Chapter 15 Problems

Problem 1

Let S denote the surface of the hemisphere with $z \geqslant 0$, S_1 denote the full hemisphere including the disk in the plane at $z = 0$, and S_2 is the disk in the plane at $z = 0$. Then,

$$\begin{aligned}
\int_S \mathbf{r} \cdot d\mathbf{a} &= \oint_{S_1} \mathbf{r} \cdot d\mathbf{a} - \int_{S_2} \mathbf{r} \cdot d\mathbf{a} \\
&= \oint_{S_1} r\hat{r} \cdot d\mathbf{a} \\
&= \int_{V_1} d\tau \\
&= \frac{2}{3}\pi R^3.
\end{aligned}$$

Problem 2 (a)

We let S be the square in the xy-plane. Then,

$$\begin{aligned}
\oint_C \left(x\hat{\imath} - y\hat{\jmath}\right) \cdot d\vec{\ell} &= \int_S \left(\nabla \times \left(x\hat{\imath} - y\hat{\jmath}\right)\right) \cdot d\mathbf{a} \\
&= 0.
\end{aligned}$$

Problem 3 (a)

Let V denote the cube of side length a. Then,

$$\begin{aligned}
\int_S \mathbf{F} \cdot d\mathbf{a} &= \oint_V \left(3x^2 + 3y^2 + 3z^2\right) d\tau \\
&= 6a^5.
\end{aligned}$$

Problem 4

(a)

$$\begin{aligned}
\oint_S \mathbf{F} \cdot d\mathbf{a} &= \int_0^{\pi} \int_0^{2\pi} (R \sin\theta \hat{r}) \cdot \hat{r} \left(R^2 \sin\theta\right) d\phi d\theta \\
&= 2\pi R^3 \int_0^{\pi} \sin^2\theta \, d\theta \\
&= 2\pi^2 R^3. \\
\oint_S \mathbf{F} \cdot d\mathbf{a} &= \int_V \sin\theta d\tau \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} r^2 \sin^2\theta \, dr \, d\phi \, d\theta \\
&= 2\pi^2 R^3.
\end{aligned}$$

(b)

$$\oint_S \mathbf{F} \cdot d\mathbf{a} = \int_0^{\pi} \int_0^{2\pi} \left(R \sin\theta \hat{\theta}\right) \cdot \hat{r} \left(R^2 \sin\theta\right) d\phi d\theta$$

$$\begin{aligned} &= 0\\ \oint_S \mathbf{F}\cdot d\mathbf{a} &= \int_V 0\ d\tau\\ &= 0. \end{aligned}$$

(c)

$$\begin{aligned} \oint_S \mathbf{F}\cdot d\mathbf{a} &= \int_0^{\pi}\int_0^{2\pi}\left(R\sin\theta\hat{\phi}\right)\cdot\hat{r}\left(R^2\sin\theta\right)d\phi d\theta\\ &= 0\\ \oint_S \mathbf{F}\cdot d\mathbf{a} &= \int_V 0\ d\tau\\ &= 0. \end{aligned}$$

Problem 9

Let $C = \partial S_1 = \partial S_2$. Let $\mathbf{B} = \nabla\times\mathbf{A}$ (which must exist as $\mathbf{B}$ is solenoidal).

(a)

$$\begin{aligned} \int_{S_1}\mathbf{B}\cdot d\mathbf{a} &= \int_{S_1}(\nabla\times\mathbf{A})\cdot d\mathbf{a}\\ &= \oint_C \mathbf{A}\cdot d\vec{\ell}\\ &= \int_{S_2}(\nabla\times\mathbf{A})\cdot d\mathbf{a}\\ &= \int_{S_2}\mathbf{B}\cdot d\mathbf{a}. \end{aligned}$$

(b) For all closed surfaces S, it is the case that $\partial S_1 = \partial S_2 = 0$. Thus,

$$\begin{aligned} \oint_{S_1}\mathbf{B}\cdot d\mathbf{a} &= \int_{V_1}\nabla\cdot\mathbf{B}\ d\tau\\ &= 0\\ &= \int_{V_2}\nabla\cdot\mathbf{B}\ d\tau\\ &= \oint_{S_2}\mathbf{B}\cdot d\mathbf{a}. \end{aligned}$$

Problem 16

We have $\mathbf{E} = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k} = r^3\hat{r}$. Thus,

$$\begin{aligned} \oint_S \mathbf{E}\cdot d\mathbf{a} &= \int_V \nabla\cdot\mathbf{E}\ d\tau\\ &= \int_0^R\int_0^{2\pi}\int_0^{\pi}\left(3r^2\right)\left(r^2\sin\theta\right)\ d\theta d\phi dr\\ &= \frac{3}{5}R^5(4\pi)\\ &= \frac{12}{5}\pi R^5 \end{aligned}$$

