Problem (Problem 1)**:** Let $T\colon V \to W$ be a linear transformation between $\mathbb{F}$-vector spaces. Show that T is injective if and only if T maps $\mathbb{F}$-linearly independent subsets of V to $\mathbb{F}$-linearly independent subsets of W.

Solution: Let T be injective. We claim that if $\{v_1, \dots, v_n\}$ is linearly independent in V, then $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W. We see that if

$$\sum_{j=1}^{n} a_j T v_j = 0_W,$$

then

$$T\left(\sum_{j=1}^{n} a_j v_j\right) = 0_W,$$

meaning that

$$\sum_{j=1}^{n} a_j v_j \in \ker(T).$$

Now, since T is injective, $\ker(T) = \{0_V\}$, meaning that $\sum_{j=1}^{n} a_j v_j = 0_V$. Yet, since $\{v_1, \dots, v_n\}$ is linearly independent, this means $a_j = 0$ for each j, so $\{Tv_1, \dots, Tv_n\}$ is linearly independent in W.

Now, let T map linearly independent subsets of V to linearly independent subsets of W. If $\mathcal{B}_V = \{v_i\}_{i\in I}$ is a basis for V, then since $\mathcal{B}_V$ is linearly independent, $\mathcal{C} = \{Tv_i\}_{i\in I}$ is a linearly independent subset of W, which can be extended to a basis $\mathcal{B}_W$. Since $\mathcal{C} \subseteq \mathcal{B}_W$, we see that any linear combination in $\mathcal{B}_W$ yields 0 if and only if every coefficient is zero, meaning that $\ker(T) = \{0_V\}$, so T is injective.

Problem (Problem 2)**:** Let $P_{n+1}(\mathbb{R})$ be the space of polynomials with real coefficients of degree $\leqslant n+1$. Prove that for any n points $a_1, \dots, a_n \in \mathbb{R}$, there exists a nonzero polynomial $f \in P_{n+1}(\mathbb{R})$ such that $f(a_j) = 0$ for each j, and $\sum_{j=1}^{n} f'(a_j) = 0$.

Solution: Based on the first condition, we see that the product $\prod_{j=1}^{n}(x - a_j)$ must divide the polynomial f, and since f has degree at most $n+1$, we must have $f(x) = (Ax + B)\prod_{j=1}^{n}(x - a_j)$ for some $a, b \in \mathbb{R}$. Writing $f'(x)$, we see that

$$f'(x) = A\prod_{j=1}^{n}(x - a_j) + (Ax + B)\sum_{i=1}^{n}\prod_{j\neq i}(x - a_j),$$

Problem: Let $T\colon V \to W$ be a linear map of finite-dimensional vector spaces, and let $W_0 \subseteq W$ be a subspace.

(a) Show that $T^{-1}(W_0) = \{v \in V \mid Tv \in W_0\}$ is a subspace of V.

(b) Assuming T is surjective, express $\dim\left(T^{-1}(W_0)\right)$ in terms of $\dim(W_0)$ and $\dim(\ker(T))$.

Solution:

(a) We see that if $v_1, v_2 \in T^{-1}(W_0)$ and $\alpha \in \mathbb{R}$, then since $Tv_1, \alpha T v_2 \in W_0$, we have $Tv_1 + \alpha T v_2 \in W_0$, so by linearity, $T(v_1 + \alpha v_2) \in W_0$, meaning $v_1 + \alpha v_2 \in T^{-1}(W_0)$, so $T^{-1}(W_0)$ is a subspace of V.

(b) First, since T is surjective, $T\left(T^{-1}(W_0)\right) = W_0$. Therefore, by restricting the map T, we get the surjective map $T'\colon T^{-1}(W_0) \to W_0$, and since $\ker(T) \subseteq T^{-1}(W_0)$, the First Isomorphism Theorem gives $T^{-1}(W_0)/\ker(T) \cong W_0$, so by rank-nullity (as each of W_0 and $T^{-1}(W_0)$ are finite-dimensional), $\dim\left(T^{-1}(W_0)\right) = \dim(\ker(T)) + \dim(W_0)$.

