Abstract

We discuss the much-celebrated Baire Category Theorem and some of its consequences.

The Theorem

The Baire Category Theorem is a very celebrated theorem in analysis and topology that bridges the gap between the two fields. In words, it allows us to understand properties of complete metric spaces via their topological properties (i.e., density of open sets).

Definition (Baire Space)**.** Let $\{A_n\}_{n\geq 1}$ be a countable collection of open, dense subsets of a topological space X. We say X is a *Baire Space* if

$$A = \bigcap_{n\geq 1} A_n$$

is dense for every such collection.

Definition. We say a topological space X is *meager* if it is a countable union of nowhere dense subsets.[I]

Theorem (Baire Category Theorem)**:** If X is a complete metric space, then X is a Baire space.

Proof. Let $\{A_n\}_{n\geq 1}$ be a countable family of open, dense subsets of X. Let U_0 be any ball of radius $r > 0$, and set $B_0 = \overline{U_0}$.

Since $A_1 \cap U_0$ is nonempty, there is a closed subset $B_1 \subseteq A_1 \cap U_0$ with radius less than $r/2$. Set $U_1 = B_1^\circ$. Similarly, find $B_2 \subseteq A_2 \cap U_1$ with radius less than $r/4$, and set $U_2 = B_2^\circ$, and continue finding $B_m \subseteq A_m \cap U_{m-1}$ with radius less than $r/2^m$. We get a chain

$$B_0 \supseteq U_0 \supseteq B_1 \supseteq U_1 \supseteq \cdots,$$

and let $(x_n)_n$ be the sequence of centers of B_n. The sequence $(x_n)_n$ is Cauchy in X, as the distance between any two x_m and x_n is at most $r/2^{m-1}$ for $n > m$. Since X is complete, we get $(x_n)_n \to x$ for some $x \in X$. We claim $x \in \bigcap_{n\geq 1} B_n$.

If not, then $x \notin B_N$ for some $N \in \mathbb{N}$, and for $n \geq N$, $x \notin B_n$, so $d(x_n, x) \geq \operatorname{dist}_{B_n}(x) > 0$, which contradicts $x_n \to x$.

Therefore, $x \in \bigcap_{n\geq 1} B_n \subseteq \bigcap_{n\geq 1} A_n$, and since $\bigcap_{n\geq 1} B_n \subseteq U_0$, we have $\left(\bigcap_{n\geq 1} A_n\right) \cap U_0 \neq \varnothing$, so $\bigcap_{n\geq 1} A_n$ is dense in X. □

Remark: By taking complements, we see that if X is a complete metric space, then X is not the countable union of nowhere dense subsets.

The Consequences

There are a variety of great results that can be found from applying the Baire Category Theorem. First, we use the theorem to establish a fundamental result in the theory of complete metric spaces.

Proposition: Let X be a complete metric space with no isolated points, and let $S \subseteq X$ be a countable and dense subset. Then, S is not a G_δ set.

Proof. Since S is countable, we may write $S = \{s_k\}_{k\in\mathbb{N}}$. Suppose S is G_δ, so that $S = \bigcap_{n\geq 1} V_n$, where the V_n are open. We note that $S^c = \bigcup_{n\geq 1} V_n^c$, and write

$$X = S^c \cup \{s_k\}_{k\in\mathbb{N}}$$

[I] Recall that a nowhere dense set $A \subseteq X$ is nowhere dense if $\overline{A}^\circ = \varnothing$.

$$= \bigcup_{n\geq 1} V_n^c \cup \{s_k\}_{k\in\mathbb{N}}.$$

We see that V_n^c and $\{s_k\}$ are closed subsets of X for each n, k, and $\{s_k\}_{k\in\mathbb{N}}^\circ = \varnothing$ as X does not contain any isolated points.

By the Baire Category Theorem, there must be some $n \in \mathbb{N}$ such that $(V_n^c)^\circ \neq \varnothing$, so there is some $\delta > 0$ and $x \in X$ such that $U(x, \delta) \subseteq V_n^c \subseteq S^c$ for some $\delta > 0$. However, since S is dense, $U(x, \delta) \cap S \neq \varnothing$, which is a contradiction. $\square$

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