Problem (Problem 1)**:** Given $z = x + iy \in \mathbb{C}$, define

$$z^* = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

(a) Show that $z^* \in S^2$.

(b) Prove that if $(x_1, x_2, x_3) \in S^2 \setminus \{(0,0,1)\}$, then there exists a unique $z \in \mathbb{C}$ such that $z^* = (x_1, x_2, x_3)$.

(c) A circle in S^2 is the intersection of a plane in $\mathbb{R}^3$ with S^2, provided this intersection is nonempty. Prove that if C is a circle in S^2, then there exists a set $\widetilde{C} \subseteq \mathbb{C}$ that is either a circle or a straight line such that $C \setminus \{(0,0,1)\} = \left\{z^* \in \mathbb{R}^3 \,\middle|\, z \in \widetilde{C}\right\}$.

Solution:

(a) Via brute force calculation, we see that

$$\begin{aligned}\frac{4x^2}{(x^2+y^2+1)^2} + \frac{4y^2}{(x^2+y^2+1)^2} + \frac{(x^2+y^2-1)^2}{(x^2+y^2+1)^2} &= \frac{(x^2+y^2)^1 + 1 - 2(x^2+y^2) + 4(x^2+y^2)}{(x^2+y^2+1)^2}\\ &= \frac{(x^2+y^2)^1 + 1 + 2(x^2+y^2)}{(x^2+y^2+1)^2}\\ &= 1.\end{aligned}$$

(b) Let $z^* = (x_1, x_2, x_3) \in S^2 \setminus \{(0,0,1)\}$, and let $L\colon [0,\infty) \to \mathbb{R}^3$ be the line parametrized such that $L(1) = (x_1, x_2, x_3)$ and $L(0) = (0,0,1)$, which is given by

$$L(t) = (tx_1, tx_2, tx_3 + (1-t)).$$

Note then that $\|L(t)\| = 1$ only when $t = 0$ or $t = 1$, meaning that $L(t)$ intersects $S^2 \setminus \{(0,0,1)\}$ exactly once. By identifying $\mathbb{C}$ with $x + iy \mapsto (x, y, 0)$, we may find $z \in \mathbb{C}$ that uniquely maps to (x_1, x_2, x_3) under the z^* identification by taking

$$\begin{aligned} tx_3 + (1-t) &= 0\\ 1 + t(x_3 - 1) &= 0\\ t &= \frac{1}{1-x_3},\end{aligned}$$

so that

$$x + iy = \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3}$$

maps to z^* under the given identification.

(c)

Problem (Problem 2)**:** Define $f\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $f(z) = \left(\frac{z+1}{z-1}\right)^2$.

(a) Is f injective on $\mathbb{D}$? Why or why not?

(b) Determine $f(\mathbb{D})$.

Solution:

(a) We consider $q(z) = \frac{z+1}{z-1}$ as a fractional linear transformation on $\hat{\mathbb{C}}$. We see that

$$\begin{aligned}
q\left(e^{i\theta}\right) &= \frac{e^{i\theta}+1}{e^{i\theta}-1}\\
&= \frac{(1+\cos(\theta)) + i\sin(\theta)}{(\cos(\theta)-1) + i\sin(\theta)}\\
&= \frac{((\cos(\theta)+1) + i\sin(\theta))((\cos(\theta)-1) - i\sin(\theta))}{(1-\cos(\theta))^2 + \sin^2(\theta)}\\
&= \frac{\left(\cos^2(\theta)-1\right) + \sin^2(\theta) + i\sin(\theta)(\cos(\theta) - 1 - (\cos(\theta)+1))}{2-2\cos(\theta)}\\
&= i\frac{\sin(\theta)}{\cos(\theta)-1},
\end{aligned}$$

and since $\frac{\sin(\theta)}{\cos(\theta)-1}$ maps $(0, 2\pi) \to \mathbb{R}$ bijectively, we see that q maps the unit circle into the imaginary axis. We also see that $q(0) = -1$, so $\mathbb{D}$ maps $\mathbb{D}$ bijectively onto the left half-plane, $\mathbb{L} = \{z \mid \operatorname{Re}(z) < 0\}$.

Now, notice that the function $h(z) = z^2$ is injective when defined on a half-plane (the arguments $(\pi/2, 3\pi/2)$ map injectively to $(\pi, 3\pi)$, and the function $|z|^2$ is clearly injective on $(0, \infty)$), so since $f = h \circ q$ is injective on $\mathbb{D}$.

(b) Since $f = h \circ q$, where q maps $\mathbb{D}$ to the left half-plane, and h maps the left half-plane to the full complex plane save for $(-\infty, 0]$, we have that f maps $\mathbb{C}$ to $\mathbb{C} \setminus (-\infty, 0]$.

Problem (Problem 3)**:** Prove that there exists a linear fractional transformation that maps the first quadrant in $\mathbb{C}$ bijectively to the top half of the unit disc, and satisfies $f(2) = i$.

Solution: We start from the Cayley transform,

$$f_1(z) = \frac{z-i}{z+i},$$

which maps the upper half-plane to the unit disc.