Partial Isometries and the Polar Decomposition

Definition: A *partial isometry* is an operator $W \in B(H)$ such that for any $h \in (\ker(W))^{\perp}$, we have $\|Wh\| = \|h\|$. The space $(\ker(W))^{\perp}$ is called the *initial space* of W, and the space $\operatorname{im}(W)$ is called the final space of W.

Proposition: If $W \in B(H)$, the following are equivalent:

(i) W is a partial isometry;

(ii) W^* is a partial isometry;

(iii) W^*W is a projection (onto the initial space of W);

(iv) WW^* is a projection (onto the final space of W);

Proof. Let W be a partial isometry, meaning that W is an isometry from $(\ker(W))^{\perp}$ to $\operatorname{im}(W)$. Since $\operatorname{im}(W)$ is dense in $\ker(W^*)^{\perp}$, it follows that we only need to show that W^* is an isometry on $\operatorname{im}(W)$. Let $k \in \operatorname{im}(W)$, so there is $h \in (\ker(W))^{\perp}$ such that $Wh = k$. Then, we have

$$\langle Wh, Wh \rangle = \langle h, h \rangle$$

so

$$\langle W^*Wh - h, h \rangle = 0,$$

meaning that $W^*W - I$ is zero on $(\ker(W))^{\perp}$, so we have

$$\begin{aligned}\|W^*k\| &= \|W^*Wh\| \\ &= \|h\| \\ &= \|Wh\| \\ &= \|k\|,\end{aligned}$$

meaning W^* is a partial isometry.

By taking adjoints, we see that (i) and (ii) are equivalent. Let $x \in H$ have the decomposition $x = y + z$ where $y \in \ker(W)$ and $z \in (\ker(W))^{\perp}$. We will show that $W^*Wx = z$. Observe that $Wx = Wz$, meaning that

$$\begin{aligned}\langle z - W^*Wx, z \rangle &= \langle z - W^*Wz, z \rangle \\ &= \langle z, z \rangle - \langle W^*Wz, z \rangle \\ &= \langle z, z \rangle - \langle Wz, Wz \rangle \\ &= 0,\end{aligned}$$

since $\|Wz\| = \|z\|$ by definition. In particular, following from the polarization identity, this means that for all $v \in H$, we have $\langle z - W^*Wx, v \rangle = 0$, so that $z = W^*Wx$. This shows that (i) implies (iii). By replacing all instances of W with W^*, we see that (ii) implies that WW^* is a projection onto the initial space of W^*, which is equal to the final space of W. ☐

Theorem (Polar Decomposition)**:** Let $A \in B(H)$. Then, there is a partial isometry W with initial space $(\ker(A))^{\perp}$ and final space $\overline{\operatorname{im}(A)}$ such that $A = W|A|$. Moreover, if $A = UP$, where P is a positive operator and U is a partial isometry with $\ker(U) = \ker(P)$, then $P = |A|$ and $U = W$.

Proof. Let $h \in H$. Then,

$$\begin{aligned}\|Ah\| &= \langle A^*Ah, h \rangle \\ &= \langle |A|h, |A|h \rangle,\end{aligned}$$

so that

$$\|Ah\| = \||A|h\|.$$

We may thus define $W\colon \operatorname{im}(|A|) \to \operatorname{im}(A)$ by taking

$$W(|A|h) = Ah.$$

Then, from above, we know that W is an isometry, so it can be extended to an isometry from $\overline{\operatorname{im}(|A|)}$ to $\overline{\operatorname{im}(A)}$. We may then extend W to all of H by defining it to be 0 on $(\operatorname{im}(|A|))^{\perp}$. This makes W a partial isometry with $W|A| = A$. We must verify that W has the correct initial space. That is, we must show that $\overline{\operatorname{im}(|A|)} = (\ker(A))^{\perp}$.

Suppose $f = A^*g$ for some $g \in (\ker(A^*))^{\perp} = \overline{\operatorname{im}(A)}$. Then, $\operatorname{im}(A^*A)$ is dense in $(\ker(A))^{\perp}$. Yet, since $A^*Ak = |A|h$, where $h = |A|k$, it follows that $\operatorname{im}(|A|)$ is dense in $(\ker(A))^{\perp}$.

