Compact Operators

Definition: A linear map $T\colon X \to Y$ between Banach spaces is called *compact* if $T(B_X) \subseteq Y$ has compact closure, where B_X denotes the closed unit ball of X. We denote the space of compact operators $K(X, Y)$.

The theory of compact operators (and the soon to arise Fredholm operators) arose from the analysis of integral equations. To start, let $I = [0, 1]$, and consider the Banach space $C(I)$ with the supremum norm. Letting $k \in C(I \times I)$, we define $u \in B(X)$ by taking

$$Tf(x) = \int_0^1 k(x, y) f(y)\, dy.$$

The fact that $Tf \in X$ follows from an application of the Dominated Convergence Theorem and the fact that, since $k(x, y)$ is jointly continuous, it is also separately continuous (see [Fol99, Theorem 2.27]). In fact, we can show something even stronger: we claim that the family $T(B_X)$ is in fact equicontinuous. This follows from the fact that, I^2 is compact, so if $\varepsilon > 0$, there is δ such that whenever $\max\{|x - x'|, |y - y'|\} < \delta$, we have $|k(x, y) - k(x', y')| < \varepsilon$. Therefore,

$$\begin{aligned} |Tf(x) - Tf(x')| &= \left|\int_0^1 (k(x, y) - k(x', y)) f(y)\, dy\right| \\ &\leq \int_0^1 |k(x, y) - k(x', y)||f(y)|\, dy \\ &\leq \sup_{y \in I} |k(x, y) - k(x', y)| \|f\|_u \\ &\leq \varepsilon \|f\|_u. \end{aligned}$$

Furthermore, since

$$|Tf(x)| \leq \|k\|_u \|f\|_u,$$

we have that $T(B_X)$ is pointwise bounded. Thus, by the Arzelà–Ascoli theorem, it follows that $T(B_X)$ is totally bounded, so T is a compact operator. We call the function k the *kernel* of the operator T.

Similarly, the operator $V \in B(X)$ given by

$$Vf(x) = \int_0^x f(y)\, dy$$

is such that $V(B_X)$ is totally bounded by Arzelà–Ascoli, so V is also compact. In fact, V has no eigenvalues as well. This follows from the fact that, if there were $\lambda \in \mathbb{C} \setminus \{0\}$ with $V(f) = \lambda f$, then $f(0) = 0$ and $f'(t) = 1/\lambda f(t)$, so that $f(t) = f(0)e^{t/\lambda} = 0$, meaning $f = 0$.

We call the operator V the *Volterra integral operator* on X.

We can see that $K(X)$ is in fact an algebraic ideal in $B(X)$ (by continuity). In fact, there is a topological dimension to $K(X) \subseteq B(X)$.

Proposition: If X, Y are Banach spaces, then $K(X, Y)$ is a closed subspace of $B(X, Y)$.

Proof. Let $(T_n)_n$ converge to $T \in B(X, Y)$. Let $\varepsilon > 0$, and select N such that $\|T_N - T\| < \varepsilon/3$. Since $T_N(B_X)$ is totally bounded, there are $x_1, \dots, x_n \in B_X$ such that for each $x \in S$, we have

$$\|T_N x - T_N x_j\| < \varepsilon/3$$

for some j. Therefore, we have

$$\begin{aligned} \|Tx - Tx_j\| &\leq \|Tx - T_N x\| + \|T_N x - T_N x_j\| + \|T_N x_j - Tx_j\| \\ &< \varepsilon. \end{aligned}$$

Therefore, $T(B_X)$ is totally bounded, so $T \in K(X, Y)$. □

Therefore, we see that $\overline{F(X,Y)} \subseteq K(X,Y)$ is, where $F(X,Y)$ denotes the finite-rank operators, but this inclusion may be strict. In the cases where $\overline{F(X)} = K(X)$, we say the Banach space X has the approximation property. There are Banach spaces that do not have the approximation property.

Recall that if $T\colon X \to Y$ is a bounded linear map between Banach spaces, the transpose is defined by $T^*\colon Y^* \to X^*$, given by $T^*\varphi = \varphi \circ T$.

Theorem: If X and Y are Banach spaces with $T \in K(X,Y)$, then $T^* \in K(Y^*, X^*)$.

