## Introduction: naive set theory

$$\mathbb{N} = \{1, 2, 3, \ldots, \}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots, \}$$
$$\mathbb{Z}_+ = \{0, 1, 2, \ldots, \}$$
$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$
$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$
$$\mathbb{C}_q = \{a + bi \mid a, b \in \mathbb{Q}\}$$

Recall: given sets $X$ and $Y$, a relation from $X$ to $Y$ is a subset of $X \times Y$, where $\times$ denotes the cartesian product of $X$ and $Y$.

A relation $f \subseteq X \times Y$ is a function from $X$ to $Y$ such that $\forall x \in X$, $\exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote $f$ as $f : X \to Y$.

$X$ is the **domain** of $f$ and $Y$ is the **codomain**. The range $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

### Examples

$$\text{id}_x : X \to X, \text{id}_X(x) = x$$

This is the identity function.

The Characteristic Function: If $A \subseteq X$

$$\mathbf{1}_A : X \to \mathbb{R}, \ \mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

### Algebra of Functions

Let $X$ be any set, and $(X; \mathbb{R}) = \{f : X \to \mathbb{R}\}$ represent the function space of $X$ with codomain $\mathbb{R}$.

Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$ (scalar multiplication). If $g(x) \neq 0 \forall x \in X$, then $\left( \frac{f}{g} \right)(x) := \frac{f(x)}{g(x)}$.

Finally, we have composition. If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f(x) = g(f(x))$.

### Injective, Subjective, and Bijective

A function $f : X \to Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. For example, the shift map $S : \mathbb{N} \to \mathbb{N}$, $S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \to \mathbb{R}$, where $I$ is any interval, is injective.

A function $f$ is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f) = \mathbb{R}$.

$f$ is **bijective** if it is injective and surjective.

### Invertibility

Let $f : X \to Y$ be a function. $f$ is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. $f$ is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$.

$f$ is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \mathrm{id}_Y$ and $k \circ f = \mathrm{id}_X$.

#### Invertibility Definition

$f$ is invertible if and only if $f$ is left and right invertible.

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Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose $g$ is a left-inverse of $f$, and $h$ is a right-inverse of $f$. Therefore, $g \circ f = \mathrm{id}_X$, and $f \circ h = \mathrm{id}_Y$. Observe that $g = g \circ \mathrm{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$.

#### Injection and Surjection Invertibility

If $f : X \to Y$ is a function:

1. $f$ is injective $\Leftrightarrow$ $f$ is left-invertible.

2. $f$ is surjective $\Leftrightarrow$ $f$ is right-invertible.

3. $f$ is bijective $\Leftrightarrow$ $f$ is invertible.

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We will prove the first proposition in the forward direction. Suppose $f$ is injective. Given $y \in \mathrm{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define $g$ as follows:

$$g(y) = \begin{cases} x_y & y \in \mathrm{ran}(f) \\ x_0 & y \notin \mathrm{ran}(f) \end{cases}$$

Where $x_0$ is an arbitrary point in $X$. We can see that $g \circ f = \mathrm{id}_X$.

For example, the function $\mathrm{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

### Cardinality and Finitude

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets $A$ and $B$, we can show that $A$ is the same size as $B$ by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

#### Cardinality

Sets $A$ and $B$ have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\mathrm{card}(A) = \mathrm{card}(B)$.

#### Equivalent Cardinalities of Intervals

Given $a < b$ and $c < d$, we know that $\mathrm{card}([a, b]) = \mathrm{card}([c, d])$.

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We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\mathrm{card}([a, b]) = \mathrm{card}([c, d])$.

## Intervals and Real Numbers

$$\text{card}\left((0,1)\right) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - $\tan$ is strictly increasing (and thus injective)
  - $\lim_{x \to \infty} \tan(x) = \infty$ and $\lim_{x \to -\infty} \tan(x) = -\infty$, and by intermediate value theorem, $\tan$ is surjective
- $\ell : (0,1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0,1) \to \mathbb{R}$.

## Finitude

A set $F$ is **finite** if $F$ is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}\left(\{1, 2, \ldots, n\}\right)$. A non-finite set is called infinite.

We can *enumerate* $F$ by creating a function $\sigma : \{1, 2, \ldots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \ldots, x_n\}$.

## Inequality of Finite Sets

If $m \neq n$, then $\text{card}\{1, 2, \ldots, m\} = \text{card}\{1, 2, \ldots, n\}$.

WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ is our bijection. This means there are $m$ "pigeons" and $n$ "holes."

One hole, $j$, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \ldots, m\}$). Since $f$ is assumed to be injective, this is a contradiction.

## Infinitude of the Naturals

$\mathbb{N}$ is infinite.

Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \ldots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \ldots, m+1\} \to \mathbb{N}$. $i$ is injective.

Then, $f \circ i : \{1, 2, \ldots, m+1\} \to \{1, 2, \ldots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

### Proposition

If $A$ is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

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$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

$\vdots$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of $A$.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. $f$ is injective as $a_n$ are distinct.

### Cardinality of Integers and Natural Numbers

$$\mathrm{card}(\mathbb{Z}) = \mathrm{card}(\mathbb{N})$$

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$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m + 1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

$f$ is a bijection as $g : \mathbb{N} \to \mathbb{Z}$, $g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of $f$.

### Power Set

Given any set $X$, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of $X$.

$2^X := \{f \mid f : X \to \{0, 1\}\}$.

### Power Set and $2^X$

$$\mathrm{card}(\mathcal{P}(X)) = \mathrm{card}(2^X)$$

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Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbf{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$.

## Cantor's Theorem

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

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Fact from calculus: $\forall \sigma \in (0,1)$, $\sigma$ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \ldots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$, and $\sigma_j(n) \in \{0, 1, \ldots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0, 1, \ldots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\ldots$. Since $r$ is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\ldots\sigma_m(m)\cdots = \tau = 0.\tau(1)\tau(2)\ldots\tau(m)\ldots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

## Comparing Cardinalities

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$

- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

### Transitive Property

If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

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The composition of two injective functions is injective.

### Canonical Set Comparisons

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

### Cardinality of the Power Set

For any set $A$, $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

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Let us construct a function: $f : A \to \mathcal{P}(A)$, where $a \mapsto \{a\}$.

$f$ is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

**Claim** $\nexists g : A \to \mathcal{P}(A)$, $g$ is surjective.

Suppose toward contradiction that such a $g$ exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since $g$ is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. $\bot$

### Equivalent Propositions

   (i) $\operatorname{card}(A) \leq \operatorname{card}(B)$

  (ii) $\exists f : A \hookrightarrow B$

 (iii) $\exists g : B \to A$, $g$ surjection.

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By definition, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) If $f : A \hookrightarrow B$, $f$ is left-invertible, and thus $\exists g : B \to A$ with $g \circ f = id_A$. So, $g$ is right-invertible, so $g$ is surjective.

(iii) $\Rightarrow$ (ii) If $g : B \to A$ is surjective, then $g$ is right-invertible, so $\exists f : A \to B$ such that $g \circ f = id_B$. So, $f$ is left-invertible, so $f$ is injective.

### Corollary

If $f : A \to B$ is any map, $\operatorname{card}(f(A)) \leq \operatorname{card}(A)$.

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Consider $g : A \to f(A)$, where $g(a) = f(a)$. So, $g$ is onto, so $\exists$ an injection $f(A) \hookrightarrow A$.

### More Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, q(m,n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\operatorname{card}(\mathbb{Q}) \leq \operatorname{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $H(m,n) = (h(m), n)$.

**Claim** $H$ is a bijection.

**Proof of Injection** If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since $h$ is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

**Proof of Surjection** Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m, n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m, n) = (k, \ell)$. Set $n = \ell$, and since $h$ is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\operatorname{card}(\mathbb{Z} \times \mathbb{N}) = \operatorname{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\operatorname{card}(\mathbb{N} \times \mathbb{N}) = \operatorname{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m, n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, $\varphi$ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned} \operatorname{card}(\mathbb{N}) &\leq \operatorname{card}(\mathbb{Q}) \\ &\leq \operatorname{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \operatorname{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \operatorname{card}(\mathbb{N}) \end{aligned}$$

## Cardinality Rules

(i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)

(ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)

(iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)

(iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

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**Proof of (iii)** We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

**Claim** We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, $g$ and $f$ are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1$. $\perp$

We let $A_n = g \circ f(A_{n-1})$ for arbitrary $n$, and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \to B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where $y_x$ is the unique element in $B$ with $g(y_x) = x$.

**Claim** We claim $h$ is the desired bijection.

**Proof of Injection** Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of $H$, $f(x_1) = f(x_2)$, $f$ is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y(x_2)) = x_2 \notin A_\infty$. This case is not possible.

Thus, $h$ is injective.

**Proof of Surjection** Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where $y_x$ is the unique element in $B$ with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since $g$ is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Therefore, we have $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$.

## Countability

A set $X$ is *countable* if $\exists f : x \hookrightarrow \mathbb{N}$ ($\text{card}(X) \leq \text{card}(\mathbb{N})$). $\text{card}(\mathbb{N}) = \aleph_0$. If $X$ is countable and infinite, $X$ is *denumerable*.

### Corollary to Cantor-Schröder-Bernstein

If $X$ is denumerable, then $\text{card}(X) = \aleph_0$.

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Since $X$ is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since $X$ is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

(as shown earlier)

### Countability under Union

The countable union of countable sets is countable. If $I$ is a countable indexing set and for each $i \in I$, $A_i$ is countable, then $\bigcup_{i \in I} A_i$ is countable.

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Since each $A_i$ is countable, $\exists \pi_i : \mathbb{N} \twoheadrightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \to \bigcup_{i \in I} A_i$$

defined as $\pi(i, j) = \pi_i(j)$.

**Claim 1** $\pi$ is a surjection.

**Proof 1** Let $x \in \bigcup_{i \in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since $\pi_{i_0}$ is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, $\pi$ is surjective.

**Claim 2** $I \times \mathbb{N}$ is countable.

**Proof 2** We know $\exists f : I \hookrightarrow \mathbb{N}$ since $I$ is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i, n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m, n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since $\pi$ is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i \in I} A_i$ is countable.

### Continuum Hypothesis

We saw that $\operatorname{card}(\mathbb{N}) < \operatorname{card}(\mathcal{P}(\mathbb{N})) = \operatorname{card}(2^{\mathbb{N}})$, where $2^{\mathbb{N}} \{f \mid f : \mathbb{N} \to \{0, 1\}\}$.

**Theorem** $\operatorname{card}(\mathbb{R}) = \operatorname{card}(I) = \operatorname{card}(2^{\mathbb{N}})$, where $I$ is any non-degenerate interval.

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**Lemma 1** $\operatorname{card}([0, 1]) \leq \operatorname{card}(2^{\mathbb{N}})$.

**Proof 1** Every $t \in [0, 1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0, 1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\varphi} [0, 1]$, defined as $\phi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Set $f : \mathbb{N} \to \{0, 1\}$, $f(k) = \sigma_k$.

Therefore, $\varphi$ is surjective, so $\exists \{0, 1\} \hookrightarrow 2^{\mathbb{N}}$, so $\operatorname{card}([0, 1]) \leq 2^{\mathbb{N}}$

**Lemma 2** $\operatorname{card}([0, 1]) = \operatorname{card}(\mathbb{R})$.

**Proof 2** We have $[0, 1] \xrightarrow{i} \mathbb{R}$ via inclusion, so $\operatorname{card}([0, 1]) \leq \operatorname{card}(\mathbb{R})$.

Also, $\operatorname{card}(\mathbb{R}) = \operatorname{card}((0, 1)) \leq \operatorname{card}([0, 1])$, so by Cantor-Schröder-Bernstein, $\operatorname{card}(\mathbb{R}) = \operatorname{card}([0, 1])$.

**Lemma 3** Any two non-degenerate intervals $I$ and $J$ have the same cardinality.

**Proof 3** We can create injections $I \hookrightarrow J$ and vice-versa.

**Lemma 4** $\operatorname{card}(2^{\mathbb{N}}) \leq \operatorname{card}([0, 1])$.

**Proof 4** $\psi : 2^{\mathbb{N}} \to [0, 1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

$\psi$ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim $\psi$ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$, $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} + \frac{1}{3^{k_0}} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$|t_f - t_g| = |\sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k}|$$
$$\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k}$$
$$\leq \sum_{k>k_0} \frac{1}{3^k}$$
$$= \frac{(1/3)^{k_0+1}}{1 - (1/3)}$$
$$= \frac{1}{2} \cdot \frac{1}{3^{k_0}}$$

$\perp$

We have thus shown:
$$\mathrm{card}(\mathbb{R}) = \mathrm{card}([0,1]) = \mathrm{card}(2^{\mathbb{N}})$$

We know that
$$\aleph_0 = \mathrm{card}(\mathbb{N}) = \mathrm{card}(\mathbb{Q}) = \mathrm{card}(\mathbb{Z}) < 2^{\aleph_0} = \mathrm{card}(2^{\mathbb{N}}) = \mathrm{card}(\mathbb{R}) = \mathrm{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than $\aleph_0$ and strictly less than $2^{\aleph_0}$ is an axiom (i.e., it can be an assumption or not).

