

Introduction

Consider the equations

$$\frac{d^2y}{dx^2} + y(x) = e^x \quad (1)$$

$$\frac{d^{17}y}{dx^{17}}(x) + \sin(y(x)) = (x^x)^x \quad (2)$$

Before we want to solve these equations, we need to understand what these equations *are*.

(1) This is a second order, inhomogeneous, linear ordinary differential equation.

(2) This is a 17th order, inhomogeneous, nonlinear ordinary differential equation.

Generally, when we have a nonlinear equation, we convert it (using the Jacobian) to the “nearest” corresponding linear equation using Taylor approximations. In this case, converting equation (2), we have

$$\frac{d^{17}y}{dx^{17}}(x) + y(x) = (x^x)^x. \quad (2')$$

Now, equation (2') is linear, so it is able to be solved. It may not be pretty,¹ but it can be solved, using Laplace Transforms or other methods.

Ordinary Differential Equations

Returning to our equation (1),

$$\frac{d^2y}{dx^2} + y(x) = e^x, \quad (1)$$

there is one more fact that we can see — this is an equation with constant coefficients. The most general form of a n th order linear ordinary differential equation is of the form

$$a_n(x)\frac{d^ny}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}(x) + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = g(x). \quad (\dagger)$$

Specifically, we also require $a_k(x) \in C(I)$, where I is some interval (specifics will be detailed later).

Theorem (Existence and Uniqueness Theorem): Any ordinary differential equation of the form (\dagger) has unique solutions in the interval I .

There are n linearly independent solutions for $g(x) = 0$.

The corresponding homogeneous equation for (1) is

$$\frac{d^2y}{dx^2} + y(x) = 0. \quad (1')$$

The equations (1) and (1') are related by the linearity principle. In particular, if $y_0(x)$ is a solution to (1'), then we can add $\alpha y_0(x)$ to any solution $y_p(x)$ of (1), then we have all the solutions for (1). In particular, the solutions to (1') are

$$\begin{aligned} y_1(x) &= \sin(x) \\ y_2(x) &= \cos(x). \end{aligned}$$

¹Citation needed.

To evaluate that these solutions are linearly independent, we consider the differential operator L from (†) defined by

$$L[y] = \sum_{k=0}^n a_k(x) \frac{d^k y}{dx^k}.$$

We rewrite (†) as

$$L[y] = g(x).$$

The operator L is linear, so L has the following properties:

- $L[y_1 + y_2]$;
- $L[cy] = cL[y]$.

Now, in (1) and (1'), if we set $L[y] = \frac{d^2 y}{dx^2} + y(x)$, then evaluating our solutions y_1 and y_2 to (1'), we get

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= c_1 L[y_1] + c_2 L[y_2] \\ &= 0. \end{aligned}$$

Now, we get

$$y_0(x) = c_1 \sin(x) + c_2 \sin(x)$$

as our general solution to (1'). By the linearity principle, all we need is one solution to $L[y] = e^x$ to find all solutions to (1).

Evaluating (†) in the most general form, we have the general solution

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{\text{homogeneous solution}} + y_p(x),$$

where $y_p(x)$ is the particular solution. In other words, our general solution is

$$y(x) = \text{span}(y_1(x), y_2(x), \dots, y_n(x)) + y_p(x).$$

For this to work, we need the set $\{y_1, \dots, y_n\}$ to be linearly independent. To do this, we evaluate the Wronskian:

$$W(x) = \det \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{pmatrix}.$$

Specifically, the set $\{y_1, \dots, y_n\}$ is linearly independent if $W(x) \neq 0$ for all $x \in I$.

Example. Consider the equation

$$\frac{d^2 y}{dx^2} - y(x) = e^x \tag{1}$$

We want to find the general solution to this constant coefficient equation.

We start by finding two linearly independent homogeneous solutions to the equation, take their span, then add a particular solution.

The characteristic equation of the homogeneous equation for (1) is

$$r^2 - 1 = 0$$

We get $r = \pm 1$, which by the definition of the characteristic equation yields $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. To verify that this solution set is linearly independent

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2 \\ &\neq 0. \end{aligned}$$

Thus, our solutions are linearly independent. We get the general form of

$$y(x) = c_1 e^x + c_2 e^{-x} + y_p(x).$$

Now, we only have to find a particular solution. This is, unfortunately, the hard part.

We begin by guessing. But, in a way that doesn't suck. Specifically, we let $y_p(x) = A x e^x$. Evaluating, we get

$$\begin{aligned} \frac{dy_p}{dx} &= A(x+1)e^x \\ \frac{d^2 y_p}{dx^2} &= A(x+2)e^x \\ \frac{d^2 y_p}{dx^2} - y_p(x) &= A(x+2)e^x - A x e^x \\ &= 2A e^x, \end{aligned}$$

so $2A = 1$, and $A = \frac{1}{2}$. Thus, we have the end result of

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x.$$

Evaluating in Mathematica, we take

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DSolve[y''[x] - y[x] == Exp[x], y[x], x]
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and we get

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{4}(2x - 1)e^x,$$

corroborating our solution.^{II}

Example. Consider the equation

$$\frac{d^3 y}{dx^3} - y(x) = 0.$$

The particular solution to this equation is $y(x) = 0$. The characteristic equation for this equation is

$$r^3 - 1 = 0.$$

^{II}Only slightly different, but they're the same solution.

Factoring, we get

$$(r-1)(r^2+r+1)=0$$

$$(r-1)(r-\zeta_3)(r-\zeta_3^2)=0.$$

Thus, we get

$$r = \left\{ 1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} \right\}.$$

Thus, our solutions are of the form

$$y(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

Recall that the most general second order constant-coefficient linear differential equation is

$$y'' + ay' + by = 0,$$

with characteristic equation

$$r^2 + ar + b = 0.$$

The solutions to the characteristic equation are

$$r = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}.$$

There are a few cases:

- (1) $r_1 \neq r_2$ with $r_1, r_2 \in \mathbb{R}$;
- (2) $r_1 = r_2$ with $r_1, r_2 \in \mathbb{R}$;
- (3) $r_1 = c + id, r_2 = c - id$, where $c, d \in \mathbb{R}$.

