Observations on Excess Area Identities and Operator Symbols in Bergman Spaces

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Summary

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Definition and Notations

- Ω : a region in $\mathbb{C}$ e.g. $\mathbb{D}, D(0,r), \mathbb{A}(0,r,1), \mathbb{C}$
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$: weight function

Definition (λ-weighted Square-Integrable Functions)

$$L^2(\Omega,\lambda) = \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(z)|^2 \lambda(z)\, dA(z) < \infty \right\}$$

- $L^2(\Omega,\lambda)$ forms a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega,\lambda)} = \int_\Omega f(z)\overline{g(z)}\lambda(z)\, dA(z)$$

inducing the norm

$$\|f\|^2_{L^2(\Omega,\lambda)} = \int_\Omega |f(z)|^2 \lambda(z)\, dA(z)$$

Definitions and Notations

Definition (Holomorphic Function on Ω)

We say h is holomorphic on Ω, or $h \in \mathcal{O}(\Omega)$, if, for all $z \in \Omega$

$$\begin{aligned}\frac{\partial h(z)}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial h(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}\right)\\ &= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)\\ &= 0.\end{aligned}$$

Definition (λ-weighted Bergman Space)

$$A^2(\Omega, \lambda) \coloneqq \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

Definitions and Notations

Definition ($A^{1,2}(\Omega, \lambda)$)

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \;\middle|\; \frac{\partial h}{\partial z} \in A^2(\Omega, \lambda) \right\}$$

Definition (Weighted Image-Area)

Let $h \in A^{1,2}(\Omega, \lambda)$.

$$\begin{aligned} A_{\Omega,\lambda}(h) &= \int_\Omega \left| \frac{\partial h}{\partial z} \right|^2 \lambda(z)\, dA(z) \\ &= \left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\Omega,\lambda)} \end{aligned}$$

Definitions and Notations

- $A^2(\Omega, \lambda)$ has a reproducing kernel i.e $\exists!\, K_\Omega^\lambda(\cdot, z) \in A^2(\Omega, \lambda)$:

$$h(z) = \left\langle h(\cdot), K_\Omega^\lambda(\cdot, z) \right\rangle_{L^2(\Omega,\lambda)}$$

- $A^2(\Omega, \lambda)$ is a closed subspace of $L^2(\Omega, \lambda)$.

Definition (Bergman Projection)

Let $P^{\Omega,\lambda} : L^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$

$$\begin{aligned}\left(P^{\Omega,\lambda} h\right)(z) &:= \left\langle h(\cdot), K_\Omega^\lambda(\cdot, z) \right\rangle_{L^2(\Omega,\lambda)} \\ &= \int_\Omega h(w) \overline{K_\Omega^\lambda(w, z)} \lambda(w)\, dA(w)\end{aligned}$$

Definitions and Notations

Definition (Multiplication Operator)

Let $M_\varphi : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda)$ where $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

Definition (Toeplitz Operator)

$T_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$, where $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega,\lambda} := P^{\Omega,\lambda} M_\varphi$$

Definitions and Notations

Definition (Commutator)

Let $\left[P^{\Omega,\lambda}, M_\varphi\right] : L^2(\Omega,\lambda) \to L^2(\Omega,\lambda)$

$$\left[P^{\Omega,\lambda}, M_\varphi\right] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

Definition (Hankel Operator)

Let $H_\varphi^{\Omega,\lambda} : A^2(\Omega,\lambda) \to (A^2(\Omega,\lambda))^\perp$

$$\begin{aligned} H_\varphi^{\Omega,\lambda} &:= -\left[P^{\Omega,\lambda}, M_\varphi\right]\Big|_{A^2(\Omega,\lambda)} \\ &= \left(I - P^{\Omega,\lambda}\right) M_\varphi \\ &= M_\varphi - P^{\Omega,\lambda} M_\varphi \\ &= M_\varphi - T_\varphi^{\Omega,\lambda} \end{aligned}$$

