Textbook Questions

Solution (4.4, Problem 2)**:** By the method of inspection, we get the general solution of

$$y(x) = c_1 \cos\left(\frac{3}{2}x\right) + c_2 \sin\left(\frac{3}{2}x\right) + \frac{5}{3}.$$

Solution (4.4, Problem 4)**:** By the method of inspection (basically just undetermined coefficients without actually going through the full steps), we get the general solution of

$$y(x) = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{3}x + \frac{1}{8}.$$

Solution (4.4, Problem 12)**:** Using the power of inspection, we have the homogeneous solution of $k_1 e^{4x} + k_2 e^{-4x}$. For the particular solution, we guess

$$y_p(x) = (ax + b)e^{4x},$$

and use the method of computation through Sage's `desolve` command to obtain the general solution of

$$y(x) = k_1 e^{4x} + k_2 e^{-4x} + \frac{1}{4}xe^{4x}$$

This can be independently verified by using undetermined coefficients on $y_p(x) = (ax + b)e^{4x}$, giving

$$\begin{aligned} y'' - 16y &= 8ae^{4x} \\ &= 2e^{4x}. \end{aligned}$$

Solution (4.6, Problem 2)**:** We guess $y_p(x) = u_1 y_1(x) + u_2 y_2(x)$, and use the variation of parameters derivation to find

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -\frac{\sin^2 x}{\cos(x)\cos(2x)} \\ \frac{\sin(x)}{\cos(2x)} \end{pmatrix}.$$

After many tedious, error-prone calculations that are better performed in Mathematica, we get the particular solution of

$$y_p(x) = -\frac{1}{2}\cos(x)\ln(\sin(x) + 1) + \frac{1}{2}\cos(x)\ln(\sin(x) - 1).$$

This adds to the homogeneous solution of $k_1 \cos(x) + k_2 \sin(x)$.

Solution (4.6, Problem 8)**:** We find homogeneous solutions of $y_1 = e^x$ and $y_2 = e^x$. We use variation of parameters to find

$$y_p(x) = \frac{1}{3}\sinh(2x),$$

giving the general solution of

$$y(x) = k_1 e^x + k_2 e^{-x} + \frac{1}{3}\sinh(2x).$$

Solution (4.6, Problem 14)**:** The solutions to the homogeneous equation $y'' - 2y' + y = 0$ are e^x and xe^x. Setting up the variation of parameters equation, we get

$$\begin{aligned}
\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= \frac{1}{e^x(xe^x + e^x) - (xe^x)e^x}\begin{pmatrix} -xe^x(e^x \arctan(x)) \\ e^{2x}\arctan(x) \end{pmatrix} \\
&= \frac{1}{e^{2x}}\begin{pmatrix} -xe^{2x}\arctan(x) \\ e^{2x}\arctan(x) \end{pmatrix} \\
&= \begin{pmatrix} -x\arctan(x) \\ \arctan(x) \end{pmatrix}.
\end{aligned}$$

Using the power of scratch work on the chalkboard, we get

$$\begin{aligned}
v_1 &= x\arctan(x) - \frac{1}{2}\ln\left(1 + x^2\right) \\
v_2 &= -\frac{1}{2}x^2\arctan(x) + \frac{1}{2}x - \frac{1}{2}\arctan(x).
\end{aligned}$$

Therefore, we get the general solution of

$$y(x) = c_1 e^x + c_2 xe^x + e^x\left(x\arctan(x) - \frac{1}{2}\ln\left(1 + x^2\right)\right) + xe^x\left(-\frac{1}{2}x^2\arctan(x) + \frac{1}{2}x - \frac{1}{2}\arctan(x)\right).$$

Solution (4.6, Problem 31)**:** We will show that $y_p(x)$ is a solution to the equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f.$$