The solutions are $y_1 = c_1 e^{r_1 x}$ and $y_2 = c_2 e^{r_2 x}$.

Example (Solving Second-Order Equations).

- (1) Let

$$y'' - 3y' + 2y = 0.$$

The characteristic equation is $r^2 - 3r + 2 = 0$, whose solutions are $r = 1, r = 2$. The general solution is, thus,

$$y(x) = c_1 e^x + c_2 e^{2x} \tag{†}$$

The Wronskian is

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \\ &= 2e^{3x} - e^{3x} \\ &= e^{3x} \\ &\neq 0. \end{aligned}$$

Thus, the solution is indeed (†).

(2) Let

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with solution $r = -3, -3$. Currently, we only have the solution $y_1(x) = c_1 e^{-3x}$.

Note that in an n th order linear ordinary differential equation, we always have n linearly independent solutions. Let's guess. Consider the equation $y_2(x) = c_2 x e^{-3x}$.

We can see that $y_2(x)$ is also a solution to this equation,^{III} but we need to verify linear independence. Taking the Wronskian, we get

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & -3x e^{-3x} + e^{-3x} \end{pmatrix} \\ &= e^{-6x} \begin{pmatrix} 1 & x \\ -3 & -3x + 1 \end{pmatrix} \\ &= e^{-6x} (-3x + 1 + 3x) \\ &= e^{-6x} \\ &\neq 0. \end{aligned}$$

Thus, we have two linearly independent solutions, with the general solution of

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

(3) Let

$$y'' + 4y' + 5 = 0.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with solutions of $r = -2 \pm i$. We then have the solutions

$$\begin{aligned} y_1(x) &= e^{(-2+i)x} \\ y_2(x) &= e^{(-2-i)x}. \end{aligned}$$

Unfortunately, we cannot just let these equations stand on their own, because we want *real* solutions. Let's use Euler's theorem, $e^{ix} = \cos x + i \sin x$. Then, we get

$$\begin{aligned} y(x) &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\ &= e^{-2x} (c_1 e^{ix} + c_2 e^{-ix}). \end{aligned}$$

Let $f(x) = c_1 e^{ix} + c_2 e^{-ix}$. Using the even/odd decomposition, we get

$$\begin{aligned} f(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\ &= (c_1 + c_2) \cos(x) + i(c_1 - c_2) \sin(x). \end{aligned}$$

We "real"-ize our solution by just dropping the value of i in $f(x)$. Thus, we get the full general solution

$$y(x) = e^{-2x} (d_1 \cos(x) + d_2 \sin(x)).$$

^{III}Exercise left for the reader.

(4) If we have the equation

$$y^{(4)} - 25y'' = 0,$$

then using a similar process, we get the solution

$$y(x) = c_1 + c_2x + c_3e^{5x} + c_4e^{-5x}.$$

(5) Considering the equation

$$y^{(5)} + 4y''' + 4y' = 0,$$

we take the characteristic equation $r^5 + 4r^3 + 4r = 0$. Factoring, we get solutions of $r = 0, r = \pm i\sqrt{2}$. Thus, we get the solution of

$$y(x) = c_1 + c_2 \cos(\sqrt{2}x) + c_3 \sin(\sqrt{2}x) + c_4x \cos(\sqrt{2}x) + c_5x \sin(\sqrt{2}x).$$

Reducing our Orders

Let

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0.$$

Suppose we know $y_1(x)$. Can we find $y_2(x)$? The answer is yes. We presume

$$y_2(x) = v(x)y_1(x).$$

Now, we have

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'', \end{aligned}$$

and inserting into the equation, we get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v \underbrace{(y_1'' + py_1' + qy_1)}_{=0} \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Now, we have

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p. \quad (*)$$

Integrating, we get

$$\ln(v') = -2\ln(y_1) - \int p(x) dx.$$

Taking powers, we get

$$\begin{aligned} v' &= e^{-2\ln(y_1) - \int p(x) dx} \\ &= y_1^{-2} e^{-\int p(x) dx} \\ &= \frac{e^{-\int p(x) dx}}{y_1(x)^2} \\ v &= \int \frac{e^{-\int p(x) dx}}{y_1(x)^2} dx \end{aligned}$$

Example. Consider the equation

$$\cos^2(x) \frac{d^2 y}{dx^2} - \sin(x) \cos(x) y' - y(x) = 0.$$

Putting our equation into standard form, we may be able to find another solution.

$$y'' - \tan(x)y' - \sec^2(x)y = 0.$$

Guessing $y(x) = \tan(x)$, we get $y' = \sec^2(x)$ and $y'' = 2\sec^2(x)\tan(x)$. This is also another solution, $y_2(x) = \tan(x)$.

We don't want to guess anymore. Let $y_2(x) = v(x)y_1(x)$. We get

$$v(x) = \int \frac{e^{-\int p(x) dx}}{y_1^2(x)} dx.$$

We have $-p(x) = \tan(x)$, so $-\int p(x) dx = \ln(\sec(x))$. Thus, $e^{-\int p(x) dx} = \sec(x)$. Thus, we get

$$\begin{aligned} v(x) &= \int \frac{\sec(x)}{\tan^2(x)} dx \\ &= \int \frac{\cos(x)}{\sin^2(x)} dx \\ &= \int \frac{1}{u^2} du \quad u = \sin(x) \\ &= -\frac{1}{u} \\ &= -\csc(x). \end{aligned}$$

Thus, we have $y_2(x) = -\csc(x) \tan(x) = -\sec(x)$.

Example. Consider the equation

$$x^2(\ln(x) - 1) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + \frac{dy}{dx} = 0.$$

We can use the power of inspection to find one solution, $y_1(x) = x$. Dividing out, we have

$$y'' - \frac{1}{x(\ln(x) - 1)} y' + \frac{1}{x^2(\ln(x) - 1)} y = 0.$$

Using the reduction of order, we guess $y_2(x) = v(x)y_1(x)$, and have

$$v(x) = \int \frac{e^{-\int p(x) dx}}{y_1^2} dx.$$

Noting that $-p(x) = \frac{1}{x(\ln(x)-1)}$, we have $\int \frac{1}{x(\ln(x)-1)} dx = \ln(\ln(x) - 1)$.