Contents

Motivations

- $\{z^n\}_{n=0}^{\infty}$ form a *complete orthogonal basis* for $A^2(\mathbb{D})$
- If h is holomorphic, then h is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

  and

$$S_N := \sum_{n=0}^{N} h_n z^n$$

  converges uniformly on compact subsets.
- Relationship between L^2 norm of h to the ℓ^2 norm of $\{h_k\}_{k=0}^{\infty}$:

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 \, dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $\left[T_{\overline{z}}^{\mathbb{D}} M_z, DM_z\right](z^m) = 0$

Problems

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

Literature Review on Previous Results I

- D'Angelo's Excess Area identity [D'A19]

  Let $h \in A^{1,2}(\mathbb{D})$. Then,

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial(zh)}{\partial z} \right\|^2_{L^2(\mathbb{D})} - \left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\mathbb{D})} \\ &= \frac{1}{2} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta \\ &= \pi \left\| Sh \right\|^2_{L^2(b\mathbb{D})} \end{aligned}$$

  where Sh is the restriction of h to the unit circle.

Literature Review on Previous Results II

- Excess Area identity with Blaschke product multiplier
- 'Excess Area' identity for harmonic functions [BÇGH22]

- Generating symbols for Toeplitz operators for a given initial p and target polynomial q on unit disc and polydisc, $T_\varphi^{\mathbb{D}}(p) = q$ and $T_\varphi^{\mathbb{D}^n}(p) = q$ [ÇDTR$^+$24]

- Substituted derivatives for Toeplitz operators in Excess Area identity [ÇDTR$^+$24]

Contents

Summary of Results

1. Results and Observations influenced by the Excess Area identity:
   i. On $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2})$, $A^2(\mathbb{D}, \lambda)$, $A^2(D(0,r))$
   ii. On convergence of identities on certain weighted discs.
2. Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
   i. On unweighted and weighted Toeplitz operators relation
   ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on $A^2(\mathbb{D})$

Methods Used

- Relation between L^2 norms of functions and ℓ^2 norms of Taylor series:
$$\|h\|_{L^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$
- Integration by parts via Stokes's theorem on forms:
$$\oint_{b\Omega} f\ dz = \int_{\Omega} \overline{\frac{\partial f}{\partial z}}\ d\overline{z} \wedge dz$$
$$\oint_{b\Omega} f\ d\overline{z} = \int_{\Omega} \frac{\partial f}{\partial z} dz \wedge d\overline{z}.$$
- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Beta, Gamma, and Hypergeometric functions

Using Integration by Parts to find Excess Area identity: Wedge Product I

The area is integrated with respect to $dA = dx \wedge dy$. When a and b are 1-forms, the wedge product has the two following properties:

$$\begin{aligned} a \wedge b &= -b \wedge a \\ a \wedge a &= 0. \end{aligned}$$

With $z = x + iy$, $\overline{z} = x - iy$, the substitution $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$ yields

$$\begin{aligned} dx \wedge dy &= \frac{1}{2i}\left(d\overline{z} \wedge dz\right) \\ &= -\frac{1}{2i}\left(dz \wedge d\overline{z}\right). \end{aligned}$$

Using Integration by Parts to find Excess Area identity: Wedge Product II

The area integral is now rewritten as:

$$\begin{aligned}\left\langle \frac{\partial h}{\partial z}, \frac{\partial h}{\partial z} \right\rangle_{L^2(\Omega,\lambda)} &= \int_\Omega \left(\overline{\frac{\partial h}{\partial z}}\right) \left(\frac{\partial h}{\partial z}\right) \lambda(|z|)\ dx \wedge dy \\ &= \frac{1}{2i} \int_\Omega \lambda(|z|)\ \left(\left(\overline{\frac{\partial h}{\partial z}}\right) d\overline{z}\right) \wedge \left(\left(\frac{\partial h}{\partial z}\right) dz\right)\end{aligned}$$

