# Normed Vector Spaces

## Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **vector space** over $\mathbb{F}$ is a nonempty set $V$ equipped with two operations: vector addition and scalar multiplication.

$$V \times V \xrightarrow{+} V$$
$$(v, w) \mapsto v + w \qquad \text{Vector Addition}$$
$$F \times V \to V$$
$$(\alpha, v) \mapsto \alpha v \qquad \text{Scalar Multiplication}$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

(i) $u + (v + w) = (u + v) + w$

(ii) $\exists 0_v \in V$ with $\forall v \in V,\ 0_v + v = v + 0_v = v$

(iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$

(iv) $\forall v, w \in V,\ v + w = w + v$

(v) $\alpha(v + w) = \alpha v + \alpha w,\ (\alpha + \beta)v = \alpha v + \beta v$

(vi) $\alpha(\beta w) = (\alpha\beta)w$

(vii) $1 \cdot v = v$

**Remarks:**

(a) $0_v$ is unique and known as the zero vector.

(b) The vector $w$ in (iii) is unique, and denoted $-v$.

(c) $0 \cdot v = 0_v$

(d) $(-1) \cdot v = -v$

(e) Property (iv) follows from all the other axioms.

(f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

## Subspaces

Let $V$ be a vector space over $\mathbb{F}$. A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

(i) $w \in W, \alpha \in \mathbb{F} \to \alpha w \in W$.

(ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

**Remark:** $0_v$ is always a member of any subspace; a subspace is also a vector space.

#### Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of $V$, then, $\bigcap W_i$ is a subspace of $V$.

#### Proposition: Union of Subspaces

It is not the case that the union of subspaces of $V$ also a subspace. For example, consider $\mathbb{R}^2$ with the traditional vector space operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$$
$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

If $W_1, W_2 \in V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

**Generated Subspaces**

Let $S \subseteq V$ be any subset of a vector space $V$. Then,

$$\operatorname{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

**Remarks:**

- $\operatorname{span}(S) \subseteq V$ is a subspace.
- $\operatorname{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\operatorname{span}(S)$ is the "smallest" subspace containing $S$, or the subspace generated by $S$.

**Proposition: Quotient Group on Vector Space**

Let $V$ be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

(1) $\sim_W$ is an equivalence relation.

(2) If $[v]_W$ denotes the equivalence class of $v$, then $[v]_W = v + W = \{v + w | w \in W\}$.

(3) $V/W := \{[v]_W | v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

**Proof of (1):**

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

**Proof of (2):**

$$\begin{aligned}
[v]_W &= \{u \in V \mid u \sim_W v\} \\
&= \{u \in V \mid u - v \in W\} \\
&= \{u \in V \mid u = v + w \text{ some } w \in W\} \\
&= \{v + w \mid w \in W\} \\
&= v + W
\end{aligned}$$

**Proof of (3):** Prove that the operation is well-defined.

## Bases

Let $V$ be a vector space and $S \subseteq V$ be a subset.

(1) $S$ is said to be spanning for $V$ if $\operatorname{span}(S) = V$.

(2) $S$ is linearly independent if, for $\sum_{j=1}^{n} \alpha_j v_j = 0_v$ with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

(3) $S$ is a basis for $V$ if $S$ is linearly independent and spanning for $V$.

**Proposition: Existence of Basis**

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that $B$ is a basis and $B \supseteq B_0$.

**Background:** A relation on a set $X$ is a subset $R \subseteq X \times X$. If $R$ is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \to x \sim z$), and antisymmetric ($x \sim y, y \sim x \to x = y$), then $R$ is an ordering, and we write $x \leq y$.

If $\leq$ is an ordering of $X$ such that $\forall x, y \in X,\ x \leq y$ or $y \leq x$, then $\leq$ is a total (or linear) ordering.

Let $\leq$ be an ordering of $X$, let $Y \subseteq X$. An upper bound for $Y$ is an element $u \in X$ such that $y \leq u\ \forall y \in Y$. A maximal element in $X$ is an element $m \in X$ such that $x \in X,\ x \geq m \to x = m$.

**Example:** $\mathbb{N}$ under the division ordering defines $a \leq b \Leftrightarrow a|b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of $A$ can be divided by 9 and 12). Meanwhile, $\mathbb{N}$ itself has no maximal elements.

This leads us to ask: given an ordered set, $(X, \leq)$, does $X$ admit maximal elements.

**Zorn's Lemma (or Axiom):** Let $(X, \leq)$ be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in $X$. Then, $X$ admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

**Proof:** Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with $D$ linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that $X$ has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i \in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \dots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \dots, v_n \in D_j$ because $Y$ is totally ordered. However, by definition, $D_j$ is a linearly independent set — therefore, $\alpha_k = 0$. Thus, $D$ is linearly independent.

Since $D$ is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. $D$ is also an upper bound for $Y$. So, by Zorn's Lemma, $X$ has a maximal element, $B$.

So, $B_0 \subseteq B \subseteq V$, $B$ is independent, and $B$ is maximal in $X$. We claim that $B$ is a basis for $V$. Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and $B'$ is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \dots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.
- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \text{span}(B)$. $\perp$

Thus, we have a linearly independent set, $B'$, with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of $B$. Therefore, $\text{span}(B) = V$, and $B$ is a basis for $V$.

## Examples: Vector Spaces

(1) $n$-Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y+n \end{pmatrix} = \begin{pmatrix} x_1 + y + 1 \\ \vdots \\ x_n + y + n \end{pmatrix}$$
$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$
$$B = \{e_1, \dots, e_n\}$$

where $e_i$ denotes the unit vector at position $i$.

(2) $m \times n$ Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$
$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$
$$\alpha(a_{ij}) = (\alpha a_{ij})$$
$$B = \{e_{ij}\}$$

where $e_{ij}$ denotes a matrix of 0 everywhere except column $i$ and row $j$.

(3) Functions with domain $\Omega$:

$$\begin{aligned}\mathcal{F}(\Omega,\mathbb{F}) &= \{f \mid f:\Omega\to\mathbb{F}\}\\ (f+g)(x) &= f(x)+g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

(4) Bounded functions with domain $\Omega$:

$$\begin{aligned}\ell_\infty(\Omega,\mathbb{F}) &= \{f\in\mathcal{F}(\Omega,\mathbb{F}) \mid \|f\|_u \leq \infty\}\\ \|f\|_u &= \sup_{x\in\Omega}|f(x)|\end{aligned}$$

Exercises:

- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha|\|f\|_u$
- Positive Definite: $\|f\|_u = 0 \Rightarrow f = \mathbb{0}$

**Proof of Triangle Inequality:** Given $x\in\Omega$,

$$\begin{aligned}|(f+g)(x)| &= |f(x)+g(x)|\\ &\leq |f(x)|+|g(x)|\\ &\leq \|f\|_u+\|g\|_u\end{aligned}$$

Therefore,

$$\begin{aligned}\sup|(f+g)(x)| &\leq \|f\|_u+\|g\|_u\\ \|f+g\|_u &\leq \|f\|_u+\|g\|_u\end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a,b],\mathbb{F}) = \{f:[a,b]\to\mathbb{F} \mid f\text{ continuous}\}$$

Check that $C([a,b],\mathbb{F})\subseteq\ell_\infty([a,b],\mathbb{F})$ is a subspace.

