Amenability: A (Somewhat) Brief Introduction

Avinash Iyer

Occidental College

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Outline

Contents

Groups

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then we call the pair $(A, \star)$ a *group*.

We (usually) abbreviate $a \star b$ as ab. If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups

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- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

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- The group $\mathrm{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group $\mathrm{E}(3)$ consists of all translations, rotations, and flips in $\mathbb{R}^3$, and is also known as the *isometry group* of $\mathbb{R}^3$.

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Every group is equipped with a family of canonical actions, $\sigma_a\colon G\to G$ for each $a\in G$, given by $x\mapsto ax$, known as *left-multiplication*.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i\in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a σ-*algebra* of subsets.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

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If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say μ is a measure.

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- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We define $F(a,b)$ to be the set of all "words" in the alphabet $\left\{a,b,a^{-1},b^{-1}\right\}$, subject to the condition that, for $w,w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

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- all words that start with b — think words that start with b^2 before you multiply b^{-1}.

Thus, all we need to do is add back $W\left(b^{-1}\right)$ to get $F(a,b)$ back.

$$F(a,b) = W\left(b^{-1}\right) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

We're able to take part of the group $F(a, b)$, take some translations, and, miraculously, obtain the entire group back.

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$;

such that

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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is *G-paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
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Paradoxical Decompositions of Sets

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such that

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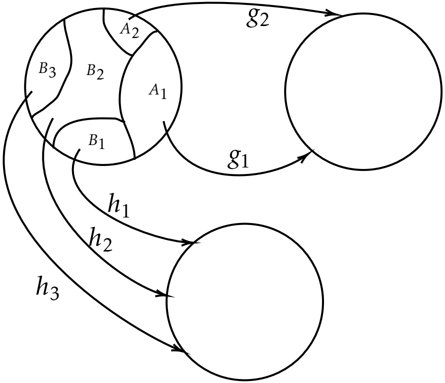
A paradoxical group is a paradoxical set under the action of left-multiplication.

Depiction



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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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Thus, SO(3) is paradoxical — can we use it to find a paradoxical decomposition?

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

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- In other words, not all subsets of $\mathbb{R}^3$ have a definite "volume" invariant under isometry.

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- group elements $g_1, \ldots, g_n \in G$

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Effectively, A and B are "equal" to each other up to the group action.

If A is G-paradoxical, then so too is B.

The Banach–Tarski Paradox: Proof Outline I

1. We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of SO(3) isomorphic to $F(a, b)$.

The Banach–Tarski Paradox: Proof Outline II

2. We use the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right) \end{aligned}$$

   to duplicate the unit sphere in $\mathbb{R}^3$, S^2, except for a countable subset D. (The *Hausdorff Paradox*.)
3. We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable — there is thus a paradoxical decomposition of S^2.
4. We show that the unit ball, $B(0,1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group E(3).

The Banach–Tarski Paradox: Proof Outline III

5. Define a relation $A \preceq B$ if A is G-equidecomposable with a subset of B, and show that if $A \preceq B$ and $B \preceq A$, then A and B are G-equidecomposable.
6. Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

Contents

Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped “create” the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu: F(a,b) \to [0,1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$1 = \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right)$$

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1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\
&= \nu(F(a,b)) + \nu(F(a,b))
\end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned}
1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\
&= \nu(F(a,b)) + \nu(F(a,b)) \\
&= 2.
\end{aligned}$$

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu \colon G \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

If G admits a mean, we say G is *amenable*.

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If G admits a mean, we say G is *amenable*.

- In other words, G is sufficiently "well-behaved."

Inheritance Properties of Amenability

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- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

  Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

  is a mean.
- Abelian groups are amenable.
- The free group, $F(a,b)$, is *not* amenable.

Paradoxical Groups and Amenability

Every paradoxical group is *not* amenable — the argument is similar to the case for $F(a,b)$.

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Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

Paradoxical Groups and Amenability

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More surprisingly, though, every *non*-paradoxical group is amenable.

Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

Contents

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Why Find Alternative Characterizations?

On first glance, it may seem like we're finished, but we're really not.

Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|: V \to [0, \infty)$ satisfying:

- definiteness: $\|v\| \geq 0$, with equality if and only if $v = 0$;
- homogeneity: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$;
- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sup_{t\in\Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sum_{t\in\Gamma} |f(t)| < \infty \right\};$$

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They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_1} := \sum_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_2} := \left(\sum_{t\in\Gamma} |f(t)|^2\right)^{1/2}$.

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

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A linear transformation $T: V \to W$ is called *bounded* if

$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted $\|T\|_{\text{op}}$.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a bounded linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}}$.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s \colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

for all $t \in \Gamma$ and fixed $s \in \Gamma$.

If $\varphi \colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

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If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

Motivating Følner's Condition

There is actually one way that working with sets makes life easier.

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Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was "significantly" "bigger" than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

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Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was "significantly" "bigger" than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

But what does "bigger" actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n\to\infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

Approximate Means

The Følner condition allows us to find an "approximate" version of a mean.

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The Følner condition allows us to find an "approximate" version of a mean.

