## Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

**Proof:** Let $(f_n)_n$ be a Cauchy sequence in $C_0(\mathbb{R})$. Since each $f_k \in C_0(\mathbb{R})$, it must be the case that each $f_k$ is uniformly continuous. For each $x \in \mathbb{R}$, it is thus the case that $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(f_n(x))_n \to f(x)$ for each $x \in \mathbb{R}$, and since each $f_k$ is uniformly continuous, it must be the case that $f(x)$ is continuous.

For $\varepsilon > 0$, there must be $N$ large such that for $m, n \geq N$ and $m \geq n$, it must be the case that $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Letting $m \to \infty$, we have $|f_n(x) - f(x)| < \varepsilon$, meaning $(f_n)_n \to f$. Thus, $f \in C_0(\mathbb{R})$.

## Problem 2

Show that $\ell_2$ is a Hilbert space.

**Proof:** Let $(x_n)_n$ be a Cauchy sequence in $\ell_2$. Let $x_n(k)$ denote the index $k$ of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$\begin{aligned} |x_n(k) - x_m(k)| &\leq c \left\|x_m(k) - x_n(k)\right\|_2 \\ &\to 0 && \text{since } (x_n)_n \text{ is Cauchy in } \ell_2. \end{aligned}$$

Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, $x_n(k) \to x(k)$ as $k \to \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \to x(k)$ for each $n$.

We must show that $\|x_n - x\|_2 \to 0$ as $n \to \infty$. This is equivalent to

$$\begin{aligned} \lim_{N\to\infty} \sum_{k=1}^{N} \lim_{m\to\infty} |x_n(k) - x_m(k)|^2 &\leq \lim_{m\to\infty} \sup_{m \geq M} \|x_n - x_m\|^2 \\ &\leq \varepsilon^2 && \text{since } (x_n)_n \text{ is Cauchy.} \end{aligned}$$

Thus, $\|x_n - x_m\| \to 0$ as $m \to \infty$ and $n \to \infty$, so $\|x_n - x\| \to 0$ as $n \to \infty$.

## Problem 3

Suppose $(X, d)$ is a complete metric space and $(x_n)_n$ is a sequence in $X$ such that there is a $\theta \in (0,1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

**Proof:** We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let $m, n$ be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq d(x_2, x_1)\left(\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}\right) \\ &= d(x_2, x_1)\theta^{n-1}\left(1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}\right) \\ &\leq d(x_2, x_1)\frac{\theta^{n-1}}{1-\theta}. \end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since $X$ is complete, $(x_n)_n$ is convergent.

## Problem 4

Let $(X, d)$ be a complete metric space, and suppose $f : X \to X$ is a contractive map — i.e., there is a $\theta \in (0,1)$ with

$$d(f(x), f(y)) \leq \theta d(x, y).$$

Prove that $f$ has a unique fixed point.

**Proof:** Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all $n$, we have

$$d(x_n, x_{n-1}) \leq \theta^n d(x_1, x_0).$$

Therefore, for $x_m, x_n$ arbitrary in $X$ with $m > n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0) \\ &= d(x_1, x_0)\theta^{n+1}\left(1 + \theta + \cdots + \theta^{m-n-1}\right) \\ &\leq d(x_1, x_0)\frac{\theta^{n+1}}{1-\theta}. \end{aligned}$$

Since $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$ as $n \to \infty$, it must be the case that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Since $X$ is complete, this means $(x_n)_n \to x$ for some $x \in X$, meaning $f(x) = x$.

Suppose it were the case that there existed $s, t$ distinct with $f(s) = s$ and $f(t) = t$. Then, $d(f(s), f(t)) = d(s, t) \leq \theta d(s, t)$, but $\theta < 1$. Thus, $x$ is unique.