Now, we have

$$\begin{aligned} v(x) &= \int \frac{\ln(x) - 1}{x^2} dx \\ &= \frac{1 - \ln(x)}{x} - \int -\frac{1}{x^2} dx \quad u = \ln(x) - 1, dv = x^{-2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{-\ln(x)}{x} - \frac{1}{x} \\
 &= -\frac{\ln(x)}{x}.
 \end{aligned}$$

Thus, we get $y_2(x) = -\ln(x)$, and the general solution of $y(x) = c_1x + c_2 \ln(x)$.

Example (Cauchy–Euler Equation). A second-order Cauchy–Euler equation is of the form

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy(x) = 0.$$

More generally,

$$\sum_{k=0}^n c_k x^k y^{(k)}(x) = 0.$$

We guess $y(x) = x^r$. Then, $\frac{dy}{dx} = rx^{r-1}$ and $\frac{d^2y}{dx^2} = r(r-1)x^{r-2}$. This yields

$$\begin{aligned}
 a(r)(r-1)x^r + brx^r + cx^r &= x^r (a(r^2 - r) + br + c) \\
 &= 0.
 \end{aligned}$$

Example (Solving a Cauchy–Euler Equation). Consider the equation

$$x^2 y'' + xy' - y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 1 = 0,$$

so our general solution is $y(x) = c_1x + c_2/x$.

Example (Solving another Cauchy–Euler Equation). Consider the equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 4r + 4 = 0,$$

so our solutions are x^2 and x^2 . This is not good enough, we need another solution.

Now, we place our equation into standard form.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0.$$

Thus, we get $p(x) = -\frac{3}{x}$. Using reduction of order, we get $y_2(x) = v(x)y_1(x)$,

$$\begin{aligned}
 v(x) &= \int \frac{e^{-\int -3/x \, dx}}{x^4} \, dx \\
 &= \int \frac{e^{3\ln(x)}}{x^4} \, dx \\
 &= \int \frac{x^3}{x^4} \, dx \\
 &= \ln(x).
 \end{aligned}$$

Thus, we have the solution $y_2(x) = \ln(x)x^2$, and the general solution of $y(x) = c_1x^2 + c_2 \ln(x)x^2$.

Example. Consider the equation

$$x^2 y'' + 3xy' + 5y = 0.$$

We get the characteristic equation of

$$0 = r^2 - 4r + 5$$

$$r = 2 \pm i.$$

Now, we need to figure out what $x^{2 \pm i}$ means.

To solve this part, we keep the positive exponent, so we only need to try to understand $y = x^{2+i}$. Now, we get $y = x^2 x^i$. To evaluate x^i , we take $x = (e^{\ln x})^i = e^{i \ln x}$. Using Euler's identity, we get

$$y = x^2 (\cos(\ln x) + i \sin(\ln x)).$$

Since our solutions are real, get

$$y = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x).$$

Example. Consider the equation

$$x^4 y^{(4)} - 2x^2 y'' + y = 2.$$

We have the particular solution $y_p(x) = 2$. Substituting into our method for the Cauchy–Euler equation, we have

$$r(r-1)(r-2)(r-3) - 2r(r-1) + 1 = 0.$$

Factoring, we have

$$r(r-1)^2(r-4) + 1 = 0.$$

Unfortunately, to go forward from here we need Mathematica.

This has the solution set of of

$$\begin{aligned} r_1 &= \frac{3}{2} - \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \\ r_2 &= \frac{3}{2} - \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad + \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \\ r_3 &= \frac{3}{2} + \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \end{aligned}$$

$$r_4 = \frac{3}{2} + \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ + \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} + \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}$$

Varying our Parameters

Given a set of n linearly independent homogeneous solutions, we want to find a particular solution.

To find this, we start with the general second-order inhomogeneous equation in standard form:

$$\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y(x) = g(x).$$

Given y_1, y_2 , we find $y_p(x)$ by taking

$$y_p = v_1 y_1 + v_2 y_2.$$

Finding the derivatives, we have

$$y'_p = v_1 y'_1 + v'_1 y_1 + v_2 y'_2 + v'_2 y_2 \\ y''_p = v_1 y''_1 + 2v'_1 y'_1 + v''_1 y_1 + v_2 y''_2 + 2v'_2 y'_2 + v''_2 y_2.$$

Substituting, we have

$$y''_p = v_1 y''_1 + 2v'_1 y'_1 + v''_1 y_1 + v_2 y''_2 + 2v'_2 y'_2 + v''_2 y_2 \\ p y'_p = p v_1 y'_1 + p v'_1 y_1 + p v_2 y'_2 + p v'_2 y_2 \\ q y_p = q v_1 y_1 + q v_2 y_2 \\ g(x) = \underbrace{v_1 (y''_1 + p y'_1 + q y_1)}_{=0} + \underbrace{v_2 (y''_2 + p y'_2 + q y_2)}_{=0} + v'_1 (2y'_1 + p y_1) + v''_1 y_1 + v_2 (2y'_2 + p y_2) + v''_2 y_2 \\ g(x) = v'_1 (2y'_1 + p y_1) + v''_1 y_1 + v_2 (2y'_2 + p y_2) + v''_2 y_2.$$

We suppose that $v'_1 y_1 + v'_2 y_2 = 0$. Then,

$$\frac{d}{dx} (v'_1 y_1 + v'_2 y_2) = 0 \\ v''_1 y_1 + v'_1 y'_1 + v''_2 y_2 + v'_2 y'_2 = 0.$$

Plugging into our earlier expression, we get the expression of

$$v'_1 y_1 + v'_2 y_2 = 0 \\ v'_2 y'_2 + v'_1 y'_1 = g(x).$$

Plugging into matrix form, we have

$$\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

Since the Wronskian is nonzero, we have

$$\begin{aligned} \begin{pmatrix} \frac{dv_1}{dx} \\ \frac{dv_2}{dx} \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g(x) \end{pmatrix} \\ &= \frac{1}{y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx}} \begin{pmatrix} -y_2(x)g(x) \\ y_1(x)g(x) \end{pmatrix} \end{aligned} \quad (\dagger)$$

Example. Let

$$y'' - 2y' + y = e^x.$$

Solving the homogeneous solution, we have the characteristic equation of $r^2 - 2r + 1 = 0$. Thus, $y_1(x) = e^x$ and $y_2(x) = xe^x$.

