Properties of $\mathbb{Z}^+$

(i)

$$m, n \in \mathbb{Z}^+ \Rightarrow m+n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$$

Since $0 \leq_a m$ and $0 \leq_a n$, it must be the case that $0 \leq_a m+n$, as n cannot be less than zero. Similarly, $m \cdot n = \underbrace{m + m + \cdots + m}_{n \text{ times}}$, meaning $0 \leq_a m \cdot n$, so $m \cdot n$ and $m+n$ are both in $\mathbb{Z}^+$

(ii)

$$m \in \mathbb{Z} \Rightarrow m \in \mathbb{Z}^+ \text{ or } -m \in \mathbb{Z}^+$$

Let $m \in \mathbb{Z}$. Then, $0 \leq_a m$ or $m \leq_a 0$. In the first case, $m \in \mathbb{Z}^+$, and in the second case, we know that $-m\ {}_a{\geq} 0$, so $-m \in \mathbb{Z}^+$.

(iii)

$$m, -m \in Z^+ \Rightarrow m = 0$$

If $m \in \mathbb{Z}^+$, then $0 \leq_a m$, and if $-m \in \mathbb{Z}^+$, $0 \leq_a -m$, or $m \leq_a 0$. Therefore, $m = 0$.

(iv)

$$m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$$

($\Rightarrow$) Let $m \leq_a n$. Then, $m - m \leq_a n - m$, so $0 \leq_a n - m$, so $n - m \in \mathbb{Z}^+$.

($\Leftarrow$) Let $n - m \in \mathbb{Z}^+$. Then, $0 \leq_a n - m$, so $m \leq_a n$.