\psi} & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \\ j \downarrow & & \downarrow j \\ \bigoplus_{k+\ell=i} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i((U_1 \cap U_2) \times N) \\ D_i \downarrow & & \downarrow D_i \\ & \vdots & \vdots \end{array}$$

Since U_1 , U_2 , and $U_1 \cap U_2$ are contractible, under the good cover assumption, it follows from the Poincaré Lemma that the following subsection of the diagram is commutative, with ψ necessarily an isomorphism in each of the columns.

$$\begin{array}{ccc} \bigoplus_{k+\ell=i} (H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) & \xrightarrow{\psi} & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \\ j \downarrow & & \downarrow j \\ \bigoplus_{k+\ell=i} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i((U_1 \cap U_2) \times N) \end{array}$$

Similarly, we have that the following diagram is commutative, following from the Mayer–Vietoris sequence.

$$\begin{array}{ccc} \bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i(M \times N) \\ i \downarrow & & \downarrow j \\ \bigoplus_{k+\ell=i} (H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) & \xrightarrow{\psi} & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \end{array}$$

Therefore, we only need to verify commutativity for the following square.

$$\begin{array}{ccc} \bigoplus_{k+\ell=i-1} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^{i-1}((U_1 \cap U_2) \times N) \\ D_{i-1} \downarrow & & \downarrow D_{i-1} \\ \bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i(M \times N) \end{array}$$

First, from the Mayer–Vietoris sequence and the fact that the coboundary map in de Rham cohomology emerges from the exterior derivative, we have that the map D_i is given by

$$D_{i-1}([\omega]) = \begin{cases} [d(-f_U \omega)] \\ [d(f_V \omega)] \end{cases}$$

for any cohomology class representative ω . Now, we observe that

$$\begin{aligned} \psi(D_{i-1}([\omega], [\eta])) &= [\pi_1^*(D_{i-1}(\omega)) \wedge \pi_2^*\eta] \\ D_{i-1}(\psi([\omega], [\eta])) &= [D_{i-1}(\pi_1^*\omega \wedge \pi_2^*\eta)] \end{aligned}$$

In particular, since $\pi_1^* f_U$ and $\pi_1^* f_V$ form a partition of unity for $M \times F$, we have

$$\begin{aligned} \pi_1^*(D_{i-1}(\omega)) \wedge \pi_2^*\eta &= \pi_1^*(d(f_V \omega)) \wedge \pi_2^*\eta \\ &= d(\pi_1^*(f_V \omega)) \wedge \pi_2^*\eta \\ D_{i-1}(\pi_1^*\omega \wedge \pi_2^*\eta) &= d(\pi_1^* f_V \pi_1^*\omega \wedge \pi_2^*\eta) \\ &= d(\pi_1^*(f_V \omega)) \wedge \pi_2^*\eta. \end{aligned}$$

Since ψ at each of U , V , and $U \cap V$ is an isomorphism, and the diagram commutes, the Five Lemma gives that ψ at M is an isomorphism.

For any finite good cover with more than 2 elements, induction gives the desired result.

Problem (Problem Set 10, Problem 7): Show that the invariant cohomology $H_L^*(G)$ is isomorphic to the de Rham cohomology of G . Conclude that the de Rham cohomology of an n -torus is isomorphic to the exterior algebra on \mathbb{R}^n .