1.1

Individual

1.1.1

Determine which complete bipartite graphs are complete graphs.

$K_{1,1}$ is the only complete bipartite graph that is complete

1.1.3

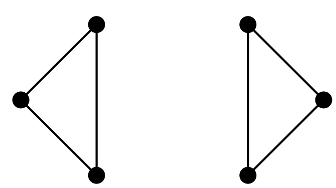
Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

$$K_{m,n} = \begin{array}{cc}
 & \begin{array}{cccccccc} a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_n \end{array} \\
\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{array} &
\begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\end{array}$$

1.1.5

Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle.

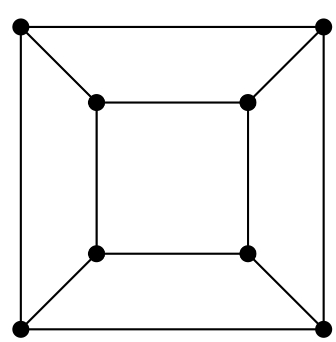
Let G be the following graph:

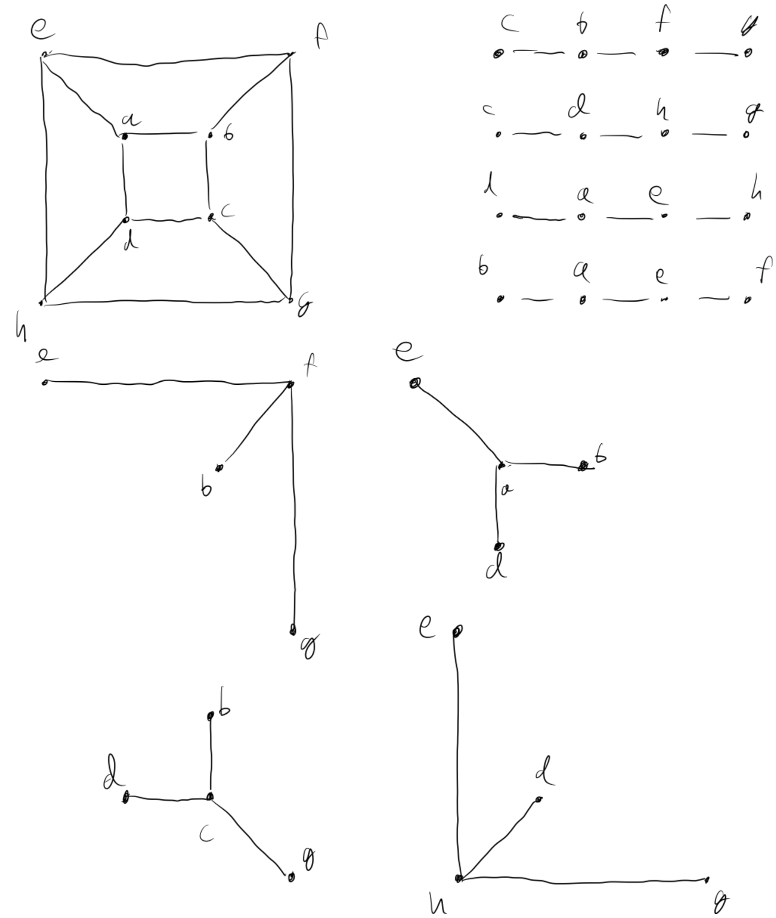


Every vertex in G has a degree 2, yet G is not a cycle.

1.1.8

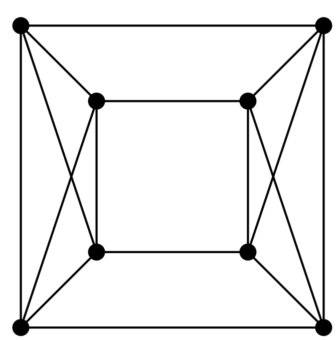
Prove that the 8 vertex graph below decomposes into copies of $K_{1,3}$ and also into copies of P_4

