

Problem 1

Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a family of subsets satisfying

- (i) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- (ii) If $\{A_k\}_{k \geq 1}$ is a countable family of pairwise disjoint members of \mathcal{A} , then $\bigsqcup_{k \geq 1} A_k \in \mathcal{A}$.

Prove that \mathcal{A} is a σ -algebra on Ω .

Problem 2

Consider the family $\mathcal{E} : \{(-\infty, b) \mid b \in \mathbb{R}\}$. Show that $\sigma(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$.

Proof: Consider the family $\mathcal{E}' := \{[a, b) \mid a, b \in \mathbb{R}\}$. We have established that $\sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

We see that for any element of \mathcal{E} , $(-\infty, b) = \bigcup_{n=1}^{\infty} [a-n, b)$, meaning $\mathcal{E} \in \sigma(\mathcal{E}')$, so $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

Additionally, $[a, b) = (-\infty, b) \setminus (-\infty, a)$, meaning $\mathcal{E}' \in \sigma(\mathcal{E})$, so $\sigma(\mathcal{E}') \subseteq \sigma(\mathcal{E})$, so $\sigma(\mathcal{E}) = \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

Problem 3

Let (Ω, \mathcal{M}) and (Λ, \mathcal{N}) be measurable spaces. We define the product σ -algebra on $\Omega \times \Lambda$ as

$$\mathcal{M} \otimes \mathcal{N} := \sigma(\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}).$$

Prove that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

Proof: For $a < b$ and $c < d$, it is the case that $(a, b) \times (c, d) \subseteq \mathbb{R}^2$ is open, meaning

$$\begin{aligned} \sigma(\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{R}\}) &= \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \\ &\subseteq \mathcal{B}_{\mathbb{R}^2}. \end{aligned}$$

Letting $U \in \mathcal{B}_{\mathbb{R}^2}$, it is the case that $U = \bigcup_{j=1}^{\infty} U(x_j, r_j)$. For each $U(x_j, r_j)$, take $I_j = (x_{jx} - r_j, x_{jx} + r_j) \times (x_{jy} - r_j, x_{jy} + r_j)$, so $U \subseteq \bigcup_{j=1}^{\infty} I_j$. Thus, $U \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, so $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Problem 4

Let (Ω, \mathcal{M}) and (Λ, \mathcal{N}) be measurable spaces. A map $f : \Omega \rightarrow \Lambda$ is \mathcal{M} - \mathcal{N} -measurable if $E \in \mathcal{N} \Rightarrow f^{-1}(E) \in \mathcal{M}$.

Let (Ω, \mathcal{M}) be a measurable space and suppose $E \in \mathcal{M}$. Show that $\mathcal{M}_E = \{M \cap E \mid M \in \mathcal{M}\}$ is a σ -algebra on E and the inclusion map $\iota : E \rightarrow \Omega$ is \mathcal{M}_E - \mathcal{M} -measurable.

Proof: Let $M \in \mathcal{M}$. Then, $\iota^{-1}(M) = E \cap M \in \mathcal{M}_E$. Thus, ι is \mathcal{M}_E - \mathcal{M} -measurable.

Problem 5

Let (Ω, \mathcal{M}) and (Λ, \mathcal{N}) be measurable spaces. Suppose \mathcal{N} is generated as a σ -algebra by a family of subsets $\mathcal{E} \subseteq \mathcal{P}(\Lambda)$. Prove that a map $f : \Omega \rightarrow \Lambda$ is \mathcal{M} - \mathcal{N} -measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$. Conclude that a continuous function $f : X \rightarrow Y$ between metric spaces is \mathcal{B}_X - \mathcal{B}_Y -measurable.

Proof: Let \mathcal{N} be generated by \mathcal{E} . Then, for any $E_1, E_2 \in \mathcal{E}$, it is the case that $E_1^c \in \mathcal{N}$ or $E_1 \cup E_2 \in \mathcal{N}$.

Let f be measurable. Then, since $\mathcal{E} \subseteq \mathcal{N}$, and for any $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$, it is the case that for any $E \in \mathcal{E}$, $f^{-1}(E) \in \mathcal{M}$.

Let f be a function such that for any $E \in \mathcal{E}$, $f^{-1}(E) \in \mathcal{M}$. So, $f^{-1}(E^c) = (f^{-1}(E))^c \in \mathcal{M}$, and $f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2) \in \mathcal{M}$. Therefore, for any $E \in \mathcal{N}$, it must be the case that $f^{-1}(E) \in \mathcal{M}$.

Since the preimage of any element of the topology on Y is the topology on X if f is continuous, it is the case that such a continuous function is \mathcal{B}_X - \mathcal{B}_Y -measurable.

Problem 6

Suppose (Ω, \mathcal{M}) is a measurable space and $f : \Omega \rightarrow \Lambda$ is a map. Show that $\mathcal{N} := \{E \subseteq \Lambda \mid f^{-1}(E) \in \mathcal{M}\}$ is a σ -algebra on Λ and f is \mathcal{M} - \mathcal{N} -measurable. \mathcal{N} is called the σ -algebra produced by f .

