

## Week 1

**Problem** (Exercise 1.17): In this exercise, you will show that the moments of a standard Gaussian variable count pair partitions.

- (i) Let  $X$  be a standard Gaussian variable. Prove that

$$\begin{aligned} \mathbb{E}(X^{2k}) &= (2k-1)!! \\ \mathbb{E}(X^{2k-1}) &= 0. \end{aligned}$$

- (ii) Prove that  $|P_2(2k)| = (2k-1)!!$  by putting  $P_2(2k)$  in explicit bijection with a set of cardinality

$$(2k-1)|P_2(2k-2)|.$$

**Remark:** I had done the first part of this exercise earlier in a separate notes document, but had not written it up here for submission.

**Solution:**

- (i) We see that

$$\begin{aligned} E[Z^m] &= \int_{-\infty}^{\infty} x^m e^{-x^2/2} dx \\ &= -x^m e^{-x^2/2} \Big|_{-\infty}^{\infty} + (m-1) \int_{-\infty}^{\infty} x^{m-2} e^{-x^2/2} dx \\ &= (m-1)E[Z^{m-2}]. \end{aligned}$$

Therefore, we recover the recursion relation for  $(2k-1)!!$  whenever  $m = 2k$  and 0 otherwise.

- (ii) Considering the set  $[2k] = \{1, 2, \dots, 2k\}$ , we see that there are  $2k-1$  ways to pair 1 with any other element, and there are then  $P_2(2k-2)$  pair partitions of the remaining  $2k-2$  elements. This gives our desired bijection.

## Week 2

**Problem** (Exercise 2.23):

- (i) Find a recursion for  $|NC_2(2k)|$ .
- (ii) Show that  $\text{Cat}(k) = \frac{1}{k+1} \binom{2k}{k}$  satisfies the same recursion relation shown in (i).
- (iii) Let  $d\mu = f(x) dx$ , where

$$f(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4-x^2} & -2 \leq x \leq 2 \\ 0 & \text{else} \end{cases}.$$

Show that

$$\int_{\mathbb{R}} x^{2k} d\mu = \text{Cat}(k)$$

$$\int_{\mathbb{R}} x^{2k-1} d\mu = 0.$$

**Solution:**

- (i) To find a recursion for  $NC_2(2k)$ , we start by counting the number of valid pairings of 1 with any other element of  $[2k]$ . For this, we observe that there must be an even number of elements between 1 and whatever it is paired with, meaning there are  $k$  valid pairings of 1.

For each of these pairings, there are two separate “sub-blocks” that we use to count the non-crossing partitions, the ones “between”  $\{1, 2\ell\}$  as  $\ell$  ranges from 1 to  $k$ , and the ones “outside” the pairing  $\{1, 2\ell\}$ . Combined, this gives the recurrence relation

$$|NC_2(2k)| = \sum_{i=1}^k |NC_2(2i-2)| |NC_2(2k-2i)|,$$

where  $NC_2(0) = 1$  vacuously.

- (ii) To evaluate the proposed expression for  $C(z)$ , we see that

$$\begin{aligned} \frac{1}{2z} (1 - \sqrt{1-4z}) &= \frac{1}{2z} \left( 1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-4z)^k \right) \\ &= \frac{1}{2z} \left( - \sum_{k=1}^{\infty} \left( \frac{1}{k!} \prod_{i=0}^k \left( \frac{1}{2} - i \right) \right) (-1)^k 2^{2k} z^k \right) \\ &= \frac{1}{2z} \left( - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k-3)!!}{2^k k!} (-1)^k 2^{2k} z^k \right) \\ &= \frac{1}{2z} \left( \sum_{k=0}^{\infty} \frac{2^{k+1} (2k-1)!!}{(k+1)!} z^{k+1} \right) \\ &= \frac{1}{2z} \left( \sum_{k=0}^{\infty} \frac{2^{k+1} (2k)!}{2^k k! (k+1)!} z^k \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^k \\ &= \sum_{k=0}^{\infty} \text{Cat}(k) z^k. \end{aligned}$$

By plugging the proposed functional equation into the closed-form expression into  $C(z)$ , we get

$$\begin{aligned} \frac{1 - \sqrt{1 - 4z}}{2z} &= 1 + z \left( \frac{1 - \sqrt{1 - 4z}}{2z} \right)^2 \\ &= 1 + z \left( \frac{1 + (1 - 4z) - 2\sqrt{1 - 4z}}{4z^2} \right) \\ &= \frac{1 - \sqrt{1 - 4z}}{2z}. \end{aligned}$$

Yet, we also observe that

$$\begin{aligned} 1 + zC(z)^2 &= z \left( \sum_{k=0}^{\infty} \text{Cat}(k) z^k \right) \left( \sum_{\ell=0}^{\infty} \text{Cat}(\ell) z^\ell \right) \\ &= 1 + z \sum_{k=0}^{\infty} \sum_{\ell=0}^k \text{Cat}(k) \text{Cat}(k - \ell) z^k \\ &= 1 + \sum_{k=0}^{\infty} \sum_{\ell=0}^k \text{Cat}(k) \text{Cat}(k - \ell) z^{k+1} \\ &= 1 + \sum_{k=1}^{\infty} \sum_{\ell=1}^k \text{Cat}(k - 1) \text{Cat}(k - \ell) z^k \\ &= \sum_{k=0}^{\infty} \text{Cat}(k) z^k, \end{aligned}$$

so that  $\text{Cat}(k)$  satisfies the same recurrence relation as the one for the noncrossing pair partitions.

(iii) The fact that

$$\begin{aligned} \int_{\mathbb{R}} x^{2k-1} d\mu &= \frac{1}{2\pi} \int_{-2}^2 x^{2k-1} \sqrt{1 - 4x^2} dx \\ &= 0, \end{aligned}$$

follows from the fact that this is an odd integrand over a symmetric interval. Else, we find

$$\begin{aligned} \frac{1}{2\pi} \int_{-2}^2 x^{2k} \sqrt{4 - x^2} dx &= \frac{1}{2\pi} \int_0^\pi 2^{2k+2} \sin^2(\theta) \cos^{2k}(\theta) d\theta \\ &= 2^{2k+1} \left( \frac{(2k-1)!!}{(2k)!!} - \frac{(2k+1)!!}{(2k+2)!!} \right) \\ &= 2^{2k+1} \left( 1 - \frac{2k+1}{2k+2} \right) \frac{(2k-1)!!}{(2k)!!} \\ &= 2^{2k+1} \frac{1}{2k+2} \frac{(2k-1)!!}{(2k)!!} \\ &= \frac{(2k)!}{(k+1)!k!} \\ &= \frac{1}{k+1} \binom{2k}{k} \\ &= \text{Cat}(k). \end{aligned}$$