Problem (Problem 1)**:** Prove that if $f\colon M \to N$ is smooth, and L is a k-dimensional submanifold of N that is transverse to f, then $f^{-1}(L)$ is either empty or a submanifold of M with codimension k.

Solution: If L is not contained in $f(M)$, then $f^{-1}(L)$ is clearly empty. Therefore, we focus on the case where $f^{-1}(L)$ is not empty.

Let L be transverse to f, $q \in L$, and $p \in M$ such that $f(p) = q$. We observe that $T_qL + D_pF\big(T_pM\big) = T_qN$, so any vector in T_qN can be written (not necessarily uniquely) as an element of $D_pF\big(T_pM\big)$ and T_qL. Next, we observe that, if we take a coordinate chart for q in U such that $\varphi(U) \cong \mathbb{R}^{n-k}$, then by the Regular Value Theorem, we may select φ such that $L \cap U = \varphi^{-1}(0)$.

Now, if we can show that 0 is a regular value for $\varphi \circ f$, then $(\varphi \circ f)^{-1}(0) = f^{-1}(L) \cap f^{-1}(U)$. First, since 0 is a regular value for φ, it follows that if $v \in T_0\mathbb{R}^{n-k}$, then there is some $w \in T_qN$ such that $D_q\varphi(w) = v$. Since f is transverse to L, there is $w_1 \in T_qL$ and $w_2 \in T_pM$ such that $w = w_1 + D_pF(w_2)$. We observe that, since φ is constant on L, we have $D_q\varphi(w_1) = 0$, so that

$$\begin{aligned} D_p(\varphi \circ f)(w_2) &= D_q\varphi \circ D_pF(w_2) \\ &= D_q\varphi\big(w_1 + D_pF(w_2)\big) \\ &= D_q\varphi(w) \\ &= v, \end{aligned}$$

so 0 is a regular value for $\varphi \circ F$.

Problem (Problem 2)**:** Let $GL_n(\mathbb{R})$ denote the space of invertible $n \times n$ matrices over $\mathbb{R}$, let $SL_n(\mathbb{R})$ denote the matrices of determinant one, and let $O(n)$ be the orthogonal group.

(a) Prove that we may identify the tangent space of $GL_n(\mathbb{R})$ at the identity with $n \times n$ matrices over $\mathbb{R}$.

(b) Prove that the tangent space of $SL_n(\mathbb{R})$ at the identity consists of matrices of trace zero.

(c) Prove that the tangent space of $O(n)$ at the identity consists of skew-symmetric matrices. What is the dimension of $O(n)$?

(d) Show that $SL_n(\mathbb{R})$ and $O(n)$ do not intersect transversely at the identity.

Solution:

(a) Let $A \in \operatorname{Mat}_n(\mathbb{R})$, and consider a path through the identity given by $\gamma(t) = I + tA$. Since the determinant is a smooth function, and $\det(I) = 1$, we have that for a small $\varepsilon > 0$ there is δ, such that $|\det(I + tA) - 1| < \varepsilon$ whenever $|t| < \delta$. In particular, this means that the tangent space at the identity of $GL_n(\mathbb{R})$ consists of all matrices.

(b) We let $\gamma(t) = I + tA$ be a curve in $SL_n(\mathbb{R})$, so that $\gamma'(0) = A$ is an element of the tangent space of $SL_n(\mathbb{R})$ at the identity. We observe that $\det(\gamma(t)) = 1$ for all (sufficiently small) t, so we see that

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)) \\ &= \overset{\prime}{\det}(\gamma(0))(\gamma'(0)) \\ &= \overset{\prime}{\det}(I)(A). \end{aligned}$$

Therefore, we must evaluate what $\det'(I)(A)$ yields. Toward this end, we compute the derivative directly from the definition, yielding

$$\overset{\prime}{\det}(I)(A) = \lim_{t \to 0} \frac{\det(I + tA) - 1}{t}.$$

The expression $\det(I + tA)$ is a polynomial in t where the constant term is 1 and the term in t is $\operatorname{tr}(A)$. Thus, we find that $0 = \operatorname{tr}(A)$, so A is traceless.

(c) If $\gamma(t) = I + tA$ is a curve in $O(n)$, then then we have that

$$\begin{aligned}(I + tA)^T(I + tA) &= I\\ I + t\left(A^T + A\right) + t^2\left(A^T A\right) &= I,\end{aligned}$$

meaning that by taking an equivalence class of this tangent curve, we have

$$i + t\left(A^T + A\right) = I,$$

so that $A^T = -A$.

We observe that the function $f\colon \operatorname{Mat}_n(\mathbb{R}) \to \operatorname{Mat}_n(\mathbb{R})_{\text{s.a.}}$, given by

$$f(A) = A^T A,$$

has I_n as a regular value. To see this, observe that curves in $T_I \operatorname{Mat}_n(\mathbb{R})_{\text{s.a.}}$ are of the form $\gamma(t) = I + tK$, where K is a self-adjoint(/symmetric) matrix. Similarly, $T_A \operatorname{Mat}_n(\mathbb{R})$ is of the form $\varepsilon(t) = A + tB$, where $B \in \operatorname{Mat}_n(\mathbb{R})$ and $t \in \mathbb{R}$. Both of these follow from the fact that $\operatorname{Mat}_n(\mathbb{R})$ and $\operatorname{Mat}_n(\mathbb{R})_{\text{s.a.}}$ are isomorphic to Euclidean spaces. Therefore, we see that the image of $\delta(t)$ is of the form $A^T A + t\left(A^T B + B^T A\right)$; if A satisfies $A^T A = I$, we can put this in the form of $I + tK$ by taking $\delta(t) = A + \frac{1}{2}tAK$. Therefore, by the Regular Value Theorem, the dimension of $O(n)$ is $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

(d) Since both skew-symmetric and traceless matrices have trace zero, it follows that the tangent spaces of $SL_n(\mathbb{R})$ and $O(n)$ cannot span the tangent space of $GL_n(\mathbb{R})$, as there are matrices with nonzero trace.