Keeping $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a "sufficient" finitely supported function g,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

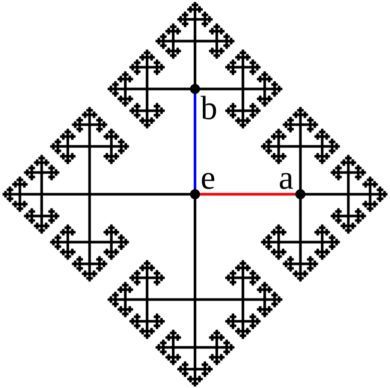
then use a "layer cake" decomposition to find our Følner sets:

$$g = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S, we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by "walking" along the generators.



Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

$$\inf\left\{ \frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), |S| \text{ finite} \right\} = 0.$$

- Essentially, the Cayley graph doesn't "get too big" "too fast."
- This is proven with the Følner condition.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ that satisfies

- $\langle x,x\rangle \geq 0$, with equality only when $x=0$;
- $\langle x_1+\alpha x_2, y\rangle = \langle x_1, y\rangle + \alpha\langle x_2, y\rangle$;
- $\langle x, y_1+\alpha y_2\rangle = \langle x, y_1\rangle + \overline{\alpha}\langle x, y_2\rangle$.

The inner product induces a norm $\|x\|^2 = \langle x,x\rangle$. If $\mathcal{H}$ is complete with respect to this norm, we call $\mathcal{H}$ a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that "plays nicely" with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

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$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle.$$

If $U\colon \mathcal{H} \to \mathcal{H}$ is such that

$$\begin{aligned} U^*U &= I \\ UU^* &= I, \end{aligned}$$

then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda \colon \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$
$$\lambda\left(s^{-1}\right) = \lambda(s)^*$$

is called a *unitary representation* of Γ.

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is called a *unitary representation* of Γ.

All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma \colon \Gamma \to \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

One special representation is defined by $s \mapsto \lambda_s$, where $\lambda_s(f)(t) = f\left(s^{-1}t\right)$.

This is known as the *left-regular representation*, and is a very useful

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* if

$$\lim_{k \to \infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

If $\lambda \colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a vector space with the following properties:

- $\|TS\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}}\|S\|_{\mathrm{op}}$;
- $\|T^*\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$;
- $\|T^*T\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}^2$.

Additionally, the adjoint map satisfies:

- $(T + \alpha S)^* = T^* + \overline{\alpha} S^*$;
- $T^{**} = T$;
- $(TS)^* = S^* T^*$.

These make $\mathbb{B}(\mathcal{H})$ a C^**-algebra*. However, there are other C^*-algebras.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t)\delta_t,$$

where δ_t is the point mass at $t \in \Gamma$.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t)\delta_t,$$

where δ_t is the point mass at $t \in \Gamma$.

This becomes a $*$-algebra when endowed with multiplication and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g\left(s^{-1} t\right)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^*-Algebra, cont'd

If we represent $\pi_\lambda \colon \mathbb{C}[\Gamma] \to \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\mathrm{op}},$$

we get the *reduced group C^*-algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

The $n \times n$ matrices, $\mathrm{Mat}_n(\mathbb{C})$, are also C^*-algebras.

Using a common tactic of finite approximations, we can define a special kind of finite-dimensional approximation for C^*-algebras using matrices.

Nuclearity

A C^*-algebra, A, is called *nuclear* if there exist two sequences of maps, $\varphi_n\colon A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n\colon \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$, such that

$$\left\| a - \psi_n \circ \varphi_n(a) \right\| \xrightarrow{n\to\infty} 0.$$

- Essentially, any $a \in A$ is "close enough" to a certain family of finite-dimensional analogues.

Nuclearity and Amenability

A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

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A group Γ is amenable if and only if the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear.

- This is also proven using the Følner condition.

What We've Learned

If Γ is a discrete group, then Γ is amenable if and only if

- Γ is non-paradoxical (Tarski's Theorem);
- Γ admits a finitely additive probability measure, $\mu\colon \Gamma \to [0,1]$ such that $\mu(E) = \mu(tE)$ (existence of means);
- $\ell_\infty(\Gamma)$ admits a state, $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$, such that $\varphi(\lambda_s(f)) = \varphi(f)$ (invariant states);
- there is a sequence of finite subsets, $(F_n)_n$, such that for all $s \in \Gamma$, $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ (Følner's Theorem);
- there is a sequence $(f_k)_k \subseteq \ell_1(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_1} \to 0$ (Approximate Means);
- the Cayley graph of Γ satisfies $\inf\left\{\frac{|N(S)|}{|S|} \;\middle|\; S \subseteq V(G), S \text{ finite}\right\} = 0$ (graph amenability);
- there is a sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_2} \to 0$ (almost-invariant vectors);
- the reduced group C^*-algebra, $C^*_\lambda(\Gamma)$, is nuclear (nuclearity).

Contents

Final Remarks

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Nuclear C^*-algebras are classified, so active research areas primarily concern whether or not certain classes of C^*-algebras are nuclear (hence classifiable).

There are also a lot of other directions that amenability can take the eager student, but I think this was a pretty nice overview of some of the ways that amenability touches all sorts of other fields of math.

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- friends, family, and acquaintances both in the math major and outside;
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