

## Problem 1

Let  $\mathcal{A} \subseteq \mathcal{P}(\Omega)$  be a family of subsets satisfying

- (i) if  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$ ;
- (ii) If  $\{A_k\}_{k \geq 1}$  is a countable family of pairwise disjoint members of  $\mathcal{A}$ , then  $\bigsqcup_{k \geq 1} A_k \in \mathcal{A}$ .

Prove that  $\mathcal{A}$  is a  $\sigma$ -algebra on  $\Omega$ .

## Problem 2

Consider the family  $\mathcal{E} : \{(-\infty, b) \mid b \in \mathbb{R}\}$ . Show that  $\sigma(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$ .

**Proof:** Consider the family  $\mathcal{E}' := \{[a, b) \mid a, b \in \mathbb{R}\}$ . We have established that  $\sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$ .

We see that for any element of  $\mathcal{E}$ ,  $(-\infty, b) = \bigcup_{n=1}^{\infty} [a-n, b)$ , meaning  $\mathcal{E} \in \sigma(\mathcal{E}')$ , so  $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$ .

Additionally,  $[a, b) = (-\infty, b) \setminus (-\infty, a)$ , meaning  $\mathcal{E}' \in \sigma(\mathcal{E})$ , so  $\sigma(\mathcal{E}') \subseteq \sigma(\mathcal{E})$ , so  $\sigma(\mathcal{E}) = \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$ .

## Problem 3

Let  $(\Omega, \mathcal{M})$  and  $(\Lambda, \mathcal{N})$  be measurable spaces. We define the product  $\sigma$ -algebra on  $\Omega \times \Lambda$  as

$$\mathcal{M} \otimes \mathcal{N} := \sigma(\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}).$$

Prove that  $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$ .

**Proof:** For  $a < b$  and  $c < d$ , it is the case that  $(a, b) \times (c, d) \subseteq \mathbb{R}^2$  is open, meaning

$$\begin{aligned} \sigma(\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{R}\}) &= \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \\ &\subseteq \mathcal{B}_{\mathbb{R}^2}. \end{aligned}$$

Letting  $U \in \mathcal{B}_{\mathbb{R}^2}$ , it is the case that  $U = \bigcup_{j=1}^{\infty} U(x_j, r_j)$ . For each  $U(x_j, r_j)$ , take  $I_j = (x_{jx} - r_j, x_{jx} + r_j) \times (x_{jy} - r_j, x_{jy} + r_j)$ , so  $U \subseteq \bigcup_{j=1}^{\infty} I_j$ . Thus,  $U \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ , so  $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ .

## Problem 4

Let  $(\Omega, \mathcal{M})$  and  $(\Lambda, \mathcal{N})$  be measurable spaces. A map  $f : \Omega \rightarrow \Lambda$  is  $\mathcal{M}$ - $\mathcal{N}$ -measurable if  $E \in \mathcal{N} \Rightarrow f^{-1}(E) \in \mathcal{M}$ .

Let  $(\Omega, \mathcal{M})$  be a measurable space and suppose  $E \in \mathcal{M}$ . Show that  $\mathcal{M}_E = \{M \cap E \mid M \in \mathcal{M}\}$  is a  $\sigma$ -algebra on  $E$  and the inclusion map  $\iota : E \rightarrow \Omega$  is  $\mathcal{M}_E$ - $\mathcal{M}$ -measurable.

**Proof:** Let  $M \in \mathcal{M}$ . Then,  $\iota^{-1}(M) = E \cap M \in \mathcal{M}_E$ . Thus,  $\iota$  is  $\mathcal{M}_E$ - $\mathcal{M}$ -measurable.

## Problem 5

Let  $(\Omega, \mathcal{M})$  and  $(\Lambda, \mathcal{N})$  be measurable spaces. Suppose  $\mathcal{N}$  is generated as a  $\sigma$ -algebra by a family of subsets  $\mathcal{E} \subseteq \mathcal{P}(\Lambda)$ . Prove that a map  $f : \Omega \rightarrow \Lambda$  is  $\mathcal{M}$ - $\mathcal{N}$ -measurable if and only if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{E}$ . Conclude that a continuous function  $f : X \rightarrow Y$  between metric spaces is  $\mathcal{B}_X$ - $\mathcal{B}_Y$ -measurable.

**Proof:**

## Problem 6

Suppose  $(\Omega, \mathcal{M})$  is a measurable space and  $f : \Omega \rightarrow \Lambda$  is a map. Show that  $\mathcal{N} := \{E \subseteq \Lambda \mid f^{-1}(E) \in \mathcal{M}\}$  is a  $\sigma$ -algebra on  $\Lambda$  and  $f$  is  $\mathcal{M}$ - $\mathcal{N}$ -measurable.  $\mathcal{N}$  is called the  $\sigma$ -algebra produced by  $f$ .

## Problem 7

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space, and suppose  $\{E_k\}_{k \geq 1}$  is a decreasing sequence of measurable sets with  $\mu(E_1) < \infty$ . Show that

$$\begin{aligned} \mu\left(\bigcap_{k \geq 1} E_k\right) &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \inf_{k \geq 1} \mu(E_k). \end{aligned}$$

## Problem 8

Let  $(\Omega, \mathcal{M})$  and  $(\Lambda, \mathcal{N})$  be measurable spaces and suppose  $f : \Omega \rightarrow \Lambda$  is measurable. If  $\mu$  is a measure on  $\mathcal{M}$ , show that

$$f * \mu : \mathcal{N} \rightarrow [0, \infty]; \quad f * \mu(E) := \mu(f^{-1}(E))$$

defines a measure on  $(\Lambda, \mathcal{N})$ . This is called the push-forward measure.

## Problem 9

A group  $G$  is paradoxical if there are pairwise disjoint subsets of  $G$ ;  $E_1, \dots, E_n, F_1, \dots, F_m$  and group elements  $t_1, \dots, t_n, s_1, \dots, s_m$  such that

$$\begin{aligned} G &= \bigsqcup_{j=1}^n t_j E_j \\ &= \bigsqcup_{k=1}^m s_k F_k. \end{aligned}$$

A mean on a group  $G$  is a finitely additive probability measure  $\nu : \mathcal{P}(G) \rightarrow [0, 1]$  that is translation invariant; that is,  $\nu(tE) = \nu(E)$  for all  $E \subseteq G$  and  $t \in G$ . A group is said to be amenable if it admits a mean.

Show that a paradoxical group is nonamenable.

## Problem 10

Let  $\Delta$  be a totally disconnected compact metric space (for example, the cantor set). Suppose  $\varphi : C(\Delta) \rightarrow \mathbb{R}$  is a state —  $\varphi$  is linear, continuous, positive, and  $\varphi(\mathbb{1}_\Delta) = 1$ .

- (i) Show that  $\mathcal{C} := \{E \mid \mathbb{1}_E \in C(\Delta)\}$  is an algebra of subsets on  $\Delta$ .
- (ii) Show that

$$\mu_0 : \mathcal{C} \rightarrow [0, 1]; \quad \mu_0(E) = \varphi(\mathbb{1}_E)$$

is a well-defined finitely additive measure.

(iii) If  $\{E_k\}_{k \geq 1}$  is a countable family of members of  $\mathcal{C}$  such that  $\bigsqcup_{k \geq 1} E_k \in \mathcal{C}$ , show that

$$\mu_0 \left( \bigsqcup_{k \geq 1} E_k \right) = \sum_{k=1}^{\infty} \mu_0(E_k).$$