Problem (Problem 1)**:** For all $n \in \mathbb{N}$, find the residue of $f(z) = (1 - e^{-z})^n$ at $z = 0$ via Cauchy's residue theorem.

Solution: Choose a square contour γ defined by

$$\begin{aligned} \gamma &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\ \gamma_1 &= 1 + iy \\ \gamma_2 &= i - x \\ \gamma_3 &= -1 - iy \\ \gamma_4 &= -i + x \end{aligned}$$

with $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$. Then,

$$\begin{aligned} 2\pi i \operatorname{Res}(f; 0) &= \oint_\gamma f(z)\, dz \\ &= \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \int_{\gamma_3} f(z)\, dz + \int_{\gamma_4} f(z)\, dz. \end{aligned}$$

We compute

$$\int_{\gamma_1} f(z)\, dz = \int_{-1}^{1} \frac{i}{(1 - e^{-1-iy})^n}\, dy.$$

Taking $u = e^{-1-iy}$, we get

$$\begin{aligned} &= -\int_{u(-1)}^{u(1)} \frac{1}{u(1-u)^n}\, du \\ &= -\int_{e^{-1+i}}^{e^{-1-i}} \frac{1}{e^{-1-iy}} + \frac{p\left(e^{-1-iy}\right)}{(1 - e^{-1-iy})^n}\, dy, \end{aligned}$$

where $p(u) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} u^{k-1}$.

$$\int_{\gamma_2} f(z)\, dz = \int_{-1}^{1} \frac{-1}{(1 - e^{-i+x})^n}\, dx.$$

Taking $v = e^{-i+x}$

$$\begin{aligned} &= -\int_{v(-1)}^{v(1)} \frac{1}{v} + \frac{p(v)}{(1-v)^n}\, dv \\ &= -\int_{e^{-1-i}}^{e^{1-i}} \frac{1}{e^{-i+x}} + \frac{p\left(e^{-i+x}\right)}{(1 - e^{-i+x})^n}\, dx \end{aligned}$$

Problem (Problem 2)**:** Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2+1}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin^2(x)}{x^2+1}\, dx.$$

Solution: We compute

$$\int_{-R}^{R} \frac{\sin^2(x)}{x^2+1}\, dx = \frac{1}{2}\int_{-R}^{R} \frac{1}{x^2+1}\, dx - \frac{1}{2}\int_{-R}^{R} \frac{\cos(2x)}{x^2+1}\, dx.$$

Calling the latter integral I, we take

$$f(z) = \frac{e^{2iz}}{z^2+1},$$

close the contour γ in the upper half-plane with the half-circle $C_R = \left\{Re^{i\theta} \mid 0 \leqslant \theta \leqslant \pi\right\}$. This gives

$$\begin{aligned}\operatorname{Re}\oint_{\gamma} f(z)\,dz &= \operatorname{Re}(I) + \operatorname{Re}\int_{C_R} f(z)\,dz\\ &= \operatorname{Re}(I) + \operatorname{Re}\int_0^{\pi} \frac{e^{2iRe^{i\theta}}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta.\end{aligned}$$

Estimating the integrand on the second integral, we see that for $R > 1$,

$$\begin{aligned}\left|\frac{iRe^{i\theta}e^{2iRe^{i\theta}}}{R^2e^{2i\theta}+1}\right| &\leqslant \frac{R}{R^2-1}\left|e^{2iR(\cos(\theta)+i\sin(\theta))}\right|\\ &\leqslant \frac{R}{(R^2-1)\left(e^{2R\sin(\theta)}\right)}\\ &\leqslant \frac{R}{R^2-1}\end{aligned}$$

whence

$$\begin{aligned}\left|\int_{C_R} f(z)\,dz\right| &\leqslant \pi\frac{R}{R^2-1}\\ &\to 0.\end{aligned}$$

Therefore, by Cauchy's residue theorem,

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos(2x)}{x^2+1}\,dx &= \operatorname{Re}(2\pi i \operatorname{Res}(f;i))\\ &= \operatorname{Re}\left(2\pi i \lim_{z\to i}\frac{(z-i)e^{2iz}}{(z-i)(z+i)}\right)\\ &= \frac{\pi}{e^2}.\end{aligned}$$

Thus, we find that

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2+1}\,dx = \frac{\pi}{2} - \frac{\pi}{2e^2}.$$

Problem (Problem 3)**:** For $\xi \in \mathbb{R}$, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+4x+5}\,dx = \lim_{R\to\infty}\frac{\cos(\xi x)}{x^2+4x+5}.$$

Solution: First, if $\xi = 0$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\,dx &= \int_{-\infty}^{\infty} \frac{1}{(x+2)^2+1}\,dx\\ &= \pi\end{aligned}$$

upon a u-substitution.