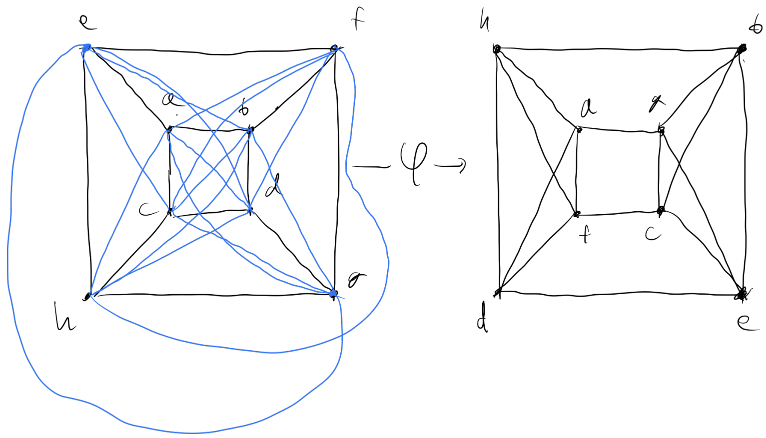




1.1.9

Prove that the graph below is isomorphic to the complement of the previous graph





1.1.10

Prove or disprove: the complement of a simple disconnected graph must be connected.

Let G be a graph that is disconnected. We want to show that $\forall x, y \in V(G), \exists xz$ path. We can split into two cases.

- Suppose $x \nleftrightarrow y$ in G. Then, in $\overline{G}$, $x \leftrightarrow y$ by the definition of a graph complement.
- Suppose $x \leftrightarrow y$ in G. Then, since G is disconnected, we know that there must be some $z \in V(G)$ such that there is no xz path. Since there is no xz path, then there is no yz path. In particular, this means $x \nleftrightarrow z$ and $y \nleftrightarrow z$ in G. Therefore, in $\overline{G}$, we have that $x \leftrightarrow z$ and $y \leftrightarrow z$, meaning there is a path between x and y.

Group

1.1.13

Let G be the graph whose vertex set is the set of k-tuples with coordinates $\{0, 1\}$, with x adjacent to y if x and y differ by exactly one position. Determine whether G is bipartite.

G is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

1.1.26

Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least $2k$ vertices. Determine all such graphs with $2k$ vertices.

Suppose G is a graph with girth 4 with every vertex of degree k. Let $v_i \in V(G)$. Then, there must be k vertices which v_i is adjacent to. However, none of these vertices can be adjacent to themselves or G would have girth 3. Thus, we can form a bipartition such that v_i is in a set of at least k vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to k vertices in a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least $2k$ vertices.

The graphs with exactly $2k$ vertices are the $K_{n,n}$ complete bipartite graphs.

1.1.27

Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with $k^2 + 1$ vertices.

Let G be a simple graph with girth 5. Suppose that every vertex of G has degree k. Let $u \in V(G)$. Then, u has k adjacent vertices, each of which is not adjacent to each other (or else the girth of G would be 3). Let this set be N. The elements of N cannot have any other common neighbors aside from u, or else the girth of G would be 4, meaning each has $k - 1$ distinct neighbors. Therefore, the total number of vertices in our graph includes u, the elements of N that are the k distinct neighbors of u, and the $k(k - 1)$ distinct vertices for each vertex in N. Therefore, our total is $1 + k + k(k - 1) = k^2 + 1$.

If there were any vertex with degree greater than k, then there would be additional vertices beyond the $k^2 + 1$ vertices necessary for a k-regular graph.

For $k = 2$, we have the graph C_5 for an example of a graph with $k^2 + 1$ vertices, and for $k = 3$ we have the Petersen graph.

1.1.30

Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry of A^2 and MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

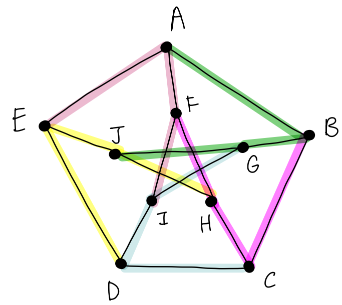
Let A be the adjacency matrix for a simple graph G. In A, every vertex's corresponding row and column are identical, meaning that the entry $A^2_{i,i}$ will be equal to $r_i c_i$ for row i and column i corresponding to v_i. Thus, $r_i c_i$ is equal to $|c_i|^2$, which is equal to the sum of the elements of c_i, which is equal to the degree of v_i.

Let M be the incidence matrix for a simple graph G. In MM^T, the diagonal element $MM^T_{i,i}$ will be equal to $r_i r_i^T$, where r_i represents the edge incidence row of v_i. This is equal to $\left|r_i^T\right|^2$, which is equal to the sum of the elements of r_i, which is equal to the number of edges incident on v_i, which is equal to the degree of v_i.

