Things You Just Gotta Know

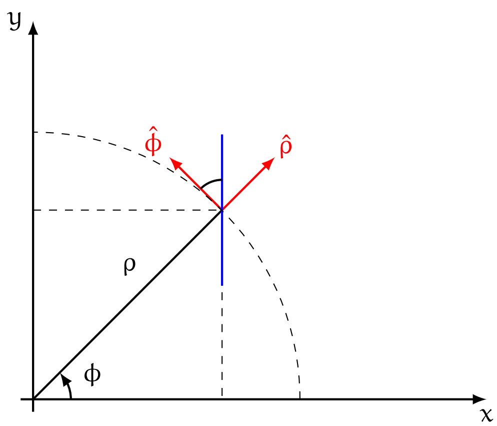
Coordinate Systems

We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x, y)\hat{\imath} + V_y(x, y)\hat{\jmath}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

Polar Coordinates



We can also express the inputs to $\vec{V}$ in polar coordinates, (ρ, ϕ).

$$\vec{V}(\mathbf{r}) = V_\rho\,(\rho, \phi)\,\hat{\imath} + V_\phi\,(\rho, \phi)\,\hat{\jmath}.$$

To extract the input functions, we take

$$\begin{aligned} V_x &= \hat{\imath} \cdot \vec{V} \\ V_y &= \hat{\jmath} \cdot \vec{V}. \end{aligned}$$

Alternatively, we can project $\vec{V}$ onto the $\hat{\rho}, \hat{\phi}$ axis:

$$\vec{V}(\mathbf{r}) = V_\rho\,(\rho, \phi)\,\hat{\rho} + V_\phi\,(\rho, \phi)\,\hat{\phi},$$

and we extract

$$\begin{aligned} V_\rho &= \hat{\rho} \cdot \vec{V} \\ V_\phi &= \hat{\phi} \cdot \vec{V}. \end{aligned}$$

Notice that $\mathbf{r}$ is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned} \mathbf{s} &= x\hat{\imath} + y\hat{\jmath} \\ &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} \\ &= \rho\hat{\rho} \\ &= \sqrt{x^2 + y^2}\hat{\rho} \end{aligned}$$

The main reason we avoided using the $\hat{\rho}, \hat{\phi}$ axis up until this point is that ρ and ϕ are *position-dependent*, while the $\hat{\imath}, \hat{\jmath}$ axis is position-independent.

Now, we must figure out the position-dependence of $\hat{\rho}$ and $\hat{\phi}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi.$$

If we hold ϕ constant, it must be the case that any change in ρ is in the $\hat{\rho}$ direction. Therefore,

$$\begin{aligned}
\hat{\rho} &= \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|} \\
&= \frac{\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}}{\left|\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\right|} \\
&= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|} \\
&= \frac{-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}}{\left\|-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}\right\|} \\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}.
\end{aligned}$$

Thus, we can see that the $\hat{\rho}, \hat{}\phi$ axis is orthogonal.

$$\begin{aligned}
\frac{\partial \hat{\rho}}{\partial \phi} &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
&= \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\rho}, \\
\frac{\partial \hat{\phi}}{\partial \rho} &= 0,
\end{aligned}$$

and

$$\frac{\partial \hat{\rho}}{\partial \rho} = 0$$

Example (Velocity)**.**

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{s}}{dt} \\
&= \frac{d}{dt}\left(x\hat{\imath}\right) + \frac{d}{dt}\left(y\hat{\jmath}\right).
\end{aligned}$$

In the case of cartesian coordinates, $\hat{\imath}$ and $\hat{\jmath}$ are constants.

$$= v_x\hat{\imath} + v_y\hat{\jmath}$$

When we examine polar coordinates, since $\hat{\rho}$ and $\hat{\phi}$ are position-dependent, we must use the chain rule.[1]