## Ordering

Let $X$ be a non-empty set. A relation on $X$ is a subset of $X \times X$.

- $R$ is *reflexive* if $\forall x \in X, \ (x, x) \in R$.

- $R$ is *transitive* if $(x, y), (y, z) \in R \to (x, z) \in R$.

- If $R$ is *antisymmetric* $(x, y), (y, x) \in R \to x = y$.

If $R$ is reflexive, transitive, and antisymmetric, then $R$ is an *ordering* of $X$.

If $R$ is an ordering of $X$, then we write:
$$(x, y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \ \forall x \in X$

- $x \leq_R y, \ y \leq_R z \to x \leq_R z$

- $x \leq_R y, \ y \leq_R x \to x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

### Algebraic ordering of $\mathbb{N}_0$

$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$ such that $n + k = m$

### $\mathbb{N}$ ordered via division

$$n \leq_D m \Leftrightarrow n \mid m$$

Under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

**Inclusion** Let $S$ be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_i B \Leftrightarrow A \subseteq B$.

**Containment** With $X$ defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

For $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \to \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x) \ \forall x \in X$.

## Types of Orderings

- An ordering $\leq$ of $X$ is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.

- An ordering is *directed* if $\forall x, y \in X$ $\exists z \in X$ such that $x \leq z$ and $y \leq z$.

If $X$ is a totally ordered set, $X$ is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D), \ (\mathcal{P}(S), \leq_i), \ (\mathcal{P}(S), \leq_c)$$

## Upper/Lower Bounds

(i) Let $(X, \leq)$ be an ordered set, $A \subseteq X$. $A$ is bounded above if $\exists v \in X$ with $a \leq v$ $\forall a \in A$. Such a $v$ is an upper bound.

(ii) $A$ is bounded below if $\exists \ell \in X$ such that $a \geq \ell$ $\forall a \in A$. Such a $w$ is a lower bound.

(iii) If $v$ is an upper bound of $A$ and $v \in A$, then $v$ is the greatest element of $A$, or $\max(A) = v$.

(iv) If $\ell$ is a lower bound for $A$ and $\ell \in A$, then $\ell$ is the least element of $A$, or $\min(A) = \ell$.

(v) If $u$ is an upper bound for $A$, and $u \leq v$ for all other upper bounds $v$ of $A$, then $u$ is the *least upper bound* of $A$, or $\sup(A) = u$ (for *supremum*).

(vi) If $\ell$ is a lower bound for $A$, and $\ell \leq g$ for all other lower bounds $g$ of $A$, then $\ell$ is the *greatest lower bound* of $A$, or $\inf(A) = \ell$ (for *infimum*).

(vii) If $A$ is bounded above and below, then $A$ is bounded.

## Well-Ordering Principle

With $(\mathbb{N}, \leq_a)$, every nonempty $A \subseteq \mathbb{N}$ has a least element.

## Examples

### Example 1

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \ldots, 12\}$, we have the following:

**Bounded Above?** Yes.

**Upper Bounds** $12, 13, 14, \ldots$

**Greatest Element** $12$

### Example 2

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \ldots, 10\}$

**Bounded Above?** Yes.

**Upper Bounds** 10!

**Greatest Element?** No.

**Supremum** $2^3 \cdot 3^2 \cdot 5 \cdot 7$

**Bounded Below?** Yes.

**Lower Bound** 1

**Least Element?** No.

**Infimum** 1

### Example 3

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$.

**Supremum** $\bigcup_{i \in I} A_i$

**Infimum** $\bigcap_{i \in I} A_i$

### Complete Sets

An ordered set $(X, \leq)$ is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, $\mathbb{Q}$ is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

### Ordering of $\mathbb{Z}$

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0, \ n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

### Properties of $\mathbb{Z}^+$

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, \ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

### Ordering of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$

Recall the ordering of $\mathbb{Z}$:

$$n \leq_a m \overset{\text{def}}{\Longleftrightarrow} \exists k \in \mathbb{N}_0 \text{ with } n + k = m$$

**Claim** $\leq_a$ is an ordering of $\mathbb{Z}$

We claim that $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$. Thus, $\mathbb{Z}^+ = \mathbb{N}_0$.

## Properties of $\mathbb{Z}^+$

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+$, $m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

## Other Properties of $\mathbb{Z}$

(1) $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$

(2) $m \leq_a n$ and $p \leq_a q \Rightarrow m + p \leq_a n + q$

(3) $m \leq_a n$ and $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$

(4) $m \leq_a n \Rightarrow -m {}_a{\geq} n$

(5) $\leq_a$ is total.

(6) If $a {}_a{>} -$, and $ab {}_a{\geq} 0$, then $b {}_a{>} 0$

(7) If $a > 0$ and $ab {}_a{\geq} ac$, then $b \geq c$.

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**Proof of (3):**
$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0$ with $m + k = n$.
$\Rightarrow pm + pk = pn$
$pk \in \mathbb{N}_0$ by the properties of $\mathbb{Z}^+$. So, $pm \leq_a pn$

**Proof of (5):**
Let $m, n \in \mathbb{Z}$. Consider $m - n$.
By (ii), $m - n \in \mathbb{Z}^+$ or $-(m - n) \in \mathbb{Z}^+$. Thus, $m - n = k$ for some $k \in \mathbb{Z}^+$, or $-(m - n) = \ell$ for some $\ell \in \mathbb{Z}^+$.
Thus, $n \leq_a m$ in the first case, or $m \leq_a n$ in the second case.

We now want an ordering on $Q$.

## Creating the Rationals

Recall that $Q = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z}, \ b \in \mathbb{Z}, \ b \neq 0\}$. Consider the equivalence relation:

$$(a, b) \sim (c, d) \xLeftrightarrow{\text{def}} ad = bc$$

We will let $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in Q\}$ be the set of all equivalence classes in $Q$. We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined: $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{a'd' + c'b'}{b'd'} = \frac{ad + bc}{bd}$.

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make $\mathbb{Q}$ a **field**:

### Fields

A ring is a nonempty set set $R$ equipped with two binary operations:

- $+ : R \times R \to R$, $(a, b) \mapsto a + b$ ("addition")

- $\cdot : R \times R \to R$, $(a, b) \mapsto a \cdot b$ ("multiplication")

such that the following hold:

(1) $(a + b) + c = a + (b + c)$

(2) $\exists z \in R$ such that $a + z = a = z + a$ $\forall a \in R$; there is at most one such $z$. Set $z = 0_R$.

(3) $\forall a \in R, \exists b \in R$ such that $a + b = 0_R = b + a$; there is at most one such $b$. Set $b = -a$.

(4) $\forall a, b \in R, \ a + b = b + a$.

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If $(R, +, \cdot)$ satisfies $ab = ba$, $R$ is a commutative ring.

If there exists $u \in R$ such that $ua = au = a$ $\forall a \in R$, $R$ is a unital ring; there is at most one unit. Set $u = 1_R$

An integral domain is a unital, commutative ring such that $ab = 0 \Rightarrow a = 0 \vee b = 0$. For example, $\mathbb{Z}$ is an integral domain. However, $c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ is a unital, commutative ring, but there exist two functions such that $f, g \neq \mathbf{0}$, but $f \cdot g = \mathbf{0}$.

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R, \ \text{with } ab = 1_R$$

There is only one such $b$. Set $b = a^{-1}$.

### Proof that $\mathbb{Q}$ is a Field:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Provided that $\frac{a}{b} \neq 0_{\mathbb{Q}}$.

Additionally, $\mathbb{Z} \xrightarrow{j} \mathbb{Q}$, $j(n) = \frac{n}{1}$ is injective.

## Ordering of $\mathbb{Q}$

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

Prove that this ordering is well-defined.

### Order Embedding

$\leq$ is a well-defined total ordering of $\mathbb{Q}$, and $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$, $j(n) = \frac{n}{1}$ is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

### Properties of $\mathbb{Q}^+$

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

(i) $q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+$, $q_1 q_2 \in \mathbb{Q}^+$

(ii) $q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$

(iii) $\pm q \in \mathbb{Q}^+, q = 0$

(iv) $x \leq y, !u \leq v \Rightarrow x + u \leq y + v$

(v) $x \leq y,\ 0 \leq z \Rightarrow zx \leq zy$

## Ordering of $\mathbb{R}$, cont'd

An **ordered field** is a field $F$ equipped with a total ordering $\leq_F$ such that:

(i) if $s \leq_F t$, and $x \leq_F y$, then $s + x \leq_F t + y$

(ii) if $s \leq_F t$ and $0 \leq_F z$, then $zs \leq_F zt$

For example, $\mathbb{Q}$ with its ordering is an ordered field.

**Proposition 1:** If $(F, \leq_F)$ is an ordered field, we define $F^+ = \{x \in F, x_F \geq 0\}$ with the following properties:

(1) $x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$

(2) $x \in F \Rightarrow x \in F^+, -x \in F^+$

(3) $\pm x \in F^+ \Rightarrow x = 0_F$

### Proofs

(1) Let $x, y \in F^+$. Then, $x \geq 0$ and $y \geq 0$, so by property $(i)$ of an ordered field, $x + y \geq 0 + 0 = 0$, so $x + y \in F^+$. Additionally, we have $x \cdot y \geq x \cdot 0 = 0$, so $xy \in F^+$.

(2) Let $x \in F$. Since the ordering on $F$ is total, $x \geq 0$ or $0 \geq x$. In the first case, $x \in F^+$. In the second case, we add $-x$ to both sides, so by $(i)$, $-x \geq 0$, so $-x \in F^+$.

(3) We have $x \geq 0$ and $-x \geq 0$. So $x \geq 0$ and $x + (-x) \geq x + 0$, so $x \geq 0$ and $0 \geq x$. So, $x = 0$ by antisymmetry.

**Note:** $x \leq_F y \Leftrightarrow y - x \in F^+$.

**Proposition 2:** Let $F$ be an ordered field. Then, the following is true:

(1) $\forall a \in F$, $a^2 \in F^+$

(2) $0, 1 \in F^+$

(3) If $n \in \mathbb{N}$, $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$

(4) If $x \in F^+$, and $x \neq 0$, then $x^{-1} \in F^+$

(5) If $xy > 0$, then $x, y \in F^+$, or $-x, -y \in F^+$

(6) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$

(7) If $x \leq y$, then $-y \leq -x$

(8) $x \geq 1 \Rightarrow x^2 \geq x \geq 1$, and $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$.

### Proofs

(1) Let $a \in F$. Then, $a \in F^+$ or $-a \in F^+$.

    CASE 1  If $a \in F^+$, then by the previous proposition, $a^2 \in F^+$.

    CASE 2  If $-a \in F^+$, then by the previous proposition, $(-a)(-a) = a^2 \in F^+$.

(2) $0 \geq 0$, so $0 \in F+$. $1 = 1 \cdot 1 = 1^2 \in F^+$ by the previous result.

(3) $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots 1_F}_{n \text{ times}} \in F^+$ by the previous proposition.

(4) Let $x \neq 0, x \in F^+$. Suppose toward contradiction that $x^{-1} \notin F^+$, then $-x^{-1} \in F^+$. Thus, $x \cdot (-x^{-1}) \in F^+$, so $-1 \in F^+$, but $1 \in F^+$, so $1 = 0$. $\perp$

(5) Let $xy > 0$, meaning $xy \in F^+$. Suppose toward contradiction that $x > 0$ and $y < 0$. So, $x > 0$ and $-y > 0$, so $(x)(-y) > 0$, so $-(xy) \in F^+0$, so $xy = 0$. $\perp$

(6) Let $0 < x \leq y$. We know $x^{-1} \in F^+$, so $x^{-1}x \leq x^{-1}y$. So $1 \leq x^{-1}y$. We also know $y \in F^+$, so $y^{-1} \in F^+$. So, $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$.

(7) Let $x \leq y$. Then, $0 \leq y - x$, so $-y \leq -x$.

(8) Let $x \geq 1$. Then, $x \cdot x \geq 1 \cdot x \geq 1$.

### Order Axiom

$\mathbb{R}$ is an ordered field. The injection $\mathbb{Q} \hookrightarrow \mathbb{R}$, $i(q) = q$ is an order embedding.

## Rational Orderings

**Proposition 1:** If $a \leq b$, then $a \leq \frac{1}{2}(a+b) \leq b$

### Proof

$2a = a + a \leq a + b \leq b + b$, all by property (i) of an ordered field.

