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## Normed Vector Spaces

### Vector Spaces

Throughout,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . A **vector space** over  $\mathbb{F}$  is a nonempty set  $V$  equipped with two operations: vector addition and scalar multiplication.

$$\begin{array}{ll}
 V \times V \xrightarrow{+} V & \\
 (v, w) \mapsto v + w & \text{Vector Addition} \\
 F \times V \rightarrow V & \\
 (\alpha, v) \mapsto \alpha v & \text{Scalar Multiplication}
 \end{array}$$

The vector space is an Abelian group, where  $u, v, w \in V$  and  $\alpha, \beta \in \mathbb{F}$ , we have:

- (i)  $u + (v + w) = (u + v) + w$
- (ii)  $\exists 0_v \in V$  with  $\forall v \in V, 0_v + v = v + 0_v = v$
- (iii)  $(\forall v \in V)(\exists w \in V)$  with  $v + w = 0_v$
- (iv)  $\forall v, w \in V, v + w = w + v$
- (v)  $\alpha(v + w) = \alpha v + \alpha w, (\alpha + \beta)v = \alpha v + \beta v$
- (vi)  $\alpha(\beta w) = (\alpha\beta)w$
- (vii)  $1 \cdot v = v$

#### Remarks:

- (a)  $0_v$  is unique and known as the zero vector.
- (b) The vector  $w$  in (iii) is unique, and denoted  $-v$ .
- (c)  $0 \cdot v = 0_v$
- (d)  $(-1) \cdot v = -v$
- (e) Property (iv) follows from all the other axioms.
- (f) For  $n \in \mathbb{N}$ ,  $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

### Subspaces

Let  $V$  be a vector space over  $\mathbb{F}$ . A **subspace** is a nonempty subset  $W \subseteq V$  satisfying the following:

- (i)  $w \in W, \alpha \in \mathbb{F} \rightarrow \alpha w \in W$ .
- (ii)  $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$ .

**Remark:**  $0_v$  is always a member of any subspace; a subspace is also a vector space.

**Proposition: Intersection of Subspaces**

If  $\{W_i\}_{i \in I}$  is a family of subspaces of  $V$ , then,  $\bigcap W_i$  is a subspace of  $V$ .

**Proposition: Union of Subspaces**

It is not the case that the union of subspaces of  $V$  also a subspace. For example, consider  $\mathbb{R}^2$  with the traditional vector space operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$$

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

If  $W_1, W_2 \subseteq V$  are subspaces such that  $W_1 \cup W_2$  is a subspace, then  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

**Generated Subspaces**

Let  $S \subseteq V$  be any subset of a vector space  $V$ . Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^n \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

**Remarks:**

- $\text{span}(S) \subseteq V$  is a subspace.
- $\text{span}(S) = \bigcap W$ , where  $S \subseteq W$  and  $W \subseteq V$  is a subspace. Thus,  $\text{span}(S)$  is the “smallest” subspace containing  $S$ , or the subspace generated by  $S$ .

**Proposition: Quotient Group on Vector Space**

Let  $V$  be a vector space, and let  $W \subseteq V$  is a subspace. Define  $u \sim_W v \leftrightarrow u - v \in W$ .

(1)  $\sim_W$  is an equivalence relation.

(2) If  $[v]_W$  denotes the equivalence class of  $v$ , then  $[v]_W = v + W = \{v + w \mid w \in W\}$ .

(3)  $V/W := \{[v]_W \mid v \in V\}$  is a vector space with  $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$  and  $\alpha[v]_W = [\alpha v]_W$ .

**Proof of (1):**

- Reflexive:  $u \sim_W u$ , since  $u - u = 0 \in W$ .
- Transitive: Suppose  $u \sim_W v$ , and  $v \sim_W z$ . Then,  $u - v \in W$ , and  $v - z \in W$ . So,  $(u - v) + (v - z) \in W$ , so  $u - z \in W$ . Whence,  $u \sim_W z$ .
- Symmetric: If  $u \sim_W v$ , then  $u - v \in W$ , so  $-1 \cdot (u - v) \in W$ , so  $v - u \in W$ . Whence,  $v \sim_W u$ .

**Proof of (2):**

$$\begin{aligned} [v]_W &= \{u \in V \mid u \sim_W v\} \\ &= \{u \in V \mid u - v \in W\} \\ &= \{u \in V \mid u = v + w \text{ some } w \in W\} \\ &= \{v + w \mid w \in W\} \\ &= v + W \end{aligned}$$

**Proof of (3):** Prove that the operation is well-defined.

## Bases

Let  $V$  be a vector space and  $S \subseteq V$  be a subset.

- (1)  $S$  is said to be spanning for  $V$  if  $\text{span}(S) = V$ .
- (2)  $S$  is linearly independent if, for  $\sum_{j=1}^n \alpha_j v_j = 0_v$  with  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ ,  $v_1, \dots, v_n \in S$ , then  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .
- (3)  $S$  is a basis for  $V$  if  $S$  is linearly independent and spanning for  $V$ .

### Proposition: Existence of Basis

Every vector space admits a basis. If  $B_0 \subseteq V$  is linearly independent,  $\exists B \subseteq V$  such that  $B$  is a basis and  $B \supseteq B_0$ .

**Background:** A relation on a set  $X$  is a subset  $R \subseteq X \times X$ . If  $R$  is reflexive ( $x \sim x$ ), transitive ( $x \sim y, y \sim z \rightarrow x \sim z$ ), and antisymmetric ( $x \sim y, y \sim x \rightarrow x = y$ ), then  $R$  is an ordering, and we write  $x \leq y$ .

If  $\leq$  is an ordering of  $X$  such that  $\forall x, y \in X$ ,  $x \leq y$  or  $y \leq x$ , then  $\leq$  is a total (or linear) ordering.

Let  $\leq$  be an ordering of  $X$ , let  $Y \subseteq X$ . An upper bound for  $Y$  is an element  $u \in X$  such that  $y \leq u \forall y \in Y$ . A maximal element in  $X$  is an element  $m \in X$  such that  $x \in X$ ,  $x \geq m \rightarrow x = m$ .

**Example:**  $\mathbb{N}$  under the division ordering defines  $a \leq b \Leftrightarrow a|b$ . If we want to find the maximal elements of  $A = \{2, 6, 9, 12\}$ , we would see that they are 9 and 12 (since no element of  $A$  can be divided by 9 and 12). Meanwhile,  $\mathbb{N}$  itself has no maximal elements.

This leads us to ask: given an ordered set,  $(X, \leq)$ , does  $X$  admit maximal elements.

**Zorn's Lemma (or Axiom):** Let  $(X, \leq)$  be an ordered set. Suppose that every totally ordered subset,  $Y \subseteq X$  has an upper bound in  $X$ . Then,  $X$  admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

**Proof:** Let  $X = \{D \mid B_0 \subseteq D \subseteq V\}$  with  $D$  linearly independent. Since  $B_0 \subseteq X$ ,  $X \neq \emptyset$ . Define  $D, E \in X$ ,  $D \leq E \Leftrightarrow D \subseteq E$ . We will show that  $X$  has a maximal element.

Consider any totally ordered subset,  $Y = \{D_i\}_{i \in I}$ . Consider  $D = \bigcup D_i$ . Clearly,  $B_0 \subseteq D \subseteq V$ . Suppose  $\sum \alpha_k v_k = 0_v$  with  $v_1, \dots, v_n \in D$ . Therefore,  $\exists D_j$  with  $v_1, \dots, v_n \in D_j$  because  $Y$  is totally ordered. However, by definition,  $D_j$  is a linearly independent set — therefore,  $\alpha_k = 0$ . Thus,  $D$  is linearly independent.

Since  $D$  is linearly independent, and  $B_0 \subseteq D$ , it must be the case that  $D \in X$ .  $D$  is also an upper bound for  $Y$ . So, by Zorn's Lemma,  $X$  has a maximal element,  $B$ .

So,  $B_0 \subseteq B \subseteq V$ ,  $B$  is independent, and  $B$  is maximal in  $X$ . We claim that  $B$  is a basis for  $V$ . Suppose toward contradiction that  $\exists v \in V$  such that  $v \notin \text{span}(B)$ . Consider  $B' = B \cup \{v\}$ .

Then,  $B_0 \subseteq B'$ , and  $B'$  is linearly independent — if  $\sum \alpha_k v_k + \alpha v = 0$ , where  $v_1, \dots, v_n \in B$ , then either:

- If  $\alpha = 0$ , then  $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$ .
- If  $\alpha \neq 0$ , then  $\sum \alpha_k v_k = -\alpha v$ , which means  $v \in \text{span}(B)$ .  $\perp$

Thus, we have a linearly independent set,  $B'$ , with  $B \subseteq B'$ , and  $B_0 \subseteq B'$ . Therefore,  $B' \in X$ . However, this contradicts the maximality of  $B$ . Therefore,  $\text{span}(B) = V$ , and  $B$  is a basis for  $V$ .

**Examples: Vector Spaces**(1)  $n$ -Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$B = \{e_1, \dots, e_n\}$$

where  $e_i$  denotes the unit vector at position  $i$ .(2)  $m \times n$  Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$\alpha(a_{ij}) = (\alpha a_{ij})$$

$$B = \{e_{ij}\}$$

where  $e_{ij}$  denotes a matrix of 0 everywhere except column  $i$  and row  $j$ .(3) Functions with domain  $\Omega$ :

$$\mathcal{F}(\Omega, \mathbb{F}) = \{f \mid f : \Omega \rightarrow \mathbb{F}\}$$

$$(f + g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

(4) Bounded functions with domain  $\Omega$ :

$$\ell_\infty(\Omega, \mathbb{F}) = \{f \in \mathcal{F}(\Omega, \mathbb{F}) \mid \|f\|_u \leq \infty\}$$

$$\|f\|_u = \sup_{x \in \Omega} |f(x)|$$

Exercises:

- Triangle Inequality:  $\|f + g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity:  $\|\alpha f\|_u = |\alpha| \|f\|_u$
- Positive Definite:  $\|f\|_u = 0 \Rightarrow f = 0$

**Proof of Triangle Inequality:** Given  $x \in \Omega$ ,

$$\begin{aligned} |(f + g)(x)| &= |f(x) + g(x)| \\ &\leq |f(x)| + |g(x)| \\ &\leq \|f\|_u + \|g\|_u \end{aligned}$$

Therefore,

$$\begin{aligned} \sup |(f + g)(x)| &\leq \|f\|_u + \|g\|_u \\ \|f + g\|_u &\leq \|f\|_u + \|g\|_u \end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a, b], \mathbb{F}) = \{f : [a, b] \rightarrow \mathbb{F} \mid f \text{ continuous}\}$$

Check that  $C([a, b], \mathbb{F}) \subseteq \ell_\infty([a, b], \mathbb{F})$  is a subspace.

(6) Let  $f : [a, b] \rightarrow \mathbb{R}$  be any function. Let  $\mathcal{P} : a = x_0 < x_1 < x_2 < \dots < x_n = b$ .

$$\text{var}(f; \mathcal{P}) := \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

$$\text{var}(f) = \sup_{\mathcal{P}} \text{var}(f; \mathcal{P})$$

$$\text{BV}([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid \text{var}(f) < \infty\}$$

$$\|f\|_{\text{BV}} = |f(a)| + \text{var}(f)$$

$\text{BV}([a, b])$  is a vector space.

**Question:** Is  $\mathbb{1}_{\mathbb{Q}} \in \text{BV}([0, 1])$ ?

(7) Suppose  $K \subseteq V$  is a *convex* subset of a vector space:  $v, w \in K, t \in [0, 1] \Rightarrow (1-t)v + tw \in K$ . Let  $\text{Aff}(K) = \{f : K \rightarrow \mathbb{R} \mid f \text{ is affine}\}$ , where  $f$  is affine if  $\forall v, w \in K, t \in [0, 1], f((1-t)v + tw) = (1-t)f(v) + tf(w)$ .

**Exercise:** Show that  $\text{Aff}(K) \subseteq \mathcal{F}(K, \mathbb{R})$  is a subspace.

(8) Let  $S$  be defined as

$$S = \{(a_k)_{k=1}^\infty \mid a_k \in \mathbb{F}\}.$$

Under pointwise operations,  $S$  is a vector space.

$$(a_k)_k + (b_k)_k = (a_k + b_k)_k$$

$$\alpha(a_k)_k = (\alpha a_k)_k$$

**Note 1:**  $S = \mathcal{F}(\mathbb{N}, \mathbb{F})$ .

**Note 2:**  $c_{00} \subseteq \ell_1 \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq S$ .

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k \neq 0\}$
- $c_0 = \{(a_k)_k \mid (a_k)_k \rightarrow 0\}$
- $c = \{(a_k)_k \mid (a_k)_k \rightarrow a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_\infty < \infty\}$
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| < \infty\}$

(9)  $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$  are all subspaces.

- $C_c(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ compactly supported}\}$ :  $f : \mathbb{R} \rightarrow \mathbb{F}$  is compactly supported if  $\exists [a, b]$  such that  $x \notin [a, b] \Rightarrow f(x) = 0$ .
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ continuous, } \lim_{x \rightarrow \pm\infty} f(x) = 0\}$

(10) Let  $S$  be any non-empty set.

$$\mathbb{F}(S) := \{f : S \rightarrow \mathbb{F} \mid f \text{ finitely supported}\}$$

$$\text{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that  $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$  is a subspace. Consider  $e_t : S \rightarrow \mathbb{F}$  defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that  $\xi = \{e_t\}_{t \in S}$  is a basis for  $\mathbb{F}(S)$ .



Indeed, given  $f \in \mathbb{F}(S)$ , we know that  $\text{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$ . Therefore,  $f = \sum_{k=1}^n f(t_k)e_{t_k} \in \text{span}(\xi)$ . Therefore,  $\xi$  is spanning for  $\mathbb{F}(S)$ . Suppose  $\sum_{k=1}^n \alpha_{t_k} e_{t_k} = 0$  for some  $\alpha_k \in \mathbb{F}$ ,  $t_k \in S$ .

$$\left( \sum_{k=1}^n \alpha_{t_k} e_{t_k} \right) = 0(t_1)$$

$$\alpha_{t_1} = 0.$$

Similarly,  $\alpha_{t_j} = 0$  for  $j = 1, \dots, n$ . Therefore,  $\xi$  is linearly independent. Since  $\xi$  is linearly independent and spanning,  $\xi$  forms a basis for  $\mathbb{F}(S)$ .

**Note:** The free vector space,  $\mathbb{F}(S)$ , displays the universal property.

There are functions  $\iota : S \rightarrow \mathbb{F}(S)$ , where  $\iota(t) = e_t$ , and given any map  $\varphi : S \rightarrow V$  for  $V$  a vector space over  $\mathbb{F}$ ,  $\exists!$  linear map  $T_\varphi : \mathbb{F}(S) \rightarrow V$  such that  $\iota \circ T_\varphi = \varphi$ .

$$\begin{array}{ccc} S & \xrightarrow{\iota} & \mathbb{F}(S) \\ & \searrow \varphi & \downarrow T_\varphi \\ & & V \end{array}$$

**Proof:** Every  $f \in \mathbb{F}(S)$  has a unique expression  $f = \sum_{k=1}^n f(t_k)e_{t_k}$ , where  $\text{supp}(f) = \{t_1, \dots, t_n\}$ . Therefore,

$$T_\varphi(f) := \sum_{k=1}^n f(t_k)\varphi(t_k)$$

**Exercise:** Show  $T_\varphi$  is linear and unique.

**Exercise 2:** Suppose  $V$  is a vector space over  $\mathbb{F}$  with basis  $B$ . Show that  $\mathbb{F}(B) \cong V$ . Remember that  $V \cong W$  if  $\exists T : V \rightarrow W$  such that  $T$  is bijective and linear.

## Normed Spaces

To every vector  $v \in V$ , we want to assign a length to  $v$ ,  $\|v\|$ .

A **norm** on a vector space  $V$  is a map

$$\begin{aligned} \|\cdot\| : V &\rightarrow \mathbb{R}^+ \\ v &\mapsto \|v\| \geq 0 \end{aligned}$$

such that

- (i) Homogeneity:  $\|\alpha v\| = |\alpha|\|v\|$
- (ii) Triangle Inequality:  $\|v + w\| \leq \|v\| + \|w\|$
- (iii) Positive definiteness:  $\|v\| = 0 \Rightarrow v = 0_V$ .

If  $p : V \rightarrow \mathbb{R}^+$  satisfies (i) and (ii), then  $p$  is a **seminorm**.

The pair  $(V, \|\cdot\|)$  is called a normed space.

Two norms,  $\|\cdot\|$  and  $\|\cdot\|'$  are called **equivalent** if  $\exists c_1, c_2 \geq 0$  with,  $\forall v \in V$ ,

$$\begin{aligned} \|v\| &\leq c_1 \|v\|' \\ \|v\|' &\leq c_2 \|v\| \end{aligned}$$

**Note:** On  $\mathbb{R}^n$ , all norms are equivalent.

**Exercise:** If  $p$  is any seminorm on  $V$ , then  $|p(v) - p(w)| \leq p(v - w)$ .

**Notation:** If  $V$  is a normed space, then  $B_V = \{v \in V \mid \|v\| \leq 1\}$ , and  $U_V = \{v \in V \mid \|v\| < 1\}$  are the closed and open unit ball respectively.

**Examples of Normed Spaces**

(1) Given  $V = \mathbb{F}^n$  and  $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ , we have different norms:

$$\|x\|_1 = \sum_{j=1}^n |x_j|$$

$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$$

$$\|x\|_2 = \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2}.$$

In general, for  $1 \leq p < \infty$ ,

$$\|x\|_p = \left( \sum_{j=1}^n |x_j|^p \right)^{1/p}.$$

**Exercise:** Show that  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are norms. Show that  $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$ .

We want to show that  $\|\cdot\|_p$  defines a norm for  $1 \leq p < \infty$ . If  $1 \leq p < \infty$ , its conjugate index  $q \in [1, \infty]$  whereby  $\frac{1}{p} + \frac{1}{q} = 1$ . For example, if  $p = 1$ , then  $q = \infty$ , and if  $p = \infty$ , then  $q = 1$ .

**Lemma 1:** For  $1 < p < \infty$ ,  $p^{-1} + q^{-1} = 1$ ,  $f : [0, \infty) \rightarrow \mathbb{R}$ ,  $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$ . Then,  $f(t) \geq 0$  for all  $t \geq 0$ .

**Proof 1:** We can see that  $f'(t) = t^{p-1} - 1$ . Then,  $f'(t) = 0$  at  $t = 1$ ;  $f'(t) > 0$  for  $t > 1$  and  $f'(t) < 0$  for  $t \in [0, 1)$ .

So, since  $f(t) \geq f(1)$  for all  $t \geq 0$ , and  $f(1) = 0$ ,  $f(t) \geq 0$  for all  $t \geq 0$ .

**Lemma 2:** For  $1 < p < \infty$ ,  $p^{-1} + q^{-1} = 1$ ,  $z, y \geq 0$ ,  $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ .

**Proof 2:** We know from Lemma 1,  $t \leq \frac{1}{p}t^p + \frac{1}{q}$ . Multiply by  $y^q$  to get

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Set  $t = xy^{1-q}$ . Then,

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq}y^q + \frac{1}{q}y^q.$$

Since  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p - pq = -q$ , so

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

With these two lemmas in mind, we get two important inequalities.

**Hölder's Inequality:** For  $1 \leq p \leq \infty$ ,  $p^{-1} + q^{-1} = 1$ . Then, for  $x, y \in \mathbb{F}^n$ ,

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \|x\|_p \|y\|_q.$$

**Proof of Hölder's Inequality:** For  $p = 1$ , the solution is as follows:

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n |x_j| \|y\|_\infty \\ &= \|x\|_1 \|y\|_\infty, \end{aligned}$$

and similarly for  $p = \infty, q = 1$ .

For  $1 < p < \infty$ , assume  $\|x\|_p = \|y\|_q = 1$ .

$$\begin{aligned}
 \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\
 &\leq \sum_{j=1}^n \left( \frac{1}{p} |x_j|^p + \frac{1}{q} |y_j|^q \right) \\
 &= \frac{1}{p} \left( \sum_{j=1}^n |x_j|^p \right) + \frac{1}{q} \left( \sum_{j=1}^n |y_j|^q \right) \\
 &= \frac{1}{p} + \frac{1}{q} \\
 &= 1.
 \end{aligned}$$

If  $\|x\|_p = 0$  or  $\|y\|_q = 0$ , then  $x = 0_{\mathbb{F}}$  or  $y = 0_{\mathbb{F}}$ , the inequality still holds.

Assume  $\|x\|_p \neq 0, \|y\|_p \neq 0$ . Set

$$\begin{aligned}
 x' &= \frac{x}{\|x\|_p} \\
 y' &= \frac{y}{\|y\|_p}.
 \end{aligned}$$

It can be verified that  $\|x'\|_p = 1 = \|y'\|_q$ . Therefore,

$$\begin{aligned}
 \left| \sum_{j=1}^n x'_j y'_j \right| &\leq 1 \\
 \left| \sum_{j=1}^n \frac{x_j}{\|x\|_p} \frac{y_j}{\|y\|_q} \right| &\leq 1 \\
 \left| \sum_{j=1}^n x_j y_j \right| &\leq \|x\|_p \|y\|_q
 \end{aligned}$$

**Minkowski's Inequality:** Given  $x, y \in \mathbb{F}^n$ ,  $1 \leq p \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

**Proof of Minkowski's Inequality:** We can verify for  $p = 1, q = \infty$ , and vice versa.

Assume  $1 < p < \infty$ . Then,

$$\begin{aligned}
 \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p \\
 &= \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\
 &\leq \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\
 &\leq \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \left( \sum_{j=1}^n |x_j + y_j|^{pq-q} \right)^{1/q} + \left( \sum_{j=1}^n |y_j|^p \right)^{1/p} \left( \sum_{j=1}^n |x_j + y_j|^{pq-q} \right)^{1/q} \\
 &\hspace{15em} \text{Hölder's Inequality} \\
 &= \|x\|_p \|x + y\|_p^{p/q} + \|y\|_p \|x + y\|_p^{p/q} \\
 &= (\|x\|_p + \|y\|_p) \|x + y\|_p^{p-1}
 \end{aligned}$$

Divide by  $\|x + y\|_p^{p-1}$  to get desired inequality.

(2)  $\ell_\infty(\Omega, \mathbb{F})$  with  $\|\cdot\|_\infty$ . This includes subspaces that inherit the norm, such as

$$\begin{aligned} C([a, b]) &\subseteq \ell_\infty(\Omega) \\ \ell_\infty(\mathbb{R}) &\supseteq C_0(\mathbb{R}) \supseteq C_c(\mathbb{R}) \end{aligned}$$

**Exercise:** Show that  $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$  is a subspace.

(3)  $\Omega = \mathbb{N}$ ,  $\ell_\infty = \ell_\infty(\mathbb{N})$  with  $\|\cdot\|_\infty$ . Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \subseteq \ell_\infty.$$

(4)  $\ell_1$  with  $\|\cdot\|_1$ ,

$$\|(a_k)_k\|_1 = \sum_{k=1}^n |a_k|.$$

(5)  $C([a, b])$  with

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

(6) Let  $1 \leq p < \infty$ .

$$\ell_p = \left\{ (a_k)_{k=1}^\infty \mid \sum_{k=1}^\infty |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left( \sum_{k=1}^\infty |a_k|^p \right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\begin{aligned} \left( \sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} &= \left\| \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \right\|_{\ell_p^n} \\ &= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\ &\leq \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\|_{\ell_p^n} + \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\ &\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p. \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  (by the definition of an infinite series), we find that  $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$ .

(7)  $BV([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid \text{Var}(f) < \infty\}$  with the norm  $\|f\|_{BV} = |f(a)| + \text{Var}(f)$  is a normed space:

$$\|f\|_{BV} = 0$$

$$|f(a)| = 0$$

$$\text{Var}(f) = 0$$

given  $t \in (a, b]$ , look at the partition  $a < t \leq b$ . Then,

$$\text{Var}(f) \geq |f(t) - f(a)| + |f(b) - f(t)|$$

$$f(t) = 0$$

$$f = 0_f.$$

(8)  $M_{m,n}(\mathbb{F})$  with

$$\|a\|_{\text{op}} = \sup_{\|\xi\|_{\ell_2^m} \leq 1} \|a\xi\|_{\ell_2^n}$$

is a normed vector space. If  $\|a\|_{\text{op}} = 0$ , then

$$ae_j = 0 \quad \forall j \in \{1, \dots, n\}.$$

take the dot product with  $i \neq j$

$$\begin{aligned} ae_j \cdot e_i &= a_{ij} \\ &= 0 \end{aligned}$$

so  $a_{ij} = 0$  for all  $a_{ij}$ , so  $a$  is the 0 matrix.

(9) Let  $V, W$  be vector spaces over  $\mathbb{F}$ . Then,  $\mathcal{L}(V, W) = \{T \mid T : V \rightarrow W \text{ linear}\}$ , where  $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ .

$\mathcal{L}(V, W)$  is a vector space with operations

$$\begin{aligned} (T + S)(v) &= T(v) + S(v) \\ (\alpha T)(v) &= \alpha T(v). \end{aligned}$$

**Notation:**  $\mathcal{L}(V) := \mathcal{L}(V, V)$  is all linear operators on  $V$ .  $\mathcal{L}(V, \mathbb{F}) = V'$  is all linear functionals.

Suppose  $V$  and  $W$  are normed vector spaces. If  $T : V \rightarrow W$ , set

$$\|T\|_{\text{op}} := \sup_{\|v\|_V \leq 1} \|T(v)\|_W,$$

$$\mathbb{B}(V, W) = \{T \in \mathcal{L}(V, W) \mid \|T\|_{\text{op}} \leq \infty\},$$

where  $\mathbb{B}(V, W)$  is referred to as the set of all bounded linear maps from  $V$  to  $W$ .  $\mathbb{B}(V, W)$  with  $\|\cdot\|_{\text{op}}$  is a normed space.

- Homogeneity:

$$\begin{aligned} \|\alpha T\|_{\text{op}} &= \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W \\ &= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W \\ &= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W \\ &= |\alpha| \|T\|_{\text{op}}. \end{aligned}$$

- Triangle Inequality: for  $\|v\|_V \leq 1$ ,

$$\begin{aligned} \|(T + S)(v)\|_W &= \|T(v) + S(v)\|_W \\ &\leq \|T(v)\|_W + \|S(v)\|_W \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

so

$$\begin{aligned} \|T + S\|_{\text{op}} &= \sup_{\|v\| \leq 1} \|T + S(v)\| \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

- Positive Definite: If  $\|T\|_{\text{op}} = 0$ , then  $T(v) = 0$  for all  $v \in V$ ,  $\|v\| \leq 1$ .

Let  $v \in V$ ,  $v \neq 0$ . Then,  $\frac{v}{\|v\|} \in B_V$ .

$$\begin{aligned} T\left(\frac{v}{\|v\|}\right) &= 0 \\ \frac{1}{\|v\|} T(v) &= 0 \\ T(v) &= 0 \end{aligned}$$

**Special Cases:**  $\mathbb{B}(V) = \mathbb{B}(V, V)$ ,  $V^* = \mathbb{B}(V, \mathbb{F})$ .

**Exercise:**  $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$ .

(10) Inner Product Spaces (expanded upon below).

### Inner Product Spaces

An inner product on a vector space  $V$  is a pairing

$$V \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}$$

that satisfies

$$(i) \quad \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle, \quad \langle \alpha v, w \rangle = \alpha \langle v, w \rangle.$$

$$(ii) \quad \langle v, w \rangle = \overline{\langle w, v \rangle}$$

$$(iii) \quad \langle v, v \rangle \geq 0.$$

$$(iv) \quad \text{If } \langle v, v \rangle = 0, \text{ then } v = 0.$$

The pair  $(V, \langle \cdot, \cdot \rangle)$  is known as an inner product space.

**Remarks:**  $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$ ,  $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$ .

If  $\langle \cdot, \cdot \rangle$  is an inner product on a linear space  $V$ , then set

$$\|v\|_2 := \langle v, v \rangle^{1/2}.$$

**Exercise:**  $\|\alpha v\|_2 = |\alpha| \|v\|_2$ ,  $\|v\|_2 = 0 \Rightarrow v = 0$ .

$v, w \in (V, \langle \cdot, \cdot \rangle)$  are *orthogonal* if  $\langle v, w \rangle = 0$ .

The Pythagorean theorem states that for  $v_1, \dots, v_n \in V$  mutually orthogonal, then

$$\left\| \sum_{i=1}^n v_i \right\|^2 = \sum_{j=1}^n \|v_j\|^2.$$

For two vectors  $v, w \in V$ ,  $P_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$ .

**Exercise:** Check that  $\langle P_w(v), v - P_w(v) \rangle$ , meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

**Cauchy-Schwarz Inequality:** In any inner product space,

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|.$$

**Proof of Cauchy-Schwarz:** From the exercise,

$$\begin{aligned} \|v\| &\geq \|P_w(v)\| \\ \|v\| &\geq \left\| \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\| \\ &= \frac{|\langle v, w \rangle|}{\|w\|^2} \|w\| \end{aligned}$$

therefore,

$$\|v\| \|w\| \geq |\langle v, w \rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

**Proof of Triangle Inequality:**

$$\begin{aligned}
\|v + w\|_2^2 &= \langle v + w, v + w \rangle \\
&= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\
&= \|v\|^2 + \|w\|^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} \\
&= \|v\|^2 + \|w\|^2 + 2\operatorname{Re} \langle v, w \rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w \rangle| \\
&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\
&= (\|v\| + \|w\|)^2.
\end{aligned}$$

Cauchy-Schwarz Inequality

Take square roots on both sides.

(1)  $\ell_2^n = \mathbb{F}^n$  with

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left| \sum_{j=1}^n x_j \overline{y_j} \right| \leq \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2} \left( \sum_{j=1}^n |y_j|^2 \right)^{1/2}.$$

(2)  $\ell_2$  with

$$\langle (a_j)_j, (b_j)_j \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

We can see that for any finite  $n$ , the Cauchy-Schwarz inequality in  $\ell_2^n$  states

$$\begin{aligned}
\left| \sum_{j=1}^n a_j \overline{b_j} \right| &\leq \left( \sum_{j=1}^n |a_j|^2 \right)^{1/2} \left( \sum_{j=1}^n |b_j|^2 \right)^{1/2} \\
&\leq \left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} |b_j|^2 \right)^{1/2}.
\end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ , we see that  $\langle (a_j)_j, (b_j)_j \rangle$  is convergent.(3)  $C([a, b])$  with

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

(4) Let  $V = \mathbb{M}_n(\mathbb{C})$ .

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let  $\operatorname{Tr} : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$ ,  $\operatorname{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$ .

- $\operatorname{Tr}(I_n) = n$
- $\operatorname{Tr}(a + \alpha b) = \operatorname{Tr}(a) + \alpha \operatorname{Tr}(b)$
- $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$

Then, if  $\text{Tr}(a^*a) = 0$ , then  $a = 0$ .

$$\begin{aligned} a^*a &= (\overline{a_{ji}})_{i,j} (a_{ij})_{i,j} \\ &= \left( \sum_{k=1}^n \overline{a_{ki}} a_{kj} \right)_{i,j} \\ \text{Tr}(a^*a) &= \sum_{i=1}^n \sum_{k=1}^n \overline{a_{ki}} a_{ki} \\ &= \sum_{i,k=1}^n |a_{ki}|^2 \\ &= \sum_{i,j=1}^n |a_{ij}|^2. \end{aligned}$$

If  $\text{Tr}(a^*a) = 0$ , then  $a_{ij} = 0$  for all  $i, j$ .

We define

$$\langle a, b \rangle_{\text{HS}} = \text{Tr}(b^*a).$$

- (i)  $(b_1 + b_2)^* = b_1^* + b_2^*$
- (ii)  $(\alpha b)^* = \overline{\alpha} b^*$
- (iii)  $(b_1 b_2)^* = b_2^* b_1^*$
- (iv)  $b^{**} = b$

The norm is defined as

$$\begin{aligned} \|a\|_{\text{HS}} &= \langle a, a \rangle_{\text{HS}}^{1/2} \\ &= \text{Tr}(a^*a)^{1/2} \\ &= \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2} \end{aligned}$$

## Metric Spaces

We looked at normed spaces, where we attach a length  $\|v\|$  to every vector  $v$ . We can also speak of the distance between two vectors, defined as  $d(v, w) = \|v - w\|$ .

Notice that the following hold:

- $d(v, w) \geq 0$
- 

$$\begin{aligned} d(v, w) &= \|v - w\| \\ &= \|(-1)(w - v)\| \\ &= |-1| \|w - v\| \\ &= \|w - v\| \end{aligned}$$

•

$$\begin{aligned} d(u, w) &= \|u - w\| \\ &= \|u - v + v - w\| \\ &\leq \|u - v\| + \|v - w\| \\ &= d(u, v) + d(v, w). \end{aligned}$$

- $d(v, v) = \|v - v\| = 0$ . If  $d(v, w) = 0$ , then  $\|v - w\| = 0$ , so  $v - w = 0$ , so  $v = w$ .

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely  $(\mathbb{R}, |\cdot|)$ . We will expand these concepts to all metric spaces.



## Definition of a Metric Space

Let  $X$  be a non-empty set. A **metric** on  $X$  is a map

$$\begin{aligned} d : X \times X &\rightarrow \mathbb{R}^+ \\ (x, y) &\mapsto d(x, y) \geq 0 \end{aligned}$$

such that

- (i) Symmetry:  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
- (ii) Triangle Inequality:  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .
- (iii) Zero Distance:  $d(x, x) = 0$
- (iv) Definite:  $d(x, y) = 0 \Rightarrow x = y$

If  $d$  satisfies (i), (ii), and (iii), then  $d$  is called a semi-metric. If  $d$  satisfies (iv) as well, then  $d$  is a metric.

If  $d$  is a (semi-)metric on  $X$ , the pair  $(X, d)$  is called a (semi-)metric space.

Two metrics,  $d$  and  $\rho$ , on  $X$ , are equivalent if  $\exists c_1, c_2 \geq 0$  such that  $d(x, y) \leq c_1 \rho(x, y)$  and  $\rho(x, y) \leq c_2 d(x, y)$  for all  $x, y$ .

## Examples of Metric Spaces

- (1) Discrete Metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for  $X$  any set.

- (2) Hamming distance: between two bit strings of equal length. Let

$$\begin{aligned} X &= \{0, 1\}^n \\ &= \{0, 1\} \underbrace{\times \cdots \times}_{n \text{ times}} \{0, 1\} \\ d_H((x_j)_1^n, (y_j)_1^n) &= |\{j \mid x_j \neq y_j\}|. \end{aligned}$$

- (3) Any normed space  $(V, \|\cdot\|)$  is a metric space.

$$d(v, w) = \|v - w\|.$$

**Exercise:** Show that if two norms are equivalent, their induced metrics are equivalent.

- (4) Subset of Metric Space: If  $(X, d)$  is a metric space, and  $Y \subseteq X$  is non-empty. Then,  $(Y, d)$  is a metric space.