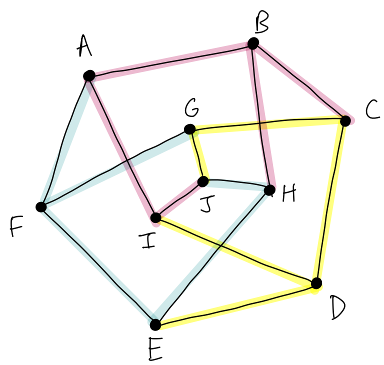
The entry in position (i, j) in both A^2 and MM^T shows whether vertices v_i and v_j are adjacent to each other.

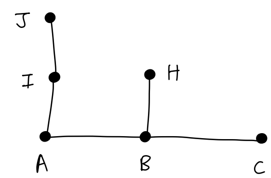
1.1.34

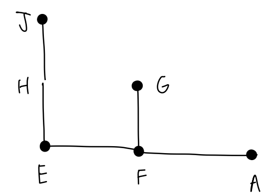
Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4.

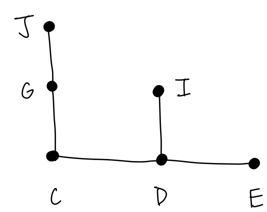


A – B – G – J
B – C – H – F
C – D – I – G
D – E – J – H
E – A – F – I









1.2

Individual

1.2.1

Determine whether the following statements are true or false:

- Every disconnected graph has an isolated vertex.
- A graph is connected if and only if some vertex is connected to all other vertices.
- The edge set of every closed trail can be partitioned into edge sets of cycles.
- If a maximal trail in a graph is not closed, then its endpoints have odd degree.

- False; we can imagine a graph with two components, each of which consists of K_3, where there are no isolated vertices.
- True; since $\forall u, v \in G, \exists u, v$ path by the definition of a connected graph, this means any vertex must have a path to any other vertex.
- True; every closed trail contains within it a cycle — we can delete the edge set of this cycle, and find cycles within remaining components until we reach isolated vertices.
- True; if there were a maximal trail with an endpoint of even degree, then we would be able to extend the trail further by re-entering the endpoint vertex.

1.2.5

Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Explain why this allows us to conclude that no graph has a cut-vertex of degree 1.

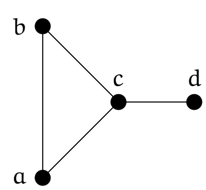
Suppose that $G - v$ is connected. Then, since G is connected, and $v \in V(G)$, it must be the case that v is connected to every component of $G - v$, meaning that it has a neighbor in every component of $G - v$ as $G - v$ is connected.

Now suppose that $G - v$ is disconnected, meaning that it has more than one component after removing v. Before, v must have been connected to every vertex in G as G was a simple connected graph, and afterwards $G - v$ is no longer connected, meaning that v is a cut-vertex. This means v must have been adjacent to a vertex in each component of $G - v$, as removing the incident edges on v along with v increased the number of components from the original 1 that was in G.

From this result, we can conclude that no cut-vertex has degree 1 as removing a vertex of degree 1 and its incident edges does not increase the number of components in G, since there is only one edge incident on a vertex of degree 1.

1.2.6

In the graph below, find all the maximal paths, maximal cliques, and maximal independent sets. Also, find all the maximum paths, cliques, and independent sets.



- The maximal paths are as follows:
  - d, c, b, a
  - d, c, a, b
  - a, b, c, d
  - b, a, c, d
  - b, c, a
  - c, b, a
  - a, c, b
- The maximal cliques are K_3 consisting of a, b, c and K_2 consisting of c, d.
- The maximal independent sets are $\{a, d\}$ and $\{b, d\}$.
- The maximum path is any of those paths listed above with length 4.
- The maximum clique is K_3.
- The maximum independent sets are those listed above with size 2.

1.2.8

Determine the values of m and n such that $K_{m,n}$ is Eulerian.

$$m, n \in 2\mathbb{Z}^+$$

1.2.10

Prove or disprove:

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

(a)

Let G be an Eulerian bipartite graph. Since G is Eulerian, it must contain an Eulerian cycle, meaning that as seen above, there are an even number of vertices, meaning that there are an even number of edges in G.

