Contents

Introduction

This is going to be the notes from my Honors Thesis project on amenability. We will be covering different results that are used to show that a discrete topological group has a translation-invariant finitely additive probability measure (i.e., a mean).

The primary source texts to inform this independent study will be Volker Runde's *Lectures on Amenability* and Timothy Rainone's *Functional Analysis-en route to Operator Algebras*, as well as various notes compiled by my professor, and some other sources that I've found on the internet.

I do not claim any of this work to be original.

Group Actions, Paradoxical Decompositions, and the Banach–Tarski Paradox

In order to introduce Tarski's theorem, which is where our first condition about the amenability of groups appears, we begin by discussing paradoxical decompositions, with the goal of this section being a proof of the Banach–Tarski Paradox. The Banach–Tarski paradox says the following:

> If A and B are any bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Basics of Group Actions

The information for these essentials about group actions will be drawn from Dummit and Foote's *Abstract Algebra*.

Definition (Group Action)**.** A (left) group action of G onto a set A is a map from $G \times A$ to A that satisfies:

- $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$;
- $e \cdot a = a$[1] for all $a \in A$.

Definition (Permutation Representation)**.** For each g, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ (the group element g acts on a) is a permutation of A. There is a homomorphism associated to these actions:

$$\varphi : G \to S_A,$$

where $\varphi(g) = \sigma_g$. Recall that S_A is the symmetric group (group of permutations) on the elements of A.

This is the permutation representation for the action.

In particular, given any nonempty set A and a homomorphism G into S_A, we can define an action of G on A by taking $g \cdot a = \varphi(g)(a)$.

Definition (Kernel)**.** The kernel of the action of G is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}$$

Note: The kernel of the action is the kernel of the permutation representation $\varphi : G \to S_A$.

Definition (Stabilizer)**.** For each $a \in A$, the stabilizer of a under G is the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

Note: The kernel of the group action is the intersection of the stabilizers of every element of G:

$$\text{kernel} = \bigcap_{a \in A} G_a.$$

Note: For each $a \in A$, G_a is a subgroup of G.

[1] The identity element is usually written as 1, but I will write it as e out of familiarity.

Definition (Faithful Action)**.** An action is faithful if the kernel of the action is e.

Definition (Free Action)**.** For a set X with G acting on X, the action of G on X is free if, for every x, $g \cdot x = x$ if and only if $g = e_G$.

If the action of G on X is a free action, we say G acts freely on X.

Proposition (Equivalence Relation on A)**:** Let G be a group that acts on a nonempty set A. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking $|G : G_a|$, which is the index of the stabilizer of a.

Proof. We can see that $a \sim a$, since $e \cdot a = a$. Similarly, we can see that if $a \sim b$, then $b = g^{-1} \cdot a$, meaning $b \sim a$. Finally, let $a \sim b$ and $b \sim c$. Then, we have $a = g \cdot b$ for some $g \in G$, and $b = h \cdot c$ for some $h \in G$. Thus, we have

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$.

We say there is a bijection between the left cosets of G_a and the elements of the equivalence class of a.

Define C_a to be the set $\{g \cdot a \mid g \in G\}$, and let $b = g \cdot a$. Define a map $g \cdot a \mapsto gG_a$. This map is surjective since $g \cdot a$ is always an element of C_a. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, meaning $h^{-1}g \in G_a$, and $h^{-1}g \in G_a$ if and only if $gG_a = hG_a$, this map is injective.

Since there is a one-to-one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we now know that the number of equivalence classes of a under the action of G is $|G : G_a|$. □

Definition (Orbit)**.** For any $a \in A$, we define the orbit under G of a by

$$G \cdot a = \{b \in A \mid \forall g \in G,\ b = g \cdot a\}$$

In particular, if $c \in G \cdot a$ for some $a \in A$, then $G \cdot c = G \cdot a$.

Paradoxical Decompositions

Most of the information from this section will be drawn from Volker Runde's *Lectures on Amenability*, as well as *Amenable Banach Algebras: A Panorama*.

Definition (Paradoxical Sets and Decompositions)**.** Let G be a group that acts on a set X. Let $E \subseteq X$.

If there exist pairwise disjoint $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j,$$

then we say that E is G-paradoxical.

In particular, a paradoxical group is one where G acts on itself by left-multiplication.

Example (Our First Paradoxical Group)**.** The free group on two generators, $\mathbb{F}(a, b)$,[II] is paradoxical. To see this, we let

$$W(x) = \{w \in \mathbb{F}(a, b) \mid w \text{ starts with } x\}.$$

Here, "starts with" refers to the left-most element. For instance, $ba^2ba^{-1} \in W(b)$.

In particular, we can see that

$$\mathbb{F}(a, b) = \left\{e_{\mathbb{F}(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For any $w \in \mathbb{F}(a, b) \setminus W(a)$, we can see that $a^{-1}w \in W\left(a^{-1}\right)$, meaning $w \in aW\left(a^{-1}\right)$. Therefore, $\mathbb{F}(a, b) = W(a) \sqcup aW\left(a^{-1}\right)$.

