

## Contents

<b>Introduction</b>	<b>1</b>
<b>Basics of Amenable Groups and Subgroups</b>	<b>1</b>
<b>Understanding Free Groups</b>	<b>3</b>
Groups specified by Generating Sets . . . . .	3
Free Groups . . . . .	3
Free Groups, Free Products, and the Ping Pong Lemma . . . . .	5
<b>States and Means on <math>\ell_\infty(G)</math></b>	<b>10</b>
<b>Using Invariant States</b>	<b>16</b>
<b>Følner's Condition and Invariant Approximate Means</b>	<b>19</b>
Equivalence between Means and Approximate Means . . . . .	23

## Introduction

This is going to be a part of my Honors thesis independent study, focused on amenability and  $C^*$ -algebras. This section of notes will be a deeper dive into group amenability. These notes will be taken from the notes my professor has prepared on group amenability, with supplementation from Volker Runde's *Lectures on Amenability* and Pierre de la Harpe's *Topics in Geometric Group Theory*.

I do not claim any of this work to be original.

## Basics of Amenable Groups and Subgroups

Let  $G$  be a group, with  $P(G)$  denoting the power set.

**Definition.** An invariant mean on  $G$  is a set function  $m: P(G) \rightarrow [0, 1]$ , which satisfies, for all  $t \in G$  and  $E, F \subseteq G$ ,

- (1)  $m(G) = 1$ ;
- (2)  $m(E \sqcup F) = m(E) + m(F)$ ;
- (3)  $m(tE) = m(E)$ .

We say  $G$  is amenable if it admits a mean.

We can also say that  $m$  is a translation-invariant probability measure on  $(G, P(G))$ .

**Proposition** (Amenability of Subgroups and Quotient Groups): Let  $G$  be amenable, with  $H \leq G$ .

- (1)  $H$  is amenable;
- (2) for  $H \trianglelefteq G$ ,  $G/H$  is amenable.

*Proof.*

- (1) Let  $R$  be a right transversal for  $H$  (i.e., selecting one element of each right coset of  $H$  to make up  $R$ ).

If  $m$  is a mean for  $G$ , we set

$$\lambda: P(H) \rightarrow [0, 1]$$

by  $\lambda(E) = m(ER)$ . We have

$$\begin{aligned}\lambda(H) &= m(HR) \\ &= m(G) \\ &= 1.\end{aligned}$$

We claim that if  $E \cap F = \emptyset$ , then  $ER \cap FR = \emptyset$ , since if we suppose toward contradiction that  $ER \cap FR \neq \emptyset$ , then  $xr_1 = yr_2$  for some  $x \in E, y \in F$  and  $r_1, r_2 \in R$ . Then, we must have  $r_2r_1^{-1} = y^{-1}x \in H$ , meaning  $r_1 = r_2$  and  $x = y$ , which means  $E \cap F \neq \emptyset$ .

Thus, we have

$$\begin{aligned}\lambda(E \sqcup F) &= m((E \sqcup F)R) \\ &= m(ER \sqcup FR) \\ &= m(ER) + m(FR) \\ &= \lambda(E) + \lambda(F),\end{aligned}$$

and

$$\begin{aligned}\lambda(sE) &= m(sER) \\ &= m(ER) \\ &= \lambda(E).\end{aligned}$$

(2) For the canonical projection map  $\pi: G \rightarrow G/H$  defined by  $\pi(t) = tH$ , we define

$$\lambda: P(G/H) \rightarrow [0, 1]$$

by  $\lambda(E) = m(\pi^{-1}(E))$ . We have

$$\begin{aligned}\lambda(G/H) &= m(\pi^{-1}(G/H)) \\ &= m(G) \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(E \sqcup F) &= m(\pi^{-1}(E \sqcup F)) \\ &= m(\pi^{-1}(E) \sqcup \pi^{-1}(F)) \\ &= m(\pi^{-1}(E)) + m(\pi^{-1}(F)) \\ &= \lambda(E) + \lambda(F).\end{aligned}$$

To show translation-invariance, we let  $sH = \pi(s) \in G/H$ , and  $E \subseteq G/H$ . Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for  $r \in s\pi^{-1}(E)$ , we have  $r = st$  for  $\pi(t) \in E$ , so  $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$ .

Additionally, if  $r \in \pi^{-1}(\pi(s)E)$ , then  $\pi(r) \in \pi(s)E$ , so  $\pi(s^{-1}r) \in E$ , and  $s^{-1}r \in \pi^{-1}(E)$ . Thus, we have

$$\begin{aligned}\lambda(\pi(s)E) &= m(\pi^{-1}(\pi(s)E)) \\ &= m(s\pi^{-1}(E)) \\ &= m(\pi^{-1}(E)) \\ &= \lambda(E).\end{aligned}$$

□

## Understanding Free Groups

In the Tarski's Theorem notes, we discussed a little bit the ramifications of the free group on two generators being paradoxical. In order to better understand free groups, we will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory* and Clara Löh's *Geometric Group Theory: An Introduction*.

### Groups specified by Generating Sets

**Definition.** Let  $G$  be a group and  $S \subseteq G$  be a subset. The subgroup generated by  $S$  is the intersection of all subgroups of  $G$  that contain  $S$ . We write  $\langle S \rangle_G$ . We say  $S$  generates  $G$  if  $\langle S \rangle_G = G$ .

A group is called finitely generated if it contains a finite subset that contains the group in question.

**Definition** (Characterization of a Generated Subgroup). We can characterize a generated subgroup by  $S$  as follows:

$$\langle S \rangle_G = \{s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n} \mid n \in \mathbb{N}, s_1, \dots, s_n \in S, \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}\}.$$

**Example** (Generating Sets).

- If  $G$  is a group, then  $G$  is a generating set of  $G$ .
- The trivial group is generated by the empty set.
- The set  $\{1\}$  generates  $\mathbb{Z}$ , as does  $\{2, 3\}$ . However,  $\{2\}$  and  $\{3\}$  alone do not generate  $\mathbb{Z}$ .
- Let  $X$  be a set. The symmetric group  $S_X$  is finitely generated if and only if  $X$  is finite.

### Free Groups

**Definition.** Let  $S$  be a set. A group  $F$  containing  $S$  is said to be freely generated if, for every group  $G$  and every map  $\varphi: S \rightarrow G$ , there is a unique group homomorphism  $\bar{\varphi}: F \rightarrow G$  extending  $\varphi$ . The following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & G \\ \downarrow \iota & \nearrow \bar{\varphi} & \\ F & & \end{array}$$

A group is free if it contains a free generating set.

**Example.**

- The additive group  $\mathbb{Z}$  is freely generated by  $\{1\}$ . The additive group  $\mathbb{Z}$  is *not* freely generated by  $\{2, 3\}$ , or  $\{2\}$ , or  $\{3\}$ . In particular, not every generating set of a group contains a free generating set.
- The trivial group is freely generated by the empty set.
- Not every group is free — the additive groups  $\mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z} \times \mathbb{Z}$  are not free.

We will use the universal property of free groups to show their uniqueness up to isomorphism.

**Proposition:** Let  $S$  be a set. Then, there is at most one group freely generated by  $S$  up to isomorphism.

*Proof.* Let  $F$  and  $F'$  be two groups freely generated by  $S$ , with inclusions of  $\varphi$  and  $\varphi'$  respectively. Because  $F$  is freely generated by  $S$ , there is a group homomorphism  $\bar{\varphi}': F \rightarrow F'$  that extends  $\varphi$  — i.e., that  $\bar{\varphi}' \circ \varphi = \varphi'$ .

Similarly, there is a group homomorphism  $\bar{\varphi}: F' \rightarrow F$  with  $\bar{\varphi} \circ \varphi' = \varphi$ .

$$\begin{array}{ccc}
 S & \xrightarrow{\varphi'} & F' \\
 \varphi \downarrow & \nearrow \overline{\varphi'} & \\
 F & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 S & \xrightarrow{\varphi} & F \\
 \varphi' \downarrow & \nearrow \overline{\varphi} & \\
 F' & & 
 \end{array}$$

We will show that  $\overline{\varphi} \circ \overline{\varphi'} = \text{id}_{F'}$  and  $\overline{\varphi'} \circ \overline{\varphi} = \text{id}_F$ . The composition  $\overline{\varphi} \circ \overline{\varphi'}$  is a group homomorphism that makes the following diagram commute.

$$\begin{array}{ccc}
 S & \xrightarrow{\varphi} & F \\
 \varphi \downarrow & \nearrow \overline{\varphi} \circ \overline{\varphi'} & \\
 F & & 
 \end{array}$$

Since  $\text{id}_F$  is a group homomorphism contained in this diagram, and  $F$  is freely generated by  $S$ , we must have  $\overline{\varphi} \circ \overline{\varphi'} = \text{id}_F$ . Similarly, we must have  $\overline{\varphi'} \circ \overline{\varphi} = \text{id}_{F'}$ .  $\square$

**Theorem** (Existence of Free Groups): Let  $S$  be a set. There exists a group freely generated by  $S$ . This group is unique up to isomorphism.

