

## Classical Mechanics

### Motion in $\mathbb{R}^1$

Let  $x(t)$  denote position. Then,  $v(t) = \frac{dx}{dt} = \dot{x}(t)$  is velocity (where the  $\cdot$  denotes derivative with respect to time),  $a(t) = \dot{v}(t) = \ddot{x}(t)$ , etc.

Considering Newton's second law,  $F(x(t)) = m\ddot{x}(t)$ , every exact solution requires initial conditions of  $x(t_0)$  and  $v(t_0)$ . Solutions to Newton's second law are known as trajectories.

Considering a spring of constant  $k$ ,  $F(x) = -kx$  yields the differential equation  $m\ddot{x} + kx = 0$ . The general solution is

$$x(t) = a \cos(\omega t) + b \sin(\omega t),$$

with  $\omega = \sqrt{k/m}$  denoting the frequency. The spring is an example of a simple harmonic oscillator.

### Conservation of Energy

For a general force function  $F(x)$ , the kinetic energy is  $\frac{1}{2}mv^2$ , and the potential energy is

$$V(x) = - \int F(x) dx,$$

meaning  $F(x) = -\frac{dV}{dx}$ . The total energy is thus found as

$$E(x, v) = \frac{1}{2}mv^2 + V(x).$$

#### Theorem: Conservation of Energy

If a particle with trajectory  $x(t)$  satisfies  $m\ddot{x} = F(x)$ , then the energy  $E$  is conserved.

**Proof:**

$$\begin{aligned} \frac{d}{dt}E(x(t), \dot{x}(t)) &= \frac{d}{dt} \left( \frac{1}{2}m(\dot{x}(t))^2 + V(x(t)) \right) \\ &= m\dot{x}(t)\ddot{x}(t) + \frac{dV}{dx}\dot{x}(t) \\ &= \dot{x}(t)(m\ddot{x}(t) - F(x(t))). \end{aligned}$$

By using the conservation of energy, we can reduce the second order differential equation  $F(x) = m\ddot{x}$  to a system of first order differential equations in  $x(t)$  and  $v(t)$  respectively:

$$\begin{aligned} \frac{dx}{dt} &= v(t) \\ \frac{dv}{dt} &= \frac{1}{m}F(x(t)). \end{aligned}$$

If  $(x(t), v(t))$  satisfies this set of equations, then  $x(t)$  satisfies Newton's second law. We say the set of all possible  $(x, v)$  forms the phase space for the particle in  $\mathbb{R}^1$ .

In phase space, conservation of energy implies that the set of all  $(x, v)$  must lie on the level curve of the energy function:  $\{ (x, v) \mid E(x, v) = E(x_0, v_0) \}$ .

Using the conservation of energy, we find that, though Newton's second law is a second order differential equation in time, it is actually a first order differential equation:

$$\frac{m}{2} (\dot{x}(t))^2 + V(x(t)) = E(x(t_0), v(t_0))$$

$$\dot{x}(t) = \sqrt{\frac{2(E_0 - V(x(t)))}{m}}$$

## Damping

Suppose we also introduce a force that depends on velocity — in the case of a damped simple harmonic oscillator, the equation for force changes from  $F = -kx$  to  $F = -kx - \gamma\dot{x}$ , with  $\gamma > 0$ . The damping force acts in the opposite direction of velocity, meaning the particle slows down.

The equation of motion is then

$$m\ddot{x} + \gamma\dot{x} + kx = 0.$$

For  $\gamma$  small, the solutions are a sum sines and cosines multiplied by some exponential decay factor, but for  $\gamma$  large, the solutions are only the exponential decay.

