Problem (Problem 1)**:**

(a) Show that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$, in which it defines an analytic function, which we denote e^z.

(b) With this as the definition of e^z, prove that $e^z e^w = e^{z+w}$.

(c) Show that for $\theta \in \mathbb{R}$, we have that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, where $\cos(\theta)$ and $\sin(\theta)$ are defined via their usual power series representations.

Solution:

(a) To compute

$$\rho = \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n},$$

we start by noticing that

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} &= \lim_{n\to\infty} \frac{1}{(n+1)}\\ &= 0.\end{aligned}$$

In particular, for $\varepsilon > 0$, there is some N such that for all $n \geqslant N$,

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} < \varepsilon,$$

so

$$\frac{1}{(n+1)!} < \frac{\varepsilon}{n!},$$

and by inductively using this approximation, we get that for any $n \geqslant N$,

$$\begin{aligned}\frac{1}{n!} &< \frac{\varepsilon^{n-N}}{N!}\\ &= \varepsilon^n \left(\frac{1}{\varepsilon^N N!}\right)\end{aligned}$$

so that

$$\limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} \leqslant \varepsilon,$$

meaning that $\rho = 0$, and thus the radius of convergence for the power series is infinite.

(b) Computing $e^z e^w$, we get

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{\ell=0}^{\infty} \frac{w^k}{k!}\right) &= \sum_{k=0}^{\infty}\sum_{\ell=k}^{\infty} \frac{1}{(\ell-k)!}\frac{1}{k!} w^k z^{\ell-k}\\ &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k=0}^{\ell} \frac{1}{k!(\ell-k)!} w^k z^{\ell-k}\\ &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(z+w)^{\ell}\\ &= e^{z+w}.\end{aligned}$$

(c) Computing $e^{i\theta}$ by direct substitution, we find that

$$\begin{aligned} e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\ &= \sum_{k \text{ even}} \frac{(-1)^{(k/2)}\theta^k}{k!} + i \sum_{k \text{ odd}} \frac{(-1)^{(k-1)/2}\theta^k}{k!} \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

Problem (Problem 2)**:** Let $U \subseteq \mathbb{C}$ be an open set, $f\colon U \to \mathbb{C}$ an analytic function. Since f is analytic, given $z_0 \in U$, there is $r > 0$ and a sequence $(a_n)_n$ such that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z \in U(z_0, r)$.

Suppose there exists $R > r$ such that $U(z_0, R) \subseteq U$ and $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence at least R. Show that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z \in U(z_0, R)$.

Solution: On the connected open set $V = U(z_0, R)$, define

$$g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Observe that $f|_V$ and g agree on the open subset $U(z_0, r) \subseteq U(z_0, R)$. By the identity theorem, this means that $f = g$ on $U(z_0, R)$.

Problem (Problem 3)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be an analytic function.

(a) Suppose f is nonconstant, $z_0 \in U$. Show that there exists some $r > 0$ for which $U(z_0, r) \subseteq U$, a positive integer $k \in \mathbb{N}$, an analytic function $g\colon U(z_0, r) \to \mathbb{C}$, and a nonconstant $\lambda \in \mathbb{C} \setminus \{0\}$ such that for $z \in U(z_0, r)$,

$$f(z) = f(z_0) + \lambda(z - z_0)^k + (z - z_0)^{k+1} g(z).$$

(b) Suppose that f is nonconstant, and $z_0 \in U$ is such that $f(z_0) \neq 0$. Show that there exists some $s > 0$ such that $U(z_0, s) \subseteq U$, and $w_1, w_2 \in U(z_0, s)$ such that $|f(w_1)| > |f(z_0)| > |f(w_2)|$.

(c) Show that if $|f|$ is constant, then f is constant.

