Problem 1

Let $X = \{0,1\}^n$. Show that the Hamming distance:

$$\begin{aligned} d_H &: X \times X \to [0,\infty) \\ d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) &= \left|\{j \mid x_j \neq y_j\}\right| \end{aligned}$$

defines a metric on X.

Proof:

- Symmetry:

$$\begin{aligned} d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) &= \left|\{j \mid x_j \neq y_j\}\right| \\ &= \left|\{j \mid y_j \neq x_j\}\right| \\ &= d_H\left((y_j)_{j=1}^n, (x_j)_{j=1}^n\right) \end{aligned}$$

- Definiteness: it is only the case that $d_H(x_j, y_j) = 0$ if $x_j = y_j$ for all j, by the definition of the distance.
- Similarly, since $x_j = x_j$ for all j, $d_H(x_j, x_j) = 0$.
- Let $(x_j)_j$, $(y_j)_j$, and $(z_j)_j$ be sequences of bits. The set $\{j \mid x_j \neq z_j\}$ is formed by taking all the values $\{j \mid x_j \neq y_j\}$ along with $\{j \mid y_j \neq z_j\}$, net of particular indices where $x_j = z_j$, but $x_j \neq y_j$. Therefore,

$$d(x,z) \leq d(x,y) + d(y,z).$$

Problem 2

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V, show that the induced metrics d and d' are equivalent.

Proof: Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms. Then, $\exists c_1, c_2 \in \mathbb{R}$ such that $\|v - w\|' \leq c_1 \|v - w\|$ and $\|v - w\| \leq c_2 \|v - w\|'$. However, this is the exact same statement as $d(v,w) \leq c_1 d'(v,w)$ and $d'(v,w) \leq c_2 d'(v,w)$. Thus, d and d' are equivalent metrics.

Problem 3

Let $\{X_k, d_k\}$ be a sequence of metric spaces with uniformly bounded metrics. Let

$$X := \prod_{k \geq 1} X_k$$

denote the product.

(a) Show that

$$\begin{aligned} D &: X \times X \to [0,\infty) \\ D(x,y) &:= \sum_{k \geq 1} 2^{-k} d_k(x_k, y_k) \end{aligned}$$

defines a metric on X.

(b) Consider the case where $\{X_k\} = \{0,2\}$ and $d_k(a,b) = |a-b|$ for every $k \geq 1$. We get the abstract Cantor set

$$\begin{aligned} \Delta &:= \prod_{k \geq 1} \{0,2\}; \\ D(x,y) &:= \sum_{k=1}^{\infty} 3^{-k} |x_k - y_k|. \end{aligned}$$

Prove that $D(x,z) = D(y,z)$ implies $x = y$.

Proof:

(i) Let D be defined as above. Then, $D((x_k)_k,(x_k)_k)$ is a sum of $d_k(x_k,x_k)$, all uniformly zero, meaning $D((x_k)_k,(x_k)_k)=0$.

Similarly, $D((x_k)_k,(y_k)_k)=0$ implies that $d_k(x_k,y_k)=0$ for all x_k,y_k. Since d_k is a metric, this means $x_k=y_k$ for all k, implying that $(x_k)_k=(y_k)_k$.

Additionally, $d_k(x_k,y_k)=d_k(y_k,x_k)$, it is the case that $D((x_k)_k,(y_k)_k)=D((y_k)_k,(x_k)_k)$.

Finally, we must show the triangle inequality:

$$\begin{aligned}
D((x_k)_k,(z_k)_k) &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_k,z_k)\\
&\leq \sum_{k=1}^{\infty} 2^{-k}(d_k(x_k,y_k)+d(y_k,z_k))\\
&= \sum_{k=1}^{\infty} 2^{-k} d_k(x_k,y_k)+\sum_{k=1}^{\infty} d(y_k,z_k)\\
&= D((x_k)_k,(y_k)_k)+D((y_k)_k,(z_k)_k).
\end{aligned}$$

(ii) Suppose $x\neq y$. Let ℓ denote the smallest index where $x_\ell\neq y_\ell$. Suppose without loss of generality that $x_\ell=2$ and $y_\ell=0$. Then, $||x_\ell-z_\ell|-|y_\ell-z_\ell||=2\cdot 3^{-\ell}$. Additionally,

$$\begin{aligned}
0 &\leq \sum_{k=\ell+1}^{\infty} 3^{-k}|x_k-z_k|\\
&\leq \sum_{k=\ell+1}^{\infty} 3^{-k}(2)\\
&= \frac{2}{3^{\ell+1}}\\
&< \frac{2}{3^{\ell}}.
\end{aligned}$$

Thus, $D(x,z)\neq D(y,z)$.

