

Normed Vector Spaces

Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . A **vector space** over \mathbb{F} is a nonempty set V equipped with two operations: vector addition and scalar multiplication.

$$\begin{aligned} V \times V &\xrightarrow{+} V \\ (v, w) &\mapsto v + w && \text{Vector Addition} \\ F \times V &\rightarrow V \\ (\alpha, v) &\mapsto \alpha v && \text{Scalar Multiplication} \end{aligned}$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

- (i) $u + (v + w) = (u + v) + w$
- (ii) $\exists 0_v \in V$ with $\forall v \in V, 0_v + v = v + 0_v = v$
- (iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$
- (iv) $\forall v, w \in V, v + w = w + v$
- (v) $\alpha(v + w) = \alpha v + \alpha w, (\alpha + \beta)v = \alpha v + \beta v$
- (vi) $\alpha(\beta w) = (\alpha\beta)w$
- (vii) $1 \cdot v = v$

Remarks:

- (a) 0_v is unique and known as the zero vector.
- (b) The vector w in (iii) is unique, and denoted $-v$.
- (c) $0 \cdot v = 0_v$
- (d) $(-1) \cdot v = -v$
- (e) Property (iv) follows from all the other axioms.
- (f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \dots + v}_{n \text{ times}}$

Subspaces

Let V be a vector space over \mathbb{F} . A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

- (i) $w \in W, \alpha \in \mathbb{F} \rightarrow \alpha w \in W$.
- (ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

Remark: 0_v is always a member of any subspace; a subspace is also a vector space.

Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of V , then, $\bigcap W_i$ is a subspace of V .

Proposition: Union of Subspaces

It is not the case that the union of subspaces of V also a subspace. For example, consider \mathbb{R}^2 with the traditional vector space operations:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} x + x' \\ y + y' \end{pmatrix} \\ \alpha \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \end{aligned}$$

If $W_1, W_2 \subseteq V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Generated Subspaces

Let $S \subseteq V$ be any subset of a vector space V . Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^n \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

Remarks:

- $\text{span}(S) \subseteq V$ is a subspace.
- $\text{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\text{span}(S)$ is the “smallest” subspace containing S , or the subspace generated by S .

Proposition: Quotient Group on Vector Space

Let V be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

- (1) \sim_W is an equivalence relation.
- (2) If $[v]_W$ denotes the equivalence class of v , then $[v]_W = v + W = \{v + w \mid w \in W\}$.
- (3) $V/W := \{[v]_W \mid v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

Proof of (1):

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

Proof of (2):

$$\begin{aligned} [v]_W &= \{u \in V \mid u \sim_W v\} \\ &= \{u \in V \mid u - v \in W\} \\ &= \{u \in V \mid u = v + w \text{ some } w \in W\} \\ &= \{v + w \mid w \in W\} \\ &= v + W \end{aligned}$$

Proof of (3): Prove that the operation is well-defined.

Bases

Let V be a vector space and $S \subseteq V$ be a subset.

- (1) S is said to be spanning for V if $\text{span}(S) = V$.
- (2) S is linearly independent if, for $\sum_{j=1}^n \alpha_j v_j = 0_V$ with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.
- (3) S is a basis for V if S is linearly independent and spanning for V .

Proposition: Existence of Basis

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that B is a basis and $B \supseteq B_0$.

Background: A relation on a set X is a subset $R \subseteq X \times X$. If R is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \rightarrow x \sim z$), and antisymmetric ($x \sim y, y \sim x \rightarrow x = y$), then R is an ordering, and we write $x \leq y$.

If \leq is an ordering of X such that $\forall x, y \in X$, $x \leq y$ or $y \leq x$, then \leq is a total (or linear) ordering.

Let \leq be an ordering of X , let $Y \subseteq X$. An upper bound for Y is an element $u \in X$ such that $y \leq u \forall y \in Y$. A maximal element in X is an element $m \in X$ such that $x \in X$, $x \geq m \rightarrow x = m$.

