Prelude

My REU mentor recently bought me the book *Banach Algebra Techniques in Operator Theory*, so I'm going to be reading through it here. Astute readers may already know that I am also reading through the book *Quantum Theory for Mathematicians*, and may be wondering if this is going to crowd out that book. The answer is yes — but I don't really care that much. If I come out of the summer knowing more things than I knew entering, then I will have succeeded.

Banach Spaces

Let X be a compact Hausdorff space, and let $C(X)$ denote the set of continuous functions $f : X \to \mathbb{C}$. For $f_1, f_2 \in C(X)$ and $\lambda \in \mathbb{C}$, we define

(1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

(2) $(\lambda f_1)(x) = \lambda f_1(x)$

(3) $(f_1 f_2)(x) = f_1(x) f_2(x)$

With these operations, $C(X)$ is a commutative algebra[i] with identity over the field $\mathbb{C}$.

For each $f \in C(X)$, f is bounded (since X is compact and f is continuous); thus, $\sup |f| < \infty$. We call this the norm of f, and denote it

$$\|f\|_\infty = \sup \{|f(x)| \mid x \in X\} .$$

Proposition (Properties of the Norm on $C(X)$)**.**

(1) Positive Definiteness: $\|f\|_\infty = 0 \Leftrightarrow f = 0$

(2) Absolute Homogeneity: $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$

(3) Subadditivity (Triangle Inequality): $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

(4) Submultiplicativity: $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$

We define a metric ρ on $C(X)$ by $\rho(f, g) = \|f - g\|_\infty$.

Proposition (Properties of the Induced Metric on $C(X)$)**.**

(1) $\rho(f, g) = 0 \Leftrightarrow f = g$

(2) $\rho(f, g) = \rho(g, f)$

(3) $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$

[i] A vector space with multiplication.

Proposition (Completeness of $C(X)$)**.** *If X is a compact Hausdorff space, then $C(X)$ is a complete metric space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be Cauchy. Then,

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq \|f_n - f_m\|_{\infty}\\ &= \rho(f_n, f_m)\end{aligned}$$

for each $x \in X$. Thus, $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy for each $x \in X$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We will need to show that this implies $\lim_{n\to\infty} \|f_n - f\|_{\infty} = 0$.

Let $\varepsilon > 0$; choose N such that $n, m \geq N$ implies $\|f_n - f_m\|_{\infty} < \varepsilon$. For $x_0 \in X$, there exists a neighborhood U such that $|f_N(x_0) - f_N(x)| < \varepsilon$ for $x \in U$.[ii] Thus,

$$\begin{aligned}|f(x_0) - f(x)| &= |f_n(x_0) - f_N(x_0) + f_N(x_0) - f_N(x) + f_N(x) - f_n(x)|\\ &\leq |f_n(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f_n(x)|\\ &\leq 3\varepsilon.\end{aligned}$$

Thus, f is continuous. Additionally, for $n \geq N$ and $x \in X$, we have

$$\begin{aligned}|f_n(x) - f(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)|\\ &\leq \lim_{m\to\infty} \|f_n - f_m\|_{\infty}\\ &\leq \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \|f_n - f\|_{\infty} = 0$, meaning $C(X)$ is complete. □

Definition (Banach Space)**.** A Banach space is a vector space over $\mathbb{C}$ with a norm $\|\cdot\|$ is complete with respect to the induced metric.

Proposition (Properties of the Banach Space Operations)**.** *Let $\mathcal{X}$ be a Banach space. The functions*

- $a : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$; $a(f, g) = f + g$,
- $s : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$; $s(\lambda, f) = \lambda f$,
- $n : \mathcal{X} \to \mathbb{R}^{+}$; $n(f) = \|f\|$

are continuous.

Definition (Directed Sets and Nets)**.** Let A be a partially ordered set with ordering $\leq$. We say A is directed if for each $\alpha, \beta \in A$, there exists a γ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

A net is a map $\alpha \mapsto \lambda_{\alpha}$, where $\alpha \in A$ for some directed set A.

[ii] This is by the continuity of $\{f_n\}_n$.

Definition (Convergence of Nets)**.** Let $\{\lambda_\alpha\}$ be a net in X. We say the net converges to $\lambda \in X$ if for every neighborhood U of λ, there exists α_U such that for $\alpha \geq \alpha_U$, every λ_α is contained in U.[iii]

Definition (Cauchy Nets in Banach Spaces)**.** A net $\{f_\alpha\}_\alpha$ in a Banach space $\mathcal{X}$ is said to be a Cauchy net if for every $\varepsilon > 0$, there exists α_0 in A such that $\alpha_1, \alpha_2 \geq \alpha_0$ implies $\|f_{\alpha_1} - f_{\alpha_2}\| < \varepsilon$.