(6) Let $f:[a,b]\to\mathbb{R}$ be any function. Let $\mathcal{P}: a=x_0<x_1<x_2<\cdots<x_n=b$.

$$\begin{aligned}\operatorname{var}(f;\mathcal{P}) &:= \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\\ \operatorname{var}(f) &= \sup_{\mathcal{P}}\operatorname{var}(f;\mathcal{P})\\ \mathrm{BV}([a,b]) &= \{f:[a,b]\to\mathbb{R} \mid \operatorname{var}(f)<\infty\}\\ \|f\|_{\mathrm{BV}} &= |f(a)|+\operatorname{var}(f)\end{aligned}$$

$\mathrm{BV}([a,b])$ is a vector space.

**Question:** Is $\mathbb{1}_{\mathbb{Q}}\in\mathrm{BV}([0,1])$?

(7) Suppose $K\subseteq V$ is a *convex* subset of a vector space: $v,w\in K, t\in[0,1]\Rightarrow(1-t)v+tw\in K$. Let $\operatorname{Aff}(K)=\{f:K\to\mathbb{R} \mid f\text{ is affine}\}$, where $f$ is affine if $\forall v,w\in K, t\in[0,1], f((1-t)v+tw)=(1-t)f(v)+tf(w)$.

**Exercise:** Show that $\operatorname{Aff}(K)\subseteq\mathcal{F}(K,\mathbb{R})$ is a subspace.

(8) Let $S$ be defined as

$$S=\{(a_k)_{k=1}^{\infty} \mid a_k\in\mathbb{F}\}.$$

Under pointwise operations, $S$ is a vector space.

$$\begin{aligned}(a_k)_k+(b_k)_k &= (a_k+b_k)_k\\ \alpha(a_k)_k &= (\alpha a_k)_k\end{aligned}$$

**Note 1:** $S=\mathcal{F}(\mathbb{N},\mathbb{F})$.

**Note 2:** $c_{00}\subseteq\ell_1\subseteq c_0\subseteq c\subseteq\ell_\infty\subseteq S$.

- $c_{00}=\{(a_k)_k \mid \text{finitely many } a_k\neq 0\}$
- $c_0=\{(a_k)k \mid (a_k)_k\to 0\}$

- $c = \{(a_k)_k \mid (a_k)_k \to a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_u < \infty\}$
- $\ell_1 = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k| = a < \infty\right\}$

(9) $C_C(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \to \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ continuous, } \lim_{x\to\pm\infty} f(x) = 0\}$

(10) Let $S$ be any non-empty set.

$$\mathbb{F}(S) := \{f : S \to \mathbb{F} \mid f \text{ finitely supported}\}$$
$$\operatorname{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \to \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t\in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\operatorname{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^{n} f(t_k)e_{t_k} \in \operatorname{span}(\xi)$. Therefore, $\xi$ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^{n} \alpha_{t_k} e_{t_k} = \mathbb{0}$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\left(\sum_{k=1}^{\alpha_{t_k}} e_{t_k}\right) = \mathbb{0}(t_1)$$
$$\alpha_{t_1} = 0.$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \dots, n$. Therefore, $\xi$ is linearly independent. Since $\xi$ is linearly independent and spanning, $\xi$ forms a basis for $\mathbb{F}(S)$.

**Note:** The free vector space, $\mathbb{F}(S)$, displays the universal property.

> There are functions $\iota : S \to \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \to V$ for $V$ a vector space over $\mathbb{F}$, $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \to V$ such that $\iota \circ T_\varphi = \varphi$.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbb{F}(S) \arrow[d, "T_\varphi"] \\ & V \end{tikzcd}$$

**Proof:** Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^{n} f(t_k)e_{t_k}$, where $\operatorname{supp}(f) = \{t_1, \dots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^{n} f(t_k)\varphi(t_k)$$

**Exercise:** Show $T_\varphi$ is linear and unique.

**Exercise 2:** Suppose $V$ is a vector space over $\mathbb{F}$ with basis $B$. Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists\, T : V \to W$ such that $T$ is bijective and linear.

## Normed Spaces

To every vector $v \in V$, we want to assign a length to $v$, $\|v\|$.

A **norm** on a vector space $V$ is a map

$$\|\cdot\| : V \to \mathbb{R}^+$$
$$v \mapsto \|v\| \geq 0$$

such that

(i) Homogeneity: $\|\alpha v\| = |\alpha|\|v\|$

(ii) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

(iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = \mathbb{0}_V$.

If $p : V \to \mathbb{R}^+$ satisfies (i) and (ii), then $p$ is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\|v\| \leq c_1\|v\|'$$
$$\|v\|' \leq c_2\|v\|$$

**Note:** On $\mathbb{R}^n$, all norms are equivalent.

**Exercise:** If $p$ is any seminorm on $V$, then $|p(v) - p(w)| \leq p(v - w)$.

**Notation:** If $V$ is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.

**Examples of Normed Spaces**

(1) Given $V = \mathbb{F}^n$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have different norms:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$
$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$$
$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}.$$

In general, for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}.$$

**Exercise:** Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

We want to show that $\|\cdot\|_p$ defines a norm for $1 \leq p < \infty$. If $1 \leq p < \infty$, its conjugate index $q \in [1, \infty]$ whereby $\frac{1}{p} + \frac{1}{q} = 1$. For example, if $p = 1$, then $q = \infty$, and if $p = \infty$, then $q = 1$.

**Lemma 1:** For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $f : [0, \infty) \to \mathbb{R}$, $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then, $f(t) \geq 0$ for all $t \geq 0$.

**Proof 1:** We can see that $f'(t) = t^{p-1} - 1$. Then, $f'(t) = 0$ at $t = 1$; $f'(t) > 0$ for $t > 1$ and $f'(t) < 0$ for $t \in [0, 1)$.

So, since $f(t) \geq f(1)$ for all $t \geq 0$, and $f(1) = 0$, $f(t) \geq 0$ for all $t \geq 0$.

**Lemma 2:** For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $z, y \geq 0$, $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

**Proof 2:** We know from Lemma 1, $t \leq \frac{1}{p}t^p + \frac{1}{q}$. Multiply by $y^q$ to get

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Set $t = xy^{1-q}$. Then,

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq} y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $p - pq = -q$, so

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

With these two lemmas in mind, we get two important inequalities.