Now, let $\xi > 0$. Using $f(z) = \frac{e^{i\xi z}}{z^2+4z+5}$ and closing the contour

$$\gamma_R = [-R, R] + \left\{ Re^{i\theta} \;\middle|\; 0 \leqslant \theta \leqslant \pi \right\}$$

in the upper half plane, we find that we get

$$\oint_{\gamma_R} f(z)\,dz = \underbrace{\int_{-R}^{R} f(x)\,dx}_{=:I} + \int_{C_R} f(z)\,dz.$$

Parametrizing the integral over C_R by $z = Re^{i\theta}$, we get

$$= I + \int_0^{\pi} \frac{e^{i\xi Re^{i\theta}}}{\left(Re^{i\theta}+2\right)^2+1} iRe^{i\theta}\,d\theta.$$

Estimating the second integral, we see that for $R > 5$,

$$\begin{aligned}
\left|\frac{iRe^{i\theta}e^{i\xi Re^{i\theta}}}{\left(Re^{i\theta}+2\right)^2+1}\right| &\leqslant \frac{R}{R^2-4R-5}\left|e^{i\xi R(\cos(\theta)+i\sin(\theta))}\right| \\
&\leqslant \frac{R}{(R^2-4R-5)\left(e^{\xi R\sin(\theta)}\right)} \\
&\leqslant \frac{R}{R^2-4R-5}
\end{aligned}$$

meaning that

$$\begin{aligned}
\left|\int_{C_R} f(z)\,dz\right| &\leqslant \pi\frac{R}{R^2-4R-5} \\
&\to 0.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
2\pi i\operatorname{Res}(-2+i) &= \lim_{R\to\infty}\oint_{\gamma_R} f(z)\,dz \\
&= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+4x+5}\,dx \\
&= 2\pi i \lim_{z\to -2+i} \frac{(z-(-2+i))e^{i\xi z}}{(z-(-2+i))(z-(-2-i))} \\
&= 2\pi i\frac{e^{i\xi(-2+i)}}{2i} \\
&= \frac{\pi}{e^{\xi}}e^{-2i\xi} \\
&= \frac{\pi}{e^{\xi}}(\cos(2\xi) - i\sin(2\xi)) \\
&= \frac{\pi}{e^{\xi}}\cos(2\xi) - i\frac{\pi}{e^{\xi}}\sin(2\xi).
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+4x+5}\,dx &= \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+4x+5}\,dx \\
&= \frac{\pi}{e^{\xi}}\cos(2\xi).
\end{aligned}$$

Now, let $\xi < 0$. We take η_R to be the contour

$$\eta_R = [-R, R] + \left\{ Re^{-i\theta} \;\middle|\; 0 \leqslant \theta \leqslant \pi \right\}.$$

We find that

$$\begin{aligned}\oint_{\eta_R} f(z)\,dz &= \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz\\ &= I + \int_0^{\pi} \frac{e^{i\xi\left(Re^{-i\theta}\right)}}{\left(Re^{-i\theta}+2\right)^2+1}\left(-iRe^{-i\theta}\right)\,d\theta.\end{aligned}$$

Estimating the second integrand, we have for $R > 5$

$$\begin{aligned}\left|\frac{-iRe^{i\theta}e^{i\xi\left(Re^{-i\theta}\right)}}{\left(Re^{-i\theta}+2\right)^2+1}\right| &\leqslant \frac{R}{R^2-4R-5}\left|e^{i\xi R(\cos(\theta)-i\sin(\theta))}\right|\\ &\leqslant \frac{R}{R^2-4R-5}e^{\xi R\sin(\theta)}\\ &\leqslant \frac{R}{R^2-4R-5}.\end{aligned}$$

Thus,

$$\left|\int_{C_R} f(z)\,dz\right| \leqslant \pi\frac{R}{R^2-4R-5},$$

whence the integral over C_R goes to zero as $R \to \infty$. Therefore, we have

$$\begin{aligned}-2\pi i\operatorname{Res}(f; -2-i) &= \lim_{R\to\infty}\int_{\eta_R} f(z)\,dz\\ &= I + \lim_{R\to\infty}\int_{C_R} f(z)\,dz\\ &= I\\ &= -2\pi i\lim_{z\to-2-i}\frac{(z-(-2-i))e^{i\xi z}}{(z-(-2-i))(z-(-2+i))}\\ &= -2\pi i\frac{e^{i\xi(-2-i)}}{-2i}\\ &= \pi e^{i\xi(-2-i)}\\ &= \pi e^{\xi}(\cos(2\xi) - i\sin(2\xi))\\ &= \pi e^{\xi}\cos(2\xi) - i\pi e^{\xi}\sin(2\xi).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos(\xi x)}{x^2+4x+5}\,dx &= \operatorname{Re}(I)\\ &= \pi e^{\xi}\cos(2\xi).\end{aligned}$$