Solution:

(a) Since f is analytic, we may find $r > 0$ and a sequence $(a_n)_n$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Observe that $f(z_0) = a_0$, so

$$= f(z_0) + \sum_{n=1}^{\infty} a_n(z - z_0)^n.$$

Next, we find the minimum value of n such that $a_n \neq 0$, which we define to be k. Such a value must exist since f is a nonconstant function, and if it were to not exist, the identity theorem would give f as a constant function on $U(z_0, r)$. This gives

$$= f(z_0) + a_k(z - z_0)^k + \sum_{n=k+1}^{\infty} a_n(z - z_0)^n.$$

Finally, by reindexing the sum and factoring out $(z-z_0)^{k+1}$, we get

$$= f(z_0) + a_k(z-z_0)^k + (z-z_0)^{k+1}\sum_{n=0}^{\infty} a_{n+k+1}(z-z_0)^n.$$

Define $g(z)$ to be equal to the sum, and define $\lambda = a_k$. Notice that since the radius of convergence of a power series is a limiting case, g and f have the same radius of convergence. This gives

$$= f(z_0) + \lambda(z-z_0)^k + (z-z_0)^{k+1}g(z).$$

(b) Let f be a nonconstant analytic function with $f(z_0) \neq 0$. Since f is nonconstant, we see that λ in the previous problem is nonzero, meaning that $|\lambda|$ is nonzero, in addition to $|f(z_0)|$.

We start by considering the case where $f(z) = f(z_0) + \lambda(z-z_0)^k$. We will reintroduce $g(z)$ later, but first we work on establishing the existence of w_1 and w_2 in this scenario. Writing $(z-z_0) = |z-z_0|e^{i\varphi}$, we thus get that

$$f(z) = |f(z_0)|e^{i\theta_0} + |\lambda||z-z_0|^k e^{i(\theta_\lambda + k\varphi)}$$

for all $z \in U(z_0, r)$. Note that the phases θ_0 and $\theta_\lambda + k\varphi$ "add" if and only if $\varphi = \frac{1}{k}(\theta_0 - \theta_\lambda)$. Therefore, if $\omega_1 \in U(z_0, r) \setminus \{z_0\}$ is such that $\omega_1 - z_0 = |\omega_1 - z_0|e^{i\varphi_1}$ with φ_2 satisfying this condition, we then have

$$|f(\omega)| = |f(z_0)| + |\lambda||\omega_1 - z_0|^k,$$

implying that $|f(\omega)| > |f(z_0)|$. Similarly, if φ_2 is such that $\varphi_2 = \frac{1}{k}(\theta_0 - \theta_\lambda + \pi)$, then if $\omega_2 \in U(z_0, r) \setminus \{z_0\}$ is such that

$$|f(\omega_2)| = |f(z_0)| - |\lambda||\omega_2 - z_0|^k.$$

Thus, in this case, we have found ω_1 and ω_2 satisfying $|f(\omega_1)| > |f(z_0)| > |f(\omega_2)|$.

Now, reintroducing our term $(z-z_0)^{k+1}g(z)$, which we write in polar form as $|z-z_0||g(z)|e^{i\psi}$, we notice that for a fixed $0 < s_0 < r$ such that $B(z_0, s_0) \subseteq U(z_0, r)$, $|g|$ is bounded on $B(z_0, s_0)$, as g is analytic and thus continuous. Call this bound M.

We may then find $0 < s < s_0$ small enough with $w_1, w_2 \in U(z_0, s)$ and arguments φ_1 and φ_2 as in the case of ω_1 and ω_2 defined earlier such that

$$\begin{aligned}
\left|f(z_0) + \lambda(w_2 - z_0)^k\right| - Ms^{k+1} &> |f(z_0)| \\
\left|f(z_0) + \lambda(w_2 - z_0)^k\right| + Ms^{k+1} &< |f(z_0)|.
\end{aligned}$$