**Hölder's Inequality:** For $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. Then, for $x, y \in \mathbb{F}^n$,

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \|x\|_p \|y\|_q.$$

**Proof of Hölder's Inequality:** For $p = 1$, the solution is as follows:

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j y_j\right| &\leq \sum_{j=1}^{n} |x_j||y_j| \\ &\leq \sum_{j=1}^{n} |x_j|\|y\|_\infty \\ &= \|x\|_q \|y\|_\infty,\end{aligned}$$

and similarly for $p = \infty, q = 1$.

For $1 < p < \infty$, assume $\|x\|_p = \|y\|_q = 1$.

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j y_j\right| &\leq \sum_{j=1}^{\infty} |x_j||y_j| \\ &\leq \sum_{j=1}^{n} \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q\right) \\ &= \frac{1}{p}\left(\sum_{j=1}^{n} |x_j|^p\right) + \frac{1}{q}\left(\sum_{j=1}^{n} |y_j|^q\right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1.\end{aligned}$$

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = \mathbb{0}_{\mathbb{F}}$ or $y = \mathbb{0}_{\mathbb{F}}$, the inequality still holds.

Assume $\|x\|_p \neq 0$, $\|y\|_p \neq 0$. Set

$$\begin{aligned}x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_p}.\end{aligned}$$

It can be verified that $\|x'\|_p = 1 = \|y'\|_q$. Therefore,

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j' y_j'\right| &\leq 1 \\ \left|\sum_{j=1}^{n} \frac{x_j}{\|x\|_p} \frac{y_j}{\|y\|_q}\right| &\leq 1 \\ \left|\sum_{j=1}^{n} x_j y_j\right| &\leq \|x\|_p \|y\|_q\end{aligned}$$

**Minkowski's Inequality:** Given $x, y \in \mathbb{F}^n$, $1 \leq p \leq \infty$, $\frac{1}{p} = \frac{1}{q} = 1$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

**Proof of Minkowski's Inequality:** We can verify for $p = 1, q = \infty$, and vice versa.

Assume $1 < p < \infty$. Then,

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{j=1}^{n} |x_j + y_j|^p \\
&= \sum_{j=1}^{\infty} |x_j + y_j||x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^{\infty} |x_j||x_j + y_j|^{p-1} + \sum_{j=1}^{n} |y_j||x_j + y_j|^{p-1} \\
&\leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} && \text{Hölder's Inequality} \\
&= \|x\|_p \|x+y\|_p^{p/q} + \|y\|_p \|x+y\|_p^{p/q} \\
&= \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p-1}
\end{aligned}$$

Divide by $\|x+y\|_p^{p-1}$ to get desired inequality.

(2) $\ell_\infty(\Omega, \mathbb{F})$ with $\|\cdot\|_u$. This includes subspaces that inherit the norm, such as

$$\begin{aligned}
C([a,b]) &\subseteq \ell_\infty(\Omega) \\
\ell_\infty(\mathbb{R}) \supseteq C_0(\mathbb{R}) &\supseteq C_C(\mathbb{R})
\end{aligned}$$

**Exercise:** Show that $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ is a subspace.

(3) $\Omega = \mathbb{N}$, $\ell_\infty = \ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$. Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \leq \ell_\infty.$$

(4) $\ell_1$ with $\|\cdot\|_1$,

$$\|(a_k)_k\|_1 = \sum_{k=1}^{n} |a_k|.$$

(5) $C([a,b])$ with

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

(6) Let $1 \leq p < \infty$.

$$\ell_p = \left\{ (a_k)_{k=1}^{\infty} \mid \sum_{k=1}^{\infty} |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\begin{aligned}
\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} &= \left\| \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \right\|_{\ell_p^n} \\
&= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
&\leq \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\| + \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^\infty} \\
&\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p.
\end{aligned}$$

Taking the limit as $n \to \infty$ (by the definition of an infinite series), we find that $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$.

(7) $\mathrm{BV}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid \mathrm{Var}(f) < \infty\}$ with the norm $\|f\|_{\mathrm{BV}} = |f(a)| + \mathrm{Var}(f)$ is a normed space:

$$\begin{aligned} \|f\|_{\mathrm{BV}} &= 0 \\ |f(a)| &= 0 \\ \mathrm{Var}(f) &= 0 \end{aligned}$$

given $t \in (a,b]$, look at the partition $a < t \leq b$. Then,

$$\begin{aligned} \mathrm{Var}(f) &\geq |f(t) - f(a)| + |f(b) - f(t)| \\ f(t) &= 0 \\ f &= \mathbb{0}_f. \end{aligned}$$

(8) $\mathbb{M}_{m,n}(\mathbb{F})$ with

$$\|a\|_{\mathrm{op}} = \sup_{\|\xi\|_{\ell_2^n} \leq 1} \|a\xi\|_{\ell_2^m}$$

is a normed vector space. If $\|a\|_{\mathrm{op}} = 0$, then

$$ae_j = 0 \qquad \forall j \in \{1, \dots, n\}.$$

take the dot product with $i \neq j$

$$\begin{aligned} ae_j \cdot e_i &= a_{ij} \\ &= 0 \end{aligned}$$

so $a_{ij} = 0$ for all $a_{ij}$, so $a$ is the $\mathbb{0}$ matrix.

(9) Let $V, W$ be vector spaces over $\mathbb{F}$. Then, $\mathcal{L}(V,W) = \{T \mid T : V \to W \text{ linear}\}$, where $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

$\mathcal{L}(V,W)$ is a vector space with operations

$$\begin{aligned} (T+S)(v) &= T(v) + S(v) \\ (\alpha T)(v) &= \alpha T(v). \end{aligned}$$

**Notation:** $\mathcal{L}(V) := \mathcal{L}(V,V)$ is all linear operators on $V$. $\mathcal{L}(V,\mathbb{F}) = V'$ is all linear functionals.

Suppose $V$ and $W$ are normed vector spaces. If $T : V \to W$, set

$$\begin{aligned} \|T\|_{\mathrm{op}} &:= \sup_{\|v\|_v \leq 1} \|T(v)\|_W, \\ \mathbb{B}(V,W) &= \{T \in \mathcal{L}(V,W) \mid \|T\|_{op} \leq \infty\}, \end{aligned}$$

where $\mathbb{B}(V,W)$ is referred to as the set of all bounded linear maps from $V$ to $W$. $\mathbb{B}(V,W)$ with $\|\cdot\|_{\mathrm{op}}$ is a normed space.