(b)

Let G be an Eulerian simple graph with an even number of vertices. Since G is Eulerian, this means there must be an Eulerian circuit C that traverses every edge exactly once in G. Every vertex in G must have even degree (or else we would require a backtrack in our Eulerian cycle, which is not a circuit); a simple pairing of the vertices would yield that we have $\lfloor n/2 \rfloor$ edges, and to complete the cycle we need $2(n/2) + 2k$ edges for n vertices and some integer k. Therefore, there must be an even number of edges.

Group

1.2.20

Let v be a cut-vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

Let $x, y, v \in V(\overline{G})$, where v is a cut-vertex of G.

Suppose x and y belong to distinct components of $G - v$. Then, $xy \notin E(G)$, meaning that $xy \in E(\overline{G})$, meaning there is an x, y path in $\overline{G}$, so there is an x, y path in $\overline{G} - v$.

Suppose x and y are in the same component of $G - v$. Since v is a cut-vertex, this means there must be at least two components in $G - v$. Let H_1 be the component that x, y are in, while $\exists w \in H_2$ is a vertex in H_2 disjoint from H_1. Since H_1 and H_2 are disjoint, this means the components do not contain any edges between them, so $x \not\leftrightarrow w$ and $y \not\leftrightarrow w$ in $G - v$ — however, this means that $x \leftrightarrow w$ and $y \leftrightarrow w$ in $\overline{G}$, meaning that $\exists x, y$ path in $\overline{G} - v$.

1.2.22

Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Let G be a graph where there exists a partition of its vertices into two non-empty sets such that there is no edge with endpoints in both sets. Call these sets A and B. By our assumptions, $\forall u \in A$ and $\forall v \in B$, $\nexists e$ such that $e = uv$. Therefore, we cannot create a path between any $u \in A$ and any $v \in B$ as there is no edge to connect any element in A and any element in B. Therefore, G is disconnected.

Suppose G is a disconnected graph. Then, G contains more than one component — we can create a partition of $V(G)$ by letting $H_1, H_2, \dots, H_k$ refer to the k components of G. Each of these components is necessarily disjoint from every other component. By taking $H = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ as one set and H_k as our other set, we know that $H_1, \dots, H_k$ are all disjoint, meaning that H and H_k are disjoint, meaning that there is no edge connecting any vertex H with any vertex in H_k, meaning we have created a partition of G such that there exists no edge between any vertex in one set and any vertex in the other set.

1.2.26

Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

Suppose G is bipartite. Then, there exists a partition of the vertices $V = X \sqcup Y$ such that X and Y are independent sets. Let H be a subgraph of G, and let $H_X = X \cap V(H)$ and $H_Y = Y \cap H$. Because H is a subgraph of G, each vertex of H must be an element of either H_X or H_Y, or that $V(H) = H_X \sqcup H_Y$. WLOG, let $|H_X| > |H_Y|$. Since $H_X \subseteq X$ and X is an independent set, H_X is an independent subset consisting of at least half of $V(H)$.

Suppose every subgraph of G has an independent set consisting of at least half of $V(H)$. We will suppose toward contradiction that G is not bipartite. Then, G must contain an odd cycle, H_1. However, an independent set of H_1 consists of at most $\left\lfloor \frac{|V(H_1)|}{2} \right\rfloor < \frac{|V(H_1)|}{2}$, otherwise two vertices would be adjacent. Because H_1 is an independent set with less than half of the elements of $V(H)$, we have reached a contradiction. Therefore, G must be bipartite.

1.2.38

Prove that every n-vertex graph with at least n edges contains a cycle.

We proceed via induction as follows:

For the base case where $|V(G)| = 1$, we know that there is a cycle with one edge that connects back on the vertex.

For the case where $|V(G)| > 1$, if $v \in V(G)$ has degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges, so by our inductive hypothesis, we know that $G - v$ contains a cycle. Meanwhile, if $\forall v \in V(G), d(v) \geq 2$, we know by Lemma 1.2.25 that G contains a cycle.