Similarly, for any $v \in \mathbb{F}(a, b) \setminus W(b)$, $b^{-1}v \in W\left(b^{-1}\right)$, so $v \in bW\left(b^{-1}\right)$. Therefore, $\mathbb{F}(a, b) = W(b) \sqcup bW\left(b^{-1}\right)$.

Proposition (Free Action of a Paradoxical Group)**:** Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$ be pairwise disjoint, with $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

[II] The set of all reduced words over $\left\{a, b, a^{-1}, b^{-1}, e_{\mathbb{F}(a,b)}\right\}$. In particular, a word is reduced when the pairs aa^{-1} and bb^{-1} are replaced with the identity $e_{\mathbb{F}(a,b)}$.

We let $M \subseteq X$ contain exactly one element from every orbit of G.

The set $\{g \cdot M \mid g \in G\}$ is a partition of X. Since M contains exactly one element from every orbit of G, it is then the case that $\bigcup_{g \in G} g \cdot M = X$, since $G \cdot M = X$.

Additionally, if $x, y \in M$ with $g \cdot x = h \cdot y$, then $\left(h^{-1}g\right) \cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Thus, we must have $h^{-1}g = e_G$, as we assume G acts freely.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$ if $g_1 \neq g_2$, meaning $\{g \cdot M \mid g \in G\}$ is a partition.

Define A_j^* to be the subset of X that is the result of the elements of A_j acting on M. In other words,

$$A_j^* = \bigcup_{g \in A_j} g \cdot M.$$

As a useful shorthand, we can say $A_j^* = A_j \cdot M$.[III] Similarly, we define

$$\begin{aligned} B_j^* &= \bigcup_{h \in B_j} h \cdot M \\ &= B_j \cdot M. \end{aligned}$$

We can see that $A_1^*, A_2^*, \dots, A_n^*, B_1^*, B_2^*, \dots, B_m^* \subseteq X$ are disjoint, since $\{g \cdot M \mid g \in G\}$ is a partition, and $A_1, \dots, A_n, B_1, \dots, B_m$ are disjoint in G.

Thus, we have

$$\begin{aligned} \bigcup_{j=1}^{n} g_j \cdot A_j^* &= \bigcup_{j=1}^{n} \left(g_j A_j\right) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Similarly,

$$\begin{aligned} \bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} \left(h_j B_j\right) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Thus, we see that X has a paradoxical decomposition, meaning X is G-paradoxical. □

Note: We invoked the axiom of choice when we defined M to contain exactly one element from each orbit in X.

[III] Yes, I know that A_j is not technically a group acting on M, but this will help illuminate the final conclusion.

Paradoxical Decompositions of the Unit Sphere and Unit Ball

We are aware of $\mathbb{F}(a, b)$ being a paradoxical group — in particular, we hope to use the properties of $\mathbb{F}(a, b)$ to yield paradoxical decompositions of the unit sphere in $\mathbb{R}^3$, denoted S^2.

Definition (Special Orthogonal Group)**.** For $n \in \mathbb{N}$, we define the special orthogonal group to consist of all real $n \times n$ matrices A such that

$$A^T A = AA^T = I,$$

with $\det(A) = 1$.

In particular, $SO(3)$ denotes the set of all rotations about some line that runs through the origin. An important fact about $SO(3)$ is that it contains a paradoxical subgroup.

Theorem: There are rotations A and B about lines through the origin in $\mathbb{R}^3$ which generate a subgroup of $SO(3)$ isomorphic to $\mathbb{F}(a, b)$.

Proof. We set

$$A^{\pm} = \begin{bmatrix} 1/3 & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$B^{\pm} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

Here, A^+ denotes A, and A^- denotes A^{-1}, and similarly with B.

Let w be a reduced word in A, B, A^{-1}, and B^{-1} which is not the empty word. We claim that w cannot be the identity. Without loss of generality, we assume w ends in A or A^{-1} — this is because w acts as the identity if and only if AwA^{-1} or $A^{-1}wA$ act as the identity.

In particular, we will show that there exist $a, b, c \in \mathbb{Z}$ with $b \not\equiv 0$ modulo 3 such that

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix},$$

where k is the length of w. The main reason we wish to show this is that, if we have $b \not\equiv 0$ modulo 3, it is the case that w necessarily cannot map $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to itself.

We start with induction on the length of w. In particular, for $w = A^{\pm}$, we have

$$w \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\ \pm 2\sqrt{2}\\ 0\end{pmatrix},$$

proving the base case.

Suppose $k > 0$, meaning $w = A^{\pm}w'$ or $w = B^{\pm}w'$, with w' not equal to the empty word. The inductive hypothesis says that

$$w' \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3^{k-1}}\begin{pmatrix}a'\\ b'\sqrt{2}\\ c'\end{pmatrix},$$

for some $a', b', c' \in \mathbb{Z}$ with $b \not\equiv 0$ modulo 3. In particular,

$$\begin{aligned} A^{\pm}w' \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} &= \frac{1}{3^{k}}\begin{pmatrix}a' \mp 4b'\\ (b' \pm 2a')\sqrt{2}\\ 3c'\end{pmatrix}\\ B^{\pm}w' \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} &= \frac{1}{3^{k}}\begin{pmatrix}3a'\\ (b' \mp 2c')\sqrt{2}\\ c' \pm 4b'\end{pmatrix}, \end{aligned}$$

where we say

$$w \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3^{k}}\begin{pmatrix}a\\ b\\ c\end{pmatrix},$$

i.e., we set the coordinates of $w \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$ through their definition in $A^{\pm}w'$ or $B^{\pm}w'$.

