Problem (Problem 1)**:** Let R be a Euclidean domain, $n \geq 2$ an integer.

(a) Use the proof of the Smith Normal Form to show that every matrix $A \in \mathrm{GL}_n(R)$ can be written as a product of elementary matrices $E_{ij}(\lambda)$, flip matrices F_{ij}, and a diagonal matrix D.

(b) Now show that the flip matrices can be eliminated from the product in (a), and one can assume that $D = \operatorname{diag}(d, 1, \dots, 1)$. That is, all diagonal entries of D except possibly the $(1, 1)$ entry are equal to 1.

(c) Deduce from (b) that $\mathrm{SL}_n(R)$ is generated by the elementary matrices $E_{ij}(\lambda)$.

Solution:

(a) Observe that a square matrix is in Smith normal form if and only if it is a diagonal matrix of the form $D = \operatorname{diag}(d_1, \dots, d_m, 0, \dots, 0)$ where $d_1 | d_2 | \cdots | d_m$. By the proof of the Smith normal form, we have that the matrix UAV in Smith normal form is the product of three invertible matrices, so it is invertible, meaning that it is necessarily diagonal with $d_1, \dots, d_n \in R^\times$. Since the inverse of any $E_{ij}(\lambda)$ is another matrix of the form $E_{ij}(\lambda)$, and the inverse of F_{ij} is itself, it follows that we may write any $A \in \mathrm{GL}_n(R)$ as

$$A = U^{-1} D V^{-1},$$

where U^{-1} and V^{-1} are collections of flips and $E_{ij}(\lambda)$ and D is a diagonal matrix with $d_1, \dots, d_n \in R^\times$.

(b) We start by computing F_{ij} in terms of elementary matrices acting on identity. Operating on a 2×2 matrix, we have

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &\xrightarrow{R_1 \mapsto R_1 + R_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &\xrightarrow{R_2 \mapsto R_2 - R_1} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\ &\xrightarrow{R_1 \mapsto R_1 + R_2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &\xrightarrow{R_2 \mapsto -R_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Letting $D_j(\lambda)$ denote the multiplication of the diagonal matrix in row j by λ, and replacing 1 and 2 with i and j, we find that this product is equal to $D_j(-1)E_{ij}(1)E_{ji}(-1)E_{ij}(1)$.

Now, if we let $D = \operatorname{diag}(d_1, \dots, d_n)$, we observe that

$$\begin{aligned} DE_{ij}(\lambda) &= \begin{cases} R_i & \mapsto d_i(R_i + \lambda R_j) \\ R_j & \mapsto d_j R_j \end{cases} \\ &= E_{ij}\left(d_i d_j^{-1} \lambda\right) D. \end{aligned}$$

Therefore, by using these procedures, we obtain a matrix of the form

$$A = \prod E_{ij}(\lambda) \operatorname{diag}(d_1, \dots, d_n).$$

Next, we will show that this diagonal matrix can be taken to be of the form $\operatorname{diag}(d_1 d_2 \cdots d_n, 1, \dots, 1)$. We show the 2×2 case.

$$\begin{aligned} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} &\xrightarrow{R_1 \mapsto R_1 + d_2^{-1} R_2} \begin{pmatrix} d_1 & 1 \\ 0 & d_2 \end{pmatrix} \\ &\xrightarrow{R_2 \mapsto R_2 + (1 - d_2) R_1} \begin{pmatrix} d_1 & 1 \\ d_1 - d_1 d_2 & 1 \end{pmatrix} \end{aligned}$$

$$\xrightarrow{C_1 \mapsto (d_1 d_2 - d_1) C_2 + C_1} \begin{pmatrix} d_1 d_2 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_1 \mapsto R_1 - R_2} \begin{pmatrix} d_1 d_2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, this diagonal matrix is equal to a diagonal matrix of the desired form multiplied by a family of $E_{ij}(\lambda)$. Inductively, we apply this to the entirety of D to obtain our desired result.

