Problem 1

Find $\sup(A)$ and $\inf(A)$ where

(a) $A := \left\{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$

(b) $A := \left\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbb{N}\right\}$

(c) $A := \left\{\frac{m}{n} \mid m, n \in \mathbb{N},\ m + n \leq 10\right\}$

(a)

$\sup(A) = 2$: For any $t \in A,\ t < 2$, we can find a_t as follows:

$$a_t := \begin{cases} 1, \ t < 1 \\ \frac{4}{3}, \ 1 \leq t < \frac{4}{3} \\ 2, \ t = \frac{4}{3} \end{cases}$$

$\inf(A) = \frac{1}{2}$: For any $t \in A$, $t > \frac{1}{2}$, we can find a_t as follows:

$$a_t := \begin{cases} 1, \ t > 1 \\ \frac{3}{4}, \ \frac{3}{4} < t \leq 1 \\ \frac{1}{2}, \ t < \frac{3}{4} \end{cases}$$

(b)

$\sup(A) = 1$: For any $t \in A$, $t < 1$, we can find $a_t > t$ as follows:

(1) Take $|t| \geq t$.

(2) If $|t| < \frac{1}{2}$, find m such that $\frac{1}{m} < |t|$ (which exists by the Archimedean Property corollary). Set $a_t = 1 - \frac{1}{m}$.

(3) If $|t| > \frac{1}{2}$, then find m such that $\frac{1}{m} < 1 - |t|$, and set $a_t = 1 - \frac{1}{m}$.

In all three cases, $a_t > t$, meaning $\sup(A) = 1$

$\inf(A) = -1$

(c)

Since A is finite, $\sup(A) = \max(A) = 9$ and $\inf(A) = \min(A) = \frac{1}{9}$

Problem 2

Suppose $u = \sup(A)$ such that $u \notin A$. Show that there is a strictly increasing sequence

$$t_1 < t_2 < t_3 < \dots$$

With $t_n \in A$ and $t_n + \frac{1}{n} > u$ for all $n \geq 1$

Let $t_n = u - \frac{1}{2n}$. t_n must be a strictly increasing sequence because $t_{n+1} = u - \frac{1}{2n+2} > u - \frac{1}{2n} = t_n$.

Additionally, $t_n + \frac{1}{n} = u - \frac{1}{n} < u$, meaning $t_n \in A$.

Problem 3

If m is a lower bound for $A \subseteq \mathbb{R}$, show that the following are equivalent:

(i) $m = \inf(A)$

(ii) $\forall t > m,\ \exists a_t \in A \ni a_t < t$

(iii) $\forall \varepsilon > 0,\ \exists a_\varepsilon \ni m + \varepsilon > a_\varepsilon$

(i) $\Rightarrow$ (ii) Let $m = \inf(A)$. Given $t > m$, if no such a with $t > a$ exists, then $t \leq a\ \forall a \in A$. However, $t > m$ and $m = \inf(A)$. $\perp$

(ii) $\Rightarrow$ (iii) Set $t = m + \varepsilon > m$. Then, by (ii), $\exists a_t \in A,\ a_t < t$. Set $a_\varepsilon = a_t$.

(iii) $\Rightarrow$ (i) Let l be a lower bound for A, and suppose $l > m$. Then, set $\varepsilon = l - m > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u + \varepsilon > a_\varepsilon$. So, $u + (l - m) > a_\varepsilon$, so $l > a_\varepsilon$, so $l \neq \inf(A)$.

Problem 4

Let $A, B \in \mathbb{R}$ be bounded subsets.

(a) Show that

$$\sup(A+B) = \sup(A) + \sup(B)$$
$$\inf(A+B) = \inf(A) + \inf(B)$$

(b) If $t > 0$, show that

$$\sup(tA) = t\sup(A)$$
$$\inf(tA) = t\inf(A)$$

(a)

Let $a = \sup(A)$ and $b = \sup(B)$, and $x_a \in A$ and $x_b \in B$. Then

$$\begin{aligned} a &\geq x_a \\ a + x_b &\geq x_a + x_b && \text{by the ordering of } \mathbb{R} \\ a + b &\geq a + x_b && \text{by the definition of } \sup(B) \\ a + b &\geq x_a + x_b && \text{by the ordering of } \mathbb{R} \\ \sup(A) + \sup(B) &= \sup(A+B) \end{aligned}$$

