

Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

Quantity	Value
\mathbf{r}	$\rho \cos \phi \hat{\mathbf{i}} + \rho \sin \phi \hat{\mathbf{j}} + z \hat{\mathbf{k}}$
u_1	ρ
u_2	ϕ
u_3	z
$\hat{\mathbf{e}}_1$	$\hat{\rho}$
$\hat{\mathbf{e}}_2$	$\hat{\phi}$
$\hat{\mathbf{e}}_3$	$\hat{\mathbf{k}}$
$\frac{\partial \mathbf{r}}{\partial u_1}$	$\hat{\rho}$
$\frac{\partial \mathbf{r}}{\partial u_2}$	$\rho \hat{\phi}$
$\frac{\partial \mathbf{r}}{\partial u_3}$	$\hat{\mathbf{k}}$
h_1	1
h_2	ρ
h_3	1

- Line element:

$$\begin{aligned}
 (ds)^2 &= \sum_i h_i^2 (du_i)^2 \\
 &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.
 \end{aligned}$$

- Area element:

$$\begin{aligned}
 d\mathbf{a} &= \left(\sum_k \epsilon_{ijk} \hat{\mathbf{e}}_k \right) h_i h_j du_i du_j \\
 &= \rho \hat{\mathbf{k}} d\rho d\phi \\
 &= \rho \hat{\phi} d\phi dz \\
 &= \hat{\phi} dz d\rho
 \end{aligned}$$

- Volume element:

$$\begin{aligned}
 d\tau &= h_1 h_2 h_3 du_1 du_2 du_3 \\
 &= \rho d\rho d\phi dz
 \end{aligned}$$

Spherical Coordinates

Quantity	Value
\mathbf{x}	$r \sin \theta \cos \phi \hat{\mathbf{i}} + r \sin \theta \sin \phi \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$
u_1	r
u_2	ϕ
u_3	θ
$\hat{\mathbf{e}}_1$	$\hat{\rho}$
$\hat{\mathbf{e}}_2$	$\hat{\phi}$
$\hat{\mathbf{e}}_3$	$\hat{\theta}$
$\frac{\partial \mathbf{x}}{\partial u_1}$	$\hat{\mathbf{r}}$
$\frac{\partial \mathbf{x}}{\partial u_2}$	$r \sin \theta \hat{\phi}$
$\frac{\partial \mathbf{x}}{\partial u_3}$	$r \hat{\theta}$
h_1	1
h_2	$r \sin \theta$
h_3	r

- Line element:

$$\begin{aligned}
 (ds)^2 &= \sum_i h_i^2 (du_i)^2 \\
 &= (dr)^2 + r^2 \sin^2 \theta (d\phi)^2 + r^2 (d\theta)^2
 \end{aligned}$$

- Area element:

$$\begin{aligned}
 d\mathbf{a} &= \left(\sum_k \epsilon_{ijk} \hat{\mathbf{e}}_k \right) h_i h_j du_i du_j \\
 &= r \sin \theta \hat{\theta} dr d\phi \\
 &= r^2 \sin \theta \hat{\mathbf{r}} d\phi d\theta \\
 &= r \hat{\phi} d\theta dr
 \end{aligned}$$

- Volume element:

$$\begin{aligned}
 d\tau &= h_1 h_2 h_3 du_1 du_2 du_3 \\
 &= r^2 \sin \theta dr d\phi d\theta
 \end{aligned}$$

4.9

We have

$$\begin{aligned}
 \epsilon_{ijk} &= (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k \\
 &= \det \begin{pmatrix} \hat{\mathbf{e}}_i & \hat{\mathbf{e}}_j & \hat{\mathbf{e}}_k \end{pmatrix} \\
 &= \det \begin{pmatrix} \hat{\mathbf{e}}_i^T \\ \hat{\mathbf{e}}_j^T \\ \hat{\mathbf{e}}_k^T \end{pmatrix}.
 \end{aligned}$$

