Problem 1

Prove the following limits:

(i) $\left(\dfrac{2n}{n+2}\right)_n \to 2$

(ii) $\left(\dfrac{\sqrt{n}}{n+1}\right)_n \to 0$

(iii) $\left(\dfrac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0$

(iv) $\left(n^k b^n\right)_n \to 0$ where $0 \leq b < 1$ and $k \in \mathbb{N}$

(v) $\left(\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \to 1/3$

(i)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \Rightarrow \left|\frac{2n}{n+2} - 2\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}\frac{2n}{n+2} &> 2 - \varepsilon \\ 2n &> (2n - \varepsilon n) - 2\varepsilon + 4 \\ n &> \frac{4-2\varepsilon}{\varepsilon}\end{aligned}$$

Proof Let $N = \left\lceil \dfrac{4-2\varepsilon}{\varepsilon} \right\rceil$. Then,

$$\begin{aligned}n &> \frac{4-2\varepsilon}{\varepsilon} \\ \varepsilon n &> 4 - 2\varepsilon \\ 0 &> 4 - 2\varepsilon - \varepsilon n \\ 2n &> 2n + 4 - \varepsilon(n+2) \\ 2n &> (2-\varepsilon)(n+2) \\ \frac{2n}{n+2} - 2 &> -\varepsilon \\ \left|\frac{2n}{n+2} - 2\right| &< \varepsilon && \frac{2n}{n+2} < 2\ \forall n \in \mathbb{N}\end{aligned}$$

(ii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \to \left|\left(\frac{\sqrt{n}}{n+1}\right)\right| < \varepsilon$$

Preliminary Work

$$\frac{\sqrt{n}}{n+1} > 0 \ \forall n \in \mathbb{N}$$

$$\begin{aligned}
\frac{\sqrt{n}}{n+1} &< \varepsilon \\
\sqrt{n} &< (n+1)\varepsilon \\
n &< (n^2+2n+1)\varepsilon^2 \\
0 &< \varepsilon^2 n^2 + (2\varepsilon^2 - 1)n + \varepsilon^2 \\
0 &< \left(n - \frac{(1-2\varepsilon^2)+\sqrt{1-4\varepsilon^2}}{2\varepsilon^2}\right)\left(n + \frac{(1-2\varepsilon^2)+\sqrt{1-4\varepsilon^2}}{2\varepsilon^2}\right)
\end{aligned}$$

(iii)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \Rightarrow \left|\frac{(-1)^n}{\sqrt{n+7}}\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}
\frac{1}{\sqrt{n+7}} &< \varepsilon \\
\frac{1}{\varepsilon} &< \sqrt{n+7} \\
n &> \frac{1}{\varepsilon^2} - 7
\end{aligned}$$

Proof Let $N = \frac{1}{\varepsilon^2} - 7$. Then,

$$\begin{aligned}
n &> \frac{1}{\varepsilon^2} - 7 \\
n + 7 &> \frac{1}{\varepsilon^2} \\
\frac{1}{\sqrt{n+7}} &< \varepsilon \\
-\varepsilon &< \frac{-1}{\sqrt{n+7}} \\
\left|\frac{(-1)^n}{\sqrt{n+7}}\right| &< \varepsilon
\end{aligned}$$

(iv)

We need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \ni n > N \to \left|n^k b^n\right| < \varepsilon$$

Preliminary Work

$$\begin{aligned}
n^k b^n &< \varepsilon \\
b^{-n} &> \frac{n^k}{\varepsilon} \\
-n &> \frac{k \ln n}{\ln b} - \frac{\ln \varepsilon}{\ln b} \\
n + \frac{k \ln n}{\ln b} &< \frac{\ln \varepsilon}{\ln b} \\
n \ln b + k \ln n &> \ln \varepsilon && \text{since } \ln b < 0 \\
n \ln b + kn &> \ln \varepsilon && n > \ln n \ \forall n \in \mathbb{N} \\
n &> \frac{\ln \varepsilon}{k + \ln b}
\end{aligned}$$