Problem (Problem 1)**:** Let F be a finite field with q elements.

(a) Find the order of the general linear group

$$GL_n(F) = \{A \in \mathrm{Mat}_n(F) \mid \det(A) \neq 0\}.$$

(b) Find the order of the special linear group

$$SL_n(F) = \{A \in \mathrm{Mat}_n(F) \mid \det(A) = 1\}.$$

Solution:

(a) In order to find the order of the general linear group, we let $A \in GL_n(F)$ be an arbitrary matrix. The first column of A can consist $q^n - 1$ nonzero vectors in F^n.

To determine the second column, we observe that it cannot be a nonzero element element of the 1-dimensional linear subspace spanned by the first column of A. In particular, this means that there are $q^n - 1 - (q-1)$ such possible elements, or $q^n - q$. Inductively, we see that if we have determined the first $k-1$ columns, then the choices of column k consist of all nonzero vectors in F^n that are not of the form

$$v = c_1 v_1 + \cdots + c_{k-1} v_{k-1}$$

for $c_1, \dots, c_{k-1} \in F$; there are $q^{k-1} - 1$ such nonzero elements of the linear subspace spanned by $v_1, \dots, v_{k-1}$, so that

$$|GL_n(F)| = \prod_{i=0}^{n-1} (q^n - q^i).$$

(b) By using the determinant homomorphism $\det\colon GL_n(F) \to F$, we find that

$$GL_n(F)/SL_n(F) \cong F^{\times},$$

whence

$$|GL_n(F)| = (q-1)|SL_n(F)|,$$

or

$$|SL_n(F)| = \frac{1}{q-1} \prod_{i=0}^{n-1} (q^n - q^i).$$

Problem (Problem 2)**:** Let F be a field with q elements, $V = F^n$ an n-dimensional F-vector space, and $m \leqslant n$. The purpose of this problem is to determine the cardinality of the set $T(m)$, the set of all m-dimensional subspaces W of V.

(a) Show that the standard action of $G = GL_n(F)$ on V induces a natural action of G on $T(m)$. Furthermore, show that this action is transitive.

(b) Let $W \in T(m)$ be the subspace spanned by the first m elements of the standard basis $\{e_1, \dots, e_n\}$ of V. Identify explicitly the stabilizer $\mathrm{stab}_G(W)$.

(c) Combine these facts with the formulas from Problem 1 (a) to determine $|T(m)|$.

Solution:

(a) We observe that G acts on V by mapping $0 \neq v \mapsto Av \neq 0$ for $A \in G$. We can extend this to an action on a m-dimensional subspace with ordered basis $(v_1, \dots, v_m)$ by mapping

$$S \cdot (v_1, \dots, v_m) = (Sv_1, \dots, Sv_m).$$

We observe that $\mathrm{id} \cdot (v_1, \dots, v_m) = (v_1, \dots, v_m)$, and

$$\begin{aligned} S \cdot (T \cdot (v_1, \dots, v_m)) &= S \cdot (Tv_1, \dots, Tv_m) \\ &= (STv_1, \dots, STv_m) \\ &= ST \cdot (v_1, \dots, v_m). \end{aligned}$$

Finally, to see that this action is transitive, we observe that for any two ordered bases $(v_1, \dots, v_m)$ and $(w_1, \dots, w_m)$ that define elements of $T(m)$, each can be extended to bases for V, $(v_1, \dots, v_m, v_{m+1}, \dots, v_n)$ and $(w_1, \dots, w_m, w_{m+1}, \dots, w_n)$, and we can specify a linear map $T \colon V \to V$ taking $v_i \mapsto w_i$ for each i. This specifies an element of $GL_n(F)$ by taking the matrix representation of this linear map. Therefore, the action is transitive.

(b) We observe that if $(e_1, \dots, e_m)$ is the basis for W, and $T \in GL_n(F)$, then

$$Te_i = c_1e_1 + \cdots + c_me_m + c_{m+1}e_{m+1} + \cdots + c_ne_n$$

for some constants $c_1, \dots, c_n$. In order for T to stabilize W, then we must have

$$Te_i = c_1e_1 + \cdots + c_me_m$$

for each $i = 1, \dots, m$. In particular, T is an invertible block matrix of the form

$$T = \begin{pmatrix} A & * \\ 0 & C \end{pmatrix},$$

where $A \in GL_m(F)$, $*$ is an arbitrary $m \times (n-m)$ matrix, and $C \in GL_{n-m}(F)$.