*Proof.* We want to construct a group consisting of “words” made up of the elements of  $S$  and their “inverses,” then modding out by the natural cancellation rules.

We consider the alphabet

$$A = S \cup \hat{S}.$$

Here,  $\hat{S} = \{\hat{s} \mid s \in S\}$  is a disjoint copy of  $S$ , such that  $\hat{s}$  will serve as the inverse of  $s$  in the group we will construct.

We define  $A^*$  to be the set of all finite sequences over the alphabet  $A$ , including the empty word  $\epsilon$ . We define the operation  $A^* \times A^* \rightarrow A^*$  by concatenation. This operation is associative with neutral element  $\epsilon$ .

We define

$$F(S) = A^* / \sim,$$

where  $\sim$  is the equivalence relation generated by, for all  $x, y \in A^*$  and  $s \in S$ ,  $xs\hat{s}y \sim xy$  and  $x\hat{s}s y \sim xy$ .

We denote the equivalence classes with respect to  $\sim$  by  $[\cdot]$ .

Concatenation induces a well-defined operation  $F(S) \times F(S) \rightarrow F(S)$  by

$$[x][y] = [xy]$$

for  $x, y \in A^*$ .

We claim that  $F(S)$  with the defined concatenation is a group. We can see that  $[\epsilon]$  is a neutral element for the operation, and associativity of the operation is inherited from the associativity of the operation on  $A^*$ .

To find inverses, we define  $I: A^* \rightarrow A^*$  by  $I(\epsilon) = \epsilon$ , and

$$\begin{aligned}
 I(sx) &= I(x)\hat{s} \\
 I(\hat{s}x) &= I(x)s
 \end{aligned}$$

for all  $x \in A^*$  and  $s \in S$ . Induction shows that  $I(I(x)) = x$ , and

$$[I(x)][x] = [I(x)x]$$

$$= [\epsilon]$$

for all  $x \in A^*$ . Thus, we must also have

$$\begin{aligned} [x] [I(x)] &= [I(I(x))] [I(x)] \\ &= [\epsilon]. \end{aligned}$$

Thus, we see that there are inverses in  $F(S)$ .

To see that  $F(S)$  is freely generated by  $S$ , we let  $\iota: S \rightarrow F(S)$  be the map given by sending a letter in  $S \subseteq A^*$  to its equivalence class in  $F(S)$ . By construction,  $F(S)$  is generated by the subset  $\iota(S) \subseteq F(S)$ .

We do not know yet that  $\iota$  is injective, so we take a bit of a detour. We show that for every group  $G$  and every map  $\varphi: S \rightarrow G$ , there is a unique group homomorphism  $\overline{\varphi}: F(S) \rightarrow G$  such that  $\overline{\varphi} \circ \iota = \varphi$ .

We construct a map  $\varphi^*: A^* \rightarrow G$  inductively by

$$\begin{aligned} \epsilon &\mapsto e \\ sx &\mapsto \varphi(s)\varphi^*(x) \\ \hat{s}x &\mapsto (\varphi(s))^{-1}\varphi^*(x) \end{aligned}$$

for all  $s \in S$  and  $x \in A^*$ . We can see that, since the definition of  $\varphi^*$  is compatible with the generating set of  $\sim$ , it is compatible with the equivalence relation of  $\sim$  on  $A^*$ . Additionally, we can see that  $\varphi^*(xy) = \varphi^*(x)\varphi^*(y)$  for all  $x, y \in A^*$ . Thus,

$$\begin{aligned} \overline{\varphi}: F(S) &\rightarrow G \\ [x] &\mapsto [\varphi^*(x)], \end{aligned}$$

is, as constructed, a group homomorphism, with  $\overline{\varphi} \circ \iota = \varphi$ . Since  $\iota(S)$  generates  $F(S)$ , this group homomorphism is unique.

We must now show that  $\iota$  is injective.

Let  $s_1, s_2 \in S$ . Consider the map  $\varphi: S \rightarrow \mathbb{Z}$  given by  $\varphi(s_1) = 1$  and  $\varphi(s_2) = -1$ . The corresponding homomorphism  $\overline{\varphi}: F(S) \rightarrow G$  satisfies

$$\begin{aligned} \overline{\varphi}(\iota(s_1)) &= \varphi(s_1) \\ &= 1 \\ &\neq -1 \\ &= \varphi(s_2) \\ &= \overline{\varphi}(\iota(s_2)), \end{aligned}$$

meaning  $\iota(s_1) \neq \iota(s_2)$ . Thus,  $\iota$  is injective. □

## Free Groups, Free Products, and the Ping Pong Lemma

We can now consider free groups in a more categorical context — specifically, as a special type of free object. Whereas the previous section drew information from Clara Löh's *Geometric Group Theory: An Introduction*, this section will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory*. Specifically, we are focused on chapter 2, which discusses free products, free groups, and the ping pong lemma (which will provide a more general proof of the paradoxicality of  $SO(3)$ ).

**Definition** (Free Monoid). A monoid is a set with multiplication that is associative and includes a neutral element.

Given a set  $A$ , the free monoid on  $A$  is the set  $W(A)$  of finite sequences of elements of  $A$  (also known as words). We write an element of  $W(A)$  as  $w = a_1 a_2 \cdots a_n$ , where each  $a_j \in A$ . We identify  $A$  with the subset of  $W(A)$  of words with length 1.

**Definition** (Free Product). Let  $(\Gamma_i)_{i \in I}$  be a family of groups. Set

$$\begin{aligned} A &= \coprod_{i \in I} \Gamma_i \\ &= \{(g_i, i) \mid g_i \in \Gamma_i, i \in I\} \end{aligned}$$

to be the coproduct of this family.

Let  $\sim$  be the equivalence relation generated by

$$\begin{aligned} we_i w' &\sim ww' && \text{where } e_i \in \Gamma_i \text{ is the neutral element} \\ wabw' &\sim wcw' && \text{where } a, b, c \in \Gamma_i, c = ab \text{ for some } i \in I \end{aligned}$$

for all  $w, w' \in W(A)$ . The quotient  $W(A)/\sim$  with the operation of concatenation is a group, which is known as the free product of the groups  $\{\Gamma_i\}_{i \in I}$ . We write it as

$$\star_{i \in I} \Gamma_i.$$

The inverse of the equivalence class for  $w = a_1 a_2 \cdots a_n$  is  $w^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$ . The neutral element is  $\epsilon$ , which is the empty word.

A word  $w = a_1 a_2 \cdots a_n \in W(A)$  with  $a_j \in \Gamma_{i_j}$  is said to be reduced if  $i_{j+1} \neq i_j$  and  $a_j$  is not the neutral element of  $\Gamma_{i_j}$ .

**Proposition** (Existence of the Free Product): Let  $\{\Gamma_i\}_{i \in I}$  be a family of groups,  $A = \coprod_{i \in I} \Gamma_i$ , and  $\star_{i \in I} \Gamma_i = W(A)/\sim$  be as above.

Then, any element in the free product  $\star_{i \in I} \Gamma_i$  is represented by a unique reduced word in  $W(A)$ .

*Proof.*

EXISTENCE: Consider an integer  $n \geq 0$  and a reduced word  $w = a_1 a_2 \cdots a_n$  in  $W(A)$ , an element  $a \in A$ , and the word  $aw \in W(A)$ . We set

$$\mathcal{R}(aw) = \begin{cases} w & a = e_i \\ aa_1 a_2 \cdots a_n & a \in \Gamma_i, a \neq e_i, i \neq k \\ ba_2 \cdots a_n & a \in \Gamma_k, aa_1 = b \neq e_k \\ a_2 \cdots a_n & a \in \Gamma_k, a_1 = a^{-1} \in \Gamma_k \end{cases},$$

where  $k$  is the index for which  $a_1 \in \Gamma_k$ .

