Revised Problems from Homework 3

Problem (Problem 1)**:** Prove that our cell complex structure for T^2 coincides with a product cell complex structure on $S^1 \times S^1$.

Solution: Consider the cell complex structure on S^1 given by one 1-cell, e^1, and one 0-cell, e^0, where the endpoints of e^1 are identified via the constant map to e^0; the corresponding characteristic map is $\Phi\colon [0,1] \to S^1$, identifying $0 \sim 1$.

Considering two copies of S^1 in this fashion, the product CW complex structure is then one consisting of

- one 0-cell, $e_1^0 \times e_2^0$
- two 1-cells, $e_1^1 \times e_2^0$ and $e_1^0 \times e_2^1$;
- one 2-cell, $e_1^1 \times e_2^1$.

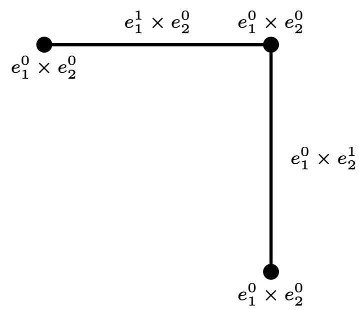
There are then characteristic maps

$$\begin{aligned}\Phi_1\colon e_1^1 \times e_2^0 &\to S^1 \times S^1\\ \Phi_2\colon e_1^0 \times e_2^1 &\to S^1 \times S^1,\end{aligned}$$

with

$$\begin{aligned}\Phi_1|_{\partial\left(e_1^1 \times e_2^0\right)}\left(e_1^1 \times e_2^0\right) &= e_1^0 \times e_2^0\\ \Phi_2|_{\partial\left(e_1^0 \times e_2^1\right)}\left(e_1^0 \times e_2^1\right) &= e_1^0 \times e_2^0.\end{aligned}$$

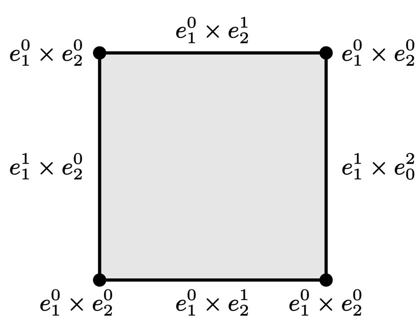
In particular, this means we may view the 1-skeleton as a wedge of two circles; since this is difficult to draw in TikZ, we instead simply label all the vertices and edges of the figure below assuming the necessary identifications.



The 2-skeleton is given by $e_1^1 \times e_2^1$, with a characteristic map Ψ given by the product of the characteristic maps of each of e_1^1 and e_2^1 (see Hatcher Theorem A.6). Observe that the attaching map is then given by the restriction of Ψ to the boundary, which is

$$\partial\left(e_1^1 \times e_2^1\right) = \left(e_1^1 \times e_2^0\right) \sqcup \left(e_1^0 \times e_2^1\right).$$

Since the 1-skeleton is precisely $\left(e_1^0 \times e_2^1\right) \sqcup \left(e_1^1 \times e_2^0\right)$, it follows that the attaching map for the 2-skeleton identifies the boundary of $e_1^1 \times e_2^1$ with the 1-skeleton, giving the following figure:



which is the cell complex structure of the torus.

Problem (Problem 2)**:** Prove that if X is a cell complex, then so is the suspension SX.

Solution: We observe that the product $X \times [0,1]$ is a cell complex, as it s a Cartesian product of a cell complex with one 1-cell (the interval itself) and two 0-cells (the endpoints of the interval). Since the characteristic maps on $X \times [0,1]$ are the products of the characteristic maps on X and $[0,1]$ (see Hatcher Theorem A.6), we see that the attaching maps for $X \times \{0\}$ and $X \times \{1\}$ are the products of the attaching maps for X and the constant maps representing $\{0\}$ and $\{1\}$.

In particular, this means that $X\times\{0\}$ and $X\times\{1\}$ are subcomplexes of $X\times[0,1]$ (as they contain all their attaching maps). Since the quotient of a cell complex by a subcomplex is a cell complex, it follows that $SX = X \times [0,1]/X \times \{0\}/X \times \{1\}$ is a cell complex.

Problem (Problem 3 (b))**:** Prove that S^∞ is contractible.

