Problem (Problem 1)**:**

(a) Determine every holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $\operatorname{Re}(f(z)) = \operatorname{Re}(z)^2 - \operatorname{Im}(z)^2$.

(b) Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) := \sqrt{|\operatorname{Re}(z)\operatorname{Im}(z)|}.$$

Show that the Cauchy–Riemann equations are satisfied for f at $z = 0$, but f is not differentiable at $z = 0$.

Solution:

(a) We want to determine $f\colon \mathbb{C} \to \mathbb{C}$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

satisfies

$$u(x, y) = x^2 - y^2,$$

and the Cauchy–Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

First, we must verify that u is indeed harmonic. This follows from the fact that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= 2 \\ \frac{\partial^2 u}{\partial y^2} &= -2. \end{aligned}$$

Furthermore, we see that u is C^3, as all of the third partial derivatives are equal to zero. Therefore, a harmonic conjugate of u exists and ensures that f is holomorphic on $\mathbb{C}$. By evaluating the Cauchy–Riemann equations separately, we find that

$$\frac{\partial v}{\partial y} = 2x,$$

or $v = 2xy + K(x)$, and

$$-\frac{\partial v}{\partial x} = -2y,$$

or $v = 2xy + L(y)$. These are only in harmony when $v = 2xy + c$, where $c \in \mathbb{C}$ is a constant. Thus, we find that

$$f(x + iy) = \left(x^2 - y^2\right) + i(2xy) + c$$

is necessarily (up to a constant) unique.

(b) We write f as

$$f(x + iy) = \sqrt{|xy|}.$$

In particular, we see that $f(x + iy) = u(x, y)$ where $u(x, y) = \sqrt{|xy|}$. Evaluating the Cauchy–Riemann equations for f at 0, we have

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{\sqrt{|0 + h||0|} - \sqrt{|0||0|}}{h}$$

$$\begin{aligned}
&= 0\\
&= \frac{\partial v}{\partial y}\\
\left.\frac{\partial u}{\partial y}\right|_{(0,0)} &= \lim_{h\to 0}\frac{\sqrt{|0||0+h|}-\sqrt{|0||0|}}{h}\\
&= 0\\
&= -\frac{\partial v}{\partial x}.
\end{aligned}$$

Yet, we observe that if we let $h \to 0$ along the line $h + ih$ with $h > 0$, then

$$\begin{aligned}
f'(0,0) &= \lim_{h\to 0}\frac{\sqrt{|h|^2}-\sqrt{|0|}}{h}\\
&= \lim_{h\to 0}\frac{h}{h}\\
&= \lim_{h\to 0} 1\\
&= 1,
\end{aligned}$$

meaning that, while the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist and satisfy the Cauchy–Riemann equations at $(0,0)$, the limit defining the complex derivative doesn't exist at $(0,0)$.

Problem (Problem 2)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be a function.

(a) Suppose that f and $\overline{f}$ are both holomorphic. Show that f is constant.

(b) Suppose that f is holomorphic and $\operatorname{Re}(f)$ is constant. Show that f is constant.

Solution:

(a) Write $f(x+iy) = u(x,y) + iv(x,y)$. Since f is holomorphic, we thus get

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.
\end{aligned}$$

Now, since $\overline{f}$ is also holomorphic, we have

$$\overline{f(x+iy)} = u(x,y) - iv(x,y),$$

meaning that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y}\\
\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}
\end{aligned}$$

or that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \pm\frac{\partial v}{\partial y}\\
\frac{\partial u}{\partial y} &= \pm\frac{\partial v}{\partial x}.
\end{aligned}$$

Considering the first equation, we then get that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, or that

$$u = c_1(y)$$

$$v = d_1(x),$$

while in the second equation, we get that $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$, meaning that u and v are thus constant. Therefore, f is constant.

(b) If f is holomorphic and $\operatorname{Re}(f)$ is constant, then $i\operatorname{Im}(f) = f - \operatorname{Re}(f)$ is holomorphic as it is the difference of two holomorphic functions, so $-i\operatorname{Im}(f)$ is holomorphic as it is a constant multiple of a holomorphic function, and thus $\operatorname{Re}(f) - i\operatorname{Im}(f)$ is holomorphic as it is the sum of two holomorphic functions. This gives $\overline{f}$ is holomorphic, so f is constant.

