This is a collection of old real analysis qualifier exam solutions.

August 2019

Problem 1

Problem: Let $\mathcal{C}$ be the Cantor set on $[0,1]$.

(a) Show that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) Find two sets $A, B \subseteq \mathbb{R}$ that are closed and have Lebesgue measure zero such that $A + B = \mathbb{R}$.

(a) Recall that the Cantor set $\mathcal{C}$ is defined to consist of all $x \in [0,1]$ such that x only contains 0 and 2 in the ternary expansion of x. Writing $a \in [0,2]$ as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where $a_k \in \{0,1,2\}$, we may then find a_k at each ternary expansion slot for k as follows:

- if $a_k = 0$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = c_k = 0$
- if $a_k = 2$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = 2$ and $c_k = 0$ or vice versa.
- if $a_k = 1$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_{k+1} = c_{k+1} = 2$.

Therefore, since every digit of every ternary expansion in $[0,2]$ can be obtained from $\mathcal{C}$, we see that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) We may set B to be the union of all integer translates of $\mathcal{C}$, and set $A = \mathcal{C}$. This yields closed subsets of $\mathbb{R}$ with Lebesgue measure zero that sum to $\mathbb{R}$.

Problem 2

Problem: Does there exist a finite measure space $(X, \mathcal{F}, \mu)$ and a sequence $(f_n)_n$ of μ-measurable functions such that

- $f_n(x) \geqslant 0$;
- $f_n(x) \longrightarrow 0$ for all x;
- $\int_X f_n(x)\, d\mu(x) \longrightarrow 0$ as $n \longrightarrow \infty$;
- $\Phi(x) = \sup_n f_n(x)$ has infinite integral?

Consider the sequence of functions

$$f_n(x) = n\mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on $[0,1]$. This sequence is pointwise convergent everywhere to zero, as $f_n(0) = 0$ and the Archimedean property give that for any $x \in (0,1]$, there is some n large enough that gives $\frac{1}{n} < x$. Furthermore, we see that

$$\begin{aligned}\int f_n\, d\mu &= n\left(\frac{1}{n} - \frac{1}{n+1}\right)\\ &= \frac{1}{n+1}\\ &\longrightarrow 0.\end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned}\int \Phi\, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1}\\ &\longrightarrow \infty.\end{aligned}$$

Problem 3

> **Problem:** Let μ be a signed measure in $\mathbb{R}^n$ that is bounded on bounded sets. Suppose that
>
> $$\int_{\mathbb{R}^n} f\, d\mu = 0$$
>
> for all continuous functions f with bounded support. Show that $\mu = 0$.

Fix $r > 0$, and consider the family of continuous functions f_n defined by

$$f_n = \begin{cases} 1 & x \in B(0,r) \\ 0 \leqslant f_n(x) \leqslant 1 & x \in B(0, r+1/n) \setminus B(0,r) \\ 0 & x \in B(0, r+1/n)^c \end{cases}$$

Since each f_n is continuous with bounded support, we see that

$$\begin{aligned} \int_{\mathbb{R}^n} f_n\, d\mu &= 0 \\ &= \int_{B(0,r)} f_n\, d\mu + \int_{B(0,r+1/n)\setminus B(0,r)} f_n\, d\mu. \end{aligned}$$

We may define $K_n = B(0, r+1/n) \setminus B(0,r)$. Therefore,

$$\begin{aligned} |\mu(B(0,r))| &= \left|\int_{B(0,r)} f_n\, d\mu\right| \\ &= \left|\int_{K_n} f_n\, d\mu\right| \\ &\leqslant \int_{K_n} f_n\, d|\mu| \\ &\leqslant \int_{K_n} d|\mu| \\ &= \mu^+(K_n) + \mu^-(K_n). \end{aligned}$$

Now, since K_n is bounded, we see that μ^+ and μ^- are both finite. Thus, since $\bigcap_{n\geqslant 1} K_n = \emptyset$, we see that $|\mu(B(0,r))| \leqslant \lim_{n\to\infty}(\mu^+(K_n) + \mu^-(K_n)) = 0$, so $\mu(B(0,r))$. Since the Borel σ-algebra is generated by the closed balls, $\mu = 0$ for all Borel sets.

Problem 4

> **Problem:** Let $L_1(\mathbb{R})$ be the space of Lebesgue integrable functions on $\mathbb{R}$. Suppose $f \in L_1(\mathbb{R})$ is positive. Show that $\frac{1}{f(x)} \notin L_1(\mathbb{R})$.

Suppose toward contradiction that both f and $1/f$ are in $L_1(\mathbb{R})$. Then, from Hölder's Inequality, we have

$$\begin{aligned} \infty &= \int 1\, d\mu \\ &\leqslant \left(\int f\, d\mu\right)^{1/2} \left(\int \frac{1}{f}\, d\mu\right)^{1/2} \\ &< \infty, \end{aligned}$$

which is a contradiction.

Problem 5

Problem: Applying the Gram–Schmidt orthogonalization to $\{1, x, x^2, \dots\}$ in the Hilbert space $L_2([-1,1])$ with Lebesgue measure, one gets the Legendre polynomials $L_n(x)$.

(a) Show that the Legendre polynomials form a basis (complete orthogonal system) in the Hilbert space $L_2([-1,1])$.

(b) Show that the Legendre polynomials are given by the formula $L_n(x) = c_n \frac{d^n}{dx^n}\left(x^2-1\right)^n$.

(a) Let $f \in L_2([-1,1])$. We may find $g \in C([-1,1])$ such that $\|f-g\|_{L_2} < \varepsilon/2$. Similarly, we may find a polynomial p such that $\|g-p\|_u < \varepsilon/4$, meaning that $|p(x)-g(x)| < \varepsilon/4$ for all $x \in [-1,1]$. This yields

$$\begin{aligned}\|p-g\|_{L_2} &= \left(\int_{-1}^{1} |p(x)-g(x)|^2\,dx\right)^{1/2}\\ &< \left(\int_{-1}^{1} \left(\frac{\varepsilon}{4}\right)^2 dx\right)^{1/2}\\ &= \left(\frac{\varepsilon^2}{8}\right)^{1/2}\\ &< \frac{\varepsilon}{2},\end{aligned}$$

so $\|f-p\|_{L_2} < \varepsilon$, meaning that the closed linear span of the monomials is dense in L_2, and the Legendre polynomials form an orthonormal system.

(b) We see that at every step in evaluating the expression

$$L_n(x) = c_n \frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{$*$}$$

the degree of the polynomial increases by 1, so each $L_n(x)$ has degree n. To verify that the polynomials generated from $(*)$ are orthogonal to each other, we let $n > m$ without loss of generality, and use integration by parts to obtain

$$\begin{aligned}\langle L_n, L_m\rangle &= \int_{-1}^{1}\left(\frac{d^n}{dx^n}\left(x^2-1\right)^n\right)\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx\\ &= \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n \frac{d^m}{dx^m}\left(x^2-1\right)^m\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n \frac{d^{m+1}}{dx^{m+1}}\left(x^2-1\right)^m dx\\ &\vdots\\ &= (-1)^n \int_{-1}^{1} \frac{d^{m+n}}{dx^{m+n}}\left(x^2-1\right)^m dx\\ &= (-1)^n \int_{-1}^{1} \frac{d^n}{dx^n}\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx\\ &= (-1)^n \int \frac{d^n}{dx^n} L_m(x)\,dx\\ &= 0,\end{aligned}$$

seeing as we are taking n derivatives of a degree $m < n$ polynomial.

January 2020

Problem 1

Problem: Let μ be the Lebesgue measure on $\mathbb{R}$, and let $A \subseteq [0,1]$ be Lebesgue-measurable.

(a) Prove or show a counterexample to the assertion that

$$\mu(A) = \sup_{\substack{U \subseteq A \\ U \text{ open}}} \mu(U).$$

(b) Prove or show a counterexample to the assertion that

$$\mu(A) = \inf_{\substack{A \subseteq U \\ U \text{ open}}} \mu(U).$$

(a) This is false. If $A \subseteq [0,1]$ is the "fat Cantor set" constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then $\mu(A) = \frac{1}{2}$, but A is nowhere dense, meaning that if $U \subseteq A$ is open, then $U = \emptyset$.

To see that A is nowhere dense, we see that A is closed, so if $x \in A \subseteq [0,1]$, and $\varepsilon > 0$, we may show that the interval $(x - \varepsilon, x + \varepsilon)$ is not contained in A. In the recursive construction of A, we may see that there is some step n_1 such that $\frac{1}{4^{n_1}} < 2\varepsilon$, implying that $(x - \varepsilon, x + \varepsilon)$ is not contained in the recursive construction at n_1. Therefore $A^\circ = \emptyset$.

(b) This is true. By the definition of the Lebesgue outer measure, for any $\varepsilon > 0$, there are $\{(a_k, b_k)\}_{k=1}^{\infty}$ such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^{\infty} (a_k, b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^{\infty} (a_k, b_k),$$

we have that U is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\left\{U \;\middle|\; A \subseteq U, U \text{ open}\right\}.$$

Remark: Part (a) can be solved by selecting $A = \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$.

Problem 3

Problem: Let X be a compact metric space, $C(X)$ the space of real-valued continuous functions on X with the supremum norm. Assume that $\mathcal{A} \subseteq C(X)$ satisfies

- (algebra) for all $f, g \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$, we have $\alpha f + \beta g \in \mathcal{A}$ and $fg \in \mathcal{A}$;
- (separates points) for any $x \neq y$ in X, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

(a) Show by example that $\mathcal{A}$ need not be dense in $C(X)$.

(b) In order to conclude that $\mathcal{A}$ is dense by the Stone–Weierstrass Theorem, what additional condition(s) should be added.

(a) Consider the algebra of polynomials on $[0,1]$ without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and $f(x) = x$ separates points in $[0,1]$, this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at $x = 0$, the uniform closure of the algebra yields all the continuous functions on $[0,1]$ with $f(0) = 0$.

(b) In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra $\mathcal{A}$ to include the constant functions.

Problem 4

Problem: Let μ be a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra. Let $\mu(\mathbb{R}) = 1$. Next, let $\mathcal{F} \subseteq \mathcal{B}$ be the sub-σ-algebra generated by symmetric intervals.

Let $f \in L_1(\mathbb{R}, \mathcal{B}, \mu)$. Find a function g such that:

- $g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$ (in particular, g is $\mathcal{F}$-measurable);
- for all $E \in \mathcal{F}$, $\int_E g \, d\mu = \int_E f \, d\mu$.

