Problem (Problem 1)**:** Given $z = x + iy \in \mathbb{C}$, define

$$z^* = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

(a) Show that $z^* \in S^2$.

(b) Prove that if $(x_1, x_2, x_3) \in S^2 \setminus \{(0,0,1)\}$, then there exists a unique $z \in \mathbb{C}$ such that $z^* = (x_1, x_2, x_3)$.

(c) A circle in S^2 is the intersection of a plane in $\mathbb{R}^3$ with S^2, provided this intersection is nonempty. Prove that if C is a circle in S^2, then there exists a set $\widetilde{C} \subseteq \mathbb{C}$ that is either a circle or a straight line such that $C \setminus \{(0,0,1)\} = \left\{ z^* \in \mathbb{R}^3 \,\middle|\, z \in \widetilde{C} \right\}$.

Solution:

(a) Via brute force calculation, we see that

$$\begin{aligned}
\frac{4x^2}{(x^2+y^2+1)^2} + \frac{4y^2}{(x^2+y^2+1)^2} + \frac{(x^2+y^2-1)^2}{(x^2+y^2+1)^2} &= \frac{(x^2+y^2)^2 + 1 - 2(x^2+y^2) + 4(x^2+y^2)}{(x^2+y^2+1)^2} \\
&= \frac{(x^2+y^2)^2 + 1 + 2(x^2+y^2)}{(x^2+y^2+1)^2} \\
&= 1.
\end{aligned}$$

(b) Let $z^* = (x_1, x_2, x_3) \in S^2 \setminus \{(0,0,1)\}$, and let $L\colon [0,\infty) \to \mathbb{R}^3$ be the line parametrized such that $L(1) = (x_1, x_2, x_3)$ and $L(0) = (0,0,1)$, which is given by

$$L(t) = (tx_1, tx_2, tx_3 + (1-t)).$$

Note then that $\|L(t)\| = 1$ only when $t = 0$ or $t = 1$, meaning that $L(t)$ intersects $S^2 \setminus \{(0,0,1)\}$ exactly once. By identifying $\mathbb{C}$ with $x + iy \mapsto (x, y, 0)$, we may find $z \in \mathbb{C}$ that uniquely maps to (x_1, x_2, x_3) under the z^* identification by taking

$$\begin{aligned}
tx_3 + (1-t) &= 0 \\
1 + t(x_3 - 1) &= 0 \\
t &= \frac{1}{1-x_3},
\end{aligned}$$

so that

$$x + iy = \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3}$$

maps to z^* under the given identification.

(c) Let $(x_1, x_2, x_3) \in S^2$ lie on the plane $ax_1 + bx_2 + cx_3 = d$. By substituting $z = x + iy \mapsto z^*$, we get

$$a\frac{2x}{x^2+y^2+1} + b\frac{2y}{x^2+y^2+1} + c\frac{x^2+y^2-1}{x^2+y^2+1} = d$$

$$\begin{aligned} 2ax + 2by + c\left(x^2 + y^2 - 1\right) &= d\left(x^2 + y^2 + 1\right) \\ (c-d)x^2 + 2ax + (c-d)y^2 + 2by &= c + d. \end{aligned}$$

This gives two cases. If $c = d$, then we get the line

$$ax + by = c.$$

Else, if $c \neq d$, we get the circle

$$\begin{aligned} x^2 + \frac{2a}{c-d}x + y^2 + \frac{2b}{c-d}y &= \frac{c+d}{c-d} \\ \left(x - \frac{a}{c-d}\right)^2 + \left(y - \frac{b}{c-d}\right)^2 &= \frac{a^2+b^2+c^2-d^2}{(c-d)^2}. \end{aligned}$$

Thus, circles in S^2 correspond to either circles or lines in $\mathbb{C}$.

Problem (Problem 2)**:** Define $f\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $f(z) = \left(\frac{z+1}{z-1}\right)^2$.

(a) Is f injective on $\mathbb{D}$? Why or why not?

(b) Determine $f(\mathbb{D})$.

Solution:

(a) We consider $q(z) = \frac{z+1}{z-1}$ as a fractional linear transformation on $\hat{\mathbb{C}}$. We see that

$$\begin{aligned} q\left(e^{i\theta}\right) &= \frac{e^{i\theta}+1}{e^{i\theta}-1} \\ &= \frac{(1+\cos(\theta)) + i\sin(\theta)}{(\cos(\theta)-1) + i\sin(\theta)} \\ &= \frac{((\cos(\theta)+1) + i\sin(\theta))((\cos(\theta)-1) - i\sin(\theta))}{(1-\cos(\theta))^2 + \sin^2(\theta)} \\ &= \frac{\left(\cos^2(\theta)-1\right) + \sin^2(\theta) + i\sin(\theta)(\cos(\theta) - 1 - (\cos(\theta)+1))}{2 - 2\cos(\theta)} \\ &= i\frac{\sin(\theta)}{\cos(\theta)-1}, \end{aligned}$$

and since $\frac{\sin(\theta)}{\cos(\theta)-1}$ maps $(0, 2\pi)$ to $\mathbb{R}$ bijectively, we see that q maps $S^1 \setminus \{1\}$ into the imaginary axis. We also see that $q(0) = -1$, so by existence/uniqueness of Möbius transformations, we see that q maps $\mathbb{D}$ bijectively onto $\mathbb{L} = \{z \mid \operatorname{Re}(z) < 0\}$.

