**Problem** (Problem 1)**:** Let $T\colon V \to W$ be a linear transformation between $\mathbb{F}$-vector spaces. Show that $T$ is injective if and only if $T$ maps $\mathbb{F}$-linearly independent subsets of $V$ to $\mathbb{F}$-linearly independent subsets of $W$.

**Solution:** Let $T$ be injective. We claim that if $\{v_1, \dots, v_n\}$ is linearly independent in $V$, then $\{Tv_1, \dots, Tv_n\}$ is linearly independent in $W$. We see that if

$$\sum_{j=1}^{n} a_j T v_j = 0_W,$$

then

$$T\left(\sum_{j=1}^{n} a_j v_j\right) = 0_W,$$

meaning that

$$\sum_{j=1}^{n} a_j v_j \in \ker(T).$$

Now, since $T$ is injective, $\ker(T) = \{0_V\}$, meaning that $\sum_{j=1}^{n} a_j v_j = 0_V$. Yet, since $\{v_1, \dots, v_n\}$ is linearly independent, this means $a_j = 0$ for each $j$, so $\{Tv_1, \dots, Tv_n\}$ is linearly independent in $W$.

Now, let $T$ map linearly independent subsets of $V$ to linearly independent subsets of $W$. If $\mathcal{B}_V = \{v_i\}_{i\in I}$ is a basis for $V$, then since $\mathcal{B}_V$ is linearly independent, $\mathcal{C} = \{Tv_i\}_{i\in I}$ is a linearly independent subset of $W$, which can be extended to a basis $\mathcal{B}_W$. Since $\mathcal{C} \subseteq \mathcal{B}_W$, we see that any linear combination in $\mathcal{B}_W$ yields 0 if and only if every coefficient is zero, meaning that $\ker(T) = \{0_V\}$, so $T$ is injective.

**Problem** (Problem 2)**:** Let $P_{n+1}(\mathbb{R})$ be the space of polynomials with real coefficients of degree $\leqslant n+1$. Prove that for any $n$ points $a_1, \dots, a_n \in \mathbb{R}$, there exists a nonzero polynomial $f \in P_{n+1}(\mathbb{R})$ such that $f(a_j) = 0$ for each $j$, and $\sum_{j=1}^{n} f'(a_j) = 0$.

**Solution:** Based on the first condition, we see that the product $\prod_{j=1}^{n}(x - a_j)$ must divide the polynomial $f$, and since $f$ has degree at most $n+1$, we must have $f(x) = (x - L)\prod_{j=1}^{n}(x - a_j)$ for some $a, b \in \mathbb{R}$. Writing $f'(x)$, we see that

$$f'(x) = \prod_{j=1}^{n}(x - a_j) + (x - L)\sum_{i=1}^{n}\prod_{j\neq i}(x - a_j),$$

implying that

$$\sum_{i=1}^{n} f'(a_i) = \sum_{i=1}^{n}(a_i - L)\prod_{j\neq i}(a_i - a_j).$$

By setting

$$0 = \sum_{i=1}^{n}(a_i - L)\prod_{j\neq i}(a_i - a_j),$$

we get

$$L = \frac{1}{\sum_{i=1}^{n}\prod_{j\neq i}(a_i - a_j)}\sum_{i=1}^{n} a_i \prod_{j\neq i}(a_i - a_j),$$

which is well-defined whenever the $a_i$ are distinct.

**Problem** (Problem 3)**:** Let $T: V \to W$ be a linear map of finite-dimensional vector spaces, and let $W_0 \subseteq W$ be a subspace.

(a) Show that $T^{-1}(W_0) = \{v \in V \mid Tv \in W_0\}$ is a subspace of $V$.

(b) Assuming $T$ is surjective, express $\dim\left(T^{-1}(W_0)\right)$ in terms of $\dim(W_0)$ and $\dim(\ker(T))$.

**Solution:**

(a) We see that if $v_1, v_2 \in T^{-1}(W_0)$ and $\alpha \in \mathbb{R}$, then since $Tv_1, \alpha Tv_2 \in W_0$, we have $Tv_1 + \alpha Tv_2 \in W_0$, so by linearity, $T(v_1 + \alpha v_2) \in W_0$, meaning $v_1 + \alpha v_2 \in T^{-1}(W_0)$, so $T^{-1}(W_0)$ is a subspace of $V$.

(b) First, since $T$ is surjective, $T\left(T^{-1}(W_0)\right) = W_0$. Therefore, by restricting the map $T$, we get the surjective map $T': T^{-1}(W_0) \to W_0$, and since $\ker(T) \subseteq T^{-1}(W_0)$, the First Isomorphism Theorem gives $T^{-1}(W_0)/\ker(T) \cong W_0$, so by rank-nullity (as each of $W_0$ and $T^{-1}(W_0)$ are finite-dimensional), $\dim\left(T^{-1}(W_0)\right) = \dim(\ker(T)) + \dim(W_0)$.

**Problem** (Problem 4)**:**

(a) Do there exist invertible matrices $A, B \in \mathrm{Mat}_2(\mathbb{R})$ such that

$$ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

(b) Do there exist matrices $A, B \in \mathrm{Mat}_2(\mathbb{R})$ such that

$$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

**Solution:**

(a) There do not. This follows from the fact that $\det\left(ABA^{-1}B^{-1}\right) = 1$, while the determinant of the latter matrix is 2.