We consider the signed measure on $\mathcal{F}$ defined by

$$\nu(E) = \int_E f \, d\mu,$$

meaning that $\nu \ll \mu$, so the function $g := \frac{d\nu}{d\mu}$, where $\frac{d\nu}{d\mu}$ denotes the Radon–Nikodym derivative of ν with respect to μ (where we restrict μ to $\mathcal{F}$), is $\mathcal{F}$-measurable (by Radon–Nikodym) and in $L_1(\mathbb{R}, \mathcal{F}, \mu)$. This gives, for all $E \in \mathcal{F}$,

$$\begin{aligned}\int_E g \, d\mu &= \int_E \frac{d\nu}{d\mu} \, d\mu \\ &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f \, d\mu.\end{aligned}$$

Problem 5

Problem: Let μ be a finite measure on $(X, \mathcal{F})$. Show that a sequence of $\mathcal{F}$-measurable functions $(f_n)_n$ converges to f in measure if and only if

$$\int_X \min\{1, |f_n - f|\} \, d\mu(x) \to 0.$$

Let $M = \mu(X)$.

Let $(f_n)_n \to f$ in measure, and let $\varepsilon > 0$. If we let

$$\begin{aligned}A &= \{x \mid |f_n(x) - f(x)| > \varepsilon/2M\} \\ B &= \{x \mid |f_n(x) - f(x)| \leqslant \varepsilon/2M\},\end{aligned}$$

we have

$$\begin{aligned}\int_X \min(1, |f_n - f|) \, d\mu &= \int_A \min(1, |f_n - f|) \, d\mu + \int_B \min(1, |f_n - f|) \, d\mu \\ &\leqslant \mu(A) + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon.\end{aligned}$$

Meanwhile, if

$$\int_X \min(1, |f_n - f|) \, d\mu \to 0,$$

then by Chebyshev's Inequality, we have, for a fixed $0 < \varepsilon \leqslant 1$,

$$\begin{aligned}\mu(\{x \mid |f_n - f| \geqslant \varepsilon\}) &= \mu(\{x \mid \min(1, |f_n - f|) \geqslant \varepsilon\}) \\ &\leqslant \frac{1}{\varepsilon} \int_X \min(1, |f_n - f|)\, d\mu \\ &\to 0,\end{aligned}$$

so $(f_n)_n \to f$ in measure.

August 2020

Problem 1

> **Problem:** Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and almost everywhere differentiable such that $f'(x) = 1$ almost everywhere. Does this imply that $f(2) - f(1) = 1$?

This is false. To see this, let $\mathfrak{C}(x)$ denote the Cantor–Lebesgue function, and let

$$h(x) = \sum_{n=-\infty}^{\infty} \mathfrak{C}(x - n) + n.$$

Then, since $\mathfrak{C}(x)$ has derivative zero almost everywhere, the sum of a number of translates of $\mathfrak{C}(x)$ still has derivative zero almost everywhere. Then, setting

$$f(x) = h(x) + x,$$

we get that $f(x)$ has derivative equal to 1 almost everywhere. However, at the same time, $f(2) - f(1) = 2$.

Problem 2

> **Problem:** Prove or provide a counterexample to the assertion that every open set in $\mathbb{R}^2$ is a countable union of closed sets.

We show the inverse problem, which is that every closed set in $\mathbb{R}^2$ is G_δ. To do this, we let $A \subseteq \mathbb{R}^2$ be closed, nonempty, and proper (if $A = \emptyset$ or $A = \mathbb{R}^2$ the answer is trivial).

Then, there is some $x \in A^c$, and specifically there is $x \in A^c$ with rational coordinates (else, select $y \in \mathbb{Q}^2$ within the ball of radius ε that allows A^c to be open). Furthermore, since $\mathbb{R}^2$ is a metric space, $\mathbb{R}^2$ is regular, so there are open U_x and V_x such that $A \subseteq U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$.

Therefore, we get

$$A = \bigcap \left\{U_x \mid x \in \mathbb{Q}^2 \setminus A\right\},$$

meaning that A is G_δ. Taking complements, we thus get that every open set is F_σ.

Problem 3

> **Problem:** Let $\mathcal{H}$ be a separable complex Hilbert space with basis $(f_n)_n$. Define $P(f_n) = f_{n+1}$.
>
> (a) Find P^*, the adjoint to P.
>
> (b) Find PP^* and P^*P.

(a) We see that

$$\left\langle Pf_i, f_j\right\rangle = \delta_{i+1,j}$$

$$\begin{aligned} &= \delta_{i,j-1} \\ &= \left\langle f_i, f_{j-1} \right\rangle \\ &= \left\langle f_i, P^* f_j \right\rangle, \end{aligned}$$

so that $Pf_n = f_{n-1}$ if $n > 1$. Else, if $n = 1$, then $P^* f_n = 0$.

(b) We see that, acting on the orthonormal basis $(f_n)_n$, $P^*P(f_n) = f_n$, and

$$PP^*(f_n) = \begin{cases} 0 & n = 1 \\ 1 & \text{else,} \end{cases}$$

so that $P^*P = I$ and PP^* is as above.

Problem 4

> **Problem:** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $f_n \colon X \to \mathbb{R}$ be measurable functions such that
>
> $$\lim_{n\to\infty} \mu(\{x \mid f_n(x) \leqslant t\}) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}.$$
>
> Show that $f_n \to 0$ in measure.

We see that

$$\mu(\{x \mid f_n(x) > t\}) = \mu(X) - \mu(\{x \mid f_n(x) \leqslant t\}),$$

so by taking limits, we find that

$$\lim_{n\to\infty} \mu(\{x \mid f_n(x) > t\}) = \begin{cases} 1 & t < 0 \\ 0 & t \geqslant 0 \end{cases}.$$

So, if $\varepsilon > 0$, then

$$\begin{aligned} \mu(\{x \mid |f_n(x)| > \varepsilon\}) &= \mu(\{x \mid f_n(x) < -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\leqslant \mu(\{x \mid f_n(x) \leqslant -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\to 0. \end{aligned}$$

January 2021

Problem 1

> **Problem:** Let $(f_n)_n$, f be measurable functions on $(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ in measure. Does this imply that there exists a measurable set $A \subseteq \Omega$ with $\mu(\Omega \setminus A) = 0$ such that $f_n(x) \to f(x)$ for all $x \in A$.

This is not true. To see this, consider the family of functions defined by

$$\begin{aligned} f_1 &= \mathbb{1}_{[0,1]} \\ f_2 &= \mathbb{1}_{[0,1/2]} \\ f_3 &= \mathbb{1}_{[1/2,1]} \\ &\vdots \end{aligned}$$

where f_n is of width $\frac{1}{2^k}$ when $2^k \leqslant n < 2^{k+1}$, moving along $[0,1]$. Then, since $\mu(\{x \mid |f_n(x)| > 0\}) = \frac{1}{2^k}$, we have that for any $\varepsilon > 0$, $(\mu(\{x \mid |f_n(x)| > \varepsilon\}))_n \leqslant (\mu(A_n))_n$, where we have defined A_n to be the particular set with width $\frac{1}{2^k}$ when $2^k \leqslant n \leqslant 2^{k+1}$. Yet, since for any $x \in [0,1]$ there are infinitely many such n such that $f_n(x) = 1$, the family $(f_n)_n$ does not converge to 0 pointwise anywhere on $[0,1]$.

Problem 2

> **Problem:** Let B be a measurable subset of the two-dimensional plane such that the intersection of B with every vertical line is either finite or countable. Find $\mu(B)$, where μ is the two-dimensional Lebesgue measure.

Note that the two-dimensional Lebesgue measure is the completion of $m \times m$, where $m \times m$ is the product measure on the product σ-algebra $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. If $B \in \mathcal{L}(\mathbb{R}^2)$, then $B = C \cup N$, where N is a μ-null set and $C \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. Therefore, if we show that $(m \times m)(C) = 0$, we then show that $\mu(B) = 0$.

To see that $(m \times m)(\mathbb{C}) = 0$, note that by our assumption, $B^x = \{y \in \mathbb{R} \mid (x, y) \in B\}$ is either finite or countable, so since $C^x \subseteq B^x$, we must have that C^x is either finite or countable. By Tonelli's Theorem, since $\mathbb{1}_C$ is positive, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbb{1}_C \, d(m \times m) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{C^x} \, dy \, dx \\ &= \int_{\mathbb{R}} m(C^x) \, dx \\ &= 0,\end{aligned}$$

so $(m \times m)(C^x) = 0$, meaning

$$\begin{aligned}\mu(B) &= \mu(C) + \mu(N) \\ &= (m \times m)(C) + \mu(N) \\ &= 0.\end{aligned}$$

Problem 3

> **Problem:** Let $(\Omega, \mathcal{F})$ be a measurable space, μ, ν, ρ finite positive measures with $\mu \ll \nu$. Show that there exists a measurable function f on Ω such that for all $E \in \mathcal{F}$,
> $$\mu(E) = \int_E f \, d\nu + \int_E (f - 1) \, d\rho.$$

Since $\mu \ll \nu$, and $\rho \ll \rho$, we have $\mu + \rho \ll \nu + \rho$, as $(\nu + \rho)(E) = 0$ if and only if $\nu(E) = 0$ and $\rho(E) = 0$, meaning that $\mu(E) = 0$ and $\rho(E) = 0$, so by Radon–Nikodym, there is some measurable f such that

$$\mu(E) + \rho(E) = \int_E f \, d(\nu + \rho),$$

so by rearranging, we get

$$\mu(E) = \int_E f \, d\nu + \int_E (f - 1) \, d\rho.$$

Problem 4

> **Problem:** Let f, g be nonnegative measurable functions on $[0, 1]$, and let $a, b, c, d \geqslant 0$ be arbitrary nonnegative numbers. Show that
> $$\left(ac + bd + \int_0^1 f(x)g(x) \, dx\right)^3 \leqslant \left(a^3 + b^3 + \int_0^1 (f(x))^3 \, dx\right)\left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx\right)^2.$$

Since all of f, g, a, b, c, d are positive, we may show

$$ac + bd + \int_0^1 f(x)g(x) \, dx \leqslant \left(a^3 + b^3 + \int_0^1 (f(x))^3 \, dx\right)^{1/3} \left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx\right)^{2/3}.$$

To do this, we use Hölder's Inequality three times:

$$\begin{aligned}
ac + bd + \int_0^1 f(x)g(x)\,dx &\leqslant \left(a^3 + b^3\right)^{1/3}\left(c^{3/2} + d^{3/2}\right)^{2/3} + \int_0^1 f(x)g(x)\,dx \\
&\leqslant \left(a^3 + b^3\right)^{1/3}\left(c^{3/2} + d^{3/2}\right)^{2/3} + \left(\int_0^1 (f(x))^3\,dx\right)^{1/3}\left(\int_0^1 (g(x))^{3/2}\,dx\right)^{2/3} \\
&\leqslant \left(a^3 + b^3 + \int_0^1 (f(x))^3\,dx\right)^{1/3}\left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2}\,dx\right)^{2/3}.
\end{aligned}$$

Problem 5

> **Problem:** Let $f(x)$ be a continuous function on $[0,1]$. Show that for every $\varepsilon > 0$ there exists $n \in \mathbb{Z}_{\geqslant 0}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$ such that for
>
> $$D := \sum_{k=0}^{n} a_k \left(\frac{d}{dx}\right)^k,$$
>
> we have
>
> $$\left|f(x) - e^{x^2}\left(De^{-x^2}\right)\right| < \varepsilon$$
>
> for all $x \in [0,1]$.