1.3

Individual

1.3.3

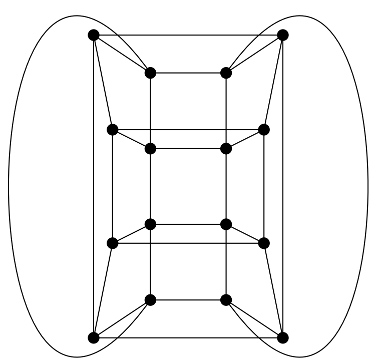
Let u and v be adjacent vertices in G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

Let $u \leftrightarrow v \in G$. In order for uv to be in a triangle in G, u and v must share a common neighbor. By using the principle of inclusion and exclusion, we can find the set as follows:

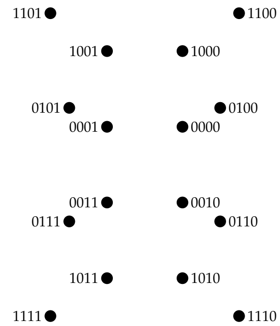
$$\begin{aligned} |N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &\geq d(u) + d(v) - n(G) \end{aligned}$$

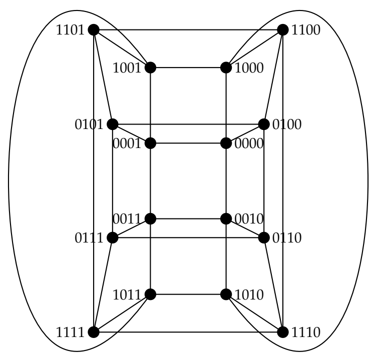
1.3.4

Prove that the graph below is isomorphic to Q_4.



We can assign tuples to the graph as follows:





1.3.6

Given graphs G and H, determine the number of components and maximum degree in $G + H$ in terms of the parameters for G and H.

We can find the number of components in $G + H$ by summing the number of components in G and the number of components in H.

The maximum degree in $G + H$ is equal to $\max\{\Delta(G), \Delta(H)\}$.

1.3.7

Determine the maximum number of edges in a bipartite subgraph of P_n, C_n, and K_n.

For the graph P_n, we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets A and B. This is a bipartition since a path does not include any repeated vertices or edges, so A and B are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of P_n is the number of edges in P_n, which is $n-1$.

For C_n, we have two values of the maximum number of edges in a bipartite subgraph of C_n:

- If n is even, then C_n is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of C_n is the number of edges in C_n, which is n.
- If n is odd, then C_n is not a bipartite graph. After one edge deletion, we get that $C_n - e = P_n$, which is bipartite, so the maximum number of edges in a bipartite subgraph of C_n is the number of edges in P_n, which is $n-1$.

For the graph K_n, there are two options for the maximum number of edges in a bipartite subgraph depending on the value of n:

- If n is even, then the subgraph $K_{\frac{n}{2},\frac{n}{2}}$ is the maximal bipartite subgraph, meaning that the number of edges is equal to $n^2/4$. We know that $K_{\frac{n}{2},\frac{n}{2}}$ is a subgraph of K_n because the vertex set is the same, and K_n is complete, so any subset of edges is a subset of the edge set of K_n.
- If n is odd, then the subgraph $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ is the maximal bipartite subgraph, because $\lfloor\frac{n}{2}\rfloor + \lceil\frac{n}{2}\rceil = n$ and K_n is complete. Therefore, the total number of edges is $\left\lfloor\frac{n^2}{4}\right\rfloor$.

1.3.26 (a)

Count the 6-cycles in Q_3.

In order to find a 6-cycle in Q_3, we select two vertices to delete and see if we can find a cycle from the graph $Q_3 - \{u, v\}$. Vertices are either adjacent, distance 2 (antipodal on the same face), or antipodal (distance 3) with each other.

Adjacent: If two vertices are adjacent, we can find one cycle from the remaining vertices after deletion. There are (8)(3)/2 sets of adjacent vertices, for a total of 12 from this selection.

Antipodal on the same face: If two vertices are antipodal, we cannot find a cycle from the remaining graph after deletion.

Antipodal: If two vertices are antipodal, we can find one cycle from the remaining vertices after deletion. There are 4 sets of antipodal vertices.

We find a total of 16 6-cycles in Q_3.

