Contents

Introduction

This is going to be a part of my Honors thesis independent study, focused on amenability and C*-algebras. This section of notes will be a deeper dive into group amenability. These notes will be taken from the notes my professor has prepared on group amenability, with supplementation from Volker Runde's *Lectures on Amenability* and Pierre de la Harpe's *Topics in Geometric Group Theory*.

I do not claim any of this work to be original.

Basics of Amenable Groups and Subgroups

Let G be a group, with $P(G)$ denoting the power set.

Definition. An invariant mean on G is a set function $m\colon P(G) \to [0,1]$, which satisfies, for all $t \in G$ and $E, F \subseteq G$,

(1) $m(G) = 1$;

(2) $m(E \sqcup F) = M(E) + m(F)$;

(3) $m(tE) = m(E)$.

We say G is amenable if it admits a mean.

We can also say that m is a translation-invariant probability measure on $(G, P(G))$.

Proposition (Amenability of Subgroups and Quotient Groups)**:** Let G be amenable, with $H \leqslant G$.

(1) H is amenable;

(2) for $H \trianglelefteq G$, G/H is amenable.

Proof.

(1) Let R be a right transversal for H (i.e., selecting one element of each right coset of H to make up R).

If m is a mean for G, we set

$$\lambda\colon \mathcal{P}(H) \to [0,1]$$

by $\lambda(E) = m(ER)$. We have

$$\begin{aligned}\lambda(H) &= m(HR)\\ &= m(G)\\ &= 1.\end{aligned}$$

We claim that if $E \cap F = \emptyset$, then $ER \cap FR = \emptyset$, since if we suppose toward contradiction that $ER \cap FR \neq \emptyset$, then $xr_1 = yr_2$ for some $x \in E, y \in F$ and $r_1, r_2 \in R$. Then, we must have $r_2 r_1^{-1} = y^{-1}x \in H$, meaning $r_1 = r_2$ and $x = y$, which means $E \cap F \neq \emptyset$.

Thus, we have

$$\begin{aligned}\lambda(E \sqcup F) &= m((E \sqcup F)R)\\ &= m(ER \sqcup FR)\\ &= m(ER) + m(FR)\\ &= \lambda(E) + \lambda(F),\end{aligned}$$

and

$$\begin{aligned}\lambda(sE) &= m(sER)\\ &= m(ER)\\ &= \lambda(E).\end{aligned}$$

(2) For the canonical projection map $\pi \colon G \to G/H$ defined by $\pi(t) = tH$, we define

$$\lambda \colon P(G/H) \to [0,1]$$

by $\lambda(E) = m\left(\pi^{-1}(E)\right)$. We have

$$\begin{aligned}\lambda(G/H) &= m\left(\pi^{-1}(G/H)\right)\\ &= m(G)\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(E \sqcup F) &= m\left(\pi^{-1}(E \sqcup F)\right)\\ &= m\left(\pi^{-1}(E) \sqcup \pi^{-1}(F)\right)\\ &= m\left(\pi^{-1}(E)\right) + m\left(\pi^{-1}(F)\right)\\ &= \lambda(E) + \lambda(F).\end{aligned}$$

To show translation-invariance, we let $sH = \pi(s) \in G/H$, and $E \subseteq G/H$. Note that

$$\pi^{-1}(\pi(s)E) = s\pi^{-1}(E),$$

since for $r \in s\pi^{-1}(E)$, we have $r = st$ for $\pi(t) \in E$, so $\pi(r) = \pi(st) = \pi(s)\pi(t) \in \pi(s)E$.

Additionally, if $r \in \pi^{-1}(\pi(s)E)$, then $\pi(r) \in \pi(s)E$, so $\pi\left(s^{-1}r\right) \in E$, and $s^{-1}r \in \pi^{-1}(E)$. Thus, we have

$$\begin{aligned}\lambda(\pi(s)E) &= m\left(\pi^{-1}(\pi(s)E)\right)\\ &= m\left(s\pi^{-1}(E)\right)\\ &= m\left(\pi^{-1}(E)\right)\\ &= \lambda(E).\end{aligned}$$

$\square$

Understanding Free Groups

In the Tarski's Theorem notes, we discussed a little bit the ramifications of the free group on two generators being paradoxical. In order to better understand free groups, we will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory* and Clara Löh's *Geometric Group Theory: An Introduction*.

Groups specified by Generating Sets

Definition. Let G be a group and $S \subseteq G$ be a subset. The subgroup generated by S is the intersection of all subgroups of G that contain S. We write $\langle S\rangle_G$. We say S generates G if $\langle S\rangle_G = G$.

A group is called finitely generated if it contains a finite subset that contains the group in question.

Definition (Characterization of a Generated Subgroup)**.** We can characterize a generated subgroup by S as follows:

$$\langle S\rangle_G = \left\{ s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n} \;\middle|\; n \in \mathbb{N},\ s_1,\dots,s_n \in S,\ \varepsilon_1,\dots,\varepsilon_n \in \{-1,1\} \right\}.$$

Example (Generating Sets)**.**

- If G is a group, then G is a generating set of G.
- The trivial group is generated by the empty set.
- The set $\{1\}$ generates $\mathbb{Z}$, as does $\{2,3\}$. However, $\{2\}$ and $\{3\}$ alone do not generate $\mathbb{Z}$.
- Let X be a set. The symmetric group S_X is finitely generated if and only if X is finite.

Free Groups

Definition. Let S be a set. A group F containing S is said to be freely generated if, for every group G and every map $\varphi\colon S \to G$, there is a unique group homomorphism $\overline{\varphi} : F \to G$ extending φ. The following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\iota"'] & G \\ F \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

A group is free if it contains a free generating set.

Example.

- The additive group $\mathbb{Z}$ is freely generated by $\{1\}$. The additive group $\mathbb{Z}$ is *not* freely generated by $\{2,3\}$, or $\{2\}$, or $\{3\}$. In particular, not every generating set of a group contains a free generating set.
- The trivial group is freely generated by the empty set.
- Not every group is free — the additive groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$ are not free.

We will use the universal property of free groups to show their uniqueness up to isomorphism.

Proposition: Let S be a set. Then, there is at most one group freely generated by S up to isomorphism.

Proof. Let F and F' be two groups freely generated by S, with inclusions of φ and φ' respectively. Because F is freely generated by S, there is a group homomorphism $\overline{\varphi}'\colon F \to F'$ that extends φ — i.e., that $\overline{\varphi}' \circ \varphi = \varphi'$.

Similarly, there is a group homomorphism $\overline{\varphi}\colon F' \to F$ with $\overline{\varphi} \circ \varphi' = \varphi$.

$$
\begin{tikzcd}
S \arrow[r, "\varphi'"] \arrow[d, hook, "\varphi"'] & F' \\
F \arrow[ur, "\overline{\varphi}'"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi'"'] & F \\
F' \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}
$$

We will show that $\overline{\varphi} \circ \overline{\varphi}' = \mathrm{id}_F$, and $\overline{\varphi}' \circ \overline{\varphi} = \mathrm{id}_{F'}$. The composition $\overline{\varphi} \circ \overline{\varphi}'$ is a group homomorphism that makes the following diagram commute.

$$
\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi"'] & F \\
F \arrow[ur, "\overline{\varphi}\circ\overline{\varphi}'"'] &
\end{tikzcd}
$$

Since id_F is a group homomorphism contained in this diagram, and F is freely generated by S, we must have $\overline{\varphi} \circ \overline{\varphi}' = \mathrm{id}_F$. Similarly, we must have $\overline{\varphi}' \circ \overline{\varphi} = \mathrm{id}_{F'}$. $\square$

Theorem (Existence of Free Groups)**:** Let S be a set. There exists a group freely generated by S. This group is unique up to isomorphism.