Let $a' = \inf(A)$ and $b' = \inf(B)$, with x_a and x_b defined as above. Then

$$\begin{aligned} a' &\leq x_a \\ a' + x_b &\leq x_a + x_b && \text{by the ordering of } \mathbb{R} \\ a' + b' &\leq a' + x_b' && \text{by the definition of } \inf(B) \\ a' + b' &\leq x_a + x_b && \text{by the ordering of } \mathbb{R} \\ \inf(A) + \inf(B) &= \inf(A+B) \end{aligned}$$

(b)

Let $a = \sup(A)$, $x_a \in A$, and $t > 0$. Then

$$\begin{aligned} a &\geq x_a \\ ta &\geq tx_a && \text{by the ordering of } \mathbb{R} \\ t\sup(A) &= \sup(tA) \end{aligned}$$

Let $a' = \inf(A)$, with x_a and t defined as above.

$$\begin{aligned} a' &\leq x_a \\ ta' &\leq tx_a && \text{by the ordering of } \mathbb{R} \\ t\inf(A) &= \inf(tA) \end{aligned}$$

Problem 5

Let $I = (0,1)$ denote the open unit interval and consider $F : I \times I \to \mathbb{R}$, $F(x,y) = 2x + y$.

Compute

$$\sup_{y\in I}\left(\inf_{x\in I} F(x,y)\right)$$

and

$$\inf_{x\in I}\left(\sup_{y\in I} F(x,y)\right)$$

We start by finding $\inf_{x\in I} F(x,y)$, which is equal to $F(x,y) = y$ (as the infimum is the greatest lower bound on $2x$, which is $2(0) = 0$). So, $\sup_{y\in I} y = 1$.

We start by finding $\sup_{y\in I} F(x,y)$, which is $\sup_{y\in I} 2x + y$, which is $2x + 1$, as $\sup I = 1$. So, by similar reasoning, $\inf_{x\in I} 2x + 1 = 1$.

These values are the same.

Problem 6

Let D be a nomempty set and consider the vector space

$$\ell_\infty(D) := \{f \mid f : D \to \mathbb{R} \text{ is bounded}\}$$

with point-wise addition and scalar multiplication. Show that

$$\|f\|_u := \sup_{x\in D} |f(x)|$$

defines a norm on $\ell_\infty(D)$.

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(1) Because $\forall x \in \mathbb{R},\ |x| \geq 0,\ \|\cdot\|_u \geq 0$.

(2) $\|f + g\|_u = \sup_{x\in D} |f(x) + g(x)| \leq \sup_{x\in D} |f(x)| + \sup_{x\in D} |g(x)|$ (by the Triangle Inequality) $= \|f\|_u + \|g\|_u$.

(3) $\|\mathbf{0}\| = \sup_{x\in D} |\mathbf{0}| = 0$.

(4) Let $\|f\|_u = 0$. Then, $\sup_{x\in D} |f(x)| = 0$, meaning that $\nexists x' \in D$ such that $f(x') \neq 0$ (or else $\sup_{x\in D} |f(x)| = f(x')$), so $f(x) = \mathbf{0}$.

(5) $\|tf\|_u = \sup_{x\in D} |tf(x)| = |t| \sup_{x\in D} |f(x)| = |t|\|f\|_u$.

Therefore, $\|\cdot\|_u$ is a norm on ℓ_∞.

Problem 7

Let $f, g : D \to \mathbb{R}$ be bounded functions. Show that

(a) $\sup_{x\in D}(f+g)(x) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$

(b) $\inf_{x\in D}(f+g)(x) \geq \inf_{x\in D} f(x) + \inf_{x\in D} g(x)$

(c) $|\sup_{x\in D} f(x) - \sup_{x\in D} g(x)| \leq \sup_{x\in D} |f(x) - g(x)|$

(a)

Let $h = f + g$, and x be such that $h(x) = \sup_{x'\in D}(h(x'))$. Let $z = \sup_{x'\in D}(f(x'))$ and $y = \sup_{x'\in D}(g(x'))$. Suppose toward contradiction that $f(z) + g(y) < h(x)$.

