Problem (Problem 1)**:** In this exercise, we prove another fundamental result in differential topology, called the tubular neighborhood theorem. Let M be a compact smooth manifold with orientable boundary N. For simplicity, assume that N is connected. The tubular neighborhood theorem asserts that N admits a neighborhood in M which is diffeomorphic to $N \times [0, 1)$.

(a) Choose a Riemannian metric on M, and show that N admits a nonvanishing vector field that is everywhere orthogonal to the tangent space of N. That is, a vector field X such that for all $p \in N$, $g(X_p, T_p N) = 0$.

(b) Use the flow generated by X to find the desired neighborhood.

Solution:

(a) If $p \in N$, then we observe that $T_p N < T_p M$ is a proper subspace with codimension 1. Letting $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for $T_p N$, then we may extend to a basis for $T_p M$ by taking a representative for a basis for $T_p M / T_p N$, and observing that such a vector necessarily has

$$g_p(e_n, e_k) = 0$$

for all $k = 1, \dots, n-1$. By smoothly varying over all points $p \in N$, we get our desired everywhere nonvanishing vector field normal to $T_p N$.

(b) Let φ_t be the one-parameter diffeomorphism group generated by X, where $\varphi_t \colon M \to M$ is such that $\varphi_0(p) = p$ for all $p \in N$. Then, $\varphi \colon (-\varepsilon, \varepsilon) \to \operatorname{diff}(M)$ restricted to $[0, \varepsilon)$ gives our desired neighborhood in M diffeomorphic to $N \times [0, 1)$.

Problem (Problem Set 7, Problem 5)**:** Suppose G is a finite group acting freely on a manifold M by diffeomorphisms.

(a) Show that M/G is a manifold.

(b) Show that the de Rham cohomology of M/G is isomorphic to the G-invariant cohomology of M.

Problem (Problem Set 8, Problem 3)**:** Compute the de Rham cohomology of $\mathbb{RP}^n$.

Solution: We will use the result related to invariant cohomology to compute this.

Problem (Problem Set 8, Problem 5)**:** Use the Mayer–Vietoris sequence to prove the Künneth Formula: if M and N are smooth manifolds, then $H^*_{DR}(M \times N)$ is the tensor product of $H^*_{DR}(M)$ and $H^*_{DR}(N)$. Specifically, in each degree ℓ, we have

$$H^\ell_{DR}(M \times N) = \bigoplus_{i+j=\ell} H^i_{DR}(M) \otimes H^j_{DR}(N).$$

Solution: For the sake of being able to solve this problem, we focus on the case where M and N are closed smooth manifolds.

Let $V = M \times N$ be the product manifold for M and N. If $\pi_1 \colon V \to M$ and $\pi_2 \colon V \to N$ are the projection maps on M and N respectively, we get the composed maps

$$\mathcal{A}^k(M) \times \mathcal{A}^\ell(N) \to \mathcal{A}^{k+\ell}(V)$$

given by $(\omega, \eta) \mapsto \pi_1^*\omega \wedge \pi_2^*\eta$. If ω and η are closed forms, then we observe that

$$\begin{aligned} d\left(\pi_1^*\omega \wedge \pi_2^*\eta\right) &= d\pi_1^*\omega \wedge \pi_2^*\eta + (-1)^k \pi_1^*\omega \wedge d\pi_2^*\eta \\ &= \pi_1^*(d\omega) \wedge \pi_2^*\eta + (-1)^k \pi_1^*\omega \wedge \pi_2^*(d\eta) \\ &= 0. \end{aligned}$$

Furthermore, if we let $\omega' = \omega + d\tau$ and $\eta' = \eta + d\rho$, then we know from earlier work that $\pi_1^*\omega' \wedge \pi_2^*\eta'$ can be expressed as $\pi_1^*\omega \wedge \pi_2^*\eta + d\sigma$ for some form σ by using the fact that d and the pullback commute.

Thus, it follows that the map descends to a map in cohomology, given by

$$\begin{aligned} H^k_{DR}(M) \times H^\ell_{DR}(N) &\to H^{k+\ell}(M \times N) \\ ([\omega], [\eta]) &\mapsto \left[\pi_1^*\omega \wedge \pi_2^*\eta\right], \end{aligned}$$

whence via the universal property of tensor products and direct sums, we get the map

$$\psi\colon H^*_{DR}(M) \otimes H^*_{DR}(N) \to H^*(M \times N).$$

Our goal now is to show that ψ is indeed an isomorphism.

