These are some notes I have taken from Gerald B. Folland's *A Course in Abstract Harmonic Analysis*, with some other textbooks for various sources.

Basic Properties of Topological Groups

Definition: A *topological group* is a group G with a topology such that the operation

$$\begin{aligned} m\colon G \times G &\to G \\ (x, y) &\mapsto xy \end{aligned}$$

is continuous with respect to the product topology on $G \times G$ and the operation

$$\begin{aligned} i\colon G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

is continuous with respect to the topology on G.

For a topological group G, we denote the unit element as 1_G, and we set

$$\begin{aligned} Ax &= \{yx \mid y \in A\} \\ xA &= \{xy \mid y \in A\} \\ A^{-1} &= \left\{y^{-1} \;\middle|\; y \in A\right\} \\ AB &= \{xy \mid x \in A, y \in B\} \end{aligned}$$

for all subsets $A, B \subseteq G$ and elements $x \in G$.

Definition: A subset $A \subseteq G$ is called *symmetric* if $A = A^{-1}$.

Proposition: Let G be a topological group.

(i) The topology of G is invariant under translations and inversion; that is, if U is open, then xU, Ux, U^{-1}, AU, UA are open for any $x \in G$ and subset $A \subseteq G$.

(ii) For every neighborhood U of 1_G, there is a symmetric neighborhood V of 1_G such that $VV \subseteq U$.

(iii) If H is a subgroup of G, so is $\overline{H}$.

(iv) Every open subgroup of G is closed.

(v) If A and B are compact sets in G, so is AB.

Proof.

(i) This is equivalent to the separate continuity of $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$; furthermore,

$$\begin{aligned} AU &= \bigcup_{x \in A} xU \\ UA &= \bigcup_{x \in A} Ux. \end{aligned}$$

(ii) Since $(x, y) \mapsto xy$ is continuous at 1_G, then for every neighborhood U of 1_G, there are neighborhoods $W_1, W_2 \subseteq U$. We may take $V = W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1}$.

(iii) For $x, y \in \overline{H}$, there are nets $(x_\alpha)_\alpha \to x$ and $(y_\alpha)_\alpha \to y$; since $(x_\alpha y_\alpha) \to xy$ and $\left(x_\alpha^{-1}\right)_\alpha \to x^{-1}$ by continuity of the operations, we have $xy, x^{-1} \in \overline{H}$.

(iv) If H is open, then so are all the cosets xH; since $G \setminus H$ is the union of all the cosets of H except for H itself, $G \setminus H$ is open, so H is closed.

(v) Since $A \times B$ is compact, and AB is the continuous image of $A \times B$ under $(x, y) \mapsto xy$, we have AB is compact.

□

Now, if H is a subgroup of G, we let G/H be the space of left cosets of H, and $q\colon G \to G/H$ is the canonical quotient map, we may impose the quotient topology on G/H, meaning that $U \subseteq G/H$ is open if and only if $q^{-1}(U)$ is open. Thus, q maps open sets in G to open sets in G/H, as if $V \subseteq G$ is open, $q^{-1}(q(V)) = VH$ is also open, so $q(V)$ is open.

Proposition: Let H be a subgroup of a topological group G.

(i) If H is closed, then G/H is Hausdorff.

(ii) If G is locally compact, so is G/H.

(iii) If H is normal, then G/H is a topological group.

Proof.

(i) If $\overline{x} = q(x)$ and $\overline{y} = q(y)$ are distinct points in G/H, and since H is closed, xHy^{-1} is a closed set that does not contain 1_G. There is a symmetric neighborhood U of 1_G such that $UU \cap xHy^{-1} = \emptyset$; since $U = U^{-1}$ and $H = HH$ (H is a subgroup), we have $1_G \notin UxH(Uy)^{-1} = (UxH)(UyH)^{-1}$, so $UxH \cap UyH = \emptyset$. Therefore, $q(Ux)$ and $q(Uy)$ are disjoint neighborhoods of $\overline{x}$ and $\overline{y}$.

(ii) If U is a compact neighborhood of 1_G, $q(Ux)$ is a compact neighborhood of $q(x)$ in G/H.