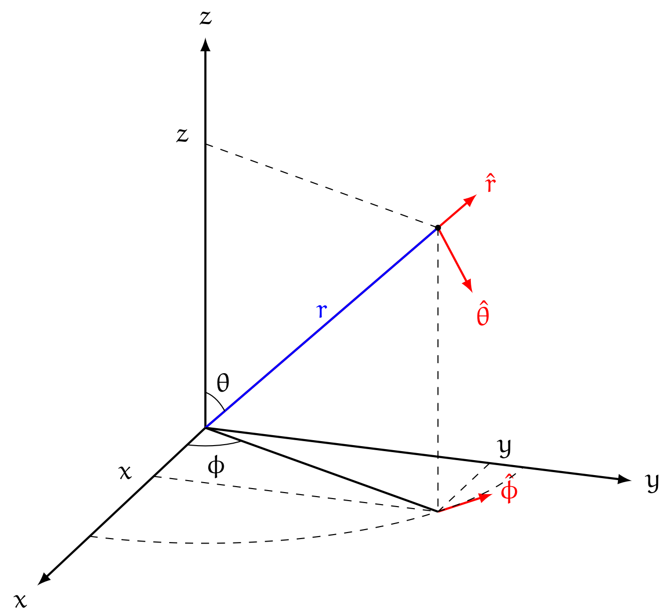
$$\mathbf{v} = \frac{d\mathbf{s}}{dt}$$

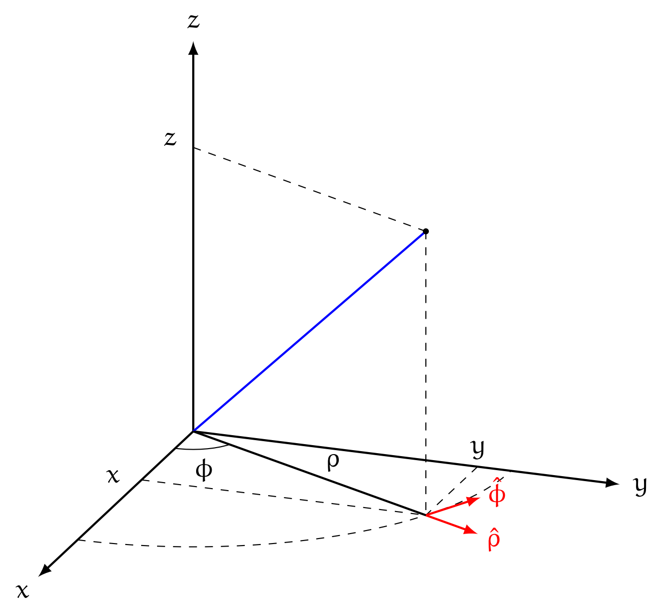
[1]Note that $\hat{\rho} = \hat{\rho}\left(\rho, \phi\right)$ and $\hat{\phi} = \hat{\phi}(\rho, \phi)$.

$$
\begin{aligned}
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} \\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\cancelto{0}{\frac{\partial\hat{\rho}}{\partial\rho}}\frac{d\rho}{dt} + \underbrace{\frac{\partial\hat{\rho}}{\partial\phi}}_{=\hat{\phi}}\frac{d\phi}{dt}\right) \\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi} \\
&= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}.
\end{aligned}
$$

Notice that $\dot{\rho}$ is the radial velocity and $\dot{\phi} = \omega$ is the angular velocity.

Spherical and Cylindrical Coordinates





Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho, \phi)$	$\mathbf{s} = s(\rho, \phi, z)$	$\mathbf{s} = s(r, \phi, \theta)$
$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$

Here,[II] ϕ denotes the polar angle and θ denotes the azimuthal angle. Notice that $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

We can see that $\hat{\rho}$, $\hat{\phi}$, and $\hat{\theta}$ in spherical coordinates are also position-dependent.

$$\begin{aligned}
\hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\|\frac{\partial \mathbf{s}}{\partial r}\right\|} \\
&= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k} \\
\hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\|\frac{\partial \mathbf{s}}{\partial \phi}\right\|} \\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
\hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\|\frac{\partial \mathbf{s}}{\partial \theta}\right\|} \\
&= \cos\phi\cos\theta\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}
\end{aligned}$$

Scale Factors and Jacobians

Coordinate System	Line Element	Area Element	Volume Element
Polar	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi$	$d\mathbf{a} = r\,drd\phi$	—
Cylindrical	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz$	—	$d\mathbf{v} = r\,drd\phi dz$
Spherical	$d\mathbf{s} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta$	$d\mathbf{a} = r^2\sin\theta d\phi d\theta$	$d\mathbf{v} = r^2\sin\theta\,drd\phi d\theta$

In cylindrical coordinates, we can use the chain rule to find the value of $d\mathbf{r}$:

$$d\mathbf{r} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz.$$

The extra factor of ρ in the expression of $\rho\hat{\phi}d\phi$ is the *scale factor* on ϕ.

Similarly, in spherical coordinates, we have

$$d\mathbf{r} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta,$$

with scale factors of $r\sin\theta$ on $\hat{\phi}d\phi$ and r on $\hat{\theta}d\theta$.

When we go from line elements (of the form $d\mathbf{r}$) to area elements (of the form $d\mathbf{a}$), we can see that the area element in polar coordinates is $d\mathbf{a} = \rho d\rho d\phi$ — we need the extra factor of ρ to account for the fact that the magnitude of the area element scales with the radius.

Similarly, the volume element in cylindrical coordinates is $d\mathbf{v} = r dr d\phi dz$ and the volume element in spherical coordinates is $r^2\sin\theta dr d\phi d\theta$.

[II]Physicists amirite?

