These are assorted exercises and solutions from Conway's *A Course in Functional Analysis.*

Spectral Theory

Problem ([Con90, Exercise IX.1.2])**:** Show that the unit ball of $B(H)$ is WOT-compact.

Solution: Consider the set

$$K = \prod_{x,y \in B_H} \overline{\mathbb{D}},$$

where $\mathbb{D}$ represents the complex unit disk and B_H denotes the closed unit ball of H. The space K is compact by Tychonoff's theorem. Let $\phi\colon B_{B(H)} \to K$ be defined by

$$\phi(T) = (\langle Tx, y\rangle)_{x,y}.$$

Observe that by Cauchy–Schwarz, we have that

$$\begin{aligned} |\langle Tx, y\rangle| &\leq \|T\|_{\mathrm{op}} \|x\| \|y\| \\ &\leq 1, \end{aligned}$$

so ϕ is indeed well-defined. Furthermore, ϕ is injective since for any two operators T and S, we have that $T = S$ if and only if $\langle Tx, y\rangle = \langle Sx, y\rangle$ for all $x, y \in B_H$, and ϕ is continuous by the definition of the weak operator topology. Therefore, we only need to show that ϕ has a closed range.

Let $(T_i)_i$ be a net of operators in $B_{B(H)}$ such that

$$\lim_{i \in I} (\langle T_i x, y\rangle)_{x,y} = (z_{x,y})_{x,y}.$$

Then, from the Cauchy–Schwarz inequality, it follows that $(z_{x,y})_{x,y} \in K$, and by the definition of convergence in the product topology, we have, for each x, y,

$$\lim_{i \in I} \langle T_i x, y\rangle = z_{x,y}.$$

Therefore, we may define a semidefinite sesquilinear form $F\colon H \times H \to \mathbb{C}$ given by

$$F(x,y) = \lim_{i \in I} \langle T_i x, y\rangle$$

for each $x, y \in H$. From the structure of sesquilinear forms, it then follows that there is some $T \in B_{B(H)}$ such that $F(x,y) = \langle Tx, y\rangle$, and thus $(T_i)_{i \in I} \to T$ in WOT.

Problem ([Con90, Exercise IX.1.13])**:** A representation $\rho\colon A \to B(H)$ is *irreducible* if the only projections in $B(H)$ that commute with every $\rho(a)$ for $a \in A$ are 0 and 1. Prove that if A is abelian and ρ is an irreducible representation, then $\dim(H) = 1$. Find the corresponding spectral measure.

Solution: Since A is abelian, so too is $\rho(A)$, meaning that $\rho(A) \subseteq \rho(A)'$. Since $\rho(A)' = \mathbb{C}1$ by the assumption of irreducibility, it follows then that $\rho(A) = \mathbb{C}1$, whence $H = [\rho(A)v] = \mathbb{C}v$.

Without loss of generality, we may assume that $A = C_0(X)$ for some locally compact Hausdorff space X, and $\rho\colon C_0(X) \to \mathbb{C}$ is a character. The characters of $C_0(X)$ are given by evaluation at $x_0 \in X$, meaning that their corresponding spectral measure is the Dirac mass δ_{x_0}.

Problem ([Con90, Exercise IX.2.1])**:** Show that $\lambda \in \sigma_p(N)$ if and only if $E(\{\lambda\}) \neq 0$. Moreover, if $\lambda \in \sigma_P(N)$, then $E(\{\lambda\})$ is the orthogonal projection onto $\ker(N - \lambda I)$.

Solution: Suppose $E(\{\lambda\}) \neq 0$, meaning that

$$E(\{\lambda\}) = \int_{\sigma(N)} \mathbb{1}_{\{\lambda\}}\, dE$$

$$\neq 0.$$

Since, for all $x \in H$, we have

$$\begin{aligned}(N-\lambda I)E(\{\lambda\})x &= \left(\int_{\sigma(N)} (z-\lambda)\mathbb{1}_{\{\lambda\}}\,dE\right)x\\ &= E(\{\lambda\})(N-\lambda I)x\\ &= 0,\end{aligned}$$

it follows that $E(\{\lambda\})x \in \ker(N-\lambda I)$, so that $E(\{\lambda\}) \leq P_\lambda$, where P_λ is the projection onto $\ker(N-\lambda I)$. Since $E(\{\lambda\}) > 0$, it follows that $P_\lambda > 0$, so P_λ is nontrivial, meaning $\ker(N-\lambda I)$ is nontrivial, so $\lambda \in \sigma_p(N)$.

Now, let $\lambda \in \sigma_p(N)$. We start by supposing that $\lambda \neq 0$. If $x \in \ker(T-\lambda I)$ is nonzero, then we have

$$\begin{aligned}Tx &= \lambda x\\ &= \left(\int_{\sigma(N)} \lambda\,dE\right)x\\ &= \left(\int_{\sigma(N)} \mathbb{1}_{\{\lambda\}} z\,dE\right)x\\ &= E(\{\lambda\})Tx\\ &= \lambda E(\{\lambda\})x,\end{aligned}$$

so $E(\{\lambda\})x = x$, meaning that $E(\{\lambda\}) \geq P_\lambda$. In particular, from what we have established above, this means that $E(\{\lambda\}) = P_\lambda$, so $E(\{\lambda\}) \neq 0$. If $\lambda = 0$, then we shift N by subtracting a factor of I, perform the same process, and shift back.

