1.2.1

Determine whether the following statements are true or false:

- Every disconnected graph has an isolated vertex.
- A graph is connected if and only if some vertex is connected to all other vertices.
- The edge set of every closed trail can be partitioned into edge sets of cycles.
- If a maximal trail in a graph is not closed, then its endpoints have odd degree.

- False
- True
- False
- True

1.2.5

Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Explain why this allows us to conclude that no graph has a cut-vertex of degree 1.

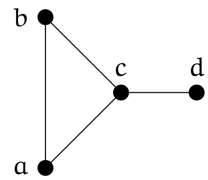
Suppose that $G - v$ is connected. Then, since G is connected, and $v \in V(G)$, it must be the case that v is connected to every component of $G - v$, meaning that it has a neighbor in every component of $G - v$ as $G - v$ is connected.

Now suppose that $G - v$ is disconnected, meaning that it has more than one component after removing v. Before, v must have been connected to every vertex in G as G was a simple connected graph, and afterwards $G - v$ is no longer connected, meaning that v is a cut-vertex. This means v must have been adjacent to a vertex in each component of $G - v$, as removing the incident edges on v along with v increased the number of components from the original 1 that was in G.

From this result, we can conclude that no cut-vertex has degree 1 as removing a vertex of degree 1 and its incident edges does not increase the number of components in G, since there is only one edge incident on a vertex of degree 1.

1.2.6

In the graph below, find all the maximal paths, maximal cliques, and maximal independent sets. Also, find all the maximum paths, cliques, and independent sets.



- The maximal paths are as follows:
  - d, c, b, a
  - d, c, a, b
  - a, b, c, d
  - b, a, c, d
  - b, c, a
  - c, b, a
  - a, c, b
- The maximal cliques are K_3 consisting of a, b, c and K_2 consisting of c, d.
- The maximal independent sets are $\{a, d\}$ and $\{b, d\}$.
- The maximum path is any of those paths listed above with length 4.
- The maximum clique is K_3.
- The maximum independent sets are those listed above with size 2.

1.2.8

Determine the values of m and n such that $K_{m,n}$ is Eulerian.

$$m, n \in 2\mathbb{Z}^+$$

1.2.10

Prove or disprove:

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

(a)

Let G be an Eulerian bipartite graph. Since G is Eulerian, it must contain an Eulerian cycle, meaning that as seen above, there are an even number of vertices, meaning that there are an even number of edges in G.

(b)

Let G be an Eulerian simple graph with an even number of vertices. Since G is Eulerian, this means there must be an Eulerian circuit C that traverses every edge exactly once in G. Every vertex in G must have even degree (or else we would require a backtrack in our Eulerian cycle, which is not a circuit); a simple pairing of the vertices would yield that we have $\lfloor n/2 \rfloor$ edges, and to complete the cycle we need $2(n/2) + 2k$ edges for n vertices and some integer k. Therefore, there must be an even number of edges.