Solution (32.20): We start by taking the recurrence relation

$$\left(1 - x^2\right)P'_n = -nxP_n + nP_{n-1}. \tag{$*$}$$

Differentiating, this gives

$$\left(1 - x^2\right)P''_n - 2xP'_n = n\left(-P_n - xP'_n + P'_{n-1}\right).$$

We seek to show that

$$-xP'_n + P'_{n-1} = -nP_n.$$

At this point, I ran out of board space to deal with the generating functions and their ensuing mess of partial derivatives.

Solution (32.21): Using $dv = P'_m(x)$, we integrate by parts to get

$$\int_{-1}^{1}\left(1 - x^2\right)P'_n(x)P'_m(x)\,dx = P_m(x)P'_n(x)\left(1 - x^2\right)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}\left(\left(1 - x^2\right)P'_n(x)\right)P_m(x)\,dx$$

$$= -\int_{-1}^{1}\left(\left(1 - x^2\right)P''_n(x) - 2xP'_n(x)\right)P_m(x)\,dx$$

$$= n(n + 1)\int_{-1}^{1}P_n(x)P_m(x)\,dx$$

$$= \frac{2n(n + 1)}{2n + 1}\delta_{mn}.$$

Solution (32.23): Upon taking m derivatives of Legendre's equation, and using the Leibniz rule for differentiation, we get

$$\left(1 - x^2\right)\frac{d^{m+2}P_\ell}{dx^{m+2}} - 2x(m + 1)\frac{d^{m+1}P_\ell}{dx^{m+2}} + ((\ell)(\ell + 1) - (m(m - 1) + 2m))\frac{d^m P_\ell}{dx^m} = 0.$$

Rewriting $u(x) = \frac{d^m P_\ell}{dx^m}$, we obtain

$$0 = \left(1 - x^2\right)\frac{d^2 u}{dx^2} - 2x(m + 1)\frac{du}{dx} + \left(\ell(\ell + 1) - m^2 - m\right)u(x).$$

Setting $u(x) = \left(1 - x^2\right)^{-m/2}v(x)$, we find

$$\frac{du}{dx} = \left(1 - x^2\right)^{-m/2}\frac{dv}{dx} + mxv(x)\left(1 - x^2\right)^{-m/2-1}$$

$$= \left(1 - x^2\right)^{-m/2}\left(\frac{dv}{dx} + \frac{mxv(x)}{1 - x^2}\right)$$

$$\frac{d^2 u}{dx^2} = -mx\left(1 - x^2\right)^{-m/2-1}\left(\frac{dv}{dx} + \frac{mxv(x)}{1 - x^2}\right) + \left(1 - x^2\right)^{-m/2}\left(\frac{d^2 v}{dx^2} + \frac{mv(x)}{1 - x^2} + \frac{mx}{1 - x^2}\frac{dv}{dx} + \frac{2mx^2v(x)}{\left(1 - x^2\right)^2}\right)$$

$$= \left(1 - x^2\right)^{-m/2}\left(\frac{d^2 v}{dx^2} + \frac{2mx}{1 - x^2}\frac{dv}{dx} + \frac{mv(x)}{1 - x^2} + \frac{2mx^2v(x)}{\left(1 - x^2\right)^2}\right).$$

Substituting, we have the equation

$$0 = \left(1 - x^2\right)\left(1 - x^2\right)^{-m/2}\left(\frac{d^2 v}{dx^2} + \frac{2mx}{1 - x^2}\frac{dv}{dx} + \frac{mv(x)}{1 - x^2} + \frac{2mx^2v(x)}{\left(1 - x^2\right)^2}\right)$$

$$- 2x(m + 1)\left(\left(1 - x^2\right)^{-m/2}\left(\frac{dv}{dx} + \frac{mxv(x)}{1 - x^2}\right)\right)$$

$$+ \left(\ell(\ell + 1) - m^2 - m\right)\left(1 - x^2\right)^{-m/2}v(x),$$

which after much more tedious algebra, yields

$$0 = \frac{d^2 v}{dx^2} - 2x \frac{dv}{dx} + \left((\ell)(\ell + 1) - \frac{m^2}{1 - x^2} \right) v(x),$$

so v satisfies the differential equation. Thus, we have

$$v(x) = \left(1 - x^2 \right)^{m/2} \frac{d^m P_\ell}{dx^m}.$$

Solution (35.4): Using the expression

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin(\gamma) - in\gamma} \, d\gamma$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin(\gamma)} e^{-in\gamma} \, d\gamma,$$

we expand the first term in a Taylor series, giving

$$J_n(x) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{i^k x^k}{k!} \int_{-\pi}^{\pi} \sin^k(\gamma) e^{-in\gamma} \, d\gamma.$$

Now, k has to be even (else we have an odd integrand over a symmetric interval).

Solution (35.5): Differentiating,

$$\frac{dJ_0}{dx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial x} \left(e^{ix \sin(\gamma)} \right) d\gamma$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (i \sin(\gamma)) e^{ix \sin(\gamma)} \, d\gamma$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} i \left(\frac{1}{2i} \left(e^{i\gamma} - e^{-i\gamma} \right) \right) d\gamma$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} e^{ix \sin(\gamma) + i\gamma} - \frac{1}{2} e^{ix \sin(\gamma) - i\gamma} \, d\gamma$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos(x \sin(\gamma) + i\gamma) + i \sin(x \sin(\gamma) + i\gamma) - (\cos(x \sin(\gamma) - i\gamma) + i \sin(x \sin(\gamma) - i\gamma))) \, d\gamma$$