(c) Observe that the determinant of A is equal to $d_1 \cdots d_n$, since the $E_{ij}(\lambda)$ are all upper-triangular or lower-triangular. In particular, this means that if $A \in \mathrm{SL}_n(R)$, then $\det(A) = 1$, so that $d_1 \cdots d_n = 1$, and the diagonal matrix in part (b) is in fact the identity matrix, so A is the product of $E_{ij}(\lambda)$. Similarly, a matrix that is the product of $E_{ij}(\lambda)$ is in $\mathrm{SL}_n(R)$, so $\mathrm{SL}_n(R)$ is generated by matrices of the form $E_{ij}(\lambda)$.

Problem (Problem 2)**:** Let R be a Euclidean domain, let $k, n \in \mathbb{N}$, and let $i \leq \min(k, n)$. Given a matrix $A \in \mathrm{Mat}_{k,n}(R)$, define $d_i(A)$ to be the greatest common divisor of all $i \times i$ minors of A. Prove that $d_i(PAQ) = d_i(A)$ for all $P \in \mathrm{GL}_k(R)$ and $Q \in \mathrm{GL}_n(R)$.

Solution: Since P and Q are invertible $k \times k$ and $n \times n$ matrices respectively, it follows from Problem 1 that we may write P and Q as

$$P = \left(\prod_{\alpha=1}^{\alpha_p} E_{i_\alpha j_\alpha}(\lambda_\alpha)\right)(\mathrm{diag}(d_p, 1, \dots, 1))$$
$$Q = (\mathrm{diag}(d_q, 1, \dots, 1))\left(\prod_{\beta=1}^{\beta_q} E_{i_\beta j_\beta}(\lambda_\beta)\right),$$

where we used the fact that diagonal matrices commute with all other matrices if R is commutative. Furthermore, since P and Q are commutative, d_p and d_q are units. We observe now that

$$PAQ = \left(\prod_{\alpha=1}^{\alpha_p} E_{i_\alpha j_\alpha}(\lambda_\alpha)\right)(\mathrm{diag}(d_p, 1, \dots, 1) A \,\mathrm{diag}(d_q, 1, \dots, 1))\left(\prod_{\beta=1}^{\beta_q} E_{i_\beta j_\beta}(\lambda_\beta)\right).$$

Focusing on the product in the middle, we find that it multiplies the first column of A by d_q and the first row of A by d_p; in particular, it does not affect any of the $i \times i$ minors of A (up to associates). Additionally, since each of the $E_{ij}(\lambda)$ are simply linear combinations of the columns and rows of A respectively, they do not affect the greatest common divisor of any of the $i \times i$ minors of A, meaning that $d_i(A) = d_i(PAQ)$.

Problem (Problem 3)**:** Let R be a commutative ring with 1.

(a) Let C be an R-algebra, and $A, B \subseteq C$ R-subalgebras that commute with each other; that is, $ab = ba$ for any $a \in A$ and $b \in B$. Prove that there is an R-algebra homomorphism $\varphi\colon A \otimes B \to C$ such that $\varphi(a \otimes b) = ab$ for each $a \in A$ and $b \in B$.

(b) Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as rings.

(c) Now assume that R is a field, and let A be a finite-dimensional R-algebra. Prove that $A \otimes A$ cannot be a field unless $\dim(A) = 1$.

Solution:

(a) Let $\phi\colon A \times B \to C$ be defined by $(a, b) \mapsto ab$. Then, ϕ is an R-bilinear map, so it induces a unique linear map on the tensor product $\varphi\colon A \otimes B \to C$. We claim that this map is compatible with the R-algebra structure of $A \otimes B$.

To see this, observe that if $a_1, a_2 \in A$ and $b_1, b_2 \in B$, then

$$\begin{aligned} \varphi((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \varphi(a_1 a_2 \otimes b_1 b_2) \\ &= a_1 a_2 b_1 b_2 \end{aligned}$$

$$
\begin{aligned}
&= a_1 b_1 a_2 b_2 \\
&= \varphi(a_1 \otimes b_1)\varphi(a_2 \otimes b_2).
\end{aligned}
$$

This gives our desired R-algebra homomorphism.

