Complex Analysis

Analyticity and Path-Independence in the Complex Plane

Baby's First Complex Function Theory

We are interested in functions of the form $f(z)$, where $z = x + iy$ is some complex number. Note that this is specifically different from a function $g\colon \mathbb{R}^2 \to \Omega$ for some domain Ω; in the latter case, we have independent variables x and y, while in the former case, we must express $z = x + iy$.

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z)\,dz &= \oint_C w(z)\,(dx + i dy)\\ &= \oint_C w(z)\,dx + i\oint_C w(z)\,dy.\end{aligned}$$

Taking $A_x = w(z)$ and $A_y = iw(z)$, we have

$$= \oint_C \mathbf{A}\cdot\, d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla\times\mathbf{A})\,d\mathbf{a},$$

and when this integral is zero. Note that $(\nabla\times\mathbf{A})\cdot\hat{n} = 0$, so $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$.

Note that we can take

$$w(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$.

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

Theorem (Cauchy–Riemann Equations)**:**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that w is analytic,[1] which leads to Cauchy's theorem.

Theorem (Cauchy's Theorem)**:** If C is a simple closed curve in a simply connected region, then w is analytic if and only if

$$\oint_C w(z)\,dz = 0. \tag{†}$$

Fact. The function $w(z)$ is analytic inside the simply connected region R if any of these hold:

- w satisfies the Cauchy–Riemann equations;

[1] Equal to its Taylor series, also holomorphic.

- $w'(z)$ is unique and exists;
- $\frac{\partial w}{\partial \overline{z}} = 0$.
- w can be expanded as $w(z) = \sum_{n\geqslant 0} c_n(z-a)^n$, convergent on some open neighborhood of a for each a on its domain;[II]
- $w(z)$ is path-independent everywhere in R: $\oint_C w(z)\,dz = 0$.

Example. Considering $w(z) = z$, we have $u = x$ and $v = y$, so it satisfies the Cauchy–Riemann equations. However, neither $\operatorname{Re}(z)$ nor $\operatorname{Im}(z)$ are analytic, and neither is $\overline{z} = x - iy$.

Remark: Whenever we say "analytic at p," we mean "analytic in a neighborhood of p."

Note that since $\mathbb{C}$ is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of $\mathbb{C}$, by adjoining a point $\{\infty\}$, $\mathbb{C}^* = \mathbb{C}\cup\{\infty\}$. This compactified $\mathbb{C}^*$ is often represented as a unit sphere with the north pole, determined by $(0,0,1)$, is the point at infinity. The correspondence between $\mathbb{C}^*\setminus\{\infty\}$ and $\mathbb{C}$ is evaluated via stereographic projection.

We define $\frac{z}{\infty} = 0$ and $\frac{z}{0} = \infty$ for any $z \neq 0, \infty$. The correspondence between $z = x + iy$ in the plane to Z on the Riemann sphere with $\mathbb{R}^3$ coordinates (ξ_1, ξ_2, ξ_3) is

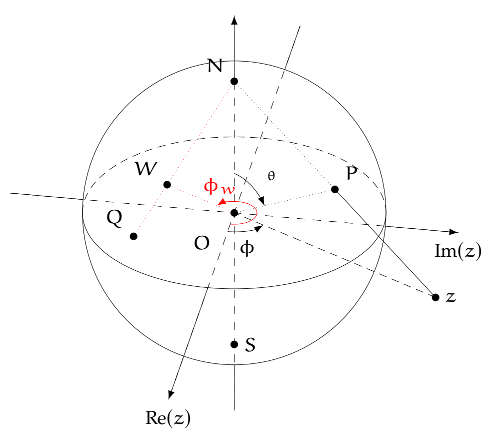
$$\begin{aligned}\xi_1 &= \frac{2\operatorname{Re}(z)}{|z|^2+1}\\ \xi_2 &= \frac{2\operatorname{Im}(z)}{|z|^2+1}\\ \xi_3 &= \frac{|z|^2-1}{|z|^2+1}.\end{aligned}$$

Inverting, we may find

$$\begin{aligned}x &= \frac{\xi_1}{1-\xi_3}\\ y &= \frac{\xi_2}{1-\xi_3},\end{aligned}$$

and with polar coordinates,

$$z = \cot(\theta/2)e^{i\phi}.$$



To determine analyticity at ∞, we set $\zeta = \frac{1}{z}$, and analyze the analyticity of $\tilde{w}(\zeta) = w(1/z)$ at 0.

[II]This is technically the real definition of analytic for the case when we're dealing with a function with domain $\mathbb{R}$.

Cauchy's Integral Formula

Consider the function $w(z) = c/z$, integrated around a circle of radius R. Then, writing $z = Re^{i\varphi}$, we get

$$\begin{aligned}\oint_{\Gamma} w(z)\,dz &= C\int_0^{2\pi} \frac{e^{-i\varphi}}{R}\underbrace{iRe^{i\varphi}\,d\varphi}_{dz}\\ &= ic\int_0^{2\pi} d\varphi\\ &= 2\pi ic.\end{aligned}$$

If our contour C runs around our singularity at $z = 0$ a total of n times, then we pick up a factor of n.

Now, when we consider

$$I = \oint_C \frac{dz}{z^n},$$

this integral actually yields 0 for any $n \neq 1$, despite the fact that 0 is a singularity for $f(z) = \frac{1}{z^n}$. This 0 is not a reflection of Cauchy's integral theorem, but of the fact that

$$z^{-n} = \frac{d}{dz}\left(\frac{z^{-n+1}}{n+1}\right),$$

meaning that z^{-n} is an exact differential, so integrating along a closed curve yields zero change. However, $\frac{1}{z} = \frac{d}{dz}(\ln z)$ may be an exact differential, but for complex z, $\ln z = \ln|z| + i\arg(z) = \ln r + i\varphi$. This yields

$$\begin{aligned}\oint_C \frac{c}{z}dz &= c\oint_C d(\ln z)\\ &= c(i(\varphi + 2\pi) - \varphi)\\ &= 2\pi ic.\end{aligned}$$

Ultimately, what this shows is that when we integrate any analytic function $f(\zeta)$ along a closed contour with a singularity at z, only the coefficient on $\frac{1}{\zeta - z}$ will remain. This coefficient is known as the residue at 0.

Theorem (Cauchy's Integral Formula)**:** If w is analytic in a simply connected region and C is a closed contour winding once around a point z in the region, then

$$w(z) = \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta. \tag{**}$$

Furthermore, this shows that any once-differentiable function is infinitely differentiable, as by differentiating under the integral sign, we get

$$\frac{d^n w}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Example (Deriving Liouville's Theorem)**.** Consider a circle C centered at radius r centered at at z, $\zeta - z = Re^{i\varphi}$. We take $d\zeta = iRe^{i\varphi}\,d\varphi$, and taking derivatives, we have

$$w'(z) = \frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi.$$

If w is bounded — i.e., $|w(z)| \leqslant M$ for all z in a given region — then

$$|w'(z)| = \left|\frac{1}{2\pi R}\int_0^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi\right|$$

$$\begin{aligned} &\leqslant \frac{1}{2\pi R}\int_{0}^{2\pi}\left|w\left(z+Re^{i\varphi}\right)\right|\,d\varphi \\ &\leqslant \frac{M}{R} \end{aligned}$$

for all R within the analytic region.

In the case where w is entire (i.e., analytic on $\mathbb{C}$), then this inequality holds for all $R \to \infty$. Thus, $|w'(z)| = 0$ for all z, meaning that w is constant.

This is known as Liouville's theorem — every bounded entire function is constant. This can be used to prove the fundamental theorem of algebra.

What Liouville's theorem tells us is that any nontrivial behavior will emerge from a function's singularities.

Singularities and Branches

To understand nontrivial behavior on the complex plane, we need to understand singularities. This will require us to develop understanding of Laurent series.

Taylor Series

We want to integrate $w(z)$ around some point a in an analytic region of $w(z)$. This yields the form

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)-(z-a)}\,d\zeta \\ &= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)\left(1-\frac{z-a}{\zeta-a}\right)}\,d\zeta. \end{aligned} \tag{‡}$$

Since ζ is on the contour and z is in the contour, $\left|\frac{z-a}{\zeta-a}\right| < 1$, we may expand as a geometric series. Thus, we get

$$= \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)}\left(\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^{n}\right)d\zeta.$$

Since the series is uniformly convergent, we are allowed to exchange sum and integral, yielding

$$\begin{aligned} &= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right)}_{=c_n}(z-a)^{n} \\ &= \sum_{n=0}^{\infty}c_n(z-a)^{n}, \end{aligned}$$

where

$$c_n = \frac{1}{n!}\left.\frac{d^n w}{dz^n}\right|_{z=a}.$$

If our Taylor series reduces to a known series on the real axis, we find this very desirable. We say this is a type of analytic continuation from the real axis to the complex plane. For example,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is an analytic continuation of e^x.

However, more interestingly,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for all $s > 1$. However, we have also shown that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

converges for complex s for all real part greater than 1. Since values of this integral agree with the series representation of $\zeta(s)$ on real axis, we have that this is an analytic continuation of $\zeta(s)$ to the subset of $\mathbb{C}$ defined by $\mathrm{Re}(s) > 1$.

Laurent Series

Now, what happens if, at (‡), we have $\left|\frac{z-a}{\zeta-a}\right| > 1$. The series as constructed would not converge, but what if we have a series that converges everywhere *outside* C? This would entail an expansion in reciprocal integer powers of $z - a$. This yields

$$\begin{aligned}
w(z) &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(z-a)\left(1 - \frac{\zeta-a}{z-a}\right)}\, d\zeta \\
&= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{z-a} \left(\sum_{n=0}^{\infty} \left(\frac{\zeta-a}{z-a}\right)^n\right) d\zeta \\
&= -\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C w(\zeta-a)^n\, d\zeta\right) \frac{1}{(z-a)^{n+1}} \\
&= \sum_{n=1}^{\infty} \underbrace{\left(-\frac{1}{2\pi i} \oint_C w(\zeta-a)^{n-1}\, d\zeta\right)}_{=c_{-n}} \frac{1}{(z-a)^n} \\
&= \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}
\end{aligned}$$

Note that this series has a singularity at $z = a$, but since our series is only defined outside a particular region, that doesn't matter. We call a series in reciprocal powers a Laurent series. More specifically, Laurent series may include expansions in negative powers as well as positive powers.

Example (Annuli)**.** If we have a point a, we want to surround a by a special contour to apply Cauchy's integral formula.

In particular, for any z in the annulus, we get

$$w(z) = \frac{1}{2\pi i} \oint_{c_1 - c_2} \frac{w(\zeta)}{\zeta - z}\, d\zeta$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\oint_{c_1}\frac{w(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{c_2}\frac{w(\zeta)}{\zeta - z}\,d\zeta\\
&= \sum_{n=-\infty}^{\infty} c_n (z-a)^n\\
&= c_0 + \sum_{n=1}^{\infty}\left(c_{-n}(z-a)^n + c_n(z-a)^n\right).
\end{aligned}$$

Example. Consider the function

$$\begin{aligned}
w(z) &= \frac{1}{z^2+z-2}\\
&= \frac{1}{(z-1)(z+2)}\\
&= \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).
\end{aligned}$$

Now, we have three regions to expand w in.

- If $|z| < 1$, then our series is in both z^n and z^n.
- If $1 < |z| < 2$, then one of our series is going to in $\frac{1}{z^n}$ and one is in z^n.
- If $|z| > 2$, then both of our series are in the form of $\frac{1}{z^n}$ and $\frac{1}{z^n}$

Via tedious, heavily error-prone calculations, we find that

$$\begin{aligned}
w_1(z) &= -\frac{1}{3}\sum_{n=0}^{\infty}\left(1 + (-1)^n\left(\frac{1}{2}\right)^{n+1}\right)z^n\\
w_2(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}} + \left(-\frac{1}{2}\right)^{n+1} z^n\right)\\
w_3(z) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(1 - (-2)^n\right)\frac{1}{z^{n+1}}.
\end{aligned}$$

Sewing all of w_1, w_2, w_3 together, then we get a full series representation of $w(z)$.

Definition. If $w(z)$ is a function that can be written as $w(z) = (z-a)^n g(a)$, where $g(a) \neq 0$, then we say w has an n-th order zero at $z = a$. If $n = 1$, then we say w has a simple zero at a.

Similarly, if we can write

$$w(z) = \frac{g(a)}{(z-a)^n}$$

with $g(a) \neq 0$, then we say w has a pole of order n at a. If $n = 1$, then we say w has a simple pole at a.

There are three types of isolated singularities (i.e., isolated points where $w(z)$ is not defined).

Definition. Let w be an analytic function with isolated singularity at a.

- If w remains bounded in any neighborhood of a, then it must be the case that $c_{-n} = 0$ for all $n > 1$, so the Laurent series is a pure Taylor expansion. We say $z = a$ is a removable singularity.

For instance, the function

$$\frac{\sin(z-a)}{z-a} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-a)^{2n}}{(2n+1)!}$$

has a removable singularity at $z = a$.

- If not all the c_{-n} are equal to zero, but there is a largest $n > 0$ such that c_{-n} is in the Laurent series expansion, then we say a is an n-th order pole. If $n = 1$, we say a is a simple pole.
- If there is no largest value of n such that c_{-n} is in the Laurent series — i.e., that $c_{-n} \neq 0$ for all n — then we say that a is an essential singularity.

  One of the most important facts about an essential singularity is that the behavior is path dependent. For instance,

  $$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

  has an essential singularity at $z = 0$. We see that $e^{1/z}$ diverges as $z \to 0$ along the positive real axis, but if $z \to 0$ along the negative real axis, we get $e^{1/z} \to 0$.

Singularities can also occur at ∞, which occurs when $w(1/z)$ has a singularity at 0.

Multivalued Function

Consider the function

$$\begin{aligned} w(z) &= z^2 \\ &= \underbrace{\left(x^2 - y^2\right)}_{u(x,y)} + i\underbrace{(2xy)}_{v(x,y)} \\ &= r^2 e^{2i\varphi}. \end{aligned}$$

Note that if we take a path around the origin going around by an angle of 2π, then the resulting path goes around twice. Note that this means the lines φ and $\varphi + \pi$ map to the same point in the w plane.

This isn't such a big deal in and of itself, but if we take $w(z) = z^{1/2}$, we get an issue. Instead of w being a two-to-one function, we now have w is a one-to-two function. This is an implicit problem in $\mathbb{R}$ with the function $w(x) = \sqrt{x}$, which we resolve by taking the "positive" square root. This is known as choosing a branch.

We have to do something similar in the complex plane. Note that if we go around by an angle of 2π in the z plane, then we only go around by an angle of π in the w-plane. As we keep going around the plane, we jump from branch to branch, which brings issues of continuity.

To resolve this, we create a "branch cut" that contours are not allowed to cross.

Example. The most common branch cut is to start from the branch point at $z = 0$, in the case of $w(z) = z^{1/2}$ or $w(z) = \ln(z)$, and extend along the real axis, meaning our branch cut is $(-\infty, 0]$.

This principal branch restricts *output* values of φ to $-\pi < \varphi \leqslant \pi$.

For instance, above the cut, we have $\varphi = \pi$, and below the branch cut, we have $\varphi = -\pi$, meaning we have

$$\sqrt{z} = \sqrt{r}e^{i\pi/2} \qquad \varphi \to \pi$$

$$\begin{aligned} &= i\sqrt{r} \\ \sqrt{z} &= \sqrt{r}e^{-i\pi/2} && \varphi \to -\pi \\ &= -i\sqrt{r}. \end{aligned}$$

This is why the branch cut "causes" a discontinuity across the branch, but in $\mathbb{C} \setminus (-\infty, 0]$.

Now, if we have

$$\begin{aligned} \sqrt{z_1}\sqrt{z_2} &= \left(r_1 e^{i\varphi_1}\right)^{1/2}\left(r_2 e^{i\varphi_2}\right)^{1/2} \\ &= \sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2}. \end{aligned}$$

However, if we want to calculate $\sqrt{z_1 z_2}$, and if $|\varphi_1 + \varphi_2| > \pi <$ then our product $z_1 z_2$ crosses the branch cut, and our discontinuity requires $\varphi_1 + \varphi_2$ to be converted to $\varphi_1 + \varphi_2 \pm 2\pi$ so as to bring the angle sum back into the principal branch. This means we have

$$\begin{aligned} \sqrt{z_1 z_2} &= \left(r_1 r_2 e^{i(\varphi_1+\varphi_2)/2}\right) \\ &= \begin{cases} \sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| \leqslant \pi \\ -\sqrt{r_1 r_2}e^{i(\varphi_1+\varphi_2)/2} & |\varphi_1 + \varphi_2| > \pi \end{cases}. \end{aligned}$$

Example. Now, if we have $z_1 = 2e^{i(3\pi/4)}$ and $z_2 = e^{i(\pi/2)}$, then we have

$$\begin{aligned} \sqrt{z_1} &= \sqrt{2}e^{i3(\pi/8)} \\ \sqrt{z_2} &= e^{i(\pi/4)}. \end{aligned}$$

Note that if we take $\sqrt{z_1 z_2}$, then the argument of $z_1 z_2$ is $5\pi/4$, so we have to change our argument to $-3\pi/4$ to return to the principal branch before we may calculate the square root. This gives

$$\begin{aligned} \sqrt{z_1 z_2} &= \sqrt{2e^{-i(3\pi/4)}} \\ &= \sqrt{2}e^{-i\pi+i(5\pi/8)} \\ &= -\sqrt{2}e^{i(5\pi/8)} \\ &= -\sqrt{z_1}\sqrt{z_2}. \end{aligned}$$

Now, it is possible to have a branch point at ∞, by determining if $w\left(\frac{1}{z}\right)$ has a branch point at zero. For instance, if $w = z^{1/2}$, this gives

$$\begin{aligned} w\left(\frac{1}{z}\right) &= \frac{1}{\zeta^{1/2}} \\ &= \frac{1}{\sqrt{r}}e^{-i\varphi/2}, \end{aligned}$$

which has the multivalued behavior around the origin. Thus, $z = \infty$ is a branch point for z, and we consider the $(-\infty, 0]$ branch cut that connects the branch points at 0 and ∞.

Example. Consider

$$w(z) = \sqrt{(z-a)(z-b)}.$$

where $a, b \in \mathbb{R}$ with $a < b$. We expect the only finite branch points to be a and b. Introducing polar coordinates, we have

$$r_1 e^{i\varphi_1} = z - a$$

$$r_2 e^{i\varphi_2} = z - b,$$

giving

$$w(z) = \sqrt{r_1 r_2} e^{i\varphi_1} e^{i\varphi_2}.$$

Closed contours around *either* a or b are double-valued. However, if our closed contour goes around *both* a and b, then both φ_1 and φ_2 add up to 2π, meaning we don't have the multivalued behavior.

Now, to select our branch cut, we need to find out if the point at infinity is a branch point. We take $\zeta = \frac{1}{z}$, and we have

$$w(\zeta) = \frac{1}{\zeta}\sqrt{(1 - a\zeta)(1 - b\zeta)},$$

which blows up at ∞, but only takes a singular value.[III]

In general, $z^{1/m}$ for integral m will require m branch cuts.

Example. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{xe^{ikx}}{\sqrt{x^2 + a^2}}\, dx.$$

This is a hard integral to evaluate. To resolve this, we extend the integrand to the complex plane, and invoke Cauchy's theorem to deform the contour.

Note that $\sqrt{x^2 + a^2}$ is multivalued, with branch points at $x = \pm ia$. We choose the branch cut such that our integration contour does not cross the branch cut — i.e., from $-ia$ to ∞ to ia.

Now, we may deform the contour so as to closely wrap around the branch cut from ia to ∞. Remembering the sign discontinuity over the branch cut, this gives the integral

$$\begin{aligned}
\int_{i\infty}^{i\infty} \frac{ze^{ikz}}{\sqrt{x^2 + a^2}}\, dz &= \int_{i\infty}^{ia} \frac{ze^{ikz}}{-i\sqrt{x^2 + a^2}}\, dz + \int_{-a}^{\infty} \frac{ze^{ikz}}{i\sqrt{z^2 + a^2}}\, dz \\
&= 2\int_{ia}^{i\infty} \frac{ze^{ikz}}{i\sqrt{z^2 + a^2}}\, dz \\
&= 2\int_{a}^{\infty} \frac{ye^{-ky}}{\sqrt{y^2 - a^2}}\, dy && z = iy \\
&= 2aK_1(ka) \\
&\sim e^{-ka}
\end{aligned}$$

Here, K_1 refers to the modified Bessel function.

Logarithms

In the complex plane, we say

$$\begin{aligned}
\ln z &= \ln\left(re^{i\varphi}\right) \\
&= \ln r + i\varphi \\
&= \ln|z| + i\arg(z).
\end{aligned}$$

[III] Alternatively, we may see that a positively-oriented contour that surrounds both a and b is a negatively-oriented contour around ∞. Since such a contour is valid, ∞ is not a branch point.

Unfortunately, this $\ln z$ is a multivalued function — a very multivalued one indeed. This yields many branch points, including 0 and ∞:

$$\ln(1/\zeta) = -\ln(\zeta).$$

However, we choose the principal branch, $\pi < \varphi \leqslant \pi$, giving

$$\operatorname{Ln} z = \operatorname{Ln}|z| + i \operatorname{Arg}(z).$$

Example. Consider $\ln(z_1 z_1)$ and $\operatorname{Ln}(z_1 z_2)$. If we have

$$\begin{aligned} z_1 &= 1 + i \\ z_2 &= i, \end{aligned}$$

then

$$\begin{aligned} \arg(z_1) &= \pi/4 \\ \arg(z_2) &= \pi/2, \end{aligned}$$

so

$$\begin{aligned} \arg(z_1 z_2) &= 3\pi/4 \\ &= \arg(z_1) + \arg(z_2) \\ &= \operatorname{Arg}(z_1 z_2). \end{aligned}$$

However, if $z_1 = z_2 = -1$, then

$$\begin{aligned} \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\ &= 2\pi \\ \operatorname{Arg}(z_1 z_2) &= \operatorname{Arg}(1) \\ &= 0. \end{aligned}$$

Thus, we get that $\operatorname{Ln}(z_1 z_2) \neq \operatorname{Ln}(z_1) + \operatorname{Ln}(z_2)$.

Example (Logarithms vs Inverse Trig)**.** Here, we will derive $\arctan(z)$ in terms of the complex logarithm.

Recall that

$$\begin{aligned} \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \end{aligned}$$

so we have

$$\begin{aligned} z &= \tan(w) \\ &= -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}, \end{aligned}$$

which after much tedious, error-prone symbolic manipulation, gives

$$e^{2iw} = \frac{i - z}{i + z}.$$

Thus, we have

$$w = \arctan(z)$$

$$= \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right).$$

Note that since ln has branch points at 0 and ∞, $\ln\left(\frac{i-z}{i+z}\right)$ has branch points when $z = \pm i$.

Now, we must choose a branch cut. Specifically, we want our branch cut to continue the real $\arctan(x)$. We dub this $\operatorname{Arctan}(x)$. Along the real axis, we have

$$\begin{aligned}
\operatorname{Arctan}(x) &= \frac{1}{2i}\operatorname{Ln}\left(\frac{i-x}{i+x}\right)\\
&= \frac{1}{2i}\left(\operatorname{Ln}\left|\frac{i-x}{i+x}\right| + i\operatorname{Arg}\left(\frac{i-x}{i+x}\right)\right)\\
&= \frac{1}{2}\operatorname{Arg}\left(\frac{i-x}{i+x}\right).
\end{aligned}$$

The principal values are from $-\pi$ to π, so the output of $\operatorname{Arctan}(x)$ ranges from $-\pi/2$ to $\pi/2$.

Conformal Maps

A conformal map is a special type of map $w\colon \mathbb{C} \to \mathbb{C}$ that "preserves angles." If, in z, we map curves whose intersections are at some angle φ, then the image of those curves also intersect at the angle φ.

Example (Our First Conformal Map)**.** Consider the map

$$\begin{aligned}
w(z) &= z^2\\
&= \left(x^2 - y^2\right) + i(2xy)\\
&= u(x,y) + iv(x,y).
\end{aligned}$$

Examining the line elements in the z and w planes, we have

$$\begin{aligned}
ds^2 &= du^2 + dv^2\\
&= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2\\
&= \left(\frac{\partial u}{\partial x}dx - \frac{\partial v}{\partial x}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy\right)^2\\
&= \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)\left(dx^2 + dy^2\right)\\
&= \left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)\left(dx^2 + dy^2\right)\\
&= 4\left(x^2 + y^2\right)\left(dx^2 + dy^2\right)
\end{aligned}$$

Note that dx^2 and dy^2 have identical scale factors. Since angles are determined by the ratio of dx and dy, it is the case that *all* angles are preserved. This is what is meant by a conformal map.