Group

1.3.17

Let G be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

(a)

Assume toward contradiction that deleting a vertex of degree $\Delta(G)$ increases the average degree.

$$\begin{aligned}
d'_{avg} &> d_{avg}\\
\frac{2e(G)-2\Delta(G)}{n(G)-1} &> \frac{2e(G)}{n(G)}\\
\frac{2e(G)-2\Delta(G)}{2e(G)} &> \frac{n(G)-1}{n(G)}\\
1-\frac{\Delta(G)}{e(G)} &> 1-\frac{1}{n(G)}\\
\frac{1}{n(G)}-\frac{\Delta(G)}{e(G)} &> 0\\
\frac{1}{n(G)}-\frac{2\Delta(G)}{n(G)d_{avg}} &> 0\\
\frac{d_{avg}-2\Delta(G)}{n(G)} &> 0\\
d_{avg}-2\Delta(G) &> 0\\
d_{avg} &> 2\Delta(G)
\end{aligned}$$

However, we have reached a contradiction — by definition, $\Delta(G) \geq d_{avg}$, meaning that $d_{avg} \not> \Delta(G)$, let alone $2\Delta(G)$.

(b)

Deleting a vertex of the graph $K_{1,1}$ yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in $K_{1,1}$.

1.3.20

Count the cycles n-cycles in K_n and the $2n$-cycles in $K_{n,n}$.

To count the cycles in K_n, we start at a vertex v and choose an edge out of $n-1$ options. After choosing the edge, there are $n-2$ options remaining that do not backtrack to v, and so on and so forth. Therefore, there are $(n-1)!$ cycles in K_n.

Let $v \in A$, where A and B are the order n sets that partition G. Then, there are n possible vertices in B which can be the second element in our cycle — afterwards, there are $n-1$ elements in A, and after that there are $n-1$ elements in B, and so on and so forth. Therefore, there are $n!(n-1)!$ possible options for cycles in $K_{n,n}$.

1.3.25

Prove that every cycle of length $2r$ in a hypercube is contained within a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r.

Let C be a cycle of length $2r$ in Q_n. Then, C contains $2r$ n-tuples. For every tuple in C, there exists a “switched” tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since C is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are $2r$ switches (corresponding to the $2r$ edges in C), this means there are at most r coordinates that are switched, then switched back sometime along the cycle's path. This means the other $n-r$ coordinates are fixed, implying that $C \subseteq Q_r$, the r-dimensional subcube of Q_k.

There is a cycle of length 8 in Q_3, defined as follows: $000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000$.

1.3.31

Using complete graphs and counting arguments, prove the following:

(a) $\binom{n}{2} = \binom{n}{k} + k(n-k) + \binom{n-k}{2}$ for $0 \leq k \leq n$.

(b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \leq \binom{n}{2}$.

(a)

We can consider a decomposition of the edges of K_n into the edge set of K_k and K_{n-k}, and some connector edges.

The edge set of K_n has cardinality $\binom{n}{2}$, the edge set of K_k has cardinality $\binom{k}{2}$, and the edge set of K_{n-k} has cardinality $\binom{n-k}{2}$. In order for this set of edges to be a full decomposition, we need a graph that connects all the vertices in K_k with all the vertices in K_{n-k}, which takes $k(n-k)$ edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph G, where $|V(G)| = n$ with maximal clique components $H_1, \dots, H_k$. Each of these components has $e(H_i) = \binom{V(H_i)}{2}$, with total $\sum_{i=1}^{k} \binom{V(H_i)}{2}$. In comparison, if we consider $e(K_G)$, where K_G is defined as the complete graph on the vertices of G, then that value is $\binom{n}{2}$, and $n = \sum_{i=1}^{k} |V(H_i)|$. Therefore, the size of the edge set of G is less than or equal to the sum of the sizes of the edge sets of maximal clique components H_i (because the maximal clique components of G could just be G itself).

1.3.41

Prove or disprove: if G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

Let $u, v \in V(G)$ and let $d(u) = \left\lceil \frac{n}{2} \right\rceil$. Then, u is adjacent to $\left\lceil \frac{n}{2} \right\rceil$ vertices and nonadjacent to $\left\lfloor \frac{n}{2} \right\rfloor$ vertices. Let $u \not\leftrightarrow v$.

We want to show that there exists some other vertex such that there exists a u, v path through that vertex. We know that $|N(u)| = d(u) = \left\lceil \frac{n}{2} \right\rceil$ and $|N(v)| = d(v) \geq \delta(G) = \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Since $u \not\leftrightarrow v$, $N(u), N(v) \subseteq V(G) - \{u, v\}$. So, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq \left(\left\lceil \frac{n}{2} \right\rceil\right) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) - (n-2) = 1$.

