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**Problem (Problem 1):** Describe the topology of the Grassmanian  $\text{Gr}(k, n)$  in a uniform way, so that  $\mathbb{R}P^n$  becomes the special case of  $\text{Gr}(1, n)$ .

**Solution:** We let elements of  $\text{Gr}(k, n)$  be defined as equivalence classes of linearly independent  $k$ -tuples of vectors in  $\mathbb{R}^n$ , where  $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$  if  $\text{span}\{v_1, \dots, v_k\} = \text{span}\{w_1, \dots, w_k\}$ .

By extending  $(v_1, \dots, v_k)$  and  $(w_1, \dots, w_k)$  to ordered bases  $\mathcal{B}_1 = (v_1, \dots, v_n)$  and  $\mathcal{B}_2 = (w_1, \dots, w_n)$ , we see that these  $k$ -tuples are equivalent if and only if there is an invertible linear transformation  $Q$  with matrix representation

$$Q = \begin{pmatrix} A & H \\ 0 & B \end{pmatrix},$$

where  $A$  is a  $k \times k$  invertible matrix, and  $B$  is a  $(n - k) \times (n - k)$  invertible matrix, so that

$$Q[v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n] = [w_1, \dots, w_k, w_{k+1}^*, \dots, w_n^*],$$

where  $\{w_{k+1}^*, \dots, w_n^*\}$  is a basis for the  $n - k$ -dimensional complementary subspace. The subgroup of all such  $Q \subseteq \text{GL}_n(\mathbb{R})$ , which we call  $P$ , is the stabilizer of  $\text{Gr}(k, n)$  as we have defined it, so by the orbit-stabilizer theorem (seeing as  $\text{GL}_n(\mathbb{R})$  acts transitively on all ordered bases of  $\mathbb{R}^n$ ), we obtain  $\text{Gr}(k, n) \cong \text{GL}_n(\mathbb{R})/P$ , where the latter coset space is given the quotient topology.

This definition comports with the definition of  $\mathbb{R}P^n$  as the space of one-dimensional subspaces, as the invertible  $1 \times 1$  matrices are precisely the nonzero scalars, so the stabilizers in the case of  $\text{Gr}(1, n)$  are the  $1 \times 1$  invertible block matrices  $A$ , or the nonzero scalars.

**Problem (Problem 2):** Fix an inner product on  $\mathbb{R}^n$ . Show that the map  $V \mapsto V^\perp$  induces a  $C^\infty$  diffeomorphism  $\text{Gr}(k, n) \rightarrow \text{Gr}(n - k, n)$ .

**Solution:** Due to the inner product, we make the identification  $v \mapsto v^*$  such that  $v^*(w) = \langle v, w \rangle$ . In particular, we have isomorphisms  $V \cong V^*$  and  $V^\perp \cong (V^\perp)^*$ . Therefore, given an element  $T \in \text{Hom}(V, V^\perp)$ , dualization gives the transpose map  $T^* \in \text{Hom}((V^\perp)^*, V^*)$ .

Now, given any chart  $(U_V, \varphi_V)$  in  $\text{Gr}(k, n)$ , we identify  $T \in \text{Hom}(V, V^\perp) \cong U_V$  to  $T^* \in \text{Hom}((V^\perp)^*, V^*) \cong U_{V^\perp}$ , and identify subspaces  $W \in U_V$  with their annihilators

$$W^0 = \{w^* \in (\mathbb{R}^n)^* \mid w^*(v) = 0 \text{ for all } v \in W\},$$

so that  $W^0 \cap V^* = 0$ . Finally, we define  $\varphi_{V^\perp}$  by

$$\varphi_{V^\perp} = P_{V^*} \circ P_{(V^\perp)^*}|_{W^0}^{-1}.$$

Since every  $W \in \text{Gr}(k, n)$  has a unique annihilator subspace  $W^0 \in \text{Gr}(n - k, n)$ , we have the series of bijective correspondences

$$\begin{aligned} \text{Hom}(V, V^\perp) &\xleftrightarrow{\varphi_V} U_V \\ &\xleftrightarrow{W \mapsto W^0} U_{V^\perp} \\ &\xleftrightarrow{\varphi_{V^\perp}} \text{Hom}((V^\perp)^*, V^*) \\ &\xleftrightarrow{\langle \cdot, \cdot \rangle} \text{Hom}(V^\perp, V), \end{aligned}$$

meaning that this identification is a  $C^\infty$  diffeomorphism.

**Problem (Problem 3):** Prove that a  $C^k$  map which is a  $C^1$  diffeomorphism is necessarily a  $C^k$  diffeomorphism.

**Solution:** Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^k$  map that is a  $C^1$  diffeomorphism. In order to show that  $f$  is a  $C^k$  diffeomorphism, we need to show that  $f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  exists and is of class  $C^k$ .