For uniqueness, we have that $A^*A = PU^*UP$, but since U^*U is the projection onto the initial space, it follows that $(\ker(U))^{\perp} = (\ker(P))^{\perp} = \overline{\operatorname{im}(P)}$, meaning $A^*A = P^2$, so $P = |A|$ by the uniqueness in the continuous functional calculus. For any $h \in H$, we have $W|A|h = Ah = U|A|h$, meaning that U and W agree on a dense subset of their initial space, so $U = W$. □

Corollary: If $T = W|T|$ is the polar decomposition for $T \in B(H)$, then $|T^*| = W|T|W^*$, and $T^* = W^*|T^*|$.

Proof. We see that $W|T|W^*$ is positive, and

$$\begin{aligned} W|T|W^*W|T|W^* &= W|T|^2W^* \\ &= WTT^*W^* \\ &= TT^* \end{aligned}$$

Therefore, by uniqueness, we have $W|T|W = |T^*|$. Furthermore, we see that

$$\begin{aligned} W^*|T^*| &= W^*W|T|W^* \\ &= |T|W^* \\ &= (W|T|)^* \\ &= T^*. \end{aligned}$$

□

Definition: If M is a von Neumann algebra in $B(H)$, then the *center* of M is given by

$$Z(M) \coloneqq M \cap M'.$$

If $Z(M) = \mathbb{C}1$, then we say M is a *factor*.

Comparison Theory of Projections in a von Neumann Algebra

Recall that an element $p \in B(H)$ is called a projection if $p = p^2 = p^*$, and projects onto a unique closed subspace.

Proposition: If P and Q are projections in $B(H)$, then the following are equivalent:

(i) $QP = P$;

(ii) $\operatorname{im}(P) \subseteq \operatorname{im}(Q)$;

(iii) $P \leq Q$.

Proof. If we assume (i), then $Px = QPx$, so that $\operatorname{im}(Q) \supseteq Q(\operatorname{im}(P)) = \operatorname{im}(P)$, giving (ii). Similarly, (ii) implies (i) from the same definition.

Now, set $M = \operatorname{im}(P)$. For any $x \in H$, write $x = y + z$ for $y \in M$ and $z \in M^{\perp}$. Then, we have $\langle Px, x\rangle = \|y\|^2$, and

$$\begin{aligned}\langle Qx, x\rangle &= \langle Qy + Qz, y + z\rangle \\ &= \langle Qy, y\rangle + \langle Qz, z\rangle + \langle Qy, z\rangle + \langle Qz, y\rangle \\ &= \|y\|^2 + \langle Qy, z\rangle + \langle Qz, y\rangle \\ &= \|y\|^2 + \langle Qy, z\rangle + \langle z, Qy\rangle \\ &= \|y\|^2 + \langle Qy, z\rangle \\ &\geq \langle Px, x\rangle,\end{aligned}$$

so $P \leq Q$.

Finally, assuming $P \leq Q$, if $Qx = 0$, then $0 \leq \langle Px, x\rangle \leq \langle Qx, x\rangle = 0$, so $Px = 0$, so $\ker(Q) \subseteq \ker(P)$, meaning $\operatorname{im}(P) \subseteq \operatorname{im}(Q)$. □

Projections form a complete lattice under the operations

$$\begin{aligned}\bigwedge_{i\in I} P_{X_i} &= P_{\bigcap_{i\in I} X_i} \\ \bigvee_{i\in I} P_i &= P_{\overline{\sum_{i\in I} X_i}}.\end{aligned}$$

Unfortunately, the primary issue here is that these operations are too restrictive; for instance, the matrix units e_{11} and e_{22} both have rank 1 in $\mathbb{M}_n(\mathbb{C})$, but project onto different subspaces and are not comparable in the traditional sense. We will introduce a different way to compare projections that successfully deals with this issue.

Definition: Let $M \subseteq B(H)$ be a von Neumann algebra, and let $P(M)$ be its projection lattice. We say that projections $p, q \in P(M)$ are *equivalent* if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. That is, the initial projection of v is p and the final projection of v is q. We write $p \sim q$.

We say p is *sub-equivalent* to q, written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. We will write $p \prec q$ if $p \preceq q$ and $p \nsim q$.

Note that the traditional ordering of projections yields that $p \leq q$ implies $p \preceq q$, but the reverse is not necessarily true.

Proposition: For a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\preceq$ is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry for $\sim$ follows from the fact that if v is a partial isometry, then so too is v^*.