Problem (Problem 4)**:** Evaluate

$$\int_0^{\infty}\frac{(\log x)^2}{x^2+1}\,dx.$$

Solution: Select the branch of the logarithm that ignores $[0,\infty)$, so that $\arg(z) \in (0, 2\pi)$ for all $z \in \mathbb{C}\setminus[0,\infty)$. Draw a keyhole contour $\gamma_{\delta,\varepsilon,R}$ with an inner *semi*circle of radius δ, an outer semicircle of radius R, and returning along the negative real axis to the start of the semicircle of radius δ.

Set $f(z) = \frac{(\log z)^2}{z^2+1}$, and observe that for $0 < \varepsilon < \delta < 1 < R$, we have

$$\begin{aligned}\oint_{\gamma_{\delta,\varepsilon,R}} f(z)\,dz &= 2\pi i(\operatorname{Res}(f;i))\\ &= 2\pi i\left(\lim_{z\to i}(z-i)\frac{(\log(z))^2}{(z-i)(z+i)}\right)\\ &= -\frac{\pi^3}{4}.\end{aligned}$$

Meanwhile, we observe that in the limit as $\varepsilon \to 0$, we are left with a few integrals

$$\oint_{\gamma_{\delta,\varepsilon,R}} f(z)\,dz = \int_\delta^R \frac{(\log(x))^2}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{(\log(x))^2}{x^2+1}\,dx \tag{$*$}$$

$$+ \int_0^\pi \frac{\log\left(\delta e^{-i\theta}\right)^2}{\delta^2 e^{-2i\theta}+1}\left(-i\delta e^{-i\theta}\right)d\theta + \int_0^\pi \frac{\log\left(Re^{i\theta}\right)^2}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta \tag{$**$}$$

We start by estimating the integrals in $(**)$ by the circles $\delta e^{-i\theta}$ and $Re^{i\theta}$. Towards this end, we observe that

$$\begin{aligned}\left|\frac{-i\delta e^{-i\theta}(\ln(\delta)-i\theta)^2}{\delta^2e^{-2i\theta}+1}\right| &\leqslant \frac{\delta|\ln(\delta)|^2 + 2\theta\delta|\ln(\delta)| + \theta^2\delta}{1-\delta^2}\\ &\leqslant \frac{\delta|\ln(\delta)|^2 + 4\pi\delta|\ln(\delta)| + 4\pi^2\delta}{1-\delta^2}\\ &\to 0\end{aligned}$$

as $\delta \to 0$. Thus,

$$\begin{aligned}\left|\int_0^{2\pi} \frac{-i\delta e^{-i\theta}(\ln(\delta)-i\theta)}{\delta^2e^{2i\theta}+1}\,d\theta\right| &\leqslant \pi\frac{\delta|\ln(\delta)|^2 + 4\pi\delta|\ln(\delta)| + 4\pi^2\delta}{1-\delta^2}\\ &\to 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\frac{Re^{i\theta}(\ln(R)+i\theta)^2}{R^2e^{2i\theta}+1}\right| &\leqslant \frac{R|\ln(R)|^2 + 2\theta R|\ln(R)| + \theta^2 R}{R^2-1}\\ &\leqslant \frac{R|\ln(R)|^2}{R^2-1} + \frac{2\pi R}{R^2-1} + \frac{4\pi^2}{R^2-1}\\ &= \frac{|\ln(R)|^2}{R-\frac{1}{R}}\frac{2\pi R}{R^2-1} + \frac{4\pi^2}{R^2-1}\\ &\to 0\end{aligned}$$

as $R \to \infty$, so the corresponding integral also goes to zero.

Now, we turn our attention to $(*)$. We observe that by the coordinate change $x \mapsto -x$, we get

$$\int_\delta^R \frac{\ln(x)}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{(\ln(x))^2}{x^2+1}\,dx = 2\int_\delta^R \frac{(\ln(x))^2}{x^2+1}\,dx + 2\pi i\int_\delta^R \frac{\ln(x)}{x^2+1}\,dx - \pi^2\int_\delta^R \frac{1}{x^2+1}\,dx.$$

As we take the limit as $\delta \to 0$ and $R \to \infty$, we observe that we get the equation

$$\frac{\pi^3}{4} = 2\underbrace{\int_0^\infty \frac{(\ln(x))^2}{x^2+1}\,dx}_{=:I_1} + 2\pi i \underbrace{\int_0^\infty \frac{\ln(x)}{x^2+1}\,dx}_{=:I_0}$$

