Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $V := \left\{re^{i\theta} \in \mathbb{C} \,\middle|\, -\pi/4 < \theta < \pi/4, r > 0\right\}$. Fix $z_0 \in U$, and let $\mathcal{F} := \{f \in H(U) \mid f(z_0) = 1, \operatorname{im}(f) \subseteq V\}$. Show that $\mathcal{F}$ is normal.

Solution: We observe that a function $f \in H(U)$ if and only if $f(z_0) = 1$ and $\operatorname{im}(f) \subseteq V$, or equivalently, that $e^{i\pi/4}f(z_0) = e^{i\pi/4}$ and $\operatorname{im}\left(e^{i\pi/4}f\right)$ is a subset of the upper half-plane. In particular, this means that we seek to establish the normality of the family

$$\mathcal{G} = \left\{f \in H(U) \,\middle|\, \operatorname{im}(f) \subseteq \mathbb{H}, f(z_0) = e^{i\pi/4}\right\}.$$

Toward this end, we use the Cayley transform, $q(z) = \frac{z-i}{z+i}$ to conformally map the upper half-plane to the unit disk, establishing that the family

$$\mathcal{D} = \{q \circ g \mid g \in \mathcal{G}\}$$

is locally bounded (indeed, globally bounded by 1). Furthermore, every element of $\mathcal{D}$ has the property that

$$\begin{aligned} q \circ f(z_0) &= \frac{e^{i\pi/4} - i}{e^{i\pi/4} + i} \\ &\in \mathbb{D}. \end{aligned}$$

Now, let $(f_n)_n \subseteq \mathcal{F}$. Then, $g_n := e^{i\pi/4}f_n$ is a sequence in $\mathcal{G}$, and $h_n := q \circ g_n$ is a sequence in $\mathcal{D}$. Since $\mathcal{D}$ is normal, there is a subsequence $(h_{n_k})_k \to h\colon U \to \overline{\mathbb{D}}$ satisfying $h(z_0) = \frac{e^{i\pi/4}-i}{e^{i\pi/4}+i}$, meaning that h is a holomorphic function mapping $U \to \mathbb{D}$. Since q and multiplication by $e^{i\pi/4}$ are conformal maps on their respective domains, it then follows that

$$\begin{aligned} (f_{n_k})_k &= \left(e^{-i\pi/4}q^{-1} \circ h_{n_k}\right)_k \\ &\to e^{-i\pi/4}q^{-1} \circ h \\ &\in H(U), \end{aligned}$$

meaning that $\mathcal{F}$ is a normal family.

Problem (Problem 2)**:** Let $\mathcal{F} = \{f \in H(\mathbb{D}) \mid \operatorname{im}(f) \subseteq \mathbb{D}\}$. Fix $z_0 \in \mathbb{D}$. Show that there exists a holomorphic function $g\colon \mathbb{D} \to \mathbb{C}$ with $\operatorname{im}(g) \subseteq \mathbb{D}$, $|g'(z_0)| = \max_{f\in\mathcal{F}}|f'(z_0)|$, and $g(z_0) = 0$.

Solution: From Montel's Theorem, we know that the set $\mathcal{F}$ is normal, meaning that $\overline{\mathcal{F}}$ is compact in $H(\mathbb{D})$.

Now, we start by showing that the family

$$\mathcal{F}' = \{f' \mid f \in \mathcal{F}\}$$

is normal, by showing that it is locally bounded. Let $z \in \mathbb{D}$, let $B(z, r) \subseteq \mathbb{D}$, and let $m = \sup_{f\in\mathcal{F}}\|f\|_{B(z,r)}$. Note that by the extended maximum modulus principle,

$$\sup_{z\in B(z,r)} |f(z)| = \sup_{z\in S(z,r)} |f(z)|.$$

For a given $f \in \mathcal{F}$, Cauchy's estimate gives

$$\begin{aligned} |f'(z)| &\leqslant \frac{1}{r} \sup_{|\xi-z|=r} |f(\xi)| \\ &= \frac{1}{r} \sup_{\xi\in S(z,r)} |f(\xi)| \\ &\leqslant \frac{m}{r}, \end{aligned}$$

