## Introduction: naive set theory

$$\begin{aligned}
\mathbb{N} &= \{1,2,3,\dots,\} \\
\mathbb{Z} &= \{0,\pm1,\pm2,\dots,\} \\
\mathbb{Z}_+ &= \{0,1,2,\dots,\} \\
\mathbb{Q} &= \left\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\right\} \\
\mathbb{C} &= \{a+bi \mid a,b\in\mathbb{R}\} \\
\mathbb{C}_q &= \{a+bi \mid a,b\in\mathbb{Q}\}
\end{aligned}$$

Recall: given sets $X$ and $Y$, a relation from $X$ to $Y$ is a subset of $X\times Y$, where $\times$ denotes the cartesian product of $X$ and $Y$.

A relation $f\subseteq X\times Y$ is a function from $X$ to $Y$ such that $\forall x\in X$, $\exists! y\in Y$ such that $(x,y)\in f$. We write $f(x)=y$, and denote $f$ as $f: X\to Y$.

$X$ is the **domain** of $f$ and $Y$ is the **codomain**. The range $\text{Ran}(f)=\{f(x)\mid x\in X\}\subseteq Y$.

The graph of a function $\text{Graph}(f)=\{(x,f(x))\mid x\in X\}\subseteq X\times Y$.

### Examples

$$\text{id}_x : X\to X, \text{id}_X(x)=x$$

This is the identity function.

The Characteristic Function: If $A\subseteq X$

$$\mathbf{1}_A : X\to\mathbb{R},\ \mathbf{1}_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$$

### Algebra of Functions

Let $X$ be any set, and $(X;\mathbb{R})=\{f: X\to\mathbb{R}\}$ represent the function space of $X$ with codomain $\mathbb{R}$.

Let $f,g\in\mathcal{F}(X;\mathbb{R})$. Then, $(f+g)(x)=f(x)+g(x)$, and $(f\cdot g)(x)=f(x)\cdot g(x)$.

If $t\in\mathbb{R}$, then $(tf)(x)=tf(x)$ (scalar multiplication). If $g(x)\neq 0\forall x\in X$, then $\left(\frac{f}{g}\right)(x):=\frac{f(x)}{g(x)}$.

Finally, we have composition. If $f: X\to Y$ and $g: Y\to Z$ are functions, then $g\circ f(x)=g(f(x))$.

### Injective, Subjective, and Bijective

A function $f: X\to Y$ is a **injective** map, then, if $f(x_1)=f(x_2)$, then $x_1=x_2$.For example, the shift map $S:\mathbb{N}\to\mathbb{N},\ S(n)=n+1$ is injective.

Any strictly increasing function $f: I\to\mathbb{R}$, where $I$ is any interval, is injective.

A function $f$ is **surjective** if $\forall y\in Y,\exists x\in X$ such that $f(x)=y$.

Consider the function $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^3-2x+1$. We can show that this function is surjective because $\lim_{x\to\infty}f(x)=\infty$, $\lim_{x\to-\infty}f(x)=-\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f)=\mathbb{R}$.

$f$ is **bijective** if it is injective and surjective.

### Invertibility

Let $f : X \to Y$ be a function. $f$ is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \text{id}_X$. $f$ is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \text{id}_Y$.

$f$ is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

#### Invertibility Definition

$f$ is invertible if and only if $f$ is left and right invertible.

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Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose $g$ is a left-inverse of $f$, and $h$ is a right-inverse of $f$. Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

#### Injection and Surjection Invertibility

If $f : X \to Y$ is a function:

1. $f$ is injective $\Leftrightarrow$ $f$ is left-invertible.
2. $f$ is surjective $\Leftrightarrow$ $f$ is right-invertible.
3. $f$ is bijective $\Leftrightarrow$ $f$ is invertible.

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We will prove the first proposition in the forward direction. Suppose $f$ is injective. Given $y \in \text{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define $g$ as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where $x_0$ is an arbitrary point in $X$. We can see that $g \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

### Cardinality and Finitude

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets $A$ and $B$, we can show that $A$ is the same size as $B$ by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

#### Cardinality

Sets $A$ and $B$ have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\text{card}(A) = \text{card}(B)$.

#### Equivalent Cardinalities of Intervals

Given $a < b$ and $c < d$, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

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We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

### Intervals and Real Numbers

$$\text{card}\left((0,1)\right) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - tan is strictly increasing (and thus injective)
  - $\lim_{x\to\infty} \tan(x) = \infty$ and $\lim_{x\to-\infty} \tan(x) = -\infty$, and by intermediate value theorem, tan is surjective
- $\ell : (0,1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0,1) \to \mathbb{R}$.

### Finitude

A set $F$ is **finite** if $F$ is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}\left(\{1, 2, \dots, n\}\right)$. A non-finite set is called infinite.

We can *enumerate* $F$ by creating a function $\sigma : \{1, 2, \dots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

### Inequality of Finite Sets

If $m \neq n$, then $\text{card}\{1, 2, \dots, m\} = \text{card}\{1, 2, \dots, n\}$.

WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is our bijection. This means there are $m$ "pigeons" and $n$ "holes."

One hole, $j$, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since $f$ is assumed to be injective, this is a contradiction.

### Infinitude of the Naturals

$\mathbb{N}$ is infinite.

Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m+1\} \to \mathbb{N}$. $i$ is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \to \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

**Proposition**

If $A$ is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

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$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

$\vdots$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of $A$.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. $f$ is injective as $a_n$ are distinct.

**Cardinality of Integers and Natural Numbers**

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

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$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

$f$ is a bijection as $g : \mathbb{N} \to \mathbb{Z}$, $g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of $f$.

**Power Set**

Given any set $X$, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of $X$.

$2^X := \{f \mid f : X \to \{0,1\}\}$.

**Power Set and $2^X$**

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

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Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbf{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$.

### Cantor's Theorem

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

Fact from calculus: $\forall \sigma \in (0,1)$, $\sigma$ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0,1,\dots,9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0,1,\dots,9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0,1,\dots,9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since $r$ is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

### Comparing Cardinalities

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

#### Transitive Property

If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

The composition of two injective functions is injective.

#### Canonical Set Comparisons

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

#### Cardinality of the Power Set

For any set $A$, $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Let us construct a function: $f : A \to \mathcal{P}(A)$, where $a \mapsto \{a\}$.

$f$ is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

**Claim** $\nexists g : A \to \mathcal{P}(A)$, $g$ is surjective.

Suppose toward contradiction that such a $g$ exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since $g$ is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. $\perp$

**Equivalent Propositions**

(i) $\text{card}(A) \leq \text{card}(B)$

(ii) $\exists f : A \hookrightarrow B$

(iii) $\exists g : B \to A$, $g$ surjection.

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By definition, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) If $f : A \hookrightarrow B$, $f$ is left-invertible, and thus $\exists g : B \to A$ with $g \circ f = id_A$. So, $g$ is right-invertible, so $g$ is surjective.

(iii) $\Rightarrow$ (ii) If $g : B \to A$ is surjective, then $g$ is right-invertible, so $\exists f : A \to B$ such that $g \circ f = id_B$. So, $f$ is left-invertible, so $f$ is injective.

**Corollary**

If $f : A \to B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$.

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Consider $g : A \to f(A)$, where $g(a) = f(a)$. So, $g$ is onto, so $\exists$ an injection $f(A) \hookrightarrow A$.

**More Cardinality of Canonical Sets**

Consider the map $q : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, q(m, n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $H(m, n) = (h(m), n)$.

**Claim** $H$ is a bijection.

**Proof of Injection** If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since $h$ is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

**Proof of Surjection** Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m, n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m, n) = (k, \ell)$. Set $n = \ell$, and since $h$ is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m, n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, $\varphi$ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned}
\text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\
&\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\
&= \text{card}(\mathbb{N} \times \mathbb{N}) \\
&\leq \text{card}(\mathbb{N})
\end{aligned}$$

Cardinality Rules

(i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)

(ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)

(iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)

(iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

**Proof of (iii)** We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

**Claim** We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, $g$ and $f$ are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1$. $\perp$

We let $A_n = g \circ f(A_{n-1})$ for arbitrary $n$, and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \to B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where $y_x$ is the unique element in $B$ with $g(y_x) = x$.

**Claim** We claim $h$ is the desired bijection.

**Proof of Injection** Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of $H$, $f(x_1) = f(x_2)$, $f$ is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y(x_2)) = x_2 \notin A_\infty$. This case is not possible.

Thus, $h$ is injective.

**Proof of Surjection** Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where $y_x$ is the unique element in $B$ with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since $g$ is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Therefore, we have $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$.

Countability

A set $X$ is *countable* if $\exists f : x \hookrightarrow \mathbb{N}$ ($\text{card}(X) \leq \text{card}(\mathbb{N})$). $\text{card}(\mathbb{N}) = \aleph_0$. If $X$ is countable and infinite, $X$ is *denumerable*.

Corollary to Cantor-Schröder-Bernstein

If $X$ is denumerable, then $\text{card}(X) = \aleph_0$.

Since $X$ is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since $X$ is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

(as shown earlier)

**Countability under Union**

The countable union of countable sets is countable. If $I$ is a countable indexing set and for each $i \in I$, $A_i$ is countable, then $\bigcup\limits_{i\in I} A_i$ is countable.

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Since each $A_i$ is countable, $\exists \pi_i : \mathbb{N} \twoheadrightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \to \bigcup_{i\in I} A_i$$

defined as $\pi(i,j) = \pi_i(j)$.

**Claim 1** $\pi$ is a surjection.

**Proof 1** Let $x \in \bigcup_{i\in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since $\pi_{i_0}$ is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, $\pi$ is surjective.

**Claim 2** $I \times \mathbb{N}$ is countable.

**Proof 2** We know $\exists f : I \hookrightarrow \mathbb{N}$ since $I$ is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i,n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m,n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since $\pi$ is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i\in I} A_i$ is countable.

**Continuum Hypothesis**

We saw that $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(2^{\mathbb{N}})$, where $2^{\mathbb{N}}\{f \mid f : \mathbb{N} \to \{0,1\}\}$.

**Theorem** $\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}})$, where $I$ is any non-degenerate interval.

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**Lemma 1** $\text{card}([0,1]) \leq \text{card}(2^{\mathbb{N}})$.

**Proof 1** Every $t \in [0,1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0,1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\varphi} [0,1]$, defined as $\phi(f) = \displaystyle\sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Set $f : \mathbb{N} \to \{0,1\}$, $f(k) = \sigma_k$.

Therefore, $\varphi$ is surjective, so $\exists \{0,1\} \hookrightarrow 2^{\mathbb{N}}$, so $\text{card}([0,1]) \leq 2^{\mathbb{N}}$

**Lemma 2** $\text{card}([0,1]) = \text{card}(\mathbb{R})$.

**Proof 2** We have $[0,1] \overset{i}{\hookrightarrow} \mathbb{R}$ via inclusion, so $\text{card}([0,1]) \leq \text{card}(\mathbb{R})$.

Also, $\text{card}(\mathbb{R}) = \text{card}((0,1)) \leq \text{card}([0,1])$, so by Cantor-Schröder-Bernstein, $\text{card}(\mathbb{R}) = \text{card}([0,1])$.

