

# Amenability: A (Somewhat) Brief Introduction

Avinash Iyer

Occidental College

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# Outline

- ① Definitions
- ② Paradoxical Decompositions
- ③ From Paradoxical Decompositions to Amenability
- ④ Equivalent Definitions and Other Criteria
  - A Taste of Functional Analysis
  - Introducing Approximations
  - Approximations with Representations and Operators
  - Review
- ⑤ Remarks and Acknowledgments

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We (usually) abbreviate  $a \star b$  as  $ab$ . If  $ab = ba$ , then we say the group is *abelian*.

## Subgroups, Quotient Groups

Let  $G$  be a group.

- If  $H \subseteq G$  is a subset that satisfies, for all  $a, b \in H$ ,  $ab^{-1} \in H$ , then we say  $H$  is a *subgroup*.

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- The equivalence classes under the relation  $g \sim_N g'$  if  $g^{-1}g' \in N$  form a group  $gN := [g]_{\sim}$  known as the *quotient group*  $G/N$ .
- The *index* of a subgroup  $H \leq G$  is the number of cosets,  $gH := \{gh \mid h \in H\}$ , written  $[G : H]$ .

## Some Groups

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- The group  $\text{SO}(n)$  consisting of  $n \times n$  orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group  $E(3)$  consists of all translations, rotations, and flips in  $\mathbb{R}^3$ , and is also known as the *isometry group* of  $\mathbb{R}^3$ .



## Group Actions

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Every group is equipped with a family of canonical actions,  $\sigma_a: G \rightarrow G$  for each  $a \in G$ , given by  $x \mapsto ax$ , known as *left-multiplication*.

## $\sigma$ -Algebras and Measures

If  $X$  is a set, then a collection of subsets  $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$  is known as an *algebra* of subsets if

- ①  $\emptyset, X \in \mathcal{A}$ ;
- ② for any  $A_i \in \mathcal{A}$ ,  $A_i^c \in \mathcal{A}$ ;
- ③ for any  $A_i, A_j \in \mathcal{A}$ ,  $A_i \cup A_j \in \mathcal{A}$ .

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The most important  $\sigma$ -algebra, and the one we will be dealing with throughout this talk, is  $P(G)$ , where  $G$  is a group.

## $\sigma$ -Algebras and Measures, Cont'd

If  $X$  is a set and  $\mathcal{A}$  is a  $\sigma$ -algebra, then a map  $\mu: \mathcal{A} \rightarrow [0, \infty]$  that satisfies:

- $\mu(\emptyset) = 0$ ;
- for disjoint sets  $A, B \in \mathcal{A}$ ,  $\mu(A \sqcup B) = \mu(A) + \mu(B)$ ,

then we say  $\mu$  is a *finitely additive* measure.

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If  $\{A_n\}_{n \geq 1}$  is a countable collection of disjoint sets, then if  $\mu$  satisfies

- $$\mu\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n),$$

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we say  $\mu$  is a measure. If  $\mu(X) = 1$ , then we say  $\mu$  is a probability measure.

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- Are these questions even related?



# Free Groups

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- We begin by considering a special group, known as  $F(a, b)$  or the *free group on two generators*.
- We define  $F(a, b)$  to be the set of all “words” in the alphabet  $\{a, b, a^{-1}, b^{-1}\}$ , subject to the condition that, for  $w, w' \in F(a, b)$ ,

$$\begin{aligned} waa^{-1}w' &\sim wa^{-1}aw' \sim ww' \\ wbb^{-1}w' &\sim wb^{-1}bw' \sim ww'. \end{aligned}$$

- Examples:  $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a, b)$ .

## A Curiosity

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Thus, all we need to do is add back  $W(b^{-1})$  to get  $F(a, b)$  back.

$$F(a, b) = W(b^{-1}) \cup b^{-1}W(b).$$

## A Curiosity, Cont'd

Similarly, we can do this for  $a$ , giving a decomposition of  $F(a, b)$  in two separate ways:

$$\begin{aligned} F(a, b) &= b^{-1}W(b) \cup W(b^{-1}) \\ &= a^{-1}W(a) \cup W(a^{-1}). \end{aligned}$$



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We're able to take part of the group  $F(a, b)$ , take some translations, and, miraculously, obtain the entire group back.