Problem 4

Let $(V,\|\cdot\|)$ be a normed space, and suppose $E\subseteq V$. Show that the following are equivalent:

(1) E is bounded — $\operatorname{diam}(E)<\infty$;

(2) $\sup_{v\in E}\|v\|<\infty$;

(3) there is an $r>0$ such that $E\subseteq B(0,r)$.

Proof: (i) $\Rightarrow$ (ii): Let E be bounded. Then,

$$\begin{aligned}
|\|v\|-\|w\|| &\leq \|v-w\|\\
\sup_{v,w\in E}|\|v\|-\|w\|| &\leq \sup_{v,w\in E}\|v-w\|\\
\sup_{v\in E}\|v\|-\inf_{w\in E}\|w\| &\leq c\\
\sup_{v\in E}\|v\| &\leq c+\inf_{w\in E}\|w\|.
\end{aligned}$$

(ii) $\Rightarrow$ (iii): Since, for $v\in E$, $\sup\|v\|<\infty$, if we set $r=\sup\|v\|+1$, then $v\in B(0,r)$, meaning $E\subseteq B(0,r)$.

(iii) $\Rightarrow$ (i): Let E be such that $E\subseteq B(0,r)$ for some r. Then, $\forall v,w\in B(0,r)$, $\|v-w\|\leq 2r$, meaning that $\forall v,w\in E, \|v-w\|\leq 2r$, meaning $\operatorname{diam}(E)<\infty$.

Problem 5

Let (X,d) be a metric space and suppose $A\subseteq X$. Show:

(i) $\overline{A^c}=(A^\circ)^c$

(ii) $(\overline{A})^c=(A^c)^\circ$

Proof:

(i) We have previously established that $\overline{A^c} \subseteq (A^\circ)^c$. Let $x \in (A^\circ)^c$. Then, $x \notin A^\circ$, meaning $\forall \delta > 0$, $U(x,\delta) \cap A^c \neq \emptyset$. Thus, $x \in \overline{A^c}$.

(ii) Let $x \in \overline{A}^c$. Then, $x \notin \overline{A}$, meaning $\exists \delta > 0$ such that $U(x,\delta) \cap A = \emptyset$. Thus, $U(x,\delta) \subseteq A^c$, meaning $x \in (A^c)^\circ$.

Let $x \in (A^c)^\circ$. Then, $\exists \delta > 0$ such that $U(x,\delta) \subseteq A^c$. Therefore, $U(x,\delta) \cap A = \emptyset$, meaning $x \notin \overline{A}$, so $x \in \overline{A}^c$.

Problem 6

In any metric space, show that open balls are open, closed balls are closed, and spheres are closed. Moreover, in a normed space, show that $\partial U(v,r) = \partial B(v,r) = S(v,r)$.

Proof:

(i) Let $\delta > 0$, and $A = U(x,\delta)$ for some $x \in X$. Then, for any $y \in A$, set $\varepsilon = \min\{d(x,y), \delta - d(x,y)\}$. Then, $U(y,\varepsilon) \subseteq A$.

(ii) Let $M = B(x,\delta)$. Let $y \in M^c$. Set $\varepsilon = d(x,y) - \delta$. Then, $U(y,\varepsilon) \subseteq M^c$, meaning M^c is open, and M is thus closed.

(iii) Let $A = S(x,\delta)$ for some $\delta > 0$. Then, $A^c = U(x,\delta) \cup (B(x,\delta))^c$, meaning A^c is a union of open sets, which is open. Thus, A is closed.

