Spectral Theory

Problem ([Con90, Exercise IX.1.2])**:** Show that the unit ball of $B(H)$ is WOT-compact.

Solution: Consider the set

$$K = \prod_{x,y \in B_H} \overline{\mathbb{D}},$$

where $\mathbb{D}$ represents the complex unit disk and B_H denotes the closed unit ball of H. The space K is compact by Tychonoff's theorem. Let $\phi\colon B_{B(H)} \to K$ be defined by

$$\phi(T) = (\langle Tx, y\rangle)_{x,y}.$$

Observe that by Cauchy–Schwarz, we have that

$$\begin{aligned} |\langle Tx, y\rangle| &\leq \|T\|_{\text{op}}\|x\|\|y\| \\ &\leq 1, \end{aligned}$$

so ϕ is indeed well-defined. Furthermore, ϕ is injective since for any two operators T and S, we have that $T = S$ if and only if $\langle Tx, y\rangle = \langle Sx, y\rangle$ for all $x, y \in B_H$, and ϕ is continuous by the definition of the weak operator topology. Therefore, we only need to show that ϕ has a closed range.

Let $(T_i)_i$ be a net of operators in $B_{B(H)}$ such that

$$\lim_{i\in I}(\langle T_i x, y\rangle)_{x,y} = (z_{x,y})_{x,y}.$$

Then, from the Cauchy–Schwarz inequality, it follows that $(z_{x,y})_{x,y} \in K$, and by the definition of convergence in the product topology, we have, for each x, y,

$$\lim_{i\in I}\langle T_i x, y\rangle = z_{x,y}.$$

Therefore, we may define a semidefinite sesquilinear form $F\colon H \times H \to \mathbb{C}$ given by

$$F(x, y) = \lim_{i\in I}\langle T_i x, y\rangle$$

for each $x, y \in H$. From the structure of sesquilinear forms, it then follows that there is some $T \in B_{B(H)}$ such that $F(x, y) = \langle Tx, y\rangle$, and thus $(T_i)_{i\in I} \to T$ in WOT.

Problem ([Con90, Exercise IX.1.13])**:** A representation $\rho\colon A \to B(H)$ is *irreducible* if the only projections in $B(H)$ that commute with every $\rho(a)$ for $a \in A$ are 0 and 1. Prove that if A is abelian and ρ is an irreducible representation, then $\dim(H) = 1$. Find the corresponding spectral measure.

Solution: Since A is abelian, so too is $\rho(A)$, meaning that $\rho(A) \subseteq \rho(A)'$. Since $\rho(A)' = \mathbb{C}1$ by the assumption of irreducibility, it follows then that $\rho(A) = \mathbb{C}1$, whence $H = [\rho(A)v] = \mathbb{C}v$.

Without loss of generality, we may assume that $A = C_0(X)$ for some locally compact Hausdorff space X, and $\rho\colon C_0(X) \to \mathbb{C}$ is a character. The characters of $C_0(X)$ are given by evaluation at $x_0 \in X$, meaning that their corresponding spectral measure is the Dirac mass δ_{x_0}.

References

[Con90] John B. Conway. *A Course in Functional Analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.