Example: \mathbb{N} under the division ordering defines $a \leq b \Leftrightarrow a \mid b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of A can be divided by 9 and 12). Meanwhile, \mathbb{N} itself has no maximal elements.

This leads us to ask: given an ordered set, (X, \leq) , does X admit maximal elements.

Zorn's Lemma (or Axiom): Let (X, \leq) be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in X . Then, X admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

Proof: Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with D linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that X has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i \in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \dots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \dots, v_n \in D_j$ because Y is totally ordered. However, by definition, D_j is a linearly independent set — therefore, $\alpha_k = 0$. Thus, D is linearly independent.

Since D is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. D is also an upper bound for Y . So, by Zorn's Lemma, X has a maximal element, B .

So, $B_0 \subseteq B \subseteq V$, B is independent, and B is maximal in X . We claim that B is a basis for V . Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and B' is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \dots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.
- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \text{span}(B)$. \perp

Thus, we have a linearly independent set, B' , with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of B . Therefore, $\text{span}(B) = V$, and B is a basis for V .

Examples: Vector Spaces

(1) n -Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$B = \{e_1, \dots, e_n\}$$

where e_i denotes the unit vector at position i .

(2) $m \times n$ Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$\alpha(a_{ij}) = (\alpha a_{ij})$$

$$B = \{e_{ij}\}$$

where e_{ij} denotes a matrix of 0 everywhere except column i and row j .

(3) Functions with domain Ω :

$$\begin{aligned}\mathcal{F}(\Omega, \mathbb{F}) &= \{f \mid f : \Omega \rightarrow \mathbb{F}\} \\ (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

(4) Bounded functions with domain Ω :

$$\begin{aligned}\ell_\infty(\Omega, \mathbb{F}) &= \{f \in \mathcal{F}(\Omega, \mathbb{F}) \mid \|f\|_u \leq \infty\} \\ \|f\|_u &= \sup_{x \in \Omega} |f(x)|\end{aligned}$$

Exercises:

- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha| \|f\|_u$
- Positive Definite: $\|f\|_u = 0 \Rightarrow f = 0$

Proof of Triangle Inequality: Given $x \in \Omega$,

$$\begin{aligned}|(f+g)(x)| &= |f(x) + g(x)| \\ &\leq |f(x)| + |g(x)| \\ &\leq \|f\|_u + \|g\|_u\end{aligned}$$

Therefore,

$$\begin{aligned}\sup |(f+g)(x)| &\leq \|f\|_u + \|g\|_u \\ \|f+g\|_u &\leq \|f\|_u + \|g\|_u\end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a, b], \mathbb{F}) = \{f : [a, b] \rightarrow \mathbb{F} \mid f \text{ continuous}\}$$

Check that $C([a, b], \mathbb{F}) \subseteq \ell_\infty([a, b], \mathbb{F})$ is a subspace.

(6) Let $f : [a, b] \rightarrow \mathbb{R}$ be any function. Let $\mathcal{P} : a = x_0 < x_1 < x_2 < \dots < x_n = b$.

$$\begin{aligned}\text{var}(f; \mathcal{P}) &:= \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \\ \text{var}(f) &= \sup_{\mathcal{P}} \text{var}(f; \mathcal{P}) \\ \text{BV}([a, b]) &= \{f : [a, b] \rightarrow \mathbb{R} \mid \text{var}(f) < \infty\} \\ \|f\|_{\text{BV}} &= |f(a)| + \text{var}(f)\end{aligned}$$

$\text{BV}([a, b])$ is a vector space.

Question: Is $\mathbb{1}_{\mathbb{Q}} \in \text{BV}([0, 1])$?

