Abstract

We detail the construction and some of the properties of the Lebesgue measure via the Lebesgue–Stieltjes Measure.

Premeasures, Outer Measures, and Measures

Consider a set-function $\lambda\colon P(\mathbb{R}) \to [0,\infty]$ that satisfies

- $\lambda(\emptyset) = 0$;
- for any finite or infinite sequence of disjoint sets, $\{E_j\}_{j=1}^{\infty}$, we have

$$\lambda\left(\bigsqcup_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty}\lambda(E_j);$$

- $\lambda(I) = b - a$, where I is an interval (either open, closed, or a half-interval);
- $\lambda(s + E) = \lambda(E)$.

Unfortunately, such a set-function doesn't exist.

In order to construct a set function on a restricted domain $\lambda\colon \mathcal{L} \to [0,\infty]$, we need to define a particular class of measurable subsets of $\mathbb{R}$. This is where the concept of an *outer measure* comes in.

Definition: Let X be a set, and let $\mu^*\colon P(X) \to [0,\infty]$ be a set function. We say μ^* is an *outer measure* if

- $\mu^*(\emptyset) = 0$;
- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$;
- $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty}\mu^*(A_j)$.

We will obtain an outer measure on the entirety of $P(X)$ by defining a notion of measure on some "satisfactory" subfamily $\mathcal{E} \subseteq P(X)$, then by approximating other subsets using this family.

Proposition: Let $\mathcal{E} \subseteq P(X)$ be a family of subsets such that $\emptyset \in \mathcal{E}$ and $X \in \mathcal{E}$, and let $\rho\colon \mathcal{E} \to [0,\infty]$ be a set function such that $\rho(\emptyset) = 0$. For any $A \subseteq X$, define

$$\mu^*(E) = \inf\left\{\sum_{j\geq 1}\rho(E_j) \;\middle|\; E_j \in \mathcal{E}, A \subseteq \bigcup_{j\geq 1} E_j\right\}.$$

Then, μ^* is an outer measure.

Proof. We start by showing well-definedness, which stems from the fact that we may select $E_j = X$ for all j.

Since we may take $E_j = \emptyset$ for all j, we must have $\mu^*(\emptyset) = 0$. Furthermore, if $A \subseteq B$, since the set over which the infimum is taken for the definition of $\mu^*(A)$ includes the corresponding set for B, we must have $\mu^*(A) \leq \mu^*(B)$.

Finally, let $\{A_j\}_{j\geq 1} \subseteq P(X)$, and let $\varepsilon > 0$. For each j, there exists $\{E_{j,k}\}_{k\geq 1} \subseteq \mathcal{E}$ such that $A_j \subseteq \bigcup_{k\geq 1} E_{j,k}$ and $\sum_{k\geq 1}\rho(E_{j,k}) \leq \mu^*(A_j) + \varepsilon 2^{-j}$.

Then, if $A = \bigcup_{j\geq 1} A_j$, we have $A \subseteq \bigcup_{j,k\geq 1} E_{j,k}$, and $\sum_{j,k\geq 1}\rho(E_{j,k}) \leq \sum_{j\geq 1}\mu^*(A_j) + \varepsilon$, so that $\mu^*(A) \leq \sum_{j\geq 1}\mu^*(A_j) + \varepsilon$. Since ε is arbitrary, we are done. $\square$

Definition: A subset $A \subseteq X$ is said to be μ^*-*measurable* if for any $E \subseteq X$, A serves as a good "cookie cutter" for E, in that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Equivalently, due to subadditivity, we have A is measurable if and only if for all $E \subseteq X$,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition: Let $\mathcal{A}$ be an algebra of subsets of X. We call a set function $\mu_0 \colon \mathcal{A} \to [0, \infty]$ a *premeasure* if

- $\mu_0(\emptyset) = 0$;
- for a collection of disjoint elements of $\mathcal{A}$, $\{A_j\}_{j=1}^{\infty}$ where $\bigcup_{j\geq 1} A_j \in \mathcal{A}$, we have

$$\mu_0\left(\bigsqcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} \mu_0(A_j).$$

Every premeasure gives rise to an outer measure by taking

$$\mu^*(E) = \inf\left\{\sum_{j\geq 1} \mu_0(A_j) \;\middle|\; A_j \in \mathcal{A}, E \subseteq \bigcup_{j\geq 1} A_j\right\}. \tag{$*$}$$

A remarkable result by Caratheodory allows us to extend premeasures from algebras to measures on σ-algebras. To start, there is a little bit of build-up.

