

**Problem (Problem 1):** Let  $R$  be a commutative ring. An  $R$ -module  $M$  is called torsion if for any  $m \in M$ , there is a nonzero  $r \in R$  such that  $rm = 0$ . An  $R$ -module  $M$  is called divisible if for any nonzero  $r \in R$ , we have  $rM = M$ . In other words,  $M$  is divisible if for any  $m \in M$  and nonzero  $r \in R$ , there is  $x \in M$  such that  $rx = m$ .

- (a) Suppose  $M$  is a torsion  $R$ -module and  $N$  is a divisible  $R$ -module. Prove that  $M \otimes_R N = \{0\}$ .
- (b) Let  $M = \mathbb{Q}/\mathbb{Z}$  considered as a  $\mathbb{Z}$ -module. Prove that  $M \otimes_{\mathbb{Z}} M = \{0\}$ .

**Solution:**

- (a) It is enough to show that any simple tensor  $m \otimes n \in M \otimes_R N$  is the zero tensor. To see this, we let  $r \in R$  be such that  $rm = 0$ , and observe that there is some  $x \in N$  such that  $rx = n$ . By using property (R3) of tensor products, we observe then that

$$\begin{aligned} m \otimes n &= m \otimes (rx) \\ &= (rm) \otimes x \\ &= 0 \otimes x \\ &= 0. \end{aligned}$$

Thus,  $M \otimes_R N = \{0\}$ .

- (b) It is enough to show that  $\mathbb{Q}/\mathbb{Z}$  is both torsion and divisible, as we may then apply (a). To see that  $\mathbb{Q}/\mathbb{Z}$  is torsion, we have that

$$\begin{aligned} b \left[ \frac{a}{b} \right] &= [a] \\ &= [0] \end{aligned}$$

for any element  $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$ . Additionally, for any  $n \in \mathbb{Z}$ , we have

$$\left[ \frac{a}{b} \right] = n \left[ \frac{a}{nb} \right],$$

so  $\mathbb{Q}/\mathbb{Z}$  is both torsion and divisible.

**Problem (Problem 2):** Let  $R$  be a commutative ring,  $\{N_{\alpha}\}_{\alpha \in A}$  a collection of  $R$ -modules, and  $M$  another  $R$ -module.

- (a) Prove that  $M \otimes (\bigoplus_{\alpha} N_{\alpha}) \cong \bigoplus_{\alpha} (M \otimes N_{\alpha})$ .
- (b) Show by example that  $M \otimes (\prod_{\alpha} N_{\alpha})$  need not be isomorphic to  $\prod_{\alpha} (M \otimes N_{\alpha})$ .

**Solution:**

- (a) Consider the map on elementary tensors

$$f: M \times \left( \bigoplus_{\alpha} N_{\alpha} \right) \rightarrow \bigoplus_{\alpha} (M \otimes N_{\alpha})$$

that takes

$$(m, (n_{\alpha})_{\alpha}) \rightarrow (m \otimes n_{\alpha})_{\alpha}.$$

We observe that, since the  $(n_{\alpha})_{\alpha}$  are nonzero for all but finitely many indices  $\alpha$ , and that the map is  $R$ -bilinear, we have a well-defined and unique  $R$ -linear map  $\bar{f}: M \otimes (\bigoplus_{\alpha} N_{\alpha}) \rightarrow \bigoplus_{\alpha} (M \otimes N_{\alpha})$  that maps  $m \otimes (n_{\alpha})_{\alpha} \mapsto (m \otimes n_{\alpha})_{\alpha}$ .

We observe that for each index  $i$ , we have an inclusion homomorphism

$$M \times N_i \hookrightarrow M \otimes \left( \bigoplus_{\alpha} N_{\alpha} \right)$$

that takes  $(m, n_\alpha) \mapsto m \otimes (n_\alpha)_\alpha$ , where  $(n_\alpha)_\alpha$  is zero everywhere except for index  $i$ . By the universal property of the direct sum, this induces a unique homomorphism  $g: \bigoplus_\alpha (M \otimes N_\alpha) \rightarrow M \otimes (\bigoplus_\alpha N_\alpha)$  given by taking

$$(m_\alpha \otimes n_\alpha)_\alpha \mapsto \sum_\alpha m_\alpha \otimes (n_\alpha)_\alpha,$$

where the summand  $(n_\alpha)_\alpha$  is defined as above, and the sum is finite by the definition of the direct sum. Since  $g$  and  $f$  are inverses of each other (as can be seen by the action on simple tensors), it follows that  $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$ .

- (b) We consider the direct product

$$M = \prod_{i=1}^{\infty} \mathbb{Z}/2^i \mathbb{Z},$$

regarded as a  $\mathbb{Z}$ -module. Notice that  $M$  is not torsion, as the element  $m = (1, 1, \dots)$  is such that there is no  $z \in \mathbb{Z}$  with  $zm = 0$ . Therefore, considering the extension of scalars

$$\mathbb{Q} \otimes M = \mathbb{Q} \otimes \left( \prod_{i=1}^{\infty} \mathbb{Z}/2^i \mathbb{Z} \right),$$

we have that this is not a zero module. Yet, since each of the individual  $\mathbb{Z}/2^i \mathbb{Z}$  has torsion, it would follow that

$$\prod_{i=1}^{\infty} (\mathbb{Q} \otimes \mathbb{Z}/2^i \mathbb{Z}) = 0,$$

so it follows that tensor products do not commute with direct sums.

**Problem (Problem 4):** Let  $R$  be commutative, and let  $I$  and  $J$  be ideals of  $R$ , so  $R/I$  and  $R/J$  are naturally  $R$ -modules.

