Distributions and Estimates

The purpose of both of these distributions is to allow for inferences about μ and σ in an unknown distribution. Both are quotients of known distributions.

Preliminaries

Sample Mean: Let $Y_1, \dots, Y_n$ be a random, independent sample from a distribution with mean μ and variance σ^2. Then,

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad \text{Sample Mean}$$

is a distribution with mean $\overline{\mu} = \mu$ and variance $\overline{\sigma}^2 = \frac{\sigma^2}{n}$. If the underlying distribution is a normal distribution, then $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ is a *standard* normal distribution.

Sample Variance: The *sample variance* is defined as

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2. \qquad \text{Sample Variance}$$

It is important to note that the sample variance is found for samples drawn from a distribution; for population standard deviation/variance, we use n instead of $n-1$ in the denominator.

When Y_i is a normal distribution, then $\frac{(n-1)S^2}{\sigma^2}$ is a χ^2 distribution with $n-1$ df — S^2 and $\overline{Y}$ are independent.

Definition of T Distribution

Let Z be a standard normal distribution, W be χ^2 with ν df, and Z and W be independent. Then,

$$T = \frac{Z}{\sqrt{W/\nu}}$$

has a T distribution with ν df.

Creating a T Distribution: Let Y_i be sampled from a normal distribution with mean μ and standard deviation σ.

Then, $Z = \frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ is a standard normal distribution, and $W = \frac{(n-1)S^2}{\sigma^2}$ is χ^2 with $n-1$ df.

So,

$$\begin{aligned}
T &= \frac{Z}{\sqrt{W/(n-1)}}\\
&= \frac{(\overline{Y}-\mu)\sqrt{n}}{\sigma}\sqrt{\frac{(n-1)\sigma^2}{S^2}}\\
&= \frac{(\overline{Y}-\mu)\sqrt{n}}{S}
\end{aligned}$$

has a T distribution with $n-1$ df.

T **Distribution:** Let $Y_1, \dots, Y_6$ be samples from a normal distribution with unknown μ, σ. Estimate $P(|\overline{Y} - \mu| < (2S/\sqrt{n}))$.

Thus, we have

$$\begin{aligned} P\left(|\overline{Y} - \mu| \leq \frac{2S}{\sqrt{n}}\right) &= P\left(-2 \leq \frac{\sqrt{n}(\overline{Y} - \mu)}{S} \leq 2\right) \\ &= P(-2 \leq T \leq 2) \end{aligned}$$

Thus, for $n = 6$, we have that our random variable T has 5 df. By looking at a T distribution table, we can find that $P \approx 0.9$. We can also use R.

Definition of F Distribution

Let W_1 and W_2 be independent χ^2 distributions with ν_1 and ν_2 df respectively. Then, the F distribution with ν_1 numerator df and ν_2 denominator df is found as follows:

$$F = \frac{W_1/\nu_1}{W_2/\nu_2}$$

Simplifying an F Distribution: Let n_1 samples be drawn from normal distribution with mean μ_1 and variance σ_1^2, and n_2 samples be drawn from normal distribution with mean μ_2 and variance σ_2^2. Both distributions are independent.

From each of these samples, we find the sample variance, and create χ^2 distributions with their respective df.

$$\begin{aligned} W_1 &= \frac{(n_1 - 1)S_1^2}{\sigma_1^2} \\ W_2 &= \frac{(n_2 - 2)S_2^2}{\sigma_2^2} \end{aligned}$$

Therefore, we have

$$\begin{aligned} F &= \frac{W_1/(n_1 - 1)}{W_2/(n_2 - 1)} \\ &= \frac{(n_1 - 1)S_1^2}{\sigma_1^2(n_1 - 1)} \frac{\sigma_2^2(n_2 - 1)}{(n_2 - 1)S_2^2} \\ &= \frac{\sigma_2^2}{\sigma_1^2} \frac{S_1^2}{S_2^2} \end{aligned}$$

as an F distribution with $n_1 - 1$ numerator df and $n_2 - 1$ denominator df.

