Part 1

3.5, Problem 18

(a)

$$\det\begin{pmatrix}2-\lambda & 4\\ 3 & 6-\lambda\end{pmatrix} = (\lambda-2)(\lambda-6)-12$$
$$\lambda(\lambda-8) = 0,$$

meaning $\lambda = 0, 8$.

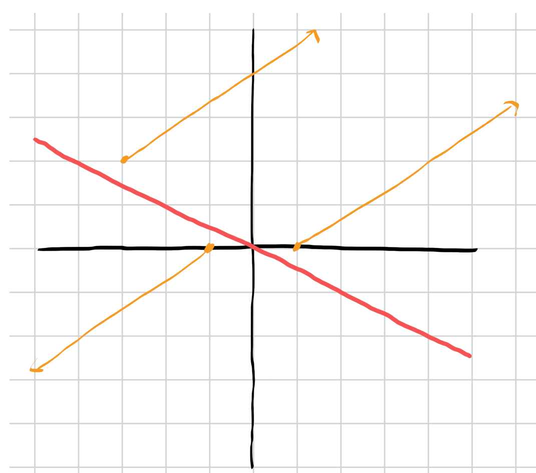
(b) The eigenvector for $\lambda = 0$ is

$$\begin{aligned}2x &= -4y\\ x &= -2y\\ \vec{v}_1 &= \begin{pmatrix}-2\\1\end{pmatrix},\end{aligned}$$

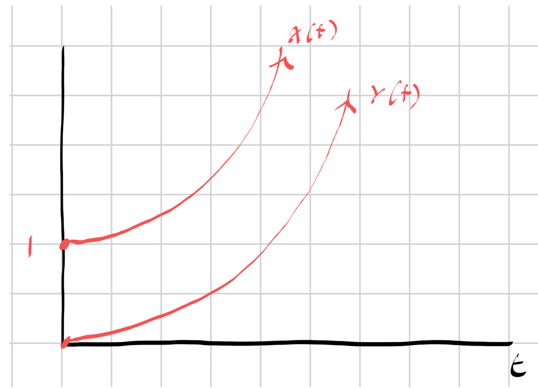
and the eigenvector for $\lambda = 8$ is

$$\begin{aligned}2x+4y &= 8x\\ 4y &= 6x\\ y &= \frac{2}{3}x\\ \vec{v}_2 &= \begin{pmatrix}3\\2\end{pmatrix}.\end{aligned}$$

(c)



(d)



(e)

$$\vec{Y}(t) = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(f) Solving for the initial condition, we have

$$\begin{aligned} 1 &= -2k_1 + 3k_2 \\ 0 &= k_1 + 2k_2, \end{aligned}$$

so $k_1 = -2k_2$, and

$$\begin{aligned} 1 &= 7k_2 \\ k_2 &= \frac{1}{7} \\ k_1 &= -\frac{2}{7}. \end{aligned}$$

Thus, we get

$$\vec{Y}_1(t) = \frac{1}{7} \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \frac{2}{7} e^{8t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

3.5, Problem 21 (a)

There are two repeated eigenvalues at 0. All solutions are of the form

$$\vec{v} = \vec{v}_0 + t \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\vec{v}_0$ is the initial condition.

