Abstract

We introduce some of the most important inequalities that are used frequently in real and functional analysis. These inequalities include Jensen's inequality and Young's inequality (concerning convex functions), which are then used to prove Hölder's inequality and Minkowski's inequality (concerning p-norms). Afterwards, we define the L_p-spaces and show some basic properties such as completeness and the L_p/L_q duality.

Contents

Introduction

This is the first of a series of expository writings, primarily focused on analysis (though they will certainly branch out), designed to complement my own knowledge and understanding of the various facets of the field. Ideally these will be shorter in length than my traditional notes documents, and they will focus on some short, coherent theme.

Here, we introduce and discuss some of the most important inequalities in analysis, primarily focusing on the context of function spaces such as $L_p(\Omega, \mu)$. Afterwards, we will establish the completeness of the $L_p(\Omega, \mu)$ spaces, and show that $L_p(\Omega, \mu)^* = L_q(\Omega, \mu)$ whenever $1 \leqslant p < \infty$. for Much of this document will be a fleshed out version of certain results and theorems discussed in Walter Rudin's *Real and Complex Analysis*, primarily from Chapter 3, as well as the discussion of duality from Royden and Fitzpatrick's *Real Analysis*.

We will assume that the reader has an understanding of measures, measurable functions, and basic integration theory (such as the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem), though I will probably end up writing expositions on those as well. My Real Analysis II notes should be sufficient as a background.

Convex Functions

Definition. A function $\varphi\colon (a, b) \to \mathbb{R}$ is called *convex* if, for all $x, y \in (a, b)$,

$$\varphi((1-t)x + ty) \leqslant (1-t)\varphi(x) + t\varphi(y).$$

Remark: A *differentiable* function $\varphi\colon \Omega \to \mathbb{R}$ is convex if and only if its second derivative is positive along its domain.

Note here that we define convexity along an open interval. This is because it is convenient to do so.

Two major examples of convex functions are

$$\begin{aligned} \varphi_1(x) &= e^x \\ \varphi_2(x) &= x^p. && 1 \leqslant p < \infty \end{aligned}$$

Both of these functions are convex along their domain.

Convex functions defined over an interval are useful precisely because they are continuous — and, thus, measurable.

Theorem: Let $\varphi\colon (a,b) \to \mathbb{R}$ be convex. Then, φ is continuous.

We follow the proof from this website.

Proof. We begin by an observation. If $a < x_1 < x_2 < x_3 < b$, then convexity gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{$*$}$$

By the characterization of an interval, for $s, t \in (a,b)$ with $s < t$, we have $[s,t] \subseteq (a,b)$. Now, note that since (a,b) is an open interval, there are $s', t' \in (a,b)$ with $s' < s$ and $t < t'$. In particular, this means that for any $x_1, x_2 \in [s,t]$ with $x_1 < x_2$, we have

$$\begin{aligned}\frac{f(s) - f(s')}{s - s'} &\leqslant \frac{f(x_1) - f(s)}{x_1 - s}\\ &\leqslant \frac{f(x_2) - f(x_1)}{x_2 - x_1}\\ &\leqslant \frac{f(t) - f(x_2)}{t - x_2}\\ &\leqslant \frac{f(t') - f(t)}{t' - t}.\end{aligned}$$

Setting $C := \max\left\{\frac{f(t')-f(t)}{t'-t}, \frac{f(s)-f(s')}{s-s'}\right\}$, we see that for any $x_1, x_2 \in [s,t]$,

$$|f(x_2) - f(x_1)| \leqslant C|x_2 - x_1|.$$

Thus, f is Lipschitz on $[s,t]$, so f is continuous on $[s,t]$.

Since, for any $x \in (a,b)$, there is some closed interval containing x, and f is continuous on said closed interval, we have that f is continuous on (a,b). □

Remark: The fact that (a,b) is an open interval is indeed load-bearing. Consider the function defined by

$$f(x) = \begin{cases} x & x > 0 \\ 1 & x = 0 \end{cases}.$$

Then, f is convex, but f is not continuous.

The most famous inequality regarding convex functions is Jensen's inequality, which effectively provides a generalization of the definition of a convex function.

