

Cardinality and Countability

Section 1.1: Countable Sets

Definition (Denumerable Set). A set S is denumerable if there exists a function $f : S \rightarrow \mathbb{N}$ with f a bijection. We also say S is countably infinite.

Definition (Countable Set). We say S is countable if S is either finite or denumerable.

Theorem (Countability of Unions). If A and B are countable sets, then $A \cup B$ is countable.

Theorem (Countability of Subsets). If $A \subseteq B$, then if B is countable, then A is countable.

Theorem (Union of Finite Sets). If A and B are finite, then $A \cup B$ is finite.

Proof. If A is finite and B has one element, then we show that $A \cup B$ is finite (with two cases).

Afterward, for $|B| > 1$, we use induction on $|B|$. □

Definition (Finite Set). A set A is finite if there exists a bijection $f : S \rightarrow \{1, 2, \dots, n\}$ for some $n \in \mathbb{N} = \{0, 1, \dots\}$.

We write $|A| = n$.

Theorem (Disjoint Union of Countable Sets). If A is denumerable, B is finite, and $A \cap B = \emptyset$, then $A \cup B$ is denumerable.

Proof. There exists a bijection $f : A \rightarrow \mathbb{N}$ (since A is denumerable), and a bijection $g : B \rightarrow \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ (since B is finite).

We create a new bijection $h : A \cup B \rightarrow \mathbb{N}$ by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since $A \cap B = \emptyset$, we know that h is well-defined.

Now, we must show that h is a bijection.

Suppose $h(x) = h(y)$.

Case 1: If $x, y \in B$, then $h(x) = g(x) - 1$, and $h(y) = g(y) - 1$, meaning $g(x) - 1 = g(y) - 1$, meaning $g(x) = g(y)$. Since g is a bijection, $x = y$.

Case 2: If $x, y \in A$, a similar argument yields that $x = y$.

Case 3: Without loss of generality, let $x \in A$ and $y \in B$. If $x \in A$, then $h(x) = f(x) + n$ and $h(y) = g(y) - 1$. Thus, $f(x) + n = g(y) - 1$. However, since $f(x) + n \geq n$ and $0 \leq g(y) - 1 \leq n - 1$. Thus, we get that $0 \leq n \leq n - 1$, which is a contradiction.

Thus, we have shown that h is injective. □

Theorem (Cartesian Product of Natural Numbers). $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof. We consider $\mathbb{N} \times \mathbb{N}$ as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \dots,$$

$$\begin{array}{rcll}
\mathbb{N} \times \{0\} : & (0, 0) & (1, 0) & (2, 0) & (3, 0) & \dots \\
\mathbb{N} \times \{1\} : & (0, 1) & (1, 1) & (2, 1) & (3, 1) & \dots \\
\mathbb{N} \times \{2\} : & (0, 2) & (1, 2) & (2, 2) & (3, 2) & \dots \\
\mathbb{N} \times \{3\} : & (0, 3) & (1, 3) & (2, 3) & (3, 3) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{rcll}
\mathbb{N} \times \{0\} : & \cancel{(0,0)}^0 & \cancel{(1,0)}^1 & \cancel{(2,0)}^2 & \cancel{(3,0)}^3 & \dots \\
\mathbb{N} \times \{1\} : & \cancel{(0,1)}^4 & \cancel{(1,1)}^5 & \cancel{(2,1)}^6 & (3,1) & \dots \\
\mathbb{N} \times \{2\} : & \cancel{(0,2)}^7 & \cancel{(1,2)}^8 & (2,2) & (3,2) & \dots \\
\mathbb{N} \times \{3\} : & \cancel{(0,3)}^9 & (1,3) & (2,3) & (3,3) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can also find a bijection $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, with

$$P(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

A fun challenge is to prove that P is a bijection. □

Theorem (Countability of the Rationals). \mathbb{Q} is denumerable.

Theorem (Countability of the Integers). The set \mathbb{Z} is denumerable.

Proof. Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

□

Definition (Cardinality). We say two sets, A and B , have the same cardinality if there exists a bijection $f : A \rightarrow B$.

Theorem (Finite Subset Cardinality). If $m, n \in \mathbb{N}$ and $m \neq n$, then $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$ do not have the same cardinality.

Theorem (Infinitude of the Natural Numbers). \mathbb{N} is not finite.

Example. If $A \subsetneq B$ and $|A| = |B|$, then both A and B are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.

Section 1.2

Definition (Uncountable Set). A set is uncountable if it is not countable.

Theorem (Uncountability of \mathbb{R}). \mathbb{R} is uncountable.

Proof. For all $x \in \mathbb{R}$, and for all $j \in \mathbb{N}$, we define $[x]_j$ to denote the $j + 1$ -th digit after the decimal point in the decimal expansion of x .

