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Problem (Problem 1)**:** Describe the topology of the Grassmanian $\mathrm{Gr}(k,n)$ in a uniform way, so that $\mathbb{RP}^n$ becomes the special case of $\mathrm{Gr}(1,n)$.

Solution: We let elements of $\mathrm{Gr}(k,n)$ be defined as equivalence classes of linearly independent k-tuples of vectors in $\mathbb{R}^n$, where $(v_1,\dots,v_k) \sim (w_1,\dots,w_k)$ if $\operatorname{span}\{v_1,\dots,v_k\} = \operatorname{span}\{w_1,\dots,w_k\}$.

By extending $(v_1,\dots,v_k)$ and $(w_1,\dots,w_k)$ to ordered bases $\mathcal{B}_1 = (v_1,\dots,v_n)$ and $\mathcal{B}_2 = (w_1,\dots,w_n)$, we see that these k-tuples are equivalent if and only if there is an invertible linear transformation Q with matrix representation

$$Q = \begin{pmatrix} A & H \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n-k)\times(n-k)$ invertible matrix, so that

$$Q[v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n] = \left[w_1, \dots, w_k, w_{k+1}^*, \dots, w_n^*\right],$$

where $\left\{w_{k+1}^*, \dots, w_n^*\right\}$ is a basis for the $n-k$-dimensional complementary subspace. The subgroup of all such $Q \subseteq \mathrm{GL}_n(\mathbb{R})$, which we call P, is the stabilizer of $\mathrm{Gr}(k,n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $\mathrm{GL}_n(\mathbb{R})$ acts transitively on all ordered bases of $\mathbb{R}^n$), we obtain $\mathrm{Gr}(k,n) \cong \mathrm{GL}_n(\mathbb{R})/P$, where the latter coset space is given the quotient topology.

This definition comports with the definition of $\mathbb{RP}^n$ as the space of one-dimensional subspaces, as the invertible 1×1 matrices are precisely the nonzero scalars, so the stabilizers in the case of $\mathrm{Gr}(1,n)$ are the 1×1 invertible block matrices A, or the nonzero scalars.

Problem (Problem 2)**:** Fix an inner product on $\mathbb{R}^n$. Show that the map $V \mapsto V^\perp$ induces a C^∞ diffeomorphism $\mathrm{Gr}(k,n) \to \mathrm{Gr}(n-k,n)$.

Solution: Due to the inner product, we make the identification $v \mapsto v^*$ such that $v^*(w) = \langle v, w\rangle$. In particular, we have isomorphisms $V \cong V^*$ and $V^\perp \cong (V^\perp)^*$. Therefore, given an element $T \in \operatorname{Hom}(V, V^\perp)$, dualization gives the transpose map $T^* \in \operatorname{Hom}\left((V^\perp)^*, V^*\right)$.

Now, given any chart (U_V, φ_V) in $\mathrm{Gr}(k,n)$, we identify $T \in \operatorname{Hom}(V, V^\perp) \cong U_V$ to $T^* \in \operatorname{Hom}\left((V^\perp)^*, V^*\right) \cong U_{V^\perp}$, and identify subspaces $W \in U_V$ with their annihilators

$$W^0 = \{w^* \in (\mathbb{R}^n)^* \mid w^*(v) = 0 \text{ for all } v \in W\},$$

so that $W^0 \cap V^* = 0$. Finally, we define $\varphi_{V^\perp}$ by

$$\varphi_{V^\perp} = P_{V^*} \circ P_{(V^\perp)^*}|_{W^0}^{-1}.$$

Since every $W \in \mathrm{Gr}(k,n)$ has a unique annihilator subspace $W^0 \in \mathrm{Gr}(n-k,n)$, we have the series of bijective correspondences

$$\begin{aligned} \operatorname{Hom}\left(V, V^\perp\right) &\xleftrightarrow{\varphi_V} U_V \\ &\xleftrightarrow{W \leftrightarrow W^0} U_{V^\perp} \\ &\xleftrightarrow{\varphi_{V^\perp}} \operatorname{Hom}\left(\left(V^\perp\right)^*, V^*\right) \\ &\xleftrightarrow{\langle\cdot,\cdot\rangle} \operatorname{Hom}\left(V^\perp, V\right), \end{aligned}$$

meaning that this identification is a C^∞ diffeomorphism.

