Introduction

Consider the equations

$$y''(x) + y(x) = e^x \tag{1}$$
$$y^{(17)}(x) + \sin(y(x)) = (x^x)^x \tag{2}$$

Before we want to solve these equations, we need to understand what these equations *are*.

(1) This is a second order, inhomogeneous, linear ordinary differential equation.

(2) This is a 17th order, inhomogeneous, nonlinear ordinary differential equation.

Generally, when we have a nonlinear equation, we convert it (using the Jacobian) to the "nearest" corresponding linear equation using Taylor approximations. In this case, converting equation (2), we have

$$y^{(17)}(x) + y(x) = (x^x)^x. \tag{2'}$$

Now, equation (2′) is linear, so it is able to be solved. It may not be pretty,[1] but it can be solved, using Laplace Transforms or other methods.

Ordinary Differential Equations

Returning to our equation (1),

$$y''(x) + y(x) = e^x, \tag{1}$$

there is one more fact that we can see — this is an equation with constant coefficients. The most general form of a nth order linear ordinary differential equation is of the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x). \tag{†}$$

Specifically, we also require $a_k(x) \in C(I)$, where I is some interval (specifics will be detailed later).

Theorem (Existence and Uniqueness Theorem)**:** Any ordinary differential equation of the form (†) has unique solutions in the interval I.

There are n linearly independent solutions for $g(x) = 0$.

The corresponding homogeneous equation for (1) is

$$y''(x) + y(x) = 0. \tag{1'}$$

The equations (1) and (1′) are related by the linearity principle. In particular, if $y_0(x)$ is a solution to (1′), then we can add $\alpha y_0(x)$ to any solution $y_p(x)$ of (1), then we have all the solutions for (1). In particular, the solutions to (1′) are

$$y_1(x) = \sin(x)$$
$$y_2(x) = \cos(x).$$

To evaluate that these solutions are linearly independent, we consider the differential operator L from (†) defined by

$$L[y] = \sum_{k=0}^{n} a_k(x)y^{(k)}(x).$$

We rewrite (†) as

$$L[y] = g(x).$$

The operator L is linear, so L has the following properties:

[1] Citation needed.

- $L[y_1 + y_2]$;
- $L[cy] = cL[y]$.

Now, in (1) and (1′), if we set $L[y] = y''(x) + y(x)$, then evaluating our solutions y_1 and y_2 to (1′), we get

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= c_1 L[y_1] + c_2 L[y_2] \\ &= 0. \end{aligned}$$

Now, we get

$$y_0(x) = c_1 \sin(x) + c_2 \sin(x)$$

as our general solution to (1′). By the linearity principle, all we need is one solution to $L[y] = e^x$ to find all solutions to (1).

Evaluating (†) in the most general form, we have the general solution

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{\text{homogeneous solution}} + y_p(x),$$

where $y_p(x)$ is the particular solution. In other words, our general solution is

$$y(x) = \operatorname{span}(y_1(x), y_2(x), \dots, y_n(x)) + y_p(x).$$

For this to work, we need the set $\{y_1, \dots, y_n\}$ to be linearly independent. To do this, we evaluate the Wronskian:

$$W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}.$$

Specifically, the set $\{y_1, \dots, y_n\}$ is linearly independent if $W(x) \neq 0$ for all $x \in I$.

Example. Consider the equation

$$y''(x) - y(x) = e^x \tag{1}$$

We want to find the general solution to this constant coefficient equation.

We start by finding two linearly independent homogeneous solutions to the equation, take their span, then add a particular solution.

The characteristic equation of the homogeneous equation for (1) is

$$r^2 - 1 = 0$$

We get $r = \pm 1$, which by the definition of the characteristic equation yields $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. To verify that this this solution set is linearly independent

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2 \end{aligned}$$

$$\neq 0.$$

Thus, our solutions are linearly independent. We get the general form of

$$y(x) = c_1 e^{x} + c_2 e^{-x} + y_p(x).$$

Now, we only have to find a particular solution. This is, unfortunately, the hard part.