Recall that the definition of an angle ϕ that subtends an arc length s is $\phi\frac{s}{r}$, where r is the radius of a circle. We can imagine a similar concept on a sphere — a *solid angle* measured in steradians is of the form $\Omega = \frac{A}{r^2}$, where A denotes the surface area subtended by the angle Ω. In particular, since $d\Omega = \frac{dA}{r^2}$, we find that $d\Omega = \sin\theta d\phi d\theta$.

When we are dealing with products of scale factors, we need to use the Jacobian to determine the proper scale factor on any given element:

$$\begin{aligned} d\mathbf{a} &= dxdy \\ &= |J|\ dudv, \end{aligned}$$

where $|J|$ denotes the determinant of the Jacobian matrix. We write the Jacobian as follows:

$$\begin{aligned} J &= \frac{\partial(x, y)}{\partial(u, v)} \\ &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}. \end{aligned}$$

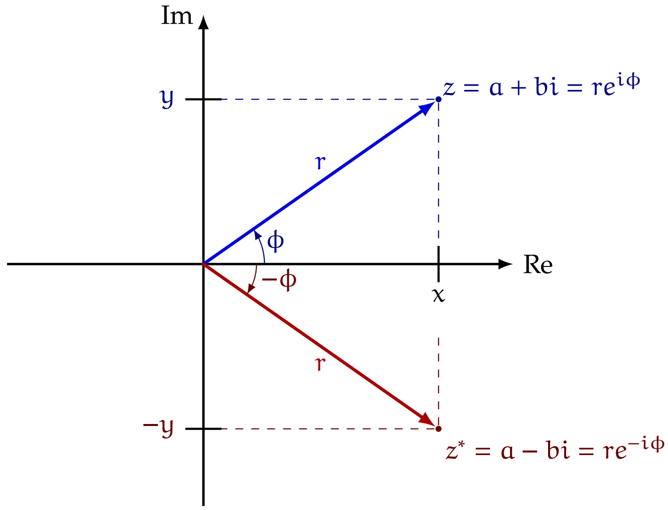
We specifically desire the determinant:

$$|J| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Complex Numbers

Quantity	Expression and/or Criterion
Cartesian form	$z = a + bi$
Polar form	$z = re^{i\phi}$
r	$\sqrt{a^2 + b^2}$
ϕ	$\arg z = \arctan\left(\frac{b}{a}\right)$
Cartesian z^*	$z^* = a - bi$
Polar z^*	$z = re^{-i\phi}$
$\|z\|$	$\sqrt{zz^*}$
$\mathrm{Re}(z)$	$\mathrm{Re}(z) = \frac{z+z^*}{2}$
$\mathrm{Im}(z)$	$\mathrm{Im}(z) = \frac{z-z^*}{2i}$
$\cos\phi$	$\frac{e^{i\phi}+e^{-i\phi}}{2}$
$\sin\phi$	$\frac{e^{i\phi}-e^{-i\phi}}{2i}$
$e^{i\phi}$	$\cos\phi + i\sin\phi$
$e^{in\phi}$	$\cos(n\phi) + i\sin(n\phi)$

Introduction



A complex number is denoted

$$z = a + bi$$

where $i^2 = -1$ and $a, b \in \mathbb{R}$. This is known as the cartesian representation. However, we can also imagine z as the polar representation:

$$z = re^{i\phi},$$

where $\phi = \arg z$ is known as the argument, and $r = |z|$ is the modulus. We can see the relation between the cartesian and polar representations through Euler's identity:[III]

$$r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

We denote the conjugate of z as z^*[IV], found by $z^* = a - bi = re^{-i\phi}$.

We find $\text{Re}(z)$ and $\text{Im}(z)$, the real and imaginary parts of z, by

$$\begin{aligned}\text{Re}(z) &= \frac{z+z^*}{2}\\ \text{Im}(z) &= \frac{z-z^*}{2i}.\end{aligned}$$

We say that a complex number of the form $e^{i\phi}$ is a *pure phase*, as $\left|e^{i\phi}\right| = 1$.

To find if some complex number z is purely real or purely imaginary, we can use the following criterion:

$$\begin{aligned}z \in \mathbb{R} &\Leftrightarrow z = z^*\\ z \in i\mathbb{R} &\Leftrightarrow z = -z^*.\end{aligned}$$

Example (Real, Imaginary, or Complex?)**.** Consider

$$z_1 = i^i.$$

To find if this is purely real or complex, we take

$$z_1^* = (-i)^{-i}$$

[III]This can be proven relatively easily through substitution into the Taylor series, which is allowed because e^z is entire.
[IV]Physicists amirite?

