Problem (Problem 1)**:** Prove that if $f\colon M \to N$ is smooth, and L is a k-codimensional submanifold of N that is transverse to f, then $f^{-1}(L)$ is either empty or a submanifold of M with codimension k.

Solution: If L is not contained in $f(M)$, then $f^{-1}(L)$ is clearly empty. Therefore, we focus on the case where $f^{-1}(L)$ is not empty.

Let L be transverse to f, $q \in L$, and $p \in M$ such that $f(p) = q$. We observe that $T_qL + D_pF(T_pM) = T_qN$, so any vector in T_qN can be written (not necessarily uniquely) as an element of $D_pF(T_pM)$ and T_qL. Next, we observe that, if we take a coordinate chart for q in U such that $\varphi(U) \cong \mathbb{R}^k$, then by the Regular Value Theorem, we may select φ such that $L \cap U = \varphi^{-1}(0)$. This follows from the assumption that L has codimension k.

Now, if we can show that 0 is a regular value for $\varphi \circ f$, then $(\varphi \circ f)^{-1}(0) = f^{-1}(L) \cap f^{-1}(U)$, meaning that $f^{-1}(L)$ is a submanifold of M with codimension k. First, since 0 is a regular value for φ, it follows that if $v \in T_0\mathbb{R}^k$, then there is some $w \in T_qN$ such that $D_q\varphi(w) = v$. Since f is transverse to L, there is $w_1 \in T_qL$ and $w_2 \in T_pM$ such that $w = w_1 + D_pF(w_2)$. We observe that, since φ is constant on L, we have $D_q\varphi(w_1) = 0$, so that

$$
\begin{aligned}
D_p(\varphi \circ f)(w_2) &= D_q\varphi \circ D_pF(w_2)\\
&= D_q\varphi\left(w_1 + D_pF(w_2)\right)\\
&= D_q\varphi(w)\\
&= v,
\end{aligned}
$$

so 0 is a regular value for $\varphi \circ F$.

Problem (Problem 2)**:** Let $GL_n(\mathbb{R})$ denote the space of invertible $n \times n$ matrices over $\mathbb{R}$, let $SL_n(\mathbb{R})$ denote the matrices of determinant one, and let $O(n)$ be the orthogonal group.

(a) Prove that we may identify the tangent space of $GL_n(\mathbb{R})$ at the identity with $n \times n$ matrices over $\mathbb{R}$.

(b) Prove that the tangent space of $SL_n(\mathbb{R})$ at the identity consists of matrices of trace zero.

(c) Prove that the tangent space of $O(n)$ at the identity consists of skew-symmetric matrices. What is the dimension of $O(n)$?

(d) Show that $SL_n(\mathbb{R})$ and $O(n)$ do not intersect transversely at the identity.

Solution:

(a) Let $A \in \operatorname{Mat}_n(\mathbb{R})$, and consider a path through the identity given by $\gamma(t) = I + tA$. Since the determinant is a smooth function, and $\det(I) = 1$, we have that for a small $\varepsilon > 0$ there is δ, such that $|\det(I + tA) - 1| < \varepsilon$ whenever $|t| < \delta$. In particular, this means that the tangent space at the identity of $GL_n(\mathbb{R})$ consists of all matrices.

(b) We let $\gamma(t) = I + tA$ be a curve in $SL_n(\mathbb{R})$, so that $\gamma'(0) = A$ is an element of the tangent space of $SL_n(\mathbb{R})$ at the identity. We observe that $\det(\gamma(t)) = 1$ for all (sufficiently small) t, so by chain rule, we find that

$$
\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t))\\
&= D_{\gamma(0)}\det(\gamma'(0))\\
&= D_I\det(A).
\end{aligned}
$$

Therefore, we must evaluate what $\det'(I)(A)$ yields. Toward this end, we see that

$$
D_I\det(A) = \lim_{t\to 0}\frac{\det(I - tA) - 1}{t}
$$

$$= \lim_{t\to 0} \frac{t^n \det\left(\frac{1}{t}I - A\right) - 1}{t}.$$

Observe that the expression $\det\left(\frac{1}{t}I - A\right)$ is the characteristic polynomial of A in $\frac{1}{t}$. This means that the $\left(\frac{1}{t}\right)^{n-1}$ term is equal to $\operatorname{tr}(A)$, so that $D_I \det(A) = \operatorname{tr}(A)$. Thus, we find that A is traceless.

(c) If $\gamma(t) = I + tA$ is a curve in $O(n)$, then then we have that

$$\begin{aligned} (I + tA)^T(I + tA) &= I \\ I + t\left(A^T + A\right) + t^2\left(A^T A\right) &= I, \end{aligned}$$

meaning that by taking an equivalence class of this tangent curve, we have

$$I + t\left(A^T + A\right) = I,$$

so that $A^T = -A$.

