Problem (Problem 1)**:** Describe the topology of the Grassmanian $\mathrm{Gr}(k, n)$ in a uniform way, so that $\mathbb{RP}^n$ becomes the special case of $\mathrm{Gr}(1, n)$.

Solution: We let elements of $\mathrm{Gr}(k, n)$ be defined as equivalence classes of linearly independent k-tuples of vectors in $\mathbb{R}^n$, where $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if $\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{w_1, \dots, w_k\}$.

By extending $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ to ordered bases $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$, we see that these k-tuples are equivalent if and only if there is a change of basis transformation Q with matrix representation

$$Q = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n-k) \times (n-k)$ matrix. The subgroup of all such $Q \subseteq \mathrm{GL}_n(\mathbb{R})$, which we call P, is the stabilizer of $\mathrm{Gr}(k, n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $\mathrm{GL}_n(\mathbb{R})$ acts transitively on all ordered bases of $\mathbb{R}^n$), we obtain $\mathrm{Gr}(k, n) \cong \mathrm{GL}_n(\mathbb{R})/P$, where the latter coset space is given the quotient topology.

Note that this definition comports with the definition of $\mathbb{RP}^n$ as the space of one-dimensional subspaces, as the invertible 1×1 matrices are precisely the nonzero scalars.

Problem (Problem 2)**:** Fix an inner product on $\mathbb{R}^n$. Show that the map $V \mapsto V^{\perp}$ induces a C^{∞} diffeomorphism $\mathrm{Gr}(k, n) \to \mathrm{Gr}(n-k, n)$.

Solution: We know that, since there is an inner product, we may express the smooth atlas of $\mathrm{Gr}(n, k)$ by $\{(U_V, \varphi_V)\}$, where

$$U_V = \left\{ W \in \mathrm{Gr}(k, n) \;\middle|\; W \cap V^{\perp} = 0 \right\},$$

and $\varphi = P_{V^{\perp}} P_V|_W^{-1}$ is the sequence of projections. By pre-composing with the map $V \mapsto V^{\perp}$, we get the atlas $\{(U_{V^{\perp}}, \varphi_{V^{\perp}})\}$ for $\mathrm{Gr}(n-k, n)$ consisting of charts of the form

$$\begin{aligned} U_{V^{\perp}} &= \{ W \in \mathrm{Gr}(n-k, n) \mid W \cap V = 0 \} \\ \varphi_{V^{\perp}} &= P_V P_{V^{\perp}}|_W^{-1}, \end{aligned}$$

Since the maps $\varphi_V \circ (V \mapsto V^{\perp}) \circ \varphi_{V^{\perp}}^{-1}$ are a composition of smooth bijections with smooth inverses, we see that this is a C^{∞} diffeomorphism between $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k, n)$.

Problem (Problem 3)**:** Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Problem (Problem 4)**:** Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either $\mathbb{R}$ or S^1, depending on whether it is compact or not.

Solution: Let M be a connected, paracompact manifold with dimension 1, and let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas for M, where φ_i are homeomorphisms between U_i and $\mathbb{R}$.

Let $\left\{V_j\right\}_{j \in J}$ be a locally finite refinement of $\{U_i\}_{i \in I}$, where the restrictions $\psi_j := \varphi_i|_{V_j}$ are homeomorphisms to $O_j \subseteq \mathbb{R}$. We see that for any $p \in M$, since the family of V_j with $p \in V_j$, which we call $\mathcal{V}_p = \left\{ V_j \;\middle|\; p \in V_j \right\}$, is finite, the intersection $\bigcap \mathcal{V}_p$ is open; similarly, the intersection $\bigcap \mathcal{O}_p \subseteq \mathbb{R}$ is open, where $\mathcal{O}_p = \left\{ \varphi|_{V_j}(V_j) \subseteq \mathbb{R} \;\middle|\; V_j \in \mathcal{V}_p \right\}$.

We see that $M = \bigcup_{p \in M} \bigcap \mathcal{V}_p$. Note that for any distinguished point p_1, the corresponding sets $\bigcap \mathcal{V}_{p_1}$ and $\bigcup_{p \neq p_1} \bigcap \mathcal{V}_p$ must have nonempty (open) intersection, by the assumption that M is connected. Thus, the corresponding union $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ is an open and connected subset of $\mathbb{R}$. We may similarly map $\bigcup_{p \in M} \bigcap \mathcal{O}_p$ into $\S^1$ by composing with the quotient map.