## Paradoxical Decompositions of Groups

Let  $G$  be a group. A *paradoxical decomposition* of  $G$  consists of

- pairwise disjoint subsets  $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$ ; and
- elements  $g_1, \dots, g_n, h_1, \dots, h_m \in G$ ;

such that

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If  $G$  admits a paradoxical decomposition, we say  $G$  is *paradoxical*.

## Paradoxical Decompositions of Sets

If  $G$  acts on a set  $X$ , then a subset  $A \subseteq X$  is *G-paradoxical* if there exist

- pairwise disjoint subsets  $A_1, \dots, A_n, B_1, \dots, B_m \subseteq A$ ; and
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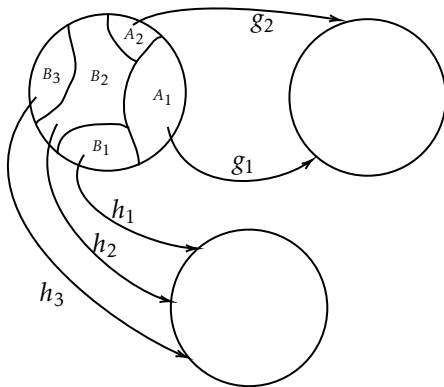
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A paradoxical group is a paradoxical set under the action of left-multiplication.

# Depiction



## Some Paradoxical Groups

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- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$ , where  $S$  is any nonempty set with more than two elements, is paradoxical.

## A Paradoxical Subgroup of $SO(3)$

The following two matrices (and their inverses) generate a subgroup of  $SO(3)$  that is isomorphic to  $F(a, b)$ .

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

Thus,  $SO(3)$  is paradoxical — can we use it to find a paradoxical decomposition?

## Introducing the Banach–Tarski Paradox

### Theorem (The Banach–Tarski Paradox)

*Let  $A$  and  $B$  be bounded subsets of  $\mathbb{R}^3$  with nonempty interior. There is a partition of  $A$  into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields  $B$ .*

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- In other words, not all subsets of  $\mathbb{R}^3$  have a definite “volume” invariant under isometry.

# Equidecomposability

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## Equidecomposability

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- finite partitions,  $A_1, \dots, A_n \subseteq A$ ,  $B_1, \dots, B_n \subseteq B$
- group elements  $g_1, \dots, g_n \in G$

such that  $g_i \cdot A_i = B_i$ , then we say  $A$  and  $B$  are  $G$ -*equidecomposable*.

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Effectively,  $A$  and  $B$  are “equal” to each other up to the group action.

If  $A$  is  $G$ -paradoxical, then so too is  $B$ .

# The Banach–Tarski Paradox: Proof Outline I

- 1 We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of  $\mathrm{SO}(3)$  isomorphic to  $F(a, b)$ .

## The Banach–Tarski Paradox: Proof Outline II

- ② We use the decomposition

$$\begin{aligned} F(a, b) &= a^{-1}W(a) \cup W(a^{-1}) \\ &= b^{-1}W(b) \cup W(b^{-1}) \end{aligned}$$

to duplicate the unit sphere in  $\mathbb{R}^3$ ,  $S^2$ , except for a countable subset  $D$ . (The *Hausdorff Paradox*.)

- ③ We show that  $S^2$  and  $S^2 \setminus D$  are  $\text{SO}(3)$ -equidecomposable — there is thus a paradoxical decomposition of  $S^2$ .
- ④ We show that the unit ball,  $B(0, 1) \subseteq \mathbb{R}^3$ , is paradoxical under the isometry group  $E(3)$ .

## The Banach–Tarski Paradox: Proof Outline III

- ⑤ Define a relation  $A \leq B$  if  $A$  is  $G$ -equidecomposable with a subset of  $B$ , and show that if  $A \leq B$  and  $B \leq A$ , then  $A$  and  $B$  are  $G$ -equidecomposable.
- ⑥ Show that  $A \subseteq \mathbb{R}^3$  is equidecomposable with a subset of  $B \subseteq \mathbb{R}^3$ .