Applying the F Distribution: Let $n_1 = 6$ and $n_2 = 10$ be two samples from independent normal distributions with the same σ^2. Find b such that $P\left(\frac{S_1^2}{S_2^2} \leq b\right) = 0.95$.

$$\frac{S_1^2}{S_2^2} = \frac{S_1^2/\sigma^2}{S_2^2/\sigma^2}$$

The given F distribution has 5 numerator df and 9 denominator df. Therefore, we want to find $0.95 = P(F_{5,9} < b)$, or find the 0.95 quantile; in R, we find this with the `qt` function.

Normal Approximation of Binomial

Recall that a binomial distribution Y with n trials and p probability of success has probabilities found below:

$$P(Y \leq \ell) = \sum_{k=0}^{\ell} \binom{n}{k} p^k (1-p)^{n-k}.$$

For very large n, this sum is hard to calculate. We could approximate with the Poisson distribution, but this still requires a lot of calculations and large factorial values. Instead, we will try the following:

$$\begin{aligned} X_i &= \begin{cases} 1 & i \text{ trial success} \\ 0 & i \text{ trial failure} \end{cases} \\ E(X_i) &= p \\ E(X_i^2) &= p \\ V(X_i) &= p(1-p) \\ \overline{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{Y}{n} \\ E(\overline{X}) &= p \\ V(\overline{X}) &= \frac{p(1-p)}{n} \end{aligned}$$

By the Central Limit Theorem, we approximate $\overline{X}$ as a normal distribution with mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.

Alternatively, we can create, for large fixed n, $Y = n\overline{X}$ with mean np and standard deviation $\sqrt{np(1-p)}$.

For example, consider $p = 0.5$, $n = 100$, $Y =$ number of successes. To find $P\left(\frac{Y}{n} > 0.55\right)$. By the Central Limit Theorem, this is approximately a normal distribution with mean 0.5 and standard deviation 0.05.

Applying Central Limit Theorem: Let Y be a binomial distribution with $n = 25$ and $p = 0.4$. Then, $\mu = np = 10$, and standard deviation $\sigma = \sqrt{\frac{p(1-p)}{n}} = 5\sqrt{0.24}$.

To find $P(Y \leq 8)$, we can potentially approximate with $P(X \leq 8.5)$ — the reason we use 8.5 instead of 8 is due to the fact that n may not be large enough, a process known as the continuity correction.

Using standardization (or R), we find that this probability is approximately 0.269.

The actual probability $P(Y \leq 8)$ is found as below:

$$\begin{aligned} P(Y \leq 8) &= \sum_{k=0}^{8} \binom{25}{k} (0.4)^k (0.6)^{1-k} \\ &= 0.274 \end{aligned}$$

The normal approximation for the binomial is adequate when $p \pm 3\sqrt{\frac{p(1-p)}{n}} \in (0,1)$. Essentially, the binomial trial needs to have an adequate sample size such that the "spread" is small. This is equivalent to $n \geq 9\frac{\max(p,1-p)}{\min(p,1-p)}$.

Estimators

Let Y be a random variable with an *unknown* distribution.

Parameter: Feature of Y's distribution that are not computable from samples.

Examples of Parameters: μ, σ, m'_k, interval $(a, b) \ni P(y \in I) = 0.95$.

Statistic: Random variable that is computable from samples.

Examples of Statistics: sample mean, $\overline{Y}$, sample variance, S^2, $Y_{(i)}$.

Estimator: a statistic intended to approximate a parameter. A point estimator estimates a single value.

Examples of Estimators: $\overline{Y}$ as an estimator for μ, and S^2 as an estimator of σ^2.

Bias and Mean Square Error of Estimators

We want to find θ, a constant parameter of the underlying distribution — $\hat{\theta}$ is a random variable.