Note that $\hat{e}_i \hat{e}_j^T = \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$. Thus,

$$\begin{aligned}
 \sum_{\ell} \epsilon_{mnl} \epsilon_{ij\ell} &= \sum_{\ell} \det \begin{pmatrix} \hat{e}_m & \hat{e}_n & \hat{e}_{\ell} \end{pmatrix} \det \begin{pmatrix} \hat{e}_i & \hat{e}_j & \hat{e}_{\ell} \end{pmatrix} \\
 &= \sum_{\ell} \det \begin{pmatrix} \hat{e}_m & \hat{e}_n & \hat{e}_{\ell} \end{pmatrix} \det \begin{pmatrix} \hat{e}_i^T \\ \hat{e}_j^T \\ \hat{e}_{\ell}^T \end{pmatrix} \\
 &= \sum_{\ell} \det \begin{pmatrix} \hat{e}_m \hat{e}_i^T & \hat{e}_m \hat{e}_j^T & \hat{e}_m \hat{e}_{\ell}^T \\ \hat{e}_n \hat{e}_i^T & \hat{e}_n \hat{e}_j^T & \hat{e}_n \hat{e}_{\ell}^T \\ \hat{e}_{\ell} \hat{e}_i^T & \hat{e}_{\ell} \hat{e}_j^T & \hat{e}_{\ell} \hat{e}_{\ell}^T \end{pmatrix} \\
 &= \sum_{\ell} \det \begin{pmatrix} \delta_{mi} & \delta_{mj} & \delta_{m\ell} \\ \delta_{ni} & \delta_{nj} & \delta_{n\ell} \\ \delta_{\ell i} & \delta_{\ell j} & \delta_{\ell\ell} \end{pmatrix} \\
 &= \sum_{\ell} \det \begin{pmatrix} \delta_{mi} & \delta_{mj} & \delta_{m\ell} \\ \delta_{ni} & \delta_{nj} & \delta_{n\ell} \\ \delta_{\ell i} & \delta_{\ell j} & 1 \end{pmatrix} \\
 &= \det \begin{pmatrix} \delta_{mi} & \delta_{mj} \\ \delta_{ni} & \delta_{nj} \end{pmatrix} \\
 &= \delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}.
 \end{aligned}$$

4.11

(a)

$$\begin{aligned}
 \mathbf{A} \times \mathbf{B} &= \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k \\
 &= - \sum_{i,j,k} \epsilon_{jik} B_j A_i \hat{e}_k \\
 &= -(\mathbf{B} \times \mathbf{A})
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= \sum_{i,j,k} (\epsilon_{ijk} A_i B_j \hat{e}_k) \cdot A_i \hat{e}_i \\
 &= \sum_{i,j,k} \delta_{ik} (\epsilon_{ijk} A_i^2 B_j) \\
 &= 0.
 \end{aligned}$$

(c)

$$\begin{aligned}
 \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_{\ell} B_i C_j \\
 &= \sum_{i,j,\ell} (\epsilon_{\ell ij} A_{\ell} B_i) C_j
 \end{aligned}$$

$$= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

and

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_\ell B_i C_j \\ &= \sum_{i,j,\ell} (\epsilon_{j\ell i} C_j A_i) B_i \\ &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}). \end{aligned}$$

(d)

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j} \epsilon_{ijk} A_i \left(\sum_{\alpha,\beta} \epsilon_{\alpha\beta j} B_\alpha C_\beta \right) \\ &= \sum_{i,j,\alpha,\beta} \epsilon_{ijk} \epsilon_{\alpha\beta j} A_i B_\alpha C_\beta \\ &= - \left(\sum_{i,j,\alpha,\beta} \epsilon_{ikj} \epsilon_{\alpha\beta j} A_i B_\alpha C_\beta \right) \\ &= - \left(\sum_{i,j,\alpha,\beta} (\delta_{i\alpha} \delta_{k\beta} - \delta_{i\beta} \delta_{k\alpha}) A_i B_\alpha C_\beta \right) \\ &= \sum_{i,j,\alpha,\beta} (\delta_{k\alpha} \delta_{i\beta} - \delta_{i\alpha} \delta_{k\beta}) A_i B_\alpha C_\beta \\ &= \sum_{i,j,\alpha,\beta} (B_\alpha \delta_{k\alpha}) (A_i C_\beta \delta_{i\beta}) - (C_\beta \delta_{k\beta}) (A_i B_\alpha \delta_{i\alpha}) \\ &= \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}). \end{aligned}$$