Example (Analyticity and Conformality)**.** Consider an analytic function $w(z)$, with its Taylor expansion about z_0.

$$w(z) = w(z_0) + w'(z_0)(z - z_0) + \cdots .$$

For a very small $\xi = z - z_0$, we may truncate it into first order, and place into polar form

$$w(z) - w(z_0) = w'(z_0)\xi$$

$$= |w'(z_0)|e^{i\alpha_0}\xi.$$

Moving from z to w, we get a magnification (or shrinkage) by $|w'(z_0)|$ and a rotation by α_0.

Since, close to z_0, $\xi_1 = z_1 - z_0$ and $\xi_2 = z_2 - z_0$ are magnified by (effectively) the same amount, and rotated by (effectively) the same amount, conformality is established.

Definition. A conformal map is an analytic function $w(z)$ defined on a domain Ω such that $w'(z_0) \neq 0$ for all $z_0 \in \Omega$.

Example (Möbius Transformations)**.** A Möbius transformation is a fractional linear transformation of the form

$$w(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. We can calculate $w'(z)$ to be

$$w'(z) = \frac{ad-bc}{(cz+d)^2}.$$

Since $w(z)$ is conformal, it is invertible, so

$$\begin{aligned} w^{-1}(z) &= z(w) \\ &= \frac{dw-b}{-cw+a}. \end{aligned}$$

The Möbius transformations include ∞, as we have $w(\infty) = \frac{a}{c}$, meaning that it is an automorphism of the Riemann sphere. Note that because of the constraint, we only need three numbers to specify a Möbius transformation.

Consider the Möbius transformation

$$w(z) = \frac{z-i}{z+i}.$$

We let $z_1 = -1$, $z_2 = 1$, and $z_3 = \infty$. Then, we have

$$\begin{aligned} w(z_2) &= \frac{-1-i}{-1+i} \\ &= \frac{2i}{2} \\ &= i. \end{aligned}$$

Similarly, this gives $w(z_3) = 1$. After a bit more playing, we can find that this is a map of the (closed) upper half-plane to the (closed) unit disk, $\mathbb{D}$.

Now, if we look at the "ribbon" between the real axis and the line $\operatorname{Im}(z) = i$, we see that it maps to the region

$$S = \mathbb{D} \setminus \left\{ z \;\middle|\; \left|z - \frac{1}{2}\right| \leqslant \frac{1}{2} \right\}.$$

Example. Consider the map $w(z) = e^z$. This gives

$$\begin{aligned} w(z) &= e^x e^{iy} \\ &= \rho e^{i\beta}. \end{aligned}$$

This sends curves of constant y to curves of constant argument, and maps curves of constant x to circles of constant radius.

Complex Potentials

Consider the analytic function

$$\Omega(z) = \Phi(x, y) + i\Psi(x, y).$$

We know that

$$\begin{aligned}\frac{\partial\Phi}{\partial x} &= \frac{\partial\Psi}{\partial y}\\ \frac{\partial\Phi}{\partial y} &= -\frac{\partial\Psi}{\partial x}.\end{aligned}$$

Thus, we separate to get

$$\begin{aligned}\frac{\partial^2\Phi}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial\Psi}{\partial y}\\ &= \frac{\partial}{\partial y}\frac{\partial\Psi}{\partial x}\\ &= -\frac{\partial^2\Phi}{\partial y^2},\end{aligned}$$

so

$$\begin{aligned}\nabla^2\Phi &= 0\\ \nabla^2\Psi &= 0.\end{aligned}$$

The converse is also true — if there is some real harmonic function $\Phi(x, y)$, there is a conjugate harmonic function $\Psi(x, y)$ such that $\Omega(z) = \Phi(x, y) + i\Psi(x, y)$ is analytic.

If Ω is analytic, then Φ and Ψ must satisfy the Cauchy–Riemann equations, meaning that

$$\begin{aligned}\Psi(x, y) &= \int \frac{\partial\Psi}{\partial y}\,dy + \frac{\partial\Psi}{\partial x}\,dx\\ &= \int \frac{\partial\Phi}{\partial x}\,dy - \frac{\partial\Phi}{\partial x}\,dx.\end{aligned}$$

For Ψ to be a proper single-valued real function, the integral must be path-independent. Using Green's theorem, we may close the path in a simply connected region, and consider it as a surface integral. This gives

$$\begin{aligned}\oint_C \frac{\partial\Phi}{\partial x}\,dy - \frac{\partial\Phi}{\partial y}\,dx &= \int_S \left(\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\Phi}{\partial y}\right)\right)dx\,dy\\ &= \int_x \left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right)dx\,dy\\ &= 0.\end{aligned}$$

We call $\Omega(z) = \Phi(x, y) + i\Psi(x, y)$ the complex potential.

This gives

$$\begin{aligned}\frac{d\Omega}{dz} &= \frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x}\\ &= \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y}\end{aligned}$$

$$\begin{aligned} &= \frac{\partial \Psi}{\partial y} + i\frac{\partial \Psi}{\partial x}. \\ &= \overline{\mathcal{E}}, \end{aligned}$$

where $\mathcal{E}$ is the complex representation of the electric field, $\mathbf{E}$. We have

$$\begin{aligned} \mathcal{E} &= \overline{\frac{\partial \Omega}{\partial z}} \\ &= \frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y}, \end{aligned}$$

with

$$E = \left|\frac{d\Omega}{dz}\right|.$$

The physics of electric fields is then determined entirely by the complex potential.

What makes harmonic functions useful is that, if there are complicated boundary conditions, we may apply a conformal map and the functions remain harmonic.

Example (Cylindrical Capacitor)**.** Consider a cylindrical capacitor with nonconcentric plates meeting at insulated point $u = 1$ and $v = 0$. The larger cylinder with radius 1 is grounded, and the smaller cylinder with radius $1/2$ is held at voltage V_0. We want to find the electric field.

We want to find $\widetilde{\Phi}(w)$ such that

$$\nabla^2\widetilde{\Phi}(u, v) = 0.$$

This domain is kind of difficult, so we will solve the problem on a simpler domain and use a conformal map. Note that from Figure 20.4 in the book, we may use the Möbius transformation

$$w(z) = \frac{z - i}{z + i}$$

to transform *to* our cylindrical capacitor *from* a two-plate infinite capacitor with one plate at $\text{Im}(z) = 1$ and one plate at $\text{Im}(z) = 0$. From physics, we know that $\Phi(x, y) = \frac{V_0 y}{d}$, where $d = 1$. Thus, the harmonic conjugate, $\Psi = -V_0 x$, gives us a complex potential of $\Phi = -iV_0 z$.

Solving

$$\frac{z - i}{z + i} = u(x, y) + iv(x, y),$$

we find

$$\begin{aligned} x(u, v) &= -\frac{2v}{(1 - u)^2 + v^2} \\ y(u, v) &= \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}. \end{aligned}$$

Now, this gives

$$\begin{aligned} \widetilde{\Phi}(u, v) &= \Phi(x(u, v), y(u, v)) \\ &= V_0\frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}. \end{aligned}$$

Example (Fluid Flow)**.** Consider fluid flow around a rock with disk of radius a; far away from the rock, we have uniform flow speed of α.

Symmetry allows us to focus only on the upper half-plane. Now, there is a conformal map in Table 20.1 of the textbook, which is the map $w(z) = z + \frac{a^2}{z} = u(x, y) + iv(x, y)$ that maps the upper half-plane to the upper half-plane. Furthermore, this map sends the boundary hugging the rock into the u-axis.

After applying the conformal map, we get the stream lines $\widetilde{\Psi}(u, v) = \beta v$, as they are streamlines of uniform horizontal flow.

Building the complex potential, we have

$$\begin{aligned}\widetilde{\Omega}(w) &= \Phi(u, v) + i\Psi(u, v)\\ \widetilde{\Omega}(w) &= \beta w,\end{aligned}$$

as we must have $\frac{d\Phi}{du} = \frac{d\Psi}{dv} = \beta$.

Mapping back into the z-plane, we have

$$\Omega(z) = \beta\left(z + \frac{a^2}{z}\right).$$

Note that as z becomes very big, the term $\frac{a^2}{z}$ goes to 0, so we must have $\beta = \alpha$.

Now, we may find the streamlines and potentials. Note that we have

$$\begin{aligned}\Phi &= \operatorname{Re}(\Omega)\\ \Psi &= \operatorname{Im}(\Omega).\end{aligned}$$

Now, we have

$$\begin{aligned}\Omega(z) &= \alpha r\left(e^{i\varphi} + \frac{a^2}{r^2}e^{-i\varphi}\right)\\ &= \alpha r\left(\cos(\varphi) + i\sin(\varphi) + \frac{a^2}{r^2}(\cos(\varphi) - i\sin(\varphi))\right).\end{aligned}$$

Taking real and imaginary parts, we have

$$\begin{aligned}\Phi &= \alpha r\left(1 + \frac{a^2}{r^2}\right)\cos(\varphi)\\ \Psi &= \alpha r\left(1 - \frac{a^2}{r^2}\right)\sin(\varphi).\end{aligned}$$

Example. Considering our conformal map

$$w(z) = z + \frac{a^2}{z}$$

again, we see that if $|z| = a$, then $|u| \leqslant 2a$. Meanwhile, if $r > a$, then

$$\begin{aligned}w(z) &= z + \frac{a^2}{z}\\ &= re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\end{aligned}$$

$$\begin{aligned}&= \left(r + \frac{a^2}{r}\right)\cos(\varphi) + i\left(r - \frac{a^2}{r}\right)\sin(\varphi)\\&= u + iv.\end{aligned}$$

This gives

$$\frac{u^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{v^2}{\left(r - \frac{a^2}{r}\right)} = 1.$$

Note that w fails to be conformal when $\frac{dw}{dz} = 0$, meaning that it fails to be conformal at $z = \pm a$.

This is occasionally used in the real world[IV] to design airfoils.

Residues

Consider a function $f(z)$ with an nth order pole. Then, f can be written as

$$f(z) = \frac{g(z)}{(z-a)^n},$$

where $g(z)$ is analytic and $g(a) \neq 0$. Recalling Cauchy's integral formula, we see that this expression for f is tantalizingly close to our desired state.

We may expand g in a Taylor series:

$$g(z) = \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!}(z-a)^m.$$

Letting C be a positively oriented contour in the analytic domain of f that encircles the singularity, we get

$$\oint_C f(z)\,dz = \sum_{m=0}^{\infty} \frac{g^{(m)(a)}}{m!} \oint_C (z-a)^{m-n}\,dz.$$

Note that if $m - n \neq -1$, then the integral on the right vanishes, so we only obtain a nonzero contribution at $m = n - 1$. Thus, we get

$$\oint_C f(z)\,dz = 2\pi i \frac{g^{(n-1)(a)}}{(n-1)!}.$$

Definition. Let $f(z)$ be an analytic function with a pole at $z = a$ with order n. We define the residue of f at a as

$$\operatorname{Res}[f(z), a] := \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}\big((z-a)^n f(z)\big).$$

This gives an alternative statement of Cauchy's integral formula, giving

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f(z), a].$$

However, when we have lots of poles for f, and C is a contour that surrounds all the poles, we may deform C such that it surrounds each pole. This gives the residue theorem.

Theorem (Residue Theorem)**:**

$$\oint_C f(z)\,dz = 2\pi i \sum_{a\in C} \operatorname{Res}[f(z), a] \tag{††}$$

Type	Method
n-th order pole	$\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right)$
simple pole	$\lim\limits_{z\to a}(z-a)f(z)$
$f=\frac{p}{q}, q(a)$ simple zero	$\dfrac{p(a)}{q'(a)}$
pole at infinity	$\lim\limits_{z\to 0}\left(-\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right)\right)$
pole at infinity, $\lim_{\lvert z\rvert\to\infty} f(z)=0$	$-\lim\limits_{\lvert z\rvert\to\infty}(zf(z))$

Table 1: Finding $\text{Res}[f(z), a]$

We can find the residue in a variety of ways.

Example. We will find the residue for $\cot(z)$ for each of the residues.

$$\begin{aligned}\text{Res}[\cot(z), n\pi] &= \lim_{z\to n\pi}(z-n\pi)\frac{\cos(z)}{\sin(z)}\\ &= (-1)^n\lim_{z\to n\pi}\frac{z-n\pi}{\sin(z)}\\ &= (-1)^n\lim_{z\to n\pi}\frac{z-n\pi}{(-1)^n\sin(z-n\pi)}\\ &= 1.\end{aligned}$$

Example. We may find

$$\begin{aligned}\text{Res}\left[\frac{z}{\sinh(z)}, in\pi\right] &= \left.\frac{z}{\frac{d}{dz}(\sinh(z))}\right|_{z=in\pi}\\ &= \frac{in\pi}{\cosh(in\pi)}\\ &= (-1)^n in\pi\end{aligned}$$

Example. Let's evaluate

$$\oint_C \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

Finding the residue at each pole, we get

$$\begin{aligned}\text{Res}[f(z),0] &= \frac{2}{3}\\ \text{Res}[f(z),-1] &= -\frac{3}{2}\\ \text{Res}[f(z),3] &= -\frac{1}{6}.\end{aligned}$$

These are evaluated using the cover-up method.

Now, we may find the integral by taking

$$\oint_{\lvert z\rvert=2} f(z)\,dz = -i\frac{5\pi}{3}.$$

[IV] I guess people do things over there.

Example. Let

$$\begin{aligned} f(z) &= \frac{1}{z^2\sinh(z)} \\ &= \frac{1}{-iz^2\sin(iz)}. \end{aligned}$$

The simple zeros of $\sinh(z)$ are at $in\pi$, so we have an order 3 pole at $z = 0$

$$\begin{aligned} \operatorname{Res}[f(z), 0] &= \frac{1}{(n-1)!}\frac{d^2}{dz^2}\left[z^3 f(z)\right]\Big|_{z=0} \\ &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z}{\sinh(z)}\right)\Big|_{z=0} \\ &= -\frac{1}{6}. \end{aligned}$$

Thus, integrating about the unit circle, we get

$$\oint_{|z|=1} = -\frac{i\pi}{3}.$$

If we were to evaluate via the Laurent series, we would have

$$\begin{aligned} \frac{1}{z^2\sinh(z)} &= \frac{1}{z^2}\left(\frac{1}{z + z^2/3 + z^5/5! + \cdots}\right) \\ &= \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right) \\ &\approx \frac{1}{z^3}\left(1 - \frac{z^2}{3!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{6z} + \cdots, \end{aligned}$$

giving a residue of $-\frac{1}{6}$.

Instead of using the contour on the unit circle, if we want to use a circle of radius 4, we get the residues at $z = \pm i\pi$. To evaluate this, we take

$$\begin{aligned} \operatorname{Res}[f(z), i\pi] &= \frac{1}{-\pi^2(-1)} \\ &= \frac{1}{\pi^2} \\ \operatorname{Res}[f(z), -i\pi] &= \frac{1}{\pi^2}. \end{aligned}$$

Evaluating the integral, we would get

$$\begin{aligned} \oint_C f(z)\,dz &= 2\pi i\left(-\frac{1}{6} + \frac{2}{\pi^2}\right) \\ &= -\frac{i\pi}{3} + \frac{4i}{\pi}. \end{aligned}$$

Example. We will now use the residue theorem to evaluate a real-valued integral. Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx.$$

Since this integral goes to zero, we will evaluate

$$I' = \oint_C \frac{1}{z^2+1}\,dz,$$

where C is a semicircle with radius r along the real axis from $-r$ to r "pointing upward," so to speak.

This gives

$$\oint_C \frac{1}{z^2+1}\,dz = \int_{C_r} f(z)\,dz + \int_{-r}^{r} f(x)\,dx,$$

which, sending r to infinity, is equal to

$$I = \int_{-\infty}^{\infty} f(x)\,dx.$$

However, since our expression $\frac{1}{z^2+1}$ has poles at i and $-i$, our semicircle gives

$$\begin{aligned}\oint_C \frac{1}{z^2+1} &= 2\pi i \operatorname{Res}[f(z), i]\\ &= 2\pi i \lim_{z\to i}\frac{1}{z+i}\\ &= 2\pi i \frac{1}{2i}\\ &= \pi.\end{aligned}$$

If we have a finite number of isolated singularities, we are always able to draw a contour that encloses all of them, which allows us to use the residue theorem.

Now, we know that we can have poles at infinity — and that any positively-oriented contour in the plane is a negatively-oriented contour around ∞. Thus, if we have a contour surrounding all our finite singularities, we get

$$\begin{aligned}\sum_i \operatorname{Res}[f(z), a_i] &= -\operatorname{Res}[f(z), \infty]\\ \operatorname{Res}[f(z),\infty] + \sum_i [f(z), a_i] &= 0,\end{aligned}$$

as we're doing the same integral, but in negative orientation about ∞ and positive orientation about our singularities.

We have

$$\operatorname{Res}[f(z),\infty] = \operatorname{Res}\left[-\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right].$$

Example. Now, recalling

$$f(z) = \frac{(z-1)(z-2)}{z(z+1)(3-z)}.$$

The residues are

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= 2/3\\ \operatorname{Res}[f(z), -1] &= -3/2\end{aligned}$$

$$\operatorname{Res}[f(z), 3] = -1/6.$$

Now, calculating the residue at infinity, we have

$$\begin{aligned}
\operatorname{Res}[f(z), \infty] &= \operatorname{Res}\left[-\frac{1}{z^2}\frac{(1/z-1)(1/z-2)}{1/z(1/z+1)(3-1/z)}, 0\right] \\
&= -\operatorname{Res}\left[\frac{(z-1)(2z-1)}{z(z+1)(3z-1)}\right] \\
&= 1.
\end{aligned}$$

Now, if $\lim_{|z|\to\infty} f(z) = 0$, then f is pure Laurent series. In that case, if there is a residue, then we find the residue by evaluating

$$\operatorname{Res}[f(z), \infty] = -\lim_{|z|\to\infty} zf(z)$$

Example. Consider functions of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where q is a higher-order polynomial than p.

If q has first-order zeros a and second-order zeros at b, then

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z-a_k} + \frac{B_k}{z-b_k} + +\frac{C_k}{(z-b_k)^2}.$$

Note that the coefficients are actually residues. This gives

$$\begin{aligned}
A_k &= \operatorname{Res}[f(z), a_k] \\
B_k &= \operatorname{Res}[f(z), b_k] \\
C_k &= \operatorname{Res}[(z-b_k)f(z), b_k].
\end{aligned}$$

For instance,

$$\frac{(z-1)(z-2)}{z(z+1)(3-z)} = \frac{2}{3}\frac{1}{z} - \frac{1}{6}\frac{1}{z-3} - \frac{3}{2}\frac{1}{z+1}.$$

Now, we may also have

$$\frac{(z-1)(z-2)}{z(z+1)^2(3-z)} = \frac{2}{3}\frac{1}{z} - \frac{1}{24}\frac{1}{z-3} - \frac{5}{8}\frac{1}{z+1} - \frac{3}{2}\frac{1}{(z+1)^2}.$$

Integrating around a Circle

We want to evaluate angular integrals of the form

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi))\, d\varphi.$$

Now, while this is a real integral over a domain, we may reformulate it about the unit circle by using the substitutions

$$\begin{aligned}
z &= e^{i\varphi} \\
d\varphi &= \frac{dz}{iz},
\end{aligned}$$

which yields

$$\begin{aligned}\sin(n\varphi) &= \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)\\ \cos(m\varphi) &= \frac{1}{2}\left(z^m + \frac{1}{z^m}\right).\end{aligned}$$

Thus, our integral becomes

$$\int_0^{2\pi} f(\sin(n\varphi), \cos(m\varphi))\,d\varphi = \oint_{|z|=1} f\left(\frac{1}{2i}\left(z^n - \frac{1}{z^n}\right), \frac{1}{2}\left(z^m + \frac{1}{z^m}\right)\right)\frac{dz}{iz}.$$

Example. Consider

$$\begin{aligned}\int_0^{2\pi} \sin^2(\varphi)\,d\varphi &= -\frac{1}{4}\oint_{|z|=1}\left(z - \frac{1}{z}\right)^2 \frac{dz}{iz}\\ &= \frac{i}{4}\oint_{|z|=1} \frac{1}{z^3}\left(z^4 - 2z^1 + 1\right)dz\\ &= -\frac{1}{2}\pi \operatorname{Res}\left[\frac{1}{z^3}\left(z^4 - 2z^1 + 1\right), 0\right].\end{aligned}$$

The residue at $z = 0$ is -2 — this can be found by dividing out by z^3.

Thus, we get the answer of

$$\int_0^{2\pi} \sin^2(\varphi)\,d\varphi = \pi.$$

Example. Using residues, we can evaluate a lot of integrals that are quite tricky on their face.

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\varphi)}{5 - 4\sin(\varphi)}\,d\varphi &= \oint_{|z|=1} \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{5 - \frac{4}{2i}\left(z - \frac{1}{z}\right)}\frac{dz}{iz}\\ &= -\oint_{|z|=1} \frac{z^4 + 1}{2z^2(2z - i)(z - 2i)}\,dz.\end{aligned}$$

Now, we have a simple pole at $i/2$, a simple pole at $2i$, and a pole of order 2 at 0. We only evaluate the residues at 0 and $i/2$. We get

$$\begin{aligned}\operatorname{Res}[f(z), 0] &= -\frac{d}{dz}\left(\frac{z^4 + 1}{2z^2(2z - i)(z - 2i)}\right)\Bigg|_{z=0}\\ &= -\frac{5i}{8}\\ \operatorname{Res}[f(z), i/2] &= \frac{17i}{24}.\end{aligned}$$

Thus, we get the result of

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\varphi)}{5 - 4\sin(\varphi)}\,d\varphi &= 2\pi i\left(-\frac{5i}{8} + \frac{17i}{24}\right)\\ &= -\frac{\pi}{6}.\end{aligned}$$

Integrating along the Real Axis

If we want to evaluate integrals along the real axis, such as

$$I = \int_{-\infty}^{\infty} f(x)\,dx,$$

we may be curious as to how we may evaluate this.

To do this, we recall that we created a contour in the upper half-plane of large enough radius r, and evaluated the residues inside the contour. We consider the contour to be equal to $C = C_r + l_r$, where l_r is along the real axis and C_r closes our contour. Thus, we get

$$\oint_C f(z)\,dz = \lim_{r\to\infty}\left(\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz\right).$$

Note that the polar coordinate Jacobian gives us the requirement that $\lim_{|z|\to\infty}|zf(z)| = 0$.

When $f(z) = \frac{p(z)}{q(z)}$, this is satisfied when q is of degree at least two more than that of p.

Example. Consider

$$\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4}\,dx = \oint_C \frac{2z+1}{z^4+5z^2+4}\,dz.$$

Factoring, we get

$$\oint_C \frac{2z+1}{z^4+5z^2+4}\,dz = \oint_C \frac{2z+1}{(z-2i)(z+2i)(z-i)(z+i)}\,dz.$$

We only care about the residues in the upper half-plane. We have residues of

$$\begin{aligned}\operatorname{Res}[f(z), 2i] &= -\frac{1}{3} + \frac{i}{12}\\ \operatorname{Res}[f(z), i] &= \frac{1}{3} - \frac{i}{6}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{2x+1}{x^4+5x^2+4}\,dx &= 2\pi i\left(\frac{1}{3} - \frac{i}{6} - \frac{1}{3} + \frac{i}{12}\right)\\ &= \frac{\pi}{6}.\end{aligned}$$

Note that if we chose our contour to be in the lower half-plane, then we would have a *negatively* oriented contour, and evaluate at the residues in the lower half-plane.

Example. Consider

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^3-i}\,dx &= \oint_C \frac{1}{z^3-i}\,dz\\ &= \oint_C \frac{1}{(z+i)\left(z-e^{i\pi/6}\right)\left(z-e^{5i\pi/6}\right)}.\end{aligned}$$

Closing C in the lower half-plane, we only need the residue at $-i$. This gives

$$\begin{aligned}\oint_C \frac{1}{z^3-i} &= -2\pi i\left(-\frac{1}{3}\right)\\ &= \frac{2\pi i}{3}.\end{aligned}$$

Consider integrals of the form

$$\int_{-\infty}^{\infty} g(x)e^{ikx}\,dx,$$

where k is real.

