Contents

First-Order Differential Equations

Introduction to First-Order Differential Equations

Recall that for $y = f(x)$, x is the independent variable and y is the dependent variable.

Definition (Differential Equation)**.** A differential equation is an equation which contains derivatives of a dependent variable with respect to one or more independent variables.

Example (A Basic Differential Equation)**.**

$$\frac{dy}{dx} - 5y - 1 = 0$$

We can classify differential equations by

- type;
- order;
- linearity.

Definition (Classification by Type)**.** There are two types of differential equations:

- ordinary differential equations (ODEs);
- partial differential equations (PDEs).

ODEs are characterized by derivatives of the dependent variable with respect to one independent variable. PDEs are characterized by derivatives of the dependent variable with respect to multiple independent variables.

Example (ODEs and PDEs)**.**

(1) An ODE:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$$

(2) A PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Definition (Classificiation by Order)**.** The order of the highest derivative in a differential equation is the order of the differential equation.

Example (Differential Equations of Varying Orders)**.**

(1)
$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) - 4y = x \qquad \text{order 2}$$

(2)
$$2\frac{dy}{dx} + y = 0 \qquad \text{order 1}$$

(3)
$$\sin(x)y''' - (\cos x)\,y' = 2 \qquad \text{order 3}$$

In general, we write a differential equation of order n in the form

$$\underbrace{F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)}_{n+2 \text{ variables}} = 0.$$

Example. Suppose we have the differential equation

$$\frac{d^3y}{dx^3} + y^2 = x.$$

Then, we rewrite as

$$\underbrace{\frac{d^3y}{dx^3} + y^2 - x}_{F\left(x,y,\frac{dx}{dy},\frac{d^2x}{dy^2},\frac{d^3x}{dy^3}\right)} = 0.$$

Alternatively, we can write as

$$\frac{d^3y}{dx^3} = \underbrace{x - y^2}_{f(x,y)}.$$

Definition (Classification by Linearity)**.** We much prefer to analyze linear differential equations over non-linear differential equations.

A differential equation is linear if it has the following form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

(1) The power of each term involving y and all its derivatives is one.

(2) All coefficients are exclusively functions of x.

A differential equation that is not linear is called nonlinear.

Example (Linear Differential Equations (or lack thereof))**.**

(1)
$$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = e^x \qquad \text{Linear}$$

(2)
$$yy'' - 2y' = x \qquad \text{Nonlinear}$$

(3)
$$\frac{d^3y}{dx^3} + y^2 = 0 \qquad \text{Nonlinear}$$

Definition (Autonomous Differential Equations)**.** An autonomous (first-order) differential equation is a differential equation in the following form:
$$\frac{dy}{dx} = f(y).$$

Definition (Solution to an ODE)**.** Consider the general ODE
$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0. \tag{$*$}$$
A solution of $(*)$ is a function $y = f(x)$ that satisfies the ODE; that is,
$$F\left(x, f(x), f'(x), f''(x), \dots, f^{(n)}(x)\right) = 0$$
for every x in the domain of $f(x)$.

Notice that f is an element of a family of functions that satisfy the differential equation.

Example (Verifying a Solution)**.** We wish to show that $y = xe^x$ is a solution to
$$y'' - 2y' + y = 0$$
on $(-\infty, \infty)$.

In order to do this, we plug the proposed solution into the ODE:
$$\begin{aligned} y'' - 2y' + y &= \frac{d^2}{dx^2}\left(xe^x\right) - 2\frac{d}{dx}\left(xe^x\right) + xe^x \\ &= \left(xe^x + 2e^x\right) - 2\left(xe^x + e^x\right) + xe^x \\ &= 0. \end{aligned}$$

Definition (Equilibrium Solution)**.** An equilibrium solution of a differential equation is a *constant function* that satisfies the differential equation.

Example. We want to find equilibrium solutions for the following equations:

(1) $\frac{dy}{dt} = 2 - y$

(2) $\frac{dy}{dt} = y^2 - 3y - 4$.

In order to find equilibrium solutions, we know that $\frac{dy}{dt} = 0$. Thus, the equilibrium solutions are, respectively,

(1) $y(t) = 2$

(2) $y(t) = -1$ or $y(t) = 4$.

For first-order ODEs of the form
$$\frac{dy}{dt} = f(t, y),$$
equilibrium solutions are found by taking $\frac{dy}{dt} = 0$, and solving for y.

Modeling with Differential Equations

Definition (Initial Value Problem)**.** An initial value problem is a problem with a given ODE and an initial condition.

Example (Initial Value Problems)**.**

(1) We want to find

$$\frac{dy}{dx} = f(x, y)$$

such that $y(x_0) = y_0$.

(2)

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

must satisfy $y(x_0) = y_0$, $y'(x_0) = y_1$.

Modeling primarily occurs via the following feedback loop:

- real-world problem;
- mathematical model;
- solution;
- result/prediction.

As predictions from the model begin to stray from real-world observations, we update the model to reflect these new observations.

Example (Vertical Motion)**.** Consider someone who throws a rock off a building.

We let $y(t)$ denote the height of the ball at time t, with y_0 denoting initial height. The acceleration due to gravity is equal to $a(t) = v'(t) = y''(t) = g$.

Our ODE is

$$y''(t) = -g \qquad \text{for } 0 \leqslant t \leqslant T.$$

We require some initial conditions:

- $y(0) = y_0$ (initial position);
- $y'(0) = v_0$ (initial velocity).

Thus, we have created our second-order initial value problem.

To solve this second-order initial value problem analytically, we start with

$$y'' = -g.$$

Taking our first integral with respect to t, we have

$$y' = -gt + c_1.$$

Now, taking our second integral,

$$y(t) = -\frac{1}{2}gt^2 + c_1 t + c_2.$$

This version of $y(t)$ is the general solution.

Applying our initial condition on $y'(t)$, we have $y'(0) = c_1$, meaning $c_1 = v_0$, and applying the initial condition to $y(t)$, we have $y(0) = c_2$, meaning $c_2 = y_0$.

Thus, the solution to this initial value problem is

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0.$$

Example (Population Growth, Exponential and Logistic)**.** Let $P(t)$ be the population of living fish in a lake at time t.

We know that the rate of growth in population is proportional to the population. In other words,

$$\frac{dP}{dt} = kP(t)$$

for some constant $k > 0$.

We can also include an initial condition, $P(0) = P_0$.