(e)

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \sum_{\alpha,\beta} (\mathbf{A} \times \mathbf{B})_\alpha (\mathbf{C} \times \mathbf{D})_\beta \delta_{\alpha\beta} \\ &= \sum_{\substack{\alpha,\beta, \\ i,j, \\ m,n}} \epsilon_{ij\alpha} \epsilon_{mn\beta} A_i B_j C_m D_n \delta_{\alpha\beta} \\ &= \sum_{\substack{\alpha, \\ i,j, \\ m,n}} \epsilon_{ij\alpha} \epsilon_{mn\alpha} A_i B_j C_m D_n \\ &= \sum_{\substack{i,j, \\ m,n}} A_i B_j C_m D_n (\delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}) \\ &= \sum_{\substack{i,j, \\ m,n}} ((A_i C_m \delta_{mi}) (B_j D_n \delta_{nj})) - ((B_j C_m \delta_{mj}) - (A_i D_n \delta_{ni})) \\ &= (\mathbf{A} \cdot \mathbf{C}) (\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C}) (\mathbf{A} \cdot \mathbf{D}). \end{aligned}$$

Chapter 5 Problems

5.1

Let $f(x) = x^n$. We use linearity for the general case.

$$\begin{aligned}\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^n + n(hx^{n-1}) + \dots + nh^{n-1}x + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} (nx^{n-1} + \dots + nh^{n-2}x + h^{n-1}) \\ &= nx^{n-1}.\end{aligned}$$

5.6

$$\begin{aligned}\cos(N\phi) + i \sin(N\phi) &= (\cos \phi + i \sin \phi)^N \\ &= \sum_{k=0}^N \binom{N}{k} (\cos \phi)^k (\sin \phi)^{N-k} \left(e^{i\frac{\pi}{2}}\right)^{N-k} \\ &= \sum_{k=0}^N \binom{N}{k} (\cos \phi)^k (\sin \phi)^{N-k} \left(\cos\left((N-k)\frac{\pi}{2}\right) + i \sin\left((N-k)\frac{\pi}{2}\right)\right) \\ &= \sum_{k=0}^N \binom{N}{k} \cos\left((N-k)\frac{\pi}{2}\right) (\cos \phi)^k (\sin \phi)^{N-k} \\ &\quad + i \left(\sum_{k=0}^N \binom{N}{k} \sin\left((N-k)\frac{\pi}{2}\right) (\cos \phi)^k (\sin \phi)^{N-k}\right).\end{aligned}$$

We get the final answer by equating real and imaginary parts.

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n}{n+1}} \right| \\ &= 1,\end{aligned}$$

which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \quad \forall n \geq 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

- (b) • Ratio test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} \right) \left(\frac{1}{2} \right) \right| \\ &= \frac{1}{2} \\ &< 1,\end{aligned}$$

meaning the series converges by the ratio test.

- Comparison test:

$$\frac{1}{n2^n} < \frac{1}{2^n} \quad \text{for all } n \geq 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.

- (c) • Ratio test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{2(n+1)^2+1}{(n+1)^\pi}}{\frac{2n^2+1}{n^\pi}} \\ &= 1,\end{aligned}$$

which means the ratio test is inconclusive.

- Comparison test:

$$\sum_{n=1}^{\infty} \frac{2n^2+1}{n^\pi} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{\pi-2}} + \sum_{n=1}^{\infty} \frac{1}{n^\pi},$$

which converges by p-series (and, thus, comparison).

- (d) • Ratio Test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln n} \\ &= 1,\end{aligned}$$

which means the ratio test is inconclusive.

- Comparison test:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n},$$

which means the sum diverges by the comparison test.

