

This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

Useful Results and Proofs

Analytic Functions

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A function $f: U \rightarrow \mathbb{C}$ is called *analytic* if, for any $z_0 \in U$, there is $r > 0$ and $(a_k)_k \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in U(z_0, r)$.

Analytic functions form a \mathbb{C} -algebra.

Theorem (Identity Theorem): Let $f, g: U \rightarrow \mathbb{C}$ be analytic functions defined on a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in U , then $f = g$ on U .

Proof. To begin, we show that if $f: U \rightarrow \mathbb{C}$ is an analytic function that is not uniformly zero, then for any $z_0 \in U$, there is $\rho > 0$ such that f is nonzero on $\dot{U}(z_0, \rho) \subseteq U$. Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all $z \in U(z_0, r)$, some $r > 0$, and since f is not uniformly zero, there is some minimal ℓ such that $a_\ell \neq 0$. This yields

$$f(z) = (z - z_0)^\ell \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function $h: U(z_0, r) \rightarrow \mathbb{C}$ given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as f and is not zero at z_0 , so that g is not zero on some $U(z_0, \rho)$ as g is continuous.

Now, we let V_1 be the set of accumulation points of A in U , and let $V_2 = U \setminus V_1$.

If $z \in V_2$, then there is some $r_1 > 0$ such that $\dot{U}(z_0, r_1) \cap A = \emptyset$, or that $\dot{U}(z_0, r_1) \subseteq A^c$. Meanwhile, since U is open, there is some $r_2 > 0$ such that $U(z_0, r_2) \subseteq U$, meaning that if $r = \min\{r_1, r_2\}$, then $U(z_0, r) \subseteq U \setminus A$. Thus, V_2 is open.

Meanwhile, if $z \in V_1$, then since $V_1 \subseteq U$, it follows that there is $r > 0$ such that $U(z, r)$ and $(a_k)_k$ such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all $w \in U(z, r)$. We claim that $f(w) - g(w)$ is uniformly zero on $U(z, r)$. Else, if there were $w_0 \in U(z, r)$ such that $f(w_0) \neq g(w_0)$, then it would follow that there is $0 < s \leq r$ such that $f(w) \neq g(w)$ for all $w \in U(w_0, s)$. Yet, this would contradict the assumption that z is an accumulation point, meaning that V_1 is open.

Since V_1 and V_2 are disjoint open sets whose union is equal to U , it follows that either $V_1 = U$ or $V_2 = U$. If $A \neq \emptyset$, then the identity theorem follows. \square

Differentiability

Definition: If $U \subseteq \mathbb{C}$ is an open set, then we say f is differentiable at $z_0 \in U$ if

$$\lim_{w \rightarrow z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of f at z_0 , and usually write $f'(z_0)$.

If f is differentiable at every $z_0 \in U$, we say f is differentiable on U .

If f is continuous and admits a continuous derivative, then we say f is *holomorphic*.

Note that the limit must be independent of direction. That is, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(w) - f(z_0)}{w - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Now, given $U \subseteq \mathbb{C}$, write $z = x + iy$ and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$. Observe then that if f is differentiable at $x_0 + iy_0 \in U$, then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(u(x+h, y) + iv(x+h, y)) - (u(x, y) + iv(x, y))}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(u(x, y+h) + iv(x, y+h)) - (u(x, y) + iv(x, y))}{ih} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

Definition: The system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if f is differentiable, then the u and v in the definition of f satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly f is differentiable or holomorphic.

Proposition: If $f = u + iv$ is a holomorphic function such that u, v are in $C^2(U)$, then u and v are harmonic. That is, u and v satisfy Laplace’s equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We call u and v *harmonic conjugates* for each other. That is, if $u: U \rightarrow \mathbb{R}$ is a harmonic function, then $v \in C^1(U)$ is called a harmonic conjugate if the Cauchy–Riemann equations hold for u and v .

Theorem: Let $U \subseteq \mathbb{R}^2$ be a ball or all of \mathbb{R}^2 . Then, every harmonic function on U has a harmonic conjugate. If $u \in C^3(U)$, then this conjugate is itself harmonic.

Lemma: Let $g: U((x_0, y_0), R) \rightarrow \mathbb{R}$ be such that g and $\frac{\partial g}{\partial x}$ are continuous. Then, $G: U((x_0, y_0), R) \rightarrow \mathbb{R}$, given by

$$G(x, y) = \int_{y_0}^y g(x, t) dt$$

satisfies

$$\frac{\partial G}{\partial x} = \int_{y_0}^y \frac{\partial g}{\partial x}(x, t) dt.$$

Proof of Lemma. Write

$$\frac{G(x+h, y) - G(x, y)}{h} - \int_{y_0}^y \frac{\partial g}{\partial x}(x, t) dt = \int_{y_0}^y \left(\frac{g(x+h, t) - g(x, t)}{h} - \frac{\partial g}{\partial x}(x, t) \right) dt.$$

By mean value theorem, the first term is equal to $\frac{\partial g}{\partial x}(x_1, t)$ for some x_1 between x and $x+h$. As $h \rightarrow 0$, $x_1 \rightarrow x$, as $\frac{\partial g}{\partial x}$ is uniformly continuous on a compact subset that contains x and $x+h$. We may exchange limit and integral to obtain the desired result. \square

Proof of Theorem. We prove for the case of $U = U((x_0, y_0), R)$. Define

$$v(x, y) = \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt + \phi(x),$$

with $\phi(x)$ to be determined later. By the fundamental theorem of calculus, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

while by differentiating under the integral sign, and using the fact that u is harmonic, we have

$$\begin{aligned} \frac{\partial v}{\partial x} &= \int_{y_0}^y \frac{\partial^2 u}{\partial x^2}(x, t) dt + \frac{d\phi}{dx} \\ &= - \int_{y_0}^y \frac{\partial^2 u}{\partial y^2}(x, t) dt + \frac{d\phi}{dx} \\ &= - \frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \frac{d\phi}{dx}. \end{aligned}$$

Defining $\phi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(x) = - \int_{x_0}^x \frac{\partial u}{\partial y}(s, y_0) ds,$$

we see that v thus satisfies all the necessary requirements to be a harmonic conjugate.

Now, if u is C^3 , then we defined v via the derivative of u , so that v is C^2 , and thus v is harmonic. \square

Cauchy's Integral Formula

Proposition: Fix $z_0 \in \mathbb{C}$, $R > 0$, and $f: U(z_0, R) \rightarrow \mathbb{C}$ holomorphic. For all $z \in U(z_0, R)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. It suffices to show that

$$\frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = 0.$$

By using the chain rule and fundamental theorem of calculus, we find

$$\begin{aligned} \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{\int_0^1 f'((1-t)z + t\zeta)(\zeta - z) dt}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{S(z_0, R)} \int_0^1 f'((1-t)z + t\zeta) dt d\zeta \\ &= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{d}{d\zeta} \left(\frac{1}{t} f((1-t)z + t\zeta) \right) d\zeta \\ &= 0. \end{aligned}$$

\square

Proposition: Let $f: U \rightarrow \mathbb{C}$ be a holomorphic function. The following all hold:

- (i) f is analytic;
- (ii) f is smooth with $f^{(n)}$ holomorphic;
- (iii) for all $z_0 \in U$, if we let $R = \sup\{r > 0 \mid U(z_0, r) \subseteq U\}$, then there is $(a_n)_n \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the power series has radius of convergence R .

Proof.

- (i) There exists $r < s$ with $U(z_0, s) \subseteq U$ and $r < r_1 < s$ such that $S(z_0, r_1) \subseteq U$. By Cauchy's Integral Formula, and a power series expansion of $\frac{1}{\xi - z}$ about z_0 , this gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{\xi - z} d\xi \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right)}_{=: a_n} \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n. \end{aligned}$$

- (ii) Analytic functions are automatically smooth, hence complex-differentiable with continuous

derivative.

(iii) If $r < r_1 < R$, then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \right),$$

and since the series converges uniformly, we have

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

Since r was arbitrary, this holds for any $0 < r_1 < R$, whence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for all $z \in U(z_0, R)$.

□

Corollary: Let $U \subseteq \mathbb{C}$ be open, let $z_0 \in U$, and $r > 0$ with $B(z_0, r) \subseteq U$. The following hold:

(i) for all $z \in U(z_0, r)$,

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi;$$

(ii) for all $n > 0$,

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{\zeta \in S(z_0, r)} |f(\zeta)|.$$

This particular result is known as the *Cauchy Estimate*.

Theorem (Liouville's Theorem): If $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and bounded in modulus, then f is constant.

Liouville's Theorem follows from applying Cauchy's estimate to f and using the fact that f is bounded to find that all higher derivatives of f vanish.

Theorem (Fundamental Theorem of Algebra): If $p(z) = a_n z^n + \cdots + a_1 z + a_0$ has $n \geq 1$ and $a_n \neq 0$, then there is at least one z_0 such that $p(z_0) = 0$.

Proof. Suppose $p(z)$ were never zero. It would follow then that $\frac{1}{p(z)}$ is also an entire function.

Since $\lim_{|z| \rightarrow \infty} |p(z)| = \infty$, it follows that $\lim_{|z| \rightarrow \infty} \frac{1}{|p(z)|} = 0$, whence $\left| \frac{1}{p(z)} \right|$ is an entire function that is bounded (as all functions that vanish at infinity are bounded). This means that $\frac{1}{p(z)}$ is constant, so $p(z)$ is constant. □

Corollary: Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a nonconstant entire function. Then, $f(\mathbb{C})$ is dense in \mathbb{C} .

Proof. Suppose there were $w \in \mathbb{C}$ and $r > 0$ such that $U(w, r) \cap f(\mathbb{C}) = \emptyset$. Then, $|f(z) - w| \geq r$ for all $z \in \mathbb{C}$, meaning that

$$g(z) = \frac{1}{f(z) - w}$$

is bounded and entire (the entirety following from the fact that $f(z) - w$ is nonvanishing). □

Cycles, Winding Numbers, and Homology

Now, we may generalize some of these results related to Cauchy's Integral Formula.