Proposition (Convergence of Cauchy Nets in Banach Spaces)**.** *In a Banach space, every Cauchy net is convergent.*

Proof. Let $\{f_\alpha\}_\alpha$ be a Cauchy net in $\mathcal{X}$. Choose α_1 such that $\alpha \geq \alpha_1$ implies $\|f_\alpha - f_{\alpha_1}\| < 1$.

We iterate this process by choosing $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies $\left\|f_\alpha - f_{\alpha_{n+1}}\right\| < \frac{1}{n+1}$.

The sequence $\{f_{\alpha_n}\}_{n=1}^{\infty}$ is Cauchy, and since $\mathcal{X}$ is complete, there exists $f \in \mathcal{X}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.

We must now prove that $\lim_{\alpha\in A} f_\alpha = f$. Let $\varepsilon > 0$. Choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$, and $\|f_{\alpha_n} - f_\alpha\| < \frac{\varepsilon}{2}$. Then, for $\alpha \geq \alpha_n$, we have

$$\begin{aligned} \|f_\alpha - f\| &\leq \|f_\alpha - f_{\alpha_n}\| + \|f_{\alpha_n} - f\| \\ &< \frac{1}{n} + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

$\square$

Definition (Convergence of Infinite Series)**.** Let $\{f_\alpha\}_\alpha$ be a set of vectors in $\mathcal{X}$. Let $\mathcal{F} = \{F \subseteq A \mid F \text{ finite}\}$.

Define the ordering $F_1 \leq F_2 \Leftrightarrow F_1 \subseteq F_2$.[iv] For each F, define

$$g_F = \sum_{\alpha\in F} f_\alpha.$$

If $\{g_F\}_{F\in\mathcal{F}}$ converges to some $g \in \mathcal{X}$, then

$$\sum_{\alpha\in A} f_\alpha$$

converges, and we write

$$g = \sum_{\alpha\in A} f_\alpha.$$

[iii] The net convergence generalizes sequence convergence in a metric space to the case where X does not have a metric.

[iv] the inclusion ordering

Proposition (Absolute Convergence of Series in Banach Space). *Let $\{f_\alpha\}_\alpha$ be a set of vectors in the Banach space $\mathcal{X}$. Suppose $\sum_{\alpha\in A}\|f_\alpha\|$ converges in $\mathbb{R}$. Then, $\sum_{\alpha\in A} f_\alpha$ converges in $\mathcal{X}$.*

Proof. All we need show is $\{g_F\}_{F\in\mathcal{F}}$ is Cauchy. Since $\sum_{\alpha\in A}\|f_\alpha\|$ converges, there exists $F_0\in\mathcal{F}$ such that $F\geq F_0$ implies

$$\sum_{\alpha\in F}\|f_\alpha\| - \sum_{\alpha\in F_0}\|f_\alpha\| < \varepsilon.$$

Thus, for $F_1, F_2 \geq F_0$, we have

$$\begin{aligned}
\|g_{F_1} - g_{F_2}\| &= \left\|\sum_{\alpha\in F_1} f_\alpha - \sum_{\alpha\in F_2} f_\alpha\right\| \\
&= \left\|\sum_{\alpha\in F_1\setminus F_2} f_\alpha - \sum_{\alpha\in F_2\setminus F_1}\right\| \\
&\leq \sum_{\alpha\in F_1\setminus F_2}\|f_\alpha\| + \sum_{\alpha\in F_2\setminus F_1}\|f_\alpha\| \\
&\leq \sum_{\alpha\in F_1\cup F_2}\|f_\alpha\| - \sum_{\alpha\in F_0}\|f_\alpha\| \\
&< \varepsilon.
\end{aligned}$$

Thus, $\{g_F\}_{F\in\mathcal{F}}$ is Cauchy, and thus the series is convergent. $\square$

Theorem (Absolute Convergence Criterion for Banach Spaces). *Let $\mathcal{X}$ be a normed vector space. Then, $\mathcal{X}$ is a Banach space if and only if for every sequence $\{f_n\}_{n=1}^\infty$ of vectors in $\mathcal{X}$,*

$$\sum_{n=1}^\infty \|f_n\| < \infty \Rightarrow \sum_{n=1}^\infty f_n \text{ convergent.}$$

Proof. The forward direction follows from the previous proposition.

