Metric Spaces and Open Sets

A **distance metric** is a way of measuring between two points in a set. The following are the requirements for the distance metric:

- $\forall x, y \in X, d(x, y) \in \mathbb{R}$ and $d(x, y) \geq 0$.
- $d(x, x) = 0$
- $d(x, y) = d(y, x)$ (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Some basic metrics on $\mathbb{R}^n$ are defined below:

- Euclidean Metric: $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.
- Discrete Metric: $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ otherwise.
- Taxicab Metric: $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.

A set with a distance metric is known as a **Metric Space**. A **Open Ball**, denoted $B_r(x) = \{y \in X \mid d(x, y) < r\}$. A set A is open if $\exists r > 0$ such that $B_r(x) \subseteq A$ for every $x \in A$.

A set is open iff it is a union of open balls.

Forward direction proof: Let A be an open set in X. Then, for all $x \in A$, $\exists r > 0$ such that $B_r(x) \subseteq A$. As $\bigcup B_r(x) = A$, this means A is the union of open balls. The backward direction proof is omitted.

Closed Sets in Metric Spaces

A **limit point** of a set is a point not necessarily in a set where $\forall r > 0, B_r(x) \cap A - \{x\} \neq \emptyset$, or that every ball around the point x intersects A at a point other than x. The set of all limit points of A is the **boundary** of A, denoted $\text{bd}(A)$. The closure of A is equal to $A \cup \text{bd}(A)$.

A set is closed if and only if it contains all its limit points.

A closed set is a set whose complement is open. A set can be closed, open, both, or neither. The proof of the statement above is as follows:

Let A be a closed set in the metric space X. Suppose A does not contain all its limit points. Then, $\exists x \in X$ such that $\forall r > 0, B_r(x) \cap (A - \{x\}) \neq \emptyset$. This means that $\overline{A}$ is not open in X, meaning that A isn't closed. Since we have reached a contradiction, we are forced to assume that every limit point of A is in A.
In the reverse direction, let A be a set in X that contains all its limit points. Then, $\forall x$ such that $\forall r > 0, B_r(x) \cap (A - \{x\}) \neq \emptyset$, $x \in A$. Then, for any point y not in A, $\exists s > 0$ such that $B_s(y) \notin A$. This means $\overline{A}$ is open in X, so A is closed.

Topology

A **topology** on a set is a definition of open subsets of the set, with the following conditions:

- The union of two open sets is open
- The finite intersection of two open sets is open
- The empty set and the whole set are open

The **discrete topology** on the set is the powerset of the set (essentially, every subset is open). The **indiscrete topology** on the set is one where only the emptyset and the whole set are open.

Functions

A **function** or a **map** corresponds elements of one set with elements of another set. They are denoted as follows:

$$f : X \to Y$$

Where X is the domain and Y is the codomain. Specifically we are interested in continuous functions, and what we can do with them. The following is the definition of a continuous function between two metric spaces:

> Let $f : X \to Y$ be a function between two metric spaces. Then, if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall y$ where $d_X(a, y) < \delta \to d_Y(f(a), f(y)) < \epsilon$, then f is continuous at a. A function is continuous is if it is continuous at every point.

We will prove that for $f : X \to Y$ that $f(B_r(k)) \subseteq B_s(f(k))$ if f is continuous. Since f is continuous, this means that $\forall \epsilon > 0, \exists \delta > 0$ such that if $d(a, y) < \delta$, then $d(f(a), f(y)) < \epsilon$. Therefore, if $y \in B_\delta(a)$ then $f(y) \in B_\epsilon(f(a))$. Therefore, $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$ for some δ.

> $f : X \to Y$ is continuous if and only if the preimage of any open set in the codomain is open in the domain.

Suppose f is continuous, and let B be an open subset of Y. Since B is open, $\forall b \in B, \exists r > 0$ such that $B_r(b) \subseteq B$. Since f is continuous, $\exists s > 0$ such that $f(B_s(a)) \subseteq B_r(b) \subseteq B$ for some A in the preimage of B. Therefore, $\exists s > 0$ such that $B_s(a) \subseteq f^{-1}(B)$ for all $a \in f^{-1}(B)$, which means $f^{-1}(B)$ is open.

