Problem 1

If F is a finite set and $k : F \to F$ is a self-map, prove that k is injective if and only if k is surjective.

Suppose k is injective. Then, $\text{card}(k(F)) = \text{card}(F)$, and since $k(F) \subseteq F$, $k(F) = F$, so k is surjective.

Let k be surjective. Since k is a function, $\text{card}(k(F)) \leq \text{card}(F)$.

Suppose $\text{card}(k(F)) < F$. Then, $k(F)$ contains at most $n-1$ elements, for $\text{card}(F) = n$, which would violate surjectivity.

Thus, $\text{card}(k(F)) = \text{card}(F)$, so k is injective.

Problem 2

Prove that a set A is infinite if and only if there is a non-surjective injection $f : A \hookrightarrow A$.

($\Rightarrow$) Let A be infinite. Then, $\exists i : \mathbb{N} \hookrightarrow A$; $\forall n \in \mathbb{N}, a_n := i(n)$. Let $f : A \to A$, $f(a_i) = a_{i+1}$. Then, for $a_{i_1} \neq a_{i_2}$, $f(a_{i_1}) = a_{i_1+1} \neq f(a_{i_2}) = a_{i_2+1}$. Therefore, f is injective, but $a_1 \notin \text{ran}(f)$, so f is not surjective.

($\Leftarrow$) Suppose A is finite. Then, by the result in Problem 1, $\forall f : A \hookrightarrow A$, f must be surjective.

Problem 3

Let A, B, and C be sets and suppose $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$. Prove that $\text{card}(A) < \text{card}(C)$.

Since $\text{card}(A) < \text{card}(B)$, $\text{card}(A) \leq \text{card}(B)$, so $\text{card}(A) \leq \text{card}(C)$, by the transitive property.

Since $\text{card}(A) \neq \text{card}(B)$, $\text{card}(A) \neq \text{card}(C)$, so $\text{card}(A) < \text{card}(C)$.

Problem 4

If $A \subseteq B$ is an inclusion of sets with A countable and B uncountable, show that $B \setminus A$ is uncountable.

Suppose toward contradiction that $B \setminus A$ is countable.

Then, $A \cup (B \setminus A)$ must be countable, by union of countable sets.

However, $A \cup (B \setminus A) = B$, and B is uncountable, meaning that $B \setminus A$ must be uncountable.

Problem 5

Is the set $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ countable?

Since $x > 0$, $t(x) = x^2$ is a bijection, as it has an inverse $t^{-1}(x) = \sqrt{x}$. Let $q : \mathbb{Q} \to \mathbb{N}$ denote the enumeration of the rationals (which is bijective).

$q \circ t : \{x \in \mathbb{R} \mid x > 0 \ \mid \text{and } x^2 \in \mathbb{Q}\} \to \mathbb{N}$ is the composition of bijections, so $q \circ t$ is a bijection, so $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ is countable.

Problem 6

Consider the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$. Is $\mathcal{F}(\mathbb{N})$ countable?

Let $f : \mathcal{F} \to \mathbb{N}$ be defined as follows, where p_n denotes the nth prime number.

$$f(\{a_1, a_2, \dots, a_n\}) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_n^{a_n}$$

By the fundamental theorem of arithmetic, every natural number is equal to a unique product of powers of prime numbers, meaning that f is injective, so $\mathcal{F}$ is countable.

Problem 7

Let $k \in \mathbb{N}$.

(i) Prove that $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \mathbb{N}}_{k \text{ times}}$ is countable.

(ii) Show that the set $\mathbb{N}^\infty := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}\}$ consisting of all sequences of natural numbers is uncountable.

(iii) Prove that the set of **finitely-supported** natural sequences $c_c(\mathbb{N}) := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}, n_k = 0 \text{ for all but finitely many } k\}$ is countable.

(i)

Let $f : \mathbb{N}^k \to \mathbb{N}$ be defined as follows, where p_n denotes the nth prime number in the sequence $\{2, 3, 5, \dots\}$

$$f((a_1, a_2, \dots, a_k)) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$$

By the fundamental theorem of arithmetic, f is an injection, so $\mathbb{N}^k$ is countable.