(b) We observe that both $\mathbb{R}$ and $\mathbb{Z}[i]$ are $\mathbb{Z}$-subalgebras of $\mathbb{C}$. Therefore, from above, we have a $\mathbb{Z}$-algebra homomorphism

$$
\begin{aligned}
\varphi \colon \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] &\to \mathbb{C} \\
t \otimes (a + bi) &\mapsto ta + tbi.
\end{aligned}
$$

To see that this map is injective, observe that $ta + tbi = 0$ if and only if $ta = 0$ and $tb = 0$, meaning either that $t = 0$ or $a, b = 0$; in either case, the corresponding element of the tensor product is the zero tensor. As for surjectivity, if we have $x + yi \in \mathbb{C}$, then we may find the element $x \otimes 1 + y \otimes i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ that maps to $x + yi$. Since this is a bijective $\mathbb{Z}$-algebra homomorphism, it follows that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as $\mathbb{Z}$-algebras, hence as rings.

(c) Suppose A is an R-algebra such that $A \otimes_R A$ is a field. Then, $A \otimes_R A$ is generated by $1 \otimes 1$. Now, consider the subalgebra $N = \{\lambda 1 \mid \lambda \in R\}$. Then, we see that $N \otimes_R A$ is also generated by $1 \otimes 1$, so it has the same dimension as A, and N commutes with A since it consists of scalar multiples of 1. This means that $N \otimes_R A$ admits a homomorphism of R-algebras

$$
\begin{aligned}
\varphi \colon N \otimes_R A &\to A \\
\lambda 1 \otimes a &\mapsto \lambda a.
\end{aligned}
$$

This homomorphism is surjective, though, meaning that $\dim_R(A) \leq 1$, so $\dim_R(A) = 1$.

Problem (Problem 4)**:**

(a) Prove that $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $B = \mathbb{C} \times \mathbb{C}$ are isomorphic as $\mathbb{C}$-algebras.

(b) Explain why $\{1 \otimes 1, 1 \otimes i\}$ is a basis for A over $\mathbb{C}$. Compute $\varphi(1 \otimes 1)$ and $\varphi(1 \otimes i)$, where $\varphi \colon A \to B$ is the isomorphism from (a).

(c) Prove that there exist precisely 2 $\mathbb{C}$-algebra isomorphisms from A to B.

Solution:

(a) We identify $\mathbb{C} \cong \mathbb{R}[x]/\langle x^2 + 1 \rangle$, where we use angle brackets instead of parentheses because we will be using a lot of them. To see this, we observe that by Euclidean division, we have, for any $p(x) \in \mathbb{R}[x]$,

$$
p(x) = q(x)\left(x^2 + 1\right) + bx + a.
$$

In particular, we may then define the map $\varphi \colon \mathbb{R}[x]/\langle x^2 + 1 \rangle \to \mathbb{C}$ taking

$$
bx + a + \langle x^2 + 1 \rangle \mapsto a + bi.
$$

This is a linear map, and we observe then that

$$
\begin{aligned}
\left(bx + a + \langle x^2 + 1 \rangle\right)\left(dx + c + \langle x^2 + 1 \rangle\right) &= bdx^2 + (bc + ad)x + ac + \langle x^2 + 1 \rangle \\
&= (bc + ad)x + (ac - bd) + \langle x^2 + 1 \rangle,
\end{aligned}
$$

so this is an algebra homomorphism.

By the Chinese remainder theorem, we then have

$$
\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \otimes_{\mathbb{R}} \left(\mathbb{R}[x]/\langle x^2 + 1 \rangle\right) \\
&\cong \mathbb{C}[x]/\langle x^2 + 1 \rangle
\end{aligned}
$$

$$\begin{aligned} &\cong \frac{\mathbb{C}[x]}{\langle x-i\rangle} \times \frac{\mathbb{C}[x]}{\langle x+i\rangle} \\ &\cong \mathbb{C}\times\mathbb{C} \end{aligned}$$

are isomorphic as rings, where the isomorphism $\mathbb{C}[x]/\langle x \pm i\rangle \cong \mathbb{C}$ is given by evaluation at $x = \mp i$ respectively. In particular, we observe that this is a homomorphism of $\mathbb{C}$-algebras, since the evaluation homomorphism is an algebra homomorphism.

