Problem 1

Let $x_1 = 1$ and inductively set $x_{n+1} = \sqrt{2 + x_n}$. Show that $(x_n)_n$ converges and find its limit.

We claim that $(x_n)_n$ is bounded above by 2 and monotone increasing.

Base case, we have $x_1 = 1 \leq 2$. Inductively assume that $x_n \leq 2$.

$$\begin{aligned} x_{n+1} &= \sqrt{2 + x_n} \\ &\leq \sqrt{2 + 2} \\ &\leq 2 \end{aligned}$$

Similarly, we know that $\forall 0 \leq x \leq 2$, it is the case that $x^2 \leq 2 + x$, as $x^2 - x - 2 \leq 0 \Rightarrow -1 \leq 0 \leq x \leq 2$. Therefore, $\forall n \in \mathbb{N}$,

$$\begin{aligned} x_n &\leq \sqrt{2 + x_n} \\ &= x_{n+1} \end{aligned}$$

Therefore, since the sequence is monotone increasing and bounded above, the sequence is convergent. Specifically, we know that $(x_n)_n \to x$ and $(x_{n+1})_n \to x$, so

$$\begin{aligned} x &= \sqrt{2 + x} \\ x^2 - x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Problem 2

Does the following sequence converge?

$$x_n := \sum_{k=n+1}^{2n} \frac{1}{k}$$

Listing out terms, we have:

$$\begin{aligned} x_1 &= \frac{1}{2} \geq \frac{1}{2} \\ x_2 &= \frac{1}{3} + \frac{1}{4} \geq x_1 \\ x_3 &= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \geq x_2 \\ x_4 &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq x_3 \end{aligned}$$

Using the contractive test, we have:

$$\begin{aligned} |x_{n+1} - x + n| &\leq \rho \, |x_n - x_{n-1}| \\ \left| \frac{1}{(2n+2)(2n+1)} \right| &\leq \rho \left| \frac{1}{(2n)(2n-1)} \right| \\ \frac{(2n)(2n-1)}{(2n+2)(2n+1)} &\leq \rho \end{aligned}$$

Since ρ cannot be constant, we have no constant of contraction, meaning the sequence is not Cauchy and thus not convergent.

Problem 3

Let $(f_n)_n$ denote the Fibonacci sequence and let

$$x_n := \frac{f_{n+1}}{f_n}.$$

Given that $(x_n)_n$ converges, find its limit.

$$\begin{aligned} x_n &= \frac{f_{n+1}}{f_n} \\ &= \frac{f_n + f_{n-1}}{f_n} \\ &= 1 + \frac{f_{n-1}}{f_n} \\ &= 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \\ &= 1 + \frac{1}{x_{n-1}} \end{aligned}$$

As $(x_n)_n \to x$, $(x_{n-1})_n \to x$, so

$$\begin{aligned} x &= 1 + \frac{1}{x} \\ x^2 - x - 1 &= 0 \\ x &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

Problem 4

If $(x_n)_n$ is an unbounded sequence of real numbers, show that there is a sequence $(x_{n_k})_k$ such that

$$\left(\frac{1}{x_{n_k}}\right)_k \xrightarrow{x\to\infty} 0.$$

Since $(x_n)_n$ is not bounded, it is not convergent — as $(x_n)_n$ is not convergent, it is not Cauchy. Set $\varepsilon_0 = 2$. Then, $\exists n_1 > 1$ such that

$$|x_{n_1} - x_1| \geq \varepsilon_0$$

Similarly, $\exists n_2 > 0$ such that

$$|x_{n_2} - x_{n_1}| \geq \varepsilon_0$$

Inductively, we have that $\forall k \in \mathbb{N}, \exists n_k > n_{k-1}$ such that

$$|x_{n_k} - x_{n_{k-1}}| \geq \varepsilon_0$$

Therefore, $|x_{n_k}| > \left|x_{n_{k-1}}\right|$, so

$$\left(\frac{1}{|x_{n_k}|}\right)_k \to 0$$

as the given subsequence is monotone decreasing bounded below by zero.

Problem 5

Suppose that every subsequence of a sequence $(x_n)_n$ has a subsequence that converges to 0. Show that $(x_n)_n \to 0$.

Every subsequence of $(x_n)_n$ has a subsequence that converges to zero — in particular, this means the sequence $(x_n)_n$ has a subsequence that converges to zero.

$$(x_{n_k})_k = (x_{n_1}, x_{n_2}, \dots)$$

consider the subsequence excluding this sequence:

$$(x_n)_n \setminus (x_{n_k})_k = (x_1, x_2, \dots, x_{n_1 - 1}, x_{n_1 + 1}, \dots)$$

This subsequence admits a subsequence that converges to zero:

$$(x_{m_k})_k = (x_{m_1}, x_{m_2}, \dots)$$

Inductively, via this process, we can assign every element of $(x_n)_n$ to a subsequence that converges to zero; therefore, $(x_n)_n$ must converge to zero.

