

Solution (32.20): We start by taking the recurrence relation

$$(1 - x^2)P'_n = -nP_n + nP_{n-1}. \quad (*)$$

Differentiating, this gives

$$(1 - x^2)P''_n - 2xP'_n = n(-P_n - xP'_n + P'_{n-1}).$$

We seek to show that

$$-xP'_n + P'_{n-1} = -nP_n.$$

At this point, I ran out of board space to deal with the generating functions and their ensuing mess of partial derivatives.

Solution (32.21): Using $dv = P'_m(x)$, we integrate by parts to get

$$\begin{aligned} \int_{-1}^1 (1 - x^2)P'_n(x)P'_m(x) dx &= P_m(x)P'_n(x)(1 - x^2) \Big|_{-1}^1 - \int_{-1}^1 \frac{d}{dx} \left((1 - x^2)P'_n(x) \right) P_m(x) dx \\ &= - \int_{-1}^1 \left((1 - x^2)P''_n(x) - 2xP'_n(x) \right) P_m(x) dx \\ &= n(n+1) \int_{-1}^1 P_n(x)P_m(x) dx \\ &= \frac{2n(n+1)}{2n+1} \delta_{mn}. \end{aligned}$$

Solution (32.23): Upon taking m derivatives of Legendre's equation, and using the Leibniz rule for differentiation, we get

$$(1 - x^2) \frac{d^{m+2}P_\ell}{dx^{m+2}} - 2x(m+1) \frac{d^{m+1}P_\ell}{dx^{m+2}} + ((\ell)(\ell+1) - (m(m-1) + 2m)) \frac{d^m P_\ell}{dx^m} = 0.$$

Rewriting $u(x) = \frac{d^m P_\ell}{dx^m}$, we obtain

$$0 = (1 - x^2) \frac{d^2 u}{dx^2} - 2x(m+1) \frac{du}{dx} + (\ell(\ell+1) - m^2 - m)u(x).$$

Setting $u(x) = (1 - x^2)^{-m/2} v(x)$, we find

$$\begin{aligned} \frac{du}{dx} &= (1 - x^2)^{-m/2} \frac{dv}{dx} + mxv(x)(1 - x^2)^{-m/2-1} \\ &= (1 - x^2)^{-m/2} \left(\frac{dv}{dx} + \frac{mxv(x)}{1 - x^2} \right) \\ \frac{d^2 u}{dx^2} &= -mx(1 - x^2)^{-m/2-1} \left(\frac{dv}{dx} + \frac{mxv(x)}{1 - x^2} \right) + (1 - x^2)^{-m/2} \left(\frac{d^2 v}{dx^2} + \frac{mv(x)}{1 - x^2} + \frac{mx}{1 - x^2} \frac{dv}{dx} + \frac{2mx^2 v(x)}{(1 - x^2)^2} \right) \\ &= (1 - x^2)^{-m/2} \left(\frac{d^2 v}{dx^2} + \frac{2mx}{1 - x^2} \frac{dv}{dx} + \frac{mv(x)}{1 - x^2} + \frac{2mx^2 v(x)}{(1 - x^2)^2} \right). \end{aligned}$$

Substituting, we have the equation

$$\begin{aligned} 0 &= (1 - x^2) \left((1 - x^2)^{-m/2} \left(\frac{d^2 v}{dx^2} + \frac{2mx}{1 - x^2} \frac{dv}{dx} + \frac{mv(x)}{1 - x^2} + \frac{2mx^2 v(x)}{(1 - x^2)^2} \right) \right. \\ &\quad \left. - 2x(m+1) \left((1 - x^2)^{-m/2} \left(\frac{dv}{dx} + \frac{mxv(x)}{1 - x^2} \right) \right) \right. \\ &\quad \left. + (\ell(\ell+1) - m^2 - m) (1 - x^2)^{-m/2} v(x) \right), \end{aligned}$$

which after much more tedious algebra, yields

$$0 = \frac{d^2 v}{dx^2} - 2x \frac{dv}{dx} + \left((\ell)(\ell + 1) - \frac{m^2}{1 - x^2} \right) v(x),$$

so v satisfies the differential equation. Thus, we have

$$v(x) = (1 - x^2)^{m/2} \frac{d^m P_\ell}{dx^m}.$$

Solution (35.4): There were many failed attempts at manipulating the integral expression(s) for J_n , but none of them bore any fruit.

