**Problem** (Problem 1)**:** Let $F$ be a field, $a(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in F[x]$ a nonconstant monic polynomial, and let $A = C_{a(x)}$ be its companion matrix. Prove by direct computation that $\mathrm{SNF}(xI - A) = \mathrm{diag}(1, \dots, 1, a(x))$.

**Solution:** We observe that

$$xI - A = \begin{pmatrix} x & 0 & \cdots & 0 & a_0 \\ -1 & x & \cdots & 0 & a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x + a_{n-1} \end{pmatrix}.$$

Focusing on the bottom 2 rows, we use the following reduction method

$$\begin{aligned} \begin{pmatrix} x & a_{n-2} \\ -1 & x + a_{n-1} \end{pmatrix} &\xrightarrow{R_{n-1} \leftarrow xR_n + R_{n-1}} \begin{pmatrix} 0 & x^2 + a_{n-1}x + a_{n-2} \\ -1 & x + a_{n-1} \end{pmatrix} \\ &\xrightarrow{C_n \leftarrow (x + a_{n-1})C_{n-1} + C_n} \begin{pmatrix} 0 & x^2 + a_{n-1}x + a_{n-2} \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Inductively repeating this reduction method, we say at step $i$ that we perform the following two operations consecutively

- $R_{n-i} \leftarrow xR_{n-i+1} + R_{n-i}$;
- $C_{n-i+1} \leftarrow \left(x^i + a_{n-1}x^{i-1} + \cdots + a_{n-i}\right)C_{n-i} + C_{n-i+1}$

Upon completion of this process at step $n$, we obtain a matrix consisting entirely of $-1$ along the subdiagonal and $a(x)$ in position $(1, n)$. Next, we perform the following procedure as $i$ ranges from $1$ to $n - 1$.

- $R_i \leftarrow (-1)R_{i+1} + R_i$;
- $R_{i+1} \leftarrow R_i + R_{i+1}$.

This gives a matrix with 1 along the diagonal and $a(x)$ along column $n$. Then, upon performing the operation

- $R_i \leftarrow (-1)R_n + R_i$

for each $1 \leq i \leq n - 1$, we obtain our desired diagonal matrix in Smith normal form, where we have $\mathrm{diag}(1, \dots, 1, a(x))$.

**Problem** (Problem 2)**:** Prove that the constant term in the characteristic polynomial of the $n \times n$ matrix $A$ is $(-1)^n \det(A)$, and that the coefficient of $x^{n-1}$ is the negative of the sum of the diagonal entries of $A$. Prove that $\det(A)$ is the product of the eigenvalues of $A$ and that the trace of $A$ is the sum of the eigenvalues of $A$.

**Solution:** We start by showing that this holds for a companion matrix, $A = C_{a(x)}$. Note that in our computation showing that $\mathrm{SNF}(xI - A) = \mathrm{diag}(1, 1, \dots, a(x))$, we exclusively used row and column operations (and employed no flips); as a result, it follows that the characteristic polynomial of a companion matrix for $a(x)$ is exactly $a(x)$. Then, we observe that

$$\begin{aligned} a_0 &= \chi_A(0) \\ &= \det(-A) \\ &= \det((-I)A) \\ &= \det(-I)\det(A) \\ &= (-1)^n \det(A), \end{aligned}$$

and that the coefficient on the $x^{n-1}$ is equal to $a_{n-1}$, or $-(-a_{n-1})$, which is the trace of the companion matrix.

