The basis of Multivariable Calculus

If a function is continuous and differentiable, on a small enough interval, the function will approximate a line (i.e., a function of x).

A similar intuition applies to functions of more than one variable (but with a plane, cube, hypercube, etc.). However, in multivariable functions, we will have to sacrifice the ability to visualize it.

For example, in multiple dimensions, it is possible for there to be a function that is both strictly decreasing (in one dimension) and strictly increasing (in another dimension).

Some Functions and Sets

$$f(x, y) = x^2 - y^2$$

Domain: $\{(x, y) \mid \exists f(x, y)\}$

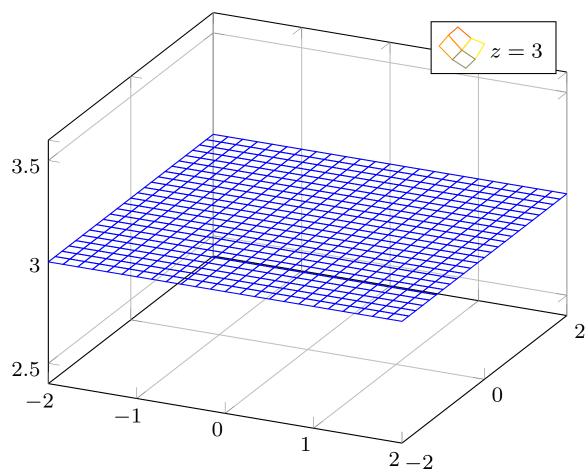
Range: $\{f(x, y) \mid (x, y) \in \text{Dom}(f)\} = \mathbb{R}$

Graph: $\text{Graph}(f) = \{x, y.f(x, y) \mid x, y \in \text{Dom}(f)\}$. For example, $(1, 3, 4) \notin \text{Graph}(f)$ since $1^2 - 3^2 \neq 4$.

Examples

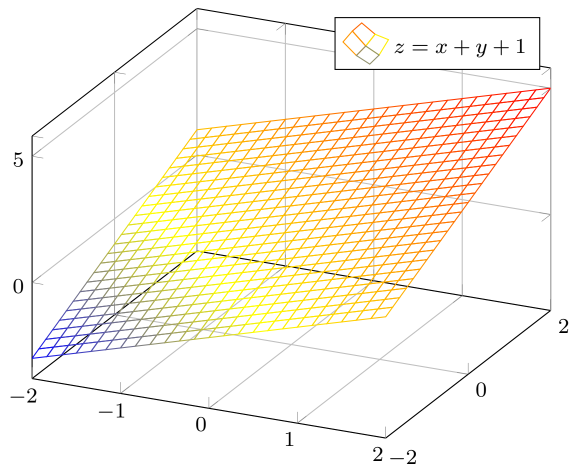
In $\mathbb{R}^3$, in x, y, z coordinates, $z = 3$ is a plane defined as follows:

- Parallel to the xy plane.
- Passes through the point $(0.0, 3)$.



Meanwhile, $y = 0$ would be a "wall" that passes through the origin that contains the line $y = 0$ in the xy plane.

Finally, $z = x + y + 1$ is a plane, as we can see below.