If $E(\hat{\theta})$ is close to θ, we can say that $\hat{\theta}$ is a good estimator — more precisely, we define the bias $B(\hat{\theta}) = E(\hat{\theta}) - \theta$, and if $B(\hat{\theta}) = 0$, then $\hat{\theta}$ is an unbiased estimator.

In addition to minimizing bias, to see whether or not an estimator is good requires minimizing the variance of the estimator — the mean squared estimator $\text{MSE}(\hat{\theta}) = V(\hat{\theta}) + B(\hat{\theta})^2$. Notice that for an *unbiased* estimator, $\text{MSE}(\hat{\theta}) = V(\hat{\theta})$.

Exercise 8.12: Let θ be the true voltage of some electronic device. The voltage test has results uniformly distributed over $[\theta, \theta + 1]$. There are n tests, $Y_1, \dots, Y_n$. Evaluate $\overline{Y}$ as an estimator for θ.

Solution: Since the voltage is uniformly distributed over $[\theta, \theta + 1]$, we have that Y_i is uniform on $[\theta, \theta + 1]$. Therefore, $E(Y_i) = \theta + 0.5$, and $V(Y_i) = \frac{1}{12}$.

Therefore, $E(\overline{Y}) = \theta + 0.5$, and $V(\overline{Y}) = \frac{1}{12n}$, meaning $\text{MSE}(\hat{\theta}) = \frac{1}{12n} + \frac{1}{4}$.

If we want an unbiased estimator for θ, we take $\hat{\theta} = \overline{Y} - \frac{1}{2}$. Then, $E(\hat{\theta}) = E(\overline{Y}) - E(1/2) - \theta = 0$. By shifting this estimator, our new MSE is $\frac{1}{12n}$.

Example 8.1: We will compare the two estimators of σ^2: sample variance and population variance.

$$
\begin{aligned}
S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 \\
S'^2 &= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2
\end{aligned}
$$

Solution: Recall $V(X) = E(X^2) - (E(X))^2$. Therefore, $E(X^2) = V(X) + (E(X))^2$.

$$
\begin{aligned}
E(Y_i^2) &= V(Y_i) + (E(Y_i))^2 \\
&= \sigma^2 + \mu^2 \\
E(\overline{Y}^2) &= V(\overline{Y}) + \left(E(\overline{Y})\right)^2 \\
&= \frac{\sigma^2}{n} + \mu^2
\end{aligned}
$$

Notice that

$$\begin{aligned}
\sum (Y_i - \overline{Y})^2 &= \sum (Y_i^2 - 2Y_i\overline{Y} + \overline{Y}^2) \\
&= \sum Y_i^2 - 2\overline{Y}\sum Y_i + \sum \overline{Y}^2 \\
&= \sum Y_i^2 - 2n\overline{Y}^2 + n\overline{Y}^2 \\
&= \sum_{Y_i}^{2} - n\overline{Y}^2 \\
E\left(\sum (Y_i - \overline{Y})^2\right) &= E(\sum Y_i^2) - nE(\overline{Y}^2) \\
&= n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) \\
&= (n-1)\sigma^2 \\
B(S'^2) &= \frac{1}{n}(n-1)\sigma^2 - \sigma^2 \\
&= -\frac{1}{n}\sigma^2 \neq 0 \\
B(S^2) &= \frac{1}{n--1}(n-1)\sigma^2 - \sigma^2 \\
&= 0
\end{aligned}$$

S'^2 is known as the *biased sample variance*, while S^2 is the unbiased sample variance.

The standard error $\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$. If $\hat{\theta}$ is unbiased, then $\sigma_{\hat{\theta}} = \sqrt{\text{MSE}(\hat{\theta})}$

Errors and Confidence Intervals

Error of Estimation: The error of estimation is $\varepsilon = |\hat{\theta} - \theta|$. Notice that while θ is a fixed value, ε is a random variable.

We say $\hat{\theta}$ is a "good" estimator if there is a high probability that ε is small. Specifically, ε being small often means $\exists b$ such that $\varepsilon < b$ — alternatively, $|\hat{\theta} - \theta| < b$, meaning $\theta - b < \hat{\theta} < \theta + b$, so $\hat{\theta} \in (\theta - b, \theta + b)$.