Proof. Let $\varepsilon > 0$. Since $T(B_X)$ is totally bounded, there exist elements $x_1, \dots, x_n$ such that if $x \in B_X$, then $\|Tx - Tx_i\| < \varepsilon/3$ for some i. Let $V \in B(Y^*, \mathbb{C}^n)$ be defined by $V\varphi = (\varphi(Tx_1), \dots, \varphi(Tx_n))$. Since V has finite rank, V is compact, so $V(B_{X^*})$ is totally bounded. Thus, there exist $\varphi_1, \dots, \varphi_m$ such that if $\varphi \in T$, then $\|V\varphi - V\varphi_j\| = \max_{i=1}^{n} |T^*\varphi(x_i) - T^*\varphi_j(x_i)|$.

Now, if $x \in B_X$, then $\|Tx - Tx_i\| < \varepsilon/3$ for some i, so thus $|T^*\varphi(x_i) - T^*\varphi_j(x_i)| < \varepsilon/3$. Thus,

$$\begin{aligned} |T^*\varphi(x) - T^*\varphi_j(x)| &\leq |T^*\varphi(x) - T^*\varphi(x_i)| + |T^*\varphi(x_i) - T^*\varphi_j(x_i)| + |T^*\varphi_j(x_i) - T^*\varphi_j(x)| \\ &< \varepsilon, \end{aligned}$$

whence $\|T^*\varphi - T^*\varphi_j\| \leq \varepsilon$, meaning $T^*(B_{X^*})$ is totally bounded, hence T^* compact. □

Recall that a linear map $T\colon X \to Y$ is called bounded below if there is $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all x. In this case, $T(X) \subseteq Y$ is necessarily closed. Every invertible linear map is bounded below, as is every isometry.

Equivalently, a map $T\colon X \to Y$ is not bounded below if and only if there is a sequence of unit vectors $(x_n)_n \subseteq X$ such that $\lim_{n\to\infty} Tx_n = 0$.

Theorem: Let T be a compact operator on a Banach space X, and let $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) The space $\ker(T - \lambda\, \mathrm{id}_X)$ is finite-dimensional.

(ii) The space $(T - \lambda\, \mathrm{id}_X)(X)$ is closed and has finite codimension in X.

Proof. Let $Z = \ker(T - \lambda\, \mathrm{id}_X)$. Then, $T(Z) \subseteq Z$, and the restriction $T|_Z$ is in $K(Z)$. Since $T|_Z = \lambda\, \mathrm{id}_Z$ with $\lambda \neq 0$, it follows that $\mathrm{id}|_Z$ is compact, meaning Z is finite-dimensional.

Since Z is finite-dimensional, there is a closed subspace Y of X such that $X = Z \oplus Y$.

Observe that $(T - \lambda\, \mathrm{id}_X)X = (T - \lambda\, \mathrm{id}_X)Y$, so to show that $(T - \lambda\, \mathrm{id}_X)X$ is closed, it suffices to show that the restriction $(T - \lambda\, \mathrm{id}_X)|_Y$ is bounded below.

Suppose otherwise. Then, there is a sequence $(x_n)_n$ of unit vectors in Y such that $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$. We may assume without loss of generality that $(Tx_n)_n$ is convergent. It follows then that, since $x_n = \frac{1}{\lambda}(Tx_n - (T - \lambda\, \mathrm{id}_X)x_n)$, we have that $(x_n)_n \to x$ for some $x \in Y$, as Y is closed. Since $Tx = \lambda x$, we have $x \in Y \cap \ker(T - \lambda\, \mathrm{id}_X)$, meaning $x = 0$. Yet, x is the limit of unit vectors, and so is also a unit vector, which means we reach a contradiction. Thus, $(T - \lambda\, \mathrm{id}_X)|_Y$ is bounded below.