We begin by guessing. But, in a way that doesn't suck. Specifically, we let $y_p(x) = Axe^{x}$. Evaluating, we get

$$\begin{aligned} y_p'(x) &= A(x+1)e^{x} \\ y_p''(x) &= A(x+2)e^{x} \\ y_p''(x) - y_p(x) &= A(x+2)e^{x} - Axe^{x} \\ &= 2Ae^{x}, \end{aligned}$$

so $2A = 1$, and $A = \frac{1}{2}$. Thus, we have the end result of

$$y(x) = c_1 e^{x} + c_2 e^{x} + \frac{1}{2}xe^{x}.$$

Evaluating in Mathematica, we take

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DSolve[y''[x] - y[x] == Exp[x], y[x], x]
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and we get

$$y(x) = c_1 e^{x} + c_2 e^{-x} + \frac{1}{4}(2x-1)e^{x},$$

corroborating our solution.[II]

Example. Consider the equation

$$y'''(x) - y(x) = 0.$$

The particular solution to this equation is $y(x) = 0$. The characteristic equation for this equation is

$$r^3 - 1 = 0.$$

Factoring, we get

$$\begin{aligned} (r-1)\left(r^2 + r + 1\right) &= 0 \\ (r-1)(r-\zeta_3)\left(r - \zeta_3^2\right) &= 0. \end{aligned}$$

Thus, we get

$$r = \left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}.$$

Thus, our solutions are of the form

$$y(x) = c_1 e^{x} + c_2 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

[II]Only slightly different, but they're the same solution.

Recall that the most general second order constant-coefficient linear differential equation is

$$y'' + ay' + by = 0,$$

with characteristic equation

$$r^2 + ar + b = 0.$$

The solutions to the characteristic equation are

$$r = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}.$$

There are a few cases:

(1) $r_1 \neq r_2$ with $r_1, r_2 \in \mathbb{R}$;

(2) $r_1 = r_2$ with $r_1, r_2 \in \mathbb{R}$;

(3) $r_1 = c + id$, $r_2 = c - id$, where $c, d \in \mathbb{R}$.

The solutions are $y_1 = c_1 e^{r_1 x}$ and $y_2 = c_2 e^{r_2 x}$.

Example (Solving Second-Order Equations)**.**

(1) Let

$$y'' - 3y' + 2y = 0.$$

The characteristic equation is $r^2 - 3r + 2 = 0$, whose solutions are $r = 1, r = 2$. The general solution is, thus,

$$y(x) = c_1 e^x + c_2 e^{2x} \tag{†}$$

The Wronskian is

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \\ &= 2e^{3x} - e^{3x} \\ &= e^{3x} \\ &\neq 0. \end{aligned}$$

Thus, the solution is indeed (†).

(2) Let

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with solution $r = -3, -3$. Currently, we only have the solution $y_1(x) = c_1 e^{-3x}$.

Note that in an nth order linear ordinary differential equation, we always have n linearly independent solutions. Let's guess. Consider the equation $y_2(x) = c_2 x e^{-3x}$.

We can see that $y_2(x)$ is also a solution to this equation,[III] but we need to verify linear independence. Taking the Wronskian, we get

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{pmatrix} \\ &= e^{-6x}\begin{pmatrix} 1 & x \\ -3 & -3x+1 \end{pmatrix} \\ &= e^{-6x}(-3x + 1 + 3x) \\ &= e^{-6x} \\ &\neq 0. \end{aligned}$$

Thus, we have two linearly independent solutions, with the general solution of

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

(3) Let

$$y'' + 4y' + 5 = 0.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with solutions of $r = -2 \pm i$. We then have the solutions

$$\begin{aligned} y_1(x) &= e^{(-2+i)x} \\ y_2(x) &= e^{(-2-i)x}. \end{aligned}$$

Unfortunately, we cannot just let these equations stand on their own, because we want *real* solutions. Let's use Euler's theorem, $e^{ix} = \cos x + i \sin x$. Then, we get

$$\begin{aligned} y(x) &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\ &= e^{-2x}\left(c_1 e^{ix} + c_2 e^{-ix}\right). \end{aligned}$$

Let $f(x) = c_1 e^{ix} + c_2 e^{-ix}$. Using the even/odd decomposition, we get

$$\begin{aligned} f(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\ &= (c_1 + c_2)\cos(x) + i(c_1 - c_2)\sin(x). \end{aligned}$$