**Lemma 3** Any two non-degenerate intervals $I$ and $J$ have the same cardinality.

**Proof 3** We can create injections $I \hookrightarrow J$ and vice-versa.

**Lemma 4** $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0,1])$.

**Proof 4** $\psi : 2^{\mathbb{N}} \to [0,1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

$\psi$ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim $\psi$ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$, $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} + \frac{1}{3^{k_0}} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$\begin{aligned}
|t_f - t_g| &= |\sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k}| \\
&\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\
&\leq \sum_{k>k_0} \frac{1}{3^k} \\
&= \frac{(1/3)^{k_0+1}}{1-(1/3)} \\
&= \frac{1}{2} \cdot \frac{1}{3^{k_0}}
\end{aligned}$$

$\perp$

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0,1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than $\aleph_0$ and strictly less than $2^{\aleph_0}$ is an axiom (i.e., it can be an assumption or not).

## Ordering

Let $X$ be a non-empty set. A relation on $X$ is a subset of $X \times X$.

- $R$ is *reflexive* if $\forall x \in X,\ (x,x) \in R$.
- $R$ is *transitive* if $(x,y),(y,z) \in R \to (x,z) \in R$.
- If $R$ is *antisymmetric* $(x,y),(y,x) \in R \to x = y$.

If $R$ is reflexive, transitive, and antisymmetric, then $R$ is an *ordering* of $X$.

If $R$ is an ordering of $X$, then we write:

$$(x,y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x\ \forall x \in X$
- $x \leq_R y,\ y \leq_R z \to x \leq_R z$
- $x \leq_R y,\ y \leq_R x \to x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

### Algebraic ordering of $\mathbb{N}_0$

$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$ such that $n + k = m$

### $\mathbb{N}$ ordered via division

$$n \leq_D m \Leftrightarrow n|m$$

Under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

**Inclusion** Let $S$ be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_i B \Leftrightarrow A \subseteq B$.

**Containment** With $X$ defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

For $\mathcal{F}(X,\mathbb{R}) = \{f \mid f : X \to \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x)\ \forall x \in X$.

### Types of Orderings

- An ordering $\leq$ of $X$ is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.
- An ordering is *directed* if $\forall x, y \in X \; \exists z \in X$ such that $x \leq z$ and $y \leq z$.

If $X$ is a totally ordered set, $X$ is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D),\ (\mathcal{P}(S), \leq_i),\ (\mathcal{P}(S), \leq_c)$$

### Upper/Lower Bounds

(i) Let $(X, \leq)$ be an ordered set, $A \subseteq X$. $A$ is bounded above if $\exists v \in X$ with $a \leq v \; \forall a \in A$. Such a $v$ is an upper bound.

(ii) $A$ is bounded below if $\exists \ell \in X$ such that $a \geq \ell \; \forall a \in A$. Such a $w$ is a lower bound.

(iii) If $v$ is an upper bound of $A$ and $v \in A$, then $v$ is the greatest element of $A$, or $\max(A) = v$.

(iv) If $\ell$ is a lower bound for $A$ and $\ell \in A$, then $\ell$ is the least element of $A$, or $\min(A) = \ell$.

(v) If $u$ is an upper bound for $A$, and $u \leq v$ for all other upper bounds $v$ of $A$, then $u$ is the *least upper bound* of $A$, or $\sup(A) = u$ (for *supremum*).

(vi) If $\ell$ is a lower bound for $A$, and $\ell \leq g$ for all other lower bounds $g$ of $A$, then $\ell$ is the *greatest lower bound* of $A$, or $\inf(A) = \ell$ (for *infimum*).

(vii) If $A$ is bounded above and below, then $A$ is bounded.

### Well-Ordering Principle

With $(\mathbb{N}, \leq_a)$, every nonempty $A \subseteq \mathbb{N}$ has a least element.

### Examples

#### Example 1

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \dots, 12\}$, we have the following:

**Bounded Above?** Yes.

**Upper Bounds** $12, 13, 14, \dots$

**Greatest Element** $12$

**Example 2**

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \dots, 10\}$

**Bounded Above?** Yes.

**Upper Bounds** 10!

**Greatest Element?** No.

**Supremum** $2^3 \cdot 3^2 \cdot 5 \cdot 7$

**Bounded Below?** Yes.

**Lower Bound** 1

**Least Element?** No.

**Infimum** 1

**Example 3**

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i\in I} \subseteq \mathcal{P}(S)$.

**Supremum** $\bigcup_{i\in I} A_i$

**Infimum** $\bigcap_{i\in I} A_i$

**Complete Sets**

An ordered set $(X, \leq)$ is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, $\mathbb{Q}$ is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

**Ordering of $\mathbb{Z}$**

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0,\ n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

**Properties of $\mathbb{Z}^+$**

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

**Ordering of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$**

Recall the ordering of $\mathbb{Z}$:

$$n \leq_a m \overset{\text{def}}{\iff} \exists k \in \mathbb{N}_0 \text{ with } n + k = m$$

**Claim** $\leq_a$ is an ordering of $\mathbb{Z}$

We claim that $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$. Thus, $\mathbb{Z}^+ = \mathbb{N}_0$.

**Properties of $\mathbb{Z}^+$**

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

**Other Properties of $\mathbb{Z}$**

(1) $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$

(2) $m \leq_a n$ and $p \leq_a q \Rightarrow m + p \leq_a n + q$

(3) $m \leq_a n$ and $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$

(4) $m \leq_a n \Rightarrow -m\ {}_a{\geq} n$

(5) $\leq_a$ is total.

(6) If $a\ {}_a{>}-$, and $ab\ {}_a{\geq}0$, then $b\ {}_a{>}0$

(7) If $a > 0$ and $ab\ {}_a{\geq}ac$, then $b \geq c$.

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**Proof of (3):**

$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0$ with $m + k = n$.
$\Rightarrow pm + pk = pn$
$pk \in \mathbb{N}_0$ by the properties of $\mathbb{Z}^+$. So, $pm \leq_a pn$

**Proof of (5):**

Let $m, n \in \mathbb{Z}$. Consider $m - n$.
By (ii), $m - n \in \mathbb{Z}^+$ or $-(m - n) \in \mathbb{Z}^+$. Thus, $m - n = k$ for some $k \in \mathbb{Z}^+$, or $-(m - n) = \ell$ for some $\ell \in \mathbb{Z}^+$.
Thus, $n \leq_a m$ in the first case, or $m \leq_a n$ in the second case.

We now want an ordering on $Q$.

### Creating the Rationals

Recall that $Q = \mathbb{Z} \times \mathbb{Z}^* = \{(a,b) \mid a \in \mathbb{Z},\ b \in \mathbb{Z},\ b \neq 0\}$. Consider the equivalence relation:

$$(a,b) \sim (c,d) \overset{\text{def}}{\Longleftrightarrow} ad = bc$$

We will let $\mathbb{Q} = \{[(a,b)] \mid (a,b) \in Q\}$ be the set of all equivalence classes in $Q$. We write:

$$[(a,b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

We must check that addition is well-defined: $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{a'd'+c'b'}{b'd'} = \frac{ad+bc}{bd}$.

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make $\mathbb{Q}$ a **field**:

#### Fields

A ring is a nonempty set set $R$ equipped with two binary operations:

- $+ : R \times R \to R$, $(a,b) \mapsto a+b$ ("addition")
- $\cdot : R \times R \to R$, $(a,b) \mapsto a \cdot b$ ("multiplication")

such that the following hold:

(1) $(a+b)+c = a+(b+c)$

(2) $\exists z \in R$ such that $a+z = a = z+a\ \forall a \in R$; there is at most one such $z$. Set $z = 0_R$.

(3) $\forall a \in R, \exists b \in R$ such that $a+b = 0_R = b+a$; there is at most one such $b$. Set $b = -a$.

(4) $\forall a,b \in R,\ a+b = b+a$.

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot (b+c) = a \cdot b + a \cdot c$, $(a+b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If $(R,+,\cdot)$ satisfies $ab = ba$, $R$ is a commutative ring.

If there exists $u \in R$ such that $ua = au = a\ \forall a \in R$, $R$ is a unital ring; there is at most one unit. Set $u = 1_R$

An integral domain is a unital, commutative ring such that $ab = 0 \Rightarrow a = 0 \vee b = 0$. For example, $\mathbb{Z}$ is an integral domain. However, $c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ is a unital, commutative ring, but there exist two functions such that $f, g \neq \mathbf{0}$, but $f \cdot g = \mathbf{0}$.

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R,\ \text{with } ab = 1_R$$

There is only one such $b$. Set $b = a^{-1}$.

#### Proof that $\mathbb{Q}$ is a Field:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Provided that $\frac{a}{b} \neq 0_{\mathbb{Q}}$.

Additionally, $\mathbb{Z} \overset{j}{\hookrightarrow} \mathbb{Q}$, $j(n) = \frac{n}{1}$ is injective.

**Ordering of $\mathbb{Q}$**

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

Prove that this ordering is well-defined.

**Order Embedding**

$\leq$ is a well-defined total ordering of $\mathbb{Q}$, and $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$, $j(n) = \frac{n}{1}$ is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

**Properties of $\mathbb{Q}^+$**

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

(i) $q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+$, $q_1 q_2 \in \mathbb{Q}^+$

(ii) $q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$

(iii) $\pm q \in \mathbb{Q}^+, q = 0$

(iv) $x \leq y, !u \leq v \Rightarrow x + u \leq y + v$

(v) $x \leq y,\ 0 \leq z \Rightarrow zx \leq zy$

**Ordering of $\mathbb{R}$, cont'd**

An **ordered field** is a field $F$ equipped with a total ordering $\leq_F$ such that:

(i) if $s \leq_F t$, and $x \leq_F y$, then $s + x \leq_F t + y$

(ii) if $s \leq_F t$ and $0 \leq_F z$, then $zs \leq_F zt$

For example, $\mathbb{Q}$ with its ordering is an ordered field.

**Proposition 1:** If $(F, \leq_F)$ is an ordered field, we define $F^+ = \{x \in F, x\ {}_F{\geq} 0\}$ with the following properties:

(1) $x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$

(2) $x \in F \Rightarrow x \in F^+, -x \in F^+$

(3) $\pm x \in F^+ \Rightarrow x = 0_F$

**Proofs**

(1) Let $x, y \in F^+$. Then, $x \geq 0$ and $y \geq 0$, so by property $(i)$ of an ordered field, $x + y \geq 0 + 0 = 0$, so $x + y \in F^+$. Additionally, we have $x \cdot y \geq x \cdot 0 = 0$, so $xy \in F^+$.

(2) Let $x \in F$. Since the ordering on $F$ is total, $x \geq 0$ or $0 \geq x$. In the first case, $x \in F^+$. In the second case, we add $-x$ to both sides, so by $(i)$, $-x \geq 0$, so $-x \in F^+$.

(3) We have $x \geq 0$ and $-x \geq 0$. So $x \geq 0$ and $x + (-x) \geq x + 0$, so $x \geq 0$ and $0 \geq x$. So, $x = 0$ by antisymmetry.