(b) Identifying $i \leftrightarrow x + \left\langle x^2+1\right\rangle$ in $\mathbb{R}[x]/\left(x^2+1\right)$, we observe that $1\otimes 1$ and $1 \otimes i$ are necessarily linearly independent, and since $\mathbb{C}$ is a dimension 2 $\mathbb{R}$-algebra, it follows that $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ is a dimension 2 $\mathbb{C}$-algebra. Computing from the definitions, we have

$$\begin{aligned} \varphi(1\otimes 1) &= (1,1) \\ \varphi(1\otimes i) &= (i,-i). \end{aligned}$$

We observe that the right-hand-side constitutes a basis, since we recover the standard basis for $\mathbb{C}\times\mathbb{C}$ by taking

$$\begin{aligned} (1,0) &= \frac{1}{2}((1,1) - i(i,-i)) \\ (0,1) &= \frac{1}{2}((1,1) + i(i,-i)). \end{aligned}$$

(c) We observe that $\mathbb{C}\times\mathbb{C}$ has a nontrivial automorphism given by $(1,0)\leftrightarrow(0,1)$, so there are at least two automorphisms (including the trivial automorphism). Additionally, the restrictions on $1\otimes 1$ and $1\otimes i$ are such that $1\otimes 1$ must map to the multiplicative identity of $\mathbb{C}\times\mathbb{C}$ as $1\otimes 1$ is the multiplicative identity of $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$, and $\varphi(1\otimes i)$ must have (exponential) order 4 in $\mathbb{C}\times\mathbb{C}$ and map to an element that is linearly independent from $(1,1)$. The set of order 4 elements are $(i,i),(i,-i),(-i,i),(-i,-i)$. Of these, only $(i,-i)$ and $(-i,i)$ are $\mathbb{C}$-linearly independent from $(1,1)$, so there are at most 2 choices for where $1\otimes i$ can be mapped to in $\mathbb{C}\times\mathbb{C}$.

Problem (Problem 5)**:** Let V and W be finite-dimensional vector spaces over F, with $\{v_1,\dots,v_n\}$ a basis for V and $\{w_1,\dots,w_m\}$ a basis for W. Let $\varphi\colon V\otimes W\to \operatorname{Mat}_{n,m}(F)$ be given by $\varphi(v_i\otimes w_j) = e_{ij}$, where e_{ij} is the matrix unit whose (i,j) entry is 1 and all other entries are 0.

(a) Prove that for a matrix $A\in\operatorname{Mat}_{n,m}(F)$, the following are equivalent:

(i) $A = \varphi(v\otimes w)$ for some elements $v\in V$ and $w\in W$;

(ii) $\operatorname{rk}(A)\leq 1$.

(b) Let $A\in\operatorname{Mat}_{n,m}(F)$. Prove that $\operatorname{rk}(A)$ is the smallest d such that $\varphi^{-1}(A)$ can be written as a sum of d simple tensors.

Solution:

(a) Suppose that $A = \varphi(v\otimes w)$ for some $v\in V$ and $w\in W$. We may write

$$\begin{aligned} v\otimes w &= \sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j e_i \otimes e_j \\ A &= \sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j e_{ij}. \end{aligned}$$

Then, using the identity

$$e_{ij}(e_k) = \delta_{jk} e_i,$$

where δ_{jk} denotes the Kronecker delta, we get that for an arbitrary vector

$$x = \sum_{k=1}^{m} r_k e_k$$

in F^m, we have

$$\begin{aligned} Ax &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j e_{ij}\right)\left(\sum_{k=1}^{m} r_k e_k\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} s_i t_j r_k e_{ij}(e_k) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} s_i t_j r_k \delta_{jk} e_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} t_j r_j s_i e_i \\ &= \sum_{j=1}^{m} t_j r_j \left(\sum_{i=1}^{n} s_i e_i\right) \\ &\in \operatorname{span}\left\{\sum_{i=1}^{n} s_i e_i\right\}. \end{aligned}$$

Therefore, $\operatorname{rk}(A) \leq 1$.