(iii) If $x, y \in G$, and U is a neighborhood of G/H, continuity of multiplication in G implies that there are neighborhoods V of x and W of y such that $VW \subseteq q^{-1}(U)$. We see that $q(V)$ and $q(W)$ are neighborhoods of $q(x)$ and $q(y)$ such that $q(V)q(W) \subseteq U$, meaning multiplication is continuous in G/H. Similarly, inversion is continuous.

□

Corollary: If G is T1, then G is Hausdorff, and if G is not T1, then $\overline{\{1_G\}}$ is a closed normal subgroup, and $G/\overline{\{1_G\}}$ is a Hausdorff topological group.

Proof. Since singletons are closed in any T1 space, the first assertion follows from part (i) in the previous proposition by taking $H = \{1_G\}$.

To see the second assertion, we note that $\overline{\{1_G\}}$ is a subgroup, and it is the smallest closed subgroup of G; it is normal, as otherwise we would obtain a smaller closed subgroup by intersection with one of the conjugates, meaning the result follows from parts (i) and (iii) in the previous proposition by taking $H = \overline{\{1_G\}}$. □

Thus, without loss of generality, we may assume that a topological group is Hausdorff (else take $G/\overline{\{1_G\}}$), and when we talk about locally compact groups, we are talking about topological groups that are locally compact and Hausdorff.

Proposition: Every locally compact group G has a subgroup G_0 that is open, closed, and σ-compact.

Proof. Let U be a symmetric compact neighborhood of 1_G, let $U_n = \prod_{i=1}^{n} U$, and let

$$G_0 = \bigcup_{n=1}^{\infty} U_n.$$

Then, G_0 is the group generated by U, so it is a subgroup; G_0 is open since U_{n+1} is a neighborhood of U_n for all n, and so G_0 is closed as all open subgroups are closed. Finally, since each U_n is a finite product of compact subsets of G, G_0 is σ-compact. □

We thus see that G_0 is the disjoint union of cosets of G_0, meaning G is a disjoint union of σ-compact spaces. In particular, if G is connected, then G is necessarily σ-compact.

Definition: Let $f\colon G \to \mathbb{C}$ be a function. The *translates* of f via $y \in G$ are defined by

$$\begin{aligned} L_y f(x) &= f\left(y^{-1}x\right) \\ R_y f(x) &= f(xy). \end{aligned}$$

Note that the maps $y \mapsto L_y$ and $y \mapsto R_y$ are group homomorphisms.

The function f is called left/right uniformly continuous if

$$\begin{aligned} \left\|L_y f - f\right\|_u &\to 0 \\ \left\|R_y f - f\right\|_u &\to 0 \end{aligned}$$

as $y \to 1_G$ respectively.

Proposition: If $f \in C_c(G)$, then f is left and right uniformly continuous.

Proof. We will prove this for $R_y f$.

If $f \in C_c(G)$, and $\varepsilon > 0$, then for every $x \in K = \operatorname{supp}(f)$, there is a neighborhood U_x of 1_G such that

$$|f(xy) - f(x)| < \frac{1}{2}\varepsilon$$

for any $y \in U_x$. Similarly, there is a symmetric neighborhood V_x of 1_G such that $V_x V_x \subseteq U_x$; the sets xV_x cover K, so there exist $x_1, \dots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} x_j V_{x_j}$.

Let $V = \bigcap_{j=1}^{n} V_{x_j}$. If $x \in K$, then there is some j such that $x_j^{-1}x \in V_{x_j}$, so $xy = x_j\left(x_j^{-1}x\right)y \in x_j U_{x_j}$, so

$$\begin{aligned} |f(xy) - f(x)| &\leqslant \left|f(xy) - f\left(x_j\right)\right| + \left|f\left(x_j\right) - f(x)\right| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon, \end{aligned}$$

for any $y \in V$, meaning that $\left\|R_y f - f\right\|_u < \varepsilon$. Similarly, if $xy \in K$, then $|f(xy) - f(x)| < \varepsilon$; meanwhile, if $x, xy \notin K$, then $f(x) = f(xy) = 0$, so we are done. $\square$

Haar Measure

Definition: We define a subset of $C_c(G)$ to be

$$C_c^+(G) = \{f \in C_c(G) \mid f \geqslant 0, f \neq 0\}.$$

Definition: A left/right Haar measure on G is a nonzero Radon measure μ on G such that $\mu(xE) = \mu(E)$ for every Borel $E \subseteq G$ and all $x \in G$.