To find $y_p(x)$, we guess $y_p(x) = x^2e^x$. Using the power of computation in Sage, we get the answer of

Avoiding Variation of Parameters

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1 de = diff(y,x,2) - 2*diff(y,x) + y - e^x
2 g = desolve(de,y)
3 latex(expand(g))

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$$y_p(x) = K_2xe^x + K_1e^x + \frac{1}{2}x^2e^x.$$

However, this is a very unsatisfying method.

Using (\dagger) , we can find a different solution. We find

$$\begin{aligned} \frac{dv_1}{dx} &= \frac{1}{e^{2x}}((-1)(xe^x)(e^x)) \\ &= -x, \end{aligned}$$

yielding

$$v_1(x) = -\frac{x^2}{2} + c_2.$$

Similarly, we get

$$\begin{aligned} \frac{dv_2}{dx} &= \frac{1}{e^{2x}}(e^x)(e^x) \\ v_2(x) &= x + c_2. \end{aligned}$$

This gives

$$y_p(x) = \frac{1}{2}x^2e^x.$$

Example. Let

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = x + e^x.$$

Using the characteristic equation, we have $y_1(x) = 1$, $y_2(x) = e^x$, and $y_3(x) = e^{-x}$.

Now, using the Wronskian, we get

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ x + e^x \end{pmatrix}.$$

This would suck, but we would be able to find a solution nonetheless.

In the general form, with linearly independent homogeneous solutions y_1, \dots, y_n , we have the solution of

$$\begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ g(x) \end{pmatrix}$$

$$y(x) = \sum_{i=1}^n c_i y_i(x) + \sum_{i=1}^n v_i(x) y_i(x).$$

Example (Solving a Coupled System). Before we can start using variation of parameters for systems, we need to recall how to solve constant-coefficient systems.

$$\begin{aligned} x'(t) &= 3x(t) + y(t) \\ y'(t) &= x(t) + 3y(t). \end{aligned}$$

Here, setting

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

we get system of linear equations

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Remark: In the matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

the eigenvalues are

$$\begin{aligned} \lambda_1 &= a + b \\ \lambda_2 &= a - b \end{aligned}$$

with eigenvectors of

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Example (General n -dimensional System of Differential Equations). Consider the system of equations defined by

$$\begin{aligned}x'_1(t) &= g_1(t, x_1(t), \dots, x_n(t)) \\&\vdots \\x'_n(t) &= g_n(t, x_1(t), \dots, x_n(t)).\end{aligned}$$

We will refine this slightly so as to be a system of *linear* equations. Let

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \\ \frac{d\mathbf{x}}{dt} &= \begin{pmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{pmatrix} \\ \mathbf{F} &= \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \\ \mathbf{x}_{t_0} &= \begin{pmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}.\end{aligned}$$

Now, we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where $\mathbf{x}(t_0) = \mathbf{x}_{t_0}$ and A is some matrix that represents some linear transformation.

Furthermore, we may make an inhomogeneous equation by

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{F}.$$

Example. Going back to our example of

$$\frac{d\mathbf{x}}{dt} = \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}_A \mathbf{x}.$$

We find eigenvalues of $\lambda_1 = 4, \lambda_2 = 2$ and eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This gives

$$\mathbf{x}_1 = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_2 = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In general, if we have two distinct eigenvalues, then our solutions are

$$\mathbf{x} = e^{\lambda t} \mathbf{v}$$

Define

$$\begin{aligned} \Phi_A(t) &= \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix}. \end{aligned}$$

We call Φ_A a fundamental matrix for A .

The general solution to the system is given by

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \\ &= c_1 \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

Example. Consider the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \tag{A}$$

Notice that we have a triple-repeated eigenvalue,

$$\begin{aligned} \lambda_1 &= 4 \\ \lambda_2 &= 4 \\ \lambda_3 &= 4. \end{aligned}$$

Unfortunately, to find the eigenvectors, this will be a bit harder.

$$\begin{aligned} (A - 4I)\mathbf{v} &= 0 \\ \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This gives

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so $b = c = 0$, and our eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We may need some more eigenvectors. Currently, our solution is

$$\mathbf{x}_1(t) = e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We need to go into the realm of generalized eigenvectors. If λ is repeated, we need to do the following.

- (1) Find all the eigenvectors for which $(A - \lambda I)\mathbf{v} = 0$. If we come up short, then we have a defective system.
- (2) For the remaining eigenvectors, we solve the system

$$(A - \lambda I)\mathbf{v}_j = \mathbf{v}_k,$$

where \mathbf{v}_k is known, and we desire \mathbf{v}_j . The \mathbf{v}_j are known as generalized eigenvectors.

- (3) Continue this process until we are done.

Now, in this case, we get

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and a generalized eigenvector of

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}.$$

Going at it again, we have

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix},$$

giving the equation

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix},$$

giving

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1/8 \\ 1/4 \end{pmatrix}.$$

Note that when we take generalized eigenvectors, we “integrate” with respect to t before adding. For instance

$$\begin{aligned} \mathbf{x}_1 &= e^{\lambda t} \mathbf{v}_1 \\ \mathbf{x}_2 &= e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) \\ \mathbf{x}_3 &= e^{\lambda t} \left(\frac{t^2}{2} \mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3 \right). \end{aligned}$$

Now, our linearly independent solutions to the system in (A) is of the form

$$\begin{aligned} \mathbf{x}_1(t) &= e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \mathbf{x}_2(t) &= e^{4t} \left(t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} \right) \\ \mathbf{x}_3(t) &= e^{4t} \left(\frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/8 \\ 1/4 \end{pmatrix} \right). \end{aligned}$$

This gives the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{4t} & te^{4t} & \frac{t^2}{2}e^{4t} \\ 0 & \frac{1}{2}e^{4t} & e^{4t} \left(\frac{t}{2} - \frac{1}{8} \right) \\ 0 & 0 & \frac{1}{4}e^{4t} \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}.$$

The general solution is, then,

$$\mathbf{x}(t) = e^{At} \mathbf{c},$$

where \mathbf{c} is a constant vector, and e^{At} is the matrix exponential of A .