Taking

$$\begin{aligned}
y_p(x) &= \int_0^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} f(t)\,dt \\
&= y_2(x)\int_0^x \frac{y_1(t)f(t)}{W(t)}\,dt - y_1(x)\int_0^x \frac{y_2(t)f(t)}{W(t)}\,dt,
\end{aligned}$$

and using the power of scratch work on the chalkboard, we get

$$\begin{aligned}
\frac{dy_p}{dx} &= \frac{dy_2}{dx}\int_0^x \frac{y_1(t)f(t)}{W(t)}\,dt - \frac{dy_1}{dx}\int_0^x \frac{y_2(t)f(t)}{W(t)}\,dt \\
\frac{d^2y_p}{dx^2} &= \frac{d^2y_2}{dx^2}\int_0^x \frac{y_1(t)f(t)}{W(t)}\,dt - \frac{d^2y_1}{dx^2}\int_0^x \frac{y_2(t)f(t)}{W(t)}\,dx + f(x).
\end{aligned}$$

Plugging these into the equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$$

gives our desired solution.

Solution (4.7, Problem 4)**:** We multiply both sides of the equation by x to get

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} = 0.$$

The auxiliary equation is, then,

$$r(r-1) - 3r = 0,$$

giving $r = 0, r = 4$. Thus, our solution is

$$y(x) = c_1x^4 + c_2.$$

Solution (4.7, Problem 10)**:** Substituting into the auxiliary equation, we have

$$\begin{aligned} 4r(r-1) + 4r - 1 &= 0 \\ 4r^2 - 1 &= 0, \end{aligned}$$

giving $r = \pm\frac{1}{2}$. Thus, our solution is

$$y(x) = c_1x^{1/2} + c_2x^{-1/2}.$$

Solution (4.7, Problem 12)**:** Substituting into the auxiliary equation, we have

$$\begin{aligned} r(r-1) + 8r + 6 &= 0 \\ r^2 + 7r + 6 &= 0, \end{aligned}$$

giving $r = -6, -1$. Thus, our solution is

$$y(x) = c_1r^{-6} + c_2r^{-1}.$$

Solution (4.7, Problem 14)**:** Substituting into the auxiliary equation, we have

$$r(r-1) - 7r + 41 = 0,$$

giving

$$r^2 - 8r + 41 = 0.$$

Completing the square and solving, we get

$$r = 4 \pm 5i,$$

so our solution is

$$y(x) = c_1x^4\cos(5\ln x) + c_2x^4\sin(5\ln x).$$

Solution (4.7, Problem 16)**:** Substituting into the auxiliary equation, we have

$$\begin{aligned} r(r-1)(r-2) + r - 1 &= 0 \\ r^3 - 3^2 + 3r - 1 &= 0. \end{aligned}$$

This gives $r = 1$ with multiplicity 3, so we have solutions of

$$y(x) = c_1 x + c_2 x \ln(x) + c_3 x(\ln(x))^2.$$

Solution (4.7, Problem 18)**:** Substituting into the auxiliary equation, we have

$$\begin{aligned} r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 9r(r-1) + 3r + 1 &= 0. \\ \left(x^2+1\right)^2 &= 0. \end{aligned}$$

Thus, we have roots of $\pm i$ with multiplicity 2, giving solutions of

$$y(x) = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) + \ln(x)(c_3 \cos(\ln(x)) + c_4 \sin(\ln(x))).$$

Solution (4.7, Problem 32)**:** Using the substitution $x = e^t$, we have

$$\begin{aligned} x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) - 9x\left(\frac{1}{x}\frac{dy}{dt}\right) + 25y &= 0 \\ \frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25y &= 0. \end{aligned}$$

Solving, we get

$$y(t) = c_1 e^{5t} + c_2 t e^{5t},$$

so

$$y(x) = c_1 x^5 + c_2 x^5 \ln(x).$$

Extra Problems

Solution: A second-order linear homogeneous ODE that has solutions of $y(x) = \tan(x)$ and $y(x) = 1$ is

$$\frac{d^2y}{dx^2} - 2\tan(x)\frac{dy}{dx} = 0.$$

Solution: A second order linear inhomogeneous ODE with solutions $y(x) = \sin^3(x)$ and $y(x) = \cos^3(x)$ is

$$\frac{d^2y}{dx^2} + 9y = 9.$$

I could not find a homogeneous ODE with these solutions.