

This is a notes document regarding essential problem-solving methods for the Real Analysis qualifier.

August 2019

Problem 1

- (a) Recall that the Cantor set \mathcal{C} is defined to consist of all $x \in [0, 1]$ such that x only contains 0 and 2 in the ternary expansion of x . Writing $a \in [0, 2]$ as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where $a_k \in \{0, 1, 2\}$, we may then find a_k at each ternary expansion slot for k as follows:

- if $a_k = 0$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = c_k = 0$
- if $a_k = 2$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = 2$ and $c_k = 0$ or vice versa.
- if $a_k = 1$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_{k+1} = c_{k+1} = 2$.

Therefore, since every digit of every ternary expansion in $[0, 2]$ can be obtained from \mathcal{C} , we see that $\mathcal{C} + \mathcal{C} = [0, 2]$.

- (b) We may set B to be the union of all integer translates of \mathcal{C} , and set $A = \mathcal{C}$. This yields closed subsets of \mathbb{R} with Lebesgue measure zero that sum to \mathbb{R} .

Problem 2

Consider the sequence of functions

$$f_n(x) = n \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on $[0, 1]$. This sequence is pointwise convergent everywhere to zero, as $f_n(0) = 0$ and the Archimedean property give that for any $x \in (0, 1]$, there is some n large enough that gives $\frac{1}{n} < x$. Furthermore, we see that

$$\begin{aligned} \int f_n \, d\mu &= n \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{n+1} \\ &\rightarrow 0. \end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned} \int \Phi \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1} \\ &\rightarrow \infty. \end{aligned}$$

Problem 4

Suppose toward contradiction that both f and $1/f$ are in $L_1(\mathbb{R})$. Then, from Hölder's Inequality, we have

$$\begin{aligned} \infty &= \int 1 \, d\mu \\ &\leq \left(\int f \, d\mu \right)^{1/2} \left(\int \frac{1}{f} \, d\mu \right)^{1/2} \\ &< \infty, \end{aligned}$$

which is a contradiction.

Problem 5

- (a) Let $f \in L_2([-1, 1])$. We may find $g \in C([-1, 1])$ such that $\|f - g\|_{L_2} < \varepsilon/2$. Similarly, we may find a polynomial p such that $\|g - p\|_{L_2} < \varepsilon/4$, meaning that $|p(x) - g(x)| < \varepsilon/4$ for all $x \in [-1, 1]$. This yields

$$\begin{aligned}\|p - g\|_{L_2} &= \left(\int_{-1}^1 |p(x) - g(x)|^2 dx \right)^{1/2} \\ &< \left(\int_{-1}^1 \left(\frac{\varepsilon}{4} \right)^2 dx \right)^{1/2} \\ &= \left(\frac{\varepsilon^2}{8} \right)^{1/2} \\ &< \frac{\varepsilon}{2},\end{aligned}$$

so $\|f - p\|_{L_2} < \varepsilon$, meaning that the closed linear span of the monomials is dense in L_2 , and the Legendre polynomials form an orthonormal system.

- (b) We see that at every step in evaluating the expression

$$L_n(x) = c_n \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (*)$$

the degree of the polynomial increases by 1, so each $L_n(x)$ has degree n . To verify that the polynomials generated from $(*)$ are orthogonal to each other, we let $n > m$ without loss of generality, and use integration by parts to obtain

$$\begin{aligned}\langle L_n, L_m \rangle &= \int_{-1}^1 \left(\frac{d^n}{dx^n} (x^2 - 1)^n \right) \left(\frac{d^m}{dx^m} (x^2 - 1)^m \right) dx \\ &= \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^m}{dx^m} (x^2 - 1)^m \Big|_{-1}^1 - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^{m+1}}{dx^{m+1}} (x^2 - 1)^m dx \\ &\vdots \\ &= (-1)^n \int_{-1}^1 \frac{d^{m+n}}{dx^{m+n}} (x^2 - 1)^m dx \\ &= (-1)^n \int_{-1}^1 \frac{d^n}{dx^n} \left(\frac{d^m}{dx^m} (x^2 - 1)^m \right) dx \\ &= (-1)^n \int_{-1}^1 \frac{d^n}{dx^n} L_m(x) dx \\ &= 0,\end{aligned}$$

seeing as we are taking n derivatives of a degree $m < n$ polynomial.