meaning in particular that

$$\sup_{f'\in\mathcal{F}'}|f'(z)| \leqslant \frac{m}{r},$$

whence $\mathcal{F}'$ is locally bounded. Thus, by Montel's Theorem, it follows that $\mathcal{F}'$ is normal. Since both evaluation and the modulus are continuous operations, we observe then that the map

$$\begin{aligned} s\colon \overline{\mathcal{F}'} &\to \mathbb{R} \\ f' &\mapsto |f'(z_0)| \end{aligned}$$

is a continuous map whose domain is a compact space, so there is some $h \in \overline{\mathcal{F}'}$ such that

$$|h(z_0)| = \sup_{f\in\mathcal{F}}|f'(z_0)|$$

Since $\mathbb{D}$ is simply connected, there is some holomorphic antiderivative for h, given by $g \in H(\mathbb{D})$. We claim that it must be the case that $g \in \overline{\mathcal{F}}$. Since $h \in \overline{\mathcal{F}'}$, there is some sequence of function $(f'_n)_n \to h = g'$ uniformly on compacts. Fix an exhaustion $(K_m)_m$ given by

$$K_m = B\left(0, \frac{m}{m+1}\right).$$

Then, we have $tz \in K_m$ for all $z \in K_m$ and all $0 \leqslant t \leqslant 1$. In particular, this means that

$$\begin{aligned} |f_n(z) - g(z)| &= \left|\int_0^1 z(f'_n(tz) - g'(tz))\, dt\right| \\ &\leqslant \int_0^1 |z(f'_n(tz) - g'(tz))|\, dt \\ &\leqslant \int_0^1 |f'_n(tz) - g'(tz)|\, dt \\ &\leqslant \int_0^1 \sup_{t\in[0,1]} |f'_n(tz) - g'(tz)|\, dt \\ &\leqslant \int_0^1 \sup_{w\in K_m} |f'_n(w) - g'(w)|\, dt \\ &= \sup_{w\in K_m} |f'_n(w) - g'(w)| \\ &= \|f'_n - g'\|_{K_m}, \end{aligned}$$

so it follows that $\|f_n - g\|_{K_m} \leqslant \|f'_n - g'\|_{K_m}$. In particular, since the latter tends to zero as K_m is compact, it follows that the former tends to zero as well. In particular, this means that $\|f_n - g\|_{H(\mathbb{D})} \to 0$, whence $f_n \to g$ uniformly on compacts. Thus, it follows that $g \in \overline{\mathcal{F}}$, so that $\operatorname{im}(g) \subseteq \overline{\mathbb{D}}$.

Furthermore, since $f(z) = z \in \mathcal{F}$, it follows that $|g'(z_0)| \geqslant 1$, meaning that g is a nonconstant holomorphic function, meaning in particular that since $g(\mathbb{D}) \subseteq \overline{\mathbb{D}}$ already, we must indeed have $g(\mathbb{D}) \subseteq \mathbb{D}$ by the open mapping principle.

Now, we claim that $g(z_0) = 0$. Suppose this were not the case. Then, there would be some $0 < r < 1$ with $|g(z_0)| = r$. We have established on a previous assignment that the map

$$h_0(z) = \frac{z - g(z_0)}{1 - \overline{g(z_0)}z}$$

is a bijective holomorphic mapping of $\mathbb{D}$ to itself, meaning that

$$h(z) = \frac{g(z) - g(z_0)}{1 - \overline{g(z_0)}g(z)}$$

maps $\mathbb{D}$ to $\mathbb{D}$, so that $h \in \mathcal{F}$, with

$$\begin{aligned} h'(z) &= \frac{g'(z)}{1 - \overline{g(z_0)}g(z)} + \overline{g(z_0)}g'(z)\frac{g(z) - g(z_0)}{\left(1 - \overline{g(z_0)}g(z)\right)^2} \\ |h'(z_0)| &= |g'(z_0)|\frac{1}{1 - |g(z_0)|^2} \\ &= |g'(z_0)|\frac{1}{1 - r^2} \\ &> |g'(z_0)|, \end{aligned}$$

which contradicts the maximality of $|g'(z_0)|$. Thus, it must be the case that $g(z_0) = 0$.