Therefore, $2a \leq a+b \leq 2b$. Since $2 = 1+1 \in \mathbb{R}^+$, $2^{-1} \in \mathbb{R}^+$, so $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$, so $a \leq \frac{1}{2}(a+b) \leq b$.

**Proposition 2:** If $a \geq 0$ and $(\forall \varepsilon > 0), a \leq \varepsilon$.

### Proof

If $a \geq 0$ and $a \neq 0$, then $a > 0$. So, we have that $\frac{1}{2}a < a$. Let $\varepsilon = \frac{1}{2}a$. We also have that $a \leq \varepsilon = \frac{1}{2}a < a$, so $a < a$. $\perp$

## Arithmetic and Geometric Means

Given $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$:

**Arithmetic Mean**

$$= \frac{\sum_{i=1}^{n} a_i}{m}$$

**Geometric Mean**

$$= \sqrt[m]{a_1 a_2 \cdots a_m}$$

### Arithmetic Mean-Geometric Mean Inequality

Let $a, b \geq 0$.

$$(ab)^{1/2} \leq \frac{1}{2}(a + b)$$

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If $x, y \geq 0$, $x \leq y \Leftrightarrow x^2 \leq y^2$.

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y \qquad\qquad \text{by property (ii) of ordered fields}$$

Therefore,

$$(ab)^{1/2} \leq \frac{1}{2}(a + b)$$
$$ab \leq \frac{1}{4}(a^2 + 2ab + b^2)$$
$$4ab \leq a^2 + 2ab + b^2$$
$$0 \leq a^2 - 2ab + b^2$$
$$0 \leq (a - b)^2 \qquad\qquad \text{by definition}$$

**Challenge:** Prove for $m$.

**Remark:** The harmonic mean is defined as:

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$$

### Bernoulli's Inequality

If $x \geq -1$, then $(1 + x)^n \geq 1 + nx$, for any $n \in \mathbb{N}_0$

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By induction, we know that for $n = 0$ and $n = 1$, this holds.

Assume the inequality holds for some $m \geq 1$.

$$(1 + x)^{m+1} = (1 + x)^m(1 + x)$$
$$\geq (1 + mx)(1 + x) \qquad\qquad \text{by the inductive hypothesis}$$
$$= 1 + x + mx + mx^2$$
$$= 1 + (m + 1)x + mx^2$$
$$\geq 1 + (m + 1)x$$

### Cauchy's Inequality

Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Then

$$\left| \sum_{j=1}^{n} a_j b_j \right| \leq \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2} \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}$$

In linear algebra language, this is equivalent to $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$.

Consider $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^{n} (a_j - b_j x)^2$$

We know that $f(x) \geq 0$ for all $x \in \mathbb{R}$

$$= \sum_{i=1}^{n} (a_j^2 - 2a_j b_j x + b_j^2 x^2)$$

$$= \left( \sum_{j=1}^{n} b_j^2 \right) x^2 + \left( \sum_{j=1}^{n} -2a_j b_j \right) x + \sum_{j=1}^{n} a_j^2$$

$$= Ax^2 + Bx + C$$

Therefore, $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\left( -2 \sum_{j=1}^{n} a_j b_j \right)^2 \leq 4 \left( \sum_{j=1}^{n} a_j \right) \left( \sum_{j=1}^{n} b_j \right)$$

$$\left| \sum_{j=1}^{n} a_j b_j \right| = \left( \sum_{j=1}^{n} a_j \right)^{1/2} \left( \sum_{j=1}^{n} b_j \right)^{1/2}$$

As we know from linear algebra, the way we get equality is when $\vec{v} = c\vec{w}$, or that $a_j = c b_j \ \forall j$ for some $c \in \mathbb{R}$.

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### Triangle Inequality

Given $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$

$$\left( \sum_{j=1}^{n} (a_j + b_j)^2 \right)^{1/2} \leq \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2} + \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}$$

In linear algebra, this is equivalent to $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

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$$\sum (a_j + b_j)^2 = \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2$$

$$\leq \sum a_j^2 + 2 \left( \sum a_j^2 \right)^{1/2} \left( \sum b_j^2 \right)^{1/2} + \sum b_j^2 \qquad \text{by Cauchy}$$

$$= \left( \left( \sum a_j^2 \right)^{1/2} + \left( \sum b_j^2 \right)^{1/2} \right)^2$$

we take square roots to get our end result

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### Metrics and Norms on $\mathbb{R}^n$

Consider $|\cdot| : \mathbb{R} \to \mathbb{R}$, defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

**Theorems about Absolute Value:**

(i) $|ab| = |a||b|$

(ii) $|a^2| = |a|^2$

(iii) $|-a| = |a|$

(iv) $|a| \in \mathbb{R}^+$

(v) $-|a| \leq a \leq |a|$

(vi) $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$ for $\delta > 0$

(vii) $|a + b| \leq |a| + |b|$, $|a - b| \leq |a| + |b|$, $||a| - |b|| \leq |a - b|$

### Proofs

**Proof of (i)**

**Case 1:** If $a, b \in \mathbb{R}^+$, then $|a| = a$, and $|b| = b$, and $ab \in \mathbb{R}^+$, so $|ab| = ab$

**Case 2:** If $a, b \notin \mathbb{R}^+$, then $|a| = -a$, and $|b| = -b$. Additionally, $(-a)(-b) = ab \in \mathbb{R}^+$, so $|ab| = ab$. The LHS $= ab$, and the RHS $= ab$.

**Case 3:** $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$. Then, $|a||b| = (a)(-b) = -ab$. Then, since $a(-b) \in \mathbb{R}^+$, $-ab \in \mathbb{R}^+$, so $|ab| = -ab$. Therefore, the LHS and RHS are equal.

**Proof of (vii)** Having established that $|a + b| \leq |a| + |b|$, we will show that $||a| - |b|| \leq |a - b|$.

$$|a| = |a - b + b|$$
$$\leq |a - b| + |b|$$
$$|a| - |b| \leq |a - b|$$

Similarly, by exchanging $a$ for $b$

$$|b| - |a| \leq |b - a|$$
$$|b| - |a| \leq |a - b|$$

Let $t = |a| - |b|$. We have shown that

$$\pm t \leq |a - b|$$
$$-|a - b| \leq t \leq |a - b|$$
$$|t| \leq |a - b|$$

### Absolute Values, cont'd

Recall:
$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

If we want to find all $x \in \mathbb{R}$ such that $|x - 1| \leq |x|$, we would split up into cases:

$x \leq 0$  $x - 1 \leq -1$, so $|x| = -x$ and $|x - 1| = 1 - x$, so $1 - x \leq -x$, so $0 \geq 1$. $\perp$

$0 < x \leq 1$  $|x| = x$ and $|x - 1| = 1 - x$, so $1 - x \leq x$, so $x \geq \frac{1}{2}$, so $\frac{1}{2} \leq x \leq 1$.

$1 < x$  $|x| = x$ and $|x - 1| = x - 1$, so $x - 1 \leq x$, so $-1 \leq 0$, which is true $\forall \mathbb{R}$ in the interval, so $x > 1$.

Therefore, we have $x \in \left(\frac{1}{2}, \infty\right)$ as that which satisfies this inequality.

### Bounded Sets

A subset $A \subseteq \mathbb{R}$ is **bounded** $\Leftrightarrow \exists c \geq 0$ such that $\forall x \in A$, $|x| \leq c$.

($\Rightarrow$) Suppose $A \subseteq \mathbb{R}$ is bounded. Then, $\exists \ell, u \in \mathbb{R}$ such that $\ell \leq x \leq u$ $\forall x \in A$. Let $c := \max\{|\ell|, |u|\}$.

Since $|u| \leq c$, we have that $x \leq c$.

Since $|\ell| \leq c$, and $-|\ell| \leq x$, we get that $-x \leq |\ell| \leq c$.

Since $x \leq c$ and $-x \leq c$, $|x| \leq c$.

($\Leftarrow$) If such a $c$ exists, then $|x| \leq c$ if and only if $-c \leq x \leq c$. Thus, $-c$ is the lower bound and $c$ is the upper bound.

## Bounded Functions

Let $D$ be any set. A function $f : D \to \mathbb{R}$ is bounded if $\mathrm{Ran}(D) \subseteq \mathbb{R}$ is bounded.

## Example

Let $f : [3, 7] \to \mathbb{R}$, $f(x) = \frac{x^2 + 2x + 1}{x - 1}$. Show that $f$ is bounded.

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$3 \leq x \leq 7 \Rightarrow 2 \leq x - 1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}$.

Also, $4 \leq x + 1 \leq 8 \Rightarrow 16 \leq x^2 + 2x + 1 \leq 64 \Rightarrow |x^2 + 2x + 1| \leq 64$.

So, $|f(x)| \leq 32$.

## Distance Metrics

For $s, t \in \mathbb{R}$, we will define $d(s, t) = |s - t|$ to be the **distance** between $s$ and $t$.

**Properties:**

    (i)

$$d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$
$$(s, t) \mapsto d(s, t) \geq 0$$

    (ii) $d(s, t) = d(t, s)$

    (iii) $d(s, r) \leq d(s, t) + d(t, r)$

    (iv) $d(s, s) = 0$

    (v) If $d(s, t) = 0$, then $s = t$.

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^{n} |x_j|$$

- $\infty$-norm:

$$\|v\|_\infty = \max_{j=1}^{n} |x_j|$$

- 2-norm:

$$\|v\|_2 = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2}$$

## Properties of the Norms

**Properties:** With $v, w$ above, let $p = 1, 2, \infty$. The following are true:

    (1) $\|v\|_p \geq 0$

    (2) $\|v + w\|_p \leq \|v\|_p + \|w\| + p$

    (3) $\|\vec{0}\|_p = 0$

    (4) $\|v\|_p = 0 \Rightarrow v = \vec{0}$

    (5) $\forall t \in \mathbb{R}, \ \|tv\|_p = |t|\|v\|_p$

### Proofs

Let $p = \infty$. We will prove (2).

Say $\|v\|_{infty} = |x_i|$ and $\|w\|_\infty = |y_k|$. We want to show that $\|v+w\|_\infty = \max_{j=1}^n |x_j + y_j| \leq |x_i| + |y_k|$.

Note that $\forall j$

$$
\begin{aligned}
|x_j + y_j| &\leq |x_j| + |y_j| &&\text{Triangle Inequality} \\
&\leq |x_i| + |y_k| \\
&= \|v\|_\infty + \|w\|_\infty
\end{aligned}
$$

Therefore, $\|v + w\|_\infty \leq \|v\|_\infty + \|w\|_\infty$.

## Distances and Norms

A **norm** on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$, $v \mapsto \|v\|$, satisfying the following properties for $v \in \mathbb{R}^n$:

(1) $\|v\| \geq 0$

(2) $\|v + w\| \leq \|v\| + \|w\|$

(3) $\|\vec{0}\| = 0$

(4) $\|v\| = 0 \Rightarrow v = \vec{0}$

(5) $\forall t \in \mathbb{R}, \ \|tv\| = |t|\|v\|$

If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$ is a norm, we define $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$, defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for $v, w \in \mathbb{R}^n$.

The properties of distance in $\mathbb{R}$ still hold for distance in $\mathbb{R}^n$:

(1) $d(v, w) = d(w, v)$

(2) $d(u, w) \leq d(u, v) + d(v, w)$

(3) $d(v, v) = 0$

(4) $d(v, w) = 0 \Rightarrow v = w$

## Metric Spaces

A **metric space** is a nonempty set $X$ equipped with a function $d : X \times X \to \mathbb{R}^+$, $(x, y) \mapsto d(x, y) \geq 0$. The metric has the following properties:

(1) $d(x, y) = d(y, x) \ \forall x, y \in X$

(2) $d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in X$

(3) $d(x, x) = 0$

(4) $d(x, y) = 0 \Leftrightarrow x = y$

The map $d$ is called a *metric* on $X$.

## Metric Spaces, Open Sets, and Closed Sets

Examples of Metric Spaces:

- $\mathbb{R}$ with $d(x, y) = |x - y|$.

- $\mathbb{R}^n$ with the *Euclidean metric*:

$$d_2(v, w) = \|v - w\|_2$$
$$= \left( \sum_{j=1}^{n} (x_j - y_j)^2 \right)^{1/2}$$

- $\mathbb{R}^n$ with the 1-norm:

$$d_1(v, w) = \|v - w\|_1$$
$$= \sum_{j=1}^{n} |x_j - y_j|$$

- $\mathbb{R}^n$ with the $\infty$-norm:

$$d_\infty(v, w) = \|v - w\|_\infty$$
$$= \max_{j=1}^{n} |x_j - y_j|$$

Let $(X, d)$ be a metric space.

