Problem (Problem 2)**:** Define $f\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $f(z) = \left(\frac{z+1}{z-1}\right)^2$.

(a) Is f injective on $\mathbb{D}$? Why or why not?

(b) Determine $f(\mathbb{D})$.

Solution:

(a) We consider $q(z) = \frac{z+1}{z-1}$ as a fractional linear transformation on $\hat{\mathbb{C}}$. We see that

$$\begin{aligned} q\left(e^{i\theta}\right) &= \frac{e^{i\theta}+1}{e^{i\theta}-1} \\ &= \frac{(1+\cos(\theta)) + i\sin(\theta)}{(\cos(\theta)-1) + i\sin(\theta)} \\ &= \frac{((\cos(\theta)+1) + i\sin(\theta))((\cos(\theta)-1) - i\sin(\theta))}{(1-\cos(\theta))^2 + \sin^2(\theta)} \\ &= \frac{\left(\cos^2(\theta)-1\right) + \sin^2(\theta) + i\sin(\theta)(\cos(\theta)-1-(\cos(\theta)+1))}{2-2\cos(\theta)} \\ &= i\frac{\sin(\theta)}{\cos(\theta)-1}, \end{aligned}$$

and since $\frac{\sin(\theta)}{\cos(\theta)-1}$ maps $(0,2\pi) \to \mathbb{R}$ bijectively, we see that q maps the unit circle into the imaginary axis. We also see that $q(0) = -1$, so $\mathbb{D}$ maps $\mathbb{D}$ bijectively onto the left half-plane, $\mathbb{L} = \{z \mid \operatorname{Re}(z) < 0\}$.

Now, notice that the function $h(z) = z^2$ is injective when defined on a half-plane (the arguments $(\pi/2, 3\pi/2)$ map injectively to $(\pi, 3\pi)$, and the function $|z|^2$ is clearly injective on $(0,\infty)$), so since $f = h \circ q$ is injective on $\mathbb{D}$.

(b) Since $f = h \circ q$, where q maps $\mathbb{D}$ to the left half-plane, and h maps the left half-plane to the full complex plane save for $(-\infty, 0]$, we have that f maps $\mathbb{C}$ to $\mathbb{C} \setminus (-\infty, 0]$.