We can see (relatively easily) that

$$P(t) = P_0 e^{kt}.$$

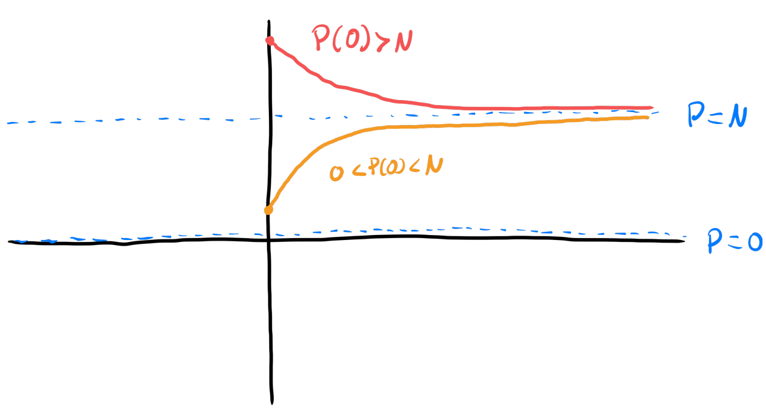
However, this is not particularly realistic; there is no theoretical upper bound on the model, even though in real life, ecosystems tend to have carrying capacities.

The logistic population model with growth rate k and carrying capacity N is is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right).$$

We can analyze this equation qualitatively first (before finding an analytical solution).

- If $P > N$, we can see that $\frac{dP}{dt} < 0$, which is expected since, if population is greater than carrying capacity, we expect population to approach carrying capacity.
- If $P < N$ (assuming P is positive), we see that $\frac{dP}{dt} > 0$, meaning population increases as it approaches carrying capacity. In particular, as population increases, the growth rate decreases.
- The equilibrium solutions occur at $P = 0$ or $P = N$.



Separable First-Order Differential Equations

Consider the first-order differential equation

$$\frac{dy}{dt} = f(t, y).$$

Definition (Separable Differential Equation)**.** A differential equation of the form

$$\frac{dy}{dt} = g(t)h(y)$$

is called separable.

Note:

$$f(t, y) = g(t)h(y).$$

Example.

(1) We can see that $\frac{dP}{dt} = kP$ is separable; $g(t) = k$, $h(y) = P$.

(2) We can see that $\frac{dy}{dt} = -\frac{t}{y}$ is also separable; $g(t) = -t$, $h(y) = \frac{1}{y}$.

(3) We can see that $\frac{dy}{dt} = y + t$ is not separable.

Method (Separation of Variables)**.** We want to solve $\frac{dy}{dt} = g(t)h(y)$.

(1) We take $\frac{dy}{h(y)} = g(t)\,dt$ by multiplying dt on both sides and dividing by $h(y)$.

(2) Integrate both sides with respect to their corresponding variable, yielding

$$\int \frac{1}{h(y)}\,dy = \int g(t)\,dt.$$

(3) We get

$$H(y) = G(t) + C,$$

where $H(y)$ and $G(t)$ are antiderivatives of $\frac{1}{h(y)}$ and $g(t)$ respectively.

Example (Solving the Exponential Population Growth Model by Separation of Variables)**.** Let $\frac{dP}{dt} = kP$, $P(0) = P_0$.

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= k\,dt \\
\int \frac{1}{P}\,dP &= \int k\,dt \\
\ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
&= e^{kt}e^{C} \\
P &= \left(\pm e^{C}\right)e^{kt} \\
&= Ae^{kt}.
\end{aligned}$$

Our solution is now of the form $P(t) = Ae^{kt}$ (where $A = \pm e^{C}$). This is not the general solution, though, since it lacks our equilibrium solution of $P = 0$. Thus, the general solution is

$$\begin{cases} P(t) = Ae^{kt} \\ P(t) = 0 \end{cases}$$

With the initial condition of $P(0) = P_0$, we have

$$\begin{aligned} P_0 = P(0) &= Ae^{k\cdot 0} \\ &= A. \end{aligned}$$

Thus, the particular solution to our initial value problem is $P(t) = P_0e^{kt}$.

Example (Solving a Sample Differential Equation by Separation of Variables)**.** Let $\frac{dy}{dt} = y^2 - 4$. Note that, even though this is not a linear equation, this is a separable equation. We start with the equilibrium solutions, which are at $y(t) = 2$ and $y(t) = -2$.

If $y \neq \pm 2$, we have

$$\begin{aligned} \frac{dy}{dt} &= y^2 - 4 \\ \frac{1}{y^2-4}\,dy &= dt \\ \int \frac{1}{4(y-2)} - \frac{1}{4(y+2)}\,dy &= \int dt \\ \frac{1}{4}\left(\ln\left|\frac{y-2}{y+2}\right|\right) &= t + C_1 \\ \ln\left|\frac{y-2}{y+2}\right| &= 4t + C_2 \\ \left|\frac{y-2}{y+2}\right| &= e^{C_2}e^{4t} \\ &= C_3e^{4t} \\ \frac{y-2}{y+2} &= \pm C_3e^{4t} \\ &= Ce^{4t} \\ y &= 2 + yCe^{4t} + 2Ce^{4t} \\ y &= \frac{2\left(1+Ce^{4t}\right)}{1-Ce^{4t}}. \end{aligned}$$

Thus, our general solution is

$$\begin{cases} y(t) = \frac{2\left(1+Ce^{4t}\right)}{1-Ce^{4t}} \\ y(t) = 2 \\ y(t) = -2 \end{cases}$$

Example (Solving the Logistic Population Growth Model)**.** Let $\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$. Our equilibrium solutions are at $P(t) = 0$ and $P(t) = N$. For non-equilibrium solutions, we have

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

$$
\begin{aligned}
\frac{1}{P\left(1-\frac{P}{N}\right)}\,dP &= k\,dt\\
\int \frac{1}{P\left(1-\frac{P}{N}\right)}\,dP &= \int k\,dt\\
\int \frac{-N}{P\,(P-N)}\,dP &= kt + C_1\\
\int \frac{1}{P} - \frac{1}{P-N}\,dP &= kt + C_1\\
\ln|P| - \ln|P-N| &= kt + C_1\\
\ln\left|\frac{P}{P-N}\right| &= kt + C_1\\
\left|\frac{P}{P-N}\right| &= e^{C_1}e^{kt}\\
\frac{P}{P-N} &= \pm e^{C_1}e^{kt}\\
&= Ce^{kt}\\
P\left(1-Ce^{kt}\right) &= -NCe^{kt}\\
P &= N\frac{Ce^{kt}}{Ce^{kt}-1}.
\end{aligned}
$$

Therefore, our general solution is

$$
\begin{cases}
P(t) = N\frac{Ce^{kt}}{Ce^{kt}-1}\\
P(t) = 0\\
P(t) = N
\end{cases}.
$$

Example (A Non-Separable Linear Differential Equation)**.** Consider the linear differential equation

$$\frac{dy}{dt} + a(t)y = b(t).$$

Notice that

$$\frac{dy}{dt} = -a(t)y + b(t),$$

which is not able to be separated.