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## Ill-Behaved Groups

- The way that our copy of  $F(a, b)$  helped “create” the Banach–Tarski paradox suggests that  $F(a, b)$  is a particularly ill-behaved group.



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## Ill-Behaved Groups

- The way that our copy of  $F(a, b)$  helped “create” the Banach–Tarski paradox suggests that  $F(a, b)$  is a particularly ill-behaved group.
- Let  $\nu: F(a, b) \rightarrow [0, 1]$  be a probability measure — we will show that  $\nu$  *cannot* be translation-invariant (i.e.,  $\nu(tE) = \nu(E)$  for all  $t \in F(a, b), E \subseteq F(a, b)$ ).

## Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant  $\nu$  exists. Taking

$$F(a, b) = W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}),$$

we have

$$1 = \nu(W(a)) + \nu(W(a^{-1})) + \nu(W(b)) + \nu(W(b^{-1}))$$

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# Amenability

Let  $G$  be a group. A *mean* is a finitely additive probability measure  $\nu: P(G) \rightarrow [0, 1]$  such that

$$\nu(tE) = \nu(E)$$

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- In other words,  $G$  is sufficiently “well-behaved.”

## Inheritance Properties of Amenability

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- If  $N \trianglelefteq G$  and  $G/N$  are amenable, then  $G$  is amenable.
- If  $(G_i, \varphi_i)_{i \in I}$  is a directed system of amenable groups, then the union  $G = \bigcup_{i \in I} G_i$  is amenable.

## Examples

- Finite groups are amenable: let  $\delta_t$  be the point mass at  $t \in G$ ,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

is a mean.

- Abelian groups are amenable.
- The free group,  $F(a, b)$ , is *not* amenable.

## Paradoxical Groups and Amenability

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### Theorem (Tarski's Theorem)

*Let  $G$  be a group. Then,  $G$  is non-paradoxical if and only if  $G$  is amenable.*

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### Theorem (Tarski's Theorem)

*Let  $G$  be a group. Then,  $G$  is non-paradoxical if and only if  $G$  is amenable.*

Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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## Why Find Alternative Characterizations?

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Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

## Normed Vector Spaces

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- homogeneity:  $\|\alpha v\| = |\alpha| \|v\|$  for all  $\alpha \in \mathbb{C}$ ;
- triangle inequality:  $\|v + w\| \leq \|v\| + \|w\|$ .

## A Normed Vector Space

The best example is that of  $\mathbb{R}^n$  or  $\mathbb{C}^n$  with the Euclidean norm,

$$\|x\| = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}$$



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However, we need a few more dimensions in order to get to where we're going.

## Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f: \Gamma \rightarrow \mathbb{C} \mid \sup_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f: \Gamma \rightarrow \mathbb{C} \mid \sum_{t \in \Gamma} |f(t)| < \infty \right\};$$

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They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t \in \Gamma} |f(t)|$ ;
- $\|f\|_{\ell_1} := \sum_{t \in \Gamma} |f(t)|$ ;
- $\|f\|_{\ell_2} := \left( \sum_{t \in \Gamma} |f(t)|^2 \right)^{1/2}$ .

# Linear Maps and Linear Functionals

A linear transformation  $T: V \rightarrow W$  is called *bounded* if

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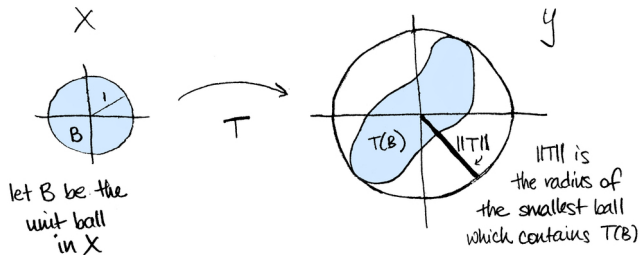
$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted  $\|T\|_{\text{op}}$ . A linear map is bounded if and only if it is continuous.