Then, $0 < (f(x)-f(z))+(g(x)-g(y))$. However, since $f(z) \geq f(x)$ and $g(y) \geq g(x)$, by the definition of the supremum, $0 \geq f(x) - f(z)$ and $0 \geq g(x) - g(y)$, so we have reached a contradiction. $\perp$

(b)

Let $h = f + g$, and x be such that $h(x) = \inf_{x'\in D}(h(x'))$. Let $z = \inf_{x'\in D}(f(x'))$ and $y = \inf_{x'\in D}(g(x'))$. Suppose toward contradiction that $f(z) + g(y) > h(x)$.

By our assumption, $0 > (f(x) - f(z)) + (g(x) - g(z))$. However, by the definition of inf, it must be the case that $f(z) \leq f(x)$ for all $x \in D$, so $f(z) - f(x) \geq 0$, and similarly, $g(x) - g(z) \geq 0$. Therefore, $0 > (f(x) - f(z)) + (g(x) - g(z)) \geq 0$. $\perp$

(c)

Let $a = \sup(f)$, $b = \sup(g)$. As $|f(x) - g(x)| = |g(x) - f(x)|$, we can say WLOG that $a \geq b$ (as otherwise we would change $\sup|f(x) - g(x)|$ to $\sup|g(x) - f(x)|$).

Suppose toward contradiction that $a - b > \sup|f - g|$. Let $\varepsilon_f = \frac{(a-b)-\sup|f-g|}{9}$, $f(x_f) > a - \varepsilon_f$, $\varepsilon_g = \frac{b-g(x_f)}{2}$, and $g(x_g) > b - \varepsilon_g$

By our assumption

$$\begin{aligned} f(x_f) - g(x_f) &< f(x_f) - g(x_g) \\ g(x_g) - g(x_f) &< 0 \end{aligned}$$

However, we also have

$$\begin{aligned} g(x_g) - g(x_f) &> b - \frac{b - g(x_f)}{2} - g(x_f) \\ &= \frac{b - g(x_f)}{2} \\ &\geq 0 \end{aligned}$$

$\perp$

Problem 8

Find $\bigcap_{n=1}^{\infty} I_n$ where

(a) $I_n = [0, 1/n]$

(b) $I_n = (0, 1/n)$

(c) $I_n = [n, \infty)$

(a)

For all $k > 1$, $\bigcap_{n=1}^{k} = [0, 1/k]$, meaning that $\bigcap_{n=1}^{\infty} = \lim_{k\to\infty}[0, 1/k] = \{0\}$.

(b)

We will show that $\bigcap_{n=1}^{\infty} = \emptyset$.

Suppose toward contradiction $\exists k \in \bigcap_{n=1}^{\infty}$. Then, $k > 0$, but $\forall n \in \mathbb{N}$, $k < 1/n$. However, by the Archimedean property, $k < 1/n \ \forall n \Rightarrow k = 0$. So $k > 0$ and $k = 0$. $\perp$

(c)

We will show that $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$.

Suppose toward contradiction that $\exists k \in \bigcap_{n=1}^{\infty}$. Then, $k \geq n \forall n$. However, since $\mathbb{N}$ is unbounded, $\nexists$ such a k. $\perp$

Problem 9

If $x > 0$, show that there is an $n \in \mathbb{N}$ with $\frac{1}{2^n} < x$.

If $x > 0$, then by the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < x$. Then, $\frac{1}{2^n} < \frac{1}{n}$, so $\frac{1}{2^n} < x$.

Problem 10

Show that the **Dyadic Rationals** are dense.

$$\mathbb{D} := \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{Z} \right\}$$

Let (a, b) be an open interval with a, b finite. Then, $b - a > 0$. We showed in the previous problem that, thus $\exists n \in \mathbb{Z}$ such that $\frac{1}{2^n} < (b - a)$, so $1 < (2^n)b - (2^n)a$.

By the Archimedean Property, $\exists m \in \mathbb{Z}$ such that $m - 1 \leq (2^n)a < m$, so $a < \frac{m}{2^n}$, and $m \leq (2^n)a + 1 < (2^n)b$, so $\frac{m}{2^n} < b$. Therefore, $\mathbb{D} \cap (a, b) \neq \emptyset$, so $\mathbb{D}$ is dense.