- (5) Paris metric: let  $(X, \rho)$  be a metric space. Let  $p \in X$  be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

- (6) Bounded metric: Let  $\rho$  be a (semi-)metric on  $X$ . Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We claim that  $d$  is a (semi-)metric. Notice that  $0 \leq d(x, y) \leq 1$ .

**Proof:** Clearly,  $d(x, y) = d(y, x)$ . Additionally,  $d(x, x) = 0$ . If  $d(x, y) = 0$  and  $\rho$  is a metric, then  $\rho(x, y) = 0$ , so  $x = y$ .

To show the triangle inequality, we examine the function

$$f(t) = \frac{t}{1+t}$$

$$f'(t) = \frac{1}{(1+t)^2} > 0.$$

Since  $\rho$  satisfies the triangle inequality,  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ . Apply  $f$  on both sides. Then,

$$\begin{aligned} \underbrace{\frac{\rho(x, z)}{1 + \rho(x, z)}}_{d(x, z)} &\leq \frac{\rho(x, y) + \rho(y, z)}{1 + (\rho(x, y) + \rho(y, z))} \\ &= \frac{\rho(x, y)}{1 + \rho(x, y) + \rho(y, z)} + \frac{\rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\ &\leq \underbrace{\frac{\rho(x, y)}{1 + \rho(x, y)}}_{d(x, y)} + \underbrace{\frac{\rho(y, z)}{1 + \rho(y, z)}}_{d(y, z)}. \end{aligned}$$

(7) If  $d_1, \dots, d_n$  are metrics on  $X$ ,  $c_1, \dots, c_n \geq 0$ . Then,

$$d(x, y) = \sum_{k=1}^n c_k d_k(x, y)$$

is a metric.

(8) Let  $\{\rho_k\}_{k=1}^\infty$  be a family of semi-metrics. Assume the family is separating — for all  $x \neq y$ , there exists  $k$  such that  $\rho_k(x, y) \neq 0$ .

Let  $d_k$  be defined as

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Note that  $\{d_k\}_{k=1}^\infty$  is also separating.

Then,

$$d(x, y) = \sum_{k=1}^\infty 2^{-k} d_k(x, y)$$

is a metric.

We will now define the Frechet Metric using this method. Let  $X = C(\mathbb{R})$ . For each  $k = 1, 2, 3, \dots$ , set  $\rho_k(f) = \sup_{x \in [-k, k]} |f(x)|$ .

We can verify that  $\rho_k$  defines a seminorm. We can then check  $\rho_k(f, g) = \rho_k(f - g)$  is a semi-metric.

We claim that  $\{\rho_k\}$  is separating: if  $f \neq g$ , then there exists  $x_0 \in \mathbb{R}$  with  $f(x_0) \neq g(x_0)$ . Since  $f$  and  $g$  are continuous, there is a neighborhood  $[x_0 - \delta, x_0 + \delta]$  such that  $f(x) \neq g(x)$  for all  $x \in [x_0 - \delta, x_0 + \delta]$ . Find  $k$  such that  $[x_0 - \delta, x_0 + \delta] \subseteq [-k, k]$ . Then,  $\rho_k(f - g) > 0$ .

Construct  $d_k$  as above, and then  $d$  as follows:

$$d_F = \sum \frac{2^{-k} \rho_k(f - g)}{1 + \rho_k(f - g)}$$

(9) Product of metric spaces: let  $(X_k, \rho_k)_{k=1}^{\infty}$  be a countable family of metric spaces. For each  $k$ , let

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

**Remark:** If the  $\rho_k$  are already uniformly bounded, let  $d_k = \rho_k$ .

Let

$$\begin{aligned} X &= \prod_{k=1}^{\infty} X_k \\ &= \{(x_k)_k \mid x_k \in X_k\} \\ &= \left\{ f : \mathbb{N} \rightarrow \bigcup_{k=1}^{\infty} X_k \mid f(k) \in X_k \right\}. \end{aligned}$$

Define  $D : X \times X \rightarrow [0, \infty)$  as

$$\begin{aligned} D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} \rho_k(x_k, y_k), \\ D(f, g) &= \sum_{k=1}^{\infty} 2^{-k} \rho(f(k), g(k)). \end{aligned}$$

For example, for each  $k$ , let  $X_k = \{0, 1\}$  with the discrete metric. Let

$$\begin{aligned} \Delta &= \prod_{k \in \mathbb{N}} \{0, 1\} \\ &= \{(x_k)_k \mid x_k \in \{0, 1\}\} \\ D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k| \quad (x_k)_k, (y_k)_k \in \Delta. \end{aligned}$$

$\Delta$  is known as the abstract Cantor set; every compact metric space is a surjective image of the abstract Cantor set.

(10) Geodesic Distance: let  $\langle \cdot, \cdot \rangle$  be the standard dot product on  $\mathbb{R}^3(\mathbb{R}^n)$ , then

$$\begin{aligned} S^2 &= \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\} \\ S^{n-1} &= \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}. \end{aligned}$$

To find the geodesic distance, we take  $d(x, y) = \arccos(\langle x, y \rangle)$ . We claim  $d$  is a metric.

- Symmetry: self-evident.
- $d(x, x) = \arccos(1) = 0$ . Suppose  $d(x, y) = 0$ . Then,  $\langle x, y \rangle = 1$ , meaning  $\|x - y\|^2 = 0$ , so  $x = y$ .
- Let  $\theta = \arccos(\langle x, y \rangle)$ ,  $\varphi = \arccos(\langle y, z \rangle)$ , where  $\theta, \varphi \in [0, \pi]$ .

$$\begin{aligned} p_x &= \frac{\langle x, y \rangle}{\langle y, y \rangle} y \\ &= \cos(\theta) y \\ x &= \cos(\theta) y + \sin(\theta) u \end{aligned}$$

where

$$u = \frac{x - p_x}{\|x - p_x\|}.$$

Similarly, we can take

$$z = \cos(\varphi) y + \sin(\varphi) v$$

where

$$v = \frac{z - p_z}{\|z - p_z\|}.$$

So,

$$\begin{aligned}\langle x, z \rangle &= \cos(\theta) \cos(\varphi) + \sin(\theta) \sin(\varphi) \langle u, v \rangle \\ &\geq \cos(\theta) \cos(\varphi) - \sin(\theta) \sin(\varphi) & \langle u, v \rangle \geq -1 \\ &= \cos(\theta + \varphi).\end{aligned}$$

Since arccos is decreasing,

$$\begin{aligned}\arccos(\langle x, z \rangle) &\leq \arccos(\cos(\theta + \varphi)) \\ &= \theta + \varphi \\ &= \arccos(\langle x, y \rangle) + \arccos(\langle y, z \rangle).\end{aligned}$$

Therefore,  $d(x, y) \leq d(x, y) + d(y, z)$ .

- Let  $\Gamma = (V, E)$  be a simple connected graph. We define  $d : V \times V \rightarrow [0, \infty)$  to be the length of the shortest path between vertices  $u$  and  $v$ .

**Exercise:** Show this is a metric.

- (11) Let  $(X, d)$  be any metric space. If  $E \subseteq X$ , define  $\text{diam}(E) = \sup_{x, y \in E} d(x, y)$ .  $E$  is bounded if  $\text{diam}(E) < \infty$ .

**Exercise:** If  $(V, \|\cdot\|)$  is a normed space,  $E \subseteq V$  is a subset, show the following are equivalent:

- (i)  $E$  is bounded (in the metric sense)
- (ii)  $\sup_{v \in E} \|v\| < \infty$
- (iii)  $\exists r > 0$  such that  $E \subseteq rB_V$ .

Let  $\Omega$  be any set. The function  $f : \Omega \rightarrow X$  is bounded if  $f(\Omega) \subseteq X$  is bounded. We let.

$$\text{Bd}(\Omega, X) = \{f : \Omega \rightarrow X \mid f \text{ is bounded}\}.$$

**Remark:**  $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty(\Omega, \mathbb{F})$ .

- (12)  $\text{Bd}(\Omega, X)$  with

$$D_u(f, g) = \sup_{x \in \Omega} d(f(x), g(x)).$$

**Exercise:** Show that  $D_u$  defines a metric.

Consider  $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty$ . Look at the subset

$$E = \{f \in \text{Bd}(\Omega, \mathbb{F}) \mid f(x) \in \{0, 1\}\}.$$

Then,

$$\begin{aligned}D_u(f, g) &= \sup_{x \in \Omega} |f(x) - g(x)|. \\ &= \begin{cases} 1 & f \neq g \\ 0 & f = g \end{cases}.\end{aligned}$$

When we take a particular subset of  $D_u(f, g)$ , we find that we get the discrete metric.

Taking an overview of the concepts we have learned so far, we see

$$\text{Inner Product Spaces} \subseteq \text{Normed Vector Spaces} \subseteq \text{Metric Spaces}$$

## Topology of Metric Spaces

Throughout this section, let  $(X, d)$  be a metric space.

(1) Let  $x_0 \in X$ ,  $\delta > 0$ .

(i) We say

$$U(x_0, \delta) = \{x \in X \mid d(x, x_0) < \delta\}$$

is the open ball centered at  $x_0$  with radius  $\delta$ .

(ii) We say

$$B(x_0, \delta) = \{x \in X \mid d(x, x_0) \leq \delta\}$$

is the closed ball.

(iii) We say

$$S(x_0, \delta) = \{x \in X \mid d(x, x_0) = \delta\}$$

is the sphere.

(2)  $U \subseteq X$  is open if

$$(\forall x \in U)(\exists \delta > 0) \text{ such that } U(x, \delta) \subseteq U.$$

Let

$$\begin{aligned} \tau_X &= \{U \subseteq X \mid U \text{ open}\} \\ &\subseteq \mathcal{P}(X). \end{aligned}$$

(3)  $D \subseteq X$  is closed if  $D^c$  is open.

(4) If  $x \in U \in \tau_X$ , then  $U$  is called an open neighborhood of  $x$ . If  $x \in U \subseteq N$ , where  $U \in \tau_X$ , then  $N$  is a neighborhood of  $x$ .

$$\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$$

(5) Let  $A \subseteq X$ . The interior of  $A$  is

$$A^\circ = \bigcup \{V \mid V \subseteq A, V \text{ open}\}.$$

The closure of  $A$  is

$$\bar{A} = \bigcap \{D \mid A \subseteq D, D \text{ closed}\}.$$

The boundary of  $A$  is

$$\partial A = \bar{A} \setminus A^\circ.$$

**Exercise:**  $\overline{A^c} = (A^\circ)^c$ ,  $(\bar{A})^c = (A^c)^\circ$ .

**Remarks:**  $A^\circ$  is the largest open set contained in  $A$ . So, if  $V$  is open and  $V \subseteq A$ , then  $V \subseteq A^\circ$ . Similarly,  $\bar{D}$  is the smallest closed set containing  $D$ . If  $C$  is closed and  $D \subseteq C$ , then  $\bar{D} \subseteq C$ .

- For example,  $(a, b]^\circ = (a, b)$ . This is because  $(a, b)$  is open and contained in  $(a, b]$ , so  $(a, b) \subseteq (a, b]^\circ$ .
- We will show that  $\overline{A^c} \subseteq (A^\circ)^c$ .

$$\begin{aligned} A^\circ &\subseteq A \\ (A^\circ)^c &\supseteq A^c \end{aligned}$$

The union of open sets is open, so  $A^\circ$  is open, so  $(A^\circ)^c$  is closed by definition. Therefore,

$$(A^\circ)^c \supseteq \overline{A^c}.$$

## Topology of Open Sets in a Metric Space

The open sets  $\tau_X$  form a topology:

- (i)  $\emptyset, X \in \tau_X$ .
- (ii) If  $\{V_i\}_{i \in I} \subseteq \tau_X$ , then

$$\bigcup_{i \in I} V_i \in \tau_X.$$

- (iii) If  $V_1, \dots, V_n \in \tau_X$ , then

$$\bigcap_{i=1}^n V_i \in \tau_X.$$

**Remark:** This is only true of finite intersections. For a counterexample, if  $V_n = (-1/n, 1/n) \subseteq \mathbb{R}$  with the Euclidean metric, then the infinite intersection yields  $\{0\}$ , which is closed in  $\mathbb{R}$  with the Euclidean metric.

**Proof:**

- (1) Clearly,  $\emptyset$  (by vacuous truth) and  $X$  are open.
- (2) Let  $x \in \bigcup_{i \in I} V_i$ . Then,  $\exists i_0 \in I$  with  $x \in V_{i_0}$ . Since  $V_{i_0}$  is open,  $\exists \varepsilon > 0$  such that  $U(x, \varepsilon) \subseteq V_{i_0} \subseteq \bigcup_{i \in I} V_i$ .
- (3) Let  $x \in \bigcap_{i=1}^n V_i$ . Then,  $x \in V_i$  for all  $i \in 1, \dots, n$ . Since each  $V_i$  is open,  $\exists \varepsilon_1, \dots, \varepsilon_n$  with  $U(x, \varepsilon_i) \subseteq V_i$  for each  $i = 1, \dots, n$ . Set  $\varepsilon = \min\{\varepsilon_i\}_{i=1}^n$ . Then,  $U(x, \varepsilon) \subseteq U(x, \varepsilon_i) \subseteq V_i$  for all  $i$ . Therefore,  $U(x, \varepsilon) \subseteq \bigcap_{i=1}^n V_i$ .

**Exercise:** Show all open balls are open. In particular, show all open intervals are open.

**Exercise:** Show the following:

- (1)  $X, \emptyset$  are closed.
- (2) If  $\{C_i\}_{i \in I}$  is a family of closed sets, then  $\bigcap_{i \in I} C_i$  is closed.
- (3) For  $C_1, \dots, C_n$  closed, then  $\bigcup_{i=1}^n C_i$  is closed.
- (4) Closed balls are closed. Spheres are closed.

Let  $x \in X$ . Recall that  $\mathcal{N}_x$  is the set of all neighborhoods of  $x$ .

- (i)  $N \in \mathcal{N}_x \Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq N$
- (ii)  $N \in \mathcal{N}_x, N \subseteq M \Rightarrow M \in \mathcal{N}_x$
- (iii)  $N_1, N_2 \in \mathcal{N}_x \Rightarrow N_1 \cap N_2 \in \mathcal{N}_x$

In this sense,  $\mathcal{N}_x$  is a directed set with reverse inclusion.

## Pointwise Characterization of Subsets

Let  $A \subseteq X$ .

- (i)  $x \in A^\circ \Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A$ .
- (ii)  $x \in \overline{A} \Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset$ .
- (iii)  $x \in \partial A \Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset$  and  $U(x, \delta) \cap A^c \neq \emptyset$ .

**Proof:** Let  $A \subseteq X$

- (i)

$$\begin{aligned} x \in A^\circ &\Leftrightarrow x \in \bigcup_{\substack{V \in \tau_X \\ V \subseteq A}} V \\ &\Leftrightarrow \exists V \in \tau_X, V \subseteq A, x \in V \\ &\Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A. \end{aligned}$$

(ii)

$$\begin{aligned}
x \notin \bar{A} &\Leftrightarrow x \in (\bar{A})^c \\
&\Leftrightarrow x \in (A^c)^\circ \\
&\Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A^c \\
&\Leftrightarrow \exists \delta > 0 : U(x, \delta) \cap A = \emptyset.
\end{aligned}$$

We negate both sides.

(iii)

$$\begin{aligned}
x \in \partial A &\Leftrightarrow x \in \bar{A} \setminus A^\circ \\
&\Leftrightarrow x \in \bar{A} \cap (A^\circ)^c \\
&\Leftrightarrow x \in \bar{A} \cap \bar{A}^c \\
&\Leftrightarrow x \in \bar{A} \text{ and } x \in \bar{A}^c \\
&\Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset, U(x, \delta) \cap A^c \neq \emptyset
\end{aligned}$$

**Remark:**  $\overline{U(v, \delta)} = B(v, \delta)$  in a normed space.  $\partial U(v, \delta) = \partial B(v, \delta) = S(v, \delta)$  in a normed space. Also,  $B(v, \delta)^\circ = U(v, \delta)$ .

**Proof:** We show that  $\overline{U(v, \delta)} = B(v, \delta)$ . Since  $B(v, \delta)$  is closed, and  $U(v, \delta) \subseteq B(v, \delta)$ , we know  $\overline{U(v, \delta)} \subseteq B(v, \delta)$ .

Let  $w \in B(v, \delta)$ . If  $\|w - v\| < \delta$ , then  $w \in U(v, \delta)$ . Assume  $\|w - v\| = \delta$ . Let  $u_t = (1 - t)v + tw$ , where  $t \in [0, 1]$ .

$$\begin{aligned}
\|w - u_t\| &= \|w - (1 - t)v - tw\| \\
&= \|(1 - t)(w - v)\| \\
&= (1 - t)\|w - v\| \\
&= (1 - t)\delta.
\end{aligned}$$

Let  $\varepsilon > 0$ . Let  $t \in (0, 1)$  such that  $(1 - t)\delta < \varepsilon$ . Then,  $u_t \in U(w, \varepsilon) \cap U(v, \delta)$ . Therefore,  $w \in \overline{U(v, \delta)}$ .

### Unions and Intersections of Closure/Interior

Let  $(X, d)$  be a metric space.

(i)

$$\left( \bigcup_{i \in I} A_i \right)^\circ \supseteq \bigcup_{i \in I} A_i^\circ$$

may be strict

(ii)

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \bar{A}_i$$

(iii)

$$\bigcap_{k=1}^n A_k^\circ = \left( \bigcap_{k=1}^n A_k \right)^\circ$$

(iv)

$$\overline{\bigcup_{k=1}^n D_k} = \bigcup_{k=1}^n \bar{D}_k$$

**Proof:**

(i)

$$\begin{aligned} A_i^\circ &\subseteq A_i \\ \bigcup_{i \in I} A_i^\circ &\subseteq \bigcup_{i \in I} A_i \\ \bigcup_{i \in I} A_i^\circ &\subseteq \left( \bigcup_{i \in I} A_i \right)^\circ \end{aligned}$$

**Remark:** We claim  $\overline{\mathbb{Q}} = \mathbb{R}$  under the absolute value metric. We know that  $\mathbb{Q} \subseteq \mathbb{R}$ ,  $\mathbb{R}$  is closed, meaning  $\overline{\mathbb{Q}} \subseteq \mathbb{R}$ . Let  $t \in \mathbb{R}$ ,  $\delta > 0$ . We know that  $(t - \delta, t + \delta) \cap \mathbb{Q} \neq \emptyset$ . Therefore,  $t \in \overline{\mathbb{Q}}$ . Thus,  $\overline{\mathbb{Q}} = \mathbb{R}$ .

### Properties of Boundary

Let  $A \subseteq X$ .

- (1)  $\partial A$  is closed.
- (2)  $\partial A = \partial A^c$
- (3)  $\overline{A} = A \cup \partial A$
- (4)  $A \setminus \partial A = A^\circ$

**Proof:**

(1)

$$\begin{aligned} \partial A &= \overline{A} \setminus A^\circ \\ &= \overline{A} \cap (A^\circ)^c. \end{aligned}$$

(2) Follows from pointwise characterization.

(3) Clearly,  $A \cup \partial A \subseteq \overline{A}$ . Let  $x \in \overline{A}$ . If  $x \in A$ , we're done. Otherwise,  $x \in \overline{A} \setminus A \subseteq \overline{A} \setminus A^\circ = \partial A$ .

(4)

$$\begin{aligned} A \setminus \partial A &= A \cap (\partial A)^c \\ &= A \cap (\overline{A} \setminus A^\circ)^c \\ &= A \cap (\overline{A} \cap (A^\circ)^c)^c \\ &= A \cap (\overline{A}^c \cup A^\circ) \\ &= (A \cap \overline{A}^c) \cup (A \cap A^\circ) \\ &= A^\circ \end{aligned}$$

### Density and Separability

Let  $(X, d)$  be a metric space.

- (1)  $A \subseteq X$  is  $d$ -dense if  $\overline{A} = X$ .
- (2)  $N \subseteq X$  is nowhere dense if  $(\overline{N})^\circ = \emptyset$ .
- (3)  $(X, d)$  is separable if there is a countable dense subset.

**Exercise:** If  $N \subseteq X$  is closed, then  $N$  is nowhere dense if and only if  $N^c$  is dense.

**Exercise:** The following are equivalent.

- (1)  $A \subseteq X$  is dense.



- (2)  $\forall \emptyset \neq U \in \tau_X, U \cap A \neq \emptyset$ .
- (3)  $\forall x \in X, \forall \varepsilon > 0, U(x, \varepsilon) \cap A \neq \emptyset$ .
- (4)  $\forall x \in X, \forall \varepsilon > 0, \exists a \in A$  such that  $d(x, a) < \varepsilon$ .

Let  $X$  be a metric space.

- (1) A base for  $\tau_X$  is a family of open subsets  $\mathcal{B}$  such that:

$$(\forall U \in \tau_X) (\forall x \in U) \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U.$$

Equivalently,

$$\forall U \in \tau_X, U = \bigcup_{i \in I} B_i. \quad B_i \in \mathcal{B}$$

- (2) We say that  $(X, d)$  is second countable if  $\tau_X$  admits a countable base.

- For any  $(X, d)$  a metric space,  $\mathcal{B} = \{U(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$  is a base. Indeed, given any  $x \in U \subseteq \tau_X$ , by definition,  $\exists \varepsilon > 0$  such that  $U(x, \varepsilon) \subseteq U$ . Alternatively,  $\mathcal{B}' = \{U(x, 1/n) \mid x \in X, n \geq 1\}$  is a topological base.
- Let  $X = \mathbb{R}^d$  with the Euclidean metric. Then, for  $\mathcal{B} = \{U(q, 1/n) \mid n \geq 1, q \in \mathbb{Q}^d\}$ , we claim this is a base.

Let  $V \subseteq \mathbb{R}^d$  be open,  $r \in V$ . Since  $V$  is open,  $\exists \delta > 0$  with  $U(r, \delta) \subseteq V$ . Find  $n$  large such that  $1/n < \delta$ . Find  $q \in \mathbb{Q}^d$  with  $\|r - q\| < 1/2n$ . This is always possible as  $\mathbb{Q}^d$  is dense in  $\mathbb{R}^d$ .

Consider  $U(q, 1/2n)$ . Then,  $r \subseteq U(q, 1/2n) \subseteq U(r, \delta) \subseteq V$  because  $\|r - q\| < 1/2n$ , and if  $t \in U(q, 1/2n)$ , then

$$\begin{aligned} \|t - r\| &\leq \|t - q\| + \|q - r\| \\ &< 1/2n + 1/2n \\ &= 1/n \\ &< \delta. \end{aligned}$$

### Separable, Non-Separable, Dense, and Non-Dense Sets

- (1)  $(\mathbb{R}^d, \|\cdot\|_p)$  is separable for any  $p \in [1, \infty]$ . Indeed,  $\mathbb{Q}^d \subseteq \mathbb{R}^d$  is the countable dense subset of  $\mathbb{R}^d$ .

Let  $r = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \in \mathbb{R}^d$ . Find  $q = \begin{bmatrix} q_1 \\ \vdots \\ q_d \end{bmatrix} \in \mathbb{Q}^d$  with  $|r_j - q_j| < \varepsilon/d$ . Then,

$$\begin{aligned} \|r - q\|_1 &= \sum_{j=1}^d |r_j - q_j| \\ &< d. \end{aligned}$$

We know that for any vector  $r \in \mathbb{R}^d$ , we can find a vector  $q$  such that

$$\|q - r\|_p \leq c \|q - r\|_1,$$

so for arbitrary  $p$ , find  $q$  such that  $\|q - r\|_1 < \varepsilon/c$ .

- (2) Similarly,  $\mathbb{C}_{\mathbb{Q}} = \{a + bi \mid a, b \in \mathbb{Q}\}$  is also countable, meaning  $\mathbb{C}_{\mathbb{Q}}^d \subseteq \mathbb{C}^d$  is dense and  $\mathbb{C}^d$  is dense.

**Proposition: Separable Subsets**

If  $(X, d)$  is separable, and  $Y \subseteq X$ , then  $(Y, d)$  is also separable.

Let  $\{a_k\}$  be a countable dense subset in  $X$ . Let  $N = \{(m, n) \mid U(a_m, 1/n) \cap Y \neq \emptyset\}$ . Clearly,  $N$  is nonempty. For each  $(m, n) \in N$ , choose  $b_{(m,n)} \in Y \cap U(a_m, 1/n)$ . We claim  $\{b_{(m,n)} \mid m, n \geq 1\}$  is dense in  $Y$ .

Let  $y \in Y$ ,  $\varepsilon > 0$ . Find  $N$  large so that  $\frac{1}{n} < \varepsilon/2$ . Since  $A \subseteq X$  is dense, find  $U(y, 1/n) \cap A \neq \emptyset$ . Suppose  $d(a_m, y) < 1/n$ . Then,

$$\begin{aligned} d(b_{(m,n)}, y) &\leq d(b_{(m,n)}, a_m) + d(a_m, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n} \\ &< \varepsilon. \end{aligned}$$

- (1)  $\ell_p^n$  is separable.
- (2)  $c_{00} = \{(a_k)_{k=1}^n \mid \text{finitely many } a_k \neq 0\}$  with  $\|\cdot\|_u$  is separable.

Recall that  $e_k = (0, 0, \dots, 1, 0, 0, \dots)$  where 1 is at position  $k$ . Consider  $E = \mathbb{Q}\text{-span}\{e_k \mid k \geq 1\}$ ,

$$E = \left\{ \sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q}, n \geq 1 \right\}.$$

The set  $E$  is countable. If we fix  $n \geq 1$ , we have

$$E_n = \left\{ \sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\}.$$

Then,  $E = \bigcup E_n$ . Note

$$\begin{aligned} \underbrace{\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}}_n &\rightarrow E_n \\ (\alpha_1, \dots, \alpha_n) &\mapsto \sum_{k=1}^n \alpha_k e_k. \end{aligned}$$

Thus,  $E_n$  is countable, and  $E$  is a countable union of countable sets.

We claim that  $E$  is dense. Given  $z \in c_{00}$ ,  $\varepsilon > 0$ , we know that  $z = \sum_{k=1}^n a_k e_k$  for some  $n$  and  $a_k \in \mathbb{R}$ . Find  $\alpha_k \in \mathbb{Q}$  such that  $|\alpha_k - a_k| < \varepsilon$ . Set  $w = \sum_{k=1}^n \alpha_k e_k$ . Then,  $\|z - w\|_u = \sup |\alpha_k - a_k| < \varepsilon$ .

- (3)  $c_0$  with  $\|\cdot\|_u$  is separable.
- (4)  $\ell_\infty$  is not separable.

Suppose  $\ell_\infty$  were separable. Consider  $E = \{(a_k)_k \in \ell_\infty \mid a_k \in \{0, 1\}\}$ . Then,  $E$  is separable. Recall that  $(E, \|\cdot\|_u)$  has the discrete metric.

In the discrete metric, every subset is open, meaning every subset is closed. Therefore, if  $X$  is separable and discrete, then  $X$  is countable.

However,  $E$  is not countable by Cantor's theorem.  $\text{card}(E) = 2^{\aleph_0}$ .

Alternatively, we can show that

$$(a_k)_k \mapsto \sum_{k=1}^{\infty} 2^{-k} a_k$$

is onto.

**Exercise:**  $\ell_p$  is separable for  $1 \leq p < \infty$ .

(5) We will show that

$$\mathbb{P}[0, 1] \left\{ \sum_{k=1}^n a_k x^k \mid a_k \in \mathbb{R}, n \geq 1 \right\}$$

is  $\|\cdot\|_u$ -dense in  $C([0, 1])$  (see: Stone-Weierstrass Theorem). Using this, we can show that  $(C([0, 1]), \|\cdot\|_u)$  is separable.

## The Cantor Set

$$\begin{aligned} C_0 &= [0, 1] \\ C_1 &= [0, 1/3] \cup [2/3, 1] \\ C_2 &= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1] \\ C_3 &= [0, 1/27] \cup [2/27, 1/9] \cup \dots \cup [26/27, 1] \\ &\vdots \end{aligned}$$

In each step, we delete the middle third of each interval. This process repeated ad infinitum yields the Cantor set.

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left( \left[ \frac{3k+0}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[ \frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right).$$

(i)  $\mathcal{C}$  is closed as it is the intersection of closed sets.

(ii)  $\text{length}(\mathcal{C}) = 0$ . Look at the total length of the removed intervals,

$$\begin{aligned} I &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots \\ &= \sum_{k=1}^{\infty} \left( \frac{2^{k-1}}{3^k} \right) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^k \\ &= 1. \end{aligned}$$

Thus,  $\text{length}(\mathcal{C}) = 0$ .

(iii)  $\mathcal{C}$  is nowhere dense —  $(\overline{\mathcal{C}})^\circ = \emptyset$ . Since  $\mathcal{C}$  is closed,  $\mathcal{C}^\circ = \emptyset$ .

Suppose  $\mathcal{C}^\circ \neq \emptyset$ . Then,  $\exists x \in \mathcal{C}, \varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}$ . So,  $(x - \varepsilon, x + \varepsilon) \subseteq C_n$  for all  $n$ .

Note  $C_n$  is the disjoint union of  $2^n$  subintervals, each with length  $1/3^n$ . Find  $m$  so large such that  $3^{-m} < \varepsilon$ . We know that  $(x - \varepsilon, x + \varepsilon) \subseteq C_m$ .

However,  $(x - \varepsilon, x + \varepsilon)$  has length  $2\varepsilon > \frac{2}{3^m}$ . Each subinterval in  $C_m$  has length  $1/3^m$ . This implies  $C_m$  contains an interval of length greater than  $\frac{2}{3^m}$ .  $\perp$

(iv)  $\text{card}(\mathcal{C}) = \text{card}(\mathbb{R})$

**Claim 1:** Given  $n \geq 1$ ,

$$E_n = \left\{ \sum_{k=1}^n \frac{w_k}{3^k} \mid w_k \in \{0, 2\} \right\}$$

is precisely the set of *left* endpoints of the subintervals of  $C_n$ .

For  $n = 1$ , if  $w_1 = 0$ , then we get 0, and  $w_1 = 2$  yields  $2/3$ . Meanwhile, if  $n = 2$ , then we have

$$\begin{aligned} w_1 = 0, w_2 = 0 &\mapsto 0 \\ w_1 = 0, w_2 = 2 &\mapsto 2/9 \\ w_1 = 2, w_2 = 0 &\mapsto 2/3 \\ w_1 = 2, w_2 = 2 &\mapsto 8/9. \end{aligned}$$

By induction, we have shown for  $n = 1, 2$ . Assume this is true for  $n$ .

$$\sum_{k=1}^{n+1} w_k 3^{-k} = \underbrace{\sum_{k=1}^n w_k 3^{-k}}_{(1)} + \underbrace{w_{n+1} 3^{-(n+1)}}_{(2)}$$

Part (1) denotes one of the left endpoints of  $C_n$ , called  $C_{n,k}$  for some  $1 \leq k \leq 2^n$ . Then, if  $w_{n+1} = 0$ , we get the left endpoint of  $C_{n+1,2k-1}$ , and if  $w_n = 2$ , we get the left endpoint of  $C_{n+1,2k}$ .

**Claim 2:**

$$C = \left\{ \sum_{k=1}^{\infty} w_k 3^{-k} \mid w_k \in \{0, 2\} \right\}$$

is precisely the Cantor set.

Let  $x = \sum_{k=1}^{\infty} w_k 3^{-k}$ . We will show that  $x \in C_n$  for all  $n$ . Fix  $n \geq 1$ . Then,

$$x = \underbrace{\sum_{k=1}^n w_k 3^{-k}}_y + \underbrace{\sum_{k>n} w_k 3^{-k}}_z.$$

From our previous claim,  $y$  is the left endpoint of some subinterval of  $C_n$ . Additionally,

$$\begin{aligned} z &= \sum_{k>n} w_k 3^{-k} \\ &\leq 2 \sum_{k>n} 3^{-k} \\ &= \frac{2}{3^{n+1}} \left( 1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) \\ &= \frac{1}{3^n}. \end{aligned}$$

Since the length of a subinterval in  $C_n$  is exactly  $3^{-n}$ , it is the case that  $x = y + z$  remains an element of  $C_{n,k}$ .

Let  $x \in C$ . Then,  $x \in C_n$  for all  $n$ . Then,  $x \in C_1$ , so let  $x_1$  be the left endpoint of the interval  $C_{1,j}$  that contains  $x$ . Then,  $|x - x_1| < \frac{1}{3}$ , and  $x_1 = w_1 3^{-1}$  for some  $w_1 \in \{0, 2\}$ .

Let  $x_2$  be the left endpoint of the subinterval  $C_{2,j}$  that contains  $x$ . Then,  $|x - x_2| < \frac{1}{3^2}$ . Therefore,

$$\begin{aligned} x_2 &= x_1 + w_2 3^{-2} \\ &= w_1 3^{-1} + w_2 3^{-2}. \end{aligned}$$

Iterating, we have  $x_n$ , the left endpoint of the subinterval  $C_{n,j}$  that contains  $x$ .

$$x_n = \sum_{k=1}^n w_k 3^{-k}.$$

We have that  $|x - x_n| < 3^{-n}$ .

Therefore,  $(x_n)_n \rightarrow x$ . Also,

$$\begin{aligned} x_n &= \sum_{k=1}^n w_k 3^{-k} \\ &\rightarrow \sum_{k=1}^{\infty} w_k 3^{-k}. \end{aligned}$$

Thus,

$$x = \sum_{k=1}^{\infty} w_k 3^{-k}.$$

To prove  $\text{card}(\mathcal{C}) = \text{card}(\mathbb{R})$ , we will show that  $\text{card}(\{0, 1\}^{\mathbb{N}}) = \text{card}(\mathcal{C})$ .

$$(a_k)_k \mapsto \sum_{k=1}^{\infty} 2a_k 3^{-k}.$$

### Relative (or Subspace) Topology

We know that if  $(X, d)$  is a metric space, and  $Y \subseteq X$  is any subset, then  $(Y, d)$  is a metric space. The question now is: what are the open sets of  $Y$ ?

For example, let  $X = \mathbb{R}$ ,  $Y = [0, 1]$ . Consider  $U = [0, 1/2]$ .  $U$  is not open in  $\mathbb{R}$ , as if  $x = 0$ , then there is no open ball completely contained in  $U$ . However, in  $Y$ ,  $U$  is open.

Let  $(X, d)$  be a metric space,  $Y \subseteq X$  any subset.  $V \subseteq Y$  is open if and only if  $\exists U \subseteq X$  open such that  $V = U \cap Y$ . That is,  $\tau_Y = \{U \cap Y \mid U \in \tau_X\}$ .

Let  $V$  be open in  $Y$ . Then,  $\forall x \in V$ ,  $\exists \delta_x > 0$  such that  $U_Y(x, \delta_x) \subseteq V$ . We have  $U_Y(x, \delta_x) = \{y \in Y \mid d(y, x) < \delta_x\}$ . Let

$$\begin{aligned} U &= \bigcup_{x \in V} U_X(x, \delta_x) \\ U \cap Y &= \left( \bigcup_{x \in V} U_X(x, \delta_x) \right) \cap Y \\ &= \bigcup_{x \in V} U_X(x, \delta_x) \cap Y \\ &= \bigcup_{x \in V} U_Y(x, \delta_x). \end{aligned}$$

Let  $U$  be open in  $X$ . Then, for  $x \in U \cap Y$ ,  $\exists \delta_x$  such that  $U(x, \delta_x) \subseteq U$ .

