Solution (40.7)**:** We have

$$\begin{aligned}\langle\psi\,|\,\mathcal{L}\phi\rangle &= \int_a^b \overline{\psi(x)}\left(\alpha(x)\frac{d^2\phi}{dx^2}+\beta(x)\frac{d\phi}{dx}+\gamma(x)\phi(x)\right)\,dx\\ &= \int_a^b \overline{\psi(x)}\alpha(x)\frac{d^2\phi}{dx^2}\,dx+\int_a^b \overline{\psi(x)}\beta(x)\frac{d\phi}{dx}\,dx+\int_a^b \overline{\psi(x)}\gamma(x)\phi(x)\,dx.\end{aligned}$$

We evaluate these integrals separately. Assuming that α,β,γ are real-valued, we have

$$\begin{aligned}\int_a^b \overline{\psi(x)}\alpha(x)\frac{d^2\phi}{dx^2}\,dx &= \frac{d\phi}{dx}\overline{\psi(x)}\alpha(x)\bigg|_a^b-\int_a^b\left(\frac{d\alpha}{dx}\overline{\psi(x)}+\overline{\frac{d\psi}{dx}}\alpha(x)\right)\frac{d\phi}{dx}\,dx\\ &= \underbrace{\left(\frac{d\phi}{dx}\alpha(x)\overline{\psi(x)}-\phi(x)\left(\frac{d\alpha}{dx}\overline{\psi(x)}+\overline{\frac{d\psi}{dx}}\alpha(x)\right)\right)\bigg|_a^b}_{S_1}\\ &\quad+\int_a^b\overline{\left(\alpha(x)\frac{d^2}{dx^2}+2\frac{d\alpha}{dx}\frac{d}{dx}+\frac{d^2\alpha}{dx^2}\right)\psi(x)}\phi(x)\,dx.\\ \int_a^b \overline{\psi(x)}\beta(x)\frac{d\phi}{dx}\,dx &= \underbrace{\left(\phi(x)\beta(x)\overline{\psi(x)}\right)\bigg|_a^b}_{S_2}-\int_a^b\phi(x)\left(\frac{d\beta}{dx}\overline{\psi(x)}+\overline{\frac{d\psi}{dx}}\beta(x)\right)\,dx.\end{aligned}$$

Thus, we have

$$\int_a^b\overline{\psi(x)}(\mathcal{L}\phi)(x)\,dx = S_1+S_2+\int_a^b\overline{\left(\alpha(x)\frac{d^2}{dx^2}+\left(2\frac{d\alpha}{dx}-\beta(x)\right)\frac{d}{dx}+\left(\frac{d^2\alpha}{dx^2}-\frac{d\beta}{dx}+\gamma(x)\right)\right)\psi(x)}\phi(x)\,dx.$$

Solution (40.23)**:**

(a) We have $p(x)=1$, and

$$\begin{aligned}\int_0^a\overline{\sin(n\pi x/a)}\sin(m\pi x/a)\,dx &= \frac{a}{m\pi-n\pi}\left(n\pi\cos(n\pi x/)\overline{\sin(m\pi x/a)}-m\pi\cos(m\pi x/a)\overline{\sin(n\pi x/a)}\right)\bigg|_0^a\\ &= 0.\end{aligned}$$

(b) With the eigenfunctions $J_0(\alpha_i r/a)$, we have

$$\int_0^a r\overline{J_0\left(\frac{\alpha_m}{a}r\right)}J_0\left(\frac{\alpha_n}{a}r\right)\,dx = \frac{r\left(\frac{\alpha_n}{a}J_0'\left(\frac{\alpha_n}{a}r\right)\right)\Big|_0^a}{\frac{\alpha_m}{a}-\frac{\alpha_n}{a}}.$$

We use the identity that

$$J_0' = -J_1$$

to use $J_1(0)=0$ and $J_0\left(\frac{\alpha_i}{a}(a)\right)=0$, so we recover the orthogonality relation.