(7) Suppose $K \subseteq V$ is a *convex* subset of a vector space: $v, w \in K, t \in [0, 1] \Rightarrow (1-t)v + tw \in K$. Let $\text{Aff}(K) = \{f : K \rightarrow \mathbb{R} \mid f \text{ is affine}\}$, where f is affine if $\forall v, w \in K, t \in [0, 1], f((1-t)v + tw) = (1-t)f(v) + tf(w)$.

Exercise: Show that $\text{Aff}(K) \subseteq \mathcal{F}(K, \mathbb{R})$ is a subspace.

(8) Let S be defined as

$$S = \{(a_k)_{k=1}^\infty \mid a_k \in \mathbb{F}\}.$$

Under pointwise operations, S is a vector space.

$$\begin{aligned}(a_k)_k + (b_k)_k &= (a_k + b_k)_k \\ \alpha(a_k)_k &= (\alpha a_k)_k\end{aligned}$$

Note 1: $S = \mathcal{F}(\mathbb{N}, \mathbb{F})$.

Note 2: $c_{00} \subseteq \ell_1 \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq S$.

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k \neq 0\}$
- $c_0 = \{(a_k)_k \mid (a_k)_k \rightarrow 0\}$

- $c = \{(a_k)_k \mid (a_k)_k \rightarrow a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_\infty < \infty\}$
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| = a < \infty\}$

(9) $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_c(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \rightarrow \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ continuous, } \lim_{x \rightarrow \pm\infty} f(x) = 0\}$

(10) Let S be any non-empty set.

$$\mathbb{F}(S) := \{f : S \rightarrow \mathbb{F} \mid f \text{ finitely supported}\}$$

$$\text{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \rightarrow \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t \in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\text{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^n f(t_k) e_{t_k} \in \text{span}(\xi)$. Therefore, ξ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^n \alpha_{t_k} e_{t_k} = 0$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\left(\sum_{k=1}^n e_{t_k} \right) = 0(t_1)$$

$$\alpha_{t_1} = 0.$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \dots, n$. Therefore, ξ is linearly independent. Since ξ is linearly independent and spanning, ξ forms a basis for $\mathbb{F}(S)$.

Note: The free vector space, $\mathbb{F}(S)$, displays the universal property.

There are functions $\iota : S \rightarrow \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \rightarrow V$ for V a vector space over \mathbb{F} , $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \rightarrow V$ such that $\iota \circ T_\varphi = \varphi$.

Proof: Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^n f(t_k) e_{t_k}$, where $\text{supp}(f) = \{t_1, \dots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^n f(t_k) \varphi(t_k)$$

Exercise: Show T_φ is linear and unique.

Exercise 2: Suppose V is a vector space over \mathbb{F} with basis B . Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists T : V \rightarrow W$ such that T is bijective and linear.

Normed Spaces

To every vector $v \in V$, we want to assign a length to v , $\|v\|$.

A **norm** on a vector space V is a map

$$\|\cdot\| : V \rightarrow \mathbb{R}^+$$

$$v \mapsto \|v\| \geq 0$$

such that

- (i) Homogeneity: $\|\alpha v\| = |\alpha| \|v\|$
- (ii) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$
- (iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = 0_V$.

If $p : V \rightarrow \mathbb{R}^+$ satisfies (i) and (ii), then p is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\begin{aligned}\|v\| &\leq c_1 \|v\|' \\ \|v\|' &\leq c_2 \|v\|\end{aligned}$$

Note: On \mathbb{R}^n , all norms are equivalent.

Exercise: If p is any seminorm on V , then $|p(v) - p(w)| \leq p(v - w)$.

Notation: If V is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.

Examples of Normed Spaces

(1) Given $V = \mathbb{F}^n$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have different norms:

$$\begin{aligned}\|x\|_1 &= \sum_{j=1}^n |x_j| \\ \|x\|_\infty &= \max_{1 \leq j \leq n} |x_j| \\ \|x\|_2 &= \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.\end{aligned}$$

In general, for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p}.$$

Exercise: Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Show that $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$.

