Contents

Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and C^*-algebras. This section of notes will be focused on the essential results in functional analysis, starting from normed vector spaces, working our way up through locally convex topological vector spaces and the Krein–Milman theorem.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

Normed Vector Spaces

Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over $\mathbb{C}$. Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

Definition (Vector Space)**.** A vector space V is a set closed under two operations

$$a : V \times V \to V,\ (v_1, v_2) \mapsto v_1 + v_2$$
$$m : \mathbb{C} \times V \to V,\ (\lambda, v) \mapsto \lambda v.$$

We refer to a as addition, and m as scalar multiplication; $(V, +)$ is an abelian ring.

Definition (Norm)**.** A norm is a function

$$\|\cdot\| : V \to \mathbb{R}^+,\ x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness: $\|v\| = 0$ if and only if $v = 0_V$.
- Triangle inequality: $\|v + w\| \leqslant \|v\| + \|w\|$.
- Absolute Homogeneity: $\|\lambda v\| = |\lambda| \|v\|$, for $\lambda \in \mathbb{C}$.

If a function $p : V \to \mathbb{R}^+$ satisfies the triangle inequality and absolute homogeneity, we say p is a seminorm.

We say the pair $(V, \|\cdot\|)$ is a normed vector space.

Definition (Balls and Spheres)**.** Let X be a normed vector space, $x \in X$, and $\delta > 0$. Then,

$$U(x, \delta) = \{y \in X \mid d(x, y) < \delta\}$$
$$B(x, \delta) = \{y \in X \mid d(x, y) \leqslant \delta\}$$
$$S(x, \delta) = \{y \in X \mid d(x, y) = \delta\}.$$

For a normed vector space, we will use the following conventions for common sets:

$$U_X = U(0, 1)$$
$$B_X = B(0, 1)$$
$$S_X = S(0, 1)$$
$$\mathbb{D} = U_{\mathbb{C}}$$
$$\mathbb{T} = S_{\mathbb{C}}.$$

Definition (Equivalent Norms)**.** Two norms on V, $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there are two constants C_1 and C_2 such that

$$\|v\|_a \leqslant C_1 \|v\|_b$$
$$\|v\|_b \leqslant C_2 \|v\|_a$$

for all $v \in V$. We say $\|\cdot\|_a \sim \|\cdot\|_b$.

Examples

Example (Finite-Dimensional Vector Spaces)**.** The vector space $\mathbb{C}^n$ is with the p-norm is denoted ℓ_p^n, where for $p \in [1, \infty]$, the p-norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

In the case with $p = 2$, this gives the traditional Euclidean norm, and with $p = \infty$, this gives the sup norm:

$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|.$$

Example (A Sequence Space)**.** We let $\ell_p = \left\{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\right\}$ be the collection of sequences in $\mathbb{C}$ with finite p-norm. Here,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

In the case with $p = \infty$, this gives the sequence space ℓ_∞, which has norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Example (A Function Space)**.** We let $\ell^\infty(\Omega)$ denote the set of all bounded functions $f : \Omega \to \mathbb{C}$, equipped with the norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|.$$

If $\Omega = (\Omega, \mathcal{M}, \mu)$ is a measure space, then we let $L^\infty(\Omega)$ be the space of μ-a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

Series Convergence and Completeness

Proposition (Criteria for Banach Spaces)**:** Let X be a normed vector space. The following are equivalent:

(i) X is a Banach space.[I]

(ii) If $(x_k)_k$ is a sequence of vectors such that $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

(iii) If $(x_k)_k$ is a sequence in X such that $\|x_k\| < 2^{-k}$, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. To show (i) implies (ii), for $n > m > N$, we have

$$\begin{aligned} \|s_n - s_m\| &= \left\|\sum_{k=m+1}^{n} x_k\right\| \\ &\leqslant \sum_{k=m+1}^{n} \|x_k\| \\ &< \epsilon, \end{aligned}$$

implying that s_n is Cauchy, and thus converges since X is complete.

[I] Complete normed vector space.

Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let $(x_n)_n$ be a Cauchy sequence in X. We only need construct a convergent subsequence in order to show that $(x_n)_n$ converges.

Chose $n_1 \in \mathbb{N}$ such that for $n, m \geqslant n_1$, $\|x_m - x_n\| < \frac{1}{2^2}$, and inductively define $n_j > n_{j-1}$ such that $n, m \geqslant n_j$ implies $\|x_m - x_n\| < \frac{1}{2^{j+1}}$.

Let $y_1 = x_{n_1}$, $y_j = x_{n_j} - x_{n_{j-1}}$. Then,

$$\begin{aligned} \left\|y_j\right\| &= \left\|x_{n_j} - x_{n_{j-1}}\right\| \\ &< \frac{1}{2^j}, \end{aligned}$$

so $\sum_{j=1}^{\infty} y_j$ converges by our assumption. By telescoping, we see that $\sum_{j=1}^{k} y_j = x_{n_k}$, so $(x_{n_k})_k$ converges. □

Quotient Spaces

Let X be a normed vector space. Then, for $E \subseteq X$ a subspace, there is a quotient space X/E with the projection map $\pi : X \to X/E$, $x \mapsto x + E$. We want to make X/E into a normed space — in order to do this, we use the distance function:

$$\operatorname{dist}_E(x) = \inf_{y \in E} d(x, y),$$

which is uniformly continuous. For E closed, then $\operatorname{dist}_E(x) = 0$ if and only if $x \in E$.

Proposition (Quotient Space Norm)**:** Let X be a normed vector space, and $E \subseteq X$ a subspace. Set

$$\|x + E\|_{X/E} = \operatorname{dist}_E(x).$$

Then,

(1) $\|\cdot\|_{X/E}$ is a well-defined seminorm on X/E.

(2) If E is closed, then $\|\cdot\|_{X/E}$ is a norm on X/E.

(3) $\|x + E\|_{X/E} \leqslant \|x\|$ for all $x \in X$.

(4) If E is closed, then $\pi : X \to X/E$ is Lipschitz.

(5) If X is a Banach space and E is closed, then X/E is also a Banach space.

Proof.

(1) We will show that $\|\cdot\|_{X/E}$ is well-defined. If $x + E = x' + E$, $x' - x \in E$, so for every $y \in E$, $x' - x + y \in E$. Thus,

$$\begin{aligned} \|x - y\| &= \|x' - (x' - x + y)\| \\ &\geqslant \inf_{z \in E} \|x' - z\| \\ &= \|x' + E\|_{X/E}. \end{aligned}$$

Thus, $\|x + E\|_{X/E} \geqslant \|x' + E\|_{X/E}$, and vice versa.

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and $x \in X$. Then,

$$\|\lambda(x + E)\|_{X/E} = \|\lambda x + E\|_{X/E}$$

$$\begin{aligned} &= \inf_{y\in E} \|\lambda x - y\| \\ &= |\lambda| \inf_{y\in E} \left\|x - \lambda^{-1}y\right\| \\ &= |\lambda| \inf_{y'\in E} \|x - y\| \\ &= |\lambda| \|x + E\|_{X/E} \end{aligned}$$

Given $x, x' \in X$ and a fixed $\varepsilon > 0$, we have

$$\|x + E\| + \frac{\varepsilon}{2} > \|x - y\|$$

for some $y \in E$, and

$$\|x' + E\| + \frac{\varepsilon}{2} > \|x' - y'\|$$

for some $y' \in E$. Thus,

$$\begin{aligned} \|(x + x') - (y + y')\| &\leqslant \|x - y\| + \|x' - y'\| \\ &< \varepsilon + \|x + E\| + \|x' + E\|. \end{aligned}$$

Since $y + y' \in E$, we have

$$\begin{aligned} \|(x + E) + (x' + E)\|_{X/E} &= \|x + x' + E\|_{X/E} \\ &\leqslant \|(x + x') - (y + y')\| \\ &< \varepsilon + \|x + E\|_{X/E} + \|x' + E\|_{X/E}, \end{aligned}$$

meaning

$$\|(x + E) + (x' + E)\| \leqslant \|x + E\| + \|x' + E\|.$$

(2) If E is closed, and $\|x + E\| = 0$, then $x \in E$ so $x + E = 0_{X/E}$.

(3) For $x \in X$,

$$\begin{aligned} \|x + E\|_{X/E} &= \inf_{y\in E} \|x - y\| \\ &\leqslant \|x\|. \end{aligned}$$

(4) We have

$$\begin{aligned} \|(x + E) - (x' + E)\|_{X/E} &= \|x - x' + E\|_{X/E} \\ &\leqslant \|x - x'\|. \end{aligned}$$

(5) Let X be complete and $E \subseteq X$ be closed. Let $(x_k + E)_k$ be a sequence in X/E with $\|x_k + E\| < 2^{-k}$. We want to show that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

For each k, since $\|x_k + E\| < 2^{-k}$, there exists $y_k \in E$ such that $\|x_k - y_k\| < 2^{-k}$. Since X is complete, $\sum_{k=1}^{\infty} x_k - y_k$ converges.

Let $\left(\sum_{k=1}^{n} x_k - y_k\right)_n \to x$ in X. Applying the canonical projection map, π, to both sides, we get

$$\sum_{k=1}^{n} (x_k + E) = \sum_{k=1}^{n} \pi(x_k)$$

$$\begin{aligned} &= \pi\left(\sum_{k=1}^{n}(x_k - y_k)\right)\\ &\to \pi(x), \end{aligned}$$

implying that $\sum_{k=1}^{\infty}(x_k + E)$ converges.

□

Exercise: Consider ℓ_∞ and its closed subspace c_0. If $\pi : \ell_\infty \to \ell_\infty/c_0$ denotes the canonical quotient map, with $(z_k)_k \in \ell_\infty$, show that

$$\|(z_k)_k + c_0\| = \limsup_{k\to\infty}|z_k|$$

Solution: Let $z = (z_k)_k \in \ell_\infty$. We define the distance

$$\operatorname{dist}_{c_0}(z) = \inf_{t\in c_0}|z_k - t_k|.$$

Let $w \in c_c$ be defined by

$$w = (z_1, z_2, \dots, z_{n-1}, 0, 0, \dots).$$

Then,

$$\begin{aligned} \|z-w\|_\infty &= \sup_{k\in\mathbb{N}}|z_k - w_k|\\ &= \sup_{k\geqslant n}|z_k - w_k|, \end{aligned}$$

meaning that

$$\operatorname{dist}_{c_c}(z) \leqslant \sup_{k\geqslant n}|z_k|.$$

Since $c_0 \supseteq c_c$, we have

$$\begin{aligned} \operatorname{dist}_{c_0}(z) &\leqslant \operatorname{dist}_{c_c}(z)\\ &\leqslant \inf_{n\geqslant 1}\left(\sup_{k\geqslant n}|z_k|\right)\\ &= \limsup_{k\to\infty}|z_k|. \end{aligned}$$

Now, we show that $\limsup_{k\to\infty}|z_k| \leqslant \operatorname{dist}_{c_c}(z)$. Given $\varepsilon > 0$, there exists $w \in c_c$ such that

$$\|z-w\| < \operatorname{dist}_{c_c}(z) + \varepsilon.$$

Additionally, for w that terminates at $n-1$ (i.e., is equal to 0 for all $k \geqslant n$), we have

$$\sup_{k\geqslant n}|z_k - w_k| \leqslant \sup_{k\in\mathbb{N}}|z_k - w_k|,$$

meaning

$$\begin{aligned} \limsup_{k\to\infty}|z_k| &= \inf_{n\geqslant 1}\left(\sup_{k\geqslant n}|z_k|\right)\\ &\leqslant \sup_{k\geqslant n}|z_k - w_k|\\ &\leqslant \sup_{k\in\mathbb{N}}|z_k - w_k|\\ &= \|z-w\|\\ &< \operatorname{dist}_{c_c}(z) + \varepsilon, \end{aligned}$$

implying that

$$\limsup_{k\to\infty} |z_k| = \operatorname{dist}_{c_c}(z).$$

For $\varepsilon > 0$, let $w \in c_0$ be such that

$$\|z - w\| < \operatorname{dist}_{c_0}(z) + \varepsilon/2.$$

Additionally, let $\lambda \in c_c$ such that $\|\lambda - w\| < \varepsilon/2$. Then, we have

$$\begin{aligned}\operatorname{dist}_{c_0}(z) + \varepsilon &> \|z - \lambda\| + \|\lambda - w\|\\ &\geqslant \operatorname{dist}_{c_c}(z) + \varepsilon/2\\ &\geqslant \limsup_{k\to\infty} |z_k|.\end{aligned}$$

Thus, $\limsup_{k\to\infty} |z_k| \leqslant \operatorname{dist}_{c_0}(z)$, meaning $\limsup_{k\to\infty} |z_k| = \operatorname{dist}_{c_0}(z)$.

Bounded Linear Operators

Definition (Continuous Functions)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is called Lipschitz if there is a constant $C > 0$ such that

$$d_Y(f(x), f(x')) \leqslant C d_x(x, x')$$

for all $x, x' \in X$.

If $C \leqslant 1$, a Lipschitz map is known as a contraction.

If

$$d_Y(f(x), f(x')) = d_X(x, x')$$

for all $x, x' \in X$, then f is known as an isometry.

Proposition (Categorization of Continuous Linear Maps)**:** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map. The following are equivalent:

(i) T is continuous at 0.

(ii) T is continuous.

(iii) T is uniformly continuous.

(iv) T is Lipschitz.

(v) There exists a constant $C > 0$ such that $\|T(x)\| \leqslant C\|x\|$ for all $x \in X$.

Definition (Bounded Linear Operator)**.** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map.

(1) T is bounded if $T(B_X)$ is bounded in Y. Equivalently, T is bounded if and only if

$$\sup_{x\in B_X} \|T(x)\| < \infty,$$

or that $\exists r > 0$ such that $T(B_X) \subseteq B_Y(0, r)$.

(2) The operator norm of T is the value

$$\|T\|_{op} = \sup_{x\in B_X} \|T(x)\|.$$

Lemma: Let $T : X \to Y$ be a linear map between normed vector spaces. Then,

$$\|T\|_{op} = \sup_{x \in S_X} \|T(x)\|$$

and for all $x \in X$,

$$\|T(x)\| \leqslant \|T\|_{op} \|x\| .$$

Lemma: Let $T : X \to Y$ be a bounded linear map between normed vector spaces. Then, for any $x \in X$ and $r > 0$,

$$r \|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|$$

Proof. Let $C = \sup_{y \in B(x,r)} \|T(y)\|$. If $z \in B(0, r)$, then $z + x, z - x \in B(x, r)$, meaning

$$2T(z) = T(z + x) + T(z - x),$$

so by the triangle inequality, we get

$$\begin{aligned} 2\|T(z)\| &\leqslant \|T(z + x)\| + \|T(z - x)\| \\ &\leqslant 2\max\{\|T(z + x)\|, \|T(z - x)\|\} \\ &\leqslant 2C. \end{aligned}$$

Thus,

$$\|T(z)\| \leqslant \sup_{y \in B(x,r)} \|T(y)\|,$$

meaning

$$r \|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\| .$$

□

Remark: For a linear map $T : X \to Y$, the following are equivalent:
(1) T is continuous.
(2) T is bounded.
(3) $\|T\|_{op} < \infty$.

Definition. Let X and Y be normed spaces, $T : X \to Y$ a linear map.

(1) T is bounded below if there exists C_2 such that $\|T(x)\| \geqslant C_2 \|x\|$ for all $x \in X$.

(2) T is bicontinuous if T is bounded and bounded below.

$$C_2 \|x\| \leqslant \|T(x)\| \leqslant C_1 \|x\|$$

(3) T is a bicontinuous isomorphism if T is bijective, linear, and bicontinuous. We say X and Y are bicontinuously isomorphic.

(4) We say T is an isometric isomorphism if T is bijective, linear, and an isometry.

Example. Let ρ be the continuous surjective wrapping function $\rho : [0, 2\pi] \to \mathbb{T}$, $\rho(t) = e^{it}$. There is an induced isometry

$$T_\rho : C(\mathbb{T}) \to C([0, 2\pi]),$$

defined by $T_\rho(f)(t) = f \circ \rho(t) = f\left(e^{it}\right)$.

The range of T_ρ is $C = \{G \in C([0, 2\pi]) \mid g(0) = g(2\pi)\}$, which means that $C(\mathbb{T})$ and C are isometrically isomorphic Banach spaces.

Proposition: Let X and Y be normed spaces, and $T : X \to Y$ be a linear map. The following are equivalent.

(i) T is bicontinuous.

(ii) $T : X \to \operatorname{Ran}(T)$ is a linear isomorphism and homeomorphism.

Proof. Let T be bicontinuous. Then, T is linear, injective, and surjective onto $\operatorname{Ran}(T)$. Since T is continuous, T is bounded. Let $S : \operatorname{Ran}(T) \to X$ be defined by $S(T(x)) = x$. We can see that S is well-defined, since $T : X \to \operatorname{Ran}(T)$ is surjective, and so has a left inverse. Similarly, since $\|S(T(x))\| = \|x\| \leqslant \frac{1}{C_2}\|T(x)\|$, S is continuous.

Let $S : \operatorname{Ran}(T) \to X$ be defined by $S(T(x)) = x$. Since T is continuous, it is bounded, so

$$\|T(x)\| \leqslant \|T\|_{op} \|x\| .$$

Since S is bounded,

$$\begin{aligned}\|x\| &= \|S(T(x))\| \\ &= \|S\|_{op} \|T(x)\| ,\end{aligned}$$

so $\frac{1}{\|S\|_{op}} \|x\| \leqslant \|T(x)\|$. □

Corollary: Let X be a vector space with $\|\cdot\|$ and $\|\cdot\|'$ two norms. The following are equivalent:

(i) The norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

(ii) The map $\mathrm{id}_X : (X, \|\cdot\|) \to (X, \|\cdot\|')$.

Proposition (Properties of Bounded Linear Operators)**:** Let X, Y, Z be normed spaces, $T : X \to Y$, $S : X \to Y$, and $R : Y \to Z$ be linear maps.

(1) $\|\alpha T\|_{op} = |\alpha| \|T\|_{op}$

(2) $\|T + S\|_{op} \leqslant \|T\|_{op} + \|S\|_{op}$

(3) $\|T\|_{op} = 0$ if and only if $T = 0$

(4) $\|R \circ T\|_{op} \leqslant \|R\|_{op} \|T\|_{op}$

(5) $\|\mathrm{id}_X\|_{op} = 1$

(6) If $E \subseteq X$ is a subspace, then $\|T|_E\|_{op} \leqslant \|T\|_{op}$

Proof. We will prove (4) here. For $x \in B_X$, we have

$$\begin{aligned}\|R \circ T(x)\| &= \|R(T(x))\| \\ &\leqslant \|R\|_{op} \|T(x)\| \\ &\leqslant \|R\|_{op} \|T\|_{op} .\end{aligned}$$

Taking the supremum, we obtain $\|R \circ T\|_{op} \leqslant \|R\|_{op} \|T\|_{op}$. □

Recall: $\mathcal{L}(X, Y)$ is the set of all linear operators with domain X and codomain Y.

Proposition: Let X and Y be normed spaces.

(1) The collection $\mathcal{B}(X, Y) = \left\{T \in \mathcal{L}(X, Y) \mid \|T\|_{op} < \infty\right\}$ equipped with the operator norm is a normed space known as the space of bounded linear operators between X and Y.

(2) If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

(3) The continuous dual space, $X^* = \mathcal{B}(X, \mathbb{C})$ is a Banach space.

Proof. We will prove (2). Let $(T_n)_n$ be Cauchy under $\|\cdot\|_{op}$. Since Cauchy sequences are bounded, there is some $C > 0$ such that $\|T_n\|_{op} \leqslant C$ for all $n \geqslant 1$. For $x \in X$,

$$\|T_n(x) - T_m(x)\| \leqslant \|T_n - T_m\|_{op} \|x\|,$$

meaning $(T_n(x))_n$ is Cauchy in Y. Since Y is complete, we define

$$T(x) = \lim_{n\to\infty} T_n(x)$$

in Y. If $x \in B_X$, we have

$$\begin{aligned} \|T(x)\| &= \left\|\lim_{n\to\infty} T_n(x)\right\| \\ &= \lim_{n\to\infty} \|T_n(x)\| \\ &\leqslant \limsup_{n\to\infty} \|T_n(x)\| \\ &\leqslant C\|x\|, \end{aligned}$$

meaning $\|T\|_{op} \leqslant C$.

Let $\varepsilon > 0$, and $N \in \mathbb{N}$ large such that $n, m \geqslant N$, $\|T_n - T_m\|_{op} \leqslant \varepsilon$. For $x \in B_X$,

$$\begin{aligned} \|T_n(x) - T(x)\| &= \lim_{m\to\infty} \|T_n(x) - T_m(x)\| \\ &\leqslant \limsup_{m\to\infty} \|T_n - T_m\|_{op} \|x\| \\ &< \varepsilon. \end{aligned}$$

Thus, $\|T - T_n\|_{op} < \varepsilon$ for all $n \geqslant N$. □

Definition (Algebras)**.** Let A be an algebra over $\mathbb{C}$.

(1) If A admits a norm $\|\cdot\|$ satisfying $\|ab\| \leqslant \|a\|\|b\|$, then A is a normed algebra. If A is unital, then $\|1_A\| = 1$.

(2) If A is complete with respect to its norm, then A is called a Banach algebra, and if A is unital, then A is a unital Banach algebra.

Lemma: In a normed algebra A, the map $\cdot : A \times A \to A, (a, b) \mapsto ab$ is continuous.

Proposition: Let X be a normed space. The set of bounded operators $\mathcal{B}(X, X) = \mathcal{B}(X)$ is a unital normed algebra. Moreover, if X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

Proposition: Let A be a unital Banach algebra, $a \in A$. The series

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges absolutely in A. We call $\exp(a)$ the exponential of a.