In order to show that $b \not\equiv 0$ modulo 3, we must examine the following four cases.

Let w^* denote the word such that

$$w^* \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3^{k-2}}\begin{pmatrix}a''\\ b''\sqrt{2}\\ c''\end{pmatrix},$$

with $a'', b'', c'' \in \mathbb{Z}$ and $b'' \not\equiv 0$ modulo 3. It is important to note here that w^* may be the empty word.

Case 1: Suppose $w = A^{\pm}B^{\pm}w^*$. Then, we have $b = b' \mp 2a'$, where $a' = 3a''$. Since $b' \not\equiv 0$ modulo 3 by the inductive hypothesis assumption, it is also the case that $b \not\equiv 0$ modulo 3.

Case 2: Suppose $w = B^{\pm}A^{\pm}w^{*}$. Then, we have $b = b' \mp 2c'$, where $c' = 3c''$. Similarly, since $b' \not\equiv 0$ modulo 3 by the inductive hypothesis assumption, it is also the case that $b \not\equiv 0$ modulo 3.

Case 3: Suppose $w = A^{\pm}A^{\pm}w^{*}$. Then, we have

$$\begin{aligned} b &= b' \pm 2a' \\ &= b' \pm 2\left(a'' \mp 4b''\right) \\ &= b' + \left(b'' \pm 2a''\right) - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Since $b', b'' \not\equiv 0$ modulo 3 by the inductive hypothesis, it is also the case that $b \not\equiv 0$ modulo 3.

Case 4: Suppose $w = B^{\pm}B^{\pm}w^{*}$. Then, we have

$$\begin{aligned} b &= b' \mp 2c' \\ &= b' \mp 2\left(c'' \pm 4b''\right) \\ &= b' + \left(b'' \mp 2c''\right) - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Since $b', b'' \not\equiv 0$ modulo 3 by the inductive hypothesis, it is also the case that $b \not\equiv 0$ modulo 3.

Thus, we have shown that any non-empty reduced word over A, A^{-1}, B, B^{-1} does not act as the identity. The subgroup of $SO(3)$ generated by A, B, A^{-1}, and B^{-1} is thus isomorphic to $\mathbb{F}(a, b)$. □

Remark: For any element of $SO(n)$ with $n \geqslant 3$, we can write A_n (denoting the $n \times n$ matrix corresponding to A) as

$$\begin{aligned} A_n &= \begin{pmatrix} A_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \\ B_n &= \begin{pmatrix} B_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \end{aligned}$$

where $\mathbf{0}$ denotes a block matrix consisting of 0 and $\mathbf{1}$ denotes a block matrix equal to the identity.

This means that our subgroup of $SO(3)$ isomorphic to $\mathbb{F}(a, b)$ embeds into $SO(n)$ via the above block matrices.

Theorem (Hausdorff Paradox)**:** There is a countable subset D of S^2 such that $S^2 \setminus D$ is paradoxical under the action of $SO(3)$.

Proof. Let A and B be the rotations in $SO(3)$ that serve as the generators of the subgroup isomorphic to $\mathbb{F}(a, b)$.

Since A and B are rotations, any word in the subgroup generated by A and B will also be a rotation — as a result, all such (non-empty) words contain two fixed points.

Let

$$F = \left\{x \in S^2 \mid x \text{ is a fixed point for some word } w\right\}.$$

Since the set of all words in A and B is countably infinite, so too is F. Therefore, the union of all these fixed points under the action of all such words w is also countable:

$$D = \bigcup_{w \in G} w \cdot F.$$

Since the set of words in A and B act freely on $S^2 \setminus D$, it must be the case that $S^2 \setminus D$ is paradoxical under the action of the group of all such words. ☐

Definition (Equidecomposable Sets)**.** Let G act on X, $A, B \subseteq X$. We say A and B are equidecomposable under G if there are $A_1, \dots, A_n \subseteq A$, $B_1, \dots, B_n \subseteq B$, and $g_1, \dots, g_n \in G$ such that

(i) $A = \bigcup_{j=1}^{n} A_j$ and $B = \bigcup_{j=1}^{n} B_j$;

(ii) the collection $\left\{A_j\right\}_{j=1}^{n}$ are pairwise disjoint and the collection $\left\{B_j\right\}_{j=1}^{n}$ are pairwise disjoint;

(iii) for each j, $g_j \cdot A_j = B_j$.

We write $A \sim_G B$ if A and B are equidecomposable under G.

Remark: The relation $\sim_G$ is an equivalence relation.

In particular, to see transitivity, we have the partitions $A_1, \dots, A_n \subseteq B$ and $B_1, \dots, B_n \subseteq B$ with $g_i \cdot A_i = B_i$, and the partitions $B_1, \dots, B_m \subseteq B$, $C_1 \cdots C_m \subseteq C$ with $h_j \cdot B_j = C_j$.