Problem ([Con90, Exercise IX.2.10])**:** Let A be a hermitian operator with spectral measure E on a separable space. For each real number t, define a projection $P(t) = E(-\infty, t)$. Show:

(a) $P(s) \leq P(t)$ for $s \leq t$;

(b) if $t_n \leq t_{n+1}$ and $(t_n)_n \to t$, then $P(t_n) \to P(t)$ in SOT;

(c) for all but a countable number of points t, $P(t_n) \to P(t)$ in SOT if $(t_n)_n \to t$;

(d) for any $f \in C(\sigma(A))$, we have

$$f(A) = \int_{-\infty}^{\infty} f(t)\,dP(t),$$

where the integral is defined in the Riemann–Stieltjes sense.

Solution: Let $x \in H$. Then, we observe that

$$\begin{aligned}\langle P(s)x, x\rangle &= \int_{\sigma(A)} \mathbb{1}_{(-\infty,s)}\,dE_{x,x}\\ &= \int_{\sigma(A)\cap(-\infty,s)} dE_{x,x}\\ &\leq \int_{\sigma(A)\cap(-\infty,t)} dE_{x,x}\\ &= \langle P(t)x, x\rangle.\end{aligned}$$

Since x is arbitrary, and the measure $E_{x,x}$ is real by the assumption that A is hermitian, it follows that $P(s) \leq P(t)$ whenever $s \leq t$. This shows (a).

To show (b), we recall that a sequence of projections $(P_n)_n \to P$ in SOT if and only if it converges in WOT, as

$$\begin{aligned}\langle (P_n - P)x, x\rangle &= \left\langle (P_n - P)^2 x, x\right\rangle \\ &= \langle (P_n - P)x, (P_n - P)x\rangle \\ &= \|(P_n - P)x\|^2.\end{aligned}$$

We thus see that if $(t_n)_n \nearrow t$, we have for any $x \in H$,

$$\begin{aligned}|\langle (P(t_n) - P(t))x, x\rangle| &= \left|\int_{\sigma(A)} \mathbb{1}_{(-\infty,t_n)} - \mathbb{1}_{(-\infty,t)}\, dE_{x,x}\right| \\ &= |E_{x,x}(\sigma(A) \cap (-\infty, t_n)) - E_{x,x}(\sigma(A) \cap (-\infty, t))|,\end{aligned}$$

which by continuity from below for measures converges to zero. Thus, $(P(t_n))_n \to P(t)$ in WOT, so it converges in SOT.

To establish (c), we fix $x \in H$. Then, the function

$$f(t) = \langle P(t)x, x\rangle$$

is a monotone increasing right-continuous function on $\mathbb{R}$. In particular, this means that $f(t)$ has at most a countable number of discontinuities, meaning that for any sequence $(t_n)_n \to t$, it follows that $P(t_n) \to P(t)$ in WOT, hence in SOT, everywhere outside these countable number of discontinuities.

Problem ([Con90, Exercise IX.2.14]): Prove that if A is hermitian, $\exp(iA)$ is unitary. Is the converse true?

Solution: Since $\sigma(A) \subseteq \mathbb{R}$, we have

$$\begin{aligned}\exp(iA) &= \int_{\sigma(A)} e^{ix}\, dE \\ \exp(iA)^* &= \int_{\sigma(A)} e^{-ix}\, dE \\ \exp(iA)\exp(iA)^* &= \int_{\sigma(A)} dE \\ &= \exp(iA)^* \exp(iA),\end{aligned}$$

so that $\exp(iA)$ is unitary.

Similarly, there is a continuous bijection $f\colon [0, 2\pi) \to S^1$ given by $t \mapsto e^{it}$, with Borel-measurable inverse g, so that if V is any unitary operator, we may define

$$A = g(V),$$

which has $\exp(iA) = V$. Since g is real-valued, it follows that A is hermitian.

Problem ([Con90, Exercise IX.2.22]): Prove that if U is any unitary operator on H, then there is a continuous function $u\colon [0, 1] \to B(H)$ such that $u(0) = U$, $u(1) = I$, and $u(t)$ is unitary for each t.

Solution: Since U is unitary, there is a hermitian operator A such that $U = \exp(iA)$. We may then define the continuous map

$$\begin{aligned}u\colon [0, 1] &\to B(H) \\ t &\mapsto \exp(i(1 - t)A).\end{aligned}$$

Since $(1 - t)A$ is also a hermitian operator, and dominated convergence gives that this is a continuous map, this is thus our desired map.

Problem ([]): If N is normal, show that there is a sequence of invertible normal operators that converges to N.

Solution: We observe that the sequence of functions

$$f_n(z) = z\mathbb{1}_{\sigma(N)\setminus\{0\}} + \frac{1}{n}\mathbb{1}_{\{0\}}$$

converges pointwise to z, is nonzero everywhere on $\sigma(N)$, and is bounded above by the necessarily integrable function

$$h(z) = \left(\sup_{z\in\sigma(N)} |z| + 1\right)\mathbb{1}_{\sigma(N)}.$$

So, the operator

$$f_n(N) = \int_{\sigma(N)} f_n(z)\, dE$$

is integrable for each N, with $f_n(N) \to N$ by dominated convergence.

References

[Con90] John B. Conway. *A Course in Functional Analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.