Solution (35.5): Differentiating,

$$\begin{aligned} \frac{dJ_0}{dx} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial x} \left(e^{ix \sin(\gamma)} \right) d\gamma \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (i \sin(\gamma)) e^{ix \sin(\gamma)} d\gamma \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} i \left(\frac{1}{2i} (e^{i\gamma} - e^{-i\gamma}) \right) d\gamma \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} e^{ix \sin(\gamma) + i\gamma} - \frac{1}{2} e^{ix \sin(\gamma) - i\gamma} d\gamma \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos(x \sin(\gamma) + i\gamma) + i \sin(x \sin(\gamma) + i\gamma) - (\cos(x \sin(\gamma) - i\gamma) + i \sin(x \sin(\gamma) - i\gamma))) d\gamma \end{aligned}$$

and with more tedious algebra, we obtain

$$\begin{aligned} &= -\frac{1}{\pi} \int_0^{\pi} \cos(x \sin(\gamma) - \gamma) d\gamma \\ &= -J_1(x). \end{aligned}$$

Evaluating

$$\frac{d}{dx}(xJ_1) = J_1 + x \frac{dJ_1}{dx},$$

we take

$$\begin{aligned} \frac{d}{dx}(xJ_1) &= \frac{1}{\pi} \int_0^{\pi} \cos(x \sin(\gamma) - \gamma) - x \sin(\gamma) \sin(x \sin(\gamma) - \gamma) d\gamma \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(x \sin(\gamma)) \cos(\gamma) + \sin(x \sin(\gamma)) \sin(\gamma) - x \sin(\gamma) \sin(x \sin(\gamma) - \gamma) d\gamma \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(\gamma) \cos(x \sin(\gamma)) + \sin(\gamma) \sin(x \sin(\gamma)) - x \sin(\gamma) (\sin(x \sin(\gamma)) \cos(\gamma) - \sin(\gamma) \cos(x \sin(\gamma))) d\gamma \\ &= \frac{1}{\pi} \int_0^{\pi} x \cos(x \sin(\gamma)) d\gamma \\ &= xJ_0. \end{aligned}$$

Solution (35.7): Solving

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + (x^2 - n^2) u(x) = 0,$$

we plug in the expression for $J_n(x)$ to get

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + (x^2 - n^2) u(x) = x^2 \left(\sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} (2m+n-1)(2m+n) \frac{(-1)^m}{m!(m+n)!} x^{2m+n-2} \right)$$

$$\begin{aligned}
& + x \left(\sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} (2m+n) \frac{(-1)^m}{m!(m+n)!} x^{2m+n-1} \right) \\
& + \sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} \frac{(-1)^m}{m!(m+n)!} x^{2m+n+2} \\
& - \sum_{m=0}^{\infty} \frac{n^2}{2^{2m+n}} \frac{(-1)^m}{m!(m+n)!} x^{2m+n} \\
& = \sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} \frac{(-1)^m}{m!(m+n)!} (x^{2m+n}) \left((2m+n-1)(2m+n) + 2m+n + x^2 - n^2 \right) \\
& = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+n} m!(m+n)!} x^{2m+n} (x^2 + 4m^2 + 4mn)
\end{aligned}$$

From here, I'm not sure how to manipulate this series to get 0 as the final answer.