Problem (Problem 3)**:** Let $U, V \subseteq \mathbb{C}$ be open sets, $f\colon V \to U$ holomorphic for which $\operatorname{Re}(f), \operatorname{Im}(f) \in C^2(V)$, and $u\colon U \to \mathbb{R}$ harmonic and $u \in C^2(U)$. Show that $u \circ f\colon V \to \mathbb{R}$ is a harmonic function.

Solution: We write $f(x+iy) = k(x,y) + \ell(x,y)$, so that $u \circ f(x+iy) = u(k(x,y), \ell(x,y))$. Observe then that this means $u \circ f$ is in $C^2(V)$, and that u is harmonic as a function of k and ℓ.

Using the fact that $u \circ f$ is in $C^2(V)$, we use the chain rule by taking

$$\begin{aligned}
\frac{\partial^2(u\circ f)}{\partial x^2} + \frac{\partial^2(u\circ f)}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial(u\circ f)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial(u\circ f)}{\partial y}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial k}\frac{\partial k}{\partial y} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial y}\right)\\
&= \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial x^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial x^2} + \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial y^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial y^2}\\
&\quad + \frac{\partial k}{\partial x}\left(\frac{\partial k}{\partial x}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial x}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial k}\right) + \frac{\partial \ell}{\partial x}\left(\frac{\partial k}{\partial x}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial x}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial \ell}\right)\\
&\quad + \frac{\partial k}{\partial y}\left(\frac{\partial k}{\partial y}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial y}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial k}\right) + \frac{\partial \ell}{\partial y}\left(\frac{\partial k}{\partial y}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial y}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial \ell}\right)\\
&= \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial x^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial x^2} + 2\frac{\partial^2 u}{\partial k\partial \ell}\frac{\partial k}{\partial x}\frac{\partial \ell}{\partial x} + \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial y^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial y^2} + 2\frac{\partial^2 u}{\partial k\partial \ell}\frac{\partial k}{\partial y}\frac{\partial \ell}{\partial y}\\
&\quad + \frac{\partial^2 u}{\partial k^2}\left(\frac{\partial k}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial \ell^2}\left(\frac{\partial \ell}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial k^2}\left(\frac{\partial k}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial \ell^2}\left(\frac{\partial \ell}{\partial y}\right)^2,
\end{aligned}$$

where we first used the fact that the mixed partials of u are equal by Clairaut's Theorem as u is in C^2. Since k and ℓ are C^2 real/imaginary components of a holomorphic function, they are harmonic, so by reducing via the Cauchy–Riemann equations, we find

$$\begin{aligned}
&= \frac{\partial u}{\partial k}\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2}\right) + \frac{\partial u}{\partial \ell}\left(\frac{\partial^2 \ell}{\partial x^2} + \frac{\partial^2 \ell}{\partial y^2}\right)\\
&\quad + \frac{\partial^2 u}{\partial k\partial \ell}\left(\frac{\partial \ell}{\partial y}\right)\frac{\partial \ell}{\partial x} + \frac{\partial^2 u}{\partial k\partial \ell}\left(-\frac{\partial \ell}{\partial x}\right)\frac{\partial \ell}{\partial y}\\
&\quad + \left(\frac{\partial k}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial k^2} + \frac{\partial^2 u}{\partial \ell^2}\right) + \left(\frac{\partial k}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial k^2} + \frac{\partial^2 u}{\partial \ell^2}\right)\\
&= 0,
\end{aligned}$$

so $u \circ f$ is harmonic.

Problem (Problem 4)**:** Define $g\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $g(z) = \frac{z+1}{z-1}$ and $f(z) = e^{g(z)}$.

(a) Prove that f is bounded in $\mathbb{D}$.

(b) Compute $\lim_{t \searrow 0} f(t + (1-t)a)$ for all $a \in \partial\mathbb{D} \setminus \{1\}$.