(1) The **open ball** centered at $x_0 \in X$ with radius $\delta$ is:

$$U(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

(2) The **closed ball** centered at $x_0 \in X$ with radius $\delta$ is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

(3) A set $U \subseteq X$ is **open** if $\forall x \in U$, $\exists \delta > 0$ such that $U(x, \delta) \subseteq U$.

(4) A set $C \subseteq X$ is **closed** if $\overline{C} = X - C \subseteq X$ is open.

### Examples

In $\mathbb{R}$ with $d(s, t) = |s - t|$:
$$U(x_0, \delta) = \{y \in \mathbb{R} \mid d(y, x_0) < \delta\}$$
$$= \{y \in \mathbb{R} \mid |y - x_0| < \delta\}$$
$$= (x_0 - \delta, x_0 + \delta)$$
$$B(x_0, \delta) = [x_0, \delta, x_0 + \delta]$$

The interval $A = [1, \infty)$ is not open, as $\forall \delta > 0$, $U(1, \delta) \not\subseteq [1, \infty)$.

However, $A$ is closed, as $\overline{A} = (-\infty, 1)$ is open: given $t \in \overline{A}$, choose $\delta = 1 - t$. Let $s \in V_\delta(t)$. Then, $s \in (t - \delta, t + \delta)$, so $s \in (t - (1 - t), t + (1 - t))$, or $s \in (2t - 1, 1)$, so $s < 1$.

### Exercises

Show that the following are open:

- $(a, b)$

- $(a, \infty)$

- $(-\infty, b)$

and that the following are closed:

- $[a, b]$

- $[a, \infty)$

- $(-\infty, b]$

In $(\mathbb{R}^2, d_2)$, $B(0_{\mathbb{R}^2}, 1)$ is the **unit disc** centered at $(0, 0)$.

However, in $(\mathbb{R}^2, d_\infty)$:

$$B(0_{\mathbb{R}^2}, 1) = \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\}$$
$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\}$$

is the **unit square**.

### Finding a Supremum

Let $0 \neq A \subseteq \mathbb{R}$. Let $u \in \mathbb{R}$ be an upper bound for $A$. The following are equivalent:

(i) $u = \sup(A)$

(ii) If $t < u$, then $\exists a_t \in A$ such that $a_t > t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $u - \varepsilon < a_\varepsilon$

#### Proofs

(i) $\Rightarrow$ (ii): Given $t < u$, if no such $a \in A$ with $t < a$ exists, then $a \leq t \, \forall a \in A$. Thus $t$ would be an upper bound. However, $t < u$ and $u$ is the supremum of $A$. $\perp$

(ii) $\Rightarrow$ (iii): Given $\varepsilon > 0$, set $t = u - \varepsilon < u$. So, by (ii), $\exists a_t$ with $t < a_t$. Thus, $u - \varepsilon \leq a_t$. Set $a_\varepsilon = a_t$.

(iii) $\Rightarrow$ (i): Let $v$ be an upper bound for $A$. Suppose $v < u$. Then, set $\varepsilon = u - v > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u - \varepsilon < a_\varepsilon$. So $u - (u - v) < a_\varepsilon$, so $v < a_\varepsilon$, meaning $v$ cannot be an upper bound. $\perp$

### Supremum Example

$\sup[0, 1) = 1$: Certainly, 1 is an upper bound for $[0, 1)$. Let $\varepsilon > 0$.

If $\varepsilon \geq 1$, pick $t = \frac{1}{2}$. Then, $1 - \varepsilon < 0 < \frac{1}{2}$

If $0 < \varepsilon < 1$, let $t = (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$. Then, $t \in [0, 1)$, and $1 - \varepsilon < 1 - \varepsilon/2 = t$

### Finding an Infimum

Let $\emptyset \neq A \subseteq \mathbb{R}$. Let $\ell \in \mathbb{R}$ be a lower bound for $A$. The following are equivalent:

(i) $\ell = \inf(A)$

(ii) If $t > \ell$, $\exists a_t$ such that $t > a_t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $\ell + \varepsilon > a_\varepsilon$

## Infimum Example

$\inf \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ : Clearly, $0 < \frac{1}{n} \ \forall n \geq 1$. Let $\varepsilon > 0$.

We need to find $a \in \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ with $\varepsilon > a$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Let $a_\varepsilon = \frac{1}{m}$.

## More on Supremum/Infimum

- If $A \subseteq \mathbb{R}$ and $\max(A) = u$, then $u = \sup(A)$: $u$ is an upper bound of $A$ by the definition of max, and if $v \neq u$ is any upper bound of $A$, then $u < v$ since $u \in A$.

- If $\min(A) = \ell$, then $\ell = \inf(A)$ (by the same logic).

- If $A$ is not bounded above, $\sup(A) = +\infty$, and if $A$ is not bounded below, then $\inf(A) = -\infty$.

- If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.

- If $A \subseteq B$, then $\inf(A) \geq \inf(B)$: Let $\ell_A = \inf(A)$ and $\ell_B = \inf(B)$. By definition, $\ell_B \leq b \ \forall b \in B$. Since $A \subseteq B$, $\ell_B \leq a \ \forall a \in A$. Thus, $\ell_B$ is a lower bound for $A$. By definition of $\ell_A$, $\ell_B \leq \ell_A$.

Let $A, B \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Then, the following are also sets:

(1) $A + B = \{a + b \mid a \in A, b \in B\}$

(2) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$

(3) $t \cdot A = \{ta \mid a \in A\}$

(4) $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$

- $\sup(A + t) = \sup(A) + t$

- $\inf(-A) = -\sup(A)$

## Completeness Axiom

If $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above, then $\sup(A)$ exists.

Well-Ordering Property: if $\emptyset \neq S \subseteq \mathbb{N}$, then $\min(S)$ exists.

Therefore, we can prove that if $F \subseteq \mathbb{Z}$ is bounded, then $F$ has a least and greatest element.

### Archimedean Property: Proof

If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x \leq n_x$.

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Suppose there exists no natural number greater than $x$, then $\mathbb{N}$ is bounded above by $X$. Let $u = \sup(\mathbb{N})$. By the Completeness Axiom, $u \in \mathbb{R}$ exists. Let $\varepsilon = 1$. Then, $\exists n \in \mathbb{N}$ with $u - 1 < n$. So, $u < n + 1$, but $n + 1 \in \mathbb{N}$, so $u$ cannot be an upper bound.

### Corollary to the Archimedean Property

$$\forall t > 0 \ \exists n \in \mathbb{N} \ni \frac{1}{n} < t$$

### Corollary: Powers of 2

$$\forall t > 0 \; \exists m \in \mathbb{N} \ni \frac{1}{2^m} < t$$

By the corollary to the Archimedean Property, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < t$. By Bernoulli's inequality with $x = 1$, we have $2^n \geq n$, so $2^{-n} < n^{-1} < t$.

### Corollary: In Between Integers

$$\forall x \in \mathbb{R} \; \exists n_x \in \mathbb{Z} \ni n_x - 1 \leq x < n_x$$

Assume $x \geq 0$. Let $S_x = \{n \mid n \in \mathbb{N} \; x < n\}$.

$S_x \neq \emptyset$ by the Archimedean Property. By the well-ordering property, let $n_x = \min(S_x)$.

Therefore, $x < n_x$. Also, $n_x - 1 \notin S_x$. Therefore $n_x - 1 \leq x$.

## Density Theorems

Let $(X, d)$ be any metric space. A subset $D \subseteq X$ is **dense** if $\forall x \in X, \; \varepsilon > 0, \; U(x, \varepsilon) \cap D \neq \emptyset$.

In the case of $X = \mathbb{R}$ and $d(s, t) = |s - t|$, $D \subseteq \mathbb{R}$ is dense if given any open interval $I$, $I \cap D \neq \emptyset$.

A metric space is **separable** if it admits a *countable* dense subset.

### Density of the Rationals

$\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Let $I = (a, b)$ be an open interval. We may assume that $a, b \in \mathbb{R}$ are finite.

Then, $b - a > 0$. By the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, meaning $1 < nb - na$.

There exists also an integer $m$ such that $m - 1 \leq na < m$, implying that $a \frac{m}{n}$. Also, $m \leq na + 1 < nb$. Therefore, $\frac{m}{n} < b$.

So, $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$.

### Density of the Irrationals

$\mathbb{R} \setminus \mathbb{Q}$ is dense.

Assume $\sqrt{2}$ exists. Let $I = (a, b)$ be any open interval. Then, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

Find $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$.

Then, $a < q\sqrt{2} < b$, where $q\sqrt{2} \in \mathbb{R}$ and $q\sqrt{2} \notin \mathbb{Q}$.

## Uniqueness of $\sqrt{2}$

$$\exists! x > 0 \ x^2 = 2$$

Existence: Let $S = \{t \in \mathbb{R} \mid 0 < t, \ t^2 < 2\}$. $S$ is nonempty because $1 \in \S$, and $S$ is bounded above because $y > 2 \Rightarrow y^2 > 4$.

So, by the completeness axiom, $x := \sup(S)$ exists, and $x \geq 1$.

Claim: $x^2 = 2$

Contradiction 1: Assume $x^2 < 2$. We want to show that $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$. By this assumption, we find that

$$0 < x + \frac{1}{n} \in S \Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2$$
$$\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2}$$

Observe:

$$x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2x}{n} + \frac{1}{n}$$
$$= x^2 + \frac{1}{n}(2x + 1)$$

We want to find $n \in \mathbb{N}$ with:

$$x^2 + \frac{1}{n}(2x + 1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

Therefore, by the Archimedean Property corollary, we know that $n$ must exist.

Contradiction 2: We know that $x^2 \geq 2$. Since $x = \sup(S)$, $\forall m \in \mathbb{N}$, $\exists t_m \in S$ with $x - \frac{1}{m} < t_m$, so $\left(x - \frac{1}{m}\right)^2 < t_m^2 < 2$.

Therefore, $x^2 - \frac{2x}{m} + \frac{1}{m^2}$, so $x^2 - \frac{2x}{m} < 2$, so $0 \leq x^2 - 2 < \frac{2x}{m}$.

So, $0 \leq \frac{x^2 - 2}{2x} < \frac{1}{m}$, so $x^2 - 2 = 0$, so $x^2 = 2$.

**Remark:** If we had set $S' = \{t' \in \mathbb{Q} \mid t^2 < 2, \ t > 0\}$, we would have still obtained $\sup(S') = \sqrt{2}$. This means that $\mathbb{Q}$ is *not* complete.

## Intervals and Nested Intervals

(*) Given any interval $I$, if $x_1, x_2 \in I$, with $x_1 < x_2$, then $[x_1, x_2] \in I$.

This seems like an obvious property, but this is the *characteristic property* of intervals.

### Characterization of Intervals

Let $S \in \mathbb{R}$ be any nonempty subset of cardinality at least 2. Suppose $S$ satisfies (*). Then, $S$ is an interval.

**Case 1:** Suppose $S$ is bounded.

Let $a = \inf(S)$ and $b = \sup(S)$. Then, $S \subseteq [a, b]$. We will show that $(a, b) \subseteq S$. Once this is shown, $S = \{(a, b), [a, b], [a, b), (a, b]\}$.

Let $t \in (a, b)$. Since $a = \inf(S)$, $\exists x_1 \in S$, $x_1 \in (a, t)$. Similarly, since $b = \sup(S)$, $\exists x_2 \in S$, $x_1 \in (t, b)$.

By the hypothesis, $[x_1, x_2] \subseteq S$. Since $t \in [x_1, x_2]$, $t \in S$.

**Case 2:** Suppose $S$ is bounded above, but not below.

Let $b = \sup(S)$. Clearly, $S \subseteq (-\infty, b]$. We will show that $(-\infty, b) \subseteq S$. Once this is shown, $S = \{(-\infty, b), (-\infty, b]\}$.

Let $t \in (-\infty, b)$. Since $b = \sup(S)$, $\exists x_2 \in S$, $x_2 \in (t, b)$.

Since $S$ is not bounded below, $\exists x_1 \in S$ such that $x_1 < t$ (or else $t$ would be a lower bound).

By the hypothesis, $[x_1, x_2] \in S$, and $t \in [x_1, x_2]$, so $t \in S$.

**Case 3, 4:** Left as an exercise for the reader.

A sequence of intervals $(I_n)_{n \geq 1}$ is called *nested* if

$$I_1 \supseteq I_2 \supseteq \ldots I_n \supseteq I_{n+1} \supseteq \ldots$$

We are primarily interested in $\bigcap I_n$.

(a) $\bigcap_{n=1}[0, 1/n] = \{0\}$.

(b) $\bigcap_{n=1}(0, 1/n) = \emptyset$

(c) $\bigcap_{n=1}[n, \infty) = \emptyset$

### Measure

The **measure** of an interval is basically its "size."

(a) $m([a, b]) = b - a$

(b) $m((a, b]) = b - a$

(c) $m((a, b)) = b - a$

(d) $m([a, b)) = b - a$

### Nested Intervals Theorem

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested sequence of intervals.

