Abstract

We introduce some of the most important inequalities that are used frequently in real and functional analysis. These inequalities include Jensen's inequality and Young's inequality (concerning convex functions), which are then used to prove Hölder's inequality and Minkowski's inequality (concerning p-norms). Afterwards, we define the L_p-spaces and show that they are complete.

Contents

Convex Functions

Definition. A function $\varphi\colon (a,b) \to \mathbb{R}$ is called *convex* if, for all $x, y \in (a,b)$,

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y).$$

Remark: A *differentiable* function $\varphi\colon \Omega \to \mathbb{R}$ is convex if and only if its second derivative is positive along its domain.

Note here that we define convexity along an open interval. This is because it is convenient to do so.

Two major examples of convex functions are

$$\begin{aligned}\varphi_1(x) &= e^x \\ \varphi_2(x) &= x^p. \qquad 1 \leq p < \infty\end{aligned}$$

Both of these functions are convex along their domain.

Convex functions defined over an open interval are useful precisely because they are continuous — and, thus, measurable.

Theorem: Let $\varphi\colon (a,b) \to \mathbb{R}$ be convex. Then, φ is continuous.

We follow the proof from this website.

Proof. We begin by an observation. If $a < x_1 < x_2 < x_3 < b$, then convexity gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{$*$}$$

By the characterization of an interval, for $s, t \in (a,b)$ with $s < t$, we have $[s,t] \subseteq (a,b)$. Now, note that since (a,b) is an open interval, there are $s', t' \in (a,b)$ with $s' < s$ and $t < t'$. In particular, this means that for any $x_1, x_2 \in [s,t]$ with $x_1 < x_2$, we have

$$\begin{aligned}\frac{f(s) - f(s')}{s - s'} &\leq \frac{f(x_1) - f(s)}{x_1 - s} \\ &\leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &\leq \frac{f(t) - f(x_2)}{t - x_2} \\ &\leq \frac{f(t') - f(t)}{t' - t}.\end{aligned}$$

Setting $C := \max\left\{\frac{f(t')-f(t)}{t'-t}, \frac{f(s)-f(s')}{s-s'}\right\}$, we see that for any $x_1, x_2 \in [s,t]$,

$$|f(x_2) - f(x_1)| \leq C|x_2 - x_1|.$$

Thus, f is Lipschitz on $[s,t]$, so f is continuous on $[s,t]$.

Since, for any $x \in (a,b)$, there is some closed interval containing x, and f is continuous on said closed interval, we have that f is continuous on (a,b). □

Remark: The fact that (a,b) is an open interval is indeed load-bearing. Consider the function defined by

$$f(x) = \begin{cases} x & x > 0 \\ 1 & x = 0 \end{cases}.$$

Then, f is convex, but f is not continuous.

The most famous inequality regarding convex functions is Jensen's inequality, which effectively provides a generalization of the definition of a convex function.

Theorem (Jensen's Inequality)**:** Let $(\Omega, \mathcal{M}, \mu)$ be a probability space, and let $f \in L_1(\Omega, \mu)$ be such that $a < f(x) < b$ for all $x \in \Omega$. Then, if $\varphi\colon (a,b) \to \mathbb{R}$ is convex,

$$\varphi\left(\int_\Omega f\, d\mu\right) \leq \int_\Omega \varphi \circ f\, d\mu$$

Proof. Set

$$t := \int_\Omega f\, d\mu,$$

and note that $a < t < b$. Note that, by a restatement of $(*)$, if $a < s < t < u < b$, then

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}.$$

Setting

$$\beta := \sup_{s \in (a,t)} \frac{\varphi(t) - \varphi(s)}{t - s},$$

it follows that

$$\beta \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

for all $u \in (t,b)$. Thus, for all $a < s < b$, we have

$$\varphi(s) \geq \varphi(t) + \beta(s - t).$$

In particular, this holds for all $s = f(x)$, where $x \in \Omega$, so that

$$\varphi(f(x)) \geq \varphi(t) + \beta f(x) - \beta t.$$

Integrating, and using the fact that t is a constant, we get

$$\int_\Omega \varphi \circ f\, d\mu \geq \varphi\left(\int_\Omega f\, d\mu\right) + \underbrace{\beta \int_\Omega f\, d\mu - \beta t \mu(\Omega)}_{=0}.$$

