Math 395: Homework 4
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Problem 15

Problem: Let $A \in \mathrm{Mat}_n(\mathbb{F})$.

(a) Assume A has eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that $\det(A) = \lambda_1 \cdots \lambda_n$ and $\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

(b) Suppose A does not have n distinct eigenvalues, but $c_A(x)$ splits into linear factors over F. Can you characterize the determinant and trace of A in terms of the eigenvalues?

Solution.

(a) If $A \in \mathrm{Mat}_n(\mathbb{F})$ has distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then there exists $P \in \mathrm{GL}_n(\mathbb{F})$ such that

$$A = P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)P^{-1},$$

where $\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)$ denote the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ at entries $a_{11}, \dots, a_{nn}$. In particular, this means

$$\begin{aligned}\det(A) &= \det\left(P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)P^{-1}\right)\\ &= \det\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)\\ &= \prod_{j=1}^{n} \lambda_j,\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\left(A\right) &= \mathrm{tr}\left(P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)P^{-1}\right)\\ &= \mathrm{tr}\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)\right)\\ &= \sum_{j=1}^{n} \lambda_j.\end{aligned}$$

(b) If $c_A(x)$ splits into linear factors over F, then the Jordan canonical form for A exists, with each of its Jordan blocks consisting of the roots of $c_A(x)$ with multiplicity.[1] Thus, we can characterize $\mathrm{tr}(A)$ to be the sum of the roots of $c_A(X)$ with multiplicity, and $\det(A)$ to be the product of the roots with multiplicity.

Problem 17

Problem: Prove that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a matrix $A \in \mathrm{Mat}_n(\mathbb{F})$, the $\lambda_1^k, \dots, \lambda_n^k$ are the eigenvalues for A^k for any $k \geqslant 0$.

Solution. Since A has eigenvalues $\lambda_1, \dots, \lambda_n$, it is the case that there exists some $P \in \mathrm{GL}_n(\mathbb{F})$ such that

$$A = P\left(\mathrm{diag}\left(\lambda_1, \dots, \lambda_k\right)\right)P^{-1}.$$

[1] Assistance from Wikipedia

For $k = 0$, we have

$$
\begin{aligned}
A^0 &= \left(P\left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right) P^{-1}\right)^0 && = I_n \\
&= P\left(\operatorname{diag}\left(\lambda_1^0, \ldots, \lambda_n^0\right)\right) P^{-1},
\end{aligned}
$$

meaning $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues for A^k.

For $k > 0$, we have

$$
\begin{aligned}
A^k &= \underbrace{\left(P\left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right) P^{-1}\right)\left(P\left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right) P^{-1}\right) \cdots \left(P\left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right) P^{-1}\right)}_{k \text{ times}} \\
&= P \underbrace{\left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right)\left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right) \cdots \left(\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)\right)}_{k \text{ times}} P^{-1} \\
&= P\left(\operatorname{diag}\left(\lambda_1^k, \ldots, \lambda_n^k\right)\right) P^{-1},
\end{aligned}
$$

meaning $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues for A^k.