Solution (32.20)**:** We start by taking the recurrence relation

$$\left(1-x^2\right)P'_n = -nxP_n + nP_{n-1}. \tag{$*$}$$

Differentiating, this gives

$$\left(1-x^2\right)P''_n - 2xP'_n = n\left(-P_n - xP'_n + P'_{n-1}\right).$$

We seek to show that

$$-xP'_n + P'_{n-1} = -nP_n.$$

At this point, I ran out of board space to deal with the generating functions and their ensuing mess of partial derivatives.

Solution (32.21)**:** Using $dv = P'_m(x)$, we integrate by parts to get

$$\begin{aligned}
\int_{-1}^{1}\left(1-x^2\right)P'_n(x)P'_m(x)\,dx &= P_m(x)P'_n(x)\left(1-x^2\right)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}\left(\left(1-x^2\right)P'_n(x)\right)P_m(x)\,dx\\
&= -\int_{-1}^{1}\left(\left(1-x^2\right)P''_n(x) - 2xP'_n(x)\right)P_m(x)\,dx\\
&= n(n+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx\\
&= \frac{2n(n+1)}{2n+1}\delta_{mn}.
\end{aligned}$$

Solution (32.23)**:** Upon taking m derivatives of Legendre's equation, and using the Leibniz rule for differentiation, we get

$$\left(1-x^2\right)\frac{d^{m+2}P_\ell}{dx^{m+2}} - 2x(m+1)\frac{d^{m+1}P_\ell}{dx^{m+2}} + ((\ell)(\ell+1) - (m(m-1)+2m))\frac{d^mP_\ell}{dx^m} = 0.$$

Rewriting $u(x) = \frac{d^mP_\ell}{dx^m}$, we obtain

$$0 = \left(1-x^2\right)\frac{d^2u}{dx^2} - 2x(m+1)\frac{du}{dx} + \left(\ell(\ell+1) - m^2 - m\right)u(x).$$

Setting $u(x) = \left(1-x^2\right)^{-m/2}v(x)$, we find

$$\begin{aligned}
\frac{du}{dx} &= \left(1-x^2\right)^{-m/2}\frac{dv}{dx} + mxv(x)\left(1-x^2\right)^{-m/2-1}\\
&= \left(1-x^2\right)^{-m/2}\left(\frac{dv}{dx} + \frac{mxv(x)}{1-x^2}\right)\\
\frac{d^2u}{dx^2} &= -mx\left(1-x^2\right)^{-m/2-1}\left(\frac{dv}{dx} + \frac{mxv(x)}{1-x^2}\right) + \left(1-x^2\right)^{-m/2}\left(\frac{d^2v}{dx^2} + \frac{mv(x)}{1-x^2} + \frac{mx}{1-x^2}\frac{dv}{dx} + \frac{2mx^2v(x)}{\left(1-x^2\right)^2}\right)\\
&= \left(1-x^2\right)^{-m/2}\left(\frac{d^2v}{dx^2} + \frac{2mx}{1-x^2}\frac{dv}{dx} + \frac{mv(x)}{1-x^2} + \frac{2mx^2v(x)}{\left(1-x^2\right)^2}\right).
\end{aligned}$$

Substituting, we have the equation

$$\begin{aligned}
0 = {}&\left(1-x^2\right)\left(1-x^2\right)^{-m/2}\left(\frac{d^2v}{dx^2} + \frac{2mx}{1-x^2}\frac{dv}{dx} + \frac{mv(x)}{1-x^2} + \frac{2mx^2v(x)}{\left(1-x^2\right)^2}\right)\\
&- 2x(m+1)\left(\left(1-x^2\right)^{-m/2}\left(\frac{dv}{dx} + \frac{mxv(x)}{1-x^2}\right)\right)\\
&+ \left(\ell(\ell+1) - m^2 - m\right)\left(1-x^2\right)^{-m/2}v(x),
\end{aligned}$$