Proposition: Let μ be a Radon measure on the locally compact group G, and let $\widetilde{\mu}(E) = \mu\left(E^{-1}\right)$. Then, the following hold:

(a) μ is a left Haar measure if and only if $\widetilde{\mu}$ is a right Haar measure.

(b) μ is a left Haar measure if and only i $\int L_y f \, d\mu = \int f \, d\mu$ for all $f \in C_c^+(G)$ and every $y \in G$.

Proof. The result in (a) follows from basic properties of the inverse.

To see (b), note that for any Radon measure μ, one has $\int L_y f\, d\mu = \int f\, d\mu_y$, where $\mu_y(E) = \mu(yE)$, which follows from approximation via simple functions. Thus, if μ is a Haar measure, then $\int L_y f\, d\mu = \int f\, d\mu$ for all $f \in C_c^+(G)$, so it holds for all $f \in C_c(G)$. The measure μ is unique from the Riesz–Markov–Kakutani Representation Theorem. □

Now, our focus turns to the question of establishing the existence and (essential) uniqueness of the Haar measure.

Theorem: Every locally compact group G possesses a left Haar measure λ.

Proof. We will construct λ as a linear functional on $C_c(G)$.

Let $f, \phi \in C_c^+(G)$. We define $(f : \phi)$ to be the infimum of all such finite sums $\sum_{j=1}^{n} c_j$ such that

$$f \leqslant \sum_{j=1}^{n} c_j L_{x_j} \phi$$

for some $x_1, \dots, x_n \in G$. Such a value necessarily exists as supp(f) can be covered by some finite number of translates of $\phi^{-1}(1/2\|\phi\|_u, \infty)$, meaning that $(f : \phi) \leqslant 2N\|f\|_u/\|\phi\|_u$. We see the following:

(i) $(f : \phi) = \left(L_y f : \phi\right)$;

(ii) $(f_1 + f_2 : \phi) \leqslant (f_1 : \phi) + (f_2 : \phi)$;

(iii) $(cf : \phi) = c(f : \phi)$ for any $c \geqslant 0$;

(iv) $(f_1 : \phi) \leqslant (f_2 : \phi)$ whenever $f_1 \leqslant f_2$;

(v) $(f : \phi) \geqslant \|f\|_u/\|\phi\|_u$;

(vi) $(f : \phi) \leqslant (f : \psi)(\psi : \phi)$ for any $\psi \in C_c^+(G)$.

To see (vi), notice that if $f \leqslant \sum_{i=1}^{n} c_i L_{x_i} \phi$ and $\psi \leqslant \sum_{j=1}^{m} b_j L_{y_j} \phi$, then $f \leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} c_i b_j L_{x_j y_j} \phi$.

We fix a function $f_0 \in C_c^+(G)$, and define

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)}.$$

This functional is left-invariant, subadditive, homogeneous, and monotone, and also satisfies

$$(f_0 : f)^{-1} \leqslant I_\phi(f) \leqslant (f : f_0).$$

Now, I_ϕ is not necessarily additive, but on a neighborhood it is very close to being so.

Lemma: If $f_1, f_2 \in C_c^+(G)$, and $\varepsilon > 0$, then there is a neighborhood V of 1_G such that $I_\phi(f_1) + I_\phi(f_2) \leqslant I_\phi(f_1 + f_2) + \varepsilon$ whenever $\operatorname{supp}(\phi) \subseteq V$.

Proof of Lemma. Fix $g \in C_c^+(G)$ such that $g = 1$ on $\operatorname{supp}(f_1 + f_2)$, and let δ be a (to be determined) positive number. Let $h = f_1 + f_2 + \delta g$, and let $h_i = f_i/h$ for each i; note that $h_i = 0$ whenever $f_i = 0$.

Then, we see that $h_i \in C_c^+(G)$, so there is a neighborhood V of 1_G such that $|h_i(x) - h_i(y)| < \delta$ for each i and all y such that $y^{-1}x \in V$.

