

**Problem (Problem 1):** Let  $(a_n)_n$  be a sequence for which  $\sum_{n=0}^{\infty} |a_n|^2$  is finite. For each positive  $N$ , define  $f_N(z) = \sum_{n=0}^{\infty} a_n z^n$ , and define  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ .

(a) Show that  $f$  is holomorphic on  $\mathbb{D}$ .

(b) For each  $r \in (0, 1)$ , determine in terms of  $(a_n)_n$  the integral

$$\frac{1}{2\pi} \int_0^{2\pi} |f_N(re^{i\theta})|^2 d\theta.$$

(c) For each  $r \in (0, 1)$ , determine in terms of  $(a_n)_n$  the integral

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

(d) Determine in terms of  $(a_n)_n$  the limit

$$\lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

**Solution:**

(a) Let  $0 < r < 1$ . Since each  $f_N$  is analytic, we can use the Cauchy Integral Formula to compute  $a_N$  explicitly, yielding

$$\begin{aligned} |a_N| &= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_N(\xi)}{\xi^{N+1}} d\xi \right| \\ &\leq \frac{1}{r^N} \sup_{|z|=r} |f_N(z)|. \end{aligned}$$

Therefore, if we are able to show that the value

$$\sup_{|z|=r} |f_N(z)|$$

is uniformly bounded by a constant for all  $N$ , we will be able to use the Cauchy–Hadamard theorem to show that  $\limsup_{N \rightarrow \infty} |a_N|^{1/N} \leq 1$ . Toward this end, we use the Cauchy–Schwarz inequality, which yields

$$\begin{aligned} \sup_{|z|=r} |f_N(z)| &= \sup_{|z|=r} \left| \sum_{n=0}^N a_n z^n \right| \\ &\leq \sup_{|z|=r} \left( \sum_{n=0}^N |a_n|^2 \right)^{1/2} \left( \sum_{n=0}^N |z|^{2n} \right)^{1/2} \\ &\leq \sup_{|z|=r} \underbrace{\left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}}_{=:K} \left( \sum_{n=0}^{\infty} |z|^{2n} \right)^{1/2} \\ &= \frac{K}{(1 - |r|^2)^{1/2}}. \end{aligned}$$

Since we have established this uniform bound, we thus find that  $\sum_{n=0}^{\infty} a_n z^n$  has radius of convergence at least 1, so  $f$  is analytic on  $\mathbb{D}$ , hence holomorphic on  $\mathbb{D}$ .

(b) We write out the integral to yield

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f_N(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=0}^N a_n r^n e^{in\theta} \right) \overline{\left( \sum_{m=0}^N a_m r^m e^{im\theta} \right)} d\theta \\ &= \frac{1}{2\pi} \sum_{n=0}^N \sum_{m=0}^N a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\ &= \sum_{n=0}^N |a_n|^2 r^{2n}. \end{aligned}$$

(c) Since  $f$  is holomorphic with radius of convergence at least 1, the series expression on  $S(0, r)$  converges uniformly, so that we may exchange sum and integral. This yields

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \end{aligned}$$

(d) Since the sequence  $(a_n)_n$  is square-summable, the limit is well-defined, and we get

$$\begin{aligned} \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \lim_{r \nearrow 1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \\ &= \sum_{n=0}^{\infty} |a_n|^2. \end{aligned}$$

**Problem (Problem 2):** Let  $\varphi: [0, 1] \rightarrow \mathbb{C}$  be continuous, and define  $f: \mathbb{C} \setminus [0, 1] \rightarrow \mathbb{C}$  by

$$f(z) = \int_0^1 \frac{\varphi(t)}{t-z} dt.$$

Show that  $f$  is holomorphic and determine the derivative of  $f$  in terms of  $\varphi$ .