Solution: We view S^∞ as the subspace of $\mathbb{R}^\infty$, which is the space of finitely supported sequences. Specifically, the space S^∞ is the set of the finitely supported sequences (x_n) such that, if k is the index of the largest nonzero element, then $(x_0, \dots, x_k)$ is an element of S^k — i.e.,

$$\sum_{i=0}^{k} x_i^2 = 1.$$

In general, we define the norm of the sequence $\|(x_n)\|$ to be the finite sum

$$\|(x_n)\| = \sum_{i=0}^{\infty} x_i^2.$$

Consider now the map $H\colon S^\infty \times [0,1] \to S^\infty$

$$H((x_n), t) = \begin{cases} (1-2t)(x_n) + 2t(x_{n+1}) & 0 \leq t \leq 1/2 \\ (2-2t)(x_{n+1}) + (2t-1)(1,0,\dots) & 1/2 \leq t \leq 1. \end{cases}$$

where we insert 0 into index 0 in the first half of the homotopy. Then, H is continuous along each of $S^\infty \times [0,1/2]$ and $S^\infty \times [1/2,1]$ as it is continuous in each variable, and since the piecewise definitions are equal at $t = 1/2$, it follows that H is continuous along $[0,1]$. Then, we see that $H(\cdot,t)/\|H(\cdot,t)\|$ is contained within S^∞, and is a homotopy between the identity and a constant map, so the identity is null-homotopic, meaning S^∞ is contractible.

Current Problems

Problem (Problem 1)**:** Show that concatenation of paths satisfies the following cancellation property: if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$, and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

Solution: Let $\overline{g_1}$ denote the reverse path for g_1. Then, since $\overline{g_0} \simeq \overline{g_1}$ as $g_0 \simeq g_1$, we have

$$\begin{aligned} c &\simeq g_0 \cdot \overline{g_0} \\ &\simeq g_0 \cdot \overline{g_1}. \end{aligned}$$

In particular, since concatenation is associative, this gives

$$\begin{aligned} f_0 &\simeq f_0 \cdot (g_0 \cdot \overline{g_0}) \\ &\simeq (f_0 \cdot g_0) \cdot \overline{g_0} \\ &\simeq (f_1 \cdot g_1) \cdot \overline{g_0} \\ &\simeq f_1 \cdot (g_1 \cdot \overline{g_0}) \\ &= f_1. \end{aligned}$$

Problem: Prove that, for a path-connected space X, the fundamental group $\pi_1(X)$ is abelian if and only if all the change-of-basepoint isomorphisms β_h depend only on the endpoints of the path h, not on the precise path.

Solution: Let X be path-connected. Suppose $\pi_1(X)$ is abelian, and let x_0, x_1 be distinct points in X with distinct paths h_1 and h_2 connecting x_0 and x_1. We will show that $\beta_{h_1}\beta_{\overline{h_2}}$ is identity on $\pi_1(X, x_0)$. Letting f be any loop based at x_0, we have

$$\begin{aligned}
\beta_{h_1}\beta_{\overline{h_2}}[f] &= \beta_{h_1}\left[\overline{h_2}\cdot f\cdot h_2\right]\\
&= \left[h_1\cdot\overline{h_2}\cdot f\cdot h_2\cdot\overline{h_1}\right]\\
&= \left[h_1\cdot\overline{h_2}\right][f]\left[h_2\cdot\overline{h_1}\right]\\
&= [f]\left[h_2\cdot\overline{h_1}\right]\left[h_1\cdot\overline{h_2}\right]\\
&= [f].
\end{aligned}$$

Thus, $\beta_{h_1} = \beta_{h_2}$.

Suppose $\beta_{h_1} = \beta_{h_2}$ for any paths h_1 and h_2 between x_0 and x_1. We see then that for two loops f, g based at x_0

$$\begin{aligned}
[f][g] &= [f\cdot g]\\
&= \beta_{h_1}\beta_{\overline{h_2}}[f\cdot g]\\
&= \beta_{h_1}\left(\beta_{\overline{h_2}}[f]\beta_{\overline{h_2}}[g]\right)\\
&= \beta_{\overline{h_1}}\left(\left(\beta_{\overline{h_2}}[f]\beta_{\overline{h_2}}[g]\right)^{-1}\right)\\
&= \beta_{\overline{h_1}}(\beta_{h_2}[g]\beta_{h_2}[f])\\
&= \beta_{\overline{h_1}}\beta_{h_2}[g\cdot f]\\
&= [g\cdot f]\\
&= [g][f].
\end{aligned}$$