(1) $\bigcap_{n \geq 1} I_n \neq \emptyset$

(2) If $\inf \{m(I_n) \mid n \geq 1\} = 0$, then $\bigcap_{n \geq 1} I_n = \{\xi\}$

### (a)

Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots$, we have that $a_1 \leq a_2 \leq a_3, \ldots$, and $b_1 \geq b_2 \geq b_3 \geq \cdots$.

We know that $\{a_n\}$ is bounded above and nonempty. Let $\xi = \sup\left(\{a_n\}_{n=1}^\infty\right)$.

We know that $\{b_n\}$ is bounded below. Let $\eta = \inf\left(\{b_n\}_{n=1}^\infty\right)$.

We claim that $\xi \leq b_n \ \forall n \geq 1$. Suppose toward contradiction that $\exists m$ such that $\xi > b_m$. Then, by the supremum property, $\exists a_i$ such that $\xi > a_i > b_m$. If $k \leq m$, $a_k \leq a_m \leq b_m < a_k$. If $m \leq k$, then $b_m \geq b_k \geq a_k > b_m$. $\perp$

Similarly, using the same argument, $a_n \leq \eta \ \forall n$.

Thus, $\xi \leq \eta$.

We claim that $\bigcap_{n \geq 1} I_n = [\xi, \eta]$. If $t \in [\xi, \eta]$, then $a_n \leq \xi \leq t \leq \eta \leq b_n$. So $t \in [a_n, b_n] \ \forall n$, so $t \in \bigcap_{n \geq 1}[a_n, b_n]$.

If $t \in \bigcap_{n \geq 1} I_n$, then $t \in [a_n, b_n] \ \forall n$, so $a_n \leq t \leq b_n \ \forall n$. So, $t$ is an upper bound on $a_n$, and a lower bound on $b_n$. So, $\xi \leq t \leq \eta$ by definition of $\xi$ and $\eta$.

### (b)

We have $\forall n \geq 1$

$$[\xi, \eta] \subseteq [a_n, b_n]$$
$$\Rightarrow 0 \leq \eta - \xi \leq b_n - a_n$$
$$= m(I_n)$$

So, if $\inf\left(\{m(I_n) \mid n \geq 1\}\right) = 0$, then $0 \leq \eta - \xi \leq 0$, so $\eta = \xi$.

### Corollary to the Nested Intervals Theorem

$[0, 1]$ is uncountable.

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Suppose it is possible to denumerate the interval $[0, 1] = \{t_1, t_2, \ldots, \}$.

We can find $[a_1, b_1] \subseteq [0, 1]$ with:

- $a_1 < b_1$

- $t_1 \notin [a_1, b_1]$.

Then, we find $[a_2, b_2] \in [a_1, b_1]$ with $a_2 < b_2$, $t_2 \notin [a_2, b_2]$.

Recursively, we find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $a_n < b_n$, $t_n \notin [a_n, b_n]$.

So, $I_n = ([a_n, b_n])_0^\infty$ is a sequence of nested intervals.

Therefore, $\exists \xi \in \bigcap I_n \subseteq [0, 1]$. However, $\xi \notin \{t_1, t_2, \ldots\}$. $\perp$

### Sequences in Metric Spaces

A sequence in a metric space $M$ is a map

$$x : \mathbb{N} \to M \qquad\qquad M = \mathbb{R}, \text{ usually}$$
$$x = (x_n)_{n=1}^\infty$$

**I.** Sequences defined by a formula:

    (i) $x_n = t$ (the constant sequence)

    (ii) $x_n = 2n + 1$

(iii) $x_n = \frac{1}{n-1}$ (here, $n \geq 2$)

(iv) $c_n = n \sin\left(\frac{1}{n}\right)$

(v) $d_n = \left(1 + \frac{1}{n}\right)^n$

(vi) Geometric *Sequence*: for $b \neq 0$, $(b^n)_{n \geq 0} = (1, b, b^2, \dots)$

(vii) $x_n = \frac{n!}{n^n}$

(viii) Given any function

$$f : [0, \infty) \to \mathbb{R}$$

we can set $x_n = f(n)$.

**II.** Sequences defined recursively:

(i) $a_1 = 1$, $a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$

(ii) Fibonacci: $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$. The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}} \left(\varphi^n - (1 - \varphi)^n\right)$$

where $\varphi = \frac{1 + \sqrt{5}}{2}$

(iii) Let $f : M \to M$ be a self-map on a metric space. Fix $x_0 \in M$.

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

**III.** New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences, $t \in \mathbb{R}$. Then, we can do the following:

- $(a_n)_n + (b_n)_n + (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n (b_n)_n = (a_n b_n)_n$
- If $b_n \neq 0$ $\forall n$, $\left(\frac{a_n}{b_n}\right)$

(ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

(iii) We can look at ratios for $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

(iv) We can look at partial sums, given $(a_k)_{k=1}^{\infty}$.

$$s_1 = a_1$$
$$s_n = s_{n-1} + a_n$$
$$= \sum_{k=1}^{n} a_k$$

The sequence $(s_n)_n$ is called the sequence of partial sums. For example, the sequence of partial sums for $(b^k)_{k=0}^{\infty}$ is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1 - b^{n+1}}{1 - b} & b \neq 1 \\ n + 1 & b = 1 \end{cases}$$

because for $b \neq 1$, $(1 - b)(1 + b + b^2 + \cdots + b^n) = 1 - b^{n+1}$

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**Exercise**

Let $a_k = \frac{1}{k(k+1)}$. Find $(s_n)_n$.

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Via partial fraction decomposition, we get that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore, $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^{\infty}$

## Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R}, \ a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k| \qquad\qquad \text{Infinity Norm}$$

This norm has the traditional properties of the norm:

$$\|(a_k)_k + (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty \qquad\qquad \text{Triangle Inequality}$$
$$\|t(a_k)_k\|_\infty = |t| \|(a_k)_k\|_\infty \qquad\qquad \text{Scalar Multiplication}$$
$$\|(a_k)_k\|_\infty = 0 \Leftrightarrow a_k = 0 \ \forall k \qquad\qquad \text{Zero Property}$$
$$\|(a_k)_k (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty \|(b_k)_k\|_\infty \qquad\qquad \text{Multiplication}$$

### Proof

Let $u = \|(a_k)_k\|_\infty$ and $v = \|(b_k)_k\|_\infty$.

Given any $k$,

$$|a_k + b_k| \leq |a_k| + |b_k| \qquad\qquad \text{Triangle Inequality on } |\cdot|$$
$$\leq u + v \qquad\qquad \text{definition of supremum}$$
$$\Rightarrow \sup_{k \geq 1} |a_k + b_k| \leq u + v$$

Thus,

$$\|(a_k)_k + (b_k)_k\|_\infty = \| ((a_k + b_k)_k)_k \|_\infty$$
$$= \sup_{k \geq 1} |a_k + b_k|$$
$$\leq u + v$$

## Monotonicity

A sequence $(x_n)_n$ is **increasing** if

$$x_1 \leq x_2 \leq \cdots \ \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \cdots \ \forall n$$

The sequence is *eventually* increasing if $\exists m \in \mathbb{N} \ni x_n \leq x_{n+1}$ for $n > m$.

Similarly, the sequence is eventually decreasing if $\exists m \in \mathbb{N} \ni x_n \geq x_{n+1}$ for $n > m$.

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

### Example

The sequence

$$a_1 = 1$$

$$a_{n+1} = \frac{1}{2}a_n + 2$$

is increasing and bounded above.

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We will prove the first statement via induction:

**Base:** $a_1 = 1$, $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

**Inductive Hypothesis** $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+1}$

**Proof:**

$$a_n \leq a_{n+1}$$

$$\frac{1}{2}a_n \leq \frac{1}{2}a_{n+1}$$

$$\frac{1}{2}a_n + 2 \leq \frac{1}{2}a_{n+1} + 2$$

$$a_{n+1} \leq a_{n+2}$$

To prove the sequence is bounded above, we do the following:

$$a_1 = 1 \leq 4$$

$$\frac{1}{2}a_1 \leq 2$$

$$\frac{1}{2}a_1 + 2 \leq 2$$

$$a_2 \leq 4$$

We claim that $\forall n$, $a_n \leq 4 \Rightarrow a_{n+1} \leq 4$, as we have shown the base case.

$$a_n \leq 4$$

$$\frac{1}{2}a_n \leq 2$$

$$\frac{1}{2}a_n + 2 \leq 4$$

$$a_{n+1} \leq 4$$

### Convergence of Sequences

Let $L \in \mathbb{R}$, $\varepsilon > 0$. Then, the $\varepsilon-$**neighborhood** of $L$ is $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$.

$$x \in V_\varepsilon(L)$$

$$\Leftrightarrow$$

$$|x - L| < \varepsilon$$

$$L - \varepsilon < x < L + \varepsilon$$

With this in mind, we know the following:

### Definition of Convergence

A real sequence $(x_n)_n$ converges to a number $x$ if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \ni n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such $L$ exists, then $(x_n)_n$ is said to **diverge**.

A sequence $(x_n)_n$ in a metric space $(X, d)$ converges to a point $x$ if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \ni d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an $N$ such that the $N$th tail (i.e., $x_N, x_{N+1}, \cdots$) are within $\varepsilon$ of $L$ for any $\varepsilon$.

**Note:** $N$ usually depends on $\varepsilon$ (the smaller the $\varepsilon$, the larger the $N$).

### Convergence Proof

$$\left(\frac{1}{n}\right)_n \xrightarrow{n \to \infty} 0$$

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We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given $\varepsilon > 0$, we want $\frac{1}{n} < \varepsilon$, meaning $n > \frac{1}{\varepsilon}$.

**Proof:** Let $\varepsilon > 0$. By the Archimedean property corollary, find $N \in \mathbb{N}$ large such that $\frac{1}{N} < \varepsilon$.

$$n \geq N$$
$$\frac{1}{n} \leq \frac{1}{N}$$
$$< \varepsilon$$

so, if $n \geq N$, then

$$|x_n - L| = \left|\frac{1}{n}\right|$$
$$= \frac{1}{n}$$
$$< \varepsilon$$

## Convergence Proof 2

Show that

$$\left(\frac{5n-1}{3-n}\right)_{n\geq 4} \xrightarrow{n\to\infty} -5$$

$$\begin{aligned}
|x_n - L| &= \left|\frac{5n-1}{3-n} + 5\right| \\
&= \frac{14}{|3-n|} \\
&= \frac{14}{n-3} \qquad\qquad\qquad\qquad < \varepsilon
\end{aligned}$$

$$\frac{14}{n-3} < \varepsilon$$

$$n > \frac{14}{\varepsilon} + 3$$

**Proof:** Let $\varepsilon > 0$. Find $N' \in \mathbb{N}$ so large that $\frac{1}{N'} < \frac{\varepsilon}{14}$ (which exists by the Archimedean property corollary). Let $N = N' + 3$. If $n \geq N$, then

$$\begin{aligned}
n - 3 &\geq \frac{1}{N'} \\
\frac{1}{n-3} &\leq \frac{1}{N'} \\
&< \frac{\varepsilon}{14}
\end{aligned}$$

whence

$$\begin{aligned}
|x_n - L| &= \frac{14}{n-3} \\
&< \frac{14\varepsilon}{14} \\
&= \varepsilon
\end{aligned}$$

## Sequences and their Limits, cont'd

### Convergence and Distance

Let $(X, d)$ be a metric space, and let $(x_n)_n$ be a sequence in the metric space. The following are equivalent:

(i) $(x_n)_n \to x$

(ii) $(d(x_n, x))_n \to 0$

(i) $\Rightarrow$ (b) Let $\varepsilon > 0$. Find $N_\varepsilon \in \mathbb{N}$ so large such that $d(x_n, x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

So, $|d(x_n, x) - 0| = d(x_n, x) < \varepsilon$ for all $\varepsilon > 0$. Whence, $(d(x_n, x))_n \to 0$.

(ii) $\Rightarrow$ (i) This direction is similar.

In $\mathbb{R}$, this implies that

$$(x_n)_n \to x$$
$$\Leftrightarrow$$
$$(|x_n - x|)_n \to 0$$

### Comparison Proposition

Let $(X, d)$ be a metric space and let $x \in X$, and suppose $(x_n)_n$ is a sequence in $X$.

If $\exists c \geq 0$, $m \in \mathbb{N}$, and a sequence $(a_n)_n \in \mathbb{R}^+$ with $(a_n)_n \to 0$ and $d(x_n, x) \leq ca_n$ $\forall n > m$. Then, $(x_n)_n \to x$.

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Let $\varepsilon > 0$. Note that $\frac{\varepsilon}{c} > 0$.