Then, by the triangle inequality, we see that

$$\begin{aligned}
|f(w_1)| &= \left|f(z_0) + \lambda(w_1 - z_0)^k + (w_1 - z_0)^{k+1}g(z)\right| \\
&\geqslant \left|f(z_0) + \lambda(w_1 - z_0)^k\right| - |w_1 - z_0|^{k+1}|g(z)| \\
&\geqslant \left|f(z_0) + \lambda(w_1 - z_0)^k\right| - Ms^{k+1} \\
&> |f(z_0)|,
\end{aligned}$$

and similarly,

$$|f(w_2)| = \left|f(z_0) + \lambda(w_2 - z_0)^k + (w_2 - z_0)^{k+1}g(z)\right|$$

$$\begin{aligned}
&\leqslant \left|f(z_0) + \lambda(w_2 - z_0)^k\right| + |g(z)||w_1 - z_0|^{k+1} \\
&\leqslant \left|f(z_0) + \lambda(w_2 - z_0)^k\right| + Ms^{k+1} \\
&< |f(z_0)|.
\end{aligned}$$

(c) Let $|f|$ be constant. Via the contrapositive of the previous part, $|f(w)| = |f(z_0)|$ for all $w \in U(z_0, s)$. In particular, this means that either $f(z_0) = 0$ or f is constant; note that if $f(z_0) = 0$, then since $|f(w)| = |f(z_0)| = 0$ for all $w \in U(z_0, s)$, the identity theorem means that $f = 0$, so either way, f is constant.

Problem (Problem 5)**:** Let $U \subseteq \mathbb{C}$ be an open set, and let $V = \{z \in \mathbb{C} \mid \overline{z} \in U\}$.

(a) Show that if $f\colon U \to \mathbb{C}$ is analytic, then $g\colon V \to \mathbb{C}$ defined by $g(z) = \overline{f(\overline{z})}$ is analytic.

(b) Show that if $f\colon U \to \mathbb{C}$ is holomorphic, then $g\colon V \to \mathbb{C}$ defined by $g(z) = \overline{f(\overline{z})}$ is holomorphic.

Solution:

(a) Let $z_0 \in V$, so that there exists $r > 0$ such that $U(\overline{z_0}, r) \subseteq U$ and $(a_n)_n \subseteq \mathbb{C}$ with

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \overline{z_0})^n.$$

Observe that the sum uniformly converges on all compact subsets of $U(\overline{z_0}, r)$, meaning that

$$f(\overline{z}) = \sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0})^n$$

uniformly converges on all compact subsets of $U(z_0, r) \subseteq V$, as conjugation is continuous. Thus, we may exchange the sum and conjugation during the following series of operations that we carry out on $f(\overline{z})$.

$$\begin{aligned}
f(\overline{z}) &= \sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0})^n \\
&= \sum_{n=0}^{\infty} a_n\overline{(z - z_0)^n} \\
&= \sum_{n=0}^{\infty} \overline{\overline{a_n}(z - z_0)^n} \\
&= \overline{\sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n}.
\end{aligned}$$

Finally, since conjugation is an involution, we have that

$$\begin{aligned}
g(z) &= \overline{f(\overline{z})} \\
&= \overline{\overline{\left(\sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n\right)}} \\
&= \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n.
\end{aligned}$$

Notice that g is defined on $U(z_0, r)$ since $U(z_0, r) \subseteq U(z_0, R)$, where R is the radius of convergence, and the radius of convergence for a power series is unchanged if all its corresponding values of $(a_n)_n$ are conjugated. Thus, g is analytic.

(b) We know that f is holomorphic, so $f'(z)$ exists and is continuous on U. If $z \in V$, we notice that $w \to z$ in V if and only if $\overline{w} \to \overline{z}$ in U, so

$$\begin{aligned}\lim_{w\to z}\frac{g(w)-g(z)}{w-z} &= \lim_{w\to z}\overline{\frac{f(\overline{w})-f(\overline{z})}{w-z}}\\ &= \lim_{w\to z}\frac{f(\overline{w})-f(\overline{z})}{\overline{w}-\overline{z}}\\ &= \lim_{\overline{w}\to\overline{z}}\frac{f(\overline{w})-f(\overline{z})}{\overline{w}-\overline{z}}\\ &= f'(\overline{z}),\end{aligned}$$

meaning that $g'(z)$ exists and is defined as $f'(\overline{z})$ whenever $z \in V$. Since f' is continuous and conjugation is continuous, so too is g', meaning g is holomorphic.