Solution (35.8):

(a) We have

$$e^{ix \sin(\phi)} = \sum_{n=-\infty}^{\infty} c_n e^{in\phi},$$

where

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin(\phi)} e^{-in\phi} d\phi \\
&= J_n(x).
\end{aligned}$$

(b) Splitting into real and imaginary parts, we have

$$e^{ix \sin(\phi)} = \cos(x \sin(\phi)) + i \sin(x \sin(\phi)),$$

so that

$$\begin{aligned}
e^{ix \sin(\phi)} &= \sum_{n=-\infty}^{\infty} c_n e^{in\phi} \\
&= \sum_{n=-\infty}^{\infty} J_n(x) (\cos(n\phi) + i \sin(n\phi)) \\
&= \sum_{n=-\infty}^{\infty} J_n(x) \cos(n\phi) + i \sum_{n=-\infty}^{\infty} J_n(x) \sin(n\phi).
\end{aligned}$$

Equating real and imaginary parts gives the desired result.

(c) We use the angle summation identity to get

$$\begin{aligned}
A \cos(\omega_c t) \cos(\beta \sin(\omega_m t)) - A \sin(\omega_c t) \sin(\beta \sin(\omega_m t)) &= A \cos(\omega_c t) \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(n\omega_m t) \\
&\quad - A \sin(\omega_c t) \sum_{n=-\infty}^{\infty} J_n(\beta) \sin(n\omega_m t) \\
&= \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c t + n\omega_m t).
\end{aligned}$$

Solution (35.10):

$$\begin{aligned}
 P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
 P_{3,1}(x) &= \frac{1}{2}(1 - x^2)^{1/2}(15x^2 - 3) \\
 P_{3,-1}(x) &= -\frac{1}{6}(1 - x^2)^{1/2}(15x^2 - 3) \\
 P_{3,2}(x) &= 15x(1 - x^2) \\
 P_{3,-2}(x) &= \frac{1}{8}x(1 - x^2). \\
 P_{3,3}(x) &= 15(1 - x^2)^{3/2} \\
 P_{3,-3}(x) &= -\frac{1}{48}(1 - x^2)^{3/2}.
 \end{aligned}$$

Solution (35.11):

$$\begin{aligned}
 Y_{\ell,m}(\pi - \theta, \phi + \pi) &= (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell,m}(\cos(\pi - \theta)) e^{im(\phi + \pi)} \\
 &= (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell,m}(-\cos(\theta)) e^{im\phi} (-1)^m \\
 &= (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} (-1)^{\ell-m} P_{\ell,m}(\cos(\theta)) (-1)^m e^{im\phi} \\
 &= (-1)^\ell \left((-1)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell,m}(\cos(\theta)) e^{im\phi} \right) \\
 &= (-1)^\ell Y_{\ell,m}(\theta, \phi).
 \end{aligned}$$

Solution (35.12): We have

$$\begin{aligned}
 Y_{\ell,0}(\hat{n}) &= (-1)^0 \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{\ell!}{\ell!}} P_{\ell,0}(\cos(\theta)) e^{i(0)\phi} \\
 &= \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos(\theta)).
 \end{aligned}$$

Furthermore, since $P_{\ell,m}(1) = \delta_{0,m}$, we have

$$\begin{aligned}
 Y_{\ell,m}(0, \phi) &= (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell,m}(1) e^{im\phi} \\
 &= \sqrt{\frac{2\ell+1}{4\pi}} \delta_{0,m}.
 \end{aligned}$$

Solution (35.16): Using the addition theorem, where $\hat{a} = \hat{b} = \hat{n}$, we get

$$\begin{aligned}
 \sum_{m=-\ell}^{\ell} |Y_{\ell,m}(\hat{n})|^2 &= \frac{2\ell+1}{4\pi} P_{\ell}(1) \\
 &= \frac{2\ell+1}{4\pi}.
 \end{aligned}$$

Solution (35.17 (c)):

Solution (35.21):

| **Solution** (35.25):