Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

Quantity	Value
$\mathbf{r}$	$\rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}$
u_1	ρ
u_2	ϕ
u_3	z
$\hat{e}_1$	$\hat{\rho}$
$\hat{e}_2$	$\hat{\phi}$
$\hat{e}_3$	$\hat{k}$
$\frac{\partial \mathbf{r}}{\partial u_1}$	$\hat{\rho}$
$\frac{\partial \mathbf{r}}{\partial u_2}$	$\rho\hat{\phi}$
$\frac{\partial \mathbf{r}}{\partial u_3}$	$\hat{k}$
h_1	1
h_2	ρ
h_3	1

- Line element:

$$\begin{aligned}(ds)^2 &= \sum_i h_i^2 (du_i)^2\\ &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.\end{aligned}$$

- Area element:

$$\begin{aligned}d\mathbf{a} &= \left(\sum_k \epsilon_{ijk}\hat{e}_k\right) h_i h_j du_i du_j\\ &= \rho\hat{k}\ d\rho d\phi\\ &= \rho\hat{\rho}\ d\phi dz\\ &= \hat{\phi}\ dz d\rho\end{aligned}$$

- Volume element:

$$\begin{aligned}d\tau &= h_1 h_2 h_3\ du_1 du_2 du_3\\ &= \rho\ d\rho d\phi dz\end{aligned}$$

Spherical Coordinates

Quantity	Value
$\mathbf{x}$	$r\sin\theta\cos\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath} + r\cos\theta\hat{k}$
u_1	r
u_2	ϕ
u_3	θ
$\hat{e}_1$	$\hat{\rho}$
$\hat{e}_2$	$\hat{\phi}$
$\hat{e}_3$	$\hat{\theta}$
$\frac{\partial\mathbf{x}}{\partial u_1}$	$\hat{r}$
$\frac{\partial\mathbf{x}}{\partial u_2}$	$r\sin\theta\hat{\phi}$
$\frac{\partial\mathbf{x}}{\partial u_3}$	$r\hat{\theta}$
h_1	1
h_2	$r\sin\theta$
h_3	r

- Line element:

$$\begin{aligned}(ds)^2 &= \sum_i h_i^2 (du_i)^2 \\ &= (dr)^2 + r^2\sin^2\theta\,(d\phi)^2 + r^2\,(d\theta)^2\end{aligned}$$

- Area element:

$$\begin{aligned}d\mathbf{a} &= \left(\sum_k \epsilon_{ijk}\hat{e}_k\right) h_i h_j\, du_i\, du_j \\ &= r\sin\theta\hat{\theta}\ dr d\phi \\ &= r^2\sin\theta\hat{r}\ d\phi d\theta \\ &= r\hat{\phi}\ d\theta dr\end{aligned}$$

- Volume element:

$$\begin{aligned}d\tau &= h_1 h_2 h_3 du_1 du_2 du_3 \\ &= r^2\sin\theta\ dr d\phi d\theta\end{aligned}$$

4.9

Without loss of generality, we have

$$\sum_\ell \epsilon_{mn\ell}\epsilon_{ij\ell} = \epsilon_{mn1}\epsilon_{ij1},$$

where $m, n, i, j = 2, 3$. If we have $m = i, n = j$, then $\epsilon_{mn1}\epsilon_{ij1} = 1$; if $m = j, n = i$, then $\epsilon_{mn1}\epsilon_{ij1} = -1$; else, $\epsilon_{mn1}\epsilon_{ij1} = 0$.