Now, notice that the function $h(z) = z^2$ is injective when defined on $\mathbb{L}$, as the arguments $(\pi/2, 3\pi/2)$ map injectively to $(\pi, 3\pi)$, and the function $|z|^2$ is injective on $(0, \infty)$, so $f = h \circ q$ is injective on $\mathbb{D}$.

(b) Since $f = h \circ q$, where q maps $\mathbb{D}$ to the left half-plane, and h maps the left half-plane to the full complex plane save for $(-\infty, 0]$, as the imaginary axis is excluded from $q(\mathbb{D})$, and so we have that f maps $\mathbb{D}$ to $\mathbb{C} \setminus (-\infty, 0]$.

Problem (Problem 3)**:** Prove that there exists a linear fractional transformation that maps the first quadrant in $\mathbb{C}$ bijectively to the top half of the unit disc, and satisfies $f(2) = i$.

Solution: We start from the Cayley transform,

$$f_1(z) = \frac{z-i}{z+i},$$

which bijectively maps the upper half-plane to the unit disc. By taking $z = x + iy$ for $x, y > 0$, we see that

$$f_1(x+iy) = \frac{1}{x^2 + (y+1)^2}\left(\left(x^2 + y^2 - 1\right) + i(-2x)\right),$$

implying that the first quadrant is mapped to the *lower* half of the unit disc. Therefore, we flip about the origin by taking $f_2(z) = -f_1(z)$, so that

$$f_2(z) = -\frac{z-i}{z+i},$$

which maps the first quadrant of the upper half plane to the top half of the unit disc. Next, we see that

$$\begin{aligned} f_2(1) &= -\frac{1-i}{1+i} \\ &= i, \end{aligned}$$

so to ensure that $f(2) = i$, we may define $f(z) = f_2(z/2)$, or

$$f(z) = \frac{-z+2i}{z+2i}.$$

Problem (Problem 4)**:** Let $f\colon \mathbb{C} \to \mathbb{C}$ be a function. We say that $\lim_{z\to\infty} f(z) = \infty$ if, for all $M > 0$, there exists $R > 0$ such that $|f(z)| > M$ whenever $|z| > R$.

(a) Show that if $f\colon \mathbb{C} \to \mathbb{C}$ is a nonconstant polynomial, then $\lim_{z\to\infty} f(z) = \infty$.

(b) Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying $\lim_{z\to\infty} f(z) = \infty$. Show that there exists some $z_0 \in \mathbb{C}$ for which $|f(z_0)| = \inf_{z\in\mathbb{C}}|f(z)|$.

Solution:

(a) If $f(z) = \sum_{k=0}^{n} a_k z^k$, with $n > 1$ and $a_n \neq 0$, then by a corollary of the triangle inequality, we see that

$$\begin{aligned} |f(z)| &= \left|\sum_{k=0}^{n} a_k z^k\right| \\ &\geqslant |a_n z^n| - \sum_{k=0}^{n-1}\left|a_k z^k\right|. \end{aligned}$$

Now, we notice a few things. First, since $|a_n|$ is nonzero, we may divide by $|a_n|$, giving

$$\frac{1}{|a_n|}|f(z)| \geqslant |z|^n - \frac{1}{|a_n|}\sum_{k=0}^{n-1}|a_k||z|^k.$$

Now, from real analysis, we know that

$$\lim_{|z|\to\infty} |z|^n = \infty,$$

as we may select $R = M^{1/n}$ to achieve this purpose. So, by using the limit comparison test, we see that

$$\lim_{|z|\to\infty} \frac{|z|^n - \sum_{k=0}^{n-1} |a_k/a_n||z|^k}{|z|^n} = 1,$$

so

$$\lim_{|z|\to\infty} \frac{1}{|a_n|}|f(z)| = \infty,$$

so

$$\lim_{z\to\infty} |f(z)| = \infty.$$

(b) Let $M > 0$ be sufficiently large such that the set $\{z \in \mathbb{C} \mid |f(z)| \leqslant M\}$ is not empty. Since $\lim_{z\to\infty} f(z) = \infty$, there exists R such that $|f(z)| > M$ whenever $|z| > R$.

We see that on $B(0, R)$, the closed disk of radius R centered at 0, the function f is continuous, and so is the function $|f(z)|$, as the modulus is also a continuous function. Since $B(0, R)$ is compact, there is some $z_0 \in B(0, R)$ such that $|f(z_0)| = \inf_{z_0 \in B(0,R)} |f(z)|$. In particular, we note that $|f(z_0)| \leqslant M$, as we have specifically selected M to be such that $\{z \in \mathbb{C} \mid |f(z)| \leqslant M\}$ is nonempty, meaning that $|f(z_0)| = \inf_{z\in\mathbb{C}} |f(z)|$, as we have selected R such that $|f(z)| > M$ for all $z \in \mathbb{C} \setminus B(0, R)$.