We will show transitivity for $\preceq$, from which it will be clear that $\sim$ is transitive. Let $p, q, r \in P(M)$ with $p \leq q$ and $q \leq r$. Then, we have partial isometries $u, v \in M$ with $u^*u = p$, $uu^* \leq q$, $v^*v = q$ and $vv^* \leq r$. Then, from

$$\begin{aligned}qu &= quu^*u \\ &= uu^*u \\ &= u,\end{aligned}$$

meaning

$$\begin{aligned}(vu)^*(vu) &= u^*v^*vu \\ &= u^*qu \\ &= u^*u\end{aligned}$$

$$= p,$$

and

$$\begin{aligned}(vu)(vu)^* &= vuu^*v\\ &\leq v^*qv\\ &= v(v^*v)v\\ &= vv^*\\ &\leq r,\end{aligned}$$

so $p \preceq r$, and $\preceq$ is a transitive relation. □

In fact, the preorder is a partial order, but this requires a bit more work and a useful lemma.

Lemma: Suppose $\tau\colon L \to L$ is an order-preserving map on a complete lattice. Then, ϕ has a fixed point.

Proof. Let $T = \{x \in L \mid x \leq \tau(x)\}$, and set x_0 to be the supremum of T. For any $x \in T$, we have $x \leq x_0$, meaning $x \leq \tau(x) \leq \tau(x_0)$, meaning $x_0 \leq \tau(x_0)$ by the definition of the supremum. Yet, this means that $\tau(x_0) \leq \tau(\tau(x_0))$, so $\tau(x_0) \in T$ with $\tau(x_0) \leq x_0$. □

Theorem (Cantor–Schröder–Bernstein for Projections)**:** Let M be a von Neumann algebra, and let $p, q \in P(M)$. If $p \preceq q$ and $q \preceq p$, then $p \sim q$.

Proof. Suppose w and v are partial isometries with $w^*w = p$, $ww^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Let L be the collection of all projections $e \in M$ with $e \leq q$. Then, L is a complete lattice; defining $\tau\colon L \to L$ by

$$\tau(e) = q - w(p - vev^*)w^*.$$

Then, τ is the composition of two order-preserving maps ($*$-conjugation by a fixed element) and two order-reversing maps (subtraction), so τ is order-preserving on L. Thus, τ has a fixed point, which we will call f. That is, there is $f \in M$ such that $f \leq q$ and $f = q - w(p - vfv^*)w^*$.

Let $v_1 = fv^*$, so that

$$\begin{aligned}v_1v_1^* &= (fv^*)(fv^*)^*\\ &= fv^*vf\\ &= f\end{aligned}$$

as $f \leq q$, and

$$v_1^*v_1 = vfv^*.$$

Therefore, $f \sim vfv^*$. Now, setting $w_1 = (p - vfv^*)w^*$, we have

$$\begin{aligned}w_1^*w_1 &= q - p\\ w_1w_2^* &= p - vfv^*,\end{aligned}$$

so $q - f \sim p - f$, meaning $q \sim p$. □

This is perhaps too slick a proof, and there is in fact a more involved proof that is similar to the proof of the Cantor–Schröder–Bernstein theorem. For this proof, we will let e, f denote the projections in question.

Alternative Proof. Let v and w be partial isometries such that $v^*v = e$, $vv^* = f_1 \leq f$, $w^*w = f$, and $ww^* = e_1 \leq e$. We inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of $f_2 \leq f_1$, it follows that we may write $f_2 = ve_1(ve_1)^*$, and since w maps the range of f_1 onto the range of $e_2 \leq e_1$, we may write $wf_1(wf_1)^* = e_2$. Furthermore, observe that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

We obtain two decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1}, and w maps the range of f_n isometrically onto that of e_{n+1}. In particular, if we let $e_\infty = \inf_n(e_n)$ and $f_\infty = \inf_n(f_n)$, we have that v maps the range of e_∞ onto that of f_∞ and w that of f_∞ onto the range of e_∞.

Similarly, $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$ as discussed earlier, so by the lemma below relating to sums of pairwise orthogonal families of projections, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) &\sim \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2}) \\ \sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) &\sin \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}).\end{aligned}$$

Thus, we have

$$\begin{aligned}e &= e_\infty + \sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) \\ &\sim f_\infty + \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}) \\ &= f.\end{aligned}$$

□

Proposition: Let $S \subseteq H$ be a subset, and let

$$[S] := P_{\overline{\text{span}(S)}}.$$

If $M \subseteq B(H)$ is a von Neumann algebra, with $x \in M$, then $[xH], [x^*H] \in M$ with $[xH] \sim_M [x^*H]$.