(ii)

Suppose toward contradiction that the set of all sequences of natural numbers is countable, so $\exists f : A_n \to \mathbb{N}$ is surjective.

$$\begin{aligned} A_1 &= a_{11}, a_{12}, a_{13}, \dots \\ A_2 &= a_{21}, a_{22}, a_{23}, \dots \\ &\vdots \end{aligned}$$

Create a new sequence N defined as follows:

$$n_k = a_{kk} + 1$$

Since f is surjective, $\exists A_m = a_{m1}, a_{m2}, \dots, a_{mm}, \dots = n_1, n_2, \dots, n_m, \dots$.However, $n_m \neq a_{mm}$, so f must not be surjective. Thus, $\mathbb{N}^\infty$ is not countable.

(iii)

Let $f : c_c(\mathbb{N}) \to \mathbb{N}$ be defined as follows, where p_n denotes the nth prime number:

$$f((n_i)) = p_1^{n_1} \cdot p_2^{n_2} \cdots p_i^{n_i} \cdots$$

By the fundamental theorem of arithmetic, f must be an injection, so $c_c(\mathbb{N})$ is countable.

(iv)

Is the set of decreasing natural sequences

$$D := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}, n_{k+1} \leq n_k,\ \forall k \geq 1\}$$

countable or uncountable?

Let $i : D \to \mathbb{N}$ be defined as follows, where p_k denotes the kth prime number:

$$i((n_i)) = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$$

where n_k is the lower bound of the sequence. k is the smallest index in the sequence with value n_k.

By the fundamental theorem of arithmetic, i must be injective, so D is countable.

Problem 8

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range $\text{ran}(f)$ cannot contain any interval.

In (a, b), $a < b$, there are countably many rational numbers (as $\mathbb{Q}$ is countable), but uncountably many irrational numbers.

$f_{(a,b)} : (a, b) \to (a, b)$ implies that there are uncountably many irrational numbers not in $\text{ran}(f_{(a,b)})$. Therefore, no interval is in $\text{ran}(f)$, as there is no interval in $\text{ran}(f_{(a,b)})$.

Problem 9

Prove that the set

$$\mathcal{P} := \left\{ \sum_{k=0}^{n} a_k x^k \mid n \subseteq \mathbb{N}_0, a_k \in \mathbb{Q} \right\}$$

consisting of all polynomials with rational coefficients, is countable.

Let $q : \mathbb{Q} \to \mathbb{N}$ be the enumeration of the rationals, and let p_n denote the nth element in the sequence of prime numbers, where $p_1 = 2, p_2 = 3$, etc.

Let $f : \mathcal{P} \to \mathbb{N}^k$ be defined as follows:

$$f(a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots) = (q(a_0), q(a_1), \ldots, q(a_k), \ldots)$$

Since $\mathbb{Q}$ is countable, $\forall a \in \mathbb{Q}$, $q(a) \in \mathbb{N}$, so the output of f is a bijection to $\mathbb{N}^k$, meaning $\mathcal{P}$ is countable.

Problem 10

A real number t is called **algebraic** if there is a nonzero polynomial p with rational coefficients such that $p(t) = 0$. If $t \in \mathbb{R}$ is not algebraic, then it is called **transcendental**. For example, $\sqrt{2}$ is algebraic, but π is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.

By the fundamental theorem of algebra, the set of real roots of a k degree polynomial has cardinality at most k.

$\forall p \in \mathcal{P}, \exists A_p = \{a_1, \dots, a_k\}$ such that $a_i \in \mathbb{R},\ \forall a_i \in \{a_1, \dots, a_k\},\ p(a_i) = 0$. Therefore, $\mathbb{A} = \bigcup_{p \in \mathcal{P}} A_p$ is a countable union of countable sets, meaning $\mathbb{A}$ is countable.

Since $\mathbb{T} = \mathbb{R} \setminus \mathbb{A}$, from Problem 4, $\mathbb{T}$ must be uncountable.