Problem 7

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and suppose $\{E_k\}_{k \geq 1}$ is a decreasing sequence of measurable sets with $\mu(E_1) < \infty$. Show that

$$\begin{aligned} \mu\left(\bigcap_{k \geq 1} E_k\right) &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \inf_{k \geq 1} \mu(E_k). \end{aligned}$$

Proof: We see that for n , $\bigcap_{k=1}^n E_k = E_n$. Therefore, $\mu\left(\bigcap_{k=1}^n E_k\right) = \mu(E_n)$, meaning

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{\infty} E_k\right) &= \lim_{n \rightarrow \infty} \mu\left(\bigcap_{k=1}^n E_k\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n). \end{aligned}$$

Problem 8

Let (Ω, \mathcal{M}) and (Λ, \mathcal{N}) be measurable spaces and suppose $f : \Omega \rightarrow \Lambda$ is measurable. If μ is a measure on \mathcal{M} , show that

$$f_*\mu : \mathcal{N} \rightarrow [0, \infty]; \quad f_*\mu(E) := \mu(f^{-1}(E))$$

defines a measure on (Λ, \mathcal{N}) . This is called the pushforward measure.

Proof: Clearly, $f_*\mu(\emptyset) = 0$. Let $E_1, E_2 \in \mathcal{N}$ be disjoint and nonempty. Note that $E_1 \sqcup E_2 \in \mathcal{N}$. Thus,

$$\begin{aligned} f_*\mu(E_1 \sqcup E_2) &= \mu(f^{-1}(E_1 \sqcup E_2)) \\ &= \mu(f^{-1}(E_1) \sqcup f^{-1}(E_2)) \\ &= \mu(f^{-1}(E_1)) + \mu(f^{-1}(E_2)) \\ &= f_*\mu(E_1) + f_*\mu(E_2), \end{aligned}$$

meaning $f_*\mu$ is a measure on (Λ, \mathcal{N}) .

Problem 9

A group G is paradoxical if there are pairwise disjoint subsets of G ; $E_1, \dots, E_n, F_1, \dots, F_m$ and group elements $t_1, \dots, t_n, s_1, \dots, s_m$ such that

$$\begin{aligned} G &= \bigsqcup_{j=1}^n t_j E_j \\ &= \bigsqcup_{k=1}^m s_k F_k. \end{aligned}$$

A mean on a group G is a finitely additive probability measure $\nu : \mathcal{P}(G) \rightarrow [0, 1]$ that is translation invariant; that is, $\nu(tE) = \nu(E)$ for all $E \subseteq G$ and $t \in G$. A group is said to be amenable if it admits a mean.

Show that a paradoxical group is nonamenable.

Proof: Let G be paradoxical. Suppose toward contradiction that there existed such a ν . Then, $\nu(G)$, and

$$\begin{aligned} \nu(G) &= \nu\left(\bigsqcup_{j=1}^n t_j E_j\right) \\ &= \sum_{j=1}^n \nu(t_j E_j) \\ &= \sum_{j=1}^n \nu(E_j). \end{aligned}$$

We know that $G \cup s_1 F_1 = G$, meaning $\nu(G) = \nu(G \cup s_1 F_1)$. However,

$$\begin{aligned} \nu(G \cup s_1 F_1) &= \nu\left(\bigsqcup_{j=1}^n t_j E_j \sqcup s_1 F_1\right) \\ &= \sum_{j=1}^n \nu(t_j E_j) + \nu(s_1 F_1) \\ &= \nu(G) + \nu(s_1 F_1) \\ &= \nu(G) + \nu(F_1) \\ &> \nu(G). \end{aligned}$$

Problem 10

Let Δ be a totally disconnected compact metric space (for example, the Cantor set). Suppose $\varphi : C(\Delta) \rightarrow \mathbb{R}$ is a state — φ is linear, continuous, positive, and $\varphi(\mathbb{1}_\Delta) = 1$.

(i) Show that $\mathcal{C} := \{E \mid \chi_E \in C(\Delta)\}$ is an algebra of subsets on Δ .

(ii) Show that

$$\mu_0 : \mathcal{C} \rightarrow [0, 1]; \quad \mu_0(E) = \varphi(\chi_E)$$

is a well-defined finitely additive measure.

(iii) If $\{E_k\}_{k \geq 1}$ is a countable family of members of \mathcal{C} such that $\bigsqcup_{k \geq 1} E_k \in \mathcal{C}$, show that

$$\mu_0\left(\bigsqcup_{k \geq 1} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

Proof:

(i) If $E \in \mathcal{C}$, then $E \subseteq \Delta$, so $E^c \subseteq \Delta$, and for $E_1, E_2 \in \mathcal{C}$, $E_1 \cup E_2 \in \Delta$.

(ii) Let $E, F \in \mathcal{C}$ with $E \cap F = \emptyset$. Then,

$$\begin{aligned}\mu_0(E \sqcup F) &= \varphi(\mathbb{1}_{E \sqcup F}) \\ &= \varphi(\mathbb{1}_E + \mathbb{1}_F) \\ &= \varphi(\mathbb{1}_E) + \varphi(\mathbb{1}_F) \\ &= \mu_0(E) + \mu_0(F).\end{aligned}$$

(iii) Let $\{E_k\}_{k \geq 1}$ be a countable family of members of \mathcal{C} with $\bigsqcup_{k \geq 1} E_k \in \mathcal{C}$.