We observe that the function $f\colon \operatorname{Mat}_n(\mathbb{R}) \to \operatorname{Mat}_n(\mathbb{R})_{\text{s.a.}}$, given by

$$f(A) = A^T A,$$

has I_n as a regular value. To see this, observe that curves in $T_I \operatorname{Mat}_n(\mathbb{R})_{\text{s.a.}}$ are of the form $\gamma(t) = I + tK$, where K is a self-adjoint(/symmetric) matrix. Similarly, $T_A \operatorname{Mat}_n(\mathbb{R})$ is of the form $\varepsilon(t) = A + tB$, where $B \in \operatorname{Mat}_n(\mathbb{R})$ and $t \in \mathbb{R}$. Both of these follow from the fact that $\operatorname{Mat}_n(\mathbb{R})$ and $\operatorname{Mat}_n(\mathbb{R})_{\text{s.a.}}$ are isomorphic to Euclidean spaces. Therefore, we see that the image of $\delta(t)$ is of the form $A^T A + t\left(A^T B + B^T A\right)$; if A satisfies $A^T A = I$, we can put this in the form of $I + tK$ by taking $\delta(t) = A + \frac{1}{2}tAK$. Therefore, by the Regular Value Theorem, the dimension of $O(n)$ is $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

(d) Since both skew-symmetric and traceless matrices have trace zero, it follows that the tangent spaces of $SL_n(\mathbb{R})$ and $O(n)$ cannot span the tangent space of $GL_n(\mathbb{R})$, as there are matrices with nonzero trace.

Problem (Problem 4)**:** Let D be a distribution on a smooth manifold of dimension n. We write $I(D)$ for the ideal of D, which consists of graded pieces $I^k(D) \subseteq \mathcal{A}^k(M)$, where $I^k(D)$ consists of forms ω such that $\omega(X_1, \dots, X_k) = 0$ for all $X_i \in D$, and

$$I(D) = \bigoplus_{k=0}^{n} I^k(D).$$

The Frobenius Theorem says that D is involutive if and only if I is *differential* — i.e., $d(I) \subseteq I$, where d is the exterior derivative.

(a) Prove that $I(D)$ is an ideal — i.e., if $\omega \in I(D)$ and η is arbitrary, then $\omega \wedge \eta \in I(D)$.

(b) Prove that $I(D)$ is locally generated by $s = n - r$ linearly independent 1-forms $\omega_1, \dots, \omega_s$, in the sense that for every point $p \in M$, there is a neighborhood U of p such that for any $\omega \in I^k(D)$ with k arbitrary, we may write

$$\omega = \sum_{i=1}^{s} \theta_i \wedge \omega_i$$

for suitable forms $\theta_1, \dots, \theta_s$.

(c) Prove that if D is involutive, then for all $\omega \in I(D)$, we have $d\omega \in I(D)$.

(d) Use this to show that if ω is a 1-form, and X, Y are vector fields, then

$$d\omega(X,Y) = \frac{1}{2}(X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])).$$

Conclude that if $\omega \in I^1(D)$, and $X, Y \in D$, then $\omega([X,Y]) = 0$. Thus, if I is a differential ideal, then D is involutive.

(e) Show that if D is defined by the vanishing of linearly independent forms $\omega_1, \dots, \omega_s$ near a point p, then D is involutive if and only if for each i there are 1-forms $\omega_{i,j}$ such that

$$d\omega_i = \sum_{j=1}^{s} \omega_{i,j} \wedge \omega_j.$$

Solution:

(a) Write

$$\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$$

so that

$$\begin{aligned}\omega(X_1, \dots, X_k) &= \det\left(\left(\alpha_i(X_j)\right)_{i,j}\right)\\ &= 0.\end{aligned}$$

for $X_1, \dots, X_k \in D$. Then, if

$$\eta = \beta_1 \wedge \cdots \wedge \beta_\ell,$$

we have the determinant of the block matrices

$$\begin{aligned}\omega \wedge \eta(X_1, \dots, X_k, \dots, X_{k+\ell}) &= \det\begin{pmatrix}\alpha_i(X_j) & \alpha_i(X_{\ell+j})\\ \beta_i(X_j) & \beta_i(X_{\ell+j})\end{pmatrix}\\ &= 0,\end{aligned}$$

so that $\omega \wedge \eta$ is contained in $I(D)$.