In order to solve such an equation, we will need to use an integrating factor.

Slope Fields

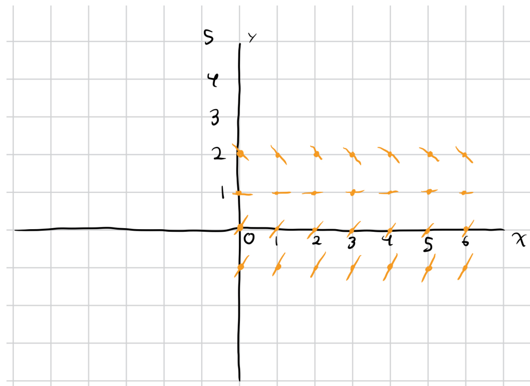
Definition. A slope field is a set of short line segments that indicate slope $\frac{dy}{dx}$ at a set of points (x, y) in the x, y-plane.

It is a graphical method of displaying the general slope and behavior of functions that satisfy $\frac{dy}{dx} = f(x, y)$.

Example. Consider $\frac{dy}{dx} = 1 - y$. We can select some samples of slopes as follows:

Point	$\frac{dy}{dx}$
$(0, 0)$	1
$(1, 0)$	1
$(0, 1)$	0
$(0, 2)$	-1

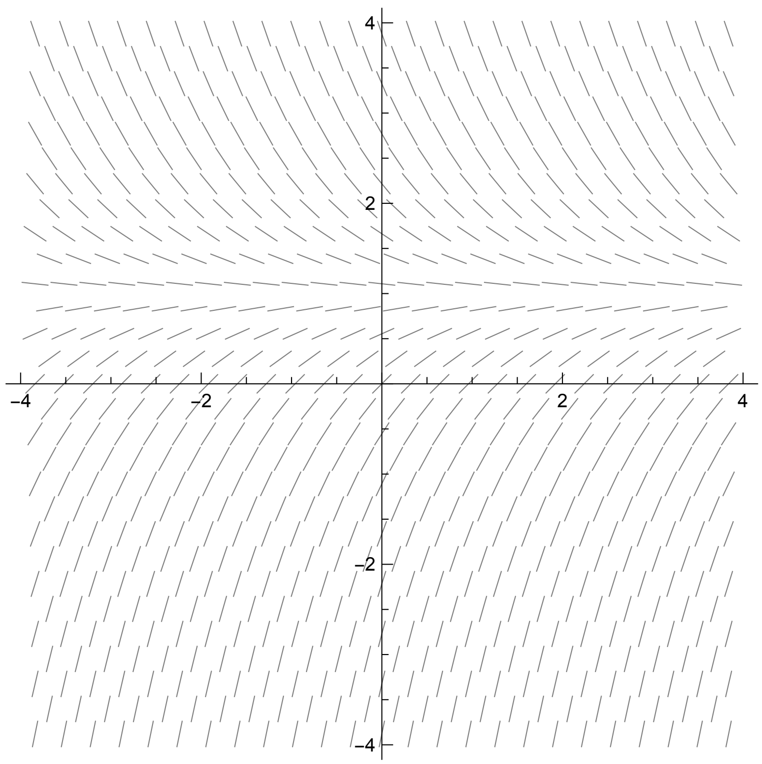
Thus, we can draw the slope field:



Qualitatively, we can see that

- at $y = 1$, all line segments are horizontal;
- for $y < 1$, all line segments have positive slope;
- for $y > 1$, all line segments have negative slope.

Using a computer, we can generate a better slope field:

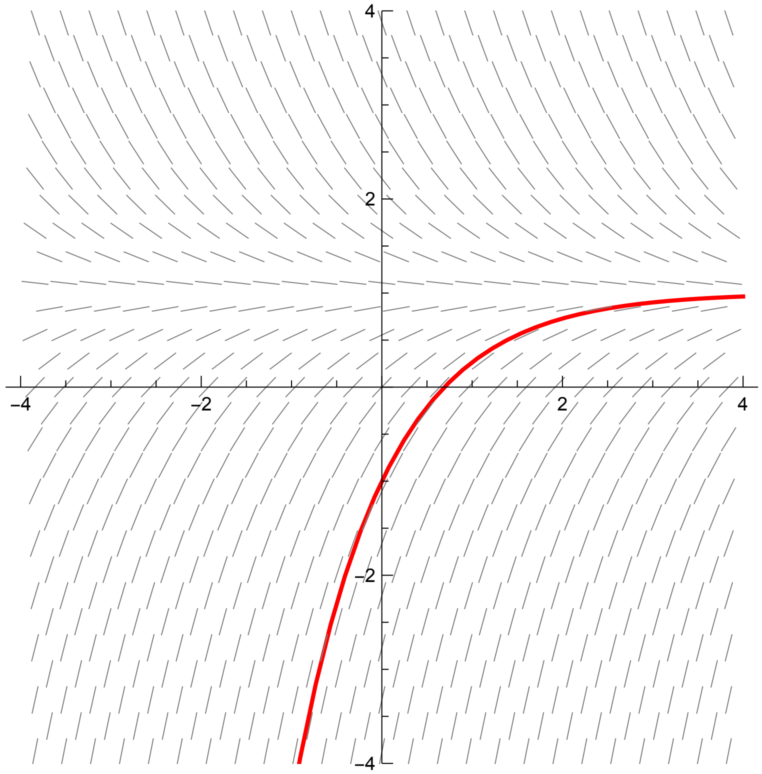


Analytically, we solve the equation by separation of variables:

