

Revised Problems

Problem (Homework 5, Problem 1): If X is a connected space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy-equivalent to a wedge sum of 1-spheres and 2-spheres.

Solution: Let A_1, \dots, A_n be the spheres in X ; we will define a graph Γ where $V(\Gamma)$ denotes each of the spheres of X and $E(\Gamma)$ is the edge set. Note that by our assumption, we have that $V(\Gamma)$ is a simple and connected graph.

First, we show that if Γ is a tree, then X is homotopy-equivalent to a wedge sum of 2-spheres. First, assign Γ a distinguished root vertex. Endow X with a CW complex structure by assigning

- 0-cells at each intersection point;
- extra 0-cells at each leaf and at the root of the tree, the latter of which we will denote a_0 ;
- 1-cells along the equator connecting all the 0-cells;
- 2-cells completing the respective spheres.

Since Γ is a tree, there is a unique path from the root of the vertex to each leaf. Traversing along a path from a_0 to the extra 0-cell on the leaf via 1-cells connecting to each intersection point on the intermediate spheres, we obtain a contractible subcomplex of X . Upon taking the quotient, we find that X is homotopy-equivalent to a wedge sum of all the spheres along this path (including the leaf) with the rest of X with connection point at a_0 . Continuing in this fashion then gives that, if Γ is a tree, then X is homotopy-equivalent to a wedge sum of spheres.

If Γ is not a tree, then Γ admits spanning tree, and the rest of Γ is then given by a collection of edges that complete some number of cycles. Therefore, we show that if Γ is a cycle, then X is homotopy-equivalent to a wedge of 2-spheres (corresponding to each vertex) and a single 1-sphere. For this, observe that if Γ is a cycle, then X is homotopy-equivalent to a line of spheres connected by 0-cells at their equator with one extra 1-cell connecting 0-cells at the endpoints of the line. Collapsing along this equator will then give all the 2-spheres as a wedge sum identified with a single point, as well as both ends of the extra 1-cell, meaning that we have that, in this case, X is a wedge sum of these 2-spheres with the 1-sphere corresponding to the extra 1-cell.

Therefore, in the general case, we may start by collapsing X along its spanning tree, which will necessarily collapse along every cycle, giving that X is a wedge sum of 1-spheres and 2-spheres.