Problem 1

Let X be a metric space and consider a subset $Y \subseteq X$ viewed as a metric space. Show that $C \subseteq Y$ is connected in Y if and only if it is connected as a subset of X.

Proof: $C \subseteq Y$ is connected if and only if any splitting $C \subseteq (Y \cap U) \sqcup (Y \cap V)$ in Y is trivial, for $U, V \subseteq X$ open. Thus, $C \subseteq Y \cap (U \sqcup V)$ is a trivial splitting, if and only if $C \subseteq U \sqcup V$ is trivial.

Problem 2

If X is a metric space, and $Y \subseteq X$ is a connected subset of X, show that for every splitting $X = X_1 \sqcup X_2$, $X_i \subseteq X$ open, we must have $Y \subseteq X_1$ or $Y \subseteq X_2$.

Problem 3

For $n = 0, 1, 2, 3 \ldots$, let $X_n := [0, 1] \times \{2^{-n}\}$, and consider the space

$$X = \{(0,0), (1,0)\} \cup \left(\bigcup_{n=1}^{\infty} X_n \right).$$

(i) List all the connected components of X.

(ii) If $X = U \sqcup V$ is a nontrivial splitting of X, show that there is a finite subset $F \subseteq \mathbb{N}$ with

$$U = \bigcup_{n \in F} X_n, \quad V = X \setminus U.$$

Problem 4

Show that the n-sphere, $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\|_2 = 1\}$ is path-connected.

Problem 5

Let X be a metric space. We define a relation on X, $x \sim y$ if and only if there exists a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Show that this defines an equivalence relation on X. Equivalence classes are called path-connected components.

Problem 6

Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Problem 7

Let V be a normed space and suppose $Y \subseteq V$ is an open and connected subset. Fix a vector $y_0 \in Y$, and set

$$W := \{w \in Y \mid \text{there is a path from } y_0 \text{ to } w\}.$$

(i) Show that W is Y.

(ii) Show that W is closed in Y.

(iii) Conclude that Y is path-connected.

Problem 8

A group is a nonempty set G with a binary operation $G \times G \to G$, $(s, t) \mapsto st$ satisfying

- $(st)r = s(tr)$;
- there is a unique neutral element $e \in G$ with $te = et$ for all $t \in G$;
- for every $t \in G$ there is a unique inverse $t^{-1} \in G$ with $t^{-1}t = tt^{-1} = e$.

A subgroup of G is a nonempty subset $H \subseteq G$ such that $s, t \in H \Rightarrow st, t^{-1} \in H$. The subgroup H is normal if $t \in G, s \in H$ implies $tst^{-1} \in H$.

Consider a group G equipped with a metric so that the operations $G \times G \to G$, $(s, t) \mapsto st$ and $G \to G$, $t \mapsto t^{-1}$ are both continuous. Show that the connected component containing the neutral element e, G_0, is a closed and normal subgroup of G.

Problem 9

Show that the Cantor set is totally disconnected.

Problem 10

A metric space X is called zero-dimensional if for any $x, y \in X$ with $x \neq y$, there are open subsets $U, V \subseteq X$ with $x \in U, y \in V$ and $X = U \sqcup V$.

(i) Show that every zero-dimensional metric space is totally disconnected.

(ii) If $Y \subseteq \mathbb{R}$ is totally disconnected, show that Y is zero-dimensional.

(iii) Conclude that $\mathbb{Q}$ and the Cantor set are zero-dimensional.

Bonus

Let X be a compact metric space. Show that X is zero-dimensional if and only if X admits a basis of compact-open subsets.