Now, to evaluate I_0, we use the same contour for $g(z) = \frac{\ln(z)}{z^2+1}$, giving

$$\begin{aligned}\int_{\gamma_{\delta,\varepsilon,R}} g(z)\,dz &= \int_\delta^R \frac{\ln(x)}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{\ln(x)}{x^2+1}\,dx \\ &\quad + \int_0^\pi \frac{\ln\left(Re^{i\theta}\right)}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta + \int_0^\pi \frac{\ln\left(\delta e^{-i\theta}\right)}{\delta^2 e^{2i\theta}+1}\left(-i\delta e^{-i\theta}\right)\,d\theta.\end{aligned}$$

The circle integrands may be estimated by

$$\begin{aligned}\left|\frac{iRe^{i\theta}|\ln(R)+i\theta|}{R^2e^{2i\theta}}\right| &\leqslant \frac{R\ln(R)+R\theta}{R^2-1} \\ &\leqslant \frac{R\ln(R)+\pi R}{R^2-1} \\ &\to 0\end{aligned}$$

as $R \to \infty$, so that

$$\begin{aligned}\left|\int_0^\pi \frac{\ln\left(Re^{i\theta}\right)}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta\right| &\leqslant \pi\frac{R\ln(R)+\pi R}{R^2-1} \\ &\to 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\frac{-i\delta e^{-i\theta}(\ln(\delta)-i\theta)}{\delta^2e^{2i\theta}+1}\right| &\leqslant \frac{\delta|\ln(\delta)|+\pi\delta}{1-\delta^2} \\ &\to 0\end{aligned}$$

as $\delta \to \infty$, so that

$$\begin{aligned}\left|\int_0^\pi \frac{\ln\left(\delta e^{-i\theta}\right)}{\delta^2e^{2i\theta}+1}\left(-i\delta e^{-i\theta}\right)\,d\theta\right| &\leqslant \pi\frac{\delta|\ln(\delta)|+\pi\delta}{1-\delta^2} \\ &\to 0.\end{aligned}$$

Thus, we must evaluate the first two integrals. Yet, by using the substitution $x \mapsto -x$, we see that

$$\int_\delta^R \frac{\ln(x)}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{\ln(x)}{x^2+1}\,dx = 2\int_\delta^R \frac{\ln(x)}{x^2+1}\,dx + i\pi\int_\delta^R \frac{1}{x^2+1}\,dx.$$

Taking limits and evaluating residues gives

$$\begin{aligned}2\pi i\operatorname{Res}(g;i) &= 2\pi i\left(\frac{i\pi/2}{2i}\right) \\ &= i\frac{\pi^2}{2} \\ &= 2\int_0^\infty \frac{\ln(x)}{x^2+1}\,dx + i\pi\int_0^\infty \frac{1}{x^2+1}\,dx\end{aligned}$$

$$= 2\int_{0}^{\infty} \frac{\ln(x)}{x^2+1}\,dx + i\frac{\pi^2}{2},$$

whence the integral for $g(z)$ is zero.

Thus, we find that

$$\int_{0}^{\infty} \frac{(\ln(x))^2}{x^2+1}\,dx = \frac{\pi^3}{8}$$

Problem (Problem 5)**:** For $\xi \in \mathbb{R}$, evaluate

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{x^3}{\left(x^2+1\right)^2} e^{-2\pi i x\xi}\,dx.$$

Solution: We write

$$\int_{-\infty}^{\infty} \frac{x^3}{\left(x^2+1\right)^2} e^{-2\pi i x\xi}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{x^3}{(x-i)^2(x+i)^2} e^{-2\pi i x\xi}\,dx.$$

Write $f(z) = \frac{z^3}{\left(z^2+1\right)^2} e^{-2\pi i z\xi}$.

Suppose $\xi \geqslant 0$. Let γ_R be the square contour in the lower half-plane with side length R sitting on the real axis. Then,

$$\begin{aligned}
-2\pi i\operatorname{Res}(f;-i) &= \int_{\gamma_R} f(z)\,dz\\
&= \int_{-R}^{R} f(x)\,dx + \int_{0}^{R} f(R-iy)\,d(R-iy) + \int_{R}^{-R} f(x-iR)\,d(x-iR) + \int_{0}^{R} f(-R+iR+iy)\,d(-R-iR+iy)
\end{aligned}$$

Writing each of the integrals not equal to the original integral, we get

$$\int_{0}^{R} f(R-iy)\,d(R-iy) = -i\int_{0}^{R} \frac{(R-iy)^3}{\left((R-iy)^2+1\right)^2} e^{-2\pi i\xi(R-iy)}\,dy$$

$$\int_{R}^{-R} f(x-iR)\,d(x-iR) =$$