We note that for each n,

$$\left(\frac{d}{dx}\right)^n \left(e^{-x^2}\right) = P_n(x)e^{-x^2}$$

where $P_n(x)$ is a degree n polynomial. To see this, using induction on n, we get

$$\begin{aligned}
\left(\frac{d}{dx}\right)^0 \left(e^{-x^2}\right) &= (1)e^{-x^2} \\
&=: P_0(x)e^{-x^2} \\
\frac{d}{dx}\left(P_n(x)e^{-x^2}\right) &= P_n'(x)e^{-x^2} - 2xP_n(x)e^{-x^2} \\
&=: P_{n+1}(x)e^{-x^2}.
\end{aligned}$$

Therefore,

$$e^{x^2}\left(\frac{d}{dx}\right)^n \left(e^{-x^2}\right) = P_n(x).$$

Since each $P_n(x)$ is linearly independent (as they have different degrees of polynomials), and consist of polynomials of each degree for all $n \geqslant 0$, they span $\mathbb{C}[x]$. Then, for any $\varepsilon > 0$, by Stone–Weierstrass, there is some polynomial $p(x)$ such that

$$\sup_{x \in [0,1]} |f(x) - p(x)| < \varepsilon.$$

Since $\{P_n(x)\}_{n \geqslant 0}$ forms a basis for $\mathbb{C}[x]$, there are $a_0, \dots, a_n$ such that $p(x) = \sum_{k=0}^{n} a_k P_k(x)$. Setting

$$D = \sum_{k=0}^{n} a_k \left(\frac{d}{dx}\right)^k,$$

we obtain that

$$\left|f(x) - e^{x^2}\left(De^{-x^2}\right)\right| < \varepsilon.$$

January 2022

Problem 1

Problem: Let $(f_n)_n, f \subseteq L_1(X, \mu)$ be nonnegative functions, and let $(f_n)_n \to f$ pointwise, as well as

$$\left(\int_X f_n \, d\mu\right)_n \to \int_X f \, d\mu.$$

Show that $(f_n)_n \to f$ in L_1.

Consider the function $g_n(x) = \min(f_n, f)$, also written as

$$g_n = \frac{1}{2}(f_n + f - |f_n - f|).$$

Note that $|g_n| \leqslant f$, and $(g_n)_n \to f$ pointwise, so by dominated convergence, we have

$$\begin{aligned}\int_X f \, d\mu &= \lim_{n\to\infty} \int_X g_n \, d\mu \\ &= \frac{1}{2} \lim_{n\to\infty}\left(\int_X f_n \, d\mu + \int_X f \, d\mu - \int_X |f_n - f| \, d\mu\right) \\ &= \int_X f \, d\mu - \frac{1}{2} \lim_{n\to\infty} \int_X |f_n - f| \, d\mu,\end{aligned}$$

so

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0,$$

and $(f_n)_n \to f$ in L_1.

Problem 2

Problem: Let $p \in [1, \infty)$.

(a) Show that if $(f_n)_n \to f$ in L_p, then there is $(f_{n_k})_k$ such that for μ-a.e. $x \in X$, $(f_{n_k})_k \to f$ pointwise.

(b) Let h be a measurable function, and let D be defined such that

$$D = \left\{f \in L_p(X, \mu) \,\middle|\, hf \in L_p(X, \mu)\right\}.$$

Suppose $(f_n)_n \to f$ in L_p, and $(hf_n)_n \to g$ in L_p. Show that $f \in D$ and $g = hf$.

(a) Since $(f_n)_n \to f$ in L_p, the sequence $(f_n)_n$ is L_p-Cauchy, so we may find a subsequence $(f_{n_k})_k$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}.$$

Defining

$$s_n = \sum_{k=1}^{n} |f_{n_{k+1}} - f_{n_k}|$$

$$s = \sum_{k=1}^{n} |f_{n_{k+1}} - f_{n_k}|,$$

we see that by Minkowski's Inequality,

$$\|s_n\| \leqslant \sum_{k=1}^{n} \|f_{n_{k+1}} - f_{n_k}\|$$

$$\leqslant 1.$$

So, by applying Fatou's Lemma to s_n^p, we see that

$$\|s\| \leqslant 1,$$

meaning that in particular, $s(x) < \infty$ almost everywhere, and $(s_n)_n$ converges absolutely almost everywhere. Defining

$$g(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})(x)$$

for all x where $s(x)$ is defined, and 0 otherwise, we see that by telescoping, $g(x) = \lim_{k\to\infty} f_{n_k}(x)$. Now, we show that $\|g - f\| = 0$, meaning that $g = f$ under the μ-a.e. equivalence relation. Computing, we have

$$\begin{aligned}\int_X |g-f|^p\,d\mu &= \int_X \liminf_{k\to\infty}|f_{n_k} - f|^p\,d\mu\\ &\leqslant \liminf_{k\to\infty}\int_X |f_{n_k} - f|^p\,d\mu\\ &= \liminf_{k\to\infty}\|f_{n_k} - f\|^p\\ &= 0,\end{aligned}$$

as for any subsequence $(f_{n_k})_k$, $(f_{n_k})_k \to f$ in L_p. Thus, $(f_{n_k})_k \to f$ for μ-almost every x.

(b) Since $(f_n)_n \to f$ in L_p, there is a subsequence $(f_{n_k})_k \to f$ pointwise almost everywhere. Thus, by multiplying $h(x)$, we see that $(hf_{n_k})_k \to hf$ pointwise almost everywhere.

Now, since $(hf_n)_n \to g$ in L_p, this applies for every subsequence of $(hf_n)_n$; in particular, it applies to $(hf_{n_k})_k$, meaning that $(hf_{n_k})_k \to g$ in L_p, and admits a subsequence $\left(hf_{n_{k_j}}\right)_j \to g$ pointwise almost everywhere.

Returning to the convergence $(hf_{n_k})_k \to hf$ pointwise almost everywhere, this applies for every subsequence, so in particular, it applies to $\left(hf_{n_{k_j}}\right)_j$.

Set

$$\begin{aligned}E_1 &= \left\{x \,\middle|\, \left(\left(hf_{n_{k_j}}\right)(x)\right)_j \nrightarrow g(x)\right\}\\ E_2 &= \left\{x \,\middle|\, \left(\left(hf_{n_{k_j}}\right)(x)\right)_j \nrightarrow (hf)(x)\right\}.\end{aligned}$$

Then, $\mu(E_1) = \mu(E_2) = 0$, so $\mu(E_1 \cup E_2) \leqslant \mu(E_1) + \mu(E_2) = 0$, and so $g(x) = (hf)(x)$ for almost every x (as $\mathbb{C}$ is Hausdorff). In particular, this means that $[g] = [hf]$ under the almost everywhere equivalence relation. Since L_p is complete, and $(hf_n)_n \to g$ in L_p, we have $g \in L_p$, so $hf \in L_p$, and $f \in D$.

Problem 3

Problem: Let μ be a Borel probability measure on $\mathbb{R}$, and define

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx}\,d\mu(x).$$

(a) Show that $\hat{\mu}(t)$ is bounded and continuous.

(b) If $\delta > 0$, show that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\,dt = \int_{\mathbb{R}} 1 - \operatorname{sinc}(\delta x)\,d\mu(x).$$

(c) Show that

$$1 - \operatorname{sinc}(u) \geqslant \frac{1}{2}\mathbb{1}_{(-\infty,2)\cup(2,\infty)}(u),$$

and deduce that

$$\mu(\{x \mid |x| > 2/\delta\}) \leqslant \frac{1}{\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\,dt.$$

(d) Let $(\mu_n)_n$ be a sequence of Borel probability measures on $\mathbb{R}$. Suppose that for all t, $\Phi(t) = \lim_{n\to\infty}\widehat{\mu_n}(t)$ exists, and $\Phi(t)$ is continuous at $t = 0$. Show that for all $\varepsilon > 0$, there is a compact K such that for all n, $\mu_n(K) \geqslant 1 - \varepsilon$.

(a) We see that $\hat{\mu}$ is bounded, as

$$\begin{aligned}|\hat{\mu}(t)| &= \left|\int_{\mathbb{R}} e^{itx}\,d\mu(x)\right| \\ &\leqslant \int_{\mathbb{R}} \left|e^{itx}\right|\,d\mu(x) \\ &\leqslant 1,\end{aligned}$$

since μ is a probability measure. Furthermore, using dominated convergence with $g(t) = 1$, we see that if $(t_n)_n \to t$, then $e^{it_nx} \to e^{itx}$ as the exponential function is continuous, so $\hat{\mu}(t_n) \to \hat{\mu}(t)$, and $\hat{\mu}$ is continuous.

(b) We note that $\operatorname{Re}(\hat{\mu}(t)) = \int_{\mathbb{R}} \cos(tx)\,d\mu(x) \leqslant 1$ for all t, meaning that $1 - \operatorname{Re}(\hat{\mu}(t)) \geqslant 0$ for all t. Writing our integral, we then get

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\,dt = \frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}\int_{\mathbb{R}} e^{itx}\,d\mu(x)\,dt$$

and using the fact that $\mu(\mathbb{R}) = 1$,

$$= \frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\mathbb{R}} 1 - \cos(tx)\,d\mu(x)\,dt.$$

By Tonelli's Theorem, we may switch the order of integration, so

$$\begin{aligned}&= \frac{1}{2\delta}\int_{\mathbb{R}}\int_{-\delta}^{\delta} 1 - \cos(tx)\,dt\,d\mu(x) \\ &= \int_{\mathbb{R}} 1 - \int_{-\delta}^{\delta}\frac{1}{2\delta}\cos(tx)\,dt\,d\mu(x).\end{aligned}$$

Now, evaluating the inner integral, we see that

$$\int_{-\delta}^{\delta}\frac{1}{2\delta}\cos(tx)\,dt = \begin{cases}1 & x = 0 \\ \frac{\sin(\delta x)}{\delta x} & x \neq 0\end{cases},$$

so

$$= \int_{\mathbb{R}} 1 - \operatorname{sinc}(\delta x)\,d\mu(x).$$

(c) We see that $1 - \operatorname{sinc}(u) \geqslant 0$ for all u, so when $|u| \leqslant 2$, the inequality is satisfied. Similarly, if $|u| > 2$, then

$$\begin{aligned} 1 - \operatorname{sinc}(u) &= 1 - \frac{\sin(u)}{u} \\ &\geqslant 1 - \frac{1}{2} \\ &= \frac{1}{2}, \end{aligned}$$

so the inequality is satisfied when $|u| > 2$. Thus, we see that

$$\begin{aligned} \frac{1}{\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\,dt &= 2\int_{\mathbb{R}} 1 - \operatorname{sinc}(\delta x)\,d\mu(x) \\ &\geqslant \int_{\mathbb{R}} \mathbb{1}_{(-\infty,2)\cup(2,\infty)}(\delta x)\,d\mu(x) \\ &= \int_{\mathbb{R}} \mathbb{1}_{(\infty,2/\delta)\cup(2/\delta,\infty)}(x)\,d\mu(x) \\ &= \mu(\{x \mid |x| > 2/\delta\}). \end{aligned}$$

(d) Let $\varepsilon > 0$. Since $\Phi(t)$ is continuous at 0, and $\Phi(0) = \lim_{n\to\infty} \widehat{\mu_n}(0) = 1$, there is δ such that whenever $|t| < \delta$, $|1 - \Phi(t)| < \varepsilon/2$. Note that this implies that $1 - \operatorname{Re}(\Phi(t)) < \varepsilon/2$ for all t with $|t| < \delta$.

