Problem 1

(a)

$$\begin{aligned}\int_{C_1}\left(x^2+y^2\right)\,d\ell &= \int_0^1 x^2\,dx + \int_0^1 y^2+1\,dy\\ &= \frac{5}{3}.\end{aligned}$$

(b)

$$\begin{aligned}\int_{C_2}\left(x^2+y^2\right)\,d\ell &= \int_0^1 2x^2\,dx\\ &= \frac{2}{3}.\end{aligned}$$

(c)

$$\begin{aligned}\int_{C_3}\left(x^2+y^2\right)\,d\ell &= \int_0^1 x^2+x^4\,dx\\ &= \frac{8}{15}.\end{aligned}$$

Problem 2

(a) Since $\oint_C\,d\ell$ "adds up" the infinitesimal lengths along C, this integral gives the length of C.

(b) Since $\oint_C\,d\vec{\ell}$ is a vector-valued integral along C, and since C is closed, this integral gives 0.

Problem 3

(a) We have $d\ell = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, and $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}}$, so

$$\begin{aligned}\int_C\,d\ell &= \int\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx\\ &= \int\frac{1}{\sqrt{a^2-x^2}}\,dx\\ &= a\arcsin\left(\frac{x}{a}\right).\end{aligned}$$

Evaluated from $x=-a$ to $x=a$, we get that $\int_C\,d\ell = \pi a$.

(b) We have $d\ell = \sqrt{dr^2+r^2d\theta^2}$, so

$$\begin{aligned}\int\,d\ell &= \int_0^\pi a\,d\theta\\ &= \pi a\end{aligned}$$

Problem 7

(a) $\oint_S\,dA$ gives the area of the sphere, as we do not have to integrate with respect to a direction.

(b) $\oint_S\,d\mathbf{A}$ yields zero, as $\hat{n}$ is symmetrical with respect to S.

Problem 11

$$\begin{aligned}\int_S \mathbf{r}\cdot d\mathbf{A} &= \int (R\hat{r})\cdot\hat{r}\left(R^2 d\Omega\right)\\ &= R^3\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\ d\phi d\theta\\ &= 2\pi R^3\end{aligned}$$

Problem 18

(a) Since $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$ are both zero, this field is both surface independent and path independent.

(b) Since both the curl and divergence of $\hat{r}$ is zero, this field is both surface independent and path independent.

(c)

$$\begin{aligned}\nabla\cdot\left(-\sin x\cosh y\hat{\imath}+\cos x\sinh y\hat{\jmath}\right) &= -\cos x\cosh y+\cos x\cosh y\\ &= 0\\ \nabla\times\left(-\sin x\cosh y\hat{\imath}+\cos x\sinh y\hat{\jmath}\right) &= \left(-\sin x\sinh y+\sin x\sinh y\right)\hat{k}\\ &= 0.\end{aligned}$$

Thus, the field is both surface independent and path independent.

(d)

$$\begin{aligned}\nabla\cdot\left(xy^2\hat{\imath}-x^2y\hat{\jmath}\right) &= 0\\ \nabla\times\left(xy^2\hat{\imath}-x^2y\hat{\jmath}\right) &= -4xy\hat{k}.\end{aligned}$$

Thus, the field is surface independent but not path independent.

(e)

$$\begin{aligned}\nabla\cdot\left(\rho z\hat{\phi}\right) &= 0\\ \nabla\times\left(\rho z\hat{\phi}\right) &\neq 0.\end{aligned}$$

Thus, the field is surface independent but not path independent.

Problem 19

The integral along the path dba is -5 and the integral along the path $ecdb$ is 3.

Problem 20

(a) Since $\nabla\times\mathbf{E}=0$, we find the integral by taking

$$\int_C \mathbf{E}\cdot d\vec{\ell} = \int_{C_1}\mathbf{E}\cdot d\vec{\ell}$$

$$
\begin{aligned}
&= \int_{C_2} \mathbf{E}\cdot d\vec{\ell}\\
&= \frac{1}{2}x^2y^2\Big|_{(0,a)}^{(a,0)}\\
&= 0.
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
\int_{C_1} \mathbf{B}\cdot d\vec{\ell} &= \int_{C_1} B_x\ dx + B_y\ dy\\
&= \sqrt{2}\int_0^a x^2\left(-x+a\right) + \sqrt{2}\int_0^a \left(-y+a\right)y^2\ dy\\
&= 2\sqrt{2}\int_0^a t^2(-t+a)\ dt\\
&= 2\sqrt{2}\int_0^a at^2 - t^3\ dt\\
&= 2\sqrt{2}\left(\frac{a^4}{3}-\frac{a^4}{4}\right)\\
&= \frac{a^4}{6}\sqrt{2}.\\
\int_{C_2} \mathbf{B}\cdot d\vec{\ell} &= \int_0^{\pi/2} \begin{pmatrix} a^3\sin^2 t\cos t\\ -a^3\sin t\cos^2 t\end{pmatrix}\cdot\begin{pmatrix} a\cos t\\ -a\sin t\end{pmatrix}\ dt\\
&= a^4\int_0^{\pi/2} 2\sin^2 t\cos^2 t\ dt\\
&= \frac{a^4}{4}\int_0^{\pi/2} \sin^2\left(2t\right)\ dt\\
&= \frac{\pi}{4}a^4.
\end{aligned}
$$

Problem 21

Problem 22

Problem 26

Problem 28