Proposition: Let μ_0 be a premeasure on $\mathcal{A}$, with μ^* defined by (**??**). Then,

(a) $\mu^*|_{\mathcal{A}} = \mu_0$;

(b) every set in $\mathcal{A}$ is μ^*-measurable.

Proof. Suppose $E \in \mathcal{A}$. If $E \subseteq \bigcup_{j\geq 1} A_j$ with $A_j \in \mathcal{A}$, we let $B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right)$. The B_n are disjoint members of $\mathcal{A}$ whose union is E, so

$$\begin{aligned}\mu_0(E) &= \sum_{j=1}^{\infty} \mu_0(B_j)\\ &\leq \sum_{j=1}^{\infty} \mu_0(A_j).\end{aligned}$$

It follows that $\mu_0(E) \leq \mu^*(E)$. The reverse inequality is clear from the fact that we may specify $A_1 = E$ and $A_{j>1} = \emptyset$.

Meanwhile, if $A \in \mathcal{A}$, $E \subseteq X$, and $\varepsilon > 0$, then there is a collection $\{B_j\}_{j\geq 1} \subseteq \mathcal{A}$ with $E \subseteq \bigcup_{j\geq 1} B_j$ and $\sum_{j\geq 1} \mu_0(B_j) \leq \mu^*(E) + \varepsilon$. By additivity on $\mathcal{A}$, we get

$$\begin{aligned}\mu^*(E) + \varepsilon &\geq \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \mu_0(B_j \cap A^c)\\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c),\end{aligned}$$

so A is measurable. $\square$

Theorem (Caratheodory's Theorem)**:** Let $\mathcal{A} \subseteq P(X)$ be an algebra, let μ_0 be a premeasure on $\mathcal{A}$, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 — namely, $\mu - \mu^*|_{\mathcal{M}}$, where μ^* is given by (**??**).

If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$, with equality for all $\mu(E) < \infty$. Furthermore, if μ_0 is σ-finite, then μ is unique.

Proof. We start by showing that if μ^* is an outer measure, then if $\mathcal{M}^*$ is the collection of μ^*-measurable sets, $\mathcal{M}^*$ is a σ-algebra and $\mu^*|_{\mathcal{M}^*}$ is a complete measure.[I]

By definition, $\mathcal{M}^*$ is closed under complements, as the definition of μ^*-measurability is symmetric in A and A^c. To show finite additivity, if $A, B \in \mathcal{M}^*$ and $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c)\\ &\quad + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

We note that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, so subadditivity gives

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B)).$$

Therefore,

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Therefore, $A \cup B \in \mathcal{M}^*$, so $\mathcal{M}^*$ is an algebra. Moreover, if $A, B \in \mathcal{M}^*$ are disjoint, then

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c)\\ &= \mu^*(A) + \mu^*(B).\end{aligned}$$

To show that $\mathcal{M}^*$ is a σ-algebra, we show that $\mathcal{M}^*$ is closed under countable *disjoint* unions. Let $\{A_j\}_{j\geq 1}$ be a sequence of disjoint sets in $\mathcal{M}^*$, and let $B_n = \bigsqcup_{j=1}^{n} A_j$, with $B = \bigsqcup_{j\geq 1} A_j$. Then, for any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}),\end{aligned}$$

so by induction, we have

$$\mu^*(E \cap B_n) = \sum_{j=1}^{n} \mu^*(E \cap A_j).$$

This gives

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c)\\ &\geq \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B^c),\end{aligned}$$

and taking $n \to \infty$, we have

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c)$$

[I]This is Theorem 1.11 in Folland's *Real Analysis*.

$$
\begin{aligned}
&\geq \mu^*\left(\bigsqcup_{j\geq 1} E\cap A_j\right) + \mu^*(E\cap B^c)\\
&= \mu^*(E\cap B) + \mu^*(E\cap B^c)\\
&\geq \mu^*(E).
\end{aligned}
$$