(iv) We have previously established that, in a normed space, $\overline{U(v,r)} = B(v,r)$. Therefore,

$$\begin{aligned}
\partial U(v,r) &= \overline{U(v,r)} \setminus U(v,r)^\circ \\
&= \{x \mid d(x,v) \leq r\} \setminus \{x \mid d(x,v) < r\} \\
&= \{x \mid d(x,v) = r\} \\
&= S(v,r).
\end{aligned}$$

Similarly, in a normed vector space, $B(v,r)^\circ = U(v,r)$. Therefore,

$$\begin{aligned}
\partial B(v,r) &= \overline{B(v,r)} \setminus B(v,r)^\circ \\
&= \{x \mid d(x,v) \leq r\} \setminus \{x \mid d(x,v) < r\} \\
&= \{x \mid d(x,v) = r\} \\
&= S(v,r).
\end{aligned}$$

Problem 7

Let (X,d) be a metric space, and suppose $A \subseteq X$. Show that the following are equivalent:

(i) A is dense in X;

(ii) For all $U \in \tau_X$, $U \cap A \neq \emptyset$;

(iii) For all $x \in X$ and for all $\varepsilon > 0$, $U(x,\varepsilon) \cap A \neq \emptyset$;

(iv) For all $x \in X$ and for all $\varepsilon > 0$, there is an $a \in A$ with $d(x,a) < \varepsilon$.

Proof:

(i) $\Leftrightarrow$ (iii): $\overline{A} = X \Leftrightarrow \forall x \in \overline{A}, \forall \varepsilon > 0, U(x,\varepsilon) \cap A \neq \emptyset \Leftrightarrow \forall x \in X, \forall \varepsilon > 0, U(x,\varepsilon) \cap A \neq \emptyset$.

(iii) $\Leftrightarrow$ (iv): $\forall x \in X, \forall \varepsilon > 0, U(x,\varepsilon) \cap A \neq \emptyset \Leftrightarrow \forall x \in X, \exists a \in A \ni a \in U(x,\varepsilon) \Leftrightarrow \forall x \in x, \exists a \in a \ni d(x,a) < \varepsilon$.

(iii) $\Rightarrow$ (ii): Suppose $\forall x \in x, U(x,\varepsilon) \cap A \neq \emptyset$. Then, since $U(x,\varepsilon) \subseteq U$ for some $U \in \tau_X$, it is the case that $U \cap A \neq \emptyset$.

(ii) $\Rightarrow$ (iii): Since $U(x,\varepsilon) \in \tau_X$, it is the case that for any $x \in X$ and any $\varepsilon > 0$, $U(x,\varepsilon) \cap A \neq \emptyset$.

Problem 9

Show that c_0 with $\|\cdot\|_u$ is separable.

Proof: Let $z \in c_0$. Set $\varepsilon_1 > 0$, then finding N_1 large such that for all $n > N_1$, $z_n < \varepsilon_1$. Set $z' \in c_{00}$ to be equal to z on $1, \dots, N_1$ and equal to 0 for all $n > N_1$.

Recall that for

$$E_n = \left\{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\},$$
$$E = \bigcup E_n,$$

E is dense in c_{00}, meaning that there exists some $w \in c_{00}$ such that $\|z' - w\| < \varepsilon$ for any $\varepsilon > 0$. However, since $z' = z$ for all n from $1, \dots, N_1$, and the index of $\|z\|_u$ is contained in $1, \dots, N_1$, this means $\|z - w\| < \varepsilon$, meaning E is dense in c_0.

Since E is countable, this means c_0 is countable.

Problem 10

Let $\mathcal{C}$ denote the Cantor set. Show that $\mathcal{C}$ is nowhere dense.

Proof: We know that $\mathcal{C}$ is closed, meaning all we need show is that $\mathcal{C}^0 = \emptyset$.

Suppose toward contradiction that $\mathcal{C}^0$ is not empty. Then, $\exists x \in \mathcal{C}$ and $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}$.

Find m so large such that $3^{-m} < \varepsilon$. Then, $(x - \varepsilon, x + \varepsilon)$ must be contained in a subinterval with length $\frac{1}{3^m}$. However, $2\varepsilon > \frac{1}{3^m}$, and every subinterval in the element $\mathcal{C}_m$ has length $\frac{1}{3^m}$.