Proof. We want to construct a group consisting of "words" made up of the elements of S and their "inverses," then modding out by the natural cancellation rules.

We consider the alphabet

$$A = S \cup \hat{S}.$$

Here, $\hat{S} = \{\hat{s} \mid s \in S\}$ is a disjoint copy of S, such that $\hat{s}$ will serve as the inverse of s in the group we will construct.

We define A^* to be the set of all finite sequences over the alphabet A, including the empty word ϵ. We define the operation $A^* \times A^* \to A^*$ by concatenation. This operation is associative with neutral element ϵ.

We define

$$F(S) = A^*/\sim,$$

where $\sim$ is the equivalence relation generated by, for all $x, y \in A^*$ and $s \in S$, $xs\hat{s}y \sim xy$ and $x\hat{s}sy \sim xy$.

We denote the equivalence classes with respect to $\sim$ by $[\cdot]$.

Concatenation induces a well-defined operation $F(S) \times F(S) \to F(S)$ by

$$[x][y] = [xy]$$

for $x, y \in A^*$.

We claim that $F(S)$ with the defined concatenation is a group. We can see that $[\epsilon]$ is a neutral element for the operation, and associativity of the operation is inherited from the associativity of the operation on A^*.

To find inverses, we define $I\colon A^* \to A^*$ by $I(\epsilon) = \epsilon$, and

$$
\begin{aligned}
I(sx) &= I(x)\hat{s} \\
I(\hat{s}x) &= I(x)s
\end{aligned}
$$

for all $x \in A^*$ and $s \in S$. Induction shows that $I(I(x)) = x$, and

$$[I(x)][x] = [I(x)x]$$

$$= [\epsilon]$$

for all $x \in A^*$. Thus, we must also have

$$\begin{aligned}[x][I(x)] &= [I(I(x))][I(x)]\\ &= [\epsilon].\end{aligned}$$

Thus, we see that there are inverses in $F(S)$.

To see that $F(S)$ is freely generated by S, we let $\iota\colon S \to F(S)$ be the map given by sending a letter in $S \subseteq A^*$ to its equivalence class in $F(S)$. By construction, $F(S)$ is generated by the subset $\iota(S) \subseteq F(S)$.

We do not know yet that ι is injective, so we take a bit of a detour. We show that for every group G and every map $\varphi\colon S \to G$, there is a unique group homomorphism $\overline{\varphi}\colon F(S) \to G$ such that $\overline{\varphi} \circ \iota = \varphi$.

We construct a map $\varphi^*\colon A^* \to G$ inductively by

$$\begin{aligned}\epsilon &\mapsto e\\ sx &\mapsto \varphi(s)\varphi^*(x)\\ \hat{s}x &\mapsto (\varphi(s))^{-1}\varphi^*(x)\end{aligned}$$

for all $s \in S$ and $x \in A^*$. We can see that, since the definition of φ^* is compatible with the generating set of $\sim$, it is compatible with the equivalence relation of $\sim$ on A^*. Additionally, we can see that $\varphi^*(xy) = \varphi^*(x)\varphi^*(y)$ for all $x, y \in A^*$. Thus,

$$\begin{aligned}\overline{\varphi}\colon F(S) &\to G\\ [x] &\mapsto [\varphi^*(x)],\end{aligned}$$

is, as constructed, a group homomorphism, with $\overline{\varphi} \circ \iota = \varphi$. Since $\iota(S)$ generates $F(S)$, this group homomorphism is unique.

We must now show that ι is injective.

Let $s_1, s_2 \in S$. Consider the map $\varphi\colon S \to \mathbb{Z}$ given by $\varphi(s_1) = 1$ and $\varphi(s_2) = -1$. The corresponding homomorphism $\overline{\varphi}\colon F(S) \to G$ satisfies

$$\begin{aligned}\overline{\varphi}(\iota(s_1)) &= \varphi(s_1)\\ &= 1\\ &\neq -1\\ &= \varphi(s_2)\\ &= \overline{\varphi}(\iota(s_2)),\end{aligned}$$

meaning $\iota(s_1) \neq \iota(s_2)$. Thus, ι is injective. $\square$

Free Groups, Free Products, and the Ping Pong Lemma

We can now consider free groups in a more categorical context — specifically, as a special type of free object. Whereas the previous section drew information from Clara Löh's *Geometric Group Theory: An Introduction*, this section will draw information from Pierre de la Harpe's *Topics in Geometric Group Theory*. Specifically, we are focused on chapter 2, which discusses free products, free groups, and the ping pong lemma (which will provide a more general proof of the paradoxicality of $SO(3)$).

Definition (Free Monoid)**.** A monoid is a set with multiplication that is associative and includes a neutral element.

Given a set A, the free monoid on A is the set $W(A)$ of finite sequences of elements of A (also known as words). We write an element of $W(A)$ as $w = a_1 a_2 \cdots a_n$, where each $a_j \in A$. We identify A with the subset of $W(A)$ of words with length 1.

Definition (Free Product)**.** Let $(\Gamma_i)_{i\in I}$ be a family of groups. Set

$$\begin{aligned} A &= \coprod_{i\in I} \Gamma_i \\ &= \{(g_i, i) \mid g_i \in \Gamma_i, i \in I\} \end{aligned}$$

to be the coproduct of this family.

Let $\sim$ be the equivalence relation generated by

$$\begin{aligned} we_i w' &\sim ww' && \text{where } e_i \in \Gamma_i \text{ is the neutral element} \\ wabw' &\sim wcw' && \text{where } a, b, c \in \Gamma_i,\ c = ab \text{ for some } i \in I \end{aligned}$$

for all $w, w' \in W(A)$. The quotient $W(A)/\sim$ with the operation of concatenation is a group, which is known as the free product of the groups $\{\Gamma_i\}_{i\in I}$. We write it as

$$\bigstar_{i\in I}\Gamma_i.$$

The inverse of the equivalence class for $w = a_1 a_2 \dots a_n$ is $w^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$. The neutral element is ϵ, which is the empty word.

A word $w = a_1 a_2 \cdots a_n \in W(A)$ with $a_j \in \Gamma_{i_j}$ is said to be reduced if $i_{j+1} \neq i_j$ and a_j is not the neutral element of Γ_{i_j}.

Proposition (Existence of the Free Product)**:** Let $\{\Gamma_i\}_{i\in I}$ be a family of groups, $A = \coprod_{i\in I} \Gamma_i$, and $\bigstar_{i\in I}\Gamma_i = W(A)/\sim$ be as above.

Then, any element in the free product $\bigstar_{i\in I}\Gamma_i$ is represented by a unique reduced word in $W(A)$.

Proof.

Existence: Consider an integer $n \geqslant 0$ and a reduced word $w = a_1 a_2 \cdots a_n$ in $W(A)$, an element $a \in A$, and the word $aw \in W(A)$. We set

$$\mathcal{R}(aw) = \begin{cases} w & a = e_i \\ a a_1 a_2 \cdots a_n & a \in \Gamma_i, a \neq e_i, i \neq k \\ b a_2 \cdots a_n & a \in \Gamma_k, a a_1 = b \neq e_k \\ a_2 \cdots a_n & a \in \Gamma_k, a_1 = a^{-1} \in \Gamma_k \end{cases},$$

where k is the index for which $a_1 \in \Gamma_k$.

Then, $\mathcal{R}(aw)$ is yet another reduced word, and $\mathcal{R}(aw) \sim aw$, meaning that any word $w \in W(A)$ is equivalent to some reduced word by inducting on the length of w.

Uniqueness: For each $a \in A$, Let $T(a) = \mathcal{R}(aw)$ be a self-map on the set of reduced words.