Then,  $\mathcal{R}(aw)$  is yet another reduced word, and  $\mathcal{R}(aw) \sim aw$ , meaning that any word  $w \in W(A)$  is equivalent to some reduced word by inducting on the length of  $w$ .

UNIQUENESS: For each  $a \in A$ , Let  $T(a) = \mathcal{R}(aw)$  be a self-map on the set of reduced words.

For each  $w = b_1 b_2 \cdots b_n$ , we set  $T(w) = T(b_1) T(b_2) \cdots T(b_n)$ . For  $a, b, c \in \Gamma_i$  with  $ab = c$ , we have  $T(a) T(b) = T(c)$ , and  $T(e_i) = \epsilon$  (the empty word) for all  $i \in I$ .

For each reduced word, notice that  $T(w) \epsilon = w$ .

Let  $w$  be some word in  $W(A)$  with  $w_1, w_2$  reduced words equivalent to  $w$ . Since  $w_1 \sim w_2$ , we have  $T(w_1) = T(w_2)$ , and

$$\begin{aligned} w_1 &= T(w_1) \epsilon \\ &= T(w_2) \epsilon \\ &= w_2. \end{aligned}$$

□

**Corollary:** Let  $\{\Gamma_i\}_{i \in I}$  and  $\Gamma = \star_{i \in I} \Gamma_i$  as above. For each  $i_0 \in I$ , the canonical inclusion

$$\iota: \Gamma_{i_0} \rightarrow \Gamma$$

is injective.

*Proof.* For any  $a \in \Gamma_{i_0} \setminus \{e_{i_0}\}$ ,  $\iota(a)$  is represented by a unique one-letter reduced word not equivalent to the empty word. □

Now that we have an understanding of free products, we can conceptualize the free group as a particular type of free product.

**Definition (Free Groups).** Let  $X$  be a set. The free group over  $X$  is the free product of a family of copies of  $\mathbb{Z}$  indexed by  $X$ , denoted  $F(X)$ .

Equivalently, the free group over  $X$  is

$$F(X) = \star_{a \in X} \langle a \rangle,$$

where  $\langle a \rangle$  denotes the cyclic group generated by the element  $a$ .

We can also identify  $F(X)$  with the set of reduced words in  $X \sqcup X^{-1}$  (as was done in the previous subsection).

The cardinality of  $X$  is called the rank of  $F(X)$ .

If  $\Gamma$  is a group, then a free subset of  $\Gamma$  is a subset  $X \subseteq \Gamma$  such that the inclusion  $X \hookrightarrow \Gamma$  extends to an isomorphism of  $\langle X \rangle_\Gamma$  onto  $F(X)$ .

We can now state and prove a universal property for free products (which naturally simplifies in the case of a free group.)

**Theorem (Universal Property for Free Products):** Let  $\Gamma$  be a group, and  $\{\Gamma_i\}_{i \in I}$  be a family of groups. Let  $\{h_i: \Gamma_i \rightarrow \Gamma\}_{i \in I}$  be a family of homomorphisms.

Then, there exists a unique homomorphism  $h: \star_{i \in I} \Gamma_i \rightarrow \Gamma$  such that the following diagram commutes for each  $i_0 \in I$ .

$$\begin{array}{ccc} \Gamma_{i_0} & \xrightarrow{h_{i_0}} & \Gamma \\ \downarrow \iota & \nearrow h & \\ \star_{i \in I} \Gamma_i & & \end{array}$$

In particular, if  $\Gamma$  is a group,  $X$  is a set, and  $\phi: X \rightarrow \Gamma$  is a set map, there exists a unique homomorphism  $\Phi: F(X) \rightarrow \Gamma$  such that  $\Phi(x) = \phi(x)$  for each  $x \in X$ .

*Proof.* For a reduced word  $w = a_1 a_2 \cdots a_n \in \star_{i \in I} \Gamma_i$  with  $a_j \in \Gamma_{i_j} \setminus \{e_{i_j}\}$ , and  $i_{j+1} \neq i_j$  for each  $j \in \{1, \dots, n-1\}$ , we set

$$h(w) = h_{i_1}(a_1) h_{i_2}(a_2) \cdots h_{i_n}(a_n),$$

which defines  $h$  uniquely in terms of  $h_i$ .  $\square$

Note that for any two sets  $X, Y$ , the universal property provides that any map  $X \rightarrow Y$  extends canonically to a group homomorphism,  $F(X) \rightarrow F(Y)$ .

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & F(Y) \end{array}$$

We can now prove an important lemma that will be useful in understanding paradoxical groups.

**Theorem** (Ping Pong Lemma): Let  $G$  be a group acting on a set  $X$ , and let  $\Gamma_1, \Gamma_2$  be subgroups of  $G$ . Let  $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ . Assume  $\Gamma_1$  contains at least 3 elements and  $\Gamma_2$  contains at least two elements.

Suppose there exist nonempty subsets  $X_1, X_2 \subseteq X$  with  $X_1 \Delta X_2 \neq \emptyset$ , such that for all  $\gamma_1 \in \Gamma_1$  with  $\gamma_1 \neq e_G$ , and for all  $\gamma_2 \in \Gamma_2$  with  $\gamma_2 \neq e_G$ ,

$$\begin{aligned} \gamma(X_2) &\subseteq X_1 \\ \gamma(X_1) &\subseteq X_2. \end{aligned}$$

Then,  $\Gamma$  is isomorphic to the free product  $\Gamma_1 \star \Gamma_2$ .

*Proof.* Let  $w$  be a nonempty reduced word spelled with letters from the disjoint union of  $\Gamma_1 \setminus \{e_G\}$  and  $\Gamma_2 \setminus \{e_G\}$ . We must show that the element of  $\Gamma$  defined by  $w$  is not the identity.

If  $w = a_1 b_1 a_2 b_2 \cdots a_k$  with  $a_1, \dots, a_k \in \Gamma_1 \setminus \{e_G\}$  and  $b_1, \dots, b_{k-1} \in \Gamma_2 \setminus \{e_G\}$ . Then,

$$\begin{aligned} w(X_2) &= a_1 b_1 \cdots a_{k-1} b_{k-1} a_k(X_2) \\ &\subseteq a_1 b_1 \cdots a_{k-1} b_{k-1}(X_1) \\ &\subseteq a_1 b_1 \cdots a_{k-1}(X_2) \\ &\vdots \\ &\subseteq a_1(X_2) \\ &\subseteq X_1. \end{aligned}$$

Since  $X_2 \not\subseteq X_1$ , this implies  $w \neq e_G$ .

If  $w = b_1 a_2 b_2 a_2 \cdots b_k$ , we select  $a \in \Gamma_1 \setminus \{e_G\}$ , and apply the previous argument to  $awa^{-1}$ . Since  $awa^{-1} \neq e_G$ , neither is  $w$ .

Similarly, if  $w = a_1 b_1 \cdots a_k b_k$ , we select  $a \in \Gamma_1 \setminus \{e_G, a_1^{-1}\}$ , and apply the argument to  $awa^{-1}$ , and if  $w = b_1 a_2 b_2 \cdots a_k$ , we select  $a \in \Gamma_1 \setminus \{e_G, a_k\}$ , and apply the argument to  $awa^{-1}$ .  $\square$

**Example.** We can use the Ping Pong Lemma to see that

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

generate a subgroup of  $SL(2, \mathbb{Z})$  which is free of rank 2.



**Corollary:** The special orthogonal group  $SO(3)$  contains a subgroup isomorphic to the free group on two generators.

To prove this, we state a different version of the Ping Pong Lemma that we will apply to a particular space.

**Theorem** (Ping Pong Lemma for Cyclic Groups): Let  $G$  act on a set  $X$ , and suppose there exist disjoint subsets  $A_+, A_-, B_+, B_- \subseteq X$  whose union is not all of  $X$ . If there exist elements  $a$  and  $b$  in  $G$  such that

$$\begin{aligned} a \cdot (X \setminus A_-) &\subseteq A_+ \\ a^{-1} \cdot (X \setminus A_+) &\subseteq A_- \\ b \cdot (X \setminus B_-) &\subseteq B_+ \\ b \cdot (X \setminus B_+) &\subseteq B_-, \end{aligned}$$

then it is the case that the group generated by  $a$  and  $b$  is free of rank 2.