4.11

(a)

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \sum_{i,j,k}\epsilon_{ijk}A_iB_j\hat{e}_k\\ &= -\sum_{i,j,k}\epsilon_{jik}B_jA_i\hat{e}_k\\ &= -\left(\mathbf{B}\times\mathbf{A}\right)\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A}\cdot\left(\mathbf{A}\times\mathbf{B}\right) &= \sum_{i,j,k}\left(\epsilon_{ijk}A_iB_j\hat{e}_k\right)\cdot A_i\hat{e}_i\\ &= \sum_{i,j,k}\delta_{ik}\left(\epsilon_{ijk}A_i^2B_j\right)\\ &= 0.\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{A}\cdot\left(\mathbf{B}\times\mathbf{C}\right) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_iC_j\\ &= \sum_{i,j,\ell}\left(\epsilon_{\ell ij}A_\ell B_i\right)C_j\\ &= \mathbf{C}\cdot\left(\mathbf{A}\times\mathbf{B}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}\cdot\left(\mathbf{B}\times\mathbf{C}\right) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_iC_j\\ &= \sum_{i,j,\ell}\left(\epsilon_{j\ell i}C_jA_i\right)B_i\\ &= \mathbf{B}\cdot\left(\mathbf{C}\times\mathbf{A}\right).\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right) &= \sum_{i,j}\epsilon_{ijk}A_i\left(\sum_{\alpha,\beta}\epsilon_{\alpha\beta j}B_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\epsilon_{ijk}\epsilon_{\alpha\beta j}A_iB_\alpha C_\beta\\ &= -\left(\sum_{i,j,\alpha,\beta}\epsilon_{ikj}\epsilon_{\alpha\beta j}A_iB_\alpha C_\beta\right)\\ &= -\left(\sum_{i,j,\alpha,\beta}\left(\delta_{i\alpha}\delta_{k\beta}-\delta_{i\beta}\delta_{k\alpha}\right)A_iB_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\left(\delta_{k\alpha}\delta_{i\beta}-\delta_{i\alpha}\delta_{k\beta}\right)A_iB_\alpha C_\beta\\ &= \sum_{i,j,\alpha,\beta}\left(B_\alpha\delta_{k\alpha}\right)\left(A_iC_\beta\delta_{i\beta}\right)-\left(C_\beta\delta_{k\beta}\right)\left(A_iB_\alpha\delta_{i\alpha}\right)\\ &= \mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right).\end{aligned}$$

(e)

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\cdot\mathbf{D}) &= \sum_{\alpha,\beta}(\mathbf{A}\times\mathbf{B})_{\alpha}(\mathbf{C}\times\mathbf{D})_{\beta}\,\delta_{\alpha\beta}\\
&= \sum_{\substack{\alpha,\beta,\\ i,j,\\ m,n}} \epsilon_{ij\alpha}\epsilon_{mn\beta}A_iB_jC_mD_n\delta_{\alpha\beta}\\
&= \sum_{\substack{\alpha,\\ i,j,\\ m,n}} \epsilon_{ij\alpha}\epsilon_{mn\alpha}A_iB_jC_mD_n\\
&= \sum_{\substack{i,j,\\ m,n}} A_iB_jC_mD_n\left(\delta_{mi}\delta_{nj}-\delta_{mj}\delta_{ni}\right)\\
&= \sum_{\substack{i,j,\\ m,n}} \left(\left(A_iC_m\delta_{mi}\right)\left(B_jD_n\delta_{nj}\right)\right)-\left(\left(B_jC_m\delta_{mj}\right)-\left(A_iD_n\delta_{ni}\right)\right)\\
&= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D}).
\end{aligned}$$

Chapter 5 Problems

5.1

Let $f(x) = x^n$. We use linearity for the general case.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\
&= \lim_{h\to 0}\frac{x^n + n\left(hx^{n-1}\right)+\cdots+nh^{n-1}x+h^n-x^n}{h}\\
&= \lim_{h\to 0}\left(nx^{n-1}+\cdots+nh^{n-2}x+h^{n-1}\right)\\
&= nx^{n-1}.
\end{aligned}$$

5.6

$$\begin{aligned}
\cos(N\phi)+i\sin(N\phi) &= (\cos\phi + i\sin\phi)^N\\
&= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k\left(\sin\phi^{N-k}\right)\left(e^{i\frac{\pi}{2}}\right)^{N-k}\\
&= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k(\sin\phi)^{N-k}\left(\cos\left((N-k)\frac{\pi}{2}\right)+i\sin\left((N-k)\frac{\pi}{2}\right)\right)\\
&= \sum_{k=0}^{N}\binom{N}{k}\cos\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\\
&\quad + i\left(\sum_{k=0}^{N}\binom{N}{k}\sin\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\right).
\end{aligned}$$

We get the final answer by equating real and imaginary parts.

