Basics

Definition: Let A be a C^*-algebra. A *representation* of A is a $*$-homomorphism $\pi\colon A \to B(H)$ for some Hilbert space H.

Definition: Two representations $\pi\colon A \to B(H_\pi)$ and $\rho\colon A \to B(H_\rho)$ are called unitarily equivalent if there is a unitary map $U\colon H_\rho \to H_\pi$ such that

$$\pi(a) = U\rho(a)U^*$$

for all $a \in A$.

Definition: If $\pi\colon A \to B(H_\pi)$ and $\rho\colon A \to B(H_\rho)$ be representations. Then, the formula

$$\pi \oplus \rho(a)(h, k) \coloneqq (\pi(a)h, \rho(a)k)$$

defines the *direct sum* of π and ρ. If π is unitarily equivalent to a direct sum $\rho_1 \oplus \rho_2$, then we consider $\rho_1 \oplus \rho_2$ to be a decomposition of π in terms of the "smaller" representations.

Definition: A closed subspace K of H_π is *invariant* under π if $\pi(a)k \in K$ for all $a \in A$ and $k \in K$.

Observe that if K is an invariant subspace, then the orthogonal complement $K^\perp$ is also invariant. This follows from the fact that if $y \in K^\perp$, then

$$\begin{aligned}\langle k, \pi(a)y\rangle &= \langle \pi(a)^*k, y\rangle \\ &= \langle \pi(a^*)k, y\rangle \\ &= 0\end{aligned}$$

for all $k \in K$.

Conversely, if K is invariant, then we can recover $\pi = \pi|_K \oplus \pi|_{K^\perp}$, via the canonical unitary isomorphism $U\colon K \oplus K^\perp \to H_\pi$ given by $(k, y) \mapsto k + y$.

Definition: A representation π is *irreducible* if there are no closed invariant subspaces apart from $\{0\}$ and H_π.

Lemma: A representation π of a C^*-algebra A is irreducible if and only if $\pi(A)' = \mathbb{C}I_H$, where $\pi(A)'$ denotes the commutant of $\pi(A)$.

Proof. Suppose V is a nontrivial invariant subspace for π. Then, the orthogonal projection P_V commutes with every $\pi(A)$ and is not a scalar multiple of I_H.

Now, suppose there is a non-scalar operator T commuting with $\pi(A)$. Then, either the real or imaginary part of T is a self-adjoint operator S that commutes with $\pi(A)$. From the continuous functional calculus, since $\sigma(S)$ is not one point, there are some nonzero continuous $f, g \in C(\sigma(S))$ such that $fg = 0$. Then, since $f(S), g(S) \in C^*(S)$, and $f(S), g(S)$ commute with $\pi(A)$, it follows that $\overline{f(S)H}$ and $\overline{g(S)H}$ are nonzero mutually orthogonal invariant subspaces, so π is reducible. $\square$

Definition: If π is a representation of the C^*-algebra A, then we call the subspace

$$[\pi(A)H_\pi] = \overline{\operatorname{span}}\{\pi(a)h \mid h \in H_\pi, a \in A\}$$

the *essential subspace* of H_π. The representation π is called *nondegenerate* if the essential subspace K is equal to H_π.

Note that the representation π being nondegenerate is equivalent to $\pi(1) = I_{H_\pi}$ if A has an identity, or $\pi(e_i) \to I_{H_\pi}$ strongly for any approximate identity $(e_i)_{i\in I}$.

The essential subspace is always invariant, and π is equivalent to $\pi|_K \oplus 0$. Generally, if I is an ideal in A, then the subspace

$$K = \overline{\operatorname{span}}\{\pi(a)h \mid h \in H_\pi, a \in I\}$$

is invariant, but π is not zero on $K^\perp$ unless I is an essential ideal.[I] Any nondegenerate representation of an ideal I extends canonically to a nondegenerate representation π of A on the same space.

