3.3.10

For every graph G, prove that $\beta(G) \leq \alpha'(G)$. For each $k \in \mathbb{N}$, construct a simple graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$.

Let M be a matching with cardinality $\alpha'(G)$. Let K be the set of vertices containing all the vertices in M — so, K is of size $2\alpha'(G)$. We posit that K is a vertex cover. Suppose toward contradiction that it were not. Then, there would exist $e = xy$ such that $e \in G$, $e \notin M$, and $x, y \notin K$. However, this would mean that M would not be a maximum matching, as we would be able to add e to it, which yields our desired contradiction. Since K is a vertex cover, we know that the minimum vertex cover must be of size less than or equal to K. Therefore, we have that $\beta(G) \leq 2\alpha'(G)$.

For every value of $k \in \mathbb{N}$, we can find a graph where $\alpha'(G) = k$ and $\beta(G) = 2k$ by using the disjoint union of k copies of C_3.

3.3.24

Let G be a simple graph of even order n with set S of size k such that $q(G - S) > k$. Prove that G has at most $\binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$ edges. Use this to determine the maximum size of a simple n-vertex graph with no 1-factor.

We consider maximum values of the cardinality of the edge set within each subset. For the k elements in S, we know that the maximum value of the edge size is $\binom{k}{2}$. Additionally, for each of the k vertices in S, there must be edges to the $n - k$ extra vertices (so as to create the components). Finally, for each component of $G - S$, we know that they must be cliques so as to yield the maximum size.

Tutte's Theorem

Do all the "justifications" in the proof of Tutte's Theorem.

Necessity The odd components of $G - S$ must have vertices matched to distinct vertices of S.

1. Let $G - S$ be a graph with $q(G - S)$ odd components. Each of these components must contain at least *one* unmatched vertex, each of which requires a distinct vertex in S that was deleted. Thus, $q(G - S) \leq |S|$ is a necessary condition for G to contain a 1-factor.

Sufficiency When we add an edge joining two components of $G - S$, the number of odd components does not increase (odd and even together become odd, two edges of the same parity become even). Hence, Tutte's Condition is preserved by addition of edges.

If $G' = G + e$ and $S \subseteq V$, then $q(G' - S) \leq q(G - S) \leq |S|$.

2. If we were to add an edge between two odd components, then the new component will become even, thereby reducing the number of odd components. Otherwise, if we were to add an edge between two even components, then the new component will be even, keeping the number of components the same. Additionally, if
   - If we were to add an edge between two odd components, then the new component will become even, thereby reducing the number of odd components.
   - If we add an edge between an odd and an even component, then the new component is odd, keeping the number of odd components the same.
   - If we were to add an edge between two even components, then the new component will be even, keeping the number of odd components the same.

Therefore, since $G' = G + e$, we must have that $q(G' - S) \leq q(G - S)$, and we are assuming that $q(G - S) \leq |S|$, so $q(G' - S) \leq |S|$.

Also, if $G' = G + e$ has no 1-factor, then G has no 1-factor.

3. If $q(G' - S) > |S|$, which is our condition for finding a 1-factor, and $q(G' - S) \leq q(G - S)$, then we know that $q(G - S) > |S|$.

Therefore, the theorem holds *unless* there exists a simple graph G such that G satisfies Tutte's condition, G has no 1-factor, and adding any missing edge to G yields a graph with a 1-factor.

4. If there exists no graph at G that satisfies Tutte's condition and lacks a 1-factor, then the theorem holds, as it proves that any graph that *does* satisfy Tutte's condition *must* contain a 1-factor.

We add as many edges to G as possible without violating Tutte's condition and lacking a 1-factor in such a way that adding an additional edge would yield a 1-factor.

Let G be a graph with the assumption. We will obtain a contradiction that G does contain a 1-factor.

Let U be the set of vertices in G that have degree $|V| - 1$ (i.e., the set of vertices in G that are adjacent to every other vertex in G). We will prove via cases.

Case 1: $G - U$ consists of disjoint complete graphs. In this case, the vertices of the odd components in $G - U$ can be paired in any way with one extra vertex remaining in the odd components. Since $q(G - U) \leq |U|$ and each vertex of U is adjacent to all of $G - U$, we can match the leftover vertices to the vertices of U.

