Problem (Problem 1): Describe the topology of the Grassmanian $Gr(k, n)$ in a uniform way, so that $\mathbb{RP}^n$ becomes the special case of $Gr(1, n)$.

Solution: We let elements of $Gr(k, n)$ be defined as equivalence classes of linearly independent k-tuples of vectors in $\mathbb{R}^n$, where $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if $span\{v_1, \dots, v_k\} = span\{w_1, \dots, w_k\}$.

By extending $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ to ordered bases $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$, we see that these k-tuples are equivalent if and only if there is a change of basis transformation Q with matrix representation

$$Q = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n-k) \times (n-k)$ matrix. The subgroup of all such $Q \subseteq GL_n(\mathbb{R})$, which we call P, is the stabilizer of $Gr(k, n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $GL_n(\mathbb{R})$ acts transitively on all ordered bases of $\mathbb{R}^n$), we have that $Gr(k, n) \cong GL_n(\mathbb{R})/P$.

Problem (Problem 2): Fix an inner product on $\mathbb{R}^n$. Show that the map $V \mapsto V^\perp$ induces a C^∞ diffeomorphism $Gr(k, n) \to Gr(n-k, n)$.

Problem (Problem 3): Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Problem (Problem 4): Recall that a topological space is paracompact if every open cover admits a locally finite subcover. Prove that a connected, paracompact manifold of dimension one is either $\mathbb{R}$ or S^1, depending on whether it is compact or not.

Solution: Let M be a connected, paracompact manifold with dimension 1, and let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas for M, where φ_i are homeomorphisms between U_i and $\mathbb{R}$.

Problem (Problem 5): In this problem, we prove a weak version of the Whitney Embedding Theorem.

(a) Find a C^∞ function λ on $\mathbb{R}^n$ with values in $[0, 1]$ such that λ takes the value 1 on the closed ball $B(0, 1)$, and vanishes outside the closed ball $B(0, 2)$.

(b) Suppose M is a compact C^k manifold of dimension n. Find a C^k atlas $\{U_i, \varphi_i\}_{i \in I}$ such that $\varphi_i(U_i)$ contains $B(0, 2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0, 1))^\circ$.

(c) Let λ_i be defined by $\lambda \circ \varphi_i$ on U_i, and 0 outside U_i. Let $f_i \colon M \to \mathbb{R}^n$ be defined by $\lambda_i \circ \varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Problem (Problem 6): Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Problem (Problem 7): Let X and Y be topological spaces, and let $C(X, Y)$ be the set of continuous maps from X to Y. Equip $C(X, Y)$ with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space, and let X be compact. Assume X is nonempty. We will show that a net $(f_\alpha)_\alpha \to f$ in the compact-open topology if and only if $(f_\alpha)_\alpha \to f$ uniformly.

If $(f_\alpha)_\alpha \to f$ uniformly, then for all $\varepsilon > 0$, there exists α_0 such that for all $\alpha \geqslant \alpha_0$ and for all $x \in X$,

$$d(f_\alpha(x), f(x)) < \varepsilon/2,$$

meaning that, in particular, $f_\alpha(X) \subseteq U(f(x), \varepsilon)$ for all $x \in X$, so by setting $V = U(f(x_0), \varepsilon)$ for a distinguished $x_0 \in X$ and the compact subset $X \subseteq X$, we see that $(f_\alpha)_\alpha \subseteq U_{X,V}$ for all $\alpha \geqslant \alpha_0$, so $(f_\alpha)_\alpha \to f$ in the compact-open topology.

Now, let $(f_\alpha)_\alpha \to f$ in the compact open topology; by definition, this means that for all basic open neighborhoods $U_{K,V}$, there exists $\alpha_{K,V}$ such that for all $\alpha \geqslant \alpha_{K,V}$, we have $f_\alpha \in U_{K,V}$.

Let $\varepsilon > 0$. Since f is uniformly continuous, for all $x \in X$, there is a pre-compact open neighborhood U_x of x such that $f(U_x) \subseteq U(f(x), \varepsilon/3)$; in particular, notice that $f\left(\overline{U_x}\right) \subseteq U(f(x), \varepsilon/2)$.

The sets $\{U_x\}_{x \in X}$ cover X, so since X is compact, there are $x_1, \dots, x_n$ such that $U_{x_1}, \dots, U_{x_n}$ cover X. We let $V_i = U(f(x_i), \varepsilon/2)$ be the corresponding open balls such that $f\left(\overline{U_{x_i}}\right) \subseteq V_i$, meaning there exist $\alpha_{U_{x_i}, V_i}$ for $i = 1, \dots, n$ such that for all $\alpha \geqslant \alpha_{\overline{U_{x_i}}, V_i}$, $f_\alpha \in U_{\overline{U_{x_i}}, V_i}$. Since the α are directed, there is $\alpha_0 \geqslant \alpha_{\overline{U_{x_i}}, V_i}$ for each $i = 1, \dots, n$, meaning that for all $\alpha \geqslant \alpha_0$, we have $f_\alpha \in \bigcap_{i=1}^n U_{\overline{U_{x_i}}, V_i}$. In particular, we see that for all $\alpha \geqslant \alpha_0$, and for all $x \in X$, $d(f_\alpha(x), f(x)) < \varepsilon/2$.