First, by the inverse function theorem, since  $f$  is a  $C^1$  diffeomorphism, we see that  $f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  exists, is continuous, and is such that  $D(f^{-1})$  is continuous.

Now, we observe that the association  $y \mapsto D_y(f^{-1})$  can be written as

$$y \mapsto f^{-1}(y) \mapsto D_y f(f^{-1}(y)) \mapsto (D_y f(f^{-1}(y)))^{-1} = D_y(f^{-1}),$$

where we observe that  $f^{-1}$  is of class  $C^1$ , the derivative  $D_y f$  is of class  $C^{k-1}$ , and matrix inversion is  $C^\infty$ ; since  $D(f^{-1})$  is a composition of  $C^1$  functions,  $D(f^{-1})$  is  $C^1$ , so  $f^{-1}$  is  $C^2$ . Inductively, we see that  $f^{-1}$  is also of class  $C^k$ , so  $f$  is a  $C^k$  diffeomorphism.

**Problem (Problem 4):** Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either  $\mathbb{R}$  or  $S^1$ , depending on whether it is compact or not.

**Problem (Problem 5):** In this problem, we prove a weak version of the Whitney Embedding Theorem.

- Find a  $C^\infty$  function  $\lambda$  on  $\mathbb{R}^n$  with values in  $[0, 1]$  such that  $\lambda$  takes the value 1 on the closed ball  $B(0, 1)$ , and vanishes outside the closed ball  $B(0, 2)$ .
- Suppose  $M$  is a compact  $C^k$  manifold of dimension  $n$ . Find a  $C^k$  atlas  $\{(U_i, \varphi_i)\}_{i \in I}$  such that  $\varphi_i(U_i)$  contains  $B(0, 2)$ , and such that  $M$  is covered by the union of  $\varphi_i^{-1}(B(0, 1))^\circ$ .
- Let  $\lambda_i$  be defined by  $\lambda \circ \varphi_i$  on  $U_i$ , and 0 outside  $U_i$ . Let  $f_i: M \rightarrow \mathbb{R}^n$  be defined by  $\lambda_i \circ \varphi_i$  on  $U_i$  and zero otherwise. Use these functions to embed  $M$  as a submanifold of some Euclidean space.

**Remark:** Most of the following solution is pieced together from Morita's *Geometry of Differential Forms*.

**Solution:** (a) Consider the function  $H: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$H(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0, \end{cases}$$

which is a  $C^\infty$  function on  $\mathbb{R}$ , as  $e^{-1/t}$  is  $C^\infty$  for all  $t > 0$ , and the derivative is well-defined at  $t = 0$ . Next, we see that the function

$$G(t) = \frac{H(4 - t^2)}{H(4 - t^2) + H(t^2 - 1)}$$

takes on the value 1 whenever  $-1 \leq t \leq 1$  and is supported on  $[-2, 2]$ . Furthermore, it is a  $C^\infty$  function, as it is a rational function of  $C^\infty$  functions where the denominator never takes the value 0. Therefore, if we replace  $t$  with  $|x|$ , when  $x \in \mathbb{R}^n$ , since the norm is a  $C^\infty$  function, we obtain a  $C^\infty$  function that is supported on  $B(0, 2)$  and is equal to 1 on  $B(0, 1)$ , given by

$$\lambda(x) = \frac{H(4 - |x|^2)}{H(4 - |x|^2) + H(|x|^2 - 1)}.$$

- Let  $M$  be a compact  $C^k$  manifold, and let  $\{(V_i, \psi_i)\}_{i \in I}$  be a  $C^k$  atlas for  $M$ , where  $\{V_i\}_{i \in I}$  is an open cover, the  $\psi_i: V_i \rightarrow \mathbb{R}^n$  are homeomorphisms, and the  $\psi_j \circ \psi_i^{-1}$  are  $C^k$  diffeomorphisms.

Since  $M$  is compact, we have a finite subcover  $\{V_j\}_{j=1}^n$  and an exhaustion by compact subsets via

$$U_j = \bigcup_{k=1}^j V_k$$

$$M = \bigcup_{j=1}^n U_j,$$

where, without loss of generality,  $\overline{U_j} \subset U_{j+1}$ .

Now, for each  $p \in \overline{U_j} \setminus U_{j-1}$  (define  $U_0 = U_1 = \emptyset$ ), we may find  $i_p$  with a corresponding  $C^k$  chart  $(V_{i_p}, \psi_{i_p})$ , where without loss of generality,  $\psi_{i_p}(p) = 0$ , and let  $W_p = \psi_{i_p}^{-1}(U(0, 1))$ .

