Math 395

Homework 6
Due: 3/28/2024

Name: Avinash Iyer

Collaborators: Antonio Cabello, Nora Manukyan

Problem 2

We will show that $\{1, \sqrt{5}, \sqrt{7}, \sqrt{35}\}$ is linearly independent.

Consider $\mathbb{Q}(\sqrt{5})$. We have $m_{\sqrt{5},\mathbb{Q}} = x^2 - 5$, which is irreducible by Eisenstein's criterion, and we have $\mathbb{Q}[x]/\langle x^2 - 5\rangle \cong \mathbb{Q}(\sqrt{5})$. Similarly, for $\mathbb{Q}(\sqrt{7})$, we have that $m_{\sqrt{7},\mathbb{Q}} = x^2 - 7$, which is also irreducible by Eisenstein's criterion, so $\mathbb{Q}[x]/\langle x^2 - 7\rangle \cong \mathbb{Q}(\sqrt{7})$.

Since $x^2 - 5 \neq x^2 - 7$, we have that $\mathbb{Q}(\sqrt{5}) \neq \mathbb{Q}(\sqrt{7})$, meaning $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\sqrt{5}, \sqrt{7})$ and $\mathbb{Q}(\sqrt{7}) \subset \mathbb{Q}(\sqrt{5}, \sqrt{7})$, meaning $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ is a simple field extension with basis $\{1, \sqrt{5}, \sqrt{7}, \sqrt{35}\}$.

Problem 4

Let $F = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$. Suppose $\alpha_i^2 \in \mathbb{Q}$ for all i. We will show that $\sqrt[3]{2} \notin F$.

Since $\alpha_i^2 \in \mathbb{Q}$, we have that $x^2 - \alpha_i^2 \in \mathbb{Q}[x]$ is irreducible, meaning each $\mathbb{Q}(\alpha_i)$ is of dimension 2 over $\mathbb{Q}$. Thus, we have $\mathbb{Q} \subset \mathbb{Q}(\alpha_1) \subset \mathbb{Q}(\alpha_1, \alpha_2) \subset \cdots \subset \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, meaning $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ is of dimension 2^n over $\mathbb{Q}$.

Suppose toward contradiction $\sqrt[3]{2} \in F$. Then, we have $x^3 - 2 \in \mathbb{Q}[x]$ is irreducible, meaning $\mathbb{Q}(\sqrt[3]{2})$ is of dimension 3 over $\mathbb{Q}$. This means $\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{Q}(\alpha_1, \dots, \alpha_n)$. However, since 3 does not divide 2^n, this cannot be the case.

Problem 5

We will show that $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$, then compute $(1+\theta)(1+\theta+\theta^2)$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$ for θ a root.

To start, we see that $x^3 - 2x - 2$ is a monic polynomial where $p = 2$, so by Eisenstein's criterion and Gauss's Lemma, $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$.