Proof. Let $x = v|x|$ be the polar decomposition, and note that $v \in M$. Since vv^* is the projection onto $\overline{xH}$ and v^*v is the projection onto $\ker(x)^\perp = \overline{x^*H}$, it follows that these projections are equivalent in M. □

Definition: If $x \in M$, we define the *central support* of x in M to be the projection

$$z(x) := \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say $p, q \in P(M)$ are *centrally orthogonal* if $z(p)z(q) = 0$.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. The central support of any $p \in P(M)$ is given by

$$\begin{aligned}z(p) &= \sup_{x \in M}[xpH] \\ &= [MpH].\end{aligned}$$

Proof. The second equality follows from the definition of the supremum. Suppose we have $w = [MpH]$. Since M is unital, we have that $p \leq w$. Since $\overline{MpH}$ is reducing for both M and M', we have that $w \in M \cap M' = Z(M)$, meaning that $z(p) \leq w$.

Conversely, if $x \in M$, then

$$\begin{aligned}xpH &= xz(p)pH \\ &= z(p)xpH,\end{aligned}$$

so that $[xpH] \leq z(p)$, meaning $w \leq z(p)$ as this holds for all $x \in M$. □

Proposition: Let M be a von Neumann algebra. For any $p, q \in P(M)$, the following are equivalent:

(i) p and q are centrally orthogonal;

(ii) $pMq = \{0\}$;

(iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ such that $p_0 \sim q_0$.

Proof. We start by showing that (i) and (ii) are equivalent. Let p and q be centrally orthogonal; then, for any $x \in M$, we have

$$\begin{aligned} pxq &= pz(p)xz(q)q \\ &= pxz(p)z(q)q \\ &= 0. \end{aligned}$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, meaning that $p \leq 1 - z(q)$, so since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, so that $z(p)z(q) = 0$.

Now, we will show that *(ii)* and (iii) are equivalent. If *(ii)* does not hold, we let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$, so if we define

$$\begin{aligned} p_0 &= [pxqH] \\ q_0 &= [qx^*pH], \end{aligned}$$

we have that p_0, q_0 are nonzero projections, with $p_0 \leq p$ and $q_0 \leq q$. Since $(pxq)^* = qx^*p$, it follows from the lemma above that $p_0 \sim q_0$.

Meanwhile, if (iii) does not hold, we let $p_0 \leq p$ and $q_0 \leq q$ be such that p_0, q_0 are nonzero and $p_0 \sim q_0$. If $v \in M$ is a partial isometry such that $v^*v = p_0$ and $vv^* = q_0$, we have that $v^* = p_0v^*q_0$, and

$$\begin{aligned} pv^*q &= pp_0v^*q_0q \\ &= p_0v^*q_0 \\ &= v^* \\ &\neq 0, \end{aligned}$$

so that $pMq \neq \{0\}$. $\square$

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i \mid i \in I\}$ and $\{q_0 \mid i \in I\}$ are pairwise orthogonal families with $p_i \preceq q_i$, then $\sum_{i\in I} p_i \preceq \sum_{i \in I} q_i$.

Proof. Let u_i be partial isometries with $u_i^*u_i = p_i$ and $r_i := u_iu_i^* \leq q_i$. Note that the r_i are pairwise orthogonal since the q_i are pairwise orthogonal. Therefore, for $i \neq j$, we have

$$\begin{aligned} u_i^*u_j &= u_i^*u_iu_i^*u_ju_js^*u_j \\ &= u_ir_ir_ju_j \\ &= 0 \\ u_iu_j^* &= u_iu_i^*u_iu_j^*u_ju_j^* \\ &= u_ip_ip_ju_j^* \\ &= 0. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \left(\sum_{i\in I} u_i\right)\left(\sum_{j \in J} u_j\right) &= \sum_{i \in I} u_i^*u_i \\ &= \sum_{i\in I} p_i \end{aligned}$$

and

$$\begin{aligned}\left(\sum_{i\in I} u_i\right)\left(\sum_{j\in J} u_j^*\right) &= \sum_{i\in I} r_i\\ &\leq \sum_{i\in I} q_i,\end{aligned}$$

so that $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$. □

Theorem (Comparison Theorem)**:** Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there exists $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ of pairwise orthogonal projections with $p_i \sim q_i$, and

$$\begin{aligned}\underbrace{\sum_{i\in I} p_i}_{=:p_0} &\leq p\\ \underbrace{\sum_{i\in I} q_i}_{=:q_0} &\leq q.\end{aligned}$$

