Problem (Problem 1): Let R be a commutative ring. An R-module M is called torsion if for any $m \in M$, there is a nonzero $r \in R$ such that $rm = 0$. An R-module M is called divisible if for any nonzero $r \in R$, we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$, there is $x \in M$ such that $rx = m$.

(a) Suppose M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Solution:

(a) It is enough to show that any simple tensor $m \otimes n \in M \otimes_R N$ is the zero tensor. To see this, we let $r \in R$ be such that $rm = 0$, and observe that there is some $x \in N$ such that $rx = n$. By using property (R3) of tensor products, we observe then that

$$\begin{aligned} m \otimes n &= m \otimes (rx) \\ &= (rm) \otimes x \\ &= 0 \otimes x \\ &= 0. \end{aligned}$$

Thus, $M \otimes_R N = \{0\}$.

(b) It is enough to show that $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible, as we may then apply (a). To see that $\mathbb{Q}/\mathbb{Z}$ is torsion, we have that

$$\begin{aligned} b\left[\frac{a}{b}\right] &= [a] \\ &= [0] \end{aligned}$$

for any element $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$. Additionally, for any $n \in \mathbb{Z}$, we have

$$\left[\frac{a}{b}\right] = n\left[\frac{a}{nb}\right],$$

so $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible.

Problem (Problem 2): Let R be a commutative ring, $\{N_\alpha\}_{\alpha \in A}$ a collection of R-modules, and M another R-module.

(a) Prove that $M \otimes \left(\bigoplus_\alpha N_\alpha\right) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) Show by example that $M \otimes \left(\prod_\alpha N_\alpha\right)$ need not be isomorphic to $\prod_\alpha (M \otimes N_\alpha)$.

Solution:

(a) Consider the map on elementary tensors

$$f \colon M \times \left(\bigoplus_\alpha N_\alpha\right) \to \bigoplus_\alpha (M \otimes N_\alpha)$$

that takes

$$(m, (n_\alpha)_\alpha) \to (m \otimes n_\alpha)_\alpha.$$

We observe that, since the $(n_\alpha)_\alpha$ are nonzero for all but finitely many indices α, and that the map is R-bilinear, we have a well-defined and unique R-linear map $\overline{f} \colon M \otimes \left(\bigoplus_\alpha N_\alpha\right) \to \bigoplus_\alpha (M \otimes N_\alpha)$ that maps $m \otimes (n_\alpha)_\alpha \mapsto (m \otimes n_\alpha))_\alpha$.

We observe that for each index i, we have an inclusion homomorphism

$$M \times N_i \hookrightarrow M \otimes \left(\bigoplus_\alpha N_\alpha\right)$$

that takes $(m, n_\alpha) \mapsto m \otimes (n_\alpha)_\alpha$, where $(n_\alpha)_\alpha$ is zero everywhere except for index i. By the universal property of the direct sum, this induces a unique homomorphism $g \colon \bigoplus_\alpha (M \otimes N_\alpha) \to M \otimes (\bigoplus_\alpha N_\alpha)$ given by taking

$$(m_\alpha \otimes n_\alpha)_\alpha \mapsto \sum_\alpha m_\alpha \otimes (n_\alpha)_\alpha,$$

where the summand $(n_\alpha)_\alpha$ is defined as above, and the sum is finite by the definition of the direct sum. Since g and f are inverses of each other (as can be seen by the action on simple tensors), it follows that $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) We consider the direct product

$$M = \prod_{i=1}^{\infty} \mathbb{Z}/2^i \mathbb{Z},$$

regarded as a $\mathbb{Z}$-module. Notice that M is not torsion, as the element $m = (1, 1, \dots)$ is such that there is no $z \in \mathbb{Z}$ with $zm = 0$. Therefore, considering the extension of scalars

$$\mathbb{Q} \otimes M = \mathbb{Q} \otimes \left(\prod_{i=1}^{\infty} \mathbb{Z}/2^i \mathbb{Z} \right),$$

we have that this is not a zero module. Yet, since each of the individual $\mathbb{Z}/2^i \mathbb{Z}$ has torsion, it would follow that

$$\prod_{i=1}^{\infty} (\mathbb{Q} \otimes \mathbb{Z}/2^i \mathbb{Z}) = 0,$$

so it follows that tensor products do not commute with direct sums.

Problem (Problem 4): Let R be commutative, and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r + I) \otimes (r' + J)$ to $rr' + (I + J)$.

Solution:

(a) By using R-bilinearity, we observe that an arbitrary simple tensor in $R/I \otimes R/J$ can be written as

$$\begin{aligned}
(r + I) \otimes (s + J) &= (r(1 + I)) \otimes (s + J) \\
&= r((1 + I) \otimes (s + J)) \\
&= (1 + I) \otimes (rs + J).
\end{aligned}$$

Since any element of $R/I \otimes_R R/J$ can be written as a sum of simple tensors, and each simple tensor can be written in the above form, it follows from bilinearity that every element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (r + J)$.

(b) We consider the map

$$f \colon R/I \times R/J \mapsto R/(I + J)$$

given by

$$(r + I, r' + J) \mapsto rr' + (I + J).$$

This map is R-bilinear by the distributive properties of multiplication, so it induces a homomorphism on the tensor product given by

$$(r + I) \otimes (r' + J) \mapsto rr' + (I + J).$$

As was established above, any element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (s + J)$, so we may establish an inverse from any element of $R/(I + J)$ to $R/I \otimes R/J$ by taking $t + (I + J) \mapsto (1 + I) \otimes (t + J)$. This establishes our desired isomorphism.