Using Integration by Parts to find Excess Area identity: Stokes's Theorem I

In particular,

$$\overline{\frac{\partial}{\partial z}}\left(\left(\lambda\left(|z|\right)\right)\overline{h}\frac{\partial h}{\partial z}\right)d\overline{z}\wedge dz=\underbrace{\left(\lambda\left(|z|\right)\right)\overline{\frac{\partial h}{\partial z}}d\overline{z}\wedge\frac{\partial h}{\partial z}dz}_{\text{area integrand}}+\left(\overline{\frac{\partial}{\partial z}}\lambda\left(|z|\right)\right)\overline{h}\wedge\frac{\partial h}{\partial z}dz$$

meaning

$$\begin{aligned}\frac{1}{2i}\int_{\Omega}\frac{\partial h}{\partial z}\overline{\frac{\partial h}{\partial z}}\lambda\left(|z|\right)\,d\overline{z}\wedge dz&=\underbrace{\frac{1}{2i}\int_{\Omega}\overline{\frac{\partial}{\partial z}}\left(\lambda\left(|z|\right)\overline{h}\frac{\partial h}{\partial z}\right)\,d\overline{z}\wedge dz}_{\text{Integral }A}\\&\quad-\frac{1}{2i}\int_{\Omega}\overline{h}\frac{\partial h}{\partial z}\left(\overline{\frac{\partial}{\partial z}}\lambda\left(|z|\right)\right)\,d\overline{z}\wedge dz.\end{aligned}$$

Using Integration by Parts to find Excess Area identity: Stokes's Theorem II

Turning our attention to Integral A,

$$\begin{aligned}\frac{1}{2i}\int \overline{\frac{\partial}{\partial z}}\left(\lambda\left(|z|\right)\overline{h}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz &= \frac{1}{2i}\int_{\Omega} d\left(\lambda\left(|z|\right)\overline{h}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz \\ &= \frac{1}{2i}\underbrace{\int_{b\Omega}\lambda\left(|z|\right)\overline{h}\frac{\partial h}{\partial z}\, dz}_{=0}.\end{aligned}$$

With this, the area integral is now

$$\frac{1}{2i}\int \frac{\partial h}{\partial z}\overline{\frac{\partial h}{\partial z}}\lambda\left(|z|\right)\, d\overline{z}\wedge dz = -\frac{1}{2i}\int \overline{h}\frac{\partial h}{\partial z}\left(\overline{\frac{\partial}{\partial z}}\lambda\left(|z|\right)\right)\, d\overline{z}\wedge dz$$

Excess Area on Fock Spaces

D'Angelo's Excess Area identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2}\int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta = \pi \left\| Sh \right\|_{L^2(b\mathbb{D})}^2$$

Excess Area on Fock Space

Given $h \in \mathcal{F}^2$ with $\frac{\partial h}{\partial z}$,

$$\begin{aligned} &A_{\mathcal{F}^2}(zh) - A_{\mathcal{F}^2}(h) \\ &= \pi \left\| z T_{\overline{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 + \pi \left\| T_{\overline{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 + \pi \left\| H_{\overline{z}}^{\mathcal{F}^2}(h) \right\|_{\mathcal{F}^2}^2 \end{aligned}$$

Here, the restriction of h to the unit circle in D'Angelo's Excess Area identity is replaced with the Bergman projection on $\mathbb{C}$.

Excess Area on $A^2(\mathbb{D}, \lambda)$

D'Angelo's Excess Area identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta = \pi \left\|Sh\right\|^2_{L^2(b\mathbb{D})}$$

Excess Area on $A^2(\mathbb{D}, \lambda)$

Let $h \in A^{1,2}(\mathbb{D}, \lambda)$, $\lambda(z) = 1 - |z|^2$. Then,

$$A_{\mathbb{D},\lambda}\left(z^{m+1}h\right) - A_{\mathbb{D},\lambda}\left(z^m h\right) = \pi \left\|z^m h\right\|^2_{L^2(\mathbb{D},\lambda)}.$$

Here, the restriction of h to the unit circle is replaced with the function itself.