From the above lemma, we know that $p_0 \sim q_0$. Let $w := z(q - q_0)$. By maximality, we must have $z(p - p_0)w = 0$, meaning that $(p - p_0)w = 0$, or $pz = p_0 z$. If v is a partial isometry such that $v^*v = p_0$ and $vv^* = q_0$, then the partial isometry vz implements the equivalence $p_0 z \sim q_0 z$. Therefore, we have $pz = p_0 z \sim q_0 z \leq qz$. Similarly, $p_0(1-z) \sim q_0(1-z)$ and we get $q(1-z) \preceq p(1-z)$. □

Corollary: If M is a factor, then any two projections can be compared.

Proof. There are no nontrivial central projections in a factor. □

Type Decomposition of von Neumann Algebras

Definition: If M is a von Neumann algebra, and $p \in B(H)$ is a projection (not necessarily in M), then

$$pMp = \{pxp \mid x \in M\}$$

is called a *corner* (or *compression*) of M.

This terminology comes from the identification that, whenever $x \in M$, we have

$$\begin{aligned}PxP &\leftrightarrow \begin{pmatrix} PxP & 0\\ 0 & 0\end{pmatrix}\\ &\in B(PH \oplus (1-P)H).\end{aligned}$$

In fact, we have $PB(H)P = B(PH)$.

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra, with $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, with $(pMp)' = M'p$ and $(M'p)' = pMp$.

Corollary: Let $M \subseteq B(H)$ be a von Neumann algebra, $p \in P(M)$. If M is a factor, then pMp and $M'p$ are factors.

Proof. Since pMp and $M'p$ are commutants of each other in $B(pH)$, they have the same center, so it suffices to show that $M'p$ is a factor. For any $y \in M'$, if we have $yp = 0$, then for all $x \in M$ and $\xi \in H$, we have

$$\begin{aligned}yxp\xi &= xyp\xi\\ &= 0.\end{aligned}$$

Since M is a factor, we have $z(p) = 1$, meaning that MpH is dense in H, and thus $y = 0$. If we have $wp \in Z(M'p)$ for $w \in M'$, then for all $y \in M'$, we have that $[w, y]p = [wp, yp] = 0$. Yet, this means that $[w, y] = 0$, meaning $w \in Z(M')$. Since M' is a factor, we have $w \in \mathbb{C}$, so $zp \in \mathbb{C}p$. Therefore, $Z(M'p) = \mathbb{C}p$, and $M'p$ is a factor. □

Proposition: Let $M \subseteq B(H)$ be a von Neumann algebra. If $p, q \in P(M)$ are equivalent in M, then pMp and qMq are spatially isomorphic.

Proof. Let $v \in M$ be a partial isometry with $v^*v = p$ and $vv^* = q$. We will show that $v|_{pH}$ is an isomorphism from pH to qH implementing the spatial isomorphism. We observe that $v = qvp$, so that $v|_{pH}$ is valued in qH, and is surjective since $q\xi = vv^*\xi = vpv^*\xi$. Furthermore, for any $p\xi, p\eta \in pH$, we have

$$\begin{aligned}\langle vp\xi, vp\eta \rangle &= \langle v^*vp\xi, p\eta \rangle\\ &= \langle p\xi, p\eta \rangle,\end{aligned}$$

so that $v|_{pH}$ is a unitary. Additionally, by using $v = qvp$ yet again, we find that

$$\begin{aligned}vpxpv^* &= vxv^*\\ &= q(vxv^*)q\\ qxq &= vv^*xvv^*\\ &= v(pv^*xvp)v^*,\end{aligned}$$

so that $v(pMp)v^* = qMq$. □

Definition: Let M be a von Neumann algebra. We say $p \in P(M)$ is

- *finite* if $q \leq p$ and $q \sim p$ implies $p = q$;
- *semi-finite* if there exists a family $\{p_i\}_{i \in I}$ of pairwise orthogonal finite projections such that $\sum_{i \in I} p_i = p$;
- *σ-finite* if every collection of pairwise orthogonal nonzero projections less than P is at most countable;
- *purely infinite* if $p \neq 0$ and there do not exist any nonzero finite projections $q \in P(M)$ with $q \leq p$;
- *properly infinite* if $p \neq 0$ and, for all nonzero $w \in P(Z(M))$, the projection wp is not finite.

