Basics

Definition: Let A be a C^*-algebra. A *representation* of A is a $*$-homomorphism $\pi\colon A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Definition: Two representations $\pi\colon A \to B(\mathcal{H}_\pi)$ and $\rho\colon A \to B(\mathcal{H}_\rho)$ are called unitarily equivalent if there is a unitary map $U\colon \mathcal{H}_\rho \to \mathcal{H}_\pi$ such that

$$\pi(a) = U\rho(a)U^*$$

for all $a \in A$.

Definition: If $\pi\colon A \to B(\mathcal{H}_\pi)$ and $\rho\colon A \to B(\mathcal{H}_\rho)$ be representations. Then, the formula

$$\pi \oplus \rho(a)(h,k) \coloneqq (\pi(a)h, \rho(a)k)$$

defines the *direct sum* of π and ρ. If π is unitarily equivalent to a direct sum $\rho_1 \oplus \rho_2$, then we consider $\rho_1 \oplus \rho_2$ to be a decomposition of π in terms of the “smaller” representations.

Definition: A closed subspace $\mathcal{K}$ of $\mathcal{H}_\pi$ is *invariant* under π if $\pi(a)k \in \mathcal{K}$ for all $a \in A$ and $k \in \mathcal{K}$.

Observe that if $\mathcal{K}$ is an invariant subspace, then the orthogonal complement $\mathcal{K}^\perp$ is also invariant. This follows from the fact that if $y \in \mathcal{K}^\perp$, then

$$\begin{aligned}\langle k, \pi(a)y\rangle &= \langle \pi(a)^*k, y\rangle\\ &= \langle \pi(a^*)k, y\rangle\\ &= 0\end{aligned}$$

for all $k \in \mathcal{K}$.

Conversely, if $\mathcal{K}$ is invariant, then we can recover $\pi = \pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp}$, via the canonical unitary isomorphism $U\colon \mathcal{K} \oplus \mathcal{K}^\perp \to \mathcal{H}_\pi$ given by $(k, y) \mapsto k + y$.

Definition: A representation π is *irreducible* if there are no closed invariant subspaces apart from $\{0\}$ and $\mathcal{H}_\pi$.

Lemma: A representation π of a C^*-algebra A is irreducible if and only if $\pi(A)' = \mathbb{C}I_{\mathcal{H}}$, where $\pi(A)'$ denotes the commutant of $\pi(A)$.

Proof. Suppose $\mathcal{V}$ is a nontrivial invariant subspace for π. Then, the orthogonal projection $P_{\mathcal{V}}$ commutes with every $\pi(A)$ and is not a scalar multiple of $I_{\mathcal{H}}$.

Now, suppose there is a non-scalar operator T commuting with $\pi(A)$. Then, either the real or imaginary part of T is a self-adjoint operator S that commutes with $\pi(A)$. From the continuous functional calculus, since $\sigma(S)$ is not one point, there are some nonzero continuous $f, g \in C(\sigma(S))$ such that $fg = 0$. Then, since $f(S), g(S) \in C^*(S)$, and $f(S), g(S)$ commute with $\pi(A)$, it follows that $\overline{f(S)\mathcal{H}}$ and $\overline{g(S)\mathcal{H}}$ are nonzero mutually orthogonal invariant subspaces, so π is reducible. □

Definition: If π is a representation of the C^*-algebra A, then we call the subspace

$$\mathcal{K} = \overline{\operatorname{span}}\{\pi(a)h \mid h \in \mathcal{H}_\pi, a \in A\}$$

the *essential subspace* of $\mathcal{H}_\pi$. The representation π is called *nondegenerate* if the essential subspace $\mathcal{K}$ is equal to $\mathcal{H}_\pi$.

Note that the representation π being nondegenerate is equivalent to $\pi(1) = I_{\mathcal{H}_\pi}$ if A has an identity, or $\pi(e_i) \to I_{\mathcal{H}_\pi}$ strongly for any approximate identity $(e_i)_{i\in I}$.

