Problem 1

Let $\mathbb{F}$ be a field. Show that the following hold:

(i) $-1(a) = -a$

(ii) $-(-a) = a$

(iii) $-(a+b) = (-a) + (-b)$

(iv) $(-a)^{-1} = -(a^{-1})$

(v) $(ab)^{-1} = a^{-1}b^{-1}$

(i)

$$\begin{aligned} 0 &= (1 + (-1)) \\ 0(a) &= (1 + (-1))a \\ 0 &= 1(a) + (-1)(a) \\ 0 &= a + (-1)(a) \\ -a &= (-1)(a) \end{aligned}$$

(ii)

$$\begin{aligned} 0 &= -(-a) + (-a) \\ a &= -(-a) + ((-a) + a) \\ a &= -(-a) \end{aligned}$$

(iii)

$$\begin{aligned} 0 &= -(a+b) + (a+b) \\ -b &= -(a+b) + a + (b-b) \\ -a + (-b) &= -(a+b) + (a-a) \\ (-a) + (-b) &= -(a+b) \end{aligned}$$

(iv)

$$\begin{aligned} 1 &= (-a)^{-1}(-a) \\ -1 &= (-a)^{-1}(a) \\ -1(a^{-1}) &= (-a)^{-1} \\ -(a^{-1}) &= (-a)^{-1} \end{aligned}$$

(v)

$$
\begin{aligned}
1 &= (ab)^{-1}(ab) \\
b^{-1} &= (ab)^{-1}(a) \\
a^{-1}b^{-1} &= (ab)^{-1}
\end{aligned}
$$

Problem 2

Consider the set

$$K := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

Show that:

(i) $x, y \in K \Rightarrow x + y \in K \hat{x} y \in K$

(ii) $x \neq 0 \Rightarrow x^{-1} \in K$

(i)

Let $x, y \in K$. Then, $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$, where $a, b, c, d \in \mathbb{Q}$.

$x + y = (a + c) + (b + d)\sqrt{2} \in K$, as $\mathbb{Q}$ is closed under addition.

$xy = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}$, as $\mathbb{Q}$ is closed under multiplication.

(ii)

Let $x = a + b\sqrt{2} \neq 0 \in K$. Thus, at least one of $a, b \neq 0$.

$$
\begin{aligned}
x^{-1} &= \frac{1}{a + b\sqrt{2}} \\
&= \frac{a - b\sqrt{2}}{a^2 - 2b^2} \\
&= \frac{a}{a^2 - 2b^2} + \frac{-b\sqrt{2}}{a^2 - 2b^2}
\end{aligned}
$$

Since $a/(a^2 - 2b^2)$ and $(-b)/(a^2 - 2b^2)$ are both in $\mathbb{Q}$, $x^{-1} \in K$.

Problem 3

Suppose F is a field admitting $P \subseteq F$ with the following properties:

(C1) If $x, y \in P$, then $x + y \in P$ and $xy \in P$

(C2) For all $x \in F$, $x \in P$ or $-x \in P$

(C3) If $x, -x \in P$, then $x = 0$.

Show that there is an ordering on F making it into an ordered field.

Let $x \leq_F y$ be defined as follows:

$$x \leq_F y \Leftrightarrow \exists p \in P \ni x + p = y$$

Symmetry: If $x \leq_F x$, that implies $p = 0 \in P$.

Transitivity: If $x \leq_F y$ and $y \leq_F z$, we let $x + p_1 = y$ and $y + p_2 = z$ for $p_1, p_2 \in P$. Then, $x + (p_1 + p_2) = z$, and since $p_1 + p_2 \in P$ by definition, $x \leq_F z$.

Antisymmetry: If $x \leq_F y$ and $y \leq_F x$, then $\exists p_1, p_2 \in P$ such that $x + p_1 = y$ and $y + p_2 = x$. Therefore, $(x + p_1) + p_2 = x$, so $p_1 = -p_2$. Since $p_1, p_2 \in P$ and $p_1 = -p_2$, $p_1, p_2 = 0$, so $x = y$.