Thus, we obtain

$$\varphi\left(\int_\Omega f\, d\mu\right) \leq \int_\Omega \varphi \circ f\, d\mu.$$

□

Jensen's inequality is incredibly powerful, as it allows us to establish a variety of other classic inequalities. For instance, if we set $\varphi(x) = e^x$, then Jensen's inequality becomes

$$e^{\int_\Omega f\,d\mu} \leq \int_\Omega e^f\,d\mu.$$

If $\Omega = \{p_1, \dots, p_n\}$, where $\mu(\{p_i\}) = \frac{1}{n}$ with $f(p_i) = x_i$, then this gives

$$e^{\frac{1}{n}(x_1 + \cdots + x_n)} \leq \frac{1}{n}(e^{x_1} + \cdots + e^{x_n}).$$

Setting $y_i \coloneqq e^{x_i}$, we recover the AM-GM inequality,

$$\left(\prod_{i=1}^n y_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n y_i.$$

More generally, if $\mu(\{p_i\}) = \alpha_i > 0$, and $\sum_{i=1}^n \alpha_i = 1$, we obtain

$$\prod_{i=1}^n y_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i y_i.$$

Definition. If $1 \leq p, q \leq \infty$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then we call p and q *conjugate exponents*. We use the convention that $\frac{1}{\infty} = 0$, so that $p = 1, q = \infty$ is a pair of conjugate exponents.

Theorem (Young's Inequality)**:** If p and q are conjugate exponents, then for any positive a, b, we have

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Proof. Note that $\frac{1}{p} = 1 - \frac{1}{q}$. Thus, since ln is a concave function,

$$\begin{aligned}\ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) &\geq \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q)\\ &= \ln(a) + \ln(b)\\ &= \ln(ab).\end{aligned}$$

Now, since e^x preserves order, we obtain Young's inequality by taking exponentials. □

In Real Analysis II, we used Young's Inequality to prove Hölder's Inequality and Minkowski's Inequality for the case of $x, y \in \mathbb{C}^n$.

Theorem (Hölder's Inequality for $\mathbb{C}^n$)**:** Let $x, y \in \mathbb{C}^n$. Then, if p and q are conjugate exponents,

$$\sum_{j=1}^n |x_j y_j| \leq \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}\left(\sum_{j=1}^n |y_j|^q\right)^{1/q}.$$

Theorem (Minkowski's Inequality for $\mathbb{C}^n$)**:** Let $x, y \in \mathbb{C}^n$. Then, for any $p \geq 1$,

$$\left(\sum_{j=1}^n |x_j + y_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^n |y_j|^p\right)^{1/p}.$$

We will prove these inequalities in the most general case — i.e., with integrals.

Hölder's Inequality

Theorem (Hölder's Inequality)**:** Let p and q be conjugate exponents with $1 < p < \infty$, and let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g\colon X \to [0, \infty]$ be measurable functions. Then,

$$\int_X fg\,d\mu \leq \left(\int_X f^p\,d\mu\right)^{1/p}\left(\int_X g^q\,d\mu\right)^{1/q}.$$

Proof. Set

$$\begin{aligned} A &:= \left(\int_X f^p\,d\mu\right)^{1/p} \\ B &:= \left(\int_X g^q\,d\mu\right)^{1/q}. \end{aligned}$$

We may safely assume that $0 < A, B < \infty$. Set

$$\begin{aligned} F &= \frac{f}{A} \\ G &= \frac{g}{B}, \end{aligned}$$

giving

$$\begin{aligned} \int_X F^p\,d\mu &= 1 \\ \int_X G^q\,d\mu &= 1. \end{aligned}$$