Problem 17

We have $\mathbf{E} = \hat{\imath} + \hat{\jmath} + z\left(x^2 + y^2\right)\hat{k}$, so $\nabla \cdot \mathbf{E} = x^2 + y^2 = r^2$. Thus,

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{a} &= \int_V \nabla \cdot \mathbf{E}\, d\tau \\ &= \int_0^1 \int_0^1 \int_0^{2\pi} \left(r^2\right) r\, d\phi\, dz\, dr \\ &= \frac{\pi}{2}.\end{aligned}$$

Problem 19 (a)

(a)

$$\begin{aligned}\oint_S f\nabla g \cdot d\mathbf{a} &= \int_V \nabla \cdot (f\nabla g)\, d\tau \\ &= \int_V \left(\nabla f \cdot \nabla g + f\nabla^2 g\right) d\tau.\end{aligned}$$

Problem 22

$$\begin{aligned}\oint_C \mathbf{A} \cdot d\vec{\ell} &= \int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} \\ &= \int_S (\nabla \times \mathbf{A} + \nabla \times \nabla\lambda) \cdot d\mathbf{a} \\ &= \int_S \nabla \times (\mathbf{A} + \nabla\lambda) \cdot d\mathbf{a} \\ &= \oint_C (\mathbf{A} + \nabla\lambda) \cdot d\vec{\ell}.\end{aligned}$$

Problem 26

(a) Note that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}.$$

We evaluate the surface integral of $-\frac{\hat{r}}{r^2}$ over a sphere of radius R centered at the origin.

$$\begin{aligned}\oint_S -\frac{\hat{r}}{r^2} \cdot d\mathbf{a} &= -\oint_S \frac{1}{r^2}\left(r^2 \sin\theta\right) d\Omega \\ &= -4\pi.\end{aligned}$$

We also evaluate the surface integral of $-\frac{\hat{r}}{r^2}$ over a surface T that does not contain the origin.

$$\oint_T -\frac{\hat{r}}{r^2} \cdot d\mathbf{a} = 0.$$

Thus, we must have $\nabla \cdot \left(\nabla\left(\frac{1}{r}\right)\right) = -4\pi\delta(\mathbf{r})$.

(b) For V a sphere that is centered at the origin, we have

$$\Phi = \int_V \nabla \cdot \left(-\frac{\hat{r}}{r^2}\right) d\tau$$

$$\begin{aligned} &= -\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\phi d\theta \\ &= -4\pi, \end{aligned}$$

while if V does not contain the origin, the divergence is zero. Thus, we get $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r})$.

Problem 32

(a)

$$\begin{aligned} \nabla\times\mathbf{B}_0 &= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{1}{3}r^3\right)\right)\hat{k} \\ &= r\hat{k} \\ (\nabla\times\mathbf{B}_0)\cdot\mathbf{B}_0 &= 0. \end{aligned}$$

(b)

$$\begin{aligned} \nabla\times\left(\mathbf{B}_0+\hat{k}\right) &= \nabla\times\mathbf{B}_0+\nabla\times\hat{k} \\ &= \nabla\times\mathbf{B}_0 \\ &= r\hat{k} \\ (\nabla\times\mathbf{B}_1)\cdot\mathbf{B}_1 &= r. \end{aligned}$$

$$\begin{aligned} \nabla\times\left(\mathbf{B}_0+z\hat{r}+r\hat{k}\right) &= \nabla\times\mathbf{B}_0+\nabla\times\left(z\hat{r}+r\hat{k}\right) \\ &= r\hat{k} \\ (\nabla\times\mathbf{B}_2)\cdot\mathbf{B}_2 &= r^2\hat{k}. \end{aligned}$$

It seems like the addition of a divergence-free component affects the orthogonality of the curl to the original vector field.

(c) With $\Lambda = -\frac{1}{3}rz$, we have

$$\begin{aligned} (\nabla\times\mathbf{B}_3)\cdot\mathbf{B}_3 &= \left(r\hat{k}+\hat{\phi}\right)\cdot\left(\frac{1}{3}r^2\hat{\phi}+z\hat{r}+\nabla\Lambda\right) \\ &= \frac{1}{3}r^2+\hat{\phi}\cdot(\nabla\Lambda)+r\hat{k}\left(-\frac{1}{3}r\hat{k}\right) \\ &= 0. \end{aligned}$$

Problem 37 (a)

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \nabla\cdot\left(\frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x})\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}d^3x\right) \\ &= \frac{1}{4\pi\epsilon_0}\int_V \nabla\cdot\left(\rho(\mathbf{x})\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\right)d^3x \\ &= \frac{1}{4\pi\epsilon_0}\oint_S -\frac{\rho(\mathbf{x})\,\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}x^2 d\Omega \\ &= \frac{\rho}{\epsilon_0}. \end{aligned}$$