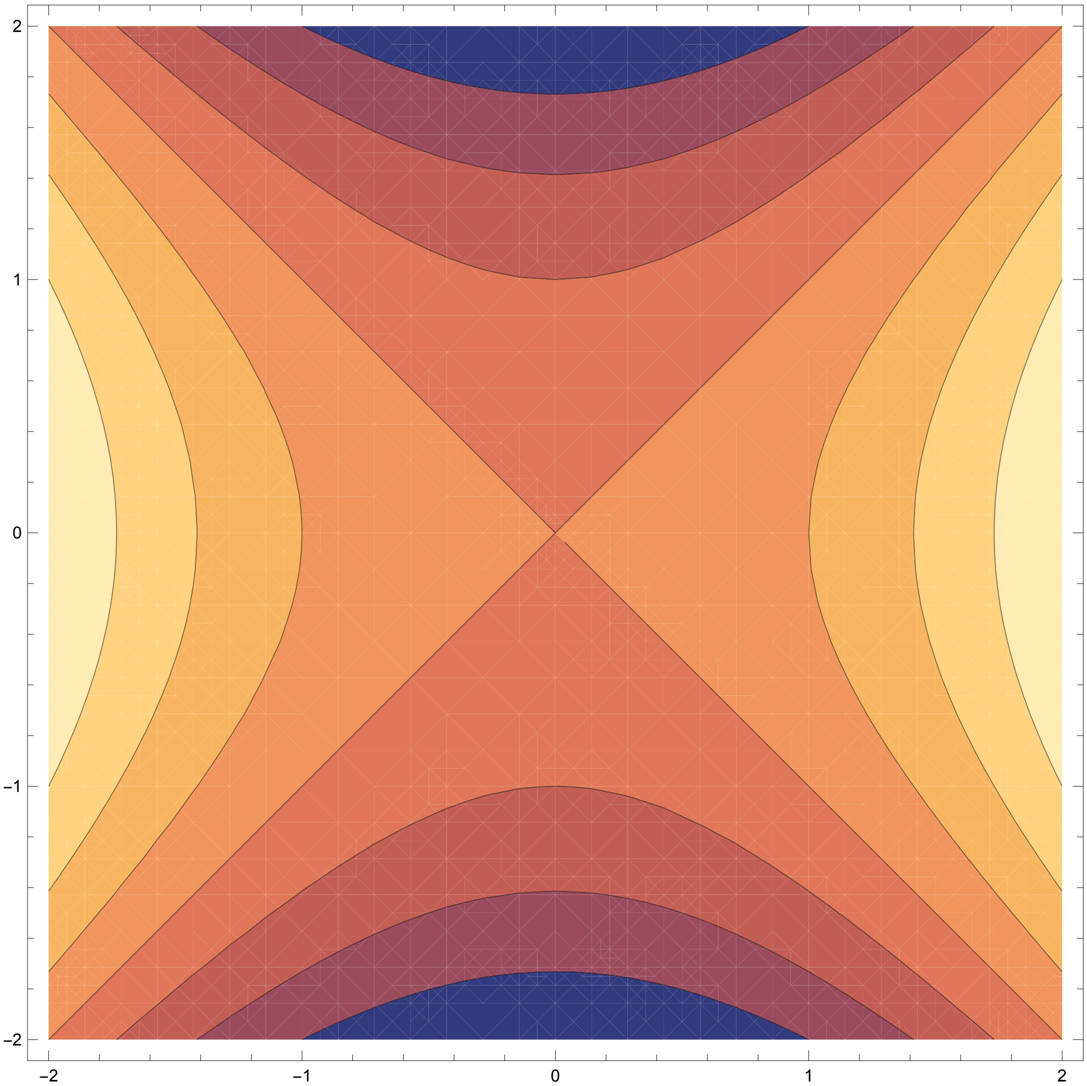
Visualizing a function of multiple variables

Consider the function $f(x, y) = x^2 - y^2$. We can try visualizing slices as follows:

- $f(-2, y) = 4 - y^2$
- $f(0, y) = -y^2$
- $f(2, y) = 4 - y^2$
- $f(x, -2) = x^2 + 4$
- $f(x, 0) = x^2$
- $f(x, 2) = x^2 + 4$

Alternatively, we can visualize via contour diagrams (i.e., everywhere that z is a certain value), as seen in mathematica as follows:

