# Contents



# Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and $C^*$-algebras. This section of notes will be focused on the essential results in functional analysis, starting from normed vector spaces, working our way up through $C^*$-algebras.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

# Normed Vector Spaces

## Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over $\mathbb{C}$. Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

**Definition** (Vector Space)**.** A vector space $V$ is a set closed under two operations

$$a : V \times V \to V,\ (v_1, v_2) \mapsto v_1 + v_2$$
$$m : \mathbb{C} \times V \to V,\ (\lambda, v) \mapsto \lambda v.$$

We refer to $a$ as addition, and $m$ as scalar multiplication; $(V, +)$ is an abelian ring.

**Definition** (Norm)**.** A norm is a function

$$\|\cdot\| : V \to \mathbb{R}^+,\ x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness: $\|v\| = 0$ if and only if $v = 0_V$.
- Triangle inequality: $\|v + w\| \leqslant \|v\| + \|w\|$.
- Absolute Homogeneity: $\|\lambda v\| = |\lambda|\,\|v\|$, for $\lambda \in \mathbb{C}$.

If a function $p : V \to \mathbb{R}^+$ satisfies the triangle inequality and absolute homogeneity, we say $p$ is a seminorm.

We say the pair $(V, \|\cdot\|)$ is a normed vector space.

**Definition** (Balls and Spheres). Let $X$ be a normed vector space, $x \in X$, and $\delta > 0$. Then,

$$\begin{aligned} U(x,\delta) &= \{y \in X \mid d(x,y) < \delta\} \\ B(x,\delta) &= \{y \in X \mid d(x,y) \leqslant \delta\} \\ S(x,\delta) &= \{y \in X \mid d(x,y) = \delta\}. \end{aligned}$$

For a normed vector space, we will use the following conventions for common sets:

$$\begin{aligned} U_X &= U(0,1) \\ B_X &= B(0,1) \\ S_X &= S(0,1) \\ \mathbb{D} &= U_{\mathbb{C}} \\ \mathbb{T} &= S_{\mathbb{C}}. \end{aligned}$$

**Definition** (Equivalent Norms). Two norms on $V$, $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there are two constants $C_1$ and $C_2$ such that

$$\begin{aligned} \|v\|_a &\leqslant C_1 \|v\|_b \\ \|v\|_b &\leqslant C_2 \|v\|_a \end{aligned}$$

for all $v \in V$. We say $\|\cdot\|_a \sim \|\cdot\|_b$.

## Examples

**Example** (Finite-Dimensional Vector Spaces). The vector space $\mathbb{C}^n$ is with the p-norm is denoted $\ell_p^n$, where for $p \in [1,\infty]$, the p-norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

In the case with $p = 2$, this gives the traditional Euclidean norm, and with $p = \infty$, this gives the sup norm:

$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|.$$

**Example** (A Sequence Space). We let $\ell_p = \left\{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\right\}$ be the collection of sequences in $\mathbb{C}$ with finite p-norm. Here,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

In the case with $p = \infty$, this gives the sequence space $\ell_\infty$, which has norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

**Example** (A Function Space). We let $\ell^\infty(\Omega)$ denote the set of all bounded functions $f : \Omega \to \mathbb{C}$, equipped with the norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|.$$

If $\Omega = (\Omega, \mathcal{M}, \mu)$ is a measure space, then we let $L^\infty(\Omega)$ be the space of $\mu$-a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

## Series Convergence and Completeness

**Proposition** (Criteria for Banach Spaces)**:** Let $X$ be a normed vector space. The following are equivalent:

(i) $X$ is a Banach space.[1]

(ii) If $(x_k)_k$ is a sequence of vectors such that $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

(iii) If $(x_k)_k$ is a sequence in $X$ such that $\|x_k\| < 2^{-k}$, then $\sum_{k=1}^{\infty} x_k$ converges.

*Proof.* To show (i) implies (ii), for $n > m > N$, we have

$$\begin{aligned}\|s_n - s_m\| &= \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\leqslant \sum_{k=m+1}^{n} \|x_k\| \\ &< \epsilon,\end{aligned}$$

implying that $s_n$ is Cauchy, and thus converges since $X$ is complete.

Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let $(x_n)_n$ be a Cauchy sequence in $X$. We only need construct a convergent subsequence in order to show that $(x_n)_n$ converges.

Chose $n_1 \in \mathbb{N}$ such that for $n, m \geqslant n_1$, $\|x_m - x_n\| < \frac{1}{2^2}$, and inductively define $n_j > n_{j-1}$ such that $n, m \geqslant n_j$ implies $\|x_m - x_n\| < \frac{1}{2^{j+1}}$.

Let $y_1 = x_{n_1}$, $y_j = x_{n_j} - x_{n_{j-1}}$. Then,

$$\begin{aligned}\|y_j\| &= \left\|x_{n_j} - x_{n_{j-1}}\right\| \\ &< \frac{1}{2^j},\end{aligned}$$

so $\sum_{j=1}^{\infty} y_j$ converges by our assumption. By telescoping, we see that $\sum_{j=1}^{k} y_j = x_{n_k}$, so $(x_{n_k})_k$ converges. □

## Quotient Spaces

Let $X$ be a normed vector space. Then, for $E \subseteq X$ a subspace, there is a quotient space $X/E$ with the projection map $\pi : X \to X/E$, $x \mapsto x + E$. We want to make $X/E$ into a normed space — in order to do this, we use the distance function:

$$\operatorname{dist}_E(x) = \inf_{y \in E} d(x, y),$$

which is uniformly continuous. For $E$ closed, then $\operatorname{dist}_E(x) = 0$ if and only if $x \in E$.

**Proposition** (Quotient Space Norm)**:** Let $X$ be a normed vector space, and $E \subseteq X$ a subspace. Set

$$\|x + E\|_{X/E} = \operatorname{dist}_E(x).$$

Then,

(1) $\|\cdot\|_{X/E}$ is a well-defined seminorm on $X/E$.

[1] Complete normed vector space.

(2) If $E$ is closed, then $\|\cdot\|_{X/E}$ is a norm on $X/E$.

(3) $\|x + E\|_{X/E} \leqslant \|x\|$ for all $x \in X$.

(4) If $E$ is closed, then $\pi : X \to X/E$ is Lipschitz.

(5) If $X$ is a Banach space and $E$ is closed, then $X/E$ is also a Banach space.