Preliminary Statements

Theorem (Definition of Countability). *A set S is countable if and only if there exists an injection $f : S \hookrightarrow \mathbb{N}$.*

Proof. Let S be countable.

Case 1: We have S is finite if and only if there is a map $f : S \to \{1, 2, \dots, n\}$, where f is a bijection. Letting $\mathrm{id} : \{1, 2, \dots, n\} \to \mathbb{N}$ be defined by $\mathrm{id}(n) = n$, it is clear that id is an injection.

Considering the map $\mathrm{id} \circ f : S \to \mathbb{N}$, since id is injective and f is injective, so too is $\mathrm{id} \circ f$, meaning our desired injection is $\mathrm{id} \circ f$.

Case 2: By definition, a set S is countably infinite if and only if there exists a bijection $g : S \to \mathbb{N}$, which is our desired injection.

□

Theorem (Injection into a Finite Set). *Let S be a nonempty set. If there exists an injection $S \hookrightarrow \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, then S is finite.*

Proof. Let $\sigma : S \hookrightarrow \{1, 2, \dots, n\}$ be an injection for some $n \in \mathbb{N}$. Define s_i by $\sigma(s_i) = i$ for $i \in \operatorname{im}(\sigma)$.

Notice that $\sigma' : S \to \sigma(S)$ is a bijection, since σ is injective and any map of the form $f : A \to f(A)$ is surjective by definition.

We define $r : \sigma(S) \hookrightarrow \mathbb{N}$ selecting i_1 to be the least element in $\sigma(S)$ (which exists by the well-ordering principle since $\{1, 2, \dots, n\} \subseteq \mathbb{N}$ is nonempty), and mapping $r(i_1) = 1$. Similarly, we inductively select i_k to be the least element in $\sigma(S) \setminus \{i_1, i_2, \dots, i_{k-1}\}$, and map $r(i_k) = k$. From this construction, it is clear that r is injective.

Then, defining $r' : \sigma(S) \to r(\sigma(S))$, we can see that r' is a bijection, with $r(\sigma(S)) = \{1, 2, \dots, j\}$ for some $j \leqslant n$ (since, by definition, σ is an injection, meaning $\sigma(s_i) \leqslant n$ for all n).

Taking $r' \circ \sigma' : S \to \{1, 2, \dots, j\}$, we see that this is a composition of bijections, meaning it is a bijection. Thus, S is finite. □

1.1

1.2

Problem. Given bijections $f : \mathbb{N} \to \mathbb{Z}$ and $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, show that the function $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ defined by $h(x, y) = P\left(f^{-1}(x), f^{-1}(y)\right)$ is bijective.

Solution. We begin by showing injectivity. Since f is bijective, so too is f^{-1}, meaning that for

$$h(x, y) = h(x', y'),$$

we have

$$\begin{aligned} P\left(f^{-1}(x), f^{-1}(y)\right) &= P\left(f^{-1}(x'), f^{-1}(y')\right) \\ f^{-1}(x) &= f^{-1}(x') \\ f^{-1}(y) &= f^{-1}(y') && \text{since } P \text{ is bijective} \end{aligned}$$

meaning

$$\begin{aligned} x &= x' \\ y &= y' && \text{since } f^{-1} \text{ is bijective.} \end{aligned}$$

Thus, h is injective.

Let $n \in \mathbb{N}$. Since P is surjective, there exist a, b such that $P(a, b) = n$. Since f^{-1} is surjective, there exists $x, y \in \mathbb{Z}$ such that $f^{-1}(x) = a$ and $f^{-1}(y) = b$. Thus, there exist $x, y \in \mathbb{Z}$ such that $h(x, y) = n$.

1.3

Problem. If A and B are countably infinite, show that $A \times B$ is countably infinite.

Solution. By the definition of countably infinite sets, there exist bijections $\alpha : A \to \mathbb{N}$ and $\beta : B \to \mathbb{N}$. Additionally, we know that there exists a bijection $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Define $h : A \times B \to \mathbb{N}$ by $h(a, b) = P(\alpha(a), \beta(b))$. Then, since h is a composition of bijections, h is a bijection between $A \times B$ and $\mathbb{N}$.

1.5

Problem. If $A_1, A_2, \dots$ is an infinite sequence of disjoint finite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

Solution. Let a_n be defined by the bijection $\alpha_n : A_n \to \{1, 2, \dots, a_n\}$.

1.6

1.7

Problem. Construct an explicit polynomial bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Solution. Let $Q : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $Q(x, y, z) = P(P(x, y), z)$, where $P(x, y) = \frac{(x+y)(x+y+1)}{2} + x$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

We know that Q is a bijection since it is a composition of bijections. I do not want to expand this expression.

Extra Problem 1

Problem. Prove that if A and B are finite sets, then $A \cup B$ is finite.

Solution. We have $A \cup B = A \setminus B \cup B \setminus A \cup A \cap B$. Since $A \setminus B \subseteq A$, $B \setminus A \subseteq B$, and $A \cap B \subseteq A$, with all three disjoint, this is a finite disjoint union of finite sets, meaning it is finite.[1]

Extra Problem 2

Problem. Prove that for every $n \in \mathbb{N}$, every subset of $\{0, 1, \dots, n\}$ is finite.

Solution. For any subset $P \subseteq \{0, 1, \dots, n\}$, the identity map is an injection into $\{0, 1, \dots, n\}$; composing the identity map with the bijection $a : \{0, 1, \dots, n\} \to \{1, 2, \dots, n+1\}$ defined by $a(m) = m + 1$, we see that there is an injection $a \circ \mathrm{id} : P \hookrightarrow \{1, 2, \dots, n+1\}$, meaning P is finite by the theorem above.

[1] In the order of my completing homework, I proved the injection to finite sets, then the subset of a finite set, then this problem.

Extra Problem 3

Problem. Prove that every subset of a finite set is finite.

Solution. Since every empty set is finite, so too is every subset of the empty set. Similarly, any empty subset of a given finite set is also finite.

Let A be a nonempty finite set. Then, there exists a bijection $\alpha : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

Let $B \subseteq A$ be nonempty. The identity map $\mathrm{id} : B \hookrightarrow A$ is an injection.

Thus, $\alpha \circ \mathrm{id} : B \hookrightarrow \{1, 2, \dots, n\}$ is an injection, as it is a composition of injections. By the established theorem above, this means B is finite.

Extra Problem 4

Problem. Prove that every infinite subset of $\mathbb{N}$ is denumerable.

Solution. Let $A \subseteq \mathbb{N}$ be infinite.

Since A is nonempty, by the well-ordering principle, there must exist a least element of A, which we label as a_0.

Consider $A \setminus \{a_0\}$. Since A is infinite, $A \setminus \{a_0\}$ must also be infinite, meaning there is a least element of $A \setminus \{a_0\}$ by the well-ordering principle. We label this element as $\{a_1\}$.

Now, we consider $A \setminus \{a_0, a_1\}$, and use the well-ordering principle to extract a_2, and inductively extract a_i by using the well ordering principle on $A \setminus \{a_0, a_1, \dots, a_{i-1}\}$.

The function $f : A \to \mathbb{N}$ defined by $f(a_i) = i$ is a bijection, since $f(a_i) = f\left(a_j\right)$ if and only if $i = j$.

Thus, f is a denumeration of A.