

Abstract

We discuss the much-celebrated Baire Category Theorem and some of its consequences.

The Theorem

The Baire Category Theorem is a very celebrated theorem in analysis and topology that bridges the gap between the two fields. In words, it allows us to understand properties of complete metric spaces via their topological properties (i.e., density of open sets).

Definition (Baire Space). Let $\{A_n\}_{n \geq 1}$ be a countable collection of open, dense subsets of a topological space X . We say X is a *Baire Space* if

$$A = \bigcap_{n \geq 1} A_n$$

is dense for every such collection.

Definition. We say a topological space X is *meager* if it is a countable union of nowhere dense subsets.^I

Theorem (Baire Category Theorem). If X is a complete metric space, then X is a Baire space.

Proof. Let $\{A_n\}_{n \geq 1}$ be a countable family of open, dense subsets of X . Let U_0 be any ball of radius $r > 0$, and set $B_0 = \overline{U_0}$.

Since $A_1 \cap U_0$ is nonempty, there is a closed subset $B_1 \subseteq A_1 \cap U_0$ with radius less than $r/2$. Set $U_1 = B_1^\circ$. Similarly, find $B_2 \subseteq A_2 \cap U_1$ with radius less than $r/4$, and set $U_2 = B_2^\circ$, and continue finding $B_m \subseteq A_m \cap U_{m-1}$ with radius less than $r/2^m$. We get a chain

$$B_0 \supseteq U_0 \supseteq B_1 \supseteq U_1 \supseteq \cdots,$$

and let $(x_n)_n$ be the sequence of centers of B_n . The sequence $(x_n)_n$ is Cauchy in X , as the distance between any two x_m and x_n is at most $r/2^{m-1}$ for $n > m$. Since X is complete, we get $(x_n)_n \rightarrow x$ for some $x \in X$. We claim $x \in \bigcap_{n \geq 1} B_n$.

If not, then $x \notin B_N$ for some $N \in \mathbb{N}$, and for $n \geq N$, $x \notin B_n$, so $d(x_n, x) \geq \text{dist}_{B_n}(x) > 0$, which contradicts $x_n \rightarrow x$.

Therefore, $x \in \bigcap_{n \geq 1} B_n \subseteq \bigcap_{n \geq 1} A_n$, and since $\bigcap_{n \geq 1} B_n \subseteq U_0$, we have $\left(\bigcap_{n \geq 1} A_n\right) \cap U_0 \neq \emptyset$, so $\bigcap_{n \geq 1} A_n$ is dense in X . \square

Remark: By taking complements, we see that if X is a complete metric space, then X is not the countable union of nowhere dense subsets.

The Consequences

There are a variety of great results that can be found from applying the Baire Category Theorem. First, we use the theorem to establish a fundamental result in the theory of complete metric spaces.

Proposition. Let X be a complete metric space with no isolated points, and let $S \subseteq X$ be a countable and dense subset. Then, S is not a G_δ set.

^IRecall that a set $A \subseteq X$ is nowhere dense if $\overline{A}^\circ = \emptyset$.

Proof. Since S is countable, we may write $S = \{s_k\}_{k \in \mathbb{N}}$. Suppose S is G_δ , so that $S = \bigcap_{n \geq 1} V_n$, where the V_n are open. We note that $S^c = \bigcup_{n \geq 1} V_n^c$, and write

$$\begin{aligned} X &= S^c \cup \{s_k\}_{k \in \mathbb{N}} \\ &= \bigcup_{n \geq 1} V_n^c \cup \{s_k\}_{k \in \mathbb{N}}. \end{aligned}$$

We see that V_n^c and $\{s_k\}$ are closed subsets of X for each n, k , and $\{s_k\}_{k \in \mathbb{N}}^\circ = \emptyset$ as X does not contain any isolated points.

By the Baire Category Theorem, there must be some $n \in \mathbb{N}$ such that $(V_n^c)^\circ \neq \emptyset$, so there is some $\delta > 0$ and $x \in X$ such that $U(x, \delta) \subseteq V_n^c \subseteq S^c$ for some $\delta > 0$. However, since S is dense, $U(x, \delta) \cap S \neq \emptyset$, which is a contradiction. \square

Bases of Banach Spaces

One of the simplest, yet surprisingly interesting results that uses the Baire Category Theorem regards the dimension of Banach spaces.

Proposition. If X is an infinite-dimensional Banach space, then X has an uncountable basis.

Proof. Suppose that X had a countable basis $\{e_k\}_{k=1}^\infty$. Then, we may consider the sequence of subspaces

$$\begin{aligned} E_1 &= \text{span}\{e_1\} \\ E_2 &= \text{span}\{e_1, e_2\} \\ &\vdots \\ E_n &= \text{span}\{e_1, \dots, e_n\}. \end{aligned}$$

We see that these finite-dimensional subspaces are closed, that $\bigcup_{n \geq 1} E_n = X$, and since X has infinitely many dimensions, each of the interiors of the E_n are empty. However, since X is a Banach space, this cannot hold, so X does not have a countable basis. \square

Example. The sequence space c_c consists of all finitely supported sequences in \mathbb{C} , equipped with the supremum norm. The canonical basis for the sequence space is $\{e_n\}_{n \geq 1}$, where e_n denotes the sequence with 1 at position n and zero elsewhere. So, c_c has a countable basis.

Yet, by taking the completion of c_c with respect to the supremum norm, we obtain c_0 , which is the sequences in \mathbb{C} that converge to zero, equipped with the supremum norm.^{II} Since c_0 is complete, it is thus equipped with an uncountable basis.

Theorems from Functional Analysis

Partitioning the Reals

^{II}This is a pretty simple exercise.