**Note:** $x \leq_F y \Leftrightarrow y - x \in F^+$.

**Proposition 2:** Let $F$ be an ordered field. Then, the following is true:

(1) $\forall a \in F$, $a^2 \in F^+$

(2) $0, 1 \in F^+$

(3) If $n \in \mathbb{N}$, $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$

(4) If $x \in F^+$, and $x \neq 0$, then $x^{-1} \in F^+$

(5) If $xy > 0$, then $x, y \in F^+$, or $-x, -y \in F^+$

(6) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$

(7) If $x \leq y$, then $-y \leq -x$

(8) $x \geq 1 \Rightarrow x^2 \geq x \geq 1$, and $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$.

**Proofs**

(1) Let $a \in F$. Then, $a \in F^+$ or $-a \in F^+$.

Case 1 If $a \in F^+$, then by the previous proposition, $a^2 \in F^+$.

Case 2 If $-a \in F^+$, then by the previous proposition, $(-a)(-a) = a^2 \in F^+$.

(2) $0 \geq 0$, so $0 \in F+$. $1 = 1 \cdot 1 = 1^2 \in F^+$ by the previous result.

(3) $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots 1_F}_{n \text{ times}} \in F^+$ by the previous proposition.

(4) Let $x \neq 0, x \in F^+$. Suppose toward contradiction that $x^{-1} \notin F^+$, then $-x^{-1} \in F^+$. Thus, $x \cdot (-x^{-1}) \in F^+$, so $-1 \in F^+$, but $1 \in F^+$, so $1 = 0$. $\perp$

(5) Let $xy > 0$, meaning $xy \in F^+$. Suppose toward contradiction that $x > 0$ and $y < 0$. So, $x > 0$ and $-y > 0$, so $(x)(-y) > 0$, so $-(xy) \in F^+0$, so $xy = 0$. $\perp$

(6) Let $0 < x \leq y$. We know $x^{-1} \in F^+$, so $x^{-1}x \leq x^{-1}y$. So $1 \leq x^{-1}y$. We also know $y \in F^+$, so $y^{-1} \in F^+$. So, $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$.

(7) Let $x \leq y$. Then, $0 \leq y - x$, so $-y \leq -x$.

(8) Let $x \geq 1$. Then, $x \cdot x \geq 1 \cdot x \geq 1$.

**Order Axiom**

$\mathbb{R}$ is an ordered field. The injection $\mathbb{Q} \hookrightarrow \mathbb{R}$, $i(q) = q$ is an order embedding.

## Rational Orderings

**Proposition 1:** If $a \leq b$, then $a \leq \frac{1}{2}(a + b) \leq b$

**Proof**

$2a = a + a \leq a + b \leq b + b$, all by property (i) of an ordered field.

Therefore, $2a \leq a+b \leq 2b$. Since $2 = 1+1 \in \mathbb{R}^+$, $2^{-1} \in \mathbb{R}^+$, so $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$, so $a \leq \frac{1}{2}(a+b) \leq b$.

**Proposition 2:** If $a \geq 0$ and $(\forall \varepsilon > 0), a \leq \varepsilon$.

**Proof**

If $a \geq 0$ and $a \neq 0$, then $a > 0$. So, we have that $\frac{1}{2}a < a$. Let $\varepsilon = \frac{1}{2}a$. We also have that $a \leq \varepsilon = \frac{1}{2}a < a$, so $a < a$. $\perp$

## Arithmetic and Geometric Means

Given $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$:

**Arithmetic Mean**

$$= \frac{\sum_{i=1}^{n} a_i}{m}$$

**Geometric Mean**

$$= \sqrt[m]{a_1 a_2 \cdots a_m}$$

Arithmetic Mean-Geometric Mean Inequality

Let $a, b \geq 0$.

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

---

If $x, y \geq 0$, $x \leq y \Leftrightarrow x^2 \leq y^2$.

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y \qquad \text{by property (ii) of ordered fields}$$

Therefore,

$$\begin{aligned}
(ab)^{1/2} &\leq \frac{1}{2}(a+b) \\
ab &\leq \frac{1}{4}(a^2 + 2ab + b^2) \\
4ab &\leq a^2 + 2ab + b^2 \\
0 &\leq a^2 - 2ab + b^2 \\
0 &\leq (a-b)^2 && \text{by definition}
\end{aligned}$$

**Challenge:** Prove for $m$.

**Remark:** The harmonic mean is defined as:

$$\frac{n}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i}}$$

Bernoulli's Inequality

If $x \geq -1$, then $(1+x)^n \geq 1 + nx$, for any $n \in \mathbb{N}_0$

---

By induction, we know that for $n = 0$ and $n = 1$, this holds.

Assume the inequality holds for some $m \geq 1$.

$$\begin{aligned}
(1+x)^{m+1} &= (1+x)^m(1+x) \\
&\geq (1+mx)(1+x) && \text{by the inductive hypothesis} \\
&= 1 + x + mx + mx^2 \\
&= 1 + (m+1)x + mx^2 \\
&\geq 1 + (m+1)x
\end{aligned}$$

Cauchy's Inequality

Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Then

$$\left| \sum_{j=1}^{n} a_j b_j \right| \leq \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2} \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}$$

In linear algebra language, this is equivalent to $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$.

Consider $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^{n} (a_j - b_j x)^2$$

We know that $f(x) \geq 0$ for all $x \in \mathbb{R}$

$$\begin{aligned} &= \sum_{i=1}^{n} (a_j^2 - 2a_j b_j x + b_j^2 x^2) \\ &= \left(\sum_{j=1}^{n} b_j^2\right) x^2 + \left(\sum_{j=1}^{n} -2a_j b_j\right) x + \sum_{j=1}^{n} a_j^2 \\ &= Ax^2 + Bx + C \end{aligned}$$

Therefore, $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned} \left(-2\sum_{j=1}^{n} a_j b_j\right)^2 &\leq 4\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right) \\ \left|\sum_{j=1}^{n} a_j b_j\right| &= \left(\sum_{j=1}^{n} a_j\right)^{1/2} \left(\sum_{j=1}^{n} b_j\right)^{1/2} \end{aligned}$$

As we know from linear algebra, the way we get equality is when $\vec{v} = c\vec{w}$, or that $a_j = cb_j \ \forall j$ for some $c \in \mathbb{R}$.

### Triangle Inequality

Given $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left(\sum_{j=1}^{n} (a_j + b_j)^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

In linear algebra, this is equivalent to $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

$$\begin{aligned} \sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\ &\leq \sum a_j^2 + 2\left(\sum a_j^2\right)^{1/2}\left(\sum b_j^2\right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\ &= \left(\left(\sum a_j^2\right)^{1/2} + \left(\sum b_j^2\right)^{1/2}\right)^2 \end{aligned}$$

we take square roots to get our end result

### Metrics and Norms on $\mathbb{R}^n$

Consider $|\cdot| : \mathbb{R} \to \mathbb{R}$, defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

**Theorems about Absolute Value:**

(i) $|ab| = |a||b|$

(ii) $|a^2| = |a|^2$

(iii) $|-a| = |a|$

(iv) $|a| \in \mathbb{R}^+$

(v) $-|a| \leq a \leq |a|$

(vi) $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$ for $\delta > 0$

(vii) $|a+b| \leq |a| + |b|$, $|a-b| \leq |a| + |b|$, $||a| - |b|| \leq |a-b|$

### Proofs

Proof of (i)

**Case 1:** If $a, b \in \mathbb{R}^+$, then $|a| = a$, and $|b| = b$, and $ab \in \mathbb{R}^+$, so $|ab| = ab$

**Case 2:** If $a, b \notin \mathbb{R}^+$, then $|a| = -a$, and $|b| = -b$. Additionally, $(-a)(-b) = ab \in \mathbb{R}^+$, so $|ab| = ab$. The LHS $= ab$, and the RHS $= ab$.

**Case 3:** $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$. Then, $|a||b| = (a)(-b) = -ab$. Then, since $a(-b) \in \mathbb{R}^+$, $-ab \in \mathbb{R}^+$, so $|ab| = -ab$. Therefore, the LHS and RHS are equal.

Proof of (vii) Having established that $|a+b| \leq |a| + |b|$, we will show that $||a| - |b|| \leq |a-b|$.

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a-b| + |b| \\ |a| - |b| &\leq |a-b| \end{aligned}$$

Similarly, by exchanging $a$ for $b$

$$\begin{aligned} |b| - |a| &\leq |b-a| \\ |b| - |a| &\leq |a-b| \end{aligned}$$

Let $t = |a| - |b|$. We have shown that

$$\begin{aligned} \pm t &\leq |a-b| \\ -|a-b| \leq t &\leq |a-b| \\ |t| &\leq |a-b| \end{aligned}$$

### Absolute Values, cont'd

Recall:

$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

If we want to find all $x \in \mathbb{R}$ such that $|x-1| \leq |x|$, we would split up into cases:

$x \leq 0$ $x - 1 \leq -1$, so $|x| = -x$ and $|x-1| = 1-x$, so $1 - x \leq -x$, so $0 \geq 1$. $\perp$

$0 < x \leq 1$ $|x| = x$ and $|x-1| = 1-x$, so $1 - x \leq x$, so $x \geq \frac{1}{2}$, so $\frac{1}{2} \leq x \leq 1$.

$1 < x$ $|x| = x$ and $|x-1| = x-1$, so $x - 1 \leq x$, so $-1 \leq 0$, which is true $\forall \mathbb{R}$ in the interval, so $x > 1$.

Therefore, we have $x \in \left(\frac{1}{2}, \infty\right)$ as that which satisfies this inequality.

### Bounded Sets

A subset $A \subseteq \mathbb{R}$ is **bounded** $\Leftrightarrow$ $\exists c \geq 0$ such that $\forall x \in A$, $|x| \leq c$.

($\Rightarrow$) Suppose $A \subseteq \mathbb{R}$ is bounded. Then, $\exists \ell, u \in \mathbb{R}$ such that $\ell \leq x \leq u$ $\forall x \in A$. Let $c := \max\{|\ell|, |u|\}$.

Since $|u| \leq c$, we have that $x \leq c$.

Since $|\ell| \leq c$, and $-|\ell| \leq x$, we get that $-x \leq |\ell| \leq c$.

Since $x \leq c$ and $-x \leq c$, $|x| \leq c$.

($\Leftarrow$) If such a $c$ exists, then $|x| \leq c$ if and only if $-c \leq x \leq c$. Thus, $-c$ is the lower bound and $c$ is the upper bound.

### Bounded Functions

Let $D$ be any set. A function $f : D \to \mathbb{R}$ is bounded if $\operatorname{Ran}(D) \subseteq \mathbb{R}$ is bounded.

### Example

Let $f : [3, 7] \to \mathbb{R}$, $f(x) = \frac{x^2+2x+1}{x-1}$. Show that $f$ is bounded.

---

$3 \leq x \leq 7 \Rightarrow 2 \leq x - 1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}$.

Also, $4 \leq x + 1 \leq 8 \Rightarrow 16 \leq x^2 + 2x + 1 \leq 64 \Rightarrow |x^2 + 2x + 1| \leq 64$.

So, $|f(x)| \leq 32$.