**Solution:** Let  $z, z+h \in \mathbb{C} \setminus [0, 1]$ , so we may calculate

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^1 \frac{\varphi(t)}{t-(z+h)} - \frac{\varphi(t)}{t-z} dt \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^1 \frac{(t-z)\varphi(t) - (t-(z+h))\varphi(t)}{(t-(z+h))(t-z)} dt \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^1 \frac{h\varphi(t)}{(t-(z+h))(t-z)} dt \\ &= \lim_{h \rightarrow 0} \int_0^1 \frac{\varphi(t)}{(t-(z+h))(t-z)} dt \\ &= \int_0^1 \frac{\varphi(t)}{(t-z)^2} dt. \end{aligned}$$

Let  $(z_n)_n \subseteq \mathbb{C} \setminus [0, 1]$  converge to  $z \in \mathbb{C} \setminus [0, 1]$ . Define the sequence of functions given by

$$(h_n)_n: [0, 1] \rightarrow \mathbb{C}$$

$$t \mapsto \frac{\varphi(t)}{(t - z_n)^2}.$$

We claim that the  $(h_n)_n$  converge uniformly to

$$h(t) = \frac{\varphi(t)}{(t - z)^2}.$$

Observe that the pointwise convergence is clear, and  $h(t)$  is well-defined for all  $t$  by definition. Now, we observe as well that the value  $K = \text{dist}_{[0,1]}(\{z_n \mid n \in \mathbb{N}\})$  is nonzero, as the closure of  $[0, 1]$  in  $\mathbb{C}$  is  $[0, 1]$ , and  $\{z_n \mid n \in \mathbb{N}\}$  are explicitly contained in  $\mathbb{C} \setminus [0, 1]$ . Similarly, since  $z$  is not contained in the closure of  $[0, 1]$ , we find that  $L = \text{dist}_{[0,1]}(\{z\})$  is also nonzero.

Thus, we find that

$$\begin{aligned} \left| \frac{\varphi(t)}{(t - z_n)^2} - \frac{\varphi(t)}{(t - z)^2} \right| &= \frac{|\varphi(t)| |2t(z_n - z) + (z^2 - z_n^2)|}{|t - z_n|^2 |t - z|^2} \\ &\leq \frac{\|\varphi\|_u |2t(z_n - z) + (z^2 - z_n^2)|}{|t - z_n|^2 |t - z|^2} \\ &\leq \|\varphi\|_u \frac{2|z_n - z| + |z^2 - z_n^2|}{K^2 L^2} \\ &= \frac{2\|\varphi\|_u}{K^2 L^2} (|z_n - z| + |z^2 - z_n^2|), \end{aligned}$$

meaning that the supremum of the left-hand side is less than or equal to a constant multiplied by  $|z_n - z| + |z^2 - z_n^2|$ . Since  $z \mapsto z^2$  is continuous, it follows that  $(h_n)_n \rightarrow h$  uniformly. Thus, we may exchange limit and integral, so that

$$\begin{aligned} \lim_{z_n \rightarrow z} f'(z_n) &= \lim_{z_n \rightarrow z} \int_0^1 \frac{\varphi(t)}{(t - z_n)^2} dt \\ &= \int_0^1 \lim_{z_n \rightarrow z} \frac{\varphi(t)}{(t - z_n)^2} dt \\ &= \int_0^1 \frac{\varphi(t)}{(t - z)^2} dt \\ &= f'(z), \end{aligned}$$

meaning  $f'(z)$  is continuous, so  $f$  is holomorphic.

**Problem** (Problem 3): Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be entire.

- Suppose there exist  $C, R > 0$  and  $n \in \mathbb{N}$  such that  $|f(z)| \leq C|z|^n$  for all  $|z| > R$ . Show that  $f$  is a polynomial of degree at most  $n$ .
- Suppose that  $g: \mathbb{C} \rightarrow \mathbb{C}$  is also entire and  $|f(z)| \leq |g(z)|$  for all  $z \in \mathbb{C}$ . Show that there exists some  $\alpha \in \mathbb{C}$  with  $|\alpha| \leq 1$  such that  $f(z) = \alpha g(z)$  for all  $z \in \mathbb{C}$ .
- Suppose that there exists some  $\theta \in \mathbb{R}$  such that  $f(\mathbb{C}) \cap \{re^{i\theta} \mid r > 0\} = \emptyset$ . Show that  $f$  is constant.