Now, we want to know when exactly we are allowed to "close up" the semicircle contour.

We start by assuming k is positive. Closing in the upper half-plane so as to ensure exponential decay, we have

$$\begin{aligned}\left|\int_{C_r} g(z)e^{ikz}\,dz\right| &\leqslant \int_{C_r}\left|g(z)e^{ikz}\right|\,dz\\ &= \int_0^{\pi}\left|g\left(re^{i\varphi}\right)\right|re^{-kr\sin(\varphi)}\,d\varphi.\end{aligned}$$

Since $\sin(\varphi) \geqslant 0$ on the range of integration, the integral vanishes as $r \to \infty$. Therefore, we are allowed to close up the contour whenever $|g(z)| \to 0$ as $|z| \to \infty$.

Example. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+4}\,dx.$$

This gives

$$\begin{aligned}I &= \mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+4}\,dx\right)\\ &= \mathrm{Re}\left(\oint_C \frac{e^{ikz}}{z^2+4}\,dz\right)\\ &= \oint_C \frac{e^{ikz}}{(z-2i)(z+2i)}\,dz.\end{aligned}$$

We assume $k > 0$. Then, evaluating at $2i$, we have

$$I = \frac{\pi}{2}e^{-2k}.$$

Now, if $k < 0$, we close our contour in the lower half-plane, we get

$$I = \frac{\pi}{2}e^{2k}.$$

Thus, our integral is always

$$I = \frac{\pi}{2}e^{-2|k|}.$$

Non-Circular Contours

Sometimes, semicircles don't work.

Example. Consider

$$\int_{-\infty}^{\infty}\frac{e^{bx}}{e^x+1}\,dx,$$

where $0 < b < 1$. Writing our integral, we have

$$I = \int \frac{e^{bz}}{e^z + 1}\, dz$$

This gives poles at $z = (2n+1)i\pi$, which means we cannot close this contour with a semicircular arc at ∞.

What may work in this case is by drawing a rectangular contour from $-a$ to a such that it encloses exactly one of the poles of our integrand. The vertical segments of this contour go to zero as we send $a \to \infty$. We call the segment of the contour along the line $a + 2\pi i$ to $a - 2\pi i$ as I'.

This gives

$$I + I' = \oint_C \frac{e^{bz}}{e^z + 1}\, dz.$$

Now, we constructed I' such that

$$\begin{aligned} I' &= \int_{\infty}^{-\infty} \frac{e^{b(x+2\pi i)}}{e^{x+2\pi i} + 1}\, dx \\ &= -e^{2\pi i b} \int_{-\infty}^{\infty} \frac{e^{bx}}{e^x + 1}\, dx \\ &= -e^{2\pi i b} I. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \oint_C \frac{e^{bz}}{e^z + 1}\, dz &= I\left(1 - e^{2\pi i b}\right) \\ &= 2\pi i \operatorname{Res}\left[\frac{e^{bz}}{e^z + 1}, i\pi\right], \end{aligned}$$

giving

$$I = \frac{\pi}{\sin(\pi b)}.$$

Example. We want to evaluate

$$\int_0^\infty \cos\left(x^2\right)\, dx$$
$$\int_0^\infty \sin\left(x^2\right)\, dx.$$

To evaluate this, we draw a slice-shaped contour going along the real axis and returning to 0 along $z = re^{i\pi/4}$. Therefore, we evaluate

$$\begin{aligned} \oint_C e^{iz^2}\, dz &= \int_0^\infty e^{ix^2}\, dx + 0 + \int_\infty^0 e^{i\left(re^{i\pi/4}\right)^2} e^{i\pi/4}\, dr \\ &= \int_0^\infty e^{ix^2}\, dx + 0 + \int_\infty^0 e^{-r^2} e^{i\pi/4}\, dr. \end{aligned}$$

Thus, we get

$$\begin{aligned} \int_0^\infty \cos\left(x^2\right)\, dx &= \int_0^\infty \sin\left(x^2\right)\, dx \\ &= \frac{1}{2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Integrating with Branch Cuts

When we're integrating with residues, branch cuts are a feature rather than a bug.

Example. Consider the integral

$$I = \int_{0}^{\infty} \frac{\sqrt{x}}{1+x^3}\,dx.$$

We need a branch cut to avoid the multivalued behavior. Our poles are at $e^{i\pi/3}, -1, e^{-i\pi/3}$. Since our integral is along the real axis, we take our branch cut along the domain $[0, \infty]$.

We draw our contour of radius R by hugging the branch without crossing it, with a small circle of radius ε just outside 0. This gives the integral

$$\oint \frac{\sqrt{z}}{1+z^3}\,dz = \int_{0}^{\infty} \frac{\sqrt{z}}{1+z^3}\,dx + \int_{C_R} \frac{\sqrt{z}}{1+z^3}\,dz + \int_{\infty}^{0} \frac{\sqrt{z}}{1+z^3}\,dz + \int_{C_\varepsilon} \frac{\sqrt{z}}{1+z^3}\,dz.$$

Note that since $\lim_{|z|\to\infty} |zf(z)| = 0$, and $\lim_{|z|\to 0} |zf(z)| = 0$, our integrals along C_R and C_ε go to zero, giving the integral

$$\begin{aligned}
I' &= \int_{\infty}^{0} \frac{\sqrt{z}}{1+z^3}\,dz \\
&= \int_{\infty}^{0} \frac{\sqrt{e^{2i\pi}x}}{1+(e^{2i\pi}x)^3}\,dx \\
&= \int_{0}^{\infty} \frac{\sqrt{x}}{1+x^3}\,dx \\
&= I.
\end{aligned}$$

Thus,

$$\oint \frac{\sqrt{z}}{1+z^3}\,dz = 2I.$$

Evaluating the residues, we have

$$\begin{aligned}
\operatorname{Res}\left[f(z), e^{i\pi/3}\right] &= \lim_{z\to e^{i\pi/3}} \frac{\sqrt{z}}{3z^2} \\
&= -\frac{i}{3} \\
\operatorname{Res}[f(z), -1] &= \lim_{z\to -1} \frac{\sqrt{z}}{3z^2} \\
&= \frac{i}{3} \\
\operatorname{Res}\left[f(z), e^{5\pi i/3}\right] &= -\frac{i}{3},
\end{aligned}$$

giving the solution of

$$\begin{aligned}
I &= \frac{1}{2} 2\pi i\left(-\frac{i}{3}\right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

Example. To evaluate

$$I = \int_0^\infty \frac{1}{1+x^3}\,dx,$$

we start by evaluating

$$\int_0^\infty \frac{\ln(x)}{1+x^3}\,dx$$

with the branch cut along the real axis. Using the keyhole contour in the previous example, we have that C_R and C_ε contribute nothing, and ln picks up a phase of $2\pi i$, so that

$$\begin{aligned}\oint_C \frac{\ln(z)}{1+z^3} &= \int_0^\infty \frac{\ln(x)}{1+x^3}\,dx + \int_\infty^0 \frac{\ln(x)+2\pi i}{1+x^3}\,dx\\ &= -2\pi i I.\end{aligned}$$

Therefore,

$$I = -\sum \operatorname{Res}\left[\frac{\ln(x)}{1+x^3}\right].$$

Thus, we get the solution of

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

Example. Consider the integral

$$I = \int_0^1 \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx.$$

The poles are around $\pm ia$.

Our problem is that we have multivalued behavior at ± 1. We may take the cut from -1 to 1 along the real axis, and our contour gives a sign flip across the cut.

We draw a dog-bone style contour hugging the cut in negative orientation to give us $2I$. Thus, we get

$$\begin{aligned}\oint_C \frac{\sqrt{1-z^2}}{z^2+a^2}\,dz &= \int_{-1}^1 \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx - \int_1^{-1} \frac{\sqrt{1-x^2}}{x^2+a^2}\,dx\\ &= 4I,\end{aligned}$$

where the sign flip in the second integral comes from crossing the branch cut.

Now, to evaluate the sum of the residues, we need to evaluate at three poles — ia, $-ia$, and the pole at ∞. Thus, we get

$$\begin{aligned}\operatorname{Res}[f(z), \pm ia] &= \frac{\sqrt{a^2+1}}{2ia}\\ -\operatorname{Res}[f(z), \infty] &= \lim_{|z|\to\infty} zf(z)\\ &= i.\end{aligned}$$

Therefore,

$$\begin{aligned}4I &= 2\pi i\left(\frac{\sqrt{a^2+1}}{ia} - i\right)\\ &= \frac{\pi}{2a}\left(\sqrt{a^2+1} - a\right).\end{aligned}$$

Poles on the axis

If we want to evaluate integrals with the pole on the contour, we need to use principal values.

$$\mathrm{PV}\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\left(\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx\right).$$

Similarly, we want to apply this for the calculus of residues. To do this, we take

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \lim_{\varepsilon\to 0} f(z)\,dz,$$

where $c_\pm$ are small semicircular contour additions of radius ε to C that hug our pole on the real axis, with c_- excluding the pole and c_+ including the pole. Thus, we have

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \lim_{\varepsilon\to 0}\int_{c_\pm} \frac{(z-x_0)f(z)}{z-x_0}\,dz.$$

Introducing $z - x_0 = \varepsilon e^{i\varphi}$, we have $dz = i\varepsilon e^{i\varphi}\,d\varphi$, giving

$$\oint_C f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \mathrm{Res}[f(z),x_0]\int_{c_\pm} i\,d\varphi.$$

Thus, we have

$$\begin{aligned}\oint_C f(z)\,dz &= \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx \pm i\pi\,\mathrm{Res}[f(z),x_0].\\ &= 2\pi i\sum_{z_i}\mathrm{Res}[f(z)-z_i].\end{aligned}$$

Thus, we have

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = \sum_{z_i \text{ in } C}\mathrm{Res}[f(z),z_i] \mp i\pi\,\mathrm{Res}[f(z),x_0].$$

Note that we only have *half* the residue when the pole is on the contour. Therefore, we have the result of

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\sum_{z_i \text{ in } C}\mathrm{Res}[f(z),z_i] + \frac{1}{2}\sum_{z_i \text{ on } C}\mathrm{Res}[f(z),z_i]\right).$$

Example. Consider

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx.$$

We have a simple pole at $x = a$.

We close our contour with a semicircle on the upper half-plane. Since we have no poles inside the contour, we have

$$\begin{aligned}\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx &= \pi i\,\mathrm{Res}\left[\frac{e^{ikx}}{x-a},a\right]\\ &= \pi i e^{ika}.\end{aligned}$$

Notice that if $k < 0$, we must close the contour in the lower half-plane, giving

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x-a}\,dx = \mathrm{sgn}(k)\pi i e^{ika}.$$

Taking real and imaginary components, we get

$$\begin{aligned}
\mathrm{PV}\int_{-\infty}^{\infty} \frac{\cos(kx)}{x-a}\,dx &= -\pi\sin(ka)\\
\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\,dx &= \pi\cos(ka).
\end{aligned}$$

Example. We will evaluate

$$I = \mathrm{PV}\int_{0}^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx.$$

We have a troublesome portion at $x = 0$, so we draw our contour to exclude 0.

We may close the contour with a large semicircle C_R. Since $\lim_{|z|\to\infty}|zf(z)| = 0$ and $\lim_{|z|\to 0}|zf(z)| = 0$, we may take these limits to give

$$\begin{aligned}
\oint_C \frac{\ln(z)}{z^2+a^2} &= \int_{-\infty}^{0} \frac{\ln\left(e^{i\pi}x\right)}{x^2+a^2}\,dx + \int_{0}^{\infty} \frac{\ln\left(e^{i0}x\right)}{x^2+a^2}\,dx\\
&= \mathrm{PV}\int_{-\infty}^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx + i\pi\int_{0}^{\infty} \frac{1}{x^2+a^2}\,dx\\
&= 2\mathrm{PV}\int_{0}^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx + \frac{i\pi^2}{2a}\\
&= 2\pi i\,\mathrm{Res}[f(z), ia].
\end{aligned}$$

Thus, we get $I = \frac{\pi}{2a}\ln(a)$.

Example. Instead of moving our contour up or down by ε to include (or exclude) a pole, we may move the pole up or down by ε. We consider

$$\int_{-\infty}^{\infty} \frac{g(x)}{x-x_0}\,dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{g(x)}{x-(x_0\pm i\varepsilon)}\,dx.$$

Breaking into real or imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{g(x)}{x-(x_0\pm i\varepsilon)^2}\,dx = \int_{-\infty}^{\infty} g(x)\frac{x-x_0}{(x-x_0)^2+\varepsilon^2}\,dx \pm i\varepsilon\int_{-\infty}^{\infty} \frac{g(x)}{(x-x_0)^2+\varepsilon^2}\,dx.$$

Now, notice that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2} = \begin{cases} 0 & x \neq x_0\\ \infty & x = x_0\end{cases}.$$

Now, we may take

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}\,dx = \varepsilon\oint_C \frac{dz}{z+\varepsilon^2},$$

where $z = (x-x_0)^2$. This gives

$$\begin{aligned}
\varepsilon\oint_C \frac{dz}{z+\varepsilon^2} &= \varepsilon(2\pi i)\left(\frac{1}{2i\varepsilon}\right)\\
&= \pi.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2}\, dx = \pi\delta(x - x_0).$$

Therefore, we recover

$$\lim_{\varepsilon \to 0} \varepsilon \int_{-\infty}^{\infty} \frac{g(x)}{(x - x_0)^2 + \varepsilon^2}\, dx = \pi g(x_0).$$

This gives the identity under the integral of

$$\frac{1}{z \mp i\varepsilon} = \mathrm{PV}\frac{1}{x} \pm i\pi\delta(x).$$

Example. Consider the integral

$$\begin{aligned}
\oint_{C_r} \frac{\cos(z)}{z}\, dz &= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\cos(x)}{x \pm i\varepsilon}\, dxj \\
&= \underbrace{\mathrm{PV}\int_{-\infty}^{\infty} \frac{\cos(x)}{x}\, dx}_{=0} \mp i\pi \int_{-\infty}^{\infty} \cos(x)\delta(x)\, dx \\
&= \mp i\pi
\end{aligned}$$

Sommerfeld–Watson Transform and Series Summation

Thus far, we've been replacing integrals with sums. Now, we're interested in going the other way around.

Consider the sum

$$S = \sum_{-\infty}^{\infty} f(n),$$

given the condition that $f(z)$ is analytic for $z \in \mathbb{R}$ and $\lim_{|z| \to \infty} \left|z^2 f(z)\right| = 0$.

We will introduce the auxiliary function

$$\begin{aligned}
g(z) &= \pi\cot(\pi z) \\
&= \pi\frac{\cos(\pi z)}{\sin(\pi z)}.
\end{aligned}$$

Note that $g(z)$ has an infinite number of poles at $z = n$ for each $n \in \mathbb{Z}$.

Now, what we will do here is integrate the product $f(z)g(z)$ around a long enough symmetric contour hugging the real axis. This gives

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C f(z)g(z)\, dz &= \sum_{n=-\infty}^{\infty} \mathrm{Res}[\pi\cot(\pi z)f(z), n] \\
&= \sum_{n=-\infty}^{\infty} f(z)\frac{\pi\cos(\pi z)}{\frac{d}{dz}(\sin(\pi z))}\bigg|_{z=n}. \\
&= \sum_{n=-\infty}^{\infty} f(n).
\end{aligned}$$

Now, this doesn't *seem* that helpful, until we remember that our contour C surrounds all the other poles of f in negative orientation.

$$\frac{1}{2\pi}\oint_C f(z)g(z)\,dz = -\sum_i \operatorname{Res}[\pi\cot(\pi z)f(z), z_i].$$

Thus, we have converted our infinite sum into a finite sum.

Similarly, if we have an alternating sign series

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty} (-1)^n f(n) \\ &= -\sum_i \operatorname{Res}[\pi\csc(\pi z)f(z), z_i] \end{aligned}$$

Example. Consider

$$S = \sum_{n=0}^{\infty} \frac{1}{n^2+a^2}.$$

Our analogous function is

$$f(z) = \frac{1}{z^2+a^2}.$$

Then,

$$\begin{aligned} S' &= -\operatorname{Res}\left[\frac{\pi\csc(\pi z)}{z^2+a^2}, \pm ia\right] \\ &= -\frac{\pi}{2a}\coth(\pi a). \end{aligned}$$

Therefore, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a).$$

Now, we write

$$S = \frac{1}{2a^2} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}.$$

Thus, we get the sum

$$\sum_{n=0}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{2a^2}(1+\pi a\coth(\pi a)).$$

Example. Now, we may consider

$$\begin{aligned} S' &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2} \\ &= -\operatorname{Res}\left[\frac{\pi\csc(\pi z)}{z^2+a^2}, \pm ia\right] \\ &= -\frac{\pi}{2a}\frac{1}{\sinh(\pi a)}, \end{aligned}$$

giving

$$S = \frac{1}{2a^2}\left(1+\frac{\pi a}{\sinh(\pi a)}\right).$$

Oscillators and Forcing

Consider a damped harmonic oscillator with position $u(t)$. Then, u obeys Newton's second law,

$$\ddot{u} + 2\beta\dot{u} + \omega_0^2 u = 0.$$

Here, β is the damping factor, and ω_0 denotes the natural frequency.

The solutions of this equation are

$$\begin{aligned} u(t) &= e^{-\beta t}\left(ae^{i\Omega t} + be^{-i\Omega t}\right) \\ &= e^{-\beta t}(a\cos(\Omega t) + b\sin(\Omega t)), \end{aligned}$$

where $\Omega^2 = \omega_0^2 - \beta^2$. This is known as a transient solution.

There are three types of motion.

- An underdamped system occurs when $\omega_0 > \beta$, so Ω is real, meaning we get oscillation that is damped out.
- An overdamped system occurs when $\beta > \omega_0$, so Ω is imaginary, and the damping slows down the return of the wave.
- When $\omega = \beta$, then $\Omega = 0$, and the solution is of the form $u(t) = e^{-\beta t}(at + b)$, and the system returns to equilibrium as quickly as possible. This is known as critical damping.

A forced system occurs when we have the differential equation

$$\ddot{u} + \beta\dot{u} + \omega_0^2 u = f(t). \tag{†}$$

We may consider a forcing function of the form $f(t) = f_0\cos(\omega t)$. We may also write

$$f(t) = f_0 \operatorname{Re}\left(e^{i\omega t}\right).$$

We expect to have a complex steady-state solution of the form

$$U_\omega = C(\omega)e^{i\omega t}.$$

We solve for U by sticking it into the differential equation of †. This will give the equation

$$U_\omega = \frac{f_0 e^{i\omega t}}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}.$$

Note that the real solution is $u = \operatorname{Re}(U_\omega)$, or

$$\begin{aligned} u(t) &= \frac{1}{2}(U_\omega + U_{-\omega}) \\ &= \frac{1}{2}\left(U_\omega + \overline{U_\omega}\right). \end{aligned}$$

Now, when we consider a generalized forcing function $f(t)$, where f is a continuum sum of forcing frequencies where the amplitudes are functions of ω, $\hat{f}(\omega)$, we get an integral:

$$u(t) = \int \frac{F(\omega)e^{i\omega t}}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}\,d\omega.$$

Plugging this solution into the differential equation, we get

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega,$$

which is a Fourier transform (see Math Methods 1).

Impulse Forcing

Consider a hammer blow forcing function, known as an impulse forcing function.

The impulse forcing is of the form

$$\begin{aligned} f(t) &= f_0\delta(\omega(t-t_0)) \\ &= \frac{\hat{f}_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t_0)}\,d\omega, \end{aligned}$$

where

$$\hat{f}_0 = \frac{f_0}{\omega_0}.$$

We want to find the impulse solution,

$$\begin{aligned} G(t) &\coloneqq u_\delta(t) \\ &= \frac{\hat{f}_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-t_0)}}{(\omega_0-\omega^2)+2i\beta\omega}\,d\omega. \end{aligned}$$

To do this integral, we will make use of residues. Writing our denominator as

$$\left(\omega_0^2-\omega^2\right)+2i\beta\omega = (\omega-\omega_+)(\omega-\omega_-),$$

where

$$\begin{aligned} \omega_\pm &= i\beta \pm \sqrt{\omega_0^2-\beta^2} \\ &= i\beta \pm \Omega. \end{aligned}$$

We close our contour in the upper half-plane so that get a decaying exponential. Evaluating the residues, we get

$$\begin{aligned} G(t) &= \hat{f}_0\frac{e^{-\beta(t-t_0)}}{2i\Omega}\left(e^{i\Omega(t-t_0)}-e^{-i\Omega(t-t_0)}\right) \\ &= \frac{f_0}{\Omega}\sin(\Omega(t-t_0))e^{-\beta(t-t_0)}, \end{aligned}$$

where $t > t_0$.

Now, if $t < t_0$, then we must close our contour in the lower half-plane, and since there are no poles in the lower half-plane, we get $G(t) = 0$ for $t < t_0$. Thus, we must have

$$G(t) = \frac{f_0}{\Omega}\sin(\Omega(t-t_0))e^{-\beta(t-t_0)}\Theta(t-t_0),$$

where Θ is the Heaviside step function.

- The imaginary and real parts of $\omega_\pm$ give the damping, β, and parameter, Ω, respectively. Now, we may interpret the different types of damping in this respect.
  - If Ω is real (i.e., underdamped motion), then $\omega_\pm$ have constant magnitude of ω_0, meaning that varying the damping only moves the poles around in a circle.
  - If $\beta = \omega_0$ (i.e., critically damped motion), then the poles converge at $i\beta$ along the imaginary axis.
  - If $\beta > \omega_0$ (i.e., overdamped mption), then the poles separate along the imaginary axis, giving non-oscillatory motion.
- The poles also encode resonance characteristics, where we have $\omega_{\text{res}}^2 = \omega_0^2 - 2\beta^2$.
- If $\beta \ll \omega_0$, then the damping is mathematically equivalent to the $i\varepsilon$ prescription moving the resonance pole at ω_0 off the real axis and into the upper half-plane.

Waves on a String

Whereas an undamped oscillator is harmonic only in time, a wave is harmonic in both space and time.

A wave satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where c is the wave speed.

The general solution is of the form

$$U(x, t) = ce^{i(kx=\omega t)}.$$

A forced wave occurs via

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(x, t).$$

Now, we start with the impulse solution,

$$\begin{aligned} f(x, t) &= f_0\delta(x - x_0)\delta(t - t_0) \\ &= f_0 \int_{-\infty}^{\infty} \frac{e^{ik(x-x_0)}}{2\pi}\, dk \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{2\pi}\, d\omega. \end{aligned}$$

Now, we have

$$G(x, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{ikx}\, dk \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2/c^2 - k^2}\, d\omega.$$

To evaluate this integral, we start with the integral in ω, given by

$$I = \frac{c^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - c^2k^2}\, d\omega.$$

Unfortunately here, our simple poles lie on the real axis at $\pm ck$.

To find the solution, we need boundary and initial conditions to know where we want to make our $i\varepsilon$ adjustment.

If there is no wave before our impulse hits, we need our integral to vanish whenever $t < 0$ which occurs when we close our contour in the lower half plane, so we subtract $i\varepsilon$ from $\pm ck$. Factoring, we have

$$I = \frac{c^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - (ck - i\varepsilon))(\omega + (ck + i\varepsilon))}\, d\omega.$$

Thus, we have

$$I(t > 0) = -\frac{c}{k}\sin(ckt)e^{-\varepsilon t},$$

and in the limit as $\varepsilon \to 0$, we have

$$I(t) = -\frac{c}{k}\sin(ckt).$$

Now, sticking our value of I into the integral in k, we have

$$\begin{aligned} G(x, t > 0) &= -\frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ictk} - e^{-ictk}}{2ik} e^{ikx}\, dk \\ &= -\frac{c}{2}\Theta(ct - |x|)\Theta(t). \end{aligned}$$

There are some comments in order.

- The factor of $\Theta(t)$ effectively states that nothing happens before $t = 0$.
- The factor of $\Theta(ct - |x|)$ denotes causality. The term $|x|$ denotes the physical symmetry, while we need $ct - |x| > 0$ in order to feel an effect.

Quantum Mechanics

The Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m} + \frac{\partial^2\psi}{\partial x^2} = 0.$$

Here, ψ denotes the wavefunction. The probability of a particle being found within dx of x is $|\psi|^2 dx$.

In quantum mechanics, we define operators for energy and momentum as

$$\begin{aligned} E &:= +i\hbar\frac{\partial}{\partial t} \\ P &:= -i\hbar\frac{\partial}{\partial x}. \end{aligned}$$

The Schrödinger equation falls from the fact that $E = \frac{P^2}{2m}$. These operators are Hermitian, so their eigenvalues are real.