### Distance Metrics

For $s, t \in \mathbb{R}$, we will define $d(s, t) = |s - t|$ to be the **distance** between $s$ and $t$.

**Properties:**

(i)

$$\begin{aligned} d :&\mathbb{R} \times \mathbb{R} \to [0, \infty) \\ &(s, t) \mapsto d(s, t) \geq 0 \end{aligned}$$

(ii) $d(s, t) = d(t, s)$

(iii) $d(s, r) \leq d(s, t) + d(t, r)$

(iv) $d(s, s) = 0$

(v) If $d(s, t) = 0$, then $s = t$.

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^{n} |x_j|$$

- $\infty$-norm:

$$\|v\|_\infty = \max_{j=1}^{n} |x_j|$$

- 2-norm:

$$\|v\|_2 = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2}$$

### Properties of the Norms

**Properties:** With $v, w$ above, let $p = 1, 2, \infty$. The following are true:

(1) $\|v\|_p \geq 0$
(2) $\|v + w\|_p \leq \|v\|_p + \|w\| + p$
(3) $\|\vec{0}\|_p = 0$
(4) $\|v\|_p = 0 \Rightarrow v = \vec{0}$
(5) $\forall t \in \mathbb{R},\ \|tv\|_p = |t|\|v\|_p$

#### Proofs

Let $p = \infty$. We will prove (2).

Say $\|v\|_{infty} = |x_i|$ and $\|w\|_\infty = |y_k|$. We want to show that $\|v+w\|_\infty = \max_{j=1}^{n} |x_j + y_j| \leq |x_i| + |y_k|$.

Note that $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| && \text{Triangle Inequality} \\ &\leq |x_i| + |y_k| \\ &= \|v\|_\infty + \|w\|_\infty \end{aligned}$$

Therefore, $\|v + w\|_\infty \leq \|v\|_\infty + \|w\|_\infty$.

### Distances and Norms

A **norm** on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$, $v \mapsto \|v\|$, satisfying the following properties for $v \in \mathbb{R}^n$:

(1) $\|v\| \geq 0$
(2) $\|v + w\| \leq \|v\| + \|w\|$
(3) $\|\vec{0}\| = 0$
(4) $\|v\| = 0 \Rightarrow v = \vec{0}$
(5) $\forall t \in \mathbb{R},\ \|tv\| = |t|\|v\|$

If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$ is a norm, we define $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$, defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for $v, w \in \mathbb{R}^n$.

The properties of distance in $\mathbb{R}$ still hold for distance in $\mathbb{R}^n$:

(1) $d(v, w) = d(w, v)$
(2) $d(u, w) \leq d(u, v) + d(v, w)$
(3) $d(v, v) = 0$
(4) $d(v, w) = 0 \Rightarrow v = w$

### Metric Spaces

A **metric space** is a nonempty set $X$ equipped with a function $d : X \times X \to \mathbb{R}^+$, $(x, y) \mapsto d(x, y) \geq 0$. The metric has the following properties:

(1) $d(x, y) = d(y, x)\ \forall x, y \in X$
(2) $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$

(3) $d(x,x) = 0$

(4) $d(x,y) = 0 \Leftrightarrow x = y$

The map $d$ is called a *metric* on $X$.

Metric Spaces, Open Sets, and Closed Sets

Examples of Metric Spaces:

- $\mathbb{R}$ with $d(x,y) = |x - y|$.
- $\mathbb{R}^n$ with the *Euclidean metric*:

$$\begin{aligned} d_2(v,w) &= \|v - w\|_2 \\ &= \left(\sum_{j=1}^{n}(x_j - y_j)^2\right)^{1/2} \end{aligned}$$

- $\mathbb{R}^n$ with the 1-norm:

$$\begin{aligned} d_1(v,w) &= \|v - w\|_1 \\ &= \sum_{j=1}^{n} |x_j - y_j| \end{aligned}$$

- $\mathbb{R}^n$ with the $\infty$-norm:

$$\begin{aligned} d_\infty(v,w) &= \|v - w\|_\infty \\ &= \max_{j=1}^{n} |x_j - y_j| \end{aligned}$$

Let $(X,d)$ be a metric space.

(1) The **open ball** centered at $x_0 \in X$ with radius $\delta$ is:

$$U(x_0,\delta) := \{x \in X \mid d(x,x_0) < \delta\}$$

(2) The **closed ball** centered at $x_0 \in X$ with radius $\delta$ is:

$$B(x_0,\delta) := \{x \in X \mid d(x,x_0) \leq \delta\}$$

(3) A set $U \subseteq X$ is **open** if $\forall x \in U$, $\exists \delta > 0$ such that $U(x,\delta) \subseteq U$.

(4) A set $C \subseteq X$ is **closed** if $\overline{C} = X - C \subseteq X$ is open.

Examples

In $\mathbb{R}$ with $d(s,t) = |s - t|$:

$$\begin{aligned} U(x_0,\delta) &= \{y \in \mathbb{R} \mid d(y,x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0,\delta) &= [x_0, \delta, x_0 + \delta] \end{aligned}$$

The interval $A = [1,\infty)$ is not open, as $\forall \delta > 0$, $U(1,\delta) \not\subseteq [1,\infty)$.

However, $A$ is closed, as $\overline{A} = (-\infty, 1)$ is open: given $t \in \overline{A}$, choose $\delta = 1 - t$. Let $s \in V_\delta(t)$. Then, $s \in (t - \delta, t + \delta)$, so $s \in (t - (1 - t), t + (1 - t))$, or $s \in (2t - 1, 1)$, so $s < 1$.

### Exercises

Show that the following are open:

- $(a, b)$
- $(a, \infty)$
- $(-\infty, b)$

and that the following are closed:

- $[a, b]$
- $[a, \infty)$
- $(-\infty, b]$

In $(\mathbb{R}^2, d_2)$, $B(0_{\mathbb{R}^2}, 1)$ is the **unit disc** centered at $(0, 0)$.

However, in $(\mathbb{R}^2, d_\infty)$:

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

### Finding a Supremum

Let $0 \neq A \subseteq \mathbb{R}$. Let $u \in \mathbb{R}$ be an upper bound for $A$. The following are equivalent:

(i) $u = \sup(A)$

(ii) If $t < u$, then $\exists a_t \in A$ such that $a_t > t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $u - \varepsilon < a_\varepsilon$

### Proofs

(i) $\Rightarrow$ (ii): Given $t < u$, if no such $a \in A$ with $t < a$ exists, then $a \leq t$ $\forall a \in A$. Thus $t$ would be an upper bound. However, $t < u$ and $u$ is the supremum of $A$. $\perp$

(ii) $\Rightarrow$ (iii): Given $\varepsilon > 0$, set $t = u - \varepsilon < u$. So, by (ii), $\exists a_t$ with $t < a_t$. Thus, $u - \varepsilon \leq a_t$. Set $a_\varepsilon = a_t$.

(iii) $\Rightarrow$ (i): Let $v$ be an upper bound for $A$. Suppose $v < u$. Then, set $\varepsilon = u - v > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u - \varepsilon < a_\varepsilon$. So $u - (u - v) < a_\varepsilon$, so $v < a_\varepsilon$, meaning $v$ cannot be an upper bound. $\perp$

### Supremum Example

**$\sup[0, 1) = 1$:** Certainly, 1 is an upper bound for $[0, 1)$. Let $\varepsilon > 0$.

If $\varepsilon \geq 1$, pick $t = \frac{1}{2}$. Then, $1 - \varepsilon < 0 < \frac{1}{2}$

If $0 < \varepsilon < 1$, let $t = (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$. Then, $t \in [0, 1)$, and $1 - \varepsilon < 1 - \varepsilon/2 = t$

### Finding an Infimum

Let $\emptyset \neq A \subseteq \mathbb{R}$. Let $\ell \in \mathbb{R}$ be a lower bound for $A$. The following are equivalent:

(i) $\ell = \inf(A)$

(ii) If $t > \ell$, $\exists a_t$ such that $t > a_t$

(iii) $(\forall\varepsilon > 0)(\exists a_\varepsilon \in A)$ with $\ell + \varepsilon > a_\varepsilon$

### Infimum Example

$\inf\left\{\frac{1}{n} \mid n \geq 1\right\}$ : Clearly, $0 < \frac{1}{n}$ $\forall n \geq 1$. Let $\varepsilon > 0$.

We need to find $a \in \left\{\frac{1}{n} \mid n \geq 1\right\}$ with $\varepsilon > a$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Let $a_\varepsilon = \frac{1}{m}$.

### More on Supremum/Infimum

- If $A \subseteq \mathbb{R}$ and $\max(A) = u$, then $u = \sup(A)$: $u$ is an upper bound of $A$ by the definition of max, and if $v \neq u$ is any upper bound of $A$, then $u < v$ since $u \in A$.
- If $\min(A) = \ell$, then $\ell = \inf(A)$ (by the same logic).
- If $A$ is not bounded above, $\sup(A) = +\infty$, and if $A$ is not bounded below, then $\inf(A) = -\infty$.
- If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.
- If $A \subseteq B$, then $\inf(A) \geq \inf(B)$: Let $\ell_A = \inf(A)$ and $\ell_B = \inf(B)$. By definition, $\ell_B \leq b$ $\forall b \in B$. Since $A \subseteq B$, $\ell_B \leq a$ $\forall a \in A$. Thus, $\ell_B$ is a lower bound for $A$. By definition of $\ell_A$, $\ell_B \leq \ell_A$.

Let $A, B \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Then, the following are also sets:

(1) $A + B = \{a + b \mid a \in A, b \in B\}$

(2) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$

(3) $t \cdot A = \{ta \mid a \in A\}$

(4) $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

### Completeness Axiom

If $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above, then $\sup(A)$ exists.

Well-Ordering Property: if $\emptyset \neq S \subseteq \mathbb{N}$, then $\min(S)$ exists.

Therefore, we can prove that if $F \subseteq \mathbb{Z}$ is bounded, then $F$ has a least and greatest element.

#### Archimedean Property: Proof

If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x \leq n_x$.

---

Suppose there exists no natural number greater than $x$, then $\mathbb{N}$ is bounded above by $X$. Let $u = \sup(\mathbb{N})$. By the Completeness Axiom, $u \in \mathbb{R}$ exists. Let $\varepsilon = 1$. Then, $\exists n \in \mathbb{N}$ with $u - 1 < n$. So, $u < n + 1$, but $n + 1 \in \mathbb{N}$, so $u$ cannot be an upper bound.

#### Corollary to the Archimedean Property

$$\forall t > 0\ \exists n \in \mathbb{N} \ni \frac{1}{n} < t$$

**Corollary: Powers of 2**

$$\forall t > 0\ \exists m \in \mathbb{N} \ni \frac{1}{2^m} < t$$

---

By the corollary to the Archimedean Property, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < t$. By Bernoulli's inequality with $x = 1$, we have $2^n \geq n$, so $2^{-n} < n^{-1} < t$.

**Corollary: In Between Integers**

$$\forall x \in \mathbb{R}\ \exists n_x \in \mathbb{Z} \ni n_x - 1 \leq x < n_x$$

---

Assume $x \geq 0$. Let $S_x = \{n \mid n \in \mathbb{N}\ x < n\}$.