If  $W = \mathbb{C}$ , then we call  $T$  a *linear functional*.

# Operator Norm Pictorial Depiction

Courtesy of Tai-Danae Bradley.





## Positive Linear Functionals on $\ell_\infty(\Gamma)$

If  $\varphi: \ell_\infty(\Gamma) \rightarrow \mathbb{C}$  is a linear functional, we say  $\varphi$  is *positive* if, for any  $f \in \ell_\infty(\Gamma)$  with  $f \geq 0$ ,  $\varphi(f) \geq 0$ .

- It can be shown that  $\varphi$  is positive if and only if  $\varphi(\mathbf{1}_\Gamma) = \|\varphi\|_{\text{op}}$ .

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- It can be shown that  $\varphi$  is positive if and only if  $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}}$ . All positive linear functionals are automatically continuous.
- If  $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}} = 1$ , then we say  $\varphi$  is a *state*.

## Translations of $\ell_\infty(\Gamma)$

If  $f \in \ell_\infty(\Gamma)$ , we define the translation  $\lambda_s: \ell_\infty(\Gamma) \rightarrow \ell_\infty(\Gamma)$  by

$$\lambda_s(f)(t) = f(s^{-1}t)$$

for all  $t \in \Gamma$  and fixed  $s \in \Gamma$ .

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If  $\varphi: \ell_\infty(\Gamma) \rightarrow \mathbb{C}$  is a state such that  $\varphi(\lambda_s(f)) = \varphi(f)$  for all  $f \in \ell_\infty(\Gamma)$ , then we say  $\varphi$  is an *invariant state*.

## Invariant States and Means

Invariant states and means are interchangeable.

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If  $\varphi$  is an invariant state on  $\ell_\infty(\Gamma)$ , define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all  $E \subseteq \Gamma$ .

# Approximations and Amenability

There is actually one way that working with sets makes life easier.

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But what does “bigger” actually mean?

## Følner's Condition

### Theorem (Følner's Theorem)

*Let  $\Gamma$  be a countable, discrete group. Then,  $\Gamma$  is amenable if and only if there exists a sequence of finite subsets  $(F_n)_n$  such that*

$$\lim_{n \rightarrow \infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

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If  $\Gamma = \langle S \rangle$  for some finite generating set  $S$ , we only need to check for all  $s \in S$ .

## $\mathbb{Z}$ are Amenable

Consider  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  under addition.

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We can find a Følner sequence by defining

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Then, since  $\mathbb{Z}$  is generated by 1,

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Thus,  $\mathbb{Z}$  is amenable.



## Approximate Means

The Følner condition allows us to find an “approximate” version of a mean.

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Keeping  $\lambda_s(f)(t) = f(s^{-1}t)$ , if  $(f_k)_k \subseteq \ell_1(\Gamma)$  is such that

$$\lim_{k \rightarrow \infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say  $(f_k)_k$  is an *approximate mean*.

## Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|} \mathbb{1}_{F_k},$$

## Approximate Means, Cont'd

In the other direction, we arbitrarily approximate  $f \in \ell_1(\Gamma)$  with a “sufficient” finitely supported function  $g$ ,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

then use a “layer cake” decomposition to find our Følner sets:

$$g = \sum_{i=1}^n c_i \mathbb{1}_{F_i},$$

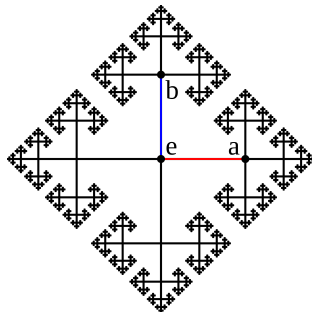
where  $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$ .

## Graphs and Amenability

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## Graphs and Amenability, cont'd

If  $S \subseteq V(G)$  is a subset of vertices of a graph  $G$ , the *neighbor vertex set*,  $N(S)$ , is the set of vertices in  $G$  that are adjacent to  $S$  (not including elements of  $S$ ).

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- This is proven with the Følner condition.