Problem 6

Let $(I_n)_n$ be a nested sequence of closed and bounded intervals. For each $n \in \mathbb{N}$ let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem for the sequence $(x_n)_n$ to give a proof of the Nested Intervals Property.

We have that

$$\begin{aligned}
x_1 &\in [a_1, b_1] \\
x_2 &\in [a_2, b_2] \subseteq [a_1, b_1] \\
&\vdots \\
x_n &\in [a_n, b_n] \subseteq [a_1, b_1]
\end{aligned}$$

So, as $a_1 \leq x_n \leq b_1$, we have that $\exists n_k$ such that

$$\left(x_{n_k}\right)_k \to x$$

for some x. However, as $k \to \infty$, this means $n \to \infty$, or equivalently, we have that

$$x \in \bigcap_{n=1}^{\infty} I_n,$$

meaning $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Problem 7

If $(x_n)_n$ is a bounded sequence and $s := \sup_n x_n$ such that $s \notin \{x_n \mid n \geq 1\}$, show that there is a subsequence $\left(x_{n_k}\right)_k$ that converges to s.

We know that since $(x_n)_n$ is bounded, $\exists (n_k)_k$ such that $\left(x_{n_k}\right)_k$ is monotone.

Suppose that $\left(x_{n_k}\right)_k$ is monotone decreasing. This means $\left(x_{n_k}\right)_k \to \inf\left(x_{n_k}\right)$. However, this means $x_{n_1} \leq x_{n_2} \leq \cdots$, meaning any upper bound on this sequence must be in the set of x_n, including the supremum.

Therefore, the monotone subsequence must be a monotone increasing subsequence. Let $\varepsilon > 0$. By the definition of supremum, $\exists n' \in \mathbb{N}$ such that $x_{n'} > s - \varepsilon$, and

$$s \geq \cdots \geq x_{n'+1} \geq x_{n'} \geq s - \varepsilon$$

We let our subsequence be the n' tail of x_n, and we have found a subsequence that converges to s.

Problem 8

Let $(x_n)_n$ and $(y_n)_n$ be bounded sequences. Show that

$$\limsup\left((x_n + y_n)_n\right) \leq \limsup\,(x_n)_n + \limsup\,(y_n)_n$$

We have shown that

$$\sup(x_n + y_n)_n \leq \sup(x_n)_n + \sup(y_n)_n$$

Additionally, we know that the sequences of these suprema are related by the following:

$$\begin{aligned}\left(\sup_{n\geq m}\,(x_n + y_n)_n\right)_m &\leq \left(\sup_{n\geq m}\,(x_n)_n + \sup_{n\geq m}\,(y_n)_n\right)_m \\ &= \left(\sup_{n\geq m}\,(x_n)_n\right)_m + \left(\sup_{n\geq m}\,(y_n)_n\right)_m\end{aligned}$$

Therefore,

$$\inf_m \left(\sup_{n\geq m}\,(x_n + y_n)_n\right)_m \leq \inf_m \left(\sup_{n\geq m}\,(x_n)_n\right)_m + \inf_m \left(\sup_{n\geq m}\,(y_n)_n\right)_m$$

So

$$\limsup\,(x_n + y_n)_n \leq \limsup\,(x_n)_n + \limsup\,(y_n)_n$$

Problem 9

Let $(x_n)_n$ be a bounded sequence. Show that

$$\liminf(x_n)_n = \sup\left\{t \mid t \in \mathbb{R},\ \{n \mid x_n < t\} \text{ is finite}\right\}$$

By the definition of $\liminf(x_n)_n$, we have

$$\liminf(x_n)_n = \sup m \geq 1 \left(\inf_{n\geq m}(x_n)_n\right)_m,$$

The parenthetical on the right is equivalent to:

$$\inf_{n\geq m}(x_n)_n = \max\{t \mid x_n \geq t\} \qquad \forall n \geq m$$

or, alternatively

$$= \sup\{t \mid x_n < t\} \qquad \forall n < m$$

which is equivalent to the following statement

$$\liminf(x_n)_n = \sup\{t \mid \{n \mid x_n < t\} \text{ is finite}\}$$

Problem 10

Let $(x_n)_n$ be a bounded sequence. Show that

$$\liminf(-x_n)_n = -\limsup(x_n)_n$$

$$\begin{aligned}
\liminf(-x_n)_n &= \sup_{m\geq 1}\left(\inf_{n\geq m}(-x_n)_n\right)_m \\
&= \sup_{m\geq 1}\left(-\sup_{n\geq m}(x_n)_n\right)_m \\
&= -\inf_{m\geq 1}\left(\sup_{n\geq m}(x_n)_n\right)_m \\
&= -\limsup(x_n)_n
\end{aligned}$$