For example, $[\pi]_0 = 1$, $[\pi]_1 = 4$, etc.

Let $f : \mathbb{N} \rightarrow \mathbb{R}$. We will show that f is not surjective.

Let $y \in [0, 1) \subseteq \mathbb{R}$ defined by $\forall j \in \mathbb{N}$,

$$[y]_j = \begin{cases} 0 & [f(j)]_j = 1 \\ 1 & [f(j)]_j \neq 1 \end{cases}.$$

We claim that $y \notin f(\mathbb{N})$. We will show that $\forall j \in \mathbb{N}$, $f(j) \neq y$.

We can see that if $[f(j)]_j = 1$, then $[y]_j = 0$. Similarly, if $[f(j)]_j \neq 1$, then $[y]_j = 1$. Either way, $[f(j)]_j \neq [y]_j$ for all $j \in \mathbb{N}$. \square

Remark: The above proof is an example of a diagonalization proof. It can be imagined as

$$\begin{array}{c|l} f(0) & *.a_1 \overset{\neq}{a_2} a_3 \dots \\ f(1) & *.b_1 b_2 \overset{\neq}{b_3} \dots \\ f(2) & *.c_1 c_2 c_3 \overset{\neq}{\dots} \\ \vdots & \vdots \end{array}$$

Note: A substantial problem that we might need to deal with is that a real number does not necessarily have a unique decimal representation. For instance, $3.999\dots = 4.000\dots$.

In order to resolve this issue, we can default to the option with trailing 0 over trailing 9.

Definition (Power Set). The power set of a set S is

$$P(S) = \{A \mid A \subseteq S\}.$$

Theorem (Power Set Surjection). Let $f : S \rightarrow P(S)$. Then, f is not surjective.

Proof. Let $T = \{x \in S \mid x \notin f(x)\}$. Then, $T \notin f(S)$.

Let $y \in S$. We want to show that $f(y) \neq T$. Suppose toward contradiction that $f(y) = T$. Then, if $y \in T$, then $y \in f(y)$, which implies that $y \notin T$.

If $y \notin T$, then $y \notin f(y)$, which implies that $y \in T$.

Thus, it cannot be the case that $f(y) = T$. \square

Definition (Cardinality Comparison). Let A and B be sets. Then, we write $\text{card}(A) \leq \text{card}(B)$ if there exists an injective map $f : A \hookrightarrow B$.

We write $\text{card}(A) < \text{card}(B)$ if there exists an injection $f : A \hookrightarrow B$ but no bijection.

Example (Cardinality of the Power Set). For every set,

$$\text{card}(S) < \text{card}(P(S)).$$

- (1) We know that $\text{card}(S) \leq \text{card}(P(S))$, defining $f : S \hookrightarrow P(S)$, $f(a) = \{a\}$, since if $f(x) = f(y)$, then $\{x\} = \{y\}$, meaning $x \in \{y\}$, so $x = y$.

In the case of $f : \emptyset \rightarrow \{\emptyset\}$, we define $\emptyset = f \subseteq \emptyset \times \{\emptyset\}$.

- (2) Since there exists no bijection $f : S \rightarrow P(S)$, it is the case that $\text{card}(S) \neq \text{card}(P(S))$.

Example (Decimal Expansion). We know that for some decimal expansion

$$\begin{aligned} 3.14159\dots &= 3 + \frac{1}{10} + \frac{4}{100} + \dots \\ &= \sum_{i=0}^{\infty} \frac{n_i}{10^i}, \end{aligned}$$

with $0 \leq n_i \leq 9$ for $i \geq 1$.

However, we can also write any real number as

$$\sum_{i=0}^{\infty} \frac{n_i}{3^i}$$

with $0 \leq n_i \leq 2$ for all $i \geq 1$.

Example (Finite Strings). Let S be the set of all finite strings of 0 and 1. S is countable.

Proof 1: We define $f : S \rightarrow \mathbb{N}$ by, for a string $x \in S$, x starts with n_1 zeroes, then has n_2 ones, then n_3 zeroes, etc. We define $f(x) := 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times 7^{n_4} \times 11^{n_5} \dots$, or

$$f(x) = \prod_i p_i^{n_i},$$

where p_i denotes the i th prime number. We can see that f is an injection.

Since S is infinite (proof omitted), we can see that $f(S)$ is also infinite.¹ Since $f(S)$ is an infinite subset of \mathbb{N} , $f(S)$ is denumerable, meaning there exists a bijection $q : f(S) \rightarrow \mathbb{N}$. Therefore, we have $q \circ f : S \rightarrow \mathbb{N}$ is a bijection, meaning S is denumerable.

Proof 2: List the elements of S by length and lexicographic order: short strings come before long strings, and 0s come before 1s.