Find $N_1 \in \mathbb{N}$ large such that $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\varepsilon}{c}$, which is always possible since $(a_n)_n \to 0$.

Let $N = \max(N_1, m)$. Then, $n \geq N \Rightarrow n \geq N_1$ and $n \geq m$. So,

$$d(x_n, x) \leq ca_n$$
$$< c\frac{\varepsilon}{c}$$
$$= \varepsilon$$

so, $n \geq N \Rightarrow d(x_n, x) < \varepsilon$, whence $(x_n)_n \to x$

### Comparison Proposition, Example 1

Prove

$$\left( \frac{\sin(n^2 - 1)}{n^2 + 3} \right)_n \to 0$$

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$$\left| \frac{\sin(n^2 - 1)}{n^2 + 3} - 0 \right| = \frac{|\sin(n^2 - 1)|}{n^2 + 3}$$
$$\leq \frac{1}{n^2 + 3}$$
$$\leq \frac{1}{n^2}$$
$$\leq \frac{1}{n}$$

We know that $a_n = \frac{1}{n}$ converges to 0. So, by our comparison proposition, we are done.

### Comparison Proposition, Example 2

Prove

$$\left( \frac{1}{2^n} \right)_n \to 0$$

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$$\begin{aligned} 2^n &= (1 + 1)^n \\ &\geq 1 + n \qquad\qquad\qquad > n \qquad\qquad\qquad \text{Bernoulli's Inequality} \end{aligned}$$

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since $a_n = \frac{1}{n}$ converges, we know that $\frac{1}{2^n}$ must converge.

### Sequence Divergence

A sequence $(x_n)_n$ is **divergent** if it does not converge. $(x_n)_n \to 0$ if and only if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni (\forall n \geq N_\varepsilon)d(x_n, x) < \varepsilon$$

So, $(x_n)_n$ diverges if and only if

$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \rightarrow d(x_n, x) \geq \varepsilon_0$$

### Diverging Sequence Proof

Show that the following sequence diverges:

$$a_n = (-1)^n$$

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**Step 1**

$$((-1)^n)_n \not\to 1$$

Take $\varepsilon_0 = 1/2$, given any $N \in \mathbb{N}$, we will find $n \geq N$ odd:

$$n = 2N + 1$$
$$d((-1)^n, 1) = 2$$
$$\geq \varepsilon_0$$

**Step 2**

$$((-1)^n)_n \not\to -1$$

by letting $\varepsilon_0 = 1/2$ and $n = 2N$.

### Diverging Sequence Proof 2

Does

$$a_n = (\sin(n))_n$$

converge?

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

It is not the case that $(\sin(n))_n \to L$ for any $L \in \mathbb{R}$.

**Case 1** If $L > 1$, set $\varepsilon_0 = \frac{L-1}{2}$. Then, given any $N \in \mathbb{N}$, pick $n = N$.

$$|\sin(n) - L| = L - \sin(n)$$
$$\geq L - 1$$
$$> \frac{L-1}{2}$$
$$= \varepsilon_0$$

**Case 2** Similarly for $L < -1$

**Case 3** Suppose $-1 < L < 1$.

    **Case 3.1** Suppose $L > 0$. Set $\varepsilon_0 = \frac{L}{2}$. Given any $N$, find $n \geq N$ with $\sin(n) < 0$.

        We find $k$ large such that $N < (2k + 1)\pi$, which we can always do because we are finding $k > \frac{1}{2}\left(\frac{N}{\pi} - 1\right)$, which is always possible by the Archimedean property.

        Note that $N < (2k + 1)\pi < (2k + 2)\pi$. Note that $\sin(x) < 0$ on the interval $((2k + 1)\pi, (2k + 2)\pi)$. Note that $|(2k + 1)\pi - (2k + 2)\pi| = \pi$. Let $n = \lceil (2k + 1)\pi \rceil$. Then, $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

    **Case 3.2** Suppose $L < 0$, set $\varepsilon_0 = \frac{-L}{2}$. Given $N$, find $n \geq N$ with $\sin(n) > 0$. Using the same strategy as above, we find $n$ such that $|L - \sin(n)| > \varepsilon_0$

    **Case 3.3** Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given $N \in \mathbb{N}$, find $n \geq N$ with $\sin(n) \geq \frac{1}{2}$. Then, $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$.

Showing that a sequence diverges is not easy — later, we will divergence with non-uniqueness of convergent subsequences.

## Alternating Series

Consider again

$$((-1)^n)_{n \geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to $-1$:

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating series diverges.

## Uniqueness of Limits

A sequence $(x_n)_n$ can converge to at most one limit.

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Suppose toward contradiction that $(x_n)_n$ converges to $L_1$ and $L_2$ with $L_1 \neq L_2$.

WLOG, let $L_2 > L_1$. Take $\varepsilon = \frac{L_2 - L_1}{3}$.

Since $(x_n)_n$ converges to $L_1$, $\exists N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \varepsilon$. Similarly, since $(x_n)_n$ converges to $L_2$, $\exists N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \varepsilon$.

Let $N = \max N_1, N_2$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

So, $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$. So, $x_n \in V_\varepsilon(L_1)$, and $x_n \in V_\varepsilon(L_2)$, meaning $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$, but $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. $\perp$

## Useful Lemmas for Convergence

### Absolutely Convergent Sequences

Let $(x_n)_n$ be a real sequence. If $x_n$ converges to $x$, then $|(x_n)_n| \to |x|$. However, the converse is not the case.

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Note that since $(x_n)_n \to x$, $d(x_n, x) \to 0$.

By the reverse triangle inequality, we have

$$||x_n| - |x|| \leq |x_n - x|$$
$$\leq 0$$

### Convergence to Zero

Let $a_n$ be a sequence.

$$(a_n)_n \to 0$$
$$\Leftrightarrow$$
$$|(a_n)| \to 0$$

---

($\Rightarrow$) If $(a_n)_n \to 0$, then we showed previously that $|(a_n)_n| \to |0| = 0$

($\Leftarrow$) Suppose $|(a_n)_n| \to 0$. Given $\varepsilon > 0$, then $\exists N$ such that $n \geq N$ implies

$$||a_n| - 0| < \varepsilon$$
$$||a_n|| < \varepsilon$$
$$|a_n| < \varepsilon$$
$$|a_n - 0| < \varepsilon$$

So, $(a_n)_n \to 0$

### Geometric Sequence

Let $b \in \mathbb{R}$. Consider

$$(b^n)_{n \geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided $-1 < b \leq 1$. Otherwise, the sequence is divergent.

---

If $b = 0$, then the sequence $(b^n)_n = (0, 0, 0, \dots)$ is convergent.

If $b = 1$, then the sequence $(b^n)_n = (1, 1, 1, \dots)$ is convergent.

If $b = -1$, then the sequence $(b^n)_n = (1, -1, 1, \dots)$ is divergent.

**Case 1** Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, so $\frac{1}{b} = 1 + a$.

So, by Bernoulli's Inequality, $\left(\frac{1}{b}\right)^n = (1 + a)^n \geq 1 + na > na$, so $b^n < \frac{1}{na}$.

$$|b^n - 0| = b^n$$
$$< \frac{1}{na}$$
$$= \frac{1}{a}\frac{1}{n}$$
$$\to 0$$

So, $(b^n)_n \to 0$.

**Case 2** Suppose $-1 < b < 0$. If we look at $|b^n| = |b|^n$, we know that $(|b|^n)_n \to 0$ by our work above. By the previous lemma, we know that since $|b^n| \to 0$, $b^n \to 0$.

**Case 3** Suppose $b > 1$. Then, $b = 1 + a$ where $a > 0$.

$$b^n = (1 + a)^n$$
$$\geq 1 + na \qquad\qquad \text{Bernoulli's Inequality}$$
$$> na$$

Suppose toward contradiction that $(b^n)_n \to L$. Let $\varepsilon_0 = 1$. Find $N \in \mathbb{N}$ such that $N > \frac{L+1}{a}$. $N$ must exist by the Archimedean property.

Therefore, $(N)(a) > L + 1$. If $n \geq N$, then $(n)(a) > (N)(a) > L + 1$, so $|b^n - L| \geq na - L \geq \varepsilon_0$. $\perp$

**Case 4** Suppose $b < -1$, and suppose toward contradiction that $(b^n)_n \to L$. By the previous lemma, we know that $|b^n| \to |L|$. So, $|b|^n \to |L|$. But, $|b| > 1$, which means our assumption contradicts the result from above. $\perp$

## Sequences and Limits, Cont'd

### $n$th Root Convergence

If $c > 0$, then $\left(c^{1/n}\right)_n \to 1$.

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**Case 1:** If $c = 1$, then we get $\left(c^{1/n}\right)_n = (1, 1, 1, \dots)$, which clearly converges to one.

**Case 2:** Assume that $c > 1$. Then, $c^{1/n} > 1$, because if $d = c^{1/n} \leq 1$, then $d^n \leq 1$, so $c \leq 1$. We can write $c^{1/n} = (1 + d_n)$, where $d_n > 0$.

$$
\begin{aligned}
c &= c^n \\
&= (1 + d_n)^n \\
&\geq 1 + n d_n \qquad\qquad\qquad\qquad \text{Bernoulli's Inequality} \\
&> n d_n
\end{aligned}
$$

So, $d_n \leq \frac{c}{n}$. Remember, $c^{1/n} = 1 + d_n$.

$$
\begin{aligned}
|c^{1/n} - 1| &= c^{1/n} - 1 \\
&= d_n \\
&\leq c \cdot \frac{1}{n} \\
&\to 0
\end{aligned}
$$

Therefore, $c^{1/n} \to 1$.

**Case 3:** Assume $0 < c < 1$. Then, $c^{1/n} < 1$, so $\frac{1}{c^{1/n}} > 1$.

So, we can write $\frac{1}{c^{1/n}} = (1 + d_n)$, where $d_n > 0$.

$$
\begin{aligned}
c^{1/n} &= \frac{1}{1 + d_n} \\
1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\
&= \frac{d_n}{1 + d_n} \\
&\leq d_n
\end{aligned}
$$

Remember, $\frac{1}{c^{1/n}} = 1 + d_n$

$$
\begin{aligned}
\frac{1}{c} &= (1 + d_n)^n \\
&\geq 1 + n d_n \\
&> n d_n
\end{aligned}
$$

So, $d_n \leq \frac{1}{cn}$

$$
\begin{aligned}
|1 - c^{1/n}| &= 1 - c^{1/n} \\
&\leq d_n \\
&\leq \frac{1}{c}\frac{1}{n} \\
&\to 0
\end{aligned}
$$

Therefore, $\left(c^{1/n}\right)_n \to 1$.

### Positive Sequence Convergence

Let $(x_n)_n$ be a sequence with $x_n \in \mathbb{R}^+ \ \forall n \in \mathbb{N}$, with $(x_n)_n \to x$. Then, $x$ is also positive, and $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

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Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0 - x|}{2}$. Since $(x_n)_n$ converges to $x$, we know that $x_n \in V_\varepsilon(x)$ for large $n$. However, every member of $V_\varepsilon(x) < 0$, and $x_n > 0$. $\perp$

**Case 1:** If $x = 0$, we will show that $\left(\sqrt{x_n}\right)_n \to 0$.

Let $\varepsilon > 0$, find $N \in \mathbb{N}$ large such that if $n \geq N$, we have

$$|x_n - 0| < \varepsilon^2$$
$$x_n < \varepsilon^2$$
$$\sqrt{x_n} < \varepsilon$$
$$|\sqrt{x_n} - 0| < \varepsilon$$

**Case 2:** If $x > 0$, we will show that $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \left|\frac{\left(\sqrt{x_n} - \sqrt{x}\right)\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x_n}}\right|$$
$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$
$$\leq \frac{1}{\sqrt{x}}|x_n - x|$$
$$\to 0$$

Therefore, $\left|\sqrt{x_n} - \sqrt{x}\right| \to 0$, so $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

### $n$th Root of $n$ Convergence

$$\left(n^{1/n}\right)_n \to 1$$

We will make use of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Note that $n^{1/n} > 1$ for $n$ past 1. So, we write

$$n^{1/n} = 1 + d_n \qquad\qquad\qquad\qquad d_n > 0$$
$$n = (1 + d_n)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} d_n^k$$
$$= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n$$
$$\geq \binom{n}{0} + \binom{n}{2} d_n^2 \qquad\qquad \text{as all terms are positive}$$
$$= 1 + \frac{n(n-1)}{2} d_n^2$$

so

$$n - 1 \geq \frac{n(n-1)}{2} d_n^2$$
$$\frac{2}{n} \geq d_n^2$$
$$\frac{\sqrt{2}}{\sqrt{n}} \geq d_n$$

So, we have

$$|n^{1/n} - 1| = n^{1/n} - 1$$
$$= d_n$$
$$\leq \sqrt{2}\, \frac{1}{\sqrt{n}}$$
$$\to 0 \qquad\qquad\qquad\qquad \text{by previous corollary}$$

So, $\left(n^{1/n}\right)_n \to 0$.

