Problem (Problem 1)**:** Let G be a Lie group, which is a topological group that is also a smooth manifold and where all group operations are smooth. For convenience, we will always assume that G is path-connected. Prove that the tangent bundle TG of G is trivial — i.e., TG composes as a direct product.

Solution: From Cayley's Theorem, we know that G acts on itself transitively by left-multiplication. That is, for any $g \in G$, there is a map $L_g \colon G \to G$ that takes $h \mapsto gh$. This is a diffeomorphism of smooth manifolds since L_g is smooth and admits the smooth inverse $L_{g^{-1}}$. In particular, this means that

$$D_e(L_g) \colon T_e G \to T_g G$$

is invertible as a linear map. Letting $T_e G \cong \mathbb{R}^n$ have a local basis $\mathcal{B}_e = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$, we then observe that $D_e(L_g)$ then maps this basis to a basis for $T_g G$ since $D_e(L_g)$ is an isomorphism, meaning that

$$
\begin{aligned}
TG &= \bigsqcup_{g \in G} T_g G \\
&= \bigsqcup_{g \in G} D_e(L_g)(T_e G) \\
&\cong \bigsqcup_{g \in G} \mathbb{R}^n \\
&\cong G \times \mathbb{R}^n.
\end{aligned}
$$

Thus, TG is trivial.

Problem (Problem 2)**:** Note that a Lie group can act on itself by left or right multiplication. A vector field on G is called *left-invariant* if it is invariant under the differential of left multiplication L_g for every $g \in G$. Prove that $T_e G$ can be identified with left invariant vector fields on G.

Solution: We observe that by definition, a left-invariant vector field X is one where $g \cdot X = X$ for every $g \in G$. In particular, this means that for any vector field $X_e \in T_e G$, there is a corresponding left-invariant vector field on G defined at each $g \in G$ by taking $X_g = D_e(L_g)(X_e)$; that such a vector field is left-invariant follows from the fact that L_g is a diffeomorphism of G onto itself. Thus, we get the correspondence between vector fields at $T_e G$ and left-invariant vector fields on G.

Problem (Problem 3)**:** Similar to invariant vector fields, invariant forms are ones for which $L_g^* \omega = \omega$. Prove that invariant forms are stable under taking d and under contraction by a left-invariant vector field.

Solution: Let ω be left-invariant. Then, by definition of the pullback,

$$
\begin{aligned}
L_g^*(d\omega) &= d(L_g^* \omega) \\
&= d\omega.
\end{aligned}
$$

Similarly, by definition of the contraction, if $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$ are a k-dimensional collection of vector fields, then

$$
\begin{aligned}
L_g^*(\iota_X(\omega))\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right) &= L_g^*\left(\omega\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right) \right) \\
&= (L_g^* \omega)\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right) \\
&= \omega\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right) \\
&= \iota_X(\omega)\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right).
\end{aligned}
$$