# Hilbert Spaces

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# Hilbert Spaces

If  $\mathcal{H}$  is a vector space, an *inner product* on  $\mathcal{H}$  is a map  $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  that satisfies

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If  $\mathcal{H}$  is complete with respect to this norm, we call  $\mathcal{H}$  a Hilbert space.



## Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces,  $T: \mathcal{H} \rightarrow \mathcal{H}$ , include a special structure called an adjoint that “plays nicely” with the inner product:

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then we call  $U$  a *unitary operator*. The space of unitary operators,  $\mathcal{U}(\mathcal{H})$ , is a group under composition.

# Representations

A map  $\lambda: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$

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All discrete groups are able to be unitarily represented by the trivial representation  $1_\Gamma: \Gamma \rightarrow \mathbb{C}$ , given by  $1_\Gamma(s) = 1$ .

## The Left-Regular Representation

As it turns out, the map  $\lambda_s(f)(t) = f(s^{-1}t)$  is a unitary operator on  $\ell_2(\Gamma)$ , where  $\lambda_s^* = \lambda_{s^{-1}}$ .

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This is because it “encodes” the group’s left-multiplication action, in the sense that  $\lambda_s(\delta_t) = \delta_{st}$ , where  $\delta_t$  is the point mass at  $t \in \Gamma$ .

# The Left-Regular Representation and Amenability

A sequence  $(f_k)_k \subseteq \ell_2(\Gamma)$  is known as an *almost-invariant vector* for  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$  if

$$\lim_{k \rightarrow \infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

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If  $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$  admits an almost-invariant vector, then  $\Gamma$  is amenable.

## Introduction to $C^*$ -Algebras

The space of *all* bounded linear operators,  $T: \mathcal{H} \rightarrow \mathcal{H}$ , written  $\mathbb{B}(\mathcal{H})$ , along with the norm  $\|\cdot\|_{\text{op}}$ , is a very special vector space.

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These make  $\mathbb{B}(\mathcal{H})$  a  $C^*$ -algebra. However, there are other  $C^*$ -algebras.

## A Group $C^*$ -Algebra

If  $\Gamma$  is a group, we may define a vector space,  $\mathbb{C}[\Gamma]$ , by finite sums

$$x = \sum_{t \in \Gamma} x(t) \delta_t,$$

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This becomes a  $*$ -algebra when endowed with multiplication (by convolution) and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g(s^{-1}t)$$
$$f^*(t) = \overline{f(t^{-1})}.$$



## A Group $C^*$ -Algebra, cont'd

If we represent  $\pi_\lambda: \mathbb{C}[\Gamma] \rightarrow \mathbb{B}(\ell_2(\Gamma))$  by mapping  $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$ , extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\text{op}},$$

we get the *reduced group  $C^*$ -algebra* on  $\Gamma$  (upon norm completion).

## Finite-Dimensional Approximations

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We can use these sufficient approximations to establish amenability.

## Nuclearity

A  $C^*$ -algebra,  $A$ , is called *nuclear* if there exist two sequences of maps,  $\varphi_n: A \rightarrow \text{Mat}_{k(n)}(\mathbb{C})$  and  $\psi_n: \text{Mat}_{k(n)}(\mathbb{C}) \rightarrow A$ , such that

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- Essentially, any  $a \in A$  is “close enough” to a certain family of finite-dimensional analogues.

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Specifically, by showing that the approximation of  $\frac{|sF_n \cap F_n|}{|F_n|} \rightarrow 1$  corresponds to the existence of maps  $\varphi_n: C_\lambda^*(\Gamma) \rightarrow \text{Mat}_{|F_n|}(\mathbb{C})$  and  $\psi_n: \text{Mat}_{|F_n|}(\mathbb{C}) \rightarrow C_\lambda^*(\Gamma)$  that satisfy

$$\|x - \psi_n \circ \varphi_n(x)\| \xrightarrow{n \rightarrow \infty} 0.$$

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- there is a sequence  $(f_k)_k \subseteq \ell_2(\Gamma)$  such that  $\|f_k - \lambda_s(f_k)\|_{\ell_2} \rightarrow 0$  (almost-invariant vectors);
- the reduced group  $C^*$ -algebra,  $C_\lambda^*(\Gamma)$ , is nuclear (nuclearity).