The Gelfand–Naimark–Segal Construction

The primary way we represent an abstract C^*-algebra is via the Gelfand–Naimark–Segal construction, which starts by using a special linear functional called a state to build a representation.

Definition: A *state* on a C^*-algebra A is a linear functional $\varphi\colon A \to \mathbb{C}$ such that $\varphi(a) \geq 0$ whenever $a \geq 0$.

Lemma: If φ is a positive linear functional, then φ satisfies the Cauchy–Schwarz inequality,

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Proof. The pairing $[a,b] = \varphi(b^*a)$ is a positive semidefinite sesquilinear form by the various operations. Therefore, it satisfies the Cauchy–Schwarz inequality. □

Lemma: If φ is a positive linear functional on a C^*-algebra A, then φ is continuous.

Furthermore, if A is unital, then $\|\varphi\| = \varphi(1)$, and if A is non-unital with approximate unit $(e_i)_i$, then $\|\varphi\| = \lim_{i\in I}\varphi(e_i)$.

Proof. If A is unital, then for any $0 \leq a \leq 1$, we have $0 \leq \varphi(a) \leq \varphi(1)$. For general $\|a\| \leq 1$, we have

$$\begin{aligned}|\varphi(a)|^2 &= |\varphi(1^*a)|^2\\ &\leq \varphi(a^*a)\varphi(1)\\ &\leq \varphi(1)^2,\end{aligned}$$

so that $\|\varphi\| = \varphi(1)$.

Now, if A is not unital, we start by showing continuity on A_+. If it were not continuous, then we would have $a_n \geq 0$ with $\|a_n\| \leq 2^{-n}$ and $\varphi(a_n) > 1$. If

$$a = \sum_{n=1}^{\infty} a_n,$$

we have that a is positive with $\|a\| \leq 1$. Yet,

$$\varphi(a) = \sum_{n=1}^{N} + \varphi\left(\sum_{n=N+1}^{\infty} a_n\right) > N$$

for all $N \geq 1$, which cannot happen, so there is some constant C such that $0 \leq \varphi(a) \leq C\|a\|$ for any $a \geq 0$. For general a, we may write

$$a = a_1 - a_2 + ia_3 - ia_4$$

with $0 \leq a_i \leq \|a\|$ for each i. Since $0 \leq \varphi(a_i) \leq C\|a\|$, we have

$$\begin{aligned}|\varphi(a)|^2 &= (\varphi(a_1) - \varphi(a_2))^2 + (\varphi(a_3) - \varphi(a_4))^2\\ &\leq 2C^2\|a\|^2,\end{aligned}$$

so φ is continuous with $\|\varphi\| \leq C\sqrt{2}$.

Let $(e_i)_i$ be an approximate identity for A. We have that

$$\varphi(e_j) = \lim_{i\in I}\varphi(e_i e_j e_i)$$

[I] An essential ideal is one that has nonzero intersection with any other closed ideal of A.

$$\begin{aligned} &\leq \liminf_{i\in I} \varphi\left(e_i^2\right) \\ &\leq \liminf_{i\in I} \varphi(e_i), \end{aligned}$$

following from the fact that $0 \leq e_i \leq 1$ for each i and that φ preserves order. Thus, we have

$$\limsup_{j\in I} \varphi(e_j) \leq \liminf_{i\in I} \varphi(e_i),$$

so we may define

$$M \coloneqq \lim_{i\in I} \varphi(e_i)$$

with $M \leq \|\varphi\|$ unambiguously. For any $a \in A$ with $\|a\| \leq 1$, we may use the Cauchy–Schwarz inequality to find

$$\begin{aligned} \left|\varphi(a)^2\right| &= \lim_{i\in I} |\varphi(e_i a)|^2 \\ &\leq \limsup_{i\in I} |\varphi(a^* a)| \varphi\left(e_i^2\right) \\ &\leq \|\varphi\| M. \end{aligned}$$

Taking suprema, we find that $\|\varphi\| \leq M$, so $\|\varphi\| = M$. □

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