We want to figure out the Green's function for the quantum hammer blow at $t = 0$, which we call the propagator.

$$i\hbar\frac{\partial G}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2} = i\hbar\delta(x)\delta(t).$$

When we try to evaluate it, we get

$$G(x, t > 0) = \Theta(t)\sqrt{\frac{m}{2\pi i\hbar t}}e^{imx^2/2\hbar t}.$$

This solution has a problem, though — there is no sense of causality. This says that, as long as $t > 0$, there is going to be a measurable reaction at all x.

The main reason is that the Schrödinger equation is not relativistic. Accounting for the relativistic relationship, we have

$$E^2 - P^2c^2 = m^2c^4.$$

This gives an equation known as the Klein–Gordon equation:

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x^2} = \frac{m^2c^2}{\hbar^2}\phi.$$

Solving the Klein–Gordon equation gives infinitely many solutions that have both positive and negative energy, the latter of which is a major issue. However, it can be shown that the Klein–Gordon equation applies to particles with integral spin, with wave function

$$\phi(x, t) = Ce^{i(kx - \omega t)}$$

so long as

$$\frac{\omega^2}{c^2} - k^2 = \frac{m^2c^2}{\hbar^2}.$$

To see how causality fares for the Klein–Gordon wavefunction, we solve it for the unit impulse acting on $x = 0$ at $t = 0$.

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right)G(x,t) = i\delta(x)\delta(t),$$

where we set $\hbar = c = 1$. We get the Green's function

$$G(x,t) = i\int_{-\infty}^{\infty} \frac{1}{2\pi}e^{ipx}\,dp \int_{-\infty}^{\infty} \frac{e^{-iEt}}{2\pi(E^2 - p^2 - m^2)}\,dE.$$

We have two poles at $\pm\sqrt{p^2+m^2}$, and when we close the contour we obtain different physical results.

When $t < 0$, we must close the contour in the upper-half plane, so we exclude the poles from the contour. Since there are no poles in the upper half-plane, we get a factor of $\Theta(t)$.

Thus, when $t > 0$, we close in the lower half-plane, giving

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{e^{-iEt}}{(E+iE)^2 - p^2 - m^2}\,dE \\ &= -\frac{1}{2\pi}2\pi i\left(\frac{e^{-i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} - \frac{e^{i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}}\right). \end{aligned}$$

Defining $E_p := \sqrt{p^2+m^2}$, we have

$$I = -\frac{i}{2E_p}\left(e^{-iE_pt} - e^{iE_pt}\right).$$

Thus, we have

$$\begin{aligned} G(x, t > 0) &= \Theta(t)\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{e^{ipx}}{2E_p}\left(e^{-iE_pt} - e^{iE_pt}\right)dp \\ &= \frac{\Theta(t)}{4\pi}\int_{-\infty}^{\infty} \frac{1}{E_p}\left(e^{-i(E_pt - px)} - e^{i(E_pt - p_x)}\right)dp, \end{aligned}$$

where we were allowed to flip the sign of p because the integral is taken over all space.

Tensors

Cartesian Tensors

Recall from intro mechanics that when we have a block on a ramp, we may choose two different orientations of our axes. However, despite our choice of basis, we still get the same result after applying Newton's second law — in that sense, the vectors represent something physical separate from their components. Using a rotation matrix, we are able to convert the equations

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y, \end{aligned}$$

into

$$\begin{aligned} F'_x &= ma'_x \\ F'_y &= ma'_y. \end{aligned}$$

This means that an expression like $\mathbf{F} = m\mathbf{a}$ is a *covariant* expression; it is the same equation regardless of reference frame.

Stress and Strain

If we have a continuous body, the stress is defined as the internal force per unit area exerted at an imaginary surface within the body. In other words, stress times area equals force.

$$dF_i = \sum_j \sigma_{ij}\, da_j,$$

where σ_{ij} denotes the stress *tensor*. The reason why it is a two-indexed object rather than a one-indexed object is because there are two different types of stress: pressure and shear.

Pressure is the normal component to the surface, while shear is the tangent component, so σ_{ij} denotes the two components; then, the diagonal components denote the pressure and the off-diagonal components the shear.

Whereas stress describes the internal forces of an object, strain describes deformations due to external forces.

For instance, in a rod of length L and cross section A, if we strain it to a length of $L + \Delta L$, the relationship between induced stress $\sigma = \frac{F}{A}$ normal to surface and strain $\varepsilon = \Delta L + L$ is given by Hooke's law:

$$\begin{aligned} F &= k\Delta L\\ \sigma &= Y\varepsilon, \end{aligned}$$

where $Y = \frac{kL}{A}$ is the elastic modulus of the material. Now, since ΔL can be negative, this equation describes both tensile stress and strain and compressive stress and strain.

Now, if stress is a tensor, then so is strain; moving beyond one dimension, we define a vector field $\mathbf{u}(\mathbf{r})$ which gives the magnitude and direction of a body's displacement at position $\mathbf{r}$. To understand internal stress, we need to focus not on the rigid body motion, but only on the relative displacement of points within the body. This means we need to consider

$$u_i(\mathbf{r} + d\mathbf{r}) = u_i(\mathbf{r}) + \nabla u_i \cdot d\mathbf{r} + \cdots,$$

where we allow $|\nabla u_i| \ll 1$, meaning we take the first order expansion. This gives

$$\mathbf{u}(\mathbf{r} + d\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \sum_j \delta_j \mathbf{u}\, dr_j,$$

where

$$\nabla\mathbf{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z}\\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z}\\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

This is where strain resides. Specifically, we may decompose $\nabla\mathbf{u}$ into symmetric and antisymmetric components,

$$\nabla\mathbf{u} = \overset{\leftrightarrow}{\varepsilon} + \overset{\leftrightarrow}{\phi},$$

where

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)\\ \phi_{ij} &= \frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right). \end{aligned}$$

The symmetric/antisymmetric decomposition is covariant, so we may express

$$\mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \overset{\leftrightarrow}{\varepsilon} \cdot \mathrm{d}\mathbf{r} + \overset{\leftrightarrow}{\phi} \cdot \mathrm{d}\mathbf{r}.$$

Turning our attention to $\overset{\leftrightarrow}{\phi}$, we may use the Levi-Civita symbol to write

$$\phi_{ij} = -\sum_{k} \epsilon_{ijk}\varphi_{k},$$

and take

$$\begin{aligned}\sum_{j} \phi_{ij}\,\mathrm{d}r_j &= -\sum_{j,k} \epsilon_{ijk}\varphi_{k}\,\mathrm{d}r_j\\ &= \sum_{j,k} \epsilon_{ikj}\varphi_{k}\,\mathrm{d}r_j,\end{aligned}$$

meaning

$$\mathrm{d}\mathbf{u} = \vec{\varphi} \times \mathrm{d}\mathbf{r},$$

describing the rotation of $\mathrm{d}\mathbf{r}$ by φ around $\hat{\varphi}$. This is rigid body motion.

Thus, deformations are described by the strain tensor, $\overset{\leftrightarrow}{\varepsilon}$. Hooke's law then becomes

$$\sigma_{ij} = \sum_{k,l} Y_{ijkl}\varepsilon_{kl},$$

where Y_{ijkl} is the elasticity tensor.

Equivalence Classes of Rotations

A tensor is an indexed object, that transforms as

$$T_{i'j'} = \sum_{i,j} R_{i',i}R_{j'j}T_{ij},$$

where the number of rotations depends on the rank of the tensor.[v]

Example (Moment of Inertia Tensor)**.** The value of the moment of inertia depends on the choice of axes, but the moment of inertia is a physical quantity unto itself. We define

$$I_{k\ell} = \int_V \left(r^2\delta_{k\ell}r_k r_\ell\right)\rho(\mathbf{r})\,\mathrm{d}\tau.$$

This is a symmetric tensor (symmetric matrix). Recalling that r^2 is a scalar, we have

$$\begin{aligned}I_{i'j'} &= \sum_{k,\ell} R_{i'k}R_{j'\ell}I_{k\ell}\\ &= \sum_{k,\ell} R_{i'k}R_{j'\ell}\left(\int_V \left(r^2\delta_{k\ell} - r_k r_\ell\right)\rho(\mathbf{r})\,\mathrm{d}\tau\right)\\ &= \int_V \left(\sum_{k,\ell} R_{i'k}R_{j'\ell}\delta_{k\ell}r^2 - \left(\sum_{k} R_{i'k}r_k\right)\left(\sum_{\ell} R_{j'\ell}r_\ell\right)\right)\rho(\mathbf{r})\,\mathrm{d}\tau\\ &= \int_V \left(\delta_{i'j'}r^2 - r_{i'}r_{j'}\right)\rho(\mathbf{r})\,\mathrm{d}\tau.\end{aligned}$$

Thus, the moment of inertia is a covariant expression.

[v]For those more mathematically inclined, they're basically linear transformations on tensor products of vector spaces and their duals. Each index adds another vector to the tensor product or the dual.

In a rank r tensor over a vector space V, there are N^r components.[VI]

Now we consider derivative operators. A Taylor expansion can be written as

$$f(\mathbf{r}+\mathbf{a}) = \left(1 + \sum_i a_i \frac{\partial}{\partial x_i} + \frac{1}{2!}\sum_{i,j} a_i a_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \cdots\right) f(\mathbf{r}).$$

This gives a tensor decomposition, where $\sum_i a_i \frac{\partial}{\partial x_i}$ is a rank 1 tensor, and $\sum_{i,j} a_i a_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$ is a rank 2 tensor.

To illustrate covariance, we consider the case of one dimension:

$$\begin{aligned} f(x+a) &= \left(1 + a\frac{d}{dx} + \frac{a^2}{2!}\frac{d^2}{dx^2}\right) f(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(a\frac{d}{dx}\right)^n f(x). \end{aligned}$$

To generalize this to higher dimensions, we can write the operator

$$\begin{aligned} a\frac{d}{dx} &= \left(a\hat{\imath}\right)\cdot\left(\hat{\imath}\frac{d}{dx}\right) \\ &= \mathbf{a}\cdot\nabla. \end{aligned}$$

Thus, using covariance of the derivative, we have

$$f(\mathbf{r}+\mathbf{a}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{a}\cdot\nabla)^n f(\mathbf{r}).$$

Tensors and Pseudotensors

We might have lied earlier[VII] when we said that tensors transform through rotations:

$$T_{i'j'} = \sum_{k,\ell} R_{i'k}R_{j'\ell}T_{k\ell}.$$

We also need to care about parity. For an illustrative example, if we have a parity flip, we have $\mathbf{r} \overset{p}{\mapsto} -\mathbf{r}$ for position, $\mathbf{v} \overset{p}{\mapsto} -\mathbf{v}$ for velocity, but we have $\omega \overset{p}{\mapsto} \omega$ for angular velocity.

We call vectors that have a sign flip under parity change *polar vectors*, but vectors that do not have a sign flip under parity change are called *pseudovectors* or *axial vectors*.

As it turns out, pseudovectors are generally brought about through a cross product between two axial vectors (in three dimensions anyway):

$$\begin{aligned} \mathbf{A}\times\mathbf{B} &\overset{p}{\mapsto} (-\mathbf{A})\times(-\mathbf{B}) \\ &= \mathbf{A}\times\mathbf{B}. \end{aligned}$$

Note that scalars generally don't change under parity transformation, but expressions of the form $\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ are *pseudoscalars* that do change under parity transformation.

[VI]This comes from the fact that that there are N^r dimensions in a r-fold tensor product of vector spaces with dimension N.
[VII]Common occurrence in physics

All of these are artifacts of the Levi-Civita symbol, ϵ_{ijk}. Note that we have

$$\epsilon_{ijk} = \sum_{\ell,m,n} R_{i,\ell}R_{j,m}R_{k,n}\epsilon_{\ell mn},$$

which yields the determinant of R if i, j, k are in cyclic order and the negative determinant of R if i, j, k are in anticyclic order. To deal with this sign issue, we need a factor of $\det(R)$,[VIII] giving

$$\epsilon_{i'j'k'} = \det(R) \qquad = \sum_{\ell,m,n} R_{i'\ell}R_{j'm}R_{k'n}\epsilon_{\ell mn}.$$

Invariants

The components of a tensor are frame-dependent, but an *invariant* is a scalar combination of components that remains the same under a rotation. For instance, if $\mathbf{v}$ is vector, then $\sum_i v_i v_i = \|\mathbf{v}\|$ is invariant under rotations.

If we have a rank 2 tensor, we have

$$\begin{aligned}\sum_{i,j} T_{ij}\delta_{ij} &= \sum_i T_{ii}\\ &= \operatorname{tr}(T).\end{aligned}$$

We may also collapse with the Levi-Civita symbol, yielding another invariant of $\frac{1}{3!}\det(T)$.

As it turns out, a rank 2 tensor in N dimensions has N independent invariants.

However, you may ask: are we actually measuring the invariance of T, or are we measuring the invariance of δ_{ij} of ϵ_{ijk}. Taking rotations, we have

$$\begin{aligned}\sum_{i,j} R_{k'i}R_{\ell'j}\delta_{ij} &= \sum_i R_{k'i}R_{\ell'i}\\ &= \sum_i R_{k'i}R^T_{i\ell'}\\ &= \delta_{k'\ell'},\end{aligned}$$

meaning that the identity matrix is invariant regardless of coordinate system. This is known as an *invariant tensor*, or an isotropic tensor.

Example. Any $\mathbb{R}^3$ vector can be rewritten as a second-rank antisymmetric tensor by taking

$$\begin{aligned}T_{ij} &= \sum_k \epsilon_{ijk}B_k\\ &= \begin{pmatrix} 0 & B_3 & -B_2\\ -B_3 & 0 & B_1\\ B_2 & -B_1 & 0\end{pmatrix}.\end{aligned}$$

Inverting, we get

$$B_i = \frac{1}{2}\sum_{j,k}\epsilon_{ijk}T_{jk}.$$

Thus, $\overleftrightarrow{T}$ is an equivalent way to package the components of B — i.e., that they are duals of each other.

[VIII] Remember that $\det(R) = \pm 1$.

Non-Cartesian Tensors

What makes a tensor a tensor is that it maintains its symmetry upon some transformation. However, we may have a shear transformation.

Consider a coordinate system with

$$\begin{aligned}\hat{e}_u &= \begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\end{pmatrix}\\ \hat{e}_v &= \begin{pmatrix}\sin(\beta)\\ \cos(\beta)\end{pmatrix},\end{aligned}$$

where α is an angle between the x axis and $\hat{e}_u$, β is an angle between the y axis and $\hat{e}_v$, with ϕ the angle between $\hat{e}_u$ and $\hat{e}_v$.

Taking orthogonal projections of $\mathbf{A}$, we have

$$\begin{aligned}A_u &= \mathbf{A}\cdot\hat{e}_u\\ A_v &= \mathbf{A}\cdot\hat{e}_v.\end{aligned}$$

Unfortunately, $\mathbf{A} \neq A_u\hat{e}_u + A_v\hat{e}_v$.

However, we can consider A^u and A^v as components *along* $\hat{e}_u$ by projecting $\mathbf{A}$ parallel to $\hat{e}_v$ to obtain A^u and projecting $\mathbf{A}$ parallel to $\hat{e}_u$ to obtain A^v, giving

$$\mathbf{A} = A^u\hat{e}_u + A^v\hat{e}_v.$$

Yet, since A_u and A_v are linearly independent components, we should still be able to construct $\mathbf{A}$ using their components. Specifically, we extend the orthogonal projections of $\mathbf{A}$ on $\hat{e}_u$ and $\hat{e}_v$ until we get a parallelogram. This gives

$$\mathbf{A} = \frac{1}{\sin(\phi)}(A_u\hat{e}^u + A_v\hat{e}^v). \qquad = A_u\vec{e}^u + A_v\vec{e}^v.$$

The sets $\{\hat{e}_u, \hat{e}_v\}$ and $\left\{\vec{e}^u, \vec{e}^v\right\}$ are intimately tied to each other:

$$\hat{e}_i\cdot\vec{e}^j = \delta_{ij}$$

The vector $\mathbf{A}$ can thus be decomposed along the "downstairs basis" and the "upstairs basis," with components defined by

$$\begin{aligned}A_a &= \hat{e}_a\cdot\mathbf{A}\\ A^a &= \vec{e}^a\cdot\mathbf{A};\end{aligned}$$

with $\vec{e}^a$ not necessarily normalized. In this case, we have

$$\begin{aligned}A_u &= \vec{e}_u\cdot\mathbf{A}\\ &= A^u + A^v\cos(\phi)\\ A_v &= \vec{e}_v\cdot\mathbf{A}\\ &= A^u\cos(\phi) + A^v.\end{aligned}$$

Metric Tensors

We want covariant index notation to apply to all bases in a straightforward manner.

We start with the Cartesian line element, $d\mathbf{s} = \sum_i \hat{e}_i dx_i$. We want a general case of $d\mathbf{s} = \sum_a \vec{e}_a du^a$, where all the scale factors are hidden in the basis $\{\vec{e}_a\}$. We calculate

$$\begin{aligned}
ds^2 &= d\mathbf{s} \cdot d\mathbf{s}\\
&= \left(\sum_a \vec{e}_a du^a\right) \cdot \left(\sum_b \vec{e}_b du^b\right)\\
&= \sum_{a,b} (\vec{e}_a \cdot \vec{e}_b) du^a du^b\\
&= \sum_{a,b} g_{ab} du^a du^b,
\end{aligned}$$

where $g_{ab} = \vec{e}_a \cdot \vec{e}_b$. We call the matrix $(g_{ab})_{ab}$ the *metric tensor*.

Example. In Cartesian coordinates, our metric is

$$\begin{aligned}
g_{ab} &= \delta_{ij}\\
&= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.
\end{aligned}$$

In spherical coordinates, our metric is

$$g_{ab} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2(\theta) \end{pmatrix},$$

after expanding on the spherical basis (see Math Methods Notes). This shows a couple things.

- The metric can be a function of position.
- The same coordinate transformation that moves us from Cartesian to spherical is the same coordinate transformation that moves from the metric tensor on Cartesian coordinates to the metric tensor on spherical coordinates.

Example. In relativity, the difference between two points is

$$ds^2 = c^2 dt^2 - dr^2.$$

The Minkowski metric is denoted $\eta_{\mu\nu}$,

$$ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu,$$

where μ and ν range from 0 to 3, and $x^0 = ct$. In Cartesian, we have

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Example. For the shear system, we have the metric of

$$\begin{aligned}
g_{ab} &= \begin{pmatrix} \vec{e}_u \cdot \vec{e}_u & \vec{e}_u \cdot \vec{e}_v \\ \vec{e}_v \cdot \vec{e}_u & \vec{e}_v \cdot \vec{e}_v \end{pmatrix}\\
&= \begin{pmatrix} 1 & \cos(\phi) \\ \cos(\phi) & 1 \end{pmatrix}.
\end{aligned}$$

In terms of the line element, we have

$$\begin{aligned}ds^2 &= \sum_{a,b} g_{ab}\,du^a du^b\\ &= du^2 + 2\cos(\phi)du\,dv + dv^2.\end{aligned}$$

Note that there is a cross term, meaning our system is not orthogonal. Furthermore, this is actually the law of cosines.

The norm of a vector **A** is

$$\begin{aligned}\|\mathbf{A}\|^2 &= \left(\sum_a A^a \vec{e}_a\right)\cdot\left(\sum_b A^b\vec{e}_b\right)\\ &= \sum_{a,b} g_{ab}A^aA^b\\ &= (A^u)^2 + 2\cos(\phi)A^uA^v + (A^v)^2.\end{aligned}$$

Streamlining Notation

In general, we can write

$$\mathbf{A}\cdot\mathbf{B} = \sum_{a,b} g_{ab}A^aB^b.$$

Setting

$$\begin{aligned}A_b &= \sum_a g_{ab}A^a\\ B_a &= \sum_b g_{ab}B^b,\end{aligned}$$

we get

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \sum_a A^aB_a\\ &= \sum_b A_bB^b.\end{aligned}$$

We can consider the metric as "lowering" the indices of A^a or B^b. The new rule is that any index must be summed over one with an "upstairs" index and one with a "downstairs" index.

From now on, we will be using the Einstein summation notation, converting

$$A_a = \sum_a M_{ab}B^b$$

to be

$$A_a = M_{ab}B^b,$$

where the sum is implied by the double index.

We should be able to convert g_{ab} to g^{ab}. We take

$$g^{da}\left(g^{cb}g_{ab}\right) = g^{da}(g^c_a)$$

$$= g^{dc}.$$

Note that if we take

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A^a B_a\\
&= \left(g^{ab}A_b\right)(g_{ac}B^c)\\
&= \left(g^{ab}g_{ac}\right)A_b B^c\\
&= A_b B^b,
\end{aligned}$$

so that $g^{ab}g_{ac} = \delta^b_c$, or that g^{ab} and g_{ab} are inverses of each other.

Example. Recall that the line element in spherical coordinates is

$$d\mathbf{s} = \hat{r}dr + r\hat{\theta}d\theta + r\sin\theta\hat{\phi}d\phi,$$

giving downstairs basis of

$$\begin{aligned}
\vec{e}_r &= \hat{r}\\
\vec{e}_\theta &= r\hat{\theta}\\
\vec{e}_\phi &= r\sin\theta\hat{\phi}.
\end{aligned}$$

We usually write this as

$$d\mathbf{s} = \vec{e}_a du^a,$$

with the implied sum on a. Calculating g_{ab}, we get

$$g_{ab} = \begin{pmatrix}1 & & \\ & r^2 & \\ & & r^2\sin^2(\theta)\end{pmatrix},$$

and

$$\begin{aligned}
A_r &= g_{ra}A^a\\
&= g_{rr}A^r + g_{r\theta}A^\theta + g_{r\phi}A^\phi\\
&= A^r\\
A_\theta &= g_{\theta a}A^a\\
&= g_{\theta r}A^r + g_{\theta\theta}A^\theta + g_{\theta\phi}A^\phi\\
&= r^2A^\theta\\
A_\phi &= g_{\phi a}A^a\\
&= g_{\phi r}A^r + g_{\phi\theta}A^\theta + g_{\phi\phi}A^\phi.\\
&= r^2\sin^2(\theta)A^\phi.
\end{aligned}$$

In particular, this gives

$$g^{ab} = \begin{pmatrix}1 & & \\ & 1/r^2 & \\ & & 1/r^2\sin^2(\theta)\end{pmatrix}.$$

Using $\vec{e}^a = g^{ab}\vec{e}_b$, we have

$$\vec{e}^r = g^{ra}\vec{e}_a$$

$$\begin{aligned}
&= \vec{e}_r \\
&= \hat{r} \\
\vec{e}^{\theta} &= \frac{1}{r}\hat{\theta} \\
\vec{e}^{\phi} &= \frac{1}{r\sin\theta}\hat{\phi}.
\end{aligned}$$

For orthogonal coordinates, the bases $\vec{e}_a$ and $\vec{e}^a$ are related by the metric to their more familiar orthonormal versions, which we write $\hat{\epsilon_a}$, by writing

$$\begin{aligned}
\vec{e}_a &= \hat{\epsilon}_a\sqrt{g_{aa}} \\
\vec{e}^a &= \frac{1}{\sqrt{g_{aa}}}\hat{\epsilon}_a,
\end{aligned}$$

without implied sum.

To transform from coordinate system to coordinate system (rather than between a coordinate system and its dual), we express transformation of components and bases by taking

$$\begin{aligned}
A^{b'} &= T^{b'}_{a}A^a \\
\vec{e}_{b'} &= \vec{e}_a S^{a}_{b'}.
\end{aligned}$$

Example. We find a transformation between coordinate systems by projecting both vector expansions

$$\begin{aligned}
\mathbf{A} &= \sum_{a'}\vec{e}_{a'}A^{a'} \\
&= \sum_{a}\vec{e}_a A^a.
\end{aligned}$$

We project both vector expansions onto the $\vec{e}^{b'}$ basis by taking dot products:

$$\begin{aligned}
\vec{e}^{b'}\cdot\mathbf{A} &= \sum_{a'}\left(\vec{e}^{b'}\cdot\vec{e}_{a'}\right)A^{a'} \\
&= \sum_{a}\left(\vec{e}^{b'}\cdot\vec{e}_a\right)A^a.
\end{aligned}$$

Using the mutual orthogonality relation

$$\begin{aligned}
\vec{e}^{b'}\cdot\vec{e}_a &= g^{b'}_{a} \\
&= \delta^{b'}_{a'},
\end{aligned}$$

we get

$$A^{b'} = \sum_a\left(\vec{e}^{b'}\cdot e_a\right)A^a$$

Example. Consider the shear system,

$$\begin{aligned}
\begin{pmatrix} A^x \\ A^y \end{pmatrix} &= \begin{pmatrix} \vec{e}^x\cdot\vec{e}_u & \vec{e}^x\cdot\vec{e}_v \\ \vec{e}^y\cdot\vec{e}_u & \vec{e}^y\cdot\vec{e}_u \end{pmatrix}\begin{pmatrix} A^u \\ A^v \end{pmatrix} \\
T^{i}_{a} &= \begin{pmatrix} \cos(\alpha) & \sin(\beta) \\ \sin(\alpha) & \cos(\beta) \end{pmatrix}.
\end{aligned}$$

It is important to emphasize that a tensor transformation between systems or frames switches between prime basis and non-prime basis.

