Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

$$d\mathbf{r} = \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}.$$

We let $\hat{e}_1 = \hat{\rho}$, $\hat{e}_2 = \hat{\phi}$, and $\hat{e}_3 = \hat{z}$, with $u_1 = \rho$, $u_2 = \phi$, and $u_3 = z$. Thus, we get

- Line element:

$$\begin{aligned}(ds)^2 &= \sum_{i,j} \frac{\partial \mathbf{r}}{\partial u_i}\frac{\partial \mathbf{r}}{\partial u_j}\left(\hat{e}_i \cdot \hat{e}_j\right) du_i\, du_j \\ &= \sum_{i=1}\left(\frac{\partial \mathbf{r}}{\partial u_i}\right)(du_i)^2 && \text{The } \hat{\rho}, \hat{\phi}, \hat{z} \text{ basis is orthogonal} \\ &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.\end{aligned}$$

- Area element:

$$d\mathbf{a} = \left(\sum_k \epsilon_{ijk}\hat{e}_k\right)\frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} du_i\, du_j$$

4.9

Without loss of generality, we have

$$\sum_\ell \epsilon_{mn\ell}\epsilon_{ij\ell} = \epsilon_{mn1}\epsilon_{ij1},$$

where $m, n, i, j = 2, 3$. If we have $m = i, n = j$, then $\epsilon_{mn1}\epsilon_{ij1} = 1$; if $m = j, n = i$, then $\epsilon_{mn1}\epsilon_{ij1} = -1$; else, $\epsilon_{mn1}\epsilon_{ij1} = 0$.

4.11

(a)

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k \\ &= -\sum_{i,j,k} \epsilon_{jik} B_j A_i \hat{e}_k \\ &= -(\mathbf{B} \times \mathbf{A})\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= \sum_{i,j,k}\left(\epsilon_{ijk} A_i B_j \hat{e}_k\right) \cdot A_i \hat{e}_i \\ &= \sum_{i,j,k} \delta_{ik}\left(\epsilon_{ijk} A_i^2 B_j\right) \\ &= 0.\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_i C_j\\ &= \sum_{i,j,\ell}\left(\epsilon_{\ell ij}A_\ell B_i\right)C_j\\ &= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_i C_j\\ &= \sum_{i,j,\ell}\left(\epsilon_{j\ell i}C_j A_i\right)B_i\\ &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})\,.\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j}\epsilon_{ijk}A_i\left(\sum_{\alpha,\beta}\epsilon_{\alpha\beta j}B_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\epsilon_{ijk}\epsilon_{\alpha\beta j}A_iB_\alpha C_\beta\\ &= -\left(\sum_{i,j,\alpha,\beta}\epsilon_{ikj}\epsilon_{\alpha\beta j}A_iB_\alpha C_\beta\right)\\ &= -\left(\sum_{i,j,\alpha,\beta}\left(\delta_{i\alpha}\delta_{k\beta}-\delta_{i\beta}\delta_{k\alpha}\right)A_iB_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\left(\delta_{k\alpha}\delta_{i\beta}-\delta_{i\alpha}\delta_{k\beta}\right)A_iB_\alpha C_\beta\\ &= \sum_{i,j,\alpha,\beta}\left(B_\alpha\delta_{k\alpha}\right)\left(A_iC_\beta\delta_{i\beta}\right)-\left(C_\beta\delta_{k\beta}\right)\left(A_iB_\alpha\delta_{i\alpha}\right)\\ &= \mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right).\end{aligned}$$

(e)

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\cdot\mathbf{D}) &= \sum_{\alpha,\beta}(\mathbf{A}\times\mathbf{B})_\alpha(\mathbf{C}\times\mathbf{D})_\beta\,\delta_{\alpha\beta}\\ &= \sum_{\substack{\alpha,\beta,\\ i,j,\\ m,n}}\epsilon_{ij\alpha}\epsilon_{mn\beta}A_iB_jC_mD_n\delta_{\alpha\beta}\\ &= \sum_{\substack{\alpha,\\ i,j,\\ m,n}}\epsilon_{ij\alpha}\epsilon_{mn\alpha}A_iB_jC_mD_n\\ &= \sum_{\substack{i,j,\\ m,n}}A_iB_jC_mD_n\left(\delta_{mi}\delta_{nj}-\delta_{mj}\delta_{ni}\right)\\ &= \sum_{\substack{i,j,\\ m,n}}\left(\left(A_iC_m\delta_{mi}\right)\left(B_jD_n\delta_{nj}\right)\right)-\left(\left(B_jC_m\delta_{mj}\right)-\left(A_iD_n\delta_{ni}\right)\right)\\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})\,.\end{aligned}$$

Chapter 5 Problems

5.1

Let $f(x) = x^n$. We use linearity for the general case.

