

Problem (Problem 1): Use de Rham cohomology to prove that if B^n is a closed ball in \mathbb{R}^n , and $f: B^n \rightarrow B^n$ is smooth, then f has a fixed point.

Solution: Suppose $f: B^n \rightarrow B^n$ is a fixed-point free self-map of the ball. It follows then that by drawing a line between v and $f(v)$, we may define a smooth retraction of the ball to the sphere S^{n-1} . Call this retraction r .

We observe then that r induces a map in cohomology $r^*: H_{\text{DR}}^*(S^{n-1}) \rightarrow H_{\text{DR}}^*(B^n)$. In particular, since r is a retraction to S^{n-1} , it follows that r is homotopic to the identity map when restricted to S^{n-1} , meaning r^* is an isomorphism in de Rham cohomology of $H_{\text{DR}}^*(S^{n-1})$ and $H_{\text{DR}}^*(B^n)$.

Yet, we recognize that $H_{\text{DR}}^{n-1}(S^{n-1}) \cong \mathbb{R}$, while $H_{\text{DR}}^{n-1}(B^n) \cong 0$, the latter emerging from the fact that B^n is contractible via the straight-line homotopy and the Poincaré lemma. Thus, no such r exists, whence f cannot have a fixed point.

Problem (Problem 2): Suppose M is a compact smooth manifold with a smooth triangulation, and let $f: M \rightarrow M$ be a smooth map preserving the triangulation. Write f_k^* for the induced map on $H_{\text{DR}}^k(M)$. Prove that if

$$L(f) = \sum_{k=0}^n (-1)^k \text{tr}(f_k^*) \neq 0,$$

then f has a fixed point.

Problem (Problem 3): Compute the de Rham cohomology of \mathbb{RP}^n .

Solution: To start, we observe that $\mathbb{RP}^1 \cong S^1$, meaning that the de Rham cohomology of \mathbb{RP}^1 is

$$H_{\text{DR}}^*(\mathbb{RP}^1) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = 1 \\ 0 & \text{else} \end{cases}.$$

In higher dimensions, we consider the family of charts defined by

$$U_k = \{[x_0 : \cdots : x_k : \cdots : x_n] \mid x_{i \neq k} \in \mathbb{R}, x_k \neq 0\}.$$

We seek to understand the picture of

$$\begin{aligned} U_{k \neq 0} &= \bigcup_{k=1}^n U_k \\ &= \bigcup_{k=1}^n \{[x_0 : \cdots : x_n] \mid x_k \neq 0\}. \end{aligned}$$

In particular, the only elements of U_0 that are not in $U_{k \neq 0}$ are the ones of the form $[1 : 0 : \cdots : 0]$, whence $U_{k \neq 0} \cong \mathbb{R}^n \setminus \{0\}$.

Next, we observe that

$$\begin{aligned} U_0 \cap U_{k \neq 0} &= \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\} \cap \bigcup_{k=1}^n \{[x_0 : \cdots : x_n] \mid x_k \neq 0\} \\ &= \{[x_0 : \cdots : x_n] \mid x_0 \neq 0, x_k \neq 0 \text{ for at least one } 1 \leq k \leq n\} \\ &= U_0 \setminus \{[1 : 0 : \cdots : 0]\} \\ &\cong \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

Thus, by Mayer–Vietoris, we obtain the following short exact sequence.

$$0 \longrightarrow H^*(\mathbb{RP}^n) \longrightarrow H^*(\mathbb{R}^n) \oplus H^*(\mathbb{R}^n \setminus \{0\}) \longrightarrow H^*(\mathbb{R}^n \setminus \{0\}) \longrightarrow 0$$

Focusing on the case of H^0 , this yields the following exact sequence, whence $H^0(\mathbb{RP}^n) \cong \mathbb{R}$.

$$0 \longrightarrow H^0(\mathbb{RP}^n) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \dots$$

Since the $H^k(\mathbb{R}^n)$ are zero for all $k \geq 1$, it follows that we have $H^k(\mathbb{RP}^n) \cong 0$ for $1 \leq k < n$.

Finally, concerning ourselves with $H^n(\mathbb{RP}^n)$, we concern ourselves with orientability; specifically, $H^n(\mathbb{RP}^n) \cong \mathbb{R}$ if n is odd and $H^n(\mathbb{RP}^n) \cong 0$ if n is even, as \mathbb{RP}^n is orientable if and only if n is odd.

Problem (Problem 4): Prove the Five Lemma. Namely, consider the following commutative diagram of vector spaces, where the horizontal sequences are exact. Show that if f_1, f_2, f_4, f_5 are isomorphisms, that f_3 is also an isomorphism.

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

Solution: We start by showing that f_3 is injective. Let $x \in \ker(f_3)$.

- By commutativity, we have

$$\begin{aligned} 0 &= \beta_3 \circ f_3(x) \\ &= f_4 \circ \alpha_3(x), \end{aligned}$$

so it follows that $\alpha_3(x) = 0$ as f_4 is injective, so $x \in \ker(\alpha_3)$. By exactness, we let $a_2 \in A_2$ be such that $\alpha_2(a_2) = x$, and define $f_2(a_2) = b_2$.

- By commutativity,

$$\begin{aligned} \beta_2(b_2) &= \beta_2(f_2(a_2)) \\ &= f_3(\alpha_2(a_2)) \\ &= f_3(x) \\ &= 0, \end{aligned}$$

so $b_2 \in \ker(\beta_2)$, meaning that by exactness, there is $b_1 \in B_1$ such that $\beta_1(b_1) = b_2$. Since f_1 is surjective, we let $a_1 \in A_1$ be such that $f_1(a_1) = b_1$.

- Finally, by commutativity, we have

$$\begin{aligned} f_2(\alpha_1(a_1)) &= \beta_2(f_1(a_1)) \\ &= \beta_1(b_1) \\ &= b_2 \\ &= f_2(a_2), \end{aligned}$$

and since f_2 is injective, we have $a_2 = \alpha_1(a_1)$.

- Thus, since $x = \alpha_2(a_2)$, we have

$$\begin{aligned} x &= \alpha_2(\alpha_1(a_1)) \\ &= (\alpha_2 \circ \alpha_1)(a_1) \\ &= 0, \end{aligned}$$

so f is injective.

Now, we show that f is surjective.