I will be using A^* to denote the adjoint of an operator A throughout this assignment.

Chapter 26 Problems

Problem 1

$$\begin{aligned} v_i &= \langle \hat{e}_i \,|\, v\rangle \\ &= \langle v \,|\, \hat{e}_i\rangle \\ &= \langle v|\, R^T \left|\hat{e}'_i\right\rangle \\ &= \sum_k \left\langle v \,\middle|\, \hat{e}'_k\right\rangle \left\langle \hat{e}'_k\right| R^T \left|\hat{e}'_i\right\rangle \\ &= \sum_k v'_k \left|\hat{e}'_k\right\rangle. \end{aligned}$$

Problem 2

$$\begin{aligned} |v'\rangle &= v'_1 |\hat{e}_1\rangle + v'_2 |\hat{e}_2\rangle \\ &= (v_1 \cos\phi - v_2 \sin\phi) |\hat{e}_1\rangle + (v_1 \sin\phi + v_2 \cos\phi) |\hat{e}_2\rangle \\ &= v_1 (\cos\phi\, |\hat{e}_1\rangle + \sin\phi\, |\hat{e}_2\rangle) + v_2 (-\sin\phi\, |\hat{e}_1\rangle + \cos\phi\, |\hat{e}_2\rangle) \\ &= v_1 \left|\hat{e}'_1\right\rangle + v_2 \left|\hat{e}'_2\right\rangle. \end{aligned}$$

This is a clockwise rotation of the unprimed basis.

Problem 4

$$\begin{aligned} (R_n(\varphi))^m &= \left(e^{-i\phi\hat{n}\cdot\mathbf{L}}\right)^m \\ &= e^{-im\phi\hat{n}\cdot\mathbf{L}} \\ &= R_n(m\varphi). \end{aligned}$$

We then have

$$\begin{aligned} R_n(3\varphi) &= (R_n(\varphi))^3 \\ &= (\cos(\varphi\hat{n}\cdot\mathbf{L}) + i\sin(\varphi\hat{n}\cdot\mathbf{L}))^3 \\ &= \left(\cos^3(\varphi\hat{n}\cdot\mathbf{L}) - 3\sin^2(\varphi\hat{n}\cdot\mathbf{L})\cos(\varphi\hat{n}\cdot\mathbf{L})\right) + i\left(\cos^2(\varphi\hat{n}\cdot\mathbf{L})\sin(\varphi\hat{n}\cdot\mathbf{L}) - \sin^3(\varphi\hat{n}\cdot\mathbf{L})\right) \\ &= \cos(3\varphi\hat{n}\cdot\mathbf{L}) + i\sin(3\varphi\hat{n}\cdot\mathbf{L}). \end{aligned}$$

Problem 5

(a) Since $A^T = A^{-1}$, this matrix is orthogonal (and unitary). However, it is not Hermitian.

(b) Since $\det(B) = 0$, this matrix is not unitary, but it is Hermitian.

(c) Since neither $C^* = C$, $C^* = C^{-1}$, nor $C^T = C^{-1}$, it is the case that C is neither orthogonal, unitary, nor Hermitian.

(d) Since

$$\operatorname{tr}\left(i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) = 0,$$

D is a unitary operator.

(e) Since $E^* = E$, E is Hermitian. Additionally, $E^* = E^{-1}$, so E is also unitary.

Problem 8

$$\begin{aligned} UU^* &= I \\ \det(UU^*) &= \det(I) \\ \det(U)\det(U^*) &= 1 \\ (\det(U))^2 &= 1 \\ &= \det(U)\det(U)^{-1}. \end{aligned}$$

meaning $\det(U) \in \mathbb{T}$, so $\det(U)$ is pure phase.

Problem 11

$$\begin{aligned} \|\mathbf{v}\|^2 &= \sum_i \langle v_i \,|\, v_i \rangle \\ &= \sum_i \left\langle \sum_k U_{ik} v_k \,\middle|\, \sum_\ell U_{i\ell} v_\ell \right\rangle \\ &= \sum_{i,k,\ell} \overline{U_{ik}} U_{i\ell} \langle v_k \,|\, v \rangle_\ell \\ &= \sum_{k,\ell} \left(\sum_i U^*_{ki} U_{i\ell} \right) \langle v_k \,|\, v_\ell \rangle, \end{aligned}$$

meaning $\sum_i U^*_{ki} U_{i\ell} = \delta_{k\ell}$, so $U^*U = I$.