Let $\{g_n\}_{n=1}^\infty$ be a Cauchy sequence in a normed vector space where

$$\sum_{n=1}^\infty \|f_n\| < \infty \Rightarrow \sum_{n=1}^\infty f_n \text{ convergent.}$$

We select a subsequence $\{g_{n_k}\}_{k=1}^\infty$ as follows. Choose n_1 such that $i,j\geq n_1$ implies $\|g_i - g_j\| < 1$; recursively, we select n_{N+1} such that $\|g_{N+1} - g_N\| < 2^{-N}$. Then,

$$\sum_{k=1}^\infty \|g_{k+1} - g_k\| < \infty.$$

Set $f_k = g_{n_k} - g_{n_{k-1}}$ for $k > 1$, with $f_1 = g_{n_1}$. Then,

$$\sum_{k=1}^{\infty} \|f_k\| < \infty,$$

meaning $\sum_{k=1}^{\infty} f_k$ converges. Thus, $\{g_{n_k}\}_{k=1}^{\infty}$ converges, meaning $\{g_n\}_{n=1}^{\infty}$ converges in $\mathcal{X}$. □

Definition (Bounded Linear Functional)**.** Let $\mathcal{X}$ be a Banach space. A function $\varphi : \mathcal{X} \to \mathbb{C}$ is known as a bounded linear functional if

(1) $\varphi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \varphi(f_1) + \lambda_2 \varphi(f_2)$ for each $\lambda_1, \lambda_2 \in \mathbb{C}$ and $f_1, f_2 \in \mathcal{X}$.

(2) There exists M such that $|\varphi(f)| \leq M \|f\|$ for each $f \in \mathcal{X}$.

Proposition (Equivalent Criteria for Bounded Linear Functionals)**.** *Let φ be a linear functional on $\mathcal{X}$. Then, the following conditions are equivalent:*

(1) φ is bounded;

(2) φ is continuous;

(3) φ is continuous at 0.

Proof. (1) $\Rightarrow$ (2): If $\{f_\alpha\}_{\alpha \in A}$ is a net in $\mathcal{X}$ converging to f, then $\lim_{\alpha \in A} \|f_\alpha - f\| = 0$. Thus,

$$\begin{aligned}
\lim_{\alpha \in A} |\varphi(f_\alpha) - \varphi(f)| &= \lim_{\alpha \in A} |\varphi(f_\alpha - f)| \\
&\leq \lim_{\alpha \in F} M \|f_\alpha - f\| \\
&= 0
\end{aligned}$$

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): If φ is continuous at 0, then there exists $\delta > 0$ such that $\|f\| < \delta \Rightarrow |\varphi(f)| < 1$. Thus, for any $g \in X$ nonzero, we have

$$\begin{aligned}
|\varphi(g)| &= \frac{2\|g\|}{\delta} \left| \varphi\left(\frac{\delta}{2\|g\|} g \right) \right| \\
&< \frac{2}{\delta} \|g\|,
\end{aligned}$$

meaning φ is bounded.

□

Definition (Dual Space)**.** Let $\mathcal{X}^*$ be the set of bounded linear functionals on $\mathcal{X}$. For each $\varphi \in \mathcal{X}^*$, define

$$\|\varphi\| = \sup_{\|f\|=1} |\varphi(f)|.$$

We say $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Proposition (Completeness of the Dual Space)**.** *For $\mathcal{X}$ a Banach space, $\mathcal{X}^*$ is a Banach space.*

Proof. Both positive definiteness and absolute homogeneity are apparent from the definition of the norm. We will now show the triangle inequality as follows. Let $\varphi_1, \varphi_2 \in \mathcal{X}^*$. Then,

$$\begin{aligned}\|\varphi_1 + \varphi_2\| &= \sup_{\|f\|=1} |\varphi_1(f) + \varphi_2(f)| \\ &\leq \sup_{\|f\|=1} |\varphi_1(f)| + \sup_{\|f\|=1} |\varphi_2(f)| \\ &= \|\varphi_1\| + \|\varphi_2\|.\end{aligned}$$