Therefore, there must be at least one element in $N(u) \cap N(v)$, meaning G is connected.

2.1

Individual

2.1.2

Let G be a graph:

(a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.

(b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

(a)

($\Rightarrow$) Let G be a tree. Thus, G is connected (by definition), and acyclic. Since G is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

($\Leftarrow$) Let G be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in G, so G is a connected acyclic graph, or a tree.

(b)

($\Rightarrow$) Let G be a tree, and let e be an edge such that $e \notin E(G)$, and $e = uv$. Then, we create a cycle from the path $uTv + e$ — since there is only one path uTv, this means that $uTv + e$ is a unique cycle.

($\Leftarrow$) Suppose toward contradiction that adding e to the tree G yielded more than one cycle in the graph $G + e$. Then, the graph $G = G + e - e$ would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that G was a tree, we have reached a contradiction, meaning that e added exactly one cycle to the tree G.

2.1.6

Let T be a tree with average degree a. In terms of a, find $n(T)$.

$$\begin{aligned} d_{avg} &= \frac{2e(T)}{n(T)} \\ a &= \frac{2(n(T)-1)}{n(T)} \\ an &= 2n - 2 \\ (a-2)n &= -2 \\ n &= \boxed{\frac{2}{2-a}} \end{aligned}$$

2.1.7

Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

Base Case If $m = 0$, then since this graph has zero edges, it has zero cycles, and since $0 \geq 1 - n$, we have proven the base case.

Inductive Hypothesis For an n-vertex graph with $0 \leq k \leq m$ vertices, then G has at least $k - n + 1$ cycles.

Proof If e is an edge within a cycle of G, then $G - e$ has $k - 1$ edges, and has seen a reduction of 1 cycle, so $G - e$ has at least $(k-1) - n + 1 = (k - n + 1) - 1$ cycles. If e is not within a cycle, then G has seen no reduction in cycles, but $G - e$ is predicted to have at least $(k - n + 1) - 1$ cycles, which it does by our assumption. Therefore, we have proven the inductive hypothesis for both cases.

2.1.12

Compute the diameter and radius of $K_{m,n}$.

The diameter of $K_{m,n}$ is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of $K_{m,n}$ is also 2 — the eccentricity of every vertex in $K_{m,n}$ is 2, so the radius must also be 2.

2.1.13

Prove that every graph with diameter d has an independent set with at least $\left\lceil \frac{1+d}{2} \right\rceil$ vertices.

Let G be a graph with diameter d, and let $u \in V(G)$ be a vertex with eccentricity d. Let P be a maximal u, v path of length d. Then, P has $d + 1$ vertices. So, P has a maximal independent set containing every other vertex, with total cardinality of $\left\lceil \frac{d+1}{2} \right\rceil$. Therefore, G has an independent set with at least $\left\lceil \frac{d+1}{2} \right\rceil$ vertices.

Group

2.1.22

Let T be an n-vertex tree with one vertex of each degree $2 \leq i \leq k$; the remaining $n - k + 1$ vertices are leaves. Determine n in terms of k.

We will find the number of vertices in T by finding the number of edges in T and adding 1. For $2 \leq i \leq k$ corresponding to each of the non-leaf vertices, summation yields $\frac{k(k+1)}{2} - 1$ edges. However, this scheme double-counts each edge, so we have to subtract the $k - 2$ edges connecting the $k - 1$ non-leaf vertices, yielding $\frac{k(k+1)}{2} - k + 1$ edges. Finally, because T is a tree, we get that T has $\frac{k(k+1)}{2} - k + 2$ vertices.

2.1.27

Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \dots, d_n$ if and only if $\sum d_i = 2n - 2$.

($\Rightarrow$) Suppose that for some tree T, $d_1, \dots, d_n$ are the degrees of the vertices of the tree. Since T is a tree, this means $e(G) = n - 1$, and $\sum d_i = 2e(G)$, meaning $\sum d_i = 2(n - 1) = 2n - 2$.