*Proof of Corollary.* We let

$$\begin{aligned} a &= \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a^{-1} &= \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ b &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \\ b^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}. \end{aligned}$$

We specify

$$X = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} A_+ &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid k \in \mathbb{Z}, x \equiv 3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ A_- &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid k \in \mathbb{Z}, x \equiv -3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ B_+ &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid k \in \mathbb{Z}, z \equiv 3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\} \\ B_- &= \left\{ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid k \in \mathbb{Z}, z \equiv -3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}. \end{aligned}$$

To verify that the conditions of the Ping Pong Lemma hold, we calculate

$$\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \frac{1}{5^{k+1}} \begin{pmatrix} 3x + 4y \\ -4x + 3y \\ 5z \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \frac{1}{5^{k+1}} \begin{pmatrix} 3x - 4y \\ 4x + 3y \\ 5z \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \frac{1}{5^{k+1}} \begin{pmatrix} 5x \\ 3y - 4z \\ 4y + 3z \end{pmatrix} \quad (3)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \frac{1}{5^{k+1}} \begin{pmatrix} 5x \\ 3y + 4z \\ -4y + 3z \end{pmatrix}. \quad (4)$$

We verify that the conditions for the Ping Pong Lemma hold for each of these four conditions.

(1) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_-,$$

we see that  $k+1 \in \mathbb{Z}$ ,  $x' = 3x + 4y \equiv 3(-4x + 3y) \pmod{5}$ , and that  $z' = 5z \equiv 0 \pmod{5}$ .

(2) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin A_+,$$

we see that  $k+1 \in \mathbb{Z}$ ,  $x' = 3x - 4y \equiv -3(4x + 3y) \pmod{5}$ , and  $z' = 5z \equiv 0 \pmod{5}$ .

(3) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_-,$$

we see that  $k+1 \in \mathbb{Z}$ ,  $z' = 4y + 3z \equiv 3(3y - 4z) \pmod{5}$ , and  $x' = 5x \equiv 0 \pmod{5}$ .

(4) For any vector

$$\frac{1}{5^k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin B_+,$$

we see that  $k+1 \in \mathbb{Z}$ ,  $z' = -4y + 3z \equiv -3(3y + 4z) \pmod{5}$ , and  $x' = 5x \equiv 0 \pmod{5}$ .

Since we have verified that the conditions for the Ping Pong Lemma hold for each of the conditions, we have that  $\{a, b\} \subseteq SO(3)$  generate a group isomorphic to the free group on two generators.  $\square$

## States and Means on $\ell_\infty(G)$

**Definition.** Let  $G$  be a group.

(1) The space  $\mathcal{F}(G, \mathbb{R})$  is defined by

$$\mathcal{F}(G, \mathbb{R}) = \{f \mid f: G \rightarrow \mathbb{R} \text{ is a function}\}.$$

(2) A function  $f \in \mathcal{F}(G, \mathbb{R})$  is positive if  $f(x) \geq 0$  for all  $x \in G$ .

(3) A function  $f \in \mathcal{F}(G, \mathbb{R})$  is simple if  $\text{Ran}(f)$  is finite. We say

$$\Sigma = \{f: \mathcal{F}(G, \mathbb{R}) \mid f \text{ is simple}\}.$$

**Fact.**  $\Sigma \subseteq \mathcal{F}(G, \mathbb{R})$  is a subspace. To see this, if  $f, g$  are such that  $\text{Ran}(f), \text{Ran}(g)$  are finite, and  $\alpha \in \mathbb{R}$ , then

$$\text{Ran}(f + \alpha g) \leq \text{Ran}(f) + \text{Ran}(g),$$

so  $f + \alpha g$  has finite range.

**Definition.** For  $E \subseteq G$ , set

$$\mathbb{1}_E: G \rightarrow \mathbb{R}$$

defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

This is the characteristic function of  $E$ .

**Fact.**

$$\text{span}\{\mathbb{1}_E \mid E \subseteq G\} = \Sigma.$$

*Proof.* We see that  $\mathbb{1}_E \in \Sigma$  for any  $E \subseteq G$ , and  $\Sigma$  is a subspace.

If  $\phi \in \Sigma$ , with  $\text{Ran}(\phi) = \{t_1, \dots, t_n\}$  with  $t_i$  distinct, we set

$$E_i = \phi^{-1}(\{t_i\}),$$

meaning

$$\phi = \sum_{i=1}^n t_i \mathbb{1}_{E_i}.$$

□

**Definition.**

(1) A function  $f \in \mathcal{F}(G, \mathbb{R})$  is bounded if there exists  $M > 0$  such that  $\text{Ran}(f) \subseteq [-M, M]$ .

(2) The space  $\ell_\infty(G)$  is defined by

$$\ell_\infty(G) = \{f \in \mathcal{F}(G, \mathbb{R}) \mid f \text{ is bounded}\}.$$

(3) The norm on  $\ell_\infty(G)$  is defined by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

**Proposition:** The space  $\ell_\infty(G)$  is complete, Additionally,  $\bar{\Sigma} = \ell_\infty(G)$ .

*Proof.* Let  $(f_n)_n$  be Cauchy. For  $x \in G$ , it is the case that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n - f_m)(x)| \\ &\leq \|f_n - f_m\|, \end{aligned}$$

meaning  $(f_n(x))_n$  is Cauchy in  $\mathbb{R}$ . We define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . We must show that  $f \in \ell_\infty(G)$  and  $\|f_n - f\| \rightarrow 0$ .

$$\begin{aligned} |f(x)| &= \left| \lim_{n \rightarrow \infty} f_n(x) \right| \\ &= \lim_{n \rightarrow \infty} |f_n(x)| \\ &\leq \limsup_{n \rightarrow \infty} \|f_n\| \\ &\leq C, \end{aligned}$$

as Cauchy sequences are always bounded. Thus,  $\sup_{x \in G} |f(x)| \leq C$ .

Given  $\varepsilon > 0$ , we find  $N$  such that for all  $m, n \geq N$ ,  $\|f_n - f_m\| \leq \varepsilon$ . Thus, for  $x \in G$ , we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq \|f_n - f_m\| \\ &\leq \varepsilon. \end{aligned}$$

Taking  $m \rightarrow \infty$ , we get  $|f_n(x) - f(x)| \leq \varepsilon$  for all  $n \geq N$ , meaning  $\|f_n - f\| \leq \varepsilon$  for all  $n \geq N$ .

Now, for  $f \in \ell_\infty(G)$ , let  $\text{Ran}(f) \subseteq [-M, M]$  for some  $M > 0$ . Let  $\varepsilon > 0$ . Since  $[-M, M]$  is compact, it is totally bounded, so we can find intervals  $I_1, \dots, I_n$  with  $[-M, M] = \bigsqcup_{k=1}^n I_k$ , with the length of each  $I_k$  less than  $\varepsilon$ .

Set  $E_k = f^{-1}(I_k)$ . Pick  $t_k \in I_k$ . Then, we set

$$\phi = \sum_{i=1}^n t_k \mathbb{1}_{E_k}.$$

We see that  $\|\phi - f\| < \varepsilon$ . □

**Corollary:** For any  $f \in \ell_\infty(G)$ , there is a sequence  $(\phi_n)_n$  in  $\Sigma$  with  $\|\phi_n - f\| \rightarrow 0$ . If  $f \geq 0$ , then it is possible to select  $\phi_n \geq 0$ .

**Proposition:** Let  $G$  be a group. There is an action

$$G \xrightarrow{\lambda_s} \text{Isom}(\ell_\infty(G))$$

defined by

$$\lambda_s(f)(t) = f(s^{-1}t).$$

*Proof.* We have

$$\begin{aligned} \lambda_s(f + \alpha g)(t) &= (f + \alpha g)(s^{-1}t) \\ &= f(s^{-1}t) + \alpha g(s^{-1}t) \\ &= \lambda_s(f)(t) + \alpha \lambda_s(g)(t) \\ &= (\lambda_s(f) + \alpha \lambda_s(g))(t). \end{aligned}$$

Thus,  $\lambda_s$  is a linear operator.