We find the partition of A by taking $A_{ij} = B_i \cap B_j$, where $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$. We then have $h_j g_i \cdot A_{ij}$ maps to a refined partition of C, yielding equidecomposability between A and C.

Remark: For equidecomposable sets A and B, there is a bijection $\phi : A \to B$ by, for each $C \subseteq A$, taking $C_i = C \cap A_i$, where $A_1, \dots, A_n$ is the partition of A, and mapping $\varphi(C_i) = g_i \cdot C_i$.

Proposition: Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \setminus D$ are equidecomposable under the action of SO(3).

Proof. Let L be a line in $\mathbb{R}^3$ with the property that $L \cap D = \emptyset$. Such a L must necessarily exist as the set of all antipodes in S^2 is uncountable.

Define $\rho_\theta \in SO(3)$ to be a rotation about L by an angle of θ. For fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{x \in D \mid \rho_\theta^n \cdot x \in D\right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

Define $W_n = \{\theta \mid R_{n,\theta} \neq \emptyset\}$. The injection $\theta \mapsto \rho_\theta^n \cdot x$ into D shows that for each n, W_n is countable. Thus, defining

$$W = \bigcup_{n \in N} W_n$$

it is evident that W is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. Define ρ to be a rotation about L by ω. Then, for every $n, m \in \mathbb{N}$,

$$\rho^n \cdot D \cap \rho^m \cdot D = \emptyset.$$

We let $\tilde{D} = \bigsqcup_{n=0}^{\infty} \rho^n \cdot D$. Notice that, in particular, $\rho \cdot \tilde{D} = \bigsqcup_{n=1}^{\infty} \rho^n \cdot D$, meaning $\tilde{D}$ and $\tilde{D} \setminus D$ are equidecomposable under $SO(3)$.

Thus, we have

$$\begin{aligned} S^2 &= \tilde{D} \sqcup \left(S^2 \setminus \tilde{D}\right) \\ &\sim_{SO(3)} \rho \cdot \tilde{D} \sqcup \left(S^2 \setminus \tilde{D}\right) \\ &= \left(\tilde{D} \setminus D\right) \sqcup \left(S^2 \setminus \tilde{D}\right) \\ &= S^2 \setminus D, \end{aligned}$$

establishing the equidecomposability of S^2 and $S^2 \setminus D$. □

Proposition: Let G act on X, with E and E' subsets of X such that $E \sim_G E'$. Then, if E is paradoxical under the action of G, so too is E'.

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m$ be pairwise disjoint subsets of E and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} E &= \bigsqcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} h_j \cdot B_j, \end{aligned}$$

which follows from the paradoxicality of E. We let

$$\begin{aligned} A &= \bigsqcup_{i=1}^{n} A_i \\ B &= \bigsqcup_{j=1}^{m} B_j. \end{aligned}$$

It follows that $A \sim_G E$ and $B \sim_G E$; to see this, set the partition of A to be $A_1, \dots, A_n$, and set the partition of E to be $g_i \cdot A_i$ for $i \in \{1, \dots, n\}$, and similarly for G.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$, implying that there exists a paradoxical decomposition of E' in $A_1, \dots, A_n$ and $B_1, \dots, B_m$. □

Since $S^2 \setminus D$ and S^2 are equidecomposable under the action of $SO(3)$, and $S^2 \setminus D$ is paradoxical under the action of $SO(3)$, the above proposition implies the following corollary.

Corollary: S^2 is paradoxical under $SO(3)$.

Definition (Euclidean Group)**.** The Euclidean group $E(n)$ consists of all isometries of a Euclidean space. An isometry of a Euclidean space consists of translations, flips about the origin, and rotation.

In particular, $E(n) = T(n) \rtimes O(n)$, where $T(n)$ denotes all translations and $O(n)$ is the orthogonal group, which denotes all rotations or flips.

We define $E_+(n)$ to be all orientation-preserving isometries of Euclidean space. In particular, $E_+(n) = T(n) \rtimes SO(n)$, where $SO(n)$ is the special orthogonal group, which denotes all orientation-preserving rotations.

Corollary (Weak Banach–Tarski Paradox)**:** Every closed ball in $\mathbb{R}^3$ is paradoxical under the Euclidean group $E(3)$.

Proof. We only need to show that the closed unit ball, $B(0,1)$, is paradoxical under the action of $E(3)$.

To start, we can show that $B(0,1) \setminus \{0\}$ is paradoxical. Since $SO(3)$ is paradoxical, there exist pairwise disjoint $A_1, \dots, A_n, B_1, \dots, B_m \subseteq S^2$ and $g_1, \dots, g_n, h_1, \dots, h_m \in SO(3)$ such that

$$\begin{aligned} S^2 &= \bigsqcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

Define

$$\begin{aligned} A_i^* &= \{tx \mid t \in (0,1], x \in A_i\} \\ B_j^* &= \left\{ty \mid t \in (0,1], y \in B_j\right\}. \end{aligned}$$

Then, $A_1^*, \dots, A_n^*, B_1^*, \dots, B_m^* \subseteq B(0,1) \setminus \{0\}$ are pairwise disjoint, and

$$\begin{aligned} B(0,1) \setminus 0 &= \bigcup_{i=1}^{n} g_i \cdot A_i^* \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j^*. \end{aligned}$$

Thus, we have established that $B(0,1) \setminus \{0\}$ is paradoxical under $SO(3) \leqslant E(3)$.[IV]

[IV]Essentially, we take the paradoxical decomposition of S^2 under $SO(3)$, and scale by t to cover all of $B(0,1) \setminus \{0\}$.