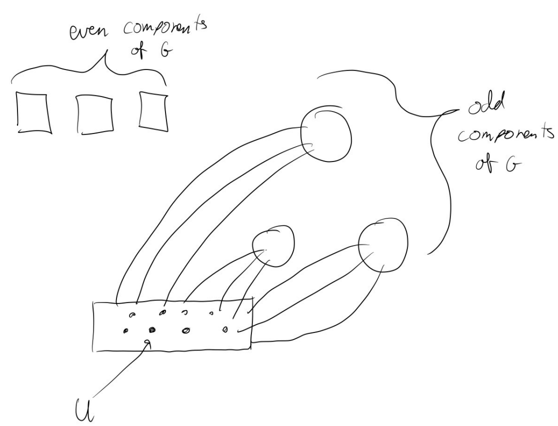
The remaining vertices are in U, which is a clique.

5. The vertices of U, by construction, are adjacent to every other vertex in G, meaning that they are adjacent to every other vertex in U, so U is a clique.

We have matched an even number of vertices in G so far, so it suffices to show that $|V|$ is even. This follows by invoking Tutte's condition for $S = \emptyset$, since a graph of odd order would contain a component of odd order.

6. If G is a graph with odd order, it then has a component of odd order, so if $S = \emptyset$, then $q(G - \emptyset) \geq 1 > |S| = 0$, which would mean Tutte's condition didn't hold.

7. A sketch is shown below:



Case 2: $G - U$ is not a disjoint union of cliques. In this case, $G - U$ has two vertices at distance 2; these are nonadjacent vertices x, z with common neighbor $y \notin U$.

8. WLOG, one of the components of $G - U$ is not a clique. Then, there is a vertex x such that x is not adjacent to z. Otherwise, $x \leftrightarrow z$ and the component would be complete. So, there must be a vertex y such that $x \leftrightarrow y$ and $z \leftrightarrow y$. So, $d(x, z) = 2$.

Furthermore, $G - U$ has another vertex w not adjacent to y, since $y \notin U$.

9. It must be the case that $\exists w \in G - U$ such that $w \not\leftrightarrow y$. Else, if $y \leftrightarrow w$, then y is adjacent to every vertex in $G - U$, which means $y \in U$, but since $y \notin U$, then it must be the case that $w \in G - U$ and $w \not\leftrightarrow y$.

By our choice of construction, we have that adding an edge to G creates a 1-factor. Let M_1 be the 1-factor in $G + xz$, and let M_2 be the 1-factor in $G + yw$. It suffices to show that $M_1 \triangle M_2 \cup \{xy, yz\}$ contains a 1-factor avoiding xz and yw, because this will be a 1-factor in G.

10. We know that the symmetric difference F contains only the edges in M_1 or the edges in M_2 exclusively, and if we exclude any edges they have in common, as well as avoid both xz and yw, then it will be a 1-factor without any edges not in G.

Let $F = M_1 \triangle M_2$. Since $xz \in M_1 - M_2$ and $yw \in M_2 - M_1$, both xz and yw are in F. Since every vertex of G has degree 1 in each of M_1 and M_2, every vertex of G has degree 2 or 0 in F.

11. Select an arbitrary vertex in G. It is either the case that M_1 and M_2 are incident on the vertex with unique edges, or they contain the same edge, in which case the edge is removed in the symmetric difference. Therefore every vertex has an even degree in F.

Hence, the components of F are even cycles and isolated vertices. Let C be the cycle of F containing xz.

If C does not also contain yw, then the desired 1-factor consists of the edges of M_2 from C and all of M_1 not in C.

12. By using the edges of M_2 in C, we avoid xz, and by using the edges of yw outside of C, we avoid yw, and since M_1 and M_2 are 1-factors, we know that this graph is a 1-factor, and since it avoids xz and yw, it is a 1-factor in G.

If C contains both yw and xz, then to avoid them we use yx or yz. In the portion of C starting from y along yw, we use the edges of M_1 to avoid using yw. When we reach either x or z, we use zy if we arrive at z, or else we use yx. In the remainder of C, we use the edges of M_2.

13. Because we have already saturated either x or z in our current 1-factor, we cannot use M_1 as it contains xz.

We have produced a 1-factor of $C + \{xy, yz\}$ that does not use xz or yw. Combined with M_1 or M_2 outside of $V(C)$, we create our desired 1-factor.