- Homogeneity:

$$\begin{aligned} \|\alpha T\|_{[op]} &= \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W \\ &= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W \\ &= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W \\ &= |\alpha| \|T\|_{\mathrm{op}}. \end{aligned}$$

- Triangle Inequality: for $\|v\|_V \leq 1$,

$$\begin{aligned} \|\left(T+S\right)(v)\|_W &= \|T(v) + S(v)\|_W \\ &\leq \|T(v)\|_W + \|S(v)\|_W \\ &\leq \|T\|_{\mathrm{op}} + \|S\|_{\mathrm{op}} \end{aligned}$$

  so

$$\begin{aligned} \|T+S\|_{\mathrm{op}} &= \sup_{\|v\| \leq 1} \|T + S(v)\| \\ &\leq \|T\|_{\mathrm{op}} + \|S\|_{\mathrm{op}} \end{aligned}$$

- Positive Definite: If $\|T\|_{\text{op}} = 0$, then $T(v) = 0$ for all $v \in V$, $\|v\| \leq 1$.

  Let $v \in V$, $v \neq 0$. Then, $\frac{v}{\|v\|} \in B_V$.

$$\begin{aligned} T\left(\frac{v}{\|v\|}\right) &= 0 \\ \frac{1}{\|v\|}T(v) &= 0 \\ T(v) &= 0 \end{aligned}$$

**Special Cases:** $\mathbb{B}(V) = \mathbb{B}(V, V)$, $V^* = \mathbb{B}(V, \mathbb{F})$.

**Exercise:** $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$.

(10) Inner Product Spaces (expanded upon below).

**Inner Product Spaces**

An inner product on a vector space $V$ is a pairing

$$V \times V \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{F}$$

that satisfies

(i) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$, $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$.

(ii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

(iii) $\langle v, v\rangle \geq 0$.

(iv) If $\langle v, v\rangle = 0$, then $v = 0$.

The pair $(V, \langle\cdot,\cdot\rangle)$ is known as an inner product space.

**Remarks:** $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$, $\langle v, \alpha w\rangle = \overline{\alpha}\langle v, w\rangle$.

If $\langle\cdot,\cdot\rangle$ is an inner product on a linear space $V$, then set

$$\|v\|_2 := \langle v, v\rangle^{1/2}.$$

**Exercise:** $\|\alpha v\|_2 = |\alpha|\|v_2\|$, $\|v\|_2 = 0 \Rightarrow v = 0$.

$v, w \in (V, \langle, \cdot, \cdot\rangle)$ are *orthogonal* if $\langle v, w\rangle = 0$.

The Pythagoran theorem states that for $v_1, \dots, v_n \in V$ mutually orthogonal, then

$$\left\|\sum_{i=1}^{n} v_i\right\|^2 = \sum_{j=1}^{n}\|v_j\|^2.$$

For two vectors $v, w \in V$, $P_w(v) = \frac{\langle v,w\rangle}{\langle w,w\rangle}w$.

**Exercise:** Check that $\langle P_w(v), v - P_w(v)\rangle$, meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

**Cauchy-Schwarz Inequality:** In any inner product space,

$$|\langle v, w\rangle| \leq \|v\| \cdot \|w\|.$$

**Proof of Cauchy-Schwarz:** From the exercise,

$$\begin{aligned} \|v\| &\geq \|P_w(v)\| \\ \|v\| &\geq \left\|\frac{\langle v, w\rangle}{\langle w, w\rangle}w\right\| \\ &= \frac{|\langle v, w\rangle|}{\|w\|^2}\|w\| \end{aligned}$$

therefore,

$$\|v\|\|w\| \geq |\langle v, w\rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

**Proof of Triangle Inequality:**

$$\begin{aligned}
\|v+w\|_2^2 &= \langle v+w, v+w\rangle\\
&= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle\\
&= \|v\|^2 + \|w\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle}\\
&= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v,w\rangle\\
&\leq \|v\|^2 + \|w\|^2 + 2|\langle v,w\rangle|\\
&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| && \text{Cauchy-Schwarz Inequality}\\
&= (\|v\| + \|w\|)^2.
\end{aligned}$$

Take square roots on both sides.

(1) $\ell_2^n = \mathbb{F}^n$ with

$$\left\langle \begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix}, \begin{bmatrix} y_1\\ \vdots\\ y_n\end{bmatrix}\right\rangle = \sum_{i=1}^n x_i\overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left|\sum_{j=1}^n x_j\overline{y_j}\right| \leq \left(\sum_{j=1}^n |x_j|^2\right)^{1/2}\left(\sum_{j=1}^n |y_j|^2\right)^{1/2}.$$

(2) $\ell_2$ with

$$\left\langle (a_j)_j, (b_j)_j\right\rangle = \sum_{j=1}^\infty a_j\overline{b}_j.$$

We can see that for any finite $n$, the Cauchy-Schwarz inequality in $\ell_2^n$ states

$$\begin{aligned}
\left|\sum_{j=1}^n a_j\overline{b_j}\right| &\leq \left(\sum_{j=1}^n |a_j|^2\right)^{1/2}\left(\sum_{j=1}^n |b_j|^2\right)^{1/2}\\
&\leq \left(\sum_{j=1}^\infty |a_j|^2\right)^{1/2}\left(\sum_{j=1}^\infty |b_j|^2\right)^{1/2}.
\end{aligned}$$

Taking the limit as $n\to\infty$, we see that $\left\langle (a_j)_j, (b_j)_j\right\rangle$ is convergent.

(3) $C([a,b])$ with

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

(4) Let $V = \mathbb{M}_n(\mathbb{C})$.

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let $\mathrm{Tr}:\mathbb{M}_n(\mathbb{C})\to\mathbb{C}$, $\mathrm{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$.

- $\mathrm{Tr}(I_n) = n$
- $\mathrm{Tr}(a+\alpha b) = \mathrm{Tr}(a) + \alpha\mathrm{Tr}(b)$
- $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$

Then, if $\text{Tr}(a^*a) = 0$, then $a = \mathbb{0}_{\mathbb{M}_n}$.

$$\begin{aligned}
a^*a &= (\overline{a_{ji}})_{i,j}(a_{ij})_{i,j}\\
&= \left(\sum_{k=1}^{n} \overline{ki}a_{kj}\right)_{i,j}\\
\text{Tr}(a^*a) &= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{a_{ki}}a_{ki}\\
&= \sum_{i,k=1}^{n} |a_{ki}|^2\\
&= \sum_{i,j=1}^{n} |a_{ij}|^2.
\end{aligned}$$

If $\text{Tr}(a^*a) = 0$, then $a_{ij} = 0$ for all $i, j$.