## Energy Conservation (or lack thereof) in Damped System

Suppose a particle moves along  $x(t)$  that satisfies  $F(x, \dot{x}) = F_1(x) - \gamma\dot{x}$ , with  $\frac{dV}{dx} = -F_1(x)$  and  $\gamma > 0$ . Then,

$$\frac{d}{dt} E(x(t), \dot{x}(t)) = -\gamma\dot{x}(t)^2.$$

**Proof:**

$$\begin{aligned} \frac{d}{dt} E(x(t), \dot{x}(t)) &= \dot{x}(t) (m\ddot{x}(t) - F_1(x(t))) \\ &= \dot{x}(t) (m\ddot{x}(t) - (m\ddot{x}(t) + \gamma\dot{x}(t))) \\ &= -\gamma\dot{x}(t)^2 \end{aligned}$$

## Motion in $\mathbb{R}^n$

The position of a particle  $\mathbf{x} = (x_1, \dots, x_n)$  lends itself to velocity  $\mathbf{v} = (v_1, \dots, v_n) = (\dot{x}_1, \dots, \dot{x}_n)$ , and  $\mathbf{a} = (\ddot{x}_1, \dots, \ddot{x}_n)$ . Similar to in  $\mathbb{R}^1$ , Newton's second law is denoted

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}(t), \dot{\mathbf{x}}(t)).$$

## Conservation of Energy in $n$ Dimensions

The energy function

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m \|\dot{\mathbf{x}}\|^2 + V(\mathbf{x})$$

is only satisfied where  $\mathbf{F} = -\nabla V$ .

**Proof:**

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} m \|\dot{\mathbf{x}}\|^2 + V(\mathbf{x}) \right) &= m \sum_{j=1}^n \dot{x}_j \ddot{x}_j + \sum_{j=1}^n \frac{\partial V}{\partial x_j} \dot{x}_j(t) \\ &= \dot{\mathbf{x}}(t) (m\ddot{\mathbf{x}}(t) + \nabla V) \\ &= \dot{\mathbf{x}}(t) (\mathbf{F}(\mathbf{x}) + \nabla V(\mathbf{x})), \end{aligned}$$

which is equal to zero only if  $-\nabla V = \mathbf{F}$ .

If  $\mathbf{F}$  is a smooth  $\mathbb{R}^n$  valued function on  $U \subset \mathbb{R}^n$ , then  $\mathbf{F}$  is conservative if there exists a smooth real-valued function  $V$  such that  $\mathbf{F} = -\nabla V$ .

In other words,  $\mathbf{F}$  is conservative if  $\mathbf{F}$  is a gradient field, implying that  $\nabla \times \mathbf{F} = 0$ .

If  $\mathbf{F}(\mathbf{x}, \mathbf{y}) = -\nabla V(\mathbf{x}) + \mathbf{F}_2(\mathbf{x}, \mathbf{y})$ , with  $\mathbf{v} \cdot \mathbf{F}_2 = 0$  for all  $\mathbf{x}$  and  $\mathbf{v}$ , then energy is conserved along a given trajectory.

## Systems of Particles

Let  $\mathbf{x}^j = (x_1^j, x_2^j, \dots, x_n^j)$  denote the  $j$ th particle of a system of  $N$  particles. Newton's second law is thus reformulated as

$$m_j \ddot{\mathbf{x}}^j = \mathbf{F}^j(\mathbf{x}^1, \dots, \mathbf{x}^N, \dot{\mathbf{x}}^1, \dots, \dot{\mathbf{x}}^N).$$

The total energy is determined by

$$E(\mathbf{x}^1, \dots, \mathbf{x}^N, \mathbf{v}^1, \dots, \mathbf{v}^N) = \left( \sum_{j=1}^N \frac{1}{2} m_j \|\mathbf{v}^j\|^2 \right) + V(\mathbf{x}^1, \dots, \mathbf{x}^N).$$

## Conservation of Energy in a System of Particles

The energy function is constant along each trajectory if  $\nabla^j V = -\mathbf{F}^j$ , where  $\nabla^j$  denotes the gradient with respect to  $\mathbf{x}^j$ .