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{x^n + n\left(hx^{n-1}\right) + \cdots + nh^{n-1}x + h^n - x^n}{h}\\ &= \lim_{h\to 0}\left(nx^{n-1} + \cdots + nh^{n-2}x + h^{n-1}\right)\\ &= nx^{n-1}.\end{aligned}$$

5.6

$$\begin{aligned}\cos(N\phi) + i\sin(N\phi) &= (\cos\phi + i\sin\phi)^N\\ &= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k\left(\sin\phi^{N-k}\right)\left(e^{i\frac{\pi}{2}}\right)^{N-k}\\ &= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k(\sin\phi)^{N-k}\left(\cos\left((N-k)\frac{\pi}{2}\right) + i\sin\left((N-k)\frac{\pi}{2}\right)\right)\\ &= \sum_{k=0}^{N}\binom{N}{k}\cos\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\\ &+ i\left(\sum_{k=0}^{N}\binom{N}{k}\sin\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\right).\end{aligned}$$

We get the final answer by equating real and imaginary parts.

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right|\\ &= 1,\end{aligned}$$

  which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \qquad \forall n \geqslant 1.$$

  Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

(b) • Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\left(\frac{1}{2}\right)\right|\\ &= \frac{1}{2}\\ &< 1,\end{aligned}$$

meaning the series converges by the ratio test.

•

$$\frac{1}{n2^n} < \frac{1}{2^n} \qquad \text{for all } n \geqslant 1,$$

and since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty}\frac{1}{n2^n}$ converges.

6.9

$$\begin{aligned}\sum_{n=-N}^{N} e^{inx} &= 1 + \sum_{n=1}^{N} e^{-inx} + \sum_{n=1}^{N} e^{inx}\\ &= 1 + e^{-ix}\sum_{n=1}^{N} e^{i(n-1)x} + e^{ix}\sum_{n=1}^{N} e^{i(n-1)x}\\ &= 1 + e^{-ix}\sum_{n=0}^{N-1} e^{-inx} + e^{ix}\sum_{n=0}^{N-1} e^{inx}\\ &= 1 + e^{-ix}\frac{1-e^{-iNx}}{1-e^{-ix}} + e^{ix}\frac{1-e^{iNx}}{1-e^{ix}}\\ &= 1 + \frac{e^{-ix}-e^{-i(N+1)x}}{1-e^{-ix}} + \frac{1-e^{iNx}}{e^{-ix}-1}\\ &= 1 + \frac{\left(e^{-ix}-1\right)+e^{iNx}-e^{-i(N+1)x}}{1-e^{-ix}}\\ &= \frac{e^{iNx}-e^{-i(N+1)x}}{1-e^{-ix}}\\ &= \frac{e^{iNx}-e^{-i(N+1)x}}{e^{-i\left(\frac{x}{2}\right)}\left(e^{i\left(\frac{x}{2}\right)}-e^{-i\left(\frac{x}{2}\right)}\right)}\\ &= \frac{e^{i\left(N+\frac{1}{2}\right)x}-e^{-i\left(N+\frac{1}{2}\right)x}}{e^{-i\left(\frac{x}{2}\right)}-e^{-i\left(\frac{x}{2}\right)}}\\ &= \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}.\end{aligned}$$

6.13

(a)

$$\begin{aligned}\frac{d}{dz}\left(\arctan(z)\right) &= 1 - z^2 + z^4 - z^6 + \cdots\\ &= \sum_{i=0}^{\infty}(-1)^n z^{2n}\\ &= \frac{1}{1+z^2}.\end{aligned}$$

(b)

$$
\begin{aligned}
\rho &= \limsup_{k\to\infty} \sqrt[k]{\left|(-1)^k\right|} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\rho &= \limsup_{k\to\infty} \left(\left|\frac{(-1)^k}{2k+1}\right|\right)^{1/k} \\
&= \limsup_{k\to\infty} \frac{1}{(2k+1)^{1/k}} \\
&= 1 \\
r &= \frac{1}{\rho} \\
&= 1.
\end{aligned}
$$

6.25

$$
\begin{aligned}
e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\
&= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k\, \theta^{2k+1}}{(2k+1)!} \\
&= \cos\theta + i\sin\theta.
\end{aligned}
$$

6.37

6.42