Theorem (Jensen's Inequality)**:** Let $(\Omega, \mathcal{M}, \mu)$ be a probability space, and let $f \in L_1(\Omega, \mu)$ be such that $a < f(x) < b$ for all $x \in \Omega$. Then, if $\varphi\colon (a,b) \to \mathbb{R}$ is convex,

$$\varphi\left(\int_\Omega f\, d\mu\right) \leqslant \int_\Omega \varphi \circ f\, d\mu$$

Proof. Set

$$t := \int_\Omega f\, d\mu,$$

and note that $a < t < b$. Note that, by a restatement of $(*)$, if $a < s < t < u < b$, then

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leqslant \frac{\varphi(u) - \varphi(t)}{u - t}.$$

Setting

$$\beta \coloneqq \sup_{s\in(a,t)} \frac{\varphi(t)-\varphi(s)}{t-s},$$

it follows that

$$\beta \leqslant \frac{\varphi(u)-\varphi(t)}{u-t}$$

for all $u \in (t,b)$. Thus, for all $a < s < b$, we have

$$\varphi(s) \geqslant \varphi(t) + \beta(s-t).$$

In particular, this holds for all $s = f(x)$, where $x \in \Omega$, so that

$$\varphi(f(x)) \geqslant \varphi(t) + \beta f(x) - \beta t.$$

Integrating, and using the fact that t is a constant, we get

$$\int_\Omega \varphi \circ f \, d\mu \geqslant \varphi\left(\int_\Omega f \, d\mu\right) + \underbrace{\beta \int_\Omega f \, d\mu - \beta t \mu(\Omega)}_{=0}.$$

Thus, we obtain

$$\varphi\left(\int_\Omega f \, d\mu\right) \leqslant \int_\Omega \varphi \circ f \, d\mu.$$

□

Jensen's inequality is incredibly powerful, as it allows us to establish a variety of other classic inequalities. For instance, if we set $\varphi(x) = e^x$, then Jensen's inequality becomes

$$e^{\int_\Omega f \, d\mu} \leqslant \int_\Omega e^f \, d\mu.$$

If $\Omega = \{p_1, \dots, p_n\}$, where $\mu(\{p_i\}) = \frac{1}{n}$ with $f(p_i) = x_i$, then this gives

$$e^{\frac{1}{n}(x_1+\cdots+x_n)} \leqslant \frac{1}{n}(e^{x_1} + \cdots + e^{x_n}).$$

Setting $y_i \coloneqq e^{x_i}$, we recover the AM-GM inequality,

$$\left(\prod_{i=1}^n y_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^n y_i.$$

More generally, if $\mu(\{p_i\}) = \alpha_i > 0$, and $\sum_{i=1}^n \alpha_i = 1$, we obtain

$$\prod_{i=1}^n y_i^{\alpha_i} \leqslant \sum_{i=1}^n \alpha_i y_i.$$

Definition. If $1 \leqslant p, q \leqslant \infty$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then we call p and q *conjugate exponents*. We use the convention that $\frac{1}{\infty} = 0$, so that $p = 1, q = \infty$ is a pair of conjugate exponents.

Theorem (Young's Inequality)**:** If p and q are conjugate exponents, then for any positive a, b, we have

$$ab \leqslant \frac{1}{p}a^p + \frac{1}{q}b^q$$

Proof. Note that $\frac{1}{p} = 1 - \frac{1}{q}$. Thus, since ln is a concave function,

$$\begin{aligned}
\ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) &\geqslant \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) \\
&= \ln(a) + \ln(b) \\
&= \ln(ab).
\end{aligned}$$

Now, since e^x preserves order, we obtain Young's inequality by taking exponentials. □

In Real Analysis II, we used Young's Inequality to prove Hölder's Inequality and Minkowski's Inequality for the case of $x, y \in \mathbb{C}^n$.

Theorem (Hölder's Inequality for $\mathbb{C}^n$)**:** Let $x, y \in \mathbb{C}^n$. Then, if p and q are conjugate exponents,

$$\sum_{j=1}^{n} |x_j y_j| \leqslant \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q\right)^{1/q}.$$

Theorem (Minkowski's Inequality for $\mathbb{C}^n$)**:** Let $x, y \in \mathbb{C}^n$. Then, for any $p \geqslant 1$,

$$\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{1/p} \leqslant \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p}.$$

We will prove these inequalities in the most general case — i.e., with integrals.