We often set b to be $2\sigma_{\hat{\theta}}$, or $2 \cdot \text{SE}(\hat{\theta})$.

When $\hat{\theta}$ is unbiased, $\mu_{\hat{\theta}} = E(\hat{\theta}) = \theta$. So, the $2\sigma_{\hat{\theta}}$ interval about θ is the same as the $2\sigma_{\hat{\theta}}$ about $\hat{\theta}$.

Finally, $\hat{\theta}$ often, but not always, has an approximate normal distribution. Therefore, the probability that $\hat{\theta}$ is within $2\sigma_{\hat{\theta}}$ of $\mu_{\hat{\theta}}$ is approximately 0.95. Specifically, $P(|\hat{\theta} - \mu_{\hat{\theta}}| < 2\sigma_{\hat{\theta}})$.

Recall that Chebyshev's Theorem states that $P(|\hat{\theta} - -\mu_{\hat{\theta}}| < 2\sigma_{\hat{\theta}}) \geq 1 - \frac{1}{2^2} = 0.75$.

Example 8.3: Suppose there are two types of tire. $n_1 = n_2 = 100$ of each type, with $Y_1 = Y_2 =$ miles tire lasts. $\overline{Y}_1 = 26400$ miles while $\overline{Y}_2 = 25100$ miles. $S_1^2 = 144000000$ and $S_2^2 = 196000000$.

Let's try to estimate how much longer tire 1 lasts than tire 2. First, we will use an unbiased estimator for the mean (sample mean).

$$\begin{aligned}
\mu_{Y_1-Y_2} &= \mu_{Y_1} - \mu_{Y_2} \\
&\approx \overline{Y}_1 - \overline{Y}_2 \\
&= 1300 \\
\sigma_{\overline{Y}_1-\overline{Y}_2} &= \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} && \text{Table 8.1} \\
&\approx \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} && S^2 \text{ an unbiased estimator for } \sigma^2 \\
&= 184.4
\end{aligned}$$

Therefore, the difference in the life expectancy between the types is about 1300 miles, and there is approximately probability 0.95 chance that the life expectancy is within 368.8 miles of 1300.

The interval $[1300 - 368.8, 1300 + 368.6]$ is called an interval estimator or confidence interval, expressed as $[\hat{\theta}_L, \hat{\theta}_H]$.

- $\hat{\theta}_L$: lower confidence limit, a left endpoint estimator.
- $\hat{\theta}_U$: upper confidence limit, a right endpoint estimator.

Example 8.4: One sample, Y, from exponential distribution with PDF

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & y \in [0, \infty) \\ 0 & y \in (-\infty, 0) \end{cases}$$

To estimate θ, we would prefer a PDF without θ. Let

$$\begin{aligned}
U &= \frac{Y}{\theta} && \text{pivotal quantity} \\
F_U(u) &= P(U \leq u) \\
&= P(Y/\theta \leq u) \\
&= F_Y(u\theta) \\
&= 1 - e^{-u} \\
f(u) &= \begin{cases} e^{-u} & u \in [0, \infty) \\ 0 & u \in (-\infty, 0) \end{cases}
\end{aligned}$$

We want a, b such that $P(a \leq \theta \leq b) = 0.9$. Pick c, d such that $P(c \leq U \leq d) = 0.9$.

By integrating, we find $c = -\ln(0.95) = 0.051$, and $d = 2.996$. Now,

$$\begin{aligned}
0.9 &= P(0.051 \leq U \leq 2.996) \\
&= P(0.051 \leq Y/\theta \leq 2.996) \\
&= P(0.051/Y \leq 1/\theta \leq 2.996/\leq) \\
&= P(Y/2.996 \leq \theta \leq Y/0.051).
\end{aligned}$$

meaning that for $Y = 2$, there is a probability 0.9 that $\theta \in [0.668, 39]$.