$$\frac{dy}{dx} = 1 - y$$
$$\frac{dy}{1-y} = dx$$

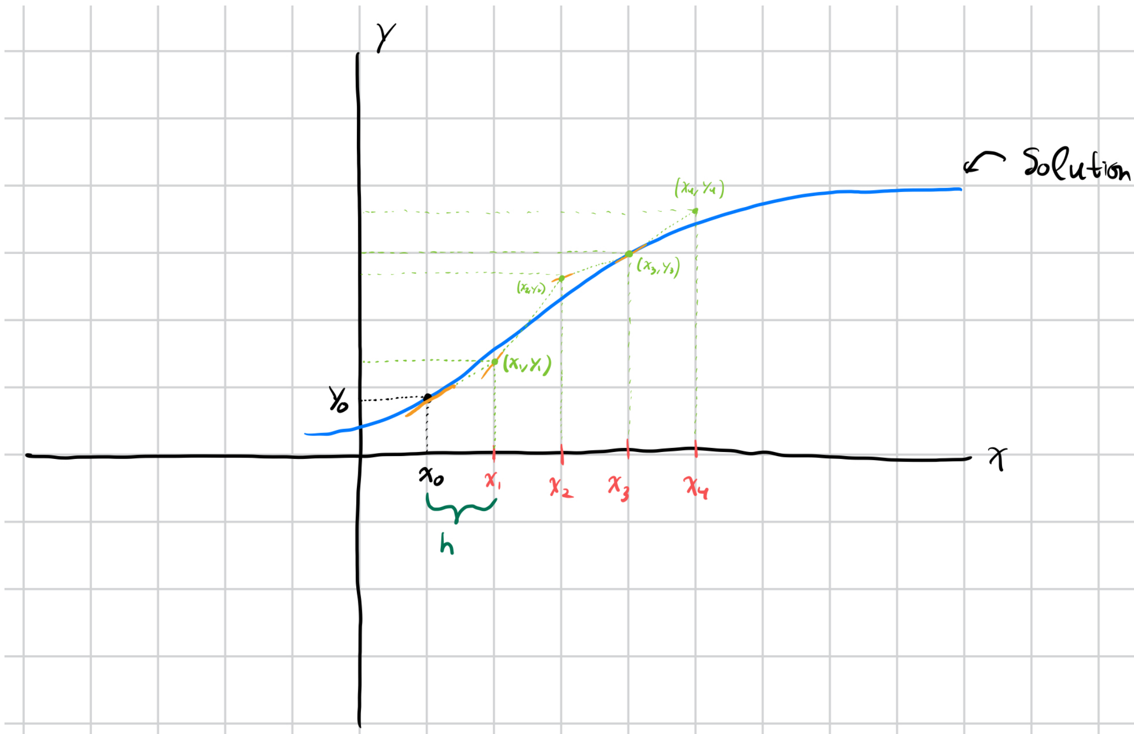
$$
\begin{aligned}
\int \frac{1}{1-y}\, dy &= \int \, dx \\
-\ln|1-y| &= x + C \\
|1-y| &= e^{-x-C_1} \\
1-y &= \pm e^{-C_1} e^{-x} \\
y &= 1 - Ae^{-x}.
\end{aligned}
$$

With the initial condition of $y(0) = -1$, we get $y = 1 - 2e^{-x}$.



Euler's Method

We want to approximate solutions to the differential equation $y' = f(x, y)$, $y(x_0) = y_0$. In the diagram, we can see the use of the slope field to calculate the approximate values of (x_i, y_i).



Method (Euler's Method)**.** To approximate the curve at $x_1 = x_0 + h$, we take the point-slope form:

$$\frac{y_1 - y_0}{(x_0 + h) - x_0} = f(x_0, y_0)$$
$$y_1 = y_0 + hf(x_0, y_0).$$

In general, we have

$$y_{k+1} = y_k + hf(x_k, y_k).$$

Example. Consider the differential equation $y' = 2y - 1$. With the step size $\Delta x = h = 0.5$ and $y(0) = 1$, we can approximate $y(1)$ by

k	x_k	y_k	$f(x_k, y_k)$
0	1	1	1
1	1.5	1.5	2
2	2	2.5	—

Thus, using Euler's method with a step size of 0.5, we find that $y(1) \approx 2.5$. The table is read left to right, changing columns after calculating $f(x_k, y_k)$, then using it to calculate y_{k+1}.

Solving the differential equation analytically, we find

$$\begin{aligned}
\frac{dy}{dx} &= 2y - 1 \\
\int \frac{1}{2y-1}\,dy &= \int dx \\
\frac{1}{2}\ln|2y-1| &= x + C_1 \\
\ln|2y-1| &= 2x + C_2 \\
|2y-1| &= e^{C_2}e^{2x}
\end{aligned}$$

$$\begin{aligned}
2y - 1 &= \pm e^{C_2} e^{2x}\\
y &= \frac{1}{2} + Ae^{2x}.
\end{aligned}$$

Plugging in our initial condition, we find $A = \frac{1}{2}$, and the exact value of $y(1)$ is $\frac{1}{2} + \frac{1}{2}e^2$, which is approximately 4.1945.

We can make our approximation via Euler's method better using a shorter step size.

For instance, by using a step size of 0.1, we find:

k	x_k	y_k	$2y_k - 1$
0	0	1	1
1	0.1	1.1	1.2
2	0.2	1.22	1.44
3	0.3	1.364	1.728
4	0.4	1.537	2.07
5	0.5	1.744	2.49
6	0.6	1.993	2.97
7	0.7	2.29	3.58
8	0.8	2.65	4.30
9	0.9	3.08	5.16
10	1	3.596	—

Note that our final approximation of 3.596 is much better than the approximation under $h = 0.5$.

In order to understand if our estimate with Euler's method is an overestimate or underestimate, we use the second derivative test.

$$\begin{aligned}
y' &= 2y - 1\\
y'' &= 2y'\\
&= 2\left(2y - 1\right)\\
&= 4y - 2.
\end{aligned}$$

In particular, our initial condition of $y(0) = 1$ suggests that $y'' = 2 > 0$, meaning Euler's method will return an underestimate (as the tangent lines will lie below the true curve).

Existence and Uniqueness

Given an initial value problem

$$\begin{aligned}
\frac{dy}{dt} &= f(t, y)\\
y\left(t_0\right) &= y_0,
\end{aligned}$$

we ask the following two questions.

(1) When does a solution to this initial value problem exist?

(2) If it does exists, is the solution unique?

Example. Consider the polynomial

$$\underbrace{2x^5 - 10x + 5}_{f} = 0,$$

and suppose want to find solutions to this equation.

Notice that f is continuous on $[-1,1]$, with $f(-1)=13$ and $f(1)=3$.

By the Intermediate Value Theorem, this must mean f takes on the value of 0 at at least one value between $x=-1$ and $x=1$.