We "real"-ize our solution by just dropping the value of i in $f(x)$. Thus, we get the full general solution

$$y(x) = e^{-2x}(d_1 \cos(x) + d_2 \sin(x)).$$

(4) If we have the equation

$$y^{(4)} - 25y'',$$

then using a similar process, we get the solution

$$y(x) = c_1 + c_2 x + c_3 e^{5x} + c_4 e^{-5x}.$$

(5) Considering the equation

$$y^{(5)} + 4y''' + 4y' = 0,$$

we take the characteristic equation $r^5 + 4r^3 + 4r = 0$. Factoring, we get solutions of $r = 0, r = \pm i\sqrt{2}$. Thus, we get the solution of

$$y(x) = c_1 + c_2 \cos\left(\sqrt{2}x\right) + c_3 \sin\left(\sqrt{2}x\right) + c_4 x \cos\left(\sqrt{2}x\right) + c_5 x \sin\left(\sqrt{2}x\right).$$

[III] Exercise left for the reader.

Reducing our Orders

Let

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Suppose we know $y_1(x)$. Can we find $y_2(x)$? The answer is yes. We presume

$$y_2(x) = v(x)y_1(x).$$

Now, we have

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'', \end{aligned}$$

and inserting into the equation, we get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{=0} \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Now, we have

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p. \tag{$*$}$$

Integrating, we get

$$\ln(v') = -2\ln(y_1) - \int p(x)\, dx.$$

Taking powers, we get

$$\begin{aligned} v' &= e^{-2\ln(y_1) - \int p(x)\, dx} \\ &= y_1^{-2}e^{-\int p(x)\, dx} \\ &= \frac{e^{-\int p(x)\, dx}}{y_1(x)^2} \\ v &= \int \frac{e^{-\int p(x)\, dx}}{y_1(x)^2}\, dx \end{aligned}$$

Example. Consider the equation

$$\cos^2(x)y''(x) - \sin(x)\cos(x)y' - y(x) = 0.$$

Putting our equation into standard form, we may be able to find another solution.

$$y'' - \tan(x)y' - \sec^2(x)y = 0.$$

Guessing $y(x) = \tan(x)$, we get $y' = \sec^2(x)$ and $y'' = 2\sec^2(x)\tan(x)$. This is also another solution, $y_2(x) = \tan(x)$.

We don't want to guess anymore. Let $y_2(x) = v(x)y_1(x)$. We get

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx.$$

We have $-p(x) = \tan(x)$, so $-\int p(x)\,dx = \ln(\sec(x))$. Thus, $e^{-\int p(x)\,dx} = \sec(x)$. Thus, we get

$$\begin{aligned} v(x) &= \int \frac{\sec(x)}{\tan^2(x)}\,dx \\ &= \int \frac{\cos(x)}{\sin^2(x)}\,dx \\ &= \int \frac{1}{u^2}\,du && u = \sin(x) \\ &= -\frac{1}{u} \\ &= -\csc(x). \end{aligned}$$

Thus, we have $y_2(x) = -\csc(x)\tan(x) = -\sec(x)$.

Example. Consider the equation

$$x^2(\ln(x) - 1)y''(x) - xy'(x) + y'(x) = 0.$$

We can use the power of inspection to find one solution, $y_1(x) = x$. Dividing out, we have

$$y'' - \frac{1}{x(\ln(x) - 1)}y' + \frac{1}{x^2(\ln(x) - 1)}y = 0.$$

Using the reduction of order, we guess $y_2(x) = v(x)y_1(x)$, and have

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

Noting that $-p(x) = \frac{1}{x(\ln(x)-1)}$, we have $\int \frac{1}{x(\ln(x)-1)}\,dx = \ln(\ln(x) - 1)$.

Now, we have

$$\begin{aligned} v(x) &= \int \frac{\ln(x) - 1}{x^2}\,dx \\ &= \frac{1 - \ln(x)}{x} - \int -\frac{1}{x^2}\,dx && u = \ln(x) - 1, dv = x^{-2} \\ &= \frac{-\ln(x)}{x} - \frac{1}{x} \\ &= -\frac{\ln(x)}{x}. \end{aligned}$$

Thus, we get $y_2(x) = -\ln(x)$, and the general solution of $y(x) = c_1 x + c_2 \ln(x)$.