Suppose $\phi \in C_c^+(G)$ and $\operatorname{supp}(\phi) \subseteq V$. If $h \leqslant \sum_{j=1}^{n} c_j L_{x_j} \phi$, then

$$f_i(x) = h(x)h_i(x)$$

$$\begin{aligned}
&\leqslant \sum_{j=1}^{m} c_j \phi\left(x_j^{-1}x\right)h_i(x)\\
&\leqslant \sum_{j=1}^{m} c_j \phi\left(x_j^{-1}x\right)\left(h_i\left(x_j\right)+\delta\right),
\end{aligned}$$

since $\left|h_i(x) - h_i\left(x_j\right)\right| < \delta$ whenever $x_j^{-1}x \in \operatorname{supp}(\phi)$. Since $h_1 + h_2 \leqslant 1$, we have

$$\begin{aligned}
(f_1 : \phi) + (f_2 : \phi) &\leqslant \sum_{j=1}^{m} c_j\left(h_1\left(x_j\right)+\delta\right) + \sum_{j=1}^{m} c_j\left(h_2\left(x_j\right)+\delta\right)\\
&\leqslant \sum_{j=1}^{m} c_j(1+2\delta).
\end{aligned}$$

Taking the infimum of all such sums, we have

$$\begin{aligned}
I_\phi(f_1) + I_\phi(f_2) &\leqslant (1+2\delta)I_\phi(h)\\
&\leqslant (1+2\delta)\left(I_\phi(f_1+f_2) + \delta I_\phi(g)\right)
\end{aligned}$$

Thus, by taking δ small enough such that

$$2\delta(f_1 + f_2 : f_0) + \delta(1+2\delta)(g : f_0) < \varepsilon,$$

we obtain our desired result. □

Now, for each $f \in C_c^+(G)$, let X_f be the interval $\left[(f_0 : f)^{-1}, (f : f_0)\right]$, and let X be the Cartesian product of all such X_f. We see that X is a compact Hausdorff space consisting of functions from $C_c^+(G)$ into $(0,\infty)$ with the value at f equal to X_f. Thus, $I_\phi(f) \in X_f$ for each $\phi \in C_c^+(G)$ via the established bound.

Now, for each $V \in \mathcal{N}_{1_G}$, we let $K(V)$ be the closure in X of the set $\left\{I_\phi \;\middle|\; \operatorname{supp}(\phi) \subseteq V\right\}$. The sets $K(V)$ have the finite intersection property, as

$$\bigcap_{j=1}^{n} K\left(V_j\right) \supseteq K\left(\bigcap_{j=1}^{n} V_j\right),$$

so by compactness, there is $I \in X$ such that I is in every $K(V)$. This means that every neighborhood of I contains a I_ϕ with $\operatorname{supp}(\phi)$ arbitrarily small, so for every neighborhood V of 1_G, any $\varepsilon > 0$, and any $f_1, \dots, f_n \in C_c^+(G)$, there is $\phi \in C_c^+(G)$ such that $\operatorname{supp}(\phi) \subseteq V$, $\left|I\left(f_j\right) - I_\phi\left(f_j\right)\right| < \varepsilon$ for all j, meaning that I commutes with left translation, addition, and multiplication by positive scalars.

Any $f \in C_c(G)$ can be written as $f = (g_1 - h_1) + i(g_2 - h_2)$ where each of g_i, h_i are positive. We thus extend $I(f) = (I(g_1) - I(h_1)) + i(I(g_2) - I(h_2))$, which gives a nonzero positive linear functional on $C_c(G)$, so by the Riesz–Markov–Kakutani Representation Theorem, we have a measure λ associated to I. □

Proposition: If λ is a left Haar measure, and U is a nonempty open set, then $\lambda(U) > 0$, and $\int f\, d\lambda > 0$ for all $f \in C_c^+(G)$.

Proof. Let U be nonempty, open, and $\lambda(U) = 0$. Then, $\lambda(xU) = 0$ for all x, and since any compact set K can be covered by finitely many such translates, $\lambda(K) = 0$ for each compact set K, meaning that $\lambda(G) = 0$ by inner regularity, contradicting λ being nonzero.

Now, if $f \in C_c^+(G)$, then $U = \left\{x \;\middle|\; f(x) > \frac{1}{2}\|f\|_u\right\}$ is open, and $\int f\, d\lambda > \frac{1}{2}\|f\|_u \lambda(U) > 0$. □