We define

$$\langle a, b\rangle_{\text{HS}} = \text{Tr}(b^*a).$$

(i) $(b_1 + b_2)^* = b_1^* + b_2^*$

(ii) $(\alpha b)^* = \overline{\alpha} b^*$

(iii) $(b_1 b_2)^* = b_2^* b_1^*$

(iv) $b^{**} = b$

The norm is defined as

$$\begin{aligned}
\|a\|_{\text{HS}} &= \langle a, a\rangle^{1/2}\\
&= \text{Tr}(a^*a)^{1/2}\\
&= \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}
\end{aligned}$$

## Metric Spaces

We looked at normed spaces, where we attach a length $\|v\|$ to very vector $v$. We can also speak of the distance between two vectors, defined as $d(v, w) = \|v - w\|$.

Notice that the following hold:

- $d(v, w) \geq 0$
- $$\begin{aligned}
d(v, w) &= \|v - w\|\\
&= \|(-1)(w - v)\|\\
&= |-1|\,\|w - v\|\\
&= \|w - v\|
\end{aligned}$$
- $$\begin{aligned}
d(u, w) &= \|u - w\|\\
&= \|u - v + v - w\|\\
&\leq \|u - v\| + \|v - w\|\\
&= d(u, v) + d(v, w).
\end{aligned}$$
- $d(v, v) = \|v - v\| = 0$. If $d(v, w) = 0$, then $\|v - w\| = 0$, so $v - w = \mathbb{0}$, so $v = w$.

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely $(\mathbb{R}, |\cdot|)$. We will expand these concepts to all metric spaces.

## Definition of a Metric Space

Let $X$ be a non-empty set. A **metric** on $X$ is a map

$$\begin{aligned} d : X \times X &\to \mathbb{R}^+ \\ (x, y) &\mapsto d(x, y) \geq 0 \end{aligned}$$

such that

(i) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.

(ii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(iii) Zero Distance: $d(x, x) = 0$

(iv) Definite: $d(x, y) = 0 \Rightarrow x = y$

If $d$ satisfies (i), (ii), and (iii), then $d$ is called a semi-metric. If $d$ satisfies (iv) as well, then $d$ is a metric.

If $d$ is a (semi-)metric on $X$, the pair $(X, d)$ is called a (semi-)metric space.

Two metrics, $d$ and $\rho$, on $X$, are equivalent if $\exists c_1, c_2 \geq 0$ such that $d(x, y) \leq c_1\rho(x, y)$ and $\rho(x, y) \leq c_2 d(x, y)$ for all $x, y$.

## Examples of Metric Spaces

(1) Discrete Metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for $X$ any set.

(2) Hamming distance: between two bit strings of equal length. Let

$$\begin{aligned} X &= \{0, 1\}^n \\ &= \{0, 1\} \underbrace{\times \cdots \times}_{n \text{ times}} \{0, 1\} \\ d_H((x_j)_1^n, (y_j)_1^n) &= \left|\{j \mid x_j \neq y_j\}\right|. \end{aligned}$$

(3) Any normed space $(V, \|\cdot\|)$ is a metric space.

$$d(v, w) = \|v - w\|.$$

**Exercise:** Show that if two norms are equivalent, their induced metrics are equivalent.

(4) Subset of Metric Space: If $(X, d)$ is a metric space, and $Y \subseteq X$ is non-empty. Then, $(Y, d)$ is a metric space.

(5) Paris metric: let $(X, \rho)$ be a metric space. Let $p \in X$ be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

(6) Bounded metric: Let $\rho$ be a (semi-)metric on $X$. Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We claim that $d$ is a (semi-)metric. Notice that $0 \leq d(x, y) \leq 1$.

**Proof:** Clearly, $d(x, y) = d(y, x)$. Additionally, $d(x, x) = 0$. If $d(x, y) = 0$ and $\rho$ is a metric, then $\rho(x, y) = 0$, so $x = y$.

To show the triangle inequality, we examine the function

$$\begin{aligned} f(t) &= \frac{t}{1 + t} \\ f'(t) &= \frac{1}{(1 + t)^2} > 0. \end{aligned}$$

Since $\rho$ satisfies the triangle inequality, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Apply $f$ on both sides. Then,

$$\begin{aligned}
\underbrace{\frac{\rho(x,z)}{1+\rho(x,z)}}_{d(x,z)} &\leq \frac{\rho(x,y)+\rho(y,z)}{1+(\rho(x,y)+\rho(y,z))}\\
&= \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)}\\
&\leq \underbrace{\frac{\rho(x,y)}{1+\rho(x,y)}}_{d(x,y)} + \underbrace{\frac{\rho(y,z)}{1+\rho(y,z)}}_{d(y,z)}.
\end{aligned}$$

(7) If $d_1, \dots, d_n$ are metrics on $X$, $c_1, \dots, c_n \geq 0$. Then,

$$d(x,y) = \sum_{k=1}^{n} c_k d_k(x,y)$$

is a metric.

(8) Let $\{\rho_k\}_{k=1}^{\infty}$ be a family of semi-metrics. Assume the family is separating — for all $x \neq y$, there exists $k$ such that $\rho_k(x,y) \neq 0$.

Let $d_k$ be defined as

$$d_k(x,y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}.$$

Note that $\{d_k\}_{k=1}^{\infty}$ is also separating.

Then,

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x,y)$$

is a metric.

We will now define the Frechet Metric using this method. Let $X = C(\mathbb{R})$. For each $k = 1,2,3,\dots$, set $p_k(f) = \sup_{x \in [-k,k]} |f(x)|$.

We can verify that $p_k$ defines a seminorm. We can then check $\rho_k(f,g) = p_k(f-g)$ is a semi-metric.

We claim that $\{\rho_k\}$ is separating: if $f \neq g$, then there exists $x_0 \in \mathbb{R}$ with $f(x_0) \neq g(x_0)$. Since $f$ and $g$ are continuous, there is a neighborhood $[x_0 - \delta, x_0 + \delta]$ such that $f(x) \neq g(x)$ for all $x \in [x_0-\delta, x_0+\delta]$. Find $k$ such that $[x_0-\delta, x_0+\delta] \subseteq [-k,k]$. Then, $\rho_k(f-g) > 0$.