Toward this end, suppose we have two open sets in the good cover for M, given by U_1 and U_2. From the Mayer–Vietoris sequence, this yields the following exact sequence in cohomology for a fixed k, where D_k denote the connecting homomorphisms from $H^k(U_1 \cap V_2)$ to $H^{k+1}(M)$.

$$\cdots \xrightarrow{D_{k-1}} H^k_{DR}(M) \xrightarrow{i} H^k_{DR}(U_1) \oplus H^k_{DR}(U_2) \xrightarrow{j} H^k_{DR}(U_1 \cap U_2) \xrightarrow{D_k} \cdots$$

Since the tensor product preserves exact sequences, we observe that by taking the tensor product with $H^\ell_{DR}(N)$, giving the following.

$$\cdots \xrightarrow{D_{k-1}} H^k_{DR}(M) \otimes H^\ell_{DR}(N) \xrightarrow{i} H^k_{DR}(U_1) \otimes H^\ell_{DR}(N) \oplus H^k_{DR}(U_2) \otimes H^\ell_{DR}(N) \xrightarrow{j} H^k(U_1 \cap U_2) \otimes H^\ell(N) \xrightarrow{D_k} \cdots$$

Taking direct sums with the same dimension, we obtain the following diagram.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\bigoplus_{k+\ell=i-1} H^k(U_1 \cap U_2) \otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "D_{i-1}"'] & H^{i-1}((U_1 \cap U_2) \times N) \arrow[d, "D_{i-1}"] \\
\bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "i"'] & H^i(M \times N) \arrow[d, "i"] \\
\bigoplus_{k+\ell=i}(H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) \arrow[r, "\psi"] \arrow[d, "j"'] & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \arrow[d, "j"] \\
\bigoplus_{k+\ell=i} H^k(U_1 \cap U_2) \otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "D_i"'] & H^i((U_1 \cap U_2) \times N) \arrow[d, "D_i"] \\
\vdots & \vdots
\end{tikzcd}$$

Since U_1, U_2, and $U_1 \cap U_2$ are contractible, under the good cover assumption, it follows from the Poincaré Lemma that the following subsection of the diagram is commutative, with ψ necessarily an isomorphism in each of the columns.

$$\begin{tikzcd}
\bigoplus_{k+\ell=i}(H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) \arrow[r, "\psi"] \arrow[d, "j"'] & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \arrow[d, "j"] \\
\bigoplus_{k+\ell=i} H^k(U_1 \cap U_2) \otimes H^\ell(N) \arrow[r, "\psi"] & H^i((U_1 \cap U_2) \times N)
\end{tikzcd}$$

Similarly, we have that the following diagram is commutative, following from the Mayer–Vietoris sequence.

$$
\begin{tikzcd}
\bigoplus_{k+\ell=i} H^k(M)\otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "i"'] & H^i(M\times N) \arrow[d, "j"] \\
\bigoplus_{k+\ell=i}(H^k(U_1)\otimes H^\ell(N)\oplus H^k(U_2)\otimes H^\ell(N)) \arrow[r, "\psi"] & H^i(U_1\times N)\oplus H^i(U_2\times N)
\end{tikzcd}
$$

Therefore, we only need to verify commutativity for the following square.

$$
\begin{tikzcd}
\bigoplus_{k+\ell=i-1} H^k(U_1\cap U_2)\otimes H^\ell(N) \arrow[r, "\psi"] \arrow[d, "D_{i-1}"'] & H^{i-1}((U_1\cap U_2)\times N) \arrow[d, "D_{i-1}"] \\
\bigoplus_{k+\ell=i} H^k(M)\otimes H^\ell(N) \arrow[r, "\psi"] & H^i(M\times N)
\end{tikzcd}
$$

First, from the Mayer–Vietoris sequence and the fact that the coboundary map in de Rham cohomology emerges from the exterior derivative, we have that the map D_i is given by

$$
D_{i-1}([\omega]) = \begin{cases}[d(-f_U\omega)]\\ [d(f_V\omega)]\end{cases}
$$

for any cohomology class representative ω. Now, we observe that

$$
\begin{aligned}
\psi(D_{i-1}([\omega],[\eta])) &= \left[\pi_1^*(D_{i-1}(\omega))\wedge\pi_2^*\eta\right]\\
D_{i-1}(\psi([\omega],[\eta])) &= \left[D_{i-1}\left(\pi_1^*\omega\wedge\pi_2^*\eta\right)\right]
\end{aligned}
$$

In particular, since $\pi_1^* f_U$ and $\pi_1^* f_V$ form a partition of unity for $M\times F$, we have

$$
\begin{aligned}
\pi_1^*(D_{i-1}(\omega))\wedge\pi_2^*\eta &= \pi_1^*(d(f_V\omega))\wedge\pi_2^*\eta\\
&= d\left(\pi_1^*(f_V\omega)\right)\wedge\pi_2^*\eta\\
D_{i-1}\left(\pi_1^*\omega\wedge\pi_2^*\eta\right) &= d\left(\pi_1^* f_V\pi_1^*\omega\wedge\pi_2^*\eta\right)\\
&= d(\pi^*(f_V\omega))\wedge\pi_2^*\eta.
\end{aligned}
$$

Since ψ at each of U, V, and $U\cap V$ is an isomorphism, and the diagram commutes, the Five Lemma gives that ψ at M is an isomorphism.

For any finite good cover with more than 2 elements, induction gives the desired result.