- (1)  $\ell_{\infty}$  is not a discrete metric space. However,  $E = \{(a_k)_k \mid a_k \in \{0, 1\}\}$  with the induced metric. Then,  $E$  is a discrete metric space.

### Convergent Sequences

Fix a metric space  $(X, d)$ . A sequence in  $X$  is a map  $x : \mathbb{N} \rightarrow X$ ,  $n \mapsto x(n) = x_n$ .

A natural sequence  $(n_k)_k$  is a sequence in  $\mathbb{N}$  with  $n_k \geq k$  for all  $k$ . A subsequence of  $(x_n)_n$  is a sequence  $(x_{n_k})_k$ , where  $(n_k)_k$  is a natural sequence.

A sequence  $(x_n)_n$  converges to  $x \in X$  if  $\forall \varepsilon > 0$ ,  $\exists N_{\varepsilon} \in \mathbb{N}$  such that  $n \geq N_{\varepsilon}$  implies  $d(x_n, x) < \varepsilon$ . We write  $(x_n)_n \xrightarrow{d} x$ .

**Exercise:** A sequence can have at most one limit, as metric spaces are Hausdorff.

**Proposition: Equivalent Definitions of Convergence**

Given  $(x_n)_n \in X$ ,  $x \in X$ , the following are equivalent.

- (i)  $(x_n)_n \rightarrow x$  in  $X$
- (ii)  $(d(x_n, x))_n \rightarrow 0$  in  $\mathbb{R}$
- (iii)  $\forall V \in \mathcal{N}_x, \exists N \in \mathbb{N}$  with  $n \geq N \Rightarrow x_n \in V$ .

**Exercise:** Let  $(X, \rho)$  be a metric space, let  $d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$ . A sequence  $(x_n)_n \xrightarrow{d} x$  if and only if  $(x_n)_n \xrightarrow{\rho} x$ .

**Proposition: Convergent Sequences are Bounded**

Let  $(x_n)_n \rightarrow x$  in  $(X, d)$ . Let  $\varepsilon = 1$ . Then,  $\exists N \in \mathbb{N}$  large such that for  $n \geq N$ ,  $d(x_n, x) < 1$ .

If  $m, n \geq N$ , then  $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < 2$ . Let  $c = \max_{1 \leq n, m \leq N} d(x_n, x_m)$ . Then,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\ &\leq 1 + c. \end{aligned}$$

Let  $k = \max\{1 + c, 2\}$ . Then,  $\text{diam}(\{x_n\}) \leq k$ .

**Convergence in Different Metric Spaces**

**Convergence for Bounded Functions:** Recall that for  $(Y, d)$  a metric space is

$$\begin{aligned} \text{Bd}(\Omega, Y) &= \{f : \Omega \rightarrow Y \mid f \text{ bounded}\} \\ D_u(f, g) &= \sup_{x \in \Omega} d(f(x), g(x)). \end{aligned}$$

Then,  $(f_n)_n \rightarrow f$  in  $\text{Bd}(\Omega, Y)$  if and only if  $D_u(f_n, f) \rightarrow 0$  in  $\mathbb{R}$ .

$$\begin{aligned} (\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow D_u(f_n, f) < \varepsilon \\ \Leftrightarrow \\ (\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow \sup_{x \in \Omega} d(f_n(x), f(x)) < \varepsilon \\ \Leftrightarrow \\ (\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow \forall x, d(f_n(x), f(x)) < \varepsilon. \end{aligned}$$

This is exactly the definition of uniform convergence.

Since  $\ell_\infty(\Omega) = \text{Bd}(\Omega, \mathbb{R})$ , convergence in  $\ell_\infty(\Omega)$  is uniform convergence. This is also the case for subspaces, such as  $c$ ,  $c_0$ , and  $c_{00}$ .

**Convergence in the Frechet Metric:** Consider a separating family of semimetrics  $\rho_k$  on a set  $X$ . Set  $d_k = \frac{\rho_k}{1 + \rho_k}$ . We saw that

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric on  $X$ .

We claim that  $(x_n)_n \rightarrow x$  in  $(X, d)$  if and only if for all  $k \geq 1$ ,  $\rho_k(x_n, x) \rightarrow 0$ .

In the forward direction, we know that  $(x_n)_n \rightarrow x$  with respect to  $d$  if and only if  $d(x_n, x) \rightarrow 0$  in  $\mathbb{R}$ . Since  $0 \leq 2^{-k} d_k(x_n, x) \leq d(x_n, x)$  for fixed  $k$ , we have that

$$0 \leq d_k(x_n, x) \leq 2^k d(x_n, x),$$

and as  $n \rightarrow \infty$ ,  $d(x_n, x) \rightarrow 0$ , meaning  $d_k(x_n, x) \rightarrow 0$ . Therefore,  $\rho_k(x_n, x) \rightarrow 0$ .

In the reverse direction, suppose  $\rho_k(x_n, x) \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$  for all  $k \geq 1$ . Thus,  $d_k(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $k \geq 1$ .

Let  $\varepsilon > 0$ . Let  $K$  be so large such that

$$\sum_{k \geq K} 2^{-k} < \varepsilon/2.$$

Therefore,  $d_k(x_n, x) \rightarrow 0$  for all  $k = 1, \dots, K$ . Therefore,  $\exists N_1, \dots, N_K$  such that for  $n \geq N_k$ ,

$$d_k(x_n, x) < \frac{\varepsilon}{2}.$$

Let  $N = \max\{N_1, \dots, N_K\}$ . Therefore, for  $n \geq N$ ,

$$d_k(x_n, x) < \frac{\varepsilon}{2}$$

for all  $k = 1, \dots, K$ .

Thus, for all  $n \geq N$ ,

$$\begin{aligned} d(x_n, x) &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_n, x) \\ &= \sum_{k=1}^K 2^{-k} d_k(x_n, x) + \sum_{k=K+1}^{\infty} 2^{-k} d_k(x_n, x) \\ &\leq \frac{\varepsilon}{2} \sum_{k=1}^K 2^{-k} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Therefore,  $(x_n)_n \rightarrow x$ .

Recall that, for the Frechet metric, our set was  $X = C(\mathbb{R})$ . For  $k = 1, 2, 3, \dots$ , we had

$$\rho_k(f) = \sup_{[-k, k]} |f(x)|$$

as our seminorm, and our semimetric was

$$\rho_k(f, g) = \rho_k(f - g).$$

We also showed that the  $\rho_k$  family is separating. We make  $d_k(f, g) = \frac{\rho_k(f, g)}{1 + \rho_k(f, g)}$  as the bounded family of separating metrics, and

$$d_F(f, g) = \sum_{k=1}^{\infty} \frac{2^{-k} \rho_k(f - g)}{1 + \rho_k(f - g)}.$$

In  $(C(\mathbb{R}), d_F)$ ,  $(f_n)_n \rightarrow f$  if and only if  $\rho_k(f_n, f) \rightarrow 0$  for all  $k$ , meaning  $(f_n)_n \rightarrow f$  uniformly on  $[-k, k]$  for all  $k$ .

This is known as convergence on compact subsets.

**Convergence in a Product Space:** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. Then,

$$\begin{aligned} X \times Y &= \{(x, y) \mid x \in X, y \in Y\}, \\ D_1((x, y), (x', y')) &= d(x, x') + \rho(y, y') \\ D_{\infty}((x, y), (x', y')) &= \max\{d(x, x'), \rho(y, y')\}. \end{aligned}$$

Both  $D_1$  and  $D_{\infty}$  are equivalent metrics.

**Exercise:**  $((x_n, y_n))_n \rightarrow (x, y)$  if and only if  $(x_n)_n \xrightarrow{d} x$  and  $(y_n)_n \xrightarrow{\rho} y$ .

## Series in a Normed Vector Space

Let  $(V, \|\cdot\|)$  be a normed vector space. Consider a sequence  $(v_k)_k$  of vectors.

$$\begin{aligned} s_1 &= v_1 \\ s_2 &= v_1 + v_2 \\ &\vdots \\ s_n &= \sum_{k=1}^n v_k. \end{aligned}$$

If  $s_n \rightarrow s$  in  $(V, \|\cdot\|)$ , meaning  $\|s_n - s\| \rightarrow 0$ , then we say the series  $\sum_{k=1}^{\infty} v_k$  converges to  $s$ . We write

$$\sum_{k=1}^{\infty} v_k = s.$$

The series converges absolutely if

$$\sum_{k=1}^{\infty} \|v_k\|$$

converges in  $\mathbb{R}$ .

### Proposition: Sequential Characterization of Closure

Let  $(X, d)$  be a metric space with  $A \subseteq X$ .  $x \in \bar{A}$  if and only if  $\exists (a_n)_n$  in  $A$  with  $(a_n)_n \rightarrow x$ .

In the forward direction, recall that  $x \in \bar{A}$  if and only if  $\forall \delta > 0$ ,  $U(x, \delta) \cap A \neq \emptyset$ . If  $x \in \bar{A}$ , then set  $\varepsilon_n = 1/n$ , and since  $U(x, 1/n) \cap A \neq \emptyset$ . Let  $a_n \in U(x, 1/n) \cap A$ . Then,  $d(a_n, x) < 1/n \rightarrow 0$ , meaning  $a_n \rightarrow x$  and  $a_n \in A$ .

In the reverse direction, if  $(a_n)_n \rightarrow x$  and  $\varepsilon > 0$ ,  $\exists N$  with  $n \geq N \Rightarrow a_n \in U(x, \varepsilon) \cap A$ . Thus,  $x \in \bar{A}$ .

### Proposition: Sequential Characterization of Closed Sets

If  $(X, d)$  is a metric space,  $A \subseteq X$ , then the following are equivalent:

- (i)  $A$  is closed.
- (ii) Whenever  $(a_n)_n$  in  $A$  with  $(a_n)_n \xrightarrow{d} x$  in  $X$ , then  $x \in A$ .

**Continuous Bounded Functions:**  $C([a, b]) \subseteq \ell_{\infty}([a, b])$  is closed under  $\|\cdot\|_{\infty}$ , since if  $(f_n)_n \rightarrow f$  uniformly, and  $f_n$  is continuous, then  $f$  is continuous.

**Sequence Closure:**  $c_0 \subseteq \ell_{\infty}$  is closed under  $\|\cdot\|_{\infty}$ . Let  $(f_n)_n$  be a sequence

$$\begin{aligned} f_1 &= (f_1(1), f_1(2), \dots) \\ f_2 &= (f_2(1), f_2(2), \dots) \\ \lim_{k \rightarrow \infty} f_n(k) &= 0 \end{aligned} \quad \forall n$$

Suppose  $(f_n)_n \xrightarrow{\|\cdot\|_{\infty}} f \in \ell_{\infty}$ .

Let  $\varepsilon > 0$ . Then,  $\exists n \in \mathbb{N}$  such that for  $n \geq N$ ,  $\|f - f_n\|_{\infty} < \varepsilon/2$ . Also,  $\lim_{k \rightarrow \infty} f_N(k) = 0$ . Then,  $\exists K \in \mathbb{N}$  such that for  $k \geq K$ ,  $|f_N(k)| < \varepsilon/2$ . Thus, for  $k \geq K$ ,

$$\begin{aligned} |f(k)| &= |f(k) - f_N(k) + f_N(k)| \\ &\leq |f(k) - f_N(k)| + |f_N(k)| \\ &\leq \|f - f_N\|_{\infty} + |f_N(k)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus,  $f \in c_0$ .



## Distance to a Set

Let  $(X, d)$  be a metric space,  $A \subseteq X$ . Then,  $\text{dist}_A : X \rightarrow [0, \infty)$  is defined as

$$\text{dist}_A(x) = \inf_{a \in A} d(x, a).$$

$$(1) \quad \bar{A} = \{x \mid \text{dist}_A(x) = 0\}$$

$$(2) \quad \text{dist}_A(\cdot) = \text{dist}_{\bar{A}}(\cdot)$$

$$(3) \quad |\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y)$$

**Proof of (1):** Let  $x \in \bar{A}$ . Then,  $\exists (a_n)_n$  such that  $(a_n)_n \rightarrow x$ . Then,  $d(a_n, x) \rightarrow 0$ . Since  $0 \leq \text{dist}_A(x) \leq d(x, a_n)$ ,  $\text{dist}_A(x) = 0$ .

Let  $x$  be such that  $\text{dist}_A(x) = 0$ . By the definition of inf, we construct  $a_n$  by finding  $a_n \in U(x, 1/n) \cap A$ . Thus,  $d(a_n, x) \rightarrow 0$ , meaning  $(a_n)_n \rightarrow x$ , so  $x \in \bar{A}$ .

**Proof of (2):** Exercise; use (1).

**Proof of (3):** For all  $a \in A$ ,

$$\begin{aligned} \text{dist}_A(x) &\leq d(x, a) \\ &\leq d(x, y) + d(y, a). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{dist}_A(x) - d(x, y) &\leq d(y, a) \\ \text{dist}_A(x) - d(x, y) &\leq \inf_{a \in A} d(y, a) \\ &= \text{dist}_A(y) \\ \text{dist}_A(x) - \text{dist}_A(y) &\leq d(x, y). \end{aligned}$$

Similarly,

$$\text{dist}_A(y) - \text{dist}_A(x) \leq d(y, x) = d(x, y)$$

meaning

$$|\text{dist}_A(y) - \text{dist}_A(x)| \leq d(x, y).$$

## Continuity

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A map  $f : X \rightarrow Y$

(1) is continuous at  $x_0 \in X$  if

$$\begin{aligned} &(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon \\ &(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in U_X(x_0, \delta) \Rightarrow f(x) \in U_Y(f(x_0), \varepsilon) \\ &(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } f(U_X(x_0, \delta)) \subseteq U_Y(f(x_0), \varepsilon). \end{aligned}$$

(2) is continuous if  $f$  is continuous at every  $x_0 \in X$ .

**Proposition: Equivalent Continuity Criteria**

Let  $f : (X, d) \rightarrow (Y, \rho)$ ,  $x_0 \in X$ . The following are equivalent:

- (1)  $f$  is continuous at  $x_0$ ;
- (2)  $(\forall V \in \mathcal{N}_{f(x_0)})(\exists U \in \mathcal{N}_{x_0})$  such that  $f(U) \subseteq V$ .
- (3)  $\forall (x_n)_n \rightarrow x_0, (f(x_n))_n \rightarrow f(x_0)$ .

(1)  $\Leftrightarrow$  (2): Clearly follows from definitions.

(1)  $\Rightarrow$  (3): Let  $(x_n)_n \rightarrow x_0$ . Let  $\varepsilon > 0$ . Then,  $\exists \delta > 0$  such that  $d(x, x_0) < \delta$  implies  $\rho(f(x), f(x_0)) < \varepsilon$ .

Thus,  $\exists N \in \mathbb{N}$  such that  $n \geq N$  implies  $d(x_n, x_0) < \delta$ . So, if  $n \geq N$ ,  $d(x_n, x_0) < \delta$ , implying  $\rho(f(x_n), f(x_0)) < \varepsilon$ . So,  $(f(x_n))_n \rightarrow f(x_0)$ .

(3)  $\Rightarrow$  (1): Suppose toward contradiction that  $\exists \varepsilon_0 > 0$  such that for  $\delta = 1/n$  where  $n \in \mathbb{N}$ ,  $\exists (x_n)_n : d(x_n, x_0) < \delta$  and  $\rho(f(x_n), f(x_0)) \geq \varepsilon_0$ . Then,  $(x_n)_n \rightarrow x_0$ , but  $(f(x_n))_n \not\rightarrow f(x_0)$ .  $\perp$

**Proposition: Topological Criterion for Continuity**

Let  $f : (X, d) \rightarrow (Y, \rho)$ . The following are equivalent:

- (1)  $f$  is continuous.
- (2)  $\forall V \in \tau_Y, f^{-1}(V) \in \tau_X$ .
- (3)  $\forall x \in X, \forall (x_n)_n \rightarrow x$ , we have  $(f(x_n))_n \rightarrow f(x)$ .

**Proof:** Exercise.

**Proposition: Composition of Functions**

Let  $(X, d) \xrightarrow{f} (Y, \rho) \xrightarrow{g} (Z, \rho)$ . If  $f$  and  $g$  are continuous, then  $g \circ f$  is continuous.

**Proof:** Exercise.

**Uniform Continuity**

Let  $f : (X, d) \rightarrow (Y, \rho)$ .

- (1)  $f$  is uniformly continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } \forall x, x' \in X, d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon$$

- (2)  $f$  is Lipschitz if  $\exists c > 0$  with

$$\rho(f(x), f(x')) \leq c d(x, x')$$

for all  $x, x' \in X$ .

- (3) If  $\rho(f(x), f(x')) = d(x, x')$ , then  $f$  is an isometry. Isometries are always injective.

**Exercise:**

$$\text{Isometry} \Rightarrow \text{Lipschitz} \Rightarrow \text{Uniformly Continuous} \Rightarrow \text{Continuous}.$$

For example,  $f(x) = x^2$  on  $[0, \infty)$  is continuous but not uniformly continuous, and  $\sqrt{x}$  on  $[0, 1]$  is uniformly continuous but not Lipschitz.

If  $(V, \|\cdot\|)$  is a normed space, we might want to care that the following operations are continuous:

- $a : V \times V \rightarrow V$ ,  $a(v, w) = v + w$ :

$$\begin{aligned} \|a(v, w) - a(v', w')\| &= \|v + w - (v' + w')\| \\ &= \|v - v' + w - w'\| \\ &\leq \|v - v'\| + \|w - w'\| \\ &= d(v, v') + d(w, w') \\ &= d_1((v, w), (v', w')), \end{aligned}$$

meaning  $a$  is Lipschitz.

- $m : \mathbb{F} \times V \rightarrow V$ ,  $m(\alpha, v) = \alpha v$ ;

$$\begin{aligned} \|m(\alpha, v) - m(\beta, w)\| &= \|\alpha v - \beta w\| \\ &= \|\alpha v - \alpha w + \alpha w - \beta w\| \\ &\leq |\alpha| \|v - w\| + |\alpha - \beta| \|w\| \end{aligned}$$

If  $(\alpha_n)_n \rightarrow \beta$  and  $(v_n)_n \rightarrow w$ , then

$$\begin{aligned} \|\alpha_n v_n - \beta w\| &\leq |\alpha_n| \|v_n - w\| + |\alpha_n - \beta| \|w\| \\ &\rightarrow 0. \end{aligned}$$

- $\|\cdot\| : V \rightarrow \mathbb{F}$ :

$$\| \|v\| - \|w\| \| \leq \|v - w\|,$$

meaning  $\|\cdot\|$  is Lipschitz.

Let  $(X, d)$  be a metric space. Then,  $\text{dist}_A : X \rightarrow [0, \infty)$ ,  $\text{dist}_A(x) = \inf_{a \in A} d(x, a)$  is continuous. We had shown

$$|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y),$$

meaning  $\text{dist}_A$  is Lipschitz.

### Proposition: Normal Property of Metric Spaces

Given  $A, B \subseteq X$  with  $A \cap B = \emptyset$ , then  $\exists U, V \in \tau_X$  with  $A \subseteq U$ ,  $B \subseteq V$ , and  $U \cap V = \emptyset$ .

**Proof:** Set

$$f(x) = \frac{\text{dist}_A(x)}{\text{dist}_A(x) + \text{dist}_B(x)}.$$

Note that  $\text{dist}_A(x) + \text{dist}_B(x) = 0$  if and only if  $x \in \overline{A} = A$  and  $x \in \overline{B} = B$ . Therefore, the denominator in  $f(x)$  is always positive.

Additionally,  $f : X \rightarrow [0, 1]$  is continuous. Note that  $f(a) = 0$  for all  $a \in A$  and  $f(b) = 1$  for all  $b \in B$ .

Let  $U = f^{-1}((-1/2, 1/2)) = f^{-1}([0, 1/2))$ , and  $V = f^{-1}((1/2, 3/2)) = f^{-1}((1/2, 1])$ . Obviously,  $U \subseteq A$  and  $V \subseteq B$ , and  $U \cap V = \emptyset$ .

### Proposition: Quotient Space

Let  $(V, \|\cdot\|)$  be a normed space, and let  $W \subseteq V$  be a closed subspace. Then,  $V/W$  is a normed space with

$$\begin{aligned} \|v + W\| &= \text{dist}_W(v) \\ &= \inf_{w \in W} \|v - w\|. \end{aligned}$$

**Proposition: Uniform Continuity of Linear Transformations**

Let  $T : V \rightarrow W$  be a linear transformation between two normed spaces. The following are equivalent:

- (1)  $T$  is continuous at  $0_V$ .
- (2)  $T$  is continuous.
- (3)  $T$  is uniformly continuous.
- (4)  $T$  is Lipschitz.
- (5)  $\exists c \geq 0$  such that  $\|T(v)\| \leq c \|v\|$  for all  $v \in V$ .
- (6)  $\|T\|_{\text{op}} = \sup_{\|v\| \leq 1} \|T(v)\| < \infty$ . In other words,  $T$  is bounded linear.

**Proof:**

(4)  $\Rightarrow$  (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1): Obvious.

(6)  $\Rightarrow$  (5) Let  $v \in V$ . If  $v = 0_V$ , then  $T(v) = 0_W$ . Suppose  $v \neq 0_V$ . We know

$$\begin{aligned} \left\| T \left( \frac{v}{\|v\|} \right) \right\| &\leq \|T\|_{\text{op}} \\ \frac{1}{\|v\|} \|T(v)\| &\leq \|T\|_{\text{op}} \\ \|T(v)\| &\leq \|T\|_{\text{op}} \|v\|. \end{aligned}$$

Therefore,  $c = \|T\|_{\text{op}}$ .

(5)  $\Rightarrow$  (6): We will have  $\|T(v)\| \leq c$  for all  $v \in B_V$ . Thus,  $\|T\|_{\text{op}} \leq c$  for such  $c$ .

(5)  $\Rightarrow$  (4): Let  $v, w \in V$ . Then,

$$\begin{aligned} \|T(v) - T(w)\| &= \|T(v - w)\| \\ &\leq c \|v - w\|, \end{aligned}$$

meaning  $T$  is Lipschitz.

(1)  $\Rightarrow$  (5): Let  $\varepsilon = 1$ . Then,  $\exists \delta$  such that

$$T(U_V(0, \delta)) \subseteq U_W(T(0), 1).$$

Since  $T$  is linear,

$$T(U_V(0, \delta)) \subseteq U_W(0, 1).$$

Let  $v \in V \neq 0_V$ . We know  $\frac{\delta v}{2\|v\|} \in U_V(0, \delta)$ . Then,

$$\begin{aligned} \left\| T \left( \frac{\delta v}{2\|v\|} \right) \right\| &\leq 1, \\ \frac{\delta}{2\|v\|} \|T(v)\| &\leq 1 \\ \|T(v)\| &\leq \frac{2}{\delta} \|v\|. \end{aligned}$$

Set  $c = \frac{2}{\delta}$ . Clearly,  $\|T(0)\| \leq \frac{2}{\delta} \|0\|$ .

A corollary to this is that any linear map  $T : \ell_p^n \rightarrow W$  for  $W$  a normed space is uniformly continuous.

**Proposition: Continuous Functions on Dense Sets**

Let  $(X, d)$ ,  $(Y, \rho)$  be metric spaces, and  $A \subseteq X$  dense. If  $f, g : X \rightarrow Y$  and  $f(A) = g(A)$ , then  $f(X) = g(X)$ .

**Proof:** Given  $x \in X$ , there exists  $(a_n)_n \rightarrow x$ . We know that  $(g(a_n))_n \rightarrow g(x)$  and  $(f(a_n))_n \rightarrow f(x)$ . Since  $f(a_n) = g(a_n)$  for all  $a_n$ , it is the case that  $f(x) = g(x)$ .

## Morphisms in the Category of Metric Spaces

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $f : X \rightarrow Y$  a map.

- (1)  $f$  is a homeomorphism if  $f$  is bijective, continuous, and has a continuous inverse. We write  $X \cong Y$  are homeomorphic.
- (2)  $f$  is a uniformism if  $f$  is bijective, uniformly continuous, and has a uniformly continuous inverse. We write  $X \cong Y$  are uniformly isomorphic.
- (3)  $f$  is a metric isomorphism if  $f$  is bijective, Lipschitz, and has a Lipschitz inverse. We write  $X \cong Y$  are metrically isomorphic.
- (4)  $f$  is an isometric isomorphism if  $f$  is bijective and isometric. We write  $X \cong Y$  are isometrically isomorphic.

For example,  $\mathbb{R} \cong (-\pi/2, \pi/2)$  are homeomorphic (using  $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ ). However,  $\mathbb{R}$  is not uniformly isomorphic to  $(-\pi/2, \pi/2)$ .

Suppose  $f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$  is a uniformism. Let  $(x_n)_n = \pi/2 - 1/n$ . Then,  $(x_n)_n$  is Cauchy. Therefore,  $(f(x_n))_n$  is Cauchy. Since  $\mathbb{R}$  is complete,  $(f(x_n))_n \rightarrow y$  for some  $y \in \mathbb{R}$ . Then,  $f^{-1}(f(x_n))_n \rightarrow f^{-1}(y)$ , meaning  $(x_n)_n \rightarrow f^{-1}(y) \in (-\pi/2, \pi/2)$ . However,  $(x_n)_n \rightarrow \pi/2 \notin (-\pi/2, \pi/2)$ .

## Completeness

### Proposition: Weierstrass M-Test

Let  $V$  be a Banach space (complete normed vector space). Suppose  $(v_k)_k$  is such that  $\sum \|v_k\|$  is convergent. Then,  $(s_n)_n = \sum_{k=1}^n v_k$  converges in  $V$ . Additionally,

$$\left\| \sum_{k=1}^{\infty} v_k \right\| \leq \sum_{k=1}^{\infty} \|v_k\|.$$

**Proof:** Let  $s_n = \sum_{k=1}^n v_k$ , and  $t_n = \sum_{k=1}^n \|v_k\|$ . Let  $n > m$ . Then,

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^n v_k \right\| \\ &\leq \sum_{k=m+1}^n \|v_k\| \\ &= |t_n - t_m|. \end{aligned}$$

Since  $(t_n)_n$  converges, it is Cauchy, and thus  $s_n$  is Cauchy. Since  $V$  is complete,  $(s_n)_n$  converges.

$$\begin{aligned} \|s_n\| &= \left\| \sum_{k=1}^n v_k \right\| \\ &\leq \sum_{k=1}^n \|v_k\| \\ &\leq \sum_{k=1}^{\infty} \|v_k\|. \end{aligned}$$

Let  $n \rightarrow \infty$ . Using the continuity of the norm, we get

$$\left\| \sum_{k=1}^{\infty} v_k \right\| \leq \sum_{k=1}^{\infty} \|v_k\|.$$

**Proposition: Convergence in Hilbert Space**

Let  $H$  be a Hilbert space (inner product space with a complete norm). Let  $(e_n)_n$  be an orthonormal sequence in  $H$ . Let  $(t_k)_k$  be a sequence in  $\ell_2$ . Then,  $\sum_{k=1}^{\infty} t_k e_k$  converges in  $H$ , but not absolutely.

**Proof:** Let  $s_n = \sum_{k=1}^n t_k e_k$ . For  $n > m$ ,

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^n t_k e_k \right\|^2 \\ &= \sum_{k=m+1}^n \|t_k e_k\|^2 && \text{Pythagorean Theorem} \\ &= \sum_{k=m+1}^n |t_k|^2 \end{aligned}$$

Since  $(t_k)_k \in \ell_2$ , we know that  $(t_k)_k$  is convergent and thus Cauchy. Thus,  $(s_n)_n$  is Cauchy.

Note that for  $t_k = \frac{1}{k}$ ,  $(t_k)_k$  is square-summable, but not summable in absolute value.

**Exercise:** Show that

$$\left\| \sum_{k=1}^{\infty} t_k e_k \right\|^2 = \sum_{k=1}^{\infty} |t_k|^2.$$

This result is known as Parseval's Theorem.

**Extensions of Continuous Functions****Lemma: Cauchy Sequences in Uniformly Continuous Functions**

Let  $f : (X, d) \rightarrow (Y, \rho)$  be uniformly continuous. If  $(x_n)_n$  is Cauchy, then  $(f(x_n))_n$  is Cauchy.

**Proof:** Let  $\varepsilon > 0$ . Then,  $\exists \delta > 0$  such that

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon.$$

Similarly, there exists  $N \in \mathbb{N}$  such that for  $p, q \geq N$ ,  $d(x_p, x_q) < \delta$ . So, for  $p, q \geq N$ ,  $d(f(x_p), f(x_q)) < \varepsilon$ .

**Remark:** This is not true for continuous functions. For example, if  $f(t) = 1/t$  on  $(0, 1)$ ,  $x_n = 1/n$  is Cauchy but not convergent.

**Theorem: Extension on a Dense Subset**

Let  $(X, d)$  be a metric space with  $A \subseteq X$  dense. Suppose  $f : A \rightarrow Y$  is uniformly continuous with  $(Y, \rho)$  complete. Then,  $\exists!$  uniformly continuous extension,  $\tilde{f} : X \rightarrow Y$  that agrees with  $f$  on  $A$ .

**Proof:** Let  $x \in X$ . Then,  $\exists (a_n)_n \in A$  with  $(a_n)_n \rightarrow x$ . Therefore,  $(a_n)_n$  is Cauchy, and since  $f$  is uniformly continuous, we know that  $(f(a_n))_n$  is Cauchy. Thus,  $\lim_{n \rightarrow \infty} (f(a_n))_n = \tilde{f}(x)$  exists.

To show  $\tilde{f}$  is well-defined, suppose  $(b_n)_n$  is another sequence in  $A$  with  $(b_n)_n \rightarrow x$ . Consider  $(c_n)_n = (a_1, b_1, a_2, b_2, \dots)$ . It must be the case that  $(c_n)_n \rightarrow x$ . Thus,  $(f(c_n))_n$  converges to  $y \in Y$ . The subsequence of  $(f(a_n))_n \rightarrow y$  and  $(f(b_n))_n \rightarrow y$ . So, we must have  $\lim f(a_n) = \lim f(b_n)$ .

Note that  $\tilde{f}(a) = f(a)$  for all  $a \in A$ , by choosing the sequence  $(a, a, a, \dots)$ .

We claim that  $\tilde{f}$  is uniformly continuous. Let  $\varepsilon > 0$ . We know  $\exists \delta > 0$  such that for any  $a, b \in A$ , with  $d(a, b) < \delta$ , then  $\rho(f(a), f(b)) < \varepsilon/2$ . Now, let  $x, x' \in X$  with  $d(x, x') < \delta/4$ . Find sequences  $(a_n)_n \rightarrow x$  and  $(b_n)_n \rightarrow x'$  with  $(a_n)_n, (b_n)_n \in A$ . Find  $N$  large such that  $n \geq N$  implies  $d(a_n, x) < \delta/4$  and  $d(b_n, x') < \delta/4$ . For  $n \geq N$ , we have

$$\begin{aligned} d(a_n, b_n) &\leq d(a_n, x) + d(x, x') + d(x', b_n) \\ &< \frac{3\delta}{4} \\ &< \delta \end{aligned}$$

Thus, for  $n \geq N$ ,  $\rho(f(a_n), f(b_n)) < \varepsilon/2$ . By continuity of  $\rho$ , taking  $n \rightarrow \infty$ , we get  $\rho(\tilde{f}(x), \tilde{f}(x')) < \varepsilon/2$ . Therefore, we have  $d(x, x') < \delta/4 \Rightarrow d(\tilde{f}(x), \tilde{f}(x')) < \varepsilon$ . Therefore,  $\tilde{f}$  is uniformly continuous.

Suppose  $g : X \rightarrow Y$  is another continuous extension of  $f$ . Therefore,  $g(a) = \tilde{f}(a)$  for all  $a \in A$ . However,  $A$  is dense. Therefore,  $g = \tilde{f}$ .

### Completion of a Metric Space

Let  $(X, d)$  be a fixed metric space. A completion of  $X$  is a pair  $((Z, \rho), i)$ .

(i)  $(Z, \rho)$  is a complete metric space.

(ii)  $i : X \rightarrow Z$  is an isometry.

(iii)  $\overline{i(X)}^\rho = Z$ .

For example, the completion of  $(0, 1)$  is  $([0, 1], |\cdot|), i(t) = t$ .

### Isometric Isomorphism of Completions

Given  $((Z, \rho), i)$  and  $((Z', \rho'), j)$  completions of  $X$ , then there exists a unique isometric isomorphism  $\varphi : Z \rightarrow Z'$  such that the following diagram commutes.

$$\begin{array}{ccc} & & Z \\ & \nearrow i & \downarrow \varphi \\ X & & Z' \\ & \searrow j & \end{array}$$

### Corollary: Isometric Map and Completion of Metric Space

If  $(X, d)$  is a metric space, and  $i : (X, d) \rightarrow (Y, \rho)$  is an isometry into a complete metric space, then  $((\overline{i(X)}), \rho, i)$  is the completion of  $X$ .

### Theorem: Every Metric Space has a Completion

Consider the Banach space  $(C_b(X), \|\cdot\|_u)$ . We embed  $X \hookrightarrow C_b(X)$  as follows. Fix  $x_0 \in X$ . Given  $x \in X$ ,  $i(x) = X \rightarrow \mathbb{F}$  where  $i(x)(t) = d(t, x) - d(t, x_0)$ .

Clearly,  $i(x)$  is continuous for all  $x$  as the distance function is continuous. Also,

$$\begin{aligned} |i(x)(t)| &= |d(t, x) - d(t, x_0)| \\ &\leq d(x, x_0) \\ \|i(x)\|_u &\leq d(x, x_0). \end{aligned}$$

We need only show that  $i(x)$  is an isometry.

$$\begin{aligned} \|i(x) - i(y)\|_u &= \sup_{t \in X} |i(x)(t) - i(y)(t)| \\ &= \sup_{t \in X} |d(t, x) - d(t, y)| \\ &= d(x, y). \end{aligned}$$

## Nowhere Dense Sets

Let  $(X, d)$  be a metric space. Recall that a subset  $A$  is nowhere dense if  $(\overline{A})^\circ = \emptyset$ . For example,  $G = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$  is nowhere dense.

### Proposition: Equivalent Conditions for Nowhere Dense Sets

For a  $A \subseteq X$ , the following are equivalent:

- (i)  $A$  is nowhere dense.
- (ii)  $\exists F \subseteq X$  closed with  $F^\circ = \emptyset$ ,  $A \subseteq F$ .
- (iii)  $\exists U \subseteq X$  open and dense with  $U \subseteq A^c$ .

**Proof:**

- (i)  $\Rightarrow$  (ii): Take  $F = \overline{A}$ .
- (ii)  $\Rightarrow$  (i):  $\overline{A} \subseteq \overline{F}$ , so  $\overline{A}^\circ \subseteq \overline{F}^\circ = \emptyset$
- (ii)  $\Rightarrow$  (iii): Take  $U = F^c$ . Note that  $U = F^c \subseteq A^c$ . Then,  $\overline{U} = \overline{F^c} = (F^\circ)^c = X$ . Therefore,  $U$  is dense and open, and  $U$  is contained in  $A^c$ .
- (iii)  $\Rightarrow$  (ii): Take  $F = U^c$ .