For each $w = b_1 b_2 \cdots b_n$, we set $T(w) = T(b_1)T(b_2)\cdots T(b_n)$. For $a, b, c \in \Gamma_i$ with $ab = c$, we have $T(a)T(b) = T(c)$, and $T(e_i) = \epsilon$ (the empty word) for all $i \in I$.

For each reduced word, notice that $T(w)\epsilon = w$.

Let w be some word in $W(A)$ with w_1, w_2 reduced words equivalent to w. Since $w_1 \sim w_2$, we have $T(w_1) = T(w_2)$, and

$$\begin{aligned} w_1 &= T(w_1)\epsilon \\ &= T(w_2)\epsilon \\ &= w_2. \end{aligned}$$

□

Corollary: Let $\{\Gamma_i\}_{i \in I}$ and $\Gamma = \bigstar_{i\in I}\Gamma_i$ as above. For each $i_0 \in I$, the canonical inclusion

$$\iota\colon \Gamma_{i_0} \to \Gamma$$

is injective.

Proof. For any $a \in \Gamma_{i_0} \setminus \{e_{i_0}\}$, $\iota(a)$ is represented by a unique one-letter reduced word not equivalent to the empty word. □

Now that we have an understanding of free products, we can conceptualize the free group as a particular type of free product.

Definition (Free Groups)**.** Let X be a set. The free group over X is the free product of a family of copies of $\mathbb{Z}$ indexed by X, denoted $F(X)$.

Equivalently, the free group over X is

$$F(X) = \bigstar_{a \in X}\langle a \rangle,$$

where $\langle a \rangle$ denotes the cyclic group generated by the element a.

We can also identify $F(X)$ with the set of reduced words in $X \sqcup X^{-1}$ (as was done in the previous subsection).

The cardinality of X is called the rank of $F(X)$.

If Γ is a group, then a free subset of Γ is a subset $X \subseteq \Gamma$ such that the inclusion $X \hookrightarrow F(X)$ extends to an isomorphism of $\langle X \rangle_\Gamma$ onto $F(X)$.

We can now state and prove a universal property for free products (which naturally simplifies in the case of a free group.)

Theorem (Universal Property for Free Products)**:** Let Γ be a group, and $\{\Gamma_i\}_{i\in I}$ be a family of groups. Let $\{h_i\colon \Gamma_i \to \Gamma\}_{i \in I}$ be a family of homomorphisms.

Then, there exists a unique homomorphism $h\colon \bigstar_{i\in I}\Gamma_i \to \Gamma$ such that the following diagram commutes for each $i_0 \in I$.

$$\begin{tikzcd} \Gamma_{i_0} \arrow[r, "h_{i_0}"] \arrow[d, hook, "\iota"'] & \Gamma \\ \bigstar_{i\in I}\Gamma_i \arrow[ur, "h"'] & \end{tikzcd}$$

In particular, if Γ is a group, X is a set, and $\phi\colon X \to \Gamma$ is a set map, there exists a unique homomorphism $\Phi\colon F(X) \to \Gamma$ such that $\Phi(x) = \phi(x)$ for each $x \in X$.

Proof. For a reduced word $w = a_1a_2\cdots a_n \in \bigstar_{i\in I}\Gamma_i$ with $a_j \in \Gamma_{i_j} \setminus \left\{e_{i_j}\right\}$, and $i_{j+1} \neq i_j$ for each $j \in \{1,\dots,n-1\}$, we set

$$h(w) = h_{i_1}(a_1)h_{i_2}(a_2)\cdots h_{i_n}(a_n),$$

which defines h uniquely in terms of h_i. □

Note that for any two sets X, Y, the universal property provides that any map $X \to Y$ extends canonically to a group homomorphism, $F(X) \to F(Y)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

We can now prove an important lemma that will be useful in understanding paradoxical groups.

Theorem (Ping Pong Lemma)**:** Let G be a group acting on a set X, and let Γ_1, Γ_2 be subgroups of G. Let $\Gamma = \langle \Gamma_1, \Gamma_2\rangle$. Assume Γ_1 contains at least 3 elements and Γ_2 contains at least two elements.

Suppose there exist nonempty subsets $X_1, X_2 \subseteq X$ with $X_1 \triangle X_2 \neq \emptyset$, such that for all $\gamma_1 \in \Gamma_1$ with $\gamma_1 \neq e_G$, and for all $\gamma_2 \in \Gamma_2$ with $\gamma_2 \neq e_G$,

$$\begin{aligned} \gamma(X_2) &\subseteq X_1 \\ \gamma(X_1) &\subseteq X_2. \end{aligned}$$

Then, Γ is isomorphic to the free product $\Gamma_1 \star \Gamma_2$.

Proof. Let w be a nonempty reduced word spelled with letters from the disjoint union of $\Gamma_1 \setminus \{e_G\}$ and $\Gamma_2 \setminus \{e_G\}$. We must show that the element of Γ defined by w is not the identity.

If $w = a_1b_1a_2b_2\cdots a_k$ with $a_1,\dots,a_k \in \Gamma_1 \setminus \{e_G\}$ and $b_1,\dots,b_{k-1} \in \Gamma_2 \setminus \{e_G\}$. Then,

$$\begin{aligned} w(X_2) &= a_1b_1\cdots a_{k-1}b_{k-1}a_k(X_2) \\ &\subseteq a_1b_1\cdots a_{k-1}b_{k-1}(X_1) \\ &\subseteq a_1b_1\cdots a_{k-1}(X_2) \\ &\vdots \\ &\subseteq a_1(X_2) \\ &\subseteq X_1. \end{aligned}$$

Since $X_2 \not\subseteq X_1$, this implies $w \neq e_G$.

If $w = b_1a_2b_2a_2\cdots b_k$, we select $a \in \Gamma_1 \setminus \{e_G\}$, and apply the previous argument to awa^{-1}. Since $awa^{-1} \neq e_G$, neither is w.

Similarly, if $w = a_1b_1\cdots a_kb_k$, we select $a \in \Gamma_1 \setminus \left\{e_G, a_1^{-1}\right\}$, and apply the argument to awa^{-1}, and if $w = b_1a_2b_2\cdots a_k$, we select $a \in \Gamma_1 \setminus \{e_G, a_k\}$, and apply the argument to awa^{-1}. □

Example. We can use the Ping Pong Lemma to see that

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

generate a subgroup of $SL(2,\mathbb{Z})$ which is free of rank 2.

Corollary: The special orthogonal group $SO(3)$ contains a subgroup isomorphic to the free group on two generators.

To prove this, we state a different version of the Ping Pong Lemma that we will apply to a particular space.

Theorem (Ping Pong Lemma for Cyclic Groups)**:** Let G act on a set X, and suppose there exist disjoint subsets $A_+, A_-, B_+, B_- \subseteq X$ whose union is not all of X. If there exist elements a and b in G such that

$$\begin{aligned} a \cdot (X \setminus A_-) &\subseteq A_+ \\ a^{-1} \cdot (X \setminus A_+) &\subseteq A_- \\ b \cdot (X \setminus B_-) &\subseteq B_+ \\ b \cdot (X \setminus B_+) &\subseteq B_-, \end{aligned}$$

then it is the case that the group generated by a and b is free of rank 2.

