**Problem** (Problem 1)**:** Let $R$ be a commutative ring. An $R$-module $M$ is called torsion if for any $m \in M$, there is a nonzero $r \in R$ such that $rm = 0$. An $R$-module $M$ is called divisible if for any nonzero $r \in R$, we have $rM = M$. In other words, $M$ is divisible if for any $m \in M$ and nonzero $r \in R$, there is $x \in M$ such that $rx = m$.

(a) Suppose $M$ is a torsion $R$-module and $N$ is a divisible $R$-module. Prove that $M \otimes_R N = \{0\}$.

(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

**Solution:**

(a) It is enough to show that any simple tensor $m \otimes n \in M \otimes_R N$ is the zero tensor. To see this, we let $r \in R$ be such that $rm = 0$, and observe that there is some $x \in N$ such that $rx = n$. By using property (R3) of tensor products, we observe then that

$$\begin{aligned} m \otimes n &= m \otimes (rx) \\ &= (rm) \otimes x \\ &= 0 \otimes x \\ &= 0. \end{aligned}$$

Thus, $M \otimes_R N = \{0\}$.

(b) It is enough to show that $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible, as we may then apply (a). To see that $\mathbb{Q}/\mathbb{Z}$ is torsion, we have that

$$\begin{aligned} b\left[\frac{a}{b}\right] &= [a] \\ &= [0] \end{aligned}$$

for any element $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$. Additionally, for any $n \in \mathbb{Z}$, we have

$$\left[\frac{a}{b}\right] = n\left[\frac{a}{nb}\right],$$

so $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible.

**Problem** (Problem 2)**:** Let $R$ be a commutative ring, $\{N_\alpha\}_{\alpha\in A}$ a collection of $R$-modules, and $M$ another $R$-module.

(a) Prove that $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) Show by example that $M \otimes (\prod_\alpha N_\alpha)$ need not be isomorphic to $\prod_\alpha (M \otimes N_\alpha)$.

**Solution:**

(a) Consider the map on elementary tensors

$$f\colon M \times \left(\bigoplus_\alpha N_\alpha\right) \to \bigoplus_\alpha (M \otimes N_\alpha)$$

that takes

$$(m, (n_\alpha)_\alpha) \to (m \otimes n_\alpha)_\alpha.$$

We observe that, since the $(n_\alpha)_\alpha$ are nonzero for all but finitely many indices $\alpha$, and that the map is $R$-bilinear, we have a well-defined and unique $R$-linear map $\overline{f}\colon M \otimes (\bigoplus_\alpha N_\alpha) \to \bigoplus_\alpha (M \otimes N_\alpha)$ that maps $m \otimes (n_\alpha)_\alpha \mapsto (m \otimes n_\alpha))_\alpha$.

We observe that for each index $i$, we have an inclusion homomorphism

$$M \times N_i \hookrightarrow M \otimes \left(\bigoplus_\alpha N_\alpha\right)$$

that takes $(m, n_\alpha) \mapsto m \otimes (n_\alpha)_\alpha$, where $(n_\alpha)_\alpha$ is zero everywhere except for index $i$. By the universal property of the direct sum, this induces a unique homomorphism $g\colon \bigoplus_\alpha (M \otimes N_\alpha) \to M \otimes (\bigoplus_\alpha N_\alpha)$ given by taking

$$(m_\alpha \otimes n_\alpha)_\alpha \mapsto \sum_\alpha m_\alpha \otimes (n_\alpha)_\alpha,$$

where the summand $(n_\alpha)_\alpha$ is defined as above, and the sum is finite by the definition of the direct sum. Since $g$ and $f$ are inverses of each other (as can be seen by the action on simple tensors), it follows that $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) We consider the direct product

$$M = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z},$$

regarded as a $\mathbb{Z}$-module. Notice that $M$ is not torsion, as the element $m = (1, 1, \dots)$ is such that there is no $z \in \mathbb{Z}$ with $zm = 0$. Therefore, considering the extension of scalars

$$\mathbb{Q} \otimes M = \mathbb{Q} \otimes \left( \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z} \right),$$

we have that this is not a zero module, since by using the same element $(1, 1, \dots)$, the Archimedean property implies that for any $\frac{a}{b}$ in lowest terms in $\mathbb{Q}$, there is some $k$ such that $2^k > a$. (Alternatively, we may use 8(d) to establish a similar result.) Yet, since each of the individual $\mathbb{Z}/2^i\mathbb{Z}$ has torsion, it would follow that

$$\prod_{i=1}^{\infty} (\mathbb{Q} \otimes \mathbb{Z}/2^i\mathbb{Z}) = 0,$$

so it follows that tensor products do not commute with direct sums.

