Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a region. Fix $z_0 \in U$. Let

$$\mathcal{F} = \{f \in H(U) \mid \operatorname{im}(f) \subseteq \mathbb{C} \setminus B(0,1), f(z_0) = 2i\}.$$

Show that $\mathcal{F}$ is normal.

Solution: Let $(f_n)_n$ be a sequence in $\mathcal{F}$. We use the conformal map $z \mapsto \frac{1}{z}$ to map $\mathbb{C} \setminus B(0,1)$ to $\mathbb{D}$, giving that the family

$$\mathcal{G} = \left\{ \frac{1}{f} \,\middle|\, f \in \mathcal{F} \right\}$$

is locally bounded (indeed, globally bounded) by 1. Thus, it follows that there is a subsequence

$$\left(\frac{1}{f_{n_k}}\right)_k \to g \colon U \to \mathbb{D}$$

for some holomorphic function $g \colon U \to \mathbb{D}$. Now, since $\frac{1}{f_n}$ has no zeros for each n, it follows from Hurwitz's theorem that either g is uniformly 0 or g also has no zeros. Yet, since $g(z_0) = -\frac{i}{2} \neq 0$, it thus follows that $\frac{1}{g}$ is holomorphic on U, whence

$$(f_{n_k})_k \to \frac{1}{g}.$$

Thus, $\mathcal{F}$ is normal.

Problem (Problem 2)**:**

(a) Using the Schwarz–Pick lemma, show that given $w \in \mathbb{D}$, there exists a holomorphic function $f \colon \mathbb{D} \to \mathbb{D}$ satisfying

$$\begin{aligned} f(w) &= 0 \\ |f'(w)| &= \sup_{\substack{g \in H(\mathbb{D}) \\ g(\mathbb{D}) \subseteq \mathbb{D}}} |g'(w)|. \end{aligned}$$

(b) Show that if $f \colon \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded, then

$$\sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)|f'(z)| \leqslant \sup_{z \in \mathbb{D}} |f(z)|.$$

(c) Let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Show that f either has at most 1 fixed point or f is the identity.

Solution:

(a) We know that the map

$$\psi_w(z) = \frac{w - z}{1 - \overline{w}z}$$

is a conformal map that takes $\psi_w(w) = 0$. Now, we know that

$$|\psi_w'(w)| = \frac{1}{1 - |w|^2}.$$

From the Schwarz–Pick Lemma, we have for all holomorphic functions $f \colon \mathbb{D} \to \mathbb{D}$

$$\frac{|f'(w)|}{1 - |f(w)|^2} \leqslant \frac{1}{1 - |w|^2}.$$

In particular, since $0 \leqslant |f(w)| < 1$, we have

$$|f'(w)| \leqslant \frac{1}{1-|w|^2},$$

whence $\psi_w(z)$ satisfies

$$\begin{aligned}\psi_w(w) &= 0\\ |\psi_w'(w)| &= \sup_{\substack{g\in H(\mathbb{D})\\ g(\mathbb{D})\subseteq\mathbb{D}}} |g'(w)|.\end{aligned}$$

(b) Let $K = \sup_{z\in\mathbb{D}}|f(z)|$. By the maximum modulus principle, $|f(z)| < K$ for all $z \in \mathbb{D}$, so it follows that $g(z) := \frac{f(z)}{K}$ is a self-map of the unit disk. By the Schwarz–Pick lemma, it then follows that

$$\frac{|g'(z)|}{1-|g(z)|^2} \leqslant \frac{1}{1-|z|^2}.$$

Simplifying, we then get

$$\begin{aligned}\left(1-|z|^2\right)|f'(z)| &\leqslant K\left(1-\frac{|f(z)|^2}{K^2}\right)\\ &\leqslant K,\end{aligned}$$

so that

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f'(z)| \leqslant \sup_{z\in\mathbb{D}}|f(z)|.$$

(c) The statement is equivalent to showing that if $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map with two fixed points, then f is the identity map. Let f be one of these maps, and let $\xi \neq \eta \in \mathbb{D}$ be such that $f(\xi) = \xi$ and $f(\eta) = \eta$.