(c) We have

$$\begin{aligned}\int_0^\infty \mathrm{Ai}(\kappa x+\alpha_n)\,\mathrm{Ai}(\kappa x+\alpha_m)\,dx &= \frac{\kappa x(\mathrm{Ai}'(\kappa x+\alpha_n)\,\mathrm{Ai}(\kappa x+\alpha_m)-\mathrm{Ai}'(\kappa x+\alpha_m)\,\mathrm{Ai}'(\kappa x+\alpha_n))\big|_0^\infty}{\kappa^2(\alpha_n-\alpha_m)}\\ &= 0.\end{aligned}$$

Solution (40.27)**:**

(a) We may express the Rayleigh quotient as

$$\rho(v) = \frac{\langle v\,|\,Av\rangle}{\langle v\,|\,v\rangle}.$$

(b) We note that if $\mathcal{L}\phi = -\lambda w(x)\phi$, then by multiplying by $\overline{\phi}$, integrating, and dividing we get

$$\begin{aligned}\lambda &= \frac{\int_a^b \overline{\phi(x)}\left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right)\phi(x)\,dx}{\int_a^b |\phi(x)|^2 w(x)\,dx}\\ &= \frac{1}{k_n}\int_a^b \overline{\phi(x)}\left(p(x)\frac{d^2\phi}{dx^2} + \frac{dp}{dx}\frac{d\phi}{dx} + q(x)\phi(x)\right)dx\end{aligned}$$

(c) Splitting things up, we get

$$\lambda = \frac{1}{k_n}\left(\int_a^b \overline{\phi(x)}p(x)\frac{d^2\phi}{dx^2}\,dx + \int_a^b \frac{dp}{dx}\frac{d\phi}{dx}\overline{\phi(x)}\,dx + \int_a^b q(x)|\phi(x)|^2\,dx\right).$$

In the "best case" scenario, we may assume that $\frac{dp}{dx}$ vanishes everywhere, so we are left with

$$\lambda \geqslant \frac{1}{k_n}\left(\int_a^b \overline{\phi(x)}p(x)\frac{d^2\phi}{dx^2}\,dx + \int_a^b q(x)|\phi(x)|^2\,dx\right).$$

Integrating the first term by parts, we may implement the condition that

$$p(x)\left(\left(\frac{d\phi}{dx}\right)\phi(x) - \overline{\phi(x)}\frac{d\phi}{dx}\right)\Big|_a^b = 0$$

to simplify down to

$$\lambda \geqslant \frac{1}{k_n}\left(-p(x)\overline{\frac{d\phi}{dx}}\phi(x)\Big|_a^b + \int_a^b q(x)|\phi(x)|^2\,dx\right).$$

Solution (41.8)**:** Using the Laplacian in spherical coordinates, we have

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right),$$

which separates

$$\psi(\mathbf{r}) = R(r)\Theta(\theta)\Phi(\phi)$$

into

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{1}{\sin^2(\theta)}\frac{d^2\Phi}{d\phi^2} + \frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = -k^2r^2.$$

The latter two terms are functions of θ, ϕ exclusively, so we have

$$\frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{d^2\Phi}{d\phi^2} = -\lambda,$$

and multiplying out by $\sin^2(\theta)$, we have

$$\frac{1}{\Theta}\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\lambda\sin^2(\theta).$$

Therefore, we recover

$$\begin{aligned}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} &= -m^2\\ \frac{1}{\Theta}\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) &= -\lambda\sin^2(\theta) + m^2\\ \frac{d^2\Phi}{d\phi^2} &= -m^2\Phi(\phi)\end{aligned}$$

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta)=0.$$

Examining the term in r, we get

$$\begin{aligned}\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)&=-k^2r^2+\lambda\\ \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(k^2r^2-\lambda\right)R(r)&=0.\end{aligned}$$

Using $\lambda=\ell(\ell+1)$, we get

$$\begin{aligned}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(k^2r^2-\ell(\ell+1)\right)R(r)&=0\\ \frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta)&=0\\ \frac{d^2\Phi}{d\phi^2}&=-m^2\Phi.\end{aligned}$$

Using $x=\cos(\theta)$ and $X(x)=\Theta(\theta)$, we have

$$\begin{aligned}\frac{dX}{dx}&=\frac{d\Theta}{d(\cos(\theta))}\\ &=-\frac{1}{\sin(\theta)}\frac{d\Theta}{d\theta},\end{aligned}$$

and

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dX}{dx}\right)=\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right).$$

Therefore, we have

$$\begin{aligned}R(r)&=a_1j_\ell(kr)+a_2n_\ell(kr)\\ \Theta(\theta)&=b_1P_{\ell,m}(\cos(\theta))+b_2Q_{\ell,m}(\cos(\theta))\\ \Phi(\phi)&=c_1e^{im\phi}+c_2e^{-im\phi}.\end{aligned}$$

Solution (41.13)**:**

Solution (41.14)**:**

Solution (41.16)**:**

Solution (41.25)**:**

Solution (41.28)**:**

Solution (42.1)**:**

Solution (42.2)**:**

Solution (42.11)**:**