The force function along a simply connected domain  $U$  in  $\mathbb{R}^{nN}$  satisfies  $\nabla^j V = -\mathbf{F}^j$  if and only if

$$\frac{\partial F_k^j}{\partial x_m^l} = \frac{\partial F_m^l}{\partial x_k^j}$$

for all  $j, k, l, m$ .

**Proof:**

$$\begin{aligned} \frac{dE}{dt} &= \sum_{j=1}^N (m_j \dot{\mathbf{x}}^j \cdot \ddot{\mathbf{x}}^j + \nabla^j V \cdot \mathbf{x}^j) \\ &= \sum_{j=1}^N \dot{\mathbf{x}}^j (m_j \ddot{\mathbf{x}}^j + \nabla^j V) \\ &= \sum_{j=1}^N \dot{\mathbf{x}}^j (\mathbf{F}^j + \nabla^j V), \end{aligned}$$

which is equal to zero if  $\nabla^j V = -\mathbf{F}^j$ .

Applying a higher dimension version of  $\nabla \times \mathbf{F}$  to each coordinate pair  $(a, b)$ , we find the identity that shows  $\mathbf{F}$  is a gradient field.

## Momentum of a System of Particles

The momentum of a particle  $\mathbf{p}^j$  is defined by

$$\mathbf{p}^j = m_j \dot{\mathbf{x}}^j.$$

Observe that  $\frac{d\mathbf{p}^j}{dt} = m_j \ddot{\mathbf{x}}^j = \mathbf{F}^j$ . The total momentum is then

$$\mathbf{p} = \sum_{j=1}^N \mathbf{p}^j.$$

Newton's third law, which states “for every action there is an equal and opposite reaction” applies if

- $\mathbf{F}^j = \sum_{k \neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{y}^j);$
- $\mathbf{F}^{j,k}(\mathbf{x}_j, \mathbf{x}_k) = -\mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j).$

If each  $\mathbf{F}^j$  is also a conservative force, then satisfying these conditions yields potential energy in the form of

$$V(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N) = \sum_{j < k} V^{j,k}(\mathbf{x}^j - \mathbf{x}^k).$$

### Newton's Third Law and Conservation of Momentum

If the system of particles satisfies the conditions of

- $\mathbf{F}^j = \sum_{k \neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{y}^j)$
- and  $\mathbf{F}^{j,k}(\mathbf{x}_j, \mathbf{x}_k) = -\mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j),$

then total momentum is conserved.

**Proof:**

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \sum_{j=1}^N \mathbf{F}^j \\ &= \sum_{j=1}^N \sum_{k \neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k), \end{aligned}$$

and since  $\mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k) + \mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j) = 0$ , we find  $\frac{d\mathbf{p}}{dt} = 0$ .

### Translation Invariance of Potential and Momentum Conservation

Let  $V$  denote the potential for a conservative force. Then, momentum is conserved if and only if  $V$  is translation invariant, meaning that for all  $\mathbf{a} \in \mathbb{R}^n$ ,

$$V(\mathbf{x}^1 + \mathbf{a}, \mathbf{x}^2 + \mathbf{a}, \dots, \mathbf{x}^N + \mathbf{a}) = V(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N).$$

**Proof:** Let  $\mathbf{a} = t\mathbf{e}_k$ . Then, differentiating at  $t = 0$  with respect to  $t$ , we find

$$\begin{aligned} 0 &= \sum_{j=1}^N \frac{\partial V}{\partial x_k^j} \\ &= - \sum_{j=1}^N F_k^j \\ &= - \sum_{j=1}^N \frac{dp_k^j}{dt} \\ &= - \frac{dp_k}{dt}, \end{aligned}$$

with  $p_k$  denoting the  $k$ th component of  $\mathbf{p}$ . Therefore,  $\mathbf{p}$  is constant in time.