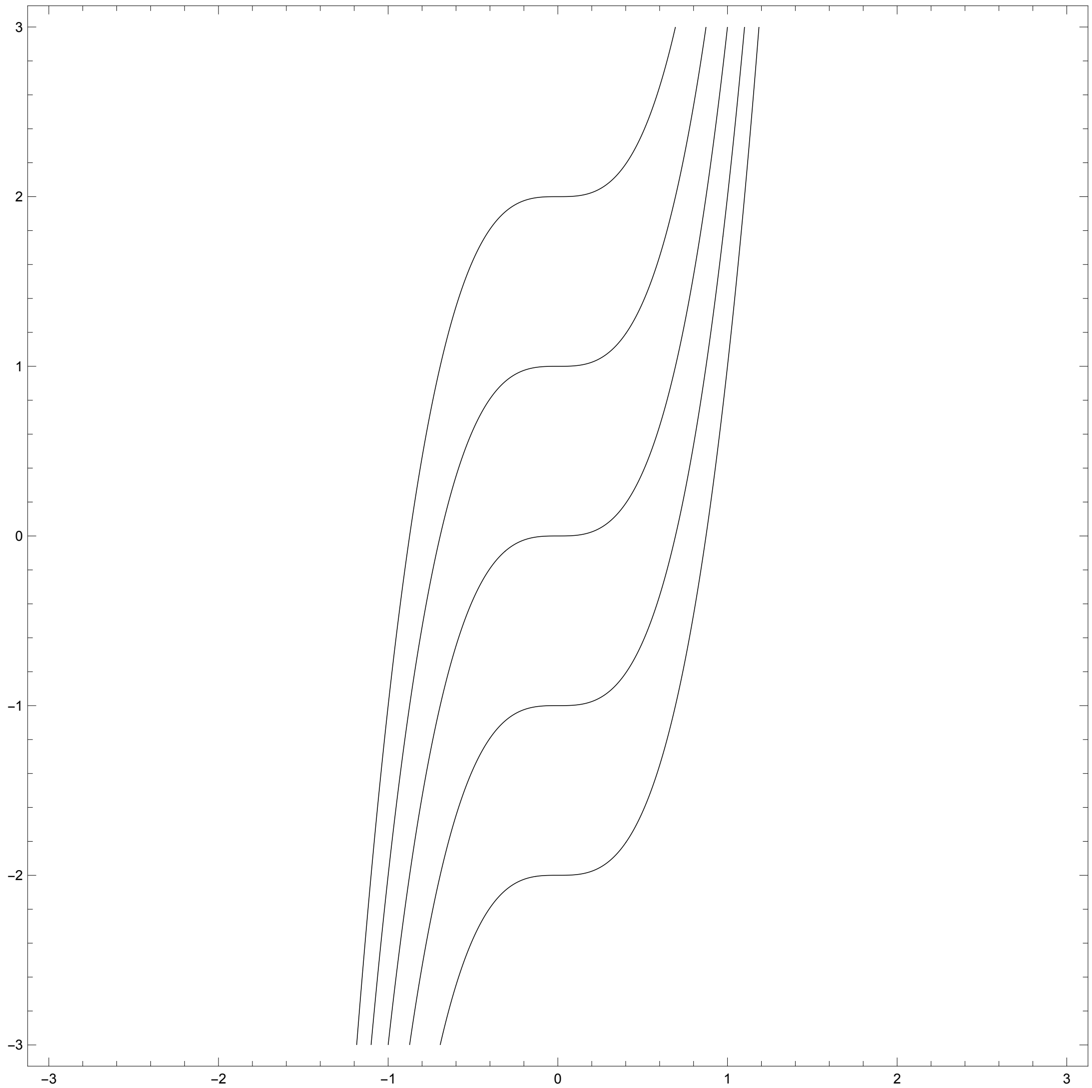
Contour Diagram



Contour Example

Consider the function $f(x, y) = y - 3x^2$. We want to find the contours.