**Problem** (Problem 5)**:**

(a) Find the inverse matrix $A^{-1}$ for the matrix

$$A = \begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}.$$

(b) Prove that

$$\begin{vmatrix} a + x_1 & a & \cdots & a \\ a & a + x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a + x_n \end{vmatrix} = x_1 x_2 \cdots x_n \left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_n}\right).$$

**Solution:**

(a) We may find $A^{-1}$ by trying to find the sequence of elementary matrices $E_1, \dots, E_n$ such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I.$$

First, we do row reduction on $A$, yielding

$$\begin{pmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ a & a+1 & a \\ a & a & a+1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ a & a & a+1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2a & a+1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3a+1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the product $E_n E_{n-1} \cdots E_2 E_1$ is our desired inverse, which we find by applying the elementary row operations to the identity matrix $I$, yielding

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - aR_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & a & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2aR_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a & -a & 2a+1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3/(3a+1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ -a/(3a+1) & 1-(a/(3a+1)) & -1+((2a+1)/(3a+1)) \\ -(a/(3a+1)) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1-a/(3a+1) & -a/(3a+1) & -1+(2a+1)/(3a+1) \\ -a/(3a+1) & 1-(a/(3a+1)) & -1+((2a+1)/(3a+1)) \\ -a/(3a+1) & -a/(3a+1) & (2a+1)/(3a+1) \end{pmatrix},$$

which is our desired inverse.

(b) We show the case for $n = 2$, then use induction from then on. By raw calculation, we see that

$$\begin{aligned} \begin{vmatrix} a+x_1 & a \\ a & a+x_2 \end{vmatrix} &= (a+x_1)(a+x_2) - a^2 \\ &= x_1x_2 + ax_1 + ax_2 \end{aligned}$$

$$= x_1x_2\left(1 + \frac{a}{x_1} + \frac{a}{x_2}\right).$$

Now, for the general $n$ case, we see that since determinants are multilinear,

$$\begin{aligned}\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x_n \end{vmatrix} &= \begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix} \\ &= a\begin{vmatrix} a+x_1 & a & \cdots & 1 \\ a & a+x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}\end{aligned}$$

and since determinants are alternating,

$$= a\begin{vmatrix} x_1 & 0 & \cdots & 1 \\ 0 & x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a+x_1 & a & \cdots & 0 \\ a & a+x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$

and by the cofactor expansion,

$$= a(x_1x_2\cdots x_{n-1}) + x_n\begin{vmatrix} a+x_1 & a & \cdots & a \\ a & a+x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_{n-1} \end{vmatrix}$$

and by the induction hypothesis,

$$\begin{aligned}&= a(x_1x_2\cdots x_{n-1}) + x_n(x_1x_2\cdots x_{n-1})\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}}\right) \\ &= x_1x_2\cdots x_n\left(1 + \frac{a}{x_1} + \cdots + \frac{a}{x_{n-1}} + \frac{a}{x_n}\right),\end{aligned}$$

we obtain our desired result.

**Problem** (Problem 6)**:** Let $A \in \operatorname{Mat}_n(\mathbb{R})$, and $(a_{ij})_{ij}$ such that $|a_{ij}| < \frac{1}{n}$ for each $i, j$. Show that $\det(I_n - A) \neq 0$.

**Solution:** Let $\|x\| = \max_{i=1}^{n} |x_i|$. Let $x_j$ be the component of $x$ such that $|x_j| = \|x\|$. Then, we see that

$$\begin{aligned}\left|(Ax)_j\right| &= \left|\sum_{i=1}^{n} a_{ij}x_i\right| \\ &\leqslant \sum_{i=1}^{n} |a_{ij}||x_i| \\ &< \sum_{i=1}^{n} \frac{1}{n}|x_i| \\ &\leqslant \sum_{i=1}^{n} \frac{1}{n}\|x\|\end{aligned}$$

$$= |x_j|,$$

which means that $Ax \neq x$ at the component $x_j$, meaning $(I_n - A)x \neq 0$.

**Problem** (Problem 7)**:**

(a) Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that $A^2 = I_n$. Show that $A$ is diagonalizable.

(b) Give an example of of $A \in \text{Mat}_2(\mathbb{C})$ satisfying $A^2 = \mathbf{0}_2$ (the zero matrix) which is not diagonalizable.

**Solution:**

(a) Since $A^2 - I_n = \mathbf{0}_n$, we see that the minimal polynomial of $A$ is $m_A(t) = t^2 - 1$, which splits over $\mathbb{C}$ to yield $m_A(t) = (t-1)(t+1)$. In particular, since the minimal polynomial splits into a product of distinct linear factors, $A$ is diagonalizable.

(b) The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfies $A^2 = \mathbf{0}_2$, but since $A \neq \mathbf{0}_2$, we see that $m_A(t) = t^2$. Since $m_A(t)$ does not split into distinct linear factors over $\mathbb{C}$, we see that $A$ is necessarily not diagonalizable.

**Problem** (Problem 8)**:** Let $A \in \text{Mat}_n(\mathbb{C})$ be a matrix such that $A^2$ has $n$ distinct eigenvalues. Show that $A$ is diagonalizable.