3.5, Problem 23

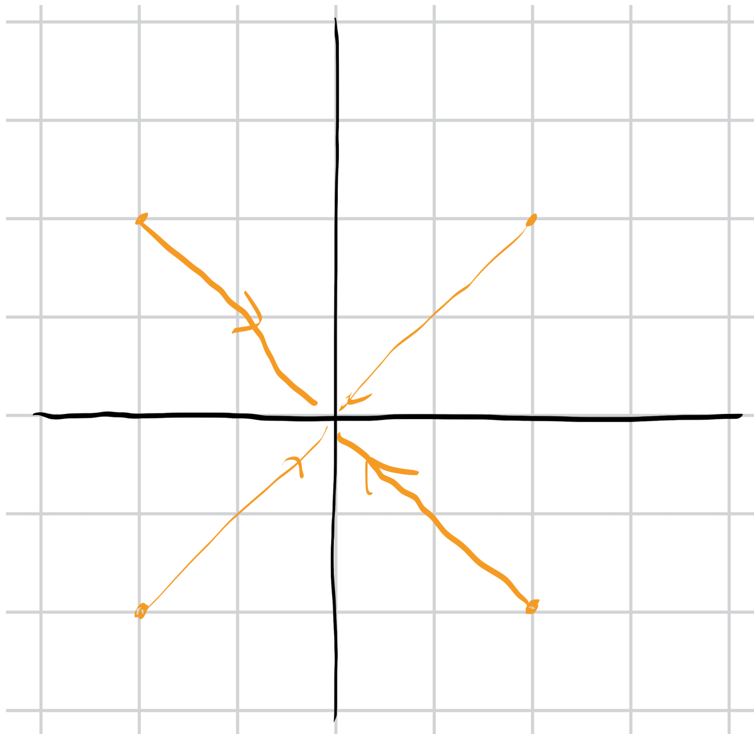
(a) The eigenvalues are $\lambda = a$ and $\lambda = d$.

(b) Every vector is an eigenvector.

(c) If $a = d < 0$, then the origin is a sink star point. The general solution is

$$\vec{Y}(t) = e^{at} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

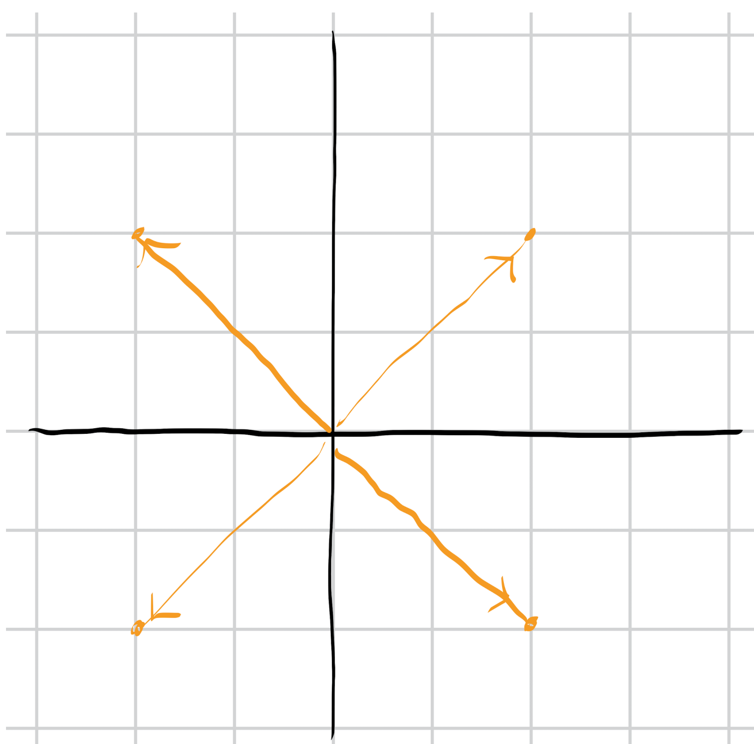
Here, the eigenvector is the original condition. The phase portrait is as follows.