# Contents

- ① Definitions
- ② Paradoxical Decompositions
- ③ From Paradoxical Decompositions to Amenability
- ④ Equivalent Definitions and Other Criteria
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  - Introducing Approximations
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## Present-Day Programs of Research

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Additionally, other questions related to finite-dimensional approximations of  $C^*$ -algebras, such as quasidiagonality, were only resolved recently. A paper in 2015 proved that if  $\Gamma$  is amenable, then  $C_\lambda^*(\Gamma)$  is quasidiagonal (the reverse direction was shown in 1987).

# Acknowledgments

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- the professors of the math department;
- friends, family, and acquaintances both in the math major and outside;
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## References I

- [AB06] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis*. Third. A Hitchhiker's Guide. Springer, Berlin, 2006, pp. xxii+703. ISBN: 978-3-540-32696-0.
- [Alu09] Paolo Aluffi. *Algebra: Chapter 0*. Vol. 104. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xx+713. ISBN: 978-0-8218-4781-7. DOI: 10.1090/gsm/104. URL: <https://doi.org/10.1090/gsm/104>.
- [BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*. Vol. 11. New Mathematical Monographs. Cambridge University Press, Cambridge, 2008, pp. xiv+472. ISBN: 978-0-521-88720-5. DOI: 10.1017/CB09780511542749. URL: <https://doi.org/10.1017/CB09780511542749>.



## References II

- [Bla06] B. Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of  $C^*$ -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4. DOI: 10.1007/3-540-28517-2. URL: <https://doi.org/10.1007/3-540-28517-2>.
- [BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004, pp. xiv+716. ISBN: 0-521-83378-7. DOI: 10.1017/CB09780511804441. URL: <https://doi.org/10.1017/CB09780511804441>.

## References III

- [BO08] Nathanial P. Brown and Narutaka Ozawa.  *$C^*$ -algebras and finite-dimensional approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+509. ISBN: 978-0-8218-4381-9. DOI: 10.1090/gsm/088. URL: <https://doi.org/10.1090/gsm/088>.
- [CE78] Man-Duen Choi and Edward G. Effros. “Nuclear  $C^*$ -Algebras and the Approximation Property”. In: *American Journal of Mathematics* 100.1 (1978), pp. 61–79. ISSN: 00029327. URL: <http://www.jstor.org/stable/2373876> (visited on 02/07/2025).
- [DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. Third. John Wiley & Sons, Inc., Hoboken, NJ, 2004, pp. xii+932. ISBN: 0-471-43334-9.

## References IV

- [Enc25] The Editors of Encyclopaedia Britannica. *Ship of Theseus*. Accessed: 2025-02-06. 2025. URL: <https://www.britannica.com/topic/Ship-of-Theseus>.
- [Fol84] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984, pp. xiv+350. ISBN: 0-471-80958-6.
- [Hal06] Paul R. Halmos. “How to write mathematics”. In: *Butl. Soc. Catalana Mat.* 21.1 (2006). Translation of Enseignement Math. (2) **16** (1970), 123–152 [MR0277319], pp. 53–79, 158. ISSN: 0214-316X, 2013-9829.

## References V

- [Hal66] James D. Halpern. “Bases in vector spaces and the axiom of choice”. In: *Proc. Amer. Math. Soc.* 17 (1966), pp. 670–673. ISSN: 0002-9939,1088-6826. DOI: 10.2307/2035388. URL: <https://doi.org/10.2307/2035388>.
- [Har00] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000, pp. vi+310. ISBN: 0-226-31719-6.
- [Jec03] Thomas Jech. *Set theory*. millennium. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