Next, we see that $1 - \operatorname{Re}(\widehat{\mu}_n(t)) \to 1 - \operatorname{Re}(\Phi(t))$, so by using the dominating function $g(t) = 2$, the dominated convergence theorem implies that

$$\begin{aligned} \frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\widehat{\mu_n}(t))\,dt &\to \frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\Phi(t))\,dt \\ &< \varepsilon/2, \end{aligned}$$

meaning that there is N such that for all $n \geqslant N$,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\widehat{\mu_n}(t))\,dt < \varepsilon/2.$$

Thus, by using part (c), we see that

$$\mu_n(\{x \mid |x| > 2/\delta\}) < \varepsilon,$$

so for all $n \geqslant N$,

$$\mu_n([-2/\delta, 2/\delta]) \geqslant 1 - \varepsilon.$$

Next, for each $n \leqslant N$, we find $k_n \in \mathbb{N}$ such that $\mu([-k_n, k_n]) \geqslant 1 - \varepsilon$; the existence of such a k_n follows from continuity from below, as for each n,

$$\begin{aligned} 1 &= \mu_n(\mathbb{R}) \\ &= \mu_n\left(\bigcup_{k\geqslant 1}[-k,k]\right) \\ &= \sup_{k\geqslant 1} \mu_n([-k,k]). \end{aligned}$$

Set $K_N = \max\left(\{k_n\}_{n=1}^{N}\right)$, and let $K = [-K_N, K_N] \cup [-2/\delta, 2/\delta]$. Then, for all n, we find that

$$\mu_n(K) \geqslant 1 - \varepsilon.$$

August 2022

Problem 1

Problem: Compute

$$\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x\left(1+x^2\right)}\,dx.$$

We note that

$$\begin{aligned}\left|\frac{n\sin(x/n)}{x(1+x^2)}\right| &\leqslant \left|\frac{n(x/n)}{x(1+x^2)}\right| \\ &= \frac{1}{1+x^2},\end{aligned}$$

and since $\frac{1}{1+x^2}$ is integrable, we may use Dominated Convergence to switch limit and integral, giving

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx &= \int_0^\infty \lim_{n\to\infty}\frac{n\sin(x/n)}{x(1+x^2)}\,dx \\ &= \int_0^\infty \lim_{h\to 0}\frac{\frac{1}{h}\sin(hx)}{x(1+x^2)}\,dx \\ &= \int_0^\infty \frac{x}{x(1+x^2)}\,dx \\ &= \frac{\pi}{2}.\end{aligned}$$

Problem 2

Problem: Fix $a < b$ in $\mathbb{R}$. For a Lipschitz function $g\colon [a,b] \to \mathbb{C}$, set

$$\|g\|_{\text{Lip}} = \sup_{x\neq y\in[a,b]} \frac{|g(x)-g(y)|}{|x-y|}.$$

(a) Show that $f\colon [a,b]\to\mathbb{C}$ is Lipschitz if and only if f is absolutely continuous and $f' \in L_\infty([a,b])$.

(b) If $f\colon [a,b]\to\mathbb{C}$ is Lipschitz, show that $\|f\|_{\text{Lip}} = \|f'\|_{L_\infty}$.

(a) Let f be Lipschitz, and let M denote the Lipschitz constant — i.e., $|f(x)-f(y)| \leqslant |x-y|$ for all $x, y \in [a,b]$. Set $\delta = \frac{\varepsilon}{M}$. Then, if $\left\{\left(a_j, b_j\right)\right\}_{j=1}^k$ is a partition such that $\sum_{j=1}^k \left|b_j - a_j\right| < \delta$, we have

$$\begin{aligned}\sum_{j=1}^k \left|f\left(b_j\right)-f\left(a_j\right)\right| &\leqslant M\sum_{j=1}^k\left|b_j-a_j\right| \\ &< \varepsilon.\end{aligned}$$

Thus, f is absolutely continuous. Now, if $x, x+h \in [a,b]$, we have that

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leqslant M,$$

meaning that

$$\begin{aligned}|f'(x)| &= \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \\ &\leqslant M,\end{aligned}$$

and since $f'(x)$ exists for a.e. $x \in [a, b]$, we have that $\operatorname{ess\,sup}_{x\in[a,b]} |f'(x)| \leqslant M$, so $f' \in L_\infty([a, b])$.

Let f be absolutely continuous with bounded derivative. Then, if M is the essential supremum of the f', the fundamental theorem of calculus gives

$$\begin{aligned} |f(y) - f(x)| &= \left|\int_x^y f'(t)\,dt\right| \\ &\leqslant \int_x^y |f'(t)|\,dt \\ &\leqslant \int_x^y M\,dx \\ &= M|y - x|, \end{aligned}$$

so f is Lipschitz.

(b) If f is such that $f'(x)$ exists, then for $x, x + h \in [a, b]$, we have

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leqslant \|f\|_{\text{Lip}},$$

so by taking limits, we have

$$|f'(x)| \leqslant \|f\|_{\text{Lip}}.$$

Thus, this ordering must respect essential suprema, meaning

$$\|f'\|_{L_\infty} \leqslant \|f\|_{\text{Lip}}.$$

Furthermore, if $\varepsilon > 0$, there are $x, y \in [a, b]$ with $x < y$ such that

$$\begin{aligned} \|f\|_{\text{Lip}} - \varepsilon &< \left|\frac{f(y) - f(x)}{y - x}\right| \\ &= \frac{1}{|y - x|}\left|\int_x^y f'(t)\,dt\right| \\ &\leqslant \frac{1}{|y - x|}\int_x^y |f'(t)|\,dt \\ &\leqslant \frac{1}{|y - x|}\int_x^y \|f'\|_{L_\infty}\,dt \\ &= \|f'\|_{L_\infty}, \end{aligned}$$

and since ε is arbitrary, we have $\|f\|_{\text{Lip}} \leqslant \|f'\|_{L_\infty}$.

Problem 3

> **Problem:** Let (X, μ) be a σ-finite measure space. Show that if $f, g \in L_1(X, \mu)$ with $0 \leqslant f, g$ almost everywhere, then
>
> $$\|f - g\|_{L_1} = \int_0^\infty \mu(\{x \mid f(x) > t\} \triangle \{x \mid g(x) > t\})\,dt.$$

We start by showing that

$$|a - b| = \int_0^\infty \left|\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)\right|\,dt$$

for all $a, b \in [0, \infty)$. Without loss of generality, $a \leqslant b$. To see this, note that there are three cases:

$$\left|1_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)\right| = \begin{cases} 0 & t < a, b \\ 1 & a \leqslant t < b, \\ 0 & a, b \leqslant t \end{cases}$$

giving

$$\begin{aligned} \int_0^\infty \mathbb{1}_{[a,b)}\, dt &= \mu([a, b)) \\ &= b - a \\ &= |a - b|. \end{aligned}$$

Now, we have

$$\begin{aligned} \|f - g\|_{L_1} &= \int_X |f(x) - g(x)|\, d\mu(x) \\ &= \int_X \int_0^\infty \left|\mathbb{1}_{(t,\infty)}(f(x)) - \mathbb{1}_{(t,\infty)}(g(x))\right| dt\, d\mu(x), \end{aligned}$$

and by Tonelli's Theorem, we have

$$\begin{aligned} &= \int_0^\infty \int_X \left|\mathbb{1}_{f^{-1}((t,\infty))} - \mathbb{1}_{g^{-1}((t,\infty))}\right| d\mu(x)\, dt \\ &= \int_0^\infty \int_X \mathbb{1}_{f^{-1}((t,\infty))\triangle g^{-1}((t,\infty))}\, d\mu(x)\, dt \\ &= \int_0^\infty \mu\Big(f^{-1}((t,\infty))\triangle g^{-1}((t,\infty))\Big)\, dt. \end{aligned}$$

Problem 4

Problem: Let (X, Σ) be a measurable space. Suppose that μ, ν are signed measures on Σ such that $\|\mu\|_{TV}, \|\nu\|_{TV} < \infty$, and $|\mu| \perp |\mu|$.

(a) If $\mu = \mu_1 - \mu_2$ and $\nu = \nu_1 - \nu_2$ with $\mu_1 \perp \mu_2$ and $\nu_1 \perp \nu_2$, show that $\mu_i \perp \nu_j$ for all $i, j \in \{1, 2\}$.

(b) Show that

$$\|\mu + \nu\|_{TV} = \|\mu\|_{TV} + \|\nu\|_{TV}.$$

(a) Since $|\mu| \perp |\nu|$, there are $U, V \subseteq X$ such that $|\mu|$ is concentrated on U and $|\nu|$ is concentrated on V, with $U \cap V = \emptyset$.

Note that by the Jordan decompositions, we have $|\mu| = \mu_1 + \mu_2 \geqslant \mu_{1,2}$ so $\mu_{1,2}$ are concentrated on U, and similarly $\nu_{1,2}$ are concentrated on V, so $\mu_i \perp \nu_j$.

(b) We show that the measures $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ are mutually singular. To see this, note the following:

- $\mu_1 = 0$ on $N_\mu \cup V$;
- $\nu_1 = 0$ on $N_\nu \cup U$;
- $\mu_2 = 0$ on $P_\mu \cup V$;
- $\nu_2 = 0$ on $P_\nu \cup U$,

so $\mu_1 + \nu_1 = 0$ on $A = \big(N_\mu \cup V\big) \cap (N_\nu \cup U)$, and $\mu_2 + \nu_2 = 0$ on $B = \big(P_\mu \cup V\big) \cap (P_\nu \cup U)$. Therefore, since

$$A \cup B = \big(N_\mu \cap N_\nu\big) \cup \big(N_\mu \cap U\big) \cup (N_\nu \cap V)$$

$$\begin{aligned}&\cup \left(P_\mu \cap P_\mu\right) \cup \left(P_\mu \cap U\right) \cup \left(P_\nu \cap V\right)\\ &= X\end{aligned}$$

$$\begin{aligned}A \cap B &= \left(N_\mu \cup V\right) \cap \left(N_\nu \cup U\right)\\ &\cap \left(P_\mu \cup V\right) \cap \left(P_\nu \cup U\right)\\ &= \emptyset,\end{aligned}$$

the measures $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ are mutually singular, so $A \sqcup B$ forms a Hahn decomposition for $\mu + \nu$ with corresponding Jordan decomposition of $(\mu_1 + \nu_1) - (\mu_2 + \nu_2)$. Thus,

$$\begin{aligned}\|\mu + \nu\|_{TV} &= |\mu + \nu|(X)\\ &= (\mu_1 + \nu_1)(X) + (\mu_2 + \nu_2)(X)\\ &= (\mu_1 + \mu_2)(X) + (\nu_1 + \nu_2)(X)\\ &= |\mu|(X) + |\nu|(X)\\ &= \|\mu\|_{TV} + \|\nu\|_{TV}.\end{aligned}$$

Problem 5

Problem:

(a) For $f \in L_1([0,1])$, let L_f be the set of all $x \in [0,1]$ such that

$$\lim_{r\to 0} \frac{1}{2r}\int_{x-r}^{x+r} |f(y) - f(x)|\,dy = 0.$$

State the conclusion of the Lebesgue differentiation theorem regarding L_f.