(1) $\exp(0) = 1_A$

(2) If A is commutative, then $\exp(a + b) = \exp(a)\exp(b)$.

(3) We have $\exp(a) \in GL(A)$ with $\exp(a)^{-1} = \exp(-a)$.

(4) $\left\|\exp(a)\right\| \leqslant \exp(\|a\|)$.

Quotient Maps

Definition. A map $f : X \to Y$ is called open if $U \subseteq X$ is open implies $f(U) \subseteq Y$ is open.

Proposition: Let X and Y be normed spaces, $T : X \to Y$ a linear map. The following are equivalent:

(i) T is surjective and open.

(ii) $T(U_X) \subseteq Y$ is open.

(iii) There exists $\delta > 0$ such that $\delta U_Y \subseteq T(U_X)$.

(iv) There exists δ such that $\delta B_Y \subseteq T(B_X)$.

(v) There exists $M > 0$ such that for all $y \in Y$, there exists $x \in X$ with $T(x) = y$ and $\|x\| \leqslant M\|y\|$.

Proof. To see (i) implies (ii), if T is surjective and open, then it is clear that $T(U_X)$, which is the image of an open set, is open.

To see (ii) implies (iii), if $T(U_X)$ is open, we have $0_Y \in T(U_X)$, so there is some δ such that $U(0,\delta) \subseteq T(U_X)$, meaning $\delta U_Y \subseteq T(U_X)$.

Assuming (iii), we see that $\frac{\delta}{2}B_Y \subseteq \delta U_Y \subseteq T(U_X) \subseteq T(B_X)$.

To see (iv) implies (v), let δ be such that $\delta B_Y \subseteq T(B_X)$, and set $M = \frac{1}{\delta}$. Note that for $y \in Y, y \neq 0$, $\frac{\delta}{\|y\|}y \in \delta B_Y$, meaning $\frac{\delta}{\|y\|}y = T(x)$ for some $x \in B_X$, implying that $T\left(\frac{\|y\|}{\delta}x\right) = y$. Finally, since $x \in B_X$, $\frac{\|y\|}{\delta}\|x\| \leqslant \frac{1}{\delta}\|y\| = M\|y\|$.

To see (v) implies (i), we can see that T is surjective by the assumption. Let $U \subseteq X$ be open, $y_0 \in T(U)$. Then, there exists x_0 such that $T(x_0) = y_0$, and $\delta > 0$ such that $U(x_0,\delta) \subseteq U$. Note that $U(x_0,\delta) = x_0 + \delta U_X$, so $x_0 + \delta U_X \subseteq U$. Applying T, we get $T(x_0 + \delta U_X) \subseteq T(U)$, or $y_0 + \delta T(U_X) \subseteq T(U)$. By assumption, since given $y \in U_Y$, there exists $x \in X$ such that $\|x\| \leqslant M\|y\|$, meaning $\|x\| \leqslant M$, we have $U_Y \subseteq T(MU_X)$. Thus, $\frac{1}{M}U_Y \subseteq T(U_X)$, meaning $y_0 + \frac{\delta}{M}U_Y \subseteq y_0\delta T(U_X) \subseteq T(U)$, so $U_Y\left(y_0, \frac{\delta}{M}\right) \subseteq T(U)$. ◻

Definition. Let X and Y be normed vector spaces.

(1) A bounded linear map $T : X \to Y$ that is surjective and open is known as a quotient map.

(2) If $T(U_X) = U_Y$, then T is called a 1-quotient map.

Exercise: If $T(B_X) = B_Y$, show that $T(U_X) = U_Y$.

Solution: Since $T(B_X) = B_Y$, it is the case that $(T(B_X))^\circ = B_Y^\circ$. Since T is an open map, T is continuous, meaning $(T(B_X))^\circ = T\left(B_X^\circ\right)$. Thus, $T(U_X) = U_Y$.

Proposition: Let X and Y be normed vector spaces with $T : X \to Y$ a quotient map. If X is a Banach space, then Y is a Banach space.

Proof. We will show that Y is complete by showing that an absolutely convergent series converges.

Let $(y_k)_k$ be a sequence in Y with $\sum_{k=1}^\infty \|y_k\| < \infty$. Since T is a quotient map, there is a universal $M > 0$ such that for all k, there is $x_k \in X$ such that $T(x_k) = y_k$ and $\|x_k\| \leqslant M\|y_k\|$. Thus,

$$\begin{aligned}\sum_{k=1}^\infty &\leqslant M\sum_{k=1}^\infty \|y_k\| \\ &< \infty.\end{aligned}$$

Since X is complete, $\sum_{k=1}^\infty x_k$ converges. Let $\sum_{k=1}^\infty x_k = x$. Then, $\left(T\left(\sum_{k=1}^n x_k\right)\right)_n \xrightarrow{n\to\infty} T(x)$, meaning $\sum_{k=1}^\infty y_k = T(x)$. Thus, $\sum_{k=1}^\infty y_k$ converges in Y, so Y is a Banach space. ◻

Proposition: Let X be a normed vector space, $E \subseteq X$ a closed subspace. The canonical quotient map, $\pi : X \to X/E$ is a 1-quotient map.

Proof. We know that $\|\pi(x)\| \leqslant \|x\|$, meaning $\pi(U_X) \subseteq U_{X/E}$.

Let $\pi(x) = x + E \subseteq U_{X/E}$. Then, $\inf_{y \in E} \|x - y\| \leqslant 1$, meaning there exists some y such that $\|x - y\| < 1$, meaning $\pi(x - y) = \pi(x)$. □

Corollary: If X is a Banach space, $E \subseteq X$ a closed subspace, then X/E is a Banach space.

Corollary: Let X be a normed vector space and $E \subseteq X$ be closed. If two of $X, E, X/E$ are complete, the third is also complete.

Proof. We have shown that if X is complete, then E is necessarily complete (since E is closed) and X/E is complete as shown above.

Let E and X/E be complete. We now want to show that X is complete. Let $(x_k)_k$ be Cauchy in X.

For each k, let $x_k = s_k + y_k$, where $y_k \in E$ and $s_k + E = \pi(x_k)$. Notice that, since x_k is Cauchy, so too is s_k, as $\|s_k\| \leqslant \|x_k\|$ for all k. Additionally, for $m, n \geqslant N$, we have

$$\begin{aligned}\|x_m - x_n\| &= \|s_m + y_m - (s_n + y_n)\| \\ &\leqslant \|s_m - s_n\| + \|y_m - y_n\| \\ &< \varepsilon,\end{aligned}$$

implying that $(y_k)_k$ is Cauchy in E. Since X/E and E are complete, we define $x = \lim_{k \to \infty} s_k + \lim_{k\to\infty} y_k$. Finally, for $m, n \geqslant N$, we have

$$\begin{aligned}\|x - x_n\| &= \lim_{m \to \infty} \|x_m - x_n\| \\ &\leqslant \varepsilon,\end{aligned}$$

meaning $(x_k)_k \xrightarrow{k \to \infty} x$, so X is complete. □

Proposition: Let X and Y be normed spaces, $E \subseteq X$ a closed subspace, and $T : X \to Y$ bounded linear with $E \subseteq \ker(T)$. Then, there exists a unique bounded linear map $\overline{T} : X/E \to Y$ such that $\overline{T} \circ \pi = T$. Moreover, $\overline{T}$ is injective if and only if $E = \ker(T)$ and $\left\|\overline{T}\right\| = \|T\|$.

Proof. The existence and uniqueness of $\overline{T} : X/E \to Y$ such that $\overline{T} \circ \pi = T$ follows from the First Isomorphism Theorem for vector spaces, as does the fact that $\overline{T}$ is injective and only if $\ker(T) = E$.

Let $x + E \in X/E$. For $y \in E$, we have

$$\begin{aligned}\left\|\overline{T}(x + E)\right\| &= \left\|\overline{T}(x - y + E)\right\| \\ &= \|T(x - y)\| \\ &\leqslant \|T\| \|x - y\|.\end{aligned}$$

Taking infimum over all $y \in E$, we get $\left\|\overline{T}(x + E)\right\| \leqslant \|T\| \|x + E\|$, meaning $\left\|\overline{T}\right\| \leqslant \|T\|$. Additionally,

$$\begin{aligned}\|T\| &= \left\|\overline{T} \circ \pi\right\| \\ &\leqslant \left\|\overline{T}\right\| \|\pi\| \\ &= \left\|\overline{T}\right\|.\end{aligned}$$

□

Theorem (First Isomorphism Theorem for Normed Vector Spaces)**:** Let X and Y be normed vector spaces, $T \in \mathcal{B}(X, Y)$.

(1) T is a quotient map if and only if $\overline{T} : X/\ker(T) \to Y$ is a bicontinuous isomorphism.

(2) T is a 1-quotient map if and only if $\overline{T} : X/\ker(T) \to Y$ is an isometric isomorphism.

Proof.

(1) Let $\overline{T} : X/\ker(T) \to Y$ be a bicontinuous isomorphism. Since $\overline{T}$ is bicontinuous, it is a homeomorphism, meaning it is open and surjective. Since π is a quotient map, so too is $T : \overline{T} \circ \pi$.

Suppose T is a quotient map. Then, T is surjective, meaning $\overline{T}$ is an isomorphism. Since T is bounded below, $\overline{T}$ is also bounded. Let $\pi(x) = x + \ker(T) \in X/\ker(T)$, with $T(x) = y$. Let M be such that $\|x\| \leqslant M\|y\|$. There is an $x' \in X$ with $T(x') = y$, and $\|x'\| \leqslant M\|y\|$. Thus, $x - x' \in \ker(T)$, so $\pi(x) = \pi(x')$, meaning

$$\begin{aligned}\left\|\overline{T} \circ \pi(x)\right\| &= \|T \circ \pi(x')\| \\ &= \|y\| \\ &\geqslant M^{-1}\|x'\| \\ &\geqslant M^{-1}\|\pi(x')\| \\ &= M^{-1}\|\pi(x)\|,\end{aligned}$$

meaning T is bounded below.

(2) Suppose $\overline{T} : X/\ker(T) \to Y$ is an isometric isomorphism. Then, $\overline{T}$ is a 1-quotient map, and since π is a 1-quotient map, so too is $T = \overline{T} \circ \pi$.

Suppose T is a 1-quotient map. Since T is surjective, $\overline{T}$ is an isomorphism. Since T is a 1-quotient map, $\|T\| = \sup_{x \in U_X}\|T(x)\| \leqslant 1$, meaning $\left\|\overline{T}\right\| \leqslant \|T\| \leqslant 1$. Consider $S = \left(\overline{T}\right)^{-1} : Y \to X/\ker(T)$; S is also an isomorphism, so $S \circ \overline{T} == \mathrm{id}_{X/\ker(T)}$. We will now show S is a contraction, meaning $\overline{T}$ is an isometry.

Let $y \in U_Y$. Since T is a 1-quotient map, there exists $x \in U_X$ such that $T(x) = y$. Then, $\overline{T}(x + \ker(T)) = T(x) = y$, meaning $S(y) = x + \ker(T)$, and

$$\begin{aligned}\|S(y)\| &= \|x + \ker(T)\| \\ &\leqslant \|x\| \\ &\leqslant 1,\end{aligned}$$

meaning $\|S\| \leqslant 1$.

$\square$

Proposition: Every separable Banach space is isometrically isomorphic to a quotient of ℓ_1.

Proof. Let X be a separable Banach space. Since X is separable, so too is S_X. Let $(z_n)_n$ be norm-dense in S_X, and define

$$\begin{aligned}T : \ell_1 &\to X \\ (\lambda_n)_n &\to \sum_{n=1}^{\infty}\lambda_n z_n.\end{aligned}$$

This series converges absolutely:

$$\begin{aligned}\sum_{n=1}^{\infty} \|\lambda_n z_n\| &= \sum_{n=1}^{\infty} |\lambda_n| \\ &< \infty,\end{aligned}$$

so this series converges in X. We can also see that T is linear; additionally, T is a contraction:

$$\begin{aligned}\|T((\lambda_n)_n)\| &= \left\|\sum_{n=1}^{\infty} \lambda_n z_n\right\| \\ &= \lim_{N\to\infty} \left\|\sum_{n=1}^{N} \lambda_n z_n\right\| \\ &\leqslant \lim_{N\to\infty} \sum_{n=1}^{N} \|\lambda_n z_n\| \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} |\lambda_n| \\ &= \|(\lambda_n)_n\|.\end{aligned}$$

Thus, $T(U_{\ell_1}) \subseteq U_X$. To show that $T(U_\ell) = U_X$, we will use the following fact (which follows from the density of z_n).

Fact. For $\delta > 0$ and $x \neq 0$ in X, and $k \in \mathbb{N}$, there exists $n > k$ such that

$$\begin{aligned}\left\|\frac{x}{\|x\|} - z_n\right\| &< \frac{\delta}{\|x\|} \\ \|x - (\|x\|)\, z_n\| &< \delta\end{aligned}$$

Let $x \in U_X$ with $x \neq 0$, and let $\varepsilon > 0$. Find n_1 such that

$$\|x - (\|x\|)\, z_{n_1}\| < \frac{\varepsilon}{2},$$

and set $\lambda_{n_1} = \|x\|$.

We find n_2 with $n_2 > n_1$ and

$$\|(x - \lambda_{n_1} z_{n_1}) - (\|x - \lambda_{n_1} z_{n_1}\|)\, z_{n_2}\| < \frac{\varepsilon}{2^2},$$

and set $\lambda_{n_2} = \|x - \lambda_{n_1} z_{n_1}\|$. We have

$$\|x - (\lambda_{n_1} z_{n_1} + \lambda_{n_2} z_{n_2})\| < \frac{\varepsilon}{2^2},$$

and $\lambda_{n_2} < \frac{\varepsilon}{2}$.

Inductively, we obtain the subsequence $(z_{n_k})_k$ in z_n and a sequence of scalars $(\lambda_{n_k})_k$ such that

$$\left\|x - \sum_{j=1}^{k} \lambda_{n_j} z_{n_j}\right\| < \frac{\varepsilon}{2^k}$$

and

$$\|\lambda_{n_k}\| < \frac{\varepsilon}{2^{k-1}}.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i = 0$ for $i \notin \{n_1, n_2, \dots\}$. We can see that

$$\begin{aligned} \|\lambda_{n_1}\| &= \left\|\lambda_{n_1} + \sum_{k=2}^{\infty} \lambda_{n_k}\right\| \\ &\leqslant \|x\| + \sum_{k=2}^{\infty} \frac{\varepsilon}{2^{k-1}} \\ &= \|x\| + \varepsilon. \end{aligned}$$

We choose ε such that $\|x\| + \varepsilon < 1$, meaning $\lambda \in U_{\ell_1}$.

We can also see that $\sum_{j=1}^{\infty} \lambda_{n_j} z_{n_j} = x$, meaning T is a 1-quotient map. □

Pillars of Functional Analysis

The five main theorems of functional analysis are:

- Baire Category Theorem;
- Open Mapping Theorem (and Bounded Inverse Theorem);
- Closed Graph Theorem;
- Uniform Boundedness Principle;
- and the Hahn Banach Theorems:
  - Hahn–Banach–Minkowski Theorem;
  - Hahn–Banach Extension Theorem;
  - Hahn–Banach Separation Theorem.

These theorems will appear time and again as we work through the fundamentals of functional analysis.

Baire Category Theorem

Definition (Baire Space)**.** Let $\{A_n\}_{n\geqslant 1}$ be a countable collection of open, dense subsets of a topological space X. We say X is a Baire space if

$$\bigcap_{n\geqslant 1} A_n$$

is dense for every such collection.

Definition (Meager Set)**.** If $X = \bigcup_{n\geqslant 1} F_n$, where $\left(\overline{F_n}\right)^{\circ} = \emptyset$ for each n, then we say X is meager.[II]

Proposition (Meager Spaces)**:** If X is a Baire space, then X is nonmeager.

Proof. Suppose toward contradiction that $X = \bigcup_{n\geqslant 1} F_n$, with F_n all nowhere dense. Then,

$$X = \bigcup_{n\geqslant 1} C_n,$$

where $C_n = \overline{F_n}$ are closed with $C_n^{\circ} = \emptyset$.

[II] In other words, X is meager if X is a countable union of nowhere dense subsets.

Let $A_n = C_n^c$. Then, A_n is open for all n, and $\overline{A_n} = \overline{C_n^c} = (C_n^c)^\circ = X$, meaning A_n are all open and dense.

Since X is a Baire space, we know that $\bigcap_{n\geqslant 1} A_n$ is dense. However, we also have

$$\begin{aligned}\emptyset &= X^c\\ &= \left(\bigcup_{n\geqslant 1} C_n\right)^c\\ &= \bigcap_{n\geqslant 1} C_n^c\\ &= \bigcap_{n\geqslant 1} A_n.\end{aligned}$$

□

Theorem (Baire Category Theorem)**:** If (X, d) is a complete metric space, then X is a Baire space.

Proof. Let $\{A_n\}_{n\geqslant 1}$ be a collection of open dense subsets of X. Let U_0 be any ball of radius $r > 0$, and set $B_0 = \overline{U_0}$. Since $A_1 \cap U_0$ is open and nonempty, it contains a closed ball B_1 with radius less than $r/2$.

Set $U_1 = B_1^\circ$. Similarly, we find a closed ball B_2 with radius less than $r/4$ such that $B_2 \subseteq A_2 \cap U_1$, and set $U_2 = B_2^\circ$.

Continuing in this manner, we find a closed ball B_n with radius less than $r/2^n$ with $B_n \subseteq A_n \cap U_{n-1}$, and the chain

$$B_0 \supseteq U_0 \supseteq B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \supseteq \cdots.$$

Letting $(x_n)_n$ be the center of B_n, we can see that x_n forms a Cauchy sequence in X, as the distance between x_m and x_n with $n > m$ is no more than $\frac{r}{2^{m-1}}$.

Since X is complete, $(x_n)_n \to x \in X$. We claim that x belongs to $\bigcap_{n\geqslant 1} B_n$.

Suppose toward contradiction that $x \notin B_N$ for some $N \in \mathbb{N}$. For $n \geqslant N$, we have $x \notin B_n$, so $d(x_n, x) \geqslant \operatorname{dist}_{B_n}(x) > 0$, which contradicts the fact that $(x_n)_n \to x$.

Thus, $x \in \bigcap_{n\geqslant 1} B_n \subseteq \bigcap_{n\geqslant 1} A_n$. Since $\bigcap_{n\geqslant 1} B_n \subseteq U_0$, we have $\left(\bigcap_{n\geqslant 1} A_n\right) \cap U_0 \neq \emptyset$, meaning $\bigcap_{n\geqslant 1} A_n$ is dense in X. □

Corollary: Let X be an infinite-dimensional Banach space. The cardinality of the Hamel basis of X is uncountable.

Proof. Suppose toward contradiction that $\{b_k\}_{k\in\mathbb{N}}$ is a Hamel basis for X. For each n, set $E_n = \operatorname{span}\{b_1, \dots, b_n\}$. Each E_n is closed, meaning $\overline{E_n} = E_n \neq X$ since X is infinite-dimensional.

Additionally, $E_n^\circ = \emptyset$ for each n, meaning the E_n are nowhere dense.

Since $\{b_k\}_{k\in\mathbb{N}}$ is a spanning set,

$$X = \bigcup_{n\geqslant 1} E_n,$$

implying that X is meager. □

Exercise: Let X be a Banach space, and $Z \subseteq X$ a subspace. Is it true that $\dim(Z) = \dim\left(\overline{Z}\right)$?

Solution: It is not the case that $\dim(Z) = \dim\left(\overline{Z}\right)$. For example, consider the subspace $c_c \subseteq \ell_\infty$. Then, the Hamel basis of c_c consists of e_n, which consists of 1 at index n and zero elsewhere, implying that $\dim(c_c) = \aleph_0$. However, $\overline{c_c} = c_0$, and c_0 is an infinite-dimensional Banach space, meaning that $\dim(\overline{c_c}) = 2^{\aleph_0} \neq \aleph_0$.

Open Mapping Theorem

A surjective continuous map between topological spaces is not necessarily an open map — however, if X and Y are Banach spaces, and $f : X \to Y$ is a surjective linear map. This is the Open Mapping theorem, which yields the result that a continuous linear bijection between Banach spaces always admits a bounded inverse.

Lemma: Let X and Y be Banach spaces, and suppose $T \in \mathcal{B}(X, Y)$.

(1) If $U_Y \subseteq \overline{T(\delta U_X)}$ for some $\delta > 0$, then $U_Y \subseteq T(2\delta U_X)$.

(2) If $\delta U_Y \subseteq \overline{T(U_X)}$ for some $\delta > 0$, then $\frac{\delta}{2} U_Y \subseteq T(U_X)$.

Proof.

(1) Let $y \in U_Y$. By our assumption, there exists $x_1 \in \delta U_X$ such that $\|y - T(x_1)\| < 1/2$. Additionally,

$$\begin{aligned} y - T(x_1) &\in \frac{1}{2} U_Y \\ &\subseteq \frac{1}{2}\overline{T(\delta U_X)} \\ &= \overline{T\left(\frac{\delta}{2} U_X\right)}. \end{aligned}$$

Thus, there exists $x_2 \in \frac{\delta}{2} U_X$ such that $\|(y - T(x_1)) - T(x_2)\| < \frac{1}{4}$, implying that

$$\begin{aligned} y - T(x_1) - T(x_2) &\in \frac{1}{4} U_Y \\ &\subseteq \overline{T\left(\frac{\delta}{4} U_X\right)}. \end{aligned}$$

Inductively, we have a sequence $(x_k)_k \in \frac{\delta}{2^{k-1}} U_X$ for each k, and

$$\left\| y - \sum_{j=1}^{k} T\left(x_j\right) \right\| < 2^{-k}.$$

We consider $\sum_{j=1}^{\infty} x_j$. Since

$$\begin{aligned} \sum_{j=1}^{\infty} \left\|x_j\right\| &\leqslant \sum_{j=1}^{\infty} \frac{\delta}{2^{j-1}} \\ &= 2\delta \\ &< \infty, \end{aligned}$$

the series converges to $x \in X$ since X is complete.