**Problem** (Problem 4)**:** Let $R$ be commutative, and let $I$ and $J$ be ideals of $R$, so $R/I$ and $R/J$ are naturally $R$-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$.

(b) Prove that there is an $R$-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r + I) \otimes (r' + J)$ to $rr' + (I + J)$.

**Solution:**

(a) By using $R$-bilinearity, we observe that an arbitrary simple tensor in $R/I \otimes R/J$ can be written as

$$\begin{aligned}(r + I) \otimes (s + J) &= (r(1 + I)) \otimes (s + J) \\ &= r((1 + I) \otimes (s + J)) \\ &= (1 + I) \otimes (rs + J).\end{aligned}$$

Since any element of $R/I \otimes_R R/J$ can be written as a sum of simple tensors, and each simple tensor can be written in the above form, it follows from bilinearity that every element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (r + J)$.

(b) We consider the map

$$f\colon R/I \times R/J \mapsto R/(I + J)$$

given by

$$(r + I, r' + J) \mapsto rr' + (I + J).$$

This map is $R$-bilinear by the distributive properties of multiplication, so it induces a homomorphism on the tensor product given by

$$(r+I)\otimes(r'+J)\mapsto rr'+(I+J).$$

As was established above, any element of $R/I\otimes R/J$ can be written as $(1+I)\otimes(s+J)$, so we may establish an inverse from any element of $R/(I+J)$ to $R/I\otimes R/J$ by taking $t+(I+J)\mapsto(1+I)\otimes(t+J)$. This establishes our desired isomorphism.

**Problem** (Problem 5)**:** Let $I=(2,x)$ be the ideal generated by 2 and $x$ in the ring $\mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z}=R/I$ is naturally an $R$-module annihilated by both 2 and $x$.

(a) Show that the map $\varphi\colon I\times I\to\mathbb{Z}/2\mathbb{Z}$ given by

$$\varphi(a_0+a_1x+\cdots+a_nx^n, b_0+b_1x+\cdots+b_mx^m)=\left[\frac{a_0}{2}b_1\right]_{\mathbb{Z}/2\mathbb{Z}}$$

is $R$-bilinear.

(b) Show that there is an $R$-module homomorphism from $I\otimes_R I\to\mathbb{Z}/2\mathbb{Z}$ mapping $p(x)\otimes q(x)$ to $\frac{p(0)}{2}q'(0)$, where $q'$ denotes the usual polynomial derivative of $q$.

(c) Show that $2\otimes x\neq x\otimes 2$ in $I\otimes_R I$.

**Solution:**

(a) By the well-definedness of addition in $R/I$, we have that $\varphi$ is additive in each variable. Now, letting $p(x)\in R$ and $a(x),b(x)\in I$ be defined by

$$\begin{aligned}a(x)&=a_0+a_1x+\cdots+a_nx^n\\ b(x)&=b_0+b_1x+\cdots+b_mx^m\\ p(x)&=p_0+p_1x+\cdots+p_\ell x^\ell,\end{aligned}$$

we note that

$$p(x)+I=[p_0]_{\mathbb{Z}/2\mathbb{Z}}.$$

Using various definitions, we see that

$$\begin{aligned}\varphi(p(x)a(x),b(x))&=\varphi(p_0a_0+O(x),b_0+b_1x+\cdots)\\ &=\left[\frac{p_0a_0}{2}b_1\right]\\ &=[p_0]\left[\frac{a_0}{2}b_1\right]\\ &=(p(x)+I)\varphi(a(x),b(x)),\end{aligned}$$

and since $b_0\in I$,

$$\begin{aligned}\varphi(a(x),p(x)b(x))&=\left[\frac{a_0}{2}(p_0b_1+p_1b_0)\right]\\ &=\left[\frac{a_0}{2}(p_0b_1)\right]\\ &=(p(x)+I)\varphi(a(x),b(x)).\end{aligned}$$

Thus, $\varphi$ is $R$-bilinear.