(b) For $n \in \mathbb{N}$, $0 \leqslant j \leqslant 2^n - 1$, set $I_{n,j} = [j2^{-n}, (j+1)2^{-n})$. For $f \in L_1([0,1])$, define

$$E_n f = \sum_{j=0}^{2^n-1}\left(\frac{1}{m(I_{n,j})}\int_{I_{n_j}} f(t)\,dt\right)\mathbb{1}_{I_{n_j}}.$$

Show that $\lim_{n\to\infty}(E_n f)(x) = f(x)$ for a.e. $x \in [0,1]$.

(a) The conclusion of the Lebesgue differentiation theorem states that $\mu([0,1] \setminus L_f) = 0$.

(b) Let $x \in [0,1]$. We note that x must be in exactly one such interval $(j2^{-n}, (j+1)2^{-n}]$ since these intervals are disjoint. If we select $r > 0$ such that $\frac{1}{2^n} < r \leqslant \frac{1}{2^{n-1}}$, then we note the following:

- $I_{n,j} \subseteq U(x,r)$ for exactly one such j;
- $m(U(x,r)) \leqslant 4\mu\left(I_{n,j}\right)$.

If $x \in L_f$, then for any $\varepsilon > 0$, there is some $\delta > 0$ such that when $r < \delta$, then

$$\frac{1}{\mu(U(x,r))}\int_{U(x,r)} |f(t) - f(x)|\,dt < \varepsilon,$$

by the Lebesgue Differentiation Theorem. If n is such that $\frac{1}{2^{n-1}} < \delta$, then when $\frac{1}{2^n} < r \leqslant \frac{1}{2^{n-1}}$, then for any $x \in L_f$, we have

$$\begin{aligned}|E_n f(x) - f(x)| &= \left|\frac{1}{m\left(I_{n,j}\right)}\int_{I_{n,j}} f(t)\,dt - f(x)\right|\\ &\leqslant \frac{1}{m\left(I_{n,j}\right)}\int_{I_{n,j}} |f(t) - f(x)|\,dt\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{m(I_{n,j})} \int_{U(x,r)} |f(t) - f(x)|\, dt \\
&\leqslant \frac{4}{U(x,r)} \int_{U(x,r)} |f(t) - f(x)|\, dt \\
&< 4\varepsilon,
\end{aligned}
$$

so $\lim_{n\to\infty} E_n f(x) = f(x)$ for all $x \in L_f$, meaning that it holds for a.e. $x \in [0,1]$.

January 2023

Problem 1

> **Problem:** Let (X, μ) be a σ-finite measure space, $p \in [1, \infty)$. Let $(f_n)_n$ be a sequence in $L_p(X, \mu)$, and suppose $\|f_n\|_{L_p} \leqslant 1$, $(f_n)_n \to f$ almost everywhere. Show that $\|f\|_p \leqslant 1$.

By using Fatou's Lemma, and assuming WLOG that $(f_n)_n \to f$ pointwise everywhere, we get

$$
\begin{aligned}
\int_X |f|^p \, d\mu &= \int_X \liminf_{n\to\infty} |f_n|^p \, d\mu \\
&\leqslant \liminf_{n\to\infty} \int_X |f_n|^p \, d\mu \\
&\leqslant 1,
\end{aligned}
$$

so $\|f\|_{L_p} \leqslant 1$.

Problem 2

> **Problem:** Let μ be an atomless Borel probability measure on $\mathbb{R}$. Suppose $E \subseteq \mathbb{R}$ is a Borel set with $\mu(E) > 0$. Show that there is $t \in \mathbb{R}$ with $\mu(E \cap (-\infty, t)) = \frac{1}{2}\mu(E)$.

Let

$$
f(t) = \mu(E \cap (-\infty, t)),
$$

and for any sequence $(t_n)_n$, define

$$
E_n = E \cap (-\infty, t_n).
$$

We will show that f is left- and right-continuous, hence continuous. To start, if $(t_n)_n \searrow t$, then

$$
\bigcap_{n\in\mathbb{N}} E_n = E \cap (-\infty, t],
$$

so

$$
\begin{aligned}
f(t) &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n \setminus \{t\}\right) \\
&= \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) - \mu(\{t\}).
\end{aligned}
$$

Since μ is atomless, we see that $\mu(\{t\}) = 0$, so since $\mu(E) < \infty$,

$$
f(t) = \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right)
$$

$$\begin{aligned}
&= \lim_{n\to\infty} \mu(E_n)\\
&= \lim_{n\to\infty} f(t_n).
\end{aligned}$$

Thus, f is right-continuous. Similarly, if f is left-continuous, and $(t_n)_n \nearrow t$, then

$$\bigcup_{n\in\mathbb{N}} E_n = E \cap (-\infty, t),$$

so by continuity from below,

$$\begin{aligned}
f(t) &= \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right)\\
&= \lim_{n\to\infty} \mu(E_n)\\
&= \lim_{n\to\infty} f(t_n).
\end{aligned}$$

Therefore, f is continuous. Since

$$\begin{aligned}
\lim_{t\to-\infty} f(t) &= 0\\
\lim_{t\to\infty} f(t) &= \mu(E)\\
&> 0,
\end{aligned}$$

the intermediate value theorem gives some $t_0 \in \mathbb{R}$ such that

$$\begin{aligned}
f(t_0) &= \mu(E \cap (-\infty, t_0))\\
&= \frac{1}{2}\mu(E).
\end{aligned}$$

Problem 3

Problem: Let X be a set equipped with a σ-algebra Σ. Suppose $\mu, \nu \colon \Sigma \to [0,\infty)$ are finite measures with $\lambda = \mu + \nu$. Define f such that

$$\nu(E) = \int_E f\, d\lambda.$$

(i) Show that $0 \leqslant f \leqslant 1$ λ-a.e.

(ii) If $F = \{x \mid f(x) = 1\}$, show that $\mu(F) = 0$.

(iii) If $A \subseteq \{x \mid 0 \leqslant f(x) < 1\}$ is such that $\mu(A) = 0$, show that $\nu(A) = 0$.

(i) Consider the sets E_n, for each $n \in \mathbb{N}$, defined by

$$E_n = \left\{x \,\middle|\, f(x) < -\frac{1}{n}\right\},$$

so that $E_n \subseteq E_{n+1}$, and

$$\begin{aligned}
E &= \bigcup_{n=1}^{\infty} E_n\\
&= \{x \mid f(x) < 0\}.
\end{aligned}$$

Then, we see that

$$0 \geqslant -\frac{1}{n}\lambda(E_n)$$

$$\begin{aligned}
&= -\frac{1}{n}\int_{E_n} d\lambda\\
&> \int_{E_n} f\,d\lambda\\
&= \nu(E_n)\\
&\geqslant 0,
\end{aligned}$$

meaning that $\lambda(E_n) = 0$ for each n, so by continuity from below, $\lambda(E) = \lim_{n\to\infty} \lambda(E_n) = 0$.

Now, the set

$$F = \{x \mid f(x) > 1\}$$

has

$$\begin{aligned}
\lambda(F) &= \int_F d\lambda\\
&< \int_F f\,d\lambda\\
&= \nu(F)\\
&\leqslant \nu(F) + \mu(F)\\
&= \lambda(F),
\end{aligned}$$

meaning that $\lambda(F) = 0$, and $0 \leqslant f \leqslant 1$ λ-a.e.

(ii) If $F = \{x \mid f(x) = 1\}$, then

$$\begin{aligned}
\lambda(F) &= \int_F d\lambda\\
&= \int_F f\,d\lambda\\
&= \nu(F),
\end{aligned}$$

so $\mu(F) = 0$.

(iii) Let $A \subseteq \{x \mid 0 \leqslant f(x) < 1\}$ be such that $\mu(A) = 0$. Then, we have

$$\begin{aligned}
\nu(A) &= \int_A f\,d\lambda\\
&= \int_A f\,d\nu + \int_A f\,d\mu\\
&< \int_A f\,d\nu + \int_A d\mu\\
&= \int_A f\,d\nu + \mu(A)\\
&= \int_A f\,d\nu\\
&\leqslant \int_A f\,d\lambda\\
&= \nu(A),
\end{aligned}$$

so $\nu(A) = 0$, else we reach a contradiction.

Problem 4

Problem: Fix $p \in [1, \infty)$. Let $W_p([0,1])$ be the space of absolutely continuous functions on $[0,1]$ such that $f' \in L_p([0,1])$. For all $f \in W_p([0,1])$, define

$$\|f\|_{W_p} = |f(0)| + \left\|f'\right\|_{L_p}.$$

Show that $\|\cdot\|_{W_p}$ is a norm that makes $W_p([0,1])$ into a Banach space. You are allowed to use the fact that $L_p([0,1])$ is a Banach space.

We start by showing that $\|\cdot\|_{W_p}$ is indeed a norm. To see that $\|\cdot\|_{W_p}$ is positive definite, if

$$\|f\|_{W_p} = 0,$$

then $|f(0)| = 0$ and $\|f'\|_{L_p} = 0$. Since $\|f'\|_{L_p} = 0$, $f' = 0$ a.e. as L_p is a Banach space. Note that, by the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t)\,dt,$$

so $f(x) = 0$ almost everywhere, hence $f(x) = 0$ in L_p.

Next, to see homogeneity, we have for all $\alpha \in \mathbb{C}$,

$$\begin{aligned}\|\alpha f\|_{W_p} &= |\alpha f(0)| + \left\|(\alpha f)'\right\|_{L_p}\\ &= |\alpha|\Big(|\alpha| + \|f'\|_{L_p}\Big)\\ &= |\alpha|\|f\|_{W_p},\end{aligned}$$

as $\|\cdot\|_{L_p}$ is a norm. Finally, we have

$$\begin{aligned}\|f+g\|_{W_p} &= |(f+g)(0)| + \left\|(f+g)'\right\|_{L_p}\\ &\leqslant |f(0)| + |g(0)| + \|f'\|_{L_p} + \|g'\|_{L_p}\\ &= \|f\|_{W_p} + \|g\|_{W_p},\end{aligned}$$

as $\|\cdot\|_{L_p}$ is a norm, so the triangle inequality holds. Thus, $\|\cdot\|_{W_p}$ is a norm.