Additionally, since $\|x\| \leqslant \sum_{j=1}^{\infty} \left\|x_j\right\| \leqslant 2\delta$, we have $x \in 2\delta U_X$, and $T(x) = y$ by the continuity of T.

(2) If $\delta U_y \subseteq \overline{T(U_X)}$, then $U_Y \subseteq \frac{1}{\delta}\overline{T(U_X)}$, so $U_Y \subseteq \overline{T\left(\frac{1}{\delta} U_X\right)}$, meaning $U_Y \subseteq T\left(\frac{2}{\delta} U_X\right)$, or $\frac{\delta}{2} U_Y \subseteq T(U_X)$.

□

Theorem (Open Mapping Theorem)**:** Let X and Y be Banach spaces, $T \in \mathcal{B}(X, Y)$ surjective. Then, T is open and thus a quotient mapping.

Proof. We will show that $\delta U_Y \subseteq T(U_X)$ for some $\delta > 0$. This is enough to show that T is a quotient mapping.

We can write

$$\begin{aligned} X &= n\bigcup_{n\geqslant 1} U_X \\ Y &= T(X) \\ &= \bigcup_{n\geqslant 1} T(nU_X) \end{aligned}$$

since T is onto. Since Y is nonmeager, there is an $m \geqslant 1$ such that $\overline{T(mU_X)}^{\circ} \neq \emptyset$. There exists $y_0 \in Y$ and $\varepsilon > 0$ such that $U_Y(y_0, \varepsilon) \subseteq \overline{T(mU_X)}$. We claim that

$$\begin{aligned} \varepsilon U_Y &= U_Y(0, \varepsilon) \\ &\subseteq T(mU_X). \end{aligned}$$

Let $z \in \varepsilon U_Y$. Note that $y_0 + z$ and $y_0 - z$ are in $U_Y(y_0, \varepsilon)$, and

$$\begin{aligned} 2z &= (y_0 + z) - (y_0 - z) \\ &\in \overline{T(mU_X)} - \overline{T(mU_X)}. \end{aligned}$$

We write $2z = z_1 - z_2$, with $z_1, z_2 \in \overline{T(mU_X)}$. We can find sequences $(T(x_k))_k$ and $\left(T\left(x_k'\right)\right)_k$ with $(T(x_k))_k \to z_1$ and $\left(T\left(x_k'\right)\right)_k \to z_2$. Thus, we have

$$\begin{aligned} 2z &= \lim_{k\to\infty} \left(T(x_k) - T\left(x_k'\right)\right) \\ &= \lim_{k\to\infty} T\left(x_k - x_k'\right), \end{aligned}$$

where $\left\|x_k - x_k'\right\| \leqslant 2m$. Thus, $2x \in \overline{T(mU_X)} = 2\overline{T(mU_X)}$, so $z \in \overline{T(mU_X)}$.

We now have

$$\frac{\varepsilon}{m} U_Y \subseteq \overline{T(U_X)},$$

so

$$\frac{\varepsilon}{2m} U_Y \subseteq T(U_X).$$

Setting $\delta = \frac{\varepsilon}{2m}$, we finish the proof. □

If $T : X \to Y$ is bijective linear, then $T^{-1} : Y \to X$ is linear. If $X = Y$, we say T is invertible in the unital algebra $\mathcal{L}(X)$. However, if X and Y are normed vector spaces, we also have to be concerned with the continuity of T^{-1}.

Corollary (Bounded Inverse Theorem)**:** Let X and Y be Banach spaces, $T : X \to Y$ is linear, bounded, and bijective. Then, $T^{-1} : Y \to X$ is also bounded.

Proof. Since T is surjective, T is open, so T^{-1} is continuous. □

Example. Consider the normed space $Y = (C([0,1]), \|\cdot\|_1)$. To show that Y is not complete, we let $X = (C([0,1]), \|\cdot\|_u)$, which we know is complete.

The identity function from X to Y is bijective and bounded linear since $\|\cdot\|_1 \leqslant \|\cdot\|_u$. If Y were to be complete, then it would imply that the inverse map is bounded. However, since there is no C such that $\|\cdot\|_u \leqslant C\|\cdot\|_1$, it is not the case that Y is complete.

Definition. Let X and Y be normed spaces. A bounded linear map $T \in \mathcal{B}(X,Y)$ is called invertible if there is a bounded linear map $S \in \mathcal{B}(Y,X)$ with $T \circ S = \mathrm{id}_Y$ and $S \circ T = \mathrm{id}_X$. We write $T^{-1} = S$.

Corollary: Let $T \in \mathcal{B}(X,Y)$ with X and Y Banach spaces. The following are equivalent.

(i) T is bounded below.

(ii) T is injective and $\mathrm{Ran}(T) \subseteq Y$ is closed.

(iii) $T : X \to \mathrm{Ran}(T)$ is a bicontinuous isomorphism.

Proof. For (i) to (ii), if T is bounded below, then $\ker T = \{0\}$, so T is injective. Additionally, since T is bounded below, if $(T(x_n))_n$ is a Cauchy sequence in Ran(T), then

$$\begin{aligned} C\|x_n - x_m\| &\leqslant \|T(x_n - x_m)\| \\ &= \|T(x_n) - T(x_m)\|, \end{aligned}$$

meaning $(x_n)_n$ is a Cauchy sequence in X. Since T is continuous, $(T(x_n))_n \to T(x) \in \mathrm{Ran}(T)$.

For (ii) to (i), since Y is complete and $\mathrm{Ran}(T) \subseteq Y$ is closed, Ran(T) is a Banach space, so $T^{-1} : \mathrm{Ran}(T) \to X$ is bounded. Thus,

$$\begin{aligned} \|x\| &= \left\|T^{-1}(T(x))\right\| \\ &\leqslant \left\|T^{-1}\right\|_{op} \|T(x)\|, \end{aligned}$$

meaning $\|T(x)\| \geqslant \left\|T^{-1}\right\|_{op}^{-1} \|x\|$ for all $x \in X$.

To show that (ii) is true if and only if (iii) is true, we can see that since T is bounded and T is bounded below, it is the case that T is a bicontinuous isomorphism. ◻

Corollary: Let X and Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then, T is invertible if and only if T is bounded below and surjective.

Complemented Subspaces and Direct Sums

For any normed vector spaces X and Y, we can form the product $X \oplus_p Y$ by defining $\|(x,y)\| = \left(\|x\|^p + \|y\|^p\right)^{1/p}$ for all $p \in [1,\infty)$.

A vector space Z with subspaces X and Y is called the direct sum of X and Y if

(a) for all $z \in Z$, there exist $x \in X$ and $y \in Y$ such that $z = x + y$;

(b) $X \cap Y = \{0\}$.

We write $Z = X \oplus Y$ for the internal direct sum.

Proposition: Let $(Z, \|\cdot\|_Z)$ be a Banach space, and suppose X and Y are closed subspaces of Z with $Z = X \oplus Y$. Then, $Z \cong X \oplus_p Y$ for all $p \in [1,\infty]$.

Proof. Let $p = 1$. Set $\phi : X \oplus_1 Y \to Z$ by taking $\phi((x, y)) = x + y$. Since $Z = X \oplus Y$, this is a bijection, hence an isomorphism. Additionally,

$$\begin{aligned}\|\phi((x,y))\|_Z &= \|x+y\|_Z \\ &\leqslant \|x\|_Z + \|y\|_Z \\ &= \|(x,y)\|_1,\end{aligned}$$

meaning ϕ is bounded. Thus, ϕ^{-1} is also bounded, meaning ϕ is bicontinuous. The proof is similar for all other $p \in (1, \infty]$. □

Definition. If Z is a normed space, X and Y are closed subspaces of Z such that $Z = X \oplus Y$, we say Z is the topological internal direct sum of X and Y.

Definition. Let Z be a normed space, and suppose X is a closed subspace of Z. We say X is complemented in Z if there is a closed $Y \subseteq Z$ with $X \oplus Y = Z$.

Not all closed subspaces are complemented.

Proposition: Let $T : X \to Y$ be a bounded linear map between Banach spaces. If $Z \subseteq Y$ is a closed subspace such that $Y = \operatorname{Ran}(T) \oplus Z$, then $\operatorname{Ran}(T)$ is closed (meaning the internal direct sum is topological).

Proof. Passing to the quotient

$$X/\ker(T) \to Y,\ x + \ker(T) \mapsto T(x),$$

we may assume that T is injective. The map $S : X \oplus_\infty Z \to Y$, $S(x, z) = T(x) + z$ is bounded and bijective. Thus, S is bounded below, so for some $C > 0$, we have

$$\begin{aligned}\|T(x)\| &= \|S(x,0)\| \\ &\geqslant C\,\|(x,0)\|_\infty \\ &= C\,\|x\|,\end{aligned}$$

meaning T is bounded below, and thus has closed range. □

Corollary: If X and Y are Banach spaces, and $T : X \to Y$ is bounded Fredholm,[III] then T has closed range.

Proof. There is a subspace $C \subseteq Y$ with C linearly isomorphic to $\operatorname{coker}(T)$, and $Y = \operatorname{Ran}(T) \oplus C$. Since T is Fredholm, $\dim(C)$ is finite, meaning C is closed. Thus, $\operatorname{Ran}(T)$ is closed. □

Closed Graph Theorem

Definition. If $f : A \to B$ is a map between arbitrary sets, then the graph of f is

$$\begin{aligned}\operatorname{graph}(f) &= \{(a, f(a)) \mid a \in A\} \\ &\subseteq A \times B.\end{aligned}$$

Proposition: If (X, d) and (Y, ρ) are metric spaces, and $f : (X, d) \to (Y, \rho)$ is continuous, then $\operatorname{graph}(f) \subseteq X \times Y$ is closed under the product topology.[IV]

Proof. Let $(x_n, f(x_n))_n$ be a sequence in $\operatorname{graph}(f)$ such that $(x_n, f(x_n))_n \to (x, y)$ in $X \times Y$. Then, $(x_n)_n \to x$ in X and $(f(x_n))_n \to y$ in Y.

By the continuity of f, we have $(f(x_n))_n \to f(x)$, and since limits are unique, we have $f(x) = y$. Thus,

$$\begin{aligned}(x, y) &= (x, f(x)) \\ &\in \operatorname{graph}(f).\end{aligned}$$

□

[III] A linear map is Fredholm if both $\ker(T)$ and $\operatorname{coker}(T)$ are finite. Here, $\operatorname{coker}(T) = Y/\operatorname{Ran}(T)$.
[IV] The product topology is the coarsest topology on $X \times Y$ such that the projection maps π_X and π_Y are continuous.

Thus, we can see that the graph of any continuous function is closed in the product topology. However, the converse fails in the general case. For instance,

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ f(x) &= \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned}$$

has a closed graph, but f is not continuous.

However, with linear maps between Banach spaces, the converse is actually true.

Theorem (Closed Graph Theorem)**:** Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear map. Then, T is continuous if and only if $\operatorname{graph}(T) \subseteq X \times Y$ is closed with respect to the product topology on $X \times Y$.

Proof. The forward direction follows from the previous proposition.

Suppose $\operatorname{graph}(T) \subseteq X \times Y$ is closed in the product topology. Note that the product topology coincides with the $\|\cdot\|_1$ topology, with $\|(x, y)\|_1 = \|x\| + \|y\|$. Thus, $\left(\operatorname{graph}(T), \|\cdot\|_1\right)$ is a Banach space.

Consider the projection map $P : \operatorname{graph}(T) \to X$ defined by $P\left((x, T(x))\right) = x$, which is bijective. We also have

$$\begin{aligned} \|P\left((x, T(x))\right)\| &= \|x\| \\ &\leqslant \|x\| + \|T(x)\| \\ &= \|(x, T(x))\|_1, \end{aligned}$$

meaning P is bounded. Thus, P is bicontinuous, meaning it is bounded below, so for some constant C, we have

$$\begin{aligned} \|x\| &= \|P\left((x, T(x))\right)\| \\ &\geqslant C\,\|(x, T(x))\|_1 \\ &\geqslant C\,\|T(x)\|, \end{aligned}$$

meaning $\|T(x)\| \leqslant \frac{1}{C}\|x\|$, so T is bounded. $\square$

Example. Consider the collection of sequences

$$X = \left\{ f : \mathbb{N} \to \mathbb{C} \mid \sum_{k \geqslant 1} k\,|f(k)| < \infty \right\}.$$

Note that $X \subseteq \ell_1$, and X is a linear subspaces. Let

$$\begin{aligned} T &: (X, \|\cdot\|_1) \to \ell_1 \\ T(f)(k) &= (kf(k))_1, \end{aligned}$$

which is well-defined and linear. We will show that T is unbounded.

Let f_n denote the element of X defined by

$$f_n = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, 0, 0, \dots).$$

For each f_n, the norm is $\|f_n\|_1 = n$. We also see that

$$T(f_n) = (1, 2, \dots, n, 0, 0, \dots),$$

with norm

$$\|T(f_n)\|_1 = \frac{n(n+1)}{2}.$$

If T were bounded, we would have a constant C such that

$$\frac{n(n+1)}{2} \leqslant Cn$$

for all n. This is not possible, meaning T is unbounded.

However, at the same time, graph(T) is closed, as for $(f_n)_n \to f$ in X and $(T(f_n))_n \to g$ in ℓ_1, we have

$$(f_n(k))_n \to f(k)$$

and

$$(kf(k))_n \to g(k)$$

for all $k \in \mathbb{N}$, implying that $g(k) = kf(k)$, or that $T(f) = g$.

The closed graph theorem thus implies that $X \subseteq \ell_1$ must not be closed.

We can use the closed graph theorem to obtain insights about the orthogonal projection operators.[v]

Proposition: Let $Z = X \oplus Y$ be a topological internal direct sum of a Banach space Z. Then, the projection operators P_X and P_Y onto the closed subspaces of X and Y respectively are bounded.

Proof. Let $(z_n)_n$ and $(P_X(z_n))_n$ be sequences in z with $(z_n)_n \to z$ and $(P_X(z_n))_n \to u$ for some $z, u \in Z$.

For each $n \in \mathbb{N}$, we may write $z_n = x_n + y_n$, with $x_n \in X$ and $y_n \in Y$. Since X is closed, and $x_n = P_X(z_n)$, it is the case that $u \in X$. Then, $y_n = z_n - x_n$ converges to $z - u$; since Y is closed, $z - u \in Y$. Setting $y = z - u$, we have

$$\begin{aligned} P_X(z) &= P_X(u + y) \\ &= P_X(u) + P_X(y) \\ &= u. \end{aligned}$$

Thus, the graph of P_X is closed. □

Proposition: Let Z be a Banach space, and let $P \in \mathcal{B}(Z)$ be a projection. Then, $\ker(P)$ and $\operatorname{Ran}(P)$ are closed subspaces, and

$$Z = \operatorname{Ran}(P) \oplus \ker(P).$$

Proof. For every $z \in Z$, we may express $z = P(z) + (z - P(z))$. Since $P(z - P(z)) = P(z) - P^2(z) = P(z) - P(z) = 0$, we can see that $Z = \operatorname{Ran}(P) \oplus \ker(P)$.

We now need to show that $\operatorname{Ran}(P)$ is closed. Let $(x_n)_n$ be a sequence in $\operatorname{Ran}(p)$ with $(x_n)_n \to z$. Then, for each $n \in \mathbb{N}$, we may write $x_n = P(z_n)$. Since P is continuous, we can see that $(P(x_n)) \to P(z)$. However, at the same time, we may write $x_n = P(z_n)$, so

$$\begin{aligned} P(x_n) &= P(P(z_n)) \\ &= P(z_n) \\ &= x_n \\ &\to z, \end{aligned}$$

Which implies that $z = P(z) \in \operatorname{Ran}(P)$ by the uniqueness of limits. □

Corollary: Let Z be a Banach space, and let $X \subseteq Z$ be a closed subspace. Then, X is complemented in Z if and only if there is a projection $P : Z \to Z$ with $\operatorname{Ran}(P) = X$.

[v] Idempotent operator, where $P^2 = P$, are also known as projections. I will refer to them as projections.

Uniform Boundedness Principle

The uniform boundedness principle is often useful when when dealing with a family of bounded operators. It says that a pointwise bounded family of operators on a Banach space is bounded.

Theorem (Uniform Boundedness Principle)**:** Let X and Y be normed vector spaces, and consider a family of bounded operators $\{T_i\}_{i\in I} \subseteq \mathcal{B}(X, Y)$.

(1) If $\sup_{T_i(x)} < \infty$ for all $x \in A \subseteq X$ with A nonmeager, then

$$\sup_{i\in I} \|T_i\|_{op} < \infty.$$

(2) If X is a Banach space with $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$, then

$$\sup_{i\in I} \|T_i\|_{op} < \infty.$$

Proof. We only need to prove (1), as (2) follows.

For each $i \in I$ and $n \in \mathbb{N}$, define

$$E_{n,i} = \{x \in X \mid \|T_i(x)\| \leqslant n\}.$$

Notice that

$$E_{n,i} = T_i^{-1} \circ \|\cdot\|^{-1} ([0, n]).$$

Since both the norm and T_i are continuous, each $E_{n,i}$ is closed. Therefore,

$$\begin{aligned} E_n &= \bigcap_{i\in I} E_{n,i} \\ &= \left\{x \in X \mid \sup_{i\in I} \|T_i(x)\| \leqslant n\right\} \end{aligned}$$

is also closed, meaning $\overline{E_n} = E_n$.

By our assumption,

$$A = \bigcup_{n\geqslant 1} E_n.$$

Since A is nonmeager, there is $m \in \mathbb{N}$, $x_0 \in X$, and $r > 0$ such that $B(x_0, r) \subseteq E_m$.[VI]

We will now prove that $B(0, r) = rB_X \subseteq E_{2m}$. Let x be such that $\|x\| < r$. Then, $x + x_0 \in B(x_0, r) \subseteq E_m$. For each i, we also have

$$\begin{aligned} \|T_i(x)\| &= \|T_i(x) - T_i(x_0) + T_i(x_0)\| \\ &\leqslant \|T_i(x) + T_i(x_0)\| + \|T_i(x_0)\| \\ &= \|T_i(x + x_0)\| + \|T_i(x_0)\| \\ &\leqslant 2m, \end{aligned}$$

meaning $\sup_{i\in I} \|T_i(x)\| \leqslant 2m$, meaning $x \in E_{2m}$.

Given $x \in B_X \setminus \{0\}$, we have $rx \in E_{2m}$, so $T_i(rx) \leqslant 2m$ for all $i \in I$, meaning

$$\|T_i(x)\| \leqslant \frac{2m}{r}$$

for all $i \in I$, so $\sup_{x\in B_X} \|T_i(x)\| \leqslant \frac{2m}{r}$. Thus, $\sup_{i\in I} \|T_i\|_{op} \leqslant \frac{2m}{r}$. □

[VI]Select δ such that $U(x_0, \delta) \subseteq E_m$, and then select $r = \delta/2$.

Alternative Proof of (2). Suppose toward contradiction that $\sup_{i \in I} \|T_i\|_{op} = \infty$. We can choose a sequence of operators $(T_{i_n})_n$ such that $\|T_{i_n}\|_{op} \geqslant 4^n$. We write $T_{i_n} := T_n$. We will construct some $x \in X$ such that $(\|T_n(x)\|)_n$ is unbounded, yielding a contradiction.

For any $x \in X$ and $r > 0$, we have

$$\begin{aligned} \sup_{y \in B(x,r)} &\geqslant r \|T\|_{op} \\ &> \frac{2}{3} r \|T\|_{op}. \end{aligned}$$

Setting $x_0 = 0$, $T = T_1$, and $r_0 = T^0$, we can find x_1 with

$$\begin{aligned} \|x_1 - x_0\| &\leqslant r_0 \\ \|T_1(x_1)\| &\geqslant \frac{2r}{3} \|T_1\|_{op}. \end{aligned}$$

We find x_2 given T_2 such that for $r_1 = 3^{-1}$

$$\|x_2 - x_1\| \leqslant r_1 \|T_2(x_2)\| \geqslant \frac{2r_1}{3} \|T_2\|_{op}.$$

Inductively, we find x_{n+1} such that for $r_n = 3^{-n}$,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leqslant r_n \\ \|T_{n+1}(x_{n+1})\| &\geqslant \frac{2r_n}{3} \|T_{n+1}\|_{op}. \end{aligned}$$

Thus, we have a sequence $(x_n)_n$ with

$$\begin{aligned} \|x_{n+1} - x_n\| &\leqslant 3^{-n} \\ \|T_n(x_n)\| &\geqslant \frac{2}{3^n} \|T_n\|_{op}. \end{aligned}$$

This means $(x_n)_n$ is Cauchy, and thus converges to $x \in X$ with $\|x - x_n\| \leqslant \frac{1}{(2)(3^{n-1})}$. Thus yields

$$\begin{aligned} \frac{1}{(2)(3^{n-1})} \|T_n\|_{op} &\geqslant \|T_n\|_{op} \|x_n - x\| \\ &\geqslant \|T_n(x_n - x)\| \\ &= \|T_n(x_n) - T_n(x)\| \\ &\geqslant \|T_n(x_n)\| - \|T_n(x)\| \\ &\geqslant \frac{2}{3^n} \|T_n\|_{op} - \|T_n(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|T_n(x)\| &\geqslant \frac{2}{3^n} \|T_n\|_{op} - \frac{1}{(2)(3^{n-1})} \|T_n\|_{op} \\ &= \frac{1}{(2)3^n} \|T_n\|_{op} \\ &\geqslant \frac{1}{2}\left(\frac{4}{3}\right)^n \end{aligned}$$

□

Corollary (Banach–Steinhaus Theorem)**:** Let X, Y be Banach spaces, and let $(T_n)_n$ be a sequence of bounded linear operators in $\mathcal{B}(X, Y)$ with $\lim_{n\to\infty}(T_n(x))$ exists for every $x \in X$. Then,

$$T(x) = \lim_{n\to\infty}(T_n(x))$$

defines a bounded linear map from X to Y.