## Multiplication by Geometric Sequence

Let $0 \leq b < 1$. Show that

$$(nb^n)_n \to 0$$

---

If $0 < b < 1$ (the 0 case is trivial). So, $\frac{1}{b} > 1$, meaning $\frac{1}{b} = 1 + d$ for some $d > 0$.

$$\frac{1}{b^n} = (1+d)^n$$
$$\geq \frac{n(n-1)}{2}d^2$$
$$\frac{2}{d^2(n)(n-1)} \geq b^n$$
$$nb^n \leq \frac{2}{d^2(n-1)}$$
$$\to 0 \qquad \qquad \text{by previous corollary}$$

Therefore, $(nb^n)_n \to 0$.

## Boundedness and Convergence

If $(x_n)_n$ is a convergent sequence, $x_n$ is bounded. The converse is false in general.

---

Suppose $(x_n)_n \to x$. Let $\varepsilon = 1$.

Then, $\exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq N$.

Let $c = \max\{|x_1|, |x_2|, \ldots, |x_N|, |x-1|, |x+1|\}$. If $n \geq N$, then $|x_n| \leq c$, because $x_n \in V_\varepsilon(x)$. If $n < N$, then $|x_n| \leq c$.

Together, we have $|x_n| \leq c$ for all $n$.

To show the converse is not true, consider $((-1)^n)_n$. This sequence is bounded but not convergent.

## Algebraic Operations on Sequences

Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$ be convergent sequences. Let $t \in \mathbb{R}$. Then, the following are all true:

(1) $(x_n \pm y_n)_n \to x \pm y$

(2) $(tx_n)_n \to tx$

(3) $(x_n y_n)_n \to xy$

(4) Assume $z_n \neq 0 \ \forall n$, and $z \neq 0$. Then, $\left(\frac{1}{z_n}\right)_n \to \frac{1}{z}$, and $\left(\frac{x_n}{z_n}\right)_n \to \frac{x}{z}$.

---

**Proof of (1)** Let $\varepsilon > 0$. Since $x_n \to x$, $y_n \to y$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \to |x_n - x| < \frac{\varepsilon}{2}$, and $n \geq N_2 \to |x_n - x| \leq \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

$$|(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

**Proof of (3)** We have $(x_n)_n \to x$ and $(y_n)_n \to y$.

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) + y(x_n - x)| \\
&\leq |x_n(y_n - y)| + |y(x_n - x)| \\
&= |x_n||y_n - y| + |x_n - x||y|
\end{aligned}$$

Since convergent sequences are bounded, $\exists c \in \mathbb{R}$ such that $|x_n| < c, \ \forall n$

$$\begin{aligned}
&\leq c|y_n - y| + |x_n - x||y| \\
&\to 0
\end{aligned}$$

Therefore, $|x_n y_n - xy| \to 0$, so $x_n y_n \to xy$.

**Proof of (4)** We have $z_n \neq 0$ and $z \neq 0$. Let $\varepsilon > 0$.

$$\begin{aligned}
\left| \frac{1}{z_n} - \frac{1}{z} \right| &= \frac{|z - z_n|}{|z_n z|} \\
&= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|}
\end{aligned}$$

Let $\varepsilon = \frac{|z|}{2}$. Since $(z_n)_n \to z$, we know that $z_n \in V_\varepsilon(z)$ for $n \geq N \in \mathbb{N}$. For $n \geq N$, $|z_n| > \frac{|z|}{2}$, so $\frac{1}{|z_n|} < \frac{2}{|z|}$.

$$\begin{aligned}
&\leq |z_n - z| \frac{2}{|z|^2} \\
&\to 0
\end{aligned}$$

So, $\left| \frac{1}{z_n} - \frac{1}{z} \right| \to 0$, so $\frac{1}{z_n} \to \frac{1}{z}$

---

### Ordering of Limits

Let $(x_n)_n \to x$ and $(y_n)_n \to y$. If $x_n \leq y_n$ for all $n$, then $x \leq y$.

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Suppose toward contradiction that $x > y$.

Let $\varepsilon = \frac{x-y}{2}$.

So, $\exists N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$.

Let $N = \max\{N_1, N_2\}$.    Then, $x_N \in V_\varepsilon(x)$ and $y_N \in V_\varepsilon(y)$.    But that means $x_N > y_N$.    $\perp$

In particular, if $(x_n)_n \to x$, and $a \leq x_n \leq b$, then $a \leq x \leq b$.

---

### Squeeze Theorem

Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$, where $x_n \leq y_n \leq z_n$ for all $n$.

If $L = x = z$, then $y = L$.

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Let $\varepsilon > 0$. Find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow V_\varepsilon(L)$, and $n \geq N_2 \Rightarrow V_\varepsilon(L)$.

Let $N = \max\{N_1, N_2\}$. Then, $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$. Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so $y_n \in V_\varepsilon(L)$, so $(y_n)_n \to L$.

---

For example, let $a_n = \frac{\sin(n)}{n}$. Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero, $a_n \to 0$

As another example, consider $a_n = (2^n + 3^n)^{1/n}$. Then,

$$3^n \leq 2^n + 3^n \leq 2 \cdot 3^n$$

$$3 \leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3$$

Since $2^{1/n} \to 1$, we have $a_n \to 3$.

## Ratio Test

Let $(x_n)$ be a sequence of strictly positive numbers, with $\left(\frac{x_{n+1}}{x_n}\right)_n \to r < 1$. Then, $(x_n)_n \to 0$.

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Since $r < 1$, then $1 - r > 0$. Let $\rho = r + \frac{1-r}{2}$, and $\varepsilon = \rho - r = \frac{1-r}{2}$.

Since the sequence converges, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\left| \frac{x_{n+1}}{x_n} - r \right| < \varepsilon$$

$$\frac{x_{n+1}}{x_n} < \rho$$

$$x_{n+1} < \rho x_n$$

In particular, $x_{N+1} < \rho x_N$, and $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$. Inductively, one can show that $\forall k \geq 1$, $x_{N+k} < \rho^k x_N$.

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as $k \to \infty$, both sides of the inequality go to 0, implying that $x_n \to 0$

## Monotone Convergence Theorem

Let $(x_n)_n$ be a monotone sequence. Then, $(x_n)_n$ is convergent if and only if it is bounded.

(a) If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n \to \sup(\{x_n \mid n \in \mathbb{N}\})$.

(b) If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n \to \inf(\{x_n \mid n \in \mathbb{N}\})$.

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We have already shown that all convergent sequences are bounded.

Assume that $(x_n)_n$ is monotonic and bounded.

**Case 1:** Suppose $(x_n)_n$ is increasing. Let $\sup\{x_n \mid n \in \mathbb{N}\} := u$. We claim that $(x_n)_n \to u$.

Let $\varepsilon > 0$. By the definition of supremum, $\exists N \in \mathbb{N}$ such that $u - \varepsilon < x_N$. Note that $\forall n \geq N$, $u - \varepsilon < x_N \leq x_n \leq u$.

Therefore, if $n \geq N$, then $|x_n - u| < \varepsilon$.

**Case 2:** Suppose $(x_n)_n$ is decreasing. Let $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$. We claim that $(x_n)_n \to \ell$.

Let $\varepsilon > 0$. By the definition of infimum, $\exists N \in \mathbb{N}$ such that $\ell + \varepsilon > x_N$. Additionally, $\forall n \geq N$, $\ell \leq x_n \leq x_N < \ell + \varepsilon$.

Therefore, if $n \geq N$, $|x_n - \ell| < \varepsilon$.

### Applications of the Monotone Convergence Theorem

#### Lemma

If $(x_n)_n$ is a convergent sequence, and $m \in \mathbb{N}$, the $m$-th tail, $x_{(m)} = (x_{m+k})_{k=1}^{\infty}$ is also convergent. If $(x_n)_n \to L$ then $x_{(m)} \to L$.

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Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. If $k \geq N$, then $m + k \geq N$, so $|x_{m+k} - L| < \varepsilon$.

Thus, $(x_{m+k})_k \to L$

Consider the inductively defined sequence

$$x_1 = 8$$
$$x_{n+1} = \frac{1}{2}x_n + 2$$
$$(x_n)_n = (8, 6, 5, 9/2, 17/4, \dots)$$

We claim that $x_n \geq 4 \; \forall n$.

$$x_1 = 8 \geq 4$$

Suppose $x_k \geq 4$. We will show that $x_{k+1} \geq 4$.

$$x_{k+1} = \frac{1}{2}x_k + 2$$
$$\geq \frac{1}{2}(4) + 2$$
$$= 4$$

Therefore, $(x_n)_n$ is bounded below by 4.

We claim that $(x_n)_n$ is decreasing. $\forall n \in \mathbb{N}$,

$$x_{n+1} \leq x_n \Leftrightarrow$$
$$\frac{1}{2}x_n + 2 \leq x_n$$
$$\Leftrightarrow 4 \leq x_n$$

By the monotone convergence theorem, we know that $(x_n)_n \to L$.

To find $L$, we use the recursive relationship and the lemma.

$$x_{n+1} = \left( \frac{1}{2}x_n + 2 \right)_{n=1}^{\infty}$$
$$L = \frac{1}{2}L + 2$$
$$L = 4$$

---

Consider the following sequence

$$x_1 = 1$$
$$x_2 = 1 + \frac{1}{4}$$
$$x_3 = 1 + \frac{1}{4} + \frac{1}{9}$$
$$x_k = \sum_{k=1}^{n} \frac{1}{k^2}$$

We will show that $(x_n)_n$, the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$k^2 \geq k(k-1) \qquad\qquad k \geq 2$$
$$\frac{1}{k^2} \leq \frac{1}{k(k-1)}$$
$$= \frac{1}{k-1} - \frac{1}{k}$$
$$\sum_{k=2}^{n} \frac{1}{k^2} \leq \sum_{k=2}^{n} \left( \frac{1}{k-1} - \frac{1}{k} \right)$$
$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \left( \frac{1}{k-1} - \frac{1}{k} \right)$$

But

$$1 + \sum_{k=2}^{n} \left( \frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{n}$$

so, we have

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$
$$< 2$$

So, $(x_n)_n$ is bounded above.

### Nested Intervals Theorem, Alternative Proof

Let $I_n = [a_n, b_n]$ be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

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Since the intervals are nested, it must be the case that $a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq b_1$.

Similarly, $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$.

So, $(a_n)_n$ is an increasing sequence bounded above by $b_1$ and $(b_n)n$ is a decreasing sequence bounded below by $a_1$.
So, $(b_n)_n \to r$ and $(a_n) \to \ell$
Note that $\ell = \sup(a_n)$ and $r = \inf(b_n)$.

Fix $n \in \mathbb{N}$, then for any $m \geq n$, $a_n \leq a_m \leq b_m \leq b_n$. So, $\sup(a_m) = \ell \leq b_n$. Unlocking $n$, we get that $\ell \leq \inf(b_n) = r$.

### Calculating Square Roots

Let $a \in \mathbb{R}^+$. We will construct a sequence $(x_n)_n \to \sqrt{a}$.

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right).$$

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We will prove that $x_n^2 \geq a$.

$$2x_{n+1} = x_n + \frac{a}{x_n}$$
$$2x_{n+1}x_n = x_n^2 + a$$
$$0 = x_n^2 - 2x_{n+1}x_n + a$$

So, $x_n$ is a real root, meaning

$$\Delta = 4x_{n+1}^2 - 4a$$
$$x_{n+1}^2 \geq a \qquad\qquad \forall n$$

So, $\forall n \geq 2$

$$x_n^2 \geq a$$

---

We will show that $x_n$ is ultimately decreasing.

$$
\begin{aligned}
x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\
&= \frac{1}{2}\underbrace{\left(\frac{x_n^2 - a}{x_n}\right)}_{\geq 0 \ \forall n \geq 2}
\end{aligned}
$$

---

So, we have that $(x_n)_n$ is decreasing and bounded below, meaning $(x_n)_n \to x$ for some $x \in \mathbb{R}$.