Proposition: Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \text{im}(\gamma)$, we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{\xi - z} d\xi \in \mathbb{Z}.$$

Proof. Let $\phi: [a, b] \rightarrow \mathbb{C}$ be defined by

$$\phi(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} ds.$$

Then, we observe

$$\phi(b) = \oint_{\gamma} \frac{1}{\xi - z} d\xi.$$

Then, define $\psi: [a, b] \rightarrow \mathbb{C}$ by

$$\psi(t) = \frac{e^{\phi(t)}}{\gamma(t) - z}.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned} \phi'(t) &= \frac{\gamma'(t)}{\gamma(t) - z} \\ \psi'(t) &= \frac{\phi'(t)e^{\phi(t)}}{\gamma(t) - z} - \frac{e^{\phi(t)}\gamma'(t)}{(\gamma(t) - z)^2} \\ &= 0, \end{aligned}$$

whence $\psi(t)$ is constant, and $\psi(t) = \psi(a)$, so

$$\psi(a) = \frac{1}{\gamma(a) - z}.$$

In particular, $\psi(b) = \psi(a)$, so

$$\begin{aligned} e^{\phi(b)} &= \psi(b)(\gamma(b) - z) \\ &= \psi(a)(\gamma(a) - z) \\ &= 1, \end{aligned}$$

so $\phi(b) = 2\pi i k$ for some $k \in \mathbb{Z}$. □

Definition: Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a piecewise C^1 loop. For all $z \in \mathbb{C} \setminus \text{im}(\gamma)$, define

$$n(\gamma; z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{\xi - z} d\xi$$

to be the *winding number* of γ about z .

Definition: A piecewise C^1 *cycle* is a formal sum

$$\Gamma = \gamma_1 + \cdots + \gamma_n,$$

where the $\gamma_j: [a_j, b_j] \rightarrow \mathbb{C}$ are piecewise C^1 loops. The *length* of Γ is the sum of the lengths of the respective γ_j .

Given a piecewise C^1 cycle Γ , define

$$\oint_{\Gamma} f(z) dz = \sum_{j=1}^n \oint_{\gamma_j} f(z) dz,$$

and

$$n(\Gamma; z) = \sum_{j=1}^n n(\gamma_j; z).$$

Proposition: The following hold for the winding number $n(\gamma; z)$:

- (i) the function $n(\Gamma; \cdot): \mathbb{C} \setminus \text{im}(\Gamma) \rightarrow \mathbb{Z}$ is continuous;
- (ii) $n(\Gamma; z)$ is constant on each connected component of $\mathbb{C} \setminus \text{im}(\Gamma)$;
- (iii) there exists a unique unbounded connected component with $n(\Gamma; z) = 0$ for all z in this unbounded connected component.

Proof.

- (i) Since $\text{im}(\Gamma)$ is compact, any $z \notin \text{im}(\Gamma)$ admits a strictly positive

$$\text{dist}_{\text{im}(\Gamma)}(z) = \inf_{w \in \text{im}(\Gamma)} |w - z|.$$

Let $w \in \mathbb{C}$ be such that

$$|w - z| < \frac{1}{2} \text{dist}_{\text{im}(\Gamma)}(z),$$

so that $w \in \mathbb{C} \setminus \text{im}(\Gamma)$. Observe then that

$$\begin{aligned} |n(\Gamma; z) - n(\Gamma; w)| &= \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\xi - z} - \frac{1}{\xi - w} d\xi \right| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^n \oint_{\gamma_j} \left| \frac{1}{\xi - z} - \frac{1}{\xi - w} \right| |d\xi| \\ &= \frac{1}{2\pi} \sum_{j=1}^n \oint_{\gamma_j} \left| \frac{z - w}{(\xi - z)(\xi - w)} \right| |d\xi| \\ &\leq \frac{1}{2\pi} \left(\frac{2}{\text{dist}_{\text{im}(\Gamma)}(z)} \right)^2 \ell(\Gamma) |z - w|, \end{aligned}$$

whence $|n(\Gamma; z) - n(\Gamma; w)|$ is sufficiently small whenever $|z - w|$ is sufficiently small.

- (ii) If C is a connected component of $\mathbb{C} \setminus \text{im}(\Gamma)$, and $n(\Gamma; \cdot): C \rightarrow \mathbb{Z}$ is continuous, then since \mathbb{Z} is discrete, $n(\Gamma; \cdot)$ is constant on C .
- (iii) For uniqueness, if there are unbounded connected components C_1 and C_2 of $\mathbb{C} \setminus \text{im}(\Gamma)$, then there exists $M > \sup_{z \in \text{im}(\Gamma)} |z|$ and $w_1 \in C_1, w_2 \in C_2$ such that $|w_1| > 2M$ and $|w_2| > 2M$. Since $\mathbb{C} \setminus \overline{U(0, 2M)}$ is path connected, there exists $\gamma: [0, 1] \rightarrow \mathbb{C}$ with $|\gamma(t)| \geq 2M$ and $\gamma(0) = w_1, \gamma(1) = w_2$. Therefore, w_1 and w_2 are in the same connected component.

Existence then follows from $\text{im}(\Gamma)$ being compact.

Finally, let $(z_n)_n \subseteq \mathbb{C}$, where \mathbb{C} is the unbounded connected component, be such that $\lim_{n \rightarrow \infty} |z_n| = \infty$. For $M > \sup_{z \in \text{im}(\gamma)} |z|$, there exists $m \in \mathbb{N}$ such that $|z_m| > M$. Then, we have

$$\begin{aligned} |n(\Gamma; z_m)| &= \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\xi - z} d\xi \right| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^k \oint_{\gamma_j} \frac{1}{|\xi - z|} |d\xi| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^k \oint_{\gamma_j} \frac{1}{|z_m| - M} |d\xi| \\ &= \frac{\ell(\Gamma)}{2\pi(|z_m| - M)}, \end{aligned}$$

whence $\lim_{m \rightarrow \infty} n(\Gamma; z_m) = 0$, meaning that there exists N such that $|n(\Gamma; z_m)| < 1$ for all $m \geq N$, meaning $n(\Gamma; z_m) = 0$ for all sufficiently large m . Since \mathbb{C} is connected, it thus follows that $n(\Gamma; z) = 0$ for all $z \in \mathbb{C}$. □

Definition: Let $U \subseteq \mathbb{C}$ be open. A cycle Γ is *homologous to zero in U* if $\text{im}(\Gamma) \subseteq U$ and for all $z \in \mathbb{C} \setminus U$, $n(\Gamma; z) = 0$.

Theorem (Cauchy's Integral Formula, General Case): Let $\Gamma = \gamma_1 + \cdots + \gamma_k$ be a piecewise C^1 cycle homologous to zero in U , and $f: U \rightarrow \mathbb{C}$ holomorphic. Then, for all $z \in U \setminus \text{im}(\Gamma)$,

$$n(\Gamma; z)f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi$$

Theorem (Cauchy's Integral Theorem): Let $U \subseteq \mathbb{C}$ be open, $f: U \rightarrow \mathbb{C}$ holomorphic, and Γ homologous to zero in U . Then,

$$\oint_{\Gamma} f(z) dz = 0.$$

Definition: A region $U \subseteq \mathbb{C}$ is called *simply connected* if its complement in the extended complex plane is connected.

Theorem: If $U \subseteq \mathbb{C}$ is simply connected, then every loop in U is homologous to zero.

Proof. Extend the function $n(\gamma; \cdot)$ to the extended complex plane by defining $n(\gamma; \infty) = 0$. This extended function is continuous on $\hat{\mathbb{C}} \setminus U$, as $n(\gamma; \cdot)$ is zero on the unique unbounded connected component of $\mathbb{C} \setminus \text{im}(\gamma)$. It follows that $n(\gamma; z)$ is equal to zero on $\hat{\mathbb{C}} \setminus U$, whence γ is homologous to zero in U . □

Proposition: Let $U \subseteq \mathbb{C}$ be a region, $f: U \rightarrow \mathbb{C}$ holomorphic. The following are equivalent:

- (i) there exists a holomorphic function $F: U \rightarrow \mathbb{C}$ such that $F'(z) = f(z)$;
- (ii) for every piecewise C^1 loop γ with $\text{im}(\gamma) \subseteq U$, we have

$$\oint_{\gamma} f(z) dz = 0.$$

Proof. The direction (i) \Rightarrow (ii) follows immediately from the fundamental theorem of calculus. In the reverse direction, we define $F: U \rightarrow \mathbb{C}$ by

$$f(z) = \int_{\sigma(z_0, z)} f(\xi) d\xi,$$

where $\sigma(z_0, z): [0, 1] \rightarrow U$ is a piecewise C^1 curve with $\sigma(0) = z_0$ and $\sigma(1) = z$. Such a curve always

exists as U is open and connected (hence path-connected). The integral is well-defined, since if γ_1 and γ_2 are any two such paths, then $\Gamma = \gamma_1 \setminus \gamma_2$ is a piecewise C^1 loop. Additionally, F is continuous.

Now, we evaluate the derivative of F . Let $z \in U$, $r > 0$ such that $U(z, r) \subseteq U$, and $h \in \mathbb{C}$ be such that $z + h \in U(z, r)$. Then,

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{\sigma(z_0, z_0+h)} f(\xi) d\xi - \frac{1}{h} \int_{\sigma(z_0, z)} f(\xi) d\xi \\ &= \frac{1}{h} \int_{\sigma(z, z+h)} f(\xi) d\xi. \end{aligned}$$

We may assume that $\sigma(z, z+h)$ is a straight line, so that

$$\int_{\sigma(z, z+h)} f(\xi) d\xi = hf(z),$$

meaning that

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \frac{1}{|h|} \left| \int_{\sigma(z, z+h)} f(\xi) d\xi - f(z) \right| \\ &\leq \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)|. \end{aligned}$$

Since f is continuous, it follows that the right hand side goes to zero as $|h|$ becomes small. Thus, F' is continuous, so f is holomorphic. \square

Observe that $\mathbb{C} \setminus \{0\}$ is not simply connected, meaning that, for instance, the function

$$f(z) = \frac{1}{z}$$

does not have a holomorphic antiderivative defined on the entirety $\mathbb{C} \setminus \{0\}$, as

$$\int_{S^1} f(z) dz = 2\pi i.$$

Yet, if we restrict $f(z)$ to a simply connected subset of \mathbb{C} , there is a holomorphic antiderivative. Choosing such a simply connected subset of \mathbb{C} is known as choosing a *branch* of the logarithm. In fact, there is more that we can say.

