Solution (18.1)**:**

(a) The function $f(z) = z^n$ is analytic on $\mathbb{C}$.

(b) The functions $f(z) = \sin(z)$ and $f(z) = \cos(z)$ are analytic on $\mathbb{C}$, while $f(z) = \tan(z)$ is analytic everywhere except for singularities at $n\pi/2$.

(c) The function $f(z) = |z|$ is analytic nowhere.

(d) The function $f(z) = \frac{z-i}{z+1}$ is analytic everywhere except for $z = -1$.

(e) The function $f(z) = \frac{z^2+1}{z}$ is analytic everywhere except for $z = 0$.

(f) The function $f(z) = \frac{p_n(z)}{q_m(z)}$ is analytic everywhere except for the roots of q.

(g) The function $x^2 + y^2$ is analytic nowhere.

(h) The function e^z is analytic on $\mathbb{C}$.

(i) The function e^{-iy} is analytic nowhere.

(j) The function $\ln(z)$ is analytic everywhere except for $(-\infty, 0]$.

Solution (18.2)**:** Let $w(z) = u(x, y) + iv(x, y)$. Then,

$$\begin{aligned} i\frac{\partial}{\partial x}(w(x+iy)) - \frac{\partial}{\partial y}(w(x+iy)) &= i\frac{\partial}{\partial x}(u(x,y) + iv(x,y)) - \frac{\partial}{\partial y}(u(x,y) + iv(x,y)) \\ &= i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \end{aligned}$$

Thus, we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

Solution (18.4)**:** We see that, when we have $w(z) = u(x, y) + iv(x, y)$, that

$$\begin{aligned} \nabla u &= \begin{pmatrix} \frac{du}{dx} \\ \frac{du}{dy} \end{pmatrix} \\ \nabla v &= \begin{pmatrix} \frac{dv}{dx} \\ \frac{dv}{dy} \end{pmatrix}. \end{aligned}$$

We note that curves of constant u and v are orthogonal if and only if the normal vectors are orthogonal, meaning

$$\begin{aligned} (\nabla u) \cdot (\nabla v) &= \frac{du}{dx}\frac{dv}{dx} + \frac{du}{dy}\frac{dv}{dy} \\ &= \frac{dv}{dy}\frac{dv}{dx} - \frac{dv}{dx}\frac{dv}{dy} \\ &= 0. \end{aligned}$$

Solution (18.5)**:** We see that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial v}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial v}{\partial x \partial y}, \end{aligned}$$

so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Symmetrically, we also have that v satisfies Laplace's equation.

Solution (18.6)**:**

(a) We start by writing everything in terms of z, so we have $w(z) = u(z) + iv(z)$. Since w is complex-differentiable, by linearity we have

$$w'(z) = \frac{du}{dz} + i\frac{dv}{dz}.$$

We write $z = x + iy$, or $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$.

$$\begin{aligned} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right). \end{aligned}$$

(b) We have that

$$\begin{aligned} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right). \end{aligned}$$

Thus, by tedious algebraic manipulations heavily prone to error, we recover the Cauchy–Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

Solution (18.7)**:** We find that

$$\begin{aligned} \frac{dw}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= 1 \\ &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \\ &= -i. \end{aligned}$$

Since these derivatives are path-dependent, we have that $w(z) = \overline{z}$ is not differentiable.

Solution (18.11)**:** Using the scale factors, we recall that $d\mathbf{x} = dr + r\,d\phi$, so the derivatives with respect to ϕ pick up a scale term of $\frac{1}{r}$. This yields our desired Cauchy–Riemann equations in polar form:

$$\begin{aligned} \frac{du}{dr} &= \frac{1}{r}\frac{dv}{d\phi} \\ \frac{1}{r}\frac{du}{d\phi} &= -\frac{dv}{dr}. \end{aligned}$$

Solution (18.14)**:** We know that $\frac{1}{z^m}$ is defined for all $z \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$, so we only need to show that if $z \neq 0$, then $\frac{1}{z^m}$ admits a derivative. We see that

$$\frac{d}{dz}\left(\frac{1}{z^m}\right) = \frac{-m}{z^{m+1}},$$

which is yet again defined for all $z \neq 0$, so $\frac{1}{z^m}$ is analytic on its domain.