Let $(f_n)_n$ be Cauchy in $W_p([0,1])$. Then, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geqslant N$,

$$\begin{aligned}\|f_n - f_m\|_{W_p} &= |f_n(0) - f_m(0)| + \|f_n' - f_m'\|_{L_p}\\ &< \varepsilon,\end{aligned}$$

meaning that both

$$\begin{aligned}|f_n(0) - f_m(0)| &< \varepsilon\\ \|f_n' - f_m'\|_{L_p} &< \varepsilon.\end{aligned}$$

Since $\mathbb{C}$ and $L_p([0,1])$ are complete, there is $c \in \mathbb{C}$ and $g \in L_p([0,1])$ such that

$$\begin{aligned}f_n(0) &\to c\\ f_n' &\to g.\end{aligned}$$

Define

$$f(x) = c + \int_0^x g(t)\,dt.$$

Then, we note that by the Fundamental Theorem of Calculus,

$$\begin{aligned} f'(x) &= g(x) \\ &\in L_p([0,1]), \end{aligned}$$

so $f \in W_p([0,1])$. Finally, we see that

$$\begin{aligned} \|f_n - f\|_{W_p([0,1])} &= |f_n(0) - f(0)| + \|f_n' - f'\|_{L_p} \\ &= |f_n(0) - c| + \|f_n' - g\|_{L_p} \\ &\to 0, \end{aligned}$$

so $(f_n)_n \to f$ in W_p, meaning W_p is complete.

Problem 5

Problem: Let m be Lebesgue measure on $\mathbb{R}$, $\Omega = \{\mathbb{1}_E \mid E \subseteq \mathbb{R} \text{ Borel}, m(E) < \infty\}$ be regarded as a subset of $L_1(\mathbb{R})$. We regard Ω as a metric space with the L_1 distance.

(i) If $a < b$ are real numbers, show that the function $\Omega \to \mathbb{R}$ given by

$$\mathbb{1}_E \mapsto m(E \cap [a,b])$$

is a continuous function.

(ii) If $a < b$ are real numbers, let $U_{a,b}$ be the subset of Ω consisting of all $\mathbb{1}_E$ where $E \subseteq \mathbb{R}$ is Borel, and

$$0 < m(E \cap [a,b]) < b - a.$$

Show that $U_{a,b}$ is open and dense in Ω.

(iii) Let D be the set of all $\mathbb{1}_E$ where $E \subseteq \mathbb{R}$ is Borel, and for every interval I of positive measure, we have

$$0 < m(E \cap I) < m(I).$$

Show that there is a countable collection $\{U_j\}_{j\in J}$ of open and dense subsets of Ω with $\bigcap_{j\in J} U_j \subseteq D$.

(i) Letting $f\colon \Omega \to \mathbb{R}$ be defined by $f(\mathbb{1}_E) = m(E \cap [a,b])$, we have

$$\begin{aligned} |m(E \cap [a,b]) - m(F \cap [a,b])| &= \left|\int_a^b \mathbb{1}_E - \mathbb{1}_F \, dm\right| \\ &\leqslant \int_a^b |\mathbb{1}_E - \mathbb{1}_F| \, dm \\ &\leqslant \int_{\mathbb{R}} |\mathbb{1}_E - \mathbb{1}_F| \, dm \\ &= \|\mathbb{1}_E - \mathbb{1}_F\|_{L_1}, \end{aligned}$$

meaning that f is Lipschitz, hence continuous.

(ii) Let $\mathbb{1}_F \in \Omega$. Then, $0 \leqslant \mu(F \cap [a,b]) \leqslant b - a$. If these inequalities are strict, then $F \in U_{a,b}$. Else, we let $\varepsilon > 0$, and see two cases:

- if $\mu(F \cap [a,b]) = b - a$, then we may set $E = F \setminus ([a, a+\varepsilon/) \cup (b - \varepsilon/2, b])$, so that $0 < \mu(E \cap [a,b]) < b - a$, and $\|\mathbb{1}_E - \mathbb{1}_F\|_{L_1} = \mu(E \triangle F) \leqslant \varepsilon$;
- if $\mu(F \cap [a,b]) = 0$, then we may set $E = F \cup ([a, a+\varepsilon/2) \cup [b - \varepsilon/2, b))$, meaning that $0 < \mu(E \cap [a,b]) < b - a$, and $\mu(E \triangle F) \leqslant \varepsilon$.

Therefore, $U_{a,b}$ is dense in Ω. To see that $U_{a,b}$ is open, notice that for any $\mathbb{1}_E \in U_{a,b}$, we may find $\varepsilon > 0$ such that $0 < \mu(E \cap [a,b]) - \varepsilon < \mu(E \cap [a,b]) < \mu(E \cap [a,b]) + \varepsilon < b - a$, and for all F with $\|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} < \varepsilon$, we have

$$|\mu(F \cap [a,b]) - \mu(E \cap [a,b])| \leqslant \|\mathbb{1}_F - \mathbb{1}_E\|_{L_1}$$

$$< \varepsilon,$$

so $0 < \mu(F \cap [a,b]) < b - a$. Thus, $U_{a,b}$ is also open.

(iii) If $\{[a_k, b_k]\}$ is an enumeration of rational-endpoint intervals in $\mathbb{R}$, then for any interval I, there is some rational-endpoint interval $[a_k, b_k] \subseteq I$ by density and the characterization of an interval. For any $\mathbb{1}_E \in U_{a_k,b_k}$, we have that for an interval $[a,b] \subseteq I$ with $a_k \geqslant a$ and $b_k \leqslant b$,

$$\begin{aligned} m(E \cap [a,b]) &= m(E \cap [a, a_k]) + m(E \cap [a_k, b_k]) + m(E \cap [b_k, b]) \\ &< a_k - a + b_k - a_k + b - b_k \\ &= b - a, \end{aligned}$$

so $U_{a_k,b_k} \subseteq D$. Thus, since this holds for all intervals of positive measure for each a_k, b_k, we get

$$\bigcap_{k=1}^{\infty} U_{a_k,b_k} \subseteq D.$$

August 2023

Problem 1

Problem: Let (X, μ) be a σ-finite Borel measure space. Let $(f_n)_n$ be a sequence in $L_2(X, \mu)$, and $f \in L_2(X, \mu)$ such that for every $g \in L_2(X, \mu)$, we have

$$\lim_{n\to\infty} \int_X f_n(x) g(x)\, d\mu(x) = \int_X f(x) g(x)\, d\mu(x).$$

Furthermore, suppose that

$$\lim_{n\to\infty} \|f_n\|_{L_2} = \|f\|_{L_2}.$$

Prove that there is a subsequence $\left(f_{n_j}\right)_j$ and a subset $E \subseteq X$ with $\mu(E) = 0$ such that for all $x \in X \setminus E$,

$$\lim_{j\to\infty} \left|f_{n_j}(x) - f(x)\right| = 0.$$

In order to show that $\left(f_{n_j}\right)_j \to f$ pointwise a.e., we show that $(f_n)_n \to f$ in measure; it has been well-established that if $(f_n)_n \to f$ in measure, then $(f_n)_n$ admits a subsequence that converges to f pointwise almost everywhere.

By Chebyshev's Inequality, we have that

$$\begin{aligned} \mu(\{x \mid |f_n(x) - f(x)| \geqslant \varepsilon\}) &\leqslant \frac{1}{\varepsilon^2} \|f_n - f\|_{L_2}^2 \\ &= \frac{1}{\varepsilon^2} \int_X |f_n - f|^2\, d\mu. \end{aligned}$$

Focusing on the integral,

$$\begin{aligned} \int_X |f_n - f|^2\, d\mu &= \int_X (f_n - f)\overline{(f_n - f)}\, d\mu \\ &= \int_X |f_n|^2 - f_n \overline{f} - \overline{f_n} f + |f|^2\, d\mu \\ &= \int_X |f_n|^2\, d\mu - \int_X f_n \overline{f}\, d\mu + \int_X |f|^2\, d\mu - \overline{\int_X f_n \overline{f}\, d\mu}. \end{aligned}$$

Now, we note the following:

- $\lim_{n\to\infty} \int_X |f_n|^2 \, d\mu = \int_X |f|^2 \, d\mu$; and
- if $f \in L_2(X, \mu)$, then so too is $\overline{f}$.

Thus, by taking limits, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_X |f_n - f|^2 \, dx &= \lim_{n\to\infty} \left(\int_X |f_n|^2 \, d\mu - \int_X f_n \overline{f} \, d\mu + \int_X |f|^2 \, d\mu - \overline{\int_X f_n \overline{f} \, d\mu} \right) \\
&= \int_X |f|^2 \, d\mu - \int_X |f|^2 \, d\mu + \int_X |f|^2 \, d\mu - \overline{\int_X |f|^2 \, d\mu} \\
&= 0,
\end{aligned}$$

so $\|f_n - f\|_{L_2}^2 \to 0$. Thus, $(f_n)_n \to f$ in measure, and thus there is a subsequence $\left(f_{n_j}\right)_j \to f$ pointwise almost everywhere.

Problem 3

> **Problem:** Let X be a LCH space. Recall that $g\colon X \to \mathbb{C}$ vanishes at infinity if for every $\varepsilon > 0$, there is a compact $K_\varepsilon \subseteq X$ such that for all $x \in X \setminus K_\varepsilon$, $|g(x)| < \varepsilon$. Show that $C_0(X)$ is complete with respect to the sup norm.

Let $(f_n)_n$ be Cauchy in the sup norm. Then, for all $\varepsilon > 0$, there is N such that for all $m, n \geqslant N$, $\|f_m - f_n\| < \varepsilon$. Therefore, for all $x \in X$, we have $|f_n(x) - f_m(x)| < \varepsilon$, meaning that the sequence $(f_n(x))_n$ is Cauchy in $\mathbb{C}$. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each x.

We must now show that

- $(f_n)_n \to f$ in the supremum norm;
- $f \in C_0(X)$.

For the first point, we see that for $\varepsilon > 0$, there is N such that for all $n, m \geqslant N$ and all $x \in X$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Taking the limit as $m \to \infty$, we have

$$|f_n(x) - f(x)| \leqslant \varepsilon.$$

Thus, by taking suprema, we get that

$$\sup_{x\in X} |f_n(x) - f(x)| \leqslant \varepsilon,$$

so $\|f_n - f\| \leqslant \varepsilon$, meaning that $(f_n)_n \to f$ in the sup norm, implying that f is continuous as it is the uniform limit of continuous functions.

Finally, we let N_1 be such that for all $n \geqslant N_1$, $\|f_n - f\| < \varepsilon/2$. Note that since $f_{N_1} \in C_0(X)$, we have a $K_{\varepsilon/2}$ such that for all $x \in X \setminus K_{\varepsilon/2}$, $|f_N(x)| < \varepsilon/2$. Therefore, for all $x \in X \setminus K_{\varepsilon/2}$, we have

$$\begin{aligned}
|f(x)| &\leqslant |f_{N_1}(x) - f(x)| + |f_{N_1}(x)| \\
&\leqslant \|f_{N_1} - f\| + |f_{N_1}(x)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon,
\end{aligned}$$

so $f \in C_0(X)$. Thus, $C_0(X)$ is complete.