We want to show that $\|\cdot\|_p$ defines a norm for $1 \leq p < \infty$. If $1 \leq p < \infty$, its conjugate index $q \in [1, \infty]$ whereby $\frac{1}{p} + \frac{1}{q} = 1$. For example, if $p = 1$, then $q = \infty$, and if $p = \infty$, then $q = 1$.

Lemma 1: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $f : [0, \infty) \rightarrow \mathbb{R}$, $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then, $f(t) \geq 0$ for all $t \geq 0$.

Proof 1: We can see that $f'(t) = t^{p-1} - 1$. Then, $f'(t) = 0$ at $t = 1$; $f'(t) > 0$ for $t > 1$ and $f'(t) < 0$ for $t \in [0, 1)$.

So, since $f(t) \geq f(1)$ for all $t \geq 0$, and $f(1) = 0$, $f(t) \geq 0$ for all $t \geq 0$.

Lemma 2: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $z, y \geq 0$, $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

Proof 2: We know from Lemma 1, $t \leq \frac{1}{p}t^p + \frac{1}{q}$. Multiply by y^q to get

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Set $t = xy^{1-q}$. Then,

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq}y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $p - pq = -q$, so

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

With these two lemmas in mind, we get two important inequalities.

Hölder's Inequality: For $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. Then, for $x, y \in \mathbb{F}^n$,

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \|x\|_p \|y\|_q.$$

Proof of Hölder's Inequality: For $p = 1$, the solution is as follows:

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n |x_j| \|y\|_\infty \\ &= \|x\|_1 \|y\|_\infty, \end{aligned}$$

and similarly for $p = \infty, q = 1$.

For $1 < p < \infty$, assume $\|x\|_p = \|y\|_q = 1$.

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n \left(\frac{1}{p} |x_j|^p + \frac{1}{q} |y_j|^q \right) \\ &= \frac{1}{p} \left(\sum_{j=1}^n |x_j|^p \right) + \frac{1}{q} \left(\sum_{j=1}^n |y_j|^q \right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = 0_{\mathbb{F}}$ or $y = 0_{\mathbb{F}}$, the inequality still holds.

Assume $\|x\|_p \neq 0, \|y\|_q \neq 0$. Set

$$\begin{aligned} x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_q}. \end{aligned}$$

It can be verified that $\|x'\|_p = 1 = \|y'\|_q$. Therefore,

$$\begin{aligned} \left| \sum_{j=1}^n x'_j y'_j \right| &\leq 1 \\ \left| \sum_{j=1}^n \frac{x_j}{\|x\|_p} \frac{y_j}{\|y\|_q} \right| &\leq 1 \\ \left| \sum_{j=1}^n x_j y_j \right| &\leq \|x\|_p \|y\|_q \end{aligned}$$

Minkowski's Inequality: Given $x, y \in \mathbb{F}^n$, $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Proof of Minkowski's Inequality: We can verify for $p = 1, q = \infty$, and vice versa.

Assume $1 < p < \infty$. Then,

$$\begin{aligned}
 \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p \\
 &= \sum_{j=1}^{\infty} |x_j + y_j| |x_j + y_j|^{p-1} \\
 &\leq \sum_{j=1}^{\infty} |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\
 &\leq \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \left(\sum_{j=1}^n |x_j + y_j|^{p(q-1)} \right)^{1/q} + \left(\sum_{j=1}^n |y_j|^p \right)^{1/p} \left(\sum_{j=1}^n |x_j + y_j|^{p(q-1)} \right)^{1/q} \quad \text{Hölder's Inequality} \\
 &= \|x\|_p \|x + y\|_p^{p/q} + \|y\|_p \|x + y\|_p^{p/q} \\
 &= (\|x\|_p + \|y\|_p) \|x + y\|_p^{p-1}
 \end{aligned}$$

Divide by $\|x + y\|_p^{p-1}$ to get desired inequality.