Corollary: Let $U \subseteq \mathbb{C}$ be simply connected, and let $f: U \rightarrow \mathbb{C} \setminus \{0\}$ be a nonvanishing holomorphic function. For each fixed pair $z_0 \in U$ and $w_0 \in \mathbb{C}$ for which $e^{w_0} = f(z_0)$, there exists a unique holomorphic function $g: U \rightarrow \mathbb{C}$ for which $g(z_0) = w_0$ and $e^{g(z)} = f(z)$.

We call g the logarithm of f , written $g(z) = \log(f(z))$.

Proof. Since f is nonvanishing and U is simply connected, it follows that $\frac{f'}{f}$ is holomorphic on U , meaning there is $\tilde{g}: U \rightarrow \mathbb{C}$ such that $\tilde{g}'(z) = \frac{f'(z)}{f(z)}$. Thus, there is some constant K such that

$$f(z) = Ke^{\tilde{g}(z)}.$$

Define

$$g(z) = \log(K) + \tilde{g}(z).$$

\square

Theorem (Morera's Theorem): Let $U \subseteq \mathbb{C}$ be open, $f: U \rightarrow \mathbb{C}$ continuous. Suppose

$$\oint_T f(z) dz = 0$$

for all triangles $T \subseteq U$ homologous to zero. Then, f is holomorphic.

Proof. Since U is open, if $z_0 \in U$, there is r such that $U(z_0, r) \subseteq U$. Define $F: U(z_0, r) \rightarrow \mathbb{C}$ by

$$F(z) = \int_{\sigma(z_0, z)} f(\xi) d\xi,$$

where σ is the straight line from z_0 to z . For $0 < |h| < r - |z - z_0|$, we construct the straight lines $\sigma(z, z + h)$ and $\sigma(z_0, z + h)$, such that

$$T = \sigma(z_0, z) + \sigma(z, z + h) - \sigma(z_0, z + h),$$

and

$$\begin{aligned} \oint_T f(z) dz &= 0 \\ &= \int_{\sigma(z_0, z)} f(\xi) d\xi + \int_{\sigma(z, z+h)} f(\xi) d\xi - \int_{\sigma(z_0, z+h)} f(\xi) d\xi \\ &= F(z) - F(z+h) + \int_{\sigma(z, z+h)} f(\xi) d\xi, \end{aligned}$$

meaning

$$\begin{aligned} F(z+h) - F(z) &= \int_{\sigma(z, z+h)} f(\xi) d\xi \\ \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{\sigma(z, z+h)} f(\xi) d\xi \\ \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \frac{1}{h} \int_{\sigma(z, z+h)} (f(\xi) - f(z)) d\xi \right| \\ &\leq \frac{1}{|h|} |h| \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)| \\ &= \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)|. \end{aligned}$$

Since f is continuous, it follows that for sufficiently small $|h|$, the right-hand-side goes to zero, whence $F'(z) = f(z)$, meaning F is holomorphic, so F is analytic, meaning f is analytic, so f is holomorphic. \square

Corollary: Let $U \subseteq \mathbb{C}$ be open, $\gamma: [a, b] \rightarrow U$ a piecewise C^1 curve, and $g: U \times \text{im}(\gamma) \rightarrow \mathbb{C}$ continuous. Suppose that for each $w \in \text{im}(\gamma)$, the function $g(\cdot, w)$ is holomorphic. Then,

$$f(z) := \int_{\gamma} g(z, w) dw$$

is holomorphic.

Proof. Let T be a triangle in U homologous to zero. Then, by Fubini's Theorem,

$$\oint_T f(z) dz = \oint_T \int_{\gamma} g(z, w) dw dz$$

$$= \int_{\gamma} \oint_{\Gamma} g(z, w) dz dw.$$

The interior integral vanishes for every w as $g(\cdot, w)$ is holomorphic. Thus, f is holomorphic. \square

Definition: Let $U \subseteq \mathbb{C}$ be open, γ_1, γ_2 piecewise C^1 loops in U . We say γ_1 and γ_2 are homotopic in U if there is a continuous function

$$H: [a, b] \times [0, 1] \rightarrow U$$

such that

$$H(s, 0) = \gamma_1(s)$$

$$H(s, 1) = \gamma_2(s)$$

$$H(a, t) = H(b, t).$$

For each t , $H(\cdot, t)$ is a continuous loop. We call H a homotopy between γ_0 and γ_1 .

Theorem: If γ_0 and γ_1 are homotopic in U , then $\Gamma = \gamma_1 - \gamma_0$ is homologous to zero in U .

Theorem: If $K \subseteq U$ is compact and U is connected, then there is some cycle Γ homologous to zero in U such that $n(\Gamma; z) = 1$ for all $z \in K$.

Corollary: Let U be a region. The following are equivalent:

- (i) U is simply connected;
- (ii) for every nonvanishing holomorphic function $f: U \rightarrow \mathbb{C} \setminus \{0\}$, there is a holomorphic function $g: U \rightarrow \mathbb{C}$ such that $f(z) = e^{g(z)}$;
- (iii) for all cycles Γ with $\text{im}(\Gamma) \subseteq U$, Γ is homologous to zero in U .

Maximum Modulus Principle

Theorem (Mean Value Property): Let $U \subseteq \mathbb{C}$ be open, $f: U \rightarrow \mathbb{C}$ holomorphic, with $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Proof. By the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{\xi - z} d\xi.$$

Parametrizing $\gamma(\theta) = z_0 + re^{i\theta}$, we get

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

\square

Corollary: If $u: \mathbb{R}^2 \supseteq U \rightarrow \mathbb{R}$ is harmonic, $(x_0, y_0) \in U$, and $r > 0$ is such that $B((x_0, y_0), r) \subseteq U$, then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos(\theta), y_0 + r \sin(\theta)) d\theta.$$

Proof. Take real parts of the mean value property for holomorphic $f = u + iv$. \square

Observe then that the triangle inequality implies that

$$|u(x_0, y_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |u(x_0 + r \cos(\theta), y_0 + r \sin(\theta))| d\theta.$$

Functions that satisfy this weaker criterion are known as *subharmonic*. It is subharmonic functions for which the most general case of the *maximum modulus principle* hold.

Theorem (Maximum Modulus Principle): Let $U \subseteq \mathbb{R}^2$ be open and connected, and let $u: U \rightarrow \mathbb{R}$ be subharmonic. Suppose there exists $(x_0, y_0) \in U$ such that $u(x_0, y_0) \geq u(x, y)$ for all $x, y \in U$. Then, u is constant.

Proof. Let $\lambda = u(x_0, y_0)$, and let $E = \{(x, y) \mid u(x, y) = \lambda\} = u^{-1}(\{\lambda\})$. We see immediately that E is closed; we claim that E is also open.

Fix $(x_1, y_1) \in E$. Then, $u(x_1, y_1) = \lambda$. Take $r > 0$ such that $U((x_1, y_1), r) \subseteq U$. Then, for all $0 < s < r$, we have $S((x_1, y_1), s) \subseteq U$, meaning that

$$\begin{aligned} \lambda &= u(x_1, y_1) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + s \cos(\theta), y_1 + s \sin(\theta)) d\theta \\ &\leq \lambda, \end{aligned}$$

with the latter inequality following from the fact that λ is a local maximum. Therefore, $u(x_1 + s \cos(\theta), y_1 + s \sin(\theta)) = \lambda$ for all $0 < s < r$, whence $U((x_1, y_1), r) \subseteq E$. Thus, E is open, so since U is connected, it follows that E is all of U , meaning u is constant. \square

Corollary: If $U \subseteq \mathbb{R}^2$ is bounded and $u: \bar{U} \rightarrow \mathbb{R}$ is continuous with $u|_U$ subharmonic, then there exists $(x_0, y_0) \in \partial U$ such that $u(x_0, y_0) = \sup_{(x,y) \in U} u(x, y)$.

Corollary: If $U \subseteq \mathbb{C}$ is open and connected, with $f: U \rightarrow \mathbb{C}$ holomorphic, then if $|f|: U \rightarrow \mathbb{R}$ has a local maximum at $z_0 \in U$, then f is constant.

Proof. Let $r > 0$ be such that $U(z_0, r) \subseteq U$. Then, restricting $|f|$ to $U(z_0, r)$, we see that $|f|$ restricted to $U(z_0, r)$ is subharmonic viewed as a function on $U(z_0, r)$, hence $|f|$ is constant on $U(z_0, r)$.

Now, by the mean value property and triangle inequality, it follows that for all $0 < s < r$, we have

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + se^{i\theta})| d\theta \\ &= |f(z_0)|, \end{aligned}$$

meaning that these are equalities. In particular, there exists some θ_s such that $e^{i\theta_s} f(z_0 + se^{i\theta}) \geq 0$, meaning that for this value of s , we have

$$\begin{aligned} |f(z_0)| &= e^{i\theta_s} \int_0^{2\pi} f(z_0 + se^{i\theta}) d\theta \\ &= e^{i\theta_s} f(z_0), \end{aligned}$$

with the latter equality following from the mean value property. Since this holds for any s , it follows that θ_s is independent of s , meaning that $f(z)e^{i\theta_s} \geq 0$ for all $z \in U(z_0, r)$, meaning that $\text{Im}(e^{i\theta_s} f(z)) = 0$ on $U(z_0, r)$, whence $f(z)e^{i\theta_s}$ is constant, meaning f is constant on $U(z_0, r)$.

Finally, by the identity theorem, it follows that f is constant on U . \square

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. We say a sequence $U \supseteq ((x_n, y_n))_n \rightarrow \partial U$ if, for every compact $K \subseteq U$, the set $\{n \in \mathbb{N} \mid (x_n, y_n) \in K\}$ is finite.

Definition: Let $U \subseteq \mathbb{R}^2$ be an open set. Given a function $u: U \rightarrow \mathbb{R}$, define

$$\limsup_{(x,y) \rightarrow \partial U} u(x,y) := \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{(x,y) \in U \setminus K} u(x,y).$$

These definitions allow us to extend the maximum modulus principle for subharmonic functions even further.

