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## Introduction

Finally, the last part of my notes on  $C^*$ -algebras and amenability as part of my Honors Thesis independent study. Specifically, I am going to focus more on the theory of  $C^*$ -algebras, discussing ideas such as amenability and nuclearity in  $C^*$ -algebras. There are a few central results I'm going to be working on understanding and proving: almost-invariant vectors, Kesten's criterion, Hulanicki's criterion, nuclearity, and the equivalence of  $C_\lambda^*(G)$  and  $C^*(G)$ .

I will be using a variety of sources more focused on amenability, including but not limited to Volker Runde's *Amenable Banach Algebras*, Kate Juschenko's *Amenability of Discrete Groups by Examples*, and Brown and Ozawa's  *$C^*$ -Algebras and Finite-Dimensional Approximations*.

## Review: Representations, the Reduced Group $C^*$ -Algebra, and the Universal Group $C^*$ -Algebra

### Left-Regular Representation

Let  $\Gamma$  be a group. Consider the space  $\ell_2(\Gamma)$ . For every  $s \in \Gamma$ , we define the operator

$$\lambda_s(\xi)(t) = \xi(s^{-1}t).$$

The map is linear, well-defined, and an isometry, as

$$\begin{aligned} \|\lambda_s(\xi)\|^2 &= \sum_{t \in \Gamma} |\lambda_s(\xi)(t)|^2 \\ &= \sum_{t \in \Gamma} \left| \xi(s^{-1}t) \right|^2 \\ &= \sum_{r \in \Gamma} |\xi(r)|^2 \\ &= \|\xi\|^2. \end{aligned}$$

Additionally, each  $\lambda_s$  admits an inverse,  $\lambda_{s^{-1}} = \lambda_s^*$ . Applying to the orthonormal basis  $\{\delta_t\}_{t \in \Gamma}$ , we get

$$\lambda_s(\delta_t) = \delta_{st}.$$

Thus,  $\lambda_s \circ \lambda_r = \lambda_{sr}$ , and we have the unitary representation of  $\Gamma$ ,  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ , where  $\lambda(s) = \lambda_s$ , for  $s \in \Gamma$ . This is the left-regular representation of  $\Gamma$ .

Note that the left regular representation is a faithful representation, hence injective.

Because the  $\lambda$  operator is linear, we may extend it to the case of any positive finitely supported function,

$$\begin{aligned}\lambda_f(\xi)(t) &= \left( \sum_{s \in \Gamma} f(s) \lambda_s(\xi) \right)(t) \\ &= \sum_{s \in \Gamma} f(s) \xi(s^{-1}t)\end{aligned}$$

Note that the space of finitely supported functions on  $\Gamma$ ,  $\mathbb{C}[\Gamma]^1$  is a  $*$ -algebra, where multiplication is given by convolution:

$$\begin{aligned}f * g(t) &= \sum_{s \in \Gamma} f(s) g(s^{-1}t) \\ &= \sum_{r \in \Gamma} f(tr^{-1}) g(r).\end{aligned}$$

Note that we are using  $*$  both to refer to the involution (when as a superscript) as well as the group operation (when not a superscript). This is to maintain coherence with the traditional way that convolution is written. The involution on  $\mathbb{C}[\Gamma]$  is given by

$$f^*(t) = \overline{f(t^{-1})}.$$

### A Bit on Representations and $C^*$ -(Semi)norms

A  $C^*$ -seminorm on a  $*$ -algebra is a seminorm such that defined by

- $\|ab\| \leq \|a\| \|b\|$ ;
- $\|a^*\| = \|a\|$ ;
- $\|a^*a\| = \|a\|^2$ .

If  $A_0$  is a  $*$ -algebra, then a representation of  $A_0$  is a pair  $(\pi_0, \mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space and  $\pi: A_0 \rightarrow \mathcal{B}(\mathcal{H})$  is a  $*$ -homomorphism.

Additionally, if  $A_0$  is a  $*$ -algebra with representation  $\pi_0$ , then we have  $C^*$ -seminorm

$$\|a\|_{\pi_0} = \|\pi_0(a)\|_{\text{op}}.$$

If  $\pi_0$  is injective, then  $\|\cdot\|_{\pi_0}$  is a  $C^*$ -norm. If  $\pi_0$  is a  $C^*$ -norm, then the completion of  $A_0$  with respect to  $\|\cdot\|_{\pi_0}$  is a  $C^*$ -algebra.