## References VI

- [Jus22] Kate Juschenko. *Amenability of discrete groups by examples*. Vol. 266. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2022, pp. xi+165. ISBN: 978-1-4704-7032-6. DOI: 10.1090/surv/266. URL: <https://doi.org/10.1090/surv/266>.
- [Kes59a] Harry Kesten. “Full Banach Mean Values on Countable Groups”. In: *Mathematica Scandinavica* 7.1 (1959), pp. 146–156. ISSN: 00255521. URL: <http://www.jstor.org/stable/24489015> (visited on 02/05/2025).
- [Kes59b] Harry Kesten. “Symmetric Random Walks on Groups”. In: *Transactions of the American Mathematical Society* 92.2 (1959), pp. 336–354. ISSN: 00029947. URL: <http://www.jstor.org/stable/1993160> (visited on 02/05/2025).
- [Knu09] Søren Knudby. “The Banach-Tarski Paradox”. 2009.

## References VII

- [Löh17] Clara Löh. *Geometric group theory*. Universitext. An introduction. Springer, Cham, 2017, pp. xi+389. ISBN: 978-3-319-72253-5. DOI: 10.1007/978-3-319-72254-2. URL: <https://doi.org/10.1007/978-3-319-72254-2>.
- [Mon17] Mehdi Sangani Monfared. “Følner’s condition and expansion of Cayley graphs for group actions”. In: *New York Journal of Mathematics* 23 (Sept. 2017), pp. 1295–1306.
- [Mon13] Nicolas Monod. “Groups of piecewise projective homeomorphisms”. In: *Proc. Natl. Acad. Sci. USA* 110.12 (2013), pp. 4524–4527. ISSN: 0027-8424,1091-6490. DOI: 10.1073/pnas.1218426110. URL: <https://doi.org/10.1073/pnas.1218426110>.

## References VIII

- [Pau02] Vern Paulsen. *Completely bounded maps and operator algebras*. Vol. 78. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+300. ISBN: 0-521-81669-6.
- [Rai23] Timothy Rainone. “Functional Analysis-En Route to Operator Algebras”. 2023.
- [Rud73] Walter Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, pp. xiii+397.
- [Run02] Volker Runde. *Lectures on amenability*. Vol. 1774. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. xiv+296. ISBN: 3-540-42852-6. DOI: 10.1007/b82937. URL: <https://doi.org/10.1007/b82937>.

## References IX

- [Run05] Volker Runde. *A taste of topology*. Universitext. Springer, New York, 2005, pp. x+176. ISBN: 978-0387-25790-7.
- [Run20] Volker Runde. *Amenable Banach algebras*. Springer Monographs in Mathematics. A panorama. Springer-Verlag, New York, 2020, pp. xvii+462. ISBN: 978-1-0716-0351-2. DOI: 10.1007/978-1-0716-0351-2. URL: <https://doi.org/10.1007/978-1-0716-0351-2>.
- [Tak64] Masamichi Takesaki. “On the cross-norm of the direct product of  $C^*$ -algebras”. In: *Tohoku Math. J. (2)* 16 (1964), pp. 111–122. ISSN: 0040-8735,2186-585X. DOI: 10.2748/tmj/1178243737. URL: <https://doi.org/10.2748/tmj/1178243737>.



## References X

- [Tao09] Terence Tao. *245B, notes 2: Amenability, the ping-pong lemma, and the Banach-Tarski paradox (optional)*.  
<https://terrytao.wordpress.com/2009/01/08/245b-notes-2-amenability-the-ping-pong-lemma-and-the-banach-tarski-paradox-optional/>. 2009.
- [Tit72] J Tits. “Free subgroups in linear groups”. In: *Journal of Algebra* 20.2 (1972), pp. 250–270. ISSN: 0021-8693. DOI: [https://doi.org/10.1016/0021-8693\(72\)90058-0](https://doi.org/10.1016/0021-8693(72)90058-0). URL: <https://www.sciencedirect.com/science/article/pii/0021869372900580>.

## References XI

- [Wei80] Joachim Weidmann. *Linear operators in Hilbert spaces*. Vol. 68. Graduate Texts in Mathematics. Translated from the German by Joseph Szücs. Springer-Verlag, New York-Berlin, 1980, pp. xiii+402. ISBN: 0-387-90427-1.