A point  $x \in X$  is isolated if  $\exists \epsilon > 0$  such that  $U(x, \epsilon) = \{x\}$ .

### Proposition: Extension of Nowhere Dense Sets

Let  $(X, d)$  be a metric space.

- (i) If  $A \subseteq X$  is nowhere dense and  $B \subseteq A$ , then  $B$  is nowhere dense.
- (ii) If  $A \subseteq X$  is nowhere dense, then  $\overline{A}$  is nowhere dense.
- (iii) Let  $A_1, \dots, A_n$  be nowhere dense. Then,  $\bigcup A_i$  is nowhere dense.
- (iv) If  $X$  has no isolated points, then every finite set is nowhere dense.

**Proof:**

- (i)  $B \subseteq A$  implies  $\overline{B} \subseteq \overline{A}$ , so  $\overline{B}^\circ \subseteq \overline{A}^\circ = \emptyset$ , so  $B$  is nowhere dense.
- (ii) If  $A$  is nowhere dense, then  $\overline{\overline{A}^\circ} = \overline{A}^\circ = \emptyset$ .
- (iii) Let  $A_1$  and  $A_2$  be nowhere dense. By the alternate characterization,  $U_1 \subseteq A_1^c$ , where  $U_1$  is open and dense. Similarly,  $U_2 \subseteq A_2^c$ , where  $U_2$  is open and dense.

$$\begin{aligned} (A_1 \cup A_2)^c &= A_1^c \cap A_2^c \\ &\supseteq U_1 \cap U_2 \end{aligned}$$

We know  $U_1 \cap U_2$  is open. We claim that  $U_1 \cap U_2$  is dense.

Let  $x \in X$ ,  $\epsilon > 0$ . We want to show that  $U(x, \epsilon) \cap (U_1 \cap U_2) \neq \emptyset$ . Since  $U_1$  is dense, we know  $U_1 \cap U(x, \epsilon) \neq \emptyset$ . Let  $z \in U_1 \cap U(x, \epsilon)$ . Therefore,  $\exists \delta > 0$  such that  $U(z, \delta) \subseteq U_1 \cap U(x, \epsilon)$ . Since  $U_2$  is dense,  $U(z, \delta) \cap U_2 \neq \emptyset$ . Therefore,  $\emptyset \neq U(z, \delta) \cap U_2 \subseteq U(x, \epsilon) \cap (U_1 \cap U_2)$ .

By induction, assuming  $A_1 \cup \dots \cup A_{n-1}$  are nowhere dense, then  $(A_1 \cup \dots \cup A_{n-1}) \cup A_n$  is nowhere dense.

- (iv) Since  $X$  has no isolated points,  $\{x\}$  is closed but not open. Therefore,  $(\overline{\{x\}})^\circ = \emptyset$ . Use (iii).

**Remark:** Note that  $\mathbb{Q}$  is not nowhere dense, but  $\mathbb{Q}$  is the countable union of nowhere dense sets.



## Meager Sets

Let  $(X, d)$  be a metric space.

- (i)  $A \subseteq X$  is meager if  $A$  is the countable union of nowhere dense sets. Or,  $A$  is of the first category.
- (ii)  $B \subseteq X$  is called residual if  $B^c$  is meager.

**Examples:**  $\mathbb{Q} \subseteq \mathbb{R}$  is meager, so  $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$  is residual.  $\mathbb{Z} \subseteq \mathbb{R}$  is meager, but  $\mathbb{Z} \subseteq \mathbb{Z}$  is not meager.

### Proposition: Extension of Meager Sets

- (i) If  $A$  is meager, and  $B \subseteq A$ , then  $B$  is meager.
- (ii) If  $A_k$  is meager for  $k = 1, \dots$ , then  $A_k$  is meager.
- (iii) If  $X$  has no isolated points, then every countable set is meager.

**Proof:**

- (i)  $A = \bigcup A_k$ , with  $A_k$  nowhere dense. Then,  $B = B \cap A = \bigcup B \cap A_k$ .
- (ii) Each  $A_k$  is meager, meaning  $A_k = \bigcup A_{k_j}$  with  $A_{k_j}$  nowhere dense. Thus,  $A = \bigcup A_k$  is the countable union of  $A_{k_j}$ . Thus,  $A$  is meager.
- (iii) Since singleton sets are nowhere dense, we write the countable set as the union of singleton sets.

### Proposition: Cantor's Intersection Theorem

Let  $(X, d)$  be a complete metric space, and  $F_1 \supseteq F_2 \supseteq \dots$  be a sequence of closed, nonempty sets with  $(\text{diam}(F_n))_n \rightarrow 0$ . Then,  $\bigcap F_n = \{x\}$  for some  $x \in X$ .

**Proof:** Let  $x_n \in F_n$  for  $n \geq 1$ . Note that  $(x_n)_n$  is Cauchy. For  $\varepsilon > 0$ , let  $N$  be large such that  $n \geq N \Rightarrow \text{diam}(F_n) < \varepsilon$ . For  $m, n \geq N$ ,  $d(x_n, x_m) < \varepsilon$  because  $x_n, x_m \in F_N$ . Therefore,  $(x_n)_n \rightarrow x$  for  $x \in X$ .

We claim that  $\{x\} = \bigcap F_n$ . To see this, fix  $m \in \mathbb{N}$ , and consider  $(x_{m+k})_k \in F_m$ . The tail sequence  $(x_{m+k})_k \rightarrow x$ . Since  $F_m$  is closed, we know  $x \in F_m$ . Therefore, since  $m$  is arbitrary,  $x \in \bigcap F_n$ .

Now, suppose  $\exists x, x' \in \bigcap F_n$  distinct. Then,  $d(x, x') > 0$ . However,  $\exists N \in \mathbb{N}$  large with  $\text{diam}(F_N) < d(x, x')$ . However,  $x, x' \in F_N$ , which is a contradiction. Therefore,  $\bigcap F_n = \{x\}$ .

## Baire's Theorem

Let  $(X, d)$  be a complete metric space.

- (i) If  $\{V_k\}_{k \geq 1}$  is a countable family of open and dense subsets, then  $\bigcap V_k$  is dense.
- (ii)  $X$  is not meager.

**Proof:**

- (i) Let  $U_0$  be any open ball. Since  $V_1$  is open and dense,  $U_0 \cap V_1$  is open and nonempty. So,  $\exists U_1$  with  $B_1 = \overline{U_1} \subseteq U_0 \cap V_1$ . We can assure that  $\text{diam}(B_1) < 1$ .

Consider  $U_1 \cap V_2$ . Since  $V_2$  is dense and open,  $U_1 \cap V_2$  is open and nonempty. Therefore, there must be  $B_2 = \overline{U_2} \subseteq U_1 \cap V_2$ . We can insure that  $\text{diam}(B_2) < 1/2$ .

Now, with  $U_2 \cap V_3$ , we have  $B_3 = \overline{U_3} \subseteq U_2 \cap V_3$ , with  $\text{diam}(B_3) < 1/3$ .

Inductively, we have  $U_1, \dots, U_{n-1}$  and  $B_1, \dots, B_{n-1}$ , we see that  $U_{n-1} \cap V_n$  is open and nonempty, so we have  $U_n$  with  $B_n = \overline{U_n} \subseteq U_{n-1} \cap V_n$ , with  $\text{diam}(B_n) < 1/n$ .

Observe that we have  $B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \cdots$ . In particular,  $\{B_n\}_{n \geq 1}$  is a nested sequence of closed sets with  $\text{diam}(B_n) \rightarrow 0$ . Therefore,  $\bigcap B_n = \{x\}$ .

We claim that  $x \in U_0 \cap (\bigcap V_k)$ . Note that  $B_n \subseteq U_{n-1} \cap V_n \subseteq V_n$ . Therefore,  $x \in \bigcap B_n$  implies  $x \in \bigcap V_n$ . Also,  $x \in B_1 = \overline{U_1} \subseteq U_0 \cap V_n \subseteq U_0$ . Therefore,  $\bigcap V_k$  is dense.

(ii) Suppose  $X = \bigcup A_k$  for  $A_k$  nowhere dense. Therefore,  $\exists V_k$  open and dense with  $V_k \subseteq A_k^c$ . Then,

$$\begin{aligned} \emptyset &= X^c \\ &= \left( \bigcup A_k \right)^c \\ &= \bigcap A_k^c \\ &\supseteq \bigcap V_k. \end{aligned}$$

Therefore, by the previous result,  $\bigcap V_k$  is open and dense, which is a contradiction. Therefore,  $X$  is not meager.

**Question:** Is  $\mathbb{Q} \subseteq \mathbb{R}$  meager? Yes,  $\mathbb{Q}$  is the countable union of singleton sets. Is  $\mathbb{R} \setminus \mathbb{Q}$  meager? The answer is no — otherwise, we would write  $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$  would be a union of meager sets, but  $\mathbb{R}$  is complete.

### Applying Baire's Theorem

Let  $(X, d)$  be a metric space.

(i)  $G \subseteq X$  is a  $G_\delta$ -set if

$$G = \bigcap_{k \geq 1} V_k$$

with  $V_k$  open.

(ii)  $F \subseteq X$  is a  $F_\sigma$ -set if

$$F = \bigcup_{k \geq 1} C_k$$

with  $C_k$  closed.

For example,  $\mathbb{Q} \subseteq \mathbb{R}$  is  $F_\sigma$ , since  $\mathbb{Q}$  is the countable union of singleton sets (which are closed in  $\mathbb{R}$ ). It can be shown that  $A$  is  $F_\sigma$  if and only if  $A^c$  is  $G_\delta$ .

We claim that  $\mathbb{Q}$  is not  $G_\delta$ .

**Proof:** If  $\mathbb{Q}$  is  $G_\delta$ , then  $\mathbb{R} \setminus \mathbb{Q}$  is  $F_\sigma$ , so

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup F_k$$

for  $F_k$  closed. Thus,

$$\begin{aligned} \mathbb{R} &= \mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q} \\ &= \bigcup \{q_k\} \cup \bigcup F_k. \end{aligned}$$

Therefore,  $\mathbb{R}$  is the countable union of closed sets. Since  $\mathbb{R}$  is complete, by Baire's Theorem, we must have  $\{q_k\}^\circ \neq \emptyset$ , or that  $F_k^\circ \neq \emptyset$  for some  $k$ . However,  $\{q_k\}^\circ = \emptyset$ , and  $F_k^\circ = \emptyset$  since  $F_k \subseteq \mathbb{R} \setminus \mathbb{Q}$ , and  $\mathbb{R} \setminus \mathbb{Q}$  cannot contain an interval. Therefore,  $\mathbb{Q}$  is not  $G_\delta$ .

Let  $(X, d)$  be a metric space. If  $A$  is closed, then  $A$  is  $G_\delta$ .

**Proof:** Recall  $\text{dist}_A : X \rightarrow \mathbb{R}$  is continuous. Therefore,  $\text{dist}_A^{-1}((-1/n, 1/n)) = \{x \mid \text{dist}_A(x) < 1/n\}$  is open. Recall that  $x \in A$  if and only if  $\text{dist}_A(x) = 0$ .

Therefore, we can write

$$A = \bigcap_{n \geq 1} \{x \mid \text{dist}_A(x) < 1/n\}.$$

Therefore,  $A$  is  $G_\delta$ .

It follows that if  $A$  is open, then  $A$  is  $F_\sigma$ .

**Theorem: Set of Continuities**

Let  $f : (X, d) \rightarrow (Y, \rho)$  be a map. Then,  $C_f := \{x \in X \mid f \text{ is continuous at } x\}$  is a  $G_\delta$  set.

**Oscillation of a Function**

Let  $f : (X, d) \rightarrow (Y, \rho)$ . Fix  $x_0 \in X$ . The oscillation  $\omega_f(x_0) = \inf_{\delta > 0} \text{diam}(f(U(x, \delta)))$ , or

$$\omega_f(x_0) = \inf_{\delta > 0} \left( \sup_{x, x' \in U(x, \delta)} \rho(f(x), f(x')) \right).$$

Note that  $\omega_f(x_0) \in [0, \infty]$ .

(i)  $f$  is continuous at  $x_0$  if and only if  $\omega_f(x_0) = 0$ .

(ii) Given  $c > 0$ ,  $\{x \mid \omega_f(x_0) < c\} \subseteq X$  is open.

**Proof:**

(i) Suppose  $f$  is continuous at  $x_0$ . Let  $\varepsilon > 0$ . Then,  $\exists \delta > 0$  such that  $d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon/2$ . Therefore,

$$\text{diam}(f(U(x_0, \delta))) \leq \varepsilon,$$

since for  $x, x' \in U(x_0, \delta)$ , we have

$$\begin{aligned} \rho(f(x), f(x')) &\leq \rho(f(x), f(x_0)) + \rho(f(x_0), f(x')) \\ &< \varepsilon. \end{aligned}$$

In particular,  $\omega_f(x_0) \leq \varepsilon$ . Since  $\varepsilon$  was arbitrary, we have  $\omega_f(x_0) = 0$ .

Suppose  $\omega_f(x_0) = 0$ . Let  $\varepsilon > 0$ . By the property of infimum, then  $\exists \delta > 0$  such that

$$\text{diam}(f(U(x_0, \delta))) < \varepsilon.$$

In particular, if  $d(x, x_0) < \delta$ , then  $\rho(f(x), f(x_0)) < \varepsilon$ . Thus,  $f$  is continuous at  $x_0$ .

(ii) Let  $V = \{x \mid \omega_f(x_0) < c\}$ . Let  $x_0 \in V$ . Since  $x_0 \in V$ ,  $\omega_f(x_0) < c$ . By the property of infimum,  $\exists \delta > 0$  such that  $\text{diam}(f(U(x_0, \delta))) < c$ . Let  $\varepsilon = \delta/2$ . We claim that  $U(x_0, \varepsilon) \subseteq V$ .

Let  $z \in U(x_0, \varepsilon)$ . Note that  $U(z, \delta/2) \subseteq U(x_0, \delta)$ . Therefore,  $f(U(z, \delta/2)) \subseteq f(U(x_0, \delta))$ . Thus,  $\text{diam}(f(U(z, \delta))) \leq \text{diam}(f(U(x_0, \delta))) < c$ .

By property of oscillation,  $\omega_f(z) < c$ . So,  $U(x_0, \varepsilon) \subseteq V$ .

**Proof of Theorem:**

$$\begin{aligned} C_f &= \{x \mid f \text{ is continuous at } x\} \\ &= \bigcap_{n \geq 1} \underbrace{\{x \mid \omega_f(x) < 1/n\}}_{\text{open sets}} \end{aligned}$$

meaning  $x \in C_f \Leftrightarrow \omega_f(x) = 0 \Leftrightarrow \omega_f(x) < 1/n$  for all  $n$ .

**Applying Set of Continuities**

There does exist a function continuous at every irrational point and discontinuous at every rational point. Recall from Real Analysis that such  $f$  is

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

However, there does not exist  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $C_f = \mathbb{Q}$ , since the set of continuities is always a  $G_\delta$  set.

## Nowhere Differentiable Functions

Does there exist a function  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $f$  is continuous on  $[0, 1]$  but differentiable nowhere? The answer is yes.

$$f(x) = \sum_{n \geq 1} a^n \cos(b^n x),$$

where  $0 < a < 1$  and  $ab > 1$  is such a function. This is known as the Weierstrass function.

Such functions are not rare at all.

In the complete normed vector space  $X = (C[0, 1], \|\cdot\|_u)$ ,  $\{f \in X \mid f \text{ differentiable nowhere}\}$  is the complement of a meager set (meaning it is topologically “big”).

**Proof:** We are aware of the fact that  $C([0, 1])$  under  $\|\cdot\|_u$  is complete with respect to the induced metric. We will show that the set of differentiable functions at any point  $x \in [0, 1]$  is meager in  $C([0, 1])$ , thus proving that differentiable nowhere functions are residual.

Let  $A_n \subseteq C([0, 1])$  denote the set of functions where there exists  $x \in (0, 1)$  with

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq n$$

for all  $y$  with  $0 < |x - y| \leq 1/n$ . All differentiable functions must be an element of  $A_n$  for some  $n$ .

To show  $A_n$  is nowhere dense, we show that  $A_n$  is closed and contains no open ball.

Let  $(f_j)_j$  be a sequence in  $A_n$  with  $(f_j)_j \rightarrow f$ . For each  $j$ , there exists an  $x_j$  such that

$$\left| \frac{f_j(x_j) - f_j(y)}{x_j - y} \right| \leq n$$

for  $0 < |x_j - y| \leq 1/n$ . By the Bolzano–Weierstrass theorem, there is a subsequence  $(x_{j_k})_k \rightarrow x$ . Suppose  $0 < |x - y| \leq 1/n$ . Then, for all  $k$  sufficiently large,  $|x_{j_k} - y| \leq 1/n$ , so by continuity and the pointwise convergence of  $f$ ,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \lim_{k \rightarrow \infty} \left| \frac{f_{j_k}(x_{j_k}) - f(y)}{x_{j_k} - y} \right| \leq n.$$

Since  $f$  is continuous (as  $C([0, 1])$  is complete), the same is true for  $|x - y| = 1/n$ .

To show that  $A_n$  has no open ball, recall from Real Analysis I that for all  $f \in C([0, 1])$ , there exists a piecewise linear function  $p_\delta$  with  $\|p_\delta - f\|_u < \delta$  for all  $\delta > 0$ .

It follows that, if  $A_n$  contains a ball  $U_\epsilon(f)$ , then  $A_n$  contains some ball  $U_{\epsilon/2}(p)$  for said piecewise linear function. Set  $M$  to be the maximum value of the slopes that  $p$  takes.

Define  $g : [0, 1] \rightarrow \mathbb{R}$  by

$$g(j/M') = \begin{cases} -1 & j \text{ odd} \\ 1 & j \text{ even} \end{cases},$$

with  $g$  linear between  $[j/M', (j+1)/M']$ .

The function  $k = f + \frac{1}{2}\epsilon g$  defines  $\|f - k\| < \epsilon$ , but its slope near any point has magnitude at least  $\frac{1}{2}\epsilon M' - n$ . If  $M'$  is large enough, this value is larger than  $M$ .

Thus, there exists  $k \in U_\epsilon(f)$  with  $k \notin A_n$  for any  $A_n$ , implying that  $A_n^\circ = \emptyset$ .

## Compactness

Compactness can best be analogized to finite dimensionality in a metric space.

Let  $(X, d)$  be a metric space, and let  $K \subseteq X$ .

- (1) A cover for  $K$  is a family of subsets  $\mathcal{U} = \{U_i\}_{i \in I} \subseteq \mathcal{P}(X)$  with  $K \subseteq \bigcup U_i$ .

The cover  $\mathcal{U}$  is called an open cover if each  $U_i \subseteq X$  is open. The cover  $\mathcal{U}$  is called finite if  $I$  is finite. If  $\mathcal{U}$  is a cover for  $K$ , a subcover of  $\mathcal{U}$  is a subfamily  $\mathcal{V} = \{U_i\}_{i \in J}$ , with  $J \subseteq I$ , and  $K \subseteq \bigcup_{i \in J} U_i$ .

- (2)  $K$  is called compact if every open cover of  $K$  admits a finite subcover. If  $\{U_i\}_{i \in I}$  is any family that covers  $K$ , then there exists a finite  $F \subseteq I$  such that  $\{U_i\}_{i \in F}$  covers  $K$ .

For example, the set  $(0, 1] \subseteq \mathbb{R}$  is not compact, because

$$(0, 1] \subseteq \bigcup_{n \in \mathbb{N}} (1/n, 3/2)$$

does not admit a finite subcover.

Any finite set is compact.

A discrete metric space is compact if and only if  $X$  is finite.

Let  $(X, d)$  be a metric space, and  $Y \subseteq X$ . Let  $K \subseteq Y$ ;  $K$  is compact in  $X$  if and only if  $K$  is compact in  $Y$ . This can be shown by taking the relative topology of  $Y$  on every open cover of  $K$  in  $X$ .

### Proposition: Properties of Compactness

Let  $(X, d)$  be a metric space.

- (1) If  $K \subseteq X$  is compact, then  $K$  is closed and bounded.  
 (2) If  $X$  is a compact metric space, and  $K \subseteq X$  is closed, then  $K$  is compact.

**Proof of (2):** Let  $K \subseteq \bigcup U_i$ , with  $U_i \subseteq X$  open. Then,  $X = (X \setminus K) \cup (\bigcup_{i \in I} U_i)$ . This is an open cover for  $X$ , meaning it admits a finite subcover  $F \subseteq I$  such that  $X = (X \setminus K) \cup \bigcup_{i \in F} U_i$ . Clearly,  $K \subseteq \bigcup_{i \in F} U_i$ . Thus,  $K$  is compact.

**Proof of (1):** Let  $K \subseteq X$  be compact. Then,

$$K \subseteq \bigcup_{x \in K} U(x, 1).$$

Since  $K$  is compact, there exist  $\{x_1, \dots, x_n\}$  with  $K \subseteq \bigcup_{j=1}^n U(x_j, 1)$ . Let  $c = \max d(x_i, x_j)$ . If  $x, y \in K$ , then  $x \in U(x_i, 1)$  and  $y \in U(x_j, 1)$  for some  $x_i, x_j$ . Then,

$$\begin{aligned} d(x, y) &\leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \\ &< 1 + c + 1 = 2 + c. \end{aligned}$$

Thus,  $\text{diam}(K) < \infty$ .

We will show that  $K^c$  is open. Let  $x_0 \notin K$ . For each  $x \in K$ , there exist  $\delta_x > 0$  with  $U(x, \delta_x) \cap U(x_0, \delta_x) = \emptyset$ . Then,

$$K \subseteq \bigcup_{x \in K} U(x, \delta_x).$$

Since  $K$  is compact, there exist  $\{x_1, \dots, x_n\}$  with  $K \subseteq \bigcup U(x_j, \delta_{x_j})$ . Let  $\delta = \min\{\delta_{x_j}\} > 0$ . Then,  $U(x_0, \delta) \subseteq K^c$ .

**Proposition: Compactness and Intersections of Closed Sets**

Let  $(X, d)$  be a metric space. The following are equivalent.

- (1)  $X$  is compact;
- (2) If  $\{C_i\}_{i \in I}$  is a family of closed sets with the finite intersection property (i.e., the intersection of finitely many elements of  $\{C_i\}$  is non-empty), then  $\bigcap_{i \in I} C_i \neq \emptyset$ .

**Proposition: Separability of Compact Metric Spaces**

Let  $(X, d)$  be a compact metric space. Then,  $(X, d)$  is separable.

**Proof:** For fixed  $n \geq 1$ , consider the cover

$$X = \bigcup U(x, 1/n).$$

By compactness, there exist  $\{x_{n,1}, \dots, x_{n,m_n}\}$  with

$$X = \bigcup_{j=1}^{m_n} U(x_{n,j}, 1/n).$$

Let  $S = \{x_{n,j} \mid n \in \mathbb{N}, 1 \leq j \leq m_n\}$ . Then,  $S$  is countable.

Let  $x \in X$ ,  $\varepsilon > 0$ . Let  $N$  be large such that  $N^{-1} < \varepsilon$ . So,

$$x \in \bigcup_{j=1}^{m_N} U(x_{N,j}, 1/N),$$

so  $x \in U(x_{N,j}, 1/N)$  for some  $j$ , whence  $d(x, x_{N,j}) < 1/N < \varepsilon$ , so  $x_{N,j} \in U(x, \varepsilon)$ . So,  $\bar{S} = X$ .

**Proposition: Sequential Compactness**

Let  $(X, d)$  be a metric space,  $K \subseteq X$ . We say  $K$  is sequentially compact if every sequence in  $K$  admits a convergent subsequence in  $K$ .

From Bolzano-Weierstrass, we know that  $[a, b] \subseteq \mathbb{R}$  is sequentially compact.

If  $K$  is compact, then  $K$  is sequentially compact.

**Proof:** Let  $(x_k)_k \in K$ . Let  $C_0 = \{x_1, x_2, \dots\}$ ,  $C_1 = \overline{\{x_2, x_3, \dots\}}$ , etc. such that  $C_n = \overline{\{x_{n+1}, x_{n+2}, \dots\}}$ .

Observe that  $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ . Additionally,  $\{C_n\}$  has the finite intersection property. Since  $K$  is compact, the previous proposition states that  $\bigcap C_n \neq \emptyset$ . Let  $x \in \bigcap C_n$ .

$x \in C_1$ , meaning  $\exists k_1 > 1$  with  $d(x, x_{k_1}) < 1$ .  $x \in C_{k_1}$ , meaning  $\exists k_2 > k_1$  with  $d(x, x_{k_2}) < 1/2$ .  $x \in C_{k_2}$ , meaning  $\exists k_3 > k_2$  with  $d(x, x_{k_3}) < 1/3$ . Continuing, we have  $(x_{k_j})_j \in K$  with  $d(x, x_{k_j}) < 1/j$ . Thus,  $(x_{k_j})_j \rightarrow x$ .

If  $(X, d)$  is sequentially compact, then  $X$  is complete.

**Lemma:** If  $(x_n)_n$  is Cauchy, and  $(x_n)_n$  admits a convergent subsequence, then  $(x_n)_n$  is convergent.

**Proof of Lemma:** Given  $\varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that for  $p, q \geq N$ ,  $d(x_p, x_q) < \varepsilon/2$ .

Also, suppose  $(x_{n_k})_k \rightarrow x$ . Then,  $\exists K \in \mathbb{N}$  large such that for  $k \geq K$ ,  $d(x_{n_k}, x) < \varepsilon/2$ .

Therefore, for  $n \geq N$ , find  $k \geq \max\{N, K\}$ , we have

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

**Proof:** If  $(X, d)$  is sequentially compact, for  $(x_n)_n$  a Cauchy sequence in  $(X, d)$ , we have that  $(x_n)_n$  admits a convergent subsequence. Then, we use the lemma.

### Total Boundedness

Let  $(X, d)$  be a metric space.  $K \subseteq X$  is totally bounded if  $\forall \delta > 0, \exists x_1, \dots, x_n \in K$  such that  $K \subseteq \bigcup_{i=1}^n U(x_i, \delta)$ .

**Exercise:** If  $K$  is totally bounded, then  $K$  is bounded. If  $L \subseteq K$ , and  $K$  is totally bounded, then  $L$  is totally bounded.

### Sequential Compactness and Total Boundedness

Let  $(X, d)$  be a metric space. Let  $K \subseteq X$  be sequentially compact. Then,  $K$  is totally bounded.

**Proof:** Suppose  $K$  is not totally bounded. Then,  $\exists \delta_0 > 0$  such that  $K \not\subseteq \bigcup_{x \in F} U(x, \delta_0)$  for any finite  $F$ .

Let  $x_1 \in K$ . Since  $K \not\subseteq U(x_1, \delta_0)$ , so let  $x_2 \in K \setminus U(x_1, \delta_0)$ . Since  $K \not\subseteq U(x_1, \delta_0) \cup U(x_2, \delta_0)$ , let  $x_3 \in K \setminus (U(x_1, \delta_0) \cup U(x_2, \delta_0))$ . Continuing, we find  $x_n \in K \setminus \bigcup_{j=1}^{n-1} U(x_j, \delta_0)$ .

Thus, we have a sequence  $(x_n)_n$ . By sequential compactness,  $(x_n)_n$  admits  $(x_{n_k})_k \rightarrow x \in K$ . Since  $(x_{n_k})_k$  is convergent,  $(x_{n_k})_k$  is Cauchy. But,  $d(x_p, x_q) \geq \delta_0$ , since, without loss of generality, for  $p > q$ ,  $x_p \notin U(x_q, \delta_0)$ .  $\perp$

### Corollary: Compact Subsets of Real Numbers

If  $K \subseteq \mathbb{R}$  is compact,  $\sup K \in K$  and  $\inf K \in K$ .

**Proof:** We can always construct sequences  $(x_n)_n \rightarrow \sup K$  and  $(y_n)_n \rightarrow \inf K$  in  $K$ . Since  $\sup K < \infty$  and  $\inf K < \infty$ , since  $K$  is compact, and thus bounded.

Since  $K$  is also closed,  $\sup K \in K$  and  $\inf K \in K$ .

### Theorem: Equivalence of Compactness Definitions

Let  $(X, d)$  be a metric space. The following are equivalent.

- (1)  $X$  is compact.
- (2)  $X$  is sequentially compact.
- (3)  $X$  is complete and totally bounded.

**Proof:** We proved that  $(1) \Rightarrow (2) \Rightarrow (3)$ . We will now prove  $(3) \Rightarrow (1)$ .

Suppose  $\mathcal{V}$  is an open cover of  $X$  that fails to admit a finite subcover. Let  $\varepsilon = 1$ . Since  $X$  is totally bounded  $X = \bigcup_{j=1}^{m_1} U_{1,j}$ , where  $U_{1,j}$  are open balls of radius 1.

There must be some open ball among the  $U_{1,j}$  not covered by finitely many members of  $\mathcal{V}$ . Call this ball  $U(x_1, 1)$ . Let  $\varepsilon = 1/2$ . By total boundedness,  $X = \bigcup_{j=1}^{m_2} U_{2,j}$ , where  $U_{2,j}$  are open balls of radius  $1/2$ . Then,  $U(x_1, 1) = \bigcup (U(x_1, 1) \cap U_{2,j})$ . So, there must be an open ball of radius  $1/2$ ,  $U(x_2, 1/2)$ , such that  $U(x_1, 1) \cap U(x_2, 1/2)$  cannot be covered by finitely many members of  $\mathcal{V}$ .

Continuing, we have a sequence  $(x_n)_n$ , where  $F_n = U(x_1, 1) \cap U(x_2, 1/2) \cap \dots \cap U(x_n, 1/n)$  cannot be covered by finitely many members of  $\mathcal{V}$ .

Let  $C_n = \overline{F_n}$ . Notice that  $F_1 \supseteq F_2 \supseteq \dots$ , meaning  $C_1 \supseteq C_2 \supseteq \dots$ . We see that  $\text{diam}(C_n) = \text{diam}(F_n) \leq 2/n$ . Applying Cantor's intersection theorem, we have  $\bigcap C_n = \{x\}$ .

Since  $\mathcal{V}$  is an open cover, locate  $V \in \mathcal{V}$  such that  $x \in V$ . Since  $V$  is open, there exists  $\varepsilon > 0$  such that  $U(x, \varepsilon) \subseteq V$ . Choose  $N$  large such that  $2/N < \varepsilon$ . Since  $x \in C_N$ ,  $d(z, x) \leq 2/N < \varepsilon$  for all  $z \in C_N$ , meaning  $F_N \subseteq C_N \subseteq U(x, \varepsilon) \subseteq V$ .

Therefore,  $\{V\}$  is a cover for  $F_N$ .  $\perp$

**Proposition: Multi-dimensional Bolzano-Weierstrass Theorem**

Let  $\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] = \prod_{j=1}^d [a_j, b_j] \subseteq \mathbb{R}^d$ . Then,  $\mathcal{R}$  is sequentially compact, so  $\mathcal{R}$  is compact.

**Proof:** The proof in  $\mathbb{R}^d$  works similarly to the proof in  $\mathbb{R}^2$ . Consider  $\pi_x : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\pi_y : \mathbb{R}^2 \rightarrow \mathbb{R}$ . We saw that  $(v_n)_n \rightarrow v$  in  $\ell_p^2$  if and only if  $(\pi_x(v_n))_n \rightarrow \pi_x(v)$  and  $(\pi_y(v_n))_n \rightarrow \pi_y(v)$ .

If  $(v_n)_n \in \mathcal{R}$ , then  $(\pi_x(v_n))_n \in [a_1, b_1]$ . By Bolzano-Weierstrass, there is a convergent subsequence  $(\pi_x(v_{n_k}))_k \rightarrow x \in [a_1, b_1]$ .

Now, consider  $(\pi_y(v_{n_k}))_k \in [a_2, b_2]$ . By Bolzano-Weierstrass, there is a convergent subsequence  $(\pi_y(v_{n_{k_j}}))_j \rightarrow y \in [a_2, b_2]$ . Thus,  $(v_{n_{k_j}})_j \rightarrow (x, y)$  in  $\mathcal{R}$ .

**Heine-Borel Theorem**

Let  $K \subseteq \mathbb{R}^d$ . The following are equivalent:

- (i)  $K$  is compact;
- (ii)  $K$  is sequentially compact;
- (iii)  $K$  is closed and bounded.

**Proof:** We have (i)  $\Leftrightarrow$  (ii), and (i)  $\Rightarrow$  (iii). We will show (iii)  $\Rightarrow$  (ii).

If  $K$  is bounded, then  $K \subseteq \mathcal{R} = \prod_{j=1}^d [a_j, b_j]$ . Let  $(v_n)_n$  be a sequence in  $K$ . By the previous proposition, there exists a subsequence  $(v_{n_k})_k \rightarrow v \in \mathcal{R}$ . Since  $K$  is closed,  $v \in K$ . Therefore,  $K$  is sequentially compact.

There are many examples of closed and bounded sets that are not compact (in infinite-dimensional vector spaces).

For example, in  $\ell_1 = \{a = (a_k)_k \mid \sum_{k=1}^{\infty} |a_k| < \infty\}$ , we have  $e_n = (0, 0, \dots, 0, 1, 0, \dots)$ , with 1 at the  $n$ th coordinate. For the sequence  $(e_n)_n$ ,  $\|e_k\|_1 = 1$  for all  $e_k$ , so  $(e_n)_n \in B_{\ell_1}$ , which is closed and bounded. Observe that  $\|e_n - e_m\| = 2$  for all  $m \neq n$ , so there does not exist a convergent subsequence. Thus,  $\ell_1$  is not sequentially compact.

**Remark:** We will show that for a normed space,  $(V, \|\cdot\|)$ ,  $B_V$  is compact if and only if  $\dim(V) < \infty$ .

**Proposition: Continuous Image of Compact Sets**

If  $f : (X, d) \rightarrow (Y, \rho)$  is continuous, and  $K \subseteq X$  is compact, then  $f(K) \subseteq Y$  is compact.

**Proof:** Let  $\bigcup_{i \in I} V_i$  be an open cover for  $f(K)$ , where  $V_i \subseteq Y$  open. Taking the preimage, we have

$$\begin{aligned} K &\subseteq f^{-1}(f(K)) \\ &\subseteq f^{-1}\left(\bigcup_{i \in I} V_i\right) \\ &= \bigcup_{i \in I} f^{-1}(V_i) \end{aligned}$$

since  $f$  is continuous,  $f^{-1}(V_i) \subseteq X$  are open. By compactness, there exists  $F \subseteq I$  finite such that

$$K \subseteq \bigcup_{i \in F} f^{-1}(V_i).$$

Taking the image, we have

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i \in F} f^{-1}(V_i)\right) \\ &= \bigcup_{i \in F} f(f^{-1}(V_i)) \\ &= \bigcup_{i \in F} V_i. \end{aligned}$$

Thus,  $f(K)$  has a finite subcover.