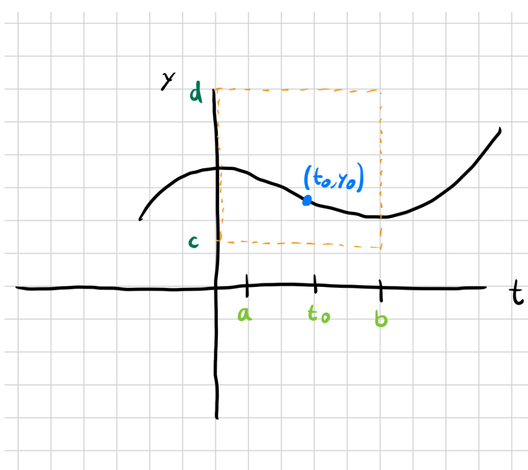
Theorem (Existence and Uniqueness)**:** Let R be a rectangular region in the t,y-plane containing the point (t_0,y_0) in the interior of R. In particular, $R=\{(t,y)\mid a<t<b, c<y<d\}$, and $(t_0,y_0)\in R$.

If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists $\varepsilon>0$ and a unique function $y(t)$ defined on some neighborhood $t_0-\varepsilon<t<t_0+\varepsilon$ contained in $a<t<b$ such that $y(t)$ is a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f(t,y) \\ y(t_0) &= y_0. \end{aligned}$$

Note the two major conditions here:

- the continuity of f on R; this guarantees existence
- the continuity of $\frac{\partial f}{\partial y}$ on R; this guarantees uniqueness.



If one of the conditions is not satisfied, we may have

- exactly one solution;
- many solutions;
- no solutions.

Example. Consider the differential equation

$$\frac{dy}{dt}=t^2y^{1/2},$$

and $y(0)=0$.

We have $f(t,y)=t^2y^{1/2}$. We have

$$\frac{\partial f}{\partial y}=\frac{1}{2}t^2y^{-1/2}$$

$$= \frac{t^2}{2\sqrt{y}}.$$

We can see that f is continuous at $(0, 0)$, but $\frac{\partial f}{\partial y}$ is not continuous at $(0, 0)$.

Additionally, since f is not defined for $y < 0$, we cannot place $(0, 0)$ in the interior of any region on the t, y-plane.

Therefore, we cannot use the existence and uniqueness theorem on f.

Going forward analytically, we start with the equilibrium condition $y(t) = 0$. Using separation of variables, we have

$$\begin{aligned}
\frac{dy}{dt} &= t^2 y^{1/2} \\
\int \frac{dy}{y^{1/2}} &= \int t^2 \, dt \\
2y^{1/2} &= \frac{t^3}{3} + C \\
y &= \left(\frac{t^3}{6} + K\right)^2 \\
0 &= \left(\frac{(0)^3}{6} + K\right)^2 \\
K &= 0
\end{aligned}$$

meaning we also have a solution of

$$\begin{aligned}
y(t) &= \left(\frac{t^3}{6}\right)^2 \\
&= \frac{t^6}{36}.
\end{aligned}$$

Example. Let $\frac{dy}{dt} = t^2 y^{1/2}$ with the (new) initial condition of $y(2) = 1$.

In particular, we can see that not only is $f(t, y)$ continuous at 0, but so too is $\frac{\partial f}{\partial y}$, and there exists a region about the point $(2, 1)$ such that f and $\frac{\partial f}{\partial y}$ are continuous.

Thus, by the existence and uniqueness theorem, there exists exactly one solution to this initial value problem.

Example. Let $\frac{dy}{dt} = 1 + y^2$, with the initial condition $y(0) = 0$.

We have $f(t, y) = 1 + y^2$, $\frac{\partial f}{\partial y} = 2y$; both of these functions are continuous on the t, y-plane. Thus, by the existence and uniqueness theorem, there exists a unique solution to this initial value problem.

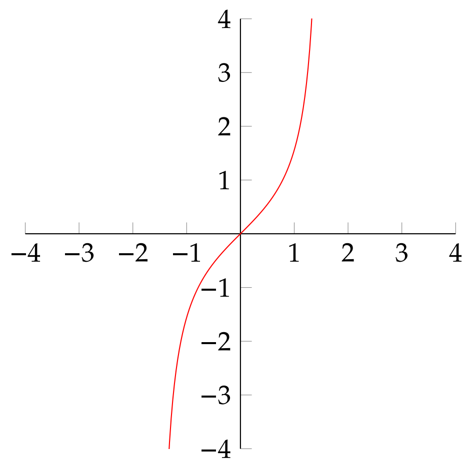
We solve by separation of variables:

$$\begin{aligned}
\frac{dy}{dt} &= 1 + y^2 \\
\int \frac{1}{1 + y^2} \, dy &= \int dt \\
\arctan(y) &= t + C
\end{aligned}$$

$$y = \tan(t + C)$$
$$0 = \tan(0 + C)$$
$$0 = \tan(C).$$

We can let $C = 0$, meaning we have the solution

$$y(t) = \tan t.$$



Example (Application of the Uniqueness Theorem)**.** Suppose $f(t, y)$ satisfies the conditions for uniqueness.

Assume $y_1(t)$ and $y_2(t)$ are solutions of the differential equation $\frac{dy}{dt} = f(t, y)$.

If these solutions intersect at some $t = t_0$, then $y_1(t)$ and $y_2(t)$ are solutions to this new initial value problem of

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_1(t_0) = y_2(t_0).$$

Since f satisfies the uniqueness theorem, it be the case that $y_1(t) = y_2(t)$.

In particular, this means that solutions to differential equations that satisfy the uniqueness conditions cannot intersect. In other words, solutions cannot equal each other at the same "place" at the same "time."

Example. Consider the initial value problem

$$\frac{dy}{dt} = \frac{2y}{t}$$
$$y(1) = 1.$$

We have $f(t, y) = \frac{2y}{t}$, $\frac{\partial f}{\partial y} = \frac{2}{t}$. We can see that f and $\frac{\partial f}{\partial y}$ are continuous at $(1, 1)$, as well as a given region with $(1, 1)$ in its interior (since continuity is only lost when $t = 0$).

Thus, there exists a unique solution to this initial value problem. We can solve the equation via separation of variables:

$$\frac{dy}{dt} = \frac{2y}{t}$$
$$\int \frac{1}{2y}\, dy = \int \frac{1}{t}\, dt$$

$$\begin{aligned}
\frac{1}{2}\ln|y| &= \ln|t| + C \\
\ln|y| &= 2\ln|t| + 2C \\
|y| &= e^{2\ln|t|}e^{2C} \\
y &= \pm e^{2C}t^2 \\
y &= Kt^2
\end{aligned}$$

We have the initial condition $y(1) = 1$, meaning we have the solution $y(t) = t^2$.

