Problem (Problem 1)**:** A topological group is a group which is also a Hausdorff topological space where the group operations are continuous.

Recall the definition of the concatenation operation on the fundamental group. Now, let G be a path-connected topological group, and let $\pi_1(G, e)$ be the fundamental group of G with base point e. Use the Hilton–Eckmann argument to prove that the concatenation operation on the fundamental group is commutative.

Solution: Define two operations, $*$ and $\cdot$, on the homotopy-classes of functions $f\colon S^1 \to (G, e)$, where $S^1 \cong [0,1]/(\{0\} \sim \{1\})$ given by

$$\begin{aligned} f * g &= \begin{cases} f(2t) & 0 \leqslant t \leqslant 1/2 \\ g(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases} \\ f \cdot g &= f(t)g(t), \end{aligned}$$

where the latter is multiplication within the group and the former is concatenation. We see that the identity map

$$\begin{aligned} \mathrm{id}\colon S^1 &\to (G, e) \\ t &\mapsto e \end{aligned}$$

is an identity for both $*$ and $\cdot$. Our task now is to show that the Hilton–Eckmann condition holds. That is, let $a, b, c, d\colon S^1 \to (G, e)$ be continuous maps with base point e. Then,

$$\begin{aligned} (a * b) \cdot (c * d) &= (a * b)(t) \cdot (c * d)(t) \\ &= \begin{cases} a(2t)c(2t) & 0 \leqslant t \leqslant 1/2 \\ b(2t-1)d(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases} \\ &= (a \cdot c) * (b \cdot d), \end{aligned}$$

whence $\cdot = *$ and the concatenation operation is commutative.

Problem (Problems 2–4)**:**

(2) Let M and N be smooth, orientable, closed manifolds of the same dimension n, and let $f\colon M \to N$ be a smooth function. Show that f induces a map $f^*\colon H^n_{DR}(N) \to H^n_{DR}(M)$ which is multiplication by an integer. This is called the degree of f and is written $\deg(f)$.

(3) Recall the definition of the degree of f from one of the previous problem sets, counting the sums of signs of determinants of the derivative of f over the preimage of a regular value of f. Prove that the two definitions of the degree agree.

(4) With the setup of the previous exercises, prove that if ω is an arbitrary n-form on N, then

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

Solution: Letting $\omega \in H^n_{DR}(N)$ be a nonvanishing top-dimensional form. By the naturality of the de Rham isomorphism, it follows that there is some $\delta \in \mathbb{R}$ such that.

$$\int_M f^*\omega = \delta \int_N \omega$$

Our task now is to show that $\delta \in \mathbb{Z}$. In particular, we will show that $\delta = \deg(f)$, where $\deg(f)$ is defined as before.

Toward this end, let q be a regular value of f. We may use a smooth bump function to restrict ω to a small open neighborhood V of q. It follows then that $f^{-1}(q) = \{p_1, \dots, p_\ell\}$ for some ℓ, with corresponding disjoint open neighborhoods $U_1, \dots, U_\ell$ locally diffeomorphic to V, whence the support of $f^*\omega$ is contained in the union of $U_1, \dots, U_\ell$. If $f^{-1}(q) = \emptyset$, then

$$\begin{aligned}\int_M f^*\omega &= \int_\emptyset f^*\omega\\ &= \delta \int_N \omega\\ &= 0,\end{aligned}$$

whence $\delta = 0$. If $f^{-1}(q) \neq \emptyset$, then we see that

$$\int_M f^*\omega = \sum_{k=1}^{\ell} \int_{U_k} f^*\omega.$$

Now, since f is a local diffeomorphism on each of the U_k, it follows that

$$\begin{aligned}\int_{U_k} f^*\omega &= \operatorname{sgn}\left(\det\left(D_{p_k} f\right)\right) \int_V \omega\\ &= \operatorname{sgn}\left(\det\left(D_{p_k} f\right)\right) \int_N \omega.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}\int_M f^*\omega &= \sum_{k=1}^{\ell} \operatorname{sgn}\left(\det\left(D_{p_k} f\right)\right) \int_N \omega\\ &= \deg(f) \int_N \omega,\end{aligned}$$

giving that $\deg(f)$ as defined via cohomology and as defined via summation over neighborhoods of preimages of a regular value are equal to each other.

Problem (Problem 5)**:** Prove that for all $d \in \mathbb{Z}$ and all n, there exists a smooth map $S^n \to S^n$ with degree d.

Solution: If $d = 0$, then we may take a constant map to have degree d.

If $d > 0$, select d points, $p_1, \dots, p_d \in S^n$ and corresponding disjoint open neighborhoods $U_1, \dots, U_d$. We define $f\colon S^n \to S^n$ to be a map that takes p_i to the north pole, a diffeomorphism between U_i and $S^n \setminus \{s\}$ (which follows from the coordinate map for U_i composed with the (inverse) south pole stereographic projection), where s is the south pole, and everything outside of $U_1 \cup \dots \cup U_d$ to the south pole.