Proof. We can see that T is linear by its definition. Since $(T_n(x))_n$ converges in Y for all $x \in X$, we have $(T_n(x))$ is bounded in Y for all $x \in X$. Thus, $\sup_{n\geqslant 1}\|T_n(x)\| < \infty$ for all $x \in X$, so $\sup_{n\geqslant 1}\|T_n\|_{op} = C$ for some $C > 0$.

Given $x \in B_X$, we can see that

$$\begin{aligned}
\|T(x)\| &= \left\|\lim_{n\to\infty}(T_n(x))\right\| \\
&= \lim_{n\to\infty}\|T_n(x)\| \\
&\leqslant \limsup_{n\to\infty}\|T_n\|_{op}\|x\| \\
&\leqslant \limsup_{n\to\infty}\|T_n\|_{op} \\
&\leqslant C,
\end{aligned}$$

meaning $\|T\|_{op} \leqslant C$. □

Example. Let $(z_k)_k$ be a sequence such that $\sum_{k=1}^{\infty} z_k y_k$ converges for every $(y_k)_k \in c_0$. We claim that $(z_k)_k \in \ell_1$.

Consider the map

$$\begin{aligned}
T_n &: c_0 \to \mathbb{C} \\
T_n(y) &= \sum_{k=1}^{n} z_k y_k.
\end{aligned}$$

We can see that

$$\begin{aligned}
|T_n(y)| &= \left|\sum_{k=1}^{n} z_k y_k\right| \\
&\leqslant \sum_{k=1}^{n}|z_k y_k| \\
&\leqslant \|y\|_\infty\left(\sum_{k=1}^{n}|z_k|\right).
\end{aligned}$$

Thus, each T_n is bounded linear with $|T_n|_{op} \leqslant \sum_{k=1}^{n}|z_k|$.[VII]

By assumption, we have that $(T_n(y))_n$ converges, meaning $\sup_{n\geqslant 1}|T_n(y)| < \infty$ for all $y \in c_0$. The Uniform Boundedness Principle gives $\sup_{n\geqslant 1}\|T_n\|_{op} < \infty$. Thus,

$$\begin{aligned}
\|z\|_1 &= \sum_{k=1}^{\infty} \\
&= \sup_{n\geqslant 1}\left(\sum_{k=1}^{n}|z_k|\right)
\end{aligned}$$

[VII]It is an exercise to show that this is an inequality. I'm not sure how to do it though.

$$= \sup_{n \geqslant 1} \|T_n\|_{op}$$
$$< \infty.$$

Example (Product Maps). Let $\{X_i\}_{i \in I}$ be a family of Banach spaces, and consider the product

$$\prod_{i \in I} X_i = \left\{ (x_i)_i \mid x_i \in X_i, \sup_{i \in I} \|x_i\| < \infty \right\}.$$

The product space is a Banach space with pointwise operations and the norm $\|(x_i)_{i \in I}\|_{\infty} = \sup_{i \in I} \|x_i\|$.

Consider a Banach space Z and a linear map $Z \to \prod_{i \in I} X_i$.

We can see that if T is continuous, then $\pi_j \circ T : Z \to X_j$, where π_j is the canonical projection map on $\prod_{i \in I} X_i$, is also continuous for each j. However, it is also the case that this is a sufficient condition — that is, if $\pi_j \circ T$ is continuous for each j, then T is continuous.

Proof. Let $\iota_j : X_j \hookrightarrow \prod_{i \in I} X_i$ be the inclusion map, and $T_j : Z \to \prod_{i \in I} X_i$ be given by

$$T_j = \iota_j \circ \pi_j \circ T.$$

Then,

$$\begin{aligned} \sup_{j \in I} \left\|T_j(z)\right\| &= \sup_{j \in I} \left\|\pi_j \circ T(z)\right\| \\ &= \|T(z)\| \\ &< \infty \end{aligned}$$

for all $z \in Z$. Thus, $\sup_{j \in I} \left\|T_j\right\|_{op} < \infty$, so $\|T\|_{op}$ is bounded. □

Hahn–Banach Theorems

The Hahn–Banach extension theorems allow us to extend continuous linear functionals from subspaces to the full vector space, while the separation theorems allow us to separate points from closed subspaces.

Definition (Algebraic Dual). Let X be a vector space over $\mathbb{F}$. Then, $X' = \mathcal{L}(X, \mathbb{F})$, which consists of *all* linear functionals from X into $\mathbb{F}$, is known as the algebraic dual.

Definition (Continuous Dual). Let X be a normed vector space over $\mathbb{F}$. Then, $X^* = \mathcal{B}(X, \mathbb{F})$, which consists of all *continuous* linear functionals from X into $\mathbb{F}$, is known as the continuous dual.[VIII]

Definition (Hyperplane). A hyperplane is a subspace $E \subseteq X$ such that $\dim(X/E) = 1$. An affine hyperplane is of the form $x_0 + E$ for some fixed $x_0 \in X$.

Proposition: Let X be a normed vector space over $\mathbb{F}$.

(1) If $\varphi \in X'$ is not zero, then φ is bounded if and only if $\ker(\varphi) \subseteq X$ is closed. In this case, $X/\ker(\varphi) \cong \mathbb{F}$ are bicontinuously isomorphic.

(2) Given a closed hyperplane $M \subseteq X$, there is a $\varphi \in X^*$ with $\ker(\varphi) = M$.

(3) If $\varphi \in X'$ is unbounded, then $\ker(\varphi) \subseteq X$ is norm-dense.

Proof.

[VIII] Since this is functional analysis, any mention of the "dual space" will refer to the continuous dual.

(1) If φ is continuous, then $\ker(\varphi) = \varphi^{-1}(0)$ is closed. Thus, from the definition of a quotient map, $\tilde{\varphi} : X/\ker(\varphi) \to \mathbb{F}$ defined by $x + \ker(\varphi) = \varphi(x)$, is a bicontinuous isomorphism from $X/\ker(\varphi)$ to $\mathbb{F}$.

Conversely, if $\ker(\varphi)$ is closed in X, then $X/\ker(\varphi)$ is a normed space with dimension 1, meaning $\tilde{\varphi}$ is continuous (as it is a linear map of dimension 1). Thus, $\varphi = \tilde{\varphi} \circ \pi$ is continuous.

(2) Since X/M is a normed space with $\dim(X/M) = 1$, there is a bicontinuous isomorphism $\psi : X/M \to \mathbb{F}$. Set $\varphi = \psi \circ \pi$, where π denotes the canonical projection map. Then, φ is continuous with $\ker(\varphi) = M$.

(3) Let $M = \ker(\varphi)$. Since φ is unbounded, then $M \subset \overline{M}$, meaning $p : X/M \to X/\overline{M}$ defined by $x + M \mapsto x + \overline{M}$ is well-defined with nontrivial kernel.

However, since $\dim(X/M) = 1$, we must have $\ker(p) = X/M$, implying that $x + \overline{M} = 0_{X/\overline{M}}$ for all $x \in X$, meaning $x \in \overline{M}$ for all $x \in X$, so $X \subseteq \overline{M}$.

□

For a vector space X over $\mathbb{C}$, we can consider X as a $\mathbb{R}$-vector space by simply "forgetting" the imaginary scalars. We can also consider the respective dual spaces over $\mathbb{R}$ and $\mathbb{C}$, which we write as follows.

$$\begin{aligned} \mathcal{L}_{\mathbb{C}}(X, \mathbb{C}) &:= \{\varphi : X \to \mathbb{C} \mid \varphi \text{ is } \mathbb{C}\text{-linear}\} \\ \mathcal{L}_{\mathbb{R}}(X, \mathbb{R}) &:= \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is } \mathbb{R}\text{-linear}\}. \end{aligned}$$

For $\varphi \in \mathcal{L}_C(X, \mathbb{C})$, we define $\operatorname{Re}(\varphi) : X \to \mathbb{R}$ and $\operatorname{Im}(\varphi) : X \to \mathbb{R}$ to denote the real and imaginary parts of φ.

Proposition: Let X be a $\mathbb{F}$-vector space.

(1) For $\varphi \in \mathcal{L}_{\mathbb{C}}(X, \mathbb{C})$, then $u := \operatorname{Re}(\varphi)$ and $v := \operatorname{Im}(\varphi)$, and $v(x) = -u(ix)$, implying $\varphi(x) = u(x) - iu(ix)$.

(2) For $u \in \mathcal{L}_{\mathbb{R}}(X, \mathbb{R})$, the map $v : X \to \mathbb{R}$ defined by $v(x) := -u(ix)$ belongs to $\mathcal{L}_{\mathbb{R}}(X, \mathbb{R})$, and the map $\varphi : X \to \mathbb{C}$ defined by $\varphi(x) = u(x) + iv(x)$ belongs to $\mathcal{L}_{\mathbb{C}}(X, \mathbb{C})$.

(3) If $(X, \|\cdot\|)$ is a normed space, with φ and u as in (1) or (2), then $\|\varphi\| = \|u\|$.

Proof.

(1) For $x, y \in X$ and $t \in \mathbb{R}$, we find

$$\begin{aligned} u(x + ty) &= \operatorname{Re}(\varphi(x + ty)) \\ &= \operatorname{Re}(\varphi(x) + t\varphi(y)) \\ &= \operatorname{Re}(\varphi(x)) + \operatorname{Re}(t\varphi(y)) \\ &= \operatorname{Re}(\varphi(x)) + t\operatorname{Re}(\varphi(y)) \\ &= u(x) + tu(y). \end{aligned}$$

For any $z \in \mathbb{C}$, we have $\operatorname{Re}(iz) = -\operatorname{Im}(z)$, meaning

$$\begin{aligned} v(x) &= \operatorname{Im}(\varphi(x)) \\ &= -\operatorname{Re}(i\varphi(x)) \\ &= -\operatorname{Re}(\varphi(ix)) \\ &= -u(ix). \end{aligned}$$

(2) Since u is $\mathbb{R}$-linear, so too are v and φ. Thus,

$$\begin{aligned}\varphi(ix) &= u(ix) - iu(-x)\\ &= u(ix) + iu(x)\\ &= i\varphi(x).\end{aligned}$$

Thus, φ is $\mathbb{C}$-linear.

(3) Note that $|u(x)| = |\text{Re}(\varphi(x))| \leqslant |\varphi(x)|$ for all $x \in X$, meaning $\|u\| \leqslant \|\phi\|$.

For $\varphi(x) \neq 0$, we set $\alpha = \frac{|\varphi(x)|}{\varphi(x)}$, meaning $|\alpha| = 1$ and $\alpha\varphi(x) = |\varphi(x)|$. Thus, we have

$$\begin{aligned}|\varphi(x)| &= \alpha\varphi(x)\\ &= \varphi(\alpha x)\\ &= \text{Re}(\varphi(\alpha x))\\ &= u(\alpha x)\\ &\leqslant \|u\|\,\|\alpha x\|\\ &= \|u\|\,\|x\|,\end{aligned}$$

meaning $\|\varphi\| \leqslant \|u\|$.

□

Definition (Minkowski Functional)**.** Let X be an $\mathbb{R}$-vector space. A Minkowski functional on X is a map $m : X \to \mathbb{R}$ such that the following are satisfied for every $x, y \in X$ and $t \geqslant 0$:

(i) $m(x+y) \leqslant m(x) + m(y)$;

(ii) $m(tx) = tm(x)$.

Before we go into the Hahn–Banach theorems, we want to consider the problem of whether, for an $\mathbb{F}$-vector space X and a subspace $E \subseteq X$, if there is a linear functional $\varphi : E \to \mathbb{F}$, is there an extension to $\psi : X \to \mathbb{F}$ such that $\psi|_E = \varphi$?

The answer is yes. If we select a basis for E, $\mathcal{B}_0$, we extend it to a basis for X, $\mathcal{B}$, and define $\psi_0 : X \to \mathbb{F}$ by sending $b \mapsto \varphi(b)$ for $b \in \mathcal{B}_0$, and 0 for elements in $\mathcal{B} \setminus \mathcal{B}_0$.

We may be interested in extending *bounded* linear functionals as well, and if we can control the norms of these various extensions. This is where the Hahn–Banach theorems start to play a major role.

Theorem (Hahn–Banach–Minkowski)**:** Let X be a real vector space with a Minkowski functional $m : X \to \mathbb{R}$, and let $E \subseteq X$ be a subspace. Suppose $\varphi : E \to \mathbb{R}$ is a linear functional with $\varphi(y) \leqslant m(y)$ for all $y \in E$.

Then, there exists a linear functional $\psi : X \to \mathbb{R}$ such that $\psi|_E = \varphi$ and $\psi(x) \leqslant m(x)$ for all $x \in X$.

Proof. Consider the collection $\mathcal{Z} = \{(Z, \phi)\}$ of all linear functionals that extend φ and are dominated by m. That is, $E \subseteq Z \subseteq X$ are subspaces, $\phi \in Z'$ with $\phi|_E = \varphi$, and $\phi(z) \leqslant m(z)$ for all $z \in Z$.

Note that (E, φ) is in $\mathcal{Z}$, meaning $\mathcal{Z}$ is nonempty. We define an ordering on $\mathcal{Z}$ by taking

$$(Z_1, \phi_1) \leqslant (Z_2, \phi_2)$$

if and only if $Z_1 \subseteq Z_2$ and $\phi_2|_{Z_1} = \phi_1$.

Consider a chain in $\mathcal{Z}$, $\mathcal{C} = (Z_i, \phi_i)_{i\in I}$. Let $Y = \bigcup_{i\in I} Z_i$. Since Z_i are totally ordered by inclusion, it is the case that Y is a subspace of X. We define $\eta : Y \to \mathbb{R}$ by $\eta(x) = \phi_i(x)$ for $x \in Z_i$. Since Z_i are totally ordered by inclusion, for $Z_j \subseteq Z_i$, it is the case that $\phi_i|_{Z_j} = \phi_i$, meaning (Y, η) is an upper bound for $\mathcal{C}$.

By Zorn's lemma,[ix] there is a maximal element (Z_0, ϕ_0). We will now show that $Z_0 = X$.

Suppose there exists $x_1 \in X \setminus Z_0$. Consider the subspace

$$Z_1 = \{x + \lambda x_1 \mid \lambda \in \mathbb{R},\ x \in Z_0\}.$$

Note that a vector in Z_1 has a unique expression $x + \lambda x_1$, since $x_1 \notin Z_0$. Thus, the linear functional $\phi_1 : Z_1 \to \mathbb{R}$ defined by $\phi_1(x + \lambda x_1) = \phi_0(x) + \lambda\alpha$ is well-defined. Note that $E \subseteq Z_1$ and $\phi_1 \in Z_1'$ extends φ, since ϕ_0 extends φ. We need to find some α such that $\phi_1(z) \leqslant m(z)$ for all $z \in Z_1$, which will contradict the maximality of (Z_0, ϕ_0).

For any $u, v \in Z_0$, we have

$$\begin{aligned}
\phi_0(u) + \phi_0(v) &= \phi_0(u + v) \\
&\leqslant m(u + v) \\
&= m(u - x_1 + v + x_1) \\
&\leqslant m(u - x_1) + m(v + x_1),
\end{aligned}$$

meaning that for all $u, v \in Z_0$,

$$\phi_0(u) - m(u - x_1) \leqslant m(v + x_1) - \phi_0(v).$$

We define

$$\alpha = \sup_{u \in Z_0} (\phi_0(u) - m(u - x_1)).$$

Then, for all $u \in Z_0$,

$$m(u - x_1) \geqslant \phi_0(u) - \alpha,$$

meaning $\alpha \leqslant m(v + x_1) - \phi_0(v)$ for all $v \in Z_0$, so

$$\phi_0(v) + \alpha \leqslant m(v + x_1).$$

Thus, for $t = \pm 1$, and $z \in Z_0$, we have

$$\phi_0(z) + t\alpha \leqslant m(z + t\alpha)$$

Scaling by $c > 0$, we find

$$\phi_0(cz) + ct\alpha \leqslant m(cz + ct\alpha).$$

Setting $c = |\lambda|$, $z = \frac{1}{c}x$, and $t = \operatorname{sgn}(\lambda)$, we find

$$\phi_1(x) + \lambda\alpha \leqslant m(x + \lambda\alpha),$$

thus contradicting maximality. □

Theorem (Hahn–Banach Extension)**:** Let X be a normed vector space, $E \subseteq X$ a subspace with a bounded linear functional $\varphi \in E^*$. Then, there exists a continuous $\psi \in X^*$ such that $\|\varphi\| = \|\psi\|$ and $\psi|_E = \varphi$.

[ix] If P is a partially ordered set, and every totally ordered $C \subseteq P$ admits an upper bound, then P has a maximal element.

Proof. Let $m(x) = \|\varphi\|\,\|x\|$ be our desired Minkowski functional. Restricting our view to $\mathbb{R}$, we find

$$\begin{aligned}\varphi(x) &\leqslant |\varphi(x)|\\ &\leqslant \|\varphi\|\,\|x\|\\ &= m(x).\end{aligned}$$

Thus, there exists $\psi \in X'$ such that $\psi|_E = \varphi$ and ψ is dominated by m on X.

We must now show that $\|\psi\| = \|\varphi\|$. We have

$$\begin{aligned}\psi(x) &\leqslant m(x)\\ &\leqslant \|\varphi\|\end{aligned}$$

for all $x \in B_X$, and we also have

$$\begin{aligned}-\psi(x) &= \psi(-x)\\ &\leqslant \|\varphi\|\,\|x\|\\ &\leqslant \|\varphi\|.\end{aligned}$$

Thus, $|\psi(x)| \leqslant \|\varphi\|$ for all $x \in B_X$. Additionally, since $\|\varphi\| \leqslant \|\psi\|$ necessarily (as ψ is defined on a larger space than φ), it is the case that $\|\varphi\| = \|\psi\|$.

Now, turning our attention to $\mathbb{C}$, we decompose $\varphi = u + iv$, where $u = \operatorname{Re}(\varphi)$ and $v(y) = -u(iy)$ for all $y \in E$. Then, $u : E \to \mathbb{R}$ is an $\mathbb{R}$-linear functional with $\|u\| = \|\varphi\|$. We extend u to $U \in X^*$ such that $\|U\| = \|u\| = \|\varphi\|$ with $U|_E = u$. Defining $V(x) = -U(ix)$ for all $x \in X$, we find $\psi = U + iV$ is a $\mathbb{C}$-linear functional with $\|\psi\| = \|U\| = \|\varphi\|$, and $\psi|_E = \varphi$. □

Thanks to the Hahn–Banach extension result, we can see that there are "enough" linear functionals in X^* for any normed vector space X.

Theorem (Hahn–Banach Separation)**:** Let X be a normed vector space.

(1) Given a nonzero $x_0 \in X$, there is a $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$. We call φ a norming functional.

(2) Given a proper closed subspace $E \subsetneq X$ and $x_0 \in X \setminus E$, there is a $\varphi \in X^*$ such that $\varphi|_E = 0$, $\|\varphi\| \leqslant 1$, and $\varphi(x_0) = \operatorname{dist}_E(x_0)$.

(3) The unit sphere in X^*, S_{X^*}, separates the points of X.

Proof.

(1) Let $E = \operatorname{span}(x_0)$, and $\varphi : E \to \mathbb{F}$ defined by $\varphi(\lambda x) = \lambda\|x\|$. This is a linear functional, with

$$\begin{aligned}|\varphi(\lambda x)| &= |\lambda|\,\|x\|\\ &= \|\lambda x\|,\end{aligned}$$

meaning $\|\varphi\| = 1$. We can extend φ to ψ on X such that $\psi|_E = \varphi$, meaning $\psi(x_0) = \varphi(x_0) = \|x_0\|$.

(2) There is a $\psi \in (X/E)^*$ with $\|\psi\| = 1$ and $\psi(x_0 + E) = \|x_0 + E\| = \operatorname{dist}_E(x_0)$. For the canonical quotient map $\pi : X \to X/E$, we set $\varphi = \psi \circ \pi$. Since π sends E to $0 + E$, φ must send any element of E to 0.

Additionally,

$$\begin{aligned}\|\varphi\| &= \|\psi \circ \pi\|\\ &\leqslant \|\psi\|\,\|\pi\|\\ &= 1.\end{aligned}$$

Thus, φ is such that $\varphi(x_0) = \operatorname{dist}_E(x_0)$.

(3) If $x \neq y$ with $x, y \in X$, then $x - y \neq 0$, meaning there is $\varphi \in S_{X^*}$ with $\varphi(x - y) = \|x - y\| \neq 0$, meaning $\varphi(x) \neq \varphi(y)$.

□

Corollary: Let X be a normed space. For every $x \in X$, we have

$$\begin{aligned} \sup_{\varphi \in B_{X^*}} |\varphi(x)| &= \sup_{\varphi \in S_{X^*}} |\varphi(x)| \\ &= \|x\|. \end{aligned}$$

Proposition: Let X be a normed vector space with separable dual X^*. Then, there is an isometric embedding $X \hookrightarrow \ell_\infty$.