For any c, we have that $c = y - 3x^3$, or $y = 3x^3 + c$. Therefore, every contour "looks like" $3x^3 + c$ for values of c. For example, in the following, we have $c = \{-2, -1, 0, 1, 2\}$

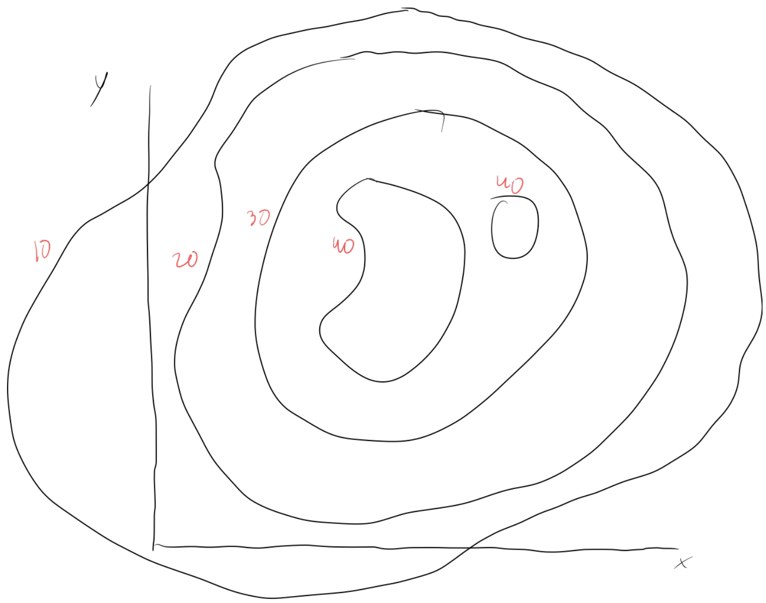


Distance

In $\mathbb{R}^5$, let $p = (3, 1, 4, 1, 5)$, and $q = (1, 0, -2, 0, 2)$. Using the Euclidean metric, we can find the distance between p and q is $d(p, q) = ((3-1)^2 + (1-0)^2 + (4-(-2))^2 + (1-0)^2 + (5-2)^2)^{1/2} = (4 + 1 + 36 + 1 + 9)^{1/2} = \sqrt{51} = 7.14$. We can also call this the 2-norm.

$$d(p, q) = \left(\sum_{k=1}^{n} (p_k - q_k)^2 \right)^{1/2}$$

Derivatives



To denote a derivative, we can't talk about one value, we must use a *partial* derivative, $\frac{\partial f}{\partial x}$, or $\frac{\partial f}{\partial y}$. The closeness of the contours specifies both resolution and steepness.

We can estimate slope by calculating the difference between two contours, divided by the distance between them along a path.

We can also analyze via a table:

x\y	0	1	2
4	5	6	7
6	8	9	10
8	11	12	13

A "linear" approximation for a function of two variables is expressed as follows:

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

Where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and is an output in $z = f(x, y)$, and $m, n \in \mathbb{R}$.

For example, with the above table, we can see that the function is linear in x and y (i.e., the slope holding the other variable constant is constant).

Limits in Multivariable Functions

Consider the following:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$$

Allow $y = mx$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2} &= \lim_{(x,y)\to(0,0)} \frac{x^2 + (mx)^2}{x^2 - (mx)^2} \\ &= \frac{1 + m^2}{1 - m^2}\end{aligned}$$

Thus, the limit must depend on the path taken. The following table shows the limits for different values of m

m	$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$
0	1
1	undefined
2	$-\frac{5}{3}$

Because the limit depends on the path of incidence, we have that the limit is **undefined**.

For graphs where the contours "approach" a particular point, we can see that the limit is defined.

Vectors

A vector is a mathematical object with direction and magnitude:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Alternatively, we can have $\vec{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$. These vectors are equivalent because they are components of $\mathbb{R}^3$.

Vector addition is *component-wise*, (i.e., you add or subtract components in order to find the new vectors).

Direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

Properties of Vectors

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Via properties of the real numbers, we know the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $c\vec{u} = \langle cu_1, cu_2, \dots, cu_k \rangle$

Additionally, we define $\vec{u} \cdot \vec{v}$ as follows:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

Partial Derivatives

Consider $f(x, y) = x^2 y + xe^y$.

$$f_x := \frac{\partial f}{\partial x}$$
$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

We know that $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, meaning f is endlessly differentiable.

Functions and Approximations

Let $f(x, y) = x^2 - y^2$, $g(x, y) = 2xy$

- $f_{xx} + f_{yy} = 0$
- $g_{xx} + g_{yy} = 0$

This is the solution to the Laplace equation:

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For $f(x, y)$ at $(a, b, f(a, b))$, we have the following:

$$\begin{aligned}\ell(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(y - b)\\ q(x, y) &= \ell(x, y) + \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right)\end{aligned}$$

In order to get a sense of the "derivative," we can use the following:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

Directional Derivative and Gradient

Given $f(x, y)$ and (a, b), where $f \in C^2(\mathbb{R}^2)$. Then, the quadratic approximation is:

$$\begin{aligned}f(x, y) &\approx f(a, b) + f_x(a, b)(x - a) + f_x(a, b)(y - b)\\ &\quad + \frac{1}{2}\left(f_{xx}(a, b)(x - a)^2 + f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b)\right)\\ df &= f_x(a, b)dx + f)y(a, b)dy && \text{a differential}\\ \Delta f &= f_x(a, b)\Delta x + f_y(a, b)\Delta y\end{aligned}$$

Evaluating $f(x, y) = xe^y$ at $(a, b) = (-1, 0)$

$$\begin{aligned}f_x &= e^y\\ f_y &= xe^y\\ f_x(-1, 0) &= 1\\ f_y(-1, 0) &= -1\\ \Delta f &= \Delta x - \Delta u\end{aligned}$$

On a given contour map, let $\vec{u} = \langle u_1, u_2\rangle$ denote a *unit* vector in a direction that we want to find the derivative of f in.

$$f_{\vec{u}}(x, y) = \nabla f(a, b) \cdot \vec{u}$$

Where

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

The directional derivative for all vectors $\vec{v}$ is as follows:

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

Chain Rule

Let $f(x, y)$ be a function where $x - x(t)$ and $y = y(t)$. We want to find

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The chain rule works in higher dimensions too. Consider $k(x_1(t), x_2(t), \dots, x_{152}(t))$. Then,

$$\frac{dk}{dt} = \sum_{i=1}^{152}\frac{\partial k}{\partial x_i}\frac{dx_i}{dt}$$

We can also view this as a vector. Let $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{152}(t) \end{pmatrix}$. Then, we can write $\frac{dk}{dt}$ more succinctly as follows:

$$\frac{dk}{dt} = \nabla k \cdot \frac{d\vec{x}}{dt}$$

For example, let $f(x, y, z) = 3x^2y + zx + 2$, where $x = x(t), y = y(t), z = z(t)$

$$\begin{aligned} \frac{df}{dt} &= \begin{pmatrix} 6xy + z \\ 3x^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} \\ &= (6xy + z)\, x'(t) + 3x^2 y'(t) + xz'(t) \end{aligned}$$

So, if we let $x(t) = \sin(t)$, $y(t) = e^t$, and $z(t) = t^2 + 1$. Then, we have

$$\frac{df}{dt} = 6\sin(t)\cos(t)e^t + t^2\cos(t) + \cos(t) + 3e^t\sin^2(t) + 2t\sin(t)$$

Alternatively, consider $f(x, y, z) = x^2 + yz + e^y$, where $x(s, t) = st, y = y(s, t) = t + s^2, z = z(s, t) = e^t$. Let

$$\vec{x} = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}$$

Then, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial t} \\ \frac{\partial f}{\partial s} &= \nabla f \cdot \frac{\partial \vec{x}}{\partial s} \end{aligned}$$

Evaluating the first expression, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix} \cdot \begin{pmatrix} s \\ 1 \\ e^t \end{pmatrix} \\ &= 2s^2t + 3^t + e^{t+s^2} + (t + s^2)e^t \end{aligned}$$

Consider $f(x, y(x))$. Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

This is the technique we use to find implicit differentiation.

