**Problem** (Problem 2)**:**

(a) Determine the Laurent series expansion of the function

$$f(z) = \frac{z}{(z-3)^2(z-4)}$$

that converges on $A(0,3,4)$.

(b) Show that there does not exist a holomorphic function $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ satisfying $|f(z)| \geqslant |z|^{-2/3}$.

**Solution:**

(a) We start by taking a partial fraction decomposition of $f$ to yield

$$\begin{aligned} f(z) &= \frac{4}{z-4} - \frac{4}{z-3} - \frac{3}{(z-3)^2} \\ &= \frac{4}{z-4} - \frac{4}{z-3} + 3\frac{d}{dz}\left(\frac{1}{z-3}\right) \end{aligned}$$

We seek to expand about $z = 0$ within the ball $U(0,4)$ and outside the closed ball $B(0,3)$. This means that the first term in our partial fraction expansion becomes

$$a_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{4^n}.$$

The expansion in the second and third terms will require a little bit more work. Dividing out by $z$, we find that the second term becomes

$$\begin{aligned} a_2(z) &= -\frac{4}{z\left(1 - \frac{3}{z}\right)} \\ &= -\frac{4}{z}\sum_{n=0}^{\infty} \frac{3^n}{z^n} \\ &= -\sum_{n=1}^{\infty} \frac{4 \cdot 3^{n-1}}{z^n} \\ &= -\sum_{n=-\infty}^{-1} 12(3^{-n})z^n. \end{aligned}$$

Finally, for the third term, we observe that, using term-by-term differentiation (allowable as the series is uniformly convergent), we have

$$\begin{aligned} 3\frac{d}{dz}\left(\frac{1}{z-3}\right) &= 3\frac{d}{dz}\left(\sum_{n=1}^{\infty} 3^{n-1}z^{-n}\right) \\ &= \sum_{n=1}^{\infty} -n3^n z^{-(n+1)} \\ &= \sum_{n=-\infty}^{-1} n3^{-n}z^{n-1}. \end{aligned}$$

This yields a Laurent series expansion of

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{4^n} + \sum_{n=-\infty}^{-1} \left(12(3^{-n})z^n + n3^{-n}z^{n-1}\right).$$

(b) Suppose toward contradiction that there were such an $f(z)$. Since $|z|^{-2/3}$ is strictly greater than zero along its domain, it would follow that $|f(z)|$ would not have any zero along its domain. This means that $g(z) = \frac{1}{f(z)}\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ would be defined on its entire domain. Furthermore, we would have

$$|g(z)| \leqslant |z|^{2/3},$$

and on $U(0,\varepsilon)$, we know that $|z|^{2/3}$ is bounded above by $\varepsilon^{2/3}$ as $|z|^{2/3}\colon \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ is an increasing function. Thus, since $g$ would be locally bounded around 0, it would follow that $g$ has a removable singularity at 0. This means that there is a holomorphic extension $h\colon \mathbb{C} \to \mathbb{C}$ that agrees with $g$ on $\mathbb{C}\setminus\{0\}$. In particular, we would have $|h(z)| \leqslant |z|^{2/3}$ for all $z \in \mathbb{C}\setminus\{0\}$.

Now, let $R > 0$. Using the Cauchy estimate on $S(0,R)$, we have, for any fixed $n > 0$,

$$\begin{aligned}\left|h^{(n)}(z)\right| &\leqslant \frac{n!}{R^n}\sup_{|z|=R}|h(z)|\\ &\leqslant \frac{n!}{R^n}\sup_{|z|=R}|z|^{2/3}\\ &= \frac{n!}{R^{n-2/3}}.\end{aligned}$$

Yet, since $R$ is arbitrary, it follows that $\left|h^{(n)}(z)\right| = 0$ for all $n > 0$, whence $h$ is constant. Yet, since $|h(z)| \leqslant |z|^{2/3}$ for all $z \in \mathbb{C}\setminus\{0\}$, it follows that $|h(z)| \leqslant \varepsilon^{2/3}$ for any $\varepsilon > 0$, whence $|h(z)| = 0$ for all $z \in \mathbb{C}$. At the same time, we explicitly defined $g(z)$ in a manner such that it could never equal zero, meaning that such an $f$ cannot exist.

**Problem** (Problem 4)**:** Show that if $f$ is entire and satisfies $\lim_{z\to\infty} f(z) = \infty$, then $f$ is a polynomial.

**Solution:** Consider the function $g\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ given by $g(z) = f\left(\frac{1}{z}\right)$. Since $f$ is entire and $\frac{1}{z}$ is holomorphic on $\mathbb{C}\setminus\{0\}$, it follows that, given the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have the Laurent series expansion

$$g(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

Observe that the limit $\lim_{z\to\infty} f(z)$ is equivalent to $\lim_{z\to 0} f\left(\frac{1}{z}\right)$, whence $\lim_{z\to 0} g(z) = \infty$. Therefore, $g$ has a pole of order $k$ at 0, whence

$$g(z) = \sum_{n=0}^{k} a_n z^{-n}.$$

Since $g\left(\frac{1}{z}\right) = f(z)$, it then follows that

$$f(z) = \sum_{n=0}^{k} a_n z^n.$$