Theorem: Let $U \subseteq \mathbb{C}$ be a region, $u: U \rightarrow \mathbb{R}$ a nonconstant subharmonic function. If $((x_n, y_n))_n \subseteq U$ is such that $u(x_n, y_n) \rightarrow \sup_{(x,y) \in U} u(x,y)$, then $((x_n, y_n))_n \rightarrow \partial U$. Moreover, $\limsup_{(x,y) \rightarrow \partial U} u(x,y) = \sup_{(x,y) \in U} u(x,y)$.

Proof. Suppose toward contradiction that $((x_n, y_n))_n \not\rightarrow \partial U$, so there exists a compact subset $K \subseteq U$ and a subset $((x_{n_k}, y_{n_k}))_k$ wholly contained in K . Since K is compact, there is a subsequence of $((x_{n_k}, y_{n_k}))_k$ converging to $(x_0, y_0) \in U$. Therefore, $u(x_0, y_0) = \sup_{(x,y) \in U} u(x,y)$, so u is constant by the maximum modulus principle, which is a contradiction.

Finally, $\limsup_{(x,y) \rightarrow \partial U} u(x,y) \leq \sup_{(x,y) \in U} u(x,y)$, while if $((x_n, y_n))_n \rightarrow \partial U$ is such that $u(x_n, y_n)$ converges to $\sup_{(x,y) \in U} u(x,y)$, then $\sup_{(x,y) \in U} u(x,y) = \lim_{n \rightarrow \infty} u(x_n, y_n) \leq \limsup_{(x,y) \rightarrow \partial U} u(x,y)$. \square

Theorem (Open Mapping Principle): Let $U \subseteq \mathbb{C}$ be a region, and let $f: U \rightarrow \mathbb{C}$ be a nonconstant holomorphic function. Then, $f(U) \subseteq \mathbb{C}$ is open.

Proof. Let $z_0 \in U$ and $r > 0$ be such that $B(z_0, r) \subseteq U$. We will show that there exists R such that $U(f(z_0), R) \subseteq f(U(z_0, r)) \subseteq U$, whence $f(U)$ is open.

Since U is a region and f is nonconstant, the zeros of $g(z) := f(z) - f(z_0)$ are isolated, so there exists some $0 < s < r$ such that

$$\delta = \inf_{|z-z_0|=s} |f(z) - f(z_0)|$$

is strictly greater than zero. We claim that $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$. Suppose this were not the case, meaning there would be some $\xi \in U(f(z_0), \delta/2) \setminus f(U(z_0, r))$, and define $h: B(z_0, s) \rightarrow \mathbb{C}$ by

$$h(z) = \frac{1}{f(z) - \xi}.$$

Since $\xi \notin f(U(z_0, r))$, this is holomorphic, while $\xi \in U(f(z_0), \delta/2)$ implies

$$\begin{aligned} \sup_{|z-z_0|=s} |h(z)| &= \sup_{|z-z_0|=s} \frac{1}{|f(z) - \xi|} \\ &\leq \sup_{|z-z_0|=s} \frac{1}{|f(z) - f(z_0)| - |f(z_0) - \xi|} \\ &\leq \frac{1}{\delta - \delta/2} \\ &= \frac{2}{\delta}. \end{aligned}$$

Yet,

$$\begin{aligned} |h(z_0)| &= \frac{1}{|f(z_0) - \xi|} \\ &> \frac{2}{\delta}, \end{aligned}$$

contradicting the maximum modulus principle. Thus, $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$. \square

In the proof of the Hadamard Three-Lines Theorem, we used the function $h_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}$ for this purpose.

Classification of Singularities

The classification of singularities seeks to answer two fundamental questions: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, and $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic,

- does f have a holomorphic extension to U including z_0 ;
- and what else can we say about the behavior of f at z_0 ?

Definition: Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ holomorphic.

- If there exists a holomorphic $g: U \rightarrow \mathbb{C}$ with $g = f$ on $U \setminus \{z_0\}$, then we say z_0 is a *removable singularity*.
- If $\lim_{z \rightarrow z_0} |f(z)| = \infty$, then we say f has a *pole* at z_0 .
- Else, we say f has an *essential singularity* at z_0 .

Theorem (Riemann's Theorem on Removable Singularities): Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ holomorphic. Then, z_0 is a removable singularity if and only if $\lim_{z \rightarrow z_0} f(z) = 0$.

Proof. If z_0 is removable, then $g(z)$ is a holomorphic function with $g(z) = f(z)$ on $U \setminus \{z_0\}$, and since g is continuous, it follows that $\lim_{z \rightarrow z_0} g(z) = g(z_0)$, whence $\lim_{z \rightarrow z_0} (z - z_0)g(z) = \lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$.

Now, if $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$, then there is r such that $B(z_0, r) \subseteq U$, and since f is locally bounded around z_0 , it follows that

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for all $z \in U(z_0, r)$. Yet, the formula extends to z_0 as it is bounded, whence we may define the holomorphic extension for f by

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\zeta)}{\zeta - z} d\zeta & z = z_0 \end{cases}.$$

□

Proposition (Existence of Laurent Series): Suppose $f: A(z_0, r, R) \rightarrow \mathbb{C}$ is holomorphic, with $0 \leq r < R$. Then, there exist holomorphic functions

$$\begin{aligned} g_1: U(z_0, R) &\rightarrow \mathbb{C} \\ g_2: \mathbb{C} \setminus B(z_0, r) &\rightarrow \mathbb{C} \end{aligned}$$

such that $f = g_1 + g_2$ on $A(z_0, r, R)$. Moreover, there exists $(a_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for all z , and the series converges uniformly on $A(z_0, \rho, s)$ with $r < \rho < s < R$.

Proof. Fix $z \in A(z_0, r, R)$. Then, for $r < \rho_1, \rho_2 < |z - z_0|$, the cycle

$$\Gamma_1 = S(z_0, \rho_1) - S(z_0, \rho_2)$$

is homologous to zero in $A(z_0, r, |z - z_0|)$. By Cauchy's Integral Theorem, it then follows that

$$\oint_{S(z_0, \rho_1)} \frac{f(\xi)}{\xi - z} d\xi = \oint_{S(z_0, \rho_2)} \frac{f(\xi)}{\xi - z} d\xi.$$

Similarly, for $|z - z_0| < s_1, s_2 < R$, we have

$$\oint_{S(z_0, s_1)} \frac{f(\xi)}{\xi - z} d\xi = \oint_{S(z_0, s_2)} \frac{f(\xi)}{\xi - z} d\xi.$$

Define $g_1: U(z_0, R) \rightarrow \mathbb{C}$ by

$$g_1(z) = \frac{1}{2\pi i} \oint_{S(z_0, s)} \frac{f(\xi)}{\xi - z} d\xi,$$

where $|z - z_0| < s < R$. This function is holomorphic by Morera's Theorem. Similarly, we may define $g: \mathbb{C} \setminus B(z_0, r) \rightarrow \mathbb{C}$ by

$$g_2(z) = -\frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{\xi - z} d\xi,$$

where $r < \rho < |z - z_0|$. We claim that $f = g_1 + g_2$ on $A(z_0, r, R)$.

For $z \in A(z_0, r, R)$, we may find, for any $r < \rho < |z - z_0| < s < R$, we let

$$\Gamma = S(z_0, s) - S(z_0, \rho),$$

homologous to zero in $A(z_0, r, R)$, whence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left(\oint_{S(z_0, s)} \frac{f(\xi)}{\xi - z} d\xi - \int_{S(z_0, \rho)} \frac{f(\xi)}{\xi - z} d\xi \right) \\ &= g_1(z) + g_2(z). \end{aligned}$$

□

Theorem: Let $U \subseteq \mathbb{C}$, $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

on $\dot{U}(z_0, R)$ for some R with $U(z_0, R) \subseteq U$. Then,

- (i) f has a removable singularity at z_0 if and only if $a_n = 0$ for all $n < 0$;
- (ii) f has a pole at z_0 if and only if

$$1 \leq |\{n < 0 \mid a_n \neq 0\}| < \infty.$$

- (iii) f has an essential singularity at z_0 if and only if

$$|\{n < 0 \mid a_n \neq 0\}| = \infty.$$

Proof.

- (i) If $a_n = 0$ for all $n < 0$, then $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$, so f has a removable singularity at z_0 .

Conversely, if f has a removable singularity at z_0 , then for $n < 0$, we have

$$a_n = \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

for any $0 < \rho < R$. Since $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$, then for any $\varepsilon > 0$, there is sufficiently small ρ such that

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right| \\ &\leq \rho^{-1-n} \sup_{|\xi - z_0| = \rho} |(\xi - z_0)f(z)| \\ &\leq \varepsilon. \end{aligned}$$

Thus, $|a_n| = 0$ for all $n < 0$.

- (ii) If $a_n \neq 0$ for a nonempty finite collection of $n < 0$, we let m be the largest number such that $a_{-m} < 0$, so that $f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$. It follows that $\lim_{z \rightarrow z_0} (z - z_0)^m f(z) = a_{-m} \neq 0$. In particular, there is some small δ such that

$$|(z - z_0)^m f(z) - a_{-m}| < \frac{|a_{-m}|}{2},$$

whence for a sufficiently small ε ,

$$\begin{aligned} |f(z)| &> \frac{|a_{-m}| - |(z - z_0)^m f(z) - a_{-m}|}{|z - z_0|^m} \\ &> \frac{|a_{-m}|}{2|z - z_0|^m} \\ &> \frac{1}{\varepsilon}, \end{aligned}$$

so f has a pole at z_0 .

Conversely, if f has a pole at z_0 , there is some $r > 0$ such that $|f(z)| \geq 1$ whenever $z \in \dot{U}(z_0, r)$. Define $g: \dot{U}(z_0, r) \rightarrow \mathbb{C}$ by

$$g(z) = \frac{1}{f(z)},$$

whence g is holomorphic and bounded on $\dot{U}(z_0, r)$. By the classification of singularities, z_0 is a removable singularity of g , so there exists a holomorphic function $h: U(z_0, r) \rightarrow \mathbb{C}$ that is equal to $\frac{1}{f(z)}$ for $z \neq z_0$ and $\lim_{z \rightarrow z_0} \frac{1}{f(z)}$ if $z = z_0$. We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

where we must have $b_0 = 0$. Let m be the smallest positive integer such that $h(z) = (z - z_0)^m \tilde{h}(z)$, where $\tilde{h}(z) = \sum_{n=0}^{\infty} b_{n+m} (z - z_0)^n$ and $b_m \neq 0$. The function h is holomorphic on $U(z_0, r)$ and nonzero on some $U(z_0, \rho)$ with $0 < \rho \leq r$, so that $\tilde{f}(z) = \frac{1}{\tilde{h}(z)}$ is holomorphic on $U(z_0, \rho)$, so

$$f(z) = (z - z_0)^{-m} \tilde{f}(z)$$

on $\dot{U}(z_0, \rho)$. Writing

$$\tilde{f}(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

we deduce that

$$f(z) = \sum_{n=-m}^{\infty} c_{n+m} (z - z_0)^n.$$

(iii) Follows from (i) and (ii). □

Definition: Let $U \subseteq \mathbb{C}$ be open.