In the general case, we observe that $A$ is similar to a matrix in rational canonical form,

$$A \sim \operatorname{diag}(A_1, \dots, A_r),$$

and has

$$\chi_A(x) = \chi_{A_1}(x) \cdots \chi_{A_r}(x),$$

where we use the fact that characteristic polynomials are invariant under similarity transformation, so that

$$\begin{aligned}
\chi_A(0) &= \chi_{A_1}(0) \cdots \chi_{A_r}(0) \\
&= a_{0,1} \cdots a_{0,r} \\
&= (-1)^{n_1} \det(A_1) \cdots (-1)^{n_r} \det(A_r) \\
&= (-1)^n \det(A_1) \cdots \det(A_r) \\
&= (-1)^n \det(A),
\end{aligned}$$

where we let $n_i$ denote the dimension of the specific companion matrix $A_i$. Additionally, we observe that the coefficient on the $n-1$ degree term on $\chi_A(x)$ is given summing the coefficient of an $n_i - 1$ degree term with the $n_j$ degree terms for all $j \neq i$. In particular, this means that we get

$$\begin{aligned}
a_{n-1} &= \sum_{i=1}^{r} a_{n_i - 1} \\
&= \sum_{i=1}^{r} -\operatorname{Tr}(A_i) \\
&= -\operatorname{Tr}(A).
\end{aligned}$$

From basic properties of polynomials, we know that the constant term of a polynomial of degree $n$ is equal to $(-1)^n$ multiplied by the product of the roots, while the coefficient on the degree $n-1$ term is equal to $-1$ multiplied by the sum of the roots. In particular, applying this to the characteristic polynomial, we get that the trace is the sum of the eigenvalues of $A$ and the determinant is the product of the eigenvalues.

**Problem** (Problem 3)**:** Determine the number of possible RCFs of $8 \times 8$ matrices over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$.

**Solution:** Factoring over $\mathbb{Q}$, we have that

$$\chi_A(x) = x^4\left(x^2+1\right)(x-1)(x+1).$$

In order to determine the possible rational canonical forms, we need to determine the possible invariant factors, $a_1(x)|a_2(x)| \cdots |a_d(x)$, subject to the constraint that $a_d(x) = \mu_A(x)$ has the same roots as $\chi_A(x)$. In particular, we must have that $\mu_A(x)$ can only be one of the following, where we observe that we cannot have $x^2+1$ anywhere in the invariant factor decomposition outside of the minimal polynomial since it has multiplicity 1:

- $p_1(x) = x\left(x^2+1\right)(x-1)(x+1)$;
- $p_1(x) = x^2\left(x^2+1\right)(x-1)(x+1)$;
- $p_2(x) = x^3\left(x^2+1\right)(x-1)(x+1)$;
- $p_4(x) = x^4\left(x^2+1\right)(x-1)(x+1)$.

We find that the possible decompositions are thus

$$\begin{aligned}
A_1 &= [x, x, x, p_1(x)] \\
A_2 &= \left[1, x, x^2, p_1(x)\right] \\
A_3 &= \left[1, 1, x^3, p_1(x)\right]
\end{aligned}$$

$$\begin{aligned} B_1 &= [x, x, p_2(x)] \\ B_2 &= \left[1, x^2, p_2(x)\right] \\ C &= [1, x, p_3(x)] \\ D &= [p_4(x)]. \end{aligned}$$

**Problem** (Problem 4)**:** Prove that two 3×3 matrices over some field $F$ are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing this does not hold for $4 \times 4$ matrices.

**Solution:** Suppose $A$ and $B$ are $3 \times 3$ matrices with characteristic polynomial $\chi(x)$ and minimal polynomial $\mu(x)$. The characteristic polynomial has degree 3, so we may consider the degree(s) of the minimal polynomial.

If $\mu(x)$ has degree 1, then it is of the form $\mu(x) = x - a$; this is a prime in $F[x]$, and since the degree of the characteristic polynomial is 3 and all the invariant factors must divide $\mu(x)$, it follows that $A$ and $B$ have invariant factors given by

$$a_i(x) = [(x-a), (x-a), (x-a)],$$

so since they have the same invariant factors, they have the same rational canonical form and are thus similar.

If $\mu(x)$ has degree 2, then the lower $2 \times 2$ submatrix of both $A$ and $B$ are equal, and both of them admit invariant factors given by

$$a_i(x) = \left[\frac{\chi(x)}{\mu(x)}, \mu(x)\right].$$

Finally, if $\mu(x)$ has degree 3, then both $A$ and $B$ admit the same rational canonical form as both of them have the invariant factor $\mu(x)$.