Problem (Problem 3)**:** Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Solution: Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^k map that is a C^1 diffeomorphism. In order to show that f is a C^k diffeomorphism, we need to show that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ exists and is of class C^k.

First, by the inverse function theorem, since f is a C^1 diffeomorphism, we see that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ exists, is continuous, and is such that $D(f^{-1})$ is continuous.

Now, we observe that the association $y \mapsto D_y(f^{-1})$ can be written as

$$y \mapsto f^{-1}(y) \mapsto D_y f\left(f^{-1}(y)\right) \mapsto \left(D_y f\left(f^{-1}(y)\right)\right)^{-1} = D_y\left(f^{-1}\right),$$

where we observe that f^{-1} is of class C^1, the derivative $D_y f$ is of class C^{k-1}, and matrix inversion is C^∞; since $D(f^{-1})$ is a composition of C^1 functions, $D(f^{-1})$ is C^1, so f^{-1} is C^2. Inductively, we see that f^{-1} is also of class C^k, so f is a C^k diffeomorphism.

Problem (Problem 4)**:** Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either $\mathbb{R}$ or S^1, depending on whether it is compact or not.

Remark: The following solution is pieced together from David Gale's "Classification of 1-Manifolds: A Take-Home Exam" published in the American Mathematical Monthly in February 1987.

Solution: Let M be a connected, paracompact manifold with dimension 1, with corresponding atlas $\{(U_i, \varphi_i)\}_{i\in I}$, where $\varphi_i(U_i) = (0, 1)$.

Observe that, for the case of two charts (U_1, φ_1) and (U_2, φ_2) that cover M (without loss of generality, their symmetric difference is nonempty), the intersection $U_1 \cap U_2$ has at least one connected component. We claim that if W is one of these connected components, then $\varphi_1(W) = (0, a_1)$ or $(b_1, 1)$, and similarly, $\varphi_2(W) = (0, a_2)$ or $(b_2, 1)$. The fact that $\varphi_{1,2}(W)$ are open intervals follows from the fact that W is an open and connected subset of $(0, 1)$. Additionally, for any sequence $(x_n)_n \subseteq W$ converging to $x \in U_1 \triangle U_2$, the continuity of the charts means that $\varphi_{1,2}(x_n) \to \varphi_{1,2}(x)$; this means that the limit points $\varphi_{1,2}(x)$ must be contained outside $\varphi_{1,2}(W)$, meaning that the limit points are contained inside the interval $(0, 1)$, and thus our claim is proven.

Now, assume that $U_1 \cap U_2$ has two connected components, which we call Z_1 and Z_2. We will write an explicit homeomorphism between M and the unit square. By some renaming, we may assume that the coordinate maps φ_1 and φ_2 act as

$$\begin{aligned}
\varphi_1(Z_1) &= (0, a_1)\\
\varphi_1(Z_2) &= (a_2, 1)\\
\varphi_2(Z_1) &= (0, b_1)\\
\varphi_2(Z_2) &= (b_2, 1).
\end{aligned}$$

The existence of these maps follows from the previous paragraph and the fact that $Z_1 \cap Z_2 = \emptyset$.

Let $f\colon [0, 1] \to \mathbb{R}^2$ be a piecewise linear map defined by

$$\begin{aligned}
f(0) &= (0, 0)\\
f(a_1) &= (1, 0)\\
f(a_2) &= (1, 1)\\
f(1) &= (0, 1),
\end{aligned}$$

and let $g\colon [b_1, b_2] \to \mathbb{R}^2$ be a linear map defined by $g(b_1) = (0, 0)$ and $g(b_2) = (0, 1)$. We define the continuous function η from M to the unit square by taking

$$\eta(x) = \begin{cases} f \circ \varphi_1(x) & x \in U_1 \\ g \circ \varphi_2(x) & x \in U_2 \setminus U_1, \end{cases}$$

which is a bijective map between a compact space and a Hausdorff space, meaning that η is a homeomorphism from M to the unit square, so by composing with another homeomorphism, we have that M is homeomorphic to S^1.

