This is a notes document regarding essential problem-solving methods for the analysis qualifiers.

Real Analysis

August 2019

Problem 1

(a) Recall that the Cantor set $\mathcal{C}$ is defined to consist of all $x \in [0,1]$ such that x only contains 0 and 2 in the ternary expansion of x. Writing $a \in [0,2]$ as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where $a_k \in \{0,1,2\}$, we may then find a_k at each ternary expansion slot for k as follows:

- if $a_k = 0$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = c_k = 0$
- if $a_k = 2$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = 2$ and $c_k = 0$ or vice versa.
- if $a_k = 1$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_{k+1} = c_{k+1} = 2$.

Therefore, since every digit of every ternary expansion in $[0,2]$ can be obtained from $\mathcal{C}$, we see that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) We may set B to be the union of all integer translates of $\mathcal{C}$, and set $A = \mathcal{C}$. This yields closed subsets of $\mathbb{R}$ with Lebesgue measure zero that sum to $\mathbb{R}$.

Problem 2

Consider the sequence of functions

$$f_n(x) = n\mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on $[0,1]$. This sequence is pointwise convergent everywhere to zero, as $f_n(0) = 0$ and the Archimedean property give that for any $x \in (0,1]$, there is some n large enough that gives $\frac{1}{n} < x$. Furthermore, we see that

$$\begin{aligned} \int f_n \, d\mu &= n\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{n+1} \\ &\to 0. \end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned} \int \Phi \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1} \\ &\to \infty. \end{aligned}$$

Problem 4

Suppose toward contradiction that both f and $1/f$ are in $L_1(\mathbb{R})$. Then, from Hölder's Inequality, we have

$$\infty = \int 1 \, d\mu$$

$$\begin{aligned} &\leqslant \left(\int f\,d\mu\right)^{1/2}\left(\int \frac{1}{f}\,d\mu\right)^{1/2} \\ &< \infty, \end{aligned}$$

which is a contradiction.

Problem 5

(a) Let $f \in L_2([-1,1])$. We may find $g \in C([-1,1])$ such that $\|f-g\|_{L_2} < \varepsilon/2$. Similarly, we may find a polynomial p such that $\|g-p\|_u < \varepsilon/4$, meaning that $|p(x)-g(x)| < \varepsilon/4$ for all $x \in [-1,1]$. This yields

$$\begin{aligned} \|p-g\|_{L_2} &= \left(\int_{-1}^{1} |p(x)-g(x)|^2\,dx\right)^{1/2} \\ &< \left(\int_{-1}^{1} \left(\frac{\varepsilon}{4}\right)^2 dx\right)^{1/2} \\ &= \left(\frac{\varepsilon^2}{8}\right)^{1/2} \\ &< \frac{\varepsilon}{2}, \end{aligned}$$

so $\|f-p\|_{L_2} < \varepsilon$, meaning that the closed linear span of the monomials is dense in L_2, and the Legendre polynomials form an orthonormal system.

(b) We see that at every step in evaluating the expression

$$L_n(x) = c_n \frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{$*$}$$

the degree of the polynomial increases by 1, so each $L_n(x)$ has degree n. To verify that the polynomials generated from $(*)$ are orthogonal to each other, we let $n > m$ without loss of generality, and use integration by parts to obtain

$$\begin{aligned} \langle L_n, L_m\rangle &= \int_{-1}^{1}\left(\frac{d^n}{dx^n}\left(x^2-1\right)^n\right)\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx \\ &= \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n \frac{d^m}{dx^m}\left(x^2-1\right)^m\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^{m+1}}{dx^{m+1}}\left(x^2-1\right)^m dx \\ &\vdots \\ &= (-1)^n\int_{-1}^{1}\frac{d^{m+n}}{dx^{m+n}}\left(x^2-1\right)^m dx \\ &= (-1)^n\int_{-1}^{1}\frac{d^n}{dx^n}\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx \\ &= (-1)^n\int \frac{d^n}{dx^n}L_m(x)\,dx \\ &= 0, \end{aligned}$$

seeing as we are taking n derivatives of a degree $m < n$ polynomial.

January 2020

Problem 1

(a) This is false. If $A \subseteq [0,1]$ is the "fat Cantor set" constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then $\mu(A) = \frac{1}{2}$, but A is nowhere dense, meaning that if $U \subseteq A$ is open, then $U = \emptyset$.

(b) This is true. By the definition of the Lebesgue outer measure, for any $\varepsilon > 0$, there are $\{(a_k, b_k)\}_{k=1}^{\infty}$ such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^{\infty}(a_k, b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^{\infty}(a_k, b_k),$$

we have that U is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\left\{U \mid A \subseteq U, U \text{ open}\right\}.$$

Problem 3

(a) Consider the algebra of polynomials on $[0,1]$ without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and $f(x) = x$ separates points in $[0,1]$, this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at $x = 0$, the uniform closure of the algebra yields all the continuous functions on $[0,1]$ with $f(0) = 0$.

(b) In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra $\mathcal{A}$ to include the constant functions.

Problem 4

We consider the signed measure on $\mathcal{F}$ defined by

$$\nu(E) = \int_E f\, d\mu,$$

meaning that $\nu \ll \mu$, so the function $g := \frac{d\nu}{d\mu}$, where $\frac{d\nu}{d\mu}$ denotes the Radon–Nikodym derivative of ν with respect to μ, is $\mathcal{F}$-measurable and in $L_1(\mathbb{R}, \mathcal{F}, \mu)$. This gives, for all $E \in \mathcal{F}$,

$$\begin{aligned}\int_E g\, d\mu &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f\, d\mu.\end{aligned}$$

August 2020

Problem 1

This is false. To see this, let $\mathfrak{C}(x)$ denote the Cantor–Lebesgue function, and let

$$h(x) = \sum_{n=-\infty}^{\infty} \mathfrak{C}(x-n) + n.$$

Then, since $\mathfrak{C}(x)$ has derivative zero almost everywhere, the sum of a number of translates of $\mathfrak{C}(x)$ still has derivative zero almost everywhere. Then, setting

$$f(x) = h(x) + x,$$

we get that $f(x)$ has derivative equal to 1 almost everywhere. However, at the same time, $f(2) - f(1) = 2$.

Problem 2

We show the inverse problem, which is that every closed set in $\mathbb{R}^2$ is G_δ. To do this, we let $A \subseteq \mathbb{R}^2$ be closed, nonempty, and proper (if $A = \emptyset$ or $A = \mathbb{R}^2$ the answer is trivial).

Then, there is some $x \in A^c$, and specifically there is $x \in A^c$ with rational coordinates (else, select $y \in \mathbb{Q}^2$ within the ball of radius ε that allows A^c to be open). Furthermore, since $\mathbb{R}^2$ is a metric space, $\mathbb{R}^2$ is regular, so there are open U_x and V_x such that $A \subseteq U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$.

Therefore, we get

$$A = \bigcap \left\{ U_x \;\middle|\; x \in \mathbb{Q}^2 \setminus A \right\},$$

meaning that A is G_δ. Taking complements, we thus get that every open set is F_σ.