The universal norm on  $A_0$  is defined as

$$\|a\|_{\text{u}} = \sup_{p \in \mathcal{P}} p(a),$$

where  $\mathcal{P}$  is the collection of all  $C^*$ -seminorms on  $A_0$ . If  $\|a\|_{\text{u}} < \infty$  for all  $a \in A_0$ , then  $\|\cdot\|_{\text{u}}$  is a  $C^*$ -seminorm on  $A_0$ . Note that if one of  $p \in \mathcal{P}$  is a norm, then  $\|\cdot\|_{\text{u}}$  defines a  $C^*$ -norm on  $A_0$ .

If we have the unitary representation  $u: \mathbb{C}[\Gamma] \rightarrow \mathcal{B}(\mathcal{H})$ , then

$$\pi_u(a) = \sum_{s \in \Gamma} u_s$$

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<sup>1</sup>Also known as the free vector space over  $\mathbb{C}$  with basis  $\Gamma$ .

is a representation of  $\mathbb{C}[\Gamma]$ . If  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$  is the left-regular representation, then the left-regular group  $C^*$ -algebra is the group  $*$ -algebra with  $C^*$ -norm defined by  $\|a\| = \|\pi_\lambda(a)\|$ .

The universal group  $C^*$ -algebra is defined as the norm completion of

$$\|a\|_u = \sup \left\{ \|\pi(a)\|_{\text{op}} \mid \pi: \mathbb{C}[\Gamma] \rightarrow \mathbb{B}(\mathcal{H}_\pi) \right\}.$$

Note that

$$\begin{aligned} \|\pi(a)\| &= \left\| \pi \left( \sum_{s \in \Gamma} a_s \delta_s \right) \right\| \\ &= \left\| \sum_{s \in \Gamma} a_s \pi(\delta_s) \right\| \\ &\leq \sum_{s \in \Gamma} \|a_s \pi(\delta_s)\| \\ &= \sum_{s \in \Gamma} |a_s|. \end{aligned}$$

Note that since  $\|\cdot\|_\lambda$  is a norm, we must have  $a = 0$  if and only if  $\|a\|_u = 0$ . The full group  $C^*$ -algebra admits a universal property.

**Proposition:** Let  $\Gamma$  be a discrete group. If  $u: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ , then there is a contractive  $*$ -homomorphism  $\pi_u: C^*(\Gamma) \rightarrow \mathbb{B}(\mathcal{H})$  that satisfies  $\pi_u(\delta_s) = u(s)$ .

## Using the Left-Regular Representation to Establish Amenability

If  $\pi: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is a unitary representation of  $\Gamma$ , then a vector  $\xi \in \mathcal{H}$  is called invariant for  $\pi$  if  $\pi(g)(\xi) = \xi$  for all  $g \in \Gamma$ .

**Proposition:** The left-regular representation for  $\Gamma$  admits an invariant vector if and only if  $\Gamma$  is finite.

*Proof.* Let  $\Gamma$  be finite. Since  $\Gamma$  is finite, all functions  $a: \Gamma \rightarrow \mathbb{C}$  are square-summable. Thus,  $\xi = \mathbb{1}_\Gamma$  is square-summable, and since  $s\Gamma = \Gamma$  for all  $s \in \Gamma$ , we have  $\mathbb{1}_\Gamma$  is invariant for  $\lambda$ .

Now, let  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$  be the left-regular representation, and suppose there is  $\xi \in \ell_2(\Gamma)$  such that for all  $s \in \Gamma$ , we have

$$\lambda_s(\xi) = \xi.$$

In particular, this means that for any  $t \in \Gamma$ , we have

$$\begin{aligned} \lambda_s(\xi)(t) &= \xi(s^{-1}t) \\ &= \xi(t). \end{aligned}$$

Since this holds for all  $s \in \Gamma$ , we have that  $\xi = c \mathbb{1}_\Gamma$  for some  $c \in \mathbb{C}$ . However, since  $\xi \in \ell_2(\Gamma)$ , we must have that  $\sum_{t \in \Gamma} |c|^2 < \infty$ , which only holds if  $\Gamma$  is finite.  $\square$

An almost-invariant vector for a representation  $\pi: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ , as the name suggests,<sup>II</sup> a sequence (or net) of unit vectors  $(\xi_i)_{i \in I}$  such that

$$\lim_{i \in I} \|\pi(g)(\xi_i) - \xi_i\| = 0.$$

<sup>II</sup>I'm only mostly being facetious here.

**Theorem:** A group  $\Gamma$  is amenable if and only if the left-regular representation has an almost-invariant vector.