Example. Consider A as the matrix with eigenvalue λ and eigenvector \mathbf{v}_1 and generalized eigenvectors \mathbf{v}_2 and \mathbf{v}_3 . Then, the solution set

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\lambda t} \mathbf{v}_1 \\ \mathbf{x}_2(t) &= e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) \\ \mathbf{x}_3(t) &= e^{\lambda t} \left(\frac{t^2}{2} \mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3 \right). \end{aligned}$$

Thus, we have

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \lambda e^{\lambda t} \mathbf{v}_1 \\ A\mathbf{x}_1(t) &= A e^{\lambda t} \mathbf{v}_1 \\ &= e^{\lambda t} A \mathbf{v}_1 \\ &= \lambda e^{\lambda t} \mathbf{v}_1. \end{aligned}$$

Now, recalling that $A\mathbf{v}_1 = \lambda\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$, we have

$$\begin{aligned} \frac{d\mathbf{x}_2}{dt} &= \lambda e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) + e^{\lambda t} \mathbf{v}_1 \\ A\mathbf{x}_2(t) &= A e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) \\ &= e^{\lambda t} (tA\mathbf{v}_1 + A\mathbf{v}_2) \\ &= e^{\lambda t} (t\lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_1) \\ &= \lambda e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) + e^{\lambda t} \mathbf{v}_1. \end{aligned}$$

Finally, we have $A\mathbf{v}_3 = \lambda\mathbf{v}_3 + \mathbf{v}_2$.

Example. We assume A is a $n \times n$ real matrix. Then, all complex eigenvalues of A come in conjugate pairs, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$.

Then, our eigenvectors are of the form $\mathbf{v}_1 = \mathbf{u} + i\mathbf{w}$ and $\mathbf{v}_2 = \mathbf{u} - i\mathbf{w}$.

Note that if we find the solution for λ_1 and \mathbf{v}_1 . This gives

$$\begin{aligned} e^{\lambda_1 t} \mathbf{v}_1 &= e^{(a+ib)t} (\mathbf{u} + i\mathbf{w}) \\ &= e^{at} (\cos(bt) + i \sin(bt)) (\mathbf{u} + i\mathbf{w}) \\ &= e^{at} ((\cos(bt)\mathbf{u} - \sin(bt)\mathbf{w}) + i(\cos(bt)\mathbf{w} + \sin(bt)\mathbf{u})). \end{aligned}$$

Example. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{pmatrix}$$

for the system of equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Using the power of computation, we have

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= 3 + 2i \\ \lambda_3 &= 3 - 2i,\end{aligned}$$

and eigenvectors of

$$\begin{aligned}\mathbf{v}_1 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} -4 \\ 0 \\ 2 + 2i \end{pmatrix} \\ \mathbf{v}_3 &= \begin{pmatrix} -4 \\ 0 \\ 2 - 2i \end{pmatrix}.\end{aligned}$$

Now, we see that

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\lambda_1 t} \mathbf{v}_1 \\ &= \begin{pmatrix} 0 \\ e^{3t} \\ 0 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_2(t) &= e^{3t} \left(\cos(2t) \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right) \\ \mathbf{x}_3(t) &= e^{3t} \left(\cos(2t) \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \sin(2t) \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \right).\end{aligned}$$

This gives the matrix

$$\Phi(t) = \begin{pmatrix} 0 & -4e^{3t} \cos(2t) & -4e^{3t} \sin(2t) \\ e^{3t} & 0 & 0 \\ 0 & 2e^{3t}(\cos(2t) - \sin(2t)) & 2e^{3t}(\sin(2t) + \cos(2t)) \end{pmatrix}$$

$$W(t) = \det(\Phi(t))$$

$$\begin{aligned}&= -e^{3t} \left(-8e^{6t} (\cos(2t) \sin(2t) + \cos^2(2t)) + 8e^{6t} (\sin(2t) \cos(2t) - \sin^2(2t)) \right) \\ &= 8e^{9t} \\ &\neq 0.\end{aligned}$$

Example. We wish to solve $\frac{dx}{dt} = Ax$, where

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

To find our eigenvalues and eigenvectors, we begin by finding

$$\begin{aligned} \det(A - \lambda I) &= (2 - \lambda)^2 \det \begin{pmatrix} 2 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 5 \\ 0 & -2 & -1 - \lambda \end{pmatrix} \\ &= (2 - \lambda)^3 \det \begin{pmatrix} 1 - \lambda & 5 \\ -2 & -1 - \lambda \end{pmatrix} \\ &= (2 - \lambda)^3 ((1 - \lambda)(-1 - \lambda) + 10). \end{aligned}$$

We have five eigenvalues,

$$\lambda = \pm 3i, 2, 2, 2.$$

For $\lambda_{1,2} = \pm 3i$, then

$$\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now for $\lambda_3 = 2$, we have

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now, we have

$$\begin{aligned} (A - 2I)\mathbf{v}_3 &= 0 \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ 3 \end{pmatrix} &= 0. \end{aligned}$$

From this equation, we have

$$\begin{aligned}c &= 0 \\ -b + 5d &= 0 \\ -2b - 3d &= 0.\end{aligned}$$

Now, we have independent a and e . This gives

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that both \mathbf{v}_3 and \mathbf{v}_4 are regular eigenvectors. Now, we wish to find one generalized eigenvector. We find this generalized eigenvector, \mathbf{w} , by observing that the 1 in entry $A_{1,3}$ effectively ties our vector \mathbf{v}_4 to vector $\mathbf{v}_{1,2}$. Thus, we get

$$(A - 2I)\mathbf{w} = \mathbf{v}_4$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, solving this, we get $c = 1$, giving the generalized eigenvector of