Totality: Let $x, y \in F$, and $x \not\leq_F y$. Then, $\forall p \in P$, $x + p \neq y$. So $x \neq y$, as $0 \in P$, but then $x = y + p'$ for some $p' \in P$. Therefore, $y \leq_F x$.

$\therefore$ the ordering is total.

Ordered Field Axiom (i)

Let $s \leq t$ and $x \leq y$. Then, for some $p_1, p_2 \in P$, we have the following:

$$\begin{aligned} t &= s + p_1 \\ y &= x + p_2 \end{aligned}$$

Adding, we have:

$$\begin{aligned} t + y &= s + x + (p_1 + p_2) \\ s + x &\leq t + y && \text{since } p_1 + p_2 \in P \end{aligned}$$

Ordered Field Axiom (ii)

Let $s \leq t$ and $z \geq 0$. Then, for some $p \in P$, the following is true:

$$\begin{aligned} t &= s + p \\ zt &= z(s + p) \\ &= zs + zp \end{aligned}$$

Since $zp \in P$ as $z \in P$ and $p \in P$, we have:

$$\begin{aligned} zt &= zs + p' && \text{where } p' = zp \\ zs &\leq zp \end{aligned}$$

Problem 4

Let $a, b \in \mathbb{R}$. Prove the following:

(i) If $0 \leq a \leq \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

(ii) If $a \leq b + \varepsilon$ for all $\varepsilon > 0$, then $a \leq b$.

(i)

Suppose toward contradiction that $a \neq 0$. Since $a \geq 0$, it must be that $a > 0$, so $\frac{1}{2}a > 0$. Let $\varepsilon = \frac{1}{2}a$. Therefore, $0 < \frac{1}{2}a < a$, which can't be true as $a \leq \varepsilon$ for all $\varepsilon > 0$. $\perp$

(ii)

Let $a > b$. Then, $a - b > 0$; let $\varepsilon = \frac{a-b}{2}$. Then, $a > b + \varepsilon$, so $a \not\leq b + \epsilon$ for all $\epsilon > 0$.

Problem 5

If $a, b \in \mathbb{R}$, show that
$$\left(\frac{1}{2}(a+b)\right)^2 \leq \frac{1}{2}(a^2 + b^2)$$

$$\left(\frac{1}{2}(a+b)\right)^2 = \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}ab$$

WLOG, let $a \geq b$. There are three cases: $a, b \in \mathbb{R}^+$, $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$, or $-a, -b \in \mathbb{R}^+$.

Case 1: If $a, b \in \mathbb{R}^+$, then $\frac{1}{2}ab \leq \frac{1}{2}a^2$. Since $a^2 \geq b^2$ (as $a \geq b$), it must be that $\frac{1}{2}a^2 \geq \frac{1}{4}a^2 + \frac{1}{4}b^2$.

$$\begin{aligned}\left(\frac{1}{2}(a+b)\right)^2 &= \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}ab \\ &\leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \\ &= \frac{1}{2}(a^2 + b^2)\end{aligned}$$

Case 2: If $a \in \mathbb{R}^+$ and $-b \in \mathbb{R}^+$, then $-\frac{1}{2}ab \in \mathbb{R}^+$, or $\frac{1}{2}ab < 0$.

$$\begin{aligned}\left(\frac{1}{2}(a+b)\right)^2 &= \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}ab \\ &\leq \frac{1}{4}a^2 + \frac{1}{4}b^2 \\ &\leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \\ &= \frac{1}{2}(a^2 + b^2)\end{aligned}$$

Case 3: If $-a, -b \in \mathbb{R}^+$, then $\frac{1}{2}ab \in \mathbb{R}^+$, so we use similar logic to Case 1.