- (2) $\ell_\infty(\Omega, \mathbb{F})$ with $\|\cdot\|_\infty$. This includes subspaces that inherit the norm, such as

$$\begin{aligned}
 C([a, b]) &\subseteq \ell_\infty(\Omega) \\
 \ell_\infty(\mathbb{R}) &\supseteq C_0(\mathbb{R}) \supseteq C_c(\mathbb{R})
 \end{aligned}$$

Exercise: Show that $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ is a subspace.

- (3) $\Omega = \mathbb{N}$, $\ell_\infty = \ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$. Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \subseteq \ell_\infty.$$

- (4) ℓ_1 with $\|\cdot\|_1$,

$$\|(a_k)_k\|_1 = \sum_{k=1}^n |a_k|.$$

- (5) $C([a, b])$ with

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

- (6) Let $1 \leq p < \infty$.

$$\ell_p = \left\{ (a_k)_{k=1}^\infty \mid \sum_{k=1}^\infty |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left(\sum_{k=1}^\infty |a_k|^p \right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\begin{aligned}
 \left(\sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} &= \left\| \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \right\|_{\ell_p^n} \\
 &= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
 &\leq \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\|_{\ell_p^n} + \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
 &\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p.
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ (by the definition of an infinite series), we find that $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$.

(7) $\mathcal{BV}([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid \text{Var}(f) < \infty\}$ with the norm $\|f\|_{\mathcal{BV}} = |f(a)| + \text{Var}(f)$ is a normed space:

$$\|f\|_{\mathcal{BV}} = 0$$

$$|f(a)| = 0$$

$$\text{Var}(f) = 0$$

given $t \in (a, b]$, look at the partition $a < t \leq b$. Then,

$$\text{Var}(f) \geq |f(t) - f(a)| + |f(b) - f(t)|$$

$$f(t) = 0$$

$$f = 0_f.$$

(8) $\mathcal{M}_{m,n}(\mathbb{F})$ with

$$\|a\|_{\text{op}} = \sup_{\|\xi\|_{\ell_2^m} \leq 1} \|a\xi\|_{\ell_2^n}$$

is a normed vector space. If $\|a\|_{\text{op}} = 0$, then

$$ae_j = 0$$

$$\forall j \in \{1, \dots, n\}.$$

take the dot product with $i \neq j$

$$\begin{aligned} ae_j \cdot e_i &= a_{ij} \\ &= 0 \end{aligned}$$

so $a_{ij} = 0$ for all a_{ij} , so a is the 0 matrix.

(9) Let V, W be vector spaces over \mathbb{F} . Then, $\mathcal{L}(V, W) = \{T \mid T : V \rightarrow W \text{ linear}\}$, where $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

$\mathcal{L}(V, W)$ is a vector space with operations

$$(T + S)(v) = T(v) + S(v)$$

$$(\alpha T)(v) = \alpha T(v).$$

Notation: $\mathcal{L}(V) := \mathcal{L}(V, V)$ is all linear operators on V . $\mathcal{L}(V, \mathbb{F}) = V'$ is all linear functionals.

Suppose V and W are normed vector spaces. If $T : V \rightarrow W$, set

$$\|T\|_{\text{op}} := \sup_{\|v\|_V \leq 1} \|T(v)\|_W,$$

$$\mathcal{B}(V, W) = \{T \in \mathcal{L}(V, W) \mid \|T\|_{\text{op}} \leq \infty\},$$

where $\mathcal{B}(V, W)$ is referred to as the set of all bounded linear maps from V to W . $\mathcal{B}(V, W)$ with $\|\cdot\|_{\text{op}}$ is a normed space.