Suppose the preimage of every open set in Y is open in X. Let $B \subseteq Y$ be open. Since by assumption, $f^{-1}(B)$ is open, $\forall a \in B, \exists s > 0$ such that $B_s(a) \subseteq f^{-1}(B)$. By the definition of preimage, we have that $f(B_s(a)) \subseteq B$. Since $f(a) \in f(B_s(a)) \subseteq B$, we have that $\exists r > 0$ such that $B_r(f(a)) \subseteq B$. By letting $\epsilon = r$ and $\delta = s$, we get that $y \in B_s(a) \to f(y) \in B_r(f(a))$, meaning that $d(a, y) < \delta \to d(f(a), f(y)) < \epsilon$.

The definition of a function between topological spaces is the above definition (the preimage of any open set is open).

Properties of Topological Spaces

The **subspace topology** is defined as follows.

> Let $A \subseteq X$ where X is a topological space. A subset $W \subseteq A$ is open if $W = A \cap V$ for some open subset V in X.

We can show the **restriction lemma**, defined as follows and proof after.

> Let $f : X \to Y$ be a continuous function. Let $A \subseteq X$ and $B \subseteq Y$ such that $f(A) \subseteq B$. Then, $f|_A : A \to B$, the function with domain restricted to A, is continuous, where A and B are given the subspace topology.

Let W be an open subset of B. Then, $W = B \cap D$ for some D open in X. So, $f^{-1}(W) = f^{-1}(B \cap D) = f^{-1}(B) \cap f^{-1}(D)$ by rules of set algebra. By definition, since $f(A) \subseteq B$, and $g(A) = f(A)$ where $g = f|_A$, we have that $A \subseteq f^{-1}(B) = g^{-1}(B)$. Therefore, $A \cap f^{-1}(D) \subseteq f^{-1}(B) \cap f^{-1}(D) = g^{-1}(B) \cap f^{-1}(D)$, and that $g^{-1}(B) \subseteq A$, so $A \cap f^{-1}(D) = f^{-1}(W)$. Since $f^{-1}(D)$ is open by the definition of a continuous function, we have that $A \cap f^{-1}(D)$ must be open, meaning that g is continuous.

Connectedness

A set X is **disconnected** if there do not exist two nonempty open subsets A, B where $A \cup B = X$ and $A \cap B = \emptyset$.

> A set is disconnected iff there exists a nonempty open proper subset which is both open and closed

Proving in the forwards direction, let X be a disconnected set. Then, $\exists A, B$ open in X where $A \cup B = X$ and $A \cap B = \emptyset$, and A, B are nonempty. This means that A is open, and $\overline{A} = B$ is also open, meaning A is closed. Additionally, A cannot equal the whole set as B must be nonempty. Therefore, A is a nonempty proper subset of X which is both open and closed.

Proving in the backward direction, suppose $A \subseteq X$ is a nonempty proper subset which is both open and closed. Then, $A \cup \overline{A} = X$ by definition, and $\overline{A}$ is open. Since A is a proper subset, $\overline{A} \neq \emptyset$, and by definition of complement, $A \cap \overline{A} = \emptyset$, so X is disconnected.

> The continuous image of a connected set is connected.

Let $f : X \to Y$ be a continuous function, and let X be connected. Suppose toward contradiction that $f(X)$ is disconnected. Then, $\exists A, B \subseteq f(X)$ where A and B are nonempty, open sets in $f(X)$ whose disjoint union is equal to $f(X)$. Therefore, $f(X) = A \cup B$ where $A \cap B = \emptyset$. Then, $f^{-1}(f(X)) = f^{-1}(A \cup B)$, and $X \subseteq f(X)$, so $X = f^{-1}(A \cup B)$. By the rules of sets, we then have that $X = f^{-1}(A) \cup f^{-1}(B)$, and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open in X and nonempty, while $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(\emptyset) = \emptyset$. So X is disconnected, and we reach a contradiction.

Compactness

The continuous image of a compact set is compact.

