Problem (Problem 1)**:**

(a) Let G be a finite group. Show that for any subgroup $H \leqslant G$, we have $n_p(H) \leqslant n_p(G)$.

(b) Let $f\colon G \to G'$ be a surjective homomorphism of finite groups, and let p be a prime. Show that every p-Sylow subgroup P' of G' is the image of some p-Sylow subgroup P of G.

Solution:

(a) Suppose $|G| = p^r m$ and $|H| = p^s \ell$, with $p \nmid m, \ell$.

First, we observe that if $s = r$, then any p-Sylow subgroup of H is a p-Sylow subgroup of G that is contained in H, whence $n_p(H) \leqslant n_p(G)$.

Now, let $s < r$. We observe that if $P \leqslant H \leqslant G$ is a p-Sylow subgroup of H, then by the second Sylow theorem, P is contained in some p-Sylow subgroup, $P' \leqslant G$. We claim that any two distinct p-Sylow subgroups of H must be contained in distinct p-Sylow subgroups of G. This follows from the fact that, if $P_1, P_2 \leqslant H$ are two distinct p-Sylow subgroups, and $P_1, P_2 \leqslant P'$, then the subgroup $\langle P_1, P_2 \rangle$ generated in H is contained in both H and P', but has strictly larger order than either P_1 or P_2, which contradicts the maximality of the orders of P_1 and P_2 respectively. Thus, any p-Sylow subgroup of H is of the form $P' \cap H$ for some p-Sylow subgroup of G, whence $n_p(H) \leqslant n_p(G)$.

Problem (Problem 8)**:** Let G be a group of order $3 \cdot 5^2 \cdot 17$.

(a) Show that $n_{17}(G) = 1$. That is, a 17-Sylow subgroup H is normal.

(b) The conjugation action of G on H defines a group homomorphism $G \to \operatorname{aut}(H)$. Show that this homomorphism is trivial, and conclude that $H \subseteq Z(G)$.

Solution:

(a) By the third Sylow theorem, we know that $n_{17}(G)|75$ and $n_{17}(G) \equiv 1$ modulo 17. Writing out the possibilities for n_{17} under the second condition explicitly gives

$$n_{17}(G) = 1, 18, 35, 52, 69, 86, \ldots$$

of which only 1 divides 75. Thus, there is only one 17-Sylow subgroup.