Construct $d_k$ as above, and then $d$ as follows:

$$d_{\text{F}} = \sum \frac{2^{-k} p_k(f-g)}{1+p_k(f-g)}$$

(9) Product of metric spaces: let $(X_k, \rho_k)_{k=1}^{\infty}$ be a countable family of metric spaces. For each $k$, let

$$d_k(x,y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}.$$

**Remark:** If the $\rho_k$ are already uniformly bounded, let $d_k = \rho_k$.

Let

$$\begin{aligned}
X &= \prod_{k=1}^{\infty} X_k\\
&= \{(x_k)_k \mid x_k \in X_k\}\\
&= \left\{ f : \mathbb{N} \to \bigsqcup_{k=1}^{\infty} X_k \mid f(k) \in X_k \right\}.
\end{aligned}$$

Define $D : X \times X \to [0,\infty)$ as

$$D(x,y) = \sum_{k=1}^{\infty} 2^{-k}\rho_k(x_k, y_k),$$
$$D(f,g) = \sum_{k=1}^{\infty} 2^{-k}\rho(f(k), g(k)).$$

For example, for each $k$, let $X_k = \{0,1\}$ with the discrete metric. Let

$$\begin{aligned} \Delta &= \prod_{k\in\mathbb{N}} \{0,1\} \\ &= \{(x_k)_k \mid x_k \in \{0,1\}\} \\ D(x,y) &= \sum_{k=1}^{\infty} 2^{-k}|x_k - y_k| && (x_k)_k, (y_k)_k \in \Delta. \end{aligned}$$

$\Delta$ is known as the abstract Cantor set; every compact metric space is a surjective image of the abstract Cantor set.

(10) Geodesic Distance: let $\langle\cdot,\cdot\rangle$ be the standard dot product on $\mathbb{R}^3(\mathbb{R}^n)$, then

$$S^2 = \left\{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\right\}$$
$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}.$$

To find the geodesic distance, we take $d(x,y) = \arccos(\langle x,y\rangle)$. We claim $d$ is a metric.

- Symmetry: self-evident.
- $d(x,x) = \arccos(1) = 0$. Suppose $d(x,y) = 0$. Then, $\langle x,y\rangle = 1$, meaning $\|x-y\|^2 = 0$, so $x = y$.
- Let $\theta = \arccos(\langle x,y\rangle)$, $\varphi = \arccos(\langle y,z\rangle)$, where $\theta,\varphi \in [0,\pi]$.

  $$\begin{aligned} p_x &= \frac{\langle x,y\rangle}{\langle y,y\rangle} y \\ &= \cos(\theta) y \\ x &= \cos(\theta)y + \sin(\theta)u \end{aligned}$$

  where

  $$u = \frac{x - p_x}{\|x - p_x\|}.$$

  Similarly, we can take

  $$z = \cos(\varphi)y + \sin(\varphi)v$$

  where

  $$v = \frac{z - p_z}{\|z - p_z\|}.$$

  So,

  $$\begin{aligned} \langle x,z\rangle &= \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)\langle u,v\rangle \\ &\geq \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) && \langle u,v\rangle \geq -1 \\ &= \cos(\theta+\varphi). \end{aligned}$$

  Since arccos is decreasing,

  $$\begin{aligned} \arccos(\langle x,z\rangle) &\leq \arccos(\cos(\theta+\varphi)) \\ &= \theta + \varphi \\ &= \arccos(\langle x,y\rangle) + \arccos(\langle y,z\rangle). \end{aligned}$$

  Therefore, $d(x,y) \leq d(x,y) + d(y,z)$.
- Let $\Gamma = (V,E)$ be a simple connected graph. We define $d : V \times V \to [0,\infty)$ to be the length of the shortest path between vertices $u$ and $v$.

  **Exercise:** Show this is a metric.

(11) Let $(X,d)$ be any metric space. If $E \subseteq X$, define $\operatorname{diam}(E) = \sup_{x,y\in E} d(x,y)$. $E$ is bounded if $\operatorname{diam}(E) < \infty$.

**Exercise:** If $(V, \|\cdot\|)$ is a normed space, $E \subseteq V$ is a subset, show the following are equivalent:

(i) $E$ is bounded (in the metric sense)
(ii) $\sup_{v\in E} \|v\| < \infty$
(iii) $\exists r > 0$ such that $E \subseteq rB_V$.

Let $\Omega$ be any set. The function $f : \Omega \to X$ is bounded if $f(\Omega) \subseteq X$ is bounded. We let $\text{Bd}(\Omega, X) = \{f : \Omega \to X \mid f \text{ is bounded}\}$.

**Remark:** $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty(\Omega, \mathbb{F})$.

(12) $\text{Bd}(\Omega, X)$ with

$$D_u(f, g) = \sup_{x\in\Omega} d(f(x), g(x)).$$

**Exercise:** Show that $D_u$ defines a metric.

Consider $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty$. Look at the subset

$$E = \{f \in \text{Bd}(\Omega, \mathbb{F}) \mid f(x) \in \{0, 1\}\}.$$

Then,

$$\begin{aligned} D_u(f, g) &= \sup_{x\in\Omega} |f(x) - g(x)|. \\ &= \begin{cases} 1 & f \neq g \\ 0 & f = g \end{cases}. \end{aligned}$$

When we take a particular subset of $D_u(f, g)$, we find that we get the discrete metric.

Taking an overview of the concepts we have learned so far, we see

Inner Product Spaces $\subseteq$ Normed Vector Spaces $\subseteq$ Metric Spaces

## Topology of Metric Spaces

Throughout this section, let $(X, d)$ be a metric space.

(1) Let $x_0 \in X$, $\delta > 0$.

(i) We say

$$U(x_0, \delta) = \{x \in X \mid d(x, x_0) < \delta\}$$

is the open ball centered at $x_0$ with radius $\delta$.

(ii) We say

$$B(x_0, \delta) = \{x \in X \mid d(x, x_0) \leq \delta\}$$

is the closed ball.

(iii) We say

$$S(x_0, \delta) = \{x \in X \mid d(x, x_0) = \delta\}$$

is the sphere.

(2) $U \subseteq X$ is open if

$$(\forall x \in U)(\exists \delta > 0) \ni U(x, \delta) \subseteq U.$$

Let

$$\begin{aligned} \tau_X &= \{U \subseteq X \mid U \text{ open}\} \\ &\subseteq \mathcal{P}(X). \end{aligned}$$

(3) $D \subseteq X$ is closed if $D^c$ is open.

(4) If $x \in U \in \tau_X$, then $U$ is called an open neighborhood of $x$. If $x \in U \subseteq N$, where $U \in \tau_X$, then $N$ is a neighborhood of $x$.

$$\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$$

(5) Let $A \subseteq X$. The interior of $A$ is

$$A^0 = \bigcup \{V \mid V \subseteq A, V \text{ open}\}.$$

The closure of $A$ is

$$\overline{A} = \bigcap \{D \mid A \subseteq D, D \text{ closed}\}.$$

The boundary of $A$ is

$$\partial A = \overline{A} \setminus A^0.$$

**Exercise:** $\overline{A^c} = (A^0)^c$, $(\overline{A})^c = (A^c)^0$.

**Remarks:** $A^0$ is the largest open set contained in $A$. So, if $V$ is open and $V \subseteq A$, then $V \subseteq A^0$. Similarly, $\overline{D}$ is the smallest closed set containing $D$. If $C$ is closed and $D \subseteq C$, then $\overline{D} \subseteq C$.