Let $f : X \to Y$ be a continuous function where X is compact. Let $F = \{B_i\}$ be an open cover of $f(X)$. Then, $f(X) = \bigcup_{B_i \in F} B_i$ where $B_i \subseteq X$ is open. Taking inverses, we get that $f^{-1}(f(X)) = f^{-1}\left(\bigcup_{B_i \in F} B_i\right)$, and by the rules of sets, we get that $X = \bigcup_{B_i \in F} f^{-1}(B_i)$. Since X is compact, this means $\exists F' \subseteq F$ where F' is finite. So, we have that $f(X) = f\left(\bigcup_{B_i \in F'} f^{-1}(B_i)\right) = \bigcup_{B_i \in F'} f(f^{-1}(B_i)) \subseteq \bigcup_{B_i \in F'} B_i$. So, every open cover of $f(X)$ has a finite subcover, meaning $f(X)$ is compact.

Any closed subset of a compact space is compact.

Let X be a compact topological space and $A \subseteq X$ be closed. We will construct an open cover $F = \{B_i\}$ of A using the second definition, or that $A \subseteq \bigcup_{B_i \in F} B_i$ where $B_i \subseteq X$ is open. By the definition of closed, we have that $\overline{A}$ is open in X, meaning that $A \cup \overline{A} = X$, or $\bigcup F \cup \overline{A} = X$. Since X is compact, we have that $\exists F'$ finite in $\bigcup F \cup \overline{A}$ such that $X = \bigcup F'$, where $\overline{A}$ is or is not in F'. Therefore, we have that $A =\subseteq \bigcup F'$ as F' is an open cover of X, so A is compact.

Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ be nested, nonempty, closed subsets of a compact topological space X. Show that their intersection, $\bigcap B_i$ is nonempty.

Suppose toward contradiction that $\bigcap B_i = \emptyset$. Then, $X = \overline{\bigcap B_i} = \bigcup \overline{B_i}$. Since X is compact, and $\overline{B_i}$ is open by definition of closed sets, this means $F = \{\overline{B_i}\}$ is an open cover of X that has a finite subcover $F' = \{B_{i_1}, B_{i_2}, \dots, B_{i_k}\}$ for some max value k. Therefore, this means $X = \bigcup F'$, meaning $\overline{X} = \emptyset = \bigcap B_i = B_{i_k}$ by the definition of intersection, meaning B_{i_k} is empty, which violates one of our assumptions and we have reached a contradiction. Therefore, we are forced to conclude that $\bigcap B_i$ is nonempty.

Every nonempty compact subset of $\mathbb{R}$ contains a minimum and maximum value.

Suppose C is a compact subset of $\mathbb{R}$ that has no minimum. Let $F = \{(b, \infty) : b \in C\}$. Then, $C \subseteq \bigcup F$, as any element has an element less than it in C, and since C is compact, $\exists F'$ finite in F such that $C = \bigcup F'$. Then, $F' = \{(b_1, \infty), (b_2, \infty), \dots, (b_n, \infty)\}$. Without loss of generality, let $b_1 = \min\{b_1, b_2, \dots, b_n\}$. Therefore, $C \subseteq (b_1, \infty)$. However, $b_1 \in C$ by definition, but $C \subseteq (b_1, \infty)$ is a subset of a set that does not contain b_1, meaning that b_1 is both an element of and not an element of C, which yields our contradiction. Therefore, C must have a minimum value.

Suppose $C \subset \mathbb{R}$ does not have a maximum. Let $F = \{(-\infty, a) : a \in C\}$. By this construction, $C \subseteq \bigcup F$, as C not having a maximum means every element has an element greater than it in C, meaning that $\exists F'$ finite in F such that $C \subseteq \bigcup F'$. Therefore, $F' = \{(-\infty, a_1), (-\infty, a_2), \dots, (-\infty, a_n)\}$. Without loss of generality, let $a_1 = \max\{a_1, a_2, \dots, a_n\}$. Then, $C \subseteq (-\infty, a_1)$ by the definition of intervals. However, $a_1 \in C$, but $C \subseteq (-\infty, a_1)$, implying C does not contain a_1, which means a_1 is both in C and not in C, which is a contradiction. Therefore, C contains a maximum value.