(c) Compute $\lim_{\theta \searrow 0} f\left(e^{i\theta}\right)$.

(d) Compute $\lim_{\theta \nearrow 0} f\left(e^{i\theta}\right)$.

Solution:

(a) We start by observing that

$$\begin{aligned}|f(z)| &= \left|e^{g(z)}\right| \\ &= e^{\operatorname{Re}(g(z))}.\end{aligned}$$

Therefore, to establish that $f(z)$ is bounded, we must establish an upper bound on $\operatorname{Re}(g(z))$ when $z \in \mathbb{D}$. To this end, we establish that g maps $\mathbb{D}$ to the left half-plane, $\{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$.

We start with the Cayley transform,

$$h_1(z) = \frac{z - i}{z + i},$$

which bijectively maps the upper half-plane to the unit disc. Therefore, the inverse of the Cayley transform, given by

$$\begin{aligned}h_2(z) &= \frac{iz + i}{-z + 1} \\ &= \frac{i(z+1)}{-(z-1)} \\ &= -i\frac{z+1}{z-1}\end{aligned}$$

bijectively maps the unit disc to the upper half-plane (since Möbius transformations are holomorphic bijections where defined, as follows from computing the derivative). Since

$$g(z) = ih_2(z),$$

it follows that $g(z)$ bijectively maps $\mathbb{D}$ to the left half-plane, as if $x + iy$ is such that $y > 0$, then $ix - y$ is in the left half-plane, meaning that $\operatorname{Re}(g(z)) < 0$ for all $z \in \mathbb{D}$, so f is bounded on $\mathbb{D}$.

(b) Since e^w is defined for all $w \in \mathbb{C}$, we may evaluate the limit in g, then apply the exponential to obtain our desired result. Additionally, g is continuous whenever $a \neq 1$, so it follows that

$$\lim_{t \to 0} g(t + (1-t)a) = \frac{a+1}{a-1},$$

and

$$\lim_{t \to 0} f(t + (1-t)a) = e^{\frac{a+1}{a-1}}.$$

(c) By computing $g\left(e^{i\theta}\right)$, we find that we get

$$\begin{aligned}g\left(e^{i\theta}\right) &= \frac{(\cos(\theta) + 1) + i\sin(\theta)}{(\cos(\theta) - 1 + i\sin(\theta))} \\ &= \frac{(\cos(\theta) + 1 + i\sin(\theta))(\cos(\theta) - 1 - i\sin(\theta))}{2 - 2\cos(\theta)} \\ &= \frac{\cos^2(\theta) - 1 + \sin^2(\theta) - 2i\sin(\theta)}{2 - 2\cos(\theta)} \\ &= -i\frac{\sin(\theta)}{1 - \cos(\theta)}\end{aligned}$$

$$= -i\cot(\theta/2).$$

Therefore,

$$\begin{aligned}\lim_{\theta\searrow 0} f\left(e^{i\theta}\right) &= \lim_{\theta\searrow 0} e^{-i\cot(\theta/2)}\\ &= \text{DNE},\end{aligned}$$

as $e^{i\cot(\theta/2)}$ is periodic, and $\lim_{\theta\searrow 0}\cot(\theta/2) = -\infty$.

(d) Similarly as above, we see that

$$\begin{aligned}\lim_{\theta\nearrow 0} f\left(e^{i\theta}\right) &= \lim_{\theta\nearrow 0} e^{-i\cot(\theta/2)}\\ &= \text{DNE},\end{aligned}$$

as $\lim_{\theta\nearrow 0}\cot(\theta/2) = \infty$.

Problem (Problem 5)**:** Define $f\colon \mathbb{C}\setminus 0 \to \mathbb{C}$ by

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

(a) Let C_r denote the circle of radius $r > 0$ centered at the origin.

(i) Show that $f(C_r)$ is an ellipse if $r \neq 1$.

(ii) Find the center and equation of this ellipse.

(iii) Show that $f(C_1) = [-1, 1]$.

(b) Show that $f|_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ is injective, and $f\left(\mathbb{C}\setminus\overline{\mathbb{D}}\right) = \mathbb{C}\setminus[-1,1]$.