Problem 16

$$\begin{aligned} \langle \delta\mathbf{r} \,|\, \mathbf{r} \rangle &= \left\langle \delta\vec{\varphi} \times \mathbf{r} \,\middle|\, \mathbf{r} \right\rangle \\ &= \sum_k \left\langle \sum_{i,j} \delta\epsilon_{ijk} \varphi_i r_j \,\middle|\, r_k \right\rangle \\ &= \sum_{i,j,k} \delta\epsilon_{ijk} \varphi_i \left\langle r_j \,\middle|\, r_k \right\rangle. \end{aligned}$$

I don't know where to go from here.

Problem 18

(a) Solving the eigenvector equation $A\hat{n} = \hat{n}$, we get

$$\hat{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} \phi &= \arccos\left(\frac{1}{2}(0) \right) \\ &= \pi/2. \end{aligned}$$

(b) This matrix is a reflection about the line $y = x$.

(c) This matrix is a flip and $\pi/6$ rotation about the y axis.

(d) This matrix is a flip and a $\pi/4$ rotation about the x axis.

Problem 20

We have

$$\begin{aligned}\phi &= \arccos\left(\frac{1}{2}\left(\frac{2}{3}\right)\right)\\ &\approx 70.53^\circ,\end{aligned}$$

and, solving

$$R_q\hat{q} = \hat{q}$$

with Mathematica, we also get

$$\hat{q} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

Problem 21

We get

$$\begin{aligned}R_n(\alpha) &= e^{-i\alpha\hat{n}\cdot\mathbf{L}}\\ &= e^{-i\alpha\left(\frac{1}{\sqrt{2}}L_2+\frac{1}{\sqrt{2}}L_3\right)}\\ &= e^{-i\frac{\alpha}{\sqrt{2}}L_2}e^{-i\frac{\alpha}{\sqrt{2}}L_3}\\ &= R_y\left(\frac{\alpha}{\sqrt{2}}\right)R_z\left(\frac{\alpha}{\sqrt{2}}\right).\end{aligned}$$

Chapter 27 Problems

Problem 1

(a)

$$\begin{aligned}\det(A-\lambda I) &= (\lambda-1)^2-4\\ (\lambda-1)^2-4 &= 0\\ \lambda_{1,2} &= -1, 3.\end{aligned}$$

We have

$$\begin{aligned}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= -1\begin{pmatrix}x\\y\end{pmatrix}\\ x+2y &= -x\\ x &= -y\\ |v_1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\\ |v_2\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.\end{aligned}$$

(b)

$$\det(A - \lambda I) = (\lambda - 1)^2 + 4$$
$$\lambda_{1,2} = 1 \pm 2i.$$

We have

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \pm 2i \begin{pmatrix} x \\ y \end{pmatrix}$$
$$x + 2y = (1 + 2i)\, x$$
$$x = -iy$$
$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$|v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

(c)

$$\det(A - \lambda I) = (\lambda - 1)^2 - 4$$
$$\lambda_{1,2} = -1, 3,$$

meaning

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$
$$|v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

(d)

$$\det(A - \lambda I) = (\lambda - 1)^2 + 4$$
$$\lambda_{1,2} = 1 \pm 2i,$$

meaning

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Problem 2

The trace is equal to $\lambda_1 + \lambda_2$, while the determinant is equal to $\lambda_1\lambda_2$, meaning we have two equations and two unknowns.

For

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

it is the case that $\operatorname{tr}(A) = 2$ and $\det(A) = -3$, so $\lambda_{1,2} = -1, 3$.

Problem 4

Computing

$$\det(S - \lambda I) = -\lambda^3 + 3\lambda - 2.$$

We find

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_{2,3} &= 1.\end{aligned}$$

The eigenvector for λ_1 is

$$|v_1\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

To solve for $|v_2\rangle$ and $|v_3\rangle$, we find

$$\begin{aligned}\frac{1}{2}\begin{pmatrix}1 & 2 & -1\\ 2 & -2 & 2\\ -1 & 2 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} &= \begin{pmatrix}x\\y\\z\end{pmatrix}\\ \frac{1}{2}x + y - \frac{1}{2}z &= x\\ 2y &= x + z.\end{aligned}$$

Two orthogonal eigenvectors corresponding to $\lambda_{2,3} = 1$ are

$$\begin{aligned}|v_2\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\\ |v_3\rangle &= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}.\end{aligned}$$

Problem 6

I don't know how to do this problem.