Now, we may define a tensor to be

$$A^{a'} = T_b^{a'} A^b.$$

Generally speaking, the rank is the number of indices (both upstairs and downstairs).

Consider a simple dot product:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{A} &= A^i A_j \\ &= \delta_{ij} A^i A^j \\ &= g_{ab} A^a A^b.\end{aligned}$$

This expression allows us to write the dot product in *any* coordinate system, just by knowing the metric.[IX]

General Covariance

The physics[X] of a system does not depend on the origin. Therefore, we need a way to talk about covariance without specifying an origin.

Therefore, rather than working in $\mathbf{r}$, or general $\mathbf{u}$, we work in $d\mathbf{r}$ and $d\mathbf{u}$, since the difference between two vectors does not depend on the origin.

The chain rule then gives us

$$du^{a'} = \frac{\partial u^{a'}}{\partial u^b} du^b,$$

with an implied sum over b. As it turns out,

$$\frac{\partial}{\partial u_b} = \partial_b,$$

or that derivative with respect to an upstairs element is a downstairs tensor. The matrix $\left(\frac{\partial u^{a'}}{\partial u^b}\right)_{ab}$ is the Jacobian matrix.

For instance, calculating the Jacobian matrix for polar coordinates, we have

$$\begin{aligned}T_b^{a'} &= \frac{\partial u^{a'}}{\partial u^b} \\ &= \frac{\partial(r,\phi)}{\partial(x,y)} \\ &= \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{pmatrix}.\end{aligned}$$

Thus, we get

$$dr = \frac{x}{r}dx + \frac{y}{r}dy$$

[IX]This is akin to the fact that a bilinear form is defined by the matrix in the expression $\varphi(x,x) = x^T A x$.

[X]Who cares about that?

$$d\phi = -\frac{x}{r^2}dx + \frac{y}{r^2}dy.$$

Therefore, we are now going to define *general covariance* of vectors to be of the form

$$A^a = \frac{\partial u^{a'}}{\partial u^b}A^b.$$

However, note that the gradient transforms as

$$\partial_{a'} = \frac{\partial u^b}{\partial u^{a'}}\partial_b.$$

Example. We want to understand a symmetry transformation of the metric. We must have

$$g_{a'b'} = \frac{\partial u^a}{\partial u^{a'}}\frac{\partial u^b}{\partial u^{b'}}g_{ab}.$$

We will now try to convert from Cartesian to polar. This gives $g_{ab} = \delta_{ij}$, so

$$\begin{aligned}
g_{rr} &= \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r}\delta_{ij}\\
&= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2\\
&= 1\\
g_{\phi\phi} &= \frac{\partial x^i}{\partial \phi}\frac{\partial x^j}{\partial \phi}\delta_{ij}\\
&= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2\\
&= r^2\\
g_{r\phi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi}\\
&= 0.
\end{aligned}$$

Tensor Derivatives

Now that we have introduced a tensor regime based on derivatives, we are now interested in understanding what happens when we take derivatives *of* tensors.

If we take $\frac{\partial}{\partial u^{b'}}A^{a'}$, we want this to yield a transformation as a mixed second-rank tensor. The tensor transformation is of the form

$$T^{a'}_{b'} = \frac{\partial u^{a'}}{\partial u^c}\frac{\partial u^d}{\partial u^{b'}}T^c_d.$$

However, taking

$$\frac{\partial}{\partial u^{b'}}A^{a'} = \frac{\partial}{\partial u^{b'}}\left(\frac{\partial u^{a'}}{\partial u^c}A^c\right)$$

and using the chain rule to convert back to the non-prime frame

$$= \frac{\partial u^d}{\partial u^{b'}}\frac{\partial}{\partial u^d}\left(\frac{\partial u^{a'}}{\partial u^c}A^c\right).$$

Using the product rule and rearranging, we take

$$= \underbrace{\frac{\partial u^{d}}{\partial u^{b'}}\frac{\partial u^{a'}}{\partial u^{c}}\left(\frac{\partial}{\partial u^{d}}A^{c}\right)}_{\text{expected}} + \underbrace{A^{c}\frac{\partial^{2}u^{a'}}{\partial u^{c}\partial u^{d}}\frac{\partial u^{d}}{\partial u^{b'}}}_{\text{why?}}.$$

Because we allowed general coordinate transformations, we created the seemingly extraneous term consisting of second derivatives (beyond what we wanted for a second-rank tensor transformation). This is because we forgot to include the change *in the basis itself*, beyond the change in component.

Then, we have

$$\frac{\partial}{\partial u^{b}}\left(A^{a}\vec{e}_{a}\right) = \frac{\partial A^{a}}{\partial u^{b}}\vec{e}_{a} + A^{a}\frac{\partial \vec{e}_{a}}{\partial u^{b}}.$$

The expression $\frac{\partial \vec{e}_{a}}{\partial u^{b}}$ is our offending character. Since this is a vector-valued integral, we may expand it on the basis

$$\frac{\partial \vec{e}_{a}}{\partial u^{b}} = \Gamma^{c}_{ab}\vec{e}_{c}, \tag{$*$}$$

with an implied sum over c. The term Γ^{c}_{ab} is known as the *Christoffel symbol* for $\frac{\partial \vec{e}_{a}}{\partial u^{b}}$.

Example (Christoffel Symbols in Polar Coordinates)**.** Recall that

$$\vec{e}_{a} = \frac{\partial \mathbf{r}}{\partial u^{a}},$$

and expanding on the Cartesian basis (yet again with implied sums), we have

$$\begin{aligned}\vec{e}_{a} &= \frac{\partial x^{j}}{\partial u^{a}}\frac{\partial \mathbf{r}}{\partial x^{j}}\\ &= \frac{\partial x^{j}}{\partial u^{a}}\hat{x}_{j}.\end{aligned}$$

Now,

$$\frac{\partial \vec{e}_{a}}{\partial u^{b}} = \frac{\partial \vec{e}^{a}}{\partial u^{b}}$$

and recalling that $\vec{e}_{a} = \frac{\partial \mathbf{r}}{\partial u^{a}}$, we have

$$= \frac{\partial^{2}}{\partial u^{a}\partial u^{b}}\mathbf{r}.$$

In polar coordinates, we have the (r, ϕ) basis, giving

$$\begin{aligned}\vec{e}_{r} &= \cos(\phi)\hat{\imath} + \sin(\phi)\hat{\jmath}\\ &= \hat{r}\\ \vec{e}_{\phi} &= -r\sin(\phi)\hat{\imath} + r\cos(\phi)\hat{\jmath}\\ &= r\hat{\phi}.\end{aligned}$$

Now, using the fact that we pull out a component by taking a dot product, we may fin

$$\Gamma^{r}_{\phi\phi} = \vec{e}_{r}\cdot\frac{\partial \vec{e}_{\phi}}{\partial \phi}$$

$$= \hat{r} \cdot (-r\hat{r})$$
$$= -r.$$

Similarly, we may find the rest of the Christoffel symbols, giving

$$\Gamma^{\phi}_{r\phi} = \frac{1}{r}$$
$$\Gamma^{\phi}_{\phi r} = \frac{1}{r},$$

and giving 0 for everything else.

Therefore, for polar coordinates, we have

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}\left(r\vec{e}_r\right)\\ &= \dot{r}\vec{e}_r + r\frac{du^j}{dt}\frac{d}{du^j}\Gamma^k_{rj}\vec{e}_k,\end{aligned}$$

with implicit sums over j and k. Plugging in our nonzero terms, we have

$$\frac{d\mathbf{r}}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}.$$

Now, we want to obtain a tensor out of this derivative. We will take the expression of $(*)$ and relabel the dummy indices

$$\begin{aligned}\frac{\partial}{\partial u^b}\left(A^a\vec{e}_a\right) &= \frac{\partial A^a}{\partial u^b}\vec{e}_a + A^a\frac{\partial \vec{e}_a}{\partial u^b}\\ &= \left(\frac{\partial A^a}{\partial u^b} + A^c\Gamma^a_{bc}\right)\vec{e}_a\end{aligned}$$

The quantity in the parentheses is the component of the derivative on the basis $\vec{e}_a$. We write

$$D_bA^a = \frac{\partial A^a}{\partial u^b} + A^c\Gamma^a_{bc},$$

with an implied sum over c. The expression D_bA^a gives the *covariant derivative*.

Therefore, we may obtain the general rule

$$D_cT^{ab} = \frac{\partial T^{ab}}{\partial u^c} + T^{db}\Gamma^a_{cd} + T^{ad}\Gamma^b_{cd}.$$

We need one Christoffel symbol summation for each tensor index. Furthermore, if we have a dual tensor, we have

$$D_cT_{ab} = \frac{\partial T_{ab}}{\partial u^c} - T_{db}\Gamma^d_{ac} - T_{ad}\Gamma^d_{bc}.$$

Now, if we take the covariant derivative of a scalar, we get the regular derivative:

$$D_a\Phi = \frac{\partial \Phi}{\partial u^a}.$$

Example. Consider the acceleration,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt},$$

Then, by the chain rule,

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \left(\frac{d\mathbf{r}}{t} \cdot \nabla\right)\mathbf{v}\\ &= (\mathbf{v} \cdot \nabla)\mathbf{v}.\end{aligned}$$

In components, we have

$$\begin{aligned}a^i &= \frac{dv^i}{dt}\\ &= v^k \frac{\partial v^i}{\partial x^k}.\end{aligned}$$

Note that since we are doing this in Cartesian, we do not need any Christoffel symbols.

When we are working in general coordinates, we may replace all the ∂ symbols with D_a, by the principle of covariance. Therefore, in any coordinate system, we have

$$\begin{aligned}a^b &= \frac{Dv^b}{dt}\\ &= v^c D_c v^b\\ &= v^c\left(\frac{\partial v^b}{\partial x^c} + v^d \Gamma^b_{cd}\right)\\ &= \frac{dv^b}{dt} + \Gamma^b_{cd} v^c v^d.\end{aligned}$$

Now, if we plug in the values of Γ and take the sum over b, c on the right for polar coordinates, we have

$$\begin{aligned}a^r &= \ddot{r} - r\dot{\phi}^2\\ a^\phi &= \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi}.\end{aligned}$$

Free Particle Motion

When we deal with free particle motion (i.e., not subject to any forces), we assume zero acceleration. In other words,

$$\begin{aligned}\frac{Dv^b}{t} &= \frac{dv^b}{dt} + \Gamma^b_{cd} v^c v^d\\ &= 0.\end{aligned}$$

This equation is also written

$$\frac{d^2u^b}{dt} + \Gamma^b_{cd}\frac{du^c}{dt}\frac{du^d}{dt} = 0,$$

and is known as the *geodesic equation*.

Now, if we work through the case of the geodesic with polar coordinates, we have

$$\begin{aligned}\ddot{r} - r\dot{\phi}^2 &= 0\\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} &= 0.\end{aligned}$$

We want to solve for the trajectory. There is only one solution to this set of equations:

$$r = a + bt$$

$$\phi = \phi_0.$$

This denotes straight line motion. Specifically, the geodesic is the "straight" line on any (smooth) manifold, or a straight line on a tangent space.

Now, in a 2-manifold that is not $\mathbb{R}^2$, the "curvature" appears as the term in the Christoffel symbol.

Specifically, the metric is what distinguishes one space from another space. We may find the metric by taking

$$\begin{aligned} D_c g_{ab} &= \frac{\partial g_{ab}}{\partial u^c} - g_{db}\Gamma^d_{ac} - g_{ad}\Gamma^d_{bc} \\ &= 0, \end{aligned}$$

since the metric in Cartesian coordinates is constant. Specifically, this allows us to express Γ as

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\frac{\partial}{\partial u^a}g_{bd} + \frac{\partial}{\partial u^d}g_{da} - \frac{\partial}{\partial u^d}g_{ab}\right),$$

with the implied sum over d.

Going back to the geodesic equation, we consider the question: if our space is curved, what does it mean to travel in a "straight line." Consider a particle moving along some curve C that covers a distance

$$\begin{aligned} S &= \int_C ds \\ &= \int_C \sqrt{dx^2 + dy^2}. \end{aligned}$$

However, since $dx^2 + dy^2$ is just the metric in $\mathbb{R}^n$, we may substitute it with other metrics, yielding

$$\begin{aligned} &= \int_C \sqrt{|g_{ab}\,du^a du^b|} \\ &= \int_{t_1}^{t_2} \sqrt{\left|g_{ab}\frac{du^a}{dt}\frac{du^b}{dt}\right|}\,dt. \end{aligned}$$

Now, the question that arises is what path yields an extremal path. Considering variations of C, we take $u(t) \mapsto u(t) + \delta u(t)$; we ask now what δu gives $\delta S = 0$.[xi] Then, using the chain rule, a lot of messy index notation, and the fact that g_{ab} is symmetric, we get

$$\delta S = \frac{1}{2}\int_{t_1}^{t_2}\left(g_{ab}\frac{du^a}{dt}\frac{du^b}{dt}\right)^{-1/2}\left(\left(\frac{\partial g_{ab}}{\partial u^c}\frac{du^c}{dt}\right)\frac{du^a}{dt}\frac{du^b}{dt} + 2g_{ab}\frac{d(\delta u^a)}{dt}\frac{du^b}{dt}\right)dt.$$

We may clean this up a little by taking $\frac{ds}{dt} = \sqrt{g_{ab}\frac{du^a}{dt}\frac{du^b}{dt}}$.

$$\begin{aligned} \delta S &= \frac{1}{2}\int_{s_1}^{s_2}\frac{dt}{ds}\left(\left(\frac{\partial g_{ab}}{\partial u^c}\frac{du^c}{dt}\right)\frac{du^a}{dt}\frac{du^b}{dt} + 2g_{ab}\frac{d(\delta u^a)}{dt}\frac{du^b}{dt}\right)\frac{dt}{ds}\,ds \\ &= \int_{s_1}^{s_2}\left(\frac{1}{2}\frac{\partial g_{ab}}{\partial u^c}\delta u^c\frac{du^a}{ds}\frac{du^b}{ds} + g_{ab}\frac{d(\delta u^a)}{ds}\frac{du^b}{ds}\right)ds. \end{aligned}$$

We stipulate that $u + \delta u$ has the same endpoints as u. Then, $\delta u(s_1) = \delta u(s_2) = 0$. To get rid of the term $\frac{d(\delta u^a)}{ds}$, we integrate the second term by parts, which we may do by paying for with a minus sign (the boundary term vanishes by our stipulation). This gives

$$\delta S = \int_{s_1}^{s_2}\left(\frac{1}{2}\frac{\partial g_{ab}}{\partial u^c}\frac{du^a}{ds}\frac{du^b}{ds} - \frac{\partial g_{bc}}{\partial u^a}\frac{du^a}{ds}\frac{du^b}{ds} - g_{ac}\frac{d^2u^a}{ds^2}\right)\delta u^c\,ds.$$

[xi] This is the Euler–Lagrange equation

Every partial derivative of the metric may be replaced by the Christoffel symbols, giving

$$\delta S = \int_{s_1}^{s_2} \left(\frac{d^2u^c}{ds^2} + \Gamma^c_{ab}\frac{du^a}{ds}\frac{du^b}{ds}\right) g_{cd}\delta u^d \, ds.$$

Since δu is fully arbitrary, the path $u(s)$ that solves $\delta S = 0$ is a solution to

$$\frac{d^2u^c}{ds^2} + \Gamma^c_{ab}\frac{du^a}{ds}\frac{du^b}{ds} = 0.$$

Thus, we get the geodesic equation back.

Orthogonal Functions

Recall the fundamental orthogonality relation

$$\langle \phi_n \mid \phi_m \rangle = k_n \delta_{nm},$$

where we refer to abstract vectors[XII] by $|\phi\rangle$ and abstract linear functionals[XIII] as $\langle\phi|$. In the case of $\{\phi_n\}_{n\geqslant 0}$ as a family of complex-valued functions defined over the interval $[a, b] \subseteq \mathbb{R}$, we have

$$\langle \phi_n \mid \phi_m \rangle = \int_a^b \overline{\phi_n(x)}\phi_m(x)w(x) \, dx,$$

where w is a weight function. Recall from Math Methods I that, if we have a family of polynomials $\{x^n\}_{n\geqslant 0}$, there are three (primary) ways to make a set of orthogonal (or orthonormal) polynomials upon using Gram–Schmidt.

- The Legendre polynomials have the weight $w(x) = 1$ and are defined along $[-1, 1]$.
- The Laguerre polynomials have the weight $w(x) = e^{-x}$ and are defined along $[0, \infty)$.
- The Hermite polynomials have the weight $w(x) = e^{-x^2}$ and are defined along $\mathbb{R}$.

We can theoretically use the Gram–Schmidt process to generate these families of polynomials, but that sucks.[XIV] Instead, we are interested in a more concrete, straightforward way to calculate these polynomials.

Recall from E&M[XV] that if q is a charge at point $\mathbf{r}'$ and we have a measurement device at point $\mathbf{r}$, both along the z axis, then the voltage is proportional to $\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}$. Orienting our axes, we may take $\frac{1}{|r-r'|}$.

If $r' < r$, we take

$$\frac{1}{r\left(1 - \frac{r'}{r}\right)} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell},$$

and if $r' > r$, we tkae

$$\frac{1}{r'\left(1 - \frac{r}{r'}\right)} = \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^{\ell}.$$

[XII] Also known as vectors.
[XIII] Also known as linear functionals.
[XIV] Citation needed.
[XV] I don't, but others might.

We may write this as

$$\begin{aligned}\frac{1}{|r - r'|} &= \frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell}\\ &= \frac{1}{r_>}\sum_{\ell=0}^{\infty} t^{\ell},\end{aligned}$$

where $t = \frac{r_<}{r_>}$. If we reorient our z, then the magnitude of the difference $|r - r'|$ doesn't change, but we need some function of θ, or that

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}} f(\theta).$$

Note that $f(\theta) \xrightarrow{\theta\to 0} 1$, and that our z axis is a function purely of $\cos\theta$, so we have

$$\begin{aligned} &= \sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}} P_{\ell}(\cos(\theta))\\ &= \frac{1}{r_>}\sum_{\ell=0}^{\infty} t^{\ell} P_{\ell}\cos(\theta).\end{aligned}$$

Using the law of cosines, we have

$$\begin{aligned}\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta)}}\\ &= \frac{1}{r_>\sqrt{1 + t^2 - 2xt}}.\end{aligned}$$

Taking $x = \cos(\theta)$, and defining

$$\begin{aligned} G(x,t) &= \frac{1}{\sqrt{1 + t^2 - 2xt}}\\ &= \sum_{\ell=0}^{\infty} t^{\ell} P_{\ell}(x)\end{aligned}$$

Then,

$$P_{\ell}(x) = \frac{1}{\ell!}\frac{\partial^{\ell}}{\partial t^{\ell}} G(x,t)\bigg|_{t=0}.$$

This is how we obtain the Legendre polynomials. Specifically, we obtain

$$\begin{aligned} P_0 &= 1\\ P_1 &= x\\ P_2 &= \frac{1}{2}\left(3x^2 - 1\right).\end{aligned}$$

The function $G(x,t)$ is the *generating function* of the Legendre polynomials.

Note that there is also a formula for the Legendre polynomials, known as the Rodrigues formula:

$$P_{\ell}(x) = \frac{(-1)^{\ell}}{2^{\ell}\ell!}\left(\frac{d}{dx}\right)^{\ell}\left(1 - x^2\right)^{\ell}.$$

We will focus on the generating function though. If we take an integral,

$$\int_{-1}^{-1} (G(x,t))^2\,dx = \sum_{n,m\geqslant 0}^{\infty} t^n t^m \int_{-1}^{1} P_n(x)P_m(x)\,dx$$
$$\int_{-1}^{1} \frac{1}{1-2xt+t^2}\,dx = \frac{1}{t}(\ln(1+t)-\ln(1-t)).$$

Now, taking an expansion in terms of Taylor series on the right side, we have

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2}\,dx = \sum_{n=0}^{\infty} \frac{1}{2n+1} t^{2n}.$$

Since the double sum equals a single sum, we must have a Kronecker delta, yielding

$$\int_{-1}^{1} (G(x,t))^2\,dx = \frac{2}{2n+1}\delta_{mn}.$$

Furthermore, since

$$G(x,t) = G(-x,-t),$$

we must have

$$\begin{aligned}\sum_{\ell=0}^{\infty} t^{\ell} P_{\ell}(x) &= \sum_{t=0}^{\infty} (-t)^{\ell} P_{\ell}(-x)\\ &= \sum_{t=0}^{\infty} t^{\ell}(-1)^{\ell} P_{\ell}(-x)\end{aligned}$$

meaning that $P_{\ell}(-x) = (-1)^{\ell} P_{\ell}(x)$.

For Hermite polynomials, the generating function is

$$\begin{aligned}G(x,t) &= e^{2xt-t^2}\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n.\end{aligned}$$

Consider

$$\frac{\partial G}{\partial x} = 2tG.$$

Then, in particular, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!}\frac{dH_m}{dx} t^n &= 2t\sum_{m=0}^{\infty} \frac{1}{m!} H_m(x) t^m\\ &= \sum_{m=0}^{\infty} \frac{1}{m!} H_m(x) t^{m+1}.\end{aligned}$$

Letting $m = n-1$, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{dH_n}{dx} - 2nH_{n-1}(x)\right) t^n = 0$$

Therefore,

$$\frac{dH_n}{dx} = 2nH_{n-1}, \tag{$*$}$$

a two-term recurrence relation. We can also take

$$\frac{\partial G}{\partial t} = 2(x-t)G,$$

from which we are able to obtain a three-term recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1} = 0.$$

Taking a derivative of ($*$) and inserting our three-term recurrence relation, we get the equation

$$\frac{d^2H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n = 0.$$

There is a similar second-order differential equation for the Legendre polynomials,

$$\left(1-x^2\right)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n = 0.$$

These equations are known as Hermite's equation and Legendre's equation (respectively). However, it is not apparent from this construction that we get orthogonality.

As it turns out, this comes from eigenvectors and eigenvalues. The equation

$$\frac{d^2f}{dx^2} = -k^2f$$

has the operator $\frac{d^2}{dx^2}$ and eigenvalue $-k^2$. Since these eigenvalues are real, we must have $\frac{d^2}{dx^2}$ be a self-adjoint (or Hermitian[xvi]) operator, admitting an orthonormal eigenbasis.

Beyond the Straight and Narrow

Recall the definition of a Fourier series:

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)\\
&= \sum_{n=-\infty}^{\infty} c_n e^{-inx}\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\\
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx
\end{aligned}$$

This is the most simple scenario, where f is defined on the interval $[-\pi,\pi]$.

Consider what happens when we want to do this in more than one dimension. Setting $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ik_xx}e^{ik_yy}$, where we have a rectangular region $L_1 \times L_2$, and setting $k_x = \frac{n\pi}{L_1}$, $k_y = \frac{m\pi}{L_2}$.

[xvi] There is a difference that I don't want to try to learn yet.

This gives

$$f(x,y) = \sum_{n,m=-\infty}^{\infty} \hat{f}_{n,m} e^{i\pi\left(\frac{n}{L_1}x + \frac{m}{L_2}y\right)}.$$

Then,

$$\hat{f}_{n,m} = \int_{-L_1}^{L_1}\int_{-L_2}^{L_2} f(x,y)e^{-i\frac{n\pi}{L_1}x}e^{-i\frac{m\pi}{L_2}y}\,dy\,dx.$$

Obviously, this becomes very difficult in more dimensions, but it can be done.

Rectangles are not very aesthetically pleasing, though. Instead, we are interested in applying it to a circular plate.