We want to find some holomorphic self-map of $\mathbb{D}$ that sends $0 \mapsto 0$. We consider the maps

$$\psi_\xi = \frac{\xi - z}{1-\overline{\xi}z},$$

which takes $0 \mapsto \xi$ and $\xi \mapsto 0$. Notice that $\psi_\xi \circ \psi_\xi = \mathrm{id}$. Therefore,

$$g = \psi_\xi \circ f \circ \psi_\xi$$

is a holomorphic self-map that sends $0 \mapsto 0$, so by Schwarz's Lemma, we have

$$|g(z)| \leqslant |z|$$

for all $z \in \mathbb{D}$. Yet, we also have

$$\begin{aligned}g(\psi_\xi(\eta)) &= \psi_\xi \circ f \circ \psi_\xi \circ \psi_\xi(\eta)\\ &= \psi_\xi(\eta).\end{aligned}$$

In particular, this means that

$$|g(\psi_\xi(\eta))| = |\psi_\xi(\eta)|,$$

so there exists $\mathbb{D} \ni w := \psi_\xi(\eta)$ such that $|g(w)| = |w|$, so that $g(w) = e^{i\theta}w$. Yet, since the identity relation holds for $\psi_\xi(\eta)$, it follows that $\theta = 0$, so $g(w) = w$. In particular, this means

$$\begin{aligned}\psi_\xi \circ f \circ \psi_\xi(z) &= z\\ f \circ \psi_\xi(z) &= \psi_\xi(z).\end{aligned}$$

Yet, since ψ_ξ is an automorphism, it follows that this relation holds for all $z \in \mathbb{D}$, so that $f(w) = w$ for all $w \in \mathbb{D}$, whence $f = \mathrm{id}$.

Problem (Problem 3)**:** Let $f: \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$.

(a) Show that $|f(z) + f(-z)| \leqslant 2|z|^2$ for all $z \in \mathbb{D}$.

(b) Show that $|f(z) + f(-z)| = 2|z|^2$ for some $z \in \mathbb{D} \setminus \{0\}$ if and only if $f(z) = e^{i\theta}z^2$.

Solution:

(a) We seek to show that the function

$$k(z) = \frac{f(z) + f(-z)}{2z}$$

maps $\mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$. We may safely assume that $z \neq 0$, as the desired inequality is certainly true for $z = 0$. We observe that since f is a self-map of $\mathbb{D}$ with $f(0) = 0$, Schwarz's Lemma gives

$$|f(z)| \leqslant |z|,$$

or that

$$\frac{|f(z)|}{|z|} \leqslant 1$$

A similar fact holds for $f(-z)$. For all $z \in \mathbb{D}$, we thus have

$$\begin{aligned}\left|\frac{f(z) + f(-z)}{2z}\right| &\leqslant \frac{1}{2}\left(\left|\frac{f(z)}{z}\right| + \left|\frac{f(-z)}{z}\right|\right) \\ &< 1.\end{aligned}$$

Therefore, since k is a self-map of $\mathbb{D}$ with $k(0) = 0$, Schwarz's Lemma gives

$$|f(z) + f(-z)| \leqslant 2|z|^2.$$

(b) Equivalently, we are assuming that

$$\left|\frac{f(z) + f(-z)}{2z}\right| = |z|$$

for some $z \in \mathbb{D} \setminus \{0\}$. From Schwarz's Lemma, we then have that

$$\frac{f(z) + f(-z)}{2z} = e^{i\theta}z$$

for some $\theta \in \mathbb{R}$. This gives

$$\frac{1}{2}(f(z) + f(-z)) = e^{i\theta}z^2.$$

Problem (Problem 4)**:**

(a) Show that if $f: \mathbb{H} \to \mathbb{D}$ is a conformal map, then there exists some $\theta \in \mathbb{R}$ and $\beta \in \mathbb{H}$ such that

$$f(z) = e^{i\theta z}\frac{z - \beta}{z - \overline{\beta}}.$$

(b) Show that if $f: \mathbb{H} \to \mathbb{H}$ is a conformal map, then there exists some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

such that

$$f(z) = \frac{az + b}{cz + d}.$$

Solution:

(a) Let $f\colon \mathbb{H} \to \mathbb{D}$ be a conformal map, and let $\beta = f^{-1}(0)$. It suffices to show that

$$g(z) = \frac{z - \beta}{z - \overline{\beta}}$$

is a conformal map from $\mathbb{H}$ to $\mathbb{D}$ that takes $\beta \mapsto 0$. The essential uniqueness of conformal maps from simply connected domains to the unit disk will give us our desired result.