Now, we want to show that $B(0,1) \setminus \{0\}$ and $B(0,1)$ are equidecomposable under $E(3)$. To do this, let $x \in B(0,1) \setminus \{0\}$, and let ρ be a rotation about x by a line that misses the origin such that $\rho^n \cdot 0 \neq \rho^m \cdot 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$.[v] Let $D = \{\rho^n \cdot 0 \mid n \in \mathbb{N}\}$. We can see that $\rho \cdot D = D \setminus \{0\}$, and that D and $\rho \cdot D$ are equidecomposable under $E(3)$.

Thus, we have

$$\begin{aligned} B(0,1) &= D \sqcup (B(0,1) \setminus D) \\ &\sim_{E(3)} (\rho \cdot D) \sqcup (B(0,1) \setminus D) \\ &= (D \setminus \{0\}) \sqcup (B(0,1) \setminus D) \\ &= B(0,1) \setminus \{0\}, \end{aligned}$$

establishing the equidecomposability of $B(0,1)$ and $B(0,1) \setminus \{0\}$.

Thus, $B(0,1)$ is paradoxical under the action of $E(3)$. □

Definition. For G acting on a set X, we write $A \preceq_G B$ if A is equidecomposable with a subset of B.

Remark: We can see that $\preceq_G$ is reflexive since A is equidecomposable with A, and $A \subseteq A$.

To show transitivity, let $A \preceq_G B$ and $B \preceq_G C$. We let $g_1, \dots, g_n \in G$ such that $A \sim_G B_\alpha$, where $B_\alpha \subseteq B$. In particular, we have $A_1, \dots, A_n \subseteq A$ and $B_{1,\alpha}, \dots, B_{n,\alpha} \subseteq B_\alpha$ such that $g_i \cdot A_i = B_{i,\alpha}$. We let $h_1, \dots, h_m \in G$ and $C_\beta \subseteq C$ such that $h_j \cdot B_j = C_{j,\beta}$ for each $j \in \{1, \dots, m\}$.

We take a refinement on B taking intersections $B_{i,j,\alpha} = B_i \cap B_{j,\alpha}$ for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Thus, taking $h_j g_i \cdot A_i$, we see that A is equidecomposable with a subset of C (namely, the subset of C "generated" by the disjoint subsets of B_α refined by B_i).

Remark: Since $A \sim_G B$ implies the existence of a bijection $\phi : A \to G$, the $\preceq_G$ relation is akin to the $\leqslant$ relation for cardinalities; in particular, $A \preceq_G B$ implies the existence of an injection $\phi : A \hookrightarrow B$.

This analogy between cardinality and the $\preceq_G$ relation naturally lends itself to the following theorem.

Theorem: Let G be a group that acts on X. Let A, B be subsets of X with $A \preceq_G B$ and $B \preceq_G A$. Then, $A \sim_G B$.

Proof. Let $B_1 \subseteq B$ with $A \sim_G B_1$, and let $A_1 \subseteq A$ with $B \sim_G A_1$.

We know that there exist bijections $\phi : A \to B_1$ and $\psi : B \to A_1$. Define $C_0 = A \setminus A_1$, $C_{n+1} = \psi(\phi(C_n))$. We set

$$C = \bigcup_{n \geqslant 1} C_n.$$

Since $\psi^{-1}(\psi(\phi(C_n))) = \phi(C_n)$, we have

$$\psi^{-1}(A \setminus C) = B \setminus \phi(C),$$

[v] This is why we need our underlying group acting on $\mathbb{R}^3$ to be the Euclidean group rather than $SO(3)$. It is still the case that $SO(3) \leqslant E(3)$, meaning that $E(3)$ is necessarily paradoxical when acting on $\mathbb{R}^3$.

meaning $A \setminus C \sim B \setminus \phi(C)$. Additionally, $C \sim \phi(C)$. Thus,

$$\begin{aligned} A &= (A \setminus C) \cup C \\ &\sim (B \setminus \phi(C)) \cup \phi(C) \\ &= B. \end{aligned}$$

□

Theorem (Banach–Tarski Paradox)**:** Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. Then, $A \sim_{E(3)} B$.

Proof. It is sufficient to show that $A \preceq_{E(3)} B$.

Since A is bounded, there is $r > 0$ such that $A \subseteq B(0, r)$. Let $x \in B^\circ$. Then, there is $\epsilon > 0$ such that $B(x, \epsilon) \subseteq B$.