Problem (Problem 5)**:**

(a) Find the inverse matrix A^{-1} for the matrix

$$A = \begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}.$$

(b) Prove that

$$\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x_n \end{vmatrix} = x_1x_2\cdots x_n\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_n}\right).$$

Solution:

(a) We may find A^{-1} by trying to find the sequence of elementary matrices $E_1, \dots, E_n$ such that

$$E_nE_{n-1}\cdots E_2E_1A = I.$$

First, we do row reduction on A, yielding

$$\begin{aligned}
\begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix} &\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}\\
&\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ a & a & a+1 \end{pmatrix}\\
&\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2a & a+1 \end{pmatrix}\\
&\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3a+1 \end{pmatrix}\\
&\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\\
&\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\\
&\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Thus, the product $E_nE_{n-1}\cdots E_2E_1$ is our desired inverse, which we find by applying the elementary row operations to the identity matrix I, yielding

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\\
&\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & a & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & -a & 2a+1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ -a/(3a+1) & 1-(a/(3a+1)) & -1+((2a+1)/(3a+1)) \\ -(a/(3a+1)) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1-a/(3a+1) & -a/(3a+1) & -1+(2a+1)/(3a+1) \\ -a/(3a+1) & 1-(a/(3a+1)) & -1+((2a+1)/(3a+1)) \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix},$$

which is our desired inverse.

(b) We show the case for $n = 2$, then use induction from then on. By raw calculation, we see that

$$\begin{aligned} \begin{vmatrix} a+x_1 & a \\ a & a+x_2 \end{vmatrix} &= (a+x_1)(a+x_2) - a^2 \\ &= x_1x_2 + ax_1 + ax_2 \\ &= x_1x_2\left(1 + \frac{a}{x_1} + \frac{a}{x_2}\right). \end{aligned}$$

Now, for the general n case, we see that since determinants are multilinear,

$$\begin{aligned} \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x_n \end{vmatrix} &= \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix} \\ &= a\begin{vmatrix} a+x_1 & a & \cdots & 1 \\ a & a+x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix} \end{aligned}$$

and since determinants are alternating,

$$= a\begin{vmatrix} x_1 & 0 & \cdots & 1 \\ 0 & x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$

and by the cofactor expansion,

$$= a(x_1x_2\cdots x_{n-1}) + x_n\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_{n-1} \end{vmatrix}$$

and by the induction hypothesis,

$$
\begin{aligned}
&= a(x_1x_2\cdots x_{n-1}) + x_n(x_1x_2\cdots x_{n-1})\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}}\right)\\
&= x_1x_2\cdots x_n\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}} + \frac{a}{x_n}\right),
\end{aligned}
$$

we obtain our desired result.

Problem (Problem 7)**:**

(a) Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be a matrix such that $A^2 = I_n$. Show that A is diagonalizable.

(b) Give an example of of $A \in \mathrm{Mat}_2(\mathbb{C})$ satisfying $A^2 = \mathbf{0}_2$ (the zero matrix) which is not diagonalizable.

Solution:

(a) Since $A^2 - I_n = \mathbf{0}_n$, we see that the minimal polynomial of A is $m_A(t) = t^2 - 1$, which splits over $\mathbb{C}$ to yield $m_A(t) = (t-1)(t+1)$. In particular, since the minimal polynomial splits into a product of distinct linear factors, A is diagonalizable.

(b) The matrix

$$
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

satisfies $A^2 = \mathbf{0}_2$, but since $A \neq \mathbf{0}_2$, we see that $m_A(t) = t^2$. Since $m_A(t)$ does not split into distinct linear factors over $\mathbb{C}$, we see that A is necessarily not diagonalizable.

Problem (Problem 8)**:** Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be a matrix such that A^2 has n distinct eigenvalues. Show that A is diagonalizable.