Therefore, $B\in \mathcal{M}^*$, and if we take $E = B$,

$$
\mu^*(B) = \sum_{j=1}^{\infty}\mu^*(A_j),
$$

and μ^* is countably additive on $\mathcal{M}^*$. Finally, if $\mu^*(A) = 0$, we have

$$
\begin{aligned}
\mu^*(E) &\leq \mu^*(E\cap A) + \mu^*(E\cap A^c)\\
&= \mu^*(E\cap A^c)\\
&\leq \mu^*(E),
\end{aligned}
$$

so $A\in\mathcal{M}^*$, and $\mu^*|_{\mathcal{M}^*}$ is a complete measure.

Returning to our premeasure, μ_0 and the corresponding outer measure μ^*, we note that since every element of $\mathcal{A}$ is μ^*-measurable, the σ-algebra of μ^*-measurable sets includes $\mathcal{A}$, so it includes $\mathcal{M} = \sigma(\mathcal{A})$.

Let ν be any other measure on $\mathcal{M}$ that extends μ_0. If $E\in\mathcal{M}$, and $E\subseteq \bigcup_{j\geq 1} A_j$ with $A_j\in\mathcal{A}$, then $\nu(E)\leq \sum_{j\geq 1}\nu(A_j) = \sum_{j\geq 1}\mu_0(A_j)$, so $\nu(E)\leq \mu(E)$.

If we set $A = \bigcup_{j\geq 1} A_j$, the properties of the premeasure give us

$$
\begin{aligned}
\nu(A) &= \lim_{n\to\infty}\nu\left(\bigcup_{j=1}^{n} A_j\right)\\
&= \lim_{n\to\infty}\mu\left(\bigcup_{j=1}^{n} A_j\right)\\
&= \mu(A).
\end{aligned}
$$

If $\mu(E) < \infty$, we may select the A_j such that $\mu(A) < \mu(E) + \varepsilon$, so $\mu(A\setminus E) < \varepsilon$, and

$$
\begin{aligned}
\mu(E) &\leq \mu(A)\\
&= \nu(A)\\
&= \nu(E) + \nu(A\setminus E)\\
&\leq \nu(E) + \mu(A\setminus E)\\
&\leq \nu(E) + \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, $\mu(E) = \nu(E)$.

Now, if μ_0 is σ-finite, we write $X = \bigsqcup_{j\geq 1} A_j$, with $\mu_0(A_j) < \infty$ and the A_j are disjoint. For any $E\in\mathcal{M}$, we have

$$
\begin{aligned}
\mu(E) &= \sum_{j\geq 1}\mu(E\cap A_j)\\
&= \sum_{j\geq 1}\nu(E\cap A_j)\\
&= \nu(E).
\end{aligned}
$$

$\square$

Construction of the Lebesgue Measure

With Caratheodory's theorem, we now know that it is possible to construct a unique measure from a suitable premeasure on a particular family of subsets. Here, we will use the family of half-open intervals, or h-intervals, of the form $(a, b]$, where $-\infty \leq a < b < \infty$, or (a, ∞).

The algebra of h-intervals, $\mathcal{A}$, generates the Borel σ-algebra, $\mathcal{B}_{\mathbb{R}}$.

Consider a finite Borel measure on $\mathbb{R}$, and let $F(x) = \mu((-\infty, x])$. We often call $F(x)$ the *distribution function* of μ. Then, we see that F is increasing and right-continuous, as

$$(-\infty, x] = \bigcap_{n\geq 1}(-\infty, x_n],$$

where x_n is a decreasing sequence convergence to x.

As it turns out, we are able to reverse this process. Given an increasing, right-continuous function $F\colon \mathbb{R} \to \mathbb{R}$, there is a corresponding Borel measure.

Proposition: Let $F\colon \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous. If $\{(a_j, b_j]\}_{j=1}^{n}$ are disjoint h-intervals, we define

$$\mu_0\left(\bigcup_{j=1}^{n}(a_j, b_j]\right) = \sum_{j=1}^{n}(F(b_j) - F(a_j)),$$

and set $\mu_0(\emptyset) = 0$. Then, μ_0 is a premeasure on $\mathcal{A}$.