- If $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ has a pole or a removable singularity at z_0 , then the order of the pole or singularity is the smallest $m \geq 0$ such that

$$f(z) = (z - z_0)^{-m} g(z)$$

with $g: U \rightarrow \mathbb{C}$ is holomorphic and has $g(z_0) \neq 0$.

- If $f: U \rightarrow \mathbb{C}$ has a zero at $z_0 \in U$, then the order of the zero at z_0 is the smallest $m \geq 0$ such that

$$f(z) = (z - z_0)^m g(z)$$

where $g: U \rightarrow \mathbb{C}$ is holomorphic and has $g(z_0) \neq 0$.

If $z_0 \in U$ is either a pole or a zero (or a removable singularity that is a zero when f is extended), then the *order* of f is the unique $m \in \mathbb{Z}$ such that

$$f(z) = (z - z_0)^m g(z)$$

with $g(z_0) \neq 0$.

Theorem (Casorati–Weierstrass): Let $U \subseteq \mathbb{C}$ be an open set, and let $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ be a holomorphic function. If z_0 is an essential singularity of f , then for any $r > 0$ with $U(z_0, r) \subseteq U$, $f(\dot{U}(z_0, r)) \subseteq \mathbb{C}$ is dense.

Proof. Suppose $f(\dot{U}(z_0, r))$ is not dense in \mathbb{C} . Then, there exists $\varepsilon > 0$ and $w \in \mathbb{C}$ such that $|f(z) - w| \geq \varepsilon$ for all $z \in \dot{U}(z_0, r)$. We will show that z_0 is either removable or a pole.

Define

$$g: \dot{U}(z_0, r) \rightarrow \mathbb{C}$$

by

$$g(z) = \frac{1}{f(z) - w}.$$

Observe that g is definitionally bounded, so z_0 is a removable singularity of g . Thus, there is a holomorphic function $h: U(z_0, r) \rightarrow \mathbb{C}$ such that $h(z) = g(z)$ for $z \neq z_0$ and $h(z) = \lim_{z \rightarrow z_0} g(z)$. We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

and take

$$h(z) = (z - z_0)^m \tilde{h}(z),$$

which exists as g is not uniformly zero, and where $\tilde{h}(z_0) \neq 0$. Consequently, there is $0 < \rho \leq r$ such that $\tilde{f}: U(z_0, \rho) \rightarrow \mathbb{C}$ given by

$$\tilde{f}(z) = \frac{1}{\tilde{h}(z)},$$

which is holomorphic. Thus,

$$f(z) = w + (z - z_0)^{-m} \tilde{f}(z).$$

Thus, we get

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n,$$

whence z_0 is either removable or a pole. □

The Argument Principle and Rouché's Theorem

Definition: If $U \subseteq \mathbb{C}$ is open, and $V \subseteq U$ is an open subset such that $U \setminus V$ consists solely of isolated points, then a function $f: V \rightarrow \mathbb{C}$ is *meromorphic* if it is holomorphic on V and every $z_0 \in U \setminus V$ is either a pole or a removable singularity. We say f is meromorphic on U .

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U . Let f be meromorphic on U with no poles or zeros on $\text{im}(\Gamma)$.

- (i) The set $\{z_0 \in U \mid \text{ord}_{z_0}(f) \neq 0, n(\Gamma; z_0) \neq 0\}$ is finite.
- (ii) We have

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = \sum_{\substack{z_0 \in U \\ \text{ord}_{z_0}(f) \neq 0 \\ n(\Gamma; z_0) \neq 0}} n(\Gamma; z_0) \text{ord}_{z_0}(f).$$

Proof.

- (i) Let $K = \{z \in U \mid n(\Gamma; z) \neq 0\} \cup \text{im}(\Gamma)$. We know that for $R > \sup_{w \in \text{im}(\Gamma)} |w|$, we have $n(\Gamma; z) = 0$ for all $z \in \mathbb{C} \setminus B(0, R)$, meaning that K is bounded. Furthermore, if $(z_n)_n \rightarrow z \in \mathbb{C}$, then either $z \in \text{im}(\Gamma) \subseteq K$ or $z \in \mathbb{C} \setminus \text{im}(\Gamma)$, so that $n(\Gamma; z) \neq 0$ by the continuity of the map $w \mapsto n(\Gamma; w)$. Thus, $z \in K$ as Γ is homologous to zero in U . It follows thus that K is compact.

Let $E = \{z_0 \in K \mid \text{ord}_{z_0}(f) \neq 0\}$, meaning that E consists of poles and zeros of f . These points are isolated, meaning E is a closed subset of K , hence compact. Since E is compact and contains isolated points only, it follows that E is finite.

- (ii) For each $z \in K$, select $\delta_z > 0$ such that $U(z, \delta_z) \subseteq U$, and $U(z, \delta_z) \cap E \subseteq \{z\}$. Such a δ_z exists since z is isolated in E and U is open. The collection $\{U(z, \delta_z) \mid z \in K\}$ is an open cover of K , so there is a finite subcover $\{U(z_1, \delta_1), \dots, U(z_m, \delta_m)\}$. Define

$$V = \bigcup_{j=1}^m U(z_j, \delta_j),$$

so that V is open with $K \subseteq V \subseteq U$. Since $\{z \in U \mid n(\Gamma; z) \neq 0\} \subseteq K \subseteq V$, it follows that Γ is homologous to zero in V .

Define

$$g(z) = f(z) \prod_{z \in E} (z - z_0)^{-\text{ord}_{z_0}(f)}.$$

Since $\text{ord}_z(g) = 0$ for all $z \in V$, all the singularities of g are removable, and $\frac{g'}{g}$ is holomorphic on V and satisfies

$$\frac{g'}{g} = \frac{f'}{f} - \sum_{z_0 \in E} \frac{1}{z - z_0} \text{ord}_{z_0}(f).$$

Cauchy's Integral theorem provides the desired result. □

Theorem (Rouché's Theorem): Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U . Let f and g be meromorphic on U with no poles or zeros on $\text{im}(\Gamma)$. If $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in \text{im}(\Gamma)$, then

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{\Gamma} \frac{g'(z)}{g(z)} dz.$$

Proof. Since $|f(z) - g(z)| < |f(z)| + |g(z)|$ on $\text{im}(\Gamma)$, coupled with the fact that $\text{ord}_z(f) = \text{ord}_z(g) = 0$ on $\text{im}(\Gamma)$ implies that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1$$

for all $z \in \text{im}(\Gamma)$. This only holds if $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in \text{im}(\Gamma)$. Since $\text{im}(\Gamma)$ is compact, there exists some $\varepsilon > 0$ such that

$$\text{dist}_{(-\infty, 0]} \left| \frac{f}{g}(\text{im}(\Gamma)) \right| \geq \varepsilon.$$

Since $\frac{f}{g}$ is continuous and $\text{im}(\Gamma)$ is compact, there also exists some $\delta > 0$ such that whenever $\text{dist}_{\text{im}(\Gamma)}(z) < \delta$, we have $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$.

Setting $V = \{z \in U \mid \text{dist}_{\text{im}(\Gamma)}(z) < \delta\}$, we let $h: V \rightarrow \mathbb{C}$ be defined by

$$h(z) = \log \left(\frac{f(z)}{g(z)} \right)$$

for the branch of the logarithm that excludes $(-\infty, 0]$, which is well-defined as $\frac{f}{g} \notin (-\infty, 0]$ on V . This satisfies

$$\frac{h'}{h} = \frac{f'}{f} - \frac{g'}{g},$$

whence by Cauchy's Integral Theorem,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{h'(z)}{h(z)} dz \\ &= \frac{1}{2\pi i} \left(\oint_{\Gamma} \frac{f'(z)}{f(z)} dz - \oint_{\Gamma} \frac{g'(z)}{g(z)} dz \right). \end{aligned}$$

□

Remark: Most use cases for Rouché's Theorem involve finding $g(z)$ such that $|f(z) - g(z)| < |g(z)|$ on $\text{im}(\Gamma)$, where both $f(z)$ and $g(z)$ have no zeros or poles on $\text{im}(\Gamma)$.

It is possible to use Rouché's Theorem to prove the fundamental theorem of algebra.

Corollary: Let $P: \mathbb{C} \rightarrow \mathbb{C}$ be a nonconstant polynomial. Then, there exists $z_0 \in \mathbb{C}$ with $P(z_0) = 0$.

Proof. Write $P(z) = a_n z^n + \cdots + a_1 z + a_0$, where $n \geq 1$, $a_n \neq 0$, and $a_1, \dots, a_n \in \mathbb{C}$.

Let $Q(z) = a_n z^n$, R sufficiently large, and $\Gamma = S(0, R)$. Then,

$$\begin{aligned} |P(z) - Q(z)| &= |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\ &\leq |a_n| R^n \left(\frac{|a_{n-1}|}{|a_n| R} + \cdots + \frac{|a_0|}{|a_n| R^n} \right) \\ &< |a_n| R^n \\ &\leq |P(z)| + |Q(z)|. \end{aligned}$$

Thus, P has the same number of zeros counted with multiplicity as Q . \square

Corollary: Let $U \subseteq \mathbb{C}$ be an open set, $f: U \rightarrow \mathbb{C}$ be a holomorphic injective function. Then, $f'(z) \neq 0$ for all $z \in U$, whence f admits a holomorphic inverse.

Proof. Suppose there is some $z_0 \in U$ such that $f'(z_0) = 0$. By the identity theorem, there must be some $r > 0$ such that $B(z_0, r) \subseteq U$ and $f'(z) \neq 0$ for all $z \in U(z_0, r)$, else f would be equal to a constant on some nonempty open set, hence not injective.