Dilation and Contraction from $A^2(D(0,r))$ to $A^2(\mathbb{D})$

Contracting $h \in A^{1,2}(\mathbb{D})$ by taking $h_r = h(rz)$ for some $0 < r < 1$,

$$A_{\mathbb{D}}(zh_r) - A_{\mathbb{D}}(h_r) = \pi \|Sh_r\|^2_{L^2(b\mathbb{D})} \tag{1}$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \|Sh\|^2_{L^2(bD(0,r))}. \tag{2}$$

Dilating $h \in A^{1,2}(D(0,r))$ by taking $h_{\frac{1}{r}} = h(\frac{z}{r})$ for some $0 < r < 1$

$$A_{D(0,r)}\left(zh_{1/r}\right) - r^2 A_{D(0,r)}\left(h_{1/r}\right) = \pi r^2 \left\|Sh_{1/r}\right\|^2_{L^2(bD(0,r))} \tag{3}$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \|Sh\|^2_{L^2(b\mathbb{D})} \tag{4}$$

Approximation for Sequences of Berezin Transform

Weighted Area on $D(0, r)$

Let $\lambda_r(z) = \chi_{D(0,r)} \left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$ where $r > 0$. Then,

$$A_{D(0,r),\lambda_r}(h) = \int_{D(0,r)} \left|h'(z)\right|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} dA(z)$$

We find that, as $r \to \infty$, $A_{D(0,r),\lambda_r}(h) \to A_{\mathcal{F}^2}(h)$.

Separately,

$$A_{\mathcal{F}^2}(h) = \left\| T_{\overline{z}}^{\mathcal{F}^2} h \right\|_{\mathcal{F}^2}^2$$

Berezin Transform Convergence, Cont'd

Reproducing Kernel on $A^2\left(D(0,r),\lambda_r\right)$

$$K^{\lambda_r}_{D(0,r)}(w,z) = \sum_{k=0}^{\infty} \frac{\overline{z}^k w^k}{\|w^k\|^2_{L^2(D(0,r),\lambda_r)}} = \frac{1}{\left(1 - \frac{\overline{z}w}{r^2}\right)^{r^2+2}}$$

Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w,z) = e^{\overline{z}w}$$

Berezin Transform Convergence, Cont'd

Definition (Berezin Transform ([Zhu07]))

Let

$$k_z^{\Omega,\lambda}(w) := \frac{K_\Omega^\lambda(w,z)}{\sqrt{K_\Omega^\lambda(z,z)}}$$

Then, for some bounded operator T on $L^2(\Omega,\lambda)$, define $\mathcal{B}^{\Omega,\lambda} : B(L^2(\Omega,\lambda)) \to L^2(\Omega,\lambda)$

$$(\mathcal{B}^{\Omega,\lambda} T)(z) := \left\langle T k_z^{\Omega,\lambda}, k_z^{\Omega,\lambda} \right\rangle_{L^2(\Omega,\lambda)}$$

Berezin Transform Convergence, Cont'd

Previous results:

- For $\varphi \in L^\infty(\Omega, \lambda)$, $\mathcal{B}^{\Omega,\lambda} T_\varphi = \mathcal{B}^{\Omega,\lambda} M_\varphi$. (see [AZ98a]).
- φ is harmonic if and only if $\mathcal{B}^{\Omega,\lambda} M_\varphi = \varphi$ (proof in [Eng94]).

New results:

- For $T_\varphi^{D(0,r),\lambda_r} = P^{D(0,r),\lambda_r} M_\varphi$, the Berezin transform $\mathcal{B}^{D(0,r),\lambda_r} T_\varphi^{D(0,r),\lambda_r}$ converges pointwise to $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$ as $r \to \infty$.
- By Dini's Theorem, this convergence is uniform on compact subsets of $\mathbb{C}$ (proof inspired by [GŞ20]).