$$= \left(\frac{1}{-i}\right)^{i}$$
$$= i^{i}.$$

Thus, $z_1 \in \mathbb{R}$. In order to determine the value of i^i, we substitute the polar form:

$$z_1 = \left(e^{i\frac{\pi}{2}}\right)^{i}$$
$$= e^{-\frac{\pi}{2}}.$$

Some Trigonometry with Complex Exponentials

Consider $z = \cos\phi + i\sin\phi$. We can see that

$$\begin{aligned}
\mathrm{Re}(z) &= \cos\phi \\
&= \frac{(\cos\phi + i\sin\phi) + (\cos\phi - i\sin\phi)}{2} \\
&= \frac{e^{i\phi} + e^{-i\phi}}{2} \\
\mathrm{Im}(z) &= \sin\phi \\
&= \frac{(\cos\phi + i\sin\phi) - (\cos\phi - i\sin\phi)}{2i} \\
&= \frac{e^{i\phi} - e^{-i\phi}}{2i}.
\end{aligned}$$

We can actually define $\sin\phi$ and $\cos\phi$ with the above derivation.

Theorem (De Moivre)**.**

$$\begin{aligned}
e^{inx} &= \cos(nx) + i\sin(nx) \\
&= \left(e^{ix}\right)^{n} \\
&= (\cos x + i\sin x)^{n}.
\end{aligned}$$

Example (Finding $\cos(2x)$ and $\sin(2x)$)**.**

$$\begin{aligned}
\cos(2x) + i\sin(2x) &= (\cos x + i\sin x)^2 \\
&= \left(\cos^2 x - \sin^2 x\right) + i\,(2\sin x\cos x).
\end{aligned}$$

Since the real parts and imaginary parts have to be equal, this means

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
\sin^2 x &= 2\sin x\cos x.
\end{aligned}$$

In particular, we can see that $e^{in\phi} = (-1)^n$ and $e^{in\frac{\pi}{2}} = i^n$.[v]

Additionally, we can see that for $z = re^{i\phi}$,

$$\begin{aligned}
z^{1/m} &= \left(re^{i\phi + 2\pi n}\right)^{1/m} \\
&= r^{1/m}e^{i\frac{1}{m}(\phi + 2\pi n)},
\end{aligned}$$

where $n \in \mathbb{N}$ and m is fixed. For $r = 1$, we call these values the m roots of unity.

[v]This will be especially useful when we get to Fourier series.

Example (Waves and Oscillations)**.** Recall that for a wave with spatial frequency k, angular frequency ω, and amplitude A, the wave is represented by

$$f(x,t) = A\cos(kx - \omega t).$$

The speed of a wave v is equal to $\frac{\omega}{k}$.

Simple harmonic motion is characterized by the solution to the differential equation $\ddot{\mathbf{x}} = -\omega^2\mathbf{x}$, where $\mathbf{x}$ denotes position. In simple harmonic motion, there is no spatial motion, meaning our function is only of time:

$$\begin{aligned} f(t) &= A\cos\omega t \\ &= \text{Re}\left(Ae^{i\omega t}\right). \end{aligned}$$

As a result of the representation of complex numbers in polar form, we can do math entirely in exponentials, then take the real part of our solution to find $f(t)$.

Unfortunately, in the real world, there is friction; as a result, our oscillation is damped by an exponential factor.

Example (Hyperbolic Sine and Hyperbolic Cosine)**.** We wish to calculate $\cos ix$ and $\sin ix$.

$$\begin{aligned} \cos ix &= \frac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right) \\ &= \frac{e^{-x}+e^{x}}{2} \end{aligned}$$

We define $\cosh x = \cos(ix)$. Additionally,

$$\begin{aligned} -i\sin ix &= -i\frac{1}{2i}\left(e^{i(ix)} - e^{-i(ix)}\right) \\ &= i\frac{e^{ix}-e^{-ix}}{2i}. \\ &= \frac{e^{x}-e^{-x}}{2}. \end{aligned}$$

We define $\sinh x = -i\sin(ix)$.

Similar to how $\cos^2 x + \sin^2 x = 1$, we can find that $\cosh^2 x - \sinh^2 x = 1$.

Index Algebra

We usually denote vectors by either $\vec{A}$, $\mathbf{A}$, or

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is defined by a basis.

If we imagine we are in n-dimensional space, we can let A_i where $i = 1, 2, \ldots, n$ denote both

- the ith component of $\vec{A}$;
- the entire vector $\vec{A}$ (since i can be arbitrary).

Contractions and Dummy Indices

Consider $C = AB$, where A, B are $n \times m$ and $m \times p$ matrices respectively.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}.$$

The index notation description of this multiplication is much more contracted than the traditional method learned in linear algebra:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

This might seem like a very convoluted system for small matrices.

However, looking at AB compared to BA, we see that

$$\begin{aligned} (AB)_{ij} &= \sum_k A_{ik} B_{kj} \\ &= \sum_k B_{kj} A_{ik}, \\ (BA)_{ij} &= \sum_k B_{ik} A_{kj} \\ &= \sum_k A_{kj} B_{ik}. \end{aligned}$$