Now, if x is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$, then there exist s, t such that $F(x) = e^{s/p}$ and $G(x) = e^{t/q}$, as $e^x\colon \mathbb{R} \to (0, \infty)$ is surjective. Since $\frac{1}{p} + \frac{1}{q} = 1$, and the exponential function is convex, we get, from Jensen's Inequality,

$$\begin{aligned} e^{s/p}e^{t/q} &= e^{s/p + t/q} \\ &\leq \frac{1}{p}e^s + \frac{1}{q}e^t, \end{aligned}$$

and substituting, we have

$$F(x)G(x) \leq \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q$$

for all $x \in X$. Integrating, we have

$$\int_X FG\,d\mu \leq 1.$$

Substituting our definition for F and G, we get

$$\int_X fg\,d\mu \leq \left(\int_X f^p\,d\mu\right)^{1/p}\left(\int_X g^q\,d\mu\right)^{1/q}.$$

$\square$

Minkowski's Inequality

Theorem (Minkowski's Inequality)**:** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f, g\colon X \to [0, \infty]$ be such that

$$\begin{aligned}\int_X f^p \, d\mu &< \infty \\ \int_X g^p \, d\mu &< \infty.\end{aligned}$$

Then, for all $1 \leq p \leq \infty$,

$$\left(\int_X (f+g)^p \, d\mu\right)^{1/p} \leq \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}.$$

Proof. Write

$$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}.$$

Then, by Hölder's inequality, we have

$$\begin{aligned}\int_X f(f+g)^{p-1} \, d\mu &\leq \left(\int_X f^p \, d\mu\right)^{1/p} \left(\int_X (f+g)^{(p-1)q} \, d\mu\right)^{1/q} \\ \int_X g(f+g)^{p-1} \, d\mu &\leq \left(\int_X g^p \, d\mu\right)^{1/p} \left(\int_X (f+g)^{(p-1)q} \, d\mu\right)^{1/q}.\end{aligned}$$

Adding, and noting that $(p-1)q = p$, we have

$$\int_X (f+g)^p \, d\mu \leq \left(\int_X (f+g)^p \, d\mu\right)^{1/q} \left(\left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}\right) \tag{$*$}$$

By the convexity of t^p, for $0 < t < \infty$, we have

$$\left(\frac{1}{2}(f+g)\right)^p \leq \frac{1}{2}(f^p + g^p),$$

so the left side of $(*)$ is finite. Dividing, we have

$$\left(\int_X (f+g)^p \, d\mu\right)^{1/p} \leq \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}.$$

In the case of $p = 1$ or $p = \infty$, Minkowski's inequality follows from the triangle inequality for $|\cdot|$. □

The L_p-Spaces

Inspired by Minkowski's inequality, we consider a special class of normed space — the L_p-spaces — and show some of its important properties.

Definition. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $1 \leq p < \infty$. We define the space $L_p(\Omega, \mu)$ to be

$$L_p(\Omega, \mu) := \left\{ f\colon \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f|^p \, d\mu < \infty \right\}.$$

The norm on $L_p(\Omega, \mu)$ is defined by

$$\|f\|_{L_p} = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}.$$

Definition. If $f\colon \Omega \to \mathbb{C}$ is a measurable function, then an *essential bound* for f is a $C > 0$ such that

$$\mu(\{x \mid |f(x)| > C\}) = 0.$$

The *essential supremum* of f is the infimum of all such essential bounds.

We define the space $L_\infty(\Omega, \mu)$ to be the space of all essentially bounded functions. The norm on $L_\infty(\Omega, \mu)$ is defined by

$$\|f\|_{L_\infty} \coloneqq \operatorname{ess\,sup}(|f|).$$

Note that by Minkowski's inequality, the norm on $L_p(\Omega, \mu)$ is a bona fide norm, so we only need to verify that the L_p spaces are complete.