Example (Cauchy–Euler Equation)**.** A second-order Cauchy–Euler equation is of the form

$$ax^2y''(x) + bxy'(x) + cy(x) = 0.$$

More generally,

$$\sum_{k=0}^{n} c_k x^k y^{(k)}(x) = 0.$$

We guess $y(x) = x^r$. Then, $y'(x) = rx^{r-1}$ and $y''(x) = r(r-1)x^{r-2}$. This yields

$$\begin{aligned} a(r)(r-1)x^r + brx^r + cx^r &= x^r\left(a\left(r^2 - r\right) + br + c\right) \\ &= 0. \end{aligned}$$

Example (Solving a Cauchy–Euler Equation)**.** Consider the equation

$$x^2y'' + xy' - y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 1 = 0,$$

so our general solution is $y(x) = c_1x + c_2/x$.

Example (Solving another Cauchy–Euler Equation)**.** Consider the equation

$$x^2y'' - 3xy' + 4y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 4r + 4 = 0,$$

so our solutions are x^2 and x^2. This is not good enough, we need another solution.

Now, we place our equation into standard form.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y' = 0.$$

Thus, we get $p(x) = -\frac{3}{x}$. Using reduction of order, we get $y_2(x) = v(x)y_1(x)$,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int -3/x \, dx}}{x^4} \, dx \\ &= \int \frac{e^{3\ln(x)}}{x^4} \, dx \\ &= \int \frac{x^3}{x^4} \, dx \\ &= \ln(x). \end{aligned}$$

Thus, we have the solution $y_2(x) = \ln(x)x^2$, and the general solution of $y(x) = c_1x^2 + c_2\ln(x)x^2$.

Example. Consider the equation

$$x^2y'' + 3xy' + 5y = 0.$$

We get the characteristic equation of

$$\begin{aligned} 0 &= r^2 - 4r + 5 \\ r &= 2 \pm i. \end{aligned}$$

Now, we need to figure out what $x^{2\pm i}$ means.

To solve this part, we keep the positive exponent, so we only need to try to understand $y = x^{2+i}$. Now, we get $y = x^2x^i$. To evaluate x^i, we take $x = \left(e^{\ln x}\right)^i = e^{i\ln x}$. Using Euler's identity, we get

$$y = x^2(\cos(\ln x) + i\sin(\ln x)).$$

Since our solutions are real, get

$$y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x).$$

Example. Consider the equation

$$x^4y^{(4)} - 2x^2y'' + y = 2.$$

We have the particular solution $y_p(x) = 2$. Substituting into our method for the Cauchy–Euler equation, we have

$$r(r-1)(r-2)(r-3) - 2r(r-1) + 1 = 0.$$

Factoring, we have

$$r(r-1)^2(r-4) + 1 = 0.$$

Unfortunately, to go forward from here we need Mathematica.

This has the solution set of of

$$\begin{aligned}
r_1 &= \frac{3}{2} - \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\
&\quad - \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\
r_2 &= \frac{3}{2} - \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\
&\quad + \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\
r_3 &= \frac{3}{2} + \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\
&\quad - \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\
r_4 &= \frac{3}{2} + \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\
&\quad + \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}}
\end{aligned}$$

Varying our Parameters

Given a set of n linearly independent homogeneous solutions, we want to find a particular solution.

To find this, we start with the general second-order inhomogeneous equation in standard form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x).$$

Given y_1, y_2, we find $y_p(x)$ by taking

$$y_p = v_1y_1 + v_2y_2.$$

Finding the derivatives, we have

$$\begin{aligned}
y_p' &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2\\
y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2.
\end{aligned}$$

Substituting, we have

$$\begin{aligned}
y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2\\
py_p' &= pv_1y_1' + pv_1'y_1 + pv_2y_2' + pv_2'y_2\\
qy_p &= qv_1y_1 + qv_2y_2
\end{aligned}$$

$$g(x) = v_1\overbrace{\left(y_1'' + py_1' + qy_1\right)}^{=0} + v_2\overbrace{\left(y_2'' + py_2' + qy_2\right)}^{=0} + v_1'\left(2y_1' + py_1\right) + v_1''y_1 + v_2\left(2y_2' + py_2\right) + v_2''y_2$$

$$g(x) = v_1'\left(2y_1' + py_1\right) + v_1''y_1 + v_2\left(2y_2' + py_2\right) + v_2''y_2.$$

