Math 395

Homework 5

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Problem 1

Let R be a commutative ring with identity. Let Σ be a multiplicative subset of R. Let $\mathcal{F} = \{(r, d) \mid r \in R, d \in \Sigma\}$. We will show by giving an explicit example that the relation $(r_1, d_1) \sim (r_2, d_2)$ if $r_1 d_2 - r_2 d_1 = 0$ is not necessarily an equivalence relation if R is not an integral domain.

Let $R = \mathbb{Z}/6\mathbb{Z}$, and consider the multiplicatively closed set $\{1, 3\}$. Then,

$$\mathcal{F} = \{(0,1), (0,3), (1,1), (1,3), (2,1), (2,3), (3,1), (3,3), (4,1), (4,3), (5,1), (5,3)\}.$$

With the given equivalence relation, we can see that $(2,1) \sim (0,3)$, as $2\cdot 3 - 0\cdot 3 \equiv 0$ modulo 6, and $(0,3) \sim (2,3)$, as $0\cdot 3 - 2\cdot 3 \equiv 0$ modulo 3, but $(2,1) \nsim (2,3)$, as $2\cdot 3 - 1\cdot 2 \not\equiv 0$ modulo 3. Thus, the relation is not transitive, and is not an equivalence relation.

Problem 3

Let R_1 and R_2 be rings with identity. We will show that if I is an ideal in $R_1 \times R_2$, then I is of the form $I_1 \times I_2$, where I_j is an ideal in R_j.

Let I be an ideal in $R_1 \times R_2$. Then, for any $(a, b), (c, d) \in I$ and any $(x, y) \in R$, then $(a, b) - (c, d) = (a - b, c - d) \in I$, $(a, b)(x, y) = (ax, by) \in I$, and $(x, y)(a, b) = (xa, yb) \in I$.

Define $I_1 = \pi_1(I)$ and $I_2 = \pi_2(I)$. We will show that I_1 and I_2 are ideals in R_1 and R_2 respectively, with $I = I_1 \times I_2$. Let $a, b \in I_1$ and $x \in R_1$. Then, $a = \pi_1((a, k))$ and $b = \pi_1((b, \ell))$for some $(a, k), (b, \ell) \in I$. Then, $a - b = \pi_1((a, k)) - \pi_1((b, \ell))$, and since the projection map is a homomorphism, this is equivalent to $\pi_1((a - b), (k - \ell))$. Since I is closed under subtraction, $(a - b, k - \ell) \in I$, so $a - b \in I_1$. Similarly, for $a, b \in I_2$, $a - b \in I_2$.

Let $x \in I_1, r \in R_1$. Then, $x = \pi_1((x, t))$ for some $(x, t) \in I$, and $r = \pi_1((r, s))$ for some $(r, s) \in R_1 \times R_2$. So,

$$\begin{aligned} xr &= \pi_1((x,t))\pi_1((r,s)) \\ &= \pi_1((x,t)(r,s)) \\ &= \pi_1((xr, ts)), \end{aligned}$$

and since $(xr, ts) \in I$,

$$\in I_1.$$

Similarly,

$$\begin{aligned} rx &= \pi_1((r,s))\pi_1(x,t) \\ &= \pi_1((rx, st)) \end{aligned}$$

and since $(rx, st) \in I$,

$$\in I_1.$$

Similar results hold for I_2. Therefore, I_1 and I_2 are ideals.

Clearly, $I \subseteq I_1 \times I_2$.

Let $(i_1, i_2) \in I_1 \times I_2$. Then, $(i_1, b) \in I$ for some $b \in R_2$ and $(a, i_2) \in I$ for some $a \in R_1$. By the definition of ideal, $(1, 0)(i_1, b) \in I$ and $(0, 1) \times (a, i_2) \in I$, so $(i_1, 0) \in I$ and $(0, i_2) \in I$. Since I is an ideal, $(i_1, i_2) \in I$. Thus, $I_1 \times I_2 \subseteq I$.

Problem 8

Let $V = \mathbb{R}^n$, and let $v = (a_1, \dots, a_n) \in V$ be fixed. We will prove that the collection $(x_1, \dots, x_n) \in V$ with $a_1x_1 + \cdots + a_nx_n = 0$ is a subspace of V.

Let $T : \mathbb{R}^n \to \mathbb{R}$ be defined by $T(y) = \langle v, y \rangle$, where $\langle v, y \rangle$ denotes the traditional inner product on $\mathbb{R}^n$. Then, $\ker T = \{y \mid \langle v, y \rangle = 0\}$, which is precisely the collection of $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $a_1x_1 + \cdots + a_nx_n = 0$. For any $y_1, y_2 \in \ker(T)$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} T(\alpha y_1 + y_2) &= \alpha T(y_1) + T(y_2) \\ &= 0, \end{aligned}$$

meaning $\ker T$ is a subspace. We know from a previous result that $\dim_{\mathbb{R}}(\ker T) + \dim_{\mathbb{R}}(\mathbb{R}) = \dim_{\mathbb{R}}(\mathbb{R}^n)$, so $\dim_{\mathbb{R}}(\ker T) = n - 1$.

To find a basis for $\ker T$, take w_1 a nonzero vector such that $\langle w_1, v \rangle = 0$ and $w_1 \notin \operatorname{span}(v)$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. From there, select $w_2 \notin \operatorname{span}(w_1) \cup \operatorname{span}(v)$ with $\langle w_2, v \rangle = 0$, and iteratively for $w_3, \dots, w_{n-1}$. The collection $\{w_i\}_{i=1}^{n-1}$ is obviously spanning for $\ker T$. To show that it is linearly independent, let

$$c_1 w_1 + \cdots + c_{n-1} w_{n-1} = 0.$$

Since $w_1, \dots, w_{n-1}$ are definitionally not in $\operatorname{span}(v)$, neither is $\{w_1, \dots, w_n\}$. However, since $0 \in \operatorname{span}(v)$, this is only the case if $c_1 = \cdots = c_{n-1} = 0$. Therefore, $\{w_1, \dots, w_n\}$ are linearly independent.

Problem 9

Let $T : \mathbb{R}^4 \to \mathbb{R}$ be the linear transformation so that

$$\begin{aligned} T((1,0,0,0)) &= 1 \\ T((1,-1,0,0)) &= 0 \\ T((1,-1,1,0)) &= 1 \\ T((1,-1,1,-1)) &= 0. \end{aligned}$$

To determine $T((a,b,c,d))$ for any $(a,b,c,d) \in \mathbb{R}^4$, we will first convert the given basis vectors into the standard basis.

$$\begin{aligned} T((1,0,0,0)) &= 1 \\ T((0,1,0,0)) &= T((1,0,0,0) - (1,-1,0,0)) \\ &= T((1,0,0,0)) - T((1,-1,0,0)) \\ &= 1 \\ T((0,0,1,0)) &= T((1,-1,1,0) - (1,-1,0,0)) \\ &= T((1,-1,1,0)) - T((1,-1,0,0)) \\ &= 1 \\ T((0,0,0,1)) &= T((1,-1,1,0) - T((1,-1,1,-1))) \\ &= T((1,-1,1,0)) - T((1,-1,1,-1)) \\ &= 1. \end{aligned}$$

Therefore,

$$\begin{aligned} T((a,b,c,d)) &= aT((1,0,0,0)) + bT((0,1,0,0)) + cT((0,0,1,0)) + dT((0,0,0,1)) \\ &= a + b + c + d. \end{aligned}$$