We know as a result that $\nabla f(a, b)$ is orthogonal to the contour curve at (a, b)

Recap

In $\mathbb{R}^3$, find the plane that contains $P = (P_1, P_2, P_3)$, Q, and R. We can find it by the following:

$$\begin{aligned} 0 &= \vec{n} \cdot \begin{pmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{pmatrix} \\ 0 &= n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3) \end{aligned}$$

where

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

Differentiability

A function $f(x)$ of one variable is differentiable at $x = a$ if

$$f(a) = \lim_{h\to 0} f(a+h)$$

and

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists and is bounded

We can also linearize the function. f is differentiable if

$$f(x) = f(a) + f'(a)(x-a) + E(x)$$

where $\lim_{h\to 0} \frac{E(a+h)}{h} = 0$.

In the multiple dimensions example, we have $f(x, y)$ is differentiable if

$$f(x, y) = \ell(x, y) + E(x, y)$$

where $\lim_{h\to 0, k\to 0} \frac{E(a+h, b+k)}{\sqrt{h^2+k^2}} = 0$

Local Maxima

Let $f(x, y) = x^2 + 2y^2$. We want to find (a, b) which are local maxima,minima, or other.

(a, b) is a local maximum if $f(a, b) \geq f(x, y)\ \forall (x, y) \in V_\varepsilon(a, b)$, where $\varepsilon > 0$.

(1) Find Critical Points for $f(x, y) : f_x(x, y), f_y(x, y) = 0,\ f_x(x, y), f_y(x, y)$ are undefined.

$$\begin{aligned} f_x(x, y) &= 2x \\ f_y(x, y) &= 4y \\ f_x(0, 0) &= 0 \\ f_y(0, 0) &= 0 \\ f(0, 0) &= 0 \\ f(x, y) &> 0 && \forall (x, y) \neq (0, 0) \end{aligned}$$

For all x, y, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 0$. Finally,

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) \cdot f_{yy}(x, y) + f_{xy}(x, y)^2 \\ &= 8 \\ &> 0 \end{aligned}$$

Since $D(x, y) > 0$, we look at the sign of f_{xx}. Since it is positive, $f(0, 0)$ has a local minimum.

Local Maxima and Minima Approach

Given $f(x, y)$, we want

(1) Find critical points:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned}$$

(2) Compute $f_{xx}, f_{yy}, f_{xy}, D = f_x x f_y y - (f_x y)^2$

(3)

f_{xx}	D	Critical Point
$+$	$+$	Local Minimum
-	$+$	Local Maximum
$\pm$	-	Saddle Point
$\pm$	0	Nothing

Consider the function

$$\begin{aligned} f(x,y) &= \ln(x^2+y^2+1)\\ f(0,0) &= 0\\ f(x,y) &> 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2x}{x^2+y^2+1}\\ \frac{\partial f}{\partial y} &= \frac{2y}{x^2+y^2+1} \end{aligned}$$

Critical Points: $(0,0)$

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} &= \frac{2(x^2+y^2+1)-4x^2}{(x^2+y^2+1)^2}\\ &= 2\\ \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} &= \frac{2(x^2+y^2+1)-4y^2}{(x^2+y^2+1)^2}\\ &= 2\\ \left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} &= \frac{-4xy}{(x^2+y^2+1)^2}\\ &= 0 \end{aligned}$$

Now, consider the function

$$\begin{aligned} f(x,y) &= x^2-2xy+y^2\\ \frac{\partial f}{\partial x} &= 2x-2y\\ \frac{\partial f}{\partial y} &= -2x+2y\\ \frac{\partial^2 f}{\partial x^2} &= 2\\ \frac{\partial^2 f}{\partial y^2} &= 2\\ \frac{\partial^2 f}{\partial x\partial y} &= -2\\ D &= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\\ &= 0 \end{aligned}$$

Therefore, the critical points of this function are indeterminate with the given approach. However, we know that $f(x,y) = (x-y)^2 = 0$ when $x = y$, so the line $y = x$ is a local minimum trough in 3-space.