- Homogeneity:

$$\begin{aligned} \|\alpha T\|_{[\text{op}]} &= \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W \\ &= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W \\ &= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W \\ &= |\alpha| \|T\|_{\text{op}}. \end{aligned}$$

- Triangle Inequality: for $\|v\|_V \leq 1$,

$$\begin{aligned} \|(T + S)(v)\|_W &= \|T(v) + S(v)\|_W \\ &\leq \|T(v)\|_W + \|S(v)\|_W \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

so

$$\begin{aligned} \|T + S\|_{\text{op}} &= \sup_{\|v\| \leq 1} \|T + S(v)\| \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

- Positive Definite: If $\|T\|_{\text{op}} = 0$, then $T(v) = 0$ for all $v \in V$, $\|v\| \leq 1$.

Let $v \in V$, $v \neq 0$. Then, $\frac{v}{\|v\|} \in B_V$.

$$\begin{aligned} T\left(\frac{v}{\|v\|}\right) &= 0 \\ \frac{1}{\|v\|} T(v) &= 0 \\ T(v) &= 0 \end{aligned}$$

Special Cases: $\mathbb{B}(V) = \mathbb{B}(V, V)$, $V^* = \mathbb{B}(V, \mathbb{F})$.

Exercise: $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$.

(10) Inner Product Spaces (expanded upon below).

Inner Product Spaces

An inner product on a vector space V is a pairing

$$V \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}$$

that satisfies

$$(i) \quad \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle, \quad \langle \alpha v, w \rangle = \alpha \langle v, w \rangle.$$

$$(ii) \quad \langle v, w \rangle = \overline{\langle w, v \rangle}$$

$$(iii) \quad \langle v, v \rangle \geq 0.$$

$$(iv) \quad \text{If } \langle v, v \rangle = 0, \text{ then } v = 0.$$

The pair $(V, \langle \cdot, \cdot \rangle)$ is known as an inner product space.

Remarks: $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$, $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$.

If $\langle \cdot, \cdot \rangle$ is an inner product on a linear space V , then set

$$\|v\|_2 := \langle v, v \rangle^{1/2}.$$

Exercise: $\|\alpha v\|_2 = |\alpha| \|v\|_2$, $\|v\|_2 = 0 \Rightarrow v = 0$.

$v, w \in (V, \langle \cdot, \cdot \rangle)$ are *orthogonal* if $\langle v, w \rangle = 0$.

The Pythagorean theorem states that for $v_1, \dots, v_n \in V$ mutually orthogonal, then

$$\left\| \sum_{i=1}^n v_i \right\|^2 = \sum_{j=1}^n \|v_j\|^2.$$

For two vectors $v, w \in V$, $P_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.

Exercise: Check that $\langle P_w(v), v - P_w(v) \rangle$, meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

Cauchy-Schwarz Inequality: In any inner product space,

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|.$$

Proof of Cauchy-Schwarz: From the exercise,

$$\begin{aligned} \|v\| &\geq \|P_w(v)\| \\ \|v\| &\geq \left\| \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\| \\ &= \frac{|\langle v, w \rangle|}{\|w\|^2} \|w\| \end{aligned}$$

therefore,

$$\|v\| \|w\| \geq |\langle v, w \rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

Proof of Triangle Inequality:

$$\begin{aligned}
 \|v + w\|_2^2 &= \langle v + w, v + w \rangle \\
 &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\
 &= \|v\|^2 + \|w\|^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} \\
 &= \|v\|^2 + \|w\|^2 + 2\operatorname{Re} \langle v, w \rangle \\
 &\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w \rangle| \\
 &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\
 &= (\|v\| + \|w\|)^2.
 \end{aligned}$$

Cauchy-Schwarz Inequality

Take square roots on both sides.

(1) $\ell_2^n = \mathbb{F}^n$ with

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left| \sum_{j=1}^n x_j \overline{y_j} \right| \leq \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |y_j|^2 \right)^{1/2}.$$

(2) ℓ_2 with

$$\langle (a_j)_j, (b_j)_j \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

We can see that for any finite n , the Cauchy-Schwarz inequality in ℓ_2^n states

$$\begin{aligned}
 \left| \sum_{j=1}^n a_j \overline{b_j} \right| &\leq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |b_j|^2 \right)^{1/2} \\
 &\leq \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |b_j|^2 \right)^{1/2}.
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we see that $\langle (a_j)_j, (b_j)_j \rangle$ is convergent.

