Problem (Problem 1)**:** Let R be a ring and M a left R-module.

(a) Prove that for every $m \in M$, the map $r \mapsto r \cdot m$ from R to M is a homomorphism of R-modules.

(b) Assume that R is commutative and M an R-module. Prove that there is an isomorphism $\hom_R(R, M) \cong M$ as left R-modules.

Solution:

(a) Let $m \in M$ be fixed, and define $\varphi_m \colon R \to M$ by

$$\varphi_m(r) = r \cdot m.$$

It follows from the axioms of left R-modules that

$$\begin{aligned}\varphi_m(r + s) &= (r + s) \cdot m \\ &= r \cdot m + s \cdot m \\ &= \varphi_m(r) + \varphi_m(s),\end{aligned}$$

and

$$\begin{aligned}\varphi_m(rs) &= (rs) \cdot m \\ &= r \cdot (s \cdot m) \\ &= r \cdot (\varphi_m(s)),\end{aligned}$$

so that φ_m is a homomorphism of left R-modules.

(b) If $\varphi_m \colon R \to M$ is the homomorphism as defined in part (a), we define a map $\varphi \colon M \to \hom_R(R, M)$ by

$$\varphi(m)(r) = \varphi_m(r).$$

First, we verify that φ is a homomorphism. If $r \in R$ is arbitrary, then

$$\begin{aligned}\varphi(m + n)(r) &= \varphi_{m+n}(r) \\ &= r \cdot (m + n) \\ &= r \cdot m + r \cdot n \\ &= \varphi_m(r) + \varphi_n(r) \\ &= (\varphi(m) + \varphi(n))(r).\end{aligned}$$

To see that φ is injective, we see that $\ker(\varphi)$ consists of all elements $m \in M$ such that $\varphi(m) = \varphi_0$, where $\varphi_0 \colon R \to M$ takes $r \mapsto 0$ for all $r \in R$. In particular, since $1 \in R$, it follows that for all $m \in \ker(\varphi)$, we have $1 \cdot m = m = 0$, so $\ker(\varphi) = \{0\}$.

To see that φ is surjective, we observe that for any $\psi \in \hom_R(R, M)$, ψ is fully determined by where it maps 1, as

$$\psi(r) = r \cdot \psi(1).$$

Therefore, if $\psi \in \hom_R(R, M)$, then we may find $m \in M$ corresponding to ψ by taking

$$m := \psi(1).$$

Thus, $M \cong \hom_R(R, M)$.

Problem (Problem 3)**:** Let R be a ring, and M a left R-module.

(a) Let N be a subset of M. The *annihilator* of N is defined to be the set

$$\mathrm{ann}_R(N) = \{r \in R \mid r \cdot n = 0 \text{ for all } n \in N\}.$$

Prove that $\mathrm{ann}_R(N)$ is a left-ideal of R.

(b) Show that if N is an R-submodule of M, then $\mathrm{ann}_R(N)$ is a two-sided ideal of R.

(c) For a subset I of R, the *annihilator* of I in M is defined to be the set

$$\mathrm{ann}_M(I) = \{m \in M \mid x \cdot m = 0 \text{ for all } x \in I\}.$$

Find a natural condition on I that guarantees $\mathrm{ann}_M(I)$ is a submodule of M.

(d) Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator.

Solution:

(a) First, we observe that $\mathrm{ann}_R(N)$ is nonempty, as $0 \in \mathrm{ann}_R(N)$. Additionally, if $s, t \in \mathrm{ann}_R(N)$, then for all $n \in N$,

$$\begin{aligned}(s-t)\cdot n &= s \cdot n - t \cdot n\\ &= 0,\end{aligned}$$

so that N is closed under subtraction. Finally, if $r \in R$ and $s \in \mathrm{ann}_R(N)$, then for all $n \in N$,

$$\begin{aligned}(rs)\cdot n &= r \cdot (s \cdot n)\\ &= r \cdot 0\\ &= 0,\end{aligned}$$

meaning that $rs \in \mathrm{ann}_R(N)$, or that $\mathrm{ann}_R(N)$ is a left-ideal of R.

(b) Let N be an R-submodule of M, and let $s \in \mathrm{ann}_R(N)$. If $r \in R$, then for all $n \in N$, $r \cdot n \in N$, so that $(sr)\cdot n = s\cdot(r \cdot n) = 0$, meaning that $sr \in \mathrm{ann}_R(N)$. Thus, $\mathrm{ann}_R(N)$ is a right-ideal, hence a two-sided ideal for R.

(c) We observe to start that $\mathrm{ann}_M(I)$ contains 0 and is additively closed, since if $m, n \in \mathrm{ann}_M(I)$ and $x \in I$ are arbitrary, then

$$\begin{aligned}x \cdot (m+n) &= x \cdot m + x \cdot n\\ &= 0.\end{aligned}$$

Therefore, if we desire for $\mathrm{ann}_M(I)$ to be a submodule of M, we would need $r \cdot m \in \mathrm{ann}_M(I)$ for all $m \in \mathrm{ann}_M(I)$ and all $r \in R$, which would mean $r \cdot m$ would have to satisfy the condition

$$\begin{aligned}0 &= x \cdot (r \cdot m)\\ &= (xr)\cdot m\end{aligned}$$

for all $x \in I$, meaning that we would require $xr \in \mathrm{ann}_M(I)$. In other words, this means that $\mathrm{ann}_M(I)$ would have to be a right-ideal for R.