Proof of Corollary. We let

$$\begin{aligned} a &= \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a^{-1} &= \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ b &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \\ b^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}. \end{aligned}$$

We specify

$$X = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} A_+ &= \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, x \equiv 3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ A_- &= \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, x \equiv -3y \text{ modulo } 5, z \equiv 0 \text{ modulo } 5 \right\} \\ B_+ &= \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, z \equiv 3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\} \\ B_- &= \left\{ \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; k \in \mathbb{Z}, z \equiv -3y \text{ modulo } 5, x \equiv 0 \text{ modulo } 5 \right\}. \end{aligned}$$

To verify that the conditions of the Ping Pong Lemma hold, we calculate

$$\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix} 3x + 4y \\ -4x + 3y \\ 5z \end{pmatrix} \tag{1}$$

$$\begin{pmatrix}3/5 & -4/5 & 0\\ 4/5 & 3/5 & 0\\ 0 & 0 & 1\end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix}3x-4y\\ 4x+3y\\ 5z\end{pmatrix} \tag{2}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 3/5 & -4/5\\ 0 & 4/5 & 3/5\end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix}5x\\ 3y-4z\\ 4y+3z\end{pmatrix} \tag{3}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 3/5 & 4/5\\ 0 & -4/5 & 3/5\end{pmatrix}\left(\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \frac{1}{5^{k+1}}\begin{pmatrix}5x\\ 3y+4z\\ -4y+3z\end{pmatrix}. \tag{4}$$

We verify that the conditions for the Ping Pong Lemma hold for each of these four conditions.

(1) For any vector

$$\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \notin A_-,$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x + 4y \equiv 3(-4x + 3y)$ modulo 5, and that $z' = 5z \equiv 0$ modulo 5.

(2) For any vector

$$\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \notin A_+,$$

we see that $k+1 \in \mathbb{Z}$, $x' = 3x - 4y \equiv -3(4x + 3y)$ modulo 5, and $z' = 5z \equiv 0$ modulo 5.

(3) For any vector

$$\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \notin B_-,$$

we see that $k+1 \in \mathbb{Z}$, $z' = 4y + 3z \equiv 3(3y - 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

(4) For any vector

$$\frac{1}{5^k}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \notin B_+,$$

we see that $k+1 \in \mathbb{Z}$, $z' = -4y + 3z \equiv -3(3y + 4z)$ modulo 5, and $x' = 5x \equiv 0$ modulo 5.

Since we have verified that the conditions for the Ping Pong Lemma hold for each of the conditions, we have that $\{a, b\} \subseteq SO(3)$ generate a group isomorphic to the free group on two generators. □

States and Means on $\ell_\infty(G)$

Definition. Let G be a group.

(1) The space $\mathcal{F}(G, \mathbb{R})$ is defined by

$$\mathcal{F}(G, \mathbb{R}) = \{f \mid f\colon G \to \mathbb{R} \text{ is a function}\}.$$

(2) A function $f \in \mathcal{F}(G, \mathbb{R})$ is positive if $f(x) \geqslant 0$ for all $x \in G$.

(3) A function $f \in \mathcal{F}(G, \mathbb{R})$ is simple if $\text{Ran}(f)$ is finite. We say

$$\Sigma = \left\{ f \colon \mathcal{F}(G, \mathbb{R}) \,\middle|\, f \text{ is simple} \right\}.$$

Fact. $\Sigma \subseteq \mathcal{F}(G, \mathbb{R})$ is a subspace. To see this, if f, g are such that $\text{Ran}(f), \text{Ran}(g)$ are finite, and $\alpha \in \mathbb{R}$, then

$$\text{Ran}(f + \alpha g) \leqslant \text{Ran}(f) + \text{Ran}(g),$$

so $f + \alpha g$ has finite range.

Definition. For $E \subseteq G$, set

$$\mathbb{1}_E \colon G \to \mathbb{R}$$

defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

This is the characteristic function of E.

Fact.

$$\text{span}\{\mathbb{1}_E \mid E \subseteq G\} = \Sigma.$$

Proof. We see that $\mathbb{1}_E \in \Sigma$ for any $E \subseteq G$, and Σ is a subspace.

If $\phi \in \Sigma$, with $\text{Ran}(\phi) = \{t_1, \dots, t_n\}$ with t_i distinct, we set

$$E_i = \phi^{-1}(\{t_i\}),$$

meaning

$$\phi = \sum_{i=1}^{n} t_i \mathbb{1}_{E_i}.$$

□

Definition.

(1) A function $f \in \mathcal{F}(G, \mathbb{R})$ is bounded if there exists $M > 0$ such that $\text{Ran}(f) \subseteq [-M, M]$.

(2) The space $\ell_\infty(G)$ is defined by

$$\ell_\infty(G) = \{f \in \mathcal{F}(G, \mathbb{R}) \mid f \text{ is bounded}\}.$$

(3) The norm on $\ell_\infty(G)$ is defined by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

Proposition: The space $\ell_\infty(G)$ is complete, Additionally, $\overline{\Sigma} = \ell_\infty(G)$.

Proof. Let $(f_n)_n$ be Cauchy. For $x \in G$, it is the case that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n - f_m)(x)| \\ &\leqslant \|f_n - f_m\|, \end{aligned}$$

meaning $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We must show that $f \in \ell_\infty(G)$ and $\|f_n - f\| \to 0$.

$$\begin{aligned} |f(x)| &= \left|\lim_{n\to\infty} f_n(x)\right| \\ &= \lim_{n\to\infty} |f_n(x)| \\ &\leqslant \limsup_{n\to\infty} \|f_n\| \\ &\leqslant C, \end{aligned}$$

as Cauchy sequences are always bounded. Thus, $\sup_{x\in G}|f(x)| \leqslant C$.

Given $\varepsilon > 0$, we find N such that for all $m, n \geqslant N$, $\|f_n - f_m\| \leqslant \varepsilon$. Thus, for $x \in G$, we have

$$\begin{aligned} |f_n(x) - f)m(x)| &\leqslant \|f_n - f_m\| \\ &\leqslant \varepsilon. \end{aligned}$$

Taking $m \to \infty$, we get $|f_n(x) - f(x)| \leqslant \varepsilon$ for all $n \geqslant N$, meaning $\|f_n - f\| \leqslant \varepsilon$ for all $n \geqslant N$.

Now, for $f \in \ell_\infty(G)$, let $\operatorname{Ran}(f) \subseteq [-M, M]$ for some $M > 0$. Let $\varepsilon > 0$. Since $[-M, M]$ is compact, it is totally bounded, so we can find intervals $I_1, \dots, I_n$ with $[-M, M] = \bigsqcup_{k=1}^{n} I_k$, with the length of each I_k less than ε.

Set $E_k = f^{-1}(I_k)$. Pick $t_k \in I_k$. Then, we set

$$\phi = \sum_{i=1}^{n} t_k \mathbb{1}_{E_k}.$$

We see that $\|\phi - f\| < \varepsilon$. □

Corollary: For any $f \in \ell_\infty(G)$, there is a sequence $(\phi_n)_n$ in Σ with $\|\phi_n - f\| \to 0$. If $f \geqslant 0$, then it is possible to select $\phi_n \geqslant 0$.

Proposition: Let G be a group. There is an action

$$G \xrightarrow{\lambda_s} \operatorname{Isom}(\ell_\infty(G))$$

defined by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right).$$

Proof. We have

$$\begin{aligned} \lambda_s(f + \alpha g)(t) &= (f + \alpha g)\left(s^{-1}t\right) \\ &= f\left(s^{-1}t\right) + \alpha g\left(s^{-1}t\right) \\ &= \lambda_s(f)(t) + \alpha\lambda_s(g)(t) \\ &= (\lambda_s(f) + \alpha\lambda_s(g))(t). \end{aligned}$$

Thus, λ_s is a linear operator.

We have

$$\|\lambda_s(f)\| = \sup_{t\in G}|\lambda_s(f)(t)|$$

$$\begin{aligned}&= \sup_{t\in G}\left|f\left(s^{-1}t\right)\right|\\&= \|f\|,\end{aligned}$$

hence

$$\begin{aligned}\|\lambda_s(f) - \lambda_s(f)\| &= \|\lambda_s(f-g)\|\\&= \|f-g\|.\end{aligned}$$

Thus, λ_s is an isometry.