If  $\mathbf{p}$  is conserved, then the sum of all forces is 0 at each point for all  $t$ , meaning that for all  $t$ ,

$$\begin{aligned} \frac{d}{dt} V(\mathbf{x}^1 + t\mathbf{a}, \mathbf{x}^2 + t\mathbf{a}, \dots, \mathbf{x}^N + t\mathbf{a}) &= \sum_{j=1}^N \nabla^j V(\mathbf{x}^1 + t\mathbf{a}, \mathbf{x}^2 + t\mathbf{a}, \dots, \mathbf{x}^N + t\mathbf{a}) \cdot \mathbf{a} \\ &= - \left( \sum_{j=1}^N \mathbf{F}^j(\mathbf{x}^1 + t\mathbf{a}, \mathbf{x}^2 + t\mathbf{a}, \dots, \mathbf{x}^N + t\mathbf{a}) \right) \cdot \mathbf{a} \\ &= 0 \end{aligned}$$

meaning  $V$  is equal at  $t = 0$  and  $t = 1$ .

## Center of Mass

For a system of  $N$  particles, the center of mass is denoted

$$\mathbf{c} = \sum_{j=1}^N \frac{m_j}{\sum_{j=1}^N m_j} \mathbf{x}_j.$$

We denote  $\sum_{j=1}^N m_j = M$ . Differentiating  $\mathbf{c}$ , we get

$$\begin{aligned} \frac{d\mathbf{c}}{dt} &= \frac{1}{M} \sum_{j=1}^N m_j \dot{\mathbf{x}}^j \\ &= \frac{\mathbf{p}}{M}. \end{aligned}$$

Notice that if  $\mathbf{p}$  is conserved, then  $\mathbf{c}(t) = \mathbf{c}(t_0) + (t - t_0) \frac{\mathbf{p}}{M}$ .

For a system of two particles, if  $V(\mathbf{x}^1, \mathbf{x}^2)$  is invariant under translation, then  $V(\mathbf{x}^1, \mathbf{x}^2) = \tilde{V}(\mathbf{x}^1 - \mathbf{x}^2)$ , and  $\tilde{V}(\mathbf{a}) = V(\mathbf{a}, 0)$ .

The positions  $\mathbf{x}^1$  and  $\mathbf{x}^2$  can be recovered from knowledge about  $\mathbf{c}$  and the relative position  $\mathbf{y} := \mathbf{x}^1 - \mathbf{x}^2$ :

$$\begin{aligned} \mathbf{x}^1 &= \frac{\mathbf{c} + m_2 \mathbf{y}}{m_1 + m_2} \\ \mathbf{x}^2 &= \frac{\mathbf{c} - m_1 \mathbf{y}}{m_1 + m_2}. \end{aligned}$$

Thus, we can calculate

$$\begin{aligned} \ddot{\mathbf{y}} &= \ddot{\mathbf{x}}^1 - \ddot{\mathbf{x}}^2 \\ &= -\frac{1}{m_1} \nabla \tilde{V}(\mathbf{x}^1 - \mathbf{x}^2) - \frac{1}{m_2} \nabla \tilde{V}(\mathbf{x}^1 - \mathbf{x}^2). \end{aligned}$$

## Motion of Relative Position under Translation Invariant Potential

For a two particle system with translation invariant potential, the relative position  $\mathbf{y} = \mathbf{x}^1 - \mathbf{x}^2$  is a solution to the differential equation

$$\mu \ddot{\mathbf{y}} = -\nabla \tilde{V}(\mathbf{y}),$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

This implies that when momentum is conserved, the relative position of the two particle system evolves as a one-particle system with effective mass  $\mu$ .

## Angular Momentum

A particle moving in  $\mathbb{R}^2$  with position  $\mathbf{x}$ , velocity  $\mathbf{v}$ , and momentum  $\mathbf{p} = m\mathbf{v}$  has angular momentum  $J$  denoted as

$$J = x_1 p_2 - x_2 p_1,$$

or  $J = \|\mathbf{x} \times \mathbf{p}\| = \|\mathbf{x}\| \|\mathbf{p}\| \sin \phi$ , with  $\phi$  measured counterclockwise. In polar coordinates, we find

$$\begin{aligned} J &= mr^2 \frac{d\theta}{dt} \\ &= 2M \frac{dA}{dt}, \end{aligned}$$

where  $A = 1/2 \int r^2 d\theta$  denotes the area swept out by  $\mathbf{x}(t)$ .