Clearly,  $B(0, 2) \subseteq \psi_{i_p}(V_{i_p})$ , and by finitely enumerating the elements  $p_{j_k}$  in  $\overline{U_j} \setminus U_{j-1}$ , we have an open cover  $\{W_{p_{j_k}}\}_{k=1}^m = \{\psi_{p_{j_k}}^{-1}(U(0, 1))\}_{k=1}^m$  of  $M$ , and  $\{(V_{i_{p_k}}, \psi_{p_k})\}_{k=1}^m$  are  $C^k$  charts such that  $B(0, 2) \subseteq \psi_{p_k}(V_{i_{p_k}})$ .

(c)

**Problem** (Problem 6): Use the ideas of the previous exercise to prove that a  $C^k$  manifold admits a  $C^k$  partition of unity subordinate to any locally finite cover.

**Solution:** Let  $\{U_i\}_{i \in I}$  be a locally finite open cover of  $M$ , and let  $\{(U_i, \varphi_i)\}_{i \in I}$  be the corresponding  $C^k$  atlas for  $M$  where  $B(0, 2) \subseteq \varphi_i(U_i)$ , and  $M$  is covered by  $\varphi_i^{-1}(U(0, 1))$ . Then, we may define

$$f_i = \begin{cases} G \circ \varphi_i & \text{on } U_i \\ 0 & \text{on } U_i^c, \end{cases}$$

where

$$G(x) = \frac{e^{\frac{1}{4-|x|^2}}}{e^{\frac{1}{4-|x|^2}} + e^{\frac{1}{|x|^2-1}}}$$

is a  $C^\infty$  function supported on  $B(0, 2)$  and equal to 1 on  $U(0, 1)$ . Defining

$$f = \sum_{i \in I} f_i,$$

we see that  $f \neq 0$  everywhere, as  $M$  is covered by the family  $\varphi_i^{-1}(U(0, 1))$ , and since  $\{U_i\}_{i \in I}$  is locally finite,  $f$  is also  $C^k$  as each  $f_i$  is the composition of a  $C^k$  diffeomorphism and a  $C^\infty$  function. The functions

$$g_i = \frac{f_i}{f}$$

are thus smooth,  $0 \leq g_i \leq 1$ , and  $\sum_{i \in I} g_i = 1$ .

**Problem** (Problem 7): Let  $X$  and  $Y$  be topological spaces, and let  $C(X, Y)$  be the set of continuous maps from  $X$  to  $Y$ . Equip  $C(X, Y)$  with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K) \subseteq V\},$$

where  $K \subseteq X$  is compact and  $V \subseteq Y$  is open.

If  $Y$  is a metric space, and if  $X$  is compact, prove that this topology is the same as the topology of uni-

form convergence.

**Solution:** Let  $Y$  be a metric space and let  $X$  be compact. We note that a neighborhood basis in the topology of uniform convergence on  $C(X, Y)$  consists of sets of the form

$$U_{f,\varepsilon} = \left\{ g \mid \sup_{x \in X} d(f(x), g(x)) < \varepsilon \right\}.$$

Similarly, a subbase for the compact open topology consists of sets of the form

$$U_{f,K,\varepsilon} = \left\{ g \mid \sup_{x \in K} d(f(x), g(x)) < \varepsilon \right\};$$

the fact that  $Y$  is a metric space allows us to take this refinement of the compact-open topology.

Thus, to prove that the compact-open topology and the topology of uniform convergence are equivalent, we show that any basis element of the topology of uniform convergence is contained in a basis element of the compact-open topology, and vice versa.

First, we see that almost by definition, if  $K \subseteq X$  is any compact subset, then

$$U_{f,\varepsilon} \subseteq U_{f,K,\varepsilon},$$

as any function whose supremum distance is less than  $\varepsilon$  over  $X$  must have that supremum distance hold over  $K \subseteq X$ .

Now, in the reverse direction, we fix  $f$  and  $\varepsilon$ . We wish to show that there is a finite family of subsets  $U_{K_i, V_i}$  with  $f \in U_{K_i, V_i}$  for each  $i$ , and their intersection lies in  $U_{f,\varepsilon}$ . We see that every point  $x \in X$  has a pre-compact open neighborhood  $U_x$  such that  $f(\overline{U_x}) \subseteq U(f(x), \varepsilon/3)$ , which follows from the fact that compact subsets of  $Y$  are bounded. The family  $\{U_x \mid x \in X\}$  is an open cover for  $X$ , so admits a finite subcover  $\{U_{x_i}\}_{i=1}^n$ . Since each  $\{\overline{U_{x_i}}\}_{i=1}^n$  is compact, and for each  $i$ ,  $f \in U_{\overline{U_{x_i}}, U(f(x_i), \varepsilon/3)}$ , we see that