- (a) Prove that every element of  $R/I \otimes_R R/J$  can be written as a simple tensor of the form  $(1+I) \otimes (r+J)$ .
- (b) Prove that there is an  $R$ -module isomorphism  $R/I \otimes_R R/J \cong R/(I+J)$  mapping  $(r+I) \otimes (r'+J)$  to  $rr' + (I+J)$ .

**Solution:**

- (a) By using  $R$ -bilinearity, we observe that an arbitrary simple tensor in  $R/I \otimes R/J$  can be written as

$$\begin{aligned} (r+I) \otimes (s+J) &= (r(1+I)) \otimes (s+J) \\ &= r((1+I) \otimes (s+J)) \\ &= (1+I) \otimes (rs+J). \end{aligned}$$

Since any element of  $R/I \otimes_R R/J$  can be written as a sum of simple tensors, and each simple tensor can be written in the above form, it follows from bilinearity that every element of  $R/I \otimes R/J$  can be written as  $(1+I) \otimes (r+J)$ .

- (b) We consider the map

$$f: R/I \times R/J \mapsto R/(I+J)$$

given by

$$(r+I, r'+J) \mapsto rr' + (I+J).$$

This map is  $R$ -bilinear by the distributive properties of multiplication, so it induces a homomorphism on the tensor product given by

$$(r + I) \otimes (r' + J) \mapsto rr' + (I + J).$$

As was established above, any element of  $R/I \otimes R/J$  can be written as  $(1 + I) \otimes (s + J)$ , so we may establish an inverse from any element of  $R/(I + J)$  to  $R/I \otimes R/J$  by taking  $t + (I + J) \mapsto (1 + I) \otimes (t + J)$ . This establishes our desired isomorphism.

**Problem (Problem 5):** Let  $I = (2, x)$  be the ideal generated by 2 and  $x$  in the ring  $\mathbb{Z}[x]$ . The ring  $\mathbb{Z}/2\mathbb{Z} = R/I$  is naturally an  $R$ -module annihilated by both 2 and  $x$ .

(a) Show that the map  $\varphi: I \times I \rightarrow \mathbb{Z}/2\mathbb{Z}$  given by

$$\varphi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = \left[ \frac{a_0}{2}b_1 \right]_{\mathbb{Z}/2\mathbb{Z}}$$

is  $R$ -bilinear.

(b) Show that there is an  $R$ -module homomorphism from  $I \otimes_R I \rightarrow \mathbb{Z}/2\mathbb{Z}$  mapping  $p(x) \otimes q(x)$  to  $\frac{p(0)}{2}q'(0)$ , where  $q'$  denotes the usual polynomial derivative of  $q$ .

(c) Show that  $2 \otimes x \neq x \otimes 2$  in  $I \otimes_R I$ .

**Solution:**

(a) By the well-definedness of addition in  $R/I$ , we have that  $\varphi$  is additive in each variable. Now, letting  $p(x) \in R$  and  $a(x), b(x) \in I$  be defined by

$$\begin{aligned} a(x) &= a_0 + a_1x + \cdots + a_nx^n \\ b(x) &= b_0 + b_1x + \cdots + b_mx^m \\ p(x) &= p_0 + p_1x + \cdots + p_\ell x^\ell, \end{aligned}$$

we note that

$$p(x) + I = [p_0]_{\mathbb{Z}/2\mathbb{Z}}.$$

Using various definitions, we see that

$$\begin{aligned} \varphi(p(x)a(x), b(x)) &= \varphi(p_0a_0 + O(x), b_0 + b_1x + \cdots) \\ &= \left[ \frac{p_0a_0}{2}b_1 \right] \\ &= [p_0] \left[ \frac{a_0}{2}b_1 \right] \\ &= (p(x) + I)\varphi(a(x), b(x)), \end{aligned}$$

and since  $b_0 \in I$ ,

$$\begin{aligned} \varphi(a(x), p(x)b(x)) &= \left[ \frac{a_0}{2}(p_0b_1 + p_1b_0) \right] \\ &= \left[ \frac{a_0}{2}(p_0b_1) \right] \\ &= (p(x) + I)\varphi(a(x), b(x)). \end{aligned}$$

Thus,  $\varphi$  is  $R$ -bilinear.

(b) Using the universal property for tensor products, there is a unique  $R$ -linear homomorphism  $\bar{\varphi}: I \otimes_R I \rightarrow \mathbb{Z}/2\mathbb{Z}$  such that

$$\begin{aligned} \bar{\varphi}(a(x) \otimes b(x)) &= \left[ \frac{a_0}{2}b_1 \right] \\ &= \left[ \frac{p(0)}{2}q'(0) \right]. \end{aligned}$$

(c) We observe that  $\bar{\varphi}(2 \otimes x) = 1$  while  $\bar{\varphi}(x \otimes 2) = 0$ , so they cannot be equal to each other in  $I \otimes_R I$ .

**Problem (Problem 6):** Suppose  $R$  is commutative, and let  $I, J$  be ideals of  $R$ .

(a) Show that there is a surjective  $R$ -module homomorphism from  $I \otimes_R J$  to the product ideal  $IJ$  mapping  $i \otimes j$  to  $ij$ .

(b) Give an example to show that the map in (a) need not be injective.

**Solution:**

(a) We define the  $R$ -bilinear map  $\varphi: I \times J \rightarrow IJ$  by

$$\varphi(i, j) = ij.$$

This induces a linear map  $\bar{\varphi}: I \otimes_R J \rightarrow IJ$  such that  $i \otimes j \mapsto ij$ . Since every element of  $I \otimes_R J$  is a finite sum of elementary tensors, this surjects onto  $IJ$  since every element of  $IJ$  is a finite sum of elements of the form  $ij$ .

(b)