**Solution:**

- Let  $r > R$ . Then, by the Cauchy estimate, we get that

$$|f^{(n+1)}(0)| \leq \frac{(n+1)!}{r^{n+1}} \sup_{|z|=r} |f(z)|$$

$$\begin{aligned}
&\leq \frac{(n+1)!}{r^{n+1}} \sup_{|z|=r} (C|z|^n) \\
&= \frac{C(n+1)!}{r},
\end{aligned}$$

so since  $r$  is arbitrary and  $f$  is entire, we find that  $f^{(n+1)}(0) = 0$ , so that the power series expansion of  $f$  about 0 terminates beyond  $n+1$ . Since  $f$  is entire, its power series expansion about any  $z_0 \in \mathbb{C}$  is equal to  $f(z)$  everywhere in  $\mathbb{C}$ , so in particular, this holds for  $f$  at 0, meaning  $f$  is a polynomial of degree at most  $n$ .

- (b) If  $g$  is 0, or  $f$  is 0, we are done. Else, assume that  $g$  and  $f$  are not identically zero. Observe that if  $g$  is everywhere non-vanishing, then the function  $\frac{f(z)}{g(z)}$  is entire, and satisfies

$$\left| \frac{f(z)}{g(z)} \right| \leq 1,$$

hence  $\frac{f(z)}{g(z)} = \alpha$  for some  $\alpha$  with  $|\alpha| \leq 1$ .

If  $k(z) = \frac{f(z)}{g(z)}$  is such that  $g(z)$  admits zeros, then they must be isolated zeros, or else by the identity theorem,  $g$  would be identically zero everywhere. Let  $a$  be one of these zeros for  $g$ . If  $\varepsilon$  is small, we observe that for  $0 < |z - a| < \varepsilon$ ,  $k(z)$  is bounded. If we let  $M$  be this bound, then we observe that the value

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{|\zeta-a|=\varepsilon} \frac{k(\zeta)}{\zeta-a} d\zeta \right| &\leq \frac{1}{2\pi} \int_{|\zeta-a|=\varepsilon} \frac{|k(\zeta)|}{|\zeta-a|} |d\zeta| \\
&\leq \frac{1}{2\pi} \int_{|\zeta-a|=\varepsilon} \frac{M}{|\zeta-a|} |d\zeta| \\
&= \frac{M}{\varepsilon},
\end{aligned}$$

so that the integral is well-defined. In particular, this means that we may define a holomorphic extension of  $k(z)$  by

$$h(z) = \begin{cases} k(z) & g(z) \neq 0 \\ \frac{1}{2\pi i} \int_{|\zeta-z|=\varepsilon} \frac{k(\zeta)}{\zeta-z} d\zeta & g(z) = 0. \end{cases}$$

The function  $h(z)$  is thus entire, and bounded by 1, so by Liouville's theorem,  $h(z) = \alpha$  for some  $\alpha$  with  $|\alpha| \leq 1$ . This means that whenever  $g(z) \neq 0$ , we have  $f(z) = \alpha g(z)$ , and clearly when  $g(z) = 0$ , we have  $f(z) = \alpha g(z)$ , so that  $f(z) = \alpha g(z)$ .

- (c) Let  $f$  be such that  $f(\mathbb{C}) \cap \{re^{i\theta} \mid r > 0\} = \emptyset$ . By adding a sufficient multiple of  $2\pi k$ , we may assume  $\theta > 0$ .

Define a branch of the logarithm  $\log_\theta(z)$  by taking

$$S_\theta = \{z \in \mathbb{C} \mid \theta < \text{Im}(z) < \theta + 2\pi\}.$$

Then, we observe that  $\sqrt{z} = e^{\frac{1}{2}\log_\theta(z)}$  maps  $S_\theta$  to the set

$$\mathbb{H}_\theta = \{z \mid \theta/2 < \arg(z) < \theta/2 + \pi\} \cup \{0\}.$$

Observe that the map  $z \mapsto e^{-i\theta/2}z$  is entire and maps  $\mathbb{H}_\theta$  to the upper half plane plus  $\{0\}$ , and the Cayley transform,  $\varphi(w) = \frac{w-i}{w+i}$ , is holomorphic on  $\mathbb{C} \setminus \{i\}$ , maps  $0 \mapsto -1$ , and maps the open upper half-plane to the unit disk. Therefore, if we fix some  $\varepsilon > 0$ , we find that the composition

$$\varphi \circ \left( z \mapsto e^{-i\theta/2}z \right) \circ \sqrt{\cdot} \circ f: \mathbb{C} \rightarrow \mathcal{U}(0, 1 + \varepsilon)$$

is an entire function that is bounded in modulus by  $1 + \varepsilon$ . In particular, since all of  $\sqrt{f}$ ,  $z \mapsto e^{-i\theta/2}z$ , and  $\varphi$  are nonconstant and holomorphic on the specified domains, this implies that  $f$  is constant.