(b) Let $p \in U \subseteq M$ be such that T_pM is spanned by $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$. Without loss of generality, the distribution may be defined to be the subset of T_pM spanned by $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}\right\}$. Then, we observe that the ideal $I^1(D)$ is then spanned by the differential forms $dx_{r+1}, \dots, dx_n$. Since $I(D)$ is an ideal, we observe that an arbitrary element of $I^k(D)$ can then be written as

$$\omega = \sum_{j=r+1}^{s} \theta_j \wedge dx_j,$$

where the θ_j are elements of $\mathcal{A}^{k-1}(M)$.

(c) Let D be involutive.

The evaluation of $d\omega$ on vector fields $(X_1, \dots, X_{k+1})$ is given by

$$d\omega(X_1, \dots, X_{k+1}) = \frac{1}{k+1}\sum_{i=1}^{k+1}(-1)^{i+1}X_i\left(\omega\left(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\right)\right)$$

$$+\frac{1}{k+1}\sum_{j=1}^{n}\sum_{i=1}^{j-1}(-1)^{i+j}\omega\left(\left[X_i,X_j\right],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}\right).$$

We verify this for the case that D is involutive, so that D may be assumed to locally be given by $(X_1,\dots,X_r) = \left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_r}\right)$. Writing $\omega \in I(D)$ as

$$\omega = \sum_{i=r+1}^{n}\theta_i \wedge dx_i,$$

where the θ_i are $(k-1)$-forms, we may then find that, by using the formula for evaluation of the derivative on k-forms that

$$\begin{aligned}
d\omega(X_1,\dots,X_{k+1}) &= \frac{1}{k+1}\sum_{i=1}^{k+1}(-1)^{i+1}\frac{\partial}{\partial x_i}\left(\omega\left(X_1,\dots,\widehat{X_i},\dots,X_{k+1}\right)\right)\\
&= 0,
\end{aligned}$$

where we see that ω evaluates to zero on each of the X_i when $1 \leqslant i \leqslant r$, and $k, k+1 \leqslant r$.

(d) If we write $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$, and

$$\omega = f\,dx + g\,dy,$$

then

$$\begin{aligned}
d\omega &= \frac{\partial f}{\partial x}\,dx\wedge dx + \frac{\partial f}{\partial y}\,dy\wedge dx + \frac{\partial g}{\partial x}\,dx\wedge dy + \frac{\partial g}{\partial y}\,dy\wedge dy\\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy.
\end{aligned}$$

Then, we see that

$$\begin{aligned}
d\omega(X,Y) &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\\
&= \frac{1}{2}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\\
&= \frac{1}{2}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\omega\left(\frac{\partial}{\partial y}\right)\right) - \frac{\partial}{\partial y}\left(\omega\left(\frac{\partial}{\partial x}\right)\right) - \omega\left(\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]\right)\right).
\end{aligned}$$

In particular, we observe that if $\omega \in I^1(D)$, and $d\omega \in I^2(D)$, then since we may locally write $X, Y \in D$ such that $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$, we find that $\omega([X,Y]) = 0$, so that if I is differential, then D is involutive.

(e) Let D be defined by the vanishing of the linearly independent 1-forms $\omega_1,\dots,\omega_s$ near $p \in U \subseteq M$. We observe that D is involutive if and only if the ideal (locally) generated by $\omega_1,\dots,\omega_s$ is differential; that is, if $d\omega_i \in I(D)$ for each ω_i. Therefore, we must have that, for $X_1,\dots,X_r$, where

$$\omega_i(X_j) = 0$$

for each $i = 1,\dots,s$ and $j = 1,\dots,r$, we have

$$d\omega_i(X_1,X_2) = \frac{1}{3}X_1(\omega_i(X_2)) - X_2(\omega_i(X_1))$$

$$= 0,$$

so that $d\omega_i$ is a sum of 2-forms $\omega_{i,j} \wedge \omega_j$ to satisfy the ideal condition. Thus, we get

$$d\omega_i = \sum_{j=1}^{s} \omega_{i,j} \wedge \omega_j.$$

Problem (Problem 5)**:** Consider the 2-form on $\mathbb{R}^{2n}$ given by

$$\omega = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}.$$

Compute ω^n, the wedge of ω with itself n times.

Solution: We start by observing that the case of $\omega \wedge \omega$, that any shift of the forms into the standard order always constitutes an even number of swaps, so that we get the result

$$\omega \wedge \omega = 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} dx_{2i-1} \wedge dx_{2i} \wedge dx_{2j-1} \wedge dx_{2j}.$$

By wedging with another copy of ω, we then get

$$\omega \wedge \omega \wedge \omega = 4\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} dx_{2i-1} \wedge dx_{2i} \wedge dx_{2j-1} \wedge dx_{2j} \wedge dx_{2k-1} \wedge dx_{2k},$$

and so on and so forth. By exhausting up to and through n, we get the result

$$\omega^n = 2^n dx_1 \wedge \cdots \wedge dx_n.$$