Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: Let $(f_n)_n$ be a Cauchy sequence in $C_0(\mathbb{R})$. Since each $f_k \in C_0(\mathbb{R})$, it must be the case that each f_k is uniformly continuous. For each $x \in \mathbb{R}$, it is thus the case that $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(f_n(x))_n \to f(x)$ for each $x \in \mathbb{R}$, and since each f_k is uniformly continuous, it must be the case that $f(x)$ is continuous.

For $\varepsilon > 0$, there must be N large such that for $m, n \geq N$ and $m \geq n$, it must be the case that $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Letting $m \to \infty$, we have $|f_n(x) - f(x)| < \varepsilon$, meaning $(f_n)_n \to f$. Thus, $f \in C_0(\mathbb{R})$.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof:

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq d(x_2, x_1)\left(\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}\right) \\
&= d(x_2, x_1)\theta^{n-1}\left(1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}\right) \\
&= d(x_2, x_1)\frac{\theta^{n-1}}{1 - \theta}.
\end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.