Chapter 27 Problems

Problem 11

(a)

$$\begin{aligned}
\lambda \langle v \mid v \rangle &= \langle v | \lambda | v \rangle && \text{(Moving } \lambda \text{ into braket.)}\\
&= \langle v | H | v \rangle && \text{(Definition of } \lambda.\text{)}\\
&= \overline{\langle v | H^* | v \rangle} && \text{(Definition of adjoint operator.)}\\
&= \overline{\langle v | H | v \rangle} && \text{(Definition of Hermitian operator.)}\\
&= \overline{\langle v | \lambda | v \rangle} && \text{(Definition of } \lambda.\text{)}\\
&= \overline{\lambda} \langle v \mid v \rangle && \text{(Moving } \lambda \text{ out of braket.)}
\end{aligned}$$

(b) It is the case that $\langle Hv_1 \mid v_2 \rangle = \overline{\lambda_1} \langle v_1 \mid v_2 \rangle$ for any operator — since our operator is Hermitian, it must be the case that $\lambda_1 = \overline{\lambda_1}$, else it would be possible for there to be $\lambda_2 - \overline{\lambda_1} = 0$ with λ_1, λ_2 distinct in (27.52b).

Problem 22

$$\begin{aligned}
M &= \sum_i \lambda_i |\hat{v}_i\rangle \langle \hat{v}_i|\\
&= (2)\left(\frac{1}{6}\right)\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}1 & 2 & 1\end{pmatrix}\\
&+ (-1)\left(\frac{1}{2}\right)\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\\
&+ (1)\left(\frac{1}{3}\right)\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}1 & -1 & 1\end{pmatrix}\\
&= \begin{pmatrix}-1 & 1 & -1\\ 1 & 1 & 1\\ -1 & 1 & -1\end{pmatrix}.
\end{aligned}$$

Problem 26

(a) Let M be a normal matrix. Then, there exists a unitary operator U such that

$$U\Lambda U^* = M,$$

where Λ is the diagonal matrix of eigenvalues. Since Λ and M are in the same similarity class, they have the same trace, so

$$\begin{aligned}
\operatorname{tr}(M) &= \operatorname{tr}(\Lambda)\\
&= \sum_i \lambda_i.
\end{aligned}$$

(b) Let M be a normal matrix. Then, there exists a unitary operator U such that

$$U\Lambda U^* = M,$$

where Λ is the diagonal matrix of eigenvalues. Since Λ and M are in the same similarity class, they have the same determinant, so

$$\begin{aligned}\det(M) &= \det(\Lambda)\\ &= \prod_i \lambda_i.\end{aligned}$$

Problem 27

I don't know what you can say about their eigenvalues.

Chapter 28 Problems

Problem 1

$$\begin{aligned}M\left|\ddot{Q}\right\rangle &= -K\left|Q\right\rangle\\ m\ddot{q}_1 &= -2kq_1 + kq_2\\ m\ddot{q}_2 &= -2kq_2 + kq_1\\ \\ m\ddot{q}_1 &= k\left(-2q_1 + q_2\right)\\ m\ddot{q}_2 &= k\left(-2q_2 + q_1\right)\end{aligned}$$

We have

$$\begin{aligned}m\left(\ddot{q}_1 + \ddot{q}_2\right) &= -k\left(q_1 + q_2\right)\\ m\left(\ddot{q}_1 - \ddot{q}_2\right) &= -3k\left(q_1 - q_2\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{d^2}{dt^2}\left(q_1 - q_2\right) &= -\frac{3k}{m}\left(q_1 - q_2\right)\\ \frac{d^2}{dt^2}\left(q_1 + q_2\right) &= -\frac{k}{m}\left(q_1 + q_2\right),\end{aligned}$$

so

$$\begin{aligned}q_1 + q_2 &= A_1\cos\left(\omega_1 t + \delta_1\right)\\ q_1 - q_2 &= A_2\cos\left(\omega_2 t + \delta_2\right)\\ q_1 &= a_1\cos\left(\omega_1 t + \delta_1\right) + a_2\cos\left(\omega_2 t + \delta_2\right)\\ q_2 &= a_1\cos\left(\omega_1 t + \delta_1\right) - a_2\cos\left(\omega_2 t + \delta_2\right).\end{aligned}$$

Problem 2

We have a matrix of eigenvectors

$$\begin{aligned}A &= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ A^{-1} &= \begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}.\end{aligned}$$

We find

$$A^{-1}MA = \begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}\begin{pmatrix}2k & k\\ k & 2k\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 3k & k \\ 3k & -k \end{pmatrix} \\
&= \begin{pmatrix} 3k & 0 \\ 0 & k \end{pmatrix}.
\end{aligned}
$$

Problem 3

We let $\delta_1 = \delta_2 = 0$, and take

$$
\begin{aligned}
q_1(0) &= a \\
&= A_1 + A_2 \\
q_2(0) &= b \\
&= A_1 - A_2.
\end{aligned}
$$

Therefore, we have $A_1 = \frac{a+b}{2}$ and $A_2 = \frac{a-b}{2}$. Inserting into their respective formula, we get

$$
\begin{aligned}
q_1(t) &= \frac{a+b}{2}\cos(\omega_1 t) + \frac{a-b}{2}\cos(\omega_2 t) \\
&= \frac{a}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right) + \frac{b}{2}\left(\cos(\omega_1 t) - \cos(\omega_2) t\right) \\
q_2(t) &= \frac{a+b}{2}\cos(\omega_1 t) - \frac{a-b}{2}\cos(\omega_2 t) \\
&= \frac{a}{2}\left(\cos(\omega_1 t) - \cos(\omega_2 t)\right) + \frac{b}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right).
\end{aligned}
$$

Problem 6

We have the matrix

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} \\
A^{-1} &= \begin{pmatrix} 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.
\end{aligned}
$$

Thus, we find

$$
\begin{aligned}
\Lambda &= \begin{pmatrix} 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{pmatrix},
\end{aligned}
$$

implying that our eigenmodes do indeed solve the generalized eigenvalue problem for this system.

Problem 7

Using Newton's second law, we get

$$
\begin{aligned}
m_1 \ddot{q}_1 &= -k_1 q_1 + k_2 (q_2 - q_1) \\
m_2 \ddot{q}_2 &= -k_2 q_2 - k_2 (q_2 - q_1).
\end{aligned}
$$

Using our initial conditions, we get the equations

$$
M\left|\ddot{Q}\right\rangle = -\begin{pmatrix} 5k & -2k \\ k & 2k \end{pmatrix} \left|Q\right\rangle,
$$

where

$$|Q\rangle = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

We then seek to solve the generalized eigenvalue equation

$$K|\Phi\rangle = \omega^2 M |\Phi\rangle.$$

We find eigenvalues of

$$\omega_1^2 = \frac{3k}{m}$$
$$\omega_2^2 = \frac{4k}{m},$$

with respective eigenvectors of

$$|\Phi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|\Phi_2\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Problem 10

Calculating

$$k = m\omega^2,$$

we get

$$k \approx 187\ \text{N/m}.$$

Problem 15

The two normal modes are the mode where both masses are swinging in the same direction, with frequency $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$, and where both masses are swinging in the opposite direction, with frequency $\frac{\sqrt{3}}{2\pi}\sqrt{\frac{g}{l}}$.