Revised Problems

Problem (Homework 8, Problem 1)**:** Let A be a path-connected subspace of a topological space X, and let $i\colon A \to X$ be the inclusion of A into X. Show that for any $x_0 \in A$, the induced map $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective if and only if every path in X with endpoints in A is homotopic to a path in A.

Solution: Suppose $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. Then, by definition, any loop in X based at x_0 is homotopic to a loop in A based at x_0.

Let γ be a path in X from $x_1 \in A$ to $x_2 \in A$. There are paths σ_1 from x_0 to x_1 and σ_2 from x_2 to x_0 in A. Composing $\sigma_1 \cdot \gamma \cdot \sigma_2$ gives a loop in X based at x_0. Therefore, this loop is homotopic to a loop in A, which we will call η. We start by showing that η is homotopic to a loop that passes through both x_1 and x_2.

Let $x_1' = \eta(1/3)$, $x_2' = \eta(2/3)$, and define $\eta|_{[0,1/3]}(3t) = \omega_1(t)$, $\eta|_{[2/3,1]}(3t-2) = \omega_2(t)$, and $\kappa(t) = \eta|_{[1/3,2/3]}(3t-2)$. Then, there are paths $\zeta_{1,2}\colon [0,1] \to A$ that go from $x_{1,2}'$ to $x_{1,2}$. We observe that, as maps, ω_1 is homotopic to the path $\omega_1 \cdot \zeta_1 \cdot \overline{\zeta_1}$, and similarly, ω_2 is homotopic to the path $\zeta_2 \cdot \overline{\zeta_2} \cdot \omega_2$. Therefore, if we take the full concatenation

$$\eta' = \left(\omega_1 \cdot \zeta_1 \cdot \overline{\zeta_1}\right) \cdot \kappa \cdot \left(\zeta_2 \cdot \overline{\zeta_2} \cdot \omega_2\right),$$

we observe that it is homotopic to η via a reparametrization, and it passes through x_1 and x_2. Therefore, by composing homotopies, we may assume that the original loop $\sigma_1 \cdot \gamma \cdot \sigma_2$ is homotopic to a loop, χ, passing through x_1 and x_2 in A. Using a reparametrization such that $\chi(1/3) = x_1$ and $\chi(2/3) = x_2$, this allows us to determine that $\chi|_{[1/3,2/3]}$ is homotopic as a path to γ.

In the reverse direction, we observe that since any loop in X with an endpoint in A is homotopic to a loop in A, it follows that every homotopy class of loops in X based at x_0 contains a representative that is a loop in A, so the induced homomorphism is surjective.

Current Problems

Problem: Suppose that Y is obtained from a space X by attaching cells of dimension n, where $n \geq 3$. Prove that the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X) \cong \pi_1(Y)$.

Solution: We let Z be a space given by attaching $I \times I$ "on top of" γ_α going from $x_0 \in X$ to points y_α on $Y \cap X$, identifying $I \times \{0\}$ to the paths γ_α, then identifying $\{0\} \times I$ with each other and $\{1\} \times I$ with sub-arcs of $D_\alpha^n \cap X$. Observe that Z deformation retracts onto Y.

Within each open cell e_α^n of Y, we select points w_α not on the sub-arcs described above, and define subsets

$$\begin{aligned} A &= Z \setminus \left(\bigcup_\alpha \{w_\alpha\}\right) \\ B &= Z \setminus X. \end{aligned}$$

We see that since each Y is contractible, each interval I, and each path, it follows that B is contractible, meaning that $\pi_1(Z)$ is given by $\pi_1(A)/N$, where N is the normal subgroup generated by the inclusion of $\pi_1(A \cap B)$ into $\pi_1(A)$. Note that A deformation retracts onto X, meaning we only need to show that $\pi_1(A \cap B)$ is trivial.

This follows from applying the van Kampen theorem to the open cover of $A \cap B$ by the subsets $V_\alpha = A \cap B \setminus \bigcup_{\beta \neq \alpha} e_\beta^n$. Each A_α deformation retracts onto $e_\alpha^n \setminus \{w_\alpha\}$, which deformation retracts to S^{n-1}. Since, for any $k \geq 2$, S^2 is contractible, it then follows that each of the A_α has trivial fundamental group, so $A \cap B$ has trivial fundamental group. Tracing back these equivalences gives

$$\begin{aligned} \pi_1(Y) &\cong \pi_1(Z) \\ &\cong (\pi_1(A) * \pi_1(B))/\langle \pi_1(A \cap B) \rangle \\ &\cong \pi_1(A) \end{aligned}$$

$$\cong \pi_1(X).$$

Problem (Problem 2)**:** Give three covering spaces of $S^1 \vee S^2$.

Solution: We consider three covering spaces of $S^1 \vee S^2$ given by $\mathbb{R} \sqcup S^2$ glued at $(1,0,0) \sim 0$, $\mathbb{R} \sqcup S^2 \sqcup S^2$ glued at $(1,0,0) \sim 0, (1,0,0) \sim 1$, and $\mathbb{R} \sqcup S^2 \sqcup S^2 \sqcup S^2$ glued at $(1,0,0) \sim 0, (1,0,0) \sim 1, (1,0,0) \sim -1$.

To consider the covering maps, we use the map $t \mapsto e^{2\pi i t}$ as the covering map taking $\mathbb{R}$ onto S^1, and consider any open subset of S^2 to have its preimage mapped to identical copies of the open subset of the copies of S^2 for the respective number of said copies.

Problem (Problem 3)**:** Prove that if $p_1 \colon \widetilde{X}_1 \to X_1$ and $p_2 \colon \widetilde{X}_2 \to X_2$ are covering spaces, then so is their product

$$p_1 \times p_2 \colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2.$$

Solution: We let $\{U_\alpha\}_\alpha$ be an open cover of X_1 satisfying the covering map criteria, and similarly for $\{V_\beta\}_\beta$ and X_2. Then, we observe that $\{U_\alpha \times V_\beta\}_{\alpha,\beta}$ forms an open cover of $X_1 \times X_2$. We claim that this map satisfies the covering map criteria. We observe that for arbitrary α and β, the definition of the product topology gives

$$(p_1 \times p_2)^{-1}(U_\alpha \times V_\beta) = p_1^{-1}(U_\alpha) \times p_2^{-1}(V_\beta).$$

This gives rise to a disjoint union

$$p_1^{-1}(U_\alpha) \times p_2^{-1}(V_\beta) = \left(\bigsqcup_{i \in I} Y_i\right) \times \left(\bigsqcup_{j \in J} W_j\right),$$

with each W_j is homeomorphic to V_β and each Y_i is homeomorphic to U_α. This means that for some distinguished $Y_i \times W_j$, we have that $Y_i \times W_j$ is homeomorphic to $U_\alpha \times V_\beta$, giving that $p_1 \times p_2$ is a covering map.