Problem (Problem 6)**:**

(a) For $a \in \mathbb{D}$, define $f_a(z) = \frac{z-a}{1-\overline{a}z}$. Prove that f_a is a bijection from $\mathbb{D}$ to $\mathbb{D}$.

(b) For $a_1, a_2 \in \mathbb{D}$, prove that there is a holomorphic bijection $f\colon \mathbb{D} \to \mathbb{D}$ satisfying $f(a_1) = a_2$.

Solution:

(a) We will show that f_a is a bijection from $\mathbb{D}$ to $\mathbb{D}$ by showing that f_a is defined for all $z \in \mathbb{D}$, that if $z \in \mathbb{D}$, then $f_a(z) \in \mathbb{D}$, then by showing that f_a admits an inverse. First, we observe that f_a is defined so long as $1 - \overline{a}z \neq 0$, meaning that f_a is undefined if

$$\begin{aligned}1-\overline{a}z &= 0\\ z &= \frac{1}{\overline{a}}\\ &= \frac{a}{|a|^2}\\ &= \frac{1}{|a|}\left(\operatorname{sgn}(a)\right),\end{aligned}$$

which necessarily has modulus greater than 1, as $|a| < 1$ and $\left|\operatorname{sgn}(a)\right| = 1$ if $a \neq 0$. Next, we see that $f_a(z)$ is a Möbius transformation that is uniquely determined by

$$\begin{aligned}a &\mapsto 0\\ 0 &\mapsto -a\\ -a &\mapsto \frac{-2a}{1+|a|^2},\end{aligned}$$

all of which stay within the unit disk (for $a \neq 0$ and $a \in \mathbb{D}$). Finally, observe that by taking

$$w = \frac{z-a}{1-\overline{a}z}$$

and solving for w, we obtain

$$z = \frac{w+a}{1+\overline{a}w}.$$

This is a left and right inverse, as

$$\begin{aligned}f_a^{-1}(f_a(z)) &= \frac{\frac{z-a}{1-\overline{a}z}+a}{1+\overline{a}\frac{z-a}{1-\overline{a}z}}\\ &= z,\end{aligned}$$

and

$$\begin{aligned} f_a\left(f_a^{-1}(w)\right) &= \frac{\frac{w+a}{1+\overline{a}w} - a}{1 - \overline{a}\frac{w+a}{1+\overline{a}w}} \\ &= w. \end{aligned}$$

Thus, f is a bijection from $\mathbb{D}$ to $\mathbb{D}$.

(b) Considering the f_a of the previous example, we observe that f_a is holomorphic. To see this, note that

$$\begin{aligned} f_a'(z) &= \lim_{h\to 0} \frac{\frac{(z+h)-a}{1-\overline{a}(z+h)} - \frac{z-a}{1-\overline{a}z}}{h} \\ &= \lim_{h\to 0} \frac{((z+h)-a)(1-\overline{a}z) - (z-a)(1-\overline{a}(z+h))}{h(1-\overline{a}z)(1-\overline{a}(z+h))} \\ &= \frac{\left(1+|a|^2\right) - \overline{a}z}{(1-\overline{a}z)^2}, \end{aligned}$$

which is continuous on $\mathbb{D}$ as it is a rational function that is not undefined on $\mathbb{D}$. Since f_a is holomorphic, it follows that the composition of f_a with any other such f_b is also holomorphic by chain rule. Finally, note that from our above calculations, $f_a^{-1} = f_{-a}$, so we may take

$$f = f_{-a_2} \circ f_{a_1}$$

to be our holomorphic bijection from $\mathbb{D}$ to $\mathbb{D}$ that maps a_1 to a_2.