January 2020**Problem 1**

- (a) This is false. If $A \subseteq [0, 1]$ is the “fat Cantor set” constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then $\mu(A) = \frac{1}{2}$, but A is nowhere dense, meaning that if $U \subseteq A$ is open, then $U = \emptyset$.

To see that A is nowhere dense, we see that A is closed, so if $x \in A \subseteq [0, 1]$, and $\varepsilon > 0$, we may show that the interval $(x - \varepsilon, x + \varepsilon)$ is not contained in A . In the recursive construction of A , we may see that there is some step n_1 such that $\frac{1}{4^{n_1}} < 2\varepsilon$, implying that $(x - \varepsilon, x + \varepsilon)$ is not contained in the recursive construction at n_1 . Therefore $A^\circ = \emptyset$.

- (b) This is true. By the definition of the Lebesgue outer measure, for any $\varepsilon > 0$, there are $\{(a_k, b_k)\}_{k=1}^\infty$ such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^\infty (a_k, b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^\infty (a_k, b_k),$$

we have that U is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\{U \mid A \subseteq U, U \text{ open}\}.$$

Remark: Part (a) can be solved by selecting $A = \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$.

Problem 3

- (a) Consider the algebra of polynomials on $[0, 1]$ without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and $f(x) = x$ separates points in $[0, 1]$, this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at $x = 0$, the uniform closure of the algebra yields all the continuous functions on $[0, 1]$ with $f(0) = 0$.
- (b) In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra \mathcal{A} to include the constant functions.

Problem 4

We consider the signed measure on \mathcal{F} defined by

$$\nu(E) = \int_E f \, d\mu,$$

meaning that $\nu \ll \mu$, so the function $g := \frac{d\nu}{d\mu}$, where $\frac{d\nu}{d\mu}$ denotes the Radon–Nikodym derivative of ν with respect to μ (where we restrict μ to \mathcal{F}), is \mathcal{F} -measurable (by Radon–Nikodym) and in $L_1(\mathbb{R}, \mathcal{F}, \mu)$. This gives, for all $E \in \mathcal{F}$,

$$\begin{aligned} \int_E g \, d\mu &= \int_E \frac{d\nu}{d\mu} \, d\mu \\ &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f \, d\mu. \end{aligned}$$

Problem 5

Let $M = \mu(X)$.

Let $(f_n)_n \rightarrow f$ in measure, and let $\varepsilon > 0$. If we let

$$\begin{aligned} A &= \{x \mid |f_n(x) - f(x)| > \varepsilon/2M\} \\ B &= \{x \mid |f_n(x) - f(x)| \leq \varepsilon/2M\}, \end{aligned}$$

we have

$$\begin{aligned} \int_X \min(1, |f_n - f|) \, d\mu &= \int_A \min(1, |f_n - f|) \, d\mu + \int_B \min(1, |f_n - f|) \, d\mu \\ &\leq \mu(A) + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Meanwhile, if

$$\int_X \min(1, |f_n - f|) \, d\mu \rightarrow 0,$$

then by Chebyshev's Inequality, we have, for a fixed $0 < \varepsilon \leq 1$,

$$\begin{aligned} \mu(\{x \mid |f_n - f| \geq \varepsilon\}) &= \mu(\{x \mid \min(1, |f_n - f|) \geq \varepsilon\}) \\ &\leq \frac{1}{\varepsilon} \int_X \min(1, |f_n - f|) \, d\mu \\ &\rightarrow 0, \end{aligned}$$

so $(f_n)_n \rightarrow f$ in measure.

August 2020**Problem 1**

This is false. To see this, let $\mathcal{C}(x)$ denote the Cantor–Lebesgue function, and let

$$h(x) = \sum_{n=-\infty}^{\infty} \mathcal{C}(x - n) + n.$$

Then, since $\mathcal{C}(x)$ has derivative zero almost everywhere, the sum of a number of translates of $\mathcal{C}(x)$ still has derivative zero almost everywhere. Then, setting

$$f(x) = h(x) + x,$$

we get that $f(x)$ has derivative equal to 1 almost everywhere. However, at the same time, $f(2) - f(1) = 2$.

