# Preliminary Statements

**Theorem** (Definition of Countability)**.** *A set $S$ is countable if and only if there exists an injection $f : S \hookrightarrow \mathbb{N}$.*

*Proof.* Let $S$ be countable.

**Case 1:** We have $S$ is finite if and only if there is a map $f : S \to \{1, 2, \dots, n\}$, where $f$ is a bijection. Letting $\text{id} : \{1, 2, \dots, n\} \to \mathbb{N}$ be defined by $\text{id}(n) = n$, it is clear that id is an injection.

Considering the map $\text{id} \circ f : S \to \mathbb{N}$, since id is injective and $f$ is injective, so too is $\text{id} \circ f$, meaning our desired injection is $\text{id} \circ f$.

**Case 2:** By definition, a set $S$ is countably infinite if and only if there exists a bijection $g : S \to \mathbb{N}$, which is our desired injection.

□

## 1.1

## 1.2

**Problem.** Given bijections $f : \mathbb{N} \to \mathbb{Z}$ and $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, show that the function $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ defined by $h(x, y) = P\left(f^{-1}(x), f^{-1}(y)\right)$ is bijective.

**Solution.** We begin by showing injectivity. Since $f$ is bijective, so too is $f^{-1}$, meaning that for

$$h(x, y) = h(x', y'),$$

we have

$$\begin{aligned} P\left(f^{-1}(x), f^{-1}(y)\right) &= P\left(f^{-1}(x'), f^{-1}(y')\right) \\ f^{-1}(x) &= f^{-1}(x') \\ f^{-1}(y) &= f^{-1}(y') && \text{since } P \text{ is bijective} \end{aligned}$$

meaning

$$\begin{aligned} x &= x' \\ y &= y' && \text{since } f^{-1} \text{ is bijective.} \end{aligned}$$

Thus, $h$ is injective.

Let $n \in \mathbb{N}$. Since $P$ is surjective, there exist $a, b$ such that $P(a, b) = n$. Since $f^{-1}$ is surjective, there exists $x, y \in \mathbb{Z}$ such that $f^{-1}(x) = a$ and $f^{-1}(y) = b$. Thus, there exist $x, y \in \mathbb{Z}$ such that $h(x, y) = n$.

## 1.3

**Problem.** If $A$ and $B$ are countably infinite, show that $A \times B$ is countably infinite.[1]

**Solution.** By the definition of countably infinite sets, there exist bijections $\alpha : A \to \mathbb{N}$ and $\beta : B \to \mathbb{N}$. Additionally, we know that there exists a bijection $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Define $h : A \times B \to \mathbb{N}$ by $h(a, b) = P(\alpha(a), \beta(b))$. Then, since $h$ is a composition of bijections, $h$ is a bijection between $A \times B$ and $\mathbb{N}$.

[1] Assuming the axiom of choice.

## 1.5

**Problem.** If $A_1, A_2, \dots$ is an infinite sequence of disjoint finite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

**Solution.** Let $a_n$ be defined by the bijection $\alpha_n : A_n \to \{1, 2, \dots, a_n\}$.

## 1.6

## 1.7

**Problem.** Construct an explicit polynomial bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

**Solution.** Let $Q : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $Q(x, y, z) = P\left(P(x, y), z\right)$, where $P(x, y) = \frac{(x+y)(x+y+1)}{2} + x$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

We know that $Q$ is a bijection since it is a composition of bijections. I do not want to expand this expression.

# Extra Problem 1

Prove that if $A$ and $B$ are finite sets, then $A \cup B$ is finite.

# Extra Problem 2

Prove that for every $n \in \mathbb{N}$, every subset of $\{0, 1, \dots, n\}$ is finite.

# Extra Problem 3

Prove that every subset of a finite set is finite.

# Extra Problem 4

**Problem.** Prove that every infinite subset of $\mathbb{N}$ is denumerable.

**Solution.** Let $A \subseteq \mathbb{N}$ be infinite.

Since $A$ is nonempty, by the well-ordering principle, there must exist a least element of $A$, which we label as $a_0$.

Consider $A \setminus \{a_0\}$. Since $A$ is infinite, $A \setminus \{a_0\}$ must also be infinite, meaning there is a least element of $A \setminus \{a_0\}$ by the well-ordering principle. We label this element as $\{a_1\}$.

Now, we consider $A \setminus \{a_0, a_1\}$, and use the well-ordering principle to extract $a_2$, and inductively extract $a_i$ by using the well ordering principle on $A \setminus \{a_0, a_1, \dots, a_{i-1}\}$.

The function $f : A \to \mathbb{N}$ defined by $f(a_i) = i$ is a bijection, since $f(a_i) = f\left(a_j\right)$ if and only if $i = j$.

Thus, $f$ is a denumeration of $A$.