Contents

Introduction

Finally, the last part of my notes on C*-algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of C*-algebras, discussing ideas such as amenability and nuclearity in C*-algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of $C^*_\lambda(G)$ and $C^*(G)$.

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's *C*-Algebras and Finite-Dimensional Approximations*.

Review: Representations, the Reduced Group $\mathbb{C}^*$-Algebra, and the Universal Group $\mathbb{C}^*$-Algebra

Let Γ be a group. Consider the space $\ell_2(\Gamma)$. For every $s \in \Gamma$, we define the operator

$$\lambda_s(\xi)(t) = \xi\left(s^{-1}t\right).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned}\|\lambda_s(\xi)\|^2 &= \sum_{t\in\Gamma}|\lambda_s(\xi)(t)|^2\\ &= \sum_{t\in\Gamma}\left|\xi\left(s^{-1}t\right)\right|^2\\ &= \sum_{r\in\Gamma}|\xi(r)|^2\\ &= \|\xi\|^2.\end{aligned}$$

Additionally, each λ_s admits an inverse, $\lambda_{s^{-1}} = \lambda_s^*$. Applying to the orthonormal basis $\{\delta_t\}_{t\in\Gamma}$, we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus, $\lambda_s \circ \lambda_r = \lambda_{sr}$, and we have the unitary representation of Γ, $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, where $\lambda(s) = \lambda_s$, for $s \in \Gamma$. This is the left-regular representation of Γ.

Note that the left regular representation is a faithful representation, hence injective.

Because the λ operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}\lambda_f(\xi)(t) &= \left(\sum_{s\in\Gamma} f(t)\lambda_s(\xi)\right)(t)\\ &= \sum_{s\in\Gamma} f(s)\xi\left(s^{-1}t\right)\end{aligned}$$

Note that the space of finitely supported functions on Γ, $\mathbb{C}[\Gamma]$,[1] is a $*$-algebra, where multiplication is given by convolution:

$$\begin{aligned} f * g(t) &= \sum_{s\in\Gamma} f(s)g\left(s^{-1}t\right) \\ &= \sum_{r\in\Gamma} f\left(tr^{-1}\right)g(r). \end{aligned}$$

Note that we are using $*$ both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on $\mathbb{C}[\Gamma]$ is given by

$$f^*(t) = \overline{f\left(t^{-1}\right)}.$$

[1] Also known as the free vector space over $\mathbb{C}$ with basis Γ.