

Problem (Homework 1, Problem 1 (c)): Show that “is homotopic to” is a transitive relation.

Solution: Suppose $f \simeq g$ and $g \simeq h$ are homotopic with homotopies defined by $F: I \times I \rightarrow X$ and $G: I \times I \rightarrow X$. We claim that the map $H: I \times I \rightarrow X$ defined by

$$H(s, t) = \begin{cases} F(s, 2t) & 0 \leq t \leq 1/2 \\ G(s, 2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

is continuous. To see this, observe that by definition, H is continuous on $I \times [0, 1/2]$ (as it is equal to F on that interval) and H is continuous on $I \times [1/2, 1]$ (as it is equal to G on that interval). Since $I \times [0, 1/2]$ and $I \times [1/2, 1]$ are closed under the product topology, and $F(s, 1) = G(s, 0)$ by assumption, it follows from the pasting lemma that H is continuous as both constituents of the piecewise definition for H are continuous on closed subsets and are equal on their intersection.

Problem (Homework 4, Problem 2): Prove that, if $f: X \rightarrow Y$ is a map, then Y is a deformation retract of the mapping cylinder M_f .

Solution: Recall that the definition of the mapping cylinder is given by

$$M_f = (X \times [0, 1]) \sqcup Y / ((x, 1) \simeq f(x)).$$

We consider the set

$$W = (X \times [0, 1]) \sqcup Y,$$

and let $q: W \rightarrow M_f$ be the quotient map. By the universal property of the quotient map, it follows that if we are able to find a deformation retract from W to $X \times \{1\} \sqcup Y$, then by composing with the quotient map, we will obtain a deformation retract from M_f to Y .

Define the homotopy

$$H: W \times [0, 1] \rightarrow W$$

on each component separately by taking

$$H(w, t) = \begin{cases} (p, \max(s, t)) & w = (p, s) \in X \times [0, 1] \\ y & w \in Y \end{cases}.$$

Since H is defined on a disjoint union, it follows that if we are able to show that H is continuous on each component, then H is continuous. First, we observe that the maximum function is continuous, so it follows that H is continuous on $(X \times [0, 1]) \times [0, 1]$. Similarly, since H is constant in t along $Y \times [0, 1]$, it follows that H is continuous. Furthermore, we observe that the image of H is equal to $X \times \{1\} \sqcup Y$, and the subset $X \times \{1\} \sqcup Y$ is constant along H as $1 \geq t$ for all $0 \leq t \leq 1$.

Therefore, upon composing with the quotient map, we have that $q \circ H$ is a deformation retract from M_f to Y .

Problem: Prove that if $f: X \rightarrow Y$ is an inclusion map, then the mapping cone C_f is homotopy equivalent to the quotient $Y/f(X)$.

Solution: Since f is an inclusion map, we will treat $f(X)$ as X . Consider the quotient $C_f/X \times [0, 1]$; we claim that this quotient is homeomorphic to $Y/X = Y/f(X)$. For this, observe that in C_f , we identify $x \times \{0\} \sim x \in Y$, and we identify all of $X \times \{1\}$ with a single point; for any element $x \times \{t\}$ in the mapping cone with $0 \leq t \leq 1$, it follows that upon identification, we have that $x \times \{t\} \sim x \times \{0\} \sim x$ and $x \times \{t\} \sim X \times \{1\}$, meaning that for any $x \in X$, we have that $x \sim X \times \{1\}$, or that all of $X \times [0, 1]$ in the mapping cone is identified with a single point; this gives that $C_f/X \times [0, 1] \cong Y/X$. Therefore, it suffices to show that $C_f/X \times [0, 1]$ is homotopy equivalent to C_f . Since $X \times [0, 1]/X \times \{1\}$ is contractible, this follows from the fact that contracting a CW complex by a contractible subcomplex is a homotopy equivalence.