Consider 4 vectors:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Is it possible to create

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

from some combination of e_1, e_2, e_3, and e_4?

We say that v is in the *span* of $\{e_1, e_2, e_3, e_4\}$. The span of a set of vectors is the set of *linear combinations* of these vectors.

Let's try writing this set of vectors differently — instead of putting them into a set, let's just create a new structure to place them all together. Put each of e_1, e_2, e_3, and e_4 into one big set of parentheses — and flip them horizontally too.

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Both I and M are examples of a matrix. But, if you look more closely, you'll see that our vector v is in the second row of M in place of e_2. This has us wondering — can we "create" M from I, just as we created v from e_2?

Consider the following matrix.

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

For any vector v that is a linear combination of the rows of M, this means we can find a vector x such that linearly combining each row of M "by" x yields v.

Rank: the minimum size of the set of vectors needed to create all the linear combinations of the rows of the matrix.

A (square) matrix is *invertible* if its rank is equal to the number of rows (or columns).

Intuitively, this means every vector with the same number of rows as M can be created by linearly combining the "constituent vectors" of M, either rows or columns.

A lot of the conditions in the Fundamental Theorem of Invertible Matrices, shown below, are essentially based on this fact of linear combinations.

Fundamental Theorem of Invertible Matrices

Let A be a $n \times n$ matrix. The following statements are equivalent.

(a) A is invertible.

(b) $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.

(c) $Ax = 0$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is a product of elementary matrices.

(f) $\text{rank}(A) = n$.

(g) $\text{nullity}(A) = 0$.

(h) The column vectors of A are linearly independent.

(i) The column vectors of A span $\mathbb{R}^n$.

(j) The column vectors of A form a basis for $\mathbb{R}^n$.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A span $\mathbb{R}^n$.

(m) The row vectors of A form a basis for $\mathbb{R}^n$.

(Adapted from David Poole's *Linear Algebra, A Modern Introduction*, pg. 206)

Thank You. Any Questions?