- For example, $(a, b]^0 = (a, b)$. This is because $(a, b)$ is open and contained in $(a, b]$, so $(a, b) \subseteq (a, b]^0$.
- We will show that $\overline{A^c} \subseteq (A^0)^c$.

$$A^0 \subseteq A$$
$$(A^0)^c \supseteq A^c$$

The union of open sets is open, so $A^0$ is open, so $(A^0)^c$ is closed by definition. Therefore,

$$(A^0)^c \supseteq \overline{A^c}.$$

## Topology of Open Sets in a Metric Space

The open sets $\tau_X$ form a topology:

(i) $\emptyset, X \in \tau_X$.

(ii) If $\{V_i\}_{i\in I} \subseteq \tau_x$, then

$$\bigcup_{i\in I} V_i \in \tau_X.$$

(iii) If $V_i, \dots, V_n \in \tau_X$, then

$$\bigcap_{i=1}^{n} V_i \in \tau_X.$$

**Remark:** This is only true of finite intersections. For a counterexample, if $V_n = (-1/n, 1/n) \subseteq \mathbb{R}$ with the Euclidean metric, then the infinite intersection yields $\{0\}$, which is closed in $\mathbb{R}$ with the Euclidean metric.

**Proof:**

(1) Clearly, $\emptyset$ (by vacuous truth) and $X$ are open.

(2) Let $x \in \bigcup_{i\in I} V_i$. Then, $\exists i_0 \in I$ with $x \in V_{i_0}$. Since $V_{i_0}$ is open, $\exists \varepsilon > 0$ such that $U(x, \varepsilon) \subseteq V_{i_0} \subseteq \bigcup V_i$.

(3) Let $x \in \bigcap_{i=1}^n V_i$. Then, $x \in V_i$ for all $i \in 1, \dots, n$. Since each $V_i$ is open, $\exists \varepsilon_1, \dots, \varepsilon_n$ with $U(x, \varepsilon_i) \subseteq V_i$ for each $i = 1, \dots, n$. Set $\varepsilon = \min\{\varepsilon_i\}_{i=1}^n$. Then, $U(x, \varepsilon) \subseteq U(x, \varepsilon_i) \subseteq V_i$ for all $i$. Therefore, $U(x, \varepsilon) \subseteq \bigcap_{i=1}^n V_i$.

**Exercise:** Show all open balls are open. In particular, show all open intervals are open.

**Exercise:** Show the following:

(1) $X, \emptyset$ are closed.

(2) If $\{C_i\}_{i\in I}$ is a family of closed sets, then $\bigcap_{i\in I} C_i$ is closed.

(3) For $C_1, \dots, C_n$ closed, then $\bigcup_{i=1}^n C_i$ is closed.

(4) Closed balls are closed. Spheres are closed.

Let $x \in X$. Recall that $\mathcal{N}_x$ is the set of all neighborhoods of $x$.

(i) $N \in \mathcal{N}_x \Leftrightarrow \exists \delta > 0 : U(x, \delta) \in N$

(ii) $N \in \mathcal{N}_x, N \subseteq M \Rightarrow M \in \mathcal{N}_x$

(iii) $N_1, N_2 \in \mathcal{N}_x \Rightarrow N_1 \cap N_2 \in \mathcal{N}_x$

In this sense, $\mathcal{N}_x$ is a directed set with reverse inclusion.

## Pointwise Characterization of Subsets

Let $A \subseteq X$.

(i) $x \in A^0 \Leftrightarrow \exists \delta > 0 : U(x,\delta) \subseteq A$.

(ii) $x \in \overline{A} \Leftrightarrow \forall \delta > 0 : U(x,\delta) \cap A \neq \emptyset$.

(iii) $x \in \partial A \Leftrightarrow \forall \delta > 0 : U(x,\delta) \cap A \neq \emptyset$ and $U(x,\delta) \cap A^c \neq \emptyset$.

**Proof:** Let $A \subseteq X$

(i)

$$\begin{aligned} x \in A^0 &\Leftrightarrow x \in \bigcup_{\substack{V \in \tau_X \\ V \subseteq A}} V \\ &\Leftrightarrow \exists V \in \tau_X, V \subseteq A, x \in V \\ &\Leftrightarrow \exists \delta > 0 : U(x,\delta) \subseteq A. \end{aligned}$$

(ii)

$$\begin{aligned} x \notin \overline{A} &\Leftrightarrow x \in (\overline{A})^c \\ &\Leftrightarrow x \in (A^c)^0 \\ &\Leftrightarrow \exists \delta > 0 : U(x,\delta) \subseteq A^c \\ &\Leftrightarrow \exists \delta > 0 : U(x,\delta) \cap A = \emptyset. \end{aligned}$$

We negate both sides.

(iii)

$$\begin{aligned} x \in \partial A &\Leftrightarrow x \in \overline{A} \setminus A^0 \\ &\Leftrightarrow x \in \overline{A} \cap (A^0)^c \\ &\Leftrightarrow x \in \overline{A} \cap \overline{A^c} \\ &\Leftrightarrow x \in \overline{A} \text{ and } x \in \overline{A^c} \\ &\Leftrightarrow \forall \delta > 0 : U(x,\delta) \cap A \neq \emptyset, U(x,\delta) \cap A^c \neq \emptyset \end{aligned}$$

**Remark:** $\overline{U(v,\delta)} = B(v,\delta)$ in a normed space. $\partial U(v,\delta) = \partial B(v,\delta) = S(v,\delta)$ in a normed space. Also, $B(v,\delta)^0 = U(v,\delta)$.

**Proof:** We show that $\overline{U}(v,\delta) = B(v,\delta)$. Since $B(v,\delta)$ is closed, and $U(v,\delta) \subseteq B(v,\delta)$, we know $\overline{U(v,\delta)} \subseteq B(v,\delta)$.

Let $w \in B(v,\delta)$. If $\|w - v\| < \delta$, then $w \in U(v,\delta)$. Assume $\|w - v\| = \delta$. Let $u_t = (1-t)v + tw$, where $t \in [0,1]$.

$$\begin{aligned} \|w - u_t\| &= \|w - (1-t)v - tw\| \\ &= \|(1-t)(w - v)\| \\ &= (1-t)\|w - v\| \\ &= (1-t)\delta. \end{aligned}$$

Let $\varepsilon > 0$. Let $t \in (0,1)$ such that $(1-t)\delta < \varepsilon$. Then, $u_t \in U(w,\varepsilon) \cap U(v,\delta)$. Therefore, $w \in \overline{U(v,\delta)}$.

## Unions and Intersections of Closure/Interior

Let $(X,d)$ be a metric space.