($\Leftarrow$) Suppose that $\sum d_i = 2n - 2$ for $d_1, \dots, d_n$ corresponding to the degrees of the vertices in G. By a previous result, we know that $\sum d_i = 2e(G)$, meaning that $\sum d_i = 2(n - 1)$, implying that $e(G) = n - 1$. We can find a tree G with $n - 1$ edges by letting G be connected with $n - 1$ edges.

2.1.29

Every tree is bipartite. Prove that every tree has a leaf in its larger partite set (or in both sets if the partite sets have equal size).

Let $T = X \sqcup Y$, and suppose without loss of generality that $|X| \geq |Y|$, and suppose that X has no vertices of degree 1 within it. Then, every vertex in X has degree at least 2, meaning that the total number of edges in T is at least $2n(X)$. However, since $n(X) \geq \frac{n(T)}{2}$, this means the number of edges in T is at least $n(T)$, which would contradict our assumption that T is a tree.

2.1.33

Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

($\Rightarrow$) Let G be a connected n-vertex graph with exactly one cycle. If we delete an edge from this cycle, then $G - e$ is acyclic, as well as connected (since e is not a cut-edge), so $G - e$ has $n - 1$ edges. Adding back e, we get that G has n edges.

($\Leftarrow$) Let G be a connected n-vertex graph with n edges. Then, G contains a spanning tree that contains all n vertices. Therefore, $T \subseteq G$ contains $n - 1$ edges. By adding another edge, we get that $e(G) = e(T) + 1 = n - 1 + 1$. Thus, G has exactly one cycle.

2.1.34

Let T be a tree with k edges, and let G be a n-vertex simple graph with more than $n(k-1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n > k$.

We will use induction to prove that $T \subseteq G$ as follows:

Base Case Suppose $n = k + 1$. Then, we can find the following:

$$\begin{aligned}
e(G) &> (k+1)(k-1) - \binom{k}{2} \\
e(G) &> (k^2 - 1) - \frac{k(k-1)}{2} \\
e(G) &> \frac{k^2 - 1}{2} + \frac{k^2 - 1 - (k^2 - k)}{2} \\
e(G) &> \frac{k^2 + k}{2} - 1 \\
e(G) &> \frac{k(k+1)}{2} - 1
\end{aligned}$$

This means $e(G) = \frac{k(k+1)}{2}$ in the base case, meaning G is the complete graph on $k + 1$ vertices, where $\delta(G) = k$. By Theorem 2.1.8, we know that $T \subseteq G$.

Inductive Hypothesis If $n > k + 1$, $e(G) > n(k-1) - \binom{k}{2}$, then either $\delta(G) \geq k$ or, if $\delta(G) < k$, then $e(G - x) > (n-1)(k-1) - \binom{k}{2}$ for $\delta(G) = d(x)$.

Proof If $\delta(G) \geq k$, then we know by Theorem 2.1.8 that $T \subseteq G$. Otherwise, suppose $\delta(G) < k$, and let $d(x) = \delta(G)$. Let $G' = G - x$.

$$\begin{aligned}
e(G') &= e(G) - \delta(G) \\
e(G') &\geq e(G) - (k-1) \\
e(G') &> n(k-1) - \binom{k}{2} - (k-1) \\
e(G') &> (n-1)(k-1) - \binom{k}{2}
\end{aligned}$$

Therefore, the inductive hypothesis is proven.

2.1.35

Let T be a tree. Prove that the vertices of T all have odd degree if and only if for all $e \in E(T)$, both components of $T - e$ are of odd order.

($\Rightarrow$) Let T be a tree. We will suppose toward contradiction with two cases:

- Case 1 Suppose $T - e$ has exactly one component of odd order. Then, T has odd order, meaning that by a previous result, we know that there must exist at least one vertex of even degree in T, otherwise $\sum d(v)$ would be odd.
- Case 2 Suppose $T - e$ has two components of even order. Let X and Y be the two components of $T - e$.
  - Subcase 2.1 Suppose every vertex in X is of odd degree. Then, $T = T - e + e$ would increase the degree of a vertex in X by 1, making that particular vertex have even degree.
  - Subcase 2.2 Suppose there is exactly one vertex in X of even degree. This would mean X has an odd number of vertices of odd degree, which we have previously shown is not possible.
  - Subcase 2.3 Suppose there is more than one vertex in X of even degree. Then, T would contain at least one vertex of even degree (as reconnecting the edge would only increase the degree of one vertex).