(d) If $a = d > 0$, then the origin is a source star point. The general solution is

$$\vec{Y}(t) = e^{at}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

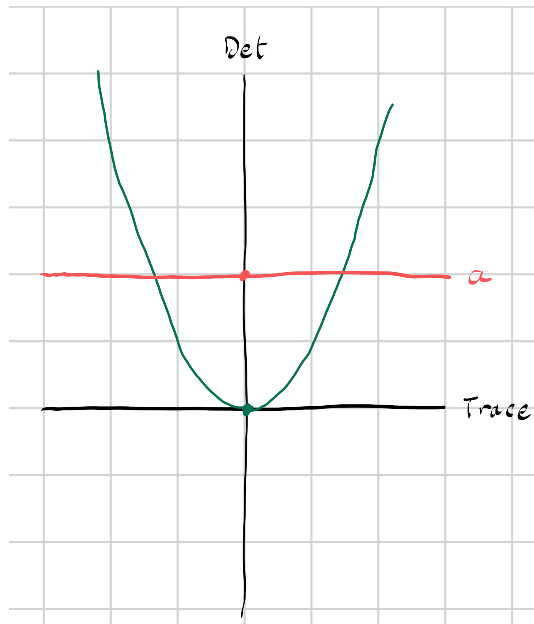
Here, the eigenvector is the original condition. The phase portrait is as follows.



3.7, Problem 2

We see that $\text{tr}(A) = a$ and $\det(A) = 2$.

(a)



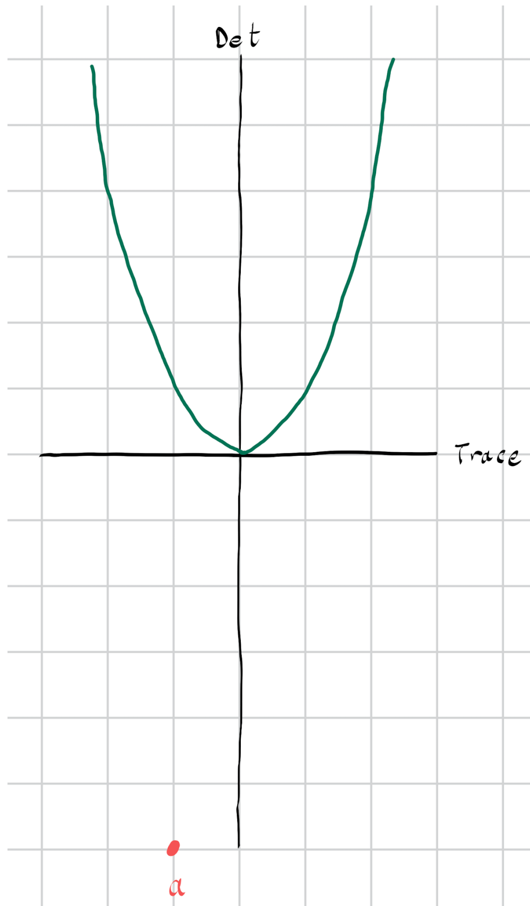
(b) If $a < -\sqrt{2}$, we have a sink. If $-\sqrt{2} < a < 0$, we have a spiral sink. If $0 < a < \sqrt{2}$, we have a spiral source. If $a > \sqrt{2}$, we have a source. At $a = -\sqrt{2}$, we have a nodal sink, at $a = 0$, we have a center, and at $a = \sqrt{2}$, we have a nodal source.

(c) The bifurcation values are at $a = -\sqrt{2}$, $a = 0$, and at $a = \sqrt{2}$.

3.7, Problem 6

We see that $\text{tr}(A) = -1$ and $\det(A) = -6$.

(a)



(b) Since a does not affect the parameter, we exclusively have a saddle.

(c) There are no bifurcation values for this system.