which after much more tedious algebra, yields

$$0 = \frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + \left((\ell)(\ell+1) - \frac{m^2}{1-x^2}\right)v(x),$$

so v satisfies the differential equation. Thus, we have

$$v(x) = \left(1 - x^2\right)^{m/2} \frac{d^m P_\ell}{dx^m}.$$

Solution (35.4)**:** Using the expression

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\gamma) - in\gamma}\,d\gamma \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\gamma)}e^{-in\gamma}\,d\gamma, \end{aligned}$$

we expand the first term in a Taylor series, giving

$$J_n(x) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{i^k x^k}{k!}\int_{-\pi}^{\pi}\sin^k(\gamma)e^{-in\gamma}\,d\gamma.$$

Now, k has to be even (else we have an odd integrand over a symmetric interval).

Solution (35.5)**:** Differentiating,

$$\begin{aligned} \frac{dJ_0}{dx} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial x}\left(e^{ix\sin(\gamma)}\right)d\gamma \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(i\sin(\gamma))e^{ix\sin(\gamma)}\,d\gamma \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} i\left(\frac{1}{2i}\left(e^{i\gamma} - e^{-i\gamma}\right)\right)d\gamma \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}e^{ix\sin(\gamma)+i\gamma} - \frac{1}{2}e^{ix\sin(\gamma)-i\gamma}\,d\gamma \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\cos(x\sin(\gamma)+i\gamma) + i\sin(x\sin(\gamma)+i\gamma) - (\cos(x\sin(\gamma)-i\gamma) + i\sin(x\sin(\gamma)-i\gamma)))\,d\gamma \end{aligned}$$

and with more tedious algebra, we obtain

$$\begin{aligned} &= -\frac{1}{\pi}\int_{0}^{\pi}\cos(x\sin(\gamma)-\gamma)\,d\gamma \\ &= -J_1(x). \end{aligned}$$

Evaluating

$$\frac{d}{dx}(xJ_1) = J_1 + x\frac{dJ_1}{dx},$$

we take

$$\begin{aligned} \frac{d}{dx}(xJ_1) &= \frac{1}{\pi}\int_{0}^{\pi}\cos(x\sin(\gamma)-\gamma) - x\sin(\gamma)\sin(x\sin(\gamma)-\gamma)\,d\gamma \\ &= \frac{1}{\pi}\int_{0}^{\pi}\cos(x\sin(\gamma))\cos(\gamma) + \sin(x\sin(\gamma))\sin(\gamma) - x\sin(\gamma)\sin(x\sin(\gamma)-\gamma)\,d\gamma \\ &= \frac{1}{\pi}\int_{0}^{\pi}\cos(\gamma)\cos(x\sin(\gamma)) + \sin(\gamma)\sin(x\sin(\gamma)) - x\sin(\gamma)(\sin(x\sin(\gamma))\cos(\gamma) - \sin(\gamma)\cos(x\sin(\gamma)))\,d\gamma \\ &= \frac{1}{\pi}\int_{0}^{\pi}x\cos(x\sin(\gamma))\,d\gamma \\ &= xJ_0. \end{aligned}$$

Solution (35.7)**:** Solving

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(x^2 - n^2\right)u(x) = 0,$$

we plug in the expression for $J_n(x)$ to get

$$\begin{aligned}
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(x^2 - n^2\right)u(x) &= x^2\left(\sum_{m=0}^{\infty}\frac{1}{2^{2m+n}}(2m+n-1)(2m+n)\frac{(-1)^m}{m!(m+n!)}x^{2m+n-2}\right)\\
&\quad + x\left(\sum_{m=0}^{\infty}\frac{1}{2^{2m+n}}(2m+n)\frac{(-1)^m}{m!(m+n)!}x^{2m+n-1}\right)\\
&\quad + \sum_{m=0}^{\infty}\frac{1}{2^{2m+n}}\frac{(-1)^m}{m!(m+n)!}x^{2m+n+2}\\
&\quad - \sum_{m=0}^{\infty}\frac{n^2}{2^{2m+n}}\frac{(-1)^m}{m!(m+n)!}x^{2m+n}\\
&= \sum_{m=0}^{\infty}\frac{1}{2^{2m+n}}\frac{(-1)^m}{m!(m+n!)}\left(x^{2m+n}\right)\left((2m+n-1)(2m+n) + 2m + n + x^2 - n^2\right)\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+n}m!(m+n!)}x^{2m+n}\left(x^2 + 4m^2 + 4mn\right)
\end{aligned}$$