Proof. Let $(\varphi_n)_{n \geqslant 1}$ be a norm-dense subset of B_{X^*}. Consider the map

$$\mu : X \to \ell_\infty$$

with $\mu(x) = (\varphi_n(x))_{n \geqslant 1}$ for $x \in X$. It is pretty clear that μ is linear. Since $\|\varphi_n\| \leqslant 1$ for each $n \geqslant 1$, we have

$$\begin{aligned} \|\mu(x)\|_\infty &= \sup_{n \geqslant 1} |\varphi_n(x)| \\ &\leqslant \sup_{n \geqslant 1} \|\varphi_n\| \|x\| \\ &\leqslant \|x\|. \end{aligned}$$

Suppose $\|\mu(x)\|_\infty < \|x\|$ for some $x \in B_X$. Then, since $x \neq 0$, we set $\delta = \|x\| - \|\mu(x)\|_\infty$.

We can find $\psi \in B_{X^*}$ with $\psi(x) = \|x\|$. Since $(\varphi_n)_{n \geqslant 1}$ is norm-dense, there is $m \in \mathbb{N}$ with

$$\|\psi - \varphi_m\| < \delta,$$

meaning

$$\begin{aligned} \|x\| &= \psi(x) \\ &\leqslant |\psi(x) - \varphi_m(x)| + |\varphi_m(x)| \\ &\leqslant \|\psi - \varphi_m\| \|x\| + \|\mu(x)\| \\ &< \delta + \|\mu(x)\|_\infty \\ &= \|x\|, \end{aligned}$$

which is a contradiction. Thus, μ is isometric on the unit ball. □

Definition. Let X be a normed vector space with $S \subseteq X$ and $T \subseteq X^*$ subsets. We define the annihilator of S by

$$S^\perp = \{\varphi \in X^* \mid \varphi(x) = 0 \ \forall x \in S\},$$

and the pre-annihilator of T by

$$^\perp T = \{x \in X \mid \varphi(x) = 0 \ \forall \varphi \in T\}.$$

Exercise: If X is a normed space and $E \subset X$ is a proper subspace, show that $E^\perp$ is nonzero. Also, show that $S^\perp \subseteq X^*$ and $^\perp T \subseteq X$ are norm-closed subspaces.

Solution:

(a) For $E \subset X$ a proper subspace, we know there exists some $\phi \in X^*$ such that $\|\phi\| \leqslant 1$ and $\|x_0\| = \operatorname{dist}_E(x_0)$ for some $x_0 \in X$, meaning $E^\perp \neq 0$.

(b) For $\varphi, \psi \in S^{\perp}$, we have $\varphi(x) = 0$ and $\psi(x) = 0$ for all $x \in \S$, meaning

$$\begin{aligned}(\varphi + \alpha\psi)(x) &= \varphi(x) + \alpha\psi(x)\\ &= 0,\end{aligned}$$

so $\varphi + \alpha\psi \in S^{\perp}$, so S is a subspace.

Additionally, for a sequence $(\varphi_n)_{n\geqslant 1}$ in $S^{\perp}$ such that $(\varphi_n)_n \to \varphi \in X^*$, we have $\varphi_n(x) = 0$ for all $x \in S$, so $\varphi(x) = 0$ for all $x \in S$.

(c) Similarly, for $x, y \in^{\perp} T$, we have

$$\begin{aligned}\varphi(x + \alpha y) &= \varphi(x) + \alpha\varphi(y)\\ &= 0,\end{aligned}$$

meaning ${}^{\perp}T$ is a subspace. Similarly, for any sequence $(x_n)_{n\geqslant 1}$ in ${}^{\perp}T$ such that $(x_n)_n \to x \in X$, we have $\varphi(x_n) = 0$ for all $\varphi \in T$, so $\varphi(x) = 0$.

Exercise: Let $T : X \to Y$ be a bounded linear map between normed vector spaces. Show

(1) $\text{Ran}(T)^{\perp} = \ker(T^*)$

(2) ${}^{\perp}\text{Ran}(T) = \ker(T)$.

Solution: Note that $T^* : Y^* \to X^*$ is defined by $T^*(\varphi)(x) = \varphi(T(x))$.

(1)

$$\begin{aligned}\text{Ran}(T)^{\perp} &= \{\varphi \in Y^* \mid \varphi(y) = 0,\ \forall y \in \text{Ran}(T)\}\\ &= \{\varphi \in Y^* \mid \varphi(T(x)) = 0,\ \forall x \in X\}\\ &= \{\varphi \in Y^* \mid T^*(\varphi)(x) = 0,\ \forall x \in X\}\\ &= \{\varphi \in Y^* \mid T^*(\varphi) = 0\}\\ &= \ker(T^*).\end{aligned}$$

(2)

$$\begin{aligned}{}^{\perp}\text{Ran}(T^*) &= \{x \in X \mid \psi(x) = 0,\ \forall \psi \in \text{Ran}(T^*)\}\\ &= \{x \in X \mid \varphi(T(x)) = 0,\ \forall \varphi \in Y^*\}\\ &= \{x \in X \mid T(x) = 0\}\\ &= \ker(T).\end{aligned}$$

Corollary: Let X be a normed vector space, and suppose $S \subseteq X$ is a subset. Then,

$${}^{\perp}\left(S^{\perp}\right) = \overline{\text{span}}(S).$$

Proof. Since $S \subseteq^{\perp} (S^{\perp})$, and the latter is a norm-closed subspace, it is the case that $Z = \overline{\text{span}}(S) \subseteq^{\perp} (S^{\perp})$.

Suppose there exists $x_0 \in^{\perp} (S^{\perp}) \setminus Z$. Then, we must have a $\varphi \in X^*$ with $\varphi|_Z = 0$ and $\varphi(x_0) \neq 0$, meaning $\varphi \in S^{\perp}$, so $\varphi(x_0) = 0$, which is a contradiction. □

Having used the Hahn–Banach theorems to understand some structure results on normed vector spaces, we now turn to an application to complex analysis.

Definition. Let X be a normed vector space and consider a function $f : \Omega \to X$, where $\Omega \subseteq \mathbb{C}$ is open. We say f is differentiable at $z \in \Omega$ if

$$\begin{aligned}f'(z) &= \lim_{w\to z} \frac{f(w) - f(z)}{w - z}\\ &= \lim_{\zeta\to 0} \frac{f(z+\zeta) - f(z)}{\zeta}\end{aligned}$$

exists in X. Here, the limit is defined through the norm of X.

If $f'(z)$ exists for all $z \in \Omega$, we say f is holomorphic on Ω. If $\Omega = \mathbb{C}$, then we say f is entire.

Example. Let A be a Banach algebra with $a \in A$ fixed. The exponential map

$$\begin{aligned} f : \mathbb{C} &\to A \\ z &\mapsto \exp(az) \end{aligned}$$

is entire.

For any $z \in \mathbb{C}$,

$$\begin{aligned} f'(z) &= \lim_{w \to 0} \frac{\exp(az + aw) - \exp(az)}{w} \\ &= \lim_{w \to 0} \frac{\exp(az)\exp(aw) - \exp(az)}{w} \\ &= \exp(az)\left(\lim_{w \to 0} \frac{\exp(aw) - 1}{w}\right) \\ &= a\exp(az). \end{aligned}$$

Lemma: Let X be a normed space, and suppose $f : \Omega \to X$ is differentiable at $z \in \Omega$. If $\varphi \in X^*$, then $\varphi \circ f$ is also differentiable at z. Similarly, if f is holomorphic on Ω, then $\varphi \circ f$ is holomorphic on Ω.

Proof. By continuity and linearity of φ, we have

$$\begin{aligned} (\varphi \circ f)'(z) &= \lim_{w \to z} \frac{\varphi(f(w)) - \varphi(f(z))}{w - z} \\ &= \lim_{w \to z} \varphi\left(\frac{f(w) - f(z)}{w - z}\right) \\ &= \varphi\left(\lim_{w \to z} \frac{f(w) - f(z)}{w - z}\right) \\ &= \varphi(f'(z)). \end{aligned}$$

□

Liouville's theorem is a fundamental result in complex function theory which provides insight into the behavior of holomorphic functions.

Theorem (Liouville's Theorem)**:** If $f : \mathbb{C} \to \mathbb{C}$ is entire, and $\sup_{z \in \mathbb{C}} |f(z)| < \infty$, then there is some $z_0 \in \mathbb{C}$ such that $f(z) = z_0$ for all $z \in \mathbb{C}$.

A proof of Liouville's theorem is presented below.

Proof. Let f be holomorphic and bounded on $\mathbb{C}$. Then, f is analytic, meaning

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

and Cauchy's integral formula gives

$$\begin{aligned} a_k &= \frac{f^{(k)}(0)}{k!} \\ &= \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w^{k+1}}\, dw, \end{aligned}$$

where C_r denotes a circle of radius r about the origin. Since f is bounded on $\mathbb{C}$, f is bounded on C_r. Thus, we have

$$|a_k| = \frac{1}{2\pi} \left| \oint_{C_r} \frac{f(w)}{w^{k+1}}\, dw \right|$$

$$\begin{aligned}
&\leqslant \frac{1}{2\pi}\oint_{C_r}\left|\frac{f(w)}{w^{k+1}}\right||dw|\\
&\leqslant \frac{1}{2\pi}\oint_{C_r}\frac{M}{r^{k+1}}|dw|\\
&= \frac{M}{2\pi r^{k+1}}\oint_{C_r}|dw|\\
&= \frac{M}{r^k}.
\end{aligned}$$

Since r was arbitrary, this means $a_k = 0$ for all $k \geqslant 1$, meaning $f(z) = a_0$ for all z. □

Corollary: Let X be a normed space. If $f : \mathbb{C} \rightarrow X$ is entire and bounded, then there is $x_0 \in X$ such that $f(z) = x_0$ for all $z \in \mathbb{C}$.

Proof. Suppose this is not the case. Then, there are $z_1, z_2 \in \mathbb{C}$ such that $f(z_1) \neq f(z_2)$. Thus, there is $\varphi \in X^*$ such that $\varphi(f(z_1)) \neq \varphi(f(z_2))$. Additionally, $\varphi \circ f : \mathbb{C} \rightarrow \mathbb{C}$ is entire, and

$$\begin{aligned}
\sup_{z\in\mathbb{C}}|\varphi\circ f| &= \sup_{z\in\mathbb{C}}|\varphi(f(z))|\\
&\leqslant \sup_{z\in\mathbb{C}}\|\varphi\||f(z)|\\
&< \infty,
\end{aligned}$$

so $\varphi \circ f$ is constant. However, this contradicts $\varphi(f(z_1)) \neq \varphi(f(z_2))$. □

Duality

If X is a normed vector spaces, we know that the continuous dual space, $X^* = \mathcal{B}(X, \mathbb{C})$, is a Banach space as $\mathbb{C}$ is complete. This section will be focused on understanding dual spaces and their operators.

Completions and the Double Dual

Definition (Double Dual)**.** We define $X^{**} = \mathcal{B}(X^*, \mathbb{C})$ to be the double dual of X.

Remark: X^{**} is complete.

Proposition: Let X be a normed vector space. If $x \in X$, the map

$$\begin{aligned}
\hat{x} &: X^* \rightarrow \mathbb{F}\\
\hat{x}(\varphi) &= \varphi(x)
\end{aligned}$$

is linear and bounded with $\hat{x}_{\text{op}} = \|x\|$. The map $\iota_X : X \hookrightarrow X^{**}$ with $x \mapsto \hat{x}$, is a linear isometry known as the canonical embedding.

Proof. First, we show that $\hat{x}$ is linear. Let $\varphi, \psi \in X^*$ and $\alpha \in \mathbb{F}$. Then,

$$\begin{aligned}
\hat{x}(\varphi + \alpha\psi) &= (\varphi + \alpha\psi)(x)\\
&= \varphi(x) + \alpha\psi(x)\\
&= \hat{x}(\varphi) + \alpha\hat{x}(\psi).
\end{aligned}$$

We can also see, by the Hahn–Banach Separation theorem,

$$\begin{aligned}
\|\hat{x}\|_{\text{op}} &= \sup_{\varphi\in B_{X^*}}|\hat{x}(\varphi)|\\
&= \sup_{\varphi\in B_{X^*}}|\varphi(x)|
\end{aligned}$$

$$= \|x\|.$$

Now, we will show that ι_X is linear. Let $x, y \in X$, $\lambda \in \mathbb{F}$. Then,

$$\begin{aligned}
\iota_X(x + \lambda y) &= \widehat{x + \lambda y} \\
&= \hat{x} + \widehat{\lambda y} \\
&= \hat{x} + \lambda \hat{y} \\
&= \iota_X(x) + \lambda \iota_X(y).
\end{aligned}$$

□

Definition (Completion of a Metric Space)**.** For a metric space X, the completion $\widetilde{X}$ is a complete metric space where $X \subseteq \widetilde{X}$ and $\overline{X} = \widetilde{X}$.

Since X^{**} is complete, we can make a completion of X using the canonical embedding.

Definition (Norm Completion)**.** Let X be a normed vector space. A norm completion of X is a pair (Z, j), with Z is a Banach space and $j : X \hookrightarrow Z$ is a linear isometry such that $\overline{\operatorname{Ran}}(j) = Z$.

First, we show the existence of a norm completion. Then, we will show that norm completions are unique up to isometric isomorphism.

Proposition: Let X be a normed vector space. Set $\widetilde{X} = \overline{\iota_X(X)}^{\|\cdot\|_{op}} \subseteq X^{**}$, where ι_X is the canonical embedding. The pair, $\left(\widetilde{X}, \iota_X\right)$, is a completion of X.

Proof. Since X^{**} is a Banach space, the closed subspace $\overline{\iota_X(X)}^{\|\cdot\|_{op}}$ is complete. Since ι_X is an isometric isomorphism, and $\overline{\operatorname{Ran}}(\iota_X) = \overline{\iota_X(X)}^{\|\cdot\|_{op}}$. □

Note: If X is a Banach space, $\operatorname{Ran}(\iota_X)$ is already norm-closed, since X is complete, so $\widetilde{X} = \iota_X(X) \cong X$.

To show that completions are unique up to isometric isomorphism, we need to be able to extend bounded linear operators from normed spaces to their completions that preserve the operator norm.

Proposition: Let X be a normed space with $E \subseteq X$ a linear subspace.

(1) The closure $\overline{E}$ is a closed linear subspace.

(2) If $T : E \to Z$ is a bounded linear map into a Banach space Z, then there is a unique bounded linear map $\widetilde{T} : \overline{E} \to Z$ such that $\widetilde{T}|_E = T$ and $\left\|\widetilde{T}\right\|_{op} = \|T\|_{op}$. If T is an isometry, then so is $\widetilde{T}$, and $\operatorname{Ran}\left(\widetilde{T}\right) = \overline{\operatorname{Ran}}(T)$.

Proof.

(1) Let $(x_n)_n \to x$ and $(y_n)_n \to y$ in $\overline{E}$. Then, $x + \alpha y = \lim_{n\to\infty}(x_n)_n + \alpha \lim_{n\to\infty}(y_n)_n$, or $x + \alpha y = \lim_{n\to\infty}(x_n + \alpha y_n)_n$, meaning $x + \alpha y \in \overline{E}$.

(2) Let $x \in \overline{E}$. There is a sequence $(x_n)_n$ in E with $(x_n)_n \to x$. Then,

$$\begin{aligned}
\|T(x_n) - T(x_m)\| &= \|T(x_n - x_m)\| \\
&\leqslant \|T\| \|x_n - x_m\|.
\end{aligned}$$

Thus, $(T(x_n))_n$ is a Cauchy sequence in Z, meaning it converges to $z \in Z$.

If $(x'_n)_n$ is a different sequence in E with $(x'_n)_n \to x$, then similarly, $(T(x'_n))_n$ is Cauchy in Z that converges to z'. We have

$$z = \lim_{n\to\infty} T(x_n)$$

$$\begin{aligned}
&= \lim_{n\to\infty} \left(T(x_n) - T(x_n') + T(x_n')\right)\\
&= \lim_{n\to\infty} \left(T(x_n) - T(x_n')\right) + \lim_{n\to\infty} T(x_n')\\
&= z'.
\end{aligned}$$

We define $\widetilde{T} : \overline{E} \to Z$ with $\widetilde{T}(x) := \lim_{n\to\infty} T(x_n)$ for any sequence $(x_n)_n$ in E that converges to x.

Linearity of $\widetilde{T}$ follows from the linearity of limits.

We now verify that $\left\|\widetilde{T}\right\|_{op} = \|T\|_{op}$.

$$\begin{aligned}
\left\|\widetilde{T}(x)\right\| &= \left\|\lim_{n\to\infty} T(x_n)\right\|\\
&= \lim_{n\to\infty} \|T(x_n)\|\\
&\leqslant \lim_{n\to\infty} \|T\| \|x_n\|\\
&= \|T\| \|x\|,
\end{aligned}$$

implying that $\left\|\widetilde{T}\right\|_{op} \leqslant \|T\|_{op}$. Additionally, since $\widetilde{T}$ is defined on a larger space than T, $\|T\|_{op} \leqslant \left\|\widetilde{T}\right\|_{op}$. Uniqueness follows from the fact that two continuous functions defined on a dense subset agree on the whole set.

Additionally, if T is an isometry, then

$$\begin{aligned}
\left\|\widetilde{T}(x)\right\| &= \left\|\lim_{n\to\infty} T(x_n)\right\|\\
&= \lim_{n\to\infty} \|T(x_n)\|\\
&= \lim_{n\to\infty} \|x_n\|\\
&= \|x\|.
\end{aligned}$$

By definition, we can see that $\operatorname{Ran}(T) \subseteq \operatorname{Ran}\left(\widetilde{T}\right) \subseteq \overline{\operatorname{Ran}}(T)$. Since $\widetilde{T}$ is an isometry, its range is closed, so we have $\operatorname{Ran}\left(\widetilde{T}\right) = \overline{\operatorname{Ran}}(T)$.

□

Theorem: Let X be a normed vector space. There is a Banach space $\widetilde{X}$ and a linear isometry $j : X \hookrightarrow \widetilde{X}$ such that $\overline{j(X)} = \widetilde{X}$. If (Z, k) is any other completion of X, then $Z \cong X$ are isometrically isomorphic.

Proof. We have shown existence with the earlier proposition.

Let (Z_1, j_1) and (Z_2, j_2) be norm completions of X. Consider the mappings $T : \operatorname{Ran}(j_1) \to Z_2$ and $S : \operatorname{Ran}(j_2) \to Z_1$ given by

$$\begin{aligned}
T(j_1(x)) &= j_2(x)\\
S(j_2(x)) &= j_1(x).
\end{aligned}$$

Since j_1 and j_2 are isometries, then T and S are well-defined linear isometries. Thus, T and S extend to isometries $T : Z_1 \to Z_2$ and $S : Z_2 \to Z_1$.

We have $S \circ T(z) = z$ for all $z \in \operatorname{Ran}(j_1)$, meaning $S \circ T = \mathrm{id}_{Z_1}$, and similarly, $T \circ S = \mathrm{id}_{Z_2}$. Thus, $Z_1 \cong Z_2$ are isometrically isomorphic. □

Corollary: Let Z be a Banach space with $X \subseteq Z$ a subspace. Then, $\widetilde{X} \cong \overline{X}$.

Remark: For a normed space $(X, \|\cdot\|)$, we let $\overline{X}^{\|\cdot\|}$ denote the norm completion $\widetilde{X}$, and $X \subseteq \overline{X}^{\|\cdot\|}$ is considered as a dense subspace.

We are now able to extend bounded linear operators to the completion.

Proposition: Let X and Y be normed spaces. If $T \in \mathcal{B}(X, Y)$, there is a unique $\widetilde{T} \in \mathcal{B}\left(\widetilde{X}, \widetilde{Y}\right)$ such that $\widetilde{T} \circ \iota_X = \iota_Y \circ T$. The following diagram commutes.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{T}"] & \widetilde{Y} \\ X \arrow[u, hook, "\iota_X"] \arrow[r, "T"] & Y \arrow[u, hook, "\iota_Y"'] \end{tikzcd}$$

Additionally, $\|T\|_{op} = \left\|\widetilde{T}\right\|_{op}$. If T is an isometry, then so is $\widetilde{T}$, and if T is an isometric isomorphism, so is $\widetilde{T}$.

Proof. Define $T_0 : \operatorname{Ran}(\iota_X) \to \tilde{Y}$ by $T_0(\iota_X(x)) = \iota_Y(T(x))$. Since ι_X is injective, T_0 is well-defined and linear. Additionally, since ι_X and ι_Y are isometries, for every $x \in X$, we have

$$\begin{aligned} \|T_0(\iota_X(x))\| &= \|\iota_Y(T(x))\| \\ &= \|T(x)\| \\ &\leqslant \|T\|_{op}\|x\| \\ &= \|T\|_{op}\|\iota_X(x)\|, \end{aligned}$$

meaning T_0 is bounded. Thus, T_0 extends to $\widetilde{T}$ defined on $\overline{\operatorname{Ran}}(\iota_X) = \tilde{X}$, such that $\left\|\widetilde{T}\right\|_{op} = \|T_0\|_{op} \leqslant \|T\|_{op}$. It is also the case that $\|T\|_{op} \leqslant \left\|\widetilde{T}\right\|_{op}$ since $\widetilde{T}$ is an extension.

Thus, the diagram commutes. Uniqueness follows from the fact that $\operatorname{Ran}(\iota_X) \subseteq \widetilde{X}$ is dense.

If T is isometric, then

$$\begin{aligned} \|T_0(\iota_X(x))\| &= \|\iota_Y(T(x))\| \\ &= \|T(x)\| \\ &= \|x\| \\ &= \|\iota_X(x)\|, \end{aligned}$$

meaning T_0 is an isometry. Thus, the extension to $\widetilde{T} : \widetilde{X} \to \widetilde{Y}$ is also an isometry.