We suppose that $v_1'y_1 + v_2'y_2 = 0$. Then,

$$\begin{aligned}
\frac{d}{dx}\left(v_1'y_1 + v_2'y_2\right) &= 0\\
v_1''y_1 + v_1'y_1' + v_2''y_2 + v_2'y_2 &= 0.
\end{aligned}$$

Plugging into our earlier expression, we get the expression of

$$\begin{aligned}
v_1'y_1 + v_2'y_2 &= 0\\
v_2'y2' + v_2'y_2' &= g(x).
\end{aligned}$$

Plugging into matrix form, we have

$$\begin{pmatrix} y_1 & y_2\\ y_1' & y_2'\end{pmatrix}\begin{pmatrix} v_1'\\ v_2'\end{pmatrix} = \begin{pmatrix} 0\\ g(x)\end{pmatrix}.$$

Since the Wronskian is nonzero, we have

$$\begin{aligned}
\begin{pmatrix} v_1'(x)\\ v_2'(x)\end{pmatrix} &= \begin{pmatrix} y_1 & y_2\\ y_1' & y_2'\end{pmatrix}^{-1}\begin{pmatrix} 0\\ g(x)\end{pmatrix}\\
&= \frac{1}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}\begin{pmatrix} -y_2(x)g(x)\\ y_1(x)g(x)\end{pmatrix}
\end{aligned} \tag{‡}$$

Example. Let

$$y'' - 2y' + y = e^x.$$

Solving the homogeneous solution, we have the characteristic equation of $r^2 - 2r + 1 = 0$. Thus, $y_1(x) = e^x$ and $y_2(x) = xe^x$.

To find $y_p(x)$, we guess $y_p(x) = x^2e^x$. Using the power of computation in Sage, we get the answer of

Avoiding Variation of Parameters

```
de = diff(y,x,2) - 2*diff(y,x) + y - e^(x)
g = desolve(de,y)
latex(expand(g))
```

$$y_p(x) = K_2xe^x + K_1e^x + \frac{1}{2}x^2e^x.$$

However, this is a very unsatisfying method.

Using (‡), we can find a different solution. We find

$$\begin{aligned} v_1'(x) &= \frac{1}{e^{2x}}((-1)(xe^x)(e^x)) \\ &= -x, \end{aligned}$$

yielding

$$v_1(x) = -\frac{x^2}{2} + c_2.$$

Similarly, we get

$$\begin{aligned} v_2'(x) &= \frac{1}{e^{2x}}(e^x)(e^x) \\ v_2(x) &= x + c_2. \end{aligned}$$

This gives

$$y_p(x) = \frac{1}{2}x^2e^x.$$

Example. Let

$$y'''(x) - y'(x) = x + e^x.$$

Using the characteristic equation, we have $y_1(x) = 1$, $y_2(x) = e^x$, and $y_3(x) = e^{-x}$.

Now, using the Wronskian, we get

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ x + e^x \end{pmatrix}.$$

This would suck, but we would be able to find a solution nonetheless.

In the general form, with linearly independent homogeneous solutions $y_1, \dots, y_n$, we have the solution of

$$\begin{aligned} \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} &= \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \vdots & g(x) \end{pmatrix} \\ y(x) &= \sum_{i=1}^{n} c_iy_i(x) + \sum_{i=1}^{n} v_i(x)y_i(x). \end{aligned}$$

Example (Solving a Coupled System)**.** Before we can start using variation of parameters for systems, we need to recall how to solve constant-coefficient systems.

$$\begin{aligned} x'(t) &= 3x(t) + y(t) \\ y'(t) &= x(t) + 3y(t). \end{aligned}$$

Here, setting

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

we get system of linear equations

$$\begin{aligned} \mathbf{x}'(t) &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\mathbf{x} \\ \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \end{aligned}$$

Remark: In the matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

the eigenvalues are

$$\begin{aligned} \lambda_1 &= a + b \\ \lambda_2 &= a - b \end{aligned}$$

with eigenvectors of

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$