Rank	String
0	0
1	1
2	00
3	01
4	10
5	11
\vdots	\vdots

This pattern yields a systematic way to map S to the natural numbers.

Proof 3: We can see that

$$S = \bigcup_{i=1}^{\infty} S_i,$$

where S_i is the set of all strings of length i , each of which contains 2^i elements.

Since each S_i is finite, and $S_i \cap S_j = \emptyset$ (by definition). Thus, S is a countable union of pairwise disjoint countable sets, so S is countable.

¹If $f(S)$ is finite, then there exists a bijection $g : f(S) \rightarrow \{1, \dots, n\}$. Composing g and f , we find S is finite as $g \circ f|_S$ is a bijection.

Example (All Possible Writings). Let W be the set of all possible writings in English. We let W_n denote the writing with n characters. Then,

$$W = \bigcup_{n=1}^{\infty} W_n,$$

which is a countable union of disjoint finite sets, which is countable.

Similarly, we can list all the writings by length and lexicographic order.

This result implies that “almost all” real numbers, in a sense, are unable to be described.

Section 1.3: Cantor–Schröder–Bernstein Theorem

Example. If we have $|A| \leq |B|$ and $|B| \leq |A|$, it does not necessarily imply $|A| = |B|$.

This is because the \leq in the cardinality comparison implies there exist injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, not that the cardinalities are necessarily “less than or equal to” each other.

However, at the same time, this fact is true — this is what is known as the Cantor–Schröder–Bernstein Theorem.

Theorem (Cantor–Schröder–Bernstein). Let $f : C \hookrightarrow D$ and $g : D \hookrightarrow C$ be injective maps. Then, $|C| = |D|$.

An Informal Proof Sketch. Consider C to be a set of cats and D to be a set of dogs. Every cat chases a dog, and every dog chases a cat, with different cats chasing different dogs and vice versa.

There are four potential arrangements:

- (1) A set of cats and dogs are chasing each other in a circle.
- (2) A chain of dogs chasing cats that starts with a dog.
- (3) A chain of cats chasing dogs that starts with a cat.
- (4) An endless chain of cats chasing dogs with no discernible start or end point.

These four cases create a bijection from C to D :

- (1) Pair each cat with the dog that it is chasing.
- (2) Pair each cat with the dog that it is chasing.
- (3) Pair each cat with the dog that *is chasing it*.
- (4) Pair each cat with the dog that it is chasing.

□

A More Formal Proof Sketch. For $C = \{c_i\}_{i \in I}$ and $D = \{d_i\}_i$, we have four types of sequences.

- (i) Circular sequence: for some $m \in \mathbb{N}$, there exist c_1, \dots, c_m and d_1, \dots, d_m such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$, where $c_{m+1} = c_1$.
- (ii) Cat sequence: there is c_1, c_2, \dots and d_1, d_2, \dots such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.
- (iii) Dog sequence: there is c_1, c_2, \dots and d_1, d_2, \dots such that $f(c_i) = d_{i+1}$ and $g(d_i) = c_i$.
- (iv) Bi-infinite sequence: $\{c_i\}_{i \in \mathbb{Z}}$ and $\{d_i\}_{i \in \mathbb{Z}}$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

Claim 1: For every $c \in C$, c is in exactly one sequence that is either a circular sequence, a cat sequence, a dog sequence, or a bi-infinite sequence.

We define our bijection $h : C \rightarrow D$ by

$$h(c) = \begin{cases} g^{-1}(c) & c \text{ in a dog sequence} \\ f(c) & \text{else} \end{cases}.$$

Claim 2: h is well-defined.

Claim 3: h is a bijection.

□

Theorem. For every set A, B , either $|A| \leq |B|$ or $|B| \leq |A|$.

In order to prove this, we need the axiom of choice.

Example (Cardinality of the Reals). Recall that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ and $|\mathbb{N}| < |\mathbb{R}|$. According to the previous theorem, it is the case that either $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$ or $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$.

In particular, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

An Informal Proof. Let S be the set of all functions $f : \mathbb{N} \rightarrow \{0, 1\}$. We will show that $|S| = |\mathcal{P}(\mathbb{N})|$ and $|S| = |\mathbb{R}|$. This will show that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ (by composing bijections).

To show that $|S| = |\mathcal{P}(\mathbb{N})|$, define a subset of \mathbb{N} by the supportⁱⁱ of some element of S . This is a bijection between $\mathcal{P}(\mathbb{N})$ and S .

To show $|S| = |\mathbb{R}|$, we place a decimal point in front of the string, and consider it as a real number in base 2, which yields a bijection between S and $[0, 1]$.

Next, we show that $|[0, 1]| = |(0, 1)|$.