Problem 2

We show the inverse problem, which is that every closed set in \mathbb{R}^2 is G_δ . To do this, we let $A \subseteq \mathbb{R}^2$ be closed, nonempty, and proper (if $A = \emptyset$ or $A = \mathbb{R}^2$ the answer is trivial).

Then, there is some $x \in A^c$, and specifically there is $x \in A^c$ with rational coordinates (else, select $y \in \mathbb{Q}^2$ within the ball of radius ε that allows A^c to be open). Furthermore, since \mathbb{R}^2 is a metric space, \mathbb{R}^2 is regular, so there are open U_x and V_x such that $A \subseteq U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$.

Therefore, we get

$$A = \bigcap \{U_x \mid x \in \mathbb{Q}^2 \setminus A\},$$

meaning that A is G_δ . Taking complements, we thus get that every open set is F_σ .

Problem 3

(a) We see that

$$\begin{aligned} \langle Pf_i, f_j \rangle &= \delta_{i+1,j} \\ &= \delta_{i,j-1} \\ &= \langle f_i, f_{j-1} \rangle \\ &= \langle f_i, P^*f_j \rangle, \end{aligned}$$

so that $Pf_n = f_{n-1}$ if $n > 1$. Else, if $n = 1$, then $P^*f_n = 0$.

(b) We see that, acting on the orthonormal basis $(f_n)_n$, $P^*P(f_n) = f_n$, and

$$PP^*(f_n) = \begin{cases} 0 & n = 1 \\ 1 & \text{else,} \end{cases}$$

so that $P^*P = I$ and PP^* is as above.

Problem 4

We see that

$$\mu(\{x \mid f_n(x) > t\}) = \mu(X) - \mu(\{x \mid f_n(x) \leq t\}),$$

so by taking limits, we find that

$$\lim_{n \rightarrow \infty} \mu(\{x \mid f_n(x) > t\}) = \begin{cases} 1 & t < 0 \\ 0 & t \geq 0 \end{cases}.$$

So, if $\varepsilon > 0$, then

$$\begin{aligned} \mu(\{x \mid |f_n(x)| > \varepsilon\}) &= \mu(\{x \mid f_n(x) < -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\leq \mu(\{x \mid f_n(x) \leq -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\rightarrow 0. \end{aligned}$$

January 2023

Problem 1

By using Fatou's Lemma, and assuming WLOG that $(f_n)_n \rightarrow f$ pointwise everywhere, we get

$$\begin{aligned} \int_X |f|^p \, d\mu &= \int_X \liminf_{n \rightarrow \infty} |f_n|^p \, d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p \, d\mu \\ &\leq 1, \end{aligned}$$

so $\|f\|_{L_p} \leq 1$.

Problem 2

Let

$$f(t) = \mu(E \cap (-\infty, t)),$$

and for any sequence $(t_n)_n$, define

$$E_n = E \cap (-\infty, t_n).$$

We will show that f is left- and right-continuous, hence continuous. To start, if $(t_n)_n \searrow t$, then

$$\bigcap_{n \in \mathbb{N}} E_n = E \cap (-\infty, t],$$

so

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n \setminus \{t\}\right) \\ &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) - \mu(\{t\}). \end{aligned}$$

Since μ is atomless, we see that $\mu(\{t\}) = 0$, so since $\mu(E) < \infty$,

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \lim_{n \rightarrow \infty} f(t_n). \end{aligned}$$

Thus, f is right-continuous. Similarly, if $(t_n)_n \nearrow t$, then

$$\bigcup_{n \in \mathbb{N}} E_n = E \cap (-\infty, t),$$

so by continuity from below,

$$\begin{aligned} f(t) &= \mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \lim_{n \rightarrow \infty} f(t_n). \end{aligned}$$

Therefore, f is continuous. Since

$$\begin{aligned} \lim_{t \rightarrow -\infty} f(t) &= 0 \\ \lim_{t \rightarrow \infty} f(t) &= \mu(E) \\ &> 0, \end{aligned}$$

the intermediate value theorem gives some $t_0 \in \mathbb{R}$ such that

$$\begin{aligned} f(t_0) &= \mu(E \cap (-\infty, t_0)) \\ &= \frac{1}{2} \mu(E). \end{aligned}$$

