Alternating Series and Conditional Convergence

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A Series

Consider the following series:

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

A Series

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1$$

A Series

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2}$$

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Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3}$$

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

A Series

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

This series appears to be related to the harmonic series, but also very different:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{Harmonic Series}$$

Divergence of the Harmonic Series

We can show that the harmonic series is divergent using the series comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

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Divergence of the Harmonic Series

We can show that the harmonic series is divergent using the series comparison test:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

Differences

However, our alternating harmonic series is different. Let's look at partial sums.

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However, our alternating harmonic series is different. Let's look at partial sums.

$$s_n = \sum_{k=1}^{n} \frac{(-1)^{n+1}}{n}$$

$$s_1 = 1$$

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$$s_3 = \frac{5}{6}$$

Differences

However, our alternating harmonic series is different. Let's look at partial sums.

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{(-1)^{n+1}}{n} \\
s_1 &= 1 \\
s_2 &= \frac{1}{2} \\
s_3 &= \frac{5}{6} \\
s_4 &= \frac{7}{12} \\
&\vdots
\end{aligned}
$$

Convergence?

Clearly, this sequence does not grow without bound — it is bounded above by 1, and doesn't seem to dip below $\frac{1}{2}$.

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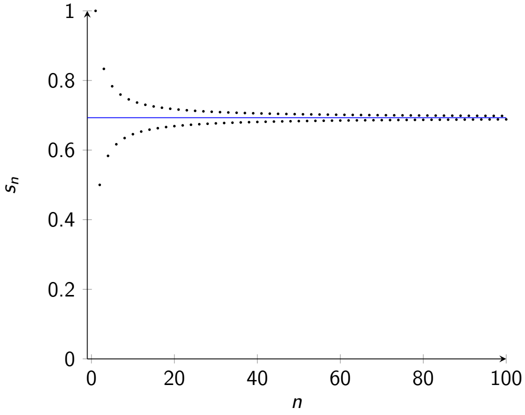
Clearly, this sequence does not grow without bound — it is bounded above by 1, and doesn't seem to dip below $\frac{1}{2}$.



Convergence? cont'd

The alternating harmonic does converge. Courtesy of Wolfram MathWorld, we know that the series converges to the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

Convergence? cont'd

The alternating harmonic does converge. Courtesy of Wolfram MathWorld, we know that the series converges to the following:

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...or does it?

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots$$

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots \end{aligned}$$

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$\begin{aligned}
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots &= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\cdots \\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \\
&= \frac{1}{2}\ln 2
\end{aligned}$$

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Introduction to Conditional Convergence

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$$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

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Introduction to Conditional Convergence

- We saw that our alternating harmonic series converges to $\ln 2$, but should it not converge to $\ln 2$ all the time?
- For example, no matter how we arrange

  $$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

  The sum should always equal 1.
- Maybe we should redefine convergence?

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- The answer is that the alternating harmonic series is *conditionally* convergent.
- We can always rearrange the terms of the alternating harmonic series to form whatever sum we want.
- In general, alternating series, of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

can be convergent, while at the same time

$$\sum_{n=1}^{\infty}a_n$$

is divergent.

Alternating Series Test

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$$0 < a_{n+1} < a_n$$

Alternating Series Test

- In general, we can find if an alternating series is *conditionally* convergent as follows:
  - The (absolute value) series terms are strictly positive and decreasing.

$$0 < a_{n+1} < a_n$$

  - The series terms tend to zero:

$$\lim_{n \to \infty} a_n = 0$$

Applying the Alternating Series Test

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Applying the Alternating Series Test

In the alternating harmonic series, we see that

$$0 < \frac{1}{n+1} < \frac{1}{n},$$

and

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

So the series is *conditionally* convergent.

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What is Absolute Convergence?

We know two facts:

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- The harmonic series diverges

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- The harmonic series diverges

We need a stronger term for series convergence — *absolute* convergence — when a series converges to a single value.

Finding Absolute Convergence

If the absolute value of the terms in the series converges, then the series converges absolutely.

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Finding Absolute Convergence, cont'd

Absolutely Convergent Alternating Series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely. Why?

Finding Absolute Convergence, cont'd

Absolutely Convergent Alternating Series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely. Why?
By the geometric series,

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2^n} \right| = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

converges.

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What We Have Learned

- The same series can converge to different values depending on the arrangement of terms — known as *conditional convergence*

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- We can use the alternating series test to find if a series converges conditionally.

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- The same series can converge to different values depending on the arrangement of terms — known as *conditional convergence*
- We can use the alternating series test to find if a series converges conditionally.
- However, we would need to use other tools to find if a series is absolutely convergent.

Questions?

Thank you for listening. Any questions?