

## Revised Problems

**Problem (Problem 1):** Prove that our cell complex structure for  $T^2$  coincides with a product cell complex structure on  $S^1 \times S^1$ .

**Solution:** Consider the cell complex structure on  $S^1$  given by one 1-cell,  $e^1$ , and one 0-cell,  $e^0$ , where the endpoints of  $e^1$  are identified via the constant map to  $e^0$ ; the corresponding characteristic map is  $\Phi: [0, 1] \rightarrow S^1$ , identifying  $0 \sim 1$ .

Considering two copies of  $S^1$  in this fashion, the product CW complex structure is then one consisting of

- one 0-cell,  $e_1^0 \times e_2^0$
- two 1-cells,  $e_1^1 \times e_2^0$  and  $e_1^0 \times e_2^1$ ;
- one 2-cell,  $e_1^1 \times e_2^1$ .

There are then characteristic maps

$$\Phi_1: e_1^1 \times e_2^0 \rightarrow S^1 \times S^1$$

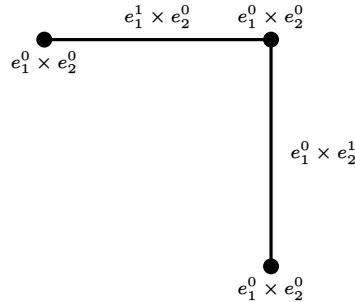
$$\Phi_2: e_1^0 \times e_2^1 \rightarrow S^1 \times S^1,$$

with

$$\Phi_1|_{\partial(e_1^1 \times e_2^0)}(e_1^1 \times e_2^0) = e_1^0 \times e_2^0$$

$$\Phi_2|_{\partial(e_1^0 \times e_2^1)}(e_1^0 \times e_2^1) = e_1^0 \times e_2^0.$$

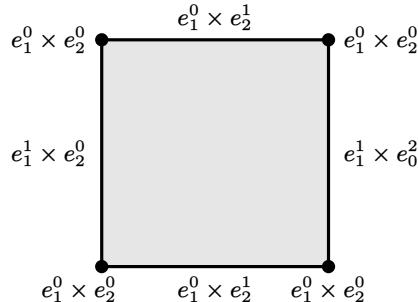
In particular, this means we may view the 1-skeleton as a wedge of two circles; since this is difficult to draw in TikZ, we instead simply label all the vertices and edges of the figure below assuming the necessary identifications.



The 2-skeleton is given by  $e_1^1 \times e_2^1$ , with a characteristic map  $\Psi$  given by the product of the characteristic maps of each of  $e_1^1$  and  $e_2^1$  (see Hatcher Theorem A.6). Observe that the attaching map is then given by the restriction of  $\Psi$  to the boundary, which is

$$\partial(e_1^1 \times e_2^1) = (e_1^1 \times e_2^0) \sqcup (e_1^0 \times e_2^1).$$

Since the 1-skeleton is precisely  $(e_1^0 \times e_2^1) \sqcup (e_1^1 \times e_2^0)$ , it follows that the attaching map for the 2-skeleton identifies the boundary of  $e_1^1 \times e_2^1$  with the 1-skeleton, giving the following figure:



which is the cell complex structure of the torus.

**Problem (Problem 2):** Prove that if  $X$  is a cell complex, then so is the suspension  $SX$ .

**Solution:** We observe that the product  $X \times [0, 1]$  is a cell complex, as it is a Cartesian product of a cell complex with one 1-cell (the interval itself) and two 0-cells (the endpoints of the interval). Since the characteristic maps on  $X \times [0, 1]$  are the products of the characteristic maps on  $X$  and  $[0, 1]$  (see Hatcher Theorem A.6), we see that the attaching maps for  $X \times \{0\}$  and  $X \times \{1\}$  are the products of the attaching maps for  $X$  and the constant maps representing  $\{0\}$  and  $\{1\}$ .

In particular, this means that  $X \times \{0\}$  and  $X \times \{1\}$  are subcomplexes of  $X \times [0, 1]$  (as they contain all their attaching maps). Since the quotient of a cell complex by a subcomplex is a cell complex, it follows that  $SX = X \times [0, 1]/X \times \{0\}/X \times \{1\}$  is a cell complex.

**Problem (Problem 3 (b)): Prove that  $S^\infty$  is contractible.**

**Solution:** We view  $S^\infty$  as the subspace of  $\mathbb{R}^\infty$ , which is the space of finitely supported sequences. Specifically, the space  $S^\infty$  is the set of the finitely supported sequences  $(x_n)$  such that, if  $k$  is the index of the largest nonzero element, then  $(x_0, \dots, x_k)$  is an element of  $S^k$  — i.e.,

$$\sum_{i=0}^k x_i^2 = 1.$$

In general, we define the norm of the sequence  $\|(x_n)\|$  to be the finite sum

$$\|(x_n)\| = \sum_{i=0}^{\infty} x_i^2.$$

Consider now the map  $H: S^\infty \times [0, 1] \rightarrow S^\infty$

$$H((x_n), t) = \begin{cases} (1 - 2t)(x_n) + 2t(x_{n+1}) & 0 \leq t \leq 1/2 \\ (2 - 2t)(x_{n+1}) + (2t - 1)(1, 0, \dots) & 1/2 \leq t \leq 1. \end{cases}$$

where we insert 0 into index 0 in the first half of the homotopy. Then,  $H$  is continuous along each of  $S^\infty \times [0, 1/2]$  and  $S^\infty \times [1/2, 1]$  as it is continuous in each variable, and since the piecewise definitions are equal at  $t = 1/2$ , it follows that  $H$  is continuous along  $[0, 1]$ . Then, we see that  $H(\cdot, t)/\|H(\cdot, t)\|$  is contained within  $S^\infty$ , and is a homotopy between the identity and a constant map, so the identity is null-homotopic, meaning  $S^\infty$  is contractible.