Problem (Problem 3)**:** Let $(a_n)_n$ be a sequence of nonnegative real numbers such that the radius of convergence of

$$\sum_{n=0}^{\infty} a_n z^n$$

is at least 1. Let

$$\mathcal{F} := \bigcap_{n=0}^{\infty}\left\{f \in H(\mathbb{D}) \,\middle|\, \left|\frac{f^{(n)}(0)}{n!}\right| \leqslant a_n\right\}.$$

Show that $\mathcal{F}$ is a normal family.

Solution: Suppose $z \in \mathbb{D}$. We wish to establish some $\delta > 0$ and some $M > 0$ such that $U(z, \delta) \subseteq \mathbb{D}$, for all $f \in \mathcal{F}$,

$$|f(z)| \leqslant M.$$

Now, let $r > 0$ be such that $B(z, r) \subseteq \mathbb{D}$. We observe that for $f \in \mathcal{F}$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)(0)}}{n!} z^n,$$

with $\left|\frac{f^{(n)}(0)}{n!}\right| \leqslant a_n$. By uniform convergence, we then see that for any $f \in \mathcal{F}$ and any $w \in B(z, r)$,

$$\begin{aligned} |f(w)| &\leqslant \sum_{n=0}^{\infty}\left|\frac{f^{(n)}(0)}{n!}\right||w|^n \\ &\leqslant \sum_{n=0}^{\infty} a_n|w|^n \\ &\leqslant \sum_{n=0}^{\infty} a_n(|z| + r)^n \\ &=: C, \end{aligned}$$

since $|z| + r$ is less than 1, while $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence at least 1. Since this holds for all $f \in \mathcal{F}$, it follows that the family $\mathcal{F}$ is locally bounded, hence normal by Montel's Theorem.

Problem (Problem 4)**:**

(a) Fix $z_0 \in \mathbb{C}$ and $r > 0$. Suppose $f\colon B(z_0, r) \to \mathbb{C}$ is continuous, and $f|_{U(z_0,r)}$ is holomorphic. Fix $0 < \rho < r$. Show that for all $z \in U(z_0, \rho)$,

$$|f(z)| \leqslant \frac{1}{\pi(r-\rho)^2} \iint_{U(z_0,r)} |f(x+iy)|\, dx dy.$$

(b) Fix $M \geqslant 0$, let $U \subseteq \mathbb{C}$ be open, and let $\mathcal{F} \subseteq H(U)$ be the family of holomorphic functions for which

$$\iint_U |f(x+iy)|\, dx dy \leqslant M.$$

Show that $\mathcal{F}$ is normal.

Solution:

(a) For each $\rho \leqslant t \leqslant r$, we parametrize $S(z_0, t)$ as $\gamma(t) = z_0 + te^{i\theta}$. In particular, by Cauchy's Integral Formula, we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{S(z_0,t)} \frac{f(w)}{w - z}\, dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f\left(z_0 + te^{i\theta}\right)te^{i\theta}}{(z_0 - z) + te^{i\theta}}\, d\theta. \end{aligned}$$

Introducing a factor of 1, then using Fubini's Theorem thus gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi(r-\rho)} \int_\rho^r \int_0^{2\pi} \frac{f\left(z_0 + te^{i\theta}\right)te^{i\theta}}{(z_0 - z) + te^{i\theta}}\, d\theta\, dt \\ &= \frac{1}{2\pi(r-\rho)} \int_0^{2\pi} \int_\rho^r \frac{f\left(z_0 + te^{i\theta}\right)te^{i\theta}}{(z_0 - z) + te^{i\theta}}\, dt\, d\theta \end{aligned}$$

