**Problem** (Problem 1)**:**

(a) Fix topological spaces $X$ and $Y$, and consider the set of all continuous maps $X \to Y$. Define a relation on this set by saying that $f$ is related to $g$ whenever $f$ is homotopic to $g$. Prove that this relation is an equivalence relation.

(b) Prove that any space $X$ is homotopy equivalent to itself, that if $X$ is homotopy equivalent to $Y$, then $Y$ is homotopy equivalent to $X$, and that if $X$ is homotopy equivalent to $Y$ and $Y$ is homotopy equivalent to $Z$, then $X$ is homotopy equivalent to $Z$.

**Solution:**

(a) For reflexivity, we may select the identity homotopy $F\colon X\times I \to Y$, given by $F(x,t) = f(x)$ for all $t \in I$ and all $x \in X$.

For symmetry, if $F\colon X \times I \to Y$ is a homotopy with $F(x,0) = f(x)$ and $F(x,1) = g(x)$, then we may define the homotopy $G\colon X \times I \to Y$ by taking $G(x,t) = F(x,1-t)$. This is a composition of continuous maps, so it is continuous, and has $G(x,0) = g(x)$ and $G(x,1) = f(x)$, so the relation is symmetric.

For transitivity, we let $F\colon X\times I \to Y$ be a homotopy between $f$ and $g$, and let $G\colon X\times I \to Y$ be a homotopy between $g$ and $h$. Define a homotopy $H\colon X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

This is a well-defined function since $G(x,0) = F(x,1)$ by the definition of the homotopies $F$ and $G$, while it is continuous since both $F$ and $G$ are continuous, the functions $2t$ and $2t-1$ are continuous, and $F$ and $G$ agree at $t = 1/2$.

Therefore, the relation is transitive, so the relation $f \sim g$ if $f$ is homotopic to $g$ is an equivalence relation.

(b) For the homotopy equivalences between $X$, $Y$, and $Z$, we will define them via the following collection of maps:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "\overline{f}"] \arrow[r, bend left, "g"] & Z \arrow[l, bend left, "\overline{g}"] \end{tikzcd}$$

where

$$\begin{aligned} \overline{f} \circ f &\simeq \mathrm{id}_X \\ f \circ \overline{f} &\simeq \mathrm{id}_Y \\ \overline{g} \circ g &\simeq \mathrm{id}_Y \\ g \circ \overline{g} &\simeq \mathrm{id}_Z\,. \end{aligned}$$

Reflexivity follows from the fact that the identity map is homotopic to itself via the identity homotopy, while symmetry follows from flipping the roles of $f$ and $\overline{f}$ in the definitions of the homotopy equivalence between $X$ and $Y$.

For transitivity, we claim that the functions $g \circ f$ and $\overline{f} \circ \overline{g}$ are the pair between $X$ and $Z$ that satisfy our desired result. That is, we claim that

$$\begin{aligned}\left(\overline{f} \circ \overline{g}\right) \circ (g \circ f) &\simeq \mathrm{id}_X \\ (g \circ f) \circ \left(\overline{f} \circ \overline{g}\right) &\simeq \mathrm{id}_Z .\end{aligned}$$

We start by claiming that

$$\overline{f} \circ \overline{g} \circ g \circ f \simeq \overline{f} \circ \mathrm{id}_Y \circ f \tag{$*$}$$

Let $H\colon Y \times I \to Y$ be the homotopy that maps $\overline{g} \circ g$ to $\mathrm{id}_Y$. Then, if we define

$$\begin{aligned}F\colon X \times I &\to X \\ (x,t) &\mapsto \overline{f} \circ H_t \circ f,\end{aligned}$$

we see that $F$ is continuous with

$$\begin{aligned}F(x,0) &= \overline{f} \circ \overline{g} \circ g \circ f \\ F(x,1) &= \overline{f} \circ \mathrm{id}_Y \circ f.\end{aligned}$$

Therefore, the claim $(*)$ is established. Collapsing with $\mathrm{id}_Y$ and using the fact that "is homotopic to" is an equivalence relation, we thus establish

$$\begin{aligned}\overline{f} \circ \overline{g} \circ g \circ f &\simeq \overline{f} \circ \mathrm{id}_Y \circ f \\ &= \overline{f} \circ f \\ &\simeq \mathrm{id}_X .\end{aligned}$$

By a similar process using the homotopy between $f \circ \overline{f}$ and $\mathrm{id}_Y$, we thus establish

$$\begin{aligned}g \circ f \circ \overline{f} \circ \overline{g} &\simeq g \circ \mathrm{id}_Y \circ \overline{g} \\ &= g \circ \overline{g} \\ &\simeq \mathrm{id}_Z .\end{aligned}$$

Therefore, homotopy equivalence is reflexive, symmetric, and transitive.