**Corollary: Compactness under Topologically Equivalent Metrics**

Let  $d_1$  and  $d_2$  be topologically equivalent ( $\text{id}_X : (X, d_1) \rightarrow (X, d_2)$  is a homeomorphism). Then,  $K \subseteq X$  is  $d_1$ -compact if and only if  $K$  is  $d_2$ -compact.

**Corollary: Heine-Borel Theorem Extension**

For  $K \subseteq \ell_p^n$ ,  $K$  is compact if and only if  $K$  is closed and bounded.

**Extreme Value Theorem**

Let  $(X, d)$  be a metric space,  $K \subseteq X$  compact, and  $f : X \rightarrow \mathbb{R}$  continuous. Then,  $\sup_{x \in X} f(x) = f(x_M)$  and  $\inf_{x \in X} f(x) = f(x_m)$  for some  $x_M, x_m \in K$ .

**Proof:** We know that  $f(K) \subseteq \mathbb{R}$  is compact. Then,  $\inf f(K)$  and  $\sup f(K)$  are elements of  $f(K)$ .

**Proposition: Compactness of Closed Unit Ball**

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ .

- (1) All norms on  $V$  are equivalent.
- (2) For any norm,  $\|\cdot\|$  on  $V$ ,  $B_{(V, \|\cdot\|)} = \{v \in V \mid \|v\| \leq 1\}$  is compact.

**Proof of (1):** Let  $\{v_1, \dots, v_n\}$  be a linear basis for  $V$ . Define

$$\left\| \sum_{j=1}^n t_j v_j \right\|_1 = \sum_{j=1}^n |t_j|.$$

This is a norm on  $V$ .

Then,  $\varphi : \ell_1^n \rightarrow V$

$$\varphi \left( \sum_{j=1}^n t_j e_j \right) = \sum_{j=1}^n t_j v_j$$

is a linear isometric isomorphism. Since  $B_{\ell_1^n}$  is compact, so too is  $\varphi(B_{\ell_1^n})$ , so  $B_{(V, \|\cdot\|)}$  is compact.

Then,  $S_1 := \{v \in V \mid \|v\|_1 = 1\}$  is compact since  $S_1 \subseteq B_{(V, \|\cdot\|)}$  is closed.

Let  $\|\cdot\|$  be any norm on  $V$ . We will show that  $\|\cdot\|$  is equivalent to  $\|\cdot\|_1$ . Note that

$$\begin{aligned} \left\| \sum_{j=1}^n t_j v_j \right\| &\leq \sum_{j=1}^n |t_j| \|v_j\| \\ &\leq c \sum_{j=1}^n |t_j| \\ &= c \left\| \sum_{j=1}^n t_j v_j \right\|_1 \end{aligned}$$

where  $c = \max \|v_j\|$ . Consider  $g : (V, \|\cdot\|_1) \rightarrow \mathbb{R}$ , with  $g(v) = \|v\|$ .

$$\begin{aligned} |g(v) - g(w)| &= |\|v\| - \|w\|| \\ &\leq \|v - w\| \\ &\leq c \|v - w\|_1 \end{aligned}$$

so  $g$  is Lipschitz, and thus continuous.  $S_1$  is compact in  $(V, \|\cdot\|_1)$ , so by the extreme value theorem,  $\inf_{v \in S_1} g(v) = g(v_0) = \|v_0\|$  for some  $v_0 \in S_1$ . Note that  $D := \|v_0\| > 0$ , else  $v_0 = 0$ . Thus,  $g(v) \geq D$  for all  $v \in S_1$

$$\|v\| \geq D$$

$$\forall v \in S_1$$

Let  $0 \neq v$ . Then,

$$\frac{v}{\|v\|_1} \in S_1$$

$$\left\| \frac{v}{\|v\|_1} \right\| \geq D$$

so

$$\|v\| \geq D \|v\|_1.$$

Therefore, we have  $\|v\|_1 \leq \frac{1}{D} \|v\|$ . Thus, any two norms on  $V$  are equivalent.

**Proof of (2):** Exercise.

### Corollary: Finite-Dimensional Subspaces

Let  $V$  be a normed space, and  $W \subseteq V$  finite-dimensional. Then,  $W \subseteq V$  is closed.

**Proof:** We know there is a linear uniformism  $\varphi : W \rightarrow \ell_1^n$ , for  $\dim(W) = n$ . If  $(w_n)_n \rightarrow v \in V$ , where  $(w_n)_n \in W$ , then  $(w_n)_n$  is Cauchy. Therefore,  $(\varphi(w_n))_n$  is Cauchy in  $\ell_1^n$ . Since  $\ell_1^n$  is complete,  $(\varphi(w_n))_n \rightarrow z \in \ell_1^n$ . Since  $\varphi^{-1}$  is uniformly continuous,  $(w_n)_n = (\varphi^{-1}(\varphi(w_n)))_n \rightarrow \varphi^{-1}(z) \in W$ . Thus,  $\varphi^{-1}(z) = v$ , so  $v \in W$ .

### Proposition: Uncountable Basis of Banach Space

If  $V$  is an infinite-dimensional Banach space, then  $\dim(V)$  is uncountable.

**Proof:** Let  $\{e_n\}$  be a linearly independent set. Let  $W_n = \text{span}\{e_1, \dots, e_n\}$ . So,  $W_n$  is closed, and  $W_n \neq V$ . We can see that  $W_1 \subseteq W_2 \subseteq \dots$ .

We claim that  $W_n^\circ = \emptyset$ . Suppose  $\exists U(x, \varepsilon) \subseteq W_n$  for some  $\varepsilon > 0$ . Given any  $v \in V$  with  $v \neq 0$ , we take  $\frac{\varepsilon}{2} \frac{v}{\|v\|} + x \in W_n$ . Thus, we have  $\frac{\varepsilon}{2} \frac{v}{\|v\|} \in W_n$ , so  $v \in W_n$ , meaning  $V \subseteq W_n$ .

By Baire's Theorem,  $\bigcup W_n \neq V$ .

### Proposition: Compact Unit Ball and Finite Dimensions

Let  $V$  be a normed space, and  $B_V := \{v \mid \|v\| \leq 1\}$ . The following are equivalent:

- (i)  $B_V$  is compact;
- (ii)  $\dim(V) < \infty$ .

**Riesz's Lemma:** Let  $V$  be a normed space, and  $W$  a proper closed subspace. For every  $t \in (0, 1)$ , there exists  $v_t \in V$  with  $\|v_t\| = 1$  and  $\text{dist}_W(v_t) \geq t$ .

**Proof of Riesz's Lemma:** Find  $v_0 \in V \setminus W$ . We know  $\text{dist}_W(v_0) := \delta > 0$ . Recall that  $\text{dist}_W(v_0) = \inf_{w \in W} \|v_0 - w\|$ . Note that  $t\delta < \delta$ . So,  $\delta < \frac{\delta}{t}$ . Find  $w_0 \in W$  with  $\delta \leq \|v_0 - w_0\| < \frac{\delta}{t}$ . Let  $v_t = \frac{v_0 - w_0}{\|v_0 - w_0\|}$ . Then,  $\|v_t\| = 1$ . We claim that  $v_t$  satisfies the lemma.

If  $w \in W$  arbitrary, then

$$\begin{aligned} \|v_t - w\| &= \left\| \frac{v_0 - w_0}{\|v_0 - w_0\|} - w \right\| \\ &= \frac{1}{\|v_0 - w_0\|} \left\| v_0 - \underbrace{(w_0 + w\|v_0 - w_0\|)}_{\in W} \right\| \\ &> \frac{t}{\delta} \cdot \delta \\ &= t. \end{aligned}$$

Thus,  $\text{dist}_W(v_t) \geq t$ .

**Proof:** To show (i)  $\Rightarrow$  (ii), we need Riesz's Lemma. Let  $B_V$  be compact. Suppose toward contradiction that  $\dim(V) = \infty$ .

Choose  $v_1 \in V$  with  $\|v_1\| = 1$ . Let  $W_1 = \text{span}\{v_1\} \subset V$ . Then,  $W_1$  is closed and proper, meaning  $\exists v_2 \in V$  with  $\|v_2\| = 1$  with  $\text{dist}_{W_1}(v_2) \geq 1/2$ . Let  $W_2 = \text{span}\{v_1, v_2\}$ . Then,  $W_2$  is a proper, closed subspace, meaning  $\exists v_3 \in V$  with  $\|v_3\| = 1$  and  $\text{dist}_{W_2}(v_3) \geq 1/2$ .

Continuing, we find  $\exists v_n \in V$  with  $\|v_n\| = 1$  and  $\text{dist}_{W_{n-1}}(v_n) \geq 1/2$ , where  $W_{n-1} = \text{span}\{v_1, \dots, v_{n-1}\}$ . We have a sequence  $(v_n)_n \in B_V$ . Since  $B_V$  is compact,  $\exists (v_{n_k})_k \rightarrow v \in B_V$ , meaning  $B_V$  is Cauchy. However, since  $\|v_n - v_m\| \geq 1/2$  for all  $n$  and  $m$ .  $\perp$

### Proposition: Compact Domain and Uniform Continuity

If  $f : (X, d) \rightarrow (Y, \rho)$  is continuous, and  $X$  is compact, then  $f$  is uniformly continuous.

**Proof:** Let  $\varepsilon > 0$ . For each  $x \in X$ , we have  $\exists \delta_x > 0$  such that for  $d(z, x) < \delta_x \Rightarrow \rho(f(z), f(x)) < \varepsilon/2$ .

Since  $X = \bigcup_{x \in X} U(x, \delta_x/2)$ , by compactness, we have  $x_1, \dots, x_n$  with  $X = \bigcup_{j=1}^n U(x_j, \delta_{x_j}/2)$ . Take  $\delta = \min\{\delta_{x_j}/2\}$ .

Let  $x, x' \in X$  arbitrary with  $d(x, x') < \delta$ . Locate  $x \in U(x_j, \delta_{x_j}/2)$  for some  $j$ . Then,

$$\begin{aligned} d(x', x_j) &\leq d(x', x) + d(x, x_j) \\ &< \delta + \delta_{x_j}/2 \\ &\leq \delta_{x_j}. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(f(x), f(x')) &\leq \rho(f(x), f(x_j)) + \rho(f(x_j), f(x')) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

### Compactness and Uniform Convergence

- (1) Let  $f_n : (0, 1) \rightarrow \mathbb{R}$  with  $f_n(t) = t^n$ . Pointwise,  $(f_n)_n \rightarrow 0$ , meaning for  $(f_n(t))_n \rightarrow 0(t) = 0$  for all  $t \in (0, 1)$ . However, the convergence is not uniform. We have  $\|f_n - 0\|_u = \|f_n\|_u = 1$ .

Note that  $f_n(t)$  decreases pointwise to 0 for all  $t \in (0, 1)$ , meaning  $f_1(t) \geq f_2(t) \geq f_3(t) \geq \dots$ .

- (2) Consider the sequence of functions defined by

$$f_n(x) = \begin{cases} 0 & x \in (-\infty, n) \\ x - n & x \in [n, n+1] \\ 1 & x \in (n+1, \infty) \end{cases}.$$

Notice that  $f_n(t)$  is decreasing in  $n$  for all  $t$  and  $(f_n)_n \rightarrow 0$  pointwise, but convergence is not uniform, as  $\|f_n\|_u = 1$  for all  $n$ .

### Dini's Theorem

If  $(X, d)$  is a compact metric space, and  $(f_n : X \rightarrow \mathbb{R})_n$  is a sequence of continuous real-valued functions with  $\forall x \in X$ ,  $(f_n(x))_n \rightarrow 0$  is decreasing. Then,  $(f_n)_n \rightarrow 0$  uniformly.

**Proof:** Let  $\varepsilon > 0$ . For each  $n \geq 1$ , take  $U_n = \{x \mid f_n(x) < \varepsilon/2\}$ . Then  $U_n = f_n^{-1}((-\infty, \varepsilon/2))$ . Since  $f_n$  is continuous, and  $(-\infty, \varepsilon/2)$ , so too is  $U_n$  in  $X$ .

Notice that  $U_1 \subseteq U_2 \subseteq \dots$ , as if  $x \in U_n$ , then  $f_{n+1}(x) \leq f_n(x) < \varepsilon/2$ , meaning  $x \in U_{n+1}$ . Then, we have that  $\bigcup U_n = X$ , as for all  $x$ ,  $f_n(x) \rightarrow 0$ . Since  $X$  is compact, we have  $X = \bigcup U_{n_k} = U_{n_K}$ . For any  $x \in X$ ,  $f_{n_K}(x) < \varepsilon/2$ . Thus,  $\|f_{n_K}\| \leq \varepsilon/2 < \varepsilon$ , so we have uniform convergence.

### Compactness in $C(X)$

If  $X$  is a compact metric space, then, by the Extreme Value Theorem,  $C(X) = C_b(X)$ . We can see that  $C_b(X)$  is complete under  $\|\cdot\|_u$ . We may ask when  $\mathcal{F} \subseteq C(X)$  is compact.

A family  $\mathcal{F} \subseteq C(X)$  is equicontinuous if and only if  $\forall \varepsilon > 0, \exists \delta > 0$  such that  $\forall x, y \in X$  with  $d(x, y) < \delta$ , then  $|f(x) - f(y)| < \varepsilon$  for all  $f \in \mathcal{F}$ .

**Exercise:** For  $\mathcal{F} \subseteq C(X)$  with  $\mathcal{F}$  finite, then  $\mathcal{F}$  is always equicontinuous.

Since every  $f \in \mathcal{F}$  is uniformly continuous, take the minimum value of  $\delta$ .

### Arzelà-Ascoli Theorem

Let  $(X, d)$  be a compact metric space. The family  $\mathcal{F} \subseteq C(X)$  is compact if and only if  $\mathcal{F}$  is closed, bounded, and equicontinuous.

**Proof:** Let  $\mathcal{F}$  be compact. Then,  $\mathcal{F}$  is complete, and thus closed and totally bounded, meaning  $\mathcal{F}$  is bounded. Thus, we need to show  $\mathcal{F}$  is equicontinuous.

Let  $\varepsilon > 0$ . By total boundedness,  $\exists f_1, \dots, f_n \in \mathcal{F}$  with  $\mathcal{F} \subseteq \bigcup_{j=1}^n U(f_j, \varepsilon/3)$ . Each  $f_j$  is uniformly continuous since  $X$  is compact. Thus,  $\exists \delta_j$  with  $x, y \in X$  and  $d(x, y) \leq \delta_j$ , then  $|f_j(x) - f_j(y)| < \varepsilon/3$ .

Let  $\delta = \min\{\delta_j\}$ . Given any  $f \in \mathcal{F}$ , we have  $f \in U(f_j, \varepsilon/3)$  for some  $j$ . For any  $x, y \in X$  with  $d(x, y) < \delta$ , we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &\leq \|f - f_j\|_u + |f_j(x) - f_j(y)| + \|f - f_j\|_u \\ &< 2\varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

Let  $\mathcal{F}$  be closed, bounded, and equicontinuous. Since  $\mathcal{F} \subseteq C(X)$  is closed,  $\mathcal{F}$  is complete. We need only show  $\mathcal{F}$  is totally bounded.

Let  $\varepsilon > 0$ . Since  $\mathcal{F}$  is equicontinuous,  $\exists \delta > 0$  such that for all  $x, y \in X$  with  $d(x, y) < \delta$ , then  $|f(x) - f(y)| < \varepsilon/4$  for any  $f \in \mathcal{F}$ .

Since  $X$  is compact,  $X$  is totally bounded, so  $\exists x_1, \dots, x_n \in X$  with  $X \subseteq \bigcup_{j=1}^n U(x_j, \delta)$ . Consider the set  $C_{\mathcal{F}} := \{(f(x_1), \dots, f(x_n)) \mid f \in \mathcal{F}\} \subseteq \mathbb{R}^n$ .

Since  $\mathcal{F}$  is bounded, we have that  $\|f\|_u \leq M$  for all  $f \in \mathcal{F}$  for some  $M > 0$ . Thus,  $|f(x_j)| \leq \|f\|_u \leq M$  for  $j = 1, \dots, n$ . Thus,  $C_{\mathcal{F}}$  is bounded in  $\mathbb{R}^n$ .

**Exercise:**  $S \subseteq \mathbb{R}^n$  is bounded if and only if  $S$  is totally bounded.

Thus,  $C_{\mathcal{F}}$  is totally bounded. Therefore,  $\exists f_1, \dots, f_m \in \mathcal{F}$  with  $C_{\mathcal{F}} \subseteq \bigcup_{i=1}^m U((f_i(x_1), \dots, f_i(x_n)), \varepsilon/4)$ .

If  $f \in \mathcal{F}$ , then  $\exists i = 1, \dots, m$  (\*) such that  $\|(f(x_1), \dots, f(x_n)) - (f_i(x_1), \dots, f_i(x_n))\|_1 < \varepsilon/4$ . Thus,

$$\sum_{j=1}^n |f(x_j) - f_i(x_j)| < \varepsilon/4.$$

We claim that  $\mathcal{F} \subseteq \bigcup_{i=1}^m U(f_i, \varepsilon)$ . Let  $f \in \mathcal{F}$  and  $x \in X$ . Pick  $i$  as in (\*), and  $j$  with  $x \in U(x_j, \delta)$ . Then,

$$\begin{aligned} |f(x) - f_i(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| \\ &< 3\varepsilon/4 \end{aligned}$$

so

$$\begin{aligned} \|f - f_i\| &\leq 3\varepsilon/4 \\ &< \varepsilon. \end{aligned}$$

### Stone-Weierstrass Theorem

Let  $(X, d)$  be a compact metric space. Suppose  $A \subseteq C(X; \mathbb{R})$  with

- $f, g \in A \Rightarrow f + g \in A$ ;
- $f \in A, \alpha \in \mathbb{F} \Rightarrow \alpha f \in A$ ;
- $f, g \in A \Rightarrow fg \in A$ ;
- $1_X \in A$ ;
- $A$  is separating — if  $x \neq y$  in  $X$ , then  $\exists f \in A$  with  $f(x) \neq f(y)$ .

We say  $A$  is a unital separating subalgebra of  $C(X)$ .

Then,  $\overline{A}^{\|\cdot\|_u} = C(X; \mathbb{R})$  ( $A$  is uniformly dense).

### Uniform Approximation by Polynomials

For example, considering  $\mathcal{P} = \{x \mapsto \sum_{k=0}^n a_k x^k \mid a_k \in \mathbb{R}\} \subseteq C([0, 1])$ . We can see that  $\mathcal{P}$  is a separating unital subalgebra. Thus,  $\mathcal{P}$  is dense.

Let  $f(x) = |x|$  on  $[-1, 1]$ . Consider the sequence  $P_n(x)$  given by

$$P_0(x) = 0$$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - (P_n(x))^2}{2}.$$

For example,  $P_1(x) = x^2/2$ ,  $P_2(x) = \frac{x^2}{2} + \frac{x^2 - x^4/4}{2}$ . Then,  $(P_n)_n \xrightarrow{\|\cdot\|_u} f$ .

**Proof:** We claim that  $0 \leq P_n(x) \leq f(x)$  for all  $x \in [-1, 1]$ . Clearly,  $0 \leq P_0(x) \leq |x|$ , and  $0 \leq P_1(x) \leq |x|$ . Assume it is the case that  $0 \leq P_n(x) \leq |x|$ . Then,

$$0 \leq P_n(x) \leq |x|$$

$$0 \leq P_n^2(x) \leq x^2$$

$$x^2 - P_n^2(x) \geq 0$$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2} \geq 0$$

and

$$|x| - P_{n+1}(x) = |x| - P_n(x) - \frac{|x|^2 - P_n(x)}{2}$$

$$= |x| - P_n(x) - \frac{(|x| - P_n(x))(|x| + P_n(x))}{2}$$

$$= (|x| - P_n(x)) \left(1 - \frac{|x| + P_n(x)}{2}\right)$$

$$\geq 0$$

Observe that  $P_n(x) \leq P_{n+1}(x)$ . For every  $x$ ,  $(P_n(x))_n$  is increasing and bounded above by  $|x|$ . So,  $P_n(x) \rightarrow L_x$ .

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

$$L_x = L_x + \frac{x^2 - L_x^2}{2}$$

$$L_x = \sqrt{x^2} = |x|.$$

Thus,  $(P_n)_n$  converges pointwise on  $[-1, 1]$ . So,  $(f - P_n) \rightarrow 0$  is decreasing pointwise. Whence, by Dini's Theorem,  $\|f - P_n\|_u \rightarrow 0$ .

## Connectedness

Let  $(X, d)$  be a metric space.

- (1) Let  $Y \subseteq X$ . A splitting for  $Y$  in  $X$  is an inclusion  $Y \subseteq U \cup V$ , where  $U, V \in \tau_X$  with  $Y \cap U \cap V = \emptyset$ .

**Remark:** If we set  $U_1 = U \cap Y$  and  $V_1 = V \cap Y$ , then  $U_1$  and  $V_1$  are open in  $Y$  with the relative topology. We have  $Y = U_1 \sqcup V_1$ . Also note that  $U_1$  and  $V_1$  are clopen in  $Y$ .

- (2) A splitting for  $Y$  is called trivial if either  $Y \cap U = \emptyset$  or  $Y \cap V = \emptyset$ .

- (3)  $Y$  is connected in  $X$  if every splitting for  $Y$  in  $X$  is trivial. Otherwise, we say  $Y$  is disconnected.

**Exercise:** Suppose  $C \subseteq Y \subseteq X$ .  $C$  is connected in  $Y$  if and only if  $C$  is connected in  $X$ .

## Connectedness of Subsets in $\mathbb{R}$

We have  $[a, b] \subseteq \mathbb{R}$  is connected.

**Proof:** Suppose  $[a, b] \subseteq U \cup V$  is a splitting.

- If  $a = b$  or  $a > b$ , clearly the splitting is trivial.
- Assume  $a < b$ . Without loss of generality,  $a \in U$ . Suppose toward contradiction that  $[a, b] \cap V \neq \emptyset$ . Set  $c = \inf[a, b] \cap V$ .

We claim that  $a < c$ ; since  $U$  is open,  $\exists \epsilon > 0$  such that  $(a - \epsilon, a + \epsilon) \subseteq U$ . So,  $V \cap [a, b] \subseteq [a + \epsilon, b]$ . Therefore,  $c \geq a + \epsilon$ . Thus,  $[a, c) \subseteq U$ .

We claim  $c \in V$ . Since  $U$  is open, we cannot have  $c < b$  and  $c \in U$ . Also, if  $c \in U$  and  $c = b$ , then  $[a, b] \cap V = \emptyset$ .

Since  $V$  is open,  $\exists \delta > 0$  with  $(c - \delta, c + \delta) \subseteq V$ . However, this means  $c \neq \inf V \cap [a, b]$ .

Thus,  $V \cap [a, b] = \emptyset$ .

We have that  $\mathbb{Q} \subseteq \mathbb{R}$  is disconnected.

**Proof:** We have  $\mathbb{Q} \subseteq (-\infty, \pi) \cup (\pi, \infty)$  is a non-trivial splitting.

## Proposition: Intervals in $\mathbb{R}$

Every interval  $I \subseteq \mathbb{R}$  is connected.

**Proof:** Let  $I \subseteq U \cup V$  be a non-trivial splitting. Therefore,  $U \cap I \neq \emptyset$ , and  $V \cap I \neq \emptyset$ . Let  $a \in I \cap U$  and  $b \in I \cap V$ . Without loss of generality,  $a < b$ . Then, by the definition of an interval,  $[a, b] \subseteq I \subseteq U \cup V$ .

However, at the same time,  $[a, b] \cap U \cap V \subseteq I \cap U \cap V = \emptyset$ . So, we have a splitting for  $[a, b]$ . This splitting for  $[a, b]$  is non-trivial, since  $[a, b] \cap U \neq \emptyset$  and  $[a, b] \cap V \neq \emptyset$ . However, we had shown that  $[a, b]$  is connected.

If  $I \subseteq \mathbb{R}$  is connected, then  $I$  is an interval.

**Proof:** Let  $a = \inf I$  and  $b = \sup I$ . It is possible for  $a$  to equal  $-\infty$  and  $b$  to equal  $+\infty$ . We claim that  $(a, b) \subseteq I$ .

If  $\exists c \in I$  with  $c \notin (a, b)$ , then we have a non-trivial splitting  $I \subseteq (-\infty, c) \cup (c, \infty)$ , which would contradict the assumption that  $I$  is connected. Thus,  $(a, b) \subseteq I$ .

If  $s, t \in I$  with  $s \leq t$ , then  $s \geq a$  or  $s > a$ , or  $t \leq b$  or  $t < b$ . By cases, we find  $[s, t] \subseteq I$ , meaning  $I$  is an interval.

**Exercise:** If  $Y \subseteq X$  is connected, then  $\bar{Y}$  is connected.

## Connected Components and Clopen Sets

Let  $(X, d)$  be a metric space. We define  $\sim_X$  on  $X$  as  $x \sim_X y$  if there is a connected  $C \subseteq X$  with  $x, y \in C$ . This is an equivalence relation.

We have that  $x \sim_X x$  by taking  $C = \{x\}$ , so the relation is reflexive. Clearly, the relation is symmetric. To show transitivity, we need the following lemma:

**Lemma:** If  $Y_1, Y_2 \subseteq X$  are connected with  $Y_1 \cap Y_2 \neq \emptyset$ , then  $Y_1 \cup Y_2$  is connected.

**Proof of Lemma:** Let  $Y_1 \cup Y_2 \subseteq U \cup V$  be a splitting. Note that  $Y_i \subseteq U \cup V$ , and  $Y_i \cap U \cap V \subseteq (Y_1 \cup Y_2) \cap U \cap V = \emptyset$ . For  $i = 1, 2$ , since  $Y_i$  are connected, so we have splittings for  $Y_i$ . Since the  $Y_i$  are connected, these splittings are trivial.

Since the splitting for  $Y_1$  is trivial,  $Y_1 \subseteq U$ , or  $Y_1 \subseteq V$ . Similarly, since the splitting for  $Y_2$  is trivial,  $Y_2 \subseteq U$  or  $Y_2 \subseteq V$ .

Suppose  $Y_1 \subseteq U$  and  $Y_2 \subseteq U$ . Then,  $Y_1 \cup Y_2 \subseteq U$ , and our original splitting is trivial.

Suppose  $Y_1 \subseteq U$  and  $Y_2 \subseteq V$ . Then,  $\emptyset \neq Y_1 \cap Y_2 = (Y_1 \cap U) \cap (Y_2 \cap V) = (Y_1 \cap Y_2) \cap (U \cap V) \subseteq (Y_1 \cup Y_2) \cap U \cap V = \emptyset$ .

Other cases follow similarly.

If  $x \sim_X y \sim_X z$ , then there exist connected subsets  $C, D \subseteq X$  with  $x, y \in C$  and  $y, z \in D$ . Since  $y \in C \cap D$ , we have that  $C \cup D$  is connected, so  $x, z \in C \cup D$ , which is connected.

The equivalence classes of  $X$  under  $\sim_X$  are called components.

**Remark:**  $[x]_{\sim} = \{y \in X \mid y \sim_X x\} = \bigcup_{x \in C} C$  with  $C$  connected. This is the largest connected subset of  $X$  containing  $x$ . We have that  $X = \bigsqcup_{i \in I} [x_i]_{\sim}$ .

If  $(X, d)$  is a metric space, and  $C \subseteq X$  is clopen and connected, then  $C$  is a component in  $X$ .

**Proof:** Let  $x \in C$ . We claim that  $C = [x]_{\sim}$ .

Clearly,  $C \subseteq [x]_{\sim}$ . Suppose  $y \in [x]_{\sim}$  and  $y \notin C$ .

Since  $y \in [x]_{\sim}$ , there is a connected  $D \subseteq X$  with  $x, y \in D$ . We have that  $D \subseteq C \cup (X \setminus C)$ . This is a non-trivial splitting for  $D$ , meaning  $D$  is disconnected.  $\perp$

## Totally Disconnected Metric Spaces

Consider the set  $X = \{0\} \cup \{1/n \mid n \geq 1\}$  with the topology inherited from  $\mathbb{R}$ . We want to find the connected components.

**Solution:** The set  $\{1/n\}$  for each  $n$  is connected in  $\mathbb{R}$ , meaning it is connected in  $X$ . Since  $\{1/n\}$  is closed in  $\mathbb{R}$ , it is also closed in  $X$ . We also have that  $\{1/n\} = X \cap (1/n - \delta_n, 1/n + \delta_n)$ , with  $\delta_n = \frac{1}{n(n+1)}$ .

Since each  $\{1/n\}$  is clopen and connected, each  $\{1/n\}$  is a component. Additionally,  $\{0\}$  is necessarily a component of  $X$  since it is left over after we take  $X \setminus \{1/n \mid n \geq 1\}$ . We see that every connected component of  $X$  is a singleton.

For  $X = \mathbb{Z}$ , we see that the components are singletons.

For  $X = \mathbb{Q}$ , we need a little bit more machinery to find the components.

**Solution:** Suppose  $q, r \in \mathbb{Q}$  with  $r \sim_{\mathbb{Q}} q$ . Then,  $\exists D \subseteq \mathbb{Q}$  connected with  $r, q \in D$ . If  $r \neq q$ , then let  $x \in \mathbb{R} \setminus \mathbb{Q}$  with  $x$  strictly between  $r$  and  $q$ . Without loss of generality,  $r < q$ . Then,  $D \subseteq ((-\infty, x) \cap \mathbb{Q}) \cup ((x, \infty) \cap \mathbb{Q})$  is a non-trivial splitting, meaning  $D$  is not connected.

Therefore,  $r = q$ , meaning the components of  $\mathbb{Q}$  are singletons.

If  $(X, d)$  is a metric space where every connected component is a singleton, then  $X$  is totally disconnected.

**Exercise:** The Cantor set is totally disconnected.

**Proposition: Open Sets in  $\mathbb{R}$** 

If  $U \subseteq \mathbb{R}$  is open, then  $U = \bigcup_{i \in I} V_i$ , where each  $V_i \subseteq \mathbb{R}$  is an open interval and  $I$  is countable.

**Proof:** Let  $U$  be the metric space with the topology inherited from  $\mathbb{R}$ . Then,  $U = \bigcup_{i \in I} V_i$ , with  $V_i \subseteq U$  are the connected components in  $U$ .

Since  $V_i$  is connected in  $U$ ,  $V_i$  is connected in  $\mathbb{R}$ . Thus,  $V_i$  is an interval. We will show that each  $V_i$  is open in  $\mathbb{R}$ .

Let  $x \in V_i$ . Since  $U$  is open,  $\exists \varepsilon > 0$  with  $(x - \varepsilon, x + \varepsilon) \subseteq U$ . Since  $x \in (x - \varepsilon, x + \varepsilon)$ , and  $(x - \varepsilon, x + \varepsilon)$ , it is the case that  $(x - \varepsilon, x + \varepsilon) \subseteq [x]_{\sim_U} = V_i$ . Thus,  $V_i$  is open.

Now, we need to show that  $I$  is countable. Consider  $N : I \rightarrow \mathbb{Q}$ ;  $N(i) = q_i \in V_i$ , with  $q_i \in \mathbb{Q}$ . If  $i \neq j$ , then  $N(i) \neq N(j)$  since  $V_i \cap V_j = \emptyset$ . Hence,  $N$  is injective, so  $I$  is countable.

**Proposition: Connectedness and Continuity**

If  $f : X_1 \rightarrow X_2$  is continuous and  $Y \subseteq X_1$  is connected, then  $f(Y) \subseteq X_2$  is connected.

**Proof:** Let  $f(Y) \subseteq U \cup V$  is a splitting of  $f(Y) \subseteq X_2$ .

Taking the preimage, we have  $Y \subseteq f^{-1}(f(Y)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ . We have that  $f^{-1}(U)$  and  $f^{-1}(V)$  are open in  $X_1$ . Additionally,

$$\begin{aligned} Y \cap f^{-1}(U) \cap f^{-1}(V) &= Y \cap f^{-1}(U \cap V) \\ &\subseteq f^{-1}(f(Y)) \cap f^{-1}(U \cap V) \\ &\subseteq f^{-1}(f(Y) \cap f^{-1}(U \cap V)) \\ &= \emptyset. \end{aligned}$$

Thus,  $Y \subseteq f^{-1}(U) \cup f^{-1}(V)$  is a splitting. Since  $Y$  is connected, the splitting is trivial, meaning without loss of generality,  $Y \subseteq f^{-1}(U)$ . So,  $f(Y) \subseteq U$ .

**Intermediate Value Theorem**

Let  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. If  $f(a) \leq \lambda \leq f(b)$ , then  $\lambda \in f([a, b])$ .

**Proof:** Since  $[a, b]$  is compact and connected, and  $f$  is continuous,  $f([a, b]) \subseteq \mathbb{R}$  is also connected. So,  $f([a, b])$  is a compact and connected interval.

Since  $f(a), f(b) \in f([a, b])$ , and  $f([a, b])$  is an interval,  $\lambda \in f([a, b])$ .

**Proposition: Continuous Map to Totally Disconnected Set**

Let  $X$  be connected,  $Y$  totally disconnected, and  $f : X \rightarrow Y$  continuous. Then,  $f$  is a constant map.

**Proof:** The continuous image of a connected set is connected, and the only connected sets in  $Y$  are singletons, meaning the image of  $X$  is a singleton.

**Path-Connectedness**

Let  $(X, d)$  be a metric space.

- (i) A path in  $X$  is a continuous map  $\gamma : [0, 1] \rightarrow X$ . If  $\gamma(0) = x_0$  and  $\gamma(1) = x_1$ , we say the path connects  $x_0$  to  $x_1$ .
- (ii)  $X$  is said to be path-connected if for any two points  $x_0$  and  $x_1$ , there exists a path.  $Y \subseteq X$  is path connected if  $Y$  is connected.
- (1) Let  $V$  be any normed space, and  $C \subseteq V$  convex. By definition,  $C$  is path-connected. Indeed,  $\gamma(t) = (1 - t)x_0 + x_1$ .
- (2) The metric space  $\mathbb{R}^2 \setminus \{0\}$  is path-connected.



**Proposition: Composition of Paths**

Let  $\gamma : [0, 1] \rightarrow X$  is a path from  $x_0$  to  $x_1$ , and  $\sigma : [0, 1] \rightarrow X$  is a path from  $x_1$  to  $x_2$ . Then, the following are all true.

- (1)  $\gamma^{-1} : [0, 1] \rightarrow X$ , with  $\gamma^{-1}(t) = \gamma(1 - t)$ , is a path from  $x_1$  to  $x_0$ .
- (2)  $\sigma \cdot \gamma : [0, 1] \rightarrow X$  is a path from  $x_0$  to  $x_2$ , with  $\sigma \cdot \gamma(t)$  defined as follows:

$$\sigma \cdot \gamma(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \sigma(2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

**Lemma: Base Point and Path-Connectedness**

Let  $(X, d)$  be a metric space, and  $x_0 \in X$  fixed. Suppose  $\forall x, \exists$  a path from  $x_0$  to  $x$ . Then,  $X$  is path-connected.