We have

$$\begin{aligned}\lambda_s \circ \lambda_r(f)(t) &= \lambda_r(f)\left(s^{-1}t\right)\\&= f\left(r^{-1}s^{-1}t\right)\\&= f\left((sr)^{-1}t\right)\\&= \lambda_{sr}(f)(t),\end{aligned}$$

meaning $\lambda_s \circ \lambda_r = \lambda_{sr}$. □

Remark: By a similar process, we find that $\lambda_s(\mathbb{1}_E) = \mathbb{1}_{sE}$ for any subset $E \subseteq G$ and $s \in G$.

Definition. A state on $\ell_\infty(G)$ is a continuous linear functional $\mu \in (\ell_\infty(G))^*$ that satisfies the following.

(1) μ is positive;

(2) $\mu(\mathbb{1}_G) = 1$.

A state is called left-invariant if

$$\mu(\lambda_s(f)) = \mu(f).$$

Example. Let G be a group.

- If $x \in G$, then $\delta_x\colon \ell_\infty(G) \to \mathbb{F}$ defined by

$$\delta_x(f) = f(x)$$

  is a state. However, note that it is not necessarily invariant.

$$\begin{aligned}\delta_x(\lambda_s(f)) &= \lambda_s(f)(x)\\&= f\left(s^{-1}x\right)\\&\neq f(x).\end{aligned}$$

- If G is finite, then

$$\mu = \frac{1}{|G|}\sum_{x\in G}\delta_x$$

  is an invariant state.

Lemma (Characterization of States)**:**

(1) If μ is a state on $\ell_\infty(G)$, then

$$\|\mu\|_{op} = 1.$$

(2) If $\mu \in (\ell_\infty(G))^*$ is such that

$$\begin{aligned}\|\mu\| &= \mu(\mathbb{1}_G)\\ &= 1,\end{aligned}$$

then μ is positive and a state.

Proof.

(1) Given $f \in \ell_\infty(G)$, we have

$$\begin{aligned}\|f\|\mathbb{1}_G - f &\geqslant 0\\ \|f\|\mathbb{1}_G + f &\geqslant 0,\end{aligned}$$

so

$$\begin{aligned}0 &\leqslant \mu(\|f\|\mathbb{1}_G - f)\\ &= \|f\|\mu(\mathbb{1}_G) - \mu(f)\\ 0 &\leqslant \mu(\|f\|\mathbb{1}_G + f)\\ &= \|f\|\mu(\mathbb{1}_G) + \mu(f).\end{aligned}$$

Thus, we have $\pm\mu(f) \leqslant \|f\|\mu(\mathbb{1}_G) = \|f\|$, so $|\mu(f)| \leqslant \|f\|$, so $\|\mu\| \leqslant 1$. Additionally, since $\mu(\mathbb{1}_G) = 1$, we must have $\|\mu\| = 1$.

(2) Suppose $\|\mu\| = \mu(\mathbb{1}_G) = 1$. Let $f \geqslant 0$. Set $g = \frac{1}{\|f\|_u} f$.

Then, $\operatorname{Ran}(g) \subseteq [0, 1]$, and $\operatorname{Ran}(g - \mathbb{1}_G) \subseteq [-1, 1]$, so $\|g - \mathbb{1}_G\|_u \leqslant 1$.

Since $\|\mu\| = 1$, we must have

$$\begin{aligned}|\mu(g - \mathbb{1}_G)| &\leqslant 1\\ |\mu(g) - 1| &\leqslant 1,\end{aligned}$$

and since $\mu(\mathbb{1}_G) = 1$, we must have $\mu(g) \in [0, 2]$, so $\mu(f) = \|f\|\mu(g) \geqslant 0$.

□

Corollary: The set of states on $(\ell_\infty(G))^*$ forms a w^*-compact subset of $B_{(\ell_\infty(G))^*}$.

Proof. It has been proven in functional analysis that a convex subset of $(\ell_\infty(G))^*$ is w^*-compact if it is norm bounded and w^*-closed. Since the set of states is convex and norm-bounded, all we need to show is that $S(\ell_\infty(G))$ is w^*-closed.

To this end, let $f \in \ell_\infty(G)$ be positive and $(\varphi_i)_i$ be a net in $S(\ell_\infty(G))$ with $(\varphi_i)_i \to \varphi$. We must show that φ is positive and satisfies $\varphi(\mathbb{1}_G) = 1$. To this end, we see that

$$\varphi_i(f) \geqslant 1$$

for all $i \in I$, so we must necessarily have $\varphi(f) \geqslant 0$, and similarly, since $\varphi_i(\mathbb{1}_G) = 1$ for each $i \in I$, we also have $\varphi(\mathbb{1}_G) = 1$. □

Proposition: If $\mu \in (\ell_\infty(G))^*$ is a state, then $m\colon P(G) \to [0, 1]$ defined by $m(E) = \mu(\mathbb{1}_E)$ is a finitely additive probability measure on G. Moreover, if μ is invariant, then m is a mean.

Proof. We have

$$\begin{aligned} m(G) &= \mu(\mathbb{1}_G) \\ &= 1 \\ m(\emptyset) &= \mu(0) \\ &= 0 \\ m(E \sqcup F) &= \mu(\mathbb{1}_{E\sqcup F}) \\ &= \mu(\mathbb{1}_E + \mathbb{1}_F) \\ &= \mu(\mathbb{1}_E) + \mu(\mathbb{1}_F) \\ &= m(E) + m(F). \end{aligned}$$

Additionally, since $0 \leqslant \mathbb{1}_E \leqslant \mathbb{1}_G$, we have $0 \leqslant \mu(\mathbb{1}_E) \leqslant 1$, so $0 \leqslant m(E) \leqslant m(G) = 1$.

If μ is invariant, then

$$\begin{aligned} m(sE) &= \mu(\mathbb{1}_{sE}) \\ &= \mu(\lambda_s(\mathbb{1}_E)) \\ &= \mu(\mathbb{1}_E) \\ &= m(E). \end{aligned}$$

□

Proposition: If G admits a mean, then $(\ell_\infty(G))^*$ admits an invariant state.

Proof. Let m be a finitely-additive probability measure. Define

$$\mu_0 \colon \Sigma \to \mathbb{R}$$

by

$$\mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right) = \sum_{k=1}^{n} t_k m(E_k).$$

Since m is finitely additive, it is the case that μ_0 is well-defined, linear, and positive.

Note that $\mu_0(\mathbb{1}_G) = m(G) = 1$.

If m is a mean, then for $f = \sum_{k=1}^{n} t_k E_k$,

$$\begin{aligned} \mu_0(\lambda_s(f)) &= \mu_0\left(\lambda_s\left(\sum_{k=1}^{n} t_k \mathbb{1}_{E_k}\right)\right) \\ &= \mu_0\left(\sum_{k=1}^{n} t_k \mathbb{1}_{sE_k}\right) \\ &= \sum_{k=1}^{n} t_k m(sE_k) \\ &= \sum_{k=1}^{n} t_k m(E_k) \\ &= \mu_0(f). \end{aligned}$$

Additionally, we see that

$$\begin{aligned}|\mu_0(f)| &= \left|\sum_{k=1}^{n} t_k m(E_k)\right|\\ &\leqslant \sum_{k=1}^{n} |t_k| m(E_k)\\ &\leqslant \sum_{k=1}^{n} \|f\| m(E_k)\\ &= \|f\| \sum_{k=1}^{n} m(E_k)\\ &\leqslant \|f\|.\end{aligned}$$

Thus, μ_0 is continuous, so μ_0 is uniformly continuous.

Since $\overline{\Sigma} = \ell_\infty(G)$, we see that μ_0 extends to a continuous linear functional $\mu \colon \ell_\infty(G) \to \mathbb{R}$, with $\mu(\mathbb{1}_G) = \mu_0(\mathbb{1}_G) = 1$.