Estimating the integral, we find that for $\rho < t \leqslant r$,

$$\begin{aligned} \left|\frac{1}{(z_0 - z) + te^{i\theta}}\right| &\leqslant \frac{1}{|t| - |z_0 - z|} \\ &\leqslant \frac{1}{|t| - \rho}. \end{aligned}$$

Since this holds for any $\rho < t \leqslant r$, it certainly holds for $t = r$, whence using this estimate and the triangle inequality gives

$$\begin{aligned} |f(z)| &\leqslant \frac{1}{2\pi(r-\rho)^2} \int_0^{2\pi} \int_\rho^r \left|f\left(z_0 + te^{i\theta}\right)te^{i\theta}\right|\, dt\, d\theta \\ &\leqslant \frac{1}{\pi(r-\rho)^2} \int_0^{2\pi} \int_0^r t\left|f\left(z_0 + te^{i\theta}\right)\right|\, dt\, d\theta \\ &= \frac{1}{\pi(r-\rho)^2} \iint_{U(z_0,r)} |f(x+iy)|\, dx dy. \end{aligned}$$

(b) Let $z_0 \in U$. Fix some $r > 0$ such that $B(z_0, r) \subseteq U$. In particular, we have

$$\begin{aligned} \iint_{U(z_0,r)} |f(x+iy)|\, dx dy &\leqslant \iint_U |f(x+iy)|\, dx dy \\ &\leqslant M, \end{aligned}$$

so for a fixed $0 < \rho < r$, we have for all $z \in U(z_0, \rho)$ and all $f \in \mathcal{F}$,

$$
\begin{aligned}
|f(z)| &\leqslant \frac{1}{\pi(r-\rho)^2} \iint_{U(z_0,r)} |f(x+iy)|\,dx\,dy \\
&\leqslant \frac{1}{\pi(r-\rho)^2} M.
\end{aligned}
$$

In particular, this means that for all $f \in \mathcal{F}$, f is bounded on $U(z_0, \rho)$, whence $\mathcal{F}$ is locally bounded, hence normal by Montel's Theorem.

Problem (Problem 5)**:** Let $(f_n)_n$ be a sequence of holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$ that is locally bounded, and suppose there exists a holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ such that the set $\{z \in \mathbb{D} \mid \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in $\mathbb{D}$. Show that $(f_n)_n$ converges uniformly on compact sets to f.

Solution: Since $(f_n)_n$ is locally bounded, it follows that the family $\{f_n \mid n \in \mathbb{N}\}$ is a normal family, by Montel's theorem. In particular, this means that for any subsequence $(f_{n_k})_k$, there is a subsequence of $(n_k)_k$, which we call $\left(n_{k_j}\right)_j$ and a holomorphic function $g_k\colon \mathbb{D} \to \mathbb{C}$ such that

$$
\left(f_{n_{k_j}}\right)_j \to g_k
$$

on compact subsets. Yet, since uniform convergence on compact subsets implies pointwise convergence, we have that $\{z \in \mathbb{D} \mid g_k(z) = f(z)\}$ has an accumulation point in $\mathbb{D}$, whence each of the g_k are equal to f by the identity theorem.

Now, if it were not the case that $(f_n)_n \to f$ uniformly on compacts, then we would be able to find some subsequence $(f_{n_k})_k$ with $\|f_{n_k} - f\| \geqslant \varepsilon_0$ for some $\varepsilon_0 > 0$ and all k. Yet, since this is a subsequence of $(f_n)_n$, it admits a subsequence converging to f, contradicting the assertion that $\|f_{n_k} - f\| \geqslant \varepsilon_0$ for all k. Therefore, we have that $(f_n)_n \to f$ uniformly on compacts.