**Problem (Problem 4):** Let  $U = \{z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1\}$ . Suppose  $f: U \rightarrow \mathbb{C}$  is holomorphic, and there exists  $C > 0$  and  $\eta \in \mathbb{R}$  such that

$$|f(z)| \leq C(1 + |z|)^\eta$$

for all  $z \in U$ . Show that for each  $n \geq 0$ , there exists a constant  $C_{n,\eta} \geq 0$  dependent only on  $n$  and  $\eta$  such that

$$|f^{(n)}(x)| \leq C_{n,\eta}(1 + |x|)^\eta$$

for all  $x \in \mathbb{R}$ .

**Solution:** Let  $x \in \mathbb{R}$ ,  $0 < r < 1$ , and to start, assume  $\eta \geq 0$ . Then, from Cauchy's estimate, a bunch of triangle inequalities, and the fact that  $\eta \geq 0$  and  $r < 1$ , we find that

$$\begin{aligned} |f^{(n)}(x)| &\leq \frac{n!}{r^n} \sup_{|w-x|=r} |f(w)| \\ &\leq \frac{n!}{r^n} \sup_{|w-x|=r} (C(1 + |w|)^\eta) \\ &\leq \frac{Cn!}{r^n} \sup_{|w-x|=r} \left(1 + \left|w - \frac{3}{2}x\right| + \frac{3}{2}|x|\right)^\eta \\ &\leq \frac{Cn!}{r^n} \sup_{|w-x|=r} (1 + |w - x| + 2|x|)^\eta \\ &\leq \frac{Cn!}{r^n} \sup_{|w-x|=r} (1 + r + 2|x|)^\eta \\ &\leq \frac{Cn!}{r^n} (2 + 2|x|)^\eta \\ &\leq \frac{C2^\eta n!}{r^n} (1 + |x|)^\eta. \end{aligned}$$

In particular, since this inequality holds for every  $0 < r < 1$ , it necessarily holds for  $r = 1$ , so that  $C_{n,\eta} = C2^\eta n!$ .

Now, if  $\eta < 0$ .

**Problem (Problem 5):** Let  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  be a polynomial of degree  $n \geq 1$ , where  $a_0, \dots, a_n \in \mathbb{C}$  with  $a_n \neq 0$ .

- Show that there exist  $n$  complex numbers  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$  not necessarily distinct such that  $P(z) = a_n(z - \alpha_1) \cdots (z - \alpha_n)$ .
- Suppose  $|\alpha_0| > |\alpha_n|$ . Show that there exists some  $\alpha \in \mathbb{C}$  for which  $|\alpha| > 1$  and  $P(\alpha) = 0$ .

**Solution:**

- Dividing out by  $a_n$ , we take

$$P(z) = a_n \left( z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_1}{a_n} z + \frac{a_0}{a_n} \right).$$

By the fundamental theorem of algebra, we can find some  $\alpha_1$  such that  $P(\alpha_1) = 0$ . Therefore, by polynomial division, we have a monic polynomial  $q(z)$  with degree  $n - 1$  such that

$$P(z) = a_n q(z)(z - \alpha_1).$$

If  $q(z)$  is a constant polynomial, it is necessarily equal to 1 and we are done. Else, inductively, we may find  $\alpha_2, \dots, \alpha_n \in \mathbb{C}$  such that  $q(z) = (z - \alpha_2) \cdots (z - \alpha_n)$ , meaning that

$$P(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

(b) If  $P$  is a polynomial, then we may factor

$$P(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

Observe that

$$a_0 = a_n \prod_{i=1}^n \alpha_i,$$

so that

$$\left| \frac{a_0}{a_n} \right| = \prod_{i=1}^n |\alpha_i|.$$

Since  $|a_0| > |a_n|$ , it follows that

$$\prod_{i=1}^n |\alpha_i| > 1.$$

By the pigeonhole principle, there must be at least one  $\alpha_i$  such that  $|\alpha_i| > 1$ .