From here, I'm not sure how to manipulate this series to get 0 as the final answer.

Solution (35.8)**:**

(a) We have

$$e^{ix\sin(\phi)} = \sum_{n=-\infty}^{\infty} c_n e^{in\phi},$$

where

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\phi)}e^{-in\phi}\,d\phi\\
&= J_n(x).
\end{aligned}$$

(b) Splitting into real and imaginary parts, we have

$$e^{ix\sin(\phi)} = \cos(x\sin(\phi)) + i\sin(x\sin(\phi)),$$

so that

$$\begin{aligned}
e^{ix\sin(\phi)} &= \sum_{n=-\infty}^{\infty} c_n e^{in\phi}\\
&= \sum_{n=-\infty}^{\infty} J_n(x)(\cos(n\phi) + i\sin(n\phi))\\
&= \sum_{n=-\infty}^{\infty} J_n(x)\cos(n\phi) + i\sum_{n=-\infty}^{\infty} J_n(x)\sin(n\phi).
\end{aligned}$$

Equating real and imaginary parts gives the desired result.

(c) We use the angle summation identity to get

$$A\cos(\omega_c t)\cos(\beta\sin(\omega_m t)) - A\sin(\omega_c t)\sin(\beta\sin(\omega_m t)) = A\cos(\omega_c t)\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(n\omega_m t)$$

$$
\begin{aligned}
&- A\sin(\omega_c t)\sum_{n=-\infty}^{\infty} J_n(\beta)\sin(n\omega_m t)\\
&= \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c t + n\omega_m t).
\end{aligned}
$$

Solution (35.10)**:**

$$
\begin{aligned}
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right)\\
P_{3,1}(x) &= \frac{1}{2}\left(1 - x^2\right)^{1/2}\left(15x^2 - 3\right)\\
P_{3,-1}(x) &= -\frac{1}{6}\left(1 - x^2\right)^{1/2}\left(15x^2 - 3\right)\\
P_{3,2}(x) &= 15x\left(1 - x^2\right)\\
P_{3,-2}(x) &= \frac{1}{8}x\left(1 - x^2\right).\\
P_{3,3}(x) &= 15\left(1 - x^2\right)^{3/2}\\
P_{3,-3}(x) &= -\frac{1}{48}\left(1 - x^2\right)^{3/2}.
\end{aligned}
$$

Solution (35.11)**:**

$$
\begin{aligned}
Y_{\ell,m}(\pi - \theta, \phi + \pi) &= (-1)^m\sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(\cos(\pi - \theta))e^{im(\phi+\pi)}\\
&= (-1)^m\sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(-\cos(\theta))e^{im\phi}(-1)^m\\
&= (-1)^m\sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}(-1)^{\ell-m}P_{\ell,m}(\cos(\theta))(-1)^m e^{im\phi}\\
&= (-1)^\ell\left((-1)^m\sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(\cos(\theta))e^{im\phi}\right)\\
&= (-1)^\ell Y_{\ell,m}(\theta, \phi).
\end{aligned}
$$

Solution (35.12)**:** We have

$$
\begin{aligned}
Y_{\ell,0}(\hat{n}) &= (-1)^0\sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{\ell!}{\ell!}}P_{\ell,0}(\cos(\theta))e^{i(0)\phi}\\
&= \sqrt{\frac{2\ell + 1}{4\pi}}P_\ell(\cos(\theta)).
\end{aligned}
$$