(c) By the orbit-stabilizer theorem, and since the action of $GL_n(F)$ on $T(m)$ is transitive, we know that

$$\begin{aligned} |T(m)| &= [GL_n(F) : \mathrm{stab}_G(W)] \\ &= \frac{|GL_n(F)|}{|\mathrm{stab}_G(W)|}. \end{aligned}$$

Our task now is to compute the order of the stabilizer. We observe that any element T of $\mathrm{stab}_G(W)$ consists of

- an arbitrary $m \times (n-m)$ matrix over F;
- an element of $GL_m(F)$;
- and an element of $GL_{n-m}(F)$.

Therefore, we find that

$$|\mathrm{stab}_G(W)| = \left(\prod_{i=0}^{m}(q^m - q^i)\right)\left(\prod_{i=0}^{n-m}(q^{n-m} - q^i)\right)q^{n(m-n)}.$$

Thus,

$$|T(m)| = \frac{\prod_{i=0}^{n}(q^n - q^i)}{\left(\prod_{i=0}^{m}(q^m - q^i)\right)\left(\prod_{i=0}^{n-m}(q^{n-m} - q^i)\right)\left(q^{m(n-m)}\right)}.$$

Problem (Problem 6)**:** Suppose a finite group G acts on a finite set X. For $g \in G$, let X^g be the set of all

$x \in X$ that are fixed by g. Prove that

$$|G \cdot X| = \frac{1}{|G|}\sum_{g\in G}|X^g| >$$

That is, the number of orbits equals the "average" number of fixed points of elements of G.

Solution: We start by showing that

$$|G \cdot X| = \frac{1}{|G|}\sum_{x\in X}|\mathrm{stab}_G(x)|.$$

From the orbit-stabilizer theorem, we know that

$$|\mathrm{stab}_G(x)| = \frac{|G|}{|G \cdot x|}$$

for each $x \in X$. Therefore, we observe that

$$\sum_{x\in X}|\mathrm{stab}_G(x)| = |G|\sum_{x\in X}\frac{1}{|G \cdot x|}.$$

Since the orbits partition X, we observe that we may split

$$\sum_{x\in X}\frac{1}{|G \cdot x|} = \sum_{i=1}^{r}|G \cdot x_i|\frac{1}{|G \cdot x_i|},$$

since for any $x \in X$ with $x \in G \cdot x_i$, there are $|G \cdot x_i|$ total elements in the same orbit, whence

$$\begin{aligned}\sum_{x\in X}|\mathrm{stab}_G(x)| &= |G|\sum_{i=1}^{r}1\\ &= |G||G \cdot X|.\end{aligned}$$

Now, let $Y = \{(g, x) \in G \times X \mid g \cdot x = x\}$. Letting π_1 and π_2 be the projections on the first and second coordinate, we observe that for a specific g_0 and x_0, we have

$$\begin{aligned}\pi_1^{-1}(\{g_0\}) &= \{(g_0, x) \in Y \mid g_0 \cdot x = x\}\\ \pi_2^{-2}(\{x_0\}) &= \{(g, x_0) \in Y \mid g \cdot x_0 = x_0\}.\end{aligned}$$

In particular, we have

$$\begin{aligned}\left|\pi_1^{-1}(\{g_0\})\right| &= |X^{g_0}|\\ \left|\pi_2^{-1}(\{x_0\})\right| &= |\mathrm{stab}_G(x_0)|,\end{aligned}$$

and

$$\begin{aligned}Y &= \bigsqcup_{g\in G}\pi_1^{-1}(\{g\})\\ &= \bigsqcup_{x\in X}\pi_2^{-1}(\{x\}),\end{aligned}$$

whence

$$|Y| = \sum_{g\in G}|X^g|$$

$$= \sum_{x \in X} |\text{stab}_G(x)|$$
$$= |G||G \cdot X|.$$

Thus,

$$|G \cdot X| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Problem (Problem 8)**:** Recall that the symmetric group S_3 consists of the following permutations: e, the transpositions $(1,2)$, $(1,3)$, and $(2,3)$, and the two 3-cycles $(1,2,3)$ and $(1,3,2)$. Also, recall that every $\sigma \in S_3$ can be written as a product of transpositions.

(a) Show that the center of S_3 is trivial, and hence $\text{inn}(S_3) \cong S_3$.

(b) Show that $\text{aut}(G) \cong G$, and hence every automorphism of G is inner.