Now, consider the function

$$\begin{aligned} f(x,y) &= (x-1)^2(y+2)\\ \frac{\partial f}{\partial x} &= 2(x-1)(y+2)\\ \frac{\partial f}{\partial y} &= (x-1)^2 \end{aligned}$$

Critical points: $\{(1,y) \mid y \in \mathbb{R}\}$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 2(y+2)\\
\frac{\partial^2 f}{\partial y^2} &= 0\\
\frac{\partial^2 f}{\partial x \partial y} &= 2(x-1)\\
D &= 0 - (2(x-1))^2\\
&= 0 && \text{Evaluating } D \text{ at critical points}
\end{aligned}$$

Finding Critical Points

Let $f(x,y) = (y^2+2)\sin(x)$. on $[-2,2] \times [-2,2]$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= (y^2+2)\cos(x)\\
&= 0\\
\frac{\partial f}{\partial y} &= 2y\sin(x)\\
&= 0\\
(x,y) &= \left(\frac{(2n+1)\,\pi}{2}, 0\right)\\
&= \{(\pi/2, 0), (-\pi/2, 0)\}\\
\frac{\partial^2 f}{\partial x^2} &= -(y^2+2)\sin(x)\\
\frac{\partial^2 f}{\partial y^2} &= 2\sin(x)\\
\frac{\partial^2 f}{\partial x \partial y} &= 2y\cos(x)\\
D(x,y) &= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\\
&= -2(y^2+2)\sin^2(x) - 4y^2\cos^2(x)\\
&< 0
\end{aligned}$$

Therefore, the critical points are saddle points. If there is no domain restriction, we have a series of saddle points all along $y = 0$.

Why Finding Critical Points Works

We create the Taylor series of $f(x,y)$ at (x_0, y_0):

$$\begin{aligned}
f(x,y) &\approx \ell(x_0,y_0) + \frac{1}{2}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2\right)\\
&= f(x_0,y_0) + \underbrace{\nabla f(x,y) \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}_{=0 \text{ at critical points}} + \frac{1}{2}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}^T \underbrace{\begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix}}_{\text{Hessian}} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}
\end{aligned}$$

If the Hessian is positive definite, then $\lambda_1, \lambda_2 > 0$ and the critical point is a local min. If the Hessian is negative definite, then $\lambda_1, \lambda_2 < 0$ and the critical point is a local max.

In any given 2×2 matrix, the eigenvalues λ_1, λ_2 are such that $\lambda_1 + \lambda_2 = \mathrm{Tr}(A)$ and $\lambda_1\lambda_2 = \mathrm{Det}(A)$.

Optimization

Let $f(x,y) = 2x - y$. We want to optimize f with respect to $g(x,y) = x^2 - y^2 - 4 = 0$.

Define $L(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$. Given $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, then $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Then, we take

$$\begin{aligned} \nabla L &= \nabla f = \lambda \nabla g \\ &= 0 && \text{critical points of } L \end{aligned}$$

We find x, y, λ for each critical point.

$$\begin{aligned} \nabla f &= \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\ \nabla g &= \begin{pmatrix} 2x \\ -2y \end{pmatrix} \\ \nabla f &= \lambda \nabla g \\ 2 &= 2\lambda x \\ -3 &= -2\lambda y \\ x^2 - y^2 &= 4 \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{1}{x} \\ \lambda &= \frac{3}{2y} \\ x &= \frac{2y}{3} \\ \frac{4y^2}{9} - y^2 &= 4 \\ -\frac{5}{9}y^2 &= 4 \end{aligned}$$