If $\operatorname{rk}(A) = 0$, then $v \otimes w$ is the zero tensor since φ is an isomorphism. Else, we assume $\operatorname{rk}(A) = 1$. Then, there are some coefficients $s_1, \dots, s_n$ such that

$$\operatorname{im}(A) = \operatorname{span}\left\{\sum_{i=1}^{n} s_i e_i\right\}.$$

Now, let

$$x = \sum_{k=1}^{m} r_k e_k.$$

We may then define

$$w = \sum_{j=1}^{m} t_j e_j$$

to be such that

$$\sum_{j=1}^{m} t_j r_j = c,$$

so that

$$\varphi\left(\left(\sum_{i=1}^{n} s_i e_i\right) \otimes \left(\sum_{j=1}^{m} t_j e_j\right)\right)\left(\sum_{k=1}^{m} r_k e_k\right) = c\sum_{i=1}^{n} s_i e_i.$$

Thus, we find $v \otimes w$ such that $A = \varphi(v \otimes w)$.

(b) From the Smith normal form, we may write

$$\operatorname{rk}(A) = \operatorname{rk}(PAQ)$$

where Q is a change of basis matrix for V taking $v_i \mapsto v_i'$ and P is a change of basis matrix for W taking $w_j \mapsto w_j'$. Define $PAQ = D$, where D is a diagonal matrix with the number of entries along the diagonal equal to its rank. We observe then that a redefined map $\psi\colon V \otimes W \to \operatorname{Mat}_{m,n}(F)$ taking $v_i' \otimes w_j' \mapsto e_{ij}$ thus has

$$\begin{aligned}\varphi^{-1}(A) &= \psi^{-1}(D)\\ &= \sum_{k=1}^{\operatorname{rk}(D)} v_i' \otimes w_i'\\ &= \sum_{k=1}^{\operatorname{rk}(A)} v_i' \otimes w_i'.\end{aligned}$$

Thus, we see that, since $\operatorname{rk}(A) = \operatorname{rk}(D)$, we have that $\operatorname{rk}(A)$ is the minimum number of elements of $V \otimes W$ necessary to write $\varphi^{-1}(A)$ as a sum of simple tensors.

Problem (Problem 6)**:** Let R be a ring with 1, and let M be a left R-module, N a submodule. Prove that M is Noetherian if and only if N and M/N are both Noetherian.

Solution: Suppose M is a Noetherian module. Then, any submodule of M is finitely generated, so since any submodule of N is a submodule of M, N is Noetherian. Similarly, since any submodule of M/N corresponds to a submodule of M that contains N by the Fourth Isomorphism Theorem, it follows that M/N is also Noetherian.

Now, suppose M is a module such that M/N and N are Noetherian. Let $P_1 \leq P_2 \leq \cdots$ be an ascending chain of submodules for M. Then, $P_1 \cap N \leq P_2 \cap N \leq \cdots$ is an ascending chain of submodules of N, so there is some index k_1 such that $P_{k_1+i} = P_{k_1}$ for all $i \in \mathbb{N}$. Similarly, the set of submodules $P_1 + N \leq P_2 + N \leq \cdots$ is an ascending chain of submodules that contains N, so the submodules $(P_1 + N)/N \leq (P_2 + N)/N \leq \cdots$ forms an ascending chain of submodules in M/N, so there is some index k_2 such that $P_{k_2+i} = P_{k_2}$ for all $i \in \mathbb{N}$. In particular, this means that for all $i \in \mathbb{N}$, $P_{k+i} = P_k$, where $k = \max(k_1, k_2)$, so M is Noetherian.