$S_x \neq \emptyset$ by the Archimedean Property. By the well-ordering property, let $n_x = \min(S_x)$.

Therefore, $x < n_x$. Also, $n_x - 1 \notin S_x$. Therefore $n_x - 1 \leq x$.

## Density Theorems

Let $(X, d)$ be any metric space. A subset $D \subseteq X$ is **dense** if $\forall x \in X,\ \varepsilon > 0,\ U(x, \varepsilon) \cap D \neq \emptyset$.

In the case of $X = \mathbb{R}$ and $d(s,t) = |s - t|$, $D \subseteq \mathbb{R}$ is dense if given any open interval $I$, $I \cap D \neq \emptyset$.

A metric space is **separable** if it admits a *countable* dense subset.

**Density of the Rationals**

$\mathbb{Q} \subseteq \mathbb{R}$ is dense.

---

Let $I = (a, b)$ be an open interval. We may assume that $a, b \in \mathbb{R}$ are finite.

Then, $b - a > 0$. By the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, meaning $1 < nb - na$.

There exists also an integer $m$ such that $m - 1 \leq na < m$, implying that $a\frac{m}{n}$. Also, $m \leq na + 1 < nb$. Therefore, $\frac{m}{n} < b$.

So, $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$.

**Density of the Irrationals**

$\mathbb{R} \setminus \mathbb{Q}$ is dense.

---

Assume $\sqrt{2}$ exists. Let $I = (a, b)$ be any open interval. Then, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

Find $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$.

Then, $a < q\sqrt{2} < b$, where $q\sqrt{2} \in \mathbb{R}$ and $q\sqrt{2} \notin \mathbb{Q}$.

### Uniqueness of $\sqrt{2}$

$$\exists! x > 0 \; x^2 = 2$$

Existence: Let $S = \{t \in \mathbb{R} \mid 0 < t,\ t^2 < 2\}$. $S$ is nonempty because $1 \in \S$, and $S$ is bounded above because $y > 2 \Rightarrow y^2 > 4$.

So, by the completeness axiom, $x := \sup(S)$ exists, and $x \geq 1$.

Claim: $x^2 = 2$

Contradiction 1: Assume $x^2 < 2$. We want to show that $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$. By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x+1) \end{aligned}$$

We want to find $n \in \mathbb{N}$ with:

$$x^2 + \frac{1}{n}(2x+1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x+1}$$

Therefore, by the Archimedean Property corollary, we know that $n$ must exist.

Contradiction 2: We know that $x^2 \geq 2$. Since $x = \sup(S)$, $\forall m \in \mathbb{N}$, $\exists t_m \in S$ with $x - \frac{1}{m} < t_m$, so $\left(x - \frac{1}{m}\right)^2 < t_m^2 < 2$.

Therefore, $x^2 - \frac{2x}{m} + \frac{1}{m^2}$, so $x^2 - \frac{2x}{m} < 2$, so $0 \leq x^2 - 2 < \frac{2x}{m}$.

So, $0 \leq \frac{x^2-2}{2x} < \frac{1}{m}$, so $x^2 - 2 = 0$, so $x^2 = 2$.

**Remark:** If we had set $S' = \{t' \in \mathbb{Q} \mid t^2 < 2,\ t > 0\}$, we would have still obtained $\sup(S') = \sqrt{2}$. This means that $\mathbb{Q}$ is *not* complete.

### Intervals and Nested Intervals

(*) Given any interval $I$, if $x_1, x_2 \in I$, with $x_1 < x_2$, then $[x_1, x_2] \in I$.

This seems like an obvious property, but this is the *characteristic property* of intervals.

**Characterization of Intervals**

Let $S \in \mathbb{R}$ be any nonempty subset of cardinality at least 2. Suppose $S$ satisfies (*). Then, $S$ is an interval.

---

**Case 1:** Suppose $S$ is bounded.

Let $a = \inf(S)$ and $b = \sup(S)$. Then, $S \subseteq [a, b]$. We will show that $(a, b) \subseteq S$. Once this is shown, $S = \{(a, b), [a, b], [a, b), (a, b]\}$.

Let $t \in (a, b)$. Since $a = \inf(S)$, $\exists x_1 \in S,\ x_1 \in (a, t)$. Similarly, since $b = \sup(S)$, $\exists x_2 \in S,\ x_1 \in (t, b)$.

By the hypothesis, $[x_1, x_2] \subseteq S$. Since $t \in [x_1, x_2]$, $t \in S$.

**Case 2:** Suppose $S$ is bounded above, but not below.

Let $b = \sup(S)$. Clearly, $S \subseteq (-\infty, b]$. We will show that $(-\infty, b) \subseteq S$. Once this is shown, $S = \{(-\infty, b), (-\infty, b]\}$.

Let $t \in (-\infty, b)$. Since $b = \sup(S)$, $\exists x_2 \in S,\ x_2 \in (t, b)$.

Since $S$ is not bounded below, $\exists x_1 \in S$ such that $x_1 < t$ (or else $t$ would be a lower bound).

By the hypothesis, $[x_1, x_2] \in S$, and $t \in [x_1, x_2]$, so $t \in S$.

**Case 3, 4:** Left as an exercise for the reader.

A sequence of intervals $(I_n)_{n\geq 1}$ is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in $\bigcap I_n$.

(a) $\bigcap_{n=1}[0, 1/n) = \{0\}$.

(b) $\bigcap_{n=1}(0, 1/n) = \emptyset$

(c) $\bigcap_{n=1}[n, \infty) = \emptyset$

**Measure**

The **measure** of an interval is basically its "size."

(a) $m([a, b]) = b - a$

(b) $m((a, b]) = b - a$

(c) $m((a, b)) = b - a$

(d) $m([a, b)) = b - a$

**Nested Intervals Theorem**

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested sequence of intervals.

(1) $\bigcap_{n\geq 1} I_n \neq \emptyset$

(2) If $\inf\{m(I_n) \mid n \geq 1\} = 0$, then $\bigcap_{n\geq 1} I_n = \{\xi\}$

---

**(a)**

Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, we have that $a_1 \leq a_2 \leq a_3, \dots$, and $b_1 \geq b_2 \geq b_3 \geq \cdots$.

We know that $\{a_n\}$ is bounded above and nonempty. Let $\xi = \sup\left(\{a_n\}_{n=1}^{\infty}\right)$.

We know that $\{b_n\}$ is bounded below. Let $\eta = \inf\left(\{b_n\}_{n=1}^{\infty}\right)$.

We claim that $\xi \leq b_n \ \forall n \geq 1$. Suppose toward contradiction that $\exists m$ such that $\xi > b_m$. Then, by the supremum property, $\exists a_i$ such that $\xi > a_i > b_m$. If $k \leq m$, $a_k \leq a_m \leq b_m < a_k$. If $m \leq k$, then $b_m \geq b_k \geq a_k > b_m$. $\perp$

Similarly, using the same argument, $a_n \leq \eta \ \forall n$.

Thus, $\xi \leq \eta$.

We claim that $\bigcap_{n\geq 1} I_n = [\xi, \eta]$. If $t \in [\xi, \eta]$, then $a_n \leq \xi \leq t \leq \eta \leq b_n$. So $t \in [a_n, b_n] \ \forall n$, so $t \in \bigcap_{n\geq 1}[a_n, b_n]$.

If $t \in \bigcap_{n\geq 1} I_n$, then $t \in [a_n, b_n] \ \forall n$, so $a_n \leq t \leq b_n \ \forall n$. So, $t$ is an upper bound on $a_n$, and a lower bound on $b_n$. So, $\xi \leq t \leq \eta$ by definition of $\xi$ and $\eta$.

**(b)**

We have $\forall n \geq 1$

$$\begin{aligned} [\xi, \eta] &\subseteq [a_n, b_n] \\ \Rightarrow 0 \leq \eta - \xi &\leq b_n - a_n \\ &= m(I_n) \end{aligned}$$

So, if $\inf\left(\{m(I_n) \mid n \geq 1\right) = 0$, then $0 \leq \eta - \xi \leq 0$, so $\eta = \xi$.

**Corollary to the Nested Intervals Theorem**

$[0, 1]$ is uncountable.

---

Suppose it is possible to denumerate the interval $[0, 1] = \{t_1, t_2, \dots, \}$.

We can find $[a_1, b_1] \subseteq [0, 1]$ with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$.

Then, we find $[a_2, b_2] \in [a_1, b_1]$ with $a_2 < b_2$, $t_2 \notin [a_2, b_2]$.

Recursively, we find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $a_n < b_n$, $t_n \notin [a_n, b_n]$.

So, $I_n = ([a_n, b_n])_0^{\infty}$ is a sequence of nested intervals.

Therefore, $\exists \xi \in \bigcap I_n \subseteq [0, 1]$. However, $\xi \notin \{t_1, t_2, \dots\}$. $\perp$

**Sequences in Metric Spaces**

A sequence in a metric space $M$ is a map

$$\begin{aligned} x &: \mathbb{N} \to M \\ x &= (x_n)_{n=1}^{\infty} \end{aligned} \qquad M = \mathbb{R}, \text{ usually}$$

**I.** Sequences defined by a formula:

(i) $x_n = t$ (the constant sequence)

(ii) $x_n = 2n + 1$

(iii) $x_n = \frac{1}{n-1}$ (here, $n \geq 2$)
(iv) $c_n = n \sin\left(\frac{1}{n}\right)$
(v) $d_n = \left(1 + \frac{1}{n}\right)^n$
(vi) Geometric *Sequence*: for $b \neq 0$, $(b^n)_{n\geq 0} = (1, b, b^2, \dots)$
(vii) $x_n = \frac{n!}{n^n}$
(viii) Given any function

$$f : [0, \infty) \to \mathbb{R}$$

we can set $x_n = f(n)$.

**II.** Sequences defined recursively:

(i) $a_1 = 1,\ a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
(ii) Fibonacci: $f_1 = 1,\ f_2 = 1,\ f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$. The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (1-\varphi)^n\right)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$
(iii) Let $f : M \to M$ be a self-map on a metric space. Fix $x_0 \in M$.

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

**III.** New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences, $t \in \mathbb{R}$. Then, we can do the following:
- $(a_n)_n + (b_n)_n + (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n(b_n)_n = (a_n b_n)_n$
- If $b_n \neq 0\ \forall n$, $\left(\frac{a_n}{b_n}\right)$

(ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

(iii) We can look at ratios for $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

(iv) We can look at partial sums, given $(a_k)_{k=1}^{\infty}$.

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^{n} a_k \end{aligned}$$

The sequence $(s_n)_n$ is called the sequence of partial sums. For example, the sequence of partial sums for $(b^k)_{k=0}^{\infty}$ is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for $b \neq 1$, $(1-b)(1+b+b^2+\cdots+b^n) = 1 - b^{n+1}$

**Exercise**

Let $a_k = \frac{1}{k(k+1)}$. Find $(s_n)_n$.