Proof. We start by verifying that μ_0 is well-defined, seeing as elements of $\mathcal{A}$ can be written in more than one way as disjoint unions of h-intervals. If $\{(a_j, b_j]\}_{j=1}^{n}$ are disjoint, and $\bigcup_{j=1}^{n}(a_j, b_j] = (a, b]$, then after relabeling indices, we must have $a = a_1 < b_1 = a_2 < \cdots < b_n = b$, so $\sum_{j=1}^{n}(F(b_j) - F(a_j)) = F(b) - F(a)$.

Generally speaking, if $\{I_i\}_{i=1}^{n}$ and $\{J_j\}_{j=1}^{m}$ are disjoint finite sets of intervals such that $\bigcup_{i=1}^{n} I_i = \bigcup_{j=1}^{m} J_j$, then

$$\begin{aligned}\sum_{i=1}^{n}\mu_0(I_i) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\mu_0(I_i \cap J_j)\\ &= \sum_{j=1}^{m}\mu_0(J_j).\end{aligned}$$

Now, we must show that if $\{I_j\}_{j=1}^{\infty}$ is a sequence of disjoint h-intervals with $\bigcup_{j\geq 1} I_j \in \mathcal{A}$, then $\mu_0\left(\bigcup_{j\geq 1} I_j\right) = \sum_{j\geq 1}\mu_0(I_j)$.

Since $\bigcup_{j\geq 1} I_j$ is a finite union of h-intervals, we may partition $\{I_j\}_{j\geq 1}$ into finitely many subfamilies such that the union in each subfamily is a single h-interval. Using the finite additivity of μ_0, we may assume that $\bigcup_{j=1}^{\infty} I_j$ is an interval $I = (a, b]$. We thus have

$$\begin{aligned}\mu_0(I) &= \mu_0\left(\bigcup_{j=1}^{n} I_j\right) + \mu_0\left(I \setminus \bigcup_{j=1}^{n} I_j\right)\\ &\geq \mu_0\left(\bigcup_{j=1}^{n} I_j\right)\\ &= \sum_{j=1}^{n}\mu_0(I_j).\end{aligned}$$

Taking limits, we get $\mu_0(I) \geq \sum_{j\geq 1} \mu_0(I_j)$.

To prove the reverse inequality, we suppose a and b are finite, and fix $\varepsilon > 0$. Since F is right-continuous, there exists $\delta > 0$ such that $F(a+\delta) - F(a) < \varepsilon$, and if $I_j = (a_j, b_j]$, then for each j there is $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \varepsilon 2^{-j}$.

The open intervals $(a_j, b_j + \delta_j)$ cover the compact set $[a+\delta, b]$, so there is a finite subcover. By discarding $(a_j, b_j + \delta_j)$ contained in larger ones, and relabeling indices, we may assume that

- the intervals $(a_1, b_1 + \delta_1), \dots, (a_N, b_N + \delta_N)$ cover $[a+\delta, b]$;
- $b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1})$ for each j.

Then,

$$\begin{aligned}
\mu_0(I) &< F(b) - F(a+\delta) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_1) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} (F(a_{j+1}) - F(a_j)) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} (F(b_j + \delta_j) - F(a_j)) + \varepsilon \\
&< \sum_{j=1}^{N} \bigl(F(b_j) + \varepsilon 2^{-j} - F(a_j)\bigr) + \varepsilon \\
&< \sum_{j=1}^{\infty} \mu_0(I_j) + 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we are done for the case that a and b are finite.

If $a = -\infty$, then for any $M < \infty$, the intervals $(a_j, b_j + \delta_j)$ cover $[-M, b]$, so the same reasoning gives $F(b) - F(-M) \leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\varepsilon$, whereas if $b = \infty$, we obtain $F(M) - F(a) \leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\varepsilon$. Our desired result follows from taking $\varepsilon \to 0$ and $M \to \infty$. $\square$

This allows us to establish the correspondence between increasing and right-continuous functions and Borel measures.

Theorem: If $F \colon \mathbb{R} \to \mathbb{R}$ is an increasing, right-continuous function, then there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all a, b. If G is another such function, then $\mu_F = \mu_G$ if and only if $F - G$ is constant.

Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on bounded sets, and we define

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases},$$

then F is increasing and right-continuous, with $\mu = \mu_F$.

Proof. We know that each F induces a premeasure on $\mathcal{A}$ by the previous proposition, and by definition, G induces the same premeasure if and only if $F - G$ is constant. These premeasures are σ-finite, since

$$\mathbb{R} = \bigcup_{j=-\infty}^{\infty} (j, j+1].$$

Therefore, the induced measure μ_F on $\mathcal{B}_{\mathbb{R}}$ is unique by the Caratheodory extension theorem.

The last assertion follows from the fact μ is monotonic, and continuous from both above and below, so that F is continuous for both $x \geq 0$ and $x < 0$. Since $\mu = \mu_F$ on $\mathcal{A}$, we have $\mu = \mu_F$ on $\mathcal{B}_{\mathbb{R}}$ by the uniqueness condition in the Caratheodory extension theorem. □

Definition: If F is an increasing and right-continuous function, then the completion of the measure μ_F, which we write λ_F, is known as the *Lebesgue–Stieltjes measure* associated to F.

We denote the σ-algebra associated to λ_F as $\mathcal{M}_\lambda$. For any $E \in \mathcal{M}_\lambda$, we have

$$\begin{aligned}\lambda_F(E) &= \inf\left\{\sum_{j=1}^{\infty}(F(b_j) - F(a_j)) \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j]\right\}\\ &= \inf\left\{\sum_{j=1}^{\infty}\lambda_F((a_j, b_j]) \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j]\right\}.\end{aligned}$$

As it turns out, we are allowed to replace the h-intervals in the formula for for $\lambda_F(E)$ with open intervals. Note that in Real Analysis II, we defined the Lebesgue measure through this method.

Lemma: For any $E \in \mathcal{M}_\lambda$, we have

$$\lambda_F(E) = \inf\left\{\sum_{j=1}^{\infty}\lambda_F((a_j, b_j)) \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)\right\}.$$

Proof. We call the quantity on the right $\nu(E)$. Let $E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)$. Each (a_j, b_j) is a countable disjoint union of h-intervals of the form $I_{j,k} = (c_{j,k}, c_{j,k+1}]$, where $(c_{j,k})_k$ is a sequence with $c_{j,1} = a_j$ and $c_{j,k} \to b_j$. Thus, $E \subseteq \bigcup_{j,k=1}^{\infty} I_{j,k}$, so

$$\begin{aligned}\sum_{j=1}^{\infty}\lambda_F((a_j, b_j)) &= \sum_{j,k=1}^{\infty}\lambda_F(I_{j,k})\\ &\geq \lambda_F(E),\end{aligned}$$

so $\nu(E) \geq \lambda_F(E)$.

On the other hand, given $\varepsilon > 0$, there exists $\{(a_j, b_j]\}_{j\geq 1}$ such that $E \subseteq \bigcup_{j\geq 1}(a_j, b_j]$ and $\sum_{j\geq 1}\lambda_F((a_j, b_j]) \leq \lambda_F(E) + \varepsilon$. For each j, right-continuity gives $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \varepsilon 2^{-j}$.

Then, $E \subseteq \bigcup_{j\geq 1}(a_j, b_j + \delta_j)$, and

$$\begin{aligned}\sum_{j\geq 1}\lambda_F((a_j, b_j + \delta_j)) &\leq \sum_{j\geq 1}\lambda_F((a_j, b_j]) + \varepsilon\\ &\leq \lambda_F(E) + 2\varepsilon,\end{aligned}$$

so $\nu(E) \leq \mu(E)$ □

Now we may understand the regularity of the Lebesgue–Stieltjes measure.

Theorem: Let λ_F be a Lebesgue–Stieltjes measure on $\mathbb{R}$, and let $E \in \mathcal{M}_\lambda$. Then, the following hold:

(a) For all $\varepsilon > 0$, there exists $U \subseteq \mathbb{R}$ open with $E \subseteq U$ and $\lambda_F(U \setminus E) < \varepsilon$.

(b) There exists $V \subseteq \mathbb{R}$ G_δ with $E \subseteq V$ and $\lambda_F(V \setminus E) < \varepsilon$.