Part 2

5.1, Problem 3

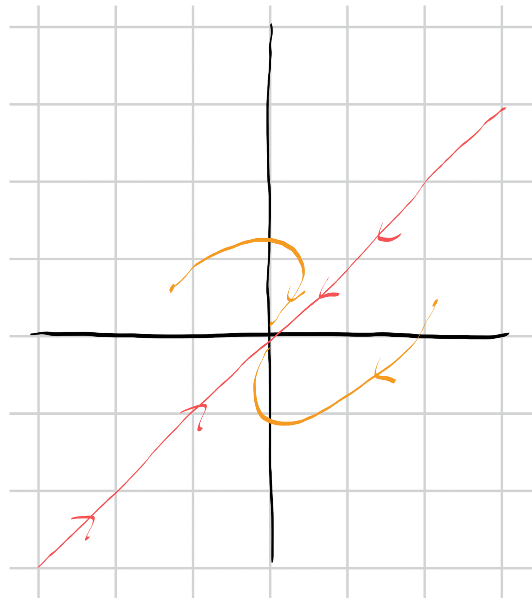
(a) We start by finding the Jacobian

$$J = \begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$$
$$J|_{(0,0)} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix},$$

meaning the linearized system at $(0, 0)$ is

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \vec{u}.$$

(b) The determinant is positive and the trace is negative, with $\det(A') = 1$ and $\operatorname{tr}(A') = -2$, so we have a nodal sink. The eigenvalue of this linearized system is at $\lambda = -1$, with corresponding eigenvector of $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

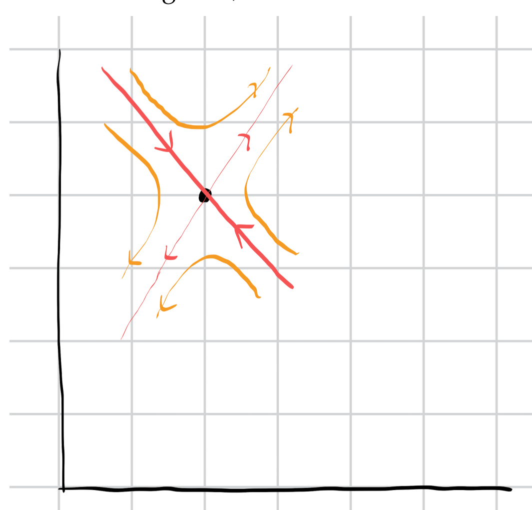


(c)

(d) (a) The Jacobian evaluated at $(2, 4)$ is

$$J|_{(2,4)} = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$$

(b) The trace and determinant are both negative, so this is a saddle.



(c)

5.1, Problem 4

(a) Since $\frac{dx}{dt} = 0$ if and only if $-x = 0$, we have that $\frac{dy}{dt} = 0$ if and only if $y = 0$. Thus $(0, 0)$ is the only equilibrium point.

(b) Taking partial derivatives, we have the linearized system

$$\frac{d\vec{U}}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) The linearized system is a saddle with eigenvectors of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

5.1, Problem 5

(a) The general solution to $\frac{dx}{dt} = -x$ is $k_1 e^{-t}$.

(b) Substituting, we have

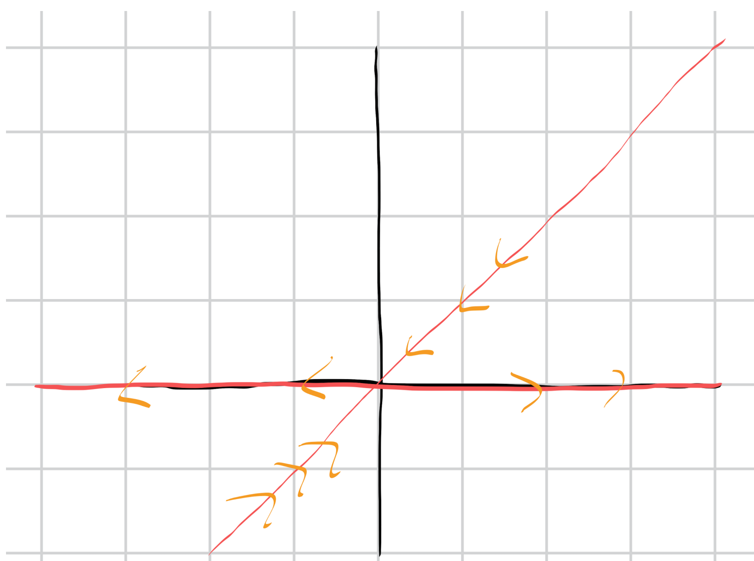
$$\begin{aligned}
\frac{dy}{dt} &= -4k_1 e^{-3t} + y\\
\frac{dy}{dt} - y &= -4k_1 e^{-3t}\\
e^{-t}\frac{dy}{dt} - e^{-t}y &= -4k_1 e^{-4t}\\
\frac{d}{dt}\left(e^{-t}y\right) &= -4k_1 e^{-4t}\\
y &= k_1 e^{-3t} + k_2 e^{t}.
\end{aligned}$$