Now, if M is compact, then $\bigcup_{p\in M}\bigcap \mathcal{V}_p$ covers M, so there is a finite subcover $M = \bigcup_{i=1}^{n}\bigcap \mathcal{V}_{p_i}$, so that $\bigcup_{i=1}^{n}\bigcap \mathcal{O}_{p_i}$ fully covers the corresponding range, meaning that, composing with the quotient map $\bigcup_{i=1}^{n}\bigcap \mathcal{O}_{p_i}$, we have that $M \cong S^1$. Similarly, if M is non-compact, then $\bigcup_{p\in M}\bigcap \mathcal{O}_p$ is an open and connected subset of $\mathbb{R}$ that does not admit any finite subcover, hence it is homeomorphic to $\mathbb{R}$.

Problem (Problem 5)**:** In this problem, we prove a weak version of the Whitney Embedding Theorem.

(a) Find a C^∞ function λ on $\mathbb{R}^n$ with values in $[0,1]$ such that λ takes the value 1 on the closed ball $B(0,1)$, and vanishes outside the closed ball $B(0,2)$.

(b) Suppose M is a compact C^k manifold of dimension n. Find a C^k atlas $\{U_i, \varphi_i\}_{i\in I}$ such that $\varphi_i(U_i)$ contains $B(0,2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0,1))^\circ$.

(c) Let λ_i be defined by $\lambda\circ\varphi_i$ on U_i, and 0 outside U_i. Let $f_i\colon M\to\mathbb{R}^n$ be defined by $\lambda_i\circ\varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Problem (Problem 6)**:** Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Problem (Problem 7)**:** Let X and Y be topological spaces, and let $C(X,Y)$ be the set of continuous maps from X to Y. Equip $C(X,Y)$ with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K)\subseteq V\},$$

where $K\subseteq X$ is compact and $V\subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space, and let X be compact. Assume X is nonempty. We will show that a net $(f_\alpha)_\alpha\to f$ in the compact-open topology if and only if $(f_\alpha)_\alpha\to f$ uniformly.

If $(f_\alpha)_\alpha\to f$ uniformly, then for all $\varepsilon>0$, there exists α_0 such that for all $\alpha\geqslant\alpha_0$ and for all $x\in X$,

$$d(f_\alpha(x), f(x)) < \varepsilon/2,$$

meaning that, in particular, $f_\alpha(X)\subseteq U(f(x),\varepsilon)$ for all $x\in X$, so by setting $V = U(f(x_0),\varepsilon)$ for a distinguished $x_0\in X$ and the compact subset $X\subseteq X$, we see that $(f_\alpha)_\alpha\subseteq U_{X,V}$ for all $\alpha\geqslant\alpha_0$, so $(f_\alpha)_\alpha\to f$ in the compact-open topology.

Now, let $(f_\alpha)_\alpha\to f$ in the compact open topology; by definition, this means that for all basic open neighborhoods $U_{K,V}$, there exists $\alpha_{K,V}$ such that for all $\alpha\geqslant\alpha_{K,V}$, we have $f_\alpha\in U_{K,V}$.

Let $\varepsilon>0$. Since f is uniformly continuous, for all $x\in X$, there is a pre-compact open neighborhood U_x of x such that $f(U_x)\subseteq U(f(x),\varepsilon/3)$; in particular, notice that $f\left(\overline{U_x}\right)\subseteq U(f(x),\varepsilon/2)$.

The sets $\{U_x\}_{x\in X}$ cover X, so since X is compact, there are $x_1,\dots,x_n$ such that $U_{x_1},\dots,U_{x_n}$ cover X. We let $V_i = U(f(x_i),\varepsilon/2)$ be the corresponding open balls such that $f\left(\overline{U_{x_i}}\right)\subseteq V_i$, meaning there exist $\alpha_{U_{x_i},V_i}$ for $i=1,\dots,n$ such that for all $\alpha\geqslant\alpha_{\overline{U_{x_i}},V_i}$, $f_\alpha\in U_{\overline{U_{x_i}},V_i}$. Since the α are directed, there is $\alpha_0\geqslant\alpha_{\overline{U_{x_i}},V_i}$ for each $i=1,\dots,n$, meaning that for all $\alpha\geqslant\alpha_0$, we have $f_\alpha\in\bigcap_{i=1}^{n}U_{\overline{U_{x_i}},V_i}$. In particular, we see that for all $\alpha\geqslant\alpha_0$, and for all $x\in X$, $d(f_\alpha(x),f(x))<\varepsilon/2$.