Abstract

We discuss and prove some fundamental results about differentiation, after which prove the fundamental theorem of calculus for Lebesgue integrals.

Preliminary

In our discussion of the Radon–Nikodym Theorem, we were able to define an abstract derivative of a (σ-finite) complex measure with respect to a different (σ-finite) measure. In Euclidean space, $\mathbb{R}^n$, we may consider trying to define a "pointwise" derivative by taking

$$F(x) = \lim_{r\to 0} \frac{\nu\left(U\left(x,r\right)\right)}{m\left(U\left(x,r\right)\right)},$$

where m is the Lebesgue measure, and ν is our given complex measure. If we take the Lebesgue–Radon–Nikodym decomposition

$$d\nu = \lambda + f\ dm,$$

we would hope that $F = f$ almost everywhere. Indeed, we will show this to be the case, after which we may prove a stronger version of the fundamental theorem of calculus, this time for Lebesgue integrals.

Note that from now on, every measure-theoretic term (i.e., integrable, almost everywhere, etc.) is taken with respect to the Lebesgue measure on $\mathbb{R}^n$.

We start with a fundamental lemma in measure theory for Euclidean spaces.

Theorem (Vitali Covering Lemma)**:** Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B\in\mathcal{C}} B$.

If $c < m\left(U\right)$, then there exist disjoint $B_1,\dots,B_k$ such that

$$3^{-n}c \leq \sum_{j=1}^{k} m\left(B_j\right).$$

Proof. By inner regularity, there is a compact $K \subseteq U$ such that $m(K) > c$; finitely many balls in $\mathcal{C}$, which we call $A_1,\dots,A_m$, cover K.

We proceed via exhaustion; select B_1 to be the largest of the A_j, B_2 to be the largest of the A_j disjoint from B_1, B_3 the largest of the A_j disjoint from B_2 and B_1, etc. According to this construction, if A_i is not among the B_j, then there is j such that $A_i \cap B_j \neq \emptyset$, and if j is the smallest such index, then the radius of A_i is at most that of B_j. Via some triangle inequality magic, we see that $A_i \subseteq B_j^*$, where B_j^* is defined to the ball with the same center as B_j and three times the radius.

Then, $K \subseteq \bigcup_{j=1}^{k} B_j^*$, so that

$$\begin{aligned}
c &< m(K)\\
&\leq \sum_{j=1}^{k} m\left(B_j^*\right)\\
&= 3^n \sum_{j=1}^{k} m\left(B_j\right).
\end{aligned}$$

$\square$

The Lebesgue Differentiation Theorem

Definition: A function $f\colon \mathbb{R}^n \to \mathbb{C}$ is called *locally integrable* if $\int_K |f|\, dm < \infty$ for every bounded measurable $K \subseteq \mathbb{R}^n$.[I]

The space of locally integrable functions is denoted $L_{1,\text{loc}}$.

Definition: If $f \in L_{1,\text{loc}}$, and $x \in \mathbb{R}^n$, and $r > 0$, define

$$A_r f(x) = \frac{1}{m\left(B\left(x,r\right)\right)} \int_{B(x,r)} f(y)\, dy$$

to be the *average* of f on $B\left(x,r\right)$.

Lemma: If $f \in L_{1,\text{loc}}$, then $A_r f$ is jointly continuous in r and x.

Proof. We know that $m\left(B\left(x,r\right)\right) = cr^n$, where $c = m\left(B\left(0,1\right)\right)$, and $m\left(S\left(x,r\right)\right) = 0$, where $S\left(x,r\right) = \{y \mid |y-x| = r\}$.

Moreover, as $r \to r_0$ and $x \to x_0$, $\mathbb{1}_{B(x,r)} \to \mathbb{1}_{B(x_0,r_0)}$ pointwise on $\mathbb{R}^n \setminus S\left(x_0,r_0\right)$, so the convergence is pointwise almost everywhere. Furthermore, note that $\left|\mathbb{1}_{B(x,r)}\right| \leq \mathbb{1}_{B(x_0,r_0+1)}$ for $r < r_0 + 1/2$ and $|x - x_0| < 1/2$. Thus, by dominated convergence, it follows that $\int_{B(x,r)} f(y)\, dy$ is continuous in r and x, and so is $A_r f(x)$. □

Definition: If $f \in L_{1,\text{loc}}$, we define the *Hardy–Littlewood Maximal Function*, Hf, by

$$\begin{aligned} Hf(x) &= \sup_{r>0} A_r\left|f\right|(x) \\ &= \sup_{r>0} \frac{1}{m\left(B\left(x,r\right)\right)} \int_{B(x,r)} \left|f(y)\right|\, dy. \end{aligned}$$

Theorem (The Maximal Theorem)**:** There is a constant $C > 0$ such that for all $f \in L_1$ and all $\alpha > 0$,

$$m\left(\{x \mid Hf(x) > \alpha\}\right) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} \left|f(x)\right|\, dx.$$

Proof. Let $E_\alpha = \{x \mid Hf(x) > \alpha\}$. For each $x \in E_\alpha$, we may find $r_x > 0$ such that $A_{r_x}\left|f\right|(x) > \alpha$. The balls $U\left(x,r_x\right)$ cover E_α, so by the Vitali Covering Lemma, if $c < m\left(E_\alpha\right)$, then there are $x_1, \dots, x_k$ such that $B_j = B\left(x_j, r_{x_j}\right)$ are disjoint and $\sum_{j=1}^{k} m\left(B_j\right) > 3^{-n}c$.

Then, we see that

$$\begin{aligned} c &< 3^n \sum_{j=1}^{k} m\left(B_j\right) \\ &\leq \frac{3^n}{\alpha} \sum_{j=1}^{k} \int_{B_j} \left|f(y)\right|\, dy \\ &\leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} \left|f(y)\right|\, dy. \end{aligned}$$

Thus, letting $c \to m\left(E_\alpha\right)$, we obtain our desired result. □

The Fundamental Theorem of Calculus for Lebesgue Integration

[I] Note that we still use the convention $0 \cdot \infty = 0$.