If $f \geqslant 0$, we find a sequence $(\phi_n)_n$ in Σ with $\phi_n \geqslant 0$, $\|\phi_n - f\| \xrightarrow{n\to\infty} 0$, and we set

$$\begin{aligned}\mu(f) &= \lim_{n\to\infty} \mu(\phi_n)\\ &= \lim_{n\to\infty} \mu_0(\phi_n)\\ &\geqslant 0,\end{aligned}$$

meaning μ is a state.

If $f \in \ell_\infty(G)$, $s \in G$, and $(\phi_n)_n$ in Σ with $(\phi_n)_n \to f$, then

$$\begin{aligned}\|\lambda_s(\phi_n) - \lambda_s(f)\| &= \|\lambda_s(\phi_n - f)\|\\ &= \|\phi_n - f\|\\ &\to 0\end{aligned}$$

Thus, we have

$$\begin{aligned}\mu(\lambda_s(\phi_n)) &= \mu_0(\lambda_s(\phi_n))\\ &= \mu_0(\phi_n)\\ &= \mu(\phi_n)\\ &\to \mu(f),\end{aligned}$$

so $\mu(f) = \mu(\lambda_s(f))$. Thus, $\mu \in (\ell_\infty(G))^*$ is an invariant state. □

Using Invariant States

Proposition: $\mathbb{Z}$ is amenable.

Proof. We know that $\lambda_1 \colon \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$, defined by

$$\lambda_1(f)(k) = f(k-1)$$

is an isometry.

We set $Y = \operatorname{Ran}(\operatorname{id} - \lambda_1) \subseteq \ell_\infty(\mathbb{Z})$.

We claim that $\operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$.

Suppose toward contradiction that there is $y \in Y$ with $\|\mathbb{1}_{\mathbb{Z}} - y\|_u = \rho < 1$. Then, $y = f - \lambda_1(f)$ for some $f \in \ell_\infty(\mathbb{Z})$, meaning

$$\|\mathbb{1} - (f - \lambda_1(f))\| = \rho.$$

Thus, for all $k \in \mathbb{Z}$, we have

$$|1 - (f(k) - f(k-1))| \leqslant \rho,$$

meaning $|f(k) - f(k-1)| \geqslant 1 - \rho > 0$. However, such an f cannot be bounded.

Since $\operatorname{dist}_{\overline{Y}}(\mathbb{1}_Z) = \operatorname{dist}_Y(\mathbb{1}_Z) \geqslant 1$, the Hahn–Banach theorem provides $\mu \in (\ell_\infty(\mathbb{Z}))^*$ with $\|\mu\| = 1$, $\mu|_{\overline{Y}} = 0$, and $\mu(\mathbb{1}) = \operatorname{dist}_Y(\mathbb{1}_{\mathbb{Z}}) \geqslant 1$.

Since $\|\mu\| = 1$ and $\mu(\mathbb{1}) \geqslant 1$, we must have $\mu(\mathbb{1}) = 1$.

Since $\|\mu\| = \mu(\mathbb{1}_{\mathbb{Z}}) = 1$, it is the case that μ is a state on $\ell_\infty(\mathbb{Z})$. Since $\mu(y) = 0$ for all $y \in Y$, we have

$$\begin{aligned} \mu(f - \lambda_1(f)) = 0 \\ \mu(f) = \mu(\lambda_1(f)), \end{aligned}$$

so inductively, we have $\mu(f) = \mu(\lambda_k(f))$ for all $k \in \mathbb{Z}$, meaning μ is an invariant state on $\ell_\infty(\mathbb{Z})$. Thus, $\mathbb{Z}$ is amenable. □

Proposition: If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.

Proof. Let $\rho \in (\ell_\infty(G/N))^*$ be an invariant state, and $p\colon P(N) \to [0,1]$. For $E \subseteq G$, we define

$$f_E\colon G/N \to \mathbb{R}$$

by $f_E(tN) = p\left(N \cap t^{-1}E\right)$.

We verify that this is well-defined — for $tN = sN$, we have $s^{-1}t \in N$, so

$$\begin{aligned} p\left(N \cap t^{-1}E\right) &= p\left(s^{-1}t\left(N \cap t^{-1}E\right)\right) \\ &= p\left(s^{-1}TN \cap s^{-1}E\right) \\ &= p\left(N \cap s^{-1}E\right). \end{aligned}$$

We also see that f_E is bounded, and

$$\begin{aligned} f_{E \sqcup F}(tN) &= p\left(N \cap t^{-1}(E \sqcup F)\right) \\ &= p\left(N \cap \left(t^{-1}E \sqcup t^{-1}F\right)\right) \\ &= p\left(\left(N \cap t^{-1}E\right) \sqcup \left(N \cap t^{-1}F\right)\right) \\ &= p(N \cap t^{1}E) + p\left(N \cap t^{-1}F\right) \\ &= f_E(tN) + f_F(tN) \\ &= (f_E + f_F)(tN). \end{aligned}$$

Thus, $f_{E \sqcup F} = f_E + f_F$.

Additionally,

$$\begin{aligned} f_{sE}(tN) &= p\left(N \cap t^{-1}sE\right) \\ &= f_E\left(s^{-1}tN\right) \\ &= \lambda_{sN}(f_E)(tN), \end{aligned}$$

so $f_{sE} = \lambda_{sN}(f_E)$.

Finally,

$$\begin{aligned} f_G(tN) &= p\left(N \cap t^{-1}G\right) \\ &= p(N) \\ &= 1, \end{aligned}$$

so $f_G = \mathbb{1}_{G/N}$.

We define $m \colon P(G) \to [0,1]$ by

$$m(E) = \rho(f_E).$$

Then, we have

$$\begin{aligned} m(E \sqcup F) &= m(E) + m(F) \\ m(G) &= 1 \\ m(sE) &= \rho(f_{sE}) \\ &= \rho(\lambda_{sN}(f_E)) \\ &= \rho(f_E) \\ &= m(E), \end{aligned}$$

meaning m is a mean. □

Corollary: The finite direct product of amenable groups is amenable.

Proof. If H and K are amenable, then we know that

$$K \cong \frac{H \times K}{H}$$

is amenable, and H is amenable, so $H \times K$ is amenable. □

Corollary: Finitely generated abelian groups are amenable.

Proof. All finitely generated abelian groups are isomorphic to $\mathbb{Z}^d \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$ by the Fundamental Theorem of Finitely Generated Abelian Groups. Since $\mathbb{Z}^d$ is a finite direct product of $\mathbb{Z}$ (which is amenable), and the torsion group $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \mathbb{Z}/n_k\mathbb{Z}$ is finite, we have that a finitely generated abelian group is amenable. □

Corollary: If $\{G_i\}_{i \in I}$ is a directed family of amenable groups — i.e., that for any two groups G_j and G_k, there is G_ℓ with $G_j \subseteq G_\ell$ and $G_k \subseteq G_\ell$ — then the direct union,

$$G = \bigcup_{i \in I} G_i,$$

is also amenable.

Proof. Let $\mu_i \in (\ell_\infty(G_i))^*$ be the respective invariant states.

Set

$$M_i = \{\mu \in S((\ell_\infty(G))^*) \mid \mu(\lambda_s(f)) = \mu(f) \text{ for all } s \in G_i\},$$

and set $\mu(f) = \mu_i(f|_{G_i})$. We see that M_i is w^*-closed in $B_{(\ell_\infty(G))^*}$, as we have established the state space as a w^*-closed subset of $B_{(\ell_\infty(G))^*}$.