Letting $\beta = a + bi$ with $b > 0$, we observe that the expression of g can be rewritten as

$$\begin{aligned} g(z) &= \frac{\left(\frac{z-a}{b}\right) - i}{\left(\frac{z-a}{b}\right) + i} \\ &= q \circ L(z), \end{aligned}$$

where $L\colon \mathbb{H} \to \mathbb{H}$ takes $z \mapsto \frac{z-a}{b}$ (and is a uniquely determined automorphism of $\mathbb{H}$ that takes $a+bi$ to i), while $q\colon \mathbb{H} \to \mathbb{D}$ is the Cayley transform. In particular, this is a composition of conformal maps, hence conformal, maps $\beta \mapsto 0$, and has $|g'(\beta)| \neq 0$, meaning that it must be the case that a general conformal map from $\mathbb{H}$ to $\mathbb{D}$ that maps $\beta \mapsto 0$ must be of the form

$$f = e^{i\theta} g(z).$$

(b) We start by showing that all conformal maps $f\colon \mathbb{H} \to \mathbb{H}$ that fix i can be expressed by fractional linear transformations from matrices in $SL_2(\mathbb{R})$. We observe then that $q \circ f\colon \mathbb{H} \to \mathbb{D}$, where q is the Cayley Transform, is necessarily of the form

$$q \circ f = e^{i\theta}\frac{z - i}{z + i},$$

following from the uniqueness we showed in part (a). This gives

$$\begin{aligned} f &= -i\frac{z\left(1 + e^{i\theta}\right) + i\left(1 - e^{i\theta}\right)}{z(e^{i\theta} - 1) - i(1 + e^{i\theta})} \\ &= -i\frac{e^{i\theta/2}(z(2\cos(\theta/2)) + 2\sin(\theta/2))}{e^{i\theta/2}(z(2i\sin(\theta/2)) - 2i\cos(\theta/2))} \\ &= \frac{z\cos(\theta/2) + \sin(\theta/2)}{-z\sin(\theta/2) + \cos(\theta/2)}. \end{aligned}$$

Since the rotation map

$$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in SL_2(\mathbb{R}),$$

it follows that any conformal map of $\mathbb{H}$ that fixes i can be expressed in this form.

In the general case, we observe that we can translate an arbitrary element of the form $z = a + bi$ with $b > 0$ to i by taking $L = \frac{z-a}{b}$, which admits a representation as an element of $SL_2(\mathbb{R})$ via the matrix

$$\begin{pmatrix} 1/\sqrt{b} & -a/\sqrt{b} \\ 0 & \sqrt{b} \end{pmatrix} \in SL_2(\mathbb{R}).$$

In particular, if $f\colon \mathbb{H} \to \mathbb{H}$ is a conformal map, then there is a unique element $a + bi$ that maps to i, meaning that we may necessarily write f as

$$f = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1/\sqrt{b} & -a/\sqrt{b} \\ 0 & \sqrt{b} \end{pmatrix} \cdot z,$$

whence any conformal map from $\mathbb{H}$ to $\mathbb{H}$ can be expressed in this fashion.

Problem (Problem 5)**:** Let f be an entire function satisfying $|f(z)| = 1$ for all $z \in S^1$. Show that there exists some $\theta \in \mathbb{R}$ and a nonnegative integer n such that $f(z) = e^{i\theta} z^n$ for all $z \in \mathbb{C}$.

Solution: To start, if f is constant, then it follows that $|f(z)| = 1$ for all $z \in \mathbb{C}$, meaning that $f(z) = e^{i\theta}$ for some θ.