Since $B(0, r)$ is compact (and thus totally bounded), there are translations $g_1, \dots, g_n$ such that

$$B(0, r) \subseteq g_1 \cdot B(x, \epsilon) \cup \cdots \cup g_n \cdot B(x, \epsilon).$$

Choose translations $h_1, \dots, h_n$ such that $h_j \cdot B(x, \epsilon) \cap h_k \cdot B(x, \epsilon) = \emptyset$ for $j \neq k$. Set

$$S = \bigcup_{j=1}^{n} h_j \cdot B(x, \epsilon).$$

Since each of the subsets $h_j \cdot B(x, \epsilon)$ is equidecomposable with any arbitrary closed ball subset of $B(x, \epsilon)$, it is the case that $S \subseteq B(x, \epsilon)$.

Thus, we have

$$\begin{aligned} A &\subseteq B(0, r) \\ &\subseteq g_1 \cdot B(x, \epsilon) \cup g_n \cdot B(x, \epsilon) \\ &\preceq S \\ &\preceq B(x, \epsilon) \\ &\subseteq B. \end{aligned}$$

□

Remark: The axiom of choice was invoked when we stated that $h_j \cdot B(x, \epsilon)$ is equidecomposable with an arbitrary closed ball subset of $B(x, \epsilon)$.

Tarski's Theorem

One of the central facts that allowed for the Banach–Tarski paradox to be true is that $\mathbb{F}(a,b)$ does not have a property known as amenability. We had also proved that $\mathbb{F}(a,b)$ is paradoxical.

In this section, we will prove paradoxicality and non-amenability are equivalent. This is formulated in Tarski's Theorem

Theorem (Tarski's Theorem)**:** Let G be a group that acts on a set X, and let E be a subset of X. There is a finitely additive set function invariant under G, $\mu : P(X) \to [0,\infty]$ with $\mu(E) \in (0,\infty)$ if and only if E is not G-paradoxical.

Remark: It is possible to see that if G is paradoxical, with $X = G$ and G acting on itself via left-multiplication, that this finitely-additive set function eventually "blows up."

Let G be paradoxical. Suppose toward contradiction that there existed such a $\nu : P(G) \to [0,\infty]$. For $E_1,\dots,E_n \subseteq G$ and $t_1,\dots,t_n \in G$, $F_1,\dots,F_m \subseteq G$ and $s_1,\dots,s_m \in G$ with $E_1,\dots,E_n,F_1,\dots,F_m$ pairwise disjoint, we have

$$\begin{aligned}
\nu(G) &= \nu\left(\bigsqcup_{j=1}^{n} t_jE_j\right)\\
&= \sum_{j=1}^{n}\nu(t_jE_j)\\
&= \sum_{j=1}^{n}\nu(E_j).
\end{aligned}$$

We know that $G \cup s_1F_1 = G$, meaning $\nu(G) = \nu(G \cup s_1F_1)$. However,

$$\begin{aligned}
\nu(G \cup s_1F_1) &= \nu\left(\bigsqcup_{j=1}^{n} t_jE_j \sqcup s_1F_1\right)\\
&= \sum_{j=1}^{n}\nu(t_jE_j) + \nu(s_1F_1)\\
&= \nu(G) + \nu(s_1F_1)\\
&= \nu(G) + \nu(F_1)\\
&> \nu(G).
\end{aligned}$$

Perfect Matchings in Infinite Bipartite Graphs

In order to prove that non-paradoxicality implies amenability, we must use some essential results from graph theory.

Definition (Graphs and Paths)**.** A graph is a triple (V, E, ϕ), where V and E are nonempty sets, and $\phi : E \to P_2(V)$ is a map from e to the set of all unordered subset pairs of V.

We say that for $\phi(e) = \{v, w\}$, we say v and w are endpoints of e, with e incident on v and w.

A path in (V, E, ϕ) is a finite sequence $(e_1, \dots, e_n)$ of edges along with a finite sequence of vertices $v_0, \dots, v_n$, with $\phi(e_k) = \{v_{k-1}, v_k\}$.

The degree of a vertex, $\deg(v)$, is the number of edges incident on the vertex.

Definition (Bipartite Graphs and k-Regularity)**.** Let (V, E, ϕ) be a graph, $k \in \mathbb{N}$.

(i) If $\deg(v) = k$ for each $v \in V$, we say (V, E, ϕ) is k-regular.

(ii) If $V = X \sqcup Y$, with each edge having an endpoint in X and an endpoint in Y, we say (V, E, ϕ) is bipartite.

Definition (Perfect Matching)**.** Let (X, Y, E, ϕ) be a bipartite graph. Let $A \subseteq X$ and $B \subseteq Y$. A perfect matching of A and B is a subset $F \subseteq E$ with

(i) each element of $A \cup B$ is an endpoint of exactly one $f \in F$

(ii) all endpoints of edges in F are in $A \cup B$.

Exercise (Hall's Theorem)**:** Let (X, Y, E, ϕ) be a bipartite graph which is k-regular for some $k \in \mathbb{N}$. Suppose $|V| < \infty$.

(i) Show that $|E| < \infty$ and that $|X| = |Y|$.

(ii) For any $M \subseteq V$, let $N(M)$ be the set of those vertices which are joined by an edge with a point in M. Show that $|N(M)| \geqslant |M|$.

(iii) Let $A \subseteq X$ and $B \subseteq Y$ be such that there is a perfect matching F of A and B with $|F|$ maximal. Show that $A = X$.

(iv) Conclude that there is a perfect matching of X and Y.

Solution.