We had

$$
\begin{aligned}
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\
x &= \frac{1}{2}\left(x + \frac{a}{x}\right) \\
x &= \frac{a}{x} \\
x^2 &= a \\
x &= \sqrt{a} \hspace{4cm} \text{remember that } x > 0
\end{aligned}
$$

## Euler's Number

Consider

$$
\begin{aligned}
(e_n)_n &= \left(1 + \frac{1}{n}\right)^n \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}
\end{aligned}
$$

Similarly,

$$
e_{n+1} = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\prod_{j=1}^{k-1}\left(1 - \frac{j}{n+1}\right)\right)
$$

$$e_{n+1} \geq e_n \hspace{6cm} \forall n$$

---

We claim that $(e_n)_n$ is bounded above.

$$e_1 = \left(1 + \frac{1}{1}\right)^1$$

$$2 \le e_n$$

$$e_n = \sum_{k=0}^{n} \left( \frac{1}{k!} \underbrace{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}_{\le 1} \right)$$

$$2^{k-1} \le k! \qquad\qquad k \ge 2$$

$$\frac{1}{k!} \le \frac{1}{2^{k-1}}$$

$$e_n = \sum_{k=0}^{n} \frac{1}{k!} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)$$

$$\le \sum_{k=0}^{n} \frac{1}{k!}$$

$$\le 2 + \sum_{\ell=1}^{n-1} \frac{1}{2^\ell}$$

$$< 3$$

so, we have

$$2 \le e_n \le 3$$

so, by the monotone convergence theorem, $(e_n)_n$ converges

$$e := \sup_n \left(1 + \frac{1}{n}\right)^n$$

### Monotone Divergence

A sequence that is increasing and *unbounded* diverges to infinity.

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Let $M > 0$. Since $(x_n)_n$ is unbounded, $\exists N \in \mathbb{N}$ such that $x_N > M$

Thus, if $n \ge N$, then $x_n \ge x_N > M$.

Consider

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

We can see that $h_n < h_{n+1}$. The primary question is as to whether $(h_n)_n$ is bounded.

$$h_1 = 1$$

$$\ge 1$$

$$h_2 = 1 + \frac{1}{2}$$

$$\ge 1 + \frac{1}{2}$$

$$h_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\ge 1 + \frac{1}{2} + \frac{1}{2}$$

$$h_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$\ge 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^{k} \frac{1}{2}$$

Let $M$ be large. Find $n$ such that $n > 2(M-1)$. In this case, $n/2 + 1 > M$. Let $N = 2^n$. Then, for $m \geq N$, $h_m > M$.

Thus, $(h_n)_n$ diverges to infinity.

---

## Natural Sequences

A **natural sequence** is a strictly increasing sequence of natural numbers, $(n_k)_{k=1}^{\infty}$

$$n_1 < n_2 < n_3 < \ldots$$

where $\forall k \in \mathbb{N}, \ n_k \in \mathbb{N}$.

### Natural Sequence Property

Given $(n_k)_k$ natural sequence, show that $(n_k) \geq k$.

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**Base Case:** We know that $n_1 \leq 1$, as $n_1 \in \mathbb{N}$.

**Inductive Step:** To be continued...

---

## Subsequences

Let $(x_n)_n$ be a sequence. A subsequence $(x_{n_k})_{k=1}^{\infty}$, where $(n_k)_k$ is a natural sequence.

For example, if $(x_n)_n = (-1)^n$. If $(n_k) = 2k$, then, $(x_{n_k}) = \left( (-1)^{2k} \right)_k = (1, 1, 1, \ldots)$. But, if $(n_k) = 2k + 1$, then $(x_{n_k}) = (-1, -1, -1, \ldots)$.

If $(x_n) = (1/n)_n$, and $(n_k)_k = k^2$, then $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \ldots)$.

If $(x_n)_n$ is a sequence, its $m$-th **tail** is $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \ldots)$, where $n_k = m + k$.

---

## Convergence of Subsequence

If $(x_n)_n \to x$, then for any natural sequence $(n_k)_k$,

$$(x_{n_k})_k \to x$$

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Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ large such that $n \geq N$, $|x_n - x| < \varepsilon$.

Take $K = N$. Then,

$$
\begin{aligned}
n_k &\geq k \\
&\geq K \\
&= N \\
\Rightarrow |x_{n_k} - x| &< \varepsilon
\end{aligned}
$$

---

## Corollary to Convergence of Subsequences

Given a sequence $(x_n)_n$, if there are two subsequences $(x_{n_k})_k \to x$, $(x_{n_\ell})_\ell \to x'$, where $x \neq x'$, then $(x_n)_n$ is divergent.

---

Recall the geometric sequence

$$(b^n)_{n=1}^{\infty} \to 0$$

if $0 < b < 1$.

The sequence $(1, b, b^2, \dots)$ is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem, $(b^n)_n \to \ell$.

Given $n = 2k$, we know that $(b^{2k})_k \to \ell$.

$$b^{2k} = \left(b^k\right)^2$$
$$\left(b^k\right)^2 \to \ell^2$$
$$\ell^2 = \ell$$
$$\ell = \{0, 1\}$$

since $b < 1$

$$\ell = 0$$

## Divergence and Subsequence

If $(x_n)_n \nrightarrow x$, then

$$(\exists \varepsilon_0 > 0)\,(\forall N \in \mathbb{N})\,(\exists n \geq N) \ni |x_n - x| \geq \varepsilon_0$$

We can use this to construct a sequence to show divergence.

---

Let $(x_n)_n$ be a sequence, and $x \in \mathbb{R}$.

$$(x_n)_n \nrightarrow x$$
$$\Leftrightarrow$$
$$(\exists \varepsilon_0 > 0)(\exists (x_{n_k})_k)$$

with

$$|x_{n_k} - x| \geq \varepsilon_0$$

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($\Rightarrow$) We know $\exists \varepsilon_0 > 0$ as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \varepsilon_0$$
$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \varepsilon_0$$
$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \varepsilon_0$$

Assume we have $n_1 < n_2 < \dots, n_k$ with

$$|x_{n_j} - x| \geq \varepsilon_0 \qquad\qquad j = 1, 2, \dots, k$$
$$N = n_k + 1 \Rightarrow n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \varepsilon_0$$

Iteratively, we have our desired subsequence $(x_{n_k})_k$.

($\Leftarrow$) If $(x_n)_n \to x$, any subsequence converges to $x$.

By assumption, $(\exists \varepsilon_0 > 0)\,(\exists (n_k)_k)$ with $|x_{n_k} - x| \geq \varepsilon_0$. Thus, $(x_{n_k})_k \nrightarrow x$.

## Bolzano-Weierstrass Theorem

If $(x_n)_n$ is a bounded sequence, then $(x_n)_n$ admits a convergent subsequence.

### Lemma

Let $(x_n)_n$ be any real sequence. Then, $\exists n_k$ such that $(x_{n_k})_k$ is monotone.

---

A **peak** of a sequence $(x_n)_n$ is an $x_m$ such that $x_m \geq x_n \ \forall n \geq m$.

**Case 1:** There are infinitely many peaks, $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$, where $n_1 < n_2 < \dots$. Then, $(x_{n_k})_k$ is decreasing.

**Case 2:** There are finitely many peaks. Let these peaks be $x_{m_1}, x_{m_2}, \dots, x_{m_r}$.

Let $n_1 = m_r + 1$. Since $x_{n_1}$ is not a peak, $\exists n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since $x_{n_2}$ is not a peak, $\exists n_3 > n_2$ such that $x_{n_3} > x_{n_2}$.

Iteratively, we have an increasing sequence of non-peaks $(x_{n_k})_k$.

Since $(x_n)_n$ admits a monotone subsequence, and $(x_{n_k})_k$ is bounded as $(x_n)_n$ is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

## Limit Superior and Limit Inferior

Let $X = (x_n)_n$ be a bounded real sequence. By Bolzano-Weierstrass, $(x_n)_n$ admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R}, \ t = \lim_{k \to \infty} x_{n_k} \right\} \qquad \text{for any subsequence } (x_{n_k})_k$$

Then, $t \in \overline{X}$ is called a **limit point** of $X$.

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Let $u_1 = \sup_{n \geq 1}(x_n)$, $\ell_1 = \inf_{n \geq 1}(x_n)$. Clearly, $\ell_1 \leq u_1$, and $\overline{X} \subseteq [\ell_1, u_1]$.

Let $u_2 = \sup_{n \geq 2}(x_n)$ and $\ell_2 = \inf_{n \geq 2}(x_n)$.

Since $u_1$ is an upper bound for $(x_n)_n$, it is an upper bound for $(x_n)_{n \geq 2}$, so $u_2 \leq u_1$. Similarly, since $\ell_1$ is a lower bound for $(x_n)_n$, it is a lower bound for $(x_n)_{n \geq 2}$, so $\ell_2 \geq \ell_1$.

As a result, we can see that $\overline{X} \subseteq [\ell_2, u_2]$.

We continue, letting $u_m = \sup_{n \geq m}(x_n)$, and $\ell_m = \inf_{n \geq m}(x_n)$. We get $\ell_1 \leq \ell_2 \leq \cdots$, and $u_1 \geq u_2 \geq \cdots$, and $\overline{X} \in [\ell_m, u_m]$, $\forall m$.

We get a nested sequence of intervals $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \cdots$. By the Nested Intervals Theorem, we know that

$$\overline{X} \subseteq \bigcap_{m \geq 1} [\ell_m, u_m]$$
$$= [\ell, u]$$

where $\ell = \sup(\ell_m)$ and $u = \inf(u_m)$.

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Given a bounded sequence $(x_n)_x = X$,

$$u = \inf_{m \geq 1}(u_m)$$

$$= \inf_{m \geq 1} \left( \sup_{n \geq m} x_n \right)$$

called the **limit superior** of $(x_n)_n$

$$u = \limsup_{n \to \infty} x_n$$

and

$$\ell = \sup_{m \geq 1} (\ell_m)$$

$$= \sup_{m \geq 1} \left( \inf_{n \geq m} (x_n) \right)$$

called the **limit inferior** of $(x_n)_n$

$$\ell = \liminf_{n \to \infty} x_n$$

## Applications of Limit Superior and Limit Inferior

Let $(x_n)_n$ be bounded. Then,

(1) $\displaystyle \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n$

(2) $(x_n)_n \to x \Leftrightarrow \displaystyle \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = x$

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(1) This was proven with the Nested Intervals Theorem

(2) Let $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$.

We know that $u_m = \sup_{n \geq m} x_n$. If $m \geq N$, then $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$. Therefore, $|u_m - x| \leq \varepsilon/2 < \varepsilon$, so $(u_m)_m \to \varepsilon x \limsup_{n \to \infty} x_n$.

Similarly, we know that $\ell_m = \inf_{n \geq m} x_n$. If $m \geq N$, then $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$. So, $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$, so $(\ell_m)_m \to x = \liminf_{n \to \infty} x_n$.

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Consider the sequence

$$x_n = \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases}$$

$$= (-1, 5/2, -1/3, 9/4, -1/5, \dots)$$

We begin by constructing the $u_m$ sequence: $(5/2, 5/2, 9/4, 9/4, \dots)$. We can see that $u_m \to 2$.

Then, we construct the $\ell_m$ sequence: $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$. We can see that $\ell_m \to 0$.

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**Exercise:** If $(x_n)_n$ and $(y_n)_n$ are sequences with $x_n \leq y_n$ $\forall n$, then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

## Ratio Test and Root Test Equivalent Convergence

If $(a_n)_n$ is a sequence of strictly positive terms such that

$$\left( \frac{a_{n+1}}{a_n} \right)_n \to \rho$$

then,

$$\left( a_n^{1/n} \right)_{n=1}^{\infty} \to \rho$$

Let $\varepsilon > 0$. Then, $\exists N$ large such that $\forall n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} - \rho \right| < \varepsilon \qquad\qquad \forall n \geq N$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < \rho + \varepsilon \qquad\qquad \forall n \geq N$$

$$a_{n+1} n a_n (\rho + \varepsilon) \qquad\qquad \forall n \geq N$$

$$a_n < a_N (\rho + \varepsilon)^{n-N} \qquad\qquad \forall n \geq N$$

$$a_n < (\rho + \varepsilon)^n \cdot \frac{a_N}{(\rho + \varepsilon)^N}$$

$$a_n^{1/n} < (\rho + \varepsilon) \left( \frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n}$$

$$\limsup a_n^{1/n} \leq \limsup (\rho + \varepsilon) \left( \frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n}$$

$$\limsup_{n \to \infty} a_n^{1/n} \leq \rho + \varepsilon$$

**Case 1:** If $\rho = 0$, the case his trivial.

**Case 2:** Suppose $\rho > 0$. Find $\varepsilon > 0$ small such that $0 < \varepsilon < \rho$.

Since $\left( \frac{a_{n+1}}{a_n} \right)_n \to \rho$, find $N$ large such that $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$. So, $\forall n \geq N$,

$$a_{n+1} \geq a_n (\rho - \varepsilon)$$

$$a_n \geq a_N (\rho - \varepsilon)^{n-N}$$

$$a_n^{1/n} \geq (\rho - \varepsilon) \left( \frac{a_N}{(\rho - \varepsilon)^N} \right)^{1/n}$$

$$\liminf a_n^{1/n} \geq \rho - \varepsilon$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together, $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$, so $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$, whence $\left( a_n^{1/n} \right) \to \rho$