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Collaborators: Gianluca Crescenzo, Noah Smith, Carly Venenciano

Problem 3

Problem. Let V be an $\mathbb{F}$-vector space.

(a) Prove that an arbitrary intersection of subspaces of V is again a subspace of V.

(b) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution.

(a) Let $U, W \subseteq V$ be subspaces. Since U and W are subspaces, $0_V \in U$ and $0_V \in W$, meaning $U \cap W$ is nonempty.

Let $u, w \in U \cap W$, and let $\alpha \in \mathbb{F}$. Then, since $u \in U$ and $w \in U$, it is the case that $u + \alpha w \in U$. Similarly, since $u \in W$ and $w \in W$, it is the case that $u + \alpha w \in W$. Thus, $u + \alpha w \in U \cap W$, meaning $U \cap W$ is a subspace.

Having shown the base case, we let $\bigcap_{k=1}^{N} U_k$ be an intersection of subspaces U_k. By the inductive hypothesis, we have $W = \bigcap_{k=1}^{N} U_k$, where W is a subspace.

(b) Let $U, W \subseteq V$ be subspaces.

In the reverse direction, if, without loss of generality, $U \subseteq W$, then it is the case that $U \cup W = W$, meaning that $U \cup W$ is a subspace of V.

In the forward direction, suppose toward contradiction that there exist subspaces $U, W \in V$ such that $U \nsubseteq W$ and $W \nsubseteq U$, but $U \cup W$ is a subspace of V. Since $U \nsubseteq W$ and $W \nsubseteq U$, there exist non-trivial vectors $w \in W \setminus U$ and $u \in U \setminus W$. Since $w + u \in W \cup U$, it is the case that $w + u$ is contained either in U or in W. If $w + u \in U$, then $(w + u) - u \in U$ (as $u \in U$ and U is a subspace), meaning $w \in U$, which is a contradiction. Similarly, if $w + u \in W$, then $(w + u) - w \in W$, or $u \in W$, which is yet again a contradiction.

Thus, it must be the case that $W \subseteq U$ or $U \subseteq W$.

Problem 4

Problem. Let $T \in \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}, \mathbb{F})$. Prove there exists $\alpha \in \mathbb{F}$ such that $T(v) = \alpha v$ for all $v \in \mathbb{F}$.

Solution. Since $\dim_{\mathbb{F}}(\mathbb{F}) = 1$, we know that the basis of $\mathbb{F}$ is $\{\beta\}$ for some $\beta \in \mathbb{F}$. For $v \in \mathbb{F}$, it is then the case that v is a linear combination of the basis of $\mathbb{F}$ over $\mathbb{F}$, meaning $v = v_0\beta$ for some $v_0 \in \mathbb{F}$, implying $\beta = \left(v_0^{-1}\right) v$.

Considering a linear transformation $T(v)$, we have

$$T(v) = T(v_0\beta).$$

Substituting $\beta = v_0^{-1}v$, and using the commutativity and associativity of multiplication under $\mathbb{F}$, we have

$$T(v) = T\left(v\left(v_0^{-1}v\right)\right).$$

Using the fact that T is linear and $v \in \mathbb{F}$, we have

$$\begin{aligned} &= vT\left(v_0^{-1}v_0\right) \\ &= vT(1). \end{aligned}$$

Thus, $\alpha = T(1)$.

Problem 6

Problem. Let V be an $\mathbb{F}$-vector space. Prove that if $\{v_1, \dots, v_n\}$ is linearly independent, then so is the set $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$.

Solution. To prove that $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ is linearly independent, we consider the sum

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_n v_n,$$

and show that this sum equals zero if and only if $a_i = 0$ for each i. Rearranging the sum, we have

$$a_1 v_1 + (a_2 - a_1)v_2 + \cdots + (a_{n-1} - a_{n-2})v_{n-1} + (a_n - a_{n-1})v_n.$$

Since the set $\{v_1, \dots, v_n\}$ are linearly independent, this linear combination equals 0_V if and only if $a_1 = (a_2 - a_1) = \cdots = a_n - a_{n-1} = 0$. In particular, since $a_1 = 0$, it must be the case that $a_2 = 0$, $a_3 = 0$, and so on.

Thus, $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ are linearly independent.

Problem 13

Problem. Let p be a prime and V a dimension n vector space over $\mathbb{F}_p$. Show there are

$$(p^n - 1)(p^n - p)\left(p^n - p^2\right)\cdots\left(p^n - p^{n-1}\right)$$

distinct bases of V.

Solution. We begin by constructing our basis by selecting $v_1 \in V \setminus \{0_V\}$. Since V is a dimension n vector space over $\mathbb{F}_p$, it is the case that there are $p^n - 1$ options to select v_1.

To select v_2, we find $v_2 \in V \setminus \operatorname{span}\{v_1\}$; since $|\operatorname{span}\{v_1\}| = p$, there are $p^n - p$ vectors that are linearly independent of v_1.

To select v_3, we find $v_3 \in V \setminus \operatorname{span}\{v_1, v_2\}$; since $\left|\operatorname{span}\{v_1, v_2\}\right| = p^2$, there are $p^n - p^2$ vectors that are linearly independent of $\{v_1, v_2\}$.

Continuing down the chain, we find that to select v_i, one can select from $p^n - p^{i-1}$ vectors that are linearly independent of $\{v_1, \dots, v_{i-1}\}$.

Thus, the number of distinct bases of V is

$$\prod_{i=0}^{n-1}\left(p^n - p^i\right).$$