Theorem: The space $L_p(\Omega, \mu)$ is complete.

Proof. Let $1 \leq p < \infty$, and let $(f_n)_n$ be a Cauchy sequence in $L_p(\Omega, \mu)$. Then, there exists a subsequence such that $\|f_{n_i+1} - f_{n_i}\|_{L_p} < 2^{-i}$ for each i. We set

$$\begin{aligned} g_k &= \sum_{i=1}^{k} |f_{n_i+1} - f_{n_i}| \\ g &= \sum_{i=1}^{\infty} |f_{n_i+1} - f_{n_i}|. \end{aligned}$$

By Minkowski's inequality, we know that $\|g_k\|_{L_p} < 1$ for each k, meaning that by Dominated Convergence, $\|g\|_{L_p} \leq 1$, meaning that $g(x) < \infty$ μ-almost everywhere. Thus, the series

$$f(x) \coloneqq f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_i+1}(x) - f_{n_i}(x))$$

converges absolutely for almost every $x \in \Omega$. Set $f(x) = 0$ where the series does not converge. Now, since

$$f_{n_k} = f_{n_1} + \sum_{i=1}^{k-1} (f_{n_i+1} - f_{n_i}),$$

we have that

$$f(x) = \lim_{i \to \infty} f_{n_i}(x)$$

almost everywhere.

Now, we prove that $(f_n)_n$ converges to f in L_p. Let $\varepsilon > 0$. Then, there is some N such that $\|f_n - f_m\|_{L_p} < \varepsilon$ for all $n, m \geq N$. By Fatou's Lemma, we have

$$\begin{aligned} \int_\Omega |f - f_m|^p \, d\mu &= \int_\Omega \liminf_{i \to \infty} |f_{n_i} - f_m|^p \, d\mu \\ &\leq \liminf_{i \to \infty} \int_\Omega |f_{n_i} - f_m|^p \, d\mu \\ &\leq \varepsilon^p. \end{aligned}$$

Thus, $f - f_m \in L_p(\Omega, \mu)$, so $f \in L_p(\Omega, \mu)$, and $\|f - f_m\|_{L_p} \xrightarrow{m \to \infty} 0$.

Now, in the case where $p = \infty$, the proof is simpler. Let $(f_n)_n$ be a Cauchy sequence in $L_\infty(\Omega, \mu)$, and let A_k be the set of all x such that $|f_k(x)| \geq \|f_k\|_{L_\infty}$, and let $B_{m,n}$ be the set of all x such that $|f_n(x) - f_m(x)| \geq$

$\|f_m - f_n\|_{L_\infty}$.

Let E be the union of A_k and $B_{m,n}$ for all $k, m, n \geq 1$. Then, $\mu(E) = 0$, and on E^c, $(f_n)_n$ converges uniformly to the bounded function f. Defining $f(x) = 0$ for all $x \in E$, we have $f \in L_\infty(\Omega, \mu)$, and $\|f_n - f\|_{L_\infty} \to 0$. □

Finally, a useful fact about L_p spaces is that certain classes of functions are dense inside them.

Theorem: Let S be the set of all simple, measurable functions that have finite measure support — i.e., that

$$\mu(\{x \mid s(x) \neq 0\}) < \infty.$$

Then, for $1 \leq p < \infty$, S is dense in $L_p(\Omega, \mu)$.

Proof. By construction, $S \subseteq L_p(\Omega, \mu)$. Let $f \geq 0$, and $(s_n)_n$ be a sequence of positive simple functions converging pointwise increasing to f. Therefore, since $0 \leq s_n \leq f$, we have $s_n \in L_p(\Omega, \mu)$, so $s_n \in S$.

Now, since $|f - s_n|^p \leq f^p$, by Dominated Convergence, we have $\|f - s_n\|_p \to 0$, so f is in the L_p closure of S.

For the general case where f is complex, we split f into positive and negative components, as well as real and imaginary components, and repeat the same procedure. □