(i) $|E| = k|X| = k|Y|$, meaning $|X| = |Y|$.

(ii) Let $M_X = M \cap X$ and $M_Y = M \cap Y$. Notice that $M = M_X \sqcup M_Y$.

Let $[M_X, N(M_X)]$ denote the set of edges with endpoints in M_X and $N(M_X)$, and similarly for $[M_Y, N(M_Y)]$. We also let $[X, N(M_X)]$ and $[Y, N(M_Y)]$ denote the set of edges with endpoints in X and $N(M_X)$ and the set of edges with endpoints in Y and $N(M_Y)$ respectively.

We can see that $[M_X, N(M_X)] \subseteq [X, N(M_X)]$, and similarly with $[M_Y, N(M_Y)] \subseteq [Y, N(M_Y)]$. Additionally, $|[M_X, N(M_X)]| = k|M_X|$, with $|[X, N(M_X)]| = k|N(M_X)|$, meaning $|M_X| \leqslant |N(M_X)|$, and similarly for $|M_Y|$ and $|N(M_Y)|$.

Thus, $|M| \leqslant |N(M)|$. For future reference, the condition $|M| \leqslant |N(M)|$ is known as Hall's marriage condition.

(iii) Let $A \subseteq X$ and $B \subseteq Y$. Let F be a perfect matching between A and B. Suppose toward contradiction that $A \neq X$.

Thus, there exists $x \in X \setminus A$. Consider the set $Z \subseteq V$ consisting of all vertices such that there exists a F-alternating path $(e_1, \dots, e_n)$ between $z \in Z$ and x.

It cannot be the case that $Z \cap Y$ is empty, since the number of neighbors of x is greater than or equal to 1 by Hall's marriage condition, and if $Z \cap Y$ were empty, then we could add an element to F consisting of x and one element of $N(\{x\})$, which would contradict the maximality of F.

Consider a path that traverses along Z, $(e_1, \dots, e_n)$. It must be the case that $e_n \in F$, as otherwise we would be able to "flip" the matching F by exchanging e_i with e_{i+1} if $e_i \in F$. Thus, we must have that every element of $Z \cap Y$ is satisfied by F, so $Z \cap Y \subseteq B$.

Additionally, since each element in $Z \cap Y$ is paired with exactly one element of $Z \cap X$, with one left over, it is the case that $|Z \cap X| = |Z \cap Y| + 1$.

Suppose toward contradiction that there exists $y \in N(Z \cap X)$ with $y \notin Z \cap Y$. Then, there exists $v \in Z \cap X$ and $e \in E$ such that $\phi(e) = \{v, y\}$. However, this means y is connected via a path to x, implying that $y \in Z$, so $y \in Z \cap Y$, which is a contradiction. Thus, we must have $N(Z \cap X) = Z \cap Y$.

Thus, we have

$$\begin{aligned} |Z \cap X| &= |Z \cap Y| + 1 \\ &= |N(Z \cap X)| + 1, \end{aligned}$$

which contradicts Hall's marriage condition we established in part (ii). Therefore, $A = X$.

(iv) By symmetry, we have $A = X$ and $B = Y$, implying that there is a perfect matching in (X, Y, E, ϕ).

Definition (Hall's Marriage Condition)**.** For a bipartite graph (X, Y, E, ϕ), there is an X-perfect matching[VI] if and only if for all $S \subseteq X$, $|N(S)| \geqslant |S|$.

Equivalently, for a finite collection of (not necessarily distinct) finite sets $\mathcal{G} = \{X_i\}_{i=1}^n$, there is a system of distinct representatives[VII] for $\mathcal{G}$ if and only if for all subcollections $\mathcal{Y}_k = \{X_{i_k}\}_{k=1}^m$, $|\mathcal{Y}_k| \leqslant \left|\bigcup_{k=1}^m X_{i_k}\right|$.

[VI] A matching on (X, Y, E, ϕ) that satisfies every vertex of X.

[VII] A set $\{x_i\}_{i=1}^n$ such that $x_i \in X_i$ and $x_i \neq x_j$ for $i \neq j$.

Remark: These two formulations of Hall's marriage condition are equivalent regarding a X-perfect matching.

In the case of a graph, this yields an injection $f : X \hookrightarrow Y$, and in the case of a collection of sets, this yields an injection $f : \mathcal{G} \hookrightarrow \bigcup_{i=1}^{n} X_i$.

Definition (Choice Function)**.** Let $\mathcal{Y} = \{X_i\}_{i\in I}$ be a collection of sets. A function $f : \mathcal{Y} \to \bigcup_{i\in I} X_i$ is known as a choice function if, for each $i \in I$, $f(X_i) \in X_i$.

We also say that $f : \mathcal{Y} \to \bigcup_{i\in I} X_i$ is a choice function if $f \in \prod_{i\in I} X_i$.

Remark: A choice function on an infinite collection $\mathcal{Y}$ is analogous to a system of distinct representatives on a finite collection $\mathcal{G}$.

Theorem (Tychonoff)**:** Let $\{X_i\}_{i\in I}$ be a family of compact topological spaces. Then, $\prod_{i\in I} X_i$ is compact.

