Problem 1

Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function with $f(0) = f(1)$. Show that there is a $c \in [0, 1/2]$ with $f(c) = f(c+1/2)$. Conclude that there are always antipodal points on the earth's equator with the same temperature.

Consider $g(x) = f(x) - f(x + 1/2)$ on $[0, 1/2]$. Then, $g(0) = f(0) - f(1/2)$, and $g(1/2) = f(1/2) - f(1)$. Since $f(0) = f(1)$, it must be the case that $g(0) = -g(1/2)$.

Therefore, on $[0, 1/2]$, if $g(0) = k$ for $k \in \mathbb{R}$, then $g(1/2) = -k$, meaning that by the Intermediate Value Theorem, $\exists c \in [0, 1/2]$ with $g(c) = 0$. This is equivalent to $f(c) = f(c + 1/2)$ by the definition of g.

For any two antipodes on the earth's equator, let $t(x)$ be the temperature at point x. Then, moving from x to $-x$, where $-x$ denotes the opposite point on the earth's equator, it must be the case that the values of t at x and $-x$ flip. Therefore, there is a point where $t(c) = t(-c)$.

Problem 2

Suppose $f : [a, b] \to \mathbb{R}$ is injective and continuous. Show that f is strictly monotone.

Let $f : [a, b] \to \mathbb{R}$ be injective and continuous. WLOG, let $p < q \in [a, b]$. Then, since $p \neq q$, $f(p) \neq f(q)$, meaning that $f(p) < f(q)$ and $f(p) > f(q)$.

Since f is continuous, f by the Intermediate Value Theorem, $\forall x \in [f(p), f(q)]$ or $[f(q), f(p)]$, $\exists! x' \in [p, q]$ or $[q, p]$ such that $f(x') = x$. Therefore, $\forall p, q \in [a, b]$, $p < q \Rightarrow f(p) < f(q)$ or $f(p) > f(q)$, so f is strictly monotone.

Problem 3

Suppose $f : [0, 1] \to \mathbb{R}$ is a map that takes on each of its values exactly twice. Show that f cannot be continuous at every point.

Problem 4

Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$ but not on $(0, \infty)$.

Let $f(x) = \frac{1}{x^2}$ defined on $[1, \infty)$. Let $\varepsilon > 0$.

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \left|\frac{x^2 - y^2}{x^2y^2}\right| \\
&= \frac{x + y}{x^2y^2}\,|x - y| \\
&\leq 2\,|x - y| \\
&< \varepsilon
\end{aligned}$$

Set $\delta = \frac{\varepsilon}{2}$.

On $(0, \infty)$, let $u_n = \frac{1}{\sqrt{n+1}}$ and $v_n = \frac{1}{\sqrt{n}}$. Then,

$$\begin{aligned}
|f(u_n) - f(v_n)| &= |n + 1 - n| \\
&= 1 \\
&= \varepsilon_0 \\
|u_n - v_n| &= \left|\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right| \\
&= \left|\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}}\right| \\
&= \left|\frac{1}{\sqrt{n(n+1)}\left(\sqrt{n+1} + \sqrt{n}\right)}\right| \\
&\to 0.
\end{aligned}$$

Therefore, f is not uniformly continuous.

Problem 5

Suppose $f:\mathbb{R}\rightarrow\mathbb{R}$ is periodic with period p; that is,

$$f(x+p)=f(x) \qquad \forall x\in\mathbb{R}$$

If f is continuous, show that f is bounded and uniformly continuous on $\mathbb{R}$.

Let $x\in\mathbb{R}$. Since f is continuous on $\mathbb{R}$, f is continuous on $[x,x+p]$, and f takes every value on $[x,x+p]$ in all of $\mathbb{R}$, since if $q\in[x,x+p]$, then $f(q+np)=f(q)$.

Since f is continuous on $[x,x+p]$, f is bounded on $[x,x+p]$, and so is bounded on $\mathbb{R}$. Additionally, f is uniformly continuous on $[x,x+p]$, and so is uniformly continuous on $\mathbb{R}$.

Problem 6

Show that $f(x)=x$ and $g(x)=\sin(x)$ are both uniformly continuous on $\mathbb{R}$, but the product

$$h(x)=x\sin(x)$$

is not uniformly continuous on $\mathbb{R}$.

Let $f(x)=x$. Setting $\delta=\varepsilon$, we have that

$$\begin{aligned}|x-y|&<\delta\\|f(x)-f(y)|&<\delta\\|f(x)-f(y)|&<\varepsilon.\end{aligned}$$

Similarly, since $\sin(x)$ is periodic and continuous, it must be uniformly continuous.

Problem 7

If $f:D\rightarrow\mathbb{R}$ is uniformly continuous and $|f(x)|\geq k>0$ for some k, show that $\frac{1}{f}$ is uniformly continuous on D.

Problem 8

If $D\subseteq\mathbb{R}$ is a bounded set and $f:D\rightarrow\mathbb{R}$ is uniformly continuous, show that f is bounded.

Problem 9

Suppose $f_n:D\rightarrow\mathbb{R}$ is a sequence of continuous functions such that $(f_n)_n\rightarrow f$ uniformly on D. Show that f is also continuous.

Problem 10

Prove that there does not exist a continuous function $f:\mathbb{R}\rightarrow\mathbb{R}$ with

$$\begin{aligned}f(\mathbb{Q})&\subseteq\mathbb{R}\setminus\mathbb{Q}\\f(\mathbb{R}\setminus\mathbb{Q})&\subseteq\mathbb{Q}.\end{aligned}$$