$$V = \bigcap_{i=1}^n U_{\overline{U_{x_i}}, U(f(x_i), \varepsilon/3)}$$

is a nonempty open subset in the compact-open topology on  $C(X, Y)$  that contains  $f$ . Now, for any  $g \in V$  and for any  $x \in X$ , we see that there is some  $U_{x_j}$  such that  $x \in U_{x_j}$ , and since  $g \in U_{\overline{U_{x_j}}, U(f(x_j), \varepsilon/3)}$ , we have that

$$\begin{aligned} d(g(x), f(x)) &\leq d(g(x), f(x_j)) + d(f(x_j), f(x)) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon, \end{aligned}$$

so  $V \subseteq U_{f,\varepsilon}$ , meaning the topologies are equal.

**Problem (Problem 8):** Let  $C^k(M, N)$  be the set of  $C^k$  maps from  $M$  to  $N$ . The compact-open topology on  $C^k(M, N)$  is defined similarly. Let  $f \in C^k(M, N)$ ,  $(U, \varphi)$  and  $(V, \psi)$  charts on  $M$  and  $N$ , let  $K \subseteq U$  be compact such that  $f(K) \subseteq V$ , and let  $\varepsilon > 0$ . We obtain a basic neighborhood  $N(f, U, \varphi, V, \psi, K, \varepsilon)$  by looking at all the maps  $g \in C^k(M, N)$  such that  $g(K) \subseteq V$ , and

$$\|D^r(\psi f \varphi^{-1})(x) - D^r(\psi g \varphi^{-1})(x)\|_{\text{op}} \leq \varepsilon \quad (*)$$

for all integers  $0 \leq r \leq k$ .

The Whitney topology is slightly different. Let  $\Phi = \{(U_i, \varphi_i)\}_{i \in I}$  be a locally finite atlas on  $M$ , let

$K_i \subseteq U_i$  be compact for all  $i$ , let  $\Psi$  be an atlas on  $N$ , and let  $\{\varepsilon_i\}_{i \in I}$  be a family of positive numbers. A basic neighborhood of  $f \in C^k(M, N)$  in this topology is given by all  $g$  such that  $g(K_i) \subseteq V_i$  for all  $i$ , and

$$\|D^r(\psi_i f \varphi_i^{-1})(x) - D^r(\psi_i g \varphi_i^{-1})(x)\|_{\text{op}} \leq \varepsilon_i \quad (**)$$

for all  $x \in \varphi_i(K_i)$  and all integers  $0 \leq r \leq k$ .

For infinite values of  $k$ , we take the compact-open and Whitney topologies on  $C^\infty(M, N)$  to be the union of these topologies via the inclusion  $C^\infty(M, N) \subseteq C^k(M, N)$ . Show the following:

- (a) these basic neighborhoods actually give a basis for a topology in both cases;
- (b) if  $M$  is compact, these two topologies coincide;
- (c) if  $M$  is compact and has no boundary, then the  $C^k$  diffeomorphisms from  $M$  to  $N$  are open in  $C^k(M, N)$  in the Whitney topology.

**Solution:**

- (a) Clearly, in both the compact open topology and the Whitney topology, the respective neighborhoods cover  $C^k(M, N)$ , so we only need to verify the condition that if  $X_1, X_2 \subseteq C^k(M, N)$  are open subsets such that  $f \in X_1 \cap X_2$ , then there is  $X_3 \subseteq C^k(M, N)$  open such that  $X_3 \subseteq X_1 \cap X_2$ .

We start with the case of the compact-open topology. Let  $f \in X_1 \cap X_2$ , where  $X_1$  and  $X_2$  are open in the compact-open topology. Since  $f \in X_1$ , there is a chart  $(U_1, \varphi_1)$  of  $M$ , a chart  $(V_1, \psi_1)$  of  $N$ ,  $K_1 \subseteq U_1$  compact such that  $f(K_1) \subseteq V_1$ , and  $\varepsilon_1 > 0$  such that  $(*)$  holds and  $N(f, U_1, \varphi_1, V_1, \psi_1, \varepsilon_1) \subseteq X_1$ . Similarly, since  $f \in X_2$ , there are charts  $(U_2, \varphi_2)$  and  $(V_2, \psi_2)$  of  $M$  and  $N$  respectively,  $K_2 \subseteq U_2$  compact with  $f(K_2) \subseteq V_2$ , and  $\varepsilon_2 > 0$  such that  $(*)$  holds, and  $N(f, U_2, \varphi_2, V_2, \psi_2, \varepsilon_2) \subseteq X_2$ . Note that by the characterization,  $(*)$  holds for the supremum over all  $x \in \varphi_j(K_j)$  for  $j = 1, 2$ .