Solution (30.1)**:** (a) This is a legal expression.

(b) This is a legal expression.

(c) This is not a legal expression; we should obtain a dual vector upon acting with B_{ij} on the vector C^i.

(d) This is not a legal expression assuming summation convention; we cannot have a repeated index on the same tensor.

(e) This is not a legal expression assuming summation convention; we cannot have a repeated index on the same tensor.

(f) This is a legal expression.

(g) This is not a legal expression; we should have a $(0,2)$ tensor in the dual space, A_{ij}, rather than a $(1,1)$ tensor of the form A_i^j.

(h) This is a legal expression.

Solution (30.3)**:** Using the chain rule, we obtain

$$\begin{aligned} A^{j'}B_{j'} &= \frac{\partial u^{j'}}{\partial u^j}A^j\frac{\partial u^j}{\partial u^{j'}}B_j\\ &= \delta_j^j A^j B_j\\ &= A^j B_j. \end{aligned}$$

Meanwhile,

$$\begin{aligned} A^{j'}B^{j'} &= \frac{\partial u^{j'}}{\partial u^j}A^j\frac{\partial u^{j'}}{\partial u^j}B^j\\ &= \frac{\partial u^{j'}}{\partial u^j}\frac{\partial u^{j'}}{\partial u^j}A^jB^j, \end{aligned}$$

which means $A^{j'}B^{j'}$ is a rank $(2,0)$ tensor.

Solution (30.5)**:** The matrix

$$g_{ab} = \begin{pmatrix} 1 & \cos(\phi) \\ \cos(\phi)1 & \end{pmatrix}$$

has inverse

$$g^{ab} = \begin{pmatrix} \csc^2(\phi) & -\cot(\phi)\csc(\phi) \\ -\cot(\phi)\csc(\phi) & \csc^2(\phi) \end{pmatrix},$$

where $\cos(\phi) = \sin(\alpha+\beta)$. Therefore, we may calculate

$$\begin{aligned} \vec{e}^a &= g^{ab}\vec{e}_b\\ &= \frac{1}{\cos(\alpha+\beta)}\begin{pmatrix} \cos(\beta) \\ -\sin(\beta) \end{pmatrix}\\ \vec{e}^b &= g^{ab}\vec{e}_a\\ &= \frac{1}{\cos(\alpha+\beta)}\begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}. \end{aligned}$$

Solution (30.6)**:**

(a) We have the downstairs basis of $\left\{\hat{r}, r\hat{\phi}, \hat{z}\right\}$ for cylindrical coordinates.

(b) Using the metric of

$$g^{ab} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & 1 \end{pmatrix},$$

we calculate

$$\begin{aligned}\vec{e}^r &= g^{ab}\vec{e}_r\\ &= \hat{r}\\ \vec{e}^\phi &= g^{ab}\vec{e}_\phi\\ &= \frac{1}{r}\hat{\phi}\\ \vec{e}^z &= g^{ab}\vec{e}_z\\ &= \hat{z}.\end{aligned}$$

(c) We calculate

$$\begin{aligned}A_r\vec{e}^r &= A_r\hat{r}\\ &= A^r\hat{r}\\ A_\phi\vec{e}^\phi &= \frac{1}{r}A_\phi\hat{\phi}\\ &= A^\phi\hat{\phi}\\ A_z\vec{e}^z &= A_z\hat{z}\\ &= A^z\hat{z}.\end{aligned}$$

Thus,

$$\begin{aligned}A_r &= A^r\\ A_\phi &= \frac{1}{r}A^\phi\\ A_z &= A^z.\end{aligned}$$

(d) We have

$$\begin{aligned}A^r\vec{e}_r &= A^r\hat{r}\\ &= A_r\hat{r}\\ &= A_r\vec{e}^r\\ A^\phi\vec{e}_\phi &= A^\phi\hat{\phi}\\ &= rA_\phi\hat{\phi}\\ &= A_\phi\vec{e}^\phi.\\ A^z\vec{e}_z &= A^z\hat{z}\\ &= A_z\hat{z}\\ &= A_z\vec{e}_z.\end{aligned}$$

Solution (30.16)**:**

(a) Let $g_{ab} = \operatorname{diag}\left(1, r^2, r^2\sin^2(\theta)\right)$ be the metric for spherical coordinates. Then,

$$\begin{aligned}g_{rr} &= \left(\frac{\partial\rho}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\theta}{\partial r}\right)^2\\ &= \frac{z^2}{r^2} + 1 + \frac{1}{z^2}\\ g_{\phi\phi} &= 1\\ g_{zz} &= \frac{r^2}{z^2} + 1 + \frac{1}{r^2}.\end{aligned}$$