(b) Using the universal property for tensor products, there is a unique $R$-linear homomorphism $\overline{\varphi}\colon I\otimes_R I\to\mathbb{Z}/2\mathbb{Z}$ such that

$$\begin{aligned}\overline{\varphi}(a(x)\otimes b(x))&=\left[\frac{a_0}{2}b_1\right]\\ &=\left[\frac{p(0)}{2}q'(0)\right].\end{aligned}$$

(c) We observe that $\overline{\varphi}(2 \otimes x) = 1$ while $\overline{\varphi}(x \otimes 2) = 0$, so they cannot be equal to each other in $I \otimes_R I$.

**Problem** (Problem 6)**:** Suppose $R$ is commutative, and let $I, J$ be ideals of $R$.

(a) Show that there is a surjective $R$-module homomorphism from $I \otimes_R J$ to the product ideal $IJ$ mapping $i \otimes j$ to $ij$.

(b) Give an example to show that the map in (a) need not be injective.

**Solution:**

(a) We define the $R$-bilinear map $\varphi\colon I \times J \to IJ$ by

$$\varphi(i, j) = ij.$$

This induces a linear map $\overline{\varphi}\colon I \otimes_R J \to IJ$ such that $i \otimes j \mapsto ij$. Since every element of $I \otimes_R J$ is a finite sum of elementary tensors, this surjects onto $IJ$ since every element of $IJ$ is a finite sum of elements of the form $ij$.

(b) The map from Problem 5, given by $I \otimes I \to I^2$ applied in the case of $2 \otimes x$ and $x \otimes 2$, is not injective, as $2 \otimes x \neq x \otimes 2$, but $R$ is commutative.

**Problem** (Problem 7)**:**

(a) Let $V$ be a finite-dimensional vector space over $\mathbb{C}$. Note that $V$ can be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.

(b) Let $R$ be an integral domain and $F$ its field of fractions. Prove that $F \otimes_R R \cong F \otimes_F F \cong F$ as $F$-modules.

**Solution:**

(a) We may consider $V \cong \mathbb{C}^k$, so that

$$\begin{aligned} V \otimes_{\mathbb{C}} V &\cong \mathbb{C}^{k^2} \\ &\cong \mathbb{R}^{2k^2} \\ V \otimes_{\mathbb{R}} V &\cong \mathbb{R}^{4k^2}. \end{aligned}$$

(b) We will show that both of the tensor products are generated by $1 \otimes 1$ as an $F$-vector space. Observe that in $F \otimes_R F$, we have

$$\begin{aligned} \frac{a}{b} \otimes \frac{c}{d} &= \frac{ad}{bd} \otimes \frac{c}{d} \\ &= \frac{a}{bd} \otimes c \\ &= \frac{ac}{bd}(1 \otimes 1), \end{aligned}$$

while in $F \otimes_F F$, $\frac{a}{b} \otimes \frac{c}{d} = \frac{ac}{bd}(1 \otimes 1)$, meaning that both tensor products are generated by $1 \otimes 1$, and thus both have dimension 1 over $F$, so that they are all isomorphic to $F$.

**Problem** (Problem 8)**:** Let $R$ be a subring of the commutative ring $S$, and let $x$ be an indeterminate over $S$. Prove that $S[x]$ and $S \otimes R[x]$ are isomorphic as $S$-algebras.

**Solution:** Using the fact that $R[x]$ is a free $R$-module with basis $\{1, x, x^2, \dots\}$, we only need to define an $R$-bilinear map on the basis elements. Consider the map $\phi\colon S \times R[x] \to S[x]$ given by

$$\phi(s, x^m) = sx^m.$$

This is an $R$-bilinear map, so it extends uniquely to a homomorphism on the tensor product

$$\varphi\colon S \otimes R[x] \to S[x] s \otimes x^m \qquad \mapsto sx^m.$$

Furthermore, this homomorphism has an inverse given by $\psi(sx^m) = s \otimes x^m$, once again extended linearly since $S[x]$ is a free $S$-module. Thus, we have that $S \otimes R[x] \cong S[x]$.