Problem 1

Let X be a metric space. Show that X is second countable if and only if X is separable. Conclude that if X is a separable metric space, then every open set is the union of countably many open balls.

Proof:

($\Rightarrow$): Let X be second countable. Then, X contains base $U_1, U_2, \dots \in \mathcal{B}$ such that each U_i is nonempty. Let $x_1 \in U_1, x_2 \in U_2, \dots$.

The set $\{x_i\}_{i\geq 1}$ is countable, as each $x_i \in U_i$. For any $U \in \tau_X$ where $U \neq \emptyset$, $U = \bigcup_{i\in I} U_i$, meaning that $U \cap \{x_i\}_{i\geq 1} \neq \emptyset$. Thus, $\{x_i\}_{i\geq 1}$ is dense in X, meaning X is separable.

($\Leftarrow$): Let X be separable, with countable dense subset $\{x_i\}_{i\geq 1}$. Let

$$\mathcal{B} = \{U(x_i, 1/n) \mid x_i \in \{x_i\}_{i\geq 1}, n \in \mathbb{N}\}.$$

Then, for every $U \in \tau_X$, since $U \cap \{x_i\}_{i\geq 1} \neq \emptyset$, and $\exists n$ such that $U(x_k, 1/n) \subseteq U$, it must be the case that $\mathcal{B}$ is a base for τ_X. Thus, X is second countable.

If X is a separable metric space, then it admits a countable base, and any element of τ_X is a union of the elements of the base, implying that any element of τ_X is a union of countably many open balls.

Problem 2

Let (X, d) be a metric space, $(x_n)_n$ a sequence in x, and $x \in X$. The following are equivalent:

(i) $(x_n)_n \to x$ in X;

(ii) $(d(x_n, x))_n \to 0$ in $\mathbb{R}$;

(iii) For every neighborhood $V \in \mathcal{N}_x$, there is an $N \in \mathbb{N}$ with $n \geq N \Rightarrow x_n \in V$.

Proof: Let $(x_n)_n \to x$ in X. Then, for any $\varepsilon > 0$, $\exists N$ large such that $n \geq N \Rightarrow d(x_n, x) < \varepsilon$. However, this is precisely the same as $|d(x_n, x) - 0| < \varepsilon$, which is true if and only if $(d(x_n, x)) \to 0$.

Problem 6

Let (X, d) be a metric space, $f, g : X \to \mathbb{F}$ continuous maps, and $\alpha \in \mathbb{F}$. Show that $f + g$, fg, and αf are continuous.

Proof: Let $(x_n)_n \to x \in X$. Then, we know that $|f(x_n) - f(x)| \to 0$ and $|g(x_n) - g(x)| \to 0$ (where $|\cdot|$ denotes absolute value in $\mathbb{F}$). Let $\varepsilon > 0$. Therefore, for N large, we know that

$$\begin{aligned} |f(x_n) + g(x_n) - (f(x) + g(x))| &\leq |f(x_n) - f(x)| + |g(x_n) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

meaning $|f(x_n) + g(x_n) - (f(x) + g(x))| \to 0$, so $(f(x_n) + g(x_n))_n \to f(x) + g(x)$. Thus, $f + g$ is continuous.

Similarly,

$$\begin{aligned} |f(x_n)g(x_n) - f(x)g(x)| &= |f(x_n)g(x_n) - f(x_n)g(x) + f(x_n)g(x) - f(x)g(x)| \\ &= |f(x_n)(g(x_n) - g(x)) + g(x)(f(x_n) - f(x))| \\ &\leq |f(x_n)||g(x_n) - g(x)| + |g(x)||f(x_n) - f(x)| \\ &\leq c|g(x_n) - g(x)| + g(x)|f(x_n) - f(x)| && \text{convergent sequences are bounded} \\ &< \varepsilon \end{aligned}$$

so $(f(x_n)g(x_n))_n \to f(x)g(x)$.

Problem 9

Suppose $T : V \to W$ is a bijective linear map between normed spaces with $\|T\|_{\text{op}} \leq 1$ and $\left\|T^{-1}\right\|_{\text{op}} \leq 1$. Show that T is an isometry.

Proof: Since the operator norm for T is less than or equal to 1, we know that for $v, w \in V$,

$$\|T(v) - T(w)\|_W \leq \|v - w\|_V$$

and

$$\left\|T^{-1}(T(v)) - T^{-1}(T(w))\right\|_V \leq \|T(v) - T(w)\|_W$$

so, since T is bijective,

$$\|v - w\|_V \leq \|T(v) - T(w)\|_W$$

meaning

$$\|T(v) - T(w)\|_W = \|v - w\|_V$$

so T is an isometry.