(d) Let $M = \langle a_1, \dots, a_n \rangle$ be a finitely generated torsion R-module. Since M has torsion, for each a_i, there is some $0 \neq r_i \in R$ such that $r_i \cdot a_i = 0$. The product

$$r = \prod_{i=1}^{n} r_i$$

is necessarily nonzero as R is an integral domain, and satisfies $r \cdot a_i = 0$ for all i by rearrangement of factors, so that $(r) \subseteq \mathrm{ann}_R(M)$ as $\mathrm{ann}_R(M)$ is an ideal containing r. Thus, $\mathrm{ann}_R(M)$ is a nonzero ideal.

Problem (Problem 4)**:** An R-module M is called *simple* if its only submodules are $\{0\}$ and M. An R-module M is called *indecomposable* if M is not isomorphic to $N \oplus Q$ for some nonzero submodules N and Q. Show that every simple R-module is indecomposable, but the converse is not true.

Solution: If R is simple, then R does not admit any nonzero proper submodules, meaning that R cannot be isomorphic to the direct sum of any nonzero proper submodules.

Now, if we let $R = \mathbb{Z}$ be our ring, then we observe that all the nonzero proper ideals (i.e., $\mathbb{Z}$-submodules) of $\mathbb{Z}$ are of the form (a) for some $a \in \mathbb{Z}$, as $\mathbb{Z}$ is a Euclidean domain (hence principal ideal domain). Observe that we can only write $\mathbb{Z}$ as a sum of submodules

$$\mathbb{Z} = (a) + (b)$$

when $\gcd(a, b) = 1$. Yet, these ideals necessarily do not intersect nontrivially, as $0 \neq ab \in (a) \cap (b)$ meaning that $\mathbb{Z}$ is indecomposable. Meanwhile, $\mathbb{Z}$ is not simple since $\mathbb{Z}$ admits nonzero proper ideals.

Problem (Problem 5)**:** Let R be a ring. An R-module M is called cyclic if it is generated as an R-module by a single element. That is, $M = R \cdot m$ for some $m \in M$.

(a) Prove that every cyclic R-module is of the form R/I for some left-ideal I of R.

(b) Show that the simple R-modules are precisely the ones which are isomorphic to $R/\mathfrak{m}$ for some maximal left-ideal $\mathfrak{m}$.

(c) Show that any nonzero homomorphism of simple R-modules is an isomorphism. Deduce that if M is simple, then its endomorphism ring

$$\operatorname{end}_R(M) := \hom_R(M, M)$$

is a division ring. This result is known as Schur's Lemma.

Solution:

(a) Let $M = \langle m \rangle$ be a cyclic R-module. Consider the map

$$\varphi \colon R \to M$$

given by $r \mapsto r \cdot m$. Since M is cyclic, this map is surjective, and admits the kernel $\operatorname{ann}_R(\{m\})$. The annihilator is a left-ideal of R as specified above, so that any such module is of the form R/I for some left-ideal I of R.

(b) If M is a simple R-module, then if $0 \neq m \in M$, we have that $R \cdot m = M$, as $R \cdot m$ is a submodule of M that contains a nonzero element. Thus, we observe that M is cyclic, so $M \cong R/I$ for some left-ideal I of R. By the fourth isomorphism theorem and the correspondence between left-R-submodules of R and left-ideals of R, we know that submodules of M correspond to left-ideals of R containing I; yet, since M does not contain any proper submodules, it follows that any submodule of M must either be isomorphic to I or to R, meaning that I is a maximal left-ideal.

(c) Let $\varphi \colon M \to N$ be a nonzero homomorphism of simple R-modules. Let $m \in M$ be nonzero, and let $\varphi(m) = n$ with $n \neq 0$. Then, for any $r \in R$, we have $\varphi(r \cdot m) = r \cdot n$. Since M and N are simple, and m and n are nonzero, it follows that $M = \langle m \rangle$ and $N = \langle n \rangle$, meaning that φ is necessarily surjective, as for any element $r \cdot n \in N$, we may find $r \cdot m \in M$ such that $\varphi(r \cdot m) = r \cdot n$. Now, considering $\ker(\varphi) \subseteq M$, we observe that $\ker(\varphi)$ is a submodule; it follows that $\ker(\varphi) = \{0\}$ or $\ker(\varphi) = M$, but we know that it cannot be the latter as φ is nonzero. Thus, φ is an isomorphism.

If M is simple, then if $\varphi \in \operatorname{end}_R(M)$ is nonzero, φ is necessarily an automorphism as we have shown that nonzero homomorphisms of simple R-modules are isomorphisms, so that φ admits an inverse. Thus, $\operatorname{end}_R(M)$ is a division ring.