Our values of k_x, k_y in the case of a rectangle arrived from imposing a periodic boundary condition. However, circles are already periodic, so we may consider a different value. We take

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos(\gamma)},$$

where $\gamma = \phi' - \phi$ is the angle between the $\mathbf{k}$ vector and the $\mathbf{r}$ vector. Since $\cos(\gamma)$ is periodic, we may expand in terms of $e^{in\gamma}$, giving

$$e^{ikr\cos(\gamma)} = \sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{-in\phi'}e^{in\phi},$$

We need the coefficient of i^n to ensure that $J_n(kr)$ is real. We may calculate

$$J_n(x) = \frac{1}{2\pi i^n}\int_{-\pi}^{\pi} e^{ix\cos(\gamma)}e^{-in\gamma}\,d\gamma$$

The family $\{J_n\}_{n\in\mathbb{Z}}$ are called (cylindrical) *Bessel functions*. We are also able to write the Bessel functions as

$$\begin{aligned}J_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\gamma)-in\gamma}\,d\gamma\\ &= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin(\gamma)-n\gamma)\,d\gamma.\end{aligned}$$

Note that in the asymptotic limit, we have

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$

and for every small x,

$$J_n(x) = \frac{x^n}{2^n n!}.$$

Now that we have expanded upon a line, rectangle, and disk, we will now expand upon a sphere. On the unit sphere, we may have a function $f(\theta,\phi)$, where $0 \leqslant \theta \leqslant \phi$ and $0 \leqslant \phi \leqslant 2\pi$. We may consider if we're allowed to expand f as follows:

$$f(\theta,\varphi) \overset{?}{=} \underbrace{\left(\sum_{\ell=0}^{\infty} c_\ell P_\ell\cos(\theta)\right)}_{\text{Legendre}}\underbrace{\left(\sum_{m=-\infty}^{\infty} d_m e^{im\phi}\right)}_{\text{Fourier}}. \qquad (*)$$

Unfortunately, this is not the right answer. It might work on the equator, but as we reduce the value of θ going towards the north pole, we pick up a factor of $\sin(\theta)$; in the limit as $\theta \to 0$, ϕ is not defined. Therefore, we need some factor of $\sin(\theta)$.

We introduce some factors of $\sin^m(\theta) = \left(1-\cos^2(\theta)\right)^{m/2}$. This gives an extra order of m in $\cos(\theta)$, so we need to reduce the order by m. We take $\left(\frac{d}{dx}\right)^m P_\ell(x)$.

We don't have Legendre polynomials anymore; instead, we have

$$P_{\ell,m}(x) = \left(1-x^2\right)^{m/2}\left(\frac{d}{dx}\right)^m P_\ell(x).$$

Substituting the Rodrigues formula, we have

$$P_{\ell,m}(x) = \frac{(-1)^\ell}{2^\ell \ell!}\left(1-x^2\right)^{m/2}\left(\frac{d}{dx}\right)^{\ell+m}\left(1-x^2\right)^\ell.$$

These are known as the *associated Legendre functions*, where $m \geqslant 0$. Note that if $m = 0$, we get the Legendre polynomials, and if $m > \ell$, this evaluates to 0. However, we are able to expand to include negative values of m by taking

$$P_{\ell,-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(x).$$

As it turns out, the family $P_{\ell,m}$ are orthogonal (holding m constant), with orthogonality relation

$$\int_{-1}^{1} P_{\ell,m}(x) P_{\ell',m}(x)\ dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}.$$

Thus, the fixed version of $(*)$, is

$$Y_{\ell,m}(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell,m}(\cos(\theta)) e^{im\phi}.$$

The family of $Y_{\ell,m}$ are indeed orthogonal functions. Furthermore, they are orthonormal in both ℓ and m:

$$\int Y_{\ell,m}(\theta,\phi)\overline{Y_{\ell',m'}(\theta,\phi)}\ d\Omega = \delta_{\ell\ell'}\delta_{mm'},$$

so that

$$a_{\ell,m} = \int f(\theta,\phi)\overline{Y_{\ell,m}(\theta,\phi)}\ d\Omega.$$

We are allowed to take any function and expand it on a basis of spherical harmonics, giving

$$\begin{aligned} f(\theta,\phi) &= f(\hat{n}) \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\infty}^{\infty} a_{\ell,m} Y_{\ell,m}(\theta,\phi). \end{aligned}$$

There are some important symmetry properties. First,

$$Y_{\ell,-m} = (-1)^m \overline{Y_{\ell,m}},$$

and under parity transformation, sending $\mathbf{r} \mapsto -\mathbf{r}$, we get

$$Y_{\ell,m}(\theta,\phi) \mapsto (-1)^\ell Y_{\ell,m}(\theta,\phi).$$

Example. In the case of $m = 0$, the $P_{\ell,m}$ are purely functions of $\cos(\theta)$, while for each increased value of m, there are factors of $\sin(\theta)$, with a factor of $e^{im\phi}$ increased.

Specifically,

$$Y_{\ell,0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_{\ell}(\cos(\theta)).$$

In other words, if $m = 0$, we have azimuthal symmetry. We also say it has "no ϕ dependence."

Example. Consider the case of $Y_{\ell,m}(\hat{z})$. Then, since $\theta = 0$ and ϕ is undefined, we must have no ϕ dependence, giving

$$\begin{aligned}Y_{\ell,m}(\hat{z}) &= \sqrt{\frac{2\ell+1}{4\pi}}P_{\ell}(1)\delta_{m0}\\ &= \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m0}.\end{aligned}$$

Now that we know that $\{Y_{\ell,m}\}_{\ell,m}$ are orthonormal, we need them to be complete too. Specifically, we need

$$\begin{aligned}f(\theta,\phi) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_{\ell,m}\\ a_{\ell,m} &= \langle \ell, m \,|\, f\rangle\\ &= \int f(\theta,\phi)\overline{Y_{\ell,m}(\theta,\phi)}\,d\Omega.\end{aligned}$$

Example. We want to expand $\delta(\hat{n}-\hat{n}')$. This gives

$$\begin{aligned}\delta(\cos(\theta)-\cos(\theta'))\delta(\phi-\phi') &= \delta(\hat{n}-\hat{n}')\\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_{\ell,m}(\theta,\phi)\\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\int \delta\left(\hat{\xi}-\hat{n}'\right)\overline{Y_{\ell,m}\left(\hat{\xi}\right)}\,d\Omega\right)Y_{\ell,m}(\hat{n})\\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\overline{Y_{\ell,m}(\hat{n}')}Y_{\ell,m}(\hat{n}).\end{aligned}$$

Now, if we let $\hat{n}' = \hat{z}$, we have

$$\begin{aligned}\delta(\hat{n}-\hat{z}) &= \sum_{\ell=0}^{\infty} Y_{\ell,0}(\hat{z})Y_{\ell,0}(\hat{n})\\ &= \sum_{\ell=0}^{\infty}\left(\sqrt{\frac{2\ell+1}{4\pi}}P_{\ell}(1)\right)\left(\sqrt{\frac{2\ell+1}{4\pi}}P_{\ell}(\cos(\theta))\right)\\ &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}P_{\ell}(\cos(\theta))\\ &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}P_{\ell}(\hat{n}\cdot\hat{z}).\end{aligned}$$

Since dot products are coordinate-independent, we must have

$$\delta(\hat{n}-\hat{n}') = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos(\gamma)),$$

where we defined

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos(\gamma)}.$$

This allows us to derive the *spherical harmonic addition theorem*. We get

$$P_\ell(\cos(\gamma)) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \overline{Y_{\ell,m}(\theta',\phi')} Y_{\ell,m}(\theta,\phi).$$

Example. Recall that

$$\begin{aligned}\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} &= \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos(\theta))\\ &= \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \sum_{m=-\ell}^{\ell} \overline{Y_{\ell,m}(\theta',\phi')} Y_{\ell,m}(\theta,\phi).\end{aligned}$$

Sticking this sum into an integral, and using the fact that $Y_{0,0} = \frac{1}{\sqrt{4\pi}}$

$$\begin{aligned}\frac{1}{4\pi}\int \frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\,d\Omega &\sim \int Y_{\ell,m}(\theta,\phi)\,d\Omega\\ &= \sqrt{4\pi}\int (Y_{0,0}(\theta,\phi))Y_{\theta,\phi}\,d\Omega\\ &= \delta_{\ell 0}\delta_{m0}.\end{aligned}$$

Moving from the sphere to the ball., we have

$$f(\mathbf{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}(r) Y_{\ell,m}(\theta,\phi).$$

Consider the case of $e^{i\mathbf{k}\cdot\mathbf{r}}$. Then,

$$\begin{aligned}e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}(r,\mathbf{k}) Y_{\ell,m}(\hat{r})\\ &= \sum_{\ell,m}\left(\sum_{\ell',m'} \hat{a}_{\ell,m,\ell',m'} Y_{\ell',m'}\left(\hat{k}\right)\right) Y_{\ell,m}(\hat{r})\end{aligned}$$

- Recalling $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos(\gamma)}$. Therefore, we must have, $a(r,k) = a(rk)$.
- Furthermore, using the identity $\cos(\gamma) = \cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta')\cos(\phi-\phi')$, meaning we must have $m = m'$.
- We use the symmetry relation

$$Y_{\ell,-m} = (-1)^m \overline{Y_{\ell,m}}.$$

- Finally, we must have $\ell = \ell'$ to remove ambiguity.

In total, we get

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_\ell(kr)\overline{Y_{\ell,m}\left(\hat{k}\right)}Y_{\ell,m}(\hat{r}).$$

Define

$$j_\ell(kr) = \frac{c_\ell(kr)}{4\pi i^\ell}.$$

This gives

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty} i^\ell i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos(\gamma)).$$

The family j_ℓ are called the *spherical Bessel functions*.

We find the expression for the spherical Bessel function by projecting onto the Legendre polynomials, giving

$$j_\ell(x) = \frac{(-i)^\ell}{2}\int_{-1}^{1} e^{ixu}P_\ell(u)\,du.$$

As it turns out the spherical Bessel functions can be written entirely in terms of sines and cosines (with some factors of x). For instance,

$$\begin{aligned} j_0 &= \frac{\sin(x)}{x}\\ j_1 &= \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}. \end{aligned}$$

Space	Expression
Rectangular	$e^{ik_x x}e^{ik_y y}e^{ik_z z}$
Cylndrical	$e^{ik_z z}\sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{in(\phi-\phi')}$
Ball (Spherical Harmonics)	$4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr)\overline{Y_{\ell,m}\left(\hat{k}\right)}Y_{\ell,m}(\hat{r})$
Ball (Legendre Polynomials)	$\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)j_\ell(kr)P_\ell(\cos(\theta))$.

Table 2: Expansions of $e^{i\mathbf{k}\cdot\mathbf{r}}$

Differential Equations

To start our investigation of differential equations, we start by discussing linearity.

Abstractly, we consider some linear operator

$$\mathcal{L}_q = \sum_{j=0}^{n} a_j(q)\frac{d^j}{dq^j}.$$

The reason we care about linearity is that if

$$\mathcal{L}_q(u) = 0$$

has solutions $u_1, \dots, u_n$, then

$$\mathcal{L}\left(\sum_{j=1}^{n} a_j u_j\right) = \sum_{j=1}^{n} a_j \mathcal{L}(u_j).$$

Equations of the form

$$\mathcal{L}_q(u) = 0$$

for some differential operator $\mathcal{L}$ are known as *homogeneous* equations. Inhomogeneous equations are of the form

$$\mathcal{L}_q(u) = r(q).$$

First-Order Equations

Consider the general free-falling mass equation,

$$m\frac{dv}{dt} = mg,$$

with $v(0) = v_0$. Then, $v(t) = v_0 + gt$.

If we add linear drag, $f_d = -bv$, we have

$$\frac{dv}{dt} = -\frac{b}{m}v + g. \tag{†}$$

Separating variables,

$$\frac{dv}{bv/m - g} = -dt,$$

and integrating, we have

$$\ln(bv/m - g) = -\frac{bt}{m} + C.$$

Exponentiating,

$$v(t) = ke^{-bt/m} + mg/b,$$

where $k = v_0 - mg/b$. We can see that terminal velocity is mg/b.

What makes these examples relatively easy is that these equations are separable. We are able to write it of the form

$$\alpha(x)dx + \beta(y)dy = 0,$$

meaning the solution is

$$\int \alpha(x)\,dx = -\int \beta(y)\,dy + C.$$

Now, in the general case, we have

$$\frac{dy}{dx} + q(x)y = r(x).$$

To solve this equation, we can introduce an integrating factor to make this equation separable. We want to find $w(x)$ for this purpose. Take

$$\frac{d}{dx}(wy) = w\frac{dy}{dx} + \frac{dw}{dx}y \tag{1*}$$

We want $\frac{dw}{dx} = qw$, so that $w(x) = \exp\left(\int q(x)\,dx\right)$. Multiplying on both sides, we have

$$\frac{d}{dx}(wy) = wr,$$

and

$$y = \frac{1}{w(x)}\int w(x)r(x)\,dx.$$

For instance, considering (†) again, we have $w(t) = e^{bt/m}$, and

$$\begin{aligned} v(t) &= e^{-bt/m}\int ge^{bt/m}\,dt \\ &= e^{-bt/m}\left(\frac{mg}{b}e^{bt/m} + k\right) \\ &= ke^{-bt} + \frac{mg}{b}. \end{aligned}$$

Example. Consider the falling raindrop. Let $m(0) = m_0$.

Now, as the raindrop falls, it increases in weight. Rewriting

$$\begin{aligned} F &= ma \\ &= \frac{dp}{dt}, \end{aligned}$$

we instead write

$$F = m\dot{v} + \dot{m}v. \tag{$*$}$$

We neglect drag, so

$$m\dot{v} + \dot{m}v = mg.$$

We assume that the rate of mass increase is proportional to $4\pi r^2$. We also assume constant density ρ proportional to m/r^3. Putting these two together, we may assume

$$\frac{dm}{dt} = km^{2/3}.$$

We are able to write our differential equation in terms of m, by using the chain rule to get

$$\frac{dv}{dt} = \frac{dm}{dt}\frac{dv}{dm}$$

$$= km^{2/3}\frac{dv}{dm}.$$

Our differential equation $(*)$ now becomes

$$\frac{dv}{dm} + \frac{v}{m} = \frac{g}{km^{2/3}},$$

with assumption that $v(m_0) = 0$. We use the integrating factor of $w(m) = \exp\left(\int \frac{1}{m}\,dm\right)$, giving

$$m\frac{dv}{dm} + v = \frac{gm^{1/3}}{k}. \tag{$**$}$$

This gives

$$\begin{aligned}\frac{d}{dm}(mv) &= \frac{gm^{1/3}}{k}\\ mv &= \frac{3}{4}\frac{gm^{4/3}}{k} + C\\ v &= \frac{3}{4}\frac{gm^{1/3}}{k} + \frac{C}{m}.\end{aligned}$$

Inputting our initial condition, we have

$$v(m) = \frac{3g}{4k}m^{1/3}\left(1 - \left(\frac{m_0}{m}\right)^{4/3}\right).$$

In particular,

$$\begin{aligned}\dot{v} &= g - \frac{\dot{m}}{m}v\\ &= \frac{g}{4}\left(1 + 3\left(\frac{m_0}{m}\right)^{3/4}\right).\end{aligned}$$

In this case, there is no terminal velocity.

Example. Consider

$$\alpha(x,y)dx + \beta(x,y)dy = 0.$$

This appears to be of the form

$$\begin{aligned}d(\Phi(x,y)) &= \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy\\ &= 0.\end{aligned}$$

In particular, if we can find Φ that matches the partials, then Φ is a constant.

Assuming that Φ is C^2, then the equality of mixed partials gives

$$\frac{\partial\alpha}{\partial y} = \frac{\partial\beta}{\partial x}. \tag{$\dagger$}$$

If solutions exist, then we say the equation is *exact*.

Returning to the falling raindrop, and rewriting $(**)$, we have

$$m\,dv + \left(v - \frac{g}{k}m^{1/3}\right)dm = 0. \tag{$\ddagger$}$$

Therefore, $\alpha(\nu, m) = m$ and $\beta(\nu, m) = \nu - \frac{g}{k}m^{1/3}$. We want to find if

$$\frac{\partial \alpha}{\partial m} = \frac{\partial \beta}{\partial \nu}.$$

The answer is yes, so it can be rendered in exact form. We will write

$$\begin{aligned}\frac{\partial \Phi}{\partial \nu} &= m\\ \frac{\partial \Phi}{\partial m} &= \nu - \frac{g}{k}m^{1/3}.\end{aligned}$$

Integrating the first equation, we have $\Phi(m, \nu) = m\nu + c_1(m)$, and integrating the second, $\Phi(m, \nu) = m\nu - \frac{3g}{4k}m^{4/3} + c_2(\nu)$. We can set $c_2(\nu) = 0$ and $c_1(m) = \frac{3g}{4k}m^{4/3}$.

Example. If our equation (†) does not hold, then we need to find an integrating factor $w(x, y)$ such that

$$\frac{\partial}{\partial y}(w\alpha) = \frac{\partial}{\partial x}(w\beta).$$

Now, if we take $w = w(x)$, we get the equation

$$\begin{aligned}\frac{dw}{dx} &= \frac{w}{\beta}\left(\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x}\right)\\ &= p(x)w(x).\end{aligned}$$

This gives $w(x) = \exp\left(\int p(x)\,dx\right)$. In particular, we need

$$p(x) = \frac{1}{\beta}\left(\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x}\right).$$

If we had $w(y)$, then

$$p(y) = \frac{1}{\alpha}\left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right).$$

Example. Consider a refinement of the falling raindrop model.

Rather than the surface area, we say that the growth rate depends on the volume swept out per unit time. In particular, this volume swept out per unit time is $A\nu$, so

$$\frac{dm}{dt} = km^{2/3}\nu.$$

This changes (‡) to

$$\nu d\nu + \left(\frac{\nu^2}{m} - \frac{g}{km^{2/3}}\right)dm = 0.$$

This is not an exact equation. We find the integrating factor

$$\begin{aligned}p(m) &= \frac{1}{\alpha}\left(\frac{\partial \beta}{\partial \nu} - \frac{\partial \alpha}{\partial m}\right)\\ &= \frac{1}{\nu}\left(\frac{2\nu}{m}\right)\\ &= \frac{2}{m}.\end{aligned}$$

Our integrating factor is $w(m) = \exp\left(\int p(m)\, dm\right)$, or $w(m) = m^2$.

With much tedious symbol-pushing, we find

$$\begin{aligned}\Phi(v, m) &= \frac{1}{2}m^2v^2 - \frac{3g}{7k}m^{7/3}\\ &= C.\end{aligned}$$

Plugging in initial conditions, with $v(m_0) = 0$, we get

$$\dot{v} = \frac{g}{7}\left(1 + 6\left(\frac{m_0}{m}\right)^{7/3}\right).$$

Second-Order Equations

When we consider second-order equations, we have the form

$$\begin{aligned}\mathcal{L}[u(x)] &= r(x)\\ \left(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right)(u(x)) &= r(x).\end{aligned}$$

We expect two linearly independent solutions, $u_1(x)$ and $u_2(x)$.

Now, we still need boundary conditions. If we have boundary conditions

$$\begin{aligned}u_1(0) &= 1\\ \left.\frac{du_1}{dx}\right|_0 &= 0\\ u_2(0) &= 0\\ \left.\frac{du_2}{dx}\right|_0 &= 1.\end{aligned}$$

Then, the solutions are of the form

$$u(x) = Au_1(x) + Bu_2(x),$$

where $u(0) = A$ and $u'(0) = B$.

Example (A Constant-Coefficient Equation)**.** Consider the constant-coefficient second-order equation

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega_0^2 u = 0.$$

This is the equation for damped simple harmonic motion.

We let $D = \frac{d}{dt}$. We have the equation

$$\begin{aligned}\left(D^2 + 2\beta D + \omega_0^2\right)u(t) &= (D - \alpha_1)(D - \alpha_2)u\\ &= 0,\end{aligned}$$

where

$$\alpha_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}.$$

We have "factored" our second-order equation into two first-order equations, yielding

$$\begin{aligned}(D-\alpha_2)u(t) &= v(t)\\ (D-\alpha_1)v(t) &= 0.\end{aligned}$$

The second equation yields

$$v(t) = Ae^{\alpha_1 t}.$$

Now, we must solve

$$(D-\alpha_2)u(t) = Ae^{\alpha_1 t}.$$

With an integrating factor of $w = e^{-\alpha_2 t}$, as in (1*), we get

$$\begin{aligned}u(t) &= e^{\alpha_2 t}\int e^{-\alpha_2 t}v(t)\,dt\\ &= e^{\alpha_2 t}\int Ae^{(\alpha_1-\alpha_2)t}\,dt\\ &= \frac{1}{\alpha_1-\alpha_2}\left(Ae^{\alpha_1 t}+Be^{\alpha_2 t}\right).\end{aligned}$$

We're fine and dandy as long as $\alpha_1 \neq \alpha_2$.

Example. To solve the equation for damped simple harmonic motion, we use the guess

$$u = e^{\alpha t}.$$

This gives the equation

$$e^{\alpha t}\left(\alpha^2+2\beta\alpha+\omega_0^2\right) = 0.$$

Example. Consider the equation

$$c_2x^2\frac{d^2u}{dx^2}+c_2x\frac{du}{dx}+c_0u = 0,$$

where c_i are constant.

To solve this equation, we use the guess $u = x^\alpha$.

The Wronskian

We want to establish that our solutions u_1, u_2 of a differential equation are indeed linearly independent.

Now, if u_1, u_2 are indeed linearly independent, then our solutions are of the form

$$\begin{aligned}u &= c_1u_1+c_2u_2\\ \frac{du}{dx} &= c_1\frac{du_1}{dx}+c_2\frac{du_2}{dx}.\end{aligned}$$

In other words, we have

$$\begin{pmatrix}u\\ \frac{du}{dx}\end{pmatrix} = \begin{pmatrix}u_1 & u_2\\ \frac{du_1}{dx} & \frac{du_2}{dx}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}.$$

This has a unique solution if and only if the matrix has a nonzero determinant. The *Wronskian* is

$$\begin{aligned} W(x) &= \det\begin{pmatrix} u_1 & u_2 \\ \frac{du_1}{dx} & \frac{du_2}{dx} \end{pmatrix} \\ &= u_1\frac{du_2}{dx} - u_2\frac{du_1}{dx}. \end{aligned}$$

As it turns out, the Wronskian only needs to be nonzero at one point. This can be seen by taking

$$\frac{dW(x)}{dx} = -p(x)W(x),$$

meaning

$$W(x) = W(a)\exp\left[\int_a^x p(s)\,ds\right].$$

This also means that we can find the Wronskian *without* knowing the solutions to the equation.

Example. With simple harmonic motion,

$$\frac{d^2u}{dx^2} + k^2u = 0,$$

we have the solutions $\cos(kx), \sin(kx)$, meaning

$$\begin{aligned} W(x) &= \det\begin{pmatrix} \cos(kx) & \sin(kx) \\ -k\sin(kx) & k\cos(kx) \end{pmatrix} \\ &= k \\ &\neq 0. \end{aligned}$$

Now, to obviate the issue of $k \to 0$ yielding a zero Wronskian (instead of linear motion), we take $u_2 = \frac{1}{k}\sin(kx)$, which gives the Wronskian of 1, and obviates this issue.

Now, for the damped oscillation, we get

$$W(x) = 2\sqrt{\beta^2 - \omega_0^2}e^{-2\beta t}.$$

Since W determines linear independence, if we're given one solution, we are able to find another solution. Taking

$$\begin{aligned} \frac{d}{dx}\left(\frac{u_2}{u_1}\right) &= \frac{u_1(x)\frac{du_2}{dx} - \frac{du_1}{dx}u_2(x)}{u_1(x)^2} \\ &= \frac{W}{u_1^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} u_2(x) &= u_1(x)\int \frac{W(x)}{u_1(x)^2}\,dx \\ &= u_1(x)\int \frac{\exp\left[-\int^x p(s)\,ds\right]}{u_1(x)^2}\,d \end{aligned}$$

Example. Consider Legendre's equation,

$$\left(1-x^2\right)\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + \ell(\ell+1)\phi = 0.$$

Here,

$$p(x) = -\frac{2x}{1-x^2},$$

so that

$$\begin{aligned} W(x) &= W(a)\exp\left[-\int -\frac{2x}{1-x^2}\,dx\right] \\ &= \frac{1}{1-x^2} && W(a) := -1 \end{aligned}$$

We know that a solution of this equation will be $P_\ell(x)$. After we use the Wronskian, we obtain the "second solution" to the equation,

$$Q_\ell(x) = P_\ell(x)\int \frac{1}{(1-x^2)P_\ell(x)^2}\,dx,$$

which are known as the *Legendre functions of the second kind*.