There is a way to extend Hall's Marriage Condition to a particular infinite case, which we will use to show König's Theorem.

Theorem (Infinite Marriage Theorem)**:** Let $\mathcal{G} = \{X_i\}_{i\in I}$ be a collection of (not necessarily distinct) finite sets. If, for every finite subcollection $\mathcal{Y} = \{X_{i_k}\}_{k=1}^{n}$,

$$n \leqslant \left|\bigcup_{k=1}^{n} X_{i_k}\right|,$$

then there is a choice function on $\mathcal{G}$.

Proof. We endow each $X_i \in \{X_i\}_{i\in I}$ with the discrete topology. Since each X_i is finite, it is the case that each X_i is compact.

By Tychonoff's theorem, it is the case that $\prod_{i\in I} X_i$ is compact.

For every finite subset Y of $\mathcal{G}$, define

$$S_Y = \left\{ f \in \prod_{i\in I} X_i \;\middle|\; f|_Y \text{ is injective} \right\}$$

In particular, the injectivity of $f|_Y$ is equivalent the existence of a system of distinct representatives on Y. Since Y satisfies Hall's marriage condition, it is the case that each S_Y is nonempty. Additionally, since $\prod_{i\in I} X_i$ is endowed with the discrete topology, it is also the case that S_Y is closed.

We define $F = \{S_Y \mid Y \subseteq \mathcal{G} \text{ finite}\}$. Notice that, for finite $Y_1, Y_2 \subseteq \mathcal{G}$, since every system of distinct representatives on $Y_1 \cup Y_2$ is a system of distinct representatives on Y_1 and a system of distinct representatives on Y_2, it is the case that $S_{Y_1 \cup Y_2} \subseteq S_{Y_1} \cap S_{Y_2}$.

Since F is a closed subset of $\prod_{i\in I} X_i$, F is compact, meaning that $\bigcap_{Y\subseteq\mathcal{G}} F$ is nonempty. Thus, for $f \in \bigcap F$, f is necessarily a choice function. □

Remark: This is equivalent to the existence of an injection $f : \mathcal{G} \hookrightarrow \bigcup_{i \in I} X_i$.

Theorem (König's Theorem)**:** Let (X, Y, E, ϕ) be a bipartite graph which is k-regular for some $k \in \mathbb{N}$. Then, there is a perfect matching of X and Y.

Proof. If $k = 1$, it is clear that there is a perfect matching in (X, Y, E, ϕ), consisting exclusively of the edges in (X, Y, E, ϕ).

Let $k \geqslant 2$. Since any finite subset of X satisfies Hall's marriage condition (as displayed in the proof that a k-regular bipartite graph has a perfect matching), it is the case that there is some X-perfect matching. We call this X-perfect matching F.

Considering $f : X \hookrightarrow Y$ by taking $x \mapsto y$, where $\{x, y\} \in F$, we see that f is an injection.

Similarly, since any finite subset of Y satisfies Hall's marriage condition (by symmetry), there is some Y-perfect matching. We call this Y-perfect matching G. Considering $g : Y \hookrightarrow X$ by taking $y \mapsto x$, where $\{x, y\} \in G$, we see that g is an injection.

Consider, now, $(X, Y, F \cup G, \phi|_{F \cup G})$. We can see that, when restricted to this subgraph of (X, Y, E, ϕ), the injections f and g between X and Y still hold. Thus, by the Cantor–Schröder-Bernstein theorem, there is a bijection $h : X \to Y$ in $(X, Y, F \cup G, \phi|_{F \cup G})$. This is our desired perfect matching. □

Definition. Let G be a group that acts on a set X.

(i) Define $X^* = X \times \mathbb{N}_0$, and

$$G^* = \left\{(g, \pi) \mid g \in G, \pi \in \operatorname{Sym}(\mathbb{N}_0)\right\}.$$

Let G^* act on X^* by

$$(g, \pi) \cdot (x, n) = (g \cdot x, \pi(n)).$$

(ii) If $A \subseteq X^*$, then the values of $n \in \mathbb{N}_0$ such that there is an element of A whose second coordinate is n are called the levels of A.

Exercise: Show that for $E_1, E_2 \subseteq X$, $E_1 \sim_G E_2$ if and only if $E_1 \times \{n\} \sim_{G^*} E_2 \{m\}$ for all $n, m \in \mathbb{N}_0$.

Solution. Let $E_1 \sim_G E_2$, with $g_1, \dots, g_n \in G$ that satisfy the definition of equidecomposability. Then, composing the permutation $\pi_1 : \mathbb{N}_0 \to \mathbb{N}_0$ where $\pi_1(n) = m$, $\pi_1(m) = n$, and $\pi_1(k) = k$ for all $k \neq n$, with $\pi_2, \dots, \pi_n = \text{id}$, we obtain elements $(g_1, \pi_1), \dots, (g_n, \pi_n) \in G^*$ that yield $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$.

Similarly, if $E_1 \times \{n\} \sim_{G^*} E_2 \times \{m\}$, we select $g_1, \dots, g_n$ from the elements $(g_1, \pi_1), \dots, (g_n, \pi_n)$ that satisfy the definition of equidecomposability. This yields $E_1 \sim_G E_2$.