(b) We convert from spherical to cylindrical by converting from spherical to Cartesian by taking g^{ab} on spherical coordinates, summing over δ_{ij}, then multiplying by the cylindrical metric, with some relabeling, giving

$$\begin{aligned} g_{rr} &= \frac{z^2}{r^2} + 1 + \frac{1}{z^2} \\ g_{\phi\phi} &= 1 \\ g_{zz} &= \frac{r^2}{z^2} + 1 + \frac{1}{r^2}. \end{aligned}$$

This approach is simpler because we get to take advantage of the properties of the Cartesian metric (i.e., that it is independent of location).

Solution (30.20)**:** Calculating

$$\sqrt{\det(g_{ab})} = \sqrt{1 - \cos^2(\varphi)},$$

we may calculate

$$\begin{aligned} \int_A d\tau' &= \int_0^b \int_0^a \sqrt{1 - \cos^2(\varphi)}\, du\, dv \\ &= ab\sqrt{1 - \cos^2(\varphi)}. \end{aligned}$$

Solution (30.21)**:**

(a) We take

$$g_{ab} = \begin{pmatrix} R^2 & \\ & R^2 \sin^2(\theta) \end{pmatrix}.$$

(b) Calculating

$$\begin{aligned} \int_C ds &= \int_C \sqrt{\left|g_{ab}\, du^a\, du^b\right|} \\ &= R \int_0^{\theta_0} d\theta \\ &= R\theta_0. \end{aligned}$$

(c) The circumference C of the circle $\theta = \theta_0$ is equal to $2\pi R \sin(\theta_0)$, which is not equal to $2\pi s$ because observers on the sphere do not perceive its curvature.

(d) Without loss of generality we may assume we are on the north pole. Then, very close to the north pole, we have very small θ_0, or that $\theta_0 \approx \sin(\theta_0)$.

Thus, a small neighborhood of the sphere with radius $r \ll R$ will appear as a disc of radius r rather than a curved section whose boundary has radius $r \sin(\theta_0)$.

Solution (30.22)**:**

(a) Assuming $r_2 > r_1 > r_s$, we calculate

$$\begin{aligned} \int_{r_1}^{r_2} ds &= \int_{r_1}^{r_2} \sqrt{\left|g_{ab}\, du^a\, du^b\right|} \\ &= \int_{r_1}^{r_2} \left(1 - \frac{r_s}{r}\right)^{-1} dr \\ &= \int_{r_1}^{r_2} \frac{r}{r - r_s}\, dr \\ &= \int_{r_1}^{r_2} 1 + \frac{r_s}{r - r_s}\, dr \end{aligned}$$

$$= (r_2 - r_1) + r_s \ln\left(\frac{r_2 - r_s}{r_1 - r_s}\right)$$
$$\geqslant r_2 - r_1.$$

(b) Fixing R, we have

$$\int dA = \int_0^{2\pi}\int_0^{\pi} R^2 \sin(\theta)\, d\theta d\phi$$
$$= 4\pi R^2.$$

In this sense, R is a measure of (warped) constant radius.

(c) I don't know how to do this.

(d) As Bob approaches $r = r_s$, Alice perceives time to slow down for Bob.

Solution (30.28)**:**

(a) Using $u^c = \hat{\theta}$ and $u^a, u^b = r\sin\theta\hat{\phi}$, with the Christoffel symbol $\Gamma^{\theta}_{\phi\phi} = -\sin(\theta)\cos(\theta)$, we substitute to obtain

$$\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 = 0.$$

Similarly, expanding the other equation, we have

$$\frac{d}{dt}\left(\sin^2(\theta)\frac{d\phi}{dt}\right) = 2\sin(\theta)\cos(\theta)\frac{d\theta}{dt}\frac{d\phi}{dt} - \sin^2(\theta)\frac{d^2\phi}{dt^2}$$
$$= 0.$$

When we take $u^c = \phi$ and $\Gamma^{\phi}_{\phi\theta} = \cot(\theta)$, we get the geodesic equation yet again.

(b) With initial velocity along $\hat{\phi}$ at the equator, the geodesic equation evaluates to yield constant $\theta = \pi/2$, constant $r = R$, and $\dot{\phi} = k$ for some constant k. In other words, this yields a great circle.

(c) Since the geodesic equation is a covariant expression, we may use a series of transformations to give any other starting position to be equal to the case in part (b), meaning that all geodesics are great circles.