Notice that as a function, the domain of y is $\mathbb{R}$, but as a solution, we cannot have $t = 0$, meaning that the domain of $y(t) = t^2$ *as a solution* is $t > 0$.

Equilibria and Phase Lines

Given a differential equation

$$\frac{dy}{dt} = f(t, y),$$

we can use slope fields and Euler's method to find an approximate solution, as well as various analytic methods to find a definite solution.

Given an autonomous differential equation, though,

$$\frac{dy}{dt} = f(y),$$

particularly one with f, $\frac{\partial f}{\partial y}$ continuous, we can use some deeper analysis.

In particular, for constant y, the slope at any point (t, y) will be the same as at any other (t, y). This means we can analyze the behavior of *the entire solution* based on one particular line.

Method (Equilibrium Analysis for Autonomous Differential Equations)**.**

(1) Solve $f(y) = 0$ to find the equilibrium solutions.

(2) Analyze the behavior of $f(y)$ around the equilibrium points.

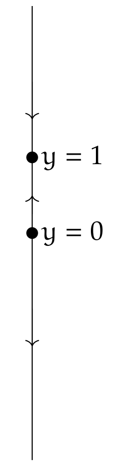
- Draw a particular line, known as a phase line, with positive t.
- Determine if $f(y)$ is positive or negative between equilibrium points.
- If $f(y) > 0$, then the solution is increasing in this region, and if $f(y) < 0$, the solution is decreasing in this region.

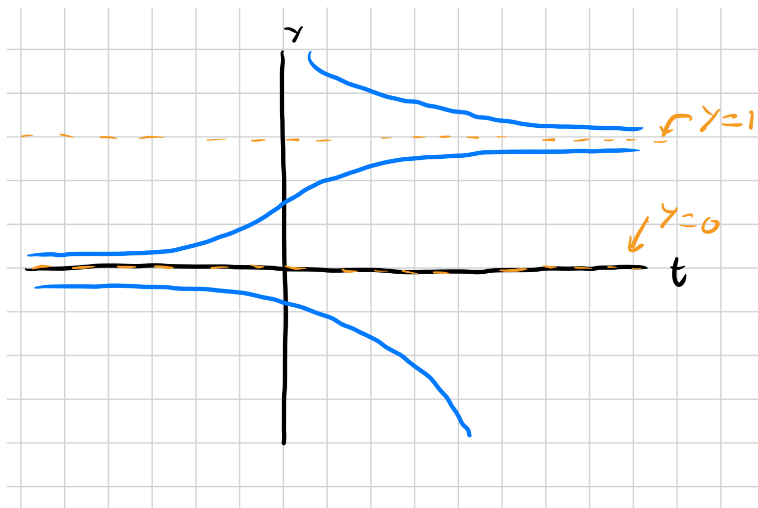
Example. Let

$$\frac{dy}{dt} = (1 - y)y.$$

Thus, we have $f(y) = (1 - y)y$, meaning $y(1 - y) = 0$ for $y = 0$ and $y = 1$.

For $y = 2$, we have $f(2) = -2 < 0$, for $y = 0.5$, $f(y) = 0.25 > 0$, and $f(-1) = -2 < 0$. Thus, our equilibrium analysis looks like the following:





Definition (Classificiation of Equilibrium Points).

(1) An equilibrium point at $y = c$ is called a sink if all the solutions with the initial condition near $y = c$ approach c as t approaches ∞.

- On the phase line, $f(y)$ goes from $+$ to $-$ on the phase line as y goes from $c - \varepsilon$ to $c + \varepsilon$.

(2) An equilibrium point at $y = c$ is called a source if all the solutions with the initial conditions near $y = c$ move away from c as t approaches ∞.

- On the phase line, $f(y)$ goes from $-$ to $+$ on the phase line as y goes from $c - \varepsilon$ to $c + \varepsilon$.

(3) Every equilibrium point that is neither a source nor a sink is called a node.

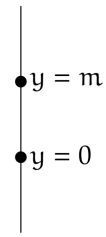
- On the phase space, $f(y)$ maintains the same sign as y goes from $c - \varepsilon$ to $c + \varepsilon$.

Example (Logistic Population Model). Recall that the logistic population model is

$$\frac{dy}{dt} = \underbrace{k\left(1 - \frac{y}{m}\right)y}_{f(y)},$$

with the conditions of $k > 0$, $m > 0$. We want to draw a phase line for this model.

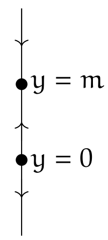
The equilibrium solutions are at $y = 0$ and $y = m$. We draw our phase line with its equilibrium points as



To find out what happens in between, we choose

- $y = -1$; $f(y) < 0$.
- $y = m/2$; $f(y) > 0$.
- $y = 2m$; $f(y) < 0$.

Therefore, our phase line is



Theorem (Linearization)**:** Suppose y_0 is an equilibrium point of the differential equation $\frac{dy}{dt} = f(y)$, where $f(y)$ is a continuously differentiable function. Then,

- if $f'(y_0) < 0$, then y_0 is a sink;
- if $f'(y_0) > 0$, then y_0 is a source;
- if $f'(y_0) = 0$, then we do not have enough information to determine the type of y_0.

Remark: In order to do an equilibrium analysis given a graph of $f(y)$, we identify all our equilibrium points by finding the roots of f, then examining the slope of f at each of the identified equilibrium points.

Example. Suppose

$$\frac{dy}{dt} = (y-1)\left(y^5 - 7y^4 + 3y^3 + 8y^2 - 11\right).$$

We want to classify the equilibrium point $y = 1$. In order to do this, we find its derivative:

$$\frac{\partial f}{\partial y} = \left(y^5 - 7y^4 + 3y^3 + 8y^2 - 11\right) + (y-1)\left(5y^4 - 28y^3 + 9y^2 + 16y\right).$$

Evaluating

$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = -6,$$

meaning our equilibrium point at $y = 1$ is a sink.

Bifurcations

Consider a first-order differential equation

$$y' = f(y, c).$$

The value $c \in \mathbb{R}$ means this differential equation is actually a family of differential equations. We say c is a parameter for the differential equation.

Example (A Parametrized in a Differential Equation)**.** Consider the differential equation modelling the population of fish in a pond, denoted

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - h.$$

Here, k is a constant, while h is a parameter denoting the harvesting rate.

In general, as c varies in our parametrized differential equation, an equilibrium solution may split into two equilibrium solutions disappear entirely.