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we have

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{v}_4 \rightarrow \mathbf{w}$ is a chain of length 2. This gives the JCF of

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1+3i & 1-3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3i & 0 & 0 & 0 & 0 \\ 0 & -3i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1+3i & 1-3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^{-1}.$$

We get the solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ -\cos(3t) - 3\sin(3t) \\ 0 \\ 2\cos(3t) \\ 0 \end{pmatrix}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} 0 \\ 3\cos(3t) - \sin(3t) \\ 0 \\ 2\sin(3t) \\ 0 \end{pmatrix}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{2t} \end{pmatrix}$$

$$\mathbf{x}_4(t) = \begin{pmatrix} e^{2t} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_5(t) = \begin{pmatrix} te^{2t} \\ 0 \\ e^{2t} \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{x}_5(t) = e^{2t}(t\mathbf{v}_4 + \mathbf{v}_5)$.

The fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} 0 & 0 & 0 & e^{2t} & te^{2t} \\ -\cos(3t) + 3\sin(3t) & 3\cos(3t) - \sin(3t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2t} \\ 2\cos(3t) & 2\sin(3t) & 0 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \end{pmatrix}.$$

Now, we want to find $\Phi(0)$, or $\Phi(t_0)$. Furthermore, we need to find $\Phi^{-1}(0)$, or $\Phi^{-1}(t_0)$.

Example (Implementing Initial Conditions). Looking back at our equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

we may apply the initial condition of

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

We use the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

with the initial condition

$$\mathbf{x}_0 = \begin{pmatrix} 4 \\ 15 \end{pmatrix}.$$

Generally our approach to solving this kind of problem, we take the eigenvectors and eigenvalues, giving

$$\lambda_1 = 4$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and associated solutions of

$$\mathbf{x}_1 = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$$

$$\mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Then, we form a fundamental matrix of solutions:

$$\Phi(t) = \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix}.$$

Note that, for any vector of constants \mathbf{c} , we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}$$

is a solution of $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

To find \mathbf{c} , we see that

$$\mathbf{x}_0 = \Phi(0)\mathbf{c},$$

so that

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0$$

is the solution to our initial value problem.

Calculating

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we find

$$\Phi^{-1}(0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus, we get the solutions of

$$\mathbf{x}(t) = \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{19}{2}e^{4t} - \frac{11}{2}e^{2t} \\ \frac{19}{2}e^{4t} + \frac{11}{2}e^{2t} \end{pmatrix}.$$

Note that we may define

$$\Psi(t) = \Phi(t)\Phi^{-1}(0),$$

giving

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0.$$

We may calculate

$$\begin{aligned} \Psi(t) &= \Phi(t)\Phi^{-1}(0) \\ &= \frac{1}{2} \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{4t} + e^{2t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{4t} + e^{2t} \end{pmatrix}. \end{aligned}$$

Example (The Matrix Exponential). When we have a single first-order equation, such as

$$\frac{dy}{dt} = 3y,$$

with initial condition $y(0)$, we solve it by taking $y(t) = \pi e^{3t}$.

Similarly, if we're given

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

we may want to know if there is an analogous $e^{A t}$.

In fact, there is. Using the Taylor expansion, we have

$$\begin{aligned} e^{A t} &= I + A t + A^2 \frac{t^2}{2} + A^3 \frac{t^3}{6} + \dots \\ &= \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}. \end{aligned}$$

Note that we may take P to be the matrix of unit eigenvectors of A , and D to be the matrix of eigenvalues corresponding to column eigenvectors

$$A = P D P^{-1}.$$

This is assuming A can be diagonalized. This gives $D = P^{-1} A P$.

Now, if A can be diagonalized, we can take

$$\begin{aligned} e^{A t} &= I + (P D P^{-1}) t + (P D P^{-1})^2 \frac{t^2}{2} + \dots \\ &= \sum_{k=0}^{\infty} (P D P^{-1})^k \frac{t^k}{k!} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} P D^k P^{-1} \frac{t^k}{k!} \\
&= P \left(\sum_{k=0}^{\infty} D^k \frac{t^k}{k!} \right) P^{-1}.
\end{aligned}$$

We can find the power on any diagonal matrix much more easily than we can on a general matrix. In particular, this gives

$$e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{pmatrix} P^{-1}$$

Given A in the system $\frac{dx}{dt} = Ax$, we wish to find e^{At} , and show that $e^{At} = \Psi(t)$, where $\Psi(t) = \Phi(t)\Phi^{-1}(0)$.

Example. Let

$$A = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}.$$

We see that A has repeated eigenvalues of 4 and 4.

Our first eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now, evaluating

$$\begin{aligned}
(A - 4I)\mathbf{v} &= 0 \\
\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

giving $2b = 0$.

Thus, we're going to need a generalized eigenvector. We have

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

giving $(A - \lambda I)\mathbf{w} = \mathbf{v}_1$. Thus, we have $2b = 1$. Thus, we have

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

Now, we have

$$\mathbf{x}_1(t) = e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} e^{4t} \\ 0 \end{pmatrix} \\
\mathbf{x}_2(t) &= te^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{4t} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \\
&= \begin{pmatrix} te^{4t} \\ \frac{1}{2}e^{4t} \end{pmatrix}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \\
&= \begin{pmatrix} e^{4t} & te^{4t} \\ 0 & \frac{1}{2}e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.
\end{aligned}$$

Now, we see that

$$\begin{aligned}
\Phi(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\
\Phi^{-1}(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.
\end{aligned}$$

Therefore, our matrix exponential is

$$\begin{aligned}
\Psi(t) &= \begin{pmatrix} e^{4t} & te^{4t} \\ 0 & \frac{1}{2}e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} e^{4t} & 2te^{4t} \\ 0 & e^{4t} \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{x}(t) &= \Psi(t) \mathbf{x}_0 \\
&= \begin{pmatrix} e^{4t} & 2te^{4t} \\ 0 & e^{4t} \end{pmatrix} \mathbf{x}_0.
\end{aligned}$$

We often refer to $\Psi(t)$ as the flow matrix.

