Compact Operators

Definition: A linear map $T\colon X \to Y$ between Banach spaces is called *compact* if $T(B_X) \subseteq Y$ has compact closure, where B_X denotes the closed unit ball of X. We denote the space of compact operators $K(X, Y)$.

The theory of compact operators (and the soon to arise Fredholm operators) arose from the analysis of integral equations. To start, let $I = [0, 1]$, and consider the Banach space $C(I)$ with the supremum norm. Letting $k \in C(I \times I)$, we define $u \in B(X)$ by taking

$$Tf(x) = \int_0^1 k(x, y)f(y)\,dy.$$

The fact that $Tf \in X$ follows from an application of the Dominated Convergence Theorem and the fact that, since $k(x, y)$ is jointly continuous, it is also separately continuous (see [Fol99, Theorem 2.27]). In fact, we can show something even stronger: we claim that the family $T(B_X)$ is in fact equicontinuous. This follows from the fact that, I^2 is compact, so if $\varepsilon > 0$, there is δ such that whenever $\max\{|x - x'|, |y - y'|\} < \delta$, we have $|k(x, y) - k(x', y')| < \varepsilon$. Therefore,

$$\begin{aligned} |Tf(x) - Tf(x')| &= \left|\int_0^1 (k(x, y) - k(x', y))f(y)\,dy\right| \\ &\leq \int_0^1 |k(x, y) - k(x', y)||f(y)|\,dy \\ &\leq \sup_{y\in I}|k(x, y) - k(x', y)|\|f\|_u \\ &\leq \varepsilon\|f\|_u. \end{aligned}$$

Furthermore, since

$$|Tf(x)| \leq \|k\|_u\|f\|_u,$$

we have that $T(B_X)$ is pointwise bounded. Thus, by the Arzelà–Ascoli theorem, it follows that $T(B_X)$ is totally bounded, so T is a compact operator. We call the function k the *kernel* of the operator T.

Similarly, the operator $V \in B(X)$ given by

$$Vf(x) = \int_0^x f(y)\,dy$$

is such that $V(B_X)$ is totally bounded by Arzelà–Ascoli, so V is also compact. In fact, V has no eigenvalues as well. This follows from the fact that, if there were $\lambda \in \mathbb{C} \setminus \{0\}$ with $V(f) = \lambda f$, then $f(0) = 0$ and $f'(t) = 1/\lambda f(t)$, so that $f(t) = f(0)e^{t/\lambda} = 0$, meaning $f = 0$.

We call the operator V the *Volterra integral operator* on X.

We can see that $K(X)$ is in fact an algebraic ideal in $B(X)$ (by continuity). In fact, there is a topological dimension to $K(X) \subseteq B(X)$.

Proposition: If X, Y are Banach spaces, then $K(X, Y)$ is a closed subspace of $B(X, Y)$.

Proof. Let $(T_n)_n$ converge to $T \in B(X, Y)$. Let $\varepsilon > 0$, and select N such that $\|T_N - T\| < \varepsilon/3$. Since $T_N(B_X)$ is totally bounded, there are $x_1, \dots, x_n \in B_X$ such that for each $x \in S$, we have

$$\|T_N x - T_N x_j\| < \varepsilon/3$$

for some j. Therefore, we have

$$\begin{aligned} \|Tx - Tx_j\| &\leq \|Tx - T_N x\| + \|T_N x - T_N x_j\| + \|T_N x_j - Tx_j\| \\ &< \varepsilon. \end{aligned}$$

Therefore, $T(B_X)$ is totally bounded, so $T \in K(X, Y)$. □

Therefore, we see that $\overline{F(X,Y)} \subseteq K(X,Y)$ is, where $F(X,Y)$ denotes the finite-rank operators, but this inclusion may be strict. In the cases where $\overline{F(X)} = K(X)$, we say the Banach space X has the approximation property. There are Banach spaces that do not have the approximation property.

References

[Mur90] Gerard J. Murphy. *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990, pp. x+286. ISBN: 0-12-511360-9.

[Fol99] Gerald B. Folland. *Real analysis.* Second. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xvi+386. ISBN: 0-471-31716-0.