Problem (Problem 1)**:** Let M be a Riemannian manifold with metric g, f a C^∞ function on M, and let X be a vector field on M. Find an expression in local coordinates for the gradient of f and the divergence of X.

Solution: Let $p \in M$ and $U \subseteq M$ a chart for p with coordinates $(x_1, \dots, x_n)$. Define

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}.$$

We observe that the 1-form ω_X defined via the Riemannian metric, $\omega_X(Y) = g(X, Y)$, is given locally by

$$\begin{aligned} g(X, Y) &= \sum_{i=1}^{n} f_i \sum_{j=1}^{n} a_{ij} g_j \\ &= \sum_{i=1}^{n} f_i \sum_{j=1}^{n} a_{ij}\ dx_j \left(\sum_{k=1}^{n} g_k \frac{\partial}{\partial x_k}\right) \end{aligned}$$

whence

$$\omega_X = \sum_{i=1}^{n} f_i \left(\sum_{j=1}^{n} a_{ij}\ dx_j\right).$$

Computing the divergence, which is given by $*d(*\omega_X)$, we find that

$$\begin{aligned} *\omega_X &= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{j-1} f_i a_{ij}\ dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ d(*\omega_X) &= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{j-1} d\left(f_i a_{ij}\right)\ dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial\left(f_i a_{ij}\right)}{\partial x_j}\ dx_1 \wedge \cdots \wedge dx_n \\ *(d(*\omega_X)) &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial f_i}{\partial x_j} + f_i \frac{\partial a_{ij}}{\partial x_j}. \end{aligned}$$

To find the gradient for f, we start by taking

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\ dx_i.$$

We wish to compute $\hat{g}^{-1}(df)$, so we need to understand what $\hat{g}^{-1}(dx_k)$ looks like. Toward this end, we observe from linear algebra that the nondegenerate symmetric bilinear map $v \mapsto v^T A \cdot$ has the inverse $y^T A^{-1}$, whence

$$\hat{g}^{-1}(df) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\ dx_i\right) \left(\left(a_{ij}\right)_{ij}\right)^{-1} \frac{\partial}{\partial x_j}.$$

Problem (Problem 2)**:** With the setup of the previous exercise, assume that M is orientable. Find an expression for the volume form of M in local coordinates.

Solution: Let θ_i be the corresponding dual orthonormal basis for $\{e_i\}_{i \in I}$, which is the orthonormal basis for g. We observe then that we may express

$$\theta_i = \sum_{j=1}^{n} a_{ij}\, dx_j,$$

whence

$$\begin{aligned}\theta_1 \wedge \cdots \wedge \theta_n &= \left(\sum_{j=1}^{n} a_{1j}\, dx_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} a_{nj}\, dx_j\right)\\ &= \left(a_{ij}\right)_{i,j}(dx_1 \wedge \cdots \wedge dx_n)\\ &= \left|\det\left(\left(a_{ij}\right)_{i,j}\right)\right| dx_1 \wedge \cdots \wedge dx_n,\end{aligned}$$

where the absolute value emerges from the fact that $\left(a_{ij}\right)_{i,j}$ is positive definite.

Problem (Problem 3)**:** Prove that the space of Riemannian metrics on a smooth manifold is a convex topological subspace of a real vector space, and in particular path-connected.

Solution: Locally, we observe that any Riemannian metric g is given by

$$g_p(x, y) = x^T G_p y$$

for some symmetric positive definite matrix G_p. Since S^2T^*M is a real vector space, it follows that we want to show that the set of all Riemannian metrics, which we will denote R, is convex; in fact, we will show that it is a cone.

Since a Riemannian metric is a section of S^2T^*M, it follows that we can work locally and extend by linearity. Therefore, we only need to show that the collection of positive definite matrices, $\mathrm{Mat}_n(\mathbb{R})_{>0}$, is a cone in $\mathrm{Mat}_n(\mathbb{R})_{\text{s.a.}}$.
Let $\alpha > 0$ in $\mathbb{R}$, $A, A_1, A_2 \in \mathrm{Mat}_n(\mathbb{R})_{>0}$ be positive definite. Then, if $x \in \mathbb{R}^n$ is an arbitrary nonzero vector, we have

$$\begin{aligned}x^T(A_1 + A_2)x &= x^T A_1 x + x^T A_2 x\\ &> 0\\ x^T(\alpha A_1)x &= \alpha x^T A x\\ &> 0,\end{aligned}$$

with equality only when $x = 0$, so the positive definite matrices are a cone in $\mathrm{Mat}_n(\mathbb{R})_{\text{s.a.}}$.

