Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: Let $(f_n)_n$ be a Cauchy sequence in $C_0(\mathbb{R})$. Since each $f_k \in C_0(\mathbb{R})$, it must be the case that each f_k is uniformly continuous. For each $x \in \mathbb{R}$, it is thus the case that $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(f_n(x))_n \to f(x)$ for each $x \in \mathbb{R}$, and since each f_k is uniformly continuous, it must be the case that $f(x)$ is continuous.

For $\varepsilon > 0$, there must be N large such that for $m, n \geq N$ and $m \geq n$, it must be the case that $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Letting $m \to \infty$, we have $|f_n(x) - f(x)| < \varepsilon$, meaning $(f_n)_n \to f$. Thus, $f \in C_0(\mathbb{R})$.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $(x_n)_n$ be a Cauchy sequence in ℓ_2. Let $x_n(k)$ denote the index k of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$|x_n(k) - x_m(k)| \leq c \, \|x_m(k) - x_n(k)\|_2$$

$$\to 0 \qquad \qquad \text{since } (x_n)_n \text{ is Cauchy in } \ell_2.$$

Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, $x_n(k) \to x(k)$ as $k \to \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \to x(k)$ for each n.

We must show that $\|x_n - x\|_2 \to 0$ as $n \to \infty$. This is equivalent to

$$\lim_{N \to \infty} \sum_{k=1}^{N} \lim_{m \to \infty} |x_n(k) - x_m(k)|^2 \leq \lim_{m \to \infty} \sup_{m \geq M} \|x_n - x_m\|^2$$

$$\leq \varepsilon^2 \qquad \qquad \text{since } (x_n)_n \text{ is Cauchy.}$$

Thus, $\|x_n - x_m\| \to 0$ as $m \to \infty$ and $n \to \infty$, so $\|x_n - x\| \to 0$ as $n \to \infty$.

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq d(x_2, x_1)\left(\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}\right) \\
&= d(x_2, x_1)\theta^{n-1}\left(1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}\right) \\
&\leq d(x_2, x_1)\frac{\theta^{n-1}}{1 - \theta}.
\end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.

Problem 4

Let (X, d) be a complete metric space, and suppose $f : X \to X$ is a contractive map — i.e., there is a $\theta \in (0, 1)$ with

$$d(f(x), f(y)) \leq \theta d(x, y).$$

Prove that f has a unique fixed point.

Proof: Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all n, we have

$$d(x_n, x_{n-1}) \leq \theta^n d(x_1, x_0).$$

Therefore, for x_m, x_n arbitrary in X with $m > n$, we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\
&\leq \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0) \\
&= d(x_1, x_0)\theta^{n+1}\left(1 + \theta + \cdots + \theta^{m-n-1}\right) \\
&\leq d(x_1, x_0)\frac{\theta^{n+1}}{1 - \theta}.
\end{aligned}$$

Since $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$ as $n \to \infty$, it must be the case that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Since X is complete, this means $(x_n)_n \to x$ for some $x \in X$, meaning $f(x) = x$.

Suppose it were the case that there existed s, t distinct with $f(s) = s$ and $f(t) = t$. Then, $d(f(s), f(t)) = d(s, t) \leq \theta d(s, t)$, but $\theta < 1$. Thus, x is unique.

Problem 5

If $(f_k)_k$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, show that the map

$$\varphi : \ell_2 \to \mathcal{H}$$

$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_k$$

is a well-defined isometry.

Proof: Let $\xi, \eta \in \ell_2$. Then,

$$d(\xi, \eta) = \|\xi - \eta\|_2$$

$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_k$$

$$\varphi(\eta) = \sum_{k=1}^{\infty} \eta(k) f_k$$

$$d(\varphi(\xi), \varphi(\eta)) = \left(\sum_{k=1}^{\infty} \langle \xi(k) - \eta(k), f_k \rangle\right)^{1/2}$$

$$= \|\xi - \eta\|_2 \qquad \text{Parseval's Identity.}$$