Déjà Vu

Consider the game in which the following stage game is repeated twice.

	L	C	R
T	$8, 8$	$0, 0$	$16, 0$
M	$0, 0$	$4, 4$	$16, -1$
B	$0, 16$	$-1, 16$	$12, 12$

For each of the action profiles below, construct a SPE in which the action profile is played in the first stage of the game, or show that no such SPE exists.

(a) (B, L)

(b) (B, C)

In order to deduce the potential action profiles, we start by finding the Nash equilibria of the game.

- R is strictly dominated by $\frac{1}{2}L + \frac{1}{2}C$.

	L	C	~~R~~
T	$8, 8$	$0, 0$	~~$16, 0$~~
M	$0, 0$	$4, 4$	~~$16, -1$~~
B	$0, 16$	$-1, 16$	~~$12, 12$~~

- Now, B is strictly dominated by $\frac{1}{2}T + \frac{1}{2}M$.

	L	C	~~R~~
T	$8, 8$	$0, 0$	~~$16, 0$~~
M	$0, 0$	$4, 4$	~~$16, -1$~~
~~B~~	~~$0, 16$~~	~~$-1, 16$~~	~~$12, 12$~~

We can see that there are three Nash equilibria:

- (T, L), with payoffs $(8, 8)$
- (M, C), with payoffs $(4, 4)$
- $\left(\frac{1}{3}T + \frac{2}{3}M, \frac{1}{3}L + \frac{2}{3}C\right)$, with payoffs $\left(\frac{16}{3}, \frac{16}{3}\right)$

Thus, we would expect that conceivable SPE play as follows:

- In stage 1, if both players cooperate, the second stage plays the (T, L) Nash equilibrium.
- In stage 1, if a player defects, the second stage plays the (M, C) Nash equilibrium.

(a) If we were to construct a SPE with (B, L) in the first stage, we have the following:

- The incentive for Player 1 to deviate to playing T yields a net payoff increase of 8
- The maximum possible punishment is a net loss of 4 — therefore, Player 1 has an incentive to deviate.

(b) If we were to construct a SPE with (B, C) in the first stage, we have the following:

- The incentive for Player 1 to deviate to playing M yields a net payoff increase of 5.
- The maximum possible punishment is a net loss of 4 — therefore, Player 1 has an incentive to deviate.

Therefore, both of the proposed strategies cannot be constructed into SPE.

Two Seagulls

Seagulls love shellfish. In order to break the shell, they need to fly high up and drop the shellfish. However, the other seagulls will steal the shellfish from the seagull that dropped it. Consider the case of two seagulls, Nina (player 1) and Irina (player 2). The seagulls have two options: Up or Down. There is a cost of 10 to going up and the value of eating the shellfish is 20. Nina and Irina repeat this game infinitely many times, with a common discount factor of δ. In particular, the stage game is given by the following payoff matrix:

	Up	Down
Up	$-10, -10$	$-10, 20$
Down	$20, -10$	$0, 0$

For each strategy profile below, find the range of discount factors (if any) such that the strategy profile is a subgame perfect equilibrium.

(a) (No Punishment) In the first stage Nina plays Up and Irina plays Down. They alternate Up and Down each day thereafter, irrespective of the history. A deviation by some player in some period does not change the prescription of play.

(b) (Grim Trigger) In the first stage Nina plays Up and Irina plays Down. They alternate Up and Down each day thereafter. If some player ever fails to follow this scheme, then both birds switch to playing Down forever.

(a)

Current period, Player i plays down:

Follow:

$$\begin{aligned} v_i &= (1-\delta)\left(-10 - 10\delta^2 - 10\delta^4 - \cdots + 20\delta + 20\delta^3 + \cdots\right) \\ &= (1-\delta)\left(\frac{20\delta - 10}{1-\delta^2}\right) \\ &= \frac{20\delta - 10}{1+\delta} \end{aligned}$$

Deviate in current period:

$$\begin{aligned} v_i &= (1-\delta)\left(0 - 10\delta^2 - 10\delta^4 - \cdots + 20\delta + 20\delta^3 + \cdots\right) \\ &= (1-\delta)\left(\frac{20\delta - 10\delta^2}{1-\delta^2}\right) \\ &= \frac{20\delta - 10\delta^2}{1+\delta} \end{aligned}$$

SPE Condition:

$$\begin{aligned} \frac{20\delta - 10}{1+\delta} &\geq \frac{20\delta - 10\delta^2}{1+\delta} \\ 20\delta - 10 &\geq 20\delta - 10\delta^2 \\ \delta^2 &\geq 1 \\ \delta &\geq 1 \end{aligned}$$

However, since $\delta < 1$ necessarily by the One Stage Deviation Property, this means the given strategy is not a SPE.