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right|\\ &= 1,\end{aligned}$$

  which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \qquad \forall n \geqslant 1.$$

  Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

(b)

- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\left(\frac{1}{2}\right)\right|\\ &= \frac{1}{2}\\ &< 1,\end{aligned}$$

  meaning the series converges by the ratio test.

-

$$\frac{1}{n2^n} < \frac{1}{2^n} \qquad \text{for all } n \geqslant 1,$$

  and since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty}\frac{1}{n2^n}$ converges.

6.9

$$\begin{aligned}\sum_{n=-N}^{N} e^{inx} &= 1 + \sum_{n=1}^{N} e^{-inx} + \sum_{n=1}^{N} e^{inx}\\ &= 1 + e^{-ix}\sum_{n=1}^{N} e^{i(n-1)x} + e^{ix}\sum_{n=1}^{N} e^{i(n-1)x}\\ &= 1 + e^{-ix}\sum_{n=0}^{N-1} e^{-inx} + e^{ix}\sum_{n=0}^{N-1} e^{inx}\\ &= 1 + e^{-ix}\frac{1-e^{-iNx}}{1-e^{-ix}} + e^{ix}\frac{1-e^{iNx}}{1-e^{ix}}\\ &= 1 + \frac{e^{-ix}-e^{-i(N+1)x}}{1-e^{-ix}} + \frac{1-e^{iNx}}{e^{-ix}-1}\\ &= 1 + \frac{\left(e^{-ix}-1\right) + e^{iNx} - e^{-i(N+1)x}}{1-e^{-ix}}\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{iNx} - e^{-i(N+1)x}}{1 - e^{-ix}} \\
&= \frac{e^{iNx} - e^{-i(N+1)x}}{e^{-i\left(\frac{x}{2}\right)}\left(e^{i\left(\frac{x}{2}\right)} - e^{-i\left(\frac{x}{2}\right)}\right)} \\
&= \frac{e^{i\left(N+\frac{1}{2}\right)x} - e^{-i\left(N+\frac{1}{2}\right)x}}{e^{-i\left(\frac{x}{2}\right)} - e^{-i\left(\frac{x}{2}\right)}} \\
&= \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}.
\end{aligned}$$

6.13

(a)

$$\begin{aligned}
\frac{d}{dz}\left(\arctan(z)\right) &= 1 - z^2 + z^4 - z^6 + \cdots \\
&= \sum_{i=0}^{\infty} (-1)^n z^{2n} \\
&= \frac{1}{1+z^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\rho &= \limsup_{k\to\infty} \sqrt[k]{\left|(-1)^k\right|} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}$$

(c)

$$\begin{aligned}
\rho &= \limsup_{k\to\infty} \left(\left|\frac{(-1)^k}{2k+1}\right|\right)^{1/k} \\
&= \limsup_{k\to\infty} \frac{1}{(2k+1)^{1/k}} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}$$

6.25

$$\begin{aligned}
e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\
&= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}$$
$$= \cos\theta + i \sin\theta.$$

6.37

$$\begin{aligned}
F &= GmM_E \left(R_E + h\right)^{-2} \\
&= \frac{GmM_E}{R_E^2}\left(1 + \frac{h}{R_E}\right)^{-2} \\
&= \frac{GmM_E}{R_E^2}\left(1 + (-2)\frac{h}{R_E} + \frac{(-2)(-3)}{2!}\left(\frac{h}{R_E}\right)^2 + \cdots\right) \\
&\approx \frac{GmM_E}{R_E^2}.
\end{aligned}$$

It would be the case that for $h = 0.05R_E$, there would need to be a 10% negative correction to the estimated force.

6.42