## Problem 1

Let $V$ be a vector space and suppose $\{W_i\}$ is a family of subspaces of $V$.

(i) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of $V$ contained in every $W_i$.

**Proof:** We will show that (a) $\bigcap_{i\in I} W_i$ is a subspace of $V$, and (b) there is is no larger subspace of $V$ contained within every $W_i$.

(a) Let $v_i, v_j \in \bigcap_{i\in I} W_i$, $\alpha, \beta \in \mathbb{F}$. We want to show that $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$. Since $v_i \in \bigcap_{i\in I} W_i$, $v_i \in W_i$ for some $W_i$, and $v_j \in W_j$ for some $W_j$. Additionally, WLOG, $v_j \in W_i$, as both $v_i$ and $v_j$ are contained within their intersection. Therefore, $\alpha v_i + \beta v_j \in W_i$, so $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$.

(b) Suppose there is a subspace $U$ of $V$ such that every $W_i$ is contained in $U$, and $U \supset \bigcap_{i\in I} W_i$.

(ii) Show that

$$\sum_{i\in I} W_i := \left\{ \sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite} \right\}$$

is the smallest subspace containing each $W_i$.

## Problem 9

Given any function $f : [0,1] \to \mathbb{C}$, we define

$$N(f) := \sup_{x\neq y, x,y\in[0,1]} \frac{|f(x) - f(y)|}{|x-y|}$$

and

$$\|f\|_\Lambda := |f(0)| + N(f).$$

Moreover, set

$$\Lambda[0,1] := \{f : [0,1] \to \mathbb{C} \mid \|f\|_\Lambda < \infty\}$$

(i) Show that $\Lambda[0,1]$ is precisely the set of Lipschitz continuous functions on $[0,1]$.

(ii) Verify that $\Lambda[0,1]$ is a vector space with norm $\|f\|_\Lambda$, which is the Lipschitz norm.

(iii) Show that $\|f\|_u \leq \|f\|_\Lambda$ for every $f : [0,1] \to \mathbb{R}$.

## Problem 10

Let $p$ be a seminorm on a vector space $V$.

(i) Show that $N_p := \{w \in V \mid p(w) = 0\}$ is a subspace of $V$.

**Proof:** Let $v, w \in N_p$. Then, $p(v) = 0$ and $p(w) = 0$. Since $p$ is a seminorm, for $\alpha, \beta \in \mathbb{F}$, we have:

$$\begin{aligned} p(\alpha v + \beta w) &\leq p(\alpha v) + p(\beta w) \\ &= |\alpha| p(v) + |\beta| p(w) \\ &= 0. \end{aligned}$$

Since $p$ is definitionally non-negative, $p(\alpha v + \beta w) = 0$. Therefore, $N_p$ is a vector space.

(ii) We form the quotient vector space $V/N_p$. Show that

$$\|[v]_{N_p}\|_p := p(v)$$

defines a norm on $V/N_p$.

(iii) If $(E, \|\cdot\|)$ is a normed space and $T : V \to E$ is a linear map, show that $p(v) := \|T(v)\|$ is a seminorm on $V$. In this case, what is $N_p$.