Problem (Problem 4)**:** Let M be a compact orientable Riemannian manifold of dimension n, ν_M its volume form, and $\nu_{\partial M}$ the volume form on ∂M. For $p \in \partial M$, choose a chart about p sending p to the origin such that the image of this chart is the upper half-space defined by $x_n \geqslant 0$.

The Riemannian metric allows one to define an outward-pointing normal vector ν_p at p. Prove that

$$\int_M \nabla \cdot X\, \nu_M = \int_{\partial M} \langle X, \nu \rangle\, \nu_{\partial M}.$$

Solution: We start by observing that the left-hand side simplifies to

$$\begin{aligned}\int_M \nabla \cdot X\, \nu_M &= \int_M ({*d*}\,\omega_X)(*1)\\ &= \int_M **\,d*\omega_X\end{aligned}$$

$$= \int_M d * \omega_X.$$

Therefore, our task is to show that

$$\int_{\partial M} *\omega_X = \int_{\partial M} \langle \omega_X, \omega_\nu \rangle (*1).$$

Considering the chart around p with orthonormal basis $\{e_1, \dots, e_n\}$ and corresponding dual forms $\{\theta_1, \dots, \theta_n\}$, we observe that $\omega_\nu = \theta_n$, and

$$\begin{aligned} \langle \omega_X, \omega_\nu \rangle &= \left\langle \sum_{i=1}^{n} f_i\, \theta_i, \theta_n \right\rangle \\ &= f_n \\ *(1) &= \theta_1 \wedge \dots \wedge \theta_{n-1}, \end{aligned}$$

while

$$\omega_X = \sum_{i=1}^{n} \langle X, e_i \rangle\, \theta_i,$$

so that

$$*\omega_X = f_n\, \theta_1 \wedge \dots \wedge \theta_{n-1}.$$

Problem (Problem 5)**:** Prove that for an orientable Riemannian manifold M and a smooth function f on M, the Laplacian Δ satisfies

$$\Delta f = -\nabla \cdot (\nabla f).$$

Solution: We observe that if f is a function, then $\nabla f = X_{df}$, and since g is nondgenerate, it follows that $\omega_{X_{df}} = df$, whence

$$*d*(\omega_{X_{df}}) = (*d*)(df),$$

while

$$*d*(f) = 0.$$

In particular, this means that

$$(-1)^1(*d*)(df) + (-1)^1(d)(*d*)(f) = -\nabla \cdot (\nabla f).$$

Problem (Problem 6)**:** Recall that the Euler characteristic $\chi(M)$ is the alternating sum of the dimensions of the de Rham cohomology spaces of M. Prove that for an odd-dimensional compact manifold M, the Euler characteristic satisfies $\frac{1}{2}\chi(\partial M) = \chi(M)$.

Solution: We have shown earlier that the Euler characteristic is equal to the alternating sum

$$\chi(M) = \sum_{i=1}^{n} k_i,$$

where k_i is the number of i-dimensional simplices in a simplicial structure for M. From the original de Rham cohomology definition, we observe that any odd-dimensional closed manifold has Euler characteristic 0.

Now, if M has boundary, then we may take two copies of the manifold M to form the manifold $\hat{M}$, given by

$$\hat{M} = \left(M \coprod M\right)/\sim,$$

where $x \sim y$ if x is the same point in M as y and both $x, y \in \partial M$. Observe that by a counting argument, we have

$$\chi\left(\hat{M}\right) = 2\chi(M) - \chi(\partial M),$$

while $\chi\left(\hat{M}\right) = 0$.

Problem (Problem 7)**:** Prove that $\mathbb{RP}^2$ is not the boundary of a compact 3-manifold.

Solution: We have already established that the top-dimensional de Rham cohomology group for $\mathbb{RP}^2$ is 0, meaning that $\chi\left(\mathbb{RP}^2\right) = 1$. Yet, if there were some compact orientable M such that $\mathbb{RP}^2$ was the boundary of M, then we would have

$$\frac{1}{2} = \chi(M),$$

but this is absurd as $\chi(M)$ is necessarily an integer.