Example. Consider the family

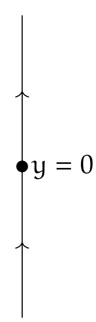
$$\frac{dy}{dt} = y^2 - c.$$

The equilibrium solutions occur when $y^2 - c = 0$, meaning that for $c > 0$, our equilibrium solutions are at $y = \pm\sqrt{c}$. If $c = 0$, then $y = 0$ is an equilibrium solution, and if $c < 0$, there are no equilibrium solutions.

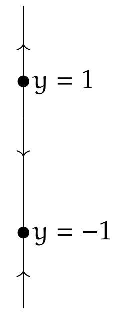
We can see that $c = 0$ is a bifurcation point.

For $c = -1$, our differential equation is $\frac{dy}{dt} = y^2 + 1$.

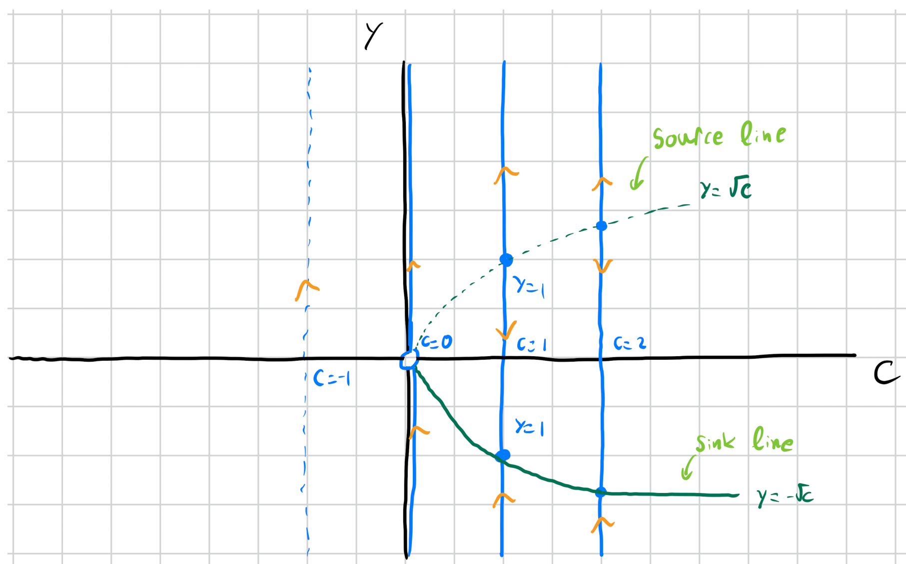
For $c = 0$, our differential equation is $\frac{dy}{dt} = y^2$.



For $c = 1$, our differential equation is $\frac{dy}{dt} = y^2 - 1$.



The bifurcation diagram is as follows.



Notice that the bifurcation point occurs when the source line and sink line meet.

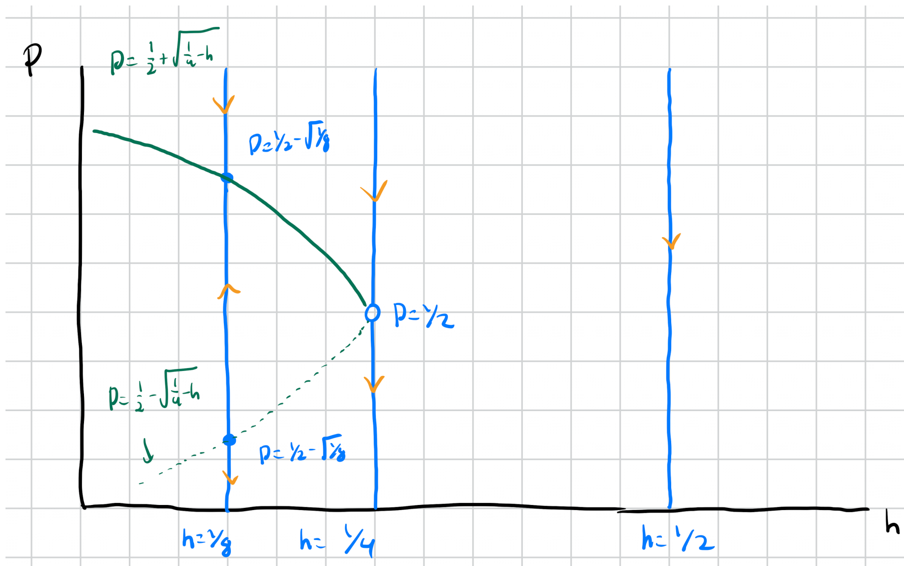
Example (Solving a Population Model)**.** Consider

$$\frac{dP}{dt} = \underbrace{P\left(1 - P\right) - h}_{=f(P,h)}$$

Equilibrium solutions occur at

$$\begin{aligned} P\left(1 - P\right) - h &= 0 \\ P^2 - P + h &= 0 \\ P^2 - P + \frac{1}{4} &= \frac{1}{4} - h \\ \left(P - \frac{1}{2}\right)^2 &= \frac{1}{4} - h \\ P &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - h}. \end{aligned}$$

Based on the values of h, we have no equilibrium solutions for $h > \frac{1}{4}$, one equilibrium solution for $h = \frac{1}{4}$, and two equilibrium solutions for $h < \frac{1}{4}$.



- If $h > 1/4$, the fish population will die out;
- If $h = 1/4$, the fish population will stabilize at $P = 1/2$;
- If $0 < h < 1/4$, the fish population will stabilize at a new, lower population.

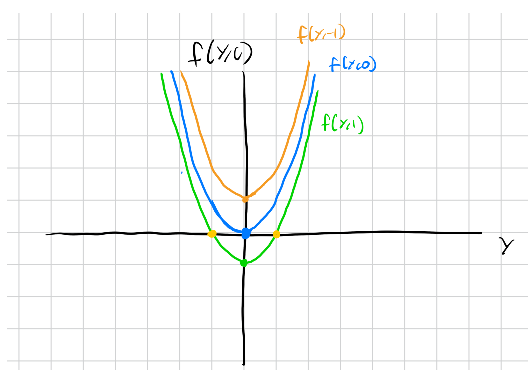
Recall: For

$$\frac{dy}{dt} = y^2 - c,$$

there are three cases.

- $c > 0$: two equilibrium solutions.
- $c = 0$: one equilibrium solution.
- $c < 0$: no equilibrium solutions.

When we graph f against y, we see the following.



In particular, we can see that the bifurcation value occurs at $c = 0$, when the root of f is tangent to the y axis. This can be formalized in the following theorem.

