Problem (Exercise 1.17)**:** In this exercise, you will show that the moments of a standard Gaussian variable count pair partitions.

(i) Let X be a standard Gaussian variable. Prove that

$$\mathbb{E}\left(X^{2k}\right) = (2k-1)!!$$
$$\mathbb{E}\left(X^{2k-1}\right) = 0.$$

(ii) Prove that $|P_2(2k)| = (2k-1)!!$ by putting $P_2(2k)$ in explicit bijection with a set of cardinality $(2k-1)|P_2(2k-2)|$.

Remark: I had done the first part of this exercise earlier in a separate notes document, but had not written it up here for submission.

Solution:

(i) We see that

$$\begin{aligned} E[Z^m] &= \int_{-\infty}^{\infty} x^m e^{-x^2/2}\,dx \\ &= -x^m e^{-x^2/2}\big|_{-\infty}^{\infty} + (m-1)\int_{-\infty}^{\infty} x^{m-2}e^{-x^2/2}\,dx \\ &= (m-1)E\left[Z^{m-2}\right]. \end{aligned}$$

Therefore, we recover the recursion relation for $(2k-1)!!$ whenever $m = 2k$ and 0 otherwise.

(ii) Considering the set $[2k] = \{1, 2, \dots, 2k\}$, we see that there are $2k-1$ ways to pair 1 with any other element, and there are then $P_2(2k-2)$ pair partitions of the remaining $2k-2$ elements. This gives our desired bijection.

Problem (Exercise 2.23)**:**

(i) Find a recursion for $|NC_2(2k)|$.

(ii) Show that $\operatorname{Cat}(k) = \frac{1}{k+1}\binom{2k}{k}$ satisfies the same recursion relation shown in (i).

(iii) Let $d\mu = f(x)\,dx$, where

$$f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2} & -2 \leq x \leq 2 \\ 0 & \text{else} \end{cases}.$$

Show that

$$\int_{\mathbb{R}} x^{2k}\,d\mu = \operatorname{Cat}(k)$$
$$\int_{\mathbb{R}} x^{2k-1}\,d\mu = 0.$$