(i)

$$\left(\bigcup_{i \in I} A_i\right)^0 \supseteq \bigcup_{i \in I} A_i^0 \qquad \text{may be strict}$$

(ii)

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$$

(iii)

$$\bigcap_{k=1}^{n} A_k^0 = \left(\bigcap_{k=1}^{n} A_k\right)^0$$

(iv)

$$\overline{\bigcup_{k=1}^{n} D_k} = \bigcup_{k=1}^{n} \overline{D_k}$$

**Proof:**

(i)

$$\begin{aligned} A_i^0 &\subseteq A_i \\ \bigcup_{i\in I} A_i^0 &\subseteq \bigcup_{i\in I} A_i \\ \bigcup_{i\in I} A_i^0 &\subseteq \left(\bigcup_{i\in I} A_i\right)^0 \end{aligned}$$

**Remark:** We claim $\overline{\mathbb{Q}} = \mathbb{R}$ under the absolute value metric. We know that $\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{R}$ is closed, meaning $\overline{\mathbb{Q}} \subseteq \mathbb{R}$. Let $t \in \mathbb{R}$, $\delta > 0$. We know that $(t-\delta, t+\delta) \cap \mathbb{Q} \neq \emptyset$. Therefore, $t \in \overline{\mathbb{Q}}$. Thus, $\overline{\mathbb{Q}} = \mathbb{R}$.

## Properties of Boundary

Let $A \subseteq X$.

(1) $\partial A$ is closed.

(2) $\partial A = \partial A^{\mathsf{c}}$

(3) $\overline{A} = A \cup \partial A$

(4) $A \setminus \partial A = A^0$

**Proof:**

(1)

$$\begin{aligned} \partial A &= \overline{A} \setminus A^0 \\ &= \overline{A} \cap (A^0)^{\mathsf{c}}. \end{aligned}$$

(2) Follows from pointwise characterization.

(3) Clearly, $A \cup \partial A \subseteq \overline{A}$. Let $x \in \overline{A}$. If $x \in A$, we're done. Otherwise, $x \in \overline{A} \setminus A \subseteq \overline{A} \setminus A^0 = \partial A$.

(4)

$$\begin{aligned} A \setminus \partial A &= A \cap (\partial A)^{\mathsf{c}} \\ &= A \cap (\overline{A} \setminus A^0)^{\mathsf{c}} \\ &= A \cap \left(\overline{A} \cap (A^0)^{\mathsf{c}}\right)^c \\ &= A \cap \left(\overline{A}^{\mathsf{c}} \cup A^0\right) \\ &= \left(A \cap \overline{A}^{\mathsf{c}}\right) \cup \left(A \cap A^0\right) \\ &= A^0 \end{aligned}$$

## Density and Separability

Let $(X, d)$ be a metric space.

(1) $A \subseteq X$ is $d$-dense if $\overline{A} = X$.

(2) $N \subseteq X$ is nowhere dense if $(\overline{N})^0 = \emptyset$.

(3) $(X, d)$ is separable if there is a countable dense subset.

**Exercise:** If $N \subseteq X$ is closed, then $N$ is nowhere dense if and only if $N^c$ is dense.

**Exercise:** The following are equivalent.

(1) $A \subseteq X$ is dense.
(2) $\forall \emptyset \neq U \in \tau_X$, $U \cap A \neq \emptyset$.
(3) $\forall x \in X, \forall \varepsilon > 0$, $U(x, \varepsilon) \cap A \neq \emptyset$.
(4) $\forall x \in X, \forall \varepsilon > 0, \exists a \in A$ such that $d(x, a) < \varepsilon$.

Let $X$ be a metric space.

(1) A base for $\tau_X$ is a family of open subsets $\mathcal{B}$ such that:

$$(\forall U \in \tau_X)\,(\forall x \in U)\exists B \in \mathcal{B} \ni x \in B \subseteq U.$$

Equivalently,

$$\forall U \in \tau_X, U = \bigcup_{i \in I} B_i. \qquad B_i \in \mathcal{B}$$

(2) We say that $(X, d)$ is second countable if $\tau_X$ admits a countable base.

- For any $(X, d)$ a metric space, $\mathcal{B} = \{U(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a base. Indeed, given any $x \in U \subseteq \tau_X$, by definition, $\exists \varepsilon > 0$ such that $U(x, \varepsilon) \subseteq U$. Alternatively, $\mathcal{B}' = \{U(x, 1/n) \mid x \in X, n \geq 1\}$ is a topological base.

- Let $X = \mathbb{R}^d$ with the Euclidean metric. Then, for $\mathcal{B} = \{U(q, 1/n) \mid n \geq 1, q \in \mathbb{Q}^d\}$, we claim this is a base.

  Let $V \subseteq \mathbb{R}^d$ be open, $r \in V$. Since $V$ is open, $\exists \delta > 0$ with $U(r, \delta) \subseteq V$. Find $n$ large such that $1/n < \delta$. Find $q \in \mathbb{Q}^d$ with $\|r - q\| < 1/2n$. This is always possible as $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$.

  Consider $U(q, 1/2n)$. Then, $r \subseteq U(q, 1/2n) \subseteq U(r, \delta) \subseteq V$ because $\|r - q\| < 1/2n$, and if $t \in U(q, 1/2n)$, then

$$\begin{aligned} \|t - r\| &\leq \|t - q\| + \|q - r\| \\ &< 1/2n + 1/2n \\ &= 1/n \\ &< \delta. \end{aligned}$$

## Separable, Non-Separable, Dense, and Non-Dense Sets

(1) $\left(R^d, \|\cdot\|_p\right)$ is separable for any $p \in [1, \infty]$. Indeed, $\mathbb{Q}^d \subseteq \mathbb{R}^d$ is the countable dense subset of $\mathbb{R}^d$.

Let $r = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \in \mathbb{R}^d$. Find $q = \begin{bmatrix} q_1 \\ \vdots \\ q_d \end{bmatrix} \in \mathbb{Q}^d$ with $|r_j - q_j| < \varepsilon/d$. Then,

$$\begin{aligned} \|r - q\|_1 &= \sum_{j=1}^{d} |r_j - q_j| \\ &< d. \end{aligned}$$

We know that for any vector $r \in \mathbb{R}^d$, we can find a vector $q$ such that

$$\|q - r\|_p \leq c\,\|q - r\|_1,$$

so for arbitrary $p$, find $q$ such that $\|q - r\|_1 < \varepsilon/c$.

(2) Similarly, $\mathbb{C}_{\mathbb{Q}} = \{a + bi \mid a, b \in \mathbb{Q}\}$ is also countable, meaning $\mathbb{C}_{\mathbb{Q}}^d \subseteq \mathbb{C}^d$ is dense and $\mathbb{C}^d$ is dense.