(3) $C([a, b])$ with

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

(4) Let $V = \mathbb{M}_n(\mathbb{C})$.

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let $\operatorname{Tr} : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$, $\operatorname{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$.

- $\operatorname{Tr}(I_n) = n$
- $\operatorname{Tr}(a + \alpha b) = \operatorname{Tr}(a) + \alpha \operatorname{Tr}(b)$
- $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$

Then, if $\text{Tr}(a^*a) = 0$, then $a = 0_{\mathbb{M}_n}$.

$$\begin{aligned} a^*a &= (\overline{a_{ji}})_{i,j} (a_{ij})_{i,j} \\ &= \left(\sum_{k=1}^n \overline{a_{ki}} a_{kj} \right)_{i,j} \\ \text{Tr}(a^*a) &= \sum_{i=1}^n \sum_{k=1}^n \overline{a_{ki}} a_{ki} \\ &= \sum_{i,k=1}^n |a_{ki}|^2 \\ &= \sum_{i,j=1}^n |a_{ij}|^2. \end{aligned}$$

If $\text{Tr}(a^*a) = 0$, then $a_{ij} = 0$ for all i, j .

We define

$$\langle a, b \rangle_{\text{HS}} = \text{Tr}(b^*a).$$

$$(i) \quad (b_1 + b_2)^* = b_1^* + b_2^*$$

$$(ii) \quad (\alpha b)^* = \overline{\alpha} b^*$$

$$(iii) \quad (b_1 b_2)^* = b_2^* b_1^*$$

$$(iv) \quad b^{**} = b$$

The norm is defined as

$$\begin{aligned} \|a\|_{\text{HS}} &= \langle a, a \rangle_{\text{HS}}^{1/2} \\ &= \text{Tr}(a^*a)^{1/2} \\ &= \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2} \end{aligned}$$

Metric Spaces

We looked at normed spaces, where we attach a length $\|v\|$ to every vector v . We can also speak of the distance between two vectors, defined as $d(v, w) = \|v - w\|$.

Notice that the following hold:

- $d(v, w) \geq 0$
-

$$\begin{aligned} d(v, w) &= \|v - w\| \\ &= \|(-1)(w - v)\| \\ &= |-1| \|w - v\| \\ &= \|w - v\| \end{aligned}$$

-

$$\begin{aligned} d(u, w) &= \|u - w\| \\ &= \|u - v + v - w\| \\ &\leq \|u - v\| + \|v - w\| \\ &= d(u, v) + d(v, w). \end{aligned}$$

- $d(v, v) = \|v - v\| = 0$. If $d(v, w) = 0$, then $\|v - w\| = 0$, so $v - w = 0$, so $v = w$.

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely $(\mathbb{R}, |\cdot|)$. We will expand these concepts to all metric spaces.

Definition of a Metric Space

Let X be a non-empty set. A **metric** on X is a map

$$\begin{aligned} d : X \times X &\rightarrow \mathbb{R}^+ \\ (x, y) &\mapsto d(x, y) \geq 0 \end{aligned}$$

such that

- (i) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
- (ii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
- (iii) Zero Distance: $d(x, x) = 0$
- (iv) Definite: $d(x, y) = 0 \Rightarrow x = y$

If d satisfies (i), (ii), and (iii), then d is called a semi-metric. If d satisfies (iv) as well, then d is a metric.

If d is a (semi-)metric on X , the pair (X, d) is called a (semi-)metric space.

Two metrics, d and ρ , on X , are equivalent if $\exists c_1, c_2 \geq 0$ such that $d(x, y) \leq c_1 \rho(x, y)$ and $\rho(x, y) \leq c_2 d(x, y)$ for all x, y .

