Problem (Problem 1)**:** Let $(a_n)_n$ be a sequence for which $\sum_{n=0}^{\infty}|a_n|^2$ is finite. For each positive N, define $f_N(z) = \sum_{n=0}^{\infty} a_n z^n$, and define $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(a) Show that f is holomorphic on $\mathbb{D}$.

(b) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f_N\left(re^{i\theta}\right)\right|^2\,d\theta.$$

(c) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta.$$

(d) Determine in terms of $(a_n)_n$ the limit

$$\lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta.$$

Solution:

(a) Let $0 < r < 1$. Since each f_N is analytic, we can use the Cauchy Integral Formula to compute a_N explicitly, yielding

$$\begin{aligned}|a_N| &= \left|\frac{1}{2\pi i}\int_0^{2\pi}\frac{f_N(\xi)}{\xi^{N+1}}\,d\xi\right| \\ &\leqslant \frac{1}{r^N}\sup_{|z|=r}|f_N(z)|.\end{aligned}$$

Therefore, if we are able to show that the value

$$\sup_{|z|=r}|f_N(z)|$$

is uniformly bounded by a constant for all N, we will be able to use the Cauchy–Hadamard theorem to show that $\limsup_{N\to\infty}|a_N|^{1/N} \leqslant 1$. Toward this end, we use the Cauchy–Schwarz inequality, which yields

$$\begin{aligned}\sup_{|z|=r}|f_N(z)| &= \sup_{|z|=r}\left|\sum_{n=0}^{N} a_n z^n\right| \\ &\leqslant \sup_{|z|=r}\left(\sum_{n=0}^{N}|a_n|^2\right)^{1/2}\left(\sup_{m=0}^{N}|z|^{2m}\right)^{1/2} \\ &\leqslant \sup_{|z|=r}\underbrace{\left(\sum_{n=0}^{\infty}|a_n|^2\right)^{1/2}}_{=:K}\left(\sum_{m=0}^{\infty}|z|^{2m}\right)^{1/2} \\ &= \frac{K}{(1-|r|^2)^{1/2}}.\end{aligned}$$

Since we have established this uniform bound, we thus find that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence at least 1, so f is analytic on $\mathbb{D}$, hence holomorphic on $\mathbb{D}$.

(b) We write out the integral to yield

$$\begin{aligned}
\frac{1}{2\pi}\int_{0}^{2\pi}\left|f_N\left(re^{i\theta}\right)\right|^2\,d\theta &= \frac{1}{2\pi}\int_{0}^{2\pi}\left(\sum_{n=0}^{N}a_n r^n e^{in\theta}\right)\overline{\left(\sum_{m=0}^{N}a_m r^m e^{im\theta}\right)}\,d\theta\\
&= \frac{1}{2\pi}\sum_{n=0}^{N}\sum_{m=0}^{N}a_n\overline{a_m}r^{m+n}\int_{0}^{2\pi}e^{i(n-m)\theta}\,d\theta\\
&= \sum_{n=0}^{N}|a_n|^2 r^{2n}.
\end{aligned}$$

(c) Since f is holomorphic with radius of convergence at least 1, the series expression on $S(0,r)$ converges uniformly, so that we may exchange sum and integral. This yields

$$\begin{aligned}
\frac{1}{2\pi}\int_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n\overline{a_m}r^{m+n}\int_{0}^{2\pi}e^{i(n-m)\theta}\,d\theta\\
&= \sum_{n=0}^{\infty}|a_n|^2 r^{2n}.
\end{aligned}$$

(d) Since the sequence $(a_n)_n$ is square-summable, the limit is well-defined, and we get

$$\begin{aligned}
\lim_{r\nearrow 1}\frac{1}{2\pi}\int_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta &= \lim_{r\nearrow 1}\sum_{n=0}^{\infty}|a_n|^2 r^{2n}\\
&= \sum_{n=0}^{\infty}|a_n|^2.
\end{aligned}$$

Problem (Problem 2)**:** Let $\varphi\colon [0,1]\to\mathbb{C}$ be continuous, and define $f\colon \mathbb{C}\setminus[0,1]\to\mathbb{C}$ by

$$f(z) = \int_{0}^{1}\frac{\varphi(t)}{t-z}\,dt.$$

Show that f is holomorphic and determine the derivative of f in terms of φ.

Problem (Problem 3)**:** Let $f\colon\mathbb{C}\to\mathbb{C}$ be entire.

(a) Suppose there exist $C,R>0$ and $n\in\mathbb{N}$ such that $|f(z)|\leqslant C|z|^n$ for all $|z|>R$. Show that f is a polynomial of degree at most n.

(b) Suppose that $g\colon\mathbb{C}\to\mathbb{C}$ is also entire and $|f(z)|\leqslant|g(z)|$ for all $z\in\mathbb{C}$. Show that there exists some $\alpha\in\mathbb{C}$ with $|\alpha|\leqslant 1$ such that $f(z)=\alpha g(z)$ for all $z\in\mathbb{C}$.

(c) Suppose that there exists some $\theta\in\mathbb{R}$ such that $f(\mathbb{C})\cap\left\{re^{i\theta}\;\middle|\;r>0\right\}=\emptyset$. Show that f is constant.

Solution:

(a) Let $r>R$. Then, by the Cauchy estimate, we get that

$$\begin{aligned}
\left|f^{(n+1)}(0)\right| &\leqslant \frac{(n+1)!}{r^{n+1}}\sup_{|z|=r}|f(z)|\\
&\leqslant \frac{(n+1)!}{r^{n+1}}\sup_{|z|=r}\left(C|z|^n\right)\\
&= \frac{C(n+1)!}{r},
\end{aligned}$$

so since r is arbitrary and f is entire, we find that $f^{(n+1)}(0) = 0$, so that the power series expansion of f about 0 terminates beyond $n+1$, meaning that f is a polynomial of degree at most n.

(b) If g is 0, then we are done. Else, assume that g is not identically zero. Observe that if g is everywhere non-vanishing, then the function $\frac{f(z)}{g(z)}$ is entire, and satisfies

$$\left|\frac{f(z)}{g(z)}\right| \leqslant 1,$$

hence $\frac{f(z)}{g(z)} = \alpha$ for some α with $|\alpha| \leqslant 1$.

Now, if $g(z)$ does admit zeros, they must be isolated zeros, or else by the identity theorem, we would have that g is identically zero on $\mathbb{C}$. Letting

$$h(z) = \frac{f(z)}{g(z)},$$

we see that h admits isolated singularities at the zeros of g. In punctured neighborhoods of these zeros, h is bounded by assumption, so each of these singularities is removable. Therefore, h can be extended to an entire function, $k(z)$, satisfying

$$|k(z)| \leqslant 1$$

for all $z \in \mathbb{C}$. Thus, by Liouville's Theorem, it follows that $k(z) = \alpha$ for some $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$. In particular, whenever $g(z) \neq 0$, we have $f(z) = \alpha g(z)$, and clearly if g is zero, so too is f. Thus, we have established the desired result.

(c) Suppose $f(\mathbb{C}) \cap \left\{re^{i\theta} \;\middle|\; r > 0\right\} = \emptyset$. For $s > 0$, Cauchy's Estimate gives

$$|f'(0)| \leqslant \frac{1}{s^2} \sup_{|z|=s} |f(z)|.$$