Problem (Problem 1)**:**

(a) Determine every holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $\operatorname{Re}(f(z)) = \operatorname{Re}(z)^2 - \operatorname{Im}(z)^2$.

(b) Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) := \sqrt{|\operatorname{Re}(z)\operatorname{Im}(z)|}.$$

Show that the Cauchy–Riemann equations are satisfied for f at $z = 0$, but f is not differentiable at $z = 0$.

Solution:

(a) We want to determine $f\colon \mathbb{C} \to \mathbb{C}$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

satisfies

$$u(x, y) = x^2 - y^2,$$

and the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

First, we must verify that u is indeed harmonic. This follows from the fact that

$$\frac{\partial^2 u}{\partial x^2} = 2$$
$$\frac{\partial^2 u}{\partial y^2} = -2.$$

Furthermore, we see that u is C^3, as all of the third partial derivatives are equal to zero. Therefore, a harmonic conjugate of u exists and ensures that f is holomorphic on $\mathbb{C}$. By evaluating the Cauchy–Riemann equations separately, we find that

$$\frac{\partial v}{\partial y} = 2x,$$

or $v = 2xy + K(x)$, and

$$-\frac{\partial v}{\partial x} = -2y,$$

or $v = 2xy + L(y)$. These are only in harmony when $v = 2xy + c$, where $c \in \mathbb{C}$ is a constant. Thus, we find that

$$f(x + iy) = \left(x^2 - y^2\right) + i(2xy) + c$$

is necessarily (up to a constant) unique.

(b) We write f as

$$f(x + iy) = \sqrt{|xy|}.$$

Problem (Problem 2)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be a function.

(a) Suppose that f and $\overline{f}$ are both holomorphic. Show that f is constant.

(b) Suppose that f is holomorphic and $\operatorname{Re}(f)$ is constant. Show that f is constant.