## Conservation of Angular Momentum

Suppose a particle of mass  $m$  is moving in  $\mathbb{R}^2$  under the influence of a conservative force with potential  $V(\mathbf{x})$ .  $V$  is invariant under rotation if and only if  $J$  is conserved.

**Proof:**

$$\begin{aligned} \frac{dJ}{dt} &= \frac{dx_1}{dt} p_2 + x_1 \frac{dp_2}{dt} - \frac{dx_2}{dt} p_1 - x_2 \frac{dp_1}{dt} \\ &= \frac{1}{m} p_1 p_2 - x_1 \frac{\partial V}{\partial x_2} - \frac{1}{m} p_2 p_1 + x_2 \frac{\partial V}{\partial x_1} \\ &= x_2 \frac{\partial V}{\partial x_1} - x_1 \frac{\partial V}{\partial x_2}. \end{aligned}$$

Alternatively, consider  $R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ . Differentiating  $V$  along  $R_\theta$ , we get

$$\begin{aligned} \left. \frac{d}{d\theta} V(R_\theta \mathbf{x}) \right|_{\theta=0} &= \frac{\partial V}{\partial x} \frac{dx}{d\theta} + \frac{\partial V}{\partial y} \frac{dy}{d\theta} \\ &= -x_2 \frac{\partial V}{\partial x_1} + x_1 \frac{\partial V}{\partial x_2} \\ &= -\frac{dJ}{dt}(\mathbf{x}) \end{aligned}$$

Thus,  $\frac{dJ}{dt} = 0$  if and only if the angular derivative of  $V$  is zero.

As a result of the conservation of angular momentum, we thus get Kepler's Second Law: if  $\mathbf{x}(t)$  is the trajectory of a particle under the influence of a force with rotationally invariant potential, then the area swept out by  $\mathbf{x}(t)$  between  $t = a$  and  $t = b$  is  $\frac{b-a}{2m} J$ .

In  $\mathbb{R}^3$ ,  $\mathbf{J}$  is a vector given by  $\mathbf{x} \times \mathbf{p}$ . Meanwhile, in  $\mathbb{R}^n$ , the angular momentum is a skew-symmetric matrix defined by

$$J_{jk} = x_j p_k - x_k p_j.$$

The total angular momentum of a system of  $N$  particles in  $\mathbb{R}^n$  is given by  $\mathbf{J}$  with entries

$$J_{jk} = \sum_{l=1}^N (x_j^l p_k^l - x_k^l p_j^l).$$

Similar to the case of linear momentum, angular momentum is constant in the presence of a conservative force if and only if the potential function  $V$  is rotationally invariant. That is,

$$V(R\mathbf{x}^1, R\mathbf{x}^2, \dots, R\mathbf{x}^N) = V(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N)$$

for all rotation matrices  $R$ .

## Hamiltonian Mechanics

The Hamiltonian is the total energy function, but formulated in terms of position and momentum rather than position and velocity. If a particle in  $\mathbb{R}^n$  has the usual energy function, we write

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m} \sum_{j=1}^n p_j^2 + V(\mathbf{x}),$$

where  $p_j = m_j \dot{x}_j$ . Observe that the equations of motion can be written as

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial H}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial x_j}. \end{aligned}$$

In the basic formulation, we can see that the first equation is just  $\dot{x}_j = p_j/m$ , and  $\dot{p}_j = F_j$ . The equations of motion written with Hamiltonians are known as Hamilton's equations.