Now, because $\Psi(0) = I$, we have

$$\Psi(t)\Psi(-t) = I,$$

meaning that $\Psi(-t) = \Psi(t)^{-1}$.

Example. Consider

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

as the set of eigenvector equations, giving

$$\begin{aligned} \begin{pmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_1\lambda_1 & \mathbf{v}_2\lambda_2 & \cdots & \mathbf{v}_n\lambda_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \end{aligned}$$

giving the expression $AP = PD$, where P is the set of eigenvectors.

Now, if we have generalized eigenvectors, we have a different case.

Consider the case of a chain. We know that

$$(A - \lambda I)\mathbf{v}_1 = 0$$

is the expression of an eigenvector. Now, if we have repeated eigenvalues, we get the second equation of

$$\begin{aligned} (A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1 \\ (A - \lambda I)\mathbf{v}_3 &= \mathbf{v}_2 \\ &\vdots \\ (A - \lambda I)\mathbf{v}_n &= \mathbf{v}_{n-1}. \end{aligned}$$

We start by changing these equations to give

$$\begin{aligned} A\mathbf{v}_1 &= \lambda\mathbf{v}_1 \\ A\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1 \\ A\mathbf{v}_3 &= \lambda\mathbf{v}_3 + \mathbf{v}_2 \\ &\vdots \\ A\mathbf{v}_n &= \lambda\mathbf{v}_n + \mathbf{v}_{n-1}. \end{aligned}$$

We see that these are effectively the eigenvalue equations with a small perturbation. Constructing the matrix, we have

$$\begin{aligned} \begin{pmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 & \cdots & A\mathbf{v}_n \end{pmatrix} &= \begin{pmatrix} \lambda\mathbf{v}_1 & \lambda\mathbf{v}_2 + \mathbf{v}_1 & \lambda\mathbf{v}_3 + \mathbf{v}_2 & \cdots & \lambda\mathbf{v}_n + \mathbf{v}_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n \end{pmatrix} \underbrace{\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}}_J. \end{aligned}$$

We call the matrix J the Jordan canonical form of A , and we get the expression $AP = PJ$, where P is the matrix of generalized eigenvectors as columns.

Now, if we have multiple chains, we get multiple blocks. For instance, if we have the chains $\mathbf{v}_1 \rightarrow \mathbf{v}_2 \rightarrow \mathbf{v}_3$, $\mathbf{v}_4 \rightarrow \mathbf{v}_5 \rightarrow \mathbf{v}_6 \rightarrow \mathbf{v}_7$, and \mathbf{v}_8 being standalone, all for the same eigenvalue λ . This gives the Jordan canonical form of

$$J = \begin{pmatrix} \lambda & 1 & 0 & & & & & \\ 0 & \lambda & 1 & 0 & & & & \\ 0 & 0 & \lambda & 0 & & & & \\ & & & \lambda & 1 & 0 & 0 & \\ & & & 0 & \lambda & 1 & 0 & \\ & & & 0 & 0 & \lambda & 1 & \\ & & & 0 & 0 & 0 & \lambda & \\ & & & & & & & \lambda \end{pmatrix}$$

The reason block matrices are useful is that they simplify calculations massively. We may consider the block matrices as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

For instance, if we have 4×4 matrices, we convert this multiplication into 8 2×2 matrix multiplications. On first glance, this doesn't seem more efficient, but if there are a lot of zeros, it does actually become more efficient.

Example. In the general case, our flow matrix is of the form

$$\begin{aligned} \Psi(t) &= e^{At} \\ &= P e^{Jt} P^{-1}. \end{aligned}$$

If

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

we now want to find e^{Jt} .

Now, if we had

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$

then

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{pmatrix}.$$

Note that the Jordan–Chevalley decomposition allows us to take

$$\begin{aligned} e^{Jt} &= e^{(D+N)t} \\ &= e^{Dt} e^{Nt}. \end{aligned}$$

where

$$\begin{aligned} N &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ N^2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ N^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We may calculate

$$\begin{aligned} e^{Nt} &= I + Nt + \frac{N^2}{2}t^2 + \frac{N^3}{6}t^3 + \dots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$e^{Jt} = \begin{pmatrix} e^{\lambda t} & & \\ & e^{\lambda t} & \\ & & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Example (Constructing Differential Equations). Consider

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

with solutions of $y_1(x) = \cos^2(x)$ and $y_2(x) = \sin^2(x)$.

We start by computing some derivatives.

$$\begin{aligned} \frac{dy_1}{dx} &= -2 \cos(x) \sin(x) \\ &= -\sin(2x) \\ \frac{d^2y_1}{dx^2} &= -2 \cos(2x) \\ &= -2 \sin^2(x) + 2 \cos^2(x) \end{aligned}$$

$$\begin{aligned}
 \frac{dy_2}{dx} &= 2 \sin(x) \cos(x) \\
 &= \sin(2x) \\
 \frac{d^2y_2}{dx^2} &= 2 \cos(2x) \\
 &= 2 \cos^2(x) - 2 \sin^2(x).
 \end{aligned}$$

Substituting, we get

$$\begin{aligned}
 -2 \sin^2(x) + 2 \cos^2(x) - 2p(x) \cos(x) \sin(x) + q(x) \cos^2(x) &= 0 \\
 2 \sin^2(x) + 2 \cos^2(x) + 2p(x) \cos(x) \sin(x) + q(x) \sin^2(x) &= 0.
 \end{aligned}$$

If we add these two equations together, we get $q(x) = 0$, meaning that a constant is a solution — note that $\cos^2(x) + \sin^2(x) = 1$, so we know this is fair.

