Abstract

We discuss compactness in topological spaces, normed spaces, and weak compactness, covering results such as Tychonoff's Theorem, relations between norm-compactness and dimension, sequential compactness, the Banach–Alaoglu Theorem, and the Eberlein–Šmulian theorem.

Compactness in Topological Spaces

Traditionally, one is introduced to compactness in their first class on topology. There, the definition of compactness appears a bit strange — but we'll see soon enough that there are a variety of simpler, equivalent ways to use compactness that are just as powerful as the original definition. However, as is customary, we start with the standard definition.

Definition. Let X be a topological space. An *open cover* of X is a family of open sets $\{U_i\}_{i\in I}$ such that

$$X \subseteq \bigcup_{i\in I} U_i.$$

Definition. Let X be a topological space. We say X is *compact* if, for any open cover of X, $\{U_i\}_{i\in I}$, there is a finite $F \subseteq I$ such that

$$X \subseteq \bigcup_{i\in F} U_i.$$

In other words, X is compact if every open cover admits a finite subcover.

Nets, Filters, and Ultrafilters

Tychonoff's Theorem

Compactness in Normed Spaces and Metric Spaces

Compactness and Dimension

Compactness and Sequential Compactness

Compactness in Continuous Function Spaces

Weak Compactness

The Banach–Alaoglu Theorem

Goldstine's Theorem

The Eberlein–Šmulian Theorem