We have

$$\|\lambda_s(f)\| = \sup_{t \in G} |\lambda_s(f)(t)|$$

$$\begin{aligned}
&= \sup_{t \in G} \left| f(s^{-1}t) \right| \\
&= \|f\|,
\end{aligned}$$

hence

$$\begin{aligned}
\|\lambda_s(f) - \lambda_s(g)\| &= \|\lambda_s(f - g)\| \\
&= \|f - g\|.
\end{aligned}$$

Thus,  $\lambda_s$  is an isometry.

We have

$$\begin{aligned}
\lambda_s \circ \lambda_r(f)(t) &= \lambda_r(f)(s^{-1}t) \\
&= f(r^{-1}s^{-1}t) \\
&= f((sr)^{-1}t) \\
&= \lambda_{sr}(f)(t),
\end{aligned}$$

meaning  $\lambda_s \circ \lambda_r = \lambda_{sr}$ . □

**Remark:** By a similar process, we find that  $\lambda_s(\mathbb{1}_E) = \mathbb{1}_{sE}$  for any subset  $E \subseteq G$  and  $s \in G$ .

**Definition.** A state on  $\ell_\infty(G)$  is a continuous linear functional  $\mu \in (\ell_\infty(G))^*$  that satisfies the following.

- (1)  $\mu$  is positive;
- (2)  $\mu(\mathbb{1}_G) = 1$ .

A state is called left-invariant if

$$\mu(\lambda_s(f)) = \mu(f).$$

**Example.** Let  $G$  be a group.

- If  $x \in G$ , then  $\delta_x: \ell_\infty(G) \rightarrow \mathbb{F}$  defined by

$$\delta_x(f) = f(x)$$

is a state. However, note that it is not necessarily invariant.

$$\begin{aligned}
\delta_x(\lambda_s(f)) &= \lambda_s(f)(x) \\
&= f(s^{-1}x) \\
&\neq f(x).
\end{aligned}$$

- If  $G$  is finite, then

$$\mu = \frac{1}{|G|} \sum_{x \in G} \delta_x$$

is an invariant state.

**Lemma** (Characterization of States):

- (1) If  $\mu$  is a state on  $\ell_\infty(G)$ , then

$$\|\mu\|_{\text{op}} = 1.$$

(2) If  $\mu \in (\ell_\infty(G))^*$  is such that

$$\begin{aligned}\|\mu\| &= \mu(\mathbb{1}_G) \\ &= 1,\end{aligned}$$

then  $\mu$  is positive and a state.

*Proof.*

(1) Given  $f \in \ell_\infty(G)$ , we have

$$\begin{aligned}\|f\| \mathbb{1}_G - f &\geq 0 \\ \|f\| \mathbb{1}_G + f &\geq 0,\end{aligned}$$

so

$$\begin{aligned}0 &\leq \mu(\|f\| \mathbb{1}_G - f) \\ &= \|f\| \mu(\mathbb{1}_G) - \mu(f) \\ 0 &\leq \mu(\|f\| \mathbb{1}_G + f) \\ &= \|f\| \mu(\mathbb{1}_G) + \mu(f).\end{aligned}$$

Thus, we have  $\pm\mu(f) \leq \|f\| \mu(\mathbb{1}_G) = \|f\|$ , so  $|\mu(f)| \leq \|f\|$ , so  $\|\mu\| \leq 1$ . Additionally, since  $\mu(\mathbb{1}_G) = 1$ , we must have  $\|\mu\| = 1$ .

(2) Suppose  $\|\mu\| = \mu(\mathbb{1}_G) = 1$ . Let  $f \geq 0$ . Set  $g = \frac{1}{\|f\|_u} f$ .

Then,  $\text{Ran}(g) \subseteq [0, 1]$ , and  $\text{Ran}(g - \mathbb{1}_G) \subseteq [-1, 1]$ , so  $\|g - \mathbb{1}_G\|_u \leq 1$ .

Since  $\|\mu\| = 1$ , we must have

$$\begin{aligned}|\mu(g - \mathbb{1}_G)| &\leq 1 \\ |\mu(g) - 1| &\leq 1,\end{aligned}$$

and since  $\mu(\mathbb{1}_G) = 1$ , we must have  $\mu(g) \in [0, 2]$ , so  $\mu(f) = \|f\| \mu(g) \geq 0$ .

□

**Corollary:** The set of states on  $(\ell_\infty(G))^*$  forms a  $w^*$ -compact subset of  $B_{(\ell_\infty(G))^*}$ .

*Proof.* It has been proven in functional analysis that a convex subset of  $(\ell_\infty(G))^*$  is  $w^*$ -compact if it is norm bounded and  $w^*$ -closed. Since the set of states is convex and norm-bounded, all we need to show is that  $S(\ell_\infty(G))$  is  $w^*$ -closed.

To this end, let  $f \in \ell_\infty(G)$  be positive and  $(\varphi_i)_i$  be a net in  $S(\ell_\infty(G))$  with  $(\varphi_i)_i \rightarrow \varphi$ . We must show that  $\varphi$  is positive and satisfies  $\varphi(\mathbb{1}_G) = 1$ . To this end, we see that

$$\varphi_i(f) \geq 1$$

for all  $i \in I$ , so we must necessarily have  $\varphi(f) \geq 0$ , and similarly, since  $\varphi_i(\mathbb{1}_G) = 1$  for each  $i \in I$ , we also have  $\varphi(\mathbb{1}_G) = 1$ . □

**Proposition:** If  $\mu \in (\ell_\infty(G))^*$  is a state, then  $m: P(G) \rightarrow [0, 1]$  defined by  $m(E) = \mu(\mathbb{1}_E)$  is a finitely additive probability measure on  $G$ . Moreover, if  $\mu$  is invariant, then  $m$  is a mean.

*Proof.* We have

$$\begin{aligned}
 m(G) &= \mu(\mathbb{1}_G) \\
 &= 1 \\
 m(\emptyset) &= \mu(0) \\
 &= 0 \\
 m(E \sqcup F) &= \mu(\mathbb{1}_{E \sqcup F}) \\
 &= \mu(\mathbb{1}_E + \mathbb{1}_F) \\
 &= \mu(\mathbb{1}_E) + \mu(\mathbb{1}_F) \\
 &= m(E) + m(F).
 \end{aligned}$$

Additionally, since  $0 \leq \mathbb{1}_E \leq \mathbb{1}_G$ , we have  $0 \leq \mu(\mathbb{1}_E) \leq 1$ , so  $0 \leq m(E) \leq m(G) = 1$ .

If  $\mu$  is invariant, then

$$\begin{aligned}
 m(sE) &= \mu(\mathbb{1}_{sE}) \\
 &= \mu(\lambda_s(\mathbb{1}_E)) \\
 &= \mu(\mathbb{1}_E) \\
 &= m(E).
 \end{aligned}$$

□

**Proposition:** If  $G$  admits a mean, then  $(\ell_\infty(G))^*$  admits an invariant state.

*Proof.* Let  $m$  be a finitely-additive probability measure. Define

$$\mu_0: \Sigma \rightarrow \mathbb{R}$$

by

$$\mu_0\left(\sum_{k=1}^n t_k \mathbb{1}_{E_k}\right) = \sum_{k=1}^n t_k m(E_k).$$

Since  $m$  is finitely additive, it is the case that  $\mu_0$  is well-defined, linear, and positive.

Note that  $\mu_0(\mathbb{1}_G) = m(G) = 1$ .

If  $m$  is a mean, then for  $f = \sum_{k=1}^n t_k E_k$ ,

$$\begin{aligned}
 \mu_0(\lambda_s(f)) &= \mu_0\left(\lambda_s\left(\sum_{k=1}^n t_k \mathbb{1}_{E_k}\right)\right) \\
 &= \mu_0\left(\sum_{k=1}^n t_k \mathbb{1}_{sE_k}\right) \\
 &= \sum_{k=1}^n t_k m(sE_k) \\
 &= \sum_{k=1}^n t_k m(E_k) \\
 &= \mu_0(f).
 \end{aligned}$$

Additionally, we see that

$$\begin{aligned}
 |\mu_0(f)| &= \left| \sum_{k=1}^n t_k m(E_k) \right| \\
 &\leq \sum_{k=1}^n |t_k| m(E_k) \\
 &\leq \sum_{k=1}^n \|f\| m(E_k) \\
 &= \|f\| \sum_{k=1}^n m(E_k) \\
 &\leq \|f\|.
 \end{aligned}$$

Thus,  $\mu_0$  is continuous, so  $\mu_0$  is uniformly continuous.