Finally, we show that $|(0, 1)| = |\mathbb{R}|$. Take $f : (0, 1) \rightarrow \mathbb{R}$ to be $\cot(\pi x)$ — or $\tan(\pi x - \pi/2)$. These are bijections from $(0, 1)$ to \mathbb{R} . □

Definition (Continuum Hypothesis). We are aware that

$$|\mathbb{N}| < |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$$

The continuum hypothesis states that there exists no set S such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

The continuum hypothesis is independent of the ZFC axioms.ⁱⁱⁱ

Exercise (Challenge Problem). Let $T = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{N}; \text{ finitely many nonzero } a_i\}$. Is T countable? We also write

$$T = \bigoplus_{i=0}^{\infty} \mathbb{N}.$$

ⁱⁱThe elements that f does not map to 0 for some $f \in S$.

ⁱⁱⁱZermelo–Fraenkel Axioms with the Axiom of Choice.

Axiomatic Set Theory

Question. Is there a set A such that $A \in A$?

Answer. Yes! There is the set $\{\cdots\{\}\cdots\}$, which contains infinitely many sets in itself. Additionally, there is the set $A = \{x \mid x \text{ is a set}\}$.

Example (Russell's Paradox). Consider the set

$$R = \{x \mid x \notin x\}.$$

The question is if $R \in R$. However, this cannot be true, because if $R \in R$, then $R \notin R$ and vice versa.

Axioms of Set Theory

We cannot just say

$$S = \{x \mid x \text{ is blah}\},$$

as evidenced by Russell's paradox. We need to carefully construct rules to create a rigorous description of formal set theory.

Axiom (Existence). The existence axiom states that there exists a set:

$$\exists a (a = a).$$

Axiom (Empty Set). The empty set axiom states that there exists a set with no elements:

$$\exists a \forall x (x \notin a).$$

Axiom (Pairing). The pairing axiom states that, given any sets a and b , there is a set c such that the only elements of c are a and b :

$$\forall a \forall b \exists c \forall x (x \in c \Leftrightarrow x = a \vee x = b)$$

Axiom (Extensionality). The axiom of extensionality states that if two sets have the same elements, they are the same sets:

$$\forall a \forall b (\forall x (x \in a \Leftrightarrow x \in b) \Rightarrow a = b)$$

Question. What is a set?

Answer. The unsatisfying answer is that "set" and "element" have no meaning *per se*. The main reason we define these axioms is to define relationships between objects (rather than objects themselves).

Example. We want to prove that for every set b , there exists a set $\{b\}$.

Symbolically, we want to show

$$\forall b \exists c \forall x (x \in c \Leftrightarrow x = b).$$

In particular, we can see that, in the pairing axiom, there is no requirement that a and b be distinct. Therefore, we can use the pairing axiom of $a = b$ and $b = b$. Therefore, the pairing axiom becomes

$$\forall b \forall b \exists c \forall x (x \in c \Leftrightarrow x = b \vee x = b),$$

which reduces to

$$\forall b \exists c \forall x (x \in c \Leftrightarrow x = b).$$

In particular, if $b = \{\}$ in the previous example, then the pairing axiom implies the uniqueness of the empty set. We will denote $\{\} = \emptyset$. We can create a tower

$$\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \dots,$$

entirely consisting of the empty set.

Axiom (Union). The axiom of union states that if a and b are sets, there exists a set c whose elements are either elements of a or elements of b , and every element of a is in c and every element of b is in c :

$$\forall a \forall b \exists c \forall x (x \in c \Leftrightarrow x \in a \vee x \in b)$$

Definition. The string $a \subseteq b$ is shorthand for

$$\forall x (x \in a \Rightarrow x \in b).$$

Axiom (Power Set). The power set axiom states that for all a , there is a set b such that all elements of b are subsets of a and all subsets of a are contained in b :

$$\forall a \exists b \forall y (y \in b \Leftrightarrow y \subseteq a).$$

Definition. We let (a, b) be shorthand for the set

$$\{a, \{a, b\}\}.$$

Exercise. If $\{a, \{a, b\}\} = \{c, \{c, d\}\}$, it is the case that $a = c$ and $b = d$.

Recall that

$$c = \{x \mid x \text{ is blah}\}$$

is a problematic definition of a set. However, if a is a set, we can define

$$c = \{x \mid x \in a \wedge x \text{ is blah}\},$$

which does not cause any contradictions. The following axiom schema formalizes this fact.

Axiom (Comprehension schema). The comprehension schema says that, given any formula $\varphi(x)$, in which x is a free variable, there exists a set c whose elements are those in a that satisfy φ :

$$\forall a \exists c \forall x (x \in c \Leftrightarrow x \in a \wedge \varphi(x)).$$

Remark: There are infinitely many axioms in the comprehension schema, one for each formula φ . This is why it is known as a schema rather than an axiom.

Remark: Since we can specify a formula $\varphi(x) : x \neq x$, the comprehension schema obviates the empty set axiom.