Furthermore, since $P_{\ell,m}(1) = \delta_{0,m}$, we have

$$
\begin{aligned}
Y_{\ell,m}(0, \phi) &= (-1)^m\sqrt{\frac{2\ell + 1}{4\pi}}\sqrt{\frac{(\ell - m)!}{(\ell + m)!}}P_{\ell,m}(1)e^{im\phi}\\
&= \sqrt{\frac{2\ell + 1}{4\pi}}\delta_{0,m}.
\end{aligned}
$$

Solution (35.16)**:** Using the addition theorem, where $\hat{a} = \hat{b} = \hat{n}$, we get

$$
\sum_{m=-\ell}^{\ell}\left|Y_{\ell,m}(\hat{n})\right|^2 = \frac{2\ell + 1}{4\pi}P_\ell(1)
$$

$$= \frac{2\ell+1}{4\pi}.$$

Solution (35.17 (c))**:** We have

$$\begin{aligned}\cos(\gamma) &= \frac{4\pi}{3}\sum_{m=-1}^{1}\overline{Y_{1,m}(\theta',\phi')}Y_{1,m}(\theta,\phi)\\ &= \frac{4\pi}{3}\left(\frac{3}{4\pi}\left(\cos(\theta')\cos(\theta)+\sin(\theta)\sin(\theta')\operatorname{Re}\left(e^{i(\phi-\phi')}\right)\right)\right)\\ &= \cos(\theta')\cos(\theta)+\sin(\theta')\sin(\theta)\cos(\phi'-\phi).\end{aligned}$$

Solution (35.21)**:**

(a) Using $L_z - -i\frac{d}{d\phi}$, we have

$$\begin{aligned}L_z(Y_{\ell,m}) &= (-1)^m\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell,m}(\cos(\theta))\left(-i\frac{d}{d\phi}e^{im\phi}\right)\\ &= m\left((-1)^m\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell,m}(\cos(\theta))e^{im\phi}\right)\\ &= mY_{\ell,m}.\end{aligned}$$

Meanwhile, defining

$$C(\ell,m) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}$$

$$\begin{aligned}L^2(Y_{\ell,m}) &= -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y_{\ell,m}}{\partial\theta}\right) - \frac{1}{\sin^2(\theta)}\frac{\partial^2 Y_{\ell,m}}{\partial\phi^2}\\ &= C(\ell,m)e^{im\phi}\left(\left(\cos^2(\theta)\right)\frac{d^2P_{\ell,m}}{dd(\cos(\theta))^2}+2\cos(\theta)\frac{dP_{\ell,m}}{dd(\cos(\theta))}+\frac{m}{1-\cos^2(\theta)}P_{\ell,m}(\cos(\theta))\right),\end{aligned}$$

which by the definition of the associated Legendre functions, gives

$$\begin{aligned}L^2(Y_{\ell,m}) &= C(\ell,m)e^{im\phi}\left(\ell(\ell+1)P_{\ell,m}\right)\\ &= (\ell)(\ell+1)Y_{\ell,m}.\end{aligned}$$

(b) I do not have enough time to calculate this out.

(c) Since the spherical harmonics are not eigenvectors of L_x and L_y, it cannot be the case that L_x and L_y commute, meaning they have a nontrivial commutator — specifically, this commutator involves L_z, as $L^2 = L_x^2 + L_y^2 + L_z^2$ has the spherical harmonics as its eigenvalues.

(d) I do not have enough time to calculate this out.

(e) I do not have enough time to calculate this out.

Solution (35.25)**:** We have

$$\begin{aligned}e^{i(0)(\cos(\gamma))} &= \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)j_\ell(0)P_\ell(\cos(\gamma))\\ &= 1,\end{aligned}$$

meaning we must have 0 at every other point in the expansion, and $j_\ell(0) = 1$ at $\ell = 0$, since $P_0(\cos(\gamma)) = 1$. Thus,

$$j_\ell(0) = \delta_{0,\ell}.$$