Since  $\bar{\Sigma} = \ell_\infty(G)$ , we see that  $\mu_0$  extends to a continuous linear functional  $\mu: \ell_\infty(G) \rightarrow \mathbb{R}$ , with  $\mu(\mathbb{1}_G) = \mu_0(\mathbb{1}_G) = 1$ .

If  $f \geq 0$ , we find a sequence  $(\phi_n)_n$  in  $\Sigma$  with  $\phi_n \geq 0$ ,  $\|\phi_n - f\| \xrightarrow{n \rightarrow \infty} 0$ , and we set

$$\begin{aligned}
 \mu(f) &= \lim_{n \rightarrow \infty} \mu(\phi_n) \\
 &= \lim_{n \rightarrow \infty} \mu_0(\phi_n) \\
 &\geq 0,
 \end{aligned}$$

meaning  $\mu$  is a state.

If  $f \in \ell_\infty(G)$ ,  $s \in G$ , and  $(\phi_n)_n$  in  $\Sigma$  with  $(\phi_n)_n \rightarrow f$ , then

$$\begin{aligned}
 \|\lambda_s(\phi_n) - \lambda_s(f)\| &= \|\lambda_s(\phi_n - f)\| \\
 &= \|\phi_n - f\| \\
 &\rightarrow 0
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 \mu(\lambda_s(\phi_n)) &= \mu_0(\lambda_s(\phi_n)) \\
 &= \mu_0(\phi_n) \\
 &= \mu(\phi_n) \\
 &\rightarrow \mu(f),
 \end{aligned}$$

so  $\mu(f) = \mu(\lambda_s(f))$ . Thus,  $\mu \in (\ell_\infty(G))^*$  is an invariant state. □

## Using Invariant States

**Proposition:**  $\mathbb{Z}$  is amenable.

*Proof.* We know that  $\lambda_1: \ell_\infty(\mathbb{Z}) \rightarrow \ell_\infty(\mathbb{Z})$ , defined by

$$\lambda_1(f)(k) = f(k-1)$$

is an isometry.



We set  $Y = \text{Ran}(\text{id} - \lambda_1) \subseteq \ell_\infty(\mathbb{Z})$ .

We claim that  $\text{dist}_Y(\mathbb{1}_\mathbb{Z}) \geq 1$ .

Suppose toward contradiction that there is  $y \in Y$  with  $\|\mathbb{1}_\mathbb{Z} - y\|_\infty = \rho < 1$ . Then,  $y = f - \lambda_1(f)$  for some  $f \in \ell_\infty(\mathbb{Z})$ , meaning

$$\|\mathbb{1} - (f - \lambda_1(f))\| = \rho.$$

Thus, for all  $k \in \mathbb{Z}$ , we have

$$|1 - (f(k) - f(k-1))| \leq \rho,$$

meaning  $|f(k) - f(k-1)| \geq 1 - \rho > 0$ . However, such an  $f$  cannot be bounded.

Since  $\text{dist}_{\overline{Y}}(\mathbb{1}_\mathbb{Z}) = \text{dist}_Y(\mathbb{1}_\mathbb{Z}) \geq 1$ , the Hahn–Banach theorem provides  $\mu \in (\ell_\infty(\mathbb{Z}))^*$  with  $\|\mu\| = 1$ ,  $\mu|_{\overline{Y}} = 0$ , and  $\mu(\mathbb{1}) = \text{dist}_Y(\mathbb{1}_\mathbb{Z}) \geq 1$ .

Since  $\|\mu\| = 1$  and  $\mu(\mathbb{1}) \geq 1$ , we must have  $\mu(\mathbb{1}) = 1$ .

Since  $\|\mu\| = \mu(\mathbb{1}_\mathbb{Z}) = 1$ , it is the case that  $\mu$  is a state on  $\ell_\infty(\mathbb{Z})$ . Since  $\mu(y) = 0$  for all  $y \in Y$ , we have

$$\begin{aligned} \mu(f - \lambda_1(f)) &= 0 \\ \mu(f) &= \mu(\lambda_1(f)), \end{aligned}$$

so inductively, we have  $\mu(f) = \mu(\lambda_k(f))$  for all  $k \in \mathbb{Z}$ , meaning  $\mu$  is an invariant state on  $\ell_\infty(\mathbb{Z})$ . Thus,  $\mathbb{Z}$  is amenable.  $\square$

**Proposition:** If  $N \trianglelefteq G$  and  $G/N$  are amenable, then  $G$  is amenable.

*Proof.* Let  $\rho \in (\ell_\infty(G/N))^*$  be an invariant state, and  $p: P(N) \rightarrow [0, 1]$ . For  $E \subseteq G$ , we define

$$f_E: G/N \rightarrow \mathbb{R}$$

by  $f_E(tN) = p(N \cap t^{-1}E)$ .

We verify that this is well-defined — for  $tN = sN$ , we have  $s^{-1}t \in N$ , so

$$\begin{aligned} p(N \cap t^{-1}E) &= p(s^{-1}t(N \cap t^{-1}E)) \\ &= p(s^{-1}tN \cap s^{-1}E) \\ &= p(N \cap s^{-1}E). \end{aligned}$$

We also see that  $f_E$  is bounded, and

$$\begin{aligned} f_{E \sqcup F}(tN) &= p(N \cap t^{-1}(E \sqcup F)) \\ &= p(N \cap (t^{-1}E \sqcup t^{-1}F)) \\ &= p((N \cap t^{-1}E) \sqcup (N \cap t^{-1}F)) \\ &= p(N \cap t^{-1}E) + p(N \cap t^{-1}F) \\ &= f_E(tN) + f_F(tN) \\ &= (f_E + f_F)(tN). \end{aligned}$$

Thus,  $f_{E \sqcup F} = f_E + f_F$ .

Additionally,

$$\begin{aligned} f_{sE}(tN) &= p\left(N \cap t^{-1}sE\right) \\ &= f_E\left(s^{-1}tN\right) \\ &= \lambda_{sN}(f_E)(tN), \end{aligned}$$

so  $f_{sE} = \lambda_{sN}(f_E)$ .

Finally,

$$\begin{aligned} f_G(tN) &= p\left(N \cap t^{-1}G\right) \\ &= p(N) \\ &= 1, \end{aligned}$$

so  $f_G = \mathbb{1}_{G/N}$ .

We define  $m: P(G) \rightarrow [0, 1]$  by

$$m(E) = \rho(f_E).$$

Then, we have

$$\begin{aligned} m(E \sqcup F) &= m(E) + m(F) \\ m(G) &= 1 \\ m(sE) &= \rho(f_{sE}) \\ &= \rho(\lambda_{sN}(f_E)) \\ &= \rho(f_E) \\ &= m(E), \end{aligned}$$

meaning  $m$  is a mean. □

**Corollary:** The finite direct product of amenable groups is amenable.

*Proof.* If  $H$  and  $K$  are amenable, then we know that

$$K \cong \frac{H \times K}{H}$$

is amenable, and  $H$  is amenable, so  $H \times K$  is amenable. □

**Corollary:** Finitely generated abelian groups are amenable.

*Proof.* All finitely generated abelian groups are isomorphic to  $\mathbb{Z}^d \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$  by the Fundamental Theorem of Finitely Generated Abelian Groups. Since  $\mathbb{Z}^d$  is a finite direct product of  $\mathbb{Z}$  (which is amenable), and the torsion group  $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$  is finite, we have that a finitely generated abelian group is amenable. □

**Corollary:** If  $\{G_i\}_{i \in I}$  is a directed family of amenable groups — i.e., that for any two groups  $G_j$  and  $G_k$ , there is  $G_\ell$  with  $G_j \subseteq G_\ell$  and  $G_k \subseteq G_\ell$  — then the direct union,

$$G = \bigcup_{i \in I} G_i,$$

is also amenable.

*Proof.* Let  $\mu_i \in (\ell_\infty(G_i))^*$  be the respective invariant states.

Set

$$M_i = \{\mu \in S((\ell_\infty(G))^*) \mid \mu(\lambda_s(f)) = \mu(f) \text{ for all } s \in G_i\},$$

and set  $\mu(f) = \mu_i(f|_{G_i})$ . We see that  $M_i$  is  $w^*$ -closed in  $B_{(\ell_\infty(G))^*}$ , as we have established the state space as a  $w^*$ -closed subset of  $B_{(\ell_\infty(G))^*}$ .