No Solution

However, if $g(x,y) = x^2 + y^2 - 4 = 0$, we have

$$\begin{aligned}
\nabla f &= \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
\nabla g &= \begin{pmatrix} 2x \\ 2y \end{pmatrix} \\
\nabla f &= \lambda \nabla g \\
2 &= 2\lambda x \\
-3 &= 2\lambda y \\
x^2 + y^2 &= 4 \\
\\
\lambda &= \frac{1}{x} \\
\lambda &= \frac{-3}{2y} \\
x &= \frac{-2y}{3} \\
\frac{4y^2}{9} + y^2 &= 4 \\
-\frac{13}{9} y^2 &= 4 \\
y &= \pm \frac{6}{\sqrt{13}} \\
x &= \mp \frac{4}{\sqrt{13}} \\
f_{\text{max}} &= 2\sqrt{13} \\
f_{\text{min}} &= -2\sqrt{13}
\end{aligned}$$

This system of Lagrange multipliers applies in the n dimensional case.

Let $f(x,y,z) = x + 2y + z^2$ subject to the constraint $g(x,y,z) = x^2 + y^2 + z^2 = 1$.

$$\begin{aligned}
\nabla f &= \lambda \nabla g \\
\begin{pmatrix} 1 \\ 2 \\ 2z \end{pmatrix} &= \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \\
2\lambda x &= 1 \\
2\lambda y &= 2 \\
2\lambda z &= 2z \quad (*) \\
x^2 + y^2 + z^2 &= 1
\end{aligned}$$

Consider (*):

$$\begin{aligned}
\lambda &= 1 \\
x &= 1/2 \\
y &= 1 \\
\frac{1}{4} + 1 + z^2 &= 1
\end{aligned}$$

no solution

$$\begin{aligned} z &= 0 \\ x^2 + y^2 &= 1 \\ \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} &= 1 \\ \frac{5}{4\lambda^2} &= 1 \\ \lambda &= \pm\frac{\sqrt{5}}{2} \end{aligned}$$

Case 1:

$$\begin{aligned} \lambda &= \frac{\sqrt{5}}{2} \\ x &= \frac{1}{\sqrt{5}} \\ y &= \frac{2}{\sqrt{5}} \end{aligned}$$

Case 2:

$$\begin{aligned} \lambda &= -\frac{\sqrt{5}}{2} \\ x &= -\frac{1}{\sqrt{5}} \\ y &= -\frac{2}{\sqrt{5}} \end{aligned}$$

Evaluating f:

x	y	z	λ	$f(x,y,z)$
$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	$\frac{\sqrt{5}}{2}$	$\sqrt{5}$
$-\frac{1}{\sqrt{5}}$	$-\frac{2}{\sqrt{5}}$	0	$-\frac{\sqrt{5}}{2}$	$-\sqrt{5}$

If we want to optimize f with respect to multiple constraint functions $g_1, g_2, g_3, \dots, g_k$, we would do:

$$\nabla f = \sum_{i=1}^{k} \lambda_i \nabla g_i$$

Integration

Consider $f(x,y)$. We want to integrate along the rectangle $D = [0,3] \times [0,2]$. We can find this as follows:

$$\begin{aligned} \int_D f(x,y) &= \int_0^3 \int_0^2 f(x,y)dydx \\ &= \int_0^3 dx \int_0^2 dyf(x,y) \end{aligned}$$

For any two regions D_1 and D_2, we have:

$$\begin{aligned} \int_{D_1} f(x,y) + \int_{D_2} f(x,y) &= \int_{D_1 \ominus D_2} f(x,y) \\ &= \int_{D_1 \cup D_2 \setminus D_1 \cap D_2} f(x,y) \end{aligned}$$