---

Via partial fraction decomposition, we get that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore, $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^{\infty}$

## Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R},\ a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k\geq 1} |a_k| \qquad \text{Infinity Norm}$$

This norm has the traditional properties of the norm:

$$\begin{aligned}
\|(a_k)_k + (b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty && \text{Triangle Inequality}\\
\|t(a_k)_k\|_\infty &= |t|\|(a_k)_k\|_\infty && \text{Scalar Multiplication}\\
\|(a_k)_k\|_\infty = 0 &\Leftrightarrow a_k = 0\ \forall k && \text{Zero Property}\\
\|(a_k)_k(b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty\|(b_k)_k\|_\infty && \text{Multiplication}
\end{aligned}$$

### Proof

Let $u = \|(a_k)_k\|_\infty$ and $v = \|(b_k)_k\|_\infty$.

Given any $k$,

$$\begin{aligned}
|a_k + b_k| &\leq |a_k| + |b_k| && \text{Triangle Inequality on } |\cdot|\\
&\leq u + v && \text{definition of supremum}\\
\Rightarrow \sup_{k\geq 1} |a_k + b_k| &\leq u + v
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(a_k)_k + (b_k)_k\|_\infty &= \|\left((a_k + b_k)_k\right)_k\|_\infty\\
&= \sup_{k\geq 1} |a_k + b_k|\\
&\leq u + v
\end{aligned}$$

## Monotonicity

A sequence $(x_n)_n$ is **increasing** if

$$x_1 \leq x_2 \leq \cdots\ \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \cdots\ \forall n$$

The sequence is *eventually* increasing if $\exists m \in \mathbb{N} \ni x_n \leq x_{n+1}$ for $n > m$.

Similarly, the sequence is eventually decreasing if $\exists m \in \mathbb{N} \ni x_n \geq x_{n+1}$ for $n > m$.

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

**Example**

The sequence

$$a_1 = 1$$
$$a_{n+1} = \frac{1}{2}a_n + 2$$

is increasing and bounded above.

---

We will prove the first statement via induction:

**Base:** $a_1 = 1$, $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

**Inductive Hypothesis** $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+1}$

**Proof:**

$$\begin{aligned} a_n &\leq a_{n+1} \\ \frac{1}{2}a_n &\leq \frac{1}{2}a_{n+1} \\ \frac{1}{2}a_n + 2 &\leq \frac{1}{2}a_{n+1} + 2 \\ a_{n+1} &\leq a_{n+2} \end{aligned}$$

To prove the sequence is bounded above, we do the following:

$$\begin{aligned} a_1 = 1 &\leq 4 \\ \frac{1}{2}a_1 &\leq 2 \\ \frac{1}{2}a_1 + 2 &\leq 2 \\ a_2 &\leq 4 \end{aligned}$$

We claim that $\forall n,\ a_n \leq 4 \Rightarrow a_{n+1} \leq 4$, as we have shown the base case.

$$\begin{aligned} a_n &\leq 4 \\ \frac{1}{2}a_n &\leq 2 \\ \frac{1}{2}a_n + 2 &\leq 4 \\ a_{n+1} &\leq 4 \end{aligned}$$

**Convergence of Sequences**

Let $L \in \mathbb{R}$, $\varepsilon > 0$. Then, the **$\varepsilon$-neighborhood** of $L$ is $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$.

$$\begin{aligned} x &\in V_\varepsilon(L) \\ &\Leftrightarrow \\ |x - L| &< \varepsilon \\ L - \varepsilon < x &< L + \varepsilon \end{aligned}$$

With this in mind, we know the following:

**Definition of Convergence**

A real sequence $(x_n)_n$ converges to a number $x$ if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \ni n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such $L$ exists, then $(x_n)_n$ is said to **diverge**.

A sequence $(x_n)_n$ in a metric space $(X, d)$ converges to a point $x$ if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \ni d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an $N$ such that the $N$th tail (i.e., $x_N, x_{N+1}, \cdots$) are within $\varepsilon$ of $L$ for any $\varepsilon$.

**Note:** $N$ usually depends on $\varepsilon$ (the smaller the $\varepsilon$, the larger the $N$).

**Convergence Proof**

$$\left(\frac{1}{n}\right)_n \xrightarrow{n\to\infty} 0$$

---

We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given $\varepsilon > 0$, we want $\frac{1}{n} < \varepsilon$, meaning $n > \frac{1}{\varepsilon}$.

**Proof:** Let $\varepsilon > 0$. By the Archimedean property corollary, find $N \in \mathbb{N}$ large such that $\frac{1}{N} < \varepsilon$.

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

so, if $n \geq N$, then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

**Convergence Proof 2**

Show that

$$\left(\frac{5n-1}{3-n}\right)_{n\geq 4} \xrightarrow{n\to\infty} -5$$

---

$$\begin{aligned} |x_n - L| &= \left|\frac{5n-1}{3-n}+5\right| \\ &= \frac{14}{|3-n|} \\ &= \frac{14}{n-3} && < \varepsilon \\ \frac{14}{n-3} &< \varepsilon \\ n &> \frac{14}{\varepsilon}+3 \end{aligned}$$

**Proof:** Let $\varepsilon > 0$. Find $N' \in \mathbb{N}$ so large that $\frac{1}{N'} < \frac{\varepsilon}{14}$ (which exists by the Archimedean property corollary). Let $N = N'+3$. If $n \geq N$, then

$$\begin{aligned} n-3 &\geq \frac{1}{N'} \\ \frac{1}{n-3} &\leq \frac{1}{N'} \\ &< \frac{\varepsilon}{14} \end{aligned}$$

whence

$$\begin{aligned} |x_n - L| &= \frac{14}{n-3} \\ &< \frac{14\varepsilon}{14} \\ &= \varepsilon \end{aligned}$$

## Sequences and their Limits, cont'd

**Convergence and Distance**

Let $(X,d)$ be a metric space, and let $(x_n)_n$ be a sequence in the metric space. The following are equivalent:

(i) $(x_n)_n \to x$

(ii) $(d(x_n,x))_n \to 0$

---

(i) $\Rightarrow$ (b) Let $\varepsilon > 0$. Find $N_\varepsilon \in \mathbb{N}$ so large such that $d(x_n,x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

So, $|d(x_n,x) - 0| = d(x_n,x) < \varepsilon$ for all $\varepsilon > 0$. Whence, $(d(x_n,x))_n \to 0$.

(ii) $\Rightarrow$ (i) This direction is similar.

In $\mathbb{R}$, this implies that

$$\begin{aligned} (x_n)_n &\to x \\ &\Leftrightarrow \\ (|x_n - x|)_n &\to 0 \end{aligned}$$

### Comparison Proposition

Let $(X, d)$ be a metric space and let $x \in X$, and suppose $(x_n)_n$ is a sequence in $X$.

If $\exists c \geq 0$, $m \in \mathbb{N}$, and a sequence $(a_n)_n \in \mathbb{R}^+$ with $(a_n)_n \to 0$ and $d(x_n, x) \leq ca_n$ $\forall n > m$. Then, $(x_n)_n \to x$.

---

Let $\varepsilon > 0$. Note that $\frac{\varepsilon}{c} > 0$.

Find $N_1 \in \mathbb{N}$ large such that $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\varepsilon}{c}$, which is always possible since $(a_n)_n \to 0$.

Let $N = \max(N_1, m)$. Then, $n \geq N \Rightarrow n \geq N_1$ and $n \geq m$. So,

$$\begin{aligned} d(x_n, x) &\leq ca_n \\ &< c\frac{\varepsilon}{c} \\ &= \varepsilon \end{aligned}$$

so, $n \geq N \Rightarrow d(x_n, x) < \varepsilon$, whence $(x_n)_n \to x$

### Comparison Proposition, Example 1

Prove

$$\left(\frac{\sin(n^2 - 1)}{n^2 + 3}\right)_n \to 0$$

---

$$\begin{aligned} \left|\frac{\sin(n^2 - 1)}{n^2 + 3} - 0\right| &= \frac{|\sin(n^2 - 1)|}{n^2 + 3} \\ &\leq \frac{1}{n^2 + 3} \\ &\leq \frac{1}{n^2} \\ &\leq \frac{1}{n} \end{aligned}$$

We know that $a_n = \frac{1}{n}$ converges to 0. So, by our comparison proposition, we are done.

### Comparison Proposition, Example 2

Prove

$$\left(\frac{1}{2^n}\right)_n \to 0$$

---

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &\geq 1 + n && > n && \text{Bernoulli's Inequality} \end{aligned}$$

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since $a_n = \frac{1}{n}$ converges, we know that $\frac{1}{2^n}$ must converge.

### Sequence Divergence

A sequence $(x_n)_n$ is **divergent** if it does not converge. $(x_n)_n \to 0$ if and only if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni (\forall n \geq N_\varepsilon) d(x_n, x) < \varepsilon$$

So, $(x_n)_n$ diverges if and only if

$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \to d(x_n, x) \geq \varepsilon_0$$

**Diverging Sequence Proof**

Show that the following sequence diverges:

$$a_n = (-1)^n$$

---

**Step 1**

$$((-1)^n)_n \not\to 1$$

Take $\varepsilon_0 = 1/2$, given any $N \in \mathbb{N}$, we will find $n \geq N$ odd:

$$\begin{aligned} n &= 2N + 1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

**Step 2**

$$((-1)^n)_n \not\to -1$$

by letting $\varepsilon_0 = 1/2$ and $n = 2N$.

**Diverging Sequence Proof 2**

Does

$$a_n = (\sin(n))_n$$

converge?

---

It is not the case that $(\sin(n))_n \to L$ for any $L \in \mathbb{R}$.

**Case 1** If $L > 1$, set $\varepsilon_0 = \frac{L-1}{2}$. Then, given any $N \in \mathbb{N}$, pick $n = N$.

$$\begin{aligned} |\sin(n) - L| &= L - \sin(n) \\ &\geq L - 1 \\ &> \frac{L-1}{2} \\ &= \varepsilon_0 \end{aligned}$$

**Case 2** Similarly for $L < -1$

**Case 3** Suppose $-1 < L < 1$.

**Case 3.1** Suppose $L > 0$. Set $\varepsilon_0 = \frac{L}{2}$. Given any $N$, find $n \geq N$ with $\sin(n) < 0$.

We find $k$ large such that $N < (2k+1)\pi$, which we can always do because we are finding $k > \frac{1}{2}\left(\frac{N}{\pi} - 1\right)$, which is always possible by the Archimedean property.

Note that $N < (2k+1)\pi < (2k+2)\pi$. Note that $\sin(x) < 0$ on the interval $((2k+1)\pi, (2k+2)\pi)$. Note that $|(2k+1)\pi - (2k+2)\pi| = \pi$. Let $n = \lceil (2k+1)\pi \rceil$. Then, $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

**Case 3.2** Suppose $L < 0$, set $\varepsilon_0 = \frac{-L}{2}$. Given $N$, find $n \geq N$ with $\sin(n) > 0$. Using the same strategy as above, we find $n$ such that $|L - \sin(n)| > \varepsilon_0$

**Case 3.3** Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given $N \in \mathbb{N}$, find $n \geq N$ with $\sin(n) \geq \frac{1}{2}$. Then, $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$.