For $i_1, \dots, i_n$, we find $G_j \supseteq G_{i_1}, \dots, G_{i_n}$, which necessarily exists as $\{G_i\}_{i\in I}$ is directed. Thus, $M_j \subseteq \bigcap_{k=1}^n M_{i_k}$, meaning $\{M_i\}_{i\in I}$ has the finite intersection property.

By compactness, there is $\mu \in \bigcap_{i\in I} M_i$, meaning μ is an invariant state. □

Corollary: All abelian groups are amenable.

Proof. Every abelian group is the direct union of its finitely generated subgroups. □

Corollary: All solvable groups are amenable.

Proof. Let $e_G = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ be such that $G_{j-1} \trianglelefteq G_j$ for $j = 1, \dots, n$, and G_j/G_{j-1} abelian.

Since G_0 is abelian, it is amenable. Similarly, G_1/G_0 is abelian, so it is amenable, so G_1 is amenable. Continuing in this fashion, we see that G is amenable. □

Følner's Condition and Invariant Approximate Means

Definition. A group G is said to satisfy the Følner condition if, for every $\varepsilon > 0$, and for all $E \subseteq G$ finite, there is a nonempty $F \subseteq G$ finite such that for all $t \in E$,

$$\frac{|tF \triangle F|}{|F|} \leqslant \varepsilon.$$

Equivalently, since

$$\frac{|tF \triangle F|}{|F|} = 2\left(1 - \frac{|tF \cap F|}{|F|}\right),$$

we have the equivalent formulation that

$$\frac{|tF \triangle F|}{|F|} \leqslant \varepsilon \text{ if and only if } 1 - \frac{|tF\cap|}{|f|} \leqslant \varepsilon/2.$$

Thus, G satisfies the Følner condition if and only if, for all $\varepsilon > 0$ and for all finite $E \subseteq G$, there exists a nonempty $F \subseteq G$ with

$$\frac{|tF \cap F|}{|f|} \geqslant 1 - \varepsilon.$$

Example. All finite groups satisfy Følner's condition by taking $F = G$ for each subset $E \subseteq G$.

Lemma: A countable group G satisfies the Følner condition if and only if G admits a Følner sequence, $(F_n)_n$ with $F_n \subseteq G$ finite, such that

$$\left(\frac{|tF_n \triangle F_n|}{|F_n|}\right)_n \xrightarrow{n\to\infty} 0,$$

or equivalently,

$$\left(\frac{|tF_n \cap F_n|}{|F_n|}\right)_n \xrightarrow{n\to\infty} 1,$$

for all t in G.

Proof. Let G admit a Følner sequence, $(F_n)_n$. Given $\varepsilon > 0$, and $E \subseteq G$ finite, find N such that for all $s \in E$ and $n \geqslant N$,

$$\frac{|sF_n \triangle F_n|}{|F_n|} \leqslant \varepsilon.$$

We take $F = F_N$.

Let G satisfy the Følner condition. We write $G = \bigcup_{n\geqslant 1} E_n$, with $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, and define F_n such that for all $t \in E_n$,

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}.$$

Then, given $t \in G$, it is the case that $t \in E_N$ for some N, so $t \in E_n$ for all $n \geqslant N$, so

$$\frac{|tF_n \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}$$

for all $n \geqslant N$, meaning that

$$\frac{|tF_n \triangle F_n|}{|F_n|} \xrightarrow{n\to\infty} 0.$$

□

Lemma: Let G be a finitely generated group with generating set S (where S may not be symmetric[1]). If $(F_n)_n$ is a sequence of finite subsets of G such that

$$\left(\frac{|sF_n \triangle F_n|}{|F_n|}\right)_n \to 0$$

for all $s \in S$, then $(F_n)_n$ is a Følner sequence for G.

Proof. We start by showing that we can assume S to be symmetric. The following are both true:

- $s(A \triangle B) = sA \triangle sB$;
- $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

Thus, if we have s^{-1} rather than s, our assumption provides, for all $s \in S$,

$$\begin{aligned}\frac{\left|s^{-1}F_n \triangle F_n\right|}{|F_n|} &= \frac{s^{-1}(F_n \triangle sF_n)}{|F_n|}\\ &= \frac{|F_n \triangle sF_n|}{|F_n|}\\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

Thus, we may assume S is symmetric.

[1] Closed under inversion.

For $s, t \in F_n$, we have

$$\begin{aligned}
\frac{|stF_n \triangle F_n|}{|F_n|} &\leqslant \frac{|StF_n \triangle sF_n|}{F_n} + \frac{|sF_n \triangle F_n|}{|F_n|}\\
&= \frac{|s(tF_n \triangle F_n)|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|}\\
&= \frac{|tF_n \triangle F_n|}{|F_n|} + \frac{|sF_n \triangle F_n|}{|F_n|}\\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

We use induction to find the general case. □

Example. Considering $\mathbb{Z}$ again, we remember that $\{1\}$ is the generating set for $\mathbb{Z}$. If we let $F_n = \{-n, -n+1, \dots, -1, 0, 1, \dots$
we have

$$\begin{aligned}
\frac{|(F_n + 1)\triangle F_n|}{|F_n|} &= \frac{2}{2n+1}\\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

Thus, we have that $\mathbb{Z}$ satisfies the Følner condition.

We now turn our attention to approximate means, from which with Følner's condition, we will be able to construct a different, equivalent condition for group amenability.

Definition. If G is a group, we define

$$\text{Prob}(G) := \left\{ f\colon G \to [0,\infty) \;\middle|\; \left|\text{supp}(f)\right| < \infty,\ \sum_{t\in G} f(t) = 1 \right\}.$$

Note that $\text{Prob}(G) \subseteq B_{\ell_1(G)}$. Given $f \in \text{Prob}(G)$, we set

$$\varphi_f\colon \ell_\infty(G) \to \mathbb{C},$$

defined by

$$\varphi_f(g) = \sum_{t\in G} g(t)f(t).$$

We claim that φ_f is a state on $\ell_\infty(G)$.

Proof. If $g \geqslant 0$, then $\varphi_f(g) \geqslant 0$, and $\varphi_f(\mathbb{1}_G) = 1$. It is also clear that φ_f is linear.

We only need show that $\|\varphi_f\| = 1$. We see

$$\begin{aligned}
|\varphi_f(g)| &= \left|\sum_{t\in G} g(t)f(t)\right|\\
&\leqslant \sum_{t\in G} |g(t)||f(t)|\\
&\leqslant \|g\|_\infty \sum_t |f_t|\\
&= \|g\|_\infty.
\end{aligned}$$

□

Proposition: There is an action $\lambda\colon G \xrightarrow{\text{Isom}} (\ell_1(G))$ such that Prob(G) is invariant.

Proof. We let $\lambda_s(f)(t) = f\left(s^{-1}t\right)$. Then,

$$\begin{aligned}
\|\lambda_s(f)\|_1 &= \sum_{t\in G}|\lambda_s(f)(t)|\\
&= \sum_{t\in G}\left|f\left(s^{-1}t\right)\right|\\
&= \sum_{r\in G}|f(r)|\\
&= \|f\|.
\end{aligned}$$

We also see that λ_s is linear.

Additionally,

$$\begin{aligned}
\lambda_r\circ\lambda_s(f)(t) &= \lambda_s(f)\left(r^{-1}t\right)\\
&= f\left(s^{-1}r^{-1}t\right)\\
&= f\left((rs)^{-1}t\right)\\
&= \lambda_{rs}(f)(t).
\end{aligned}$$

We see that if $f \in \operatorname{Prob}(G)$, then for $f \geqslant 0$, we have $\lambda_s(f) \geqslant 0$, and

$$\operatorname{supp}(\lambda_s(f)) = s\left(\operatorname{supp}(f)\right),$$

which is also finite.

