**Math 395: Homework 2**
**Name: Avinash Iyer**
**Due: 09/12/2024**
**Collaborators: Noah Smith**

# Problem 10

**Problem.** Let $T \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Prove that if $v \in V$ is not in $\ker(T)$, then

$$V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.$$

**Solution.** Since $T(v) \neq 0$, there exists $(T(v))^{-1} \in \mathbb{F}$. Let $w \in V$. Then,

$$T(w) = \left(T(w)\,(T(v))^{-1}\right) T(v).$$

We let $c = T(w)\,(T(v))^{-1}$. We have

$$\begin{aligned} T(w) &= cT(v) \\ &= T(cv), \end{aligned}$$

meaning

$$T(w - cv) = 0,$$

so $w - cv \in \ker(T)$, or $w \in [cv]_\sim$, where $\sim$ is the equivalence relation defining $V/\ker(T)$.

Thus, we have $w \in \ker(T) + \{cv \mid c \in \mathbb{F}\}$, implying that $V \subseteq \ker(T) + \{cv \mid c \in \mathbb{F}\}$, so $V = \ker(T) + \{cv \mid c \in \mathbb{F}\}$.

For $k \in \ker(T)$, suppose

$$cv + k = 0.$$

Then,

$$\begin{aligned} T(cv + k) &= 0_V \\ cT(v) + T(k) &= 0 \\ cT(v) &= 0. \end{aligned}$$

Since $T(v) \neq 0$ by the definition of $v$, it must be the case that $c = 0$, meaning $cv = 0_V$. Thus, it is the case that $\ker(T)$ and $\{cv \mid c \in \mathbb{F}\}$ are independent subspaces, meaning

$$V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.$$