Theorem: Let y_0 be an equilibrium solution. The point c_0 is a bifurcation value for the autonomous differential equation

$$\frac{dy}{dt} = f(y, c)$$

if and only if $f(y_0, c_0) = 0$ and $\left.\frac{\partial f}{\partial y}\right|_{(y_0, c_0)} = 0$.

First Order Linear Differential Equations

Definition. A first-order differential equation of the form

$$\frac{dy}{dt} = a(t)y + b(t)$$
$$\frac{dy}{dt} - a(t)y = b(t)$$

with $a(t), b(t)$ arbitrary functions of t is known as a (first order) linear differential equation.

Example.

(1) $\frac{dy}{dt} + 2y = e^{-t}$ is linear.

Definition (Homogeneous Linear Differential Equations)**.** For

$$\frac{dy}{dt} - a(t)y = b(t),$$

if $b(t) = 0$ for all t, the linear differential equation is called homogeneous (or unforced). If $b(t) \neq 0$, we call this a non-homogeneous differential equation.

Definition (Linearity Principle for Homogeneous Equations)**.** Consider the homogeneous differential equation

$$\frac{dy}{dt} - a(t)y = 0.$$

(1) If $y_h(t)$ is a solution of $\frac{dy}{dt} - a(t)y = 0$, then $cy_h(t)$ is a solution for $\frac{dy}{dt} - a(t)y = 0$, for any $c \in \mathbb{R}$.

(2) If $y_1(t)$ and $y_2(t)$ are solutions of $\frac{dy}{dt} - a(t)y = 0$, then $(y_1 + y_2)(t)$ is a solution to $\frac{dy}{dt} - a(t)y = 0$.

The proof of the linearity principle follows from the linearity of the derivative.

Example. Let

$$\frac{dy}{dt} = a(t)y + b(t).$$

Note that the associated homogeneous differential equation is

$$\frac{dy}{dt} = a(t)y.$$

We can solve the associated homogeneous differential equation using separation of variables:

$$\begin{aligned}
\frac{dy}{dt} &= a(t)y \\
\int \frac{1}{y}\,dy &= \int a(t)\,dt \\
\ln(y) &= \int a(t)dt + C \\
y &= Ke^{\int a(t)\,dt}. && C \in \mathbb{R}
\end{aligned}$$

Note that the linearity principle does *not* (necessarily) work for non-homogeneous differential equations.

Example (Failure of the Linearity Principle)**.** Consider

$$\frac{dy}{dt} = -y + 2. \tag{$*$}$$

Note that $b(t) = 2 \neq 0$.

One of the solutions to this equation is

$$y(t) = 2 - e^{-t}.$$

We will show that $2y(t)$ is not a solution of $(*)$.

$$\begin{aligned} \frac{d}{dt}\left(2\left(2 - e^{-t}\right)\right) &= 2e^{-t} \\ -y(t) + 2 &= 2e^{-t} - 2. \end{aligned}$$

Theorem (Extended Linearity Principle)**:** Let

$$\frac{dy}{dt} = a(t)y + b(t). \tag{$**$}$$

(1) If $y_h(t)$ is any solution of the homogeneous differential equation

$$\frac{dy}{dt} = a(t)y,$$

and $y_p(t)$ is any solution of $(**)$. Then, $y_h(t) + y_p(t)$ is a solution to $(**)$.

(2) If $y_1(t)$ and $y_2(t)$ are solutions to $(**)$, $y_1(t) - y_2(t)$ provides a solution to the homogeneous differential equation

$$\frac{dy}{dt} = a(t)y.$$

Proof.

(1)

$$\begin{aligned} \frac{d}{dt}\left(y_h(t) + y_p(t)\right) &= \frac{d}{dt}\left(y_h(t)\right) + \frac{d}{dt}\left(y_p(t)\right) \\ &= a(t)y_h(t) + a(t)y_p(t) + b(t) \\ &= a(t)\left(y_h(t) + y_p(t)\right) + b(t). \end{aligned}$$

(2)

$$\begin{aligned} \frac{d}{dt}\left(y_1(t) - y_2(t)\right) &= \frac{d}{dt}\left(y_1(t)\right) - \frac{d}{dt}\left(y_2(t)\right) \\ &= \left(a(t)y_1(t) + b(t)\right) - \left(a(t)y_2(t) - b(t)\right) \\ &= a(t)\left(y_1(t) - y_2(t)\right). \end{aligned}$$

□

Note that, as a result of the extended linearity principle, all solutions to a non-homogeneous first-order linear equation are of the form $y(t) = cy_h(t) + y_p(t)$, where $y_p(t)$ is *any* solution to the equation.

Example. Let

$$\frac{dy}{dt} = -2y + e^t.$$

We can see that $b(t) = e^t$, and the general solution to $\frac{dy}{dt} = -2y$ is Ke^{-2t} for $K \in \mathbb{R}$.

Now, we look at

$$\frac{dy}{dt} + 2y = e^t$$

We make a guess that

$$y_p(t) = \alpha e^t.$$

Then,

$$\begin{aligned}\frac{dy}{dt} + 2y &= \alpha e^t + 2\alpha e^t \\ &= e^t.\end{aligned}$$

Thus, $\alpha = \frac{1}{3}$, implying that $y_p(t) = \frac{1}{3}e^t$.

Thus, the general solution is

$$y(t) = \frac{1}{3}e^t + Ke^{-2t}.$$

Example. Let's try to find the general solution to

$$\frac{dy}{dt} = -2y + \cos(3t).$$

We are aware of the general solution of the homogeneous equation $\frac{dy}{dt} = -2y$, which is $y_h(t) = Ke^{-2t}$.

Now, we look at

$$\frac{dy}{dt} + 2y = \cos(3t)$$

to find a particular solution. We take a guess of $y_p(t) = A\cos(3t) + B\sin(3t)$. Then,

$$\frac{dy}{dt} + 2y = (-3A + 2B)\sin(3t) + (2A + 3B)\cos(3t).$$

Thus, $3A = 2B$ and $2A + 3B = 1$, yielding $B = \frac{3}{13}$ and $A = \frac{2}{13}$.

Therefore, the general solution is

$$y(t) = \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t) + Ke^{-2t}.$$

$b(t)$	Guess for $y_p(t)$
$ae^{\alpha t}$	$Ae^{\alpha t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\cos(\beta t)$
$a\sin(\beta t)$	$A\cos(\beta t) + B\cos(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\cos(\beta t)$
n-th degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$