Let $g(z) = f(z) - f(z_0)$. Define $m := \text{ord}_{z_0}(g)$. We observe that $m \geq 2$, as $g(z_0) = 0$ and $g'(z_0) = 0$. We may write

$$g(z) = \lambda(z - z_0)^m + (z - z_0)^{k+1}h(z)$$

for some holomorphic function $h: U \rightarrow \mathbb{C}$ and a constant $\lambda \in \mathbb{C} \setminus \{0\}$, meaning

$$f(z) = f(z_0) + \lambda(z - z_0)^m + (z - z_0)^{k+1}h(z)$$

for all $z \in U$.

Let $C = \sup_{z \in B(z_0, r)} |h(z)|$, which is finite as $B(z_0, r)$ is compact. Letting $\rho = \min\left(r, \frac{|\lambda|}{2C}\right)$ and $\eta = \frac{|\lambda|\rho^m}{2}$, and fixing $w \in U(f(z_0), \eta)$, we observe that if $z \in S(z_0, \rho)$,

$$\begin{aligned} |f(z) - w - \lambda(z - z_0)^m| &= |f(z_0) - w + (z - z_0)^{m+1}h(z)| \\ &\leq |f(z_0) - w| + |z - z_0|^{m+1}|h(z)| \\ &< \eta + \rho^{m+1}C \\ &< \rho^m|\lambda| \\ &= |(z - z_0)^m\lambda|. \end{aligned}$$

Therefore, by Rouché's theorem, the number of zeros in $U(z_0, \rho)$ for $f(z) - w$ is equal to the number of zeros counted with multiplicity in $U(z_0, \rho)$ of $(z - z_0)^m\lambda$. Since the latter $m \geq 2$, it follows that the former is also $m \geq 2$. Since $f'(z) \neq 0$ for all $z \in U(z_0, \rho)$, no zero of $f(z) - w$ can have order at least 2, meaning that there are at least two distinct zeros of $f(z) - w$ in $U(z_0, \rho)$, whence f is not injective. \square

Residues

Definition: Let $U \subseteq \mathbb{C}$, $z_0 \in U$, $r > 0$ such that $U(z_0, r) \subseteq U$, and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on $U(z_0, r)$. The *residue* of f at z_0 is the coefficient a_{-1} of $(z - z_0)^{-1}$. We write $\text{Res}(f; z_0)$.

Proposition: Let $U \subseteq \mathbb{C}$ be an open set, and let f be meromorphic on U with a pole of order $m \geq 1$ at $z_0 \in U$. Then,

$$\text{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z).$$

Proof. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Differentiating term by term, we find that

$$\frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = a_{-1}.$$

□

Proposition: Let $f, g: U \rightarrow \mathbb{C}$ be holomorphic with z_0 a simple zero for g that is not a zero or pole for f . Then,

$$\text{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. We compute the residue directly, using the fact that $g(z_0) = 0$ to find

$$\begin{aligned} \lim_{z \rightarrow z_0} (z - z_0) \frac{f(z)}{g(z)} &= \frac{f(z_0)}{\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0}} \\ &= \frac{f(z_0)}{g'(z_0)}. \end{aligned}$$

□

Example:

- If $f(z) = \frac{e^{iz}}{z}$, then

$$\text{Res}(f; 0) = 1.$$

- If $f(z) = \pi \cot(\pi z)$, then

$$\begin{aligned} \text{Res}(f; n) &= \frac{\pi \cos(\pi n)}{\frac{d}{dz} \big|_{z=n} \sin(\pi z)} \\ &= \frac{\pi \cos(\pi n)}{\pi \cos(\pi n)} \\ &= 1. \end{aligned}$$

- Let $f(z) = \frac{e^{3z}}{(z-2)^2}$. To compute the residue at $z = 2$, we may directly find

$$\begin{aligned} \text{Res}(f; 2) &= \lim_{z \rightarrow 2} \frac{d}{dz} (z-2)^2 \frac{e^{3z}}{(z-2)^2} \\ &= 3e^6. \end{aligned}$$

Theorem: Let $f: \dot{U}(z_0, r) \rightarrow \mathbb{C}$ be holomorphic. Then, for all $0 < \rho < r$, we have

$$\oint_{S(z_0, \rho)} f(z) dz = 2\pi i \text{Res}(f; z_0).$$

Proof. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The series converges uniformly on compact sets, so we may exchange order of integration and summation to find

$$\begin{aligned} \oint_{S(z_0, \rho)} f(z) dz &= \sum_{n=-\infty}^{\infty} a_n \oint_{S(z_0, \rho)} (z - z_0)^n dz \\ &= 2\pi i a_{-1}. \end{aligned}$$

□

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, Γ a piecewise C^1 cycle homologous to zero in U . Let f be meromorphic on U with no poles on $\text{im}(\Gamma)$. Then,

$$\oint_{\Gamma} f(z) dz = 2\pi i \sum_{z_0 \in E} n(\Gamma; z_0) \text{Res}(f; z_0),$$

where

$$E = \{z_0 \in U \mid \text{ord}_{z_0}(f) < 0, n(\Gamma; z_0) \neq 0\}.$$

Proof. As in the proof of the argument principle, we see that E is finite, which we write $\{z_1, \dots, z_m\}$. Select $r > 0$ such that $B(z_j, r)$ are pairwise disjoint and contained in U . Set $\tilde{\Gamma} = \Gamma - \sum_{j=1}^m S(z_j, r)$. Then, $\tilde{\Gamma}$ is homologous to zero in $U \setminus \{z_1, \dots, z_m\}$, while f is holomorphic on $U \setminus \{z_1, \dots, z_m\}$, whence

$$\begin{aligned} 0 &= \oint_{\tilde{\Gamma}} f(z) dz \\ &= \oint_{\Gamma} f(z) dz - \sum_{j=1}^m \oint_{S(z_j, r)} f(z) dz \\ &= \oint_{\Gamma} f(z) dz - 2\pi i \sum_{z_0 \in E} n(\Gamma; z_0) \text{Res}(f; z_0). \end{aligned}$$

□

Conformal Maps and Spaces of Holomorphic Functions

Given an open subset $U \subseteq \mathbb{C}$, we define $H(U)$ to be the set of all holomorphic functions $f: U \rightarrow \mathbb{C}$. Similarly, we write $C(U)$ for the continuous functions $f: U \rightarrow \mathbb{C}$.

Definition: A sequence $(f_n)_n \subseteq C(U)$ converges uniformly on compacts to $f \in C(U)$ if, for every compact $K \subseteq U$, the sequence $(f_n|_K)_n \rightarrow f|_K$ uniformly.

Proposition: Let $U \subseteq \mathbb{C}$ be open. If a sequence $(f_n)_n \subseteq H(U)$ converges uniformly on compacts to $f \in C(U)$, then $f \in H(U)$. Moreover, the sequence $(f'_n)_n \rightarrow f'$.

Proof. Since any $(f_n)_n$ converges on compacts to f , it converges uniformly on any triangle T homologous to zero in U , so that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \oint_T f_n(z) dz \\ &= \oint_T \lim_{n \rightarrow \infty} f_n(z) dz \\ &= \oint_T f(z) dz. \end{aligned}$$

To show that $(f'_n)_n \rightarrow f'$ converges uniformly on compacts, we use Cauchy's integral formula to find that

$$f'_n(z) - f'(z) = \frac{1}{2\pi i} \oint_{S(z_0, R)} \frac{f_n(w) - f(w)}{(w - z)^2} dw,$$

where $z \in U(z_0, R)$ and $B(z_0, R) \subseteq U$. Cauchy's estimate then show that this tends to 0 as $n \rightarrow \infty$ uniformly for all $z \in U(z_0, R)$. Since compact sets are totally bounded, it thus follows that the convergence is uniform on compact subsets. \square

Definition: Let $U \subseteq \mathbb{C}$ be an open set. An exhaustion of U is a collection of compacts $(K_m)_m$ for which $K_m \subseteq K_{m+1}^\circ$ and

$$U = \bigcup_{m=1}^{\infty} K_m.$$

The primary example we will use is

$$K_m := \left\{ z \in U \mid |z| \leq m, \text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{m} \right\}.$$

Definition: Let $U \subseteq \mathbb{C}$ be an open set, and let $(K_m)_m$ be an Exhaustion of U . For $f, g \in C(U)$, we define

$$d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f - g\|_{K_m}}{1 + \|f - g\|_{K_m}}.$$

Now, despite the fact that the metric space $(C(U), d)$ depends on the choice of exhaustion, it can be shown that any two metrics based on exhaustions $(K_m)_m$ and $(K'_m)_m$ are uniformly equivalent.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, $(K_m)_m$ an exhaustion of U , and let

$$d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f - g\|_{K_m}}{1 + \|f - g\|_{K_m}}$$

for $f, g \in C(U)$. Then, $(H(U), d)$ is a complete metric space.

Proposition: The topology on $(H(U), d)$ is equal to the topology of uniform convergence on compact subsets.

Proof. Let $(K_m)_m$ be an exhaustion of U . Let $(X_m, d_m) = (C(K_m), \|\cdot - \cdot\|_{K_m})$, and set

$$X = \prod_{m=1}^{\infty} C(K_m).$$

The isometry $\iota: C(U) \rightarrow X$, given by $\iota(f) = (f|_{K_m})_m$ yields that a sequence $(f_n)_n \rightarrow f$ if and only if $(f_n|_{K_m})_n \rightarrow f|_{K_m}$ for each K_m . In particular, if $(f_n)_n \rightarrow f$ uniformly on compact subsets, then it

converges uniformly on each K_m , hence $\lim_{n \rightarrow \infty} d(f_n, f) = 0$.

Conversely, if $K \subseteq \mathbb{C}$, $\{K_m^\circ\}_{m=1}^\infty$ is an open cover of K , so there exists a finite subcover. Since $K_m \subseteq K_{m+1}^\circ$, this means there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$. In particular, if $\lim_{n \rightarrow \infty} d(f_n, f) = 0$, it then follows that $(f_n)_n \rightarrow f$ uniformly on K . \square

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A family $\mathcal{F} \subseteq H(U)$ is called *normal* if its closure $\overline{\mathcal{F}}$ is compact in $H(U)$.