Problem 4

Problem: Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that for any $n \geqslant 1$, and any $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{A}$,

$$\mu((A_1 \cup \cdots \cup A_n) \triangle (B_1 \cup \cdots \cup B_n)) \leqslant \sum_{j=1}^{n} \mu(A_j \triangle B_j).$$

We start off by noting that the symmetric difference $A \triangle B$ can be written as

$$A \triangle B = A \cup B \setminus (A \cap B).$$

This is evident from unwinding the definition $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Now, writing the left-hand side of our desired inequality, we get

$$\mu((A_1 \cup \cdots \cup A_n) \triangle (B_1 \cup \cdots \cup B_n)) = \mu(A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n) - \mu((A_1 \cup \cdots \cup A_n) \cap (B_1 \cup \cdots \cup B_n)).$$

Distributing the second term on the right-hand side and rearranging the first term, we get

$$= \mu\left(\bigcup_{j=1}^{n} (A_j \cup B_j)\right) - \mu\left(\bigcup_{j=1}^{n} (A_1 \cup \cdots \cup A_n) \cap B_j\right).$$

Using subadditivity on the first term, we get

$$\leqslant \sum_{j=1}^{n} \mu(A_j \cup B_j) - \mu\left(\bigcup_{j=1}^{n} (A_1 \cup \cdots \cup A_n) \cap B_j\right).$$

Finally, using monotonicity and subadditivity on the second term, and exercising the fact that

$$A_j \cap B_j \subseteq \bigcap_{j=1}^{n} (A_1 \cup \cdots \cup A_n) \cap B_j,$$

we get

$$\begin{aligned} &\leqslant \sum_{j=1}^{n} \mu(A_j \cup B_j) - \sum_{j=1}^{n} \mu(A_j \cap B_j) \\ &= \sum_{j=1}^{n} \mu(A_j \triangle B_j). \end{aligned}$$

Problem 5

Problem: Let (X, μ) be a nonnegative measure space and f a measurable function on (X, μ) such that

$$\sup_{\lambda > 0} \mu(\{x \mid |f(x)| > \lambda\}) < \infty.$$

Prove that there is a finite constant C such that for every finite measure subset, we have

$$\int_E |f(x)| \, d\mu(x) \leqslant C\mu(E)^{1/2}.$$

Lemma (Cavalieri's Principle)**:**

$$\int_X |f| \, d\mu = \int_0^\infty \mu(\{x \in X \mid |f| > \lambda\}) \, d\lambda.$$

Using Cavalieri's Principle, we get

$$\int_E |f| \, d\mu \leqslant \int_0^\alpha \mu(\{x \in E \mid |f| > \lambda\}) \, d\lambda + \int_\alpha^\infty \mu(\{x \in E \mid |f| > \lambda\}) \, d\lambda$$

$$\begin{aligned}
&\leqslant \alpha\mu(E) + \int_{\alpha}^{\infty} \frac{M}{\lambda^2}\,d\lambda \\
&= \alpha\mu(E) + \frac{M}{\alpha} \\
&\leqslant (M+1)\mu(E)^{1/2},
\end{aligned}$$

where we selected $\alpha = \frac{1}{\mu(E)^{1/2}}$, and M denotes the given supremum.

January 2024

Problem 1

Problem: Let (X, μ) be a σ-finite measure space, and suppose $(f_n)_n$ is a sequence in $L_2(X, \mu)$ such that $\sup_{n\geqslant 1}\|f_n\|_{L_2} < \infty$ and $(f_n)_n \to f$ μ-almost everywhere. Prove that $f \in L_2(X, \mu)$.

Applying Fatou's Lemma, we find that

$$\begin{aligned}
\int_X |f|^2\,d\mu &= \int_X \liminf_{n\to\infty}|f_n|^2\,d\mu \\
&\leqslant \liminf_{n\to\infty}\int_X |f_n|^2\,d\mu \\
&\leqslant \limsup_{n\to\infty}\int_X |f_n|^2\,d\mu \\
&\leqslant \sup_{n\geqslant 1}\int_X |f_n|^2\,d\mu \\
&< \infty.
\end{aligned}$$

Problem 2

Problem: Let (X, μ) be a measure space, and let $p \in [1, \infty)$. Let $(f_n)_n \to f$ in L_p.

(i) Prove that there exists a subsequence (f_{n_k}) such that $\|f_{n_{k+1}} - f_{n_k}\|_{L_p} < 2^{-k}$.

(ii) Show that for μ-almost every x, we have $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

(i) Since $(f_n)_n \to f$ in L_p, we see that $(f_n)_n$ is L_p-Cauchy, so we may extract a subsequence as follows. Let $f_{n_1} = f_1$, and find f_{n_2} with $n_2 > 1$ such that

$$\|f_{n_2} - f_{n_1}\| < \frac{1}{2}.$$

Inductively, we may use the fact that $(f_n)_n$ is Cauchy to find $n_{k+1} > n_k$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}.$$

(ii) Consider the sequence $(s_n)_n$ given by

$$s_n = \sum_{k=1}^{n} |f_{n_{k+1}} - f_{n_k}|.$$

Then, by Minkowski's Inequality, we find that

$$\|s_n\|_{L_p} \leqslant \sum_{k=1}^{n} \|f_{n_{k+1}} - f_{n_k}\|_{L_p}.$$

In particular, $\|s_n\|_{L_p} \leqslant 1$ for all n, meaning that by dominated convergence, $s = \lim_{n\to\infty} s_n$ is in L_p, and in particular, $s(x) < \infty$ for almost every x. Notice that this means that

$$h(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges for almost every x. Defining $h(x) = 0$ for all x where this sum does not converge absolutely, we notice that

$$f_{n_1}(x) + \sum_{k=1}^{m} (f_{n_{k+1}}(x) - f_{n_k}(x)) = f_{n_{m+1}}(x),$$

meaning that h is the pointwise (almost everywhere) limit of the sequence $(f_{n_k})_k$; by Minkowski's Inequality, and applying Fatou's Lemma, as earlier, we also find that

$$\begin{aligned}
\|h\|_{L_p} &\leqslant \|f_{n_1}\|_{L_p} + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_{L_p} \\
&\leqslant \|f_{n_1}\|_{L_p} + 1 \\
&< \infty,
\end{aligned}$$

meaning $h \in L_p(X, \mu)$. All we need to do now is show that $\|f - h\|_{L_p} = 0$, meaning that $[f] = [h]$ under the pointwise almost everywhere equivalence relation. To see this,

$$\begin{aligned}
\int_X |h - f|^p \, d\mu &= \int_X \liminf_{k\to\infty} |f_{n_k} - f|^p \, d\mu \\
&\leqslant \liminf_{k\to\infty} \int_X |f_{n_k} - f|^p \, d\mu \\
&= \liminf_{k\to\infty} \|f_{n_k} - f\|_{L_p}^p \\
&= 0,
\end{aligned}$$

where the last equality is derived from the fact that $(f_n)_n \to f$ in L_p, so every subsequence of $(f_n)_n$ converges to f in L_p.

Problem 3

Problem: Let f be Lebesgue-integrable on $\mathbb{R}$, and let g be a bounded continuous function on $\mathbb{R}$. Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x - y) \, dy$$

is a continuous function on $\mathbb{R}$.

Let $M = \sup_{x\in\mathbb{R}} |g(x)|$. Now, since $f \in L_1$, there is a compactly supported continuous function $h \in C_c(\mathbb{R})$ such that $\|h - f\|_{L_1} < \frac{\varepsilon}{3M}$. If we let $K = \operatorname{supp}(h)$, then since h is compactly supported, h is uniformly continuous, so there is $\delta > 0$ such that whenever $|x - y| < \delta$, we have

$$|h(x) - h(y)| < \frac{\varepsilon}{3Mm(K)},$$

where $m(K)$ is the Lebesgue measure of K in $\mathbb{R}$. Therefore, if $|x - y| < \delta$, we have

$$|(f * g)(x) - (f * g)(y)| = \left| \int_{\mathbb{R}} (f(x - t) - f(y - t))g(t) \, dt \right|$$

$$\begin{aligned}
&\leqslant \int_{\mathbb{R}} |f(x-t) - f(y-t)||g(t)|\,dt \\
&\leqslant \int_{\mathbb{R}} |f(x-t) - h(x-t)||g(t)|\,dt \\
&\quad + \int_{\mathbb{R}} |h(x-t) - h(y-t)||g(t)|\,dt \\
&\quad + \int_{\mathbb{R}} |h(y-t) - f(y-t)||g(t)|\,dt.
\end{aligned}$$

Using Hölder's Inequality on the first and third integrals, we get

$$\leqslant M\left(\frac{\varepsilon}{3M}\right) + \int_{\mathbb{R}} |h(x-t) - h(y-t)||g(t)|\,dt + M\left(\frac{\varepsilon}{3M}\right),$$

and using the uniform continuity of h, we get

$$\begin{aligned}
&\leqslant \frac{2\varepsilon}{3} + M(m(K))\frac{\varepsilon}{3M(m(K))} \\
&= \varepsilon.
\end{aligned}$$

Alternative Solution

We know that f is integrable on $\mathbb{R}$, and g is bounded and continuous. We will show that if $(x_n)_n \to x_0$, then $((f*g)(x_n))_n \to (f*g)(x_0)$.

Now, if $(x_n)_n \to x_0$, then $g(x_n) \to g(x_0)$, since g is continuous. Since f is integrable, f is finite almost everywhere, meaning that $f(y)g(x_n - y) \to f(y)g(x_0 - y)$ almost everywhere.

Furthermore, since g is bounded, we have $|g| \leqslant M$ for some $M > 0$. Writing our convolution integrand, we have

$$|f(y)g(x_n - y)| \leqslant M|f(y)|.$$

Since f is integrable, we may use the dominated convergence theorem to find that

$$\lim_{n\to\infty} \int f(y)g(x_n - y)\,dy = \int f(y)g(x_0 - y)\,dy.$$

Problem 4

> **Problem:** Let $(a_n)_n$ be a sequence of complex numbers such that $|a_n| < 1$ for all n and $\lim_{n\to\infty} a_n = 0$.
>
> (i) Show that if $\sum_{n\geqslant 1}|a_n| < \infty$, then the sequence $(p_n)_n$ defined by $p_n = \prod_{i=1}^{n}(1 + a_i)$ is convergent.
>
> (ii) Does the converse hold? In other words, is it true that if $(p_n)_n$ is convergent, we must have $\sum_{n\geqslant 1}|a_n| < \infty$? Recall the conditions that $|a_n| < 1$ for all n and $\lim_{n\to\infty} a_n = 0$.