A continuous function maps limit points to limit points.

Let $f : X \to Y$ be a function that has the property where p is a limit point of A implies $f(p)$ is a limit point of $f(A)$, and $B \subseteq Y$ be closed. Suppose toward contradiction that $f^{-1}(B)$ is not closed. Then, there must be a limit point $p \in X$ of $f^{-1}(B)$ where $p \notin f^{-1}(B)$. Therefore, $p \in \overline{f^{-1}(B)} \to p \in f^{-1}(\overline{B})$. So, $f(p) \in \overline{B}$. However, since p is a limit point of $f^{-1}(B)$, $f(p)$ must be a limit point of $f(f^{-1}(B))$. As $f(f^{-1}(B)) \subseteq B$, $f(p)$ must be a limit point of B, and since B is closed, $f(p) \in B$. Thus, $f(p) \in B \wedge f(p) \in \overline{B}$, which is a contradiction. Therefore, $f^{-1}(B)$ must be closed, so f is continuous by the continuous map property.

A function where the preimage of every closed set is closed is continuous.

Let $f : X \to Y$ be a function where if $B \subseteq Y$ is closed, $f^{-1}(B)$ is closed in X. Since B is closed, $\overline{B}$ is open in Y by definition. Then, $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ is open in X as $f^{-1}(B)$ is closed in X by assumption. Therefore, f is continuous.

If X and Y are homeomorphic topological spaces, then X is simply connected if and only if Y is simply connected.

Let $f : X \to Y$ be a homeomorphism, and suppose X is simply connected. Then, X is path connected and every loop is null-homotopic. Since X is path connected, $\exists p : I \to X$ such that p is continuous. Then, $f \circ p : I \to Y$ is continuous as it is a composition of continuous functions, meaning that Y is path connected. Similarly, since every loop in X is null-homotopic, this means that for $\ell : S^1 \to X$, $\exists H : S^1 \times I \to X$ such that $H_0(x) = \ell(x)$ and $H_1(x) = b$. Therefore, for $f \circ \ell : S^1 \to Y$, $\exists G : S^1 \times I \to Y$ defined as $G_t(x) = f(H_t(x))$. Since $G_0(x) = f(H_0(x)) = f \circ \ell(x)$ and $G_1(x) = f(H_1(x)) = f(b)$, we have that $f \circ \ell$ is null-homotopic, meaning that Y is simply connected. Since f is a homeomorphism, we have that f^{-1} is continuous, meaning that the previous two proofs also apply to the reverse direction by substituting f^{-1} for f.

If $\sim$ is an equivalence relation on X and X is path connected, then $X/\sim$ is path connected

Since X is path connected, we have that $\exists f : I \to X$ for all $a, b \in X$ where f is continuous.

The quotient map $q : X \to X/\sim$ is continuous by assumption.

So, $q \circ f : I \to X/\sim$ is continuous as it is the composition of continuous functions, meaning that $X/\sim$ is path connected.

> Prove that for metric spaces (X, d_X) and (Y, d_Y) and X is compact, then every continuous function $f : X \to Y$ is uniformly continuous.

Let X and Y be metric spaces, $f : X \to Y$ be continuous, and X is compact. Since f is continuous, we have from a previous result that $\forall \epsilon > 0,\ \exists \delta > 0$ such that for any $x \in X$, $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. Consider the set defined as the following: $A = \bigcup_{y \in X} B_\delta(x)$ where $f(y) \in B_\epsilon(f(x))$ with the previous rules. Then, A is an open cover of X as every element of X is in A, and vice versa. Since A is an open cover of X and X is compact, this means X has a finite subcover, implying there is a set $F = \{y_1, y_2, \dots, y_n\}$ such that $\bigcup_{y \in F} B_\delta(x) = X$. So, there are values $\Delta = \{\delta_1, \delta_2, \dots, \delta_n\}$ corresponding to $d(x, y_1), d(x, y_2), \dots, d(x, y_n)$. By the set construction, this set must contain values of δ_i such that $y \in B_{\delta_i}(x) \to f(y) \in B_\epsilon(f(x))$ for every value of ϵ greater than zero. So, we can pick a value $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$, so f is uniformly continuous.