- (1) The unitary group is path-connected.

$$U_n(\mathbb{C}) = \{U \in M_n(\mathbb{C}) \mid U^*U = I_n = UU^*\}$$

$$d(U, V) = \|U - V\|_{\text{op}}$$

Let  $U \in U_n(\mathbb{C})$ . By the spectral theorem via a unitary; there exists  $V \in U_n(\mathbb{C})$  with  $V^*UV = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ , with  $|\lambda_j| = 1$ . Write  $\lambda_j = e^{i\theta_j}$ , with  $\theta_j \in [0, 2\pi)$ .

Consider  $U_t = V \text{diag}(e^{it\theta_1}, \dots, e^{it\theta_n}) V^*$ . Clearly,  $U_t \in M_n(\mathbb{C})$ . Additionally,  $U_0 = I_n$ , and  $U_1 = U$ . We have

$$\begin{aligned} \|U_s - U_t\| &= \|V^* \Lambda_s V - V \Lambda_t V^*\| \\ &= \|V(\Lambda_s - \Lambda_t)V^*\| \\ &\leq \|V\| \|\Lambda_s - \Lambda_t\| \|V^*\| \\ &= \|\Lambda_s - \Lambda_t\| \\ &\rightarrow 0. \end{aligned}$$

Thus,  $U_t$  is continuous, meaning we have a path from  $I_n$  to  $U$ . Thus,  $U_n(\mathbb{C})$  is path-connected.

**Proposition: Path-Connectedness implies Connectedness**

If  $(X, d)$  is a path-connected metric space, then  $X$  is connected.

**Proof:** Let  $X = U \sqcup V$  be a splitting. Suppose  $\exists x_0 \in U$  and  $x_1 \in V$ . We know  $\exists \gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = x_0$  and  $\gamma(1) = x_1$ . Since  $[0, 1]$  is connected and  $\gamma$  is continuous,  $\gamma([0, 1]) \subseteq X$  is connected. However,  $\gamma([0, 1]) \subseteq U \cup V$  is a non-trivial splitting.  $\perp$

**Exercise:** If  $f : X_1 \rightarrow X_2$  is continuous, and  $Y \subseteq X_1$  is path-connected, then  $f(Y) \subseteq X_2$  is path-connected.

**Proof of Exercise:** Let  $f(y_1), f(y_2) \in f(Y)$ . We have that  $\gamma : [0, 1] \rightarrow Y$  is a path. Thus,  $f \circ \gamma : [0, 1] \rightarrow f(Y)$  is a path.

**A Connected Space that is not Path-Connected**

Set  $Y_0 = \{0\} \times [-1, 1] \subseteq \mathbb{R}^2$ , and  $Y_1 = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$ . Let  $Y = Y_0 \cup Y_1$ . This space is known as the topologist's sine curve, and it is connected but not path-connected.

**Proof:** We can see that  $Y_1$  is the continuous image of a connected set, so  $Y_1$  is connected.

We also see that  $Y$  is connected, as  $Y = \overline{Y_1}$ .

We claim that  $Y$  is not path-connected. There does not exist a path  $\gamma : [0, 1] \rightarrow Y$  with  $\gamma(0) \in Y_0$  and  $\gamma(1) \in Y_1$ . Suppose toward contradiction that such a path existed. Let  $\gamma^{-1}(Y_0) := F$ , with  $\gamma^{-1}$  being the inverse image (not inverse path). Since  $Y_0$  is closed, we have  $F \subseteq [0, 1]$  is closed, so  $u = \sup F \in F$ , and  $u < 1$ .

By replacing  $[0, 1]$  by  $[u, 1]$ , we may assume a new path  $\gamma' : [0, 1] \rightarrow Y$  is a path with  $\gamma_1(t) \in (0, 1]$ , for  $\gamma'(t) = (\gamma'_1(t), \gamma'_2(t))$ .

Let  $r > 0$  be small such that  $[-1, 1] \supset [\gamma'_2(0) - r, \gamma'_2(0) + r]$ . Since  $\gamma'_2$  is continuous at  $t = 0$ , we know  $\exists \varepsilon > 0$  with  $\gamma'_2([0, \varepsilon]) \subseteq (\gamma'_2(0) - r, \gamma'_2(0) + r)$ .

Since  $\gamma'_1([0, \varepsilon])$  is connected, and hence an interval, and  $\gamma'_1(t) > 0$  for all  $t \in (0, 1]$ , we can find  $\delta$  small such that  $[0, \delta] \subseteq \gamma'_1([0, \varepsilon])$ .

We have that  $\gamma'_2(t) = \sin\left(\frac{1}{\gamma'_1(t)}\right)$  for  $t > 0$ . Therefore,

$$\begin{aligned} [-1, 1] &= \left\{ \sin\left(\frac{1}{x}\right) \mid 0 < x < \delta \right\} \\ &\subseteq \left\{ \sin\left(\frac{1}{\gamma'_1(t)}\right) \mid 0 < t < \varepsilon \right\} \\ &= \gamma'_2([0, \varepsilon)) \\ &\subseteq (\gamma'_2(0) - r, \gamma'_2(0) + r) \\ &\subset [-1, 1]. \end{aligned}$$

### Proposition: Connectedness in a Normed Space

Let  $V$  be a normed space, and  $Y \subseteq V$  is open and connected, then  $Y$  is path-connected.

**Proof:** Fix  $y_0 \in Y$ . Consider the set  $W = \{y \in Y \mid \exists \gamma \text{ from } y_0 \text{ to } y\}$ . We claim that  $W$  is open in  $Y$ .

Let  $y \in W$ . Since  $Y$  is open,  $\exists \delta > 0$  with  $U(y, \delta) \subseteq Y$ . If  $w \in U(y, \delta)$ ,  $\exists \gamma$  from  $y$  to  $w$ . Concatenating, we get a path from  $y_0$  to  $w$ . Thus,  $U(y, \delta) \subseteq W$ .

We also claim  $W$  is closed in  $Y$ .

### Measurable Spaces

The theory of integration is tied to notions of length, area, volume, etc. The Riemann integral

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right),$$

is defined through the length of a subinterval. We took the interval  $[0, 1]$ , calculated base multiplied by height, and found the area of the rectangle.

It's easy to compute the length of an interval. However, Lebesgue integration does the opposite; it subdivides the range of  $f$  into subintervals  $I_k$ , and calculates the "length" of  $f^{-1}(I_k)$ .

We need a more rigorous treatment of length (or area, or volume) to deal with Lebesgue integration.

Given  $E \subseteq \mathbb{R}^n$ , with  $E$  "sufficiently nice," we want to assign an extended positive real number  $\lambda(E) \in [0, \infty]$ , such that certain natural properties are satisfied.

- $\lambda(\emptyset) = 0$
- $\lambda\left(\prod_{j=1}^n [a_j, b_j]\right) = \prod_{j=1}^n (b_j - a_j)$
- $\lambda(x + E) = \lambda(E)$
- $\lambda\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda(E_k)$
- if  $E \subseteq F$ , then  $\lambda(E) \leq \lambda(F)$

**Proposition: Non-existence of  $\lambda$** 

There is no  $\lambda : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$  that satisfies the properties above.

**Proof:** Consider the equivalence relation on  $[0, 1]$ , with  $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ .

So,  $[0, 1] = \bigsqcup_{i \in I} [x_i]$ , with  $x_i \in [0, 1]$ . Let  $\{r_k\}_{k=1}^{\infty}$  be an enumeration of  $\mathbb{Q} \cap [-1, 1]$ . Let  $N = \{x_i\}_{i \in I}$  (possible with the axiom of choice).

Consider the set  $E_k = r_k + N$ .

- $E_k$  are pairwise disjoint; if  $r_k + x_i = r_\ell + x_j$ , then  $x_j - x_i = r_k - r_\ell \in \mathbb{Q}$ , meaning  $x_i \sim x_j$ .
- $E_k \subseteq [-1, 2]$ .

If  $t \in [0, 1]$ , then  $t \sim x_i$  for some  $i \in I$ . So,  $t - x_i \in \mathbb{Q}$ , and  $t - x_i \in [-1, 1]$ , so  $t - x_i = r_k$  for some  $k$ . Thus,  $t \in E_k$ . Thus, we have shown that  $[0, 1] \subseteq \bigsqcup E_k \subseteq [-1, 2]$ .

If  $\lambda$  were such a mapping, we have

$$\begin{aligned} 1 &= \lambda([0, 1]) \\ &\leq \lambda\left(\bigsqcup E_k\right) \\ &= \sum \lambda(E_k) \\ &= \sum \lambda(r_k + N) \\ &= \sum \lambda(N). \end{aligned}$$

If  $E = \bigsqcup E_k$ , then  $\lambda(E) \leq 3$  and  $\lambda(E) = \sum \lambda(N)$ .  $\perp$ .

Thus, we conclude that some sets are not measurable. We might then ask what sets are able to be measured.

- Intervals;
- open sets;
- closed sets.

We will eventually define a class of measurable sets,  $\mathcal{L}$ , and we will also construct a measure  $\lambda : \mathcal{L} \rightarrow [0, \infty]$  satisfying the above properties.

**Measurable Spaces and  $\sigma$ -Algebras**

Let  $\Omega \neq \emptyset$ .

(1) An algebra of subsets of  $\Omega$  is a nonempty family  $\mathcal{M} \subseteq \mathcal{P}(\Omega)$  such that

- If  $E \in \mathcal{M}$ , then  $E^c \in \mathcal{M}$ ;
- If  $E, F \in \mathcal{M}$ , then  $E \cup F \in \mathcal{M}$

(2) A nonempty collection  $\mathcal{M} \subseteq \mathcal{P}(\Omega)$  is a  $\sigma$ -algebra of subsets of  $\Omega$  if

- (i) If  $E \in \mathcal{M}$ , then  $E^c \in \mathcal{M}$ ;
- (ii) If  $\{E_k\}_{k=1}^{\infty} \in \mathcal{M}$ , then  $\bigcup E_k \in \mathcal{M}$ .

(3) A measurable space is a pair  $(\Omega, \mathcal{M})$  with  $\Omega \neq \emptyset$  a set and  $\mathcal{M}$  is a  $\sigma$ -algebra.

Let  $\mathcal{M}$  be an algebra of subsets of  $\Omega$ . Then, the following are true.

- (i)  $\emptyset, \Omega \in \mathcal{M}$ ;
- (ii) If  $E_1, \dots, E_n \in \mathcal{M}$ , then  $\bigcup E_k \in \mathcal{M}$ ;
- (iii) If  $E_1, \dots, E_n \in \mathcal{M}$ , then  $\bigcap E_k \in \mathcal{M}$ ;

(iv) If  $E, F \in \mathcal{M}$ , then  $E \setminus F \in \mathcal{M}$ .

**Proof:**

(i) Since  $\mathcal{M}$  is not empty, there is an  $E \in \mathcal{M}$ , so  $E^c \in \mathcal{M}$ , so  $E \cup E^c = \Omega \in \mathcal{M}$ , and  $(E \cup E^c)^c = \emptyset \in \mathcal{M}$ .

(ii) Induction.

(iii) We have  $\bigcap E_k = (\bigcup_{i=1}^{\infty} E_k^c)^c \in \mathcal{M}$ .

(iv) We have  $E \setminus F = E \cap F^c \in \mathcal{M}$ .

If  $\mathcal{M}$  is a  $\sigma$ -algebra, then (1) through (4) hold for countable families as well.

(1)  $(\Omega, \mathcal{P}(\Omega))$  is a measurable space.

(2)  $(\Omega, \{\emptyset, \Omega\})$  is a measurable space.

(3) For  $\Omega$  uncountable, let  $\mathcal{M} = \{E \subseteq \Omega \mid E \text{ countable or } E^c \text{ countable}\}$ . Then,  $(\Omega, \mathcal{M})$  is a measurable space.

(4) If  $\{\mathcal{M}_i\}_{i \in I}$  is a family of  $\sigma$ -algebras on  $\Omega$ , then  $\bigcap \mathcal{M}_i$  is a  $\sigma$ -algebra on  $\Omega$ .

If  $0 \neq \mathcal{E} \subseteq \mathcal{P}(\Omega)$ , the  $\sigma$ -algebra generated by  $\mathcal{E}$  is

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{M}_i \text{ } \sigma\text{-algebra} \\ \mathcal{E} \subseteq \mathcal{M}_i}} \mathcal{M}_i.$$

### Borel $\sigma$ -Algebra

Let  $(X, d)$  be a metric space. Let  $\tau_d = \{U \mid U \subseteq X \text{ open}\}$ . The Borel  $\sigma$ -algebra on  $X$  is

$$\mathcal{B}_X = \sigma(\tau_d).$$

**Remark:**  $\mathcal{B}_X$  contains all open sets, closed sets,  $F_\sigma$  sets,  $G_\delta$  sets, etc.

### Proposition: Borel $\sigma$ -Algebra on $\mathbb{R}$

Consider the families of  $\mathcal{P}(\mathbb{R})$ ,

$$\begin{aligned} \mathcal{E}_1 &= \{(a, b) \mid a < b\} \\ \mathcal{E}_2 &= \{[a, b) \mid a < b\} \\ \mathcal{E}_3 &= \{(a, b] \mid a < b\} \\ \mathcal{E}_4 &= \{[a, b] \mid a < b\} \\ \mathcal{E}_5 &= \{(-\infty, b) \mid b \in \mathbb{R}\} \\ \mathcal{E}_6 &= \{(-\infty, b] \mid b \in \mathbb{R}\} \\ \mathcal{E}_7 &= \{(a, \infty) \mid a \in \mathbb{R}\} \\ \mathcal{E}_8 &= \{[a, \infty) \mid a \in \mathbb{R}\}. \end{aligned}$$

For  $i = 1, \dots, 8$ , we have  $\sigma(\mathcal{E}_i) = \mathcal{B}_{\mathbb{R}}$ .

**Proof:** Note that  $\mathcal{E}_1 \subseteq \tau_d \subseteq \sigma(\tau_d) \subseteq \mathcal{B}_{\mathbb{R}}$ . Thus,  $\sigma(\mathcal{E}_1) \subseteq \mathcal{B}_{\mathbb{R}}$ . Let  $U \in \mathbb{R}$  be open. Then,  $U = \bigcup I_j$ , with  $I_j$  open. Consider any open interval  $I$ . If  $I$  is bounded, then  $I \in \mathcal{E}_1$ . If  $I$  is not bounded, then  $I = \bigcup_{k=1}^{\infty} J_k$  with  $J_k$  bounded open intervals. Since each  $J_k \in \mathcal{E}_1$ , then  $I \in \sigma(\mathcal{E}_1)$ . Therefore, each  $I_j \in \sigma(\mathcal{E}_1)$ , so  $U \in \sigma(\mathcal{E}_1)$ . Thus,  $\tau_d \subseteq \sigma(\mathcal{E}_1)$ , so  $\mathcal{B}_{\mathbb{R}} \subseteq \sigma(\mathcal{E}_1)$ .

Thus,  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{E}_1)$ .

We have that  $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b) \in \sigma(\mathcal{E}_1)$ . Therefore,  $\mathcal{E}_4 \in \sigma(\mathcal{E}_1)$ , thus  $\sigma(\mathcal{E}_4) \subseteq \sigma(\mathcal{E}_1)$ . Additionally,  $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b] \in \sigma(\mathcal{E}_4)$ . So,  $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_4) = \mathcal{B}_{\mathbb{R}}$ .

## Measure and Measure Spaces

Let  $(\Omega, \mathcal{M})$  be a measurable space.

- (1) A function  $\mu : \mathcal{M} \rightarrow [0, \infty]$  is a measure on  $(\Omega, \mathcal{M})$  if
  - (i)  $\mu(\emptyset) = 0$ ;
  - (ii) if  $\{E_k\}_{k \geq 1} \in \mathcal{M}$  are pairwise disjoint, then  $\mu(\bigsqcup E_k) = \sum \mu(E_k)$ . Notice that  $\mu(E_k) \geq 0$  for all  $E_k$ , so the order of the sum does not matter.
- (2) If  $\mathcal{M}$  is an algebra (or  $\sigma$ -algebra), and  $\mu$  satisfies  $\mu(E \sqcup F) = \mu(E) + \mu(F)$  for  $E, F \in \mathcal{M}$ , then  $\mu$  is called a finitely additive measure.
- (3) A measure space is a triple  $(\Omega, \mathcal{M}, \mu)$ , where  $(\Omega, \mathcal{M})$  is a measurable space and  $\mu$  is a measure.
- (4) A measure space  $(\Omega, \mathcal{M}, \mu)$  is called finite if  $\mu(\Omega) < \infty$ . If  $\mu(\Omega) = 1$ , then  $(\Omega, \mathcal{M}, \mu)$  is called a probability space, with  $\Omega$  the sample space and  $\mathcal{M}$  the collection of events.
- (5) A measure  $\mu$  is  $\sigma$ -finite if there exists  $\{E_k\}_{k \geq 1} \subseteq \mathcal{M}$  with  $\Omega = \bigcup E_k$  and  $\mu(E_k) < \infty$  for each  $k$ .
- (6) A measure  $\mu$  on  $(\Omega, \mathcal{M})$  is semi-finite if  $\forall E \in \mathcal{M}$  with  $\mu(E) = \infty$ ,  $\exists F \subseteq E$  with  $0 < \mu(F) < \infty$ .

**Exercise:** Show that  $\sigma$ -finite implies semi-finite.

## Examples of Measure Spaces

- (i) Consider  $(\Omega, \mathcal{P}(\Omega))$ . Fix  $x \in \Omega$ , with  $\delta_x : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ , with

$$\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

We can see that  $\delta_x$  is a probability measure, known as the Dirac measure.

- (ii) If  $\mu$  is a measure on  $(\Omega, \mathcal{M})$ , and  $t \in [0, \infty)$ , then  $(t\mu)(E) = t(\mu(E))$  is a measure.
- (iii) If  $\mu_1, \dots, \mu_n$  are measures on  $(\Omega, \mathcal{M})$ , then  $\mu(E) = \sum \mu_j(E)$  is a measure.
- (iv) If  $0 \leq t_1, \dots, t_n \leq 1$  with  $\sum t_j = 1$ , and  $x_1, \dots, x_n \in X$ , we have

$$\mu(E) = \sum t_j \delta_{x_j}$$

is a probability measure on  $(\Omega, \mathcal{P}(\Omega))$ .

- (v) Suppose  $f : \Omega \rightarrow [0, \infty]$  is any function. We get a measure on  $(\Omega, \mathcal{P}(\Omega))$ . We get that

$$\mu(E) = \sum_{x \in E} f(x) := \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq \Omega \text{ finite} \right\}.$$

If  $f(x) = 1$  for all elements of  $\Omega$ , then  $\mu$  is called the counting measure, with  $\mu(E) = \text{card}(E)$ .

## Proposition: Properties of Measures

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space.

- (i) Monotonicity: let  $E, F \subseteq \mathcal{M}$  with  $E \subseteq F$ , then  $\mu(E) \leq \mu(F)$
- (ii) Subadditivity: let  $\{E_k\}_{k \geq 1} \subseteq \mathcal{M}$ . Then,  $\mu(\bigcup E_k) \leq \sum \mu(E_k)$ .
- (iii) Continuity (from below): say  $\{E_k\}_{k \geq 1} \subseteq \mathcal{M}$  with  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ . Then,

$$\begin{aligned} \mu\left(\bigcup E_k\right) &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \sup \mu(E_k). \end{aligned}$$

(iv) Set subtraction: if  $E, F \subseteq \mathcal{M}$  with  $E \subseteq F$  and  $\mu(F) < \infty$ , then  $\mu(F \setminus E) = \mu(F) - \mu(E)$ .

(v) Continuity (from above): let  $\{E_k\}_{k \geq 1} \subseteq \mathcal{M}$  with  $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$  and  $\mu(E_1) < \infty$ . Then,

$$\begin{aligned}\mu\left(\bigcap E_k\right) &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \inf \mu(E_k).\end{aligned}$$

**Proof:**

(i) We have that  $\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$ .

(ii) Let  $F_1 = E_1$ ,  $F_2 = E_2 \setminus E_1$ ,  $F_3 = E_3 \setminus (E_1 \cup E_2)$ . Continuing, we have  $F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k$ . Notice

$$\begin{aligned}\bigsqcup F_k &= \bigcup E_k \\ \mu\left(\bigcup E_k\right) &= \mu\left(\bigsqcup F_k\right) \\ &= \sum \mu(F_k) \\ &\leq \sum \mu(E_k).\end{aligned}$$

(iii) Let  $F_1 = E_1$ ,  $F_2 = E_2 \setminus E_1$ , etc. with  $F_k = E_k \setminus E_{k-1}$ . Notice that

$$\begin{aligned}\mu\left(\bigsqcup F_k\right) &= \mu\left(\bigcup E_k\right) \\ \mu\left(\bigcup E_k\right) &= \sum \mu(F_k) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(F_k) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigsqcup_{k=1}^n F_k\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n).\end{aligned}$$

(iv) For  $E \subseteq F$ , we have  $\mu(F) = \mu(E) + \mu(F \setminus E)$ . Subtracting, we have  $\mu(F) \setminus \mu(E) = \mu(F \setminus E)$ , provided  $\mu(F)$  is finite.

(v) Exercise.

## Complete Measure Spaces

If  $(\Omega, \mathcal{M}, \mu)$  is a measures space, a subset  $N \subseteq \Omega$  is  $\mu$ -null if  $N \in \mathcal{M}$  and  $\mu(N) = 0$ .

**Remark:** If  $N$  is  $\mu$ -null, and  $M \subseteq N$ , then  $M$  is not necessarily  $\mu$ -null, because we do not know if  $M \in \mathcal{M}$ .

A measure space  $(\Omega, \mathcal{M}, \mu)$  is said to be complete if for any  $N$   $\mu$ -null and  $M \subseteq N$ , then  $M$  is  $\mu$ -null.

If  $(\Omega, \mathcal{M}, \mu)$ , and  $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$ , we set

$$\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M}, F \subseteq N \in \mathcal{N} \text{ for some } N \in \mathcal{N}\}.$$

We have that  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra with  $\mathcal{M} \subseteq \overline{\mathcal{M}}$  and  $\exists! \bar{\mu} : \overline{\mathcal{M}} \rightarrow [0, \infty]$ , with  $\bar{\mu}(E) = \mu(E)$  for all  $E \in \mathcal{M}$ , such that  $(\Omega, \overline{\mathcal{M}}, \bar{\mu})$  is a complete measure space.

## Outer Measures

An outer measure on a set  $\Omega$  is a map  $\theta : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  such that

(i)  $\theta(\emptyset) = 0$

(ii)  $E \subseteq F \Rightarrow \theta(E) \leq \theta(F)$

(iii)  $\theta(\bigcup E_k) \leq \sum \theta(E_k)$

**Remark:** Any measure is an outer measure.

We will construct outer measures from covering families equipped with a notion of measure.

**Proposition: Constructing an Outer Measure**

Let  $\mathcal{E} \subseteq \mathcal{P}(\Omega)$  be a "covering family" —  $\forall A \subseteq \Omega, A \subseteq \bigcup_{k \geq 1} E_k$ , where  $E_k \in \mathcal{E}$ . Let  $\rho : \mathcal{E} \rightarrow [0, \infty]$  such that  $\rho(\emptyset) = 0$ .

Set  $\theta_\rho : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ ; set

$$\theta_\rho(A) = \inf \left\{ \sum \rho(E_k) \mid A \subseteq \bigcup E_k, E_k \in \mathcal{E} \right\}.$$

Then,  $\theta_\rho$  is an outer measure.

**Proof:** Clearly,  $\theta_\rho(\emptyset) = 0$ .

Suppose  $A \subseteq B$ . If  $B \subseteq \bigcup E_k$ , then  $A \subseteq \bigcup E_k$ . Therefore,  $\theta_\rho(A) \leq \sum \rho(E_k)$ . By definition, it is then the case that  $\theta_\rho(A) \leq \theta_\rho(B)$ .

Let  $\{A_k\}_{k \geq 1} \subseteq \mathcal{P}(\Omega)$ . Let  $\varepsilon > 0$ . For each  $k$ , we can find a cover  $A_k \subseteq \bigcup_{j=1}^{\infty} E_{k,j}$  such that

$$\begin{aligned} \theta_\rho(A_k) + \frac{\varepsilon}{2^k} &> \sum_{j=1}^{\infty} \rho(E_{k,j}) \\ \sum_{k=1}^{\infty} \theta_\rho(A_k) + \varepsilon &> \sum_{j,k=1}^{\infty} \rho(E_{k,j}). \end{aligned}$$

Since  $\bigcup A_k \subseteq \bigcup_{k,j=1}^{\infty} E_{k,j}$ , it must be the case that

$$\theta_\rho \left( \bigcup A_k \right) \leq \sum_{k,j=1}^{\infty} \rho(E_{k,j}).$$

Therefore, we have

$$\theta_\rho \left( \bigcup A_k \right) \leq \sum \theta_\rho(A_k) + \varepsilon.$$

Since  $\varepsilon$  was arbitrary, it must be the case that we get countable subadditivity.

**Measurable Sets in Outer Measures**

Let  $\theta$  be an outer measure on  $\Omega$ .

A subset  $M \subseteq \Omega$  is said to be  $\theta$ -measurable if  $\forall E \subseteq \Omega, \theta(E \cap M) + \theta(E \cap M^c) = \theta(E)$ . Essentially,  $M$  is a good "cookie-cutter" for any subset of  $\Omega$ .

**Remark:** We always have  $\theta(E) = \theta((E \cap M) \cup (E \cap M^c)) \leq \theta(E \cap M) + \theta(E \cap M^c)$ . So, in order to show  $M$  is  $\theta$ -measurable, all we need show is that  $\theta(E \cap M) + \theta(E \cap M^c) \leq \theta(E)$ .

This inequality always holds if  $\theta(E) = \infty$ .

**Carathéodory's Theorem**

Let  $\theta : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  be an outer measure on  $\Omega$ .

- (i)  $\mathcal{M}_\theta = \{M \subseteq \Omega \mid M \text{ is } \theta\text{-measurable}\}$  is a  $\sigma$ -algebra.
- (ii)  $\theta|_{\mathcal{M}_\theta} : \mathcal{M}_\theta \rightarrow [0, \infty]$  is a complete measure.

**Proof:** We will show systematically via a series of claims.

**Claim 1:**  $\mathcal{M}_\theta$  is an algebra of subsets.

- We have that  $\emptyset \in \mathcal{M}_\theta$ .

$$\begin{aligned} \theta(E) &\geq \theta(E \cap \emptyset) + \theta(E \cap \emptyset^c) \\ &= 0 + \theta(E). \end{aligned}$$

- Let  $M \in \mathcal{M}_\theta$ . Clearly,  $M^c$  is measurable, since the definition of measurable is symmetric.
- Suppose  $M_1, M_2$  are measurable. We will show that  $M_1 \cap M_2$  is measurable.

$$\begin{aligned}
\theta(E) &\geq \theta(E \cap M_1) + \theta(E \cap M_1^c) \\
&\geq \theta(E \cap M_1 \cap M_2) + \theta(E \cap M_1 \cap M_2^c) + \theta(E \cap M_1^c \cap M_2) + \theta(E \cap M_1^c \cap M_2^c). \\
&\geq \theta(E \cap M_1 \cap M_2) + \theta(E \cap ((M_1 \cap M_2^c) \cup (M_1^c \cap M_2) \cup (M_1^c \cap M_2^c))) \\
&= \theta(E \cap M_1 \cap M_2) + \theta(E \cap (M_1 \cap M_2)^c).
\end{aligned}$$

Thus,  $M_1 \cap M_2$  is measurable.

**Claim 2:**  $\theta|_{\mathcal{M}_\theta}$  is a finitely additive measure. Let  $M_1, M_2 \in \mathcal{M}_\theta$  with  $M_1 \cap M_2 = \emptyset$ .

$$\begin{aligned}
\theta(M_1 \sqcup M_2) &= \theta(M_1 \sqcup M_2 \cap M_1) + \theta(M_1 \sqcup M_2 \cap M_1^c) \\
&= \theta(M_1) + \theta(M_2).
\end{aligned}$$

**Claim 3:** If  $\{M_k\}_{k \geq 1} \subseteq \mathcal{M}_\theta$  are pairwise disjoint, then  $\forall E \subseteq \Omega$ ,  $\theta(E \cap \bigcup M_k) = \sum \theta(E \cap M_k)$ .

$$\theta\left(E \cap \bigcup_{k=1}^n M_k\right) = \theta\left(\bigcup_{k=1}^n E \cap M_k\right)$$

cutting with  $M_n$ , we have

$$\begin{aligned}
&= \theta\left(\bigcup_{k=1}^n E \cap M_k \cap M_n\right) + \theta\left(\bigcup_{k=1}^n E \cap M_k \cap M_n^c\right) \\
&= \theta(E \cap M_n) + \theta\left(\bigcup_{k=1}^{n-1} E \cap M_k\right)
\end{aligned}$$

cutting with  $M_{n-1}$ , we get

$$= \theta(E \cap M_n) + \theta(E \cap M_{n-1}) + \theta\left(\bigcup_{k=1}^{n-2} E \cap M_k\right).$$

Continuing inductively, we have

$$\theta\left(E \cap \bigcup_{k=1}^n M_k\right) = \sum_{k=1}^n \theta(E \cap M_k).$$

In the infinite case,

$$\begin{aligned}
\sum \theta(E \cap M_k) &\geq \theta\left(\bigcup E \cap M_k\right) \\
&= \bigcup \theta\left(E \cap \bigcup M_k\right) \\
&\geq \theta\left(E \cap \bigcup_{k=1}^n M_k\right) \\
&= \sum_{k=1}^n \theta(E \cap M_k).
\end{aligned}$$

Letting  $n \rightarrow \infty$ , we are done.

**Claim 4:**  $\mathcal{M}_\theta$  is a  $\sigma$ -algebra. Additionally,  $\theta|_{\mathcal{M}_\theta}$  is a measure.

Let  $\{M_k\}_{k \geq 1} \subseteq \mathcal{M}_\theta$  be pairwise disjoint. Let  $M = \bigcup M_k$ . We will show  $M$  is measurable. Let  $P_n = \bigcup_{k=1}^n M_k$ . For  $E \subseteq \Omega$ ,

$$\theta(E) \geq \theta(E \cap P_n) + \theta(E \cap P_n^c) \quad \text{Claim 1}$$

$$\geq \sum_{k=1}^n \theta(E \cap M_k) + \theta(E \cap M^c). \quad \text{Monotonicity}$$



Letting  $n \rightarrow \infty$ ,

$$\begin{aligned}\theta(E) &\geq \sum_{k=1}^{\infty} \theta(E \cap M_k) + \theta(E \cap M_c) \\ &= \theta(E \cap M) + \theta(E \cap M^c)\end{aligned}\tag{Claim 3}$$

Thus,  $M$  is measurable. Taking  $E = \Omega$  in Claim 3, we show that  $\theta|_{\mathcal{M}_\theta}$  is a measure.

**Claim 5:**  $\theta|_{\mathcal{M}_\theta}$  is complete.

Let  $N \subseteq \Omega$  with  $\theta(N) = 0$ . Then, for all  $E \subseteq \Omega$ ,

$$\begin{aligned}\theta(E \cap N) + \theta(E \cap N^c) &\leq \theta(N) + \theta(E) \\ &= \theta(E).\end{aligned}$$

Thus,  $N \in \mathcal{M}_\theta$ . If  $M \in \mathcal{M}_\theta$  and  $\theta(M) = 0$ , and  $N \subseteq M$ , then by monotonicity we have  $\theta(N) = 0$ , so  $N \in \mathcal{M}_\theta$ .

**Remark:** If  $\theta(N) = 0$ , then  $N \in \mathcal{M}_\theta$ , and  $\theta(E \cup N) = \theta(E)$  and  $\theta(E \setminus N) = \theta(E)$ .

$$\begin{aligned}\theta(E) &\leq \theta(E \cup N) \\ &\leq \theta(E) + \theta(N) \\ &= \theta(E) \\ \theta(E) &= \theta(N \cup (E \setminus N)) \\ &\leq \theta(N) + \theta(E \setminus N) \\ &= \theta(E \setminus N) \\ &\leq \theta(E)\end{aligned}$$

## Lebesgue Measure over $\mathbb{R}$

Consider the family  $\mathcal{E} = \{(a, b) \mid a \leq b\}$ . Let  $\lambda_0 : \mathcal{E} \rightarrow [0, \infty]$ , with  $\lambda_0((a, b)) = b - a$ .

We see that  $\mathcal{E}$  is a covering family with  $\emptyset \in \mathcal{E}$ . Notice that  $\lambda_0(\emptyset) = 0$ . As a result, we get the Lebesgue *outer* measure,  $\lambda^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ , with

$$\lambda^*(E) := \inf \left\{ \sum_{k=1}^{\infty} \lambda_0(I_k) \mid E \subseteq \bigcup_{k=1}^{\infty} I_k, I_k \in \mathcal{E} \right\}.$$

We thus define the Lebesgue  $\sigma$ -algebra as

$$\mathcal{L} = \{E \subseteq \mathbb{R} \mid E \text{ is } \lambda^*\text{-measurable}\}.$$

The Lebesgue measure is  $\lambda := \lambda^*|_{\mathcal{L}}$ . We know from Carathéodory's theorem that  $\lambda$  is complete.

## Properties of the Lebesgue Measure

### Proposition: Countable Subsets are Lebesgue Measurable

If  $D \subseteq \mathbb{R}$  is countable, then  $D \in \mathcal{L}$  and  $\lambda(D) = 0$ .

**Proof:** It suffices to show that for  $t \in \mathbb{R}$ ,  $\{t\}$  is Lebesgue measurable.

We have, for any  $\varepsilon > 0$ ,

$$\{t\} \subseteq \left(t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}\right) \in \mathcal{E}.$$

Thus,  $\lambda^*(\{t\}) \leq \lambda_0\left(\left(t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}\right)\right) = \varepsilon$ . Since  $\varepsilon$  was arbitrary, we have that  $\lambda^*(\{t\}) = 0$ .

Thus, we have  $\{t\} \in \mathcal{L}$ . If  $D = \{t_k\}_{k \geq 1}$  is countable, since each  $\{t_k\} \in \mathcal{L}$ , we have

$$\begin{aligned} D &= \bigcup_{k=1}^{\infty} \{t_k\} \in \mathcal{L}, \\ \lambda(D) &= \sum_{k=1}^{\infty} \lambda(\{t_k\}) \\ &= 0. \end{aligned}$$

The converse is not true: the Cantor set has measure 0.

**Proposition: Borel Sets are Lebesgue Measurable**

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}.$$

**Proof:** We show that  $(-\infty, b) = I \in \mathcal{L}$  for any  $b \in \mathbb{R}$ . This is because  $\sigma(\{(-\infty, b) \mid b \in \mathbb{R}\}) = \mathcal{B}_{\mathbb{R}}$ , we will have that  $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$ .

Let  $E \subseteq \mathbb{R}$ . Let  $F = E \setminus \{b\}$ . Let  $F_1 = F \cap I = F \cap (-\infty, b)$ ,  $F_2 = F \cap I^c = F \cap [b, \infty) = F \cap (b, \infty)$ . Assume  $F \subseteq \bigcup_{k=1}^{\infty} I_k$ , with  $I_k$  open.