We say M is finite, semi-finite, purely infinite, or properly infinite if the projection 1 has the corresponding property in M.

An equivalent criterion for semi-finiteness is that, if w is any central projection with $wp \neq 0$, then there is a finite projection $0 < q \leq wp$.

Observe that for a von Neumann algebra, we have that finite implies semi-finite, and that any semi-finite von Neumann algebra is not purely infinite; similarly, any purely infinite von Neumann algebra is properly infinite.

Proposition: A von Neumann algebra $M \subseteq B(H)$ is finite if and only if all isometries are unitaries.

Proof. Let M be finite, and let $v \in M$ be an isometry, $v^*v = 1$. Then, $vv^* \leq 1$, so by finiteness, $vv^* = 1$, meaning v is a unitary.

Conversely, suppose every isometry is a unitary, and suppose $p \leq 1$ satisfies $p \sim 1$. Then, if $v \in M$ is a partial isometry with $v^*v = 1$ and $vv^* = p$, we have that v is an isometry, hence a unitary, so $vv^* = 1$, so 1 is finite in M. □

Definition: Let $M \subseteq B(H)$ be a von Neumann algebra. We say $p \in P(M)$ is minimal in M if $p \neq 0$ and $pMp = \mathbb{C}p$.

We say p is abelian in M if pMp is abelian.

Definition: We say a von Neumann algebra M is discrete if for every nonzero central projection z, there is a nonzero abelian projection f with $f \leq z$.

We say M is continuous if it contains no nonzero abelian projections.

We say a projection $e \in M$ is discrete or continuous if the compression algebra eMe is discrete or continuous, respectively.

Proposition: Let M be a von Neumann algebra.

(i) Suppose $0 < p_1 \leq p$, and p is abelian, (resp. finite, semi-finite, purely infinite). Then, p_1 is abelian (resp. finite, semi-finite, purely infinite)

(ii) If $\{p_i\}_{i\in I}$ are purely infinite (resp. semifinite) projections, then $p = \sup_i p_i$ is purely infinite (resp. semifinite). Thus, there is a largest purely infinite (resp. semifinite) projection, and it lies in $Z(M)$.

(iii) If p is a properly infinite projection, then it decomposes uniquely as a sum $p = p_s + p_p$ of a semifinite projection and a purely infinite projection, with $z(p_s)z(p_p) = 0$.

(iv) If $\{p_i\}_{i\in I}$ are abelian (resp. finite) projections with $z(p_i)z(p_j) = 0$ for each $i \neq j$, then $p = \sum_{i\in I} p_i$ is an abelian (resp. finite) projection. There is a smallest central projection w that majorizes all abelian (resp. finite) projections, and there is an abelian (resp. finite) projection p with $z(p) = w$. Additionally, there is a largest finite central projection.

(v) If e and f are abelian projections with $z(e) = z(f)$, then $e \sim f$.

Proof.

(i) Pretty much by definition, we have that the statement holds if p is abelian (subprojections yield subalgebras of compressions) or purely infinite.

If p is finite, with $p_2 \leq p_1 \leq p$ and $p_2 \sim p_1$, then $p_2 + (p - p_1) \leq p$, and

$$\begin{aligned} p_2 + (p - p_1) &\sim p_1 + (p - p_1) \\ &= p, \end{aligned}$$

so since p is finite, we have $p_2 + (p - p_1) = p$, so $p_2 = p_1$, and p_1 is finite.

Now, suppose P is semifinite, and let $0 < p_1 \leq p$. Set $z = z(p_1)$. There is a finite projection $0 < q \leq zp$; since z is the supremum of all projections equivalent to p_1, there is a projection $q_1 \sim q$ such that $p_1 q_1 \neq 0$. Thus, we have a projection $0 < q_2 \leq q_1$ with $q_2 \sim p_2 \leq p_1$, so q_2, and thus p_2, are finite as desired.

(ii) Suppose each p_i is purely infinite, and let q be a finite projection with $q \leq p$. Then, for some i, we have $qp_i \neq 0$. Thus, there is some projection $0 < q_1 \leq q$ with $q_1 \sim p_1 \leq p_i$. From (i), we have that q_1 is finite and p_1 is finite. Yet, this contradicts the fact that p_i is purely infinite. Thus, $p = \sup_i p_i$ is purely infinite.

