Problem (Problem 1)**:** In this exercise, we prove another fundamental result in differential topology, called the tubular neighborhood theorem. Let M be a compact smooth manifold with orientable boundary N. For simplicity, assume that N is connected. The tubular neighborhood theorem asserts that N admits a neighborhood in M which is diffeomorphic to $N \times [0, 1)$.

(a) Choose a Riemannian metric on M, and show that N admits a nonvanishing vector field that is everywhere orthogonal to the tangent space of N. That is, a vector field X such that for all $p \in N$, $g(X_p, T_p N) = 0$.

(b) Use the flow generated by X to find the desired neighborhood.

Solution:

(a) If $p \in N$, then we observe that $T_p N < T_p M$ is a proper subspace with codimension 1. Letting $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for $T_p N$, then we may extend to a basis for $T_p M$ by taking a representative for a basis for $T_p M/T_p N$, and observing that such a vector necessarily has

$$g_p(e_n, e_k) = 0$$

for all $k = 1, \dots, n-1$. By smoothly varying over all points $p \in N$, we get our desired everywhere nonvanishing vector field normal to $T_p N$.

(b) Let φ_t be the one-parameter diffeomorphism group generated by X, where $\varphi_t \colon M \to M$ is such that $\varphi_0(p) = p$ for all $p \in N$. Then, $\varphi \colon (-\varepsilon, \varepsilon) \to \operatorname{diff}(M)$ restricted to $[0, \varepsilon)$ gives our desired neighborhood in M diffeomorphic to $N \times [0, 1)$.

Problem (Problem Set 7, Problem 5)**:** Suppose G is a finite group acting freely on a manifold M by diffeomorphisms.

(a) Show that M/G is a manifold.

(b) Show that the de Rham cohomology of M/G is isomorphic to the G-invariant cohomology of M.

Problem (Problem Set 8, Problem 3)**:** Compute the de Rham cohomology of $\mathbb{RP}^n$.

Solution: We will use the result related to invariant cohomology to compute this.

Problem (Problem Set 8, Problem 5)**:** Use the Mayer–Vietoris sequence to prove the Künneth Formula: if M and N are smooth manifolds, then $H^*_{DR}(M \times N)$ is the tensor product of $H^*_{DR}(M)$ and $H^*_{DR}(N)$. Specifically, in each degree ℓ, we have

$$H^\ell_{DR}(M \times N) = \bigoplus_{i+j=\ell} H^i_{DR}(M) \otimes H^j_{DR}(N).$$