August 2022

Problem 1

We note that

$$\begin{aligned} \left| \frac{n \sin(x/n)}{x(1+x^2)} \right| &\leq \left| \frac{n(x/n)}{x(1+x^2)} \right| \\ &= \frac{1}{1+x^2}, \end{aligned}$$

and since $\frac{1}{1+x^2}$ is integrable, we may use Dominated Convergence to switch limit and integral, giving

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx &= \int_0^\infty \lim_{n \rightarrow \infty} \frac{n \sin(x/n)}{x(1+x^2)} dx \\ &= \int_0^\infty \lim_{h \rightarrow 0} \frac{\frac{1}{h} \sin(hx)}{x(1+x^2)} dx \\ &= \int_0^\infty \frac{x}{x(1+x^2)} dx \\ &= \frac{\pi}{2}. \end{aligned}$$

Problem 2

- (a) Let f be Lipschitz, and let M denote the Lipschitz constant — i.e., $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [a, b]$. Set $\delta = \frac{\varepsilon}{M}$. Then, if $\{(a_j, b_j)\}_{j=1}^k$ is a partition such that $\sum_{j=1}^k |b_j - a_j| < \delta$, we have

$$\begin{aligned} \sum_{j=1}^k |f(b_j) - f(a_j)| &\leq M \sum_{j=1}^k |b_j - a_j| \\ &< \varepsilon. \end{aligned}$$

Thus, f is absolutely continuous. Now, if $x, x+h \in [a, b]$, we have that

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq M,$$

meaning that

$$\begin{aligned} |f'(x)| &= \lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x)}{h} \right| \\ &\leq M, \end{aligned}$$

and since $f'(x)$ exists for a.e. $x \in [a, b]$, we have that $\text{ess sup}_{x \in [a, b]} |f'(x)| \leq M$, so $f' \in L_\infty([a, b])$.

Let f be absolutely continuous with bounded derivative. Then, if M is the essential supremum of the f' , the fundamental theorem of calculus gives

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_x^y f'(t) dt \right| \\ &\leq \int_x^y |f'(t)| dt \\ &\leq \int_x^y M dx \end{aligned}$$

$$= M|y - x|,$$

so f is Lipschitz.

(b) If f is such that $f'(x)$ exists, then for $x, x + h \in [a, b]$, we have

$$\left| \frac{f(x + h) - f(x)}{h} \right| \leq \|f\|_{\text{Lip}},$$

so by taking limits, we have

$$|f'(x)| \leq \|f\|_{\text{Lip}}.$$

Thus, this ordering must respect essential suprema, meaning

$$\|f'\|_{L_\infty} \leq \|f\|_{\text{Lip}}.$$

Furthermore, if $\varepsilon > 0$, there are $x, y \in [a, b]$ with $x < y$ such that

$$\begin{aligned} \|f\|_{\text{Lip}} - \varepsilon &< \left| \frac{f(y) - f(x)}{y - x} \right| \\ &= \frac{1}{|y - x|} \left| \int_x^y f'(t) dt \right| \\ &\leq \frac{1}{|y - x|} \int_x^y |f'(t)| dt \\ &\leq \frac{1}{|y - x|} \int_x^y \|f'\|_{L_\infty} dt \\ &= \|f'\|_{L_\infty}, \end{aligned}$$

and since ε is arbitrary, we have $\|f\|_{\text{Lip}} \leq \|f'\|_{L_\infty}$.

Problem 4

(a) We let $U, V \subseteq X$ be such that $|\mu|$ is concentrated on U , $|\nu|$ is concentrated on V , and $U \cap V = \emptyset$.

Let P_μ, N_μ partition X such that U_P is positive for μ and U_N is negative for μ (as follows from the Hahn Decomposition). Note that since $|\mu|$ is concentrated on U , so too is μ , meaning that $U_P = P_\mu \cap U$ and $U_N = N_\mu \cap U$ are the positive and negative sets for μ , and similarly for V_P and V_N for ν .

Thus, since $U_P \cap V_{P,N} = \emptyset$, $\mu_1 \perp \nu_{1,2}$, and similarly for $U_N \cap V_{P,N}$ implying $\mu_2 \perp \nu_{1,2}$.