Furthermore, all the purely infinite projections are invariant under unitary conjugation. Therefore, the supremum of all purely infinite projections satisfies $q = uqu^*$ for all unitary $u \in M$. Since all elements of M are linear combinations of unitaries, it follows that $q \in Z(M)$, and it necessarily majorizes all purely infinite projections.

Now, suppose each p_i is semifinite. Let $q \leq p$. Then, for some i, we have $qp_i \neq 0$, so there is a projection $0 < q_1 \leq q$ such that $q_1 \sim p_1 \leq p_i$. Since p_i is semifinite, then so is p_1. Since $q_1 = uu^*$ and $p_1 = u^*u$, we have $q_2 = up_2u^* \leq q_1 \leq q$ is finite. Therefore, $p = \sup_i p_i$ is semifinite. Similarly to the unitary case, since the set of semifinite projections is invariant under unitary conjugation, so we have that the supremum over all semifinite projections is central and majorizes all semifinite projections.

(iii) Suppose p is properly infinite. Let p_s be the sum of all finite projections less than p. Then, p_s is semifinite, and there are no finite projections less than $p_p = p - p_s$, so it is purely infinite.

First, we observe that $z(p_s)$ is spanned by the projections $q_1 \sim p_1 \leq p$, and each of these are finite.

If $z(p_s)z(p_p) \neq 0$, then there is a finite projection q_1 with $q_1 p_p \neq 0$. Then, there is a (necessarily finite) projection $0 < q_2 \leq q_1$ with $q_2 \sim p_2 < p_p$, which contradicts the fact that p_p is purely infinite. Therefore, p_s and p_p are centrally orthogonal.

(iv) If each p_i is abelian with pairwise orthogonal central supports, then

$$pMp = \sum_{i \in I} z(p_i) p_i M p_i$$

is a direct sum of abelian von Neumann algebras, and so is abelian. By Zorn's lemma, we may find a maximal family of these pairwise orthogonal projections. If there were some abelian projection $q \leq 1 - z(p)$, then the collection would not be maximal, meaning that $z = c(p)$ dominates every abelian projection and is the smallest central projection with this property.

Now, if each p_i is finite, and $q \leq p$ with $q \sim p$, then we have $q_i := qz(p_i) \leq p_i$ and $q_i \sim p_i$. Since each i is finite, it follows that $q_i \sim p_i$ for all $i \in I$, so $q = p$ and p is finite. Similarly to the abelian case, we could take a maximal family of finite projections with pairwise orthogonal central supports, so that $z = c(p)$ majorizes every finite projection q, else $q(1-z)$ would be finite. Since p is finite, it follows that z is the smallest central projection with this property.

Now, let $(z_i)_i$ be a maximal family of pairwise orthogonal finite central projections. Then, $z = \sum_{i \in I} z_i$ is finite and central. If there were some finite central projection p such that $p \nleq z$, then $(1-z)p$ would be a nonzero finite central projection orthogonal to z, contradicting the maximality of our family. Thus, z is the largest finite central projection.

(v) We suppose that e and f are abelian projections with $z(e) = z(f) = 1$. By the comparison theorem, there is a central projection w such that $we \sim wf_0 \leq wf$ and $(1-w)f \sim (1-w)e_0 \leq (1-w)e$. Note that $z(we) = z(wf_0) = z(wf) = w$. If it were the case that $wf_0 < wf$, then there is some $a \in M$ such that

$$\begin{aligned} 0 &\neq w(f - f_0)awf_0 \\ &= w(f-f_0)(wfawf)wf_0 \\ &= w(f-f_0)wf_0(wfawf) \\ &= 0, \end{aligned}$$

where we use the fact that wf is an abelian projection. Therefore, we have $wf_0 = wf$, and similarly, $(1-w)e_0 = (1-w)e$. Therefore, $e = (1-w)e + we \sim wf + (1-w)f = f$.

□

Corollary: Let p be an abelian projection in a von Neumann algebra M. Then, $z(p)Z(m) \cong pMp$ via the map $z(p)w \mapsto pw$.