and with more tedious algebra, we obtain

$$= -\frac{1}{\pi} \int_{0}^{\pi} \cos(x \sin(\gamma) - \gamma) \, d\gamma$$

$$= -J_1(x).$$

Evaluating

$$\frac{d}{dx} (xJ_1) = J_1 + x \frac{dJ_1}{dx},$$

we take

$$\frac{d}{dx} (xJ_1) = \frac{1}{\pi} \int_{0}^{\pi} \cos(x \sin(\gamma) - \gamma) - x \sin(\gamma) \sin(x \sin(\gamma) - \gamma) \, d\gamma$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(x \sin(\gamma)) \cos(\gamma) + \sin(x \sin(\gamma)) \sin(\gamma) - x \sin(\gamma) \sin(x \sin(\gamma) - \gamma) \, d\gamma$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(\gamma) \cos(x \sin(\gamma)) + \sin(\gamma) \sin(x \sin(\gamma)) - x \sin(\gamma)(\sin(x \sin(\gamma)) \cos(\gamma) - \sin(\gamma) \cos(x \sin(\gamma))) \, d\gamma$$

$$= \frac{1}{\pi} \int_{0}^{\pi} x \cos(x \sin(\gamma)) \, d\gamma$$

$$= xJ_0.$$

Solution (35.7): Solving

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + \left(x^2 - n^2\right) u(x) = 0,$$

we plug in the expression for $J_n(x)$ to get

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + \left(x^2 - n^2\right) u(x) = x^2 \left(\sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} (2m+n-1)(2m+n) \frac{(-1)^m}{m!(m+n!)} x^{2m+n-2} \right)$$

$$+ x \left(\sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} (2m+n) \frac{(-1)^m}{m!(m+n)!} x^{2m+n-1} \right)$$

$$+ \sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} \frac{(-1)^m}{m!(m+n)!} x^{2m+n+2}$$

$$- \sum_{m=0}^{\infty} \frac{n^2}{2^{2m+n}} \frac{(-1)^m}{m!(m+n)!} x^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^{2m+n}} \frac{(-1)^m}{m!(m+n!)} \left(x^{2m+n}\right) \left((2m+n-1)(2m+n) + 2m+n + x^2 - n^2\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+n} m!(m+n!)} x^{2m+n} \left(x^2 + 4m^2 + 4mn\right)$$

From here, I'm not sure how to manipulate this series to get 0 as the final answer.

Solution (35.8):

(a) We have

$$e^{ix \sin(\phi)} = \sum_{n=-\infty}^{\infty} c_n e^{in\phi},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin(\phi)} e^{-in\phi} \, d\phi$$

$$= J_n(x).$$

(b) Splitting into real and imaginary parts, we have

$$e^{ix \sin(\phi)} = \cos(x \sin(\phi)) + i \sin(x \sin(\phi)),$$

so that

$$e^{ix \sin(\phi)} = \sum_{n=-\infty}^{\infty} c_n e^{in\phi}$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)(\cos(n\phi) + i \sin(n\phi))$$

$$= \sum_{n=-\infty}^{\infty} J_n(x) \cos(n\phi) + i \sum_{n=-\infty}^{\infty} J_n(x) \sin(n\phi).$$

Equating real and imaginary parts gives the desired result.

(c) We use the angle summation identity to get

$$A \cos(\omega_c t) \cos(\beta \sin(\omega_m t)) - A \sin(\omega_c t) \sin(\beta \sin(\omega_m t)) = A \cos(\omega_c t) \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(n\omega_m t)$$

$$- A\sin(\omega_c t) \sum_{n=-\infty}^{\infty} J_n(\beta)\sin(n\omega_m t)$$

$$= \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c t + n\omega_m t).$$

Solution (35.10):

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_{3,1}(x) = \frac{1}{2}\left(1 - x^2\right)^{1/2}\left(15x^2 - 3\right)$$

$$P_{3,-1}(x) = -\frac{1}{6}\left(1 - x^2\right)^{1/2}\left(15x^2 - 3\right)$$

$$P_{3,2}(x) = 15x\left(1 - x^2\right)$$

$$P_{3,-2}(x) = \frac{1}{8}x\left(1 - x^2\right).$$

$$P_{3,3}(x) = 15\left(1 - x^2\right)^{3/2}$$

$$P_{3,-3}(x) = -\frac{1}{48}\left(1 - x^2\right)^{3/2}.$$

Solution (35.11):

$$Y_{\ell,m}(\pi - \theta, \phi + \pi) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(\cos(\pi - \theta))e^{im(\phi+\pi)}$$

$$= (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(-\cos(\theta))e^{im\phi}(-1)^m$$

$$= (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}(-1)^{\ell-m}P_{\ell,m}(\cos(\theta))(-1)^m e^{im\phi}$$

$$= (-1)^\ell \left((-1)^m \sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(\cos(\theta))e^{im\phi}\right)$$

$$= (-1)^\ell Y_{\ell,m}(\theta, \phi).$$

Solution (35.12): We have

$$Y_{\ell,0}(\hat{n}) = (-1)^0 \sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{\ell!}{\ell!}}P_{\ell,0}(\cos(\theta))e^{i(0)\phi}$$

$$= \sqrt{\frac{2\ell + 1}{4\pi}}P_\ell(\cos(\theta)).$$

Furthermore, since $P_{\ell,m}(1) = \delta_{0,m}$, we have

$$Y_{\ell,m}(0, \phi) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(1)e^{im\phi}$$

$$= \sqrt{\frac{2\ell + 1}{4\pi}}\delta_{0,m}.$$

Solution (35.16): Using the addition theorem, where $\hat{a} = \hat{b} = \hat{n}$, we get

$$\sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\hat{n})\right|^2 = \frac{2\ell + 1}{4\pi}P_\ell(1)$$

$$= \frac{2\ell + 1}{4\pi}.$$

Solution (35.17 (c)):

Solution (35.21):

Solution (35.25):