If we subtract the second equation from the first equation, we get

$$\begin{aligned}
 -4 \sin^2(x) + 4 \cos^2(x) - 4p(x) \cos(x) \sin(x) &= 0 \\
 4 \cos(2x) - 2p(x) \sin(2x) &= 0.
 \end{aligned}$$

Thus, we want

$$2p(x) \sin(2x) = 4 \cos(2x),$$

giving

$$p(x) = 2 \cot(2x).$$

Therefore, our equation is

$$\frac{d^2y}{dx^2} + 2 \cot(2x) \frac{dy}{dx} = 0.$$

Example. Consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix} \mathbf{x}.$$

We see that

$$\begin{aligned}
 \lambda &= 2 \pm 3i \\
 \mathbf{v} &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \pm i \begin{pmatrix} 3 \\ 0 \end{pmatrix}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 e^{\lambda t} &= e^{2t}(\cos(3t) + i \sin(3t)) \\
 e^{\lambda t} \mathbf{v} &= e^{2t}(\cos(3t) + i \sin(3t)) \left(\begin{pmatrix} -1 \\ 5 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) \\
 \mathbf{x}_1(t) &= e^{2t} \begin{pmatrix} -\cos(3t) - 3 \sin(3t) \\ 5 \cos(3t) \end{pmatrix}
 \end{aligned}$$

$$\mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 3 \cos(3t) - \sin(3t) \\ 5 \sin(3t) \end{pmatrix}.$$

Now, we want to find the fundamental matrix, $\Psi_A(t)$, and the matrix exponential e^{At} .

We start by finding $\Phi_A(t)$ to give

$$\Phi_A(t) = \begin{pmatrix} -e^{2t} \cos(3t) - 3e^{2t} \sin(3t) & 3e^{2t} \cos(3t) - e^{2t} \sin(3t) \\ 5e^{2t} \cos(3t) & 5e^{2t} \sin(3t) \end{pmatrix}$$

We have already found the solution

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}.$$

However, if we want to apply initial conditions, we need $\Psi_A(t)$ to obtain $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$.

$$\begin{aligned} \Psi_A(t) &= \Phi(t)\Phi_A^{-1}(0) \\ &= \begin{pmatrix} -e^{2t} \cos(3t) - 3e^{2t} \sin(3t) & 3e^{2t} \cos(3t) - e^{2t} \sin(3t) \\ 5e^{2t} \cos(3t) & 5e^{2t} \sin(3t) \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}^{-1}. \end{aligned}$$

Notice that $\Phi_A(0)$ has columns equal to the real and imaginary eigenvectors.

$$= -\frac{1}{15} \begin{pmatrix} -e^{2t} \cos(3t) - 3e^{2t} \sin(3t) & 3e^{2t} \cos(3t) - e^{2t} \sin(3t) \\ 5e^{2t} \cos(3t) & 5e^{2t} \sin(3t) \end{pmatrix} \begin{pmatrix} 0 & -3 \\ -5 & -1 \end{pmatrix}.$$

After many error-prone computations, we obtain

$$\Psi_A(t) = \begin{pmatrix} e^{2t} \cos(3t) - \frac{1}{3}e^{2t} \sin(3t) & -\frac{2}{3}e^{2t} \sin(3t) \\ \frac{5}{3}e^{2t} \sin(3t) & e^{2t} \cos(3t) + \frac{1}{3}e^{2t} \sin(3t) \end{pmatrix}.$$

To ensure that this is a (plausible) $\Psi(t)$, we check $\Psi_A(0)$, and see that $\Psi_A(0) = I$. Also, it can be verified^{iv} that

$$\frac{d\Psi_A}{dt} = A\Psi_A(t).$$

Now, finding Ψ_A was quite difficult. What if there's a better way?

Recall from the eigenvector with conjugate eigenvalues $\lambda = a \pm ib$ equation that

$$A(\mathbf{u} + i\mathbf{w}) = (e^{at} \cos(bt) + ie^{at} \sin(bt))(\mathbf{u} + i\mathbf{w}).$$

Now, examining the real part, we have

$$\begin{aligned} A\mathbf{u} &= e^{at} \cos(bt)\mathbf{u} - e^{at} \sin(bt)\mathbf{w} \\ A\mathbf{w} &= e^{at} \cos(bt)\mathbf{w} + e^{at} \sin(bt)\mathbf{u}. \end{aligned}$$

We will construct a similarity transform

$$AP = PD.$$

^{iv}I don't think this is a fun thing to do, but it can indeed be verified.

Now, we may take

$$\begin{aligned} P &= \begin{pmatrix} \operatorname{Re}(\mathbf{v}) & \operatorname{Im}(\mathbf{v}) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u} & \mathbf{w} \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We want to find $e^{At} = Pe^{Dt}P^{-1}$. Note that, after much tedious calculation (or noticing that our vector D is a matrix expression for the complex number $a + bi$), we obtain

$$e^{Dt} = \begin{pmatrix} e^{at} \cos(bt) & e^{at} \sin(bt) \\ -e^{at} \sin(bt) & e^{at} \cos(bt) \end{pmatrix}.$$

Example. Now, we discuss inhomogeneous systems. Consider the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{F}(t).$$

We look at the homogeneous system,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

which has the solutions of

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{c} \\ &= e^{At}\mathbf{x}_0. \end{aligned}$$

When we deal with inhomogeneous solutions in the one-dimensional case, we use methods like variation of parameters or undetermined coefficients.

Here, we will assume a perturbation — i.e., our solution is close enough to our fundamental solution.

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{u}(t) \\ \frac{d\mathbf{x}}{dt} &= \frac{d\Phi}{dt}\mathbf{u}(t) + \Phi(t)\frac{d\mathbf{u}}{dt}. \end{aligned}$$

Plugging in, we have

$$\frac{d\Phi}{dt}\mathbf{u}(t) + \Phi(t)\frac{d\mathbf{u}}{dt} = A\Phi(t)\mathbf{u}(t) + \mathbf{F}(t).$$

We want to cancel some things out. Recall that $\frac{d\Phi}{dt} = A\Phi$. Therefore, we obtain

$$\Phi(t)\frac{d\mathbf{u}}{dt} = \mathbf{F}(t).$$

Multiplying on both sides by $\Phi^{-1}(t)$, we get

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \Phi^{-1}(t)\mathbf{F}(t) \\ \mathbf{u}(t) &= \int \Phi^{-1}(t)\mathbf{F}(t) dt, \end{aligned}$$

where the integral is taken coordinatewise. Thus,

$$\mathbf{x}(t) = \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t) dt.$$