(c) For all $\varepsilon > 0$, there exists $C \subseteq \mathbb{R}$ closed with $C \subseteq E$ and $\lambda_F(E \setminus C) < \varepsilon$.

(d) There exists $F \subseteq \mathbb{R}$ F_σ with $E \subseteq F$ and $\lambda_F(F \setminus E) < \varepsilon$.

Proof.

(a) Let $\varepsilon > 0$. By the previous theorem, and the definition of the outer measure, we have a set $\{(a_j, b_j)\}_{j=1}^\infty$ such that $E \subseteq \bigcup_{j\geq 1}(a_j, b_j)$, and

$$\begin{aligned}\lambda_F(E) + \varepsilon &> \sum_{j=1}^{\infty} \lambda_F((a_j, b_j)) \\ &\geq \lambda_F\left(\bigcup_{j=1}^{\infty}(a_j, b_j)\right),\end{aligned}$$

so we set $U = \bigcup_{j\geq 1}(a_j, b_j)$.

Now, if $\lambda_F(E) < \infty$, then $\lambda_F(U \setminus E) = \lambda_F(U) - \lambda_F(E) < \varepsilon$. Meanwhile, if $\lambda_F(E) = \infty$, we partition to get $E = \bigsqcup_{k\geq 1} E_k$ with $\lambda_F(E_k) < \infty$, and find U_k such that $\lambda_F(U_k \setminus E_k) < \varepsilon 2^{-k}$. Setting $U = \bigcup_{k\geq 1} U_k$, we get

$$\begin{aligned}\lambda_F(U \setminus E) &= \lambda_F\left(\bigcup_{k\geq 1}(U_k \setminus E_k)\right) \\ &\leq \sum_{k\geq 1} \lambda_F(U_k \setminus E_k) \\ &< \sum_{k\geq 1} \varepsilon 2^{-k} \\ &= \varepsilon.\end{aligned}$$

(b) For each n, we find $U_n \subseteq \mathbb{R}$ such that $E \subseteq U_n$ and $\lambda_F(U_n \setminus E) < 1/n$. Setting $V = \bigcap_{n\geq 1} U_n$, we find

$$\begin{aligned}\lambda_F(V \setminus E) &= \lambda_F\left(\bigcap_{n\geq 1}(U_n \setminus E)\right) \\ &\leq \lambda_F(U_k \setminus E) && \text{for all } k \\ &< 1/k,\end{aligned}$$

so $\lambda_F(V \setminus E) = 0$.

(c) We may use the same methodology on E^c, and take complements.

(d) We may use the same methodology on E^c, and take complements, using the fact that the complement of a G_δ set is a F_σ set.

□

Theorem: Let $E \in \mathcal{M}_\lambda$. Then,

$$\begin{aligned}\lambda_F(E) &= \inf\{\lambda_F(U) \mid E \subseteq U, U \text{ open}\} \\ &= \sup\{\lambda_F(K) \mid K \subseteq E, K \text{ compact}\}.\end{aligned}$$

The former equality is known as *outer regularity*, and the latter equality is known as *inner regularity*.

Proof. We know that for all $\varepsilon > 0$, there is $E \subseteq \bigcup_{j\geq 1}(a_j, b_j)$, and $\sum_{j\geq 1} \lambda_F((a_j, b_j)) \leq \lambda_F(E) + \varepsilon$. Setting $U = \bigcup_{j\geq 1}(a_j, b_j)$, we have $\lambda_F(U) \leq \lambda_F(E) + \varepsilon$. Since $E \subseteq U$, we also have $\lambda_F(E) \leq \lambda_F(U)$, so the definition of outer regularity is satisfied.

We now show inner regularity. If E is bounded, given $\varepsilon > 0$, there is $C \subseteq E$ closed with $\lambda_F(E \setminus C) < \varepsilon$. Since C is bounded, C is compact, so $\lambda_F(E) - \varepsilon < \lambda_F(C)$, and so we have inner regularity whenever E is bounded.