Hölder's Inequality

Theorem (Hölder's Inequality)**:** Let p and q be conjugate exponents with $1 < p < \infty$, and let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g \colon X \to [0, \infty]$ be measurable functions. Then,

$$\int_X fg \, d\mu \leqslant \left(\int_X f^p \, d\mu\right)^{1/p} \left(\int_X g^q \, d\mu\right)^{1/q}.$$

Proof. Set

$$\begin{aligned}
A &:= \left(\int_X f^p \, d\mu\right)^{1/p} \\
B &:= \left(\int_X g^q \, d\mu\right)^{1/q}.
\end{aligned}$$

We may safely assume that $0 < A, B < \infty$. Set

$$\begin{aligned}
F &= \frac{f}{A} \\
G &= \frac{g}{B},
\end{aligned}$$

giving

$$\int_X F^p \, d\mu = 1$$

$$\int_X G^q \, d\mu = 1.$$

Now, if x is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$, then there exist s, t such that $F(x) = e^{s/p}$ and $G(x) = e^{t/q}$, as $e^x \colon \mathbb{R} \to (0, \infty)$ is surjective. Since $\frac{1}{p} + \frac{1}{q} = 1$, and the exponential function is convex, we get, from Jensen's Inequality,

$$\begin{aligned} e^{s/p} e^{t/q} &= e^{s/p + t/q} \\ &\leqslant \frac{1}{p} e^s + \frac{1}{q} e^t, \end{aligned}$$

and substituting, we have

$$F(x)G(x) \leqslant \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q$$

for all $x \in X$. Integrating, we have

$$\int_X FG \, d\mu \leqslant 1.$$

Substituting our definition for F and G, we get

$$\int_X fg \, d\mu \leqslant \left(\int_X f^p \, d\mu \right)^{1/p} \left(\int_X g^q \, d\mu \right)^{1/q}.$$

□

Minkowski's Inequality

Theorem (Minkowski's Inequality)**:** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f, g \colon X \to [0, \infty]$ be such that

$$\begin{aligned} \int_X f^p \, d\mu &< \infty \\ \int_X g^p \, d\mu &< \infty. \end{aligned}$$

Then, for all $1 \leqslant p \leqslant \infty$,

$$\left(\int_X (f+g)^p \, d\mu \right)^{1/p} \leqslant \left(\int_X f^p \, d\mu \right)^{1/p} + \left(\int_X g^p \, d\mu \right)^{1/p}.$$

Proof. Write

$$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}.$$

Then, by Hölder's inequality, we have

$$\begin{aligned} \int_X f(f+g)^{p-1} \, d\mu &\leqslant \left(\int_X f^p \, d\mu \right)^{1/p} \left(\int_X (f+g)^{(p-1)q} \, d\mu \right)^{1/q} \\ \int_X g(f+g)^{p-1} \, d\mu &\leqslant \left(\int_X g^p \, d\mu \right)^{1/p} \left(\int_X (f+g)^{(p-1)q} \, d\mu \right)^{1/q}. \end{aligned}$$

Adding, and noting that $(p-1)q = p$, we have

$$\int_X (f+g)^p \, d\mu \leqslant \left(\int_X (f+g)^p \, d\mu \right)^{1/q} \left(\left(\int_X f^p \, d\mu \right)^{1/p} + \left(\int_X g^p \, d\mu \right)^{1/p} \right) \tag{$*$}$$

By the convexity of t^p, for $0 < t < \infty$, we have

$$\left(\frac{1}{2}(f+g)\right)^p \leqslant \frac{1}{2}(f^p + g^p),$$

so the left side of $(*)$ is finite. Dividing, we have

$$\left(\int_X (f+g)^p \, d\mu\right)^{1/p} \leqslant \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}.$$

In the case of $p = 1$ or $p = \infty$, Minkowski's inequality follows from the triangle inequality for $|\cdot|$. $\square$

The L_p-Spaces

Inspired by Minkowski's inequality, we consider a special class of normed space — the L_p-spaces — and show some of its important properties.

Definition. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. We define the space $L_p(\Omega, \mu)$ to be

$$L_p(\Omega, \mu) := \left\{ f\colon \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f|^p \, d\mu < \infty \right\}.$$

The norm on $L_p(\Omega, \mu)$ is defined by

$$\|f\|_{L_p} = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}.$$

Note that by Minkowski's inequality, the norm on $L_p(\Omega, \mu)$ is a bona fide norm.

In this section, we will establish the two major properties of the L_p-spaces — first, their completeness, and second, the duality between L_p and L_q.