Theorem: Let $U \subseteq \mathbb{C}$ be an open set, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

- (i) \mathcal{F} is normal;
- (ii) for each $K \subseteq U$, the family $\mathcal{F}|_K$ has compact closure in $C(K)$;
- (iii) for each $z \in U$, there exists a bounded open set $W_z \subseteq U$ containing z such that $\mathcal{F}|_{W_z}$ has compact closure in $C(\overline{W_z})$.

Proof. We start by showing that (i) implies (iii). For $z \in U$, let $R = R_z > 0$ such that $B(z, R) \subseteq U$, and let $g_n \in \mathcal{F}|_{B(z, R)}$. Choose $f_n \in \mathcal{F}$ such that $f_n|_{B(z, R)} = g_n$. If \mathcal{F} is normal, there exists a subsequence $(f_{n_k})_k \rightarrow f \in \overline{\mathcal{F}}$. In particular, $f_{n_k}|_{B(z, R)} \rightarrow f|_{B(z, R)}$. Thus, $\mathcal{F}|_{B(z, R)}$ has compact closure in $C(B(z, R))$.

Next, we show that (iii) implies (ii). Let W_z be as above. Given $K \subseteq U$, the collection $\{W_z\}_{z \in U}$ is an open cover of K that has a finite subcover $\{W_1, \dots, W_\ell\}$. Given $g_n \in \mathcal{F}|_K$, write $g_n = f_n|_K$ for some $f_n \in \mathcal{F}$. There then exists a subsequence $(f_{n_k})_k$ such that $(f_{n_k}|_{W_j}) \rightarrow f_j \in C(\overline{W_j})$ for each j . The function $f := \lim_{k \rightarrow \infty} f_{n_k}(z)$ is well-defined and satisfies $f|_{\overline{W_j}} = f_j$ for each j . Moreover, since

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|f_{n_k} - h\|_K &\leq \limsup_{k \rightarrow \infty} \max_{1 \leq j \leq \ell} \|f_{n_k} - f_j\|_{\overline{W_j}} \\ &= 0, \end{aligned}$$

it follows that $\mathcal{F}|_K$ has compact closure in $C(K)$.

Finally, we show that (ii) implies (i). Let $(K_m)_m$ be an exhaustion of U . By Tychonoff's Theorem, $\prod_{m=1}^\infty \overline{\{f|_{K_m} \mid f \in \mathcal{F}\}}$ is compact. Thus, given a sequence $(f_n)_n$ in \mathcal{F} , there is a subsequence $(f_{n_k})_k$ such that for each $m \in \mathbb{N}$, there is some $g_m \in C(K_m)$ such that $f_{n_k}|_{K_m}$ converges uniformly to g_m . Thus, the function $f(z) = \lim_{k \rightarrow \infty} f_{n_k}(z)$ is well-defined and satisfies $f|_{K_m} = g_m$ for all m . Given a compact $K \subseteq U$, there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$, and consequently, $f_{n_k}|_K \rightarrow f|_K$ uniformly, so \mathcal{F} is normal. \square

We will now create a much more workable criterion for normality.

Definition: Let (X, d) be a compact metric space. A family $\mathcal{F} \subseteq C(X)$ is (uniformly) equicontinuous if, for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$.

Recall the Arzelà–Ascoli theorem.

Theorem (Arzelà–Ascoli): Let $K \subseteq \mathbb{C}$ be compact, and let $\mathcal{F} \subseteq C(K)$ be a family of continuous functions. The following are equivalent:

- (i) $\overline{\mathcal{F}}$ is compact;
- (ii) \mathcal{F} is bounded and, for all $z \in K$, $\overline{\{f(z) \mid f \in \mathcal{F}\}} \subseteq \mathbb{C}$ is compact.

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A family $\mathcal{F} \subseteq H(U)$ is called locally (uniformly) bounded on U if, for all $z_0 \in U$, there is some $\delta > 0$ such that $U(z_0, \delta) \subseteq U$ and there is some $C \geq 0$ such that $|f(z)| \leq C$ for all $z \in U(z_0, \delta)$ and all $f \in \mathcal{F}$.

Proposition: Let $U \subseteq \mathbb{C}$ be open, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

- (i) \mathcal{F} is locally bounded;
- (ii) for every compact $K \subseteq U$, $\sup_{f \in \mathcal{F}} \|f\|_K$ is finite.

Proof. We start by showing that (i) implies (ii). For each $z \in U$, there is $\delta_z > 0$ and $C_z > 0$ such that $U(z, \delta_z) \subseteq U$ and $|f(w)| \leq C_z$ for all $w \in U(z, \delta_z)$. Let $K \subseteq U$ be compact. There is a finite subcover $\{U(z_1, \delta_1), \dots, U(z_k, \delta_k)\}$, with corresponding bounds C_1, \dots, C_k , so by defining $C = \max\{C_1, \dots, C_k\}$, it follows that $\sup_{f \in \mathcal{F}} \|f\|_K \leq C < \infty$.

Now, we show that (ii) implies (i). Given $z_0 \in U$, there is $r > 0$ such that $B(z_0, r) \subseteq U$. By taking $K = B(z_0, r)$, we are done. \square

Theorem (Montel's Theorem): Let $U \subseteq \mathbb{C}$ be open, $\mathcal{F} \subseteq H(U)$ a family of holomorphic functions. The following are equivalent:

- (i) \mathcal{F} is normal;
- (ii) \mathcal{F} is locally bounded.

Proof. We start by showing that (i) implies (ii). Given $K \subseteq U$, the map $\|\cdot\|_K$ is continuous, so $\sup_{f \in \mathcal{F}} \|f\|_K \leq \sup_{f \in \overline{\mathcal{F}}} \|f\|_K$, which is finite as $\overline{\mathcal{F}}$ is compact.

Now, we show (ii) implies (i). Fix $z_0 \in U$, and choose $R_0 > 0$ such that $B(z_0, R_0) \subseteq U$. Let $W_0 = U(z_0, R_0/2)$. It suffices to show that $\mathcal{F}|_{\overline{W_0}}$ has compact closure in $C(\overline{W_0})$, which by the Arzelà–Ascoli theorem, is equivalent to showing that $\mathcal{F}|_{\overline{W_0}}$ is equicontinuous. Since \mathcal{F} is locally bounded, there is some $C_0 \geq 0$ such that $\sup_{f \in \mathcal{F}} \|f\|_{B(z_0, R_0)} \leq C_0$. By Cauchy's Integral Formula, we then have for all $z \in \overline{W_0}$ and all $f \in \mathcal{F}$,

$$\begin{aligned} |f'(z)| &= \left| \frac{1}{2\pi i} \oint_{S(z_0, R_0)} \frac{f(\xi)}{(\xi - z)^2} d\xi \right| \\ &\leq \frac{C_0}{2\pi} \oint_{S(z_0, R_0)} \frac{1}{|\xi - z|^2} |d\xi| \\ &\leq \frac{4C_0}{R_0} \\ &=: A_0. \end{aligned}$$

We have thus shown that $|f'(z)| \leq A_0$ for all $z \in \overline{W_0}$, meaning that each $f|_{\overline{W_0}} \in \mathcal{F}|_{\overline{W_0}}$ is A_0 -Lipschitz, hence $\mathcal{F}|_{\overline{W_0}}$ is equicontinuous, hence normal. \square

Worked Examples and Problem-Solving Methods

Example: Suppose U is a region in \mathbb{C} that contains 0, and suppose $f: U \rightarrow \mathbb{C}$ is a holomorphic function satisfying

$$\left| f\left(\frac{1}{n}\right) \right| < e^{-n}$$

for sufficiently large n . We will show that this means f is 0 everywhere.

Toward this end, since U is open, there is some $r > 0$ such that $U(0, r) \subseteq U$. Since f is holomorphic, on

$U(0, r)$, we may write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for some sequence $(a_n)_n \subseteq \mathbb{C}$. Now, we also observe that

$$\begin{aligned} |f(0)| &= \lim_{n \rightarrow \infty} \left| f\left(\frac{1}{n}\right) \right| \\ &\leq \lim_{n \rightarrow \infty} e^{-n} \\ &= 0. \end{aligned}$$

Suppose toward contradiction that f were nonconstant. Then, there would be some minimal positive value ℓ such that

$$f(z) = z^\ell \sum_{n=0}^{\infty} a_{n+\ell} z^n$$

has $a_\ell \neq 0$. Thus, defining

$$g(z) = \sum_{n=0}^{\infty} a_{n+\ell} z^n,$$

we observe that $g(0) \neq 0$, meaning that on some sufficiently small ball $U(0, \delta) \subseteq U(0, r)$, we have $|g(z)| > \left|\frac{a_\ell}{2}\right|$ for all $z \in U(0, \delta)$. In particular, this means that for n with $\frac{1}{n} < \delta$,

$$\begin{aligned} e^{-n} &\geq \left| f\left(\frac{1}{n}\right) \right| \\ &= n^{-\ell} \left| g\left(\frac{1}{n}\right) \right| \\ &\geq \frac{|a_\ell|}{2n^\ell}, \end{aligned}$$

whence

$$|a_\ell| \leq \frac{2n^\ell}{e^n}.$$

Yet, since n is arbitrary and ℓ is constant, this implies that $|a_\ell| = 0$, contradicting the assumption that there were such a g . Thus, in particular, we have that $f(z) = 0$ on $U(0, r)$, whence f is zero everywhere by the identity theorem.

Cauchy Estimate Problems

Example: Suppose f is an entire function, and suppose there exists a constant C such that for all $z \in \mathbb{C}$,

$$|f(z)| \leq C(1 + |z|)^{1/2}.$$

We will show that f is then constant. Toward this end, we will be able to use the Cauchy estimate by taking

$$|f^{(n)}(z)| \leq \frac{n!}{R^n} \sup_{|z|=R} |f(z)|$$

$$\begin{aligned} &\leq \frac{Cn!}{R^n} \sup_{|z|=R} (1 + |z|)^{1/2} \\ &= \frac{Cn!}{R^n} (1 + R)^{1/2}, \end{aligned}$$

whence for all $n \geq 1$, since R is arbitrary, we have $|f^{(n)}(z)| = 0$, so f is constant.