If E is not bounded, we set $E_n = E \cap [-n, n]$. We have $E_1 \subseteq E_2 \subseteq \cdots$, and $E = \bigcup_{n\geq 1} E_n$. Therefore, $\lambda_F(E) = \sup(\lambda_F(E_n))$. There are two cases.

If $\lambda_F(E) = \infty$, then we may find compact $K_n \subseteq E_n$ such that $\lambda_F(E_n) - 1 < \lambda_F(K - n)$, so that $\lambda_F(K_n) \to \infty$.

If $\lambda_F(E) < \infty$, then given $\varepsilon > 0$, we find N such that $\lambda_F(E) - \varepsilon/2 < \lambda_F(E_n)$. We find compact K with $K \subseteq E_n$ and $\lambda_F(E_n) - \varepsilon/2 < \lambda_F(K)$. Thus, $K \subseteq E$ is compact, with $\lambda_F(E) - \varepsilon < \lambda_F(K)$. □

Proposition: If $E \in \mathcal{M}_\lambda$, and $\lambda_F(E) < \infty$, then for every $\varepsilon > 0$, there is a set A that is a finite union of open intervals such that $\lambda_F(E \triangle A) < \varepsilon$.

Proof. By outer regularity, there is $U \subseteq \mathbb{R}$ open such that $E \subseteq U$, and $\lambda_F(U \setminus E) < \varepsilon/2$. Every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals, so that $\lambda_F\left(\bigsqcup_{j\geq 1}((a_j, b_j) \setminus E)\right) < \varepsilon$.

We find N such that $\sum_{j=N+1}^{\infty} \lambda_F((a_j, b_j)) < \varepsilon/2$, and set $A = \bigsqcup_{j=1}^{N}(a_j, b_j)$. □

Definition: *The* Lebesgue measure is defined to be the Lebesgue–Stieltjes measure associated to the function $F(x) = x$. We denote it by m.

The domain of m is known as the class of *Lebesgue-measurable* sets, denoted $\mathcal{L}$.

Theorem: If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$, and $m(rE) = |r|m(E)$.

Proof. Since open intervals are invariant under translations and dilations, so is $\mathcal{B}_\mathbb{R}$. For $E \in \mathcal{B}_\mathbb{R}$, we let $m_s(E) = m(E + s)$, and $m^r(E) = m(rE)$.

Since m_s and m^r agree with m and $|r|m$ on finite unions of intervals, they agree on $\mathcal{B}_\mathbb{R}$ by the Caratheodory extension theorem. In particular, whenever $E \in \mathcal{B}_\mathbb{R}$, and $m(E) = 0$, then $m(E + s) = m(rE) = 0$, so it follows that the class of Lebesgue-null sets is preserved under translations and dilations. Since all members of $\mathcal{L}$ are unions of a null set and a Borel set, it follows that $\mathcal{L}$ is preserved under translations and dilations. Therefore, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$ for all $E \in \mathcal{L}$. □

There are indeed elements of $\mathcal{L}$ that are not elements of $\mathcal{B}_\mathbb{R}$.

First, recall that the Cantor set, Δ consists of all $x \in [0, 1]$ such that the base 3 expansion $x = \sum_{j\geq 1} a_j 3^{-j}$ is such that $a_j \in \{0, 2\}$.

Since we may map Δ onto $[0, 1]$ by taking $a_j \mapsto a_j/2$ for each $j \geq 0$, we see that Δ is uncountable, and that $m(\Delta) = 0$. Therefore, every subset of Δ is of measure zero (since Lebesgue measure is complete), meaning that the cardinality of $\mathcal{L}$ is $2^{2^{\aleph_0}}$. Meanwhile, a result from descriptive set theory shows that $\mathcal{B}_\mathbb{R}$ has cardinality $2^{\aleph_0}$,[II] so there exists some Lebesgue-measurable set that isn't Borel-measurable.

[II] It turns out that the σ-algebra generated by a particular family $\mathcal{E}$ has cardinality $\aleph_1 \cdot |\mathcal{E}|$. Since $\tau_\mathbb{R}$ has cardinality $2^{\aleph_0}$, and $2^{\aleph_0} \geq \aleph_1$ (depending on whether or not you accept the Continuum Hypothesis), $\aleph_1 \cdot 2^{\aleph_0} = 2^{\aleph_0}$.