Let $W = X/(T - \lambda\, \mathrm{id}_X)X$. To show that $(T - \lambda\, \mathrm{id}_X)X$ has finite codimension, we show that W is finite-dimensional, by showing that W^* is finite-dimensional. Let $\pi\colon X \to W$ be the quotient map. Then, $\ker(\pi^*) \subseteq \ker(T^* - \lambda\, \mathrm{id}_{X^*})$. Letting $\sigma \in \ker(T^* - \lambda\, \mathrm{id}_{X^*})$, we have that σ annihilates $(T - \lambda\, \mathrm{id}_X)X$, so it induces a bounded linear functional $\tau\colon W \to \mathbb{C}$ such that $\sigma = \tau \circ \pi = \pi^*(\tau)$. Since T^* is compact, $\ker(T^* - \lambda\, \mathrm{id}_{X^*})$ is finite-dimensional, so π^* has finite-dimensional range, and since π^* is injective, W^* is thus finite-dimensional, so W is finite-dimensional. □

Note that if $T\colon X \to X$ is a linear map on a vector space, then the sequence of spaces $(\ker(T^n))_n$ is increasing; if $\ker(T^n) \neq \ker(T^{n+1})$ for all n, we say that T has infinite *ascent*, and write $\mathrm{asc}(T) = \infty$. Otherwise, we say T has finite ascent, and define $\mathrm{asc}(T)$ to be the smallest p such that $\ker(T^p) = \ker(T^{p+1})$ for all $n \geq p$.

Similarly, the sequence of spaces $T^n(X)$ is decreasing. We say T has infinite *descent* if $T^n(X) \neq T^{n+1}(X)$ for all n, and we write $\mathrm{desc}(T) = \infty$. Else, we say T has finite descent, and define $\mathrm{desc}(T)$ to be the smallest p such that $T^{p+1}(X) = T^p(X)$.

To prove the next theorem, we recall the Riesz lemma.

Lemma (Riesz Lemma)**:** Let Y be a proper closed subspace of a normed vector space X. Then, for any $\varepsilon > 0$, there is a unit vector $x \in X$ such that $\|x + Y\| > 1 - \varepsilon$.

Theorem: Let T be a compact operator on a Banach space X. Suppose $\lambda \in \mathbb{C} \setminus \{0\}$. Then, $T - \lambda I$ has finite ascent and descent.

Proof. Suppose toward contradiction that the ascent is infinite. Letting $N_n = \ker(T - \lambda I)^n$, we observe then that N_{n-1} is a proper subspace of N_n, so by the Riesz Lemma, there is a unit vector $x_n \in N_n$ such that $\|x_n + N_{n-1}\| \geq 1/2$. For any $m < n$, we have

$$\begin{aligned} Tx_n - Tx_m &= \lambda x_n + (T - \lambda)x_n - (T - \lambda)x_m - \lambda x_m \\ &= \lambda x_n - z \end{aligned}$$

for some $z \in N_{n-1}$. Thus, $\|Tx_n - Tx_m\| = \|\lambda x_n - z\| \geq |\lambda|/2$. It follows that $(Tx_n)_n$ has no convergent subsequence, which contradicts the compactness of T.

Similarly, if we let $V_n = \operatorname{Im}(T - \lambda I)^n$, and suppose toward contradiction that $\operatorname{desc}(T - \lambda I) = \infty$, then we have that $V_n \leq V_{n-1}$ is a proper subspace, so there is some unit vector x_n such that $\|x_n + V_{n-1}\| \geq 1/2$. By a similar process, we find a sequence $(Tx_n)_n$ with no convergent subsequence, contradicting the assumption of compactness of the operator T. □

Fredholm Operators and Connections

Definition: Let X, Y be Banach spaces, and let $T \in B(X, Y)$. We say T is *Fredholm* if both $\dim\ker(T)$ and $\dim\operatorname{coker}(T)$ are finite. The index of T is given by

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\operatorname{coker}(T).$$

Theorem: Let X, Y be Banach spaces, and let $T \in B(X, Y)$. Suppose there is a closed subspace Z of Y such that $T(X) \oplus Z = Y$. Then, $T(X)$ is closed in Y.

Proof. From the first isomorphism theorem, we may descend to the map $X/\ker(T) \to Y$ given by $x + \ker(T) \mapsto Tx$, so we may assume without loss of generality that T is injective.