6.9

$$\begin{aligned}\sum_{n=-N}^N e^{inx} &= 1 + \sum_{n=1}^N e^{-inx} + \sum_{n=1}^N e^{inx} \\ &= 1 + e^{-ix} \sum_{n=1}^N e^{i(n-1)x} + e^{ix} \sum_{n=1}^N e^{i(n-1)x}\end{aligned}$$

$$\begin{aligned}
&= 1 + e^{-ix} \sum_{n=0}^{N-1} e^{-inx} + e^{ix} \sum_{n=0}^{N-1} e^{inx} \\
&= 1 + e^{-ix} \frac{1 - e^{-iNx}}{1 - e^{-ix}} + e^{ix} \frac{1 - e^{iNx}}{1 - e^{ix}} \\
&= 1 + \frac{e^{-ix} - e^{-i(N+1)x}}{1 - e^{-ix}} + \frac{1 - e^{iNx}}{e^{-ix} - 1} \\
&= 1 + \frac{(e^{-ix} - 1) + e^{iNx} - e^{-i(N+1)x}}{1 - e^{-ix}} \\
&= \frac{e^{iNx} - e^{-i(N+1)x}}{1 - e^{-ix}} \\
&= \frac{e^{iNx} - e^{-i(N+1)x}}{e^{-i(\frac{x}{2})} (e^{i(\frac{x}{2})} - e^{-i(\frac{x}{2})})} \\
&= \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{-i(\frac{x}{2})} - e^{-i(\frac{x}{2})}} \\
&= \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})}.
\end{aligned}$$

6.13

(a)

$$\begin{aligned}
\frac{d}{dz} (\arctan(z)) &= 1 - z^2 + z^4 - z^6 + \dots \\
&= \sum_{i=0}^{\infty} (-1)^i z^{2i} \\
&= \frac{1}{1 + z^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\rho &= \limsup_{k \rightarrow \infty} \sqrt[k]{|(-1)^k|} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}$$

(c)

$$\begin{aligned}
\rho &= \limsup_{k \rightarrow \infty} \left(\left| \frac{(-1)^k}{2k+1} \right| \right)^{1/k} \\
&= \limsup_{k \rightarrow \infty} \frac{1}{(2k+1)^{1/k}} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}$$

6.25

$$\begin{aligned}
e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\
&= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \\
&= \cos \theta + i \sin \theta.
\end{aligned}$$

6.37

$$\begin{aligned}
F &= GmM_E (R_E + h)^{-2} \\
&= \frac{GmM_E}{R_E^2} \left(1 + \frac{h}{R_E}\right)^{-2} \\
&= \frac{GmM_E}{R_E^2} \left(1 + (-2)\frac{h}{R_E} + \frac{(-2)(-3)}{2!} \left(\frac{h}{R_E}\right)^2 + \dots\right) \\
&\approx \frac{GmM_E}{R_E^2}.
\end{aligned}$$

It would be the case that for $h = (0.05) R_E$, there would need to be a 10% negative correction to the estimated force.

6.42

$$\begin{aligned}
(1 - k^2 \sin^2 \phi)^{-1/2} &= \left(1 + (-k^2 \sin^2 \phi)\right)^{-1/2} \\
&= 1 + \left(-\frac{1}{2}\right) (-k^2 \sin^2 \phi) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} (-k^2 \sin^2 \phi)^2 + \dots \\
&\approx 1.
\end{aligned}$$

The integrand then expands to

$$\begin{aligned}
T &\approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\phi \\
&= 2\pi\sqrt{\frac{L}{g}}.
\end{aligned}$$

With a first-order correction and $\alpha = \pi/3$, the fractional change in T is

$$\begin{aligned}
T &\approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} 1 + k^2 \sin^2 \phi \, d\phi \\
&= 4\sqrt{\frac{L}{g}} \left(\pi/2 + \frac{1}{4} \left(\frac{\pi}{4}\right)\right) \\
&= \left(2 + \frac{1}{4}\right) \pi\sqrt{\frac{L}{g}}.
\end{aligned}$$

Thus, the error in $T \approx 2\pi\sqrt{\frac{L}{g}}$ with $\alpha = \pi/3$ is an underestimate of 11%.