Let  $L_k = (-\infty, b) \cap I_k$ ,  $U_k = (b, \infty) \cap I_k$ . Notice that  $L_k$  and  $U_k$  are open intervals, and  $F_1 \subseteq \bigcup_{k=1}^{\infty} L_k$ ,  $F_2 \subseteq \bigcup_{k=1}^{\infty} U_k$ .

$$\begin{aligned} \lambda^*(F \cap I) + \lambda^*(F \cap I^c) &= \lambda^*(F_1) + \lambda^*(F_2) \\ &\leq \sum_{k=1}^{\infty} \lambda_0(L_k) + \sum_{k=1}^{\infty} \lambda_0(U_k) \\ &= \sum_{k=1}^{\infty} (\lambda_0(L_k) + \lambda_0(U_k)) \\ &= \sum_{k=1}^{\infty} \lambda_0(I_k) \end{aligned}$$

meaning

$$\lambda^*(F \cap I) + \lambda^*(F \cap I^c) \leq \lambda^*(F).$$

Therefore,  $F$  is  $\lambda^*$ -measurable. Notice that  $E \cap I = F \cap I = F_1$ , and  $E \cap I^c = E \cap [b, \infty) \subseteq F_2 \cup \{b\}$ . We have

$$\begin{aligned} \lambda^*(E \cap I) + \lambda^*(E \cap I^c) &\leq \lambda^*(F_1) + \lambda^*(F_2 \cup \{b\}) \\ &\leq \lambda^*(F_1) + \lambda^*(F_2) + \lambda^*(\{b\}) \\ &= \lambda^*(F_1) + \lambda^*(F_2) \\ &\leq \lambda^*(F) \\ &\leq \lambda^*(E). \end{aligned}$$

Thus,  $E$  is  $\lambda^*$ -measurable.

**Remark:** Every Borel set, including closed sets, open sets, compact sets,  $F_{\sigma}$ -sets,  $G_{\delta}$ -sets, etc., is Lebesgue measurable.

**Proposition: Measure of an Interval**

If  $I$  is any interval, then  $\lambda(I)$  is equal to the length of  $I$ .

**Proof:** Let  $I = [a, b]$ . For all  $\varepsilon > 0$ , we have

$$I \subseteq \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right),$$

meaning  $\lambda^*(I) \leq (b - a) + \varepsilon$ . Thus, we have  $\lambda(I) = \lambda^*(I) \leq b - a$ . To show the reverse direction, let

$$I \subseteq \bigcup_{k=1}^{\infty} I_k \quad I_k \text{ open.}$$

It suffices to show that

$$\sum_{k=1}^{\infty} \lambda_0(I_k) \geq b - a.$$

Since  $I$  is compact,  $\exists n$  with

$$I \subseteq \bigcup_{k=1}^n I_k.$$

Let  $\ell = \sum_{k=1}^n \lambda_0(I_k)$  (\*).

Without loss of generality, let  $a \in I_1 = (a_1, b_1)$ . If  $b_1 \geq b$ , we are done. If not, we have  $a_1 < a < b_1 < b$ .

Now,  $b_1 \in I \setminus I_1$ . Without loss of generality,  $b_1 \in I_2 = (a_2, b_2)$ . If  $b_2 \geq b$ , we are done, as

$$\begin{aligned} \ell &\geq (b_1 - a_1) + (b_2 - a_2) \\ &= b_2 - (a_2 - b_1) - a_1 \\ &\geq b - a_1 \\ &\geq b - a. \end{aligned}$$

We continue this process; it must terminate, as there are finitely many such intervals, meaning  $b_m \geq b$  for some  $m$ . We have a subcollection  $\{(a_k, b_k)\}_{k=1}^m$ , with  $a_1 < a$ ,  $a_2 < b_1 < b_2$ , etc. all the way to  $a_m < b_{m-1} < b_m$ , and  $b_m \geq b$ .

$$\begin{aligned} \ell &\geq \sum_{k=1}^m \lambda_0(a_k - b_k) \\ &= (b_m - a_m) + (b_{m-1} - a_{m-1}) + \cdots + (b_1 - a_1) \\ &= b_m - (a_m - b_{m-1}) - (a_{m-1} - b_{m-2}) - \cdots - (a_2 - b_1) - a_1 \\ &= b_m + (b_{m-1} - a - m) + (b_{m-2} + a_{m-1}) + \cdots + (b_1 - a_2) - a_1 \\ &\geq b_m - a_1 \\ &\geq b - a_1 \\ &\geq b - a. \end{aligned}$$

Thus,  $\lambda^*([a, b]) = \lambda([a, b]) = b - a$ .

Let  $I = (a, b]$ . Let  $I_n = [a + 1/n, b]$ . Then,  $I = \bigcup_{n=1}^{\infty} I_n$ .

$$\begin{aligned} \lambda(I) &= \lambda\left(\bigcup_{n=1}^{\infty} I_n\right) \\ &= \lim_{n \rightarrow \infty} \lambda(I_n) \\ &= \lim_{n \rightarrow \infty} (b - a) - (1/n) \\ &= b - a. \end{aligned}$$

Similarly for  $\lambda([a, b)) = b - a$ , and  $\lambda((a, b)) = b - a$ .

If  $I$  is unbounded, for every  $n$ , we can find a closed and bounded  $I_n \subseteq I$  with  $\lambda(I_n) = n$ . Therefore,  $\lambda(I) \geq \lambda(I_n) = n$ . Therefore,  $\lambda(I) = \infty$ .

**Lemma: Translation-Invariance of the Outer Measure**

For  $E \subseteq \mathbb{R}$ ,  $t \in \mathbb{R}$ ,  $\lambda^*(E + t) = \lambda^*(E)$ .

**Proof:** Given that  $E \subseteq \bigcup_{k \geq 1} I_k$ , then  $E + t \subseteq \bigcup_{k \geq 1} (I_k + t)$ . We have that  $I_k + t$  are still open intervals. Note that  $\lambda_0(I_k + t) = \lambda_0(I_k)$ .

Therefore,

$$\begin{aligned} \lambda^*(E + t) &\leq \sum_{k=1}^{\infty} \lambda_0(I_k + t) \\ &= \sum_{k=1}^{\infty} \lambda_0(I_k). \end{aligned}$$

By definition,  $\lambda^*(E + t) \leq \lambda^*(E)$ .

Additionally,

$$\begin{aligned} \lambda^*(E) &= \lambda^*(E + t - t) \\ &\leq \lambda^*(E + t). \end{aligned}$$

**Proposition: Translation-Invariance of the Lebesgue Measure**

If  $M \in \mathcal{L}$ , and  $t \in \mathbb{R}$ , then  $M + t \in \mathcal{L}$  and  $\lambda(M + t) = \lambda(M)$ .

**Proof:** Let  $E \subseteq \mathbb{R}$ .

$$\begin{aligned} \lambda^*(E) &= \lambda^*(E - t) \\ &= \lambda^*((E - t) \cap M) + \lambda^*((E - t) \cap M^c) \\ &= \lambda^*((E - t) \cap M + t) + \lambda^*((E - t) \cap M^c + t) \\ &= \lambda^*(E \cap (M + t)) + \lambda^*(E \cap (M + t)^c). \end{aligned} \quad M \in \mathcal{L}$$

Therefore,  $M + t \in \mathcal{L}$ , and

$$\begin{aligned} \lambda(M + t) &= \lambda^*(M + t) \\ &= \lambda^*(M) \\ &= \lambda(M). \end{aligned}$$

Thus, we have our measure space,  $(\mathbb{R}, \mathcal{L}, \lambda)$ , with

- $\lambda$  complete;
- $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$ ;
- $\lambda(I) = \text{length}(I)$ ;
- $\lambda(E + t) = \lambda(E)$ ;
- $\lambda$  is  $\sigma$ -finite.

**Regularity of the Lebesgue Measure****Theorem: Approximating a Measurable Set**

Let  $M \in \mathcal{L}$ .

- (1)  $\forall \epsilon > 0$ ,  $\exists U \subseteq \mathbb{R}$  open with  $M \subseteq U$  and  $\lambda(U \setminus M) < \epsilon$ .
- (2) There is a  $G_{\delta}$  set  $V \subseteq \mathbb{R}$  with  $M \subseteq V$  and  $\lambda(V \setminus M) = 0$ .
- (3)  $\forall \epsilon > 0$ ,  $\exists C \subseteq \mathbb{R}$  closed with  $C \subseteq M$  and  $\lambda(M \setminus C) < \epsilon$ .

(4) There is a  $F_\sigma$  set with  $F \subseteq M$  and  $\lambda(M \setminus F) = 0$ .

**Proof of (1):** If  $M \in \mathcal{L}$ , then  $\lambda(M) = \lambda^*(M)$ . By definition, given  $\varepsilon > 0$ ,  $\exists M \subseteq \bigcup_{k=1}^{\infty} I_k$ , with  $I_k$  open, and

$$\begin{aligned} \lambda(M) + \varepsilon &> \sum_{k=1}^{\infty} \lambda_0(I_k) \\ &= \sum_{k=1}^{\infty} \lambda(I_k) \\ &\geq \lambda\left(\bigcup_{k=1}^{\infty} I_k\right). \end{aligned}$$

Set  $U = \bigcup_{k=1}^{\infty} I_k$ .

If  $\lambda(M) < \infty$ , then  $\lambda(U \setminus M) = \lambda(U) - \lambda(M) < \varepsilon$ . Otherwise, if  $\lambda(M) = \infty$ , then  $M = \bigsqcup_{k=1}^{\infty} M_k$ , where each  $\lambda(M_k) < \infty$ .

For each  $M_k$ , find  $U_k$  open with  $U_k \supseteq M_k$  and  $\lambda(U_k \setminus M_k) = \lambda(U_k) - \lambda(M_k) < \varepsilon \cdot 2^{-k}$ . Set  $U = \bigcup U_k \supseteq M$ . Then,

$$\begin{aligned} \lambda(U \setminus M) &= \lambda\left(\bigcup_{k=1}^{\infty} U_k \setminus \bigsqcup_{k=1}^{\infty} M_k\right) \\ &= \lambda\left(\bigcup_{k=1}^{\infty} (U_k \setminus M_k)\right) \\ &\leq \sum_{k=1}^{\infty} \lambda(U_k \setminus M_k) \\ &\leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} \\ &= \varepsilon. \end{aligned}$$

**Proof of (2):** For every  $n \geq 1$ , find an open  $U_n \subseteq \mathbb{R}$  with  $U_n \supseteq M$  and  $\lambda(U_n \setminus M) < 1/n$ . Set  $V = \bigcap_{n=1}^{\infty} U_n$ .

$$\begin{aligned} \lambda(V \setminus M) &= \lambda(V \cap M^c) \\ &= \lambda\left(\bigcap_{n=1}^{\infty} (U_n \setminus M)\right) \\ &\leq \lambda(U_n \setminus M) \\ &< 1/n \end{aligned} \quad \forall n$$

meaning

$$\lambda(V \setminus M) = 0.$$

**Proof of (3):**  $M^c \in \mathcal{L}$ . Use (1) to prove.

**Proof of (4):** Use (3).

**Corollary: Completion of the Borel Measure Space**

$$(\mathbb{R}, \mathcal{L}, \lambda) = (\mathbb{R}, \overline{\mathcal{B}}_{\mathbb{R}}, \overline{\mu}),$$

where  $\mu = \lambda|_{\mathcal{B}_{\mathbb{R}}}$ .

**Proof:** We want to show that if  $M \in \mathcal{L}$ , then  $M = B \cup E$ , where  $E \subseteq N \in \mathcal{B}_{\mathbb{R}}$ , where  $\mu(N) = 0$ .

We know  $\exists V \in \mathcal{G}_\delta$  and  $F \in F_\sigma$ , with  $F \subseteq M \subseteq V$ ,  $\lambda(M \setminus F) = \lambda(V \setminus M) = 0$ .

Set  $M = F \cup (M \setminus F)$ . We have  $F$  Borel, and  $M \setminus F \subseteq V \setminus F$ . We know that  $\mu(V \setminus F) = 0$ .

**Corollary: Inner and Outer Regularity**

Let  $M \in \mathcal{L}$ . Then,

- Outer regularity:  $\lambda(M) = \inf \{ \lambda(U) \mid U \supseteq M, U \text{ open} \}$
- Inner regularity:  $\lambda(M) = \sup \{ \lambda(K) \mid K \subseteq M, K \text{ compact} \}$

**Proof of (1):** If  $\lambda(M) = \infty$ , the proof is clear.

Assume  $\lambda(M) < \infty$ . The  $\leq$  direction is clear.

Let  $\varepsilon > 0$ ; we have  $\exists U \subseteq \mathbb{R}$  with  $M \subseteq U$  and  $\lambda(U \setminus M) < \varepsilon$ , so  $\lambda(U) < \lambda(M) + \varepsilon$ .

**Proof of (2):** Assume  $M$  is bounded. Given  $\varepsilon > 0$ , find  $C$  closed with  $C \subseteq M$  and  $\lambda(M \setminus C) < \varepsilon$ . Since  $C$  is bounded, we have  $C$  is compact. Since  $M$  is bounded,  $\lambda(M) < \infty$ . Therefore,  $\lambda(C) < \infty$ , meaning  $\lambda(M) - \lambda(C) < \varepsilon$ , meaning  $\lambda(M) - \varepsilon < \lambda(C)$ . Since  $\lambda(M)$  is an upper bound for the right hand side, we are done.

Suppose  $M$  is not bounded. Set  $M_n = M \cap [-n, n]$ . Notice that  $M_1 \subseteq M_2 \subseteq \dots$ , with  $\bigcup M_n = M$ . Therefore,

$$\lambda(M) = \sup \lambda(M_n).$$

**Case 1:**  $\lambda(M) = +\infty$ . For every  $n$ , find a compact  $K_n \subseteq M_n$  (which is possible as the  $M_n$  are bounded) and  $\lambda(M_n) - 1 < \lambda(K_n)$ . Letting  $n \rightarrow \infty$ , we have  $\lambda(K_n) \rightarrow \infty$ . Therefore,  $\sup \lambda(K_n) = \infty$ .

**Case 2:**  $\lambda(M) < \infty$ . Given  $\varepsilon > 0$ , find  $n$  with  $\lambda(M) - \varepsilon/2 < \lambda(M_n)$ . There is a compact  $K$  with  $K \subseteq M_n$  and  $\lambda(M_n) - \varepsilon/2 < \lambda(K)$ . Therefore,  $K \subseteq M$  with  $\lambda(M) - \varepsilon < \lambda(K)$ .

**Proposition: Symmetric Difference Approximation**

Let  $M \in \mathcal{L}$  with  $\lambda(M) < \infty$ . Given  $\varepsilon > 0$ , there is an open  $V = \bigcup_{j=1}^n (a_j, b_j)$  such that  $\lambda(M \Delta V) < \varepsilon$ .

**Proof:** There is an open set  $U$  with  $M \subseteq U$  and  $\lambda(U \setminus M) = \lambda(U) - \lambda(M) < \varepsilon/2$ . Since every open set is a disjoint union of open intervals, we have  $U = \bigcup_{j=1}^{\infty} (a_j, b_j)$ . Therefore,  $\sum \lambda(a_j, b_j) \leq \lambda(M)$ . Thus,  $\exists n$  large such that  $\sum_{j=n+1}^{\infty} \lambda(a_j, b_j) < \varepsilon/2$ . Set  $V = \bigcup_{j=1}^n (a_j, b_j)$ .

**Vitali's Theorem**

Given  $E \subseteq \mathbb{R}$  with  $\lambda^*(E) > 0$ ,  $\exists N \subseteq E$  with  $N \notin \mathcal{L}$ .

**Proof:** Assume  $E$  is bounded;  $E \subseteq [-a, a]$ . Put an equivalence relation on  $E$ :  $x \sim y$  if and only if  $x - y \in \mathbb{Q}$ . Therefore,  $E = \bigsqcup_{i \in I} [x_i]$ , with  $x_i \in E$ . Set  $N = \{x_i\}_{i \in I}$ . We claim that  $N$  is not measurable.

Let  $\{r_k\}_{k=1}^{\infty}$  be an enumeration of the rationals inside  $\mathbb{Q} \cap [-2a, 2a]$ . Notice that  $\{r_k + N\}_{k=1}^{\infty}$  are pairwise disjoint. Also,  $E \subseteq \bigcup_{k=1}^{\infty} r_k + N$ , since, given  $x \in E$ , we have  $x \sim x_i$  for some  $x_i \in N$ , so  $x - x_i \in \mathbb{Q} \cap [-2a, 2a]$ , meaning  $x - x_i = r_k$  for some  $r_k \in \mathbb{Q} \cap [-2a, 2a]$ , meaning  $x \in r_k + N$ .

If  $N$  were measurable, then

$$\begin{aligned} 0 &< \lambda^*(E) \\ &\leq \lambda^*\left(\bigcup_{k=1}^{\infty} r_k + N\right) \\ &= \sum_{k=1}^{\infty} \lambda(r_k + N) \\ &= \sum_{k=1}^{\infty} \lambda(N). \end{aligned}$$

We also have  $\lambda(N) = \lambda^*(N) \leq \lambda^*(E) \leq 2a$ . Together, we arrive at a contradiction.

If  $E$  is not bounded, let  $E_n = E \cap [-n, n]$ . Then,

$$\begin{aligned} 0 &< \lambda^*(E) \\ &= \lambda^*(E_n) \\ &\leq \sum_{n=1}^{\infty} \lambda^*(E_n). \end{aligned}$$

Since  $\lambda^*(E) > 0$ , there must exist some  $E_n$  with  $\lambda^*(E_n) > 0$ , meaning  $E_n$  contains a non-measurable subset, so  $E$  has a non-measurable subset.

### Cantor-Lebesgue Function

To find a non-Borel, Lebesgue-measurable set, we must construct and explore the properties of the Cantor-Lebesgue function.

**Proof:** Consider the Cantor set:

$$\begin{aligned} C_0 &= [0, 1] \\ C_1 &= [0, 1/3] \cup [2/3, 1] \\ C_2 &= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1] \\ C_n &= \frac{1}{3} (C_{n-1} \cup (2 + C_{n-1})) \\ \mathcal{C} &= \bigcap_{n=0}^{\infty} C_n. \end{aligned}$$

To find  $\lambda(\mathcal{C})$ , notice that  $\mathcal{C}$  is closed (and thus Borel), meaning  $\lambda(\mathcal{C}) = \lim_{n \rightarrow \infty} \lambda(C_n)$ , meaning  $\lambda(\mathcal{C}) = 0$ .

We will build a function from the removed intervals of the Cantor set. Let

- $G_1 = C_0 \setminus C_1 = \underbrace{(1/3, 2/3)}_{I_{1,1}}$
- $G_2 = C_1 \setminus C_2 = \underbrace{(1/9, 2/9)}_{I_{2,1}} \sqcup \underbrace{(7/9, 8/9)}_{I_{2,2}}$
- $G_3 = C_2 \setminus C_3 = \underbrace{(1/27, 2/27)}_{I_{3,1}} \sqcup \underbrace{(7/27, 8/27)}_{I_{3,2}} \sqcup \underbrace{(19/27, 20/27)}_{I_{3,3}} \sqcup \underbrace{(25/27, 26/27)}_{I_{3,4}}$ .

At each step, we have  $G_k = C_{k-1} \setminus C_k = \bigsqcup_{j=1}^{2^{k-1}} I_{k,j}$ . If we let  $L_k = \bigsqcup_{j=1}^k G_j$ . Notice that  $L_k \sqcup C_k = [0, 1]$ .

Let

$$\begin{aligned} g_k &= \sum_{j=1}^{2^{k-1}} \frac{2j-1}{2^k} \mathbb{1}_{I_{k,j}} \\ g_1 &= \frac{1}{2} \mathbb{1}_{(1/3, 2/3)} \\ g_2 &= \frac{1}{4} \mathbb{1}_{(1/9, 2/9)} + \frac{3}{4} \mathbb{1}_{(7/9, 8/9)} \\ g_3 &= \frac{1}{8} \mathbb{1}_{(1/27, 2/27)} + \frac{3}{8} \mathbb{1}_{(7/27, 8/27)} + \frac{5}{8} \mathbb{1}_{(19/27, 20/27)} + \frac{7}{8} \mathbb{1}_{(25/27, 26/27)}. \end{aligned}$$

Now, let  $f_n = \sum_{k=1}^n g_k$ .

Let  $\varphi_n : [0, 1] \rightarrow [0, 1]$  be the unique continuous extension of  $f_n$ , where  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , and  $\varphi_n$  is linear on  $C_n$ .

We claim that  $(\varphi_n)_n$  are uniformly Cauchy. Note that

$$|\varphi_{k+1}(x) - \varphi_k(x)| < \frac{1}{2^k}.$$

So, for  $m > n$ ,

$$\begin{aligned} |\varphi_m(x) - \varphi_n(x)| &\leq |\varphi_m(x) - \varphi_{m-1}(x)| + \cdots + |\varphi_{n+1}(x) - \varphi_n(x)| \\ &\leq 2^{1-m} + \cdots + 2^{-n} \\ &\leq 2^{1-n} \\ \|\varphi_m(x) - \varphi_n(x)\|_u &\leq 2^{1-n}. \end{aligned}$$

Since  $C([0, 1])$  is complete, we must have that  $(\varphi_n)_n \xrightarrow{\|\cdot\|_u} \varphi \in C([0, 1])$ . We call  $\varphi$  the Cantor-Lebesgue Function.

#### Properties of the Cantor-Lebesgue Function:

- (1)  $\varphi$  is increasing;
- (2)  $\varphi$  is constant on  $[0, 1] \setminus \mathcal{C}$ ;
- (3)  $\varphi([0, 1]) = [0, 1]$ ;
- (4)  $\varphi(\mathcal{C}) = [0, 1]$ .

#### Proof of Properties of Cantor-Lebesgue Function:

- (1) If  $x \leq y$ , then  $\varphi_n(x) \leq \varphi_n(y)$ , meaning  $\varphi(x) \leq \varphi(y)$  as  $n \rightarrow \infty$ .
- (2) If  $x \notin \mathcal{C}$ , then  $x \in \bigcup_{k=1}^{\infty} L_k$ . Let  $x \in L_\ell$ . Thus,  $\varphi(x) = \lim_{n \rightarrow \infty} \varphi_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f_\ell(x)$ .
- (3) Intermediate value theorem.
- (4) We can see that  $\mathcal{C} \sqcup \underbrace{[0, 1] \setminus \mathcal{C}}_L = [0, 1]$ . Thus,

$$\varphi([0, 1]) = \varphi(\mathcal{C}) \sqcup \varphi(L).$$

We can see that  $\lambda(\varphi(L)) = 0$ , since  $\varphi(L)$  is a countable set. Thus,  $\lambda(\varphi(\mathcal{C})) = 1$ .

Since  $\mathcal{C}$  is compact,  $\varphi(\mathcal{C})$  is compact, and thus closed.

If  $\exists t \in [0, 1] \setminus \varphi(\mathcal{C})$ , then  $\exists \delta > 0$  such that  $(t - \delta, t + \delta) \in [0, 1] \setminus \varphi(\mathcal{C})$ , implying  $\lambda(\varphi(\mathcal{C})) < 1$ .

**Properties of a New Function:** Let  $\psi : [0, 1] \rightarrow [0, 2]$ ,  $\psi(x) = x + \varphi(x)$ .

- (1)  $\psi$  is strictly increasing
- (2)  $\psi : [0, 1] \rightarrow [0, 2]$  is bijective
- (3)  $\lambda(\psi(\mathcal{C})) = 1$

#### Proof of Properties of New Function:

- (1) Trivial.
- (2) Intermediate Value Theorem.
- (3)

$$\begin{aligned} \psi(L) &= \psi\left(\bigcup L_k\right) \\ &= \bigcup \psi(L_k). \end{aligned}$$

Notice that

$$\begin{aligned} \psi(I_{k,j}) &= \{x + \varphi(x) \mid x \in I_{k,j}\} \\ &= \left\{x + \frac{2^j - 1}{2^k} \mid x \in I_{k,j}\right\} \\ \lambda(\psi(I_{k,j})) &= \lambda(I_{k,j}). \end{aligned}$$



Therefore, we see that

$$\begin{aligned}
 \lambda(\psi(G_k)) &= \lambda\left(\psi\left(\bigsqcup_{j=1}^{2k-1} I_{k,j}\right)\right) \\
 &= \lambda\left(\bigsqcup \psi(I_{k,j})\right) \\
 &= \sum_{j=1}^{2k-1} \lambda(\psi(I_{k,j})) \\
 &= \sum_{j=1}^{2k-1} \lambda(I_{k,j}) \\
 &= \lambda(G_k).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \lambda(\psi(L)) &= \sum \lambda(\psi(G_k)) \\
 &= \sum \lambda(G_k) \\
 &= \sum \lambda\left(\bigsqcup G_k\right) \\
 &= 1,
 \end{aligned}$$

meaning

$$\begin{aligned}
 [0, 2] &= \psi([0, 1]) \\
 &= \psi(\mathcal{C} \sqcup L) \\
 &= \psi(\mathcal{C}) \sqcup \psi(L) \\
 &\Rightarrow \lambda(\psi(\mathcal{C})) = 1.
 \end{aligned}$$

**Proposition:** There is a set  $E \in \mathcal{L} \setminus \mathcal{B}_{\mathbb{R}}$ .

**Proof of Proposition:** We had  $\psi : [0, 1] \rightarrow [0, 2]$  a continuous bijection with  $\lambda(\psi(\mathcal{C})) = 1$ . By Vitali's theorem,  $\exists N \in \psi(\mathcal{C})$  with  $N \notin \mathcal{L}$ .

Set  $E = \psi^{-1}(N) \subseteq \mathcal{C}$ . Since  $\lambda(\mathcal{C}) = 0$ , and  $E \subseteq \mathcal{C}$ ,  $E \in \mathcal{L}$  (the Lebesgue measure is complete). Assume  $E \in \mathcal{B}_{\mathbb{R}}$ .

Since  $\beta = \psi^{-1}$  is a continuous bijection, we have  $N = \beta^{-1}(E)$  is Borel.  $\perp$

**Exercise:** Let  $f : X \rightarrow Y$  is a continuous map between two metric spaces. If  $B \in \mathcal{B}_Y$ , then  $f^{-1}(B) \in \mathcal{B}_X$ .

## Measurable Functions

Measurable functions are morphisms in the category of measurable spaces.

Let  $(\omega, \mathcal{M})$  and  $(\Lambda, \mathcal{N})$  be measurable spaces.  $f : \Omega \rightarrow \Lambda$  is called  $\mathcal{M}$ - $\mathcal{N}$ -measurable if  $E \subseteq \mathcal{N}$  implies  $f^{-1}(E) \in \mathcal{M}$ .

When mapping into  $\mathbb{R}$  or  $\mathbb{C}$ , we assume the codomain is equipped with the Borel  $\sigma$ -algebra.

(1)  $f : \Omega \rightarrow \mathbb{F}$  is measurable if  $f^{-1}(B) \in \mathcal{M} \forall B \in \mathcal{B}_{\mathbb{F}}$ .

(2) For  $S \subseteq \mathbb{R}$ ,  $f : S \rightarrow \mathbb{F}$  is measurable if it is  $\mathcal{L}_S - \mathcal{B}_{\mathbb{F}}$ -measurable.

**Proposition: Measurability On a Generated  $\sigma$ -Algebra**

Suppose  $\sigma(\mathcal{E}) = \mathcal{N}$ . Then,  $f : \Omega \Rightarrow *$  is measurable if and only if  $f^{-1}(E) \in \mathcal{M} \forall E \in \mathcal{E}$ .

**Proof:** The forward direction is clear.

In the reverse direction, let  $\mathcal{F} = \{F \subseteq \Lambda \mid f^{-1}(F) \in \mathcal{M}\}$ . We have that  $\mathcal{E} \subseteq \mathcal{F}$ . All we need do is show that  $\mathcal{F}$  is a  $\sigma$ -algebra, so  $\mathcal{N} \subseteq \mathcal{F}$ .

Thus, if  $f : X \rightarrow Y$  is continuous between metric spaces, then  $f$  is  $\mathcal{B}_X$ - $\mathcal{B}_Y$ -measurable.

Additionally,  $f : \Omega \rightarrow \mathbb{R}$  is measurable if and only if  $[f < b] := \{x \in \Omega \mid f(x) < b\}$ , or  $[f < b] = f^{-1}((-\infty, b))$  is in  $\mathcal{M}$  for all  $b \in \mathbb{R}$ .

## Extended Real Numbers

We sometimes work with the extended real numbers  $\bar{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$ . It isn't a field, but  $-\infty \leq a \leq \infty$  for all  $a \in \bar{\mathbb{R}}$ .

**Exercise:**  $\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subseteq \bar{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$  is a  $\sigma$ -algebra on  $\bar{\mathbb{R}}$ .

A member of  $\mathcal{B}_{\bar{\mathbb{R}}}$  looks like  $B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{\pm\infty\}$  where  $B \in \mathcal{B}_{\mathbb{R}}$ .

- (1)  $f : \Omega \rightarrow \bar{\mathbb{R}}$  is measurable if it is  $\mathcal{M}$ - $\mathcal{B}_{\bar{\mathbb{R}}}$ -measurable.
- (2)  $S \subseteq \mathbb{R}, f : S \rightarrow \bar{\mathbb{R}}$  is measurable if  $f$  is  $\mathcal{L}_S$ - $\mathcal{B}_{\bar{\mathbb{R}}}$ -measurable.

## Proposition: Preservation of Measurability under Operations

If  $f, g : \Omega \rightarrow \mathbb{R}$  are measurable, then

- (1)  $\alpha \in \mathbb{R}, \alpha f$  is measurable;
- (2)  $f \pm g$  is measurable;
- (3)  $fg$  is measurable;
- (4)  $\frac{f}{g}$  is measurable provided  $g \neq 0$  on  $\Omega$ .

**Proof of (2):** Fix  $b \in \mathbb{R}$ . We want to show that  $[f + g < b] \in \mathcal{M}$ . Let  $x \in \Omega$  such that  $f(x) + g(x) < b$ . Then,  $f(x) < b - g(x)$ .

So,  $\exists r \in \mathbb{Q}$  with  $f(x) < r < b - g(x)$ . So,  $g(x) < b - r$ . Therefore,  $[f + g < b] \subseteq \bigcup_{r \in \mathbb{Q}} ([f < r] \cap [g < b - r])$ . Reverse inclusion is straightforward.

**Proof of (3):** First, we will show that  $f^2$  is measurable.

If  $b \leq 0$ , then  $[f^2 < b] = \emptyset$ .

Let  $b > 0$ . Then,  $[f^2 < b] = [-\sqrt{b} < f < \sqrt{b}] = f^{-1}((-\sqrt{b}, \sqrt{b}))$ .

We have  $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$ , so from (1), (2), and above, we have  $fg$  is measurable.

**Exercise:**  $\sigma(\{[-\infty, b] \mid b \in \mathbb{R}\}) = \sigma(\{[-\infty, b] \mid b \in \mathbb{R}\}) = \mathcal{B}_{\bar{\mathbb{R}}}$ . When checking if  $f : \Omega \rightarrow \bar{\mathbb{R}}$  is measurable, we need only check  $f^{-1}([-\infty, b])$  is measurable.

## Proposition: More Preservation of Measurability

Let  $f, g : \Omega \rightarrow \bar{\mathbb{R}}$ . Then,

- (1)  $\max(f, g)$  is measurable;
- (2)  $\min(f, g)$  is measurable;
- (3)  $f^+ = \max(f, 0)$  and  $f^- = -\min(f, 0)$  are measurable;

(4)  $|f|$  is measurable.

**Proof:** Fix  $b \in \mathbb{R}$ .

(1)

$$[\max(f, g) < b] = [f < b] \cap [g < b].$$

(2)

$$[\min(f, g) < b] = [f < b] \cup [g < b].$$

(3)

$$\begin{aligned} [\max(f, 0) < b] &= [f < b] \cap [0 < b] \\ [-\min(f, 0) < b] &= [-f < b] \cup [0 < b]. \end{aligned}$$

(4)  $|f| = f^+ + f^-$ .

### Proposition: Sequence of Measurable Functions

Let  $(f_n : \Omega \rightarrow \overline{\mathbb{R}})_n$  be a sequence of measurable functions. Then,

(1)  $\sup f_n$  is measurable;

(2)  $\inf f_n$  is measurable;

(3)  $\limsup f_n$  is measurable;

(4)  $\liminf f_n$  is measurable.

**Proof:** Let  $b \in \mathbb{R}$ .

(1)

$$[\sup f_n \leq b] = \bigcap_{n=1}^{\infty} [f_n \leq b]$$

(2)

$$[\inf f_n < b] = \bigcup_{n=1}^{\infty} [f_n < b]$$

(3)

$$\limsup f_n = \inf_{m \geq 1} \left( \sup_{n \geq m} f_n \right)$$

(4)

$$\liminf f_n = \sup_{m \geq 1} \left( \inf_{n \geq m} f_n \right)$$

### Proposition: Pointwise Convergence of Measurable Functions

Let  $(f_n : \Omega \rightarrow \overline{\mathbb{R}})_n$  be a sequence of measurable functions with  $(f_n)_n \rightarrow f$  pointwise. Then,  $f$  is measurable.

**Proof:** If  $(f_n)_n \rightarrow f$  pointwise, then  $f = \limsup f_n = \liminf f_n$ .

## Simple Functions

(1) for  $E \subseteq \Omega$ , then  $\mathbb{1}_E : \Omega \rightarrow \mathbb{R}$ , the characteristic function of  $E$ , is defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

If  $E = \{x_0\}$ , then we write  $\mathbb{1}_E = \delta_{x_0}$ .

(2) A simple function  $\phi : \Omega \rightarrow \mathbb{R}$  is a linear combination of characteristic functions.

$$\phi = \sum_{k=1}^n c_k \mathbb{1}_{E_k}. \quad c_k \in \mathbb{R}, E_k \subseteq \Omega$$

**Remark:**  $\phi$  can assume finitely many values, specifically at most  $2^n$ .

If  $\text{ran}(\phi) = \{d_1, \dots, d_m\}$ , where  $d_j$  are distinct. Write  $D_j = \phi^{-1}(\{d_j\})$ . Then,

$$\phi = \sum_{j=1}^m d_j \mathbb{1}_{D_j}$$

is known as the *standard form*, where  $d_j$  are distinct, and  $\bigcup D_j = \Omega$ .

**Exercise 1:** Given  $\phi = \sum_{k=1}^n c_k \mathbb{1}_{E_k}$ ,  $\phi$  is measurable if and only if  $E_k \in \mathcal{M}$ .

**Exercise 2:** If  $X$  is a metric space,  $\mathbb{1}_E$  is continuous if and only if  $E$  is clopen in  $X$ .

## Proposition: Properties of Characteristic Functions

(1)

$$\mathbb{1}_{\bigcup D_j} = \sum_{j=1}^m \mathbb{1}_{D_j}$$

(2)

$$\mathbb{1}_E \cdot \mathbb{1}_F = \mathbb{1}_{E \cap F}$$

(3)

$$\begin{aligned} \Sigma(\Omega) &:= \{\phi \mid \phi : \Omega \rightarrow \mathbb{R} \text{ simple}\} \\ \Sigma(\Omega, \mathcal{M}) &:= \{\phi \mid \phi : \Omega \rightarrow \mathbb{R} \text{ simple and measurable}\} \end{aligned}$$

is a unital separating subalgebra of  $\mathcal{F}(\Omega, \mathbb{R})$ .

