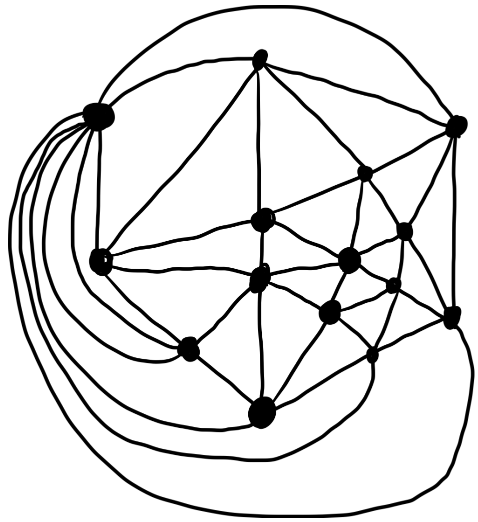
4

Show that Corollary 10.3 cannot be improved by giving an example of a planar graph that contains no vertex of degree 4 or less.



5

(a) Show that the Petersen graph does not contain a subdivision of K_5.

(b) Show that the Petersen graph is nonplanar.

(a)

Since all the vertices of the Petersen graph are of degree 3, and any subdivision of K_5 must contain a vertex of at least degree 4, this means the Petersen graph cannot contain a subdivision of K_5.

(b)

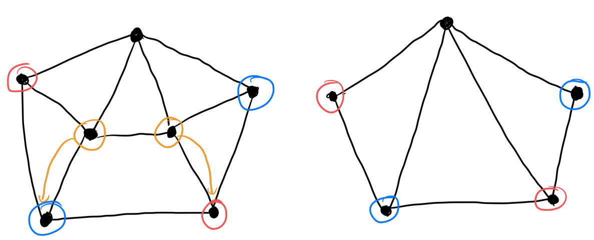
Suppose that the Petersen graph is planar. Then, by Euler's formula,

$$(10) - (15) + F = 2,$$

meaning there are 7 faces. However, since the girth of the Petersen graph is 5, this means every face in the supposed planar configuration is made of pentagons — implying that the Petersen graph must have at least $\lfloor 35/2 \rfloor$ or 17 edges. $\perp$

6

Does there exist a 4-regular planar graph of order 7?



In the left image above of an incomplete construction of our supposed planar 4-regular graph of order 7, we see that the blue vertices still require an edge between them, and similarly, the red edges still require an edge between them.

Creating a minor by retracting the two orange vertices using the above arrows, we see that the connection would yield a graph of K_5 — and since K_5 is a minor of our 4-regular graph of order 7, by Wagner's theorem, it must be non-planar.

7

Find all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that $G - e$ is planar for every edge e of G.

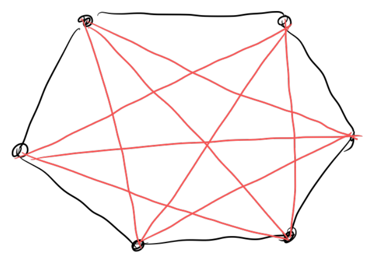
I don't know how to do this problem.

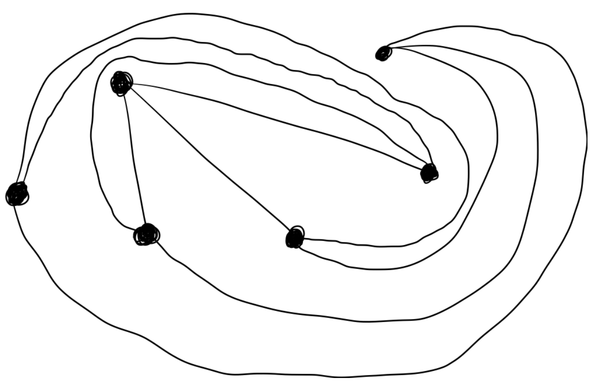
8

Find all cycles C_n of order $n \geq 3$ for which $\overline{C_n}$ is a nonplanar graph.

The following graphs are planar:

- $\overline{C_3}$ is three vertices without any edges.
- $\overline{C_4}$ is two disconnected K_2 graphs.
- $\overline{C_5}$ is one-to-one with C_5
- $\overline{C_6}$ is planar:





For $n = 7$, $\overline{C_7}$ must be a 4 regular graph of order 7, which we showed was nonplanar.

For every $n \geq 8$, $|E(\overline{C_n})| = \frac{n(n-3)}{2} > 3n - 6$.

Therefore, $\overline{C_n}$ is nonplanar for every $n \geq 7$.

18

Show that both K_5 and $K_{3,3}$ are minors of the graph G in Figure 10.13

19

Suppose that a connected graph H is a minor of a tree T. Show that H is also a tree.

Let T be a tree. Then, $|E(T)| = n - 1$, where $n = |V(T)|$. For any contraction performed on T in the process of yielding H, it must be the case that both $|E(T)|$ and $|V(T)|$ are reduced by the same quantity — thus, $|E(H)| = k - 1$ where $k = |V(H)|$.