If T is an isometric isomorphism, then

$$\begin{aligned} \operatorname{Ran}\left(\widetilde{T}\right) &= \overline{\operatorname{Ran}}(T_0) \\ &= \overline{\iota_Y(\operatorname{Ran}(T))} \\ &= \overline{\iota_Y(Y)} \\ &= \widetilde{Y}, \end{aligned}$$

meaning $\widetilde{T}$ is an isomorphism as well. □

Corollary: Let X be a vector space and Y a Banach space. Suppose there is an injective linear map $T : X \to Y$, and we write $\|\cdot\|_X$ to be the norm on X induced by T, defined by $\|x\|_X = \|T(x)\|_Y$. If $Z := \overline{T(X)}$ is the closure of the range of T in Y, then $\widetilde{X} \cong Z$.

Proof. By the definition of the norm, $T : X \to Z$ is an isometry into the Banach space Z, and so extends to $\widetilde{T} : \widetilde{X} \to Z$, where $\widetilde{T}|_X = T$. Since T is an isometry and $\widetilde{X}$ is complete, the range $\widetilde{T}\left(\widetilde{X}\right)$ is closed in Y, meaning

$$\begin{aligned} T(X) &\subseteq \widetilde{T}\left(\widetilde{X}\right) \\ \overline{T(X)} &\subseteq \widetilde{T}\left(\widetilde{X}\right). \end{aligned}$$

Thus, $\widetilde{T}$ is surjective. □

Dual Spaces

Definition. A normed vector space X is called a dual space if there is a normed vector space Z such that $Z^* \cong X$ are isometrically isomorphic. The space Z is known as the pre-dual of X.

It is generally not easy to find whether or not X is a dual space. However, we can attempt to find some dual spaces that for various Banach spaces.

A linear map is uniquely defined on the Hamel basis of its domain. However, when dealing with infinite-dimensional Banach spaces, we know that they cannot have a countable Hamel basis, which makes working with them very difficult. However, we can use a different notion of basis to assist us in our study.

Definition. Let X be a Banach space over $\mathbb{F}$. A sequence $(e_n)_n$ in X is called a Schauder basi for X if every $x \in X$ admits a unique sequence $(\lambda_n)_n$ in $\mathbb{F}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n$$

is a norm convergent sum in X.

Proposition: The canonical coordinate vectors $(e_n)_n$[x] is a Schauder basis for c_0 and ℓ_p for $1 \leqslant p < \infty$.

Proof. We will show this for c_0. Let $x \in c_0$, $x = (\lambda_n)_n$. For each $N \geqslant 1$, define

$$x_n = \sum_{n=1}^{N} \lambda_n e_n.$$

Whenever $N \geqslant M$, we have

$$\begin{aligned} \|x_N - x_M\|_\infty &= \left\| \sum_{n=M+1}^{N} \lambda_n e_n \right\|_\infty \\ &= \sup_{M+1 \leqslant n \leqslant N} |\lambda_n|. \end{aligned}$$

Since $(\lambda_n)_n \to 0$ as $x \in c_0$, we have $\|x_N - x_M\|_\infty$ is small for large N, M. Thus, $(x_N)_N$ is Cauchy.

By the completeness of c_0, we know that $\sum_{n=1}^{\infty} \lambda_n e_n$ thus converges. Since

$$\begin{aligned} \|x - x_N\|_\infty &= \sup_{n \geqslant N+1} |\lambda_n| \\ &\to 0, \end{aligned}$$

we have $x = \sum_{n=1}^{\infty} \lambda_n e_n$. Thus, we have shown existence.

[x]The sequence with 1 at position n and 0 everywhere else.

To show uniqueness, let $\varphi_k \in c_0^*$ be given by $\varphi_k\left((z_n)_n\right) = z_k$ for each $k \in \mathbb{N}$. Suppose that there exist two sequences of coefficients $(\lambda_n)_n$ and $(\mu_n)_n$ such that

$$\begin{aligned} x &= \sum_{n=1}^{\infty} \lambda_n e_n \\ &= \sum_{n=1}^{\infty} \mu_n e_n. \end{aligned}$$

For each k, since φ_k is continuous, we have

$$\begin{aligned} \lambda_k &= \sum_{n=1}^{\infty} \lambda_n \varphi_k\left(e_n\right) \\ &= \varphi_k\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) \\ &= \varphi_k\left(x\right) \\ &= \varphi_k\left(\sum_{n=1}^{\infty} \mu_n e_n\right) \\ &= \sum_{n=1}^{\infty} \mu_n \varphi_k\left(e_n\right) \\ &= \mu_k. \end{aligned}$$

Thus, $(\lambda_n)_n = (\mu_n)_n$. □

Exercise: Every Banach space that admits a Schauder basis is separable. Conclude that the canonical sequence of coordinate vectors $(e_n)_n$ is not a Schauder basis for ℓ_∞.

Solution: Let X admit a Schauder basis $(e_n)_n$, and let $(\lambda_n)_n$ be a sequence of coefficients in $\mathbb{C}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n.$$

Define

$$E = \left\{ \sum_{k=1}^{\infty} \alpha_k e_k \mid \alpha_k \in \mathbb{C}_{\mathbb{Q}} \right\}.$$

Since the coefficients in E are defined over $\mathbb{C}_{\mathbb{Q}}$, E is countable. We claim that $\overline{E} = X$.

Let $\varepsilon > 0$. Since $\overline{\mathbb{C}_{\mathbb{Q}}} = \mathbb{C}$, we find $\alpha_1 \in \mathbb{C}_{\mathbb{Q}}$ such that $\|(\alpha_1 - \lambda_1) e_1\| < \frac{\varepsilon}{2}$, and inductively find $\alpha_k \in \mathbb{C}_{\mathbb{Q}}$ such that

$$\|(\alpha_k - \lambda_k) e_k\| < \frac{\varepsilon}{2^k}.$$

Define $\mu = \sum_{k=1}^{\infty} \alpha_k e_k$. Then,

$$\begin{aligned} \|\mu - x\| &= \left\| \sum_{k=1}^{\infty} (\alpha_k - \lambda_k) e_k \right\| \\ &\leqslant \sum_{k=1}^{\infty} \|(\alpha_k - \lambda_k) e_k\| \\ &< \varepsilon. \end{aligned}$$

Since ℓ_∞ is not separable,[xi] it is the case that ℓ_∞ does not admit a Schauder basis.

[xi] The set $E = \{(a_k)_k \mid a_k \in \{0,1\}\}$ under $\|\cdot\|_\infty$ has the discrete metric but is uncountable (hence, not separable), but any subset of a separable set is necessarily separable.

We can now find some dual spaces.

Proposition: The dual space of c_0 is isometrically isomorphic to ℓ_1. That is, $c_0^* \cong \ell_1$.

Proof. Let $\theta : \ell_1 \to c_0^*$ be defined by $\theta(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n z_n$, where $\lambda = (\lambda_n)_n \in \ell_1$ and $z = (z_n)_n \in c_0$. We will show that θ is an isometric isomorphism.

We start by showing that θ is well-defined. To see this, note that

$$\begin{aligned}\sum_{n=1}^{\infty} |\lambda_n z_n| &\leqslant \sum_{n=1}^{\infty} |\lambda_n| \, \|z\|_{\infty} \\ &= \|\lambda\|_1 \, \|z\|_{\infty},\end{aligned}$$

which is finite since $\lambda \in \ell_1$ and $z \in \ell_\infty$. Thus, $\sum_{n=1}^{\infty} \lambda_n z_n$ converges absolutely, and thus is convergent.

For each $\lambda \in \ell_1$, the map $\theta(\lambda) : c_0 \to \mathbb{F}$ is linear. We now need to show that θ is bounded.

Let $z \in c_0$. Then,

$$\begin{aligned}|\theta(\lambda)(z)| &= \left|\sum_{n=1}^{\infty} \lambda_n z_n\right| \\ &\leqslant \sum_{n=1}^{\infty} |\lambda_n| \, |z_n| \\ &\leqslant \sum_{n=1}^{\infty} |\lambda_n| \, \|z\|_{\infty} \\ &= \|\lambda\|_1 \, \|z\|_{\infty}.\end{aligned}$$

Thus, $\|\theta(\lambda)\|_{op} \leqslant \|\lambda\|_1$, so $\theta(\lambda) \in c_0^*$.

The map $\lambda \mapsto \theta(\lambda)$ is linear. Now, we show that θ is an isometry. We know that $\|\theta(\lambda)\|_{op} \leqslant \|\lambda\|_1$. For $(\lambda_n)_n \in \ell_1$, we know know that

$$z_\lambda = \left(\operatorname{sgn}(\lambda_1), \operatorname{sgn}(\lambda_1), \dots, \operatorname{sgn}(\lambda_n)\right)$$

is a member of B_{c_0}, as its nonzero terms have modulus at most 1. Here, $\operatorname{sgn}(z)$ is the complex number of modulus 1 defined by $\operatorname{sgn}(z) = \frac{|z|}{z}$. Thus,

$$\begin{aligned}\|\theta(\lambda)\|_{op} &\geqslant |\theta(\lambda)(z_1)| \\ &= \left|\sum_{i=1}^{n} \lambda_i \operatorname{sgn}(\lambda_i)\right| \\ &= \sum_{i=1}^{n} |\lambda_i|.\end{aligned}$$

Sending $n \to \infty$, we have $\|\theta(\lambda)\|_{op} \geqslant \|\lambda\|_1$. Thus, θ is an isometry.

Finally, we must show that θ is onto. Let $\varphi \in c_0^*$, and let $\lambda_n = \varphi(e_n)$. We set $\omega_n = \operatorname{sgn}(\varphi(e_n))$ Then,

$$\sum_{n=1}^{N} |\lambda_n| = \sum_{n=1}^{N} |\varphi(e_n)|$$

$$\begin{aligned}
&= \sum_{n=1}^{N} \varphi_n \operatorname{sgn}\left(\varphi\left(e_n\right)\right)\\
&= \sum_{n=1}^{N} \varphi_n \left(e_n \omega_n\right)\\
&= \varphi\left(\sum_{n=1}^{N} \omega_n e_n\right)\\
&\leqslant \|\varphi\|_{op} \left\|\sum_{n=1}^{N} \omega_n e_n\right\|_{\infty}\\
&\leqslant \|\varphi\|_{op}.
\end{aligned}$$

Sending $N \to \infty$, we find that $(\lambda_n)_n \in \ell_1$. For an arbitrary $(z_n)_n \in c_0$, we have

$$\begin{aligned}
\varphi(z) &= \varphi\left(\sum_{z=1}^{\infty} z_n e_n\right)\\
&= \sum_{n=1}^{\infty} z_n \varphi\left(e_n\right)\\
&= \sum_{n=1}^{\infty} z_n \lambda_n\\
&= \theta(\lambda)(z),
\end{aligned}$$

thus meaning φ and $\theta(\lambda)$ agree on an arbitrary member of c_0. □

Remark: What made this problem so much easier than it might have been is the fact that since c_0 has a Schauder basis, we could define how $\varphi \in c_0^*$ behaved entirely through its action on the basis elements. This is similar to how we can find properties of linear maps in linear algebra by finding their actions on the Hamel bases of the vector space.

Exercise: Consider the map $\phi : \ell_1 \to \ell_\infty^*$ defined by $\phi(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n z_n$ for $\lambda = (\lambda_n)_n \in \ell_1$ and $z = (z_n)_n \in \ell_\infty$. Prove that ϕ is a linear isometry, but ϕ is *not* onto.

Solution: A similar method to the proof that $c_0^* \cong \ell_1$ gives ϕ is an isometry. We will replicate it here.

First, we show that ϕ is bounded and well-defined.

$$\begin{aligned}
|\phi(\lambda)(z)| &= \left|\sum_{n=1}^{\infty} \lambda_n z_n\right|\\
&\leqslant \sum_{n=1}^{\infty} |\lambda_n z_n|\\
&\leqslant \sum_{n=1}^{\infty} |\lambda_n| \|z\|_{\infty}\\
&= \|\lambda\|_1 \|z\|_{\infty}\\
&< \infty.
\end{aligned}$$

Thus, $\|\phi(\lambda)\|_{op} \leqslant \|\lambda\|_1$, so ϕ is bounded. It is also clear that ϕ is linear in λ and in z. Similarly, for $\lambda = (\lambda_n)_n \in \ell_1$, we define

$$\begin{aligned}
z_\lambda &= \left(\operatorname{sgn}(\lambda_1), \dots, \operatorname{sgn}(\lambda_n), 0, 0, \dots\right)\\
&\in B_{c_0}.
\end{aligned}$$

Thus,

$$\|\phi(\lambda)\|_{op} \geqslant |\phi(\lambda)(z_\lambda)|$$

$$= \sum_{k=1}^{n} |\lambda_k|,$$

meaning $\|\phi(\lambda)\|_{op} \geqslant \|\lambda\|_1$, so ϕ is an isometry.

In particular, ϕ is an isometric isomorphism from ℓ_1 to $c_0^* \subset \ell_\infty^*$, so ϕ cannot be onto ℓ_∞^*.

Proposition: The dual of ℓ_1 is isometrically isomorphic to ℓ_∞.

Proof. Let $\theta : \ell_\infty \to \ell_1^*$ be defined by $\theta(z)(\lambda) = \sum_{n=1}^{\infty} z_n \lambda_n$ for $z = (z_n)_n \in \ell_\infty$ and $(\lambda_n)_n \in \ell_1$.

We can see that θ is well-defined with $\|\theta(z)\|_{op} \leqslant \|z\|_\infty$ by a similar reasoning to the case of $\theta : \ell_1 \to c_0^*$. Additionally, since

$$\begin{aligned} \|\theta(z)\|_{op} &\geqslant |\theta(z)(e_n)| \\ &= |z_n| \end{aligned}$$

for every $n \geqslant 1$, we have $\|z\|_\infty \leqslant \|\theta(z)\|_{op}$, meaning θ is an isometry.

Suppose $\varphi \in \ell_1^*$. Set $z_n = \varphi(e_n)$. Note that $z \in \ell_\infty$ since $|z_n| = |\varphi(e_n)| \leqslant \|\varphi\|_{op}$ for all $n \geqslant 1$. For $\lambda = (\lambda_n)_n \in \ell_1$, then $\lambda = \sum_{n=1}^{\infty} \lambda_n e_n$ is convergent in the 1-norm, as

$$\begin{aligned} \left\|\sum_{n=1}^{\infty} \lambda_n e_n\right\| &\leqslant \sum_{n=1}^{\infty} \|\lambda_n e_n\| \\ &= \sum_{n=1}^{\infty} |\lambda_n| \|e_n\| \\ &= \sum_{n=1}^{\infty} |\lambda_n| \\ &< \infty. \end{aligned}$$

Thus,

$$\begin{aligned} \varphi(\lambda) &= \varphi\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) \\ &= \sum_{n=1}^{\infty} \lambda_n \varphi(e_n) \\ &= \sum_{n=1}^{\infty} \lambda_n z_n \\ &= \theta(z)(\lambda). \end{aligned}$$

□

This result can be generalized to L_p spaces, which are spaces of measurable functions satisfying some integration property.

Theorem: Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(1) If μ is a σ-finite measure, and $p, q \in (1, \infty)$ satisfying $p^{-1} + q^{-1} = 1$, then the map $L_q(\Omega, \mu) \to L_p(\Omega, \mu)^*$ given by $f \mapsto \varphi_f$, with

$$\varphi_f(g) = \int_\Omega gf \, d\mu,$$

is an isometric isomorphism.

(2) If μ is semi-finite, then $L_\infty \to L_1(\Omega,\mu)^*$ given by $f \mapsto \varphi_f$ is an isometric isomorphism.

We will now turn our attention to spaces of continuous functions. Let Ω be a locally compact Hausdorff space. We want to understand $C_0(\Omega)^*$, the space of bounded linear functionals on $C_0(\Omega)$.

Note that $C_0(\Omega)$ is a $*$-algebra, and admits a cone of positive elements, $C_0(\Omega)_+$.[XII] We start by understanding the properties of positive linear functionals.

Definition. Let Ω be a locally compact Hausdorff space.

(1) A linear functional $\varphi : C_0(\Omega) \to \mathbb{C}$ is called positive if $\varphi(f) \geqslant 0$ whenever $f \in C_0(\Omega)_+$.

(2) A positive linear functional $\varphi : C_0(\Omega) \to \mathbb{C}$ is called faithful if

$$\ker(\varphi) \cap C_0(\Omega)_+ = \{0\}.$$

(3) A state on $C_0(\Omega)$ is a positive linear functional $\varphi \in C_0(\Omega)'$ with $\|\varphi\|_{op} = 1$. The collection of states

$$\begin{aligned} S(C_0(\Omega)) &= \{\varphi \in C_0(\Omega)^* \mid \varphi \text{ is a state}\} \\ &\subseteq B_{C_0(\Omega)^*}. \end{aligned}$$

is called the state space of Ω, which we abbreviate $S(\Omega)$.

Lemma: Let Ω be a locally compact Hausdorff space, and suppose $\varphi : C_0(\varphi) \to \mathbb{C}$ is a positive linear functional.

(1) If $h \in C_0(\Omega,\mathbb{R})$, then $\varphi(h) \in \mathbb{R}$.

(2) If $h \leqslant k$ in $C_0(\Omega,\mathbb{R})$, then $\varphi(h) \leqslant \varphi(k)$.

(3) If $f \in C_0(\Omega)$, then $\varphi(f^*) = \overline{\varphi(f)}$.

Proof.

(1) We write $h = h_+ - h_-$, and see that $\varphi(h) = \varphi(h_+) - \varphi(h_-) \in \mathbb{R}$.

(2) If $h \leqslant k$ in $C_0(\Omega,\mathbb{R})$, then $k - h \geqslant 0$, so $\varphi(k-h) \geqslant 0$, so $\varphi(h) \leqslant \varphi(k)$.

(3) We write f as the Cartesian decomposition, $f = h + ik$, where $h, k \in C_0(\Omega,\mathbb{R})$. Thus,

$$\begin{aligned} \varphi(f^*) &= \varphi(h - ik) \\ &= \varphi(h) - i\varphi(k) \\ &= \overline{\varphi(h) + i\varphi k} \\ &= \overline{\varphi(f)}. \end{aligned}$$

□

Proposition: If $\varphi : C_0(\Omega) \to \mathbb{C}$ is positive and linear, then φ is bounded.

Proof. Let

$$M = \sup\{\varphi(f) \mid f \in C_0(\Omega)_+, \|f\|_u \leqslant 1\}.$$

We claim that M is finite. If $M = \infty<$ then for each $n \in \mathbb{N}$, there exists $f_n \in C_0(\Omega)_+$ with $\|f_n\|_u \leqslant 1$ and $\varphi(f_n) \geqslant 2^n$. Note that

$$\sum_{n=1}^{\infty} \|2^{-n} f_n\|_u \leqslant \sum_{n=1}^{\infty} 2^{-n}$$

[XII] $C_0(\Omega)_+ = \{f \in C_0(\Omega) \mid f(x) \geqslant 0 \; \forall x \in \Omega\}$.

$$= 1,$$

meaning the series $f = \sum_{n=1}^{\infty} 2^{-n} f_n$ converges uniformly in $C_0(\Omega)$.

For $N \in \mathbb{N}$, we can see that $f \geqslant \sum_{n=1}^{N} 2^{-n} f_n \geqslant 0$, as f_n are all positive. Since φ preserves the order on $C_0(\Omega)$, we have

$$\begin{aligned}
\varphi(f) &\geqslant \varphi\left(\sum_{n=1}^{N} 2^{-n} f_n\right)\\
&= \sum_{n=1}^{n} 2^{-n} \varphi(f_n)\\
&\geqslant \sum_{n=1}^{N} 2^{-n} 2^{n}\\
&= N.
\end{aligned}$$

This is absurd since N is arbitrary. Thus, $M < \infty$.

Let $f \in C_0(\Omega)$ with $\|f\|_u \leqslant 1$. Let $f = h + ik$ be the Cartesian decomposition. Then, we can write $f = (h_+ - h_-) + i(k_+ - k_-)$. Since φ is linear, $|\varphi(f)| \leqslant 4M$, meaning $\|\varphi\|_{op} \leqslant 4M$. □

Proposition: If Ω is a compact Hausdorff space, and $\varphi : C(\Omega) \to \mathbb{F}$ is linear, then φ is positive if and only if $\|\varphi\| = \varphi(\mathbb{1}_\Omega)$.

Proof. Let $h \in C(\Omega, \mathbb{R})$. Then,

$$-\|h\|\, \mathbb{1}_\Omega \leqslant h \leqslant \|h\|\, \mathbb{1}_\Omega.$$

Thus, since φ is positive,

$$-\|h\|\, \varphi(\mathbb{1}_\Omega) \leqslant \varphi(h) \leqslant \|h\|\, \varphi(\mathbb{1}_\Omega).$$

Thus, $|\varphi(h)| \leqslant \|h\|\, \varphi(\mathbb{1}_\Omega)$.

Let $f \in C(\Omega)$, and find $\omega \in \mathbb{C}$ such that $|\omega| = 1$ and $\omega\varphi(f) = |\varphi(f)|$. Set $g = \omega f$. Note that $\|g\|_u = \|f\|_u$. Then,

$$\begin{aligned}
|\varphi(f)| &= \operatorname{Re}(|\varphi(f)|)\\
&= \operatorname{Re}(\omega\varphi(f))\\
&= \operatorname{Re}(\varphi(\omega f))\\
&= \operatorname{Re}(\varphi(g))\\
&= \frac{1}{2}\left(\varphi(g) + \overline{\varphi(g)}\right)\\
&= \varphi\left(\frac{g + \overline{g}}{2}\right)\\
&= \varphi(\operatorname{Re}(g))\\
&\leqslant \varphi(\mathbb{1}_\Omega)\, \|\operatorname{Re}(g)\|_u\\
&\leqslant \varphi(\mathbb{1}_\Omega)\, \|g\|\\
&= \varphi(\mathbb{1}_\Omega)\, \|f\|_u,
\end{aligned}$$

meaning $\|\varphi\|_{op} \leqslant \varphi(\mathbb{1}_\Omega)$.