(c) Use f to find a conformal map from $\mathbb{C}\setminus[-1,1]$ to $\mathbb{D}\setminus\{0\}$.

(d) Show that $f\left(\left\{re^{i\theta}\;\middle|\; r > 0\right\}\right)$ is a hyperbola for each $\theta\in\mathbb{R}\setminus\frac{\pi}{2}\mathbb{Z}$, and $f\left(\left\{re^{i\theta}\;\middle|\; r > 0\right\}\right)$ is a ray for each $\theta\in\frac{\pi}{2}\mathbb{Z}$.

Solution:

(a) We write

$$C_r = \left\{x + iy \;\middle|\; x^2 + y^2 = r^2\right\}.$$

(i) Letting $z = x + iy$ where $z \in C_r$ with $\neq 1$, we find that

$$\begin{aligned}f(z) &= f(x+iy)\\ &= \frac{1}{2}\left(x + iy + \frac{1}{x+iy}\right)\\ &= \frac{1}{2}\left(x + iy + \frac{x - iy}{r^2}\right)\\ &= \frac{1}{2}\left(\frac{\left(r^2+1\right)x + \left(r^2-1\right)iy}{r^2}\right)\\ &= \frac{1}{2r^2}\left(\left(r^1+1\right)x + \left(r^2-1\right)iy\right),\end{aligned}$$

meaning that if we write a scaling transformation $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $g(x,y) =$

$(\operatorname{Re}(f(x+iy)), \operatorname{Im}(f(x+iy)))$ if $(x,y) \neq (0,0)$ and $(0,0)$ otherwise, we find that

$$\begin{aligned} g(z) &= \left(\frac{r^2+1}{2r^2}x, \frac{r^2-1}{2r^2}y\right) \\ &= (s_1(r)x, s_2(r)y), \end{aligned}$$

where s_1 and s_2 are nonzero scaling factors (constants that depend on r) for x and y. Thus, $f(C_r)$ is an ellipse.

(ii) Since there are no translations in the transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that g defines, the center of $f(C_r)$ is zero. Therefore, the transformations $x \mapsto \frac{r^2+1}{2r^2}x$ and $y \mapsto \frac{r^2-1}{2r^2}y$ induce the transformation on the ellipse given by

$$x^2 + y^2 = r^2$$

maps to

$$\left(\frac{2r^2}{r^2+1}x\right)^2 + \left(\frac{2r^2}{r^2-1}y\right)^2 = r^2$$

which equals

$$\frac{x^2}{(r^2+1)^2} + \frac{y^2}{(r^2-1)^2} = \frac{1}{4r^2}.$$

(iii) We observe that in the transformation that, if $x^2 + y^2 = 1$, that since $r^2 - 1 = 0$, we have that for $z = x + iy$ contained on S^1,

$$g(z) = (x, 0).$$

Since the x coordinate in $x + iy$ ranges from -1 to 1 inclusive, we have that $f(z) = [-1, 1]$.

(b) Consider a circle C_r with $r > 1$. From above, we know that $g(C_r)$ is an ellipse in $\mathbb{R}^2$ defined by the equation

$$\frac{x^2}{(r^2+1)^2} + \frac{y^2}{(r^2-1)^2} = \frac{1}{4r^2}.$$

In particular, since $r > 1$, the maps $r \mapsto r^2 - 1$ and $r \mapsto r^2 + 1$ are injective, so the ellipse defined $f(C_r) \subseteq \mathbb{C}$ is uniquely defined. It remains to be shown that if there is $w \in \mathbb{C} \setminus [-1, 1]$, there is a unique $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $f(z) = w$. Toward this end, we simply compute z, yielding

$$\begin{aligned} w &= \frac{1}{2}\left(z + \frac{1}{z}\right) \\ z^2 - 2wz &= -1 \\ (z-w)^2 &= w^2 - 1 \\ z &= w + \sqrt{w^2-1}. \end{aligned}$$

Notice that the square root has branch points at -1 and 1, meaning that it is not well-defined along the line $[-1, 1]$. Else, we may take the standard branch of the logarithm that defines the square root function, so that the square root is well-defined.