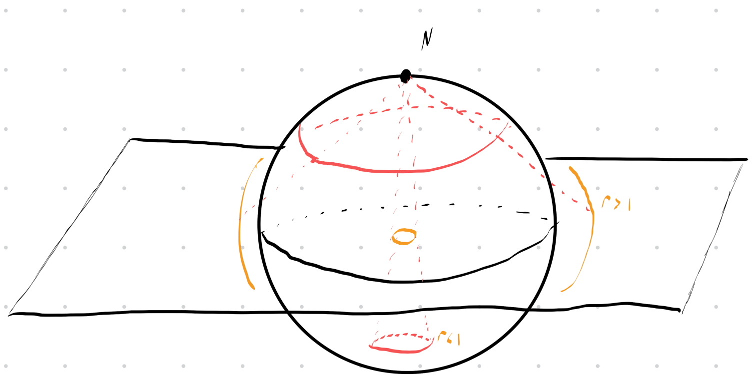
Solution (18.15)**:**

$$\oint_C \frac{1}{z^n}\,dz = \oint_C \left(r^{-n}e^{-in\varphi}\right) ire^{i\varphi}\,d\varphi$$

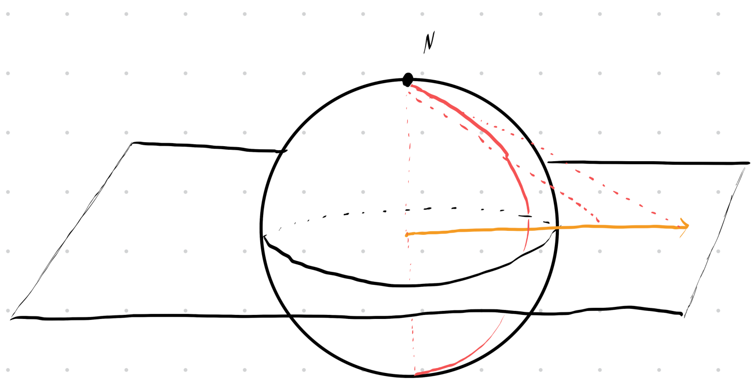
$$= \frac{1}{r^{n-1}} \int_0^{2\pi} ie^{-i(n-1)\varphi}\, d\varphi$$
$$= 0.$$

Solution (18.18)**:**

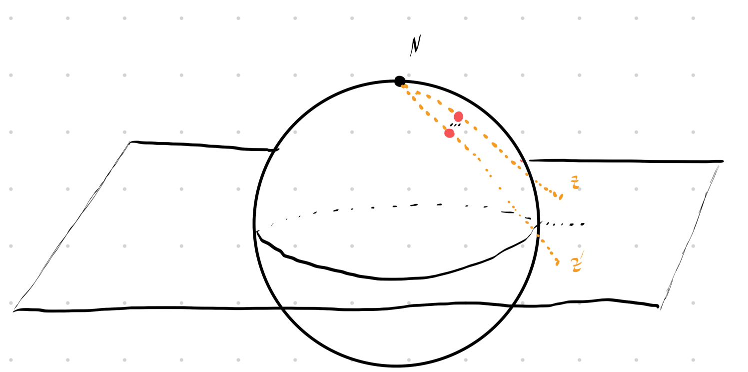
(a)



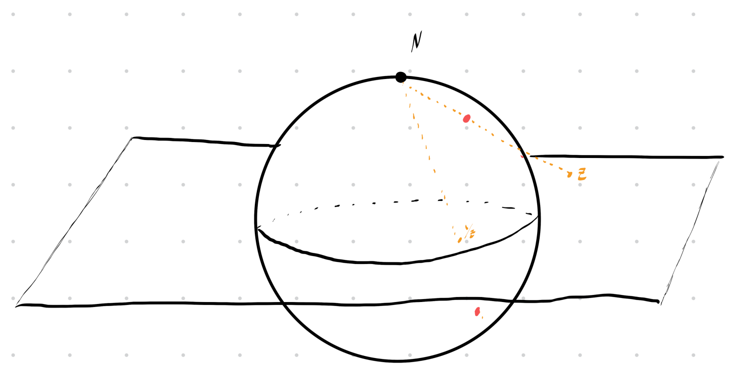
(b)



(c)



(d)



(e)