Problem 1

Let V be a vector space and suppose $\{W_i\}$ is a family of subspaces of V.

(i) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of V contained in every W_i.

Proof: We will show that (a) $\bigcap_{i\in I} W_i$ is a subspace of V, and (b) there is is no larger subspace of V contained within every W_i.

(a) Let $v_i, v_j \in \bigcap_{i\in I} W_i$, $\alpha, \beta \in \mathbb{F}$. We want to show that $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$. Since $v_i \in \bigcap_{i\in I} W_i$, $v_i \in W_i$ for some W_i, and $v_j \in W_j$ for some W_j. Additionally, WLOG, $v_j \in W_i$, as both v_i and v_j are contained within their intersection. Therefore, $\alpha v_i + \beta v_j \in W_i$, so $\alpha v_i + \beta v_j \in \bigcap_{i\in I} W_i$.

(b) Suppose there is a subspace U of V such that every W_i is contained in U, and $U \supset \bigcap_{i\in I} W_i$.

(ii) Show that

$$\sum_{i\in I} W_i := \left\{ \sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite} \right\}$$

is the smallest subspace containing each W_i.

Problem 9

Given any function $f : [0,1] \to \mathbb{C}$, we define

$$N(f) := \sup_{x\neq y, x,y\in[0,1]} \frac{|f(x) - f(y)|}{|x-y|}$$

and

$$\|f\|_\Lambda := |f(0)| + N(f).$$

Moreover, set

$$\Lambda[0,1] := \{f : [0,1] \to \mathbb{C} \mid \|f\|_\Lambda < \infty\}$$

(i) Show that $\Lambda[0,1]$ is precisely the set of Lipschitz continuous functions on $[0,1]$.

Proof: Let $f \in \Lambda[0,1]$. Then, $\|f\|_\Lambda = c$ for some finite c. Then, for $x, y \in [0,1]$

$$\begin{aligned} \frac{|f(x) - f(y)|}{|x-y|} &\leq N(f) \\ &\leq \|f\|_\Lambda \\ &= c. \end{aligned}$$

So,

$$|f(x) - f(y)| \leq c|x-y|,$$

which defines a Lipschitz continuous function.

(ii) Verify that $\Lambda[0,1]$ is a vector space with norm $\|f\|_\Lambda$, which is the Lipschitz norm.

Proof of Vector Space: Let $f, g \in \Lambda[0,1]$. Then, f and g are Lipschitz continuous. Let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned} |(\alpha f)(x) - (\alpha f)(y)| &= |\alpha||f(x) - f(y)| \\ &\leq |alpha|c|x-y| \\ &= h|x-y|, \end{aligned}$$

and

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq c|x-y| + d|x-y| \\ &= \ell|x-y|, \end{aligned}$$

meaning that $\Lambda[0,1]$ is closed under addition and scalar multiplication.

Proof of Norm:

Non-Negativity: Since, for any f, $|f(0)| \geq 0$, and $\|f\|_\Lambda \geq |f(0)|$, it must be the case that $\|f\|_\Lambda \geq 0$.

Positive Definiteness:

$$\|f\|_\Lambda = 0$$
$$|f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|} = 0,$$

meaning that for $x, y \in [0,1]$ and $x \neq y$

$$f(x) = f(y)$$

and

$$f(0) = 0$$

so $f = \mathbb{0}_f$. Additionally, if $f = \mathbb{0}_f$, then $\|f\|_\Lambda = 0$ since $|f(0)| = 0$ and $f(x) = f(y) = 0$ for all $x, y \in [0,1]$.

Absolute Homogeneity: Let $\alpha \in \mathbb{C}$.

$$\begin{aligned}
\|\alpha f\| &= |\alpha f(0)| + N(\alpha f) \\
&= |\alpha||f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|\alpha f(x) - \alpha f(y)}{|x-y|} \\
&= |\alpha| \left(|f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|} \right) \\
&= |\alpha|\|f\|_\Lambda
\end{aligned}$$

Triangle Inequality: Let $f, g \in \Lambda[0,1]$. Then,

$$\begin{aligned}
\|f+g\| &= |f(0)+g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)+g(x)-(f(y)+g(y))}{|x-y|} \\
&\leq |f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|} + |g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|g(x)-g(y)|}{|x-y|} \\
&= \|f\|_\Lambda + \|g\|_\Lambda
\end{aligned}$$

Therefore, $\Lambda[0,1]$ is a normed vector space with $\|\cdot\|_\Lambda$ as the Lipschitz norm.

(iii) Show that $\|f\|_u \leq \|f\|_\Lambda$ for every $f : [0,1] \to \mathbb{R}$.

Problem 10

Let p be a seminorm on a vector space V.

(i) Show that $N_p := \{w \in V \mid p(w) = 0\}$ is a subspace of V.

Proof: Let $v, w \in N_p$. Then, $p(v) = 0$ and $p(w) = 0$. Since p is a seminorm, for $\alpha, \beta \in \mathbb{F}$, we have:

$$\begin{aligned}
p(\alpha v + \beta w) &\leq p(\alpha v) + p(\beta w) \\
&= |\alpha| p(v) + |\beta| p(w) \\
&= 0.
\end{aligned}$$

Since p is definitionally non-negative, $p(\alpha v + \beta w) = 0$. Therefore, N_p is a vector space.

(ii) We form the quotient vector space V/N_p. Show that

$$\|[v]_{N_p}\|_p := p(v)$$

defines a norm on V/N_p.

(iii) If $(E, \|\cdot\|)$ is a normed space and $T : V \to E$ is a linear map, show that $p(v) := \|T(v)\|$ is a seminorm on V. In this case, what is N_p.