Problem (Problem 1)**:** Prove that our cell complex structure for T^2 coincides with a product cell complex structure on $S^1 \times S^1$ for some cell complex structure on S^1.

Solution: We consider the cell complex structure on S^1 given by one 0-cell and one 1-cell with characteristic map $\Phi \colon [0,1] \hookrightarrow S^1$ identifying $0 \sim 1$. If we consider two copies of S^1 in this fashion, this gives rise to a CW complex structure with

- 1 0-cell, which we label as e^0;

- 2 1-cells, which we label as e_1^1 and e_2^1;

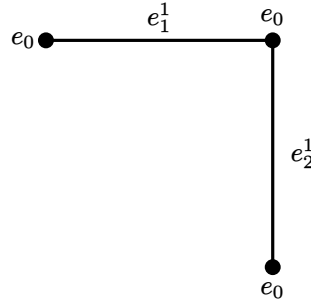- and 1 2-cell, which we label as e^2.

Since the attaching maps (viewed as restrictions of the characteristic maps at the boundary) for both e_1^1 and e_2^1 coincide at e^0, it follows that there are characteristic maps $\Phi_{1,2}|_{\partial e_{1,2}^1} = e^0$

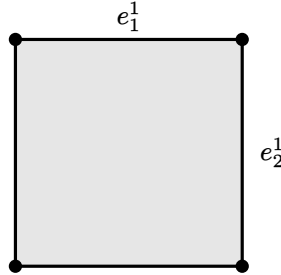$$\Phi_{1,2} \colon e_{1,2}^1 \hookrightarrow S^1 \times S^1$$

such that

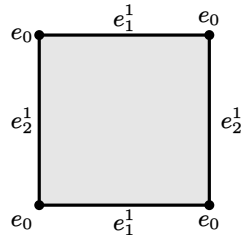$$\Phi_{1,2}|_{\partial e_{1,2}^1}\left(\partial e_{1,2}^1\right) = e^0.$$

Therefore, we may draw the 1-skeleton as the following identification.

We consider the 2-cell e^2 as a square given by $e_1^1 \times e_2^1$. That is, it is given by the following diagram.

We observe that the characteristic map for e^2, $\Psi \colon e^2 \hookrightarrow S^1 \times S^1$, is such that ∂e^2 is identified with the 1-skeleton. Therefore, we obtain the following diagram.

This is precisely the cell complex structure of the torus.

Problem (Problem 2): Prove that if X is a cell complex, then so is the suspension SX.

Solution: We observe that the product $X \times [0, 1]$ is a cell complex as it is a Cartesian product of a cell complex with one 1-cell, $e^1 \cong [0, 1]$, and two 0-cells, which are the endpoints of the interval. Similarly, $X \times \{0\}$ and $X \times \{1\}$ are cell complexes as they are the Cartesian product of a cell complex with the 0-cell, $e^0 \cong \{0\} = \{1\}$.

Since $X \times \{0\}$ is a subcomplex of $X \times [0, 1]$ with the attaching map given by restricting the product to $\{0\}$, it follows that $X \times [0, 1]/(X \times \{0\})$ is a cell complex, so $SX = X \times [0, 1]/(X \times \{0\})/(X \times \{1\})$ is a cell complex.

Problem (Problem 3): Consider the infinite-dimensional sphere $S^\infty = \bigcup_n S^n$, given by attaching two cells of arbitrarily high dimension.

   (a) Describe all the subcomplexes of S^∞ with this cell complex structure.

   (b) Prove that S^∞ is contractible.

Solution:

   (a) The n-skeleton for X is given by

$$X^n = 2e^n \cup 2e^{n-1} \cup \cdots \cup 2e^1 \cup 2e^0,$$

where we let e^i represent the i-cell. Each of the n-skeleta is a subcomplex of S^∞. Additionally, we observe that within the n-skeleton, the attaching maps for each e^n have it such that ∂e^n is identified with X^{n-1}, so there are two sub-skeleta defined by

$$A_1 = e^n \cup 2e^{n-1} \cup \cdots \cup 2e^1 \cup 2e^0$$
$$A_2 = e^n \cup 2e^{n-1} \cup \cdots \cup 2e^1 \cup 2e^0.$$

This holds for each n-skeleton, so these are the subcomplexes of S^∞.

   (b) We view $S^\infty \subseteq \mathbb{R}^\infty$, where the latter denotes finitely supported sequences in $\mathbb{R}$ endowed with the disjoint union topology. Then, for an element $(x_n)_n \in S^\infty$, we have that $(x_n)_n$ is finitely supported. Furthermore, from the definition of the union topology, we have that the homotopy

$$H \colon [0, 1] \times S^\infty \to S^\infty$$
$$(t, (x_n)_n) \mapsto (1 - t)(x_n)_n + t(x_{n-1})_n$$

is necessarily continuous, as in each coordinate it is simply a straight line in $\mathbb{R}^n$. Letting f_t denote the latter function, we observe that

$$f_t = (1 - t)\,\mathrm{id} + tV,$$

where V denotes the left unilateral shift. Now, we will define a homotopy on S^∞, $G \colon [0, 1] \times S^\infty \to S^\infty$ by taking

$$G(t, x) = \begin{cases} H(2t, x) & t \in [0, 1/2] \\ H(2(2t - 1), x) & t \in [1/2, 3/4] \\ H(4(2t - 1), x) & t \in [3/4, 7/8] \\ \vdots \end{cases}.$$

By this arbitrary composition of left unilateral shifts, it follows that every element $x \in S^\infty$ will eventually homotope to 0, so that the identity map on S^∞ is null-homotopic.