Thus,

$$\begin{aligned}
\sum_{t\in G}\lambda_s(f)(t) &= \sum_{t\in G} f\left(s^{-1}t\right)\\
&= \sum_{r\in G} f(r)\\
&= 1
\end{aligned}$$

for $f \in \operatorname{Prob}(G)$. □

Definition. For a countable group G, a sequence $(f_k)_k$ in $\operatorname{Prob}(G)$ is an approximate invariant mean if, for all $s \in G$,

$$\|f_k - \lambda_s(f_k)\|_1 \xrightarrow{k\to\infty} 0.$$

Proposition: If G admits a Følner sequence $(F_k)_k$, then it admits an approximate mean.

Proof. Set $f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k} \in \operatorname{Prob}(G)$. Then,

$$\begin{aligned}
\|f_k - \lambda_s(f_k)\|_1 &= \frac{1}{|F_k|}\|\mathbb{1}_{F_k} - \lambda_s(\mathbb{1}_{F_k})\|\\
&= \frac{1}{|F_k|}\|\mathbb{1}_{F_k} - \mathbb{1}_{sF_k}\|\\
&= \frac{|F_k\triangle sF_k|}{|F_k|},
\end{aligned}$$

which thus converges to 0 as $k \to \infty$. □

Proposition: If G has an approximate mean, then G is amenable.

Proof. Let $(f_k)_k$ be an approximate mean. We define $\varphi_k = (\varphi_{f_k})_k$ to be a sequence of states on $\ell_\infty(G)$.

Since the state space on $\ell_\infty(G)$ is w^*-compact, there is a state μ and a subnet $\left(\varphi_{k_j}\right)_j$ with $\left(\varphi_{k_j}\right)_j \xrightarrow{w^*} \mu$.

We only need to show that μ is invariant. Note that

$$|\mu(g) - \mu(\lambda_s(g))| \leqslant \left|\mu_g - \varphi_{k_j}(g)\right| + \left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| + \left|\varphi_{k_j}(\lambda_s(g)) - \mu(\lambda_s(g))\right|$$

holds for all $g \in \ell_\infty(G)$, $s \in G$, and for all j.

Given $\varepsilon > 0$, we find J such that for $j \geqslant J$, we have

$$\begin{aligned} \left|\mu(g) - \varphi_{k_j}(g)\right| &< \varepsilon/3 \\ \left|\mu(\lambda_s(g)) - \varphi_{k_j}(\lambda_s(g))\right| &< \varepsilon/3. \end{aligned}$$

We see that

$$\begin{aligned} \left|\varphi_{k_j}(g) - \varphi_{k_j}(\lambda_s(g))\right| &= \left|\sum_{t\in G} g(t) f_{k_j}(t) - \sum_{t \in G} g\left(s^{-1}t\right) f_{k_j}(t)\right| \\ &= \left|\sum_{t\in G} g(t) f_{k_j}(t) - \sum_{r\in G} g(r) f_{k_j}(sr)\right| && r = s^{-1}t \\ &= \left|\sum_{t\in G} g(t)\left(f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right)\right| \\ &\leqslant \|g\|_\infty \sum_{t\in G}\left|f_{k_j}(t) - \lambda_{s^{-1}}\left(f_{k_j}\right)(t)\right| \\ &= \|g\|_\infty \left\|f_{k_j} - \lambda_{s^{-1}}\left(f_{k_j}\right)\right\| \\ &< \varepsilon/3 \end{aligned}$$

for large j. Thus, we have

$$|\mu(g) - \mu(\lambda_s(g))| < \varepsilon,$$

for all $\varepsilon > 0$, so $\mu(g) = \mu(\lambda_s(g))$. ◻

Equivalence between Means and Approximate Means

We wish to show that if G is amenable, then G has an approximate mean.

Theorem: Let G be amenable. Then, G has an approximate mean.

Proof. Recall that a net $(f_i)_{i\in I}$ in $\mathrm{Prob}(G)$ has its domain as the set

$$I = \{(E, \varepsilon) \mid E \subseteq G \text{ finite}, \varepsilon > 0\},$$

directed by $(E, \varepsilon) \leqslant (E', \varepsilon')$ if $E \subseteq E'$ and $\varepsilon \geqslant \varepsilon'$.

Suppose there is no approximate mean. Then, there exists a finite subset $E_0 \subseteq G$ and $\varepsilon_0 > 0$ such that for all $s \in E_0$ and $f \in \mathrm{Prob}(G)$, we have

$$\|f - \lambda_s(f)\| \geqslant \varepsilon_0.$$

Let $X = \bigoplus_{|E_0|} \ell_1(G)$ be endowed with the 1-norm.

Consider the set

$$C = \left\{ (f - \lambda_s(f))_{s \in E_0} \mid f \in \text{Prob}(G) \right\}.$$

Since Prob(G) is convex, C is convex, and since $|E_0|$ is finite, C is necessarily bounded. Additionally, note that it is the case that $0 \notin \overline{C}$.

By the Hahn–Banach separation theorem, there is a real-valued $\varphi \in X^*$ such that $\varphi(C) \geqslant 1$.

Note also that

$$\begin{aligned} X^* &\cong \bigoplus_{|E_0|} \ell_1(G)^* \\ &\cong \bigoplus_{|E_0|} \ell_\infty(G) \end{aligned}$$

with the ∞ norm. We let $\varphi = \left(\varphi_{g_s}\right)_{s \in E_0}$ with $g_s \in \ell_\infty(G)$. From the duality between $\ell_1(G)$ and $\ell_\infty(G)$, for any $f \in \ell_1(G)$ and $s \in E_0$, we have

$$\varphi_{g_s}(f) = \sum_{t \in G} f(t)g(t).$$

Thus, for all $f \in \text{Prob}(G)$, we have

$$\begin{aligned} 1 &\leqslant \varphi\left((f - \lambda_s(f))_{s \in E_0}\right) \\ &= \sum_{s \in E_0} \varphi_{g_s}(f - \lambda_s(f)) \\ &= \sum_{s \in E_0} \sum_{t \in G} (f - \lambda_s(f))(t) g_s(t) \\ &= \sum_{s \in E_0} \left(\sum_{t \in G} f(t) g_s(t) - \sum_{t \in G} f\left(s^{-1}t\right) g_s(t) \right) \\ &= \sum_{s \in E_0} \left(\sum_{t \in G} f(t) g_s(t) - \sum_{r \in G} f(r) g(sr) \right) && s^{-1}t = r \\ &= \sum_{s \in E_0} \left(\sum_{r \in T} f(r) g_s(r) - \sum_{r \in G} f(r) \lambda_{s^{-1}}(g)(r) \right) \\ &= \sum_{s \in E_0} \sum_{r \in G} f(r)(g_s - \lambda_{s^{-1}}(g))(r). \end{aligned}$$

Note that this holds for any $f \in \text{Prob}(G)$. In particular, if $f = \delta_t$ for a given $t \in \text{Prob}(G)$, then we must have

$$\begin{aligned} \sum_{s \in E_0} \sum_{r \in G} f(r)(g_s(r) - \lambda_{s^{-1}}(g_s)(r)) &= \sum_{s \in E_0} \sum_{r \in G} \delta_t(r)(g_s(r) - \lambda_{s^{-1}}(g_s)(r)) \\ &= \sum_{s \in E_0} (g_s - \lambda_{s^{-1}} g_s)(t) \\ &\geqslant 1. \end{aligned}$$

Since G is amenable, there is a mean $\mu : \ell_\infty(G) \to \mathbb{C}$ with $\mu(g_s) = \mu(\lambda_{s^{-1}}(g_s))$, meaning

$$\mu\left(\sum_{s \in E_0} (g_s - \lambda_{s^{-1}}(g_s))(t) \right) = 0$$

$$\geqslant 1,$$

which is a contradiction. □