Showing that a sequence diverges is not easy — later, we will divergence with non-uniqueness of convergent subsequences.

### Alternating Series

Consider again

$$((-1)^n)_{n\geq 0} = (1,-1,1,-1,\dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1,1,1,\dots)$$

Similarly, the odd entries converge to $-1$:

$$((-1)^n)_{2n+1} = (-1,-1,-1,\dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating series diverges.

### Uniqueness of Limits

A sequence $(x_n)_n$ can converge to at most one limit.

---

Suppose toward contradiction that $(x_n)_n$ converges to $L_1$ and $L_2$ with $L_1 \neq L_2$.

WLOG, let $L_2 > L_1$. Take $\varepsilon = \frac{L_2 - L_1}{3}$.

Since $(x_n)_n$ converges to $L_1$, $\exists N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \varepsilon$. Similarly, since $(x_n)_n$ converges to $L_2$, $\exists N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \varepsilon$.

Let $N = \max N_1, N_2$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

So, $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$. So, $x_n \in V_\varepsilon(L_1)$, and $x_n \in V_\varepsilon(L_2)$, meaning $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$, but $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. $\perp$

### Useful Lemmas for Convergence

#### Absolutely Convergent Sequences

Let $(x_n)_n$ be a real sequence. If $x_n$ converges to $x$, then $|(x_n)_n| \to |x|$. However, the converse is not the case.

---

Note that since $(x_n)_n \to x$, $d(x_n, x) \to 0$.

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

**Convergence to Zero**

Let $a_n$ be a sequence.

$$(a_n)_n \to 0 \\ \Leftrightarrow \\ |(a_n)| \to 0$$

---

($\Rightarrow$) If $(a_n)_n \to 0$, then we showed previously that $|(a_n)_n| \to |0| = 0$

($\Leftarrow$) Suppose $|(a_n)_n| \to 0$. Given $\varepsilon > 0$, then $\exists N$ such that $n \geq N$ implies

$$\begin{aligned} ||a_n| - 0| &< \varepsilon \\ ||a_n|| &< \varepsilon \\ |a_n| &< \varepsilon \\ |a_n - 0| &< \varepsilon \end{aligned}$$

So, $(a_n)_n \to 0$

**Geometric Sequence**

Let $b \in \mathbb{R}$. Consider

$$(b^n)_{n\geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided $-1 < b \leq 1$. Otherwise, the sequence is divergent.

---

If $b = 0$, then the sequence $(b^n)_n = (0, 0, 0, \dots)$ is convergent.

If $b = 1$, then the sequence $(b^n)_n = (1, 1, 1, \dots)$ is convergent.

If $b = -1$, then the sequence $(b^n)_n = (1, -1, 1, \dots)$ is divergent.

**Case 1** Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, so $\frac{1}{b} = 1 + a$.

So, by Bernoulli's Inequality, $\left(\frac{1}{b}\right)^n = (1+a)^n \geq 1 + na > na$, so $b^n < \frac{1}{na}$.

$$\begin{aligned} |b^n - 0| &= b^n \\ &< \frac{1}{na} \\ &= \frac{1}{a}\frac{1}{n} \\ &\to 0 \end{aligned}$$

So, $(b^n)_n \to 0$.

**Case 2** Suppose $-1 < b < 0$. If we look at $|b^n| = |b|^n$, we know that $(|b|^n)_n \to 0$ by our work above. By the previous lemma, we know that since $|b^n| \to 0$, $b^n \to 0$.

**Case 3** Suppose $b > 1$. Then, $b = 1 + a$ where $a > 0$.

$$\begin{aligned} b^n &= (1+a)^n \\ &\geq 1 + na && \text{Bernoulli's Inequality} \\ &> na \end{aligned}$$

Suppose toward contradiction that $(b^n)_n \to L$. Let $\varepsilon_0 = 1$. Find $N \in \mathbb{N}$ such that $N > \frac{L+1}{a}$. $N$ must exist by the Archimedean property.

Therefore, $(N)(a) > L + 1$. If $n \geq N$, then $(n)(a) > (N)(a) > L + 1$, so $|b^n - L| \geq na - L \geq \varepsilon_0$. $\perp$

**Case 4** Suppose $b < -1$, and suppose toward contradiction that $(b^n)_n \to L$. By the previous lemma, we know that $|b^n| \to |L|$. So, $|b|^n \to |L|$. But, $|b| > 1$, which means our assumption contradicts the result from above. $\perp$

Sequences and Limits, Cont'd

$n$th Root Convergence

If $c > 0$, then $\left(c^{1/n}\right)_n \to 1$.

**Case 1:** If $c = 1$, then we get $\left(c^{1/n}\right)_n = (1, 1, 1, \dots)$, which clearly converges to one.

**Case 2:** Assume that $c > 1$. Then, $c^{1/n} > 1$, because if $d = c^{1/n} \leq 1$, then $d^n \leq 1$, so $c \leq 1$. We can write $c^{1/n} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c &= c^n \\ &= (1 + d_n)^n \\ &\geq 1 + nd_n && \text{Bernoulli's Inequality} \\ &> nd_n \end{aligned}$$

So, $d_n \leq \frac{c}{n}$. Remember, $c^{1/n} = 1 + d_n$.

$$\begin{aligned} |c^{1/n} - 1| &= c^{1/n} - 1 \\ &= d_n \\ &\leq c \cdot \frac{1}{n} \\ &\to 0 \end{aligned}$$

Therefore, $c^{1/n} \to 1$.

**Case 3:** Assume $0 < c < 1$. Then, $c^{1/n} < 1$, so $\frac{1}{c^{1/n}} > 1$.

So, we can write $\frac{1}{c^{1/n}} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c^{1/n} &= \frac{1}{1 + d_n} \\ 1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\ &= \frac{d_n}{1 + d_n} \\ &\leq d_n \end{aligned}$$

Remember, $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned} \frac{1}{c} &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned}$$

So, $d_n \leq \frac{1}{cn}$

$$\begin{aligned} |1 - c^{1/n}| &= 1 - c^{1/n} \\ &\leq d_n \\ &\leq \frac{1}{c}\frac{1}{n} \\ &\to 0 \end{aligned}$$

Therefore, $\left(c^{1/n}\right)_n \to 1$.

**Positive Sequence Convergence**

Let $(x_n)_n$ be a sequence with $x_n \in \mathbb{R}^+ \ \forall n \in \mathbb{N}$, with $(x_n)_n \to x$. Then, $x$ is also positive, and $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

---

Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0-x|}{2}$. Since $(x_n)_n$ converges to $x$, we know that $x_n \in V_\varepsilon(x)$ for large $n$. However, every member of $V_\varepsilon(x) < 0$, and $x_n > 0$. $\perp$

**Case 1:** If $x = 0$, we will show that $\left(\sqrt{x_n}\right)_n \to 0$.

Let $\varepsilon > 0$, find $N \in \mathbb{N}$ large such that if $n \geq N$, we have

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ x_n &< \varepsilon^2 \\ \sqrt{x_n} &< \varepsilon \\ |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

**Case 2:** If $x > 0$, we will show that $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

$$\begin{aligned} \left|\sqrt{x_n} - \sqrt{x}\right| &= \left| \frac{\left(\sqrt{x_n} - \sqrt{x}\right)\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x_n}} \right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{1}{\sqrt{x}} |x_n - x| \\ &\to 0 \end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \to 0$, so $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

**$n$th Root of $n$ Convergence**

$$\left(n^{1/n}\right)_n \to 1$$

---

We will make use of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Note that $n^{1/n} > 1$ for $n$ past 1. So, we write

$$\begin{aligned}
n^{1/n} &= 1 + d_n && d_n > 0\\
n &= (1+d_n)^n\\
&= \sum_{k=0}^{n} \binom{n}{k} d_n^k\\
&= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n\\
&\geq \binom{n}{0} + \binom{n}{2} d_n^2 && \text{as all terms are positive}\\
&= 1 + \frac{n(n-1)}{2} d_n^2
\end{aligned}$$

so

$$\begin{aligned}
n - 1 &\geq \frac{n(n-1)}{2} d_n^2\\
\frac{2}{n} &\geq d_n^2\\
\frac{\sqrt{2}}{\sqrt{n}} &\geq d_n
\end{aligned}$$

So, we have

$$\begin{aligned}
|n^{1/n} - 1| &= n^{1/n} - 1\\
&= d_n\\
&\leq \sqrt{2} \frac{1}{\sqrt{n}}\\
&\to 0 && \text{by previous corollary}
\end{aligned}$$

So, $\left(n^{1/n}\right)_n \to 0$.

Multiplication by Geometric Sequence

Let $0 \leq b < 1$. Show that

$$(nb^n)_n \to 0$$

If $0 < b < 1$ (the 0 case is trivial). So, $\frac{1}{b} > 1$, meaning $\frac{1}{b} = 1 + d$ for some $d > 0$.

$$\begin{aligned} \frac{1}{b^n} &= (1+d)^n \\ &\geq \frac{n(n-1)}{2}d^2 \\ \frac{2}{d^2(n)(n-1)} &\geq b^n \\ nb^n &\leq \frac{2}{d^2(n-1)} \\ &\to 0 && \text{by previous corollary} \end{aligned}$$

Therefore, $(nb^n)_n \to 0$.

Boundedness and Convergence

If $(x_n)_n$ is a convergent sequence, $x_n$ is bounded. The converse is false in general.

Suppose $(x_n)_n \to x$. Let $\varepsilon = 1$.

Then, $\exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq N$.

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$. If $n \geq N$, then $|x_n| \leq c$, because $x_n \in V_\varepsilon(x)$. If $n < N$, then $|x_n| \leq c$.

Together, we have $|x_n| \leq c$ for all $n$.

To show the converse is not true, consider $((-1)^n)_n$. This sequence is bounded but not convergent.

Algebraic Operations on Sequences

Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$ be convergent sequences. Let $t \in \mathbb{R}$. Then, the following are all true:

(1) $(x_n \pm y_n)_n \to x \pm y$

(2) $(tx_n)_n \to tx$

(3) $(x_n y_n)_n \to xy$

(4) Assume $z_n \neq 0\ \forall n$, and $z \neq 0$. Then, $\left(\frac{1}{z_n}\right)_n \to \frac{1}{z}$, and $\left(\frac{x_n}{z_n}\right)_n \to \frac{x}{z}$.

**Proof of (1)** Let $\varepsilon > 0$. Since $x_n \to x$, $y_n \to y$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \to |x_n - x| < \frac{\varepsilon}{2}$, and $n \geq N_2 \to |x_n - x| \leq \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

$$\begin{aligned} |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

**Proof of (3)** We have $(x_n)_n \to x$ and $(y_n)_n \to y$.

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |x_n - x||y| \end{aligned}$$

Since convergent sequences are bounded, $\exists c \in \mathbb{R}$ such that $|x_n| < c,\ \forall n$

$$\begin{aligned} &\leq c|y_n - y| + |x_n - x||y| \\ &\to 0 \end{aligned}$$

Therefore, $|x_n y_n - xy| \to 0$, so $x_n y_n \to xy$.

