Abstract

We discuss the much celebrated Regular Value Theorem and Sard's Theorem, and discuss some of the consequences and applications of these results.

A smooth map between manifolds $f\colon M \to N$ includes a certain family of local information; for instance, the derivative $D_p f\colon T_pM \to T_{f(p)}N$, which is a linear map between tangent spaces at p and q, is defined on a coordinate chart $U \subseteq M$ for p and a corresponding coordinate chart $V \subseteq N$ for $f(p)$. Yet, the properties of this linear map can give us information about the underlying map f.

To understand this, we need to dive into the world of regular and critical values.

Sard's Theorem

Definition: Let $f\colon M \to N$ be a smooth map, and let $p \in M$. We say p is a *critical point* for f if $D_p f$ does not have the same rank as the dimension of $T_{f(p)}N$. If $D_p f$ has the same rank as the dimension of $T_{f(p)}N$, then we say that p is a *regular point* of f.

We say $q \in N$ is a *critical value* for f if $f^{-1}(\{q\})$ contains a critical point for f. Else,we say that q is a *regular value*.

We start with the case of Sard's Theorem on $\mathbb{R}^n$. Then, we will expand this to the case of any arbitrary manifold by means of a technical lemma.

Theorem (Sard's Theorem)**:** Let $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be a smooth map. Then, if C is the set of critical points for f, we have $f(C) \subseteq \mathbb{R}^m$ has measure zero.

Proof. We use induction on n. The statement only makes sense for $n \geq 0$ and $p \geq 1$. Clearly, the theorem is true for $n = 0$.

Let $C_1 \subseteq C$ be the set of all $x \in U$ such that $D_x f$ is zero, and similarly, let C_i be the set of all x such that $(D_x)^j f$ is zero for all $j \leq i$. We obtain a descending sequence of closed sets $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$.

We start by showing that $f(C \setminus C_1)$ has measure zero. For each $x \in C \setminus C_1$, we find an open neighborhood $V \subseteq \mathbb{R}^n$ such that $f(V \cap C)$ has measure zero. Since $\mathbb{R}^n$ is second countable, $C \setminus C_1$ is covered by countably many such open neighborhoods, it follows that $f(C \setminus C_1)$ has measure zero.

Since $x \notin C_1$, there is some partial derivative, which we use change of coordinates to write as $\frac{\partial f}{\partial x_1}$, that is not zero at x. Let

$$h(x) = (f_1(x), x_2, \ldots, x_n).$$

Then, since $D_x h$ is nonsingular, by the inverse function theorem, h maps some neighborhood V of x diffeomorphically onto an open set $V' \subseteq \mathbb{R}^n$. The composition $f \circ h^{-1}$ then maps V' to R^m then maps V' to $\mathbb{R}^m$.

Observe that the set of critical points of g is precisely $h(V \cap C)$, so the set $g(C')$ is equal to $f(V \cap C)$. □