Maximum Modulus Principle Problems

Example: We show that if $f: U \rightarrow \mathbb{C}$ is holomorphic on a connected open set, and

$$u(x, y) = |f(x + iy)|$$

is harmonic on U , then f is constant.

Toward this end, we let $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. For any $0 < s < r$, the mean value property gives

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + se^{i\theta})| d\theta \\ &= |f(z_0)|. \end{aligned}$$

In particular, for any $0 < s < r$, we have the equality

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + se^{i\theta})| d\theta.$$

Since f is continuous, there is some θ_s such that $|f(z_0 + se^{i\theta})| = e^{i\theta_s} f(z_0 + se^{i\theta})$, whence

$$\begin{aligned} |f(z_0)| &= e^{i\theta_s} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + se^{i\theta}) d\theta \\ &= e^{i\theta_s} f(z_0), \end{aligned}$$

meaning that $\theta_s =: \theta_0$ is independent of s . Yet, this means that $e^{i\theta_0} f(z)$ is holomorphic on $U(z_0, r)$ and has $\text{Im}(e^{i\theta_0} f(z)) = 0$, meaning that by the open mapping principle, $f(z)$ is constant on $U(z_0, r)$, and so f is constant on U by the identity theorem.

The Phragmén–Lindelöf Method

The maximum modulus principle is primarily useful in the case where f is continuous on the closure of a bounded open set U and holomorphic on the interior. Yet, this fails to be true if U is unbounded.

For instance, if

$$U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \right\},$$

and $f(z) = e^{e^z}$, then

$$f\left(x \pm \frac{\pi}{2}i\right) = e^{\pm ie^x},$$

whence $|f(z)| = 1$ for $z \in \partial U$. Yet, $f(z) \rightarrow \infty$ very rapidly along the positive real axis, which is contained in U .

Yet, all hope is not lost in the case that U is unbounded. If U is unbounded and there is $g: U \rightarrow \mathbb{C}$ such that $|f| < |g|$, and $g \rightarrow \infty$ “slowly” (so to speak) as $z \rightarrow \infty$, then it turns out that f is actually bounded in U , and we can use the maximum modulus principle to obtain other conclusions about f .

Finding such a g is part of the *Phragmén–Lindelöf* method, which we expand upon here.

Example: From the Cauchy estimates, we know that if f is entire and

$$|f(z)| \leq C(1 + |z|^{1/2}),$$

then f is constant.

Theorem (Hadamard Three-Lines Theorem): Let $a, b \in \mathbb{R}$ be fixed with $a < b$. Let $U = \{z \mid a < \operatorname{Re}(z) < b\}$. Suppose $|f(z)| < B$ for all $z \in U$ and some fixed $B < \infty$. Define

$$M(x) = \sup\{|f(z)| \mid z \in \bar{U}\}.$$

Then,

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}.$$

Proof. Suppose $M(a) = M(b) = 1$. Our task now is to show that $|f(z)| \leq 1$ for all $z \in U$. Toward this end, define

$$h_\varepsilon(z) = \frac{1}{1 + \varepsilon(z - a)}$$

for $z \in \bar{U}$. We have $|h_\varepsilon| \leq 1$ in \bar{U} , so that

$$|f(z)h_\varepsilon(z)| \leq 1$$

for all $z \in \partial U$. Furthermore, since $|1 + \varepsilon(z - a)| \geq \varepsilon|\operatorname{Im}(z)|$, we have

$$|f(z)h_\varepsilon(z)| \leq \frac{B}{\varepsilon|\operatorname{Im}(z)|}$$

for all $z \in \bar{U}$. Cut out a (closed) rectangle R from \bar{U} via the lines $\operatorname{Im}(z) = \pm \frac{B}{\varepsilon}$. Thus, along ∂R , we have $|f(z)h_\varepsilon(z)| \leq 1$, so that $|f(z)h_\varepsilon(z)| \leq 1$ on R by the maximum modulus principle.

Yet, since $|f(z)h_\varepsilon(z)| \leq \frac{B}{\varepsilon|\operatorname{Im}(z)|}$ on the entirety of \bar{U} , and $\frac{B}{\varepsilon|\operatorname{Im}(z)|} < 1$ outside R , it follows that $|fh_\varepsilon| \leq 1$ on \bar{U} , so $|f(z)h_\varepsilon(z)| \leq 1$ for all $z \in U$ and all $\varepsilon > 0$. Taking the limit as $\varepsilon \rightarrow 0$, we obtain the desired result, that $|f(z)| \leq 1$.

In the general case, we define

$$g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)},$$

where for all $M > 0$ and complex w , we have $M^w = e^{w \ln(M)}$. Then, g is entire, g is always nonzero, $\frac{1}{g}$ is bounded on \bar{U} , and has

$$\begin{aligned} |g(a + iy)| &= M(a) \\ |g(b + iy)| &= M(b), \end{aligned}$$

meaning that $\frac{f}{g}$ satisfies the previous assumptions, so that $|f/g| \leq 1$ in U . □

In the Phragmén–Lindelöf method, we seek to find a particular ε -dependent function $h_\varepsilon: U \rightarrow \mathbb{C}$ such

that the following hold:

- $|fh_\varepsilon(z)| \leq M$ for all $z \in \partial U$;
- $\lim_{\varepsilon \rightarrow 0} h_\varepsilon(z) = 1$;
- there exists a *bounded* $V \subseteq U$ such that $|fh_\varepsilon| \leq M$ on ∂V and on $U \setminus \bar{V}$.

Rouché's Theorem Problems

Example: We show that if f and g are holomorphic on a neighborhood of $B(0, 1)$, and $f(z)$ has a simple zero at $z = 0$ and no other zero in $B(0, 1)$, then $f_\varepsilon(z) = f(z) + \varepsilon g(z)$ has exactly one zero in \mathbb{D} for sufficiently small ε .

To show this, we start by showing that the conditions of Rouché's Theorem are satisfied for both $f_\varepsilon(z)$ and $f(z)$. It is clear from the fact that f has no other zeros in $B(0, 1)$ that f has no zeros on $S(0, 1)$, while we may find ε small enough such that

$$\begin{aligned} |f_\varepsilon(z)| &\geq |f(z)| - \varepsilon |g(z)| \\ &> 0 \end{aligned}$$

by selecting ε such that $\varepsilon \inf_{z \in S(0,1)} |g(z)| < \inf_{z \in S(0,1)} |f(z)|$. Thus, if we set $m_1 = \inf_{z \in S(0,1)} |f(z)|$ and $m_2 = \sup_{z \in S(0,1)} |g(z)|$, we have for $\varepsilon < \frac{m_1}{m_2}$ and all $z \in S(0, 1)$,

$$\begin{aligned} |f_\varepsilon(z) - f(z)| &\leq |\varepsilon g(z)| \\ &\leq \varepsilon m_2 \\ &< m_1 \\ &< |f(z)|, \end{aligned}$$

whence $f(z)$ and $f_\varepsilon(z)$ have the same number of zeros in \mathbb{D} counted with multiplicity.

Residue Integrals

Example: We will evaluate

$$I = \int_0^\infty \frac{1}{1+x^n} dx$$

via contour integration. Toward this end, let $f(z) = \frac{1}{1+z^n}$. We observe that $z^n + 1$ has roots for $e^{i\theta}$ at values $\theta = \frac{\pi+2\pi k}{n}$ for $0 \leq k < n$.

We take the closed contour γ_R given by

$$\oint_{\gamma_R} f(z) dz = \int_0^R f(x) dx + \int_0^{\frac{2\pi}{n}} f(Re^{i\theta}) d(Re^{i\theta}) + \int_R^0 f(xe^{i(2\pi/n)}) d(xe^{i(2\pi/n)}).$$

The left-hand side encloses the residue of f at $e^{i\pi/n}$, so by the Residue Theorem, since f has a simple pole at $e^{i\pi/n}$,

$$\begin{aligned} \oint_{\gamma_R} f(z) dz &= 2\pi i \operatorname{Res}\left(f; e^{i\pi/n}\right) \\ &= 2\pi i \frac{1}{n(e^{i\pi/n})^{n-1}} \\ &= 2\pi i \frac{1}{n\left(e^{i\frac{(n-1)\pi}{n}}\right)} \end{aligned}$$

$$\begin{aligned}
&= \frac{2\pi}{n} \frac{e^{i\pi/2}}{e^{i\pi \frac{n-1}{n}}} \\
&= \frac{2\pi}{n} e^{i\frac{\pi}{n} - \frac{\pi}{2}}.
\end{aligned}$$

We now observe that the original integral expression equals

$$\begin{aligned}
\oint_{\gamma_R} f(z) dz &= \int_0^R \frac{1}{1+x^n} dx + \int_0^{\frac{2\pi}{n}} \frac{1}{1+R^n e^{in\theta}} iR e^{i\theta} d\theta + e^{i(2\pi/n)} \int_R^0 \frac{1}{1+x^n} dx \\
&= (1 - e^{i(2\pi/n)})I + \int_0^{2\pi/n} \frac{1}{1+R^n e^{in\theta}} iR e^{i\theta} d\theta.
\end{aligned}$$

Estimating the second integral, we get for $R > 1$,

$$\left| \int_0^R \frac{iR e^{i\theta}}{1+R^n e^{in\theta}} d\theta \right| \leq \frac{\frac{2\pi}{n} R}{R^n - 1},$$

so that the integral tends to 0 as $R \rightarrow \infty$. Thus,

$$\begin{aligned}
2\pi i \operatorname{Res}(f; e^{i\pi/n}) &= \lim_{R \rightarrow \infty} \oint_{\gamma_R} f(z) dz \\
&= (1 - e^{i(2\pi/n)})I,
\end{aligned}$$

whence

$$\begin{aligned}
I &= \frac{2\pi}{n} \frac{e^{i(\frac{\pi}{n} - \frac{\pi}{2})}}{1 - e^{i(2\pi/n)}} \\
&= \frac{2\pi}{n} \frac{e^{-i\frac{\pi}{2}}}{e^{-i\pi/n} - e^{i\pi/n}} \\
&= \frac{2\pi}{n} \frac{-i}{-2i \sin(\frac{\pi}{n})} \\
&= \frac{\pi}{n \sin(\frac{\pi}{n})}.
\end{aligned}$$

Old Exams

Notation

- $U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$
- $S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$
- $\dot{U}(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$
- $A(z_0, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$