(a) Let $\log(z)$ be defined on the branch $-\pi < \arg(z) \leqslant \pi$, then $\log(1 + a_n)$ is well-defined for all $n \in \mathbb{N}$, as $\operatorname{Re}(a_n) \leqslant |a_n| < 1$. Since log is continuous on the branch,

$$\begin{aligned}
\log(p_n) &= \sum_{k=1}^{n} \log(1 + a_k) \\
\log\left(\lim_{n\to\infty} p_n\right) &= \lim_{n\to\infty} \sum_{k=1}^{\infty} \log(1 + a_k).
\end{aligned}$$

Now, we notice that taking the limit $\lim_{k\to\infty} a_k = 0$, the limit comparison test (or by seeing that the singularity at $\log(z)/z$ is removable), that $\lim_{k\to\infty} \frac{\log(1+a_k)}{a_k} = 1$, and since $\sum_{k=1}^{\infty} a_k$ converges, so too does $\sum_{k=1}^{\infty} \log(1 + a_k)$, meaning p_n converges.

August 2024

Problem 1

Problem: Let $A \subseteq \mathbb{R}$ be a Lebesgue-measurable subset of finite measure. For $r \in \mathbb{R}\setminus\{0\}$, let $rA = \left\{x \in \mathbb{R} \;\middle|\; r^{-1}x \in A\right\}$, and let $A \triangle rA = (A \setminus rA) \cup (rA \setminus A)$. Show that

$$\lim_{r\to 1} m(A \triangle rA) = 0.$$

Problem 2

Problem: Let μ be a finite Borel measure on $\mathbb{R}$. For $\xi \in \mathbb{R}$, define

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi}\, d\mu(x).$$

Suppose

$$\lim_{\xi\to 0} \frac{\widehat{\mu}(\xi) - \widehat{\mu}(0)}{\xi^2} = 0.$$

(a) Show that

$$\int_{\mathbb{R}} x^2\, d\mu(x) = 0.$$

(b) Deduce that for any open interval $(a,b) \subseteq \mathbb{R}$,

$$\mu((a,b)) = \begin{cases} \mu(\mathbb{R}) & 0 \in (a,b) \\ 0 & 0 \notin (a,b) \end{cases}.$$

January 2025

Problem 1

Problem: Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_n$ and $(g_n)_n$ sequences of functions in $L_1(X,\mathcal{M},\mu)$ in $L_1(X,\mathcal{M},\mu)$ that converge pointwise almost everywhere to $f, g \in L_1(X,\mathcal{M},\mu)$. Suppose $|f_n| \leqslant g_n$ almost everywhere, and

$$\lim_{n\to\infty} \int_X g_n\, d\mu = \int_X g\, d\mu.$$

Show that

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu.$$

We see that

$$\begin{aligned}
\int_X g + f\, d\mu &= \int_X \liminf_{n\to\infty}(g_n + f_n)\, d\mu \\
&\leqslant \liminf_{n\to\infty}\left(\int_X g_n\, d\mu + \int_X f_n\, d\mu\right) \\
&= \int_X g\, d\mu + \liminf_{n\to\infty} \int_X f_n\, d\mu,
\end{aligned}$$

so

$$\int_X f\, d\mu \leqslant \liminf_{n\to\infty} \int_X f_n\, d\mu.$$

Similarly, we also see that

$$\begin{aligned}\int_X g - f\, d\mu &= \int_X \liminf_{n\to\infty}(g_n - f_n)\, d\mu\\ &\leqslant \liminf_{n\to\infty}\left(\int_X g_n\, d\mu - \int_X f_n\, d\mu\right)\\ &= \int_X g\, d\mu - \limsup_{n\to\infty}\int_X f_n\, d\mu,\end{aligned}$$

meaning that

$$\limsup_{n\to\infty}\int_X f_n\, d\mu \leqslant \int_X f\, d\mu,$$

meaning that

$$\int_X f\, d\mu = \lim_{n\to\infty}\int_X f_n\, d\mu.$$

Problem 2

Problem:

(a) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show that if $p, p' \in [1, \infty]$ with $p < p'$, then $L_p(X, \mathcal{M}, \mu) \supseteq L_{p'}(X, \mathcal{M}, \mu)$.

(b) Show that if $p, p' \in [1, \infty]$ are such that $p < p'$, then $L_p(\mathbb{R}) \setminus L_{p'}(\mathbb{R})$ and $L_{p'}(\mathbb{R}) \setminus L_p(\mathbb{R})$ are both nonempty.

(a) Let $f \in L_{p'}(X, \mathcal{M}, \mu)$. Then, we use Hölder's inequality on with $\frac{p'}{p}$ and $1 - \frac{p'}{p}$ as our Hölder conjugates to obtain

$$\begin{aligned}\int_X |f|^p\, d\mu &= \int_X |f|^p(1)\, d\mu\\ &\leqslant \left(\int_X |f|^{p(p'/p)}\, d\mu\right)^{p/p'}\left(\int_X d\mu\right)^{1-p/p'}\\ &= \left(\int_X |f|^{p'}\, d\mu\right)^{p/p'}\mu(X)^{1-p/p'}\\ &< \infty,\end{aligned}$$

so $f \in L_{p'}$.

(b) To see that $L_{p'} \setminus L_p$ is nonempty, we consider a function given by

$$f = \sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\mathbb{1}_{[n,n+1)},$$

where we see that

$$\begin{aligned}\int_{\mathbb{R}} |f|^p\, d\mu &= \sum_{n=1}^{\infty}\frac{1}{n}\\ &= \infty\\ \int_X |f|^{p'}\, d\mu &= \sum_{n=1}^{\infty}\frac{1}{n^{p'/p}}\\ &< \infty.\end{aligned}$$

Now, to see that $L_p \setminus L_{p'}$ is nonempty, we consider the function

$$f = \sum_{n=1}^{\infty} \left(3^{n/q}\right)(2^{-n})\mathbb{1}_{[2^n, 2^{n+1}-1)},$$

where q is such that $3^{p/q} < 2$ and $3^{p'/q} > 2$. This yields a function in L_p that is not in $L_{p'}$.

Problem 3

Problem: Let $\mathcal{H}$ be a separable Hilbert space. A sequence $(v_m)_m$ is said to converge weakly to $v \in \mathcal{H}$ if

$$\lim_{m\to\infty} \langle v_m, w\rangle = \langle v, w\rangle$$

for every $w \in \mathcal{H}$. Show that for any sequence $(v_m)_m \subseteq \mathcal{H}$ for which $\sup_{m\in\mathbb{N}} \|v_m\|$ is finite, there exists a subsequence $(v_{m_k})_k \to v \in \mathcal{H}$ weakly.

We see that, since $\mathcal{H}$ is a Hilbert space, $\mathcal{H} \cong \mathcal{H}^{**}$, where $\mathcal{H}^{**}$ is the double dual of $\mathcal{H}$ (Hilbert spaces are reflexive). This means that, if $(v_m)_m \subseteq \mathcal{H}$, there is an isometric isomorphism $(\hat{v}_m)_m \subseteq \mathcal{H}^{**}$, where $\hat{v}_m$ is the linear functional such that $\hat{v}_m(\varphi) = \varphi(v_m)$ for all $\varphi \in \mathcal{H}$.

Letting $M = \sup_{m\in\mathbb{N}} \|v_m\|$, we see that $\frac{1}{M}(\hat{v}_m)_m \subseteq B_{\mathcal{H}^{**}}$. By the Banach–Alaoglu theorem, there is a subsequence $\frac{1}{M}(\hat{v}_{m_k})_k \to \frac{1}{M}\hat{v}$, where the convergence is in the w^{**} topology — i.e., for all $\varphi \in \mathcal{H}^*$, $\hat{v}_{m_k}(\varphi) \to \hat{v}(\varphi)$; rewriting, we then get that $\varphi(v_{m_k}) \to \varphi(v)$ for all $\varphi \in \mathcal{H}^*$.

By the Riesz Representation Theorem for Hilbert Spaces, we thus have

$$\langle v_{m_k}, w\rangle \to \langle v, w\rangle$$

for all $w \in \mathcal{H}$, so $(v_{m_k})_k \to v$ weakly.

Problem 4

Problem: Let δ_0 be the Dirac measure at 0, defined by

$$\delta_0(A) = \begin{cases} 1 & 0 \in A \\ 0 & \text{else} \end{cases}.$$

For each $r > 0$, define ν_r to be the measure defined by

$$\nu_r(A) = \frac{1}{2r} m(A \cap [-r, r]).$$

Show that for every continuous function $f\colon \mathbb{R} \to \mathbb{C}$, we have

$$\lim_{r\searrow 0} \int_{\mathbb{R}} f(x)\, d\nu_r(x) = \int_{\mathbb{R}} f(x)\, d\delta_0(x).$$

We consider the family $\{E_r\}_{r>0}$ defined by $E_r = [-r, r]$. We notice that $\frac{1}{2r} m(A \cap E_r) = \nu(E_r)$. Furthermore, we also see that $E_r \subseteq (-4/3r, 4/3r)$, and $E_r \supseteq \frac{3}{8}(-4/3r, 4/3r)$, so that by a scaling argument, $\{E_r\}_{r>0}$ is a family that shrinks nicely to $x = 0$.

Furthermore, we see that

$$\begin{aligned}\int_{\mathbb{R}} f(x)\, d\delta_0(x) &= \int_{\{0\}} f(x)\, d\delta_0(x)\\ &= f(0).\end{aligned}$$

Finally, since f is continuous, for any compact $K \subseteq \mathbb{R}$, f is bounded, so that

$$\begin{aligned}\int_K |f(x)|\,dx &\leqslant \int_K \sup_{x\in K}|f(x)|\,dx\\ &\leqslant m(K)\sup_{x\in K}|f(x)|\\ &< \infty,\end{aligned}$$

as m is regular. Thus, f is locally integrable, meaning that by the Lebesgue Differentiation Theorem,

$$\begin{aligned}\lim_{r\to 0}\int_{\mathbb{R}} f(x)\,d\nu_r(x) &= \lim_{r\to 0}\frac{1}{2r}\int_{E_r} f(x)\,dx\\ &= f(0),\end{aligned}$$

so

$$\int_{\mathbb{R}} f(x)\,d\nu_r(x) = \int_{\mathbb{R}} f(x)\,d\delta_0(x).$$

Problem 5

Problem:

(a) State the Riemann–Lebesgue lemma for the Fourier transform on $\mathbb{R}^n$.

(b) Show that there does not exist a function $g \in L_1(\mathbb{R}^n)$ such that $f * g = f$ for all $f \in L_1(\mathbb{R}^n)$.

(a) The Riemann–Lebesgue Lemma for the Fourier transform on $\mathbb{R}^n$ states that if $f \in L_1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

(b) Suppose toward contradiction that there were such a g. Then, it would also be the case that $g * g = g$, and since the Fourier transform on $L_1(\mathbb{R}^n)$ is injective, by the convolution property of the Fourier transform, we have $\hat{g}(k)\hat{g}(k) = \hat{g}(k)$ for all $k \in \mathbb{R}^n$, implying that $\hat{g}(k) = 0$ or $\hat{g}(k) = 1$ for all k, depending on if $\hat{g}(k)$ is zero or not.

However, by the Riemann–Lebesgue Lemma, we must have $\hat{g}(k) = 0$ for all k, implying that $g = 0$; yet, this is absurd, as $f * 0 = 0$, yet there are nonzero functions in $L_1(\mathbb{R}^n)$.