Suppose $\|\varphi\| = \varphi(\mathbb{1}_\Omega)$. We start by claiming that if $h \in C(\Omega, \mathbb{R})$, then $\varphi(h) \in \mathbb{R}$. Let $\varphi(h) = a + bi$; we will show that $b = 0$. Let $t \in \mathbb{R}$ be arbitrary.

$$\begin{aligned}\|h + it\mathbb{1}_\Omega\|_u^2 &= \left(\sup_{x\in\Omega} |h(x) + it|\right)^2 \\ &= \sup_{x\in\Omega} |h(x) + it|^2 \\ &= \sup_{x\in\Omega} \left((h(x))^2 + t^2\right) \\ &= \|h\|_u^2 + t^2.\end{aligned}$$

Thus,

$$\begin{aligned}|\varphi(h + it\mathbb{1}_\Omega)|^2 &\leqslant (\varphi(\mathbb{1}_\Omega))^2 \|h + it\mathbb{1}_\Omega\|^2 \\ &= (\varphi(\mathbb{1}_\Omega))^2 \left(\|h\|^2 + t^2\right).\end{aligned}$$

Alternatively,

$$\begin{aligned}|\varphi(h + it\mathbb{1}_\Omega)|^2 &= |\varphi(h) + it\varphi(\mathbb{1}_\Omega)|^2 \\ &= |a + i(b + t\varphi(\mathbb{1}_\Omega))|^2 \\ &= a^2 + (b + t\varphi(\mathbb{1}_\Omega))^2 \\ &= a^2 + b^2 + t^2(\varphi(\mathbb{1}_\Omega))^2 + 2bt\varphi(\mathbb{1}_\Omega).\end{aligned}$$

Thus, we get $a^2 + b^2 + 2bt\varphi(\mathbb{1}_\Omega) \leqslant (\varphi(\mathbb{1}_\Omega))\|h\|^2$. If $b \neq 0$, sending $t \to \infty$ yields a contradiction. Thus, $b = 0$.

Assume $f \in C(\Omega)_+$. Observe that $\operatorname{Ran}(f - \|f\| \mathbb{1}_\Omega) \subseteq [-\|f\|, \|f\|]$, meaning $\|f - \|f\| \mathbb{1}_\Omega\| \leqslant \|f\|$. Thus,

$$\begin{aligned}|\varphi(f) - \|f\| \varphi(\mathbb{1}_\Omega)| &= |\varphi(f - \|f\| \mathbb{1}_\Omega)| \\ &\leqslant \varphi(\mathbb{1}_\Omega) \|f - \|f\| \mathbb{1}_\Omega\| \\ &\leqslant \varphi(\mathbb{1}_\Omega) \|f\|.\end{aligned}$$

Since $\varphi(f) \in \mathbb{R}$, we have $\varphi(f) \geqslant 0$, so φ is positive. □

Corollary: Let Ω be a compact Hausdorff space. The state space

$$S(\Omega) = \{\varphi \in C(\Omega)^* \mid \varphi(\mathbb{1}_\Omega) = 1 = \|\varphi\|\}$$

is convex.

Proof. Suppose $\varphi_1, \varphi_2 \in S(\Omega)$, $t \in [0, 1]$. Then, $(1-t)\varphi_1 + t\varphi_2$ is a positive functional, meaning

$$\begin{aligned}\|(1-t)\varphi_1 + t\varphi_2\| &= (1-t)\varphi(\mathbb{1}_\Omega) + t\varphi(\mathbb{1}_\Omega) \\ &= (1-t) + t \\ &= 1,\end{aligned}$$

so $S(\Omega)$ is convex. □

Now, we can associate bounded linear functionals with regular measures.

Theorem (Riesz Representation Theorem)**:** Let Ω be a locally compact Hausdorff space. If $\varphi : C_c(\Omega) \to \mathbb{C}$ is a positive linear functional, then there exists a unique Radon measure μ such that

$$\varphi(f) = \int_\Omega f \, d\mu,$$

with $|\varphi(f)| \leqslant \|f\| \mu(\operatorname{supp}(f))$. The Radon measure μ also satisfies the following.

(1) For every open $U \subseteq \Omega$, we have

$$\mu(U) = \sup\left\{\varphi(f) \mid f \in C_c\left(\Omega, [0,1]\right), \operatorname{supp}(f) \subseteq U\right\}.$$

(2) For every compact $K \subseteq \Omega$,

$$\mu(K) = \inf\left\{\varphi(f) \mid f \geqslant \mathbb{1}_K\right\}.$$

The following proof is adapted from Folland's *Real Analysis*.

Notation: For an open set U, we let $f \prec U$ denote $f \in C_c\left(\Omega, [0,1]\right)$ and $\operatorname{supp}(f) \subset U$.

Proof. We start by establishing uniqueness.

Let μ be a Radon measure such that $\varphi(f) = \int_\Omega f\, d\mu$ for all $f \in C_c(\Omega)$. For $U \subseteq X$ open, it is the case that $\varphi(f) \leqslant \mu(U)$ for $f \prec U$.

If $K \subseteq U$ is compact, Urysohn's lemma gives $f \in C_c(\Omega)$ such that $f \prec U$ and $f = 1$ on K, meaning $\mu(K) \leqslant \int_\Omega f\, d\mu = \varphi(f)$. Since μ is inner regular on U, we satisfy condition (1). Thus, μ is determined on all open sets by φ, and so is determined on all Borel sets since μ is outer regular.

To show existence, we start by defining

$$\mu(U) = \sup\left\{\varphi\left(f\right) \mid f \in C_c\left(\Omega\right), f \prec U\right\}.$$

For any $E \subseteq X$, we define

$$\mu^*(E) = \inf\left\{\mu\left(U\right) \mid E \subseteq U,\ U \text{ open}\right\}.$$

Note that by the way we define μ, if $U \subseteq V$, then $\mu(U) \leqslant \mu(V)$, meaning $\mu^*(U) = \mu(U)$ if U is open.

We begin by showing that μ^* is an outer measure. It suffices to show that for a sequence of open sets $\left\{U_j\right\}_{j=1}^n$ such that $U = \bigcup_{j=1}^\infty U_j$, $\mu(U) \leqslant \sum_{j=1}^\infty \mu\left(U_j\right)$. From this, we can see that, for any $E \subseteq X$,

$$\begin{aligned}\mu^*(E) &= \inf\left\{\sum_{j=1}^\infty \mu\left(U_j\right) \mid U_j \text{ open}, E \subseteq \bigcup_{j=1}^\infty U_j\right\}\\ &= \inf\left\{\mu(U) \mid E \subseteq U,\ U \text{ open}\right\}\end{aligned}$$

defines an outer measure.

Let $U = \bigcup_{j=1}^\infty U_j$ and $f \in C_c(\Omega)$ with $f \prec U$. Let $K = \operatorname{supp}(f)$. Since K is compact, $K \subseteq \bigcup_{i=1}^n U_i$ for some n.

Proposition: If K is a compact set in a locally compact Hausdorff space Ω with an open cover $\left\{U_j\right\}_{j=1}^n$, then there is a partition of unity[XIII] on K consisting of compactly supported functions g_j such that $\operatorname{supp}\left(g_j\right) \subseteq U_j$.

Proof. For each $x \in K$, there is a compact neighborhood N_x such that $N_x \subseteq U_j$ for some j. The set $\{N_x^\circ \mid x \in K\}$ is an open cover of K, meaning there exist $x_1, \dots, x_m$ such that $K \subseteq \bigcup_{k=1}^m N_{x_k}$.

Let F_j be the union of the set of N_{x_k} that are a subset of U_j. Then, $F_j \subseteq U_j$ is compact, so by Urysohn's lemma, there are $g_1, \dots, g_n \in C_c(\Omega)$ such that $g_j = 1$ on F_j and $\operatorname{supp}\left(g_j\right) \subseteq U_j$.

[XIII] A collection of functions in $C\left(X, [0,1]\right)$ such that their sum is 1 on E and every neighborhood has only finitely many nonzero functions.

Since F_j covers K, it is the case that $\sum_{k=1}^{n} g_k \geqslant 1$ on K. By Urysohn's Lemma, there $f \in C_c(\Omega, [0,1])$ such that $f = 1$ on K and $\operatorname{supp}(f) \subseteq \left\{x \mid \sum_{k=1}^{n} g_k(x) > 0\right\}$.

Define $g_{n+1} = 1 - f$, meaning $\sum_{k=1}^{n+1} g_k > 0$ everywhere. For $j = 1, \dots, n$, let $h_j = \frac{g_j}{\sum_{k=1}^{n+1} g_k}$. Then, $\operatorname{supp}(h_j) = \operatorname{supp}(g_j)$, $\operatorname{supp}(h_j) \subseteq U_j$, and $\sum_{j=1}^{n} h_j = 1$ on K. □

With this proposition, we can construct $g_i \prec U_i$ with $\sum_{i=1}^{n} g_j = 1$ on K. Then, $f = \sum_{i=1}^{n} fg_i$, with $fg_i \prec U_i$. Thus,

$$\begin{aligned}
\varphi(f) &= \sum_{i=1}^{n} \varphi(fg_i) \\
&\leqslant \sum_{i=1}^{n} \mu(U_i) \\
&\leqslant \sum_{j=1}^{\infty} \mu(U_j).
\end{aligned}$$

Thus, $\sup_{f \in C_c(\Omega)} (\varphi(f)) = \mu(U) \leqslant \sum_{j=1}^{\infty} \mu(U_j)$. Since this holds for any $f \prec U$, we conclude that $\mu(U) \leqslant \sum_{j=1}^{\infty} \mu(U_j)$.

We now show that for any open set $U \subseteq \Omega$, U is μ^*-measurable.

Let U be open, $E \subseteq \Omega$ with $\mu^*(E) < \infty$. Suppose E is open. Then, $E \cap U$ is open, so for $\varepsilon > 0$, we can find $f \in C_c(\Omega)$ such that $f \prec E \cap U$ and $\varphi(f) > \mu(E \cap U) - \varepsilon$. Additionally, $E \setminus \operatorname{supp}(f)$ is open, so we can find $g \in C_c(\Omega)$ such that $g \prec E \setminus \operatorname{supp}(f)$, and $\varphi(g) > \mu(E \setminus \operatorname{supp}(f)) - \varepsilon$. However, $f + g \prec E$, so

$$\begin{aligned}
\mu(E) &\geqslant \varphi(f) + \varphi(g) \\
&> \mu(E \cap U) + \mu(E \setminus \operatorname{supp}(f)) - 2\varepsilon \\
&\geqslant \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon.
\end{aligned}$$

We obtain the desired inequality as $\varepsilon \to 0$.

For any $E \subseteq \Omega$, we find V open with $E \subseteq V$ and $\mu(V) < \mu^*(E) + \varepsilon$ (by our definition of μ^*). Thus,

$$\begin{aligned}
\mu^*(E) + \varepsilon &> \mu(V) \\
&\geqslant \mu^*(V \cap U) + \mu^*(V \setminus U) \\
&\geqslant \mu^*(E \cap U) + \mu^*(E \setminus U).
\end{aligned}$$

We let $\varepsilon \to 0$ to obtain our desired inequality.

Now, we show that for any compact $K \subseteq \Omega$, $\mu(K) = \inf\{\varphi(f) \mid f \in C_c(\Omega), f \geqslant \mathbb{1}_K\}$.

Let K be compact, $f \in C_c(\Omega)$, and $f \geqslant \mathbb{1}_K$. Let $U_\varepsilon = \{x \mid f(x) > 1 - \varepsilon\}$. Then, U_ε is open, and for $g \prec U_\varepsilon$, we have $(1-\varepsilon)^{-1} f - g \geqslant 0$.

Thus, $\varphi(g) \leqslant (1-\varepsilon)^{-1}\varphi(f)$, meaning $\mu(K) \leqslant \mu(U_\varepsilon) \leqslant (1-\varepsilon)^{-1}\varphi(f)$. Letting $\varepsilon \to 0$, we find $\mu(K) \leqslant \varphi(f)$.

For any $K \subseteq U$, Urysohn's lemma gives $f \in C_c(\Omega)$ such that $f \geqslant \mathbb{1}_K$ and $f \prec U$, meaning $\varphi(f) \leqslant \mu(U)$. Since μ is outer regular, we obtain condition (2).

With our measure μ defined, we only need to show that $\varphi(f) = \int_\Omega f\, d\mu$ for $f \in C_c(\Omega, [0,1])$, as $C_c(\Omega)$ is the linear span of $C_c(\Omega, [0,1])$.

Set $f \in C_c(\Omega, [0,1])$. Given $N \in \mathbb{N}$, for $1 \leqslant j \leqslant N$, let $K_j = \left\{x \mid f(x) \geqslant \frac{j}{N}\right\}$. Let $K_0 = \operatorname{supp}(f)$. Define $f_1, \dots, f_N \in C_c(\Omega)$ with $f_j(x) = 0$ for $x \notin K_{j-1}$, $f_j(x) = f(x) - \frac{j-1}{N}$ for $x \in K_{j-1} \setminus K_j$, and $f_j = \frac{1}{N}$ for $x \in K_j$.

We, thus have $\frac{\mathbb{1}_{K_j}}{N} \leqslant f_j \leqslant \frac{\mathbb{1}_{K_{j-1}}}{N}$, meaning

$$\frac{1}{N}\mu(K_j) \leqslant \int_\Omega f_j \, d\mu \leqslant \frac{1}{N}\mu(K_{j-1}).$$

If U is an open set containing K_{j-1}, then $Nf_j \prec U$, meaning $\varphi(f_j) \leqslant \frac{\mu(U)}{N}$. Thus, by condition (2), and outer regularity,

$$\frac{1}{N}\mu(K_j) \leqslant \varphi(f_j) \leqslant \frac{1}{N}\mu(K_{j-1}).$$

Since $f = \sum_{j=1}^N f_j$, we have

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leqslant \int_\Omega f \, d\mu \leqslant \frac{1}{N}\sum_{j=1}^{N}\mu(K_{j-1}),$$

and

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leqslant \varphi(f) \leqslant \frac{1}{N}\sum_{j=1}^{N-1}\mu(K_j).$$

Thus,

$$\begin{aligned}\left|\varphi(f) - \int_\Omega f \, d\mu\right| &\leqslant \frac{\mu(K_0) - \mu(K_N)}{N}\\ &\leqslant \frac{\mu(\operatorname{supp}(f))}{N}.\end{aligned}$$

Since $\mu(\operatorname{supp}(f)) < \infty$, and N is arbitrary, we have $I(f) = \int_\Omega f \, d\mu$. $\square$

Having proven the Riesz Representation Theorem, we can begin to describe all bounded linear functionals on continuous function spaces, starting with the positive ones.

Proposition: Let Ω be a locally compact Hausdorff space.

(1) For μ a positive and finite regular measure on Ω,

$$\begin{aligned}\varphi_\mu &: C_0(\Omega) \to \mathbb{C}\\ \varphi_\mu(f) &= \int_\Omega f \, d\mu\end{aligned}$$

defines a positive linear functional with $\|\varphi_\mu\|_{op} = \mu(\Omega)$. Moreover, φ_μ is faithful if μ has full support.

(2) If $\varphi : C_0(\Omega) \to \mathbb{C}$ is a positive linear functional, then there exists a unique positive and finite regular measure μ_φ such that $\varphi - \varphi_{\mu_\varphi}$. That is,

$$\varphi(f) = \int_\Omega f \, d\mu_\varphi.$$

Moreover, $\|\varphi\|_{op} = \mu(\Omega)$, and μ has full support if φ is faithful.

Proof. The map

$$\varphi_\mu(f) = \int_\Omega f\, d\mu$$

is positive, linear, and well=defined, since

$$\begin{aligned}\left|\varphi_\mu(f)\right| &\leqslant \left|\int_\Omega f\, d\mu\right|\\ &\leqslant \int_\Omega |f|\, d\mu\\ &\leqslant \|f\|_u\, \mu(\Omega).\end{aligned}$$

This also shows that $\left\|\varphi_\mu\right\|_{op} \leqslant \mu(\Omega)$. Given $\varepsilon > 0$, the regularity of μ allows us to find $K \subseteq \Omega$ with $\mu(\Omega) - \varepsilon < \mu(K)$. Urysohn's lemma gives a continuous $f \in C_c(\Omega, [0, 1])$ with $f = 1$ on K. Thus,

$$\begin{aligned}\left\|\varphi_\mu\right\|_{op} &\geqslant \varphi_\mu(f)\\ &= \int_\Omega f\, d\mu\\ &\geqslant \int_\Omega \mathbb{1}_K\, d\mu\\ &= \mu(K)\\ &> \mu(\Omega) - \varepsilon.\end{aligned}$$

Since ε was arbitrary, $\mu(\Omega) \leqslant \left\|\varphi_\mu\right\|_{op}$.

Suppose $g \in C_0(\Omega)$ with $g \geqslant 0$ and $g \neq 0$. Then, there is a nonempty open subset $U \subseteq \Omega$ and $\delta > 0$ such that $g(x) \geqslant \delta$ for all $x \in U$. If μ has full support, $\mu(U) > 0$, meaning

$$\begin{aligned}\varphi(g) &= \int_\Omega g\, d\mu\\ &\geqslant \int_\Omega \delta\mathbb{1}_U\, d\mu\\ &= \delta\mu(U)\\ &> 0.\end{aligned}$$

Thus, we have proven (1).

If $\varphi : C_0(\Omega) \to \mathbb{C}$ is a positive linear functional, then φ is bounded. The restriction $\varphi_0 = \varphi|_{C_c(\Omega)} : C_c(\Omega) \to \mathbb{C}$ is positive and linear, meaning $\varphi_0 = I_\mu$ for some Radon measure μ by the Riesz Representation Theorem. We set $\mu_\varphi = \mu$.

By the Riesz Representation theorem, we have

$$\begin{aligned}\mu(\Omega) &= \sup\left\{I_\mu(f) \mid f \in C_c(\Omega, [0, 1]),\ \operatorname{supp}(f) \subseteq \Omega\right\}\\ &\leqslant \sup\left\{\varphi(f) \mid f \in C_0(\Omega), \|f\| \leqslant 1\right\}\\ &= \|\varphi\|_{op}\\ &< \infty.\end{aligned}$$

Additionally, if $f \in C_c(\Omega)$, then

$$|\varphi(f)| = |\varphi_0(f)|$$

$$
\begin{aligned}
&\leqslant \left|\int_{\Omega} f\,d\mu\right|\\
&\leqslant \int_{\Omega} |f|\,d\mu\\
&\leqslant \|f\|_{u}\,\mu(\Omega).
\end{aligned}
$$

Since $C_c(\Omega) \subseteq C_0(\Omega)$ is dense, we have $|\varphi(f)| \leqslant \|f\|_u\,\mu(\Omega)$ for all $f \in C_0(\Omega)$, meaning $\|\varphi\|_{op} \leqslant \mu(\Omega)$.

Since μ is finite and Radon, μ is regular. We can see that φ and φ_u are bounded functionals that agree on $C_c(\Omega)$, so $\varphi = \varphi_\mu$.

Suppose φ is faithful. Let $U \subseteq \Omega$ be open and nonempty. By Urysohn's lemma, there is $f : \Omega \to [0,1]$ that is nonzero and compactly supported with $\operatorname{supp}(f) \subseteq U$. Thus,

$$
\begin{aligned}
\mu(U) &= \int_{\Omega} \mathbb{1}_U\,d\mu\\
&\geqslant \int_{\Omega} f\,d\mu\\
&= \varphi_\mu(f)\\
&= \varphi(f)\\
&> 0,
\end{aligned}
$$

meaning μ has full support. □

We let $\mathcal{P}_r(\Omega)$ be the convex set of all regular probability measures on $(\Omega, \mathcal{B}_\Omega)$. The following corollary follows from the previous proposition.

Corollary: Let Ω be a compact Hausdorff space. There exists a bijective affine map $T : S(\Omega) \to \mathcal{P}_r(\Omega)$ given by $T(\varphi) = \mu_\varphi$.

We can use the positive linear functionals to generate all bounded linear functionals on a function space.

Theorem (Jordan Decomposition)**:** Let Ω be a locally compact Hausdorff space, and suppose $\varphi : C_0(\Omega) \to \mathbb{C}$ be bounded linear. Then, there exists positive linear functionals $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ such that

$$
\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4),
$$

and $\|\varphi_j\|_{op} \leqslant \|\varphi\|_{op}$.

Remark: The Jordan decomposition for positive linear functionals follows from the Hahn decomposition for complex measures.

If $\varphi \in C_0(\Omega)^*$, then the Jordan decomposition provides $\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4)$, which gives the positive regular measures $\mu_1, \dots, \mu_4$ such that $\varphi_j(f) = \int_\Omega f\,d\mu_j$ for $f \in C_0(\Omega)$.

Then, $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ has total variation $|\mu|$, meaning it is regular itself. Thus,

$$
\begin{aligned}
\varphi(f) &= \varphi_1(f) - \varphi_2(f) + i(\varphi_3(f) - \varphi_4(f))\\
&= \int_{\Omega} f\,d\mu_1 - \int_{\Omega} f\,d\mu_2 + i\left(\int_{\Omega} f\,d\mu_4 - \int_{\Omega} f\,d\mu_4\right)\\
&= \int_{\Omega} f\,d\mu\\
&= \varphi_\mu(f).
\end{aligned}
$$

Thus, we get the following characterization of $C_0(\Omega)^*$.