> Every path connected space is connected.

Suppose X is a disconnected set. Then, $X = A \cup B$ where $A, B \subseteq X$ are nonempty and open, and $A \cap B = \emptyset$. Consider the function $f : I \to X$. Since I is connected, f can only be continuous if $f(I)$ is also connected for every $a, b \in X$. Let $C = f(I) \cap A$ and $D = f(I) \cap B$. Since $X = A \cup B$ and A and B are nonempty, there must be $a \in A, b \in B$ such that $a, b \in f(I)$ for some f. Then, C and D are nonempty, and as A and B are open in X, C and D must be open in $f(I)$, and $C \cup D = (A \cup B) \cap f(I) = X \cap f(I) = f(I)$. Additionally, $C \cap D = A \cap B = \emptyset$, meaning that C and D are disjoint. So, $f(I)$ is disjoint, meaning f is not continuous, so X is not path connected.

> Let X_1 and X_2 be simply connected topological spaces. Show that $X_1 \times X_2$ is simply connected.

To show path connectedness, we need to show that $\exists f : I \to X_1 \times X_2$ such that f is continuous for any distinct $(a_1, a_2), (b_1, b_2) \in X_1 \times X_2$. We know that $\exists p_1 : I \to X_1$ that is continuous for any distinct $a_1, b_1 \in X_1$, and similarly for $p_2 : I \to X_2$ for any distinct $a_2, b_2 \in X_2$. We can define $f : I \to X_1 \times X_2$ as $f(x) = (p_1(x), p_2(x))$. Since both of the "component functions" of f are continuous for any distinct $(a_1, a_2), (b_1, b_2) \in X_1 \times X_2$, we know that f is continuous, meaning that $X_1 \times X_2$ is path connected.

Since X_1 is simply connected, we know that for any $\ell_1 : S^1 \to X_1$ and constant map $g_1 : S^1 \to X_1$, $\exists H : S^1 \times I \to X_1$ where $H_0(x) = \ell_1(x)$ and $H_1(x) = g_1(x)$. Similarly, since X_2 is simply connected, we know that for any $\ell_2 : S^1 \to X_2$ and constant map $g_2 : S^1 \to X_2$, $\exists G : S^1 \times I \to X_2$ such that $G_0(x) = \ell_2(x)$ and $G_1(x) = g_2(x)$. We can then construct a homotopy as follows. Let $F : S^1 \times I \to X_1 \times X_2$ be defined such that $F_0(x) = (\ell_1(x), \ell_2(x))$ and $F_1(x) = (g_1(x), g_2(x))$. F is a null homotopy as at F_0, we have a loop in $X_1 \times X_2$, and at F_1, we have a constant map. Therefore, $X_1 \times X_2$ is null-homotopic, meaning $X_1 \times X_2$ is simply connected.

Every compact set in a metric space is bounded and closed.

Let A be a compact subset of (X, d). For some $x \in A$, we have that $A \subseteq \bigcup_{k\in\mathbb{Z}^+} B_k(x)$, and since A is compact, this means there is a finite set $F = \{k_1, k_2, \dots, k_n\}$ such that $A \subseteq \bigcup_{k\in F} B_k(x)$. By the definition of open balls, this means $A \subseteq B_{k_n}(x)$ where $k_n = \max(F)$. So, A is bounded.

Let A be a compact subset of (X, d) and $p \in \overline{A}$. We can construct an open cover of A by the following: $A \subseteq \bigcup_{k\in\mathbb{Z}^+} \overline{\text{cl}\left(B_{1/k}(p)\right)}$. Since A is compact, we know that this must have a finite subcover, meaning that there is a maximum value of k, k', such that $A \subseteq \overline{\text{cl}(B_{1/k'}(p))}$. Therefore, we have that $B_{1/k'}(p) \subseteq \overline{A}$, meaning that $\overline{A}$ is open, so A is closed.