The essential subspace is always invariant, and π is equivalent to $\pi|_{\mathcal{K}} \oplus 0$. Generally, if I is an ideal in A, then the subspace

$$\mathcal{K} = \overline{\operatorname{span}}\{\pi(a)h \mid h \in \mathcal{H}_\pi, a \in I\}$$

is invariant, but π is not zero on $\mathcal{K}^{\perp}$ unless I is an essential ideal.[I] Any nondegenerate representation of an ideal I extends canonically to a nondegenerate representation π of A on the same space.

The Gelfand–Naimark–Segal Construction

Definition: An element a of a C^*-algebra A is called *positive* if there is $b \in A$ with $a = b^*b$. Equivalently, a is positive if and only if $\sigma(a) \subseteq [0, \infty)$.

There are a few useful identities for positive elements. Specifically, the following hold:

$$\begin{aligned} \|a\|^2 1_A &\geq a^*a \\ \|a\|^2 b^*b - b^*a^*ab &\geq 0. \end{aligned}$$

Definition: A linear functional $\rho\colon A \to \mathbb{C}$ is called *positive* if $\rho(a) \geq 0$ whenever $a \geq 0$. A positive linear functional of norm 1 is called a *state*.

Lemma: Let f be a positive linear functional on a C^*-algebra A. Then, for all $a, b \in A$, we have

$$f(b^*a) = \overline{f(a^*b)}$$

and

$$|f(b^*a)|^2 \leq f(b^*b)f(a^*a).$$

Proof. To see the first identity, we let $\lambda \in \mathbb{C}$, and observe that

$$\begin{aligned} 0 &\leq f\big((\lambda a + b)^*(\lambda a + b)\big) \\ &= |\lambda|^2 f(a^*a) + \overline{\lambda} f(a^*b) + \lambda f(b^*a) + f(b^*b). \end{aligned}$$

Now, since $|\lambda|^2 f(a^*a) + f(b^*b)$ is always real, we must have

$$\operatorname{Im}\Big(\overline{\lambda} f(a^*b) + \lambda f(b^*a)\Big) = 0$$

for all λ. By taking $\lambda = 1$ and $\lambda = i$, we get equality of imaginary and real parts of $f(a^*b)$ and $\overline{f(b^*a)}$.

As for the Cauchy–Schwarz inequality, we observe that if $\lambda = x\overline{f(b^*a)}$ for some $x \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leq x^2|f(b^*a)|^2 f(a^*a) + x|f(a^*b)|^2 + x|f(b^*a)|^2 + f(b^*b) \\ &= x^2|f(b^*a)|^2 f(a^*a) + 2x|f(b^*a)|^2 + f(b^*a). \end{aligned}$$

The right-hand side is a quadratic in x that is always greater than or equal to 0, so

$$4|f(b^*a)|^4 - 4|f(b^*a)|^2 f(a^*a)f(b^*b) \leq 0.$$

□

To understand the GNS construction, we start by taking a state τ on a C^*-algebra A. Then, defining

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\},$$

we observe that $\tau(b^*a) = 0$ if either a or b are in N_τ. In particular, we get the inner product on A/N_τ given by

$$\langle a + N_\tau, b + N_\tau \rangle = \tau(b^*a).$$

Define $\mathcal{H}_\tau$ to be the Hilbert space completion of A/N_τ. Since $\|a\|^2 b^*b - b^*a^*ab$ is of the form c^*c, we have

$$\|a(b + N_\tau)\|^2 = \tau(b^*a^*ab)$$

[I] An essential ideal is one that has nonzero intersection with any other closed ideal of A.

$$\begin{aligned} &= \|a\|^2\tau(b^*b) - \tau(c^*c) \\ &\leq \|a\|^2\tau(b^*b) \\ &= \|a\|^2\|b + N_\tau\|^2. \end{aligned}$$

In particular, this means that the elements of A act as bounded operators on A/N_τ, which we extend to operators $\pi_\tau(a)$ in the completion. This gives a nondegenerate representation π_τ of A on the Hilbert space $\mathcal{H}_\tau$.

Lemma: Suppose A is a non-unital C^*-algebra, and $\rho \in S(A)$. Then, if $(e_i)_{i\in I}$ is an approximate identity for A, $\rho(e_i) \to 1$. Furthermore, the formula $\tau(a + \lambda 1) = \rho(a) + \lambda$ defines a state τ on the unitization $\widetilde{A}$.