For  $i_1, \dots, i_n$ , we find  $G_j \supseteq G_{i_1}, \dots, G_{i_n}$ , which necessarily exists as  $\{G_i\}_{i \in I}$  is directed. Thus,  $M_j \subseteq \bigcap_{k=1}^n M_{i_k}$ , meaning  $\{M_i\}_{i \in I}$  has the finite intersection property.

By compactness, there is  $\mu \in \bigcap_{i \in I} M_i$ , meaning  $\mu$  is an invariant state.  $\square$

**Corollary:** All abelian groups are amenable.

*Proof.* Every abelian group is the direct union of its finitely generated subgroups.  $\square$

**Corollary:** All solvable groups are amenable.

*Proof.* Let  $e_G = G_0 \leq G_1 \leq \dots \leq G_n = G$  be such that  $G_{j-1} \trianglelefteq G_j$  for  $j = 1, \dots, n$ , and  $G_j/G_{j-1}$  abelian.

Since  $G_0$  is abelian, it is amenable. Similarly,  $G_1/G_0$  is abelian, so it is amenable, so  $G_1$  is amenable. Continuing in this fashion, we see that  $G$  is amenable.  $\square$

## Følner's Condition and Invariant Approximate Means

**Definition.** A group  $G$  is said to satisfy the Følner condition if, for every  $\varepsilon > 0$ , and for all  $E \subseteq G$  finite, there is a nonempty  $F \subseteq G$  finite such that for all  $t \in E$ ,

$$\frac{|tF \Delta F|}{|F|} \leq \varepsilon.$$

Equivalently, since

$$\frac{|tF \Delta F|}{|F|} = 2 \left( 1 - \frac{|tF \cap F|}{|F|} \right),$$

we have the equivalent formulation that

$$\frac{|tF \Delta F|}{|F|} \leq \varepsilon \text{ if and only if } 1 - \frac{|tF \cap F|}{|F|} \leq \varepsilon/2.$$

Thus,  $G$  satisfies the Følner condition if and only if, for all  $\varepsilon > 0$  and for all finite  $E \subseteq G$ , there exists a nonempty  $F \subseteq G$  with

$$\frac{|tF \cap F|}{|F|} \geq 1 - \varepsilon.$$

**Example.** All finite groups satisfy Følner's condition by taking  $F = G$  for each subset  $E \subseteq G$ .

**Lemma:** A countable group  $G$  satisfies the Følner condition if and only if  $G$  admits a Følner sequence,  $(F_n)_n$  with  $F_n \subseteq G$  finite, such that

$$\left( \frac{|tF_n \Delta F_n|}{|F_n|} \right)_n \xrightarrow{n \rightarrow \infty} 0,$$

or equivalently,

$$\left( \frac{|tF_n \cap F_n|}{|F_n|} \right)_n \xrightarrow{n \rightarrow \infty} 1,$$

for all  $t$  in  $G$ .

*Proof.* Let  $G$  admit a Følner sequence,  $(F_n)_n$ . Given  $\varepsilon > 0$ , and  $E \subseteq G$  finite, find  $N$  such that for all  $s \in E$  and  $n \geq N$ ,

$$\frac{|sF_n \Delta F_n|}{|F_n|} \leq \varepsilon.$$

We take  $F = F_N$ .

Let  $G$  satisfy the Følner condition. We write  $G = \bigcup_{n \geq 1} E_n$ , with  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ , and define  $F_n$  such that for all  $t \in E_n$ ,

$$\frac{|tF_n \Delta F_n|}{|F_n|} \leq \frac{1}{n}.$$

Then, given  $t \in G$ , it is the case that  $t \in E_N$  for some  $N$ , so  $t \in E_n$  for all  $n \geq N$ , so

$$\frac{|tF_n \Delta F_n|}{|F_n|} \leq \frac{1}{n}$$

for all  $n \geq N$ , meaning that

$$\frac{|tF_n \Delta F_n|}{|F_n|} \xrightarrow{n \rightarrow \infty} 0.$$

□

**Lemma:** Let  $G$  be a finitely generated group with generating set  $S$  (where  $S$  may not be symmetric<sup>1</sup>). If  $(F_n)_n$  is a sequence of finite subsets of  $G$  such that

$$\left( \frac{|sF_n \Delta F_n|}{|F_n|} \right)_n \rightarrow 0$$

for all  $s \in S$ , then  $(F_n)_n$  is a Følner sequence for  $G$ .

*Proof.* We start by showing that we can assume  $S$  to be symmetric. The following are both true:

- $s(A \Delta B) = sA \Delta sB$ ;
- $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$ .

Thus, if we have  $s^{-1}$  rather than  $s$ , our assumption provides, for all  $s \in S$ ,

$$\begin{aligned} \frac{|s^{-1}F_n \Delta F_n|}{|F_n|} &= \frac{s^{-1}(F_n \Delta sF_n)}{|F_n|} \\ &= \frac{|F_n \Delta sF_n|}{|F_n|} \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus, we may assume  $S$  is symmetric.

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<sup>1</sup>Closed under inversion.

For  $s, t \in F_n$ , we have

$$\begin{aligned} \frac{|stF_n \Delta F_n|}{|F_n|} &\leq \frac{|StF_n \Delta sF_n|}{|F_n|} + \frac{|sF_n \Delta F_n|}{|F_n|} \\ &= \frac{|s(tF_n \Delta F_n)|}{|F_n|} + \frac{|sF_n \Delta F_n|}{|F_n|} \\ &= \frac{|tF_n \Delta F_n|}{|F_n|} + \frac{|sF_n \Delta F_n|}{|F_n|} \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

We use induction to find the general case. □

**Example.** Considering  $\mathbb{Z}$  again, we remember that  $\{1\}$  is the generating set for  $\mathbb{Z}$ . If we let  $F_n = \{-n, -n+1, \dots, -1, 0, 1, \dots, n\}$ , we have

$$\frac{|(F_n + 1) \Delta F_n|}{|F_n|} = \frac{2}{2n+1} \xrightarrow{n \rightarrow \infty} 0.$$

Thus, we have that  $\mathbb{Z}$  satisfies the Følner condition.

We now turn our attention to approximate means, from which with Følner's condition, we will be able to construct a different, equivalent condition for group amenability.

**Definition.** If  $G$  is a group, we define

$$\text{Prob}(G) := \left\{ f: G \rightarrow [0, \infty) \mid |\text{supp}(f)| < \infty, \sum_{t \in G} f(t) = 1 \right\}.$$

Note that  $\text{Prob}(G) \subseteq B_{\ell_1(G)}$ . Given  $f \in \text{Prob}(G)$ , we set

$$\varphi_f: \ell_\infty(G) \rightarrow \mathbb{C},$$

defined by

$$\varphi_f(g) = \sum_{t \in G} g(t)f(t).$$

We claim that  $\varphi_f$  is a state on  $\ell_\infty(G)$ .

*Proof.* If  $g \geq 0$ , then  $\varphi_f(g) \geq 0$ , and  $\varphi_f(\mathbf{1}_G) = 1$ . It is also clear that  $\varphi_f$  is linear.

We only need show that  $\|\varphi_f\| = 1$ . We see

$$\begin{aligned} |\varphi_f(g)| &= \left| \sum_{t \in G} g(t)f(t) \right| \\ &\leq \sum_{t \in G} |g(t)| |f(t)| \\ &\leq \|g\|_\infty \sum_t |f_t| \\ &= \|g\|_\infty. \end{aligned}$$

□

**Proposition:** There is an action  $\lambda: G \xrightarrow{\text{Isom}} (\ell_1(G))$  such that  $\text{Prob}(G)$  is invariant.

*Proof.* We let  $\lambda_s(f)(t) = f(s^{-1}t)$ . Then,

$$\begin{aligned} \|\lambda_s(f)\|_1 &= \sum_{t \in G} |\lambda_s(f)(t)| \\ &= \sum_{t \in G} |f(s^{-1}t)| \\ &= \sum_{r \in G} |f(r)| \\ &= \|f\|. \end{aligned}$$

We also see that  $\lambda_s$  is linear.