Theorem (Riesz–Markov Theorem)**:** Let Ω be a locally compact Hausdorff space. Then, $M_r(\Omega) \cong C_0(\Omega)^*$. The map $\Phi : M_r(\Omega) \to C_0(\Omega)^*$ defined by $\Phi(\mu) = \varphi_\mu$, where

$$\varphi_\mu = \int_\Omega f \, d\mu,$$

is an isometric isomorphism.

The Banach Space Adjoint

Definition. Let X, Y be normed vector spaces, and let $T \in \mathcal{B}(X, Y)$. Then, the map $T^* : Y^* \to X^*$, defined by $T^*(\varphi) = \varphi \circ T$ is called the Banach space adjoint (or adjoint) of T.

We can see that T^* is well-defined since it is the composition of two bounded linear operators. Additionally,

$$\begin{aligned} \|T^*(\varphi)\|_{op} &= \|\varphi \circ T\|_{op} \\ &\leqslant \|\varphi\|_{op} \|T\|_{op}. \end{aligned}$$

Proposition: If X and Y are normed vector spaces spaces, $T \in \mathcal{B}(X, Y)$, and T^* be the adjoint of T. Then, $T^* \in \mathcal{B}(Y^*, X^*)$, and $\|T^*\|_{op} = \|T\|_{op}$.

Proof. Let $\varphi_1, \varphi_2 \in Y^*$, $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned} T^*(\varphi_1 + \alpha\varphi_2) &= (\varphi_1 + \alpha\varphi_2) \circ T \\ &= \varphi_1 \circ T + \alpha\varphi_2 \circ T \\ &= T^*(\varphi_1) + \alpha T^*(\varphi_2), \end{aligned}$$

meaning T^* is linear. Recall that for any normed space Z and $z \in Z$, we have $\|z\| = \sup_{\varphi \in B_{Z^*}} |\varphi(z)|$. Thus, for $x \in X$, we have

$$\begin{aligned} \|T(x)\| &= \sup_{\varphi \in B_{Y^*}} |\varphi(T(x))| \\ &= \sup_{\varphi \in B_{Y^*}} |T^*(\varphi(x))| \\ &= \sup_{\varphi \in B_{Y^*}} |T^*(\varphi)| \|x\| \\ &\leqslant \|T^*\|_{op}. \end{aligned}$$

Thus, we have $\|T\|_{op} \leqslant \|T^*\|_{op}$. $\square$

We can now understand some of the analytic and algebraic properties of the adjoint.

Theorem: Let X, Y, Z be normed spaces, and let $T, S : X \to Y$, $R : Y \to Z$ be bounded linear operators. The following are true.

(1) $(T + S)^* = T^* + S^*$

(2) $(\alpha T)^* = \alpha T^*$

(3) $(R \circ T)^* = T^* \circ R^*$

(4) $(\mathrm{id}_X)^* = \mathrm{id}_{X^*}$

(5) The following diagram commutes.

$$\begin{tikzcd} X^{**} \arrow[r, "T^{**}"] & Y^{**} \\ X \arrow[u, hook, "\iota_X"] \arrow[r, "T"'] & Y \arrow[u, hook, "\iota_Y"'] \end{tikzcd}$$

Proof of (5).

$$\begin{aligned} T^{**}\left(\iota_X(x)\right)(\varphi) &= T^{**}(\hat{x})(\varphi) \\ &= \hat{x} \circ T^{*}(\varphi) \\ &= \hat{x}(\varphi \circ T) \\ &= \varphi \circ T(x) \\ &= \widehat{T(x)}(\varphi) \\ &= \iota_Y(T(x))(\varphi). \end{aligned}$$

Thus, $T^{**}(\iota_X(x)) = \iota_Y(T(x))$, meaning $T^{**} \circ \iota_X = \iota_Y \circ T$. ◻

Theorem: Let X and Y be normed spaces.

(1) If $T : X \to Y$ is an isometric isomorphism, then $T^* : Y^* \to X^*$ is an isometric isomorphism.

(2) If $T : X \to Y$ is a bicontinuous isomorphism, then $\left(T^{-1}\right)^* = (T^*)^{-1}$, so T^* is a bicontinuous isomorphism.

(3) If $T : X \to Y$ is an isometry, then $T^* : Y^* \to X^*$ is a 1-quotient map. We have $T^*(B_{Y^*}) = B_{X^*}$.

(4) If $T : X \to Y$ is a 1-quotient map, then $T^* : Y^* \to X^*$.

(5) $T : X \to Y$ is an isometry if and only if $T^{**} : X^{**} \to Y^{**}$ is an isometry.

(6) If $T \in \mathcal{B}(X, Y)$, and T^* is a 1-quotient map, then T is an isometry.

(7) If X, Y are Banach spaces, and $T \in \mathcal{B}(X, Y)$, and T^* is an isometry, then T is a 1-quotient map.

Proof. Starting by proving (2), we have

$$\begin{aligned} \mathrm{id}_{X^*} &= (\mathrm{id}_X)^* \\ &= \left(T^{-1} \circ T\right)^* \\ &= T^* \circ \left(T^{-1}\right)^* \end{aligned}$$

and

$$\mathrm{id}_{X^*} = T^* \circ \left(T^{-1}\right)^*,$$

meaning $(T^*)^{-1} = \left(T^{-1}\right)^*$.

Now, turning our attention to (4), if T is a 1-quotient map, then $T(U_X) = U_Y$. For $\psi \in Y^*$, we have

$$\begin{aligned} \|T^*(\psi)\|_{op} &= \|\psi \circ T\|_{op} \\ &= \sup_{x \in U_X} \|\psi \circ T(x)\| \\ &= \sup_{y \in Y} \|\psi(y)\| \\ &= \|\psi\|_{op}. \end{aligned}$$

Thus, T^* is an isometry.

Focusing on (3), we have $\|T^*\|_{op} = \|T\|_{op} = 1$, so $T^*(B_{Y^*}) \subseteq B_{X^*}$. Let $\varphi \in B_{X^*}$. We want to find $\psi \in B_{Y^*}$ such that $T^*(\psi) = \varphi$. Let $\psi_0 : \operatorname{Ran}(T) \to \mathbb{F}$ be defined by $\psi_0(T(x)) = \varphi(x)$. THis is well-defined since T is injective. For $x \in B_X$, we have

$$\begin{aligned}|\psi_0(T(x))| &= |\varphi(x)| \\ &\leqslant \|\varphi\|_{op}\|x\| \\ &\leqslant \|\varphi\|_{op}.\end{aligned}$$

Thus, $\|\psi_0\|_{op} \leqslant \|\varphi\|_{op} \leqslant 1$. There is an extension $\psi \in Y^*$ of ψ_0 such that $\|\psi\|_{op} = \|\psi_0\|_{op}$, and $\psi \circ T = \varphi$. Thus, $T^*(B_{Y^*}) = B_{X^*}$.

The veracity of (1) follows from (2), (3), and (4).

To show (5), we let T be isometric, which implies that T^* is a 1-quotient map. Thus, we have T^{**} is isometric. Conversely, if T^{**} is isometric, then

$$\begin{aligned}\|T(x)\| &= \|\iota_Y(T(x))\| \\ &= \|T^{**}(\iota_X(x))\| \\ &= \|\iota_X(x)\| \\ &= \|x\|.\end{aligned}$$

To see (6), if T^* is a 1-quotient map, then T^{**} is isometric, so T is isometric.

We will show (7) later. □

Example. Let $\tau : \Omega \to \Lambda$ be a proper map[XIV] between locally compact Hausdorff spaces. The induced contractive map $T_\tau : C_0(\Lambda) \to C_0(\Omega)$ has an adjoint map

$$T_\tau^* : C_0(\Omega) \to C_0(\Lambda)^*,$$

where $T_\tau^*(\varphi) = \varphi \circ T_\tau$. By the Riesz–Markov theorem, we can identify $C_0(\Omega)^* \cong M_r(\Omega)$.

The push forward measure is an induced map

$$\tau_* : M(\Omega) \to M(\Lambda),$$

defined by

$$\tau_*\mu(E) := \mu\left(\tau^{-1}(E)\right)$$

for any $E \in \mathcal{B}_\Lambda$. We can verify that $\tau_*\mu$ is a complex measure on $(\Lambda, \mathcal{B}_\Lambda)$, and that $\tau_*(\mathcal{P}(\Omega)) \subseteq \mathcal{P}(\Lambda)$.

We can also see that for all $f \in C_0(\Lambda)$,

$$\int_\Omega f \circ \tau \, d\mu = \int_\Lambda f \, d(\tau_*\mu).$$

This is because, for some $E \in \mathcal{B}_\Lambda$, we have $\mathbb{1}_E \circ \tau = \mathbb{1}_{\tau^{-1}(E)}$, so

$$\int_\Omega \mathbb{1}_E \circ \tau \, d\mu = \int_\Omega \mathbb{1}_{\tau^{-1}(E)} \, d\mu$$

[XIV] Inverse image of a compact subset of Λ is compact in Ω

$$\begin{aligned}
&= \mu\left(\tau^{-1}(E)\right)\\
&= \tau_{*}\mu(E)\\
&= \int_{\Lambda} \mathbb{1}_{E}\, d(\tau_{*}\mu).
\end{aligned}$$

Thus, since this identity holds for all simple functions, and every bounded function is the uniform limit of a simiple function, we get that

$$\int_{\Omega} f \circ \tau \, d\mu = \int_{\Lambda} f \, d(\tau_{*}\mu)$$

for all $f \in C_0(\Lambda)$ by Dominated Convergence.

Thus, we can see that for any $\tau : \Omega \to \Lambda$ that is a proper map between locally compact Hausdorff spaces, then $T_{\tau}^{*}(\varphi_{\mu}) = \varphi_{\tau_{*}\mu}$ for all $\mu \in M_r(\Omega)$.

$$\begin{aligned}
T_{\tau}^{*}(\varphi_{\mu})(f) &= \varphi_{\mu} \circ T_{\tau}(f)\\
&= \varphi_{\mu}(f \circ \tau)\\
&= \int_{\Omega} f \circ \tau \, d\mu\\
&= \int_{\Lambda} f \, d(\tau_{*}\mu)\\
&= \varphi_{\tau_{*}\mu}(f).
\end{aligned}$$

This also shows that the push forward of a regular complex Borel measure is regular, or $\tau_{*}(M_r(\Omega)) \subseteq M_r(\Lambda)$.

Suppose we are dealing with the special case of $\tau : \Omega \to \Lambda$ is a homeomorphism. Then, T_{τ} is an isometric isomorphism of Banach spaces, meaning

$$T_{\tau}^{*} : C_0(\Omega)^{*} \to C_0(\Lambda)^{*},$$

or

$$\tau_{*} : M_r(\Omega) \to M_r(\Lambda),$$

is an isometric isomorphism of Banach spaces. The restriction to the state space

$$T_{\tau}^{*}|_{S(\Omega)} : S(\Omega) \to S(\Lambda),$$

or

$$\tau_{*}|_{\mathcal{P}_r(\Omega)} : \mathcal{P}_r(\Omega) \to \mathcal{P}_r(\Lambda)$$

is an affine bijection.

This allows us to consider the idea of an *invariant* measure (or, equivalently, invariant state).

Definition. Let Ω be a locally compact state, and let $\tau : \Omega \to \Omega$ be a continuous transformation. A regular Borel probability measure, $\mu \in \mathcal{P}_r(\Omega)$ is called τ-invariant if $\tau_{*}\mu = \mu$, or $T_{\tau}^{*}(\varphi_{\mu}) = \varphi_{\mu}$. We write

$$\mathcal{P}_r(\Omega, \tau) := \{\mu \in \mathcal{P}_r(\Omega) \mid \tau_{*}\mu = \mu\}$$

to denote the set of τ-invariant measures.

We can now use the analytic and algebraic properties of adjoints to identify duals of subspaces and quotient spaces. Recall that for a normed vector space X and $E \subseteq X$ a subset, then

$$E^{\perp} = \{\varphi \in X^* \mid \varphi|_E = 0\}$$

is the annihilator of E.

Proposition: Let X be a normed vector space, $E \subseteq X$ a subspace. The dual space of E is isometrically isomorphic to $X^*/E^{\perp}$.

Proof. The embedding $\iota : E \hookrightarrow X$ is an isometry, so $\iota^* : X^* \to E^*$ is a 1-quotient map. The first isomorphism theorem gives $X^*/\ker(\iota^*) \cong E^*$. Thus,

$$\begin{aligned}\ker(\iota^*) &= \{\varphi \in X^* \mid \iota^*(\varphi) = 0\}\\ &= \{\varphi \in X^* \mid \varphi \circ \iota = 0\}\\ &= \{\varphi \in X^* \mid \varphi|_E = 0\}\\ &= E^{\perp}.\end{aligned}$$

□

Now, we can look at the dual of a quotient space.

Proposition: Let X be a normed vector space and $E \subseteq X$ a closed subspace. Then, $(X/E)^* \cong E^{\perp}$.

Proof. Let $\pi : X \to X/E$ be the canonical projection map. Since π is a 1-quotient map, we have $\pi^* : (X/E)^* \to X^*$ is an isometry, so $(X/E)^* \cong \operatorname{Ran}(\pi^*)$. We have

$$\begin{aligned}\operatorname{Ran}(\pi^*) &= \{\pi^*(\psi) \mid \psi \in (X/E)^*\}\\ &= \{\psi \circ \pi \mid \psi \in (X/E)^*\}.\end{aligned}$$

We claim that $\operatorname{Ran}(\pi^*) = E^{\perp}$. For any $\psi \in (X/E)^*$, we have $\psi \circ \pi \in X^*$, and $\psi \circ \pi|_E = 0$, meaning $\operatorname{Ran}(\pi^*) \subseteq E^{\perp}$.

For the reverse direction, if $\varphi \in E^{\perp}$, and $\varphi : X \to \mathbb{C}$ sends E to 0, then φ factors through X/E, meaning $\varphi(x) = \widetilde{\varphi}(x + E)$ for some bounded linear operator $\widetilde{\varphi}$ such that $\|\widetilde{\varphi}\|_{op} = \|\varphi\|_{op}$. Thus, $\widetilde{\varphi} \circ \pi = \varphi$, so $\varphi \in \operatorname{Ran}(\pi^*)$, so $\operatorname{Ran}(\pi^*) = E^{\perp}$.

Thus, we have $(X/E)^* \cong \operatorname{Ran}(\pi^*) \cong E^{\perp}$. □

Proposition (Existence of Auerbach Basis)**:** Let X be a finite dimensional normed vector space. Then, there exists a collection $\{u_1, \dots, u_n\} \subseteq S_X$ that forms a basis for X, and a collection $\{\varphi_1, \dots, \varphi_n\} \subseteq S_{X^*}$ that forms a basis for X^*, such that $\varphi_u\left(u_j\right) = \delta_{ij}$. We call the collection $\{(u_i, \varphi_i)\}_{j=1}^{n}$ and Auerbach basis for X.

Proof. Let $\theta : X \to \ell_\infty^n$ be a bicontinuous isomorphism. Consider the map

$$D : \underbrace{B_X \times \cdots \times B_X}_{n \text{ times}} \to \mathbb{C}$$

defined by

$$D(x_1, \dots, x_n) = \det\begin{pmatrix}\theta(x_1) & \theta(x_2) & \cdots & \theta(x_n)\end{pmatrix},$$

where $\theta(x_n)$ are column vectors. It is the case that D is continuous, as D is a polynomial over its entries, and since $B_X \times \cdots \times B_X$ is compact (as B_X is compact, and the finite product of compact sets is compact), we must have

$$\max_{x_j \in B_X} |D(x_1, \dots, x_n)| = |D(u_1, \dots, u_n)|$$

$$:= M$$

for some $u_1, \ldots, u_n \in B_X \times \cdots \times B_X$. Define $\varphi : X \to \mathbb{C}$ by

$$\varphi_k(x) = \frac{1}{M} D(u_1, \ldots, x, \ldots, u_n),$$

where x is in the kth index. Since D is multilinear, φ is linear. Additionally, $|\varphi_k(x)| \leqslant 1$ for each B_x, so $\|\varphi_k\|_{op} \leqslant 1$, while $|\varphi_k(u_k)| = 1$, so $\|\varphi_k\|_{op} = 1$.

Let $\omega_k = \operatorname{sgn}(\varphi_k(u_k))$, and set $v_k = \omega_k u_k$. Then, $\varphi_k(v_j) = \delta_{jk}$.

If $D(x_1, \ldots, x_n) = 0$, then $\{\theta(x_1), \ldots, \theta(x_n)\}$ is linearly dependent in $\mathbb{F}^n$, and since θ is a linear isomorphism, then $\{x_1, \ldots, x_n\}$ is linearly dependent. Thus, $\{v_1, \ldots, v_n\}$ is a basis for X.

Similarly, if $x = \sum_{i=1}^{n} \alpha_i v_i$ with at least one $\alpha_i \neq 0$, then

$$(\varphi_1 + \cdots + \varphi_n)(x) \neq 0$$

so $\{\varphi_i\}_{i=1}^{n}$ is a basis for X^*.

□

Proposition: Let $E \subseteq X$ be a subspace of a normed vector space X. If E is finite-dimensional, then E is topologically complemented — i.e., there exists $F \subseteq X$ closed such that $X = E \oplus F$, or, equivalently, there exists $P \in \mathcal{B}(X)$ with $P^2 = P$ and $\operatorname{Ran}(P) = E$.

Proof. Since $\dim(E) = n$, E admits an Auerbach basis, $\{(u_i, \varphi_i)\}_{i=1}^{n}$. For each $i \in \{1, \ldots, n\}$, the Hahn–Banach theorem allows an extension to $\psi_i \in X^*$ with $\|\varphi_i\|_{op} = \|\psi_i\|_{op} = 1$.

We will define $P : X \to X$ by

$$P(x) = \sum_{k=1}^{n} \psi_k(x) u_k.$$

We can see that P is linear, and $\operatorname{Ran}(P) \subseteq E$. Note that $P(y) = y$ for all $y \in E$, since

$$\begin{aligned} P(u_j) &= \sum_{k=1}^{n} \psi_k(u_j) u_j \\ &= \sum_{k=1}^{n} \varphi_k(u_j) u_j \\ &= \sum_{k=1}^{n} \delta_{kj} u_j \\ &= u_j \end{aligned}$$

for each j. It follows that, for $y = \sum_{j=1}^{n} \alpha_j u_j$,

$$\begin{aligned} P(y) &= \sum_{j=1}^{n} \alpha_j P(u_j) \\ &= \sum_{j=1}^{n} \alpha_j u_j \\ &= y. \end{aligned}$$

Thus, $\operatorname{Ran}(P) = E$ and $P^2 = P$. □

Exercise: If Z is a normed vector space with $z_0 \in Z$ and $\psi \in Z^*$ fixed, prove that the map $T : Z \to Z$ defined by $T(z) = \psi(z)z_0$ is continuous.

Solution:

$$\begin{aligned}\|T(z)\| &= \|\psi(z)z_0\| \\ &= \|z_0\| \, \|\psi(z)\| \\ &\leqslant \|z_0\| \, \|\psi\|_{op} \, \|z\|,\end{aligned}$$

meaning $\|T\|_{op} \leqslant \|z_0\| \, \|\psi\|_{op}$, so T is bounded.

Properties of Reflexive Spaces

Definition. A normed vector space X is reflesive if $\iota_X : X \hookrightarrow X^{**}$ is surjective.

Proposition: Let X be a normed vector space, $E \subseteq X$ a subspace. The following are true.

(1) If X is finite-dimensional, then X is reflexive.

(2) If X is reflexive, then E is reflexive.

(3) If X is reflexive, then X/E is reflexive.

(4) X is reflexive if and only if X^* is reflexive.

Proof. To prove (1), let $\mathcal{B} = \{u_1, \dots, u_n\}$ be a basis for X. Since X is finite-dimensional, all linear operators on X are continuous. In particular, $X^* = X' = \mathcal{L}(X, \mathbb{F})$. Consider the linear functional $\varepsilon_I : X \to \mathbb{F}$ defined by

$$\varepsilon_i \left(\sum_{j=1}^{n} \alpha_j u_j \right) = \alpha_i.$$

Then $\mathcal{B}' = \{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for X^*, so $\dim(X) = \dim(X^*)$. Similarly, $\dim(X^*) = \dim(X^{**})$. Since $\iota_X : X \to X^{**}$ is injective, the rank-nullity theorem has ι_X is also surjective. Thus, X is reflexive.

To prove (2), we can see that if $\iota_X : X \to X^{**}$ is a bijection, then $\iota_X|_E : E \to E^{**}$ is also a bijection, so E is reflexive.

To see the forward direction of (4), we let X be reflexive, and consider $\iota_{X^*} : X^* \hookrightarrow X^{***}$. We will show that ι_{X^*} is surjective.

Let $h \in X^{***}$, so $h : X^{**} \to \mathbb{F}$ is bounded linear. Let $\varphi = h \circ \iota_X \in X^*$.

We claim that $\iota_{X^*}(\varphi) = h$. In order to do this, we will show that $\iota_{X^*}(\varphi)(\xi) = h(\xi)$ for all $\xi \in X^{**}$. For each $\xi \in X^{**}$, we know that $\xi = \iota_X(x) = \hat{x}$ for some $x \in X$ since X is reflexive. Thus,

$$\begin{aligned}\iota_{X^*}(\varphi)(\xi) &= \hat{\varphi}(\xi) \\ &= \xi(\varphi) \\ &= \hat{x}(\varphi) \\ &= \varphi(x).\end{aligned}$$

Alternatively, we have

$$\begin{aligned}h(\xi) &= h(\hat{x}) \\ &= h(\iota_X(x)) \\ &- h \circ \iota_X(x) \\ &= \varphi(x).\end{aligned}$$

For the converse, assuming X^* is reflexive, we have shown that X^{**} is reflexive. However, since $\iota_X(X) \subseteq X^{**}$, it is the case that X is reflexive. $\square$