These functions are not particularly well-behaved. For instance,

$$\begin{aligned} Q_0(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\ Q_1(x) &= \frac{1}{2}x\ln\left(\frac{1+x}{1-x}\right) - 1. \end{aligned}$$

These functions blow up at the boundary.

Example. When we have an inhomogeneous equation, we will write $u_p(x)$ for the inhomogeneous solution. If we want to know the value of $u_p(x_0)$, then since u_1, u_2 are linearly independent, we should be able to write

$$u_p(x_0) = c_1u_1(x_0) + c_2u_2(x_0).$$

Now, furthermore, we also have

$$\left.\frac{du_p}{dx}\right|_{x_0} = c_1\left.\frac{du_1}{dx}\right|_{x_0} + c_2\left.\frac{du_2}{dx}\right|_{x_0}.$$

This cannot hold for all values of x_0. We can modify this by taking c_1, c_2 to be functions of x, giving an expression of the form

$$u_p(x) = a_1(x)u_1(x) + a_2(x)u_2(x).$$

This procedure is known as *variation of parameters*. Now, if this is supposed to work, we must also have

$$\frac{du_p}{dx} = a_1(x)\frac{du_1}{dx} + a_2(x)\frac{du_2}{dx}.$$

This can only be consistent if

$$u_1(x)\frac{da_1}{dx} + u_2(x)\frac{da_2}{dx} = 0.$$

We need one more condition. We're tempted to take another derivative of u_p, but this doesn't give us any more information. As it turns out, by playing around with the differential equation, we are able to obtain

$$\begin{aligned} a_1(x) &= -\int \frac{u_2(x)r(x)}{W(x)}\,dx \\ a_2(x) &= \int \frac{u_1(x)r(x)}{W(x)}\,dx. \end{aligned}$$

Series Solutions

If our solution $u(x)$ is sufficiently smooth, we should be able to write

$$u(x) = \sum_{m=0}^{\infty} c_m x^m,$$

where

$$c_m = \frac{1}{m!}\frac{d^m u}{dx^m}\bigg|_{x=0}.$$

We'll do this backwards.

To solve for the c_m, we'll throw this expression for u into the differential equation. We'll tweak our guess just a little bit, where we take

$$u(x) = x^{\alpha}\sum_{m=0}^{\infty} c_m x^m, \tag{‡}$$

where $c_0 \neq 0$.

Example. Consider the equation

$$4xu'' + 2u' - u = 0.$$

Upon sticking in our guess, we have

$$\sum_{m=0}^{\infty} c_m 2(m+\alpha)(2m+2\alpha-1)x^{m+\alpha-1} - \sum_{m=0}^{\infty} c_m x^{m+\alpha} = 0.$$

We match powers to obtain our guess. The lowest order is $\alpha - 1$, meaning

$$c_0\alpha(2\alpha - 1) = 0. \tag{$*$}$$

For all higher powers, we have

$$c_m = \frac{1}{2(m+\alpha)(2m+2\alpha-1)}c_{m-1}. \tag{†}$$

We call $(*)$ the *indicial equation* so that we may find α. We get the solutions of $\alpha = 0$ and $\alpha = \frac{1}{2}$.

Taking $\alpha = \frac{1}{2}$, we insert it into (†) to take

$$\begin{aligned} c_m &= \frac{1}{(2m+1)(2m)}c_{m-1} \\ &= \frac{1}{(2m+1)(2m)}\frac{1}{(2m-1)(2m-2)}c_{m-2} \end{aligned}$$

$$\begin{aligned}&\vdots\\&= \frac{c_0}{(2m+1)!}.\end{aligned}$$

We get our first solution,

$$u_1(x) = \sqrt{x}\left(1 + \frac{1}{3!}x + \frac{1}{5!}x^2 + \cdots\right).$$

Next, we take $\alpha = 0$, inserting into (†) to get

$$\begin{aligned}c_m &= \frac{1}{2m(2m-1)}c_{m-1}\\&= \frac{1}{2m(2m-1)(2m-2)(2m-3)}c_{m-2}\\&\vdots\\&= \frac{c_0}{(2m)!}.\end{aligned}$$

Thus, we get the second solution of

$$u_2(x) = x^0\left(1 + \frac{x}{2!} + \frac{x^2}{4!} + \cdots\right).$$

As it turns out, our expressions for u_1 and u_2 have closed-form solutions,

$$\begin{aligned}u_1(x) &= \sinh\left(\sqrt{x}\right)\\u_2(x) &= \cosh\left(\sqrt{x}\right).\end{aligned}$$

Example. Consider the equation

$$x^2\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + x^2u = 0. \tag{**}$$

Sticking in (‡), we have

$$\sum_{m=0}^{\infty} c_m(m+\alpha)(m+\alpha+1)x^{m+\alpha} + \sum_{m=0}^{\infty} c_m x^{m+\alpha+2} = 0.$$

Taking the two lowest orders out,we write explicitly

$$c_0(\alpha)(\alpha+1)x^{\alpha} + c_1(\alpha+1)(\alpha+2)x^{\alpha+1} + \sum_{m=2}^{\infty}(c_m(m+\alpha)(m+\alpha+1) + c_{m-2})x^{m+\alpha} = 0.$$

Since c_0 cannot equal zero, the first indicial equation yields $\alpha = 0, -1$.

If $\alpha = 0$, then $c_1 = 0$, and our recursion relation is

$$c_m = \frac{(-1)^{m/2}}{(m+1)!}c_0$$

for m even. Plugging this in, we get

$$u_1(x) = x^0\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots\right)$$

$$= \frac{\sin(x)}{x}.$$

Now, if $\alpha = -1$, then we may choose c_1 to be any value we desire. We will choose $c_1 = 0$. We get the even power recursion relation

$$u_2(x) = \frac{1}{x}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$
$$= \frac{\cos(x)}{x}.$$

Now, if we had chosen any other value for c_1, the result would have been a linear combination of u_1 and u_2, which would have been more unwieldy.

In (**), we note that the differential operator

$$\mathcal{L}_x = x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2$$

has definite parity. Therefore, we get even and odd solutions.

Practically, we want to know whether or not we even have a convergent power series. In the general form,

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0,$$

there is *at least one* series solution centered at x_0 if and only if

$$\lim_{x\to x_0}(x - x_0)p(x) = 0$$
$$\lim_{x\to x_0}(x - x_0)^2q(x) = 0.$$

Now, if we want a second solution, there is a bit more work to be done, but we can show that if we have two roots of the indicial equation, α_1, α_2, then if $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, there exists a series solution. Meanwhile, if $\alpha_1 - \alpha_2 \in \mathbb{Z}$, we use the guess of

$$u_2 = Au_1\ln|x| + x^{\alpha_2}\sum_{m=0}^{\infty}c_m x^m.$$

Example (Bessel's Equation)**.** Bessel's equation is is a one-parameter family of equations of the form

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(x^2 - \lambda^2\right)u = 0.$$

We modify our guess from (‡) by taking

$$u(x) = x^{\alpha}\sum_{m=0}^{\infty}\frac{1}{m!}c_m x^m.$$

Plugging this into the equation, we get

$$\sum_{m=0}^{\infty}\frac{1}{m!}c_m\left((m+\alpha)^2 - \lambda^2\right)x^{m+\alpha} + \sum_{m=0}^{\infty}\frac{1}{m!}c_m x^{m+\alpha+2} = 0.$$

Shifting indices, we get

$$c_0\left(\alpha^2 - \lambda^2\right)x^{\alpha} + c_1\left((\alpha+1)^2 - \lambda^2\right)x^{\alpha+1} = \sum_{m=2}^{\infty}\left(\frac{c_m}{m!}\left((m+\alpha)^2 - \lambda^2\right) - \frac{c_{m-2}}{(m-2)!}\right)x^{m+\alpha}$$

$$= 0.$$

Since $c_0 \neq 0$, we have $\alpha = \pm\lambda$, and $c_1(1 \pm 2\lambda) = 0$. Our recurrence relation is

$$c_m = \frac{-(m-1)}{(m \pm 2\lambda)} c_{m-2}.$$

Expanding, we get Equation (39.75) in the book. If we divide out by $2^\lambda \lambda!$, we get the equation

$$J_\lambda(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\lambda+1)} \left(\frac{x}{2}\right)^{m+\lambda}.$$

Example. Legendre's equation is

$$\left(1 - x^2\right)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \lambda u = 0.$$

This equation has definite parity, and we expect an "every-other" recurrence relation. The indicial equations are

$$c_0\alpha(\alpha - 1) = 0$$
$$c_1\alpha(\alpha + 1) = 0.$$

We get $\alpha = 1, 0$, and the recurrence relation for $\alpha = 1$ of

$$c_m = \frac{m(m-1) - \lambda}{(m+1)m} c_{m-2}.$$

Evaluating this expansion, we get

$$u_1(x) = x + \frac{(2-\lambda)}{3!}x^3 + \frac{(2-\lambda)(12-\lambda)}{5!}x^5 + \frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!}x^7 + \cdots$$

Meanwhile, if $\alpha = 0$, we select $c_1 = 0$, and get the recurrence relation of

$$c_m = \frac{(m-1)(m-2) - \lambda}{m(m-1)} c_{m-2},$$

and series of

$$u_2 = 1 - \frac{\lambda}{2!}x^2 - \frac{\lambda(6-\lambda)}{4!}x^4 - \frac{\lambda(6-\lambda)(20-\lambda)}{6!}x^6 + \cdots$$

Evaluating the radius of convergence,

$$\begin{aligned} x^2 &< \lim_{m\to\infty} \left|\frac{c_{m-2}}{c_m}\right| \\ &= 1. \end{aligned}$$

Note that for large m, though, evaluated at ± 1, we have

$$\left|\frac{c_{m-2}}{c_m}\right| \approx 1 - \frac{2}{m}.$$

This is effectively a harmonic series, which diverges. In order to cause our series to converge at ± 1, we need specific values of λ to cause the series to truncate.

If we set $\lambda = \ell(\ell+1)$, then selecting $\ell = 0, 1, 2, \ldots$, either the even or odd series necessarily truncates.

Sturm–Liouville Problems

How does a differential equation yield orthogonal solutions?

We have dealt with orthogonal polynomials via Gram–Schmidt and generating functions, but originally they were defined by differential equations.

Consider the differential equation

$$\frac{d^2\phi}{dx^2} = -k^2\phi. \tag{†}$$

The solutions here are sines and cosines. Now, if we define $D = \frac{d}{dx}$, then

$$D^2\phi = -k^2\phi$$

looks suspiciously like an eigenvalue problem.

Now, this eigenvalue equation does not necessarily guarantee orthogonality. We need one more thing here in order to be able to find orthogonality.

Recall from linear algebra that if an operator is self-adjoint, then the operator has real eigenvalues and orthogonal eigenvectors. Since L_2 is an inner product space, we want to be able to find out the definition of D^* — ideally, it is equal to D.

Recall that if $\mathcal{L}$ is a linear operator, the definition of $\mathcal{L}^*$ is the unique operator such that

$$\langle\psi \mid \mathcal{L}\phi\rangle = \langle\mathcal{L}^*\psi \mid \phi\rangle.$$

A self-adjoint (or Hermitian) operator is such that $\mathcal{L}^* = \mathcal{L}$. We want to be able to find this in the context of our L_2 space.

We note that

$$\begin{aligned}\langle\psi \mid \mathcal{L}\phi\rangle &= \int_a^b \overline{\psi}(x)[\mathcal{L}(\phi)(x)]\,dx\\ &= \int_a^b \overline{\mathcal{L}^*(\psi)(x)}\phi(x)\,dx.\end{aligned}$$

Now, if we take $D = \frac{d}{dx}$, then

$$\begin{aligned}\langle\psi \mid D\phi\rangle &= \int_a^b \overline{\psi(x)}\frac{d\phi}{dx}\,dx\\ &= \overline{\psi(x)}\phi(x)\Big|_a^b - \int_a^b \overline{\frac{d\psi}{dx}}\phi(x)\,dx\\ &= \overline{\psi(x)}\phi(x)\Big| - a^b + \int_a^b \overline{-\frac{d\psi}{dx}}\phi(x)\,dx.\end{aligned}$$

If we restrict our view to functions that vanish at the boundary, we have $D^* = -\frac{d}{dx}$. The derivative is thus an anti-Hermitian operator.

Note that if we apply a factor of i, then $i\frac{d}{dx}$ is a Hermitian operator.

Furthermore, note that D^2 is Hermitian, so it has real eigenvalues and orthogonal eigenvectors.

Sturm–Liouville Operators

The equation (†) is very simple. We will generalize by defining

$$\begin{aligned}\mathcal{L}\phi &= \left[\alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x)\right]\phi(x)\\ &= \lambda\phi(x).\end{aligned}$$

In order to have $\mathcal{L}$ as Hermitian, we set up our inner product and integrate by parts to yield

$$\begin{aligned}\langle \psi \mid \mathcal{L}\phi \rangle &= \int_a^b \overline{\psi}(x)\left(\alpha\frac{d^2}{dx^2} + \beta\frac{d}{dx} + \gamma\right)\phi(x)\,dx\\ &= \int \overline{\left(\alpha\frac{d}{dx} + \left(2\frac{d\alpha}{dx} - \beta\right)\frac{d}{dx} + \left(\frac{d^2\alpha}{dx^2} - \frac{d\beta}{dx} + \gamma\right)\right)\psi(x)}\phi(x)\,dx + \text{boundary terms.}\end{aligned}$$

Note that the operator $\mathcal{L}^*$ does reduce to $\mathcal{L}$ if $\frac{d\alpha}{dx} = \beta(x)$. If we introduce the weight factor

$$w(x) = \frac{1}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right),$$

then we will define

$$\begin{aligned}\widetilde{\alpha}(x) &= \alpha(x)w(x)\\ &= \exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right),\end{aligned}$$

and similarly, $\widetilde{\beta} = \beta w$, $\widetilde{\gamma} = \gamma w$. Note that

$$\begin{aligned}\frac{d\widetilde{\alpha}}{dx} &= \frac{\beta(x)}{\alpha(x)}\exp\left(\int \frac{\beta(x)}{\alpha(x)}\,dx\right)\\ &= \beta(x)w(x)\\ &= \widetilde{\beta}(x).\end{aligned}$$

We will replace $\widetilde{\alpha} = -p$ and $\widetilde{\gamma} = q$. This yields an operator in *Sturm–Liouville* form:

$$\begin{aligned}\mathcal{L} &= -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\\ &= -p(x)\frac{d^2}{dx^2} - \frac{dp}{dx}\frac{d}{dx} + q(x),\end{aligned}$$

where we assume $p, \frac{dp}{dx}, q, w$ are real and well-behaved between a and b. The *Sturm–Liouville equation* is of the form

$$\mathcal{L}\phi = \lambda w(x)\phi(x).$$

Note that our boundary term is

$$p(x)\left(\overline{\frac{d\psi}{dx}}\phi(x) - \overline{\psi(x)}\frac{d\phi}{dx}\Big|_a^b\right) = 0.$$

The easiest way to deal with this is by making sure $p(x) = 0$ at a and b.

Example. Consider Legendre's equation,

$$\left(1 - x^2\right)\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + \ell(\ell+1)\phi = 0.$$

We have $p(x) = 1 - x^2 > 0$ between ± 1, and $w(x) = 1$, meaning this equation yields orthogonal solutions between $[-1, 1]$.

Laguerre's equation

$$x\frac{d^2\phi}{dx^2} + (1-x)\frac{d\phi}{dx} + n\phi = 0$$

is not in Sturm–Liouville form. Multiplying by

$$\begin{aligned} w(x) &= \frac{1}{x}\exp\left(\int \frac{1-x}{x}\,dx\right) \\ &= \frac{1}{x}e^{\ln(x)-x} \\ &= e^{-x}. \end{aligned}$$

We thus get the Sturm–Liouville problem of

$$\begin{aligned} xe^{-x}\frac{d^2\phi}{dx^2} + (1-x)e^{-x}\frac{d\phi}{dx} + ne^{-x}\phi = 0 \\ -\frac{d}{dx}\left(xe^{-x}\frac{d\phi}{dx}\right) = ne^{-x}\phi. \end{aligned}$$

Note that $p = xe^{-x}$ vanishes at 0 and ∞.

Note that we need to apply conditions on ϕ and ψ such that all the solutions χ form a vector space. The *homogeneous* restrictions

$$\begin{aligned} c_1\chi(a) + d_1\chi'(a) &= 0 \\ c_2\chi(b) + d_2\chi'(b) &= 0, \end{aligned}$$

where at least one of c_1, d_1 and c_2, d_2 are nonzero.

- The Dirichlet boundary conditions are of the form $\chi(a) = \chi(b) = 0$.
- The Neumann boundary conditions are of the form $\chi'(a) = \chi'(b) = 0$.
- Periodic boundary conditions are of the form $\chi(a) = \chi(b)$ and $\chi'(a) = \chi'(b)$.

Example. Consider the equation

$$\frac{d^2\phi}{dx^2} = -k^2,$$

which is the Schrödinger equation for a free particle with momentum $\hbar k$. We will implement special periodic boundary conditions

$$\begin{aligned} \phi(0) &= \phi(L) \\ \phi'(0) &= \phi'(L), \end{aligned}$$

which denotes a circle of circle L.

We will end up finding that

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i x/L}.$$

The Properties of Sturm–Liouville Problems

Any Sturm–Liouville equation has the following features.

- The eigenvectors form an *orthogonal* basis:

$$\int_a^b \overline{\phi_m(x)}\phi_n(x)w(x)\,dx = k_n\delta_{mn},$$

  complete in L_2.

  Any well-behaved function defined in $[a, b]$ can be expanded on the eigenbasis

$$\psi(x) = \sum_n c_n\phi_n(x),$$

  where

$$\begin{aligned} c_m &= \langle \phi_m \,|\, f\rangle \\ &= \frac{1}{k_n}\int_a^b \overline{\phi_m(x)}f(x)w(x)\,dx. \end{aligned}$$

- The eigenvalues λ are real.
- If $[b - a]$ is finite, then the spectrum forms a discrete, countably infinite ordered set, such that

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots,$$

  which is used to label ϕ_n. Since $\lim_{n\to\infty} \lambda_n = \infty$, there is no largest eigenvalue. The lower bound is

$$\lambda_n \geqslant \frac{1}{k_n}\left[-p(x)\overline{\phi_n(x)}\frac{d\phi_n}{dx}\bigg|_a^b + \int_a^b q(x)|\phi_n(x)|^2\,dx\right].$$

  If both q and $-p\overline{\phi_n}\frac{d\phi_n}{dx}\Big|_a^b$ are positive, then all eigenvalues are positive.
- The real eigenvectors are oscillatory: between a and b, there are $n - 1$ nodes α_i such that $\phi_n(\alpha_0) = 0$. Note that oscillatory does not imply periodic, nor do the ϕ_n have fixed amplitude.
- For separated boundary conditions, the spectrum is non-degenerate. With periodic conditions, at most a double degeneracy can occur.

Sturm–Liouville equations occur all the time, especially in quantum mechanics. For instance, the time-independent Schrödinger equation,

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\phi(x) = E\psi(x)$$

is a Sturm–Liouville equation.

Example. We will contain a particle in a box of radius a, with potential

$$V(r) = \begin{cases} 0 & 0 \leqslant r < a \\ \infty & r \geqslant a \end{cases}.$$

We will have to replace the $\frac{d^2}{dx^2}$ in the Schrödinger equation with the Laplacian, ∇^2, giving

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = -k^2 u,$$

where $k^2 = \frac{2mE}{\hbar^2}$. We will let $u(\mathbf{r})$ reduce to a purely radial function $R(r)$, which allows us to take

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -k^2 r R(r).$$

This is Bessel's equation. There are two solutions: the Bessel function, $J_0(r)$, and the Neumann function, $N_0(r)$.[XVII] Note that Neumann functions diverge at 0, so our solutions are the Bessel functions.

Even though these solutions are not periodic, they are oscillatory. These solutions must be continuous at $r = a$, and confining a particle at $R(a) = 0$, Thus, we have $k_n = \frac{\alpha_n}{a}$, where α_n are zeros of the Bessel function.

The circular well has energies of

$$E_n = \frac{\hbar^2\alpha_n^2}{2ma^2}.$$

We get the orthogonality condition

$$\int_0^a J_0\left(\frac{\alpha_n r}{a}\right)J_0\left(\frac{\alpha_m r}{a}\right)r\,dr = \frac{a^2}{2}J_0'(\alpha_n)^2\delta_{mn}.$$

We have the eigenfunctions

$$\begin{aligned}\phi(r) &= \sum_n c_n R_n(r)\\ &= \sum_n c_n J_0\left(\frac{\alpha_n r}{a}\right),\end{aligned}$$

where

$$c_n = \frac{2}{a^2 J_1(\alpha_n)^2}\int_0^a J_0\left(\frac{\alpha_n r}{a}\right)\psi(r)\,dr.$$

This is known as the *Fourier–Bessel* series.

Partial Differential Equations

We will now start talking about PDEs, which allow us to describe phenomena that vary in both space and time.

Separation of Variables

We will deal with the Helmholtz equation:

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) = 0.$$

We will always be working with the ansatz of separated variables:

$$\psi(\mathbf{r}) = X(x)Y(y)Z(z)$$

[XVII] Also denoted Y_0.

$$\begin{aligned} &= R(r)\Phi(\phi)Z(z) \\ &= R(r)\Phi(\phi)\Theta(\theta). \end{aligned}$$

As it turns out, there are 11 different coordinate systems where the Laplacian separates.

We start with Cartesian. Upon plugging in the ansatz, the equation becomes

$$\begin{aligned} YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} &= -k^2 XYZ \\ \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2 &= 0. \end{aligned}$$

Rewriting, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2\right).$$

Note that the left hand side is entirely a function of x, while the right hand side is entirely a function of not x. Therefore, both sides must equal some constant. We will write this constant as $-\alpha^2$, giving the equation

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\alpha^2$$

We set

$$-\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2\right) = -\alpha^2,$$

and get

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - k^2 + \alpha^2.$$

Yet again, the left hand side is entirely a function of y,and the right hand side is entirely a function of z, so we set both equal to $-\beta^2$. Therefore, we get the separated system

$$\begin{aligned} \frac{1}{X}\frac{\partial^2 X}{\partial x^2} &= -\alpha^2 \\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} &= -\beta^2 \\ \frac{1}{X}\frac{\partial^2 Z}{\partial z^2} &= -\gamma^2, \end{aligned}$$

under the condition that $\alpha^2 + \beta^2 + \gamma^2 = k^2$. We may solve this system to yield

$$\begin{aligned} X(x) &= a_1 e^{i\alpha x} + a_2 e^{-i\alpha x} \\ Y(y) &= b_1 e^{i\beta y} + b_2 e^{-i\beta y} \\ Z(z) &= c_1 e^{i\gamma z} + c_2 e^{-i\gamma z}. \end{aligned}$$

Cartesian is, unsurprisingly, the simplest. In cylindrical, we have

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$$

Separating, we get

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{1}{r^2}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2.$$

We start by setting the z term to $-\gamma^2$, giving

$$\begin{aligned}\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} &= -k^2 + \gamma^2\\ &= -\beta^2.\end{aligned}$$

Multiplying out and rearranging, we get

$$\begin{aligned}\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \beta^2 r^2 &= -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}\\ &= m^2.\end{aligned}$$

Therefore, we get the system

$$\frac{d^2Z}{dz^2} = -\gamma^2 Z \tag{1}$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\beta^2r^2 - m^2\right)R = 0 \tag{2}$$

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi. \tag{3}$$

Notice that (2) is actually Bessel's equation. Furthermore, in (3), we must have $m \in \mathbb{Z}$, else we would not return as we rotated around the cylinder by 2π. We get the solutions

$$\begin{aligned}R(r) &= a_1 J_m(\beta r) + a_2 N_m(\beta r)\\ \Phi(\phi) &= b_1\cos(m\phi) + b_2\sin(m\phi)\\ Z(z) &= c_1 e^{i\gamma z} + c_2 e^{-i\gamma z},\end{aligned}$$

where J_m denotes the Bessel function with parameter m, and N_m is the linearly independent second solution to Bessel's equation (known as Neumann's equation).