The map

$$\begin{aligned} V \colon X \oplus Z &\to Y \\ (x, z) &\mapsto Tx + z \end{aligned}$$

is a continuous isomorphism between Banach spaces, so by the open mapping theorem, v^{-1} is also continuous. Letting $x \in X$, we have $\|x\| = \left\|V^{-1}Tx\right\| \leq \left\|V^{-1}\right\|\|Tx\|$, so that $\|Tx\| \geq \left\|V^{-1}\right\|^{-1}\|x\|$, meaning T is bounded below, and thus $T(X)$ is closed in Y. □

Theorem: Let

$$X \xrightarrow{T} Y \xrightarrow{S} Z$$

be Fredholm linear maps between Banach spaces X, Y, Z. Then, ST is Fredholm with

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T).$$

Proof. Set $Y_2 = \ker(S) \cap T(X)$, and let Y_1, Y_3, Y_4 be such that $T(X) = Y_2 \oplus Y_3$, $\ker(S) = Y_1 \oplus Y_2$, and $Y = T_1 \oplus T(X) \oplus Y_4$, where Y_1, Y_2, Y_4 are finite-dimensional. We have that the map $\ker(ST) \to Y_2$, $x \mapsto Tx$ is surjective and has the same kernel as T, so $\ker(ST)$ is finite-dimensional with $\dim\ker(ST) = \dim\ker(T) + \dim(Y_2)$.

Next, since $S(Y) = S(Y_3) \oplus S(Y_4)$ and $S(Y_3) = ST(X)$, we have $S(Y) = ST(X) \oplus S(Y_4)$. Let Z' be a finite-dimensional subspace of Z such that $S(Y) \oplus Z' = Z$, so $Z = ST(X) \oplus S(Y_4) \oplus Z'$. Since $S(Y_4) \oplus Z'$ is finite-dimensional, $ST(X)$ has finite codimension in Z, so ST is Fredholm.

The map $Y_4 \to S(Y_4)$ given by $y \mapsto Sy$ is a linear isomorphism, so $\dim(Y_4) = \dim(S(Y_4))$, and thus

$$\begin{aligned}\dim \operatorname{coker}(ST) &= \dim(Y_4) + \dim(Z') \\ &= \dim(Y_4) + \dim \operatorname{coker}(S).\end{aligned}$$

Thus, we have

$$\dim \ker(ST) + \dim \operatorname{coker}(T) + \dim \operatorname{coker}(S) = \dim \ker(T) + \dim \ker(S) + \dim \operatorname{coker}(ST),$$

so that $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$. □

Theorem: Let T be a compact operator on a Banach space X, and let $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) The operator $T - \lambda I$ is Fredholm of index 0.

(ii) If p is the ascent of $T - \lambda I$, then

$$X = \ker(T - \lambda I)^p \oplus (T - \lambda I)^p(X).$$

Proof.

(i) We know from the above theorem that $\ker(T - \lambda I)$ and $\operatorname{coker}(T - \lambda I)$ are finite-dimensional. If m, n are integers greater than the maximum of the ascent and descent of $T - \lambda I$, then we have $\dim \ker(T - \lambda I)^n = \dim \ker(T - \lambda I)^m$, and analogously for the cokernel, whence $\operatorname{ind}(T - \lambda I)^m = \operatorname{ind}(T - \lambda I)^n$ for all such m, n. Thus, $m \operatorname{ind}(T - \lambda I) = n \operatorname{ind}(T - \lambda I)$, so that $\operatorname{ind}(T - \lambda I) = 0$.

(ii) Let $x \in \ker(T - \lambda I)^p \cap (T - \lambda I)^p(X)$. Then, there is $y \in X$ such that $x = (T - \lambda I)^p y$ with $(T - \lambda I)^{2p} y = 0$. Since $\ker(T - \lambda I)^p = \ker(T - \lambda I)^{2p}$, it follow that $(T - \lambda I)^p y = 0$, whence $x = 0$. Moreover, since $\dim \ker(T - \lambda I)^p = \dim \operatorname{coker}(T - \lambda I)^p$, we have $X = \ker(T - \lambda I)^p \oplus (T - \lambda I)^p(X)$.

□

Corollary (Fredholm Alternative)**:** The operator $T - \lambda I$ is injective if and only if it is surjective.

Proof. We have that $\dim \ker(T - \lambda I) = 0$ if and only if $\dim \operatorname{coker}(T - \lambda I) = 0$, meaning that $T - \lambda I$ is injective if and only if it is surjective. □

References

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