Now, let f be nonconstant. We claim that $\inf_{z \in B(0,1)} |f(z)| = 0$. If it were not the case, then by applying the maximum modulus principle to both f and $1/f$, we would reach a contradiction claiming that $f = e^{i\theta}$ on $\mathbb{D}$, hence on all of $\mathbb{C}$ by the identity theorem, contradicting the fact that f is nonconstant.

Since all the zeros of f are isolated, we have finitely many contained in $B(0,1)$, hence finitely many in $\mathbb{D}$. Call these zeros $\{z_j\}_{j=1}^n$ (with multiplicity). Let

$$B(z) = \prod_{j=1}^{n} \frac{z_j - z}{1 - \overline{z_j} z}.$$

Since B is a Blaschke product, it follows that $|B(z)| = 1$ on S^1. Furthermore, evaluating

$$\begin{aligned} \lim_{z \to z_j} \frac{f(z)}{B(z)} &= \lim_{z \to z_j} \frac{(z - z_j)^k g(z)}{(-1)^k (z - z_j)^k H(z)} \\ &= (-1)^k \frac{g(z_j)}{H(z_j)}, \end{aligned}$$

where g and H are holomorphic functions that are nonzero on $\mathbb{D}$. In particular, this means that the function $\frac{f}{B}$ has no zeros in $\mathbb{D}$ and $\left|\frac{f}{B}\right| = 1$. Therefore, by the reasoning above, we must have that

$$f(z) = e^{i\theta} B(z).$$

The only Blaschke factors that are holomorphic on $\mathbb{C}$ are the ones of the form z^n, meaning that we have $f(z) = e^{i\theta} z^n$.

Problem (Problem 6)**:**

(a) Show that there does not exist a continuous function $f\colon \overline{\mathbb{H}} \to \mathbb{C}$ for which $f|_{\mathbb{H}}$ is holomorphic, $f(\mathbb{R}) \subseteq (-\infty, 0)$, and $f(\mathbb{H}) \subseteq \mathbb{H}$.

(b) Let $f\colon \overline{\mathbb{H}} \to \mathbb{C}$ be a continuous function for which $f|_{\mathbb{H}}$ is holomorphic, $f(\mathbb{R}) \subseteq \mathbb{R}$, and $0 \leqslant \operatorname{Im}(f(z)) \leqslant \operatorname{Re}(f(z))$ for all $z \in \overline{\mathbb{H}}$. Show that f is constant.

Solution:

(a) We extend f to a holomorphic function on all of $\mathbb{C}$ using the Schwarz reflection principle. Observe then that the range of the extension for f (which we call g) is contained in $\mathbb{C} \setminus [0, \infty)$. We may thus define a branch of the square root that maps $\mathbb{C} \setminus [0, \infty)$ to the upper half-plane, so that

$$v(z) = \frac{\sqrt{g(z)} - i}{\sqrt{g(z)} + i}$$

is an entire function whose range is contained within $\mathbb{D}$. Thus, it follows that g (and thus f) is constant by Liouville's Theorem. Yet, this would lead to a contradiction, since the condition that $f(\mathbb{R}) \subseteq (0, \infty)$ would imply that the constant value for f is some element of $\mathbb{R}$, while the condition that $f(\mathbb{H}) \subseteq \mathbb{H}$ would imply that the constant value for f is an element of $\mathbb{H}$, which cannot be in $\mathbb{R}$.

(b) Using the Schwarz reflection principle, we extend f to be holomorphic on $\mathbb{C}$; we call this extension g. To understand the behavior of g with respect to the alternative condition on f, we observe that on the lower half-plane, we have that $g(z) = \overline{f(\overline{z})}$; in particular, we have

$$\operatorname{Im}\left(\overline{f(\overline{z})}\right) \leqslant 0 \leqslant \operatorname{Re}\left(\overline{f(\overline{z})}\right),$$

so we have that g maps $\mathbb{C}$ to the right half-plane. Thus, we have that $U(-1, 1/2) \not\subseteq g(\mathbb{C})$, meaning that by the converse to a corollary of Liouville's Theorem, we have that g is constant. Since g is a holomorphic extension of f, it follows that f is thus constant.