**Proof of (4)** We have $z_n \neq 0$ and $z \neq 0$. Let $\varepsilon > 0$.

$$\begin{aligned} \left|\frac{1}{z_n} - \frac{1}{z}\right| &= \frac{|z - z_n|}{|z_n z|} \\ &= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|} \end{aligned}$$

Let $\varepsilon = \frac{|z|}{2}$. Since $(z_n)_n \to z$, we know that $z_n \in V_\varepsilon(z)$ for $n \geq N \in \mathbb{N}$. For $n \geq N$, $|z_n| > \frac{|z|}{2}$, so $\frac{1}{|z_n|} < \frac{2}{|z|}$.

$$\begin{aligned} &\leq |z_n - z| \frac{2}{|z|^2} \\ &\to 0 \end{aligned}$$

So, $\left|\frac{1}{z_n} - \frac{1}{z}\right| \to 0$, so $\frac{1}{z_n} \to \frac{1}{z}$

**Ordering of Limits**

Let $(x_n)_n \to x$ and $(y_n)_n \to y$. If $x_n \leq y_n$ for all $n$, then $x \leq y$.

Suppose toward contradiction that $x > y$.

Let $\varepsilon = \frac{x-y}{2}$.

So, $\exists N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$.

Let $N = \max\{N_1, N_2\}$. Then, $x_N \in V_\varepsilon(x)$ and $y_N \in V_\varepsilon(y)$. But that means $x_N > y_N$. $\perp$

In particular, if $(x_n)_n \to x$, and $a \leq x_n \leq b$, then $a \leq x \leq b$.

**Squeeze Theorem**

Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$, where $x_n \leq y_n \leq z_n$ for all $n$.

If $L = x = z$, then $y = L$.

Let $\varepsilon > 0$. Find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow V_\varepsilon(L)$, and $n \geq N_2 \Rightarrow V_\varepsilon(L)$.

Let $N = \max\{N_1, N_2\}$. Then, $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$. Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so $y_n \in V_\varepsilon(L)$, so $(y_n)_n \to L$.

For example, let $a_n = \frac{\sin(n)}{n}$. Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero, $a_n \to 0$

---

As another example, consider $a_n = (2^n + 3^n)^{1/n}$. Then,

$$3^n \leq 2^n + 3^n \leq 2 \cdot 3^n$$
$$3 \leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3$$

Since $2^{1/n} \to 1$, we have $a_n \to 3$.

**Ratio Test**

Let $(x_n)$ be a sequence of strictly positive numbers, with $\left(\frac{x_{n+1}}{x_n}\right)_n \to r < 1$. Then, $(x_n)_n \to 0$.

---

Since $r < 1$, then $1 - r > 0$. Let $\rho = r + \frac{1-r}{2}$, and $\varepsilon = \rho - r = \frac{1-r}{2}$.

Since the sequence converges, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\begin{aligned} \left|\frac{x_{n+1}}{x_n} - r\right| &< \varepsilon \\ \frac{x_{n+1}}{x_n} &< \rho \\ x_{n+1} &< \rho x_n \end{aligned}$$

In particular, $x_{N+1} < \rho x_N$, and $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$. Inductively, one can show that $\forall k \geq 1,\ x_{N+k} < \rho^k x_N$.

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as $k \to \infty$, both sides of the inequality go to 0, implying that $x_n \to 0$

**Monotone Convergence Theorem**

Let $(x_n)_n$ be a monotone sequence. Then, $(x_n)_n$ is convergent if and only if it is bounded.

(a) If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n \to \sup(\{x_n \mid n \in \mathbb{N}\})$.

(b) If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n \to \inf(\{x_n \mid n \in \mathbb{N}\})$.

---

We have already shown that all convergent sequences are bounded.

Assume that $(x_n)_n$ is monotonic and bounded.

**Case 1:** Suppose $(x_n)_n$ is increasing. Let $\sup\{x_n \mid n \in \mathbb{N}\} := u$. We claim that $(x_n)_n \to u$.

Let $\varepsilon > 0$. By the definition of supremum, $\exists N \in \mathbb{N}$ such that $u - \varepsilon < x_N$. Note that $\forall n \geq N$, $u - \varepsilon < x_N \leq x_n \leq u$.

Therefore, if $n \geq N$, then $|x_n - u| < \varepsilon$.

**Case 2:** Suppose $(x_n)_n$ is decreasing. Let $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$. We claim that $(x_n)_n \to \ell$.

Let $\varepsilon > 0$. By the definition of infimum, $\exists N \in \mathbb{N}$ such that $\ell + \varepsilon > x_N$. Additionally, $\forall n \geq N$, $\ell \leq x_n \leq x_N < \ell + \varepsilon$.

Therefore, if $n \geq N$, $|x_n - \ell| < \varepsilon$.

## Applications of the Monotone Convergence Theorem

### Lemma

If $(x_n)_n$ is a convergent sequence, and $m \in \mathbb{N}$, the $m$-th tail, $x_{(m)} = (x_{m+k})_{k=1}^{\infty}$ is also convergent. If $(x_n)_n \to L$ then $x_{(m)} \to L$.

Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. If $k \geq N$, then $m + k \geq N$, so $|x_{m+k} - L| < \varepsilon$.

Thus, $(x_{m+k})_k \to L$

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that $x_n \geq 4\ \forall n$.

$$x_1 = 8 \geq 4$$

Suppose $x_k \geq 4$. We will show that $x_{k+1} \geq 4$.

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore, $(x_n)_n$ is bounded below by 4.

We claim that $(x_n)_n$ is decreasing. $\forall n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} \leq x_n &\Leftrightarrow \\ \frac{1}{2}x_n + 2 \leq x_n \\ &\Leftrightarrow 4 \leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that $(x_n)_n \to L$.

To find $L$, we use the recursive relationship and the lemma.

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2}x_n + 2\right)_{n=1}^{\infty} \\ L &= \frac{1}{2}L + 2 \\ L &= 4 \end{aligned}$$

---

Consider the following sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{4} \\ x_3 &= 1 + \frac{1}{4} + \frac{1}{9} \\ x_k &= \sum_{k=1}^{n} \frac{1}{k^2} \end{aligned}$$

We will show that $(x_n)_n$, the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$\begin{aligned} k^2 &\geq k(k-1) && k \geq 2 \\ \frac{1}{k^2} &\leq \frac{1}{k(k-1)} \\ &= \frac{1}{k-1} - \frac{1}{k} \\ \sum_{k=2}^{n} \frac{1}{k^2} &\leq \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ \sum_{k=1}^{n} \frac{1}{k^2} &\leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \end{aligned}$$

But

$$1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 2 - \frac{1}{n}$$

so, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k^2} &\leq 2 - \frac{1}{n} \\ &< 2 \end{aligned}$$

So, $(x_n)_n$ is bounded above.

Nested Intervals Theorem, Alternative Proof

Let $I_n = [a_n, b_n]$ be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

---

Since the intervals are nested, it must be the case that $a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq b_1$.

Similarly, $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$.

So, $(a_n)_n$ is an increasing sequence bounded above by $b_1$ and $(b_n)\,n$ is a decreasing sequence bounded below by $a_1$.
So, $(b_n)_n \to r$ and $(a_n) \to \ell$
Note that $\ell = \sup(a_n)$ and $r = \inf(b_n)$.

Fix $n \in \mathbb{N}$, then for any $m \geq n$, $a_n \leq a_m \leq b_m \leq b_n$. So, $\sup(a_m) = \ell \leq b_n$. Unlocking $n$, we get that $\ell \leq \inf(b_n) = r$.

Calculating Square Roots

Let $a \in \mathbb{R}^+$. We will construct a sequence $(x_n)_n \to \sqrt{a}$.

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

---

We will prove that $x_n^2 \geq a$.

$$\begin{aligned} 2x_{n+1} &= x_n + \frac{a}{x_n} \\ 2x_{n+1}x_n &= x_n^2 + a \\ 0 &= x_n^2 - 2x_{n+1}x_n + a \end{aligned}$$

So, $x_n$ is a real root, meaning

$$\begin{aligned} \Delta &= 4x_{n+1}^2 - 4a \\ x_{n+1}^2 &\geq a && \forall n \end{aligned}$$

So, $\forall n \geq 2$

$$x_n^2 \geq a$$

---

We will show that $x_n$ is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &= \frac{1}{2}\underbrace{\left(\frac{x_n^2 - a}{x_n}\right)}_{\geq 0\ \forall n \geq 2} \end{aligned}$$

---

So, we have that $(x_n)_n$ is decreasing and bounded below, meaning $(x_n)_n \to x$ for some $x \in \mathbb{R}$.

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ x &= \frac{1}{2}\left(x + \frac{a}{x}\right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} && \text{remember that } x > 0 \end{aligned}$$

Euler's Number

Consider

$$\begin{aligned} (e_n)_n &= \left(1 + \frac{1}{n}\right)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \end{aligned}$$

Similarly,

$$\begin{aligned} e_{n+1} &= \sum_{k=0}^{\infty}\left(\frac{1}{k!}\prod_{j=1}^{k-1}\left(1 - \frac{j}{n+1}\right)\right) \\ e_{n+1} &\geq e_n && \forall n \end{aligned}$$

---

We claim that $(e_n)_n$ is bounded above.

$$\begin{aligned}
e_1 &= \left(1+\frac{1}{1}\right)^1\\
2 &\leq e_n\\
e_n &= \sum_{k=0}^{n}\left(\frac{1}{k!}\underbrace{\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)}_{\leq 1}\right)\\
2^{k-1} &\leq k! && k\geq 2\\
\frac{1}{k!} &\leq \frac{1}{2^{k-1}}\\
e_n &= \sum_{k=0}^{n}\frac{1}{k!}\cdot\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)\\
&\leq \sum_{k=0}^{n}\frac{1}{k!}\\
&\leq 2+\sum_{\ell=1}^{n-1}\frac{1}{2^{\ell}}\\
&< 3
\end{aligned}$$

so, we have

$$2\leq e_n\leq 3$$

so, by the monotone convergence theorem, $(e_n)_n$ converges

$$e := \sup_n\left(1+\frac{1}{n}\right)^n$$

**Monotone Divergence**

A sequence that is increasing and *unbounded* diverges to infinity.

---

Let $M>0$. Since $(x_n)_n$ is unbounded, $\exists N\in\mathbb{N}$ such that $x_N>M$

Thus, if $n\geq N$, then $x_n\geq x_N>M$.

---

Consider

$$h_n=\sum_{k=1}^{n}\frac{1}{k}$$

We can see that $h_n<h_{n+1}$. The primary question is as to whether $(h_n)_n$ is bounded.

$$\begin{aligned}
h_1 &= 1\\
&\geq 1\\
h_2 &= 1+\frac{1}{2}\\
&\geq 1+\frac{1}{2}\\
h_4 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\\
&\geq 1+\frac{1}{2}+\frac{1}{2}\\
h_8 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\\
&\geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}
\end{aligned}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^{k} \frac{1}{2}$$

Let $M$ be large. Find $n$ such that $n > 2(M-1)$. In this case, $n/2 + 1 > M$. Let $N = 2^n$. Then, for $m \geq N$, $h_m > M$.

Thus, $(h_n)_n$ diverges to infinity.