Using an extension of [ÇDTR^{+}24, Lemma 2.1]

For weight $\lambda(z) = \left(1 - |z|^2\right)^{\alpha}$ ($\alpha \geq 0$) on the unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

$$\frac{T_{\overline{z}^m}^{\mathbb{D},\lambda_\alpha}(z^n)}{T_{\overline{z}^m}^{\mathbb{D}}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leq n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T_{\overline{z}^m}^{\mathbb{D},\lambda_\alpha}(z^n) = s_{n,m,\alpha}\, T_{\overline{z}^m}^{\mathbb{D}}(z^n), \text{ and } \lim_{n\to\infty} s_{n,m,\alpha} = 1$$

Unweighted and Weighted Commutator on $A^2(\mathbb{D})$

Existence of Commutator Symbols

Given p and q are harmonic polynomials and $\frac{\partial}{\partial z}(p) \neq 0$, there does not exist a polynomial symbol ϕ, such that $\left[P^{\mathbb{D}}, M_\phi\right](p) = q$ or $\left[P^{\mathbb{D},\lambda}, M_\phi\right](p) = q$.

Compare to [ÇDTR+24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

Unweighted and Weighted Hankel Operator on $A^2(\mathbb{D})$

Existence of Hankel Operator Symbols

Given some holomorphic polynomials p, q where p is not constant, there does not exist a polynomial symbol ϕ such that $H_\phi^{\mathbb{D}}(p) = \overline{q}$ or $H_\phi^{\mathbb{D},\lambda}(p) = \overline{q}$

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Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0,r,1))$

For all integers m and n,

$$T_{\overline{z}^m}^{\mathbb{A}(0,r,1)}(z^n) = \begin{cases} \frac{2mr^{2m}\ln(r)}{(r^{2m}-1)} z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m}-1}{2m\ln(r)} z^{-1} & \text{if } n = m-1 \\ \frac{(n-m+1)\left(1-r^{2n+2}\right)}{(n+1)(1-r^{2n-2m+2})} z^{n-m} & \text{else} \end{cases}.$$

We attempted to find an algorithm to generate $\varphi \in L^\infty(\mathbb{A}(0,r,1))$ such that $T_\varphi^{\mathbb{A}(0,r,1)}(p) = q$ for given holomorphic Laurent polynomials p and q, but could not prove lack of existence if p has roots inside $\overline{\mathbb{A}(r,0,1)}$.

Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial p and target polynomial q on $\mathbb{A}(0, r, 1)$, $T_{\varphi}^{\mathbb{A}(0,r,1)}(p) = q$
- Extension of 'Excess Area' identity to harmonic functions in $L^2\left(\mathbb{C}, e^{-|z|^2}\right)$.
- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential, $(1-|z|^2)^A e^{\frac{-B}{(1-|z|^2)^\alpha}}(A \geq 0, B > 0, \alpha > 0)$.

Contents

What is Research Like?

- The complex analysis group consisted of myself, Jennifer Yuan (NYU Abu Dhabi), and Sakia Akamah (Rose–Hulman Institute of Technology).
- Weeks were 9am to 5pm, mostly doing various calculations and updating our collected results document.
- Every weekday morning, our mentor met with us to go over the previous day's calculations and suggest new ideas/directions.
- We rotated project directions every day after the morning's meeting so we could bring fresh perspectives and ideas to the work the previous person had done yesterday.
- We did not fully understand what we were doing a lot of the time (especially in the beginning), but that's honestly very expected; our mentor was always there to help us along the way and answer any questions we had.

Activities and Life Outside Research

- In addition to the research itself, we also received visitors from academia and industry to provide some guidance on our potential future paths.
- We also visited the University of North Texas in Denton to talk with PhD students and faculty who gave us important advice on how to succeed as a graduate student.
- We also had some fun — usually after work was over for the day we'd watch anime in the university housing common room, or play badminton at the gym, and occasionally played board games.
- There were four project cohorts at the REU, but ours was the coolest since it wasn't applied.

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Acknowledgments

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