(c) The general solution is

$$\vec{Y}_1(t) = \begin{pmatrix} k_1 e^{-3t} \\ k_1 e^{-3t} + k_2 e^{t} \end{pmatrix}.$$

(d) The solution curves that tend toward the origin as $t \to \infty$ are all the ones that have $k_2 = 0$.

(e) The solution curves that tend toward the origin as $t \to -\infty$ are all the ones that have $k_1 = 0$.

(f) 

(g) The linearized system and the separatrix solutions appear very similar, albeit that the vectors along which the solution curves tend to zero are different than the phase portrait of the linearized system.

5.1, Problem 8

(a) We find the Jacobian to be

$$J = \begin{pmatrix} -2x + 10 - y & -x \\ 2y & -2x - 2y + 30 \end{pmatrix}.$$

The equilibrium points are at $(0,0)$, $(10,0)$, $(0,30)$, $(20,-10)$. The respective Jacobians are

$$J|_{(0,0)} = \begin{pmatrix} 10 & 0 \\ 0 & 30 \end{pmatrix}$$

$$J|_{(10,0)} = \begin{pmatrix} -10 & -10 \\ 0 & 10 \end{pmatrix}$$
$$J|_{(0,30)} = \begin{pmatrix} -20 & 0 \\ 60 & -30 \end{pmatrix}$$
$$J|_{(20,-10)} = \begin{pmatrix} -20 & -20 \\ -20 & 10 \end{pmatrix}.$$

- The equilibrium point at $(0,0)$ is a source.
- The equilibrium point at $(10,0)$ is a saddle.
- The equilibrium point at $(0,30)$ is a nodal sink.
- The equilibrium point at $(20,-10)$ is a spiral sink.

5.1, Problem 18

(a) If $a < 0$, then $\frac{dx}{dt} = x^2 - a$ is always greater than zero, so $\frac{dx}{dt}$ is never zero.

(b) If $a > 0$, then $\frac{dx}{dt} = x^2 - a$ is zero when $x = \pm\sqrt{a}$. Since $x^2 + 1$ is always greater than zero, $\frac{dy}{dt} = 0$ only when $y = 0$, so there are two equilibrium points at $x = \pm\sqrt{a}, y = 0$.

(c) If $a = 0$, then the only equilibrium point is at $(0,0)$, as $\frac{dx}{dt} = 0$ only when $x = 0$ and $\frac{dy}{dt} = 0$ only when $y = 0$.

(d) The linearization at $a = 0$ is

$$J|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

The eigenvalues are $\lambda = 0$ and $\lambda = -1$.

5.1, Problem 20

(a) The equilibrium point for y occurs at $y = a$, so for $a > 0$ we have equilibrium points at $\left(-\sqrt{a}, a\right), \left(\sqrt{a}, a\right)$; if $a = 0$, then the equilibrium point is at $(0,0)$, and if $a < 0$, there is no equilibrium point.

(b) The bifurcation occurs at $a = 0$.

(c) For $a < 0$, there are no equilibrium points, and at $a = 0$, there is a line of repelling fixed points. At $\left(-\sqrt{a}, a\right)$, there is a saddle point, and at $\left(\sqrt{a}, a\right)$ there is either a spiral source (as depicted below) or a source, depending on the value of a.