Problem 1

Let $v_1, \dots, v_n$ be mutually orthogonal vectors in an inner product space V. Show that

$$\left\| \sum_{k=1}^{n} v_k \right\|^2 = \sum_{k=1}^{n} \|v_k\|^2 .$$

Proof:

$$\begin{aligned}
\left\| \sum_{k=1}^{n} v_k \right\|^2 &= \left\langle \sum_{k=1}^{n} v_k, \sum_{k=1}^{n} v_k \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \sum_{k=1}^{n} v_k, v_i \right\rangle \\
&= \sum_{i=1}^{n} \langle v_i, v_i \rangle && \text{since for } i \neq j,\ \langle v_i, v_j \rangle = 0 \\
&= \sum_{i=1}^{n} \|v_i\|^2
\end{aligned}$$

Problem 2

Let V be an inner product space and fix $w \neq 0$ in V. We define the one-dimensional projection

$$P_w : V \to V; P_w(v) := \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

(i) Prove that $v - P_w(v) \perp P_w(v)$.

(ii) Show that $P_w : V \to V$ is a linear operator with $\|P_w\|_{\text{op}} = 1$.

(iii) Show that $P_w \circ P_w = P_w$.

Problem 3

Let V be an inner product space. Prove the reverse Cauchy-Schwarz Inequality which states

$$v, w \in V, \text{ and } |\langle v, w \rangle| = \|v\| \|w\| \Rightarrow v = \alpha w.$$

Problem 4

Let V be an inner product space. Then, for any $v, w \in V$, show that

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

Problem 5

Let $\lambda = (\lambda_k)_k$ belong to ℓ_∞. Show that the map

$$D_\lambda : \ell_2 \to \ell_2; D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$

Problem 6

Consider the vector space $C([0,2\pi])$ equipped with

$$\langle f,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt.$$

(i) Show that this pairing defines an inner product on $C([0,2\pi])$.

(ii) For $n\in\mathbb{Z}$, set $e_n(t)=\cos(nt)+i\sin(nt)$. Show that the family $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal.

Problem 7

Let V be any normed space, $p\in[1,\infty]$, and suppose $T:\ell_p^n\to V$ is linear. Show that T is bounded.

Problem 8

Let $\mathbb{P}[0,1]=\left\{\sum_0^n a_k x^k \mid a_k\in\mathbb{C}\right\}\subseteq C([0,1])$ denote the linear subspace of all polynomial functions equipped with the uniform norm $\|\cdot\|_u$ inherited from $C([0,1])$. We define the map

$$D:\mathbb{P}[0,1]\to\mathbb{P}[0,1]; D(p(x))=p'(x).$$

Show that D is unbounded.

Problem 9

Let V be an infinite-dimensional normed space. Show that there is a linear functional $\varphi:V\to\mathbb{F}$ that is unbounded.

Problem 10

Let $a,b\in\mathbb{M}_n$. Show the following properties of the operator norm.

(i) $\|a\|_{\text{op}}=\sup\left\{|\langle a\xi,\eta\rangle| \mid \xi,\eta\in B_{\ell_2^n}\right\}$

(ii) $\|a^*\|_{\text{op}}=\|a\|_{\text{op}}$

(iii) $\|ab\|_{\text{op}}\le\|a\|_{\text{op}}\|b\|_{\text{op}}$

(iv) $\|a^*a\|_{\text{op}}=\|a\|_{\text{op}}^2$