

Problem (Problem 1): Let R be a commutative ring. An R -module M is called torsion if for any $m \in M$, there is a nonzero $r \in R$ such that $rm = 0$. An R -module M is called divisible if for any nonzero $r \in R$, we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$, there is $x \in M$ such that $rx = m$.

(a) Suppose M is a torsion R -module and N is a divisible R -module. Prove that $M \otimes_R N = \{0\}$.

(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a \mathbb{Z} -module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Solution:

(a) It is enough to show that any simple tensor $m \otimes n \in M \otimes_R N$ is the zero tensor. To see this, we let $r \in R$ be such that $rm = 0$, and observe that there is some $x \in N$ such that $rx = n$. By using property (R3) of tensor products, we observe then that

$$\begin{aligned} m \otimes n &= m \otimes (rx) \\ &= (rm) \otimes x \\ &= 0 \otimes x \\ &= 0. \end{aligned}$$

Thus, $M \otimes_R N = \{0\}$.

(b) It is enough to show that \mathbb{Q}/\mathbb{Z} is both torsion and divisible, as we may then apply (a). To see that \mathbb{Q}/\mathbb{Z} is torsion, we have that

$$\begin{aligned} b \left[\frac{a}{b} \right] &= [a] \\ &= [0] \end{aligned}$$

for any element $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$. Additionally, for any $n \in \mathbb{Z}$, we have

$$\left[\frac{a}{b} \right] = n \left[\frac{a}{nb} \right],$$

so \mathbb{Q}/\mathbb{Z} is both torsion and divisible.

Problem (Problem 2): Let R be a commutative ring, $\{N_\alpha\}_{\alpha \in A}$ a collection of R -modules, and M another R -module.

(a) Prove that $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) Show by example that $M \otimes (\prod_\alpha N_\alpha)$ need not be isomorphic to $\prod_\alpha (M \otimes N_\alpha)$.

Solution:

(a) Consider the map on elementary tensors

$$f: M \times \left(\bigoplus_\alpha N_\alpha \right) \rightarrow \bigoplus_\alpha (M \otimes N_\alpha)$$

that takes

$$(m, (n_\alpha)_\alpha) \rightarrow (m \otimes n_\alpha)_\alpha.$$

We observe that, since the $(n_\alpha)_\alpha$ are nonzero for all but finitely many indices α , and that the map is R -bilinear, we have a well-defined and unique R -linear map $\bar{f}: M \otimes (\bigoplus_\alpha N_\alpha) \rightarrow \bigoplus_\alpha (M \otimes N_\alpha)$ that maps $m \otimes (n_\alpha)_\alpha \mapsto (m \otimes n_\alpha)_\alpha$.

We observe that for each index i , we have an inclusion homomorphism

$$M \times N_i \hookrightarrow M \otimes \left(\bigoplus_\alpha N_\alpha \right)$$

that takes $(m, n_\alpha) \mapsto m \otimes (n_\alpha)_\alpha$, where $(n_\alpha)_\alpha$ is zero everywhere except for index i . By the universal property of the direct sum, this induces a unique homomorphism $g: \bigoplus_\alpha (M \otimes N_\alpha) \rightarrow M \otimes (\bigoplus_\alpha N_\alpha)$ given by taking

$$(m_\alpha \otimes n_\alpha)_\alpha \mapsto \sum_\alpha m_\alpha \otimes (n_\alpha)_\alpha,$$

where the summand $(n_\alpha)_\alpha$ is defined as above, and the sum is finite by the definition of the direct sum. Since g and f are inverses of each other (as can be seen by the action on simple tensors), it follows that $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) We consider the direct product

$$M = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z},$$

regarded as a \mathbb{Z} -module. Notice that M is not torsion, as the element $m = (1, 1, \dots)$ is such that there is no $z \in \mathbb{Z}$ with $zm = 0$. Therefore, considering the extension of scalars

$$\mathbb{Q} \otimes M = \mathbb{Q} \otimes \left(\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z} \right),$$

we have that this is not a zero module. Yet, since each of the individual $\mathbb{Z}/2^i\mathbb{Z}$ has torsion, it would follow that

$$\prod_{i=1}^{\infty} (\mathbb{Q} \otimes \mathbb{Z}/2^i\mathbb{Z}) = 0,$$

so it follows that tensor products do not commute with direct sums.

Problem (Problem 4): Let R be commutative, and let I and J be ideals of R , so R/I and R/J are naturally R -modules.

- Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$.
- Prove that there is an R -module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r + I) \otimes (r' + J)$ to $rr' + (I + J)$.

Solution:

- By using R -bilinearity, we observe that an arbitrary simple tensor in $R/I \otimes R/J$ can be written as

$$\begin{aligned} (r + I) \otimes (s + J) &= (r(1 + I)) \otimes (s + J) \\ &= r((1 + I) \otimes (s + J)) \\ &= (1 + I) \otimes (rs + J). \end{aligned}$$

Since any element of $R/I \otimes_R R/J$ can be written as a sum of simple tensors, and each simple tensor can be written in the above form, it follows from bilinearity that every element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (r + J)$.

- We consider the map

$$f: R/I \times R/J \mapsto R/(I + J)$$

given by

$$(r + I, r' + J) \mapsto rr' + (I + J).$$

This map is R -bilinear by the distributive properties of multiplication, so it induces a homomorphism on the tensor product given by

$$(r + I) \otimes (r' + J) \mapsto rr' + (I + J).$$

As was established above, any element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (s + J)$, so we may establish an inverse from any element of $R/(I + J)$ to $R/I \otimes R/J$ by taking $t + (I + J) \mapsto (1 + I) \otimes (t + J)$. This establishes our desired isomorphism.

Problem (Problem 5): Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $\mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an R -module annihilated by both 2 and x .

(a) Show that the map $\varphi: I \times I \rightarrow \mathbb{Z}/2\mathbb{Z}$ given by

$$\varphi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = \left[\frac{a_0}{2} b_1 \right]_{\mathbb{Z}/2\mathbb{Z}}$$

is R -bilinear.

(b) Show that there is an R -module homomorphism from $I \otimes_R I \rightarrow \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2} q'(0)$, where q' denotes the usual polynomial derivative of q .

(c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Solution:

(a) By the well-definedness of addition in R/I , we have that φ is additive in each variable. Now, letting $p(x) \in R$ and $a(x), b(x) \in I$ be defined by

$$\begin{aligned} a(x) &= a_0 + a_1x + \cdots + a_nx^n \\ b(x) &= b_0 + b_1x + \cdots + b_mx^m \\ p(x) &= p_0 + p_1x + \cdots + p_\ell x^\ell, \end{aligned}$$

we note that

$$p(x) + I = [p_0]_{\mathbb{Z}/2\mathbb{Z}}.$$

Using various definitions, we see that

$$\begin{aligned} \varphi(p(x)a(x), b(x)) &= \varphi(p_0a_0 + O(x), b_0 + b_1x + \cdots) \\ &= \left[\frac{p_0a_0}{2} b_1 \right] \\ &= [p_0] \left[\frac{a_0}{2} b_1 \right] \\ &= (p(x) + I) \varphi(a(x), b(x)), \end{aligned}$$

and since $b_0 \in I$,

$$\begin{aligned} \varphi(a(x), p(x)b(x)) &= \left[\frac{a_0}{2} (p_0b_1 + p_1b_0) \right] \\ &= \left[\frac{a_0}{2} (p_0b_1) \right] \\ &= (p(x) + I) \varphi(a(x), b(x)). \end{aligned}$$

Thus, φ is R -bilinear.

(b) Using the universal property for tensor products, there is a unique R -linear homomorphism $\bar{\varphi}: I \otimes_R I \rightarrow \mathbb{Z}/2\mathbb{Z}$ such that

$$\begin{aligned} \bar{\varphi}(a(x) \otimes b(x)) &= \left[\frac{a_0}{2} b_1 \right] \\ &= \left[\frac{p(0)}{2} q'(0) \right]. \end{aligned}$$

- (c) We observe that $\overline{\varphi}(2 \otimes x) = 1$ while $\overline{\varphi}(x \otimes 2) = 0$, so they cannot be equal to each other in $I \otimes_R I$.

Problem (Problem 6): Suppose R is commutative, and let I, J be ideals of R .

- (a) Show that there is a surjective R -module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $i \otimes j$ to ij .
- (b) Give an example to show that the map in (a) need not be injective.

Solution:

- (a) We define the R -bilinear map $\varphi: I \times J \rightarrow IJ$ by

$$\varphi(i, j) = ij.$$

This induces a linear map $\overline{\varphi}: I \otimes_R J \rightarrow IJ$ such that $i \otimes j \mapsto ij$. Since every element of $I \otimes_R J$ is a finite sum of elementary tensors, this surjects onto IJ since every element of IJ is a finite sum of elements of the form ij .

- (b)