(4) Let  $X$  be a compact, totally disconnected metric space. Then,

$$\mathfrak{C} := \text{span}\{\mathbb{1}_E \mid E \subseteq X \text{ clopen}\}$$

is a unital separating subalgebra for  $C(X)$ .

Therefore,  $\overline{\mathfrak{C}}^{\|\cdot\|_\infty} = C(X)$ .

**Theorem: Pointwise Convergence of Simple Measurable Functions**

If  $(\Omega, \mathcal{M})$  is a measurable space, and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  is measurable, there is a sequence  $(\phi_n)_n$  of simple measurable functions such that  $\phi_n(x) \rightarrow f(x)$  for all  $x \in \Omega$ .

If  $f \geq 0$ , we can take  $(\phi_n)_n$  to be pointwise increasing.

If  $f$  is bounded, then  $(\|f - \phi_n\|_u)_n \rightarrow 0$ , and  $\phi_n$  are uniformly bounded:  $\sup \|\phi_n\|_u < \infty$ .

**Proof:** Assume that  $f \geq 0$ . For each  $n$ , partition  $[0, 2^n]$  into subintervals of length  $2^{-n}$ . We will have  $2^{2n}$  subintervals:

$$I_{n,0} = \left[0, \frac{1}{2^n}\right]$$

$$I_{n,k} = \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right],$$

with  $k = 1, 2, \dots, 2^{2n} - 1$ . We define  $J_n = (2^n, \infty]$ .

Let  $E_{n,k} = f^{-1}(I_{n,k})$ , with  $k = 1, 2, \dots, 2^{2n} - 1$ . Let  $F_n = f^{-1}(J_n)$ .

Notice that  $\left(\bigcup_{k=1}^{2^{2n}-1} E_{n,k}\right) \sqcup F_n = \Omega$ , and  $E_{n,k}, F_n$  are measurable.

Let

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n}.$$

We see that  $\phi_n$  are simple, measurable, and positive.

Fix  $x \in \Omega$ . If  $f(x)$  is finite, there is a large  $N$  with  $f(x) \leq 2^N$ . Fix  $n \geq N$ . Then,  $\exists! k$  with  $x \in E_{n,k}$ , meaning that  $\frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}$ .

Thus, we have

$$|f(x) - \phi_n(x)| = \left|f(x) - \frac{k}{2^n}\right| \leq \frac{1}{2^n}.$$

Thus, as  $n \rightarrow \infty$ , we have  $(\phi_n(x)) \rightarrow f(x)$ .

If  $f(x) = +\infty$ , then  $x \in F_n$  for all  $n$ . So,  $\phi_n(x) = 2^n$  for all  $n$ , which converges to  $f(x)$ .

If  $f$  is bounded, then for large  $n$ ,  $F_n = \emptyset$ . So,  $\|f - \phi_n\|_u \leq 2^{-n}$ , since our choice of  $N$  above works for all  $x$ . Thus,  $(\phi_n)_n \xrightarrow{\|\cdot\|_u} f$ , and clearly  $\sup \|\phi_n\|_u \leq \|f\|_u$ .

If  $f : \Omega \rightarrow \overline{\mathbb{R}}$  is measurable, then  $f = f^+ - f^-$ , where  $f^+$  and  $f^-$  are positive and measurable. Perform the above procedure for  $f^+$  and  $f^-$ , and subtract.

**Proposition: Measure on set of Measurable Functions**

Let  $(\Omega, \mathcal{M})$  be a measurable space.

$$L_0(\Omega, \mathcal{M}) := \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable}\}$$

is a unital, commutative algebra. Let  $\mu$  be a measure on  $(\Omega, \mathcal{M})$ . Define a relation on  $L_0(\Omega, \mathcal{M})$ :

$$f \sim_\mu g \Leftrightarrow \mu \left( \underbrace{\{x \mid f(x) \neq g(x)\}}_{((f-g)^{-1}(\{0\}))^c} \right) = 0.$$

Then,  $\sim_\mu$  is an equivalence relation.

We define

$$L(\Omega, \mathcal{M}, \mu) := L_0(\Omega, \mathcal{M}) / \sim_\mu$$

is a unital, commutative algebra.

$$\begin{aligned} [f]_\mu + [g]_\mu &= [f + g]_\mu \\ \alpha[f]_\mu &= [\alpha f]_\mu \\ [f]_\mu \cdot [g]_\mu &= [fg]_\mu. \end{aligned}$$

**Proof:** Reflexivity and symmetry are clear.

Let  $f \sim_\mu g \sim_\mu h$ . Let  $N := \{x \mid f(x) \neq g(x)\}$  and  $M = \{x \mid g(x) \neq h(x)\}$ . We know that  $\mu(N) = 0 = \mu(M)$ .

$$\begin{aligned} N^c \cap M^c &\subseteq \{x \mid f(x) = h(x)\}. \\ \{x \mid f(x) \neq h(x)\} &\subseteq N \cup M. \end{aligned}$$

Since  $\mu(N \cup M) = 0$ , so too is  $\mu(\{x \mid f(x) \neq h(x)\})$ .

### Essentially Bounded Functions

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space. Suppose  $f \in L_0(\Omega, \mathcal{M})$ .

- (1)  $c \geq 0$  is an essential bound for  $f$  if  $\mu(\{x \mid |f(x)| > c\}) = 0$ . If  $f$  admits an essential bound,  $f$  is called essentially bounded.
- (2) The essential supremum,  $\text{ess sup}(f) = \inf(\{c \mid c \text{ is an essential bound}\})$ . We say  $\text{ess sup}(f) = \infty$  if  $f$  has no essential bound.

For example, if  $f = \mathbb{1}_Q$ , then  $\text{ess sup}(f) = 0$ . At the same time,  $\|f\|_u = 1$ .

### Lemma: Essential Supremum Property

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space. For  $f \in L_0(\Omega, \mathcal{M})$ ,  $|f(x)| \leq \text{ess sup}(f)$  for almost every  $x \in \Omega$  ( $\mu$ -almost everywhere). We say  $\mu$ -a.e. if  $x \in \Omega$  means  $\forall x \in \Omega \setminus N$ , where  $\mu(N) = 0$ .

**Proof:** If  $\text{ess sup}(f) = \infty$ , then we are done.

Suppose  $c_f = \text{ess sup}(f) < \infty$ . For  $n \geq 1$ ,  $\exists$  essential bound  $c_n$  for  $f$  such that  $c_f + 1/n > c_n$ .

Let  $N_n = \{x \mid |f(x)| > c_n\}$ . Since  $c_n$  is an essential bound,  $\mu(N_n) = 0$ .

$$\begin{aligned} \mu(\{x \mid |f(x)| \leq c_f\}^c) &= \mu(\{x \mid |f(x)| > c_f\}^c) \\ &= \mu \left( \bigcup_{n \geq 1} \{x \mid |f(x)| > c_f + 1/n\} \right) \\ &\subseteq \mu \left( \bigcup_{n \geq 1} \{x \mid |f(x)| > c_n\} \right) \\ &= \mu \left( \bigcup_{n \geq 1} N_n \right) \\ &= 0. \end{aligned}$$

**Proposition: Arithmetic Operations of Essential Supremum**

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space, and  $f, g \in L_0(\Omega, \mathcal{M})$ . Then,

- (1)  $\text{ess sup}(f + g) \leq \text{ess sup}(f) + \text{ess sup}(g)$
- (2)  $\text{ess sup}(\alpha f) = |\alpha| \text{ess sup}(f)$
- (3)  $\text{ess sup}(fg) \leq (\text{ess sup}(f))(\text{ess sup}(g))$
- (4)  $\text{ess sup}(f) = 0 \Rightarrow f = 0 \text{ } \mu\text{-a.e.}, \text{ so } [f]_0 = L(\Omega, \mathcal{M}, \mu)$
- (5)  $\text{ess sup}(\mathbb{1}_\Omega) = 1$
- (6)  $\text{ess sup}(f) \leq \|f\|_\mu$
- (7)  $f \sim_\mu g \Rightarrow \text{ess sup}(f) = \text{ess sup}(g)$ .

**Proof of (1):** Assume  $c_f = \text{ess sup}(f)$ ,  $c_g = \text{ess sup}(g)$ , with  $c_f, c_g < \infty$ .

Let  $N = \{x \mid |f(x)| > c_f\}$  and  $M = \{x \mid |g(x)| > c_g\}$ . Both  $N$  and  $M$  are  $\mu$ -null, by the lemma.

$$\underbrace{\{x \mid |(f+g)(x)| > c_f + c_g\}}_{\mu\text{-null } (f+g \text{ measurable})} \subseteq N \cup M.$$

Therefore,  $c_f + c_g$  is an essential bound for  $f + g$ . Thus,  $\text{ess sup}(f + g) \leq c_f + c_g$ .

**Proof of (7):** Let  $N = \{x \mid f(x) \neq g(x)\}$ . It is the case that  $\mu(N) = 0$ . Let  $c_f = \text{ess sup}(f)$  and  $N_f = \{x \mid |f(x)| > c_f\}$ , which is  $\mu$ -null by the lemma.

Then,  $\{x \mid |g(x)| > c_f\} \subseteq N_f \cup N$  is  $\mu$ -null.

Therefore,  $c_f$  is an essential bound for  $g$ . Thus,  $\text{ess sup}(g) \leq c_f$ .

Similarly,  $\text{ess sup}(f) \leq c_g$ .

**Proposition: Properties of  $L_\infty$** 

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space.

$$\{[f] \in L(\Omega, \mathcal{M}, \mu) \mid \text{ess sup}(f) < \infty\}$$

is a unital commutative Banach algebra with norm  $\|[f]_\mu\|_\infty = \text{ess sup}(f)$ . It is denoted  $L_\infty(\Omega, \mu)$ .

**Proof:** All we need show is completeness.

Let  $(f_n)_n$  be Cauchy in  $L_\infty(\Omega, \mu)$ . Then,  $|f_n(x)| \leq \|f_n\|_\infty$  for all  $x \in N_n^c$ , where  $\mu(N_n) = 0$ . Let  $N = \bigcup_{n \geq 1} N_n$ . Then,  $\mu(N) = 0$ .

For all  $x \in N^c$ , we have  $|f_n(x)| \leq \|f_n\|_\infty$  for all  $n$ . Set

$$g_n(x) = \begin{cases} f_n(x) & x \in N^c \\ 0 & x \in N \end{cases}.$$

Then,  $g_n = f_n$  in  $L_\infty(\Omega, \mu)$ . Note that  $(g_n : \Omega \rightarrow \mathbb{R})_{n \geq 1}$  are uniformly Cauchy in  $\ell_\infty(\Omega)$  (in  $N^c$ ,  $|g_n - g_m| = |f_n - f_m| < \varepsilon$ , and in  $N$ ,  $|g_n - g_m| = 0$ ).

Since  $\ell_\infty(\Omega)$  is complete, we know  $(g_n)_n \rightarrow g$  in  $\ell_\infty(\Omega)$ . Certainly,  $g \in L_\infty(\Omega, \mu)$ . Thus,

$$\begin{aligned} \|f_n - g\|_\infty &= \|g_n - g\|_\infty \\ &\leq \|g_n - g\|_\mu \\ &\rightarrow 0, \end{aligned}$$

so  $L_\infty(\Omega, \mu)$  is complete.

## Lebesgue Integration

Fix a measure space  $(\Omega, \mathcal{M}, \mu)$ .

Define  $\phi : \Omega \rightarrow [0, \infty)$  be simple, positive, and measurable, given by

$$\phi = \sum_{k=1}^n d_k \mathbb{1}_{D_k}.$$

Standard Form

Then,

$$\int_{\Omega} \phi \, d\mu := \sum_{k=1}^n d_k \mu(D_k),$$

with the convention that  $0 \cdot \infty = 0$ .

**Fact:** If  $\phi = \sum_{j=1}^m c_j \mathbb{1}_{E_j}$ , with  $c_j \geq 0$  and  $E_j \in \mathcal{M}$ , not necessarily in standard form. Then,

$$\int_{\Omega} \phi \, d\mu = \sum_{j=1}^m c_j \mu(E_j).$$

## Properties of Integral of Simple Functions

Let  $\phi, \psi : \Omega \rightarrow [0, \infty)$  be simple, measurable, and positive. Then,

(i)

$$\int_{\Omega} (\phi + \psi) \, d\mu = \int_{\Omega} \phi \, d\mu + \int_{\Omega} \psi \, d\mu$$

(ii) For  $\alpha \geq 0$

$$\int_{\Omega} \alpha \phi \, d\mu = \alpha \int_{\Omega} \phi \, d\mu.$$

(iii) If  $0 \leq \phi \leq \psi$ , then

$$\int_{\Omega} \phi \, d\mu \leq \int_{\Omega} \psi \, d\mu$$

**Proof of (iii):** Let

$$\phi = \sum_{k=1}^n c_k \mathbb{1}_{E_k}$$

$$\psi = \sum_{\ell=1}^m d_{\ell} \mathbb{1}_{F_{\ell}}$$

be standard representations. Consider a common refinement  $\{E_k \cap F_{\ell}\}_{k,\ell}$ . Then,

$$\begin{aligned} \mathbb{1}_{E_k} &= \mathbb{1}_{\bigcup_{\ell} E_k \cap F_{\ell}} \\ &= \sum_{\ell=1}^m \mathbb{1}_{E_k \cap F_{\ell}}. \end{aligned}$$

Thus,

$$\begin{aligned} \phi &= \sum_{k=1}^n c_k \sum_{\ell=1}^m \mathbb{1}_{E_k \cap F_{\ell}} \\ &= \sum_{k,\ell} c_{k,\ell} \mathbb{1}_{E_k \cap F_{\ell}}, \end{aligned}$$



where

$$c_{k,\ell} = \begin{cases} 0 & E_k \cap F_\ell = \emptyset \\ c_k & E_k \cap F_\ell \neq \emptyset \end{cases}.$$

Similarly,

$$\begin{aligned} \psi &= \sum_{k,\ell} d_{k,\ell} \mathbb{1}_{E_k \cap F_\ell}, \\ d_{k,\ell} &= \begin{cases} d_\ell & E_k \cap F_\ell \neq \emptyset \\ 0 & E_k \cap F_\ell = \emptyset \end{cases}. \end{aligned}$$

Then,  $c_{k,\ell} \leq d_{k,\ell}$ .

$$\begin{aligned} \int_{\Omega} \phi \, d\mu &= \sum_{k,\ell} c_{k,\ell} \mu(E_k \cap F_\ell) \\ &\leq \sum_{k,\ell} d_{k,\ell} \mu(E_k \cap F_\ell) \\ &= \int_{\Omega} \psi \, d\mu. \end{aligned}$$

### Definition of the Lebesgue Integral

Let  $f : \Omega \rightarrow [0, \infty]$  be measurable. Then,

$$\int_{\Omega} f \, d\mu := \sup \left\{ \int_{\Omega} \phi \, d\mu \mid 0 \leq \phi \leq f, \text{ simple, measurable} \right\}$$

For  $E \in \mathcal{M}$ , we define

$$\int_E f \, d\mu = \int_{\Omega} (f)(\mathbb{1}_E) \, d\mu.$$

We say  $f$  is (Lebesgue) integrable if  $\int_{\Omega} f \, d\mu < \infty$ .

**Exercise:**

$$\int_{(0,1]} \frac{1}{x} \, d\lambda = +\infty.$$

### Proposition: Properties of the Lebesgue Integral

The following follow from the results about simple functions. Let  $f, g : \Omega \rightarrow [0, \infty]$  measurable.

(1) For  $\alpha \geq 0$

$$\int_{\Omega} (\alpha f) \, d\mu = \alpha \int_{\Omega} f \, d\mu;$$

(2) For  $0 \leq f \leq g$

$$\int_{\Omega} f \, d\mu \leq \int_{\Omega} g \, d\mu.$$

### Theorem: Monotone Convergence of Lebesgue Integral

Suppose  $(f_n : \Omega \rightarrow [0, \infty])_{n \geq 1}$  are positive, measurable, and pointwise increasing. Let  $f : \Omega \rightarrow [0, \infty]$  defined by  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Then,

$$\begin{aligned} \int_{\Omega} f \, d\mu &= \lim_{n \rightarrow \infty} \int_{\Omega} f_n \, d\mu \\ &= \sup \int_{\Omega} f_n \, d\mu. \end{aligned}$$

**Proof:** Note that  $\lim_{n \rightarrow \infty} f_n(x) \in [0, \infty]$  always exists, since  $(f_n(x))_n$  is an increasing sequence.

Also,  $f$  is measurable (pointwise limit of measurable functions). Moreover,

$$\int_{\Omega} f_n d\mu \leq \int_{\Omega} f_{n+1} d\mu.$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \sup \int_{\Omega} f_n d\mu$$

exists in  $[0, \infty]$ . Note that  $\int_{\Omega} f_n d\mu \leq \int_{\Omega} f d\mu$ , since  $f_n \leq f$ . Thus,  $\sup \int_{\Omega} f_n d\mu \leq \int_{\Omega} f d\mu$ .

Let  $0 < t < 1$ . Let  $\phi : \Omega \rightarrow [0, \infty)$  be simple and measurable with  $0 \leq \phi \leq f$ .

Set  $E_n = \{x \in \Omega \mid f_n(x) \geq t\phi(x)\}$ . Note  $E_1 \subseteq E_2 \subseteq \dots$  (since  $f_n$  are increasing). Additionally,

$$\bigcup_{n \geq 1} E_n = \Omega.$$

Notice that  $E_n$  are also measurable.

If  $A \subseteq \Omega$  is any measurable set. Then,  $A \cap E_1 \subseteq A \cap E_2 \subseteq \dots$ , and  $\bigcup_{n \geq 1} (A \cap E_n) = A$ . Therefore,  $(\mu(A \cap E_n))_n \rightarrow \mu(A)$  by continuity of  $\mu$ .

Suppose  $\phi = \sum_{k=1}^m a_k \mathbb{1}_{A_k}$ . Then,

$$\begin{aligned} \phi \mathbb{1}_{E_n} &= \sum_{k=1}^m a_k \mathbb{1}_{A_k \cap E_n} \\ \int_{\Omega} \phi \mathbb{1}_{E_n} d\mu &= \sum_{k=1}^m a_k \mu(A_k \cap E_n) \\ &\rightarrow \sum_{k=1}^m a_k \mu(A_k) \\ &= \int_{\Omega} \phi d\mu. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Omega} f_n d\mu &\geq \int_{\Omega} f_n \mathbb{1}_{E_n} d\mu \\ &\geq \int_{\Omega} t\phi \mathbb{1}_{E_n} d\mu \\ &= t \int_{\Omega} \phi \mathbb{1}_{E_n} d\mu \\ \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu &\geq t \int_{\Omega} \phi d\mu. \end{aligned}$$

Taking the supremum over all  $\phi$ ,

$$t \int_{\Omega} f d\mu \leq \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu,$$

and taking the supremum over all  $t$ , we get

$$\int_{\Omega} f d\mu \leq \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

**Remark:** Given  $f : \Omega \rightarrow [0, \infty]$  measurable, we proved that there exists a sequence  $(\phi_n)_n$  of positive, simple, measurable functions with  $(\phi_n)_n \rightarrow \phi$  pointwise increasing. Thus, by the monotone convergence theorem,  $\int_{\Omega} \phi d\mu \rightarrow \int_{\Omega} f d\mu$ .

**Linearity of the Lebesgue Integral over  $[0, \infty]$** 

Let  $f, g : \Omega \rightarrow [0, \infty]$  be measurable. Then,

$$\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu.$$

**Proof:** Use the Monotone Convergence Theorem and the earlier remark.

**Lebesgue Integral over  $\bar{\mathbb{R}}$** 

Let  $f : \Omega \rightarrow \bar{\mathbb{R}}$  be measurable.

(1) If either  $f^+$  or  $f^-$  is measurable, then

$$\int_{\Omega} f d\mu := \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu.$$

(2)  $f$  is said to be integrable if both  $f^+$  and  $f^-$  are integrable.

**Lemma: Absolute Value of Integrable Function**

$f$  is integrable if and only if  $|f|$  is integrable.

**Proof:** If  $f$  is integrable, then  $f^+$  and  $f^-$  are integrable, meaning

$$\begin{aligned} |f| &= f^+ + f^- \\ \int_{\Omega} |f| d\mu &= \int_{\Omega} f^+ d\mu + \int_{\Omega} f^- d\mu. \end{aligned}$$

If  $|f|$  is integrable, then  $\int_{\Omega} f d\mu \leq \int_{\Omega} |f| d\mu < \infty$ .

**Proposition: Linearity of the Lebesgue Integral over  $\mathbb{R}$** 

Let  $f, g : \Omega \rightarrow \mathbb{R}$  be integrable.

(1)

$$\int_{\Omega} \alpha f d\mu = \alpha \int_{\Omega} f d\mu$$

(2)

$$\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu.$$

(3) If  $f \leq g$ , then

$$\int_{\Omega} f d\mu \leq \int_{\Omega} g d\mu$$

(4)

$$\left| \int_{\Omega} f d\mu \right| \leq \int_{\Omega} |f| d\mu$$

**Proof of (2):** Write  $h$  as  $f + g$ . Note that  $|h| \leq |f| + |g|$ , so  $h$  is integrable. Then,

$$\begin{aligned} h^+ - h^- &= f^+ - f^- + g^+ - g^- \\ h^+ + f^- + g^- &= f^+ + g^+ + h^-. \end{aligned}$$

Integrating and using linearity, we get

$$\begin{aligned} \int h^+ d\mu + \int g^- d\mu + \int f^- d\mu &= \int f^+ d\mu + \int g^+ d\mu + \int h^- d\mu \\ \int h^+ d\mu - \int h^- d\mu &= \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu \\ \int h d\mu &= \int f d\mu + \int g d\mu. \end{aligned}$$

**Proof of (3):** If  $f \leq g$ , then  $g - f \geq 0$ , so  $\int (g - f) d\mu \geq 0$ , meaning  $\int g d\mu - \int f d\mu \geq 0$ .

**Proof of (4):**  $-|f| \leq f \leq |f|$ . Using (3) and (1),

$$\begin{aligned} -\int |f| d\mu &\leq \int f d\mu \leq \int |f| d\mu \\ \left| \int f d\mu \right| &\leq \int |f| d\mu. \end{aligned}$$

**Proposition: Integrable Function over Extended Real Line**

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be integrable. Then,  $f$  is finite  $\mu$ -almost everywhere.

**Proof:** Let  $E = \{x \mid |f(x)| = \infty\}$ . For any  $n \in \mathbb{N}$ ,  $|f| \geq n\mathbb{1}_E$ .

Therefore, we have  $\infty > \int_{\Omega} f d\mu \geq \int_{\Omega} n\mathbb{1}_E = n\mu(E)$ . Since this is true for any  $n$ , it must be the case that  $\mu(E) = 0$ .

**Proposition: Chebyshev's Inequality**

Let  $f : \Omega \rightarrow [0, \infty]$  be integrable. Then,  $\mu(\{x \mid f(x) \geq t\}) \leq \frac{1}{t} \int_{\Omega} f d\mu$ .

**Proof:** Let  $E_t = \{x \mid f(x) \geq t\}$ . Thus,  $f\mathbb{1}_{E_t} \geq t\mathbb{1}_{E_t}$ . Thus,

$$\begin{aligned} \int_{\Omega} f d\mu &\geq \int_{\Omega} f\mathbb{1}_{E_t} d\mu \\ &\geq \int_{\Omega} t\mathbb{1}_{E_t} d\mu \\ &= t\mu(E_t). \end{aligned}$$

**Proposition: Zero Definiteness**

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be measurable. Then,

$$\int_{\Omega} |f| d\mu = 0 \Leftrightarrow f = 0 \text{ } \mu\text{-a.e.}$$

**Proof:** Suppose  $f = 0$   $\mu$ -a.e., and we can assume  $f$  is positive.

Suppose  $0 \leq \phi \leq f$  with  $\phi$  simple and measurable. Then,  $\phi = 0$   $\mu$ -a.e. If

$$\phi = \sum_{k=1}^n c_k \mathbb{1}_{E_k},$$

then

$$\int_{\Omega} \phi d\mu = \sum_{k=1}^n c_k \mu(E_k).$$

If  $c_k \neq 0$ , then  $\mu(E_k) = 0$  (by the definition of  $\mu$ -a.e.) Thus,  $\int_{\Omega} \phi d\mu = 0$ , meaning  $\int_{\Omega} f d\mu = 0$ .

Suppose  $\int_{\Omega} |f| d\mu = 0$ . Consider  $\{x \mid f(x) > 0\} = \bigcup_{n \geq 1} \{x \mid f(x) > 1/n\}$ , meaning

$$\mu(\{x \mid f(x) \geq 1/n\}) \leq n \int_{\Omega} f d\mu = 0,$$

Chebyshev's Inequality

so  $\mu(\{x \mid f(x) > 0\}) = 0$ .

**Exercise:** If  $f : \Omega \rightarrow \overline{\mathbb{R}}$  is integrable, and  $\mu(N) = 0$ , then  $\int_N f d\mu = 0$ .

**Proposition: Equivalent Integrals**

Let  $f, g : \Omega \rightarrow \bar{\mathbb{R}}$  integrable, and  $f = g$   $\mu$ -a.e. Then,  $\int_{\Omega} f \, d\mu = \int_{\Omega} g \, d\mu$ .

**Proof:** We find  $\int_{\Omega} (f - g) \, d\mu = 0$ . Since  $\int$  is linear, we are done.

**Defining  $L_1$** 

Let  $(\Omega, \mathcal{M}, \mu)$  be a measure space. Then,

$$L_1(\Omega, \mu) := \{[f]_{\mu} \in L(\Omega, \mu) \mid f \text{ integrable}\}.$$

**Proposition: Properties of  $L^1$** 

(1)  $L_1(\Omega, \mu) \rightarrow \mathbb{R}$ ,  $[f]_{\mu} \mapsto \int_{\Omega} f \, d\mu$  and  $[f]_{\mu} \mapsto \int_{\Omega} |f| \, d\mu$  are well-defined maps.

(2)  $L_1(\Omega, \mu)$  equipped with

$$\|[f]_{\mu}\|_1 = \int_{\Omega} |f| \, d\mu$$

is a Banach space.

**Proof:**

(1) Exercise.

(2) We have already shown that  $\|[f]_{\mu}\|_1$  is a norm.

**Convergence of Integrable Functions**

Fix a measure space  $(\Omega, \mathcal{M}, \mu)$ . If  $(f_n : \Omega \rightarrow \mathbb{R})_n$  is a sequence of integrable functions with  $(f_n)_n \rightarrow f$  pointwise, with  $f$  integrable, can we ensure

$$\int_{\Omega} f_n \, d\mu \rightarrow \int_{\Omega} f \, d\mu.$$

Alternatively, can we ensure

$$\|f_n - f\|_1 \rightarrow 0.$$

One way we can ensure it is if  $(f_n)_n \rightarrow f$  uniformly and  $\mu(\Omega) < \infty$ , we find

$$\begin{aligned} \int |f_n - f| \, d\mu &\leq \int_{\Omega} \|f_n - f\|_{\infty} \, d\mu \\ &\leq \mu(\Omega) \|f_n - f\|_{\infty} \\ &\rightarrow 0. \end{aligned}$$

However, this condition is too strong.

(1) Consider  $(f_n : \mathbb{R} \rightarrow \mathbb{R})_n$ ,  $f_n = n\mathbb{1}_{(0,1/n)}$ . We see that  $(f_n)_n \rightarrow 0$  pointwise, but  $\int_{\mathbb{R}} f_n \, d\mu = 1$ .

(2) Let  $g_n = \mathbb{1}_{(0,1/n)}$ . We see that  $(g_n)_n \rightarrow 0$  pointwise, but  $\int_{\mathbb{R}} g_n \, d\mu = n$ .

(3) Let  $h_n = \mathbb{1}_{(n,n+1)}$ . Then,  $(h_n)_n \rightarrow 0$  pointwise, but  $\int_{\mathbb{R}} h_n \, d\mu = 1$ .

(4) Let  $(k_n : \mathbb{R} \rightarrow \mathbb{R})_n$  be defined by  $k_n = \frac{1}{n}\mathbb{1}_{(0,n)}$ . Then,  $(k_n)_n \rightarrow 0$  uniformly, but  $\int_{\mathbb{R}} k_n \, d\mu = 1$ .

**Convergence in Measure**

Let  $f : \Omega \rightarrow \mathbb{R}$  be measurable, and  $(f_n : \Omega \rightarrow \mathbb{R})_n$  be a sequence of measurable functions. Then,  $(f_n)_n \rightarrow f$  in measure if  $\forall \delta > 0$ ,  $\mu(\{x \mid |f_n(x) - f(x)| \geq \delta\}) \xrightarrow{n \rightarrow \infty} 0$ .

In all the above examples, we can see that  $(f_n)_n$  and  $(g_n)_n$  converge to 0 in measure. Additionally, we see  $(k_n)_n \rightarrow 0$  in measure (for any  $\delta > 0$ ,  $\exists N$  large such that  $1/N < \delta$ ). However,  $(h_n)_n$  does not converge to 0 in measure (set  $\delta = 1/2$ ).

**Exercise:** If  $(f_n)_n \rightarrow f$  in  $L_1$ , then  $(f_n)_n \rightarrow f$  in measure. (Use Chebyshev's Inequality)

### Fatou's Lemma

Let  $(f_k : \Omega \rightarrow [0, \infty])_k$  be a sequence of measurable functions. Then,

$$\int_{\Omega} \liminf f_k \, d\mu \leq \liminf \int_{\Omega} f_k \, d\mu.$$

**Proof:** Let  $f := \liminf f_k = \sup_{n \geq 1} \left( \underbrace{\inf_{k \geq n} f_k}_{g_n} \right)$ .

Notice that  $(g_n)_n$  increase pointwise to  $f$ . By monotone convergence,

$$\sup_{n \geq 1} \int_{\Omega} g_n \, d\mu = \int_{\Omega} f \, d\mu.$$

Now, fix  $n$ . Note that  $g_n \leq f_k$  for all  $k \geq n$ , meaning

$$\begin{aligned} \int_{\Omega} g_n \, d\mu &\leq \int_{\Omega} f_k \, d\mu \\ \int_{\Omega} g_n \, d\mu &\leq \inf_{k \geq n} \int_{\Omega} f_k \, d\mu \\ \sup_{n \geq 1} \int_{\Omega} g_n \, d\mu &\leq \sup_{n \geq 1} \left( \inf_{k \geq n} \int_{\Omega} f_k \, d\mu \right) \\ \int_{\Omega} \liminf f_k &\leq \liminf \int_{\Omega} f_k \, d\mu. \end{aligned}$$

### Theorem: Lebesgue's Dominated Convergence

Let  $(f_n : \Omega \rightarrow \mathbb{R})_n$  be a sequence of measurable functions with  $(f_n)_n \rightarrow f$  pointwise. Suppose  $\exists g : \Omega \rightarrow [0, \infty]$  integrable with  $|f_n| \leq g$  for all  $n^1$ . Then,  $(f_n)_n$  and  $f$  are integrable, and

$$\int_{\Omega} f_n \, d\mu \rightarrow \int_{\Omega} f \, d\mu,$$

and  $\|f_n - f\|_1 \rightarrow 0$ .

**Proof:** The proof amounts to applying Fatou's Lemma to  $g - f_n$  and  $g + f_n$ .

First,  $|f_n| \leq g$ , so each  $f_n$  is integrable, meaning  $|f| \leq g$  and  $f$  is integrable.

$$\begin{aligned} \int_{\Omega} g \, d\mu + \int_{\Omega} f \, d\mu &= \int_{\Omega} (g + f) \, d\mu \\ &= \int_{\Omega} \liminf (g + f_n) \, d\mu \\ &\leq \liminf \int_{\Omega} (g + f_n) \, d\mu \\ &\leq \liminf \left( \int_{\Omega} g \, d\mu + \int_{\Omega} f_n \, d\mu \right) \\ &\leq \overbrace{\int_{\Omega} g \, d\mu}^{< \infty} + \liminf \int_{\Omega} f_n \, d\mu. \end{aligned}$$

Thus,

$$\int_{\Omega} f \, d\mu \leq \liminf \int_{\Omega} f_n \, d\mu.$$

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<sup>1</sup>Both pointwise convergence and  $|f_n| \leq g$  apply  $\mu$ -a.e.

Additionally,

$$\begin{aligned}
 \int_{\Omega} g \, d\mu - \int_{\Omega} f \, d\mu &= \int_{\Omega} (g - f) \, d\mu \\
 &= \int_{\Omega} \liminf (g - f_n) \, d\mu \\
 &\leq \liminf \int_{\Omega} g - f_n \, d\mu \\
 &= \liminf \left( \int_{\Omega} g \, d\mu + \int_{\Omega} -f_n \, d\mu \right) \\
 &\leq \int_{\Omega} g \, d\mu + \liminf \left( - \int_{\Omega} f_n \, d\mu \right) \\
 &= \int_{\Omega} g \, d\mu - \limsup \int_{\Omega} f_n \, d\mu,
 \end{aligned}$$

meaning

$$\limsup \int_{\Omega} f_n \, d\mu \leq \int_{\Omega} f \, d\mu.$$

Thus,  $\int_{\Omega} f_n \, d\mu = \int_{\Omega} f \, d\mu$ .

Note that  $|f - f_n| \rightarrow 0$  pointwise, and  $|f - f_n| \leq 2g$ , with  $2g$  still integrable. By above, we see that  $\int_{\Omega} |f - f_n| \, d\mu \rightarrow 0$ , which is convergence in  $L^1$ .

### Addenda

Using Dominated Convergence, we find that  $C_c(\mathbb{R}) \subseteq L^1(\mathbb{R}, \lambda)$  is  $\|\cdot\|_1$ -dense.

- $f \geq 0$  integrable implies the existence of  $0 \leq \phi_n \leq f$  a sequence of simple functions increasing to  $f$ .
- If  $\phi$  is simple and integrable, then  $\phi = \sum_{k=1}^n c_k \mathbb{1}_{E_k}$ , with  $E_k$  bounded. We see that there exist  $\lambda(U \Delta E_k) < \varepsilon$  with  $U$  open in  $\mathbb{R}$ .
- We approximate each  $\mathbb{1}_{E_k}$  with a continuous bump function on  $U$ .

### Riesz–Markov–Kakutani Representation Theorem

Let  $X$  be a compact metric space. Let  $\varphi : C(X) \rightarrow \mathbb{R}$  be bounded linear, positive ( $f \geq 0 \Rightarrow \varphi(f) \geq 0$ ), and  $\varphi(\mathbb{1}_X) = 1$ . We call  $\varphi$  a state.

Then, there exists a unique regular Borel probability measure  $\mu : \mathcal{B}_X \rightarrow [0, 1]$  with

$$\varphi(f) = \int_{\Omega} f \, d\mu.$$

**Proof:** We will prove this for  $X = \Delta$ , where  $\Delta$  is the Cantor set.