Similarly, if $d < 0$, perform the same process, but f should be orientation-reversing on each of the U_i.

Problem (Problem 6)**:** Let K and L be smooth knots in $\mathbb{R}^3$, which is to say smooth embeddings of S^1 into $\mathbb{R}^3$. Assume that K and L are disjoint subsets of $\mathbb{R}^3$. Choosing an orientation on S^1 orients K and L. Write $K = f(s)$ and $L = g(t)$, where $t, s \in S^1$, and let $F(s,t) = g(t) - f(s)$ as a difference of vectors in $\mathbb{R}^3$. Notice that F never takes on the value 0. Renormalizing the vectors in $\mathbb{R}^3 \setminus \{0\}$ gives a map $\pi\colon \mathbb{R}^3 \setminus \{0\} \to S^2$. Write $G = \pi \circ F$. Notice that $G\colon S^1 \times S^1 \to S^2$ is a smooth map.

Define the linking number $\operatorname{lk}(K, L) \equiv \deg(G)$. Now, let

$$\omega = \frac{1}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}\left(x_1\, dx_2 \wedge dx_3 - x_2\, dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2\right),$$

where x_1, x_2, x_3 are (Cartesian) coordinates on $\mathbb{R}^3$.

(a) Show that $d\omega = 0$.

(b) Let $\eta = \omega|_{S^2}$. Prove that

$$\int_{S^2} \eta = 4\pi.$$

Conclude that

$$\operatorname{lk}(K, L) = \frac{1}{4\pi}\int_{S^1\times S^1} F^*\omega.$$

(c) Compute $F^* dx_i$ for $i = 1, 2, 3$.

(d) Find an explicit expression for $\operatorname{lk}(K.L)$ as an integral over $S^1 \times S^1$, in terms of f and g.

Solution:

(a) We compute

$$d\omega = \left(\frac{\partial}{\partial x_1}\left(\frac{x_1}{\left(x_1^2+x_2^2+x_3^2\right)^{3/2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{x_2}{\left(x_1^2+x_2^2+x_3^2\right)^{3/2}}\right) + \frac{\partial}{\partial x_3}\left(\frac{x_3}{\left(x_1^2+x_2^2+x_3^2\right)^{3/2}}\right)\right) dx_1 \wedge dx_2 \wedge dx_3.$$

Symmetrically, we observe that

$$\begin{aligned}\frac{\partial}{\partial x_1}\left(\frac{x_1}{\left(x_1^2+x_2^2+x_3^2\right)^{3/2}}\right) &= \frac{1}{\left(x_1^2+x_2^2+x_3^2\right)^{3/2}} - \frac{3}{2}\frac{2x_1(x_1)}{\left(x_1^2+x_2^2+x_3^2\right)^{5/2}}\\ &= \frac{-2x_1^2+x_2^2+x_3^2}{\left(x_1^2+x_2^2+x_3^2\right)^{5/2}},\end{aligned}$$

whence their sum is zero.

(b) We observe that the restriction to S^1 yields $x_1^2 + x_2^2 + x_3^2 = 1$, meaning that the restriction is

$$\iota^*\omega = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2,$$

where $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$ is inclusion. In particular, we observe that this is precisely the area form on $\mathbb{R}^3$ with respect to the vector

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

whence

$$\int_{S^2} \iota^*\omega = 4\pi.$$

Now, since $G^* = F^* \circ \pi^*$, we see that

$$\begin{aligned}\deg(G)\int_{S^2}\omega &= \int_{S^1\times S^1} G^*\omega\\ &= \int_{S^1\times S^1} F^*(\pi^*\omega)\\ &= \int_{S^1\times S^1} F^*\omega,\end{aligned}$$

where the latter equality emerges from the fact that $\pi|_{S^2} = \mathrm{id}$.

(c) We write

$$F(t, s) = \begin{pmatrix} f_1(t) - g_1(s) \\ f_2(t) - g_2(s) \\ f_3(t) - g_3(s) \end{pmatrix}.$$

Then,

$$\begin{aligned} F^* dx_i &= d(x_i \circ F) \\ &= d(f_i(t) - g_i(s)) \\ &= \frac{df_i}{dt} dt - \frac{dg_i}{ds} ds. \end{aligned}$$

(d) Inserting these expressions into ω and using the natural symmetry therein, we find

$$\begin{aligned} F^* \omega &= \frac{1}{\|F\|^3}(f_1(t) - g_1(s))\left(\frac{df_3}{dt}\frac{dg_2}{ds} - \frac{df_2}{dt}\frac{dg_3}{ds}\right) \\ &\quad + \frac{1}{\|F\|^3}(f_2(t) - g_2(s))\left(\frac{df_1}{dt}\frac{dg_3}{ds} - \frac{df_3}{dt}\frac{dg_1}{ds}\right) \\ &\quad + \frac{1}{\|F\|^3}(f_3(t) - g_3(s))\left(\frac{df_2}{dt}\frac{dg_1}{ds} - \frac{df_1}{dt}\frac{dg_2}{ds}\right) \end{aligned}$$