Additionally,

$$\begin{aligned} \lambda_r \circ \lambda_s(f)(t) &= \lambda_s(f)(r^{-1}t) \\ &= f(s^{-1}r^{-1}t) \\ &= f((rs)^{-1}t) \\ &= \lambda_{rs}(f)(t). \end{aligned}$$

We see that if  $f \in \text{Prob}(G)$ , then for  $f \geq 0$ , we have  $\lambda_s(f) \geq 0$ , and

$$\text{supp}(\lambda_s(f)) = s(\text{supp}(f)),$$

which is also finite.

Thus,

$$\begin{aligned} \sum_{t \in G} \lambda_s(f)(t) &= \sum_{t \in G} f(s^{-1}t) \\ &= \sum_{r \in G} f(r) \\ &= 1 \end{aligned}$$

for  $f \in \text{Prob}(G)$ . □

**Definition.** For a countable group  $G$ , a sequence  $(f_k)_k$  in  $\text{Prob}(G)$  is an approximate invariant mean if, for all  $s \in G$ ,

$$\|f_k - \lambda_s(f_k)\|_1 \xrightarrow{k \rightarrow \infty} 0.$$

**Proposition:** If  $G$  admits a Følner sequence  $(F_k)_k$ , then it admits an approximate mean.

*Proof.* Set  $f_k = \frac{1}{|F_k|} \mathbb{1}_{F_k} \in \text{Prob}(G)$ . Then,

$$\begin{aligned} \|f_k - \lambda_s(f_k)\|_1 &= \frac{1}{|F_k|} \|\mathbb{1}_{F_k} - \lambda_s(\mathbb{1}_{F_k})\| \\ &= \frac{1}{|F_k|} \|\mathbb{1}_{F_k} - \mathbb{1}_{sF_k}\| \\ &= \frac{|F_k \Delta sF_k|}{|F_k|}, \end{aligned}$$

which thus converges to 0 as  $k \rightarrow \infty$ . □

**Proposition:** If  $G$  has an approximate mean, then  $G$  is amenable.

*Proof.* Let  $(f_k)_k$  be an approximate mean. We define  $\varphi_k = (\varphi_{f_k})_k$  to be a sequence of states on  $\ell_\infty(G)$ .

Since the state space on  $\ell_\infty(G)$  is  $w^*$ -compact, there is a state  $\mu$  and a subnet  $(\varphi_{k_j})_j$  with  $(\varphi_{k_j})_j \xrightarrow{w^*} \mu$ .

We only need to show that  $\mu$  is invariant. Note that

$$|\mu(g) - \mu(\lambda_s(g))| \leq |\mu_g - \varphi_{k_j}(g)| + |\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))| + |\varphi_{k_j}(\lambda_s(g)) - \mu(\lambda_s(g))|$$

holds for all  $g \in \ell_\infty(G)$ ,  $s \in G$ , and for all  $j$ .

Given  $\varepsilon > 0$ , we find  $J$  such that for  $j \geq J$ , we have

$$\begin{aligned} |\mu(g) - \varphi_{k_j}(g)| &< \varepsilon/3 \\ |\mu(\lambda_s(g)) - \varphi_{k_j}(\lambda_s(g))| &< \varepsilon/3. \end{aligned}$$

We see that

$$\begin{aligned} |\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))| &= \left| \sum_{t \in G} g(t) f_{k_j}(t) - \sum_{t \in G} g(s^{-1}t) f_{k_j}(t) \right| \\ &= \left| \sum_{t \in G} g(t) f_{k_j}(t) - \sum_{r \in G} g(r) f_{k_j}(sr) \right| & r = s^{-1}t \\ &= \left| \sum_{t \in G} g(t) (f_{k_j}(t) - \lambda_{s^{-1}}(f_{k_j})(t)) \right| \\ &\leq \|g\|_\infty \sum_{t \in G} |f_{k_j}(t) - \lambda_{s^{-1}}(f_{k_j})(t)| \\ &= \|g\|_\infty \|f_{k_j} - \lambda_{s^{-1}}(f_{k_j})\| \\ &< \varepsilon/3 \end{aligned}$$

for large  $j$ . Thus, we have

$$|\mu(g) - \mu(\lambda_s(g))| < \varepsilon,$$

for all  $\varepsilon > 0$ , so  $\mu(g) = \mu(\lambda_s(g))$ . □

## Equivalence between Means and Approximate Means

We wish to show that if  $G$  is amenable, then  $G$  has an approximate mean.

**Theorem:** Let  $G$  be amenable. Then,  $G$  has an approximate mean.

*Proof.* Recall that a net  $(f_i)_{i \in I}$  in  $\text{Prob}(G)$  has its domain as the set

$$I = \{(E, \varepsilon) \mid E \subseteq G \text{ finite}, \varepsilon > 0\},$$

directed by  $(E, \varepsilon) \leq (E', \varepsilon')$  if  $E \subseteq E'$  and  $\varepsilon \geq \varepsilon'$ .

Suppose there is no approximate mean. Then, there exists a finite subset  $E_0 \subseteq G$  and  $\varepsilon_0 > 0$  such that for all  $s \in E_0$  and  $f \in \text{Prob}(G)$ , we have

$$\|f - \lambda_s(f)\| \geq \varepsilon_0.$$

Let  $X = \bigoplus_{|E_0|} \ell_1(G)$  be endowed with the 1-norm.

Consider the set

$$C = \left\{ (f - \lambda_s(f))_{s \in E_0} \mid f \in \text{Prob}(G) \right\}.$$

Since  $\text{Prob}(G)$  is convex,  $C$  is convex, and since  $|E_0|$  is finite,  $C$  is necessarily bounded. Additionally, note that it is the case that  $0 \notin \overline{C}$ .

By the Hahn–Banach separation theorem, there is a real-valued  $\varphi \in X^*$  such that  $\varphi(C) \geq 1$ .

Note also that

$$\begin{aligned} X^* &\cong \bigoplus_{|E_0|} \ell_1(G)^* \\ &\cong \bigoplus_{|E_0|} \ell_\infty(G) \end{aligned}$$

with the  $\infty$  norm. We let  $\varphi = (\varphi_{g_s})_{s \in E_0}$  with  $g_s \in \ell_\infty(G)$ . From the duality between  $\ell_1(G)$  and  $\ell_\infty(G)$ , for any  $f \in \ell_1(G)$  and  $s \in E_0$ , we have

$$\varphi_{g_s}(f) = \sum_{t \in G} f(t)g_s(t).$$

Thus, for all  $f \in \text{Prob}(G)$ , we have

$$\begin{aligned} 1 &\leq \varphi((f - \lambda_s(f))_{s \in E_0}) \\ &= \sum_{s \in E_0} \varphi_{g_s}(f - \lambda_s(f)) \\ &= \sum_{s \in E_0} \sum_{t \in G} (f - \lambda_s(f))(t) g_s(t) \\ &= \sum_{s \in E_0} \left( \sum_{t \in G} f(t)g_s(t) - \sum_{t \in G} f(s^{-1}t)g_s(t) \right) \\ &= \sum_{s \in E_0} \left( \sum_{t \in G} f(t)g_s(t) - \sum_{r \in G} f(r)g(sr) \right) & s^{-1}t = r \\ &= \sum_{s \in E_0} \left( \sum_{r \in T} f(r)g_s(r) - \sum_{r \in G} f(r)\lambda_{s^{-1}}(g)(r) \right) \\ &= \sum_{s \in E_0} \sum_{r \in G} f(r)(g_s - \lambda_{s^{-1}}(g))(r). \end{aligned}$$

Note that this holds for any  $f \in \text{Prob}(G)$ . In particular, if  $f = \delta_t$  for a given  $t \in \text{Prob}(G)$ , then we must have

$$\begin{aligned} \sum_{s \in E_0} \sum_{r \in G} f(r)(g_s(r) - \lambda_{s^{-1}}(g_s)(r)) &= \sum_{s \in E_0} \sum_{r \in G} \delta_t(r)(g_s(r) - \lambda_{s^{-1}}(g_s)(r)) \\ &= \sum_{s \in E_0} (g_s - \lambda_{s^{-1}}g_s)(t) \\ &\geq 1. \end{aligned}$$

Since  $G$  is amenable, there is a mean  $\mu : \ell_\infty(G) \rightarrow \mathbb{C}$  with  $\mu(g_s) = \mu(\lambda_{s^{-1}}(g_s))$ , meaning

$$\mu \left( \sum_{s \in E_0} (g_s - \lambda_{s^{-1}}(g_s))(t) \right) = 0$$



$$\geq 1,$$

which is a contradiction.

□