Problem 6

For $x \in \mathbb{R}$, show that $\sqrt{x^2} = |x|$.

Recall:
$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

Suppose $x \in \mathbb{R}^+$. Then, since $\sqrt{x^2} \in \mathbb{R}^+$, and $y^2 = x^2 \Rightarrow y = \pm x$, it must be the case that $\sqrt{x^2} = x$.

Suppose $x \notin \mathbb{R}^+$. Then, $x^2 \in \mathbb{R}^+$, so $\sqrt{x^2} \in \mathbb{R}^+$, so $\sqrt{x^2} = -x$.

Thus, $\sqrt{x^2} = |x|$.

Problem 7

Let $x, y, a, b \in \mathbb{R}$ and $\varepsilon > 0$.

(i) Show that $|x - a| < \varepsilon$ if and only if $a - \varepsilon < x < a + \varepsilon$

(ii) If $a < x < b$ and $a < y < b$, show that $|x - y| < b - a$. What does this mean geometrically?

(i)

($\Rightarrow$) Let $|x - a| < \varepsilon$. Then, $x - a < \varepsilon$ and $-(x - a) < \varepsilon$. Thus, $x < a + \varepsilon$ and $-x < \varepsilon - a$, so $a - \varepsilon < x < a + \varepsilon$.

($\Leftarrow$) Let $a - \varepsilon < x < a + \varepsilon$. Then, $-\varepsilon < (x - a) < \varepsilon$. Therefore, $|x - a| < \varepsilon$.

(ii)

Let $a < x < b$ and $a < y < b$. In the second case, we have that $-b < -y < -a$ (by multiplying all the inequalities by -1). Adding, we get $a - b < x - y < b - a$, or $-(b - a) < x - y < b - a$. Therefore, $|x - y| < b - a$.

Problem 8

Find all $x \in \mathbb{R}$ that satisfy:

$$4 < |x + 2| + |x - 1| < 5$$

Case 1: $x < -2$

$$\begin{aligned} 4 < -(x + 2) + -(x - 1) < 5 \\ -5 < (x + 2) + (x - 1) < -4 \\ -5 < 2x + 1 < -4 \\ -6 < 2x < -5 \\ -3 < x < -2.5 \end{aligned}$$

Case 2: $-2 \leq x < 1$

$$\begin{aligned} 4 < (x + 2) + -(x - 1) < 5 \\ 4 < 2 < 5 \end{aligned} \qquad \perp$$

Case 3: $1 \leq x$

$$\begin{aligned} 4 < (x + 2) + (x - 1) < 5 \\ 4 < 2x + 1 < 5 \\ 1.5 < x < 2 \end{aligned}$$

So the solution is:

$$x \in (-3, -2.5) \cup (1.5, 2)$$

Problem 9

Let $a, b \in \mathbb{R}$. Show that

$$\max(a,b) = \frac{1}{2}(a + b + |a-b|)$$
$$\min(a,b) = \frac{1}{2}(a + b - |a-b|)$$

WLOG, let $a > b$. Then:

$$\begin{aligned}\frac{1}{2}(a+b+|a-b|) &= \frac{1}{2}(a+b+(a-b)) \\ &= a \\ \frac{1}{2}(a+b-|a-b|) &= \frac{1}{2}(a+b-(a-b)) \\ &= b\end{aligned}$$

Similarly, if $a = b$, then we have that $\max(a,b) = \min(a,b) = a = b$.

Problem 10

If $x \neq y$ in $\mathbb{R}$, show that there is a $\delta > 0$ such that $V_\delta(x) \cap V_\delta(y) = \emptyset$.

Let $\delta = \frac{1}{2}|x-y|$. Then

$$V_\delta(x) \cap V_\delta(y) = \left(x - \frac{1}{2}|x-y|, x + \frac{1}{2}|x-y|\right) \cap \left(y - \frac{1}{2}|x-y|, y + \frac{1}{2}|x-y|\right) = \emptyset$$