## Poisson Bracket

Let  $f$  and  $g$  be two smooth functions on  $\mathbb{R}^{2n}$ , with each element of  $\mathbb{R}^{2n}$  being denoted by  $(\mathbf{x}, \mathbf{p})$ . The Poisson bracket of  $f$  and  $g$  is equal to

$$\{f, g\}(\mathbf{x}, \mathbf{p}) = \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right).$$

The Poisson bracket satisfies the following properties:

- Linearity:  $\{f, g + ch\} = \{f, g\} + c\{f, h\}$
- Antisymmetry:  $\{g, f\} = -\{f, g\}$
- Product Rule:  $\{f, gh\} = \{f, g\}h + g\{f, h\}$
- Jacobi Identity:  $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ .

It can be easily verified that the following Poisson bracket relations hold:

$$\begin{aligned} \{x_j, x_k\} &= 0 \\ \{p_j, p_k\} &= 0 \\ \{x_j, p_k\} &= \delta_{jk}, \end{aligned}$$

where  $\delta_{jk}$  denotes the Kronecker delta function.

## Functions of Solutions to Hamilton's Equations

If  $(\mathbf{x}(t), \mathbf{p}(t))$  is a solution to Hamilton's Equations, then for any smooth  $f$  on  $\mathbb{R}^{2n}$ , we have

$$\frac{df}{dt} = \{f, H\}.$$

**Proof:**

$$\begin{aligned} \frac{df}{dt} &= \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{dx_j}{dt} + \frac{\partial f}{\partial p_j} \frac{dp_j}{dt} \right) \\ &= \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{\partial H}{\partial p_j} + \frac{\partial f}{\partial p_j} \left( -\frac{\partial H}{\partial x_j} \right) \right) \\ &= \{f, H\}. \end{aligned}$$

### Conserved Quantities

Let  $f \in C^1(\mathbb{R}^{2n})$  be called conserved if  $f(\mathbf{x}(t), \mathbf{p}(t))$  is independent of  $t$  for each solution to Hamilton's equation. Then,  $f$  is a conserved quantity if and only if

$$\{f, H\} = 0.$$

Note that  $H$  is also a conserved quantity.

### Flow and Liouville's Theorem

Solving Hamilton's equations on  $\mathbb{R}^{2n}$  yields a flow  $\Phi_t^1$  with  $\Phi_t(\mathbf{x}, \mathbf{p})$  equal to the solution at time  $t$  with initial condition  $(\mathbf{x}, \mathbf{p})$ .

The  $\Phi_t$  aren't necessarily defined on all of  $\mathbb{R}^{2n}$ , but if  $\Phi_t$  is defined on  $\mathbb{R}^{2n}$  for all  $t$ , then we say  $\Phi_t$  is complete.

Liouville's Theorem<sup>2</sup> states that the flow preserves the  $2n$ -dimensional measure

$$dx_1 dx_2 \cdots dx_n dp_1 dp_2 \cdots dp_n.$$

More specifically, if  $E$  is a measurable subset of the domain of  $\Phi_t$ , then  $\mu(\Phi_t(E)) = \mu(E)$ .

**Proof:** Hamilton's equations can be written as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial x_1} \\ \vdots \\ -\frac{\partial H}{\partial x_n} \end{bmatrix}.$$

Hamilton's equations describe the flow along the vector field appearing on the right side — by a result in vector calculus,<sup>3</sup> the flow preserves the  $2n$ -dimensional area measure if and only if the divergence of the vector field is zero.

$$\begin{aligned} \nabla \cdot \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial x_1} \\ \vdots \\ -\frac{\partial H}{\partial x_n} \end{bmatrix} &= \sum_{k=1}^n \frac{\partial}{\partial x_k} \frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_k} \frac{\partial H}{\partial x_k} \\ &= \sum_{k=1}^n \frac{\partial^2 H}{\partial x_k \partial p_k} - \frac{\partial^2 H}{\partial p_k \partial x_k} \\ &= 0 \end{aligned}$$

The condition of zero divergence is equivalent to  $\Phi_t$  preserving a particular symplectic form  $\omega$  defined by

$$\omega((\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}')) = \mathbf{x} \cdot \mathbf{p}' - \mathbf{p} \cdot \mathbf{x}',$$

meaning that for any  $t$  and any  $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n}$ , the partial derivatives of  $\Phi_t$  preserves  $\omega$ .