Proof. We have that $M'p = pMp$, and since p is an abelian projection, we have

$$\begin{aligned} pMp &= Z(pMp) \\ &= pMp \cap M'p \\ &= p(M \cap M')p \\ &= pZ(M). \end{aligned}$$

Since $w \mapsto pw$ is a $*$-homomorphism with $(1 - z(p))Z(m)$, we have an isomorphism with $z(p)Z(M)$. □

Definition: A von Neumann algebra is called

- type I if every nonzero projection majorizes a nonzero abelian projection;
- type II_1 if it is finite and there are no nonzero abelian projections;
- type II_∞ if it is semifinite, properly infinite, and there are no nonzero abelian projections;
- type III if it is purely infinite.

Furthermore, we call a von Neumann algebra atomic if every nonzero projection majorizes a minimal projection. A von Neumann algebra with no minimal projections is called diffuse.

Theorem (Type Decomposition)**:** Let M be a von Neumann algebra. Then, there are unique central projections z_1, z_2, z_3, z_4 such that $1 = z_1 + z_2 + z_3 + z_4$, Mz_1 is type I, Mz_2 is type II_1, Mz_3 is type II_∞, and Mz_4 is type III respectively.

Note that all abelian von Neumann algebras are type I, as well as $B(H)$, the former since the identity is an abelian projection and the latter since every projection majorizes a rank 1, projection, which is a minimal (and hence abelian) projection. Furthermore, a factor is only one of these types (since its center is trivial).

Tensor Products of Hilbert Spaces and von Neumann Algebras

Before we can discuss different von Neumann algebras of the various types, we need to discuss the tensor product of von Neumann algebras.

Definition: Let H and K be Hilbert spaces. Then, there is an inner product on the algebraic tensor product $H \odot K$ given by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle$$

whenever $\xi_1, \eta_1 \in H$ and $\xi_2, \eta_2 \in K$. The Hilbert space tensor product of H and K, denoted $H \otimes K$, is the completion of $H \odot K$ with respect to the norm induced by this inner product.

Proposition: Let H and K be Hilbert spaces with orthonormal bases $(e_i)_i$ and $(f_j)_j$. Then,

(i) $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal basis for $H \otimes K$;

(ii) if $|J| = \alpha$, then $H \otimes K \cong H^{(\alpha)} \cong \bigoplus_{j \in J} H$;

(iii) if $H = L_2(\mu)$ for some σ-finite regular Borel measure space (X, μ), and K is separable, then if $L_2(X, \mu, K)$ denotes the space of square-integrable Borel functions, we have $H \otimes K \cong L_2(X, \mu, K)$.

Proof.

(i) We observe that the set $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal set; the spans of these elementary tensors are all vectors of the form $x \otimes y$, provided x is a finite linear combination of the e_i and y is a finite linear combination of the f_j. Therefore, the completion is equal to the completion of $H \otimes K$, meaning the set if an orthonormal basis.

(ii) Considering

$$H \otimes K \cong \bigoplus_{j \in J} H \otimes \mathbb{C} f_j,$$

we find that this is an ℓ_2 direct sum of $|J|$ copies of H.

(iii) Define maps $Y_j \colon L_2(X, \mu, K) \to L_2(\mu)$ by

$$(Y_j f)(x) = \langle f(x), f_j \rangle$$

modulo μ. Then, we may define

$$\begin{aligned} Y \colon L_2(X, \mu, K) &\to \bigoplus_{j \in J} H \otimes \mathbb{C} f_j \\ f &\mapsto \sum_{j \in J} Y_j f. \end{aligned}$$

Using Tonelli's theorem and Parseval's identity, we may then compute

$$\begin{aligned} \|Yf\|^2 &= \sum_{j \in J} \|Y_j f\|^2 \\ &= \sum_{j \in J} \int |\langle f(x), f_j \rangle|^2 \, d\mu \end{aligned}$$

$$
\begin{aligned}
&= \int \sum_{j\in J} |\langle f(x), f_j\rangle|^2 \, d\mu \\
&= \int \|f(x)\|^2 \, d\mu(x) \\
&= \|f\|^2.
\end{aligned}
$$

Therefore, Y is an isometry. The range is necessarily dense, since if $h \in L_2(\mu)$, then $f = hf_j$ is mapped to the vector with h in position j and 0 elsewhere. Thus, Y is a unitary map of $L_2(X, \mu, K)$ onto $H \otimes K$.

□

Type I von Neumann Algebras

Type II$_1$ von Neumann Algebras