As a counter-example in the $4 \times 4$ case, consider the matrices with minimal polynomial $\mu(x) = (x-1)^1$ and characteristic polynomial $\chi(x) = (x-1)^4$. These matrices have invariant factor decompositions

$$\begin{aligned} a_i(x) &= \left[(x-1), (x-1), (x-1)^2\right] \\ b_i(x) &= \left[(x-1)^2, (x-1)^2\right], \end{aligned}$$

admitting rational canonical forms

$$\begin{aligned} A &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}. \end{aligned}$$

Since these rational canonical forms differ, these matrices are necessarily not similar.

**Problem** (Problem 5)**:** Find the number of distinct conjugacy classes in the group $\mathrm{GL}_3(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the field with two elements, and specify one element in each conjugacy class.

**Solution:** We start by finding all the polynomials of degree 3 (representing all the possible characteristic polynomials) over $\mathbb{F}_2$ as follows:

(i) $x^3$;

(ii) $x^3+1=(x+1)\left(x^2+x+1\right)$;

(iii) $x^3+x=x(x+1)^2$;

(iv) $x^3+x^2=x^2(x+1)$;

(v) $x^3+x+1$;

(vi) $x^3+x^2+1$;

(vii) $x^3+x^2+x=x\left(x^2+x+1\right)$;

(viii) $x^3+x^2+x+1=(x+1)^3$.

The irreducible polynomials of this list, which are (i), (v), and (vi), must admit exactly one minimal polynomial in their invariant factor decomposition, meaning they admit the rational canonical forms listed below:

(C1) $\left[x^3\right]=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$;

(C2) $\left[x^3+x+1\right]=\begin{pmatrix}0&0&1\\1&0&1\\0&1&0\end{pmatrix}$;

(C3) $\left[x^3+x^2+1\right]=\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$.

Next, we observe that the invariant factors for (ii) must divide either $(x+1)$ or $\left(x^2+x+1\right)$, but since both of these are irreducible in $\mathbb{F}_2[x]$, and their product is of degree 3, it follows that the minimal polynomial is equal to $(x+1)\left(x^2+x+1\right)$, meaning that we get the following rational canonical form:

(C4) $\left[x^3+1\right]=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$.

A similar story as (ii) occurs in (vii), where since both $x$ and $x^2+x+1$ are irreducible and all the invariant factors must divide one of these, it follows that the only possible minimal polynomial is $x\left(x^2+x+1\right)$, so that the following rational canonical form holds:

(C5) $\left[x^3+x^2+x\right]=\begin{pmatrix}0&0&0\\1&0&1\\0&1&1\end{pmatrix}$.

**Problem** (Problem 6)**:** Prove that the matrices

$$A=\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$$

$$B=\begin{pmatrix}5&2&-8&-8\\-6&-3&8&8\\-3&-1&3&4\\3&1&-4&-5\end{pmatrix}$$

both have characteristic polynomial $(x-3)(x+1)^3$. Determine whether they are similar and determine the Jordan canonical form for each matrix.

**Solution:** We observe that

$$xI - A = \begin{pmatrix} x & -1 & -1 & -1\\ -1 & x & -1 & -1\\ -1 & -1 & x & -1\\ -1 & -1 & -1 & x \end{pmatrix}$$

$$xI - B = \begin{pmatrix} x-5 & -2 & 8 & 8\\ 6 & x+3 & -8 & -8\\ 3 & 1 & x-3 & -4\\ -3 & -1 & 4 & x+5 \end{pmatrix}$$

To resolve these determinants, we row reduce.

$$\begin{pmatrix} x & -1 & -1 & -1\\ -1 & x & -1 & -1\\ -1 & -1 & x & -1\\ -1 & -1 & -1 & x \end{pmatrix} \xmapsto{R_2 \leftarrow -R_1 + R_2} \begin{pmatrix} x & -1 & -1 & -1\\ -x-1 & x+1 & 0 & 0\\ -1 & -1 & x & -1\\ -1 & -1 & -1 & x \end{pmatrix}$$