In the special case where $k = 0$, we get the solution of

$$\begin{aligned}R(r) &= a_1 r^m + a_2 r^{-m}\\ \Phi(\phi) &= b_1\cos(m\phi) + b_2\sin(m\phi)\\ Z(z) &= 1.\end{aligned}$$

Now, in spherical coordinates, separation of variables yields

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \tag{1}$$

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2(\theta)}\right) = 0 \tag{2}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2r^2 - \lambda\right)R = 0. \tag{3}$$

We get three Sturm–Liouville problems. Specifically, (3) is the spherical Bessel equation whenever $\lambda = \ell(\ell+1)$, labeled j_ℓ, with linearly independent solutions of the spherical Neumann equations, labeled n_ℓ.

Remark: The spherical and cylindrical Bessel functions are related via

$$j_\ell(\xi) = \sqrt{\frac{\pi}{2\xi}} J_{\ell+\frac{1}{2}}(\xi)$$

$$n_\ell(\xi)\sqrt{\frac{\pi}{2\xi}} N_{\ell+\frac{1}{2}}(\xi).$$

Transforming (2) with $\xi = \cos(\theta)$ gives the equation in $\chi(\xi)$ of

$$\frac{d}{d\xi}\left(\left(1-\xi^2\right)\frac{d}{d\xi}\right) + \left(\lambda - \frac{m^2}{1-\xi^2}\right)\chi = 0,$$

which is the associated Legendre equation with solutions $P_{\ell,m}(\xi)$ and $Q_{\ell,m}(\xi)$ when $\lambda = \ell(\ell+1)$. Thus, we get the solutions of

$$\begin{aligned} R(r) &= a_1 j_\ell(kr) + a_2 n_\ell(kr) \\ \Theta(\theta) &= b_1 P_{\ell,m}(\cos(\theta)) + b_2 Q_{\ell,m}(\cos(\theta)) \\ \Phi(\phi) &= c_1 e^{im\phi} + c_2 e^{-im\phi}. \end{aligned}$$

Laplace's equation replaces the expression in j_ℓ and n_ℓ with linear combinations in terms of r^ℓ and $r^{-\ell}$. This yields the expansion in the spherical harmonics once we drop the $Q_{\ell,m}$:

$$\psi_{\ell,m}(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell,m}r^\ell + \frac{B_{\ell,m}}{r^{\ell+1}}\right)Y_{\ell,m}(\theta,\phi).$$

Now, if we have azimuthal symmetry, the spherical harmonics reduce to an expansion in Legendre functions:

$$\psi_\ell(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos(\theta)).$$

Boundary Value Problems

There are two parts of solving a PDE: the calculus and the boundary conditions. There are three primary boundary conditions:

- Dirichlet: the value of Ψ is specified on the boundary;
- Neumann: the normal derivative $\frac{\partial \Psi}{\partial n} = \nabla\Psi \cdot \hat{n}$ is specified on the boundary;
- Cauchy: both Ψ and $\nabla\Psi \cdot \hat{n}$ are specified everywhere on the boundary.

Example. We'll consider a Laplace equation in the Cartesian plane: $\nabla^2 V = 0$.

Specifically, we will consider it for a gutter with width a where we examine a cross-section at the center. Let

$$\begin{aligned} V(0,y) &= 0 \\ V(a,y) &= 0. \end{aligned}$$

We will include an initial condition $V(x,0) = V_0$, and $V(x,\infty) = 0$.

Separating variables, we get

$$\begin{aligned} X(x) &= a_1 e^{i\alpha x} + a_2 e^{-i\alpha x} \\ Y(y) &= b_1 e^{i\beta x} + b_2 e^{-i\beta x}. \end{aligned}$$

We must have $\alpha^2 = -\beta^2$, so from the boundary conditions, we get $\alpha = \frac{n\pi}{a}$, yielding

$$\begin{aligned} X(x) &= a_n \sin\left(\frac{n\pi}{a}x\right) \\ Y(y) &= b_n e^{-\frac{n\pi}{a}y}. \end{aligned}$$

We note that, since this is a homogeneous linear differential equation, we must take linear combinations, giving

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{n\pi}{a}y}.$$

We note that

$$\begin{aligned} V(x, 0) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) \\ &= V_0. \end{aligned}$$

Upon using the inner product expansions, we get

$$\begin{aligned} c_n &= \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi}{a}x\right) dx \\ &= \frac{4v_0}{\pi n}. && \text{for odd } n \end{aligned}$$

Note that we are always able to add boundary conditions together; this makes solving the more hairy boundary value problems easier.

Example. Consider a cylinder with height L and radius R. We apply ground to the top and bottom, and place the rest of the surface at a voltage of V_0. This gives the boundary conditions of

$$\begin{aligned} V(r, \phi, 0) &= 0 \\ V(r, \phi, L) &= 0 \\ V(R, \phi, z) &= V_0. \end{aligned}$$

We want to find the solution of Laplace's equation.

The general solution of Laplace's equation in cylindrical coordinates is

$$\begin{aligned} R(r) &= a_1 J_m(\beta r) + a_2 N_m(\beta r) \\ \Phi(\phi) &= b_1 \cos(m\phi) + b_2 \sin(m\phi) \\ Z(z) &= c_1 e^{\beta z} + c_2 e^{-\beta z}. \end{aligned}$$

Note that, since solutions to Laplace's equation follow the maximum principle, we must have $\beta = ik$ such that $e^{\pm \beta z}$ are oscillatory in z. Therefore, $\beta = ik$, and we write

$$\begin{aligned} J(ix) &= I(x) \\ N(ix) &= K(x), \end{aligned}$$

where I is the *modified Bessel function*, and K is the *modified Neumann function*.

This gives solutions of the form

$$\begin{aligned} R(r) &= a_1 I_m(kr) + a_2 K_m(kr) \\ \Phi(\phi) &= b_1 \cos(m\phi) + b_2 \sin(m\phi) \end{aligned}$$

$$Z(z) = c_1 \cos(kz) + c_2 \sin(kz).$$

Now, we apply our boundary conditions. To start, since the K_m blow up at the origin, we must have $a_2 = 0$. Furthermore, since at both $z = 0$ and $z = L$, we have zero potential, we have $c_1 = 0$ and $k_n = \frac{n\pi}{L}$. Thus, we have the potential of the form

$$V(r, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m(k_n r)(a_{mn} \cos(m\phi) + b_{mn} \sin(m\phi)) \sin(k_n z).$$

The family $\{I_m\}_{m \geqslant 0}$ are *not* orthogonal. However, the family $\{\sin(m\phi), \cos(m\phi)\}_{m \geqslant 0}$ *are* orthogonal, so we may calculate

$$\frac{2}{L} \int_0^L V_0 \sin(k_\ell z)\, dz = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m(k_n R)(a_{mn} \cos(m\phi) + b_{mn} \sin(m\phi)) \left(\frac{2}{L} \int_0^R \sin(k_\ell z) \sin(k_n z)\, dz\right).$$

Note that since we have azimuthal symmetry, we must have no ϕ dependence. Therefore, we only concern ourselves with a_{0n}, giving the solution

$$V(r, \phi, z) = \frac{4V_0}{\pi} \sum_{n \text{ odd}} \frac{\sin(k_n z) I_0(k_n r)}{n I_0(k_n R)}.$$

The Drums

The eigenvectors of a Sturm–Liouville problem are normal modes. Specifically, we will investigate this in the form of drums.

Imagine a rectangular membrane of the form $L_1 \times L_2$. We will use the Dirichlet boundary conditions,

$$\begin{aligned} \Psi(0, y) &= 0 \\ \Psi(x, 0) &= 0 \\ \Psi(L_1, y) &= 0 \\ \Psi(x, L_2) &= 0. \end{aligned}$$

We have the normal modes

$$\Psi_{mn}(x, y) = \sin\left(\frac{n\pi x}{L_1}\right) \sin\left(\frac{n\pi y}{L_2}\right).$$

The frequency ω is related to the wave number k by $\omega = kv$. Thus, we have the general case of $\omega_{mn} = k_{mn} v$. By the Pythagorean theorem,

$$k_{mn}^2 = \pi^2 \left(\frac{n^2}{L_1^2} + \frac{m^2}{L_2^2}\right).$$

This gives a problem: the frequencies are not evenly spaced in m and n. In other words, a square drum would not sound particularly nice.

If we have a circular drum, we have the solutions of the Helmholtz equation in cylindrical coordinates. However, since we have no z dependence, we have solutions of the form

$$\Psi_m(r, \phi) = J_m(kr)(a_1 \cos(m\phi) + a_2 \sin(m\phi)).$$

The boundary conditions mean that $\psi(R, \phi) = 0$, meaning $J_m(kR) = 0$. This means we need the zeros of the cylindrical Bessel functions. In particular, we get

$$k_{jm} = \frac{\alpha_{mj}}{R},$$

where α_{jm} is the jth zero of J_m. Our normal modes are of the form

$$\Psi_m(r,\phi) = J_m\left(\frac{\alpha_{jm}}{R}r\right)(a_1\cos(m\phi) + a_2\sin(m\phi)).$$

Note that, asymptotically, the Bessel functions appear to be like cosines.

Furthermore, at very high harmonics the normal modes cluster close to the edge of the drum head — this is displayed in the whispering gallery of, for instance, St. Paul's cathedral.

Now, we consider a fascinating question: can you hear the shape of a drum? We know that drums in one dimension[XVIII] can be heard, but as it turns out, the only other dimension for which this can be solved is the dimension $n = 16$.

This is because the eigenvalues of the N-dimensional Laplacian scale like

$$\lambda \approx 4\pi^2\left(\frac{n}{B_N V}\right)^{2/N},$$

where B_N is the volume of an N-dimensional ball.

Green's Functions

Consider a linear differential operator $\mathcal{L}$ such that

$$\mathcal{L}\psi(\mathbf{r}) = \rho(\mathbf{r}), \tag{$*$}$$

where $\rho(\mathbf{r}) \neq 0$. We know that if we have a homogeneous equation, the solutions are only reflective of $\mathcal{L}$. We will find the next best thing — a point source. The function $G(\mathbf{r},\mathbf{s})$, known as the *Green's function* for the equation $(*)$ is such that

$$\mathcal{L}G(\mathbf{r},\mathbf{s}) = \delta(\mathbf{r}-\mathbf{s}).$$

The function $G(\mathbf{r},\mathbf{s})$ is the field due to the point source at $\mathbf{r} = \mathbf{s}$.

An infinitesimal source $\rho(\mathbf{s})\,d^3s$ produces a field $G(\mathbf{r},\mathbf{s})\,d^3s$. Therefore,

$$\psi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{s})\rho(\mathbf{s})\,d^3s.$$

Applying $\mathcal{L}$, we note that $\mathcal{L}$ acts only at the "field point" $\mathbf{r}$, rather than the source point $\mathbf{s}$. Assuming everything is well-behaved, this gives

$$\begin{aligned}\mathcal{L}\psi(\mathbf{r}) &= \mathcal{L}\int_V G(\mathbf{r},\mathbf{s})\rho(\mathbf{s})\,d^3s\\ &= \int_V (\mathcal{L}G(\mathbf{r},\mathbf{s}))\rho(\mathbf{s})\,d^3s\\ &= \int_V \delta(\mathbf{r}-\mathbf{s})\rho(\mathbf{s})\,d^3s\\ &= \rho(\mathbf{r})\end{aligned}$$

Example (The Physicists' Green's Function)**.** For a point charge $V(\mathbf{r})$, we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\,d\mathbf{s},$$

[XVIII] also known as strings

which is a solution to

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

We see that $G(\mathbf{r}, \mathbf{s})$ is

$$G(\mathbf{r}, \mathbf{s}) = \frac{1}{4\pi} \frac{1}{\|\mathbf{r} - \mathbf{s}\|}.$$

Note that we have shown that

$$\nabla^2 \left(\frac{1}{\|\mathbf{r} - \mathbf{s}\|} \right) = 4\pi\delta(\mathbf{r} - \mathbf{s}).$$

Example (Poisson, One-Dimensional)**.** Poisson's equation in one dimension is

$$\frac{d^2\psi}{dx^2} = \rho(x).$$

Before we go further, note that the Green's Function must obey the boundary (or initial) conditions that we apply to the equation.

We will evaluate the equation over $0 \leqslant x \leqslant L$, with Dirichlet conditions, $\psi(0) = \psi(L) = 0$.

We need to find

$$\frac{d^2}{dx^2} G(x, t) = \delta(x - t).$$

We know that the solution is of the form $Ax + B$; however, we also need a discontinuity *in the derivative* such that we obtain a Delta function. In other words, we have $A_1 x + B_1$ when $x < t$ and $A_2 x + B_2$ when $x > t$.

For the Dirichlet boundary conditions, we note that since $\psi(0) = 0$, we must have $B_1 = 0$, and since $\psi(L) = 0$, we have the Green's function

$$G(x, s) = \begin{cases} ax & 0 < x < t \\ b(x - L) & t < x < L \end{cases}.$$

We understand the discontinuity at t by mandating

$$\begin{aligned} \int_{t-\varepsilon}^{t+\varepsilon} \frac{d^2}{dx^2} G(x, t)\, dx &= \frac{d}{dx} G(x, s) \bigg|_{t-\varepsilon}^{t+\varepsilon} \\ &= 1. \end{aligned}$$

Thus, we have the conditions

$$\begin{aligned} at &= b(t - L) \\ b - a &= 1. \end{aligned}$$

We are able to then calculate

$$G(x, t) = \frac{1}{L} \begin{cases} x(t - L) & x < t \\ t(x - L) & x > t \end{cases}.$$

Taking $\rho(x) = x$, we have

$$\begin{aligned}\psi(x) &= \int_0^L G(x,t)\rho(t)\,dt\\ &= \frac{1}{L}\int_0^x (t(x-L))t\,dt + \frac{1}{L}\int_x^L x(t-L)t\,dt\\ &= \frac{1}{6}x\left(x^2 - L^2\right).\end{aligned}$$

Example. Consider an initial value problem on the damped forced oscillator

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)u(t) = f(t),$$

with $u(0) = u_0$ and $\dot{u}(0) = v_0$.

To start solving this problem, we may write

$$f(t) = \int f(s)\delta(t-s)\,ds.$$

This approach is known as Duhamel's principle. Then, we may find

$$u_p(t) = \int_0^t f(s)G(t,s)\,ds.$$

Generally speaking, $G(t,s) = g^*(t-s)$. We note that the integral must be from 0 to t, implying a particular kind of causality.

Note that we had solved this back in the complex analysis unit. The underdamped had the Green's function of

$$G(t,s) = \frac{1}{\Omega}\Theta(t-s)\sin(\Omega(t-s))e^{-\beta(t-s)},$$

where $\Omega = \sqrt{\omega_0^2 - \beta^2}$.

Example (Poisson's Equation in Multiple Dimensions)**.** Consider the equation

$$\nabla^2 G(\mathbf{r},\mathbf{s}) = -\delta(\mathbf{r}-\mathbf{s})$$

in unbounded $\mathbb{R}^3$. We will evaluate this in spherical coordinates, mandating the Green's function be finite at 0 and ∞.

The Green's function is then a sum over spherical harmonics:

$$G(\mathbf{r},\mathbf{s}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\begin{cases}a_{\ell,m}r^{\ell}Y_{\ell,m}(\theta,\phi) & 0 < r < s\\ \frac{b_{\ell,m}}{r^{\ell+1}}Y_{\ell,m}(\theta,\phi) & s < r < \infty\end{cases}$$

We need G to be continuous at $r = s$, so that $b_{\ell,m} = s^{2\ell+1}a_{\ell,m}$.

Now, at the kink discontinuity with $r = s$, we examine a solid angle Ω and a piece on the surface with width 2ε. Using the divergence theorem, we get

$$-1 = \int_V \nabla^2 G(\mathbf{r},\mathbf{s})\,d^3s$$

$$\begin{aligned}
&= \oint_S \nabla G(\mathbf{r},\mathbf{s})\cdot d\mathbf{a}\\
&= \int \left(\frac{\partial G}{\partial r}\bigg|_{s+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{s-\varepsilon}\right) r^2\, d\Omega.
\end{aligned}$$

Therefore,[xix]

$$\begin{aligned}
\frac{\partial G}{\partial r}\bigg|_{s+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{s-\varepsilon} &= -\frac{1}{r^2}\delta(\cos(\theta_r)-\cos(\theta_s))\delta(\phi_r-\phi_s)\\
&= -\frac{1}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\overline{Y_{\ell,m}(\theta_s,\phi_s)}Y_{\ell,m}(\theta_r,\phi_r).
\end{aligned}$$

After much more suffering, we obtain

$$\begin{aligned}
G(\mathbf{r},\mathbf{s}) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\overline{Y_{\ell,m}(\theta_s,\phi_s)}Y_{\ell,m}(\theta_r,\phi_r)\begin{cases}\frac{r^\ell}{s^{\ell+1}} & 0<r<s\\ \frac{s^\ell}{r^{\ell+1}} & s<r<\infty\end{cases}\\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\overline{Y_{\ell,m}(\theta_s,\phi_s)}Y_{\ell,m}(\theta_r,\phi_r)\\
&= \frac{1}{4\pi}\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\hat{r}\cdot\hat{s})
\end{aligned}$$

After even more suffering, we may find

$$G(\mathbf{r},\mathbf{s}) = \frac{1}{4\pi}\frac{1}{\|\mathbf{r}-\mathbf{s}\|}.$$

Eigenfunctions and Green's Functions

Consider a Hermitian operator and the standard eigenvalue equation,

$$(\mathcal{L}-\lambda)|\phi\rangle = 0.$$

If $|\phi\rangle$ is to be nonzero, then we need $\mathcal{L}-\lambda I$ to be non-invertible. Meanwhile, the inhomogeneous equation

$$(\mathcal{L}-\lambda)|\psi\rangle = |\rho\rangle$$

has a solution only if $\mathcal{L}-\lambda I$ has an inverse. We will see that the inverse of a *differential* operator $\mathcal{L}-\lambda$ is the Green's function.

We may expand any homogeneous solution $|\Phi\rangle$ on the (orthonormal) basis of eigenfunctions

$$|\Phi\rangle = \sum_{n=1}^{\infty} c_n\big|\hat{\phi}_n\big\rangle.$$

Then,

$$\begin{aligned}
(L-\lambda)|\Phi\rangle &= (L-\lambda)\left(\sum_{n=1}^{\infty} c_n\big|\hat{\phi}_n\big\rangle\right)\\
&= \sum_{n=1}^{\infty} c_n(\lambda_n-\lambda)\big|\hat{\phi}_n\big\rangle
\end{aligned}$$

[xix]Recall that $d\Omega = d(\cos(\theta))\,d\phi$

$$= 0$$

only if $\lambda = \lambda_n$.

Now, if our equation is inhomogeneous, then we expand

$$\begin{aligned}
|\psi\rangle &= \sum_{n=1}^{\infty} a_n \left|\hat{\phi}_n\right\rangle \\
|\rho\rangle &= \sum_{n=1}^{\infty} b_n \left|\hat{\phi}_n\right\rangle.
\end{aligned}$$

We insert these expansions into the equation $(\mathcal{L} - \lambda)|\psi\rangle = |\rho\rangle$, so that

$$\sum_{n=1}^{\infty} (a_n(\lambda_n - \lambda) - b_n)\left|\hat{\phi}_n\right\rangle = 0,$$

and

$$\begin{aligned}
a_n &= \frac{b_n}{\lambda_n - \lambda} \\
&= \frac{\left\langle \hat{\phi}_n \,\middle|\, \rho \right\rangle}{\lambda_n - \lambda},
\end{aligned}$$

where we use the definition of b_n. Therefore,

$$\begin{aligned}
|\psi\rangle &= \sum_{n=1}^{\infty} \frac{\left\langle \hat{\phi}_n \,\middle|\, \rho \right\rangle}{\lambda_n - \lambda} \left|\hat{\phi}_n\right\rangle \\
&= \left(\sum_{n=1}^{\infty} \frac{\left|\hat{\phi}_n\right\rangle\left\langle\hat{\phi}_n\right|}{\lambda_n - \lambda} \right) |\rho\rangle \\
&= G|\rho\rangle,
\end{aligned}$$

where we define

$$G \coloneqq \sum_{n=1}^{\infty} \frac{\left|\hat{\phi}_n\right\rangle\left\langle\hat{\phi}_n\right|}{\lambda_n - \lambda}.$$

Note that we may use the completeness relation to find

$$\begin{aligned}
(\mathcal{L} - \lambda)G &= \sum_{n=1}^{\infty} (\lambda_n \lambda) \frac{\left|\hat{\phi}_n\right\rangle\left\langle\hat{\phi}_n\right|}{\lambda_n - \lambda} \\
&= \sum_{n=1}^{\infty} \left|\hat{\phi}_n\right\rangle\left\langle\hat{\phi}_n\right| \\
&= I.
\end{aligned}$$

If we put this in terms of functions, we use the completeness relation

$$\sum_{n=1}^{\infty} \overline{\hat{\phi}_n(\mathbf{s})}\hat{\phi}_n(\mathbf{t}) = \delta(\mathbf{t} - \mathbf{s}),$$

and get

$$G(s,t) = \sum_{n=1}^{\infty} \frac{\hat{\phi}_n(\mathbf{s})\overline{\hat{\phi}_n(\mathbf{t})}}{\lambda_n - \lambda}.$$

Thus, we get

$$(\mathcal{L}-\lambda)G(\mathbf{s},\mathbf{t}) = \delta(\mathbf{s}-\mathbf{t}).$$

For instance, if we have a one-dimensional string of length L, the eigenfunctions are

$$\hat{\phi}_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right),$$

with $\lambda_n = -n^2\pi^2/L^2$. Thus, the Green's function is

$$G(x,t) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}t\right).$$

Example. When we want to solve a wave equation for a fixed string with driving force $\cos(\omega t)$, we separate by taking

$$\Psi(x,t) = \psi(x)\cos(\omega t),$$

and obtain the Helmholtz equation in x. We solve for the Green's function by taking

$$\left(\frac{d^2}{dx^2}+k^2\right)\psi(x) = \delta(x-s),$$

with Dirichlet conditions $\psi(0) = \psi(L) = 0$. Therefore, we get

$$G(x,s) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}s\right)}{n^2\pi^2/L^2-k^2}.$$

Note that if $\lambda = \lambda_n$, then the Green's function *does not exist*. Thus, we would need a modification.

We may take

$$\widetilde{G}(\mathbf{x},\mathbf{s}) = \sum_{n\notin\Omega}\frac{\hat{\phi}_n(\mathbf{x})\overline{\hat{\phi}_n(\mathbf{s})}}{\lambda_n-\lambda}, \tag{MGF}$$

where Ω is the subspace of eigenfunctions with eigenvalue λ_n.

This gives

$$(\mathcal{L}-\lambda)\widetilde{G} = \delta(\mathbf{x}-\mathbf{s}) - \sum_{n\in\Omega}\hat{\phi}_n(\mathbf{x})\overline{\hat{\phi}_n(\mathbf{s})}.$$

Now, if we take

$$\begin{aligned}
\left\langle \hat{\phi}_m \,\middle|\, \rho\right\rangle &= \left\langle \hat{\phi}_m\right|\left(\sum_{n=1}^{\infty}(\mathcal{L}-\lambda)\left|\hat{\phi}_n\right\rangle\right)\\
&= \sum_{n=1}\infty\left\langle \hat{\phi}_m\right|(\mathcal{L}-\lambda)\left|\hat{\phi}_n\right\rangle\\
&= \sum_{n=1}^{\infty}(\lambda_n-\lambda)\left\langle \hat{\phi}_m \,\middle|\, \hat{\phi}_n\right\rangle\\
&= a_m(\lambda_m-\lambda).
\end{aligned}$$

Now, if $a_m = 0$, then $\hat{\phi}_m$ does not appear in the orthogonal expansion of $|\rho\rangle$, therefore, if

$$\left\langle \hat{\phi}_m \,\middle|\, \rho\right\rangle = 0$$

for all $n \notin \Omega$, then (MGF) is a valid expression.

Green's Functions and the Fourier Transform

Recall that

$$\mathcal{F}\left[\frac{d}{dx}\right] = ik,$$

so that Poisson's equation transforms into

$$\begin{aligned} \mathcal{F}\left[\nabla^2 V\right] &= \mathcal{F}[\rho] \\ -k^2\hat{V} &= \hat{\rho} \\ \hat{V} &= -\frac{1}{k^2}\hat{\rho}. \end{aligned}$$

We may take the inverse Fourier transform to solve for our Green's function for the differential operator.