We must now show completeness. Let $\{\varphi_n\}_n$ be a sequence in $\mathcal{X}^*$. Then, for every $f \in \mathcal{X}$, it is the case that

$$|\varphi_n(f) - \varphi_m(f)| \leq \|\varphi_n - \varphi_m\| \|f\|,$$

meaning $\{\varphi_n(f)\}_n$ is Cauchy for each f. Define $\varphi(f) = \lim_{n\to\infty} \varphi_n(f)$. It is clear that $\varphi(f)$ is linear, and for N such that $n, m \geq N \Rightarrow \|\varphi_n - \varphi_m\| < 1$,

$$\begin{aligned}|\varphi(f)| &\leq |\varphi(f) - \varphi_N(f)| + |\varphi_N(f)| \\ &\leq \lim_{n\to\infty} |\varphi_n(f) - \varphi_N(f)| + |\varphi_N(f)| \\ &\leq \left(\lim_{n\to\infty} \|\varphi_n - \varphi_N\| + \|\varphi_N\|\right) \|f\| \\ &\leq (1 + \|\varphi_N\|) \|f\|,\end{aligned}$$

so φ is bounded. Thus, we must show that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$. Let $\varepsilon > 0$. Set N such that $n, m \geq N \Rightarrow \|\varphi_n - \varphi_m\| < \varepsilon$. Then, for $f \in \mathcal{X}$,

$$\begin{aligned}|\varphi(f) - \varphi_n(f)| &\leq |\varphi(f) - \varphi_m(f)| + |\varphi_m(f) - \varphi_n(f)| \\ &\leq |(\varphi - \varphi_m)(f)| + \varepsilon \|f\|.\end{aligned}$$

Since $\lim_{m\to\infty} |(\varphi - \varphi_m)(f)| = 0$, we have $\|\varphi - \varphi_m\| < \varepsilon$. □

Proposition (Banach Spaces and their Duals)**.**

(1) The space ℓ^∞ consists of the set of bounded sequences. For $f \in \ell^\infty$, the norm on f is computed as $\|f\|_\infty = \sup_n |f(n)|$.

(2) The subspace $c_0 \subseteq \ell^\infty$ consists of all sequences that vanish at ∞. The norm on c_0 is inherited from the norm on ℓ_∞.

(3) The space ℓ^1 consists of the set of all absolutely summable sequences. For $f \in \ell^1$, the norm on f is computed as $\|f\| = \sum_{n=1}^{\infty} |f(n)|$.

We claim that these are all Banach spaces.

We also claim that $c_0^* = \ell^1$, and $(\ell^1)^* = \ell^\infty$.

Proof of Banach Space.

ℓ^∞:

Proof of Normed Vector Space: Let $a, b \in \ell^\infty$, and $\lambda \in \mathbb{C}$. Then,

$$\sup_n |a(n)| = 0$$

if and only if a is the zero sequence. Additionally, we have that

$$\begin{aligned} \|\lambda a\|_\infty &= \sup_n |\lambda a(n)| \\ &= |\lambda| sup_n |a(n)| \\ &= |\lambda| \, \|a\|_\infty, \end{aligned}$$

meaning $\|\cdot\|_\infty$ is absolutely homogeneous. Finally,

$$\begin{aligned} \|a + b\|_\infty &= \sup_n |a(n) + b(n)| \\ &\leq \sup_n |a(n)| + \sup_n |b(n)| \\ &= \|a\|_\infty + \|b\|_\infty. \end{aligned}$$

Proof of Completeness: Let $\{a_n\}_{n=1}^\infty$ be a Cauchy sequence of elements of ℓ^∞. Let $\varepsilon > 0$, and let N be such that $\|a_n - a_m\|_\infty < \varepsilon$ for $n, m \geq N$. Then, for each k,

$$\begin{aligned} |a_n(k) - a_m(k)| &= |(a_n - a_m)(k)| \\ &\leq \|a_n - a_m\| \\ &< \varepsilon, \end{aligned}$$

meaning that $a_n(k)$ is Cauchy in $\mathbb{C}$ for each k.

Set $a(k) = \lim_{n\to\infty} a_n(k)$. We must now show that $\lim_{n\to\infty} \|a - a_n\| = 0$. Let $\varepsilon > 0$, and set N such that for $n, m \geq N$, $\|a_m - a_n\| < \varepsilon$. Then,

$$\begin{aligned} |a(k) - a_n(k)| &\leq |a(k) - a_m(k)| + |a_m(k) - a_n(k)| \\ &\leq |a(k) - a_m(k)| + \|a_m - a_n\| \\ &< |a(k) - a_m(k)| + \varepsilon. \end{aligned}$$

Since $\lim_{m\to\infty} |a(k) - a_m(k)| = 0$, we have $\|a - a_n\| < \varepsilon$.[v]

□

[v] The reason we had to go about it like this was that we defined the sequence *a* pointwise; however, we need to show convergence *in norm*, not only pointwise.