Examples of Metric Spaces

- (1) Discrete Metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for X any set.

- (2) Hamming distance: between two bit strings of equal length. Let

$$\begin{aligned} X &= \{0, 1\}^n \\ &= \{0, 1\} \underbrace{\times \cdots \times}_{n \text{ times}} \{0, 1\} \\ d_H((x_j)_1^n, (y_j)_1^n) &= |\{j \mid x_j \neq y_j\}|. \end{aligned}$$

- (3) Any normed space $(V, \|\cdot\|)$ is a metric space.

$$d(v, w) = \|v - w\|.$$

Exercise: Show that if two norms are equivalent, their induced metrics are equivalent.

- (4) Subset of Metric Space: If (X, d) is a metric space, and $Y \subseteq X$ is non-empty. Then, (Y, d) is a metric space.

- (5) Paris metric: let (X, ρ) be a metric space. Let $p \in X$ be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

- (6) Bounded metric: Let ρ be a (semi-)metric on X . Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We claim that d is a (semi-)metric. Notice that $0 \leq d(x, y) \leq 1$.

Proof: Clearly, $d(x, y) = d(y, x)$. Additionally, $d(x, x) = 0$. If $d(x, y) = 0$ and ρ is a metric, then $\rho(x, y) = 0$, so $x = y$.

To show the triangle inequality, we examine the function

$$\begin{aligned} f(t) &= \frac{t}{1+t} \\ f'(t) &= \frac{1}{(1+t)^2} > 0. \end{aligned}$$

Since ρ satisfies the triangle inequality, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Apply f on both sides. Then,

$$\begin{aligned} \underbrace{\frac{\rho(x, z)}{1 + \rho(x, z)}}_{d(x, z)} &\leq \frac{\rho(x, y) + \rho(y, z)}{1 + (\rho(x, y) + \rho(y, z))} \\ &= \frac{\rho(x, y)}{1 + \rho(x, y) + \rho(y, z)} + \frac{\rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\ &\leq \underbrace{\frac{\rho(x, y)}{1 + \rho(x, y)}}_{d(x, y)} + \underbrace{\frac{\rho(y, z)}{1 + \rho(y, z)}}_{d(y, z)}. \end{aligned}$$

(7) If d_1, \dots, d_n are metrics on X , $c_1, \dots, c_n \geq 0$. Then,

$$d(x, y) = \sum_{k=1}^n c_k d_k(x, y)$$

is a metric.

(8) Let $\{\rho_k\}_{k=1}^\infty$ be a family of semi-metrics. Assume the family is separating — for all $x \neq y$, there exists k such that $\rho_k(x, y) \neq 0$.

Let d_k be defined as

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Note that $\{d_k\}_{k=1}^\infty$ is also separating.

Then,

$$d(x, y) = \sum_{k=1}^\infty 2^{-k} d_k(x, y)$$

is a metric.

(9) Frechet Metric: Let $X = C(\mathbb{R})$. For each $k = 1, 2, 3, \dots$, set $p_k(f) = \sup_{x \in [-k, k]} |f(x)|$.

We can verify that p_k defines a seminorm. We can then check $\rho_k(f, g) = p_k(f - g)$ is a semi-metric.

We claim that $\{\rho_k\}$ is separating: if $f \neq g$, then there exists $x_0 \in \mathbb{R}$ with $f(x_0) \neq g(x_0)$. Since f and g are continuous, there is a neighborhood $[x_0 - \delta, x_0 + \delta]$ such that $f(x) \neq g(x)$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Find k such that $[x_0 - \delta, x_0 + \delta] \subseteq [-k, k]$. Then, $\rho_k(f - g) > 0$.

Construct d_k as above, and then d as follows:

$$d_F = \sum \frac{2^{-k} p_k(f - g)}{1 + p_k(f - g)}$$