<sup>1</sup>the  $\Phi_t$  are diffeomorphisms, or differentiable isomorphisms with differentiable inverses

<sup>2</sup>not the one from complex analysis

<sup>3</sup>Author's Note: I do not know this result yet, but hopefully I will soon!



Alternatively, this is equivalent to  $\Phi_t$  preserving Poisson brackets:

$$\{f \circ \Phi_t, g \circ \Phi_t\} = \{f, g\} \circ \Phi_t.$$

Thus,  $\Phi_t$  is an example of a symplectomorphism.

### Hamiltonian Flow and Hamiltonian Generators

We say  $f \in C^1(\mathbb{R}^{2n})$  is the Hamiltonian generator of the flow that results from solving Hamilton's equations with  $f$  substituted for  $H$ :

$$\begin{aligned}\frac{dx_j}{dt} &= \frac{\partial f}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial f}{\partial x_j}.\end{aligned}$$

It is possible to see that

$$f_a(\mathbf{x}, \mathbf{p}) = \mathbf{a} \cdot \mathbf{p}$$

yields the flow

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0 + t\mathbf{a} \\ \mathbf{p}(t) &= \mathbf{p}_0.\end{aligned}$$

and

$$g_b(\mathbf{x}, \mathbf{p}) = \mathbf{b} \cdot \mathbf{x}$$

yields the flow

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0 \\ \mathbf{p}(t) &= \mathbf{p}_0 - t\mathbf{b}.\end{aligned}$$

Thus, the Hamiltonian flow generated by momentum yields translation in position, and the Hamiltonian flow generated by position yields translation in momentum.

In this light, we can think of *the* Hamiltonian as the Hamiltonian generator that yields time evolution. Other Hamiltonian generators represent some other family of symmetries of the system.

### Hamiltonian Flow generated by Angular Momentum

For a particle moving in  $\mathbb{R}^2$ , the Hamiltonian flow generated by

$$J(\mathbf{x}, \mathbf{p}) = x_1 p_2 - x_2 p_1$$

consists of simultaneous rotations of  $\mathbf{x}$  and  $\mathbf{p}$ .

$$\begin{aligned}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.\end{aligned}$$

**Proof:** Plugging  $J$  Hamilton's equations, we get

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\partial J}{\partial p_1} = -x_2 \\ \frac{dp_1}{dt} &= -\frac{\partial J}{\partial x_1} = -p_2 \\ \frac{dx_2}{dt} &= \frac{\partial J}{\partial p_2} = x_1 \\ \frac{dp_2}{dt} &= -\frac{\partial J}{\partial x_2} = p_1.\end{aligned}$$

It's important to note that the parameter  $t$  in the Hamiltonian flow for  $J$  is the rotation, not time. That is,  $J$  is the Hamiltonian generator of rotations.

If  $f$  is any smooth function, it is the case that the time derivative of any other function  $g$  along the Hamiltonian flow generated by  $f$  is  $\frac{dg}{dt} = \{g, f\}$ . In particular, the derivative of  $H$  along the flow generated by  $f$  is  $\{H, f\}$ , meaning that  $f$  is constant along the flow generated by  $H$  if and only if  $\{f, H\} = 0$ , which is true if and only if  $H$  is constant along the flow generated by  $H$ .

Thus, we find that  $f$  is conserved for solutions of Hamilton's equations if and only if  $H$  is invariant under the Hamiltonian flow generated by  $f$ . Of particular note, we find that  $J$  is conserved if and only if  $H$  is invariant under rotations of  $\mathbf{x}$  and  $\mathbf{p}$ .