*Proof.* Let  $\Gamma$  be amenable, and let  $F_i$  be a Følner sequence —  $\frac{|sF_i \Delta F_i|}{|F_i|} \rightarrow 0$  for all  $s \in \Gamma$ . Define  $\xi_i = \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}$ . Thus,

$$\begin{aligned} \|\lambda_s(\xi_i) - \xi_i\|^2 &= \sum_{t \in \Gamma} |\lambda_s(\xi_i)(t) - \xi_i(t)|^2 \\ &= \sum_{t \in \Gamma} \left| \lambda_s \left( \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i} \right)(t) - \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}(t) \right|^2 \\ &= \sum_{t \in \Gamma} \left| \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{sF_i}(t) - \frac{1}{\sqrt{|F_i|}} \mathbb{1}_{F_i}(t) \right|^2 \\ &= \frac{|sF_i \Delta F_i|}{|F_i|}. \end{aligned}$$

Thus,  $\lambda$  has an almost-invariant vector.

Suppose there exists an almost-invariant vector  $(\xi_i)_i \in \ell_2(\Gamma)$ . It is sufficient to construct an approximate mean. Since  $\xi_i \in \ell_2(\Gamma)$ , we have that  $\xi_i^2 \in \ell_1(\Gamma)$ . Setting  $\mu_i = \xi_i^2$ , we plug this into the expression for an approximate mean, and obtain

$$\begin{aligned} \|\lambda_s(\mu_i) - \mu_i\|_{\ell_1} &= \sum_{t \in \Gamma} \left| \lambda_s(\xi_i^2)(t) - \xi_i^2(t) \right| \\ &= \sum_{t \in \Gamma} |(\lambda_s(\xi_i)(t) - \xi_i(t))(\lambda_s(\xi_i)(t) + \xi_i(t))| \\ &= \|(\lambda_s(\xi_i) - \xi_i)(\lambda_s(\xi_i) + \xi_i)\|_{\ell_1} \\ &\leq \|\lambda_s(\xi_i) - \xi_i\|_{\ell_2} \|\lambda_s(\xi_i) + \xi_i\| \\ &\leq 2\|\lambda_s(\xi_i) - \xi_i\| \\ &\rightarrow 0. \end{aligned}$$

Thus,  $\mu_i$  is an approximate mean. □

Using the criterion of almost invariant vectors, we may show that a group is amenable if and only if the trivial representation — defined by  $1_\Gamma: \Gamma \rightarrow \mathbb{C}$ ,  $1_\Gamma(g) = 1$  is what is known as weakly contained in the left-regular representation.

A representation  $\pi: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is weakly contained in another representation  $\rho: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ , denoted  $\pi < \rho$ , if for every  $\xi \in \mathcal{H}$ , finite  $E \subseteq \Gamma$ , and  $\varepsilon > 0$ , then there are  $\eta_1, \dots, \eta_n \in \mathcal{H}$  such that

$$\left| \langle \pi(g)(\xi), \xi \rangle - \sum_{i=1}^n \langle \rho(g)(\eta_i), \eta_i \rangle \right| < \varepsilon.$$

**Theorem:** A discrete group  $\Gamma$  is amenable if and only if  $1_\Gamma < \lambda$ , where  $\lambda$  is the left-regular representation.

*Proof.* We show that  $1_\Gamma < \lambda$  is equivalent to the existence of an almost invariant vector for  $\lambda$ . We assume  $\lambda$  admits an almost-invariant vector. It is sufficient to show that for every  $\varepsilon > 0$  and every finite set  $E \subseteq \Gamma$ , there are  $\eta_1, \dots, \eta_n \in \ell_2(\Gamma)$  such that

$$\left| 1 - \sum_{i=1}^n \langle \lambda_t(\eta_i), \eta_i \rangle \right| < \varepsilon$$

for every  $t \in E$ . If we take  $n = 1$  and  $\eta_1 = \xi$ , where  $\xi$  is almost-invariant for all  $g \in E$  — i.e.,  $\|\lambda_g(\xi) - \xi\|_{\ell_2} < \varepsilon$  for all  $g \in E$ . Note that we have

$$\begin{aligned}
 \|\lambda_g(\xi) - \xi\|^2 &= \langle \lambda_g(\xi) - \xi, \lambda_g(\xi) - \xi \rangle \\
 &= \langle \lambda_g(\xi), \lambda_g(\xi) \rangle + \langle \xi, \xi \rangle - 2 \operatorname{Re} \langle \lambda_g(\xi), \xi \rangle \\
 &= 2 - 2 \operatorname{Re} \langle \lambda_g(\xi), \xi \rangle \\
 &= 2 \operatorname{Re}(1 - \langle \lambda_g(\xi), \xi \rangle) \\
 &\leq 2|1 - \langle \lambda_g(\xi), \xi \rangle|.
 \end{aligned}$$

□