(c) We observe that $f|_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ is conformally equivalent to $\mathbb{C}\setminus[-1,1]$, so there is a well-defined holomorphic inverse, which we call g, where $g \colon \mathbb{C}\setminus[-1,1] \to \mathbb{C}\setminus\overline{\mathbb{D}}$. We observe that, for $re^{i\theta} \in \mathbb{C}\setminus\mathbb{D}$, the function $q(z) = \frac{1}{z}$ is holomorphic, and

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta},$$

meaning that $\frac{1}{z}$ is a bijection to $\mathbb{D} \setminus \{0\}$. In particular, it has the holomorphic inverse $\frac{1}{z}\colon \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$. Therefore, we have $h\colon \mathbb{C} \setminus [-1,1] \to \mathbb{D} \setminus \{0\}$ given by $\frac{1}{g}$ where g is defined as above.

(d) Let $\theta \in \mathbb{R} \setminus \frac{\pi}{2}\mathbb{Z}$. Then,

$$\begin{aligned} f\left(re^{i\theta}\right) &= \frac{1}{2}\left(r\cos(\theta) + ir\sin(\theta) + \frac{1}{r\cos(\theta) + ir\sin(\theta)}\right) \\ &= \frac{1}{2}\left(r\cos(\theta) + ir\sin(\theta)\frac{\cos(\theta) - i\sin(\theta)}{r}\right) \\ &= \frac{1}{2}\left(\frac{r^2\cos(\theta) + ir^2\sin(\theta) + \cos(\theta) - i\sin(\theta)}{r}\right) \\ &= \frac{1}{2}\left(\cos(\theta)\frac{\left(r^2+1\right)}{r} + i\sin(\theta)\frac{r^2-1}{r}\right). \end{aligned}$$

This yields a curve in $\mathbb{C} \cong \mathbb{R}^2$ parametrized by

$$\gamma(r) = \left(\cos(\theta)\frac{r^2+1}{2r}, \sin(\theta)\frac{r^2-1}{2r}\right).$$

If we let x and y be as in those two coordinates, we desire to find a relationship between x and y in the form of a hyperbola. Toward this end, we examine

$$\begin{aligned} \cos^2(\theta)\frac{\left(r^2+1\right)^2}{4r^2} - \sin^2(\theta)\frac{\left(r^2-1\right)^2}{4r^2} &= \frac{\cos^2(\theta)\left(r^4+2r^2+1\right) - \sin^2(\theta)\left(r^4-2r^2+1\right)}{4r^4} \\ &= \frac{\left(\cos^2(\theta) - \sin^2(\theta)\right)4r^2}{4r^2} \\ &= \cos(2\theta) \end{aligned}$$

meaning that, since θ is fixed and is such that $\cos(2\theta) \neq 0$, these coordinates for $\gamma(r)$ do indeed satisfy

$$x^2 - y^2 = 1,$$

so that $\operatorname{Im}(\gamma)$ is a hyperbola.

If $\theta \in \frac{\pi}{2}\mathbb{Z}$, then we have two cases.

- If $\theta = \pi k$ for some $k \in \mathbb{Z}$, then $\cos(\theta) = (-1)^k$ and $\sin(\theta) = 0$, so that

$$\begin{aligned} f\left(re^{i\theta}\right) &= \frac{1}{2}\left(r(-1)^k + \frac{1}{r(-1)^k}\right) \\ &= \frac{(-1)^k}{2}\frac{r^2+1}{r} \end{aligned}$$

  which is a ray in $\mathbb{C}$ so long as $r > 0$.
- Similarly, if $\theta = \frac{\pi}{2} + \pi k$ for some $k \in \mathbb{Z}$, then $\sin(\theta) = (-1)^k$ and $\cos(\theta) = 0$, so that

$$\begin{aligned} f\left(re^{i\theta}\right) &= \frac{1}{2}\left(ir(-1)^k + \frac{1}{ir(-1)^k}\right) \\ &= i\frac{(-1)^k}{2}\frac{r^2-1}{r}, \end{aligned}$$

  which is yet again a ray in $\mathbb{C}$ so long as $r > 0$.