Now, if $\{(U_i, \varphi_i)\}_{i \in I}$ is any atlas of a compact, connected 1-manifold M, we have a finite subcover $\{(U_k, \varphi_k)\}_{k=1}^{n}$ that covers M. We may amalgamate any $U_k, U_{k'}$ with nonempty and connected intersection into a chart $\varphi_{k^*}\colon U_k \cup U_{k'} \to (0,1)$ from the fact that $\varphi_k \circ \varphi_{k'}^{-1}\colon \varphi_{k'}(U_k \cap U_{k'}) \hookrightarrow (0,1)$ is a topological embedding, so by taking these amalgamations we reduce to the case of two charts, meaning that M is homeomorphic to S^1.

If M is not compact, then

Problem (Problem 5)**:** In this problem, we prove a weak version of the Whitney Embedding Theorem.

(a) Find a C^∞ function λ on $\mathbb{R}^n$ with values in $[0,1]$ such that λ takes the value 1 on the closed ball $B(0,1)$, and vanishes outside the closed ball $B(0,2)$.

(b) Suppose M is a compact C^k manifold of dimension n. Find a C^k atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that $\varphi_i(U_i)$ contains $B(0,2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0,1))^\circ$.

(c) Let λ_i be defined by $\lambda \circ \varphi_i$ on U_i, and 0 outside U_i. Let $f_i\colon M \to \mathbb{R}^n$ be defined by $\lambda_i \circ \varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Remark: Most of the following solution is pieced together from Morita's *Geometry of Differential Forms*.

Solution: (a) Consider the function $H\colon \mathbb{R} \to \mathbb{R}$ given by

$$H(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leqslant 0, \end{cases}$$

which is a C^∞ function on $\mathbb{R}$, as $e^{-1/t}$ is C^∞ for all $t > 0$, and the derivative is well-defined at $t = 0$. Next, we see that the function

$$G(t) = \frac{H\left(4 - t^2\right)}{H(4 - t^2) + H(t^2 - 1)}$$

takes on the value 1 whenever $-1 \leqslant t \leqslant 1$ and is supported on $[-2,2]$. Furthermore, it is a C^∞ function, as it is a rational function of C^∞ functions where the denominator never takes the value 0. Therefore, if we replace t with $|x|$, when $x \in \mathbb{R}^n$, since the norm is a C^∞ function, we obtain a C^∞ function that is supported on $B(0,2)$ and is equal to 1 on $B(0,1)$, given by

$$\lambda(x) = \frac{H\left(4 - |x|^2\right)}{H\left(4 - |x|^2\right) + H\left(|x|^2 - 1\right)}.$$

(b) Let M be a compact C^k manifold, and let $\{(V_i, \psi_i)\}_{i \in I}$ be a C^k atlas for M, where $\{V_i\}_{i \in I}$ is an open cover, the $\psi_i\colon V_i \to \mathbb{R}^n$ are homeomorphisms, and the $\psi_j \circ \psi_i^{-1}$ are C^k diffeomorphisms.

Since M is compact, we have a finite subcover $\left\{V_j\right\}_{j=1}^{n}$ and an exhaustion by compact subsets via

$$U_j = \bigcup_{k=1}^{j} V_k$$

$$M = \bigcup_{j=1}^{n} U_j,$$

where, without loss of generality, $\overline{U_j} \subsetneq U_{j+1}$.

Now, for each $p \in \overline{U_j} \setminus U_{j-1}$ (define $U_0 = U_1 = \emptyset$), we may find i_p with a corresponding C^k chart (V_{i_p}, ψ_{i_p}), where without loss of generality, $\psi_{i_p}(p) = 0$, and let $W_p = \psi_{i_p}^{-1}(U(0,1))$.

Clearly, $B(0,2) \subseteq \psi_{i_p}(V_{i_p})$, and by finitely enumerating the elements p_{j_k} in $\overline{U_j} \setminus U_{j-1}$, we have an open cover $\left\{W_{p_{j_k}}\right\}_{k=1}^{m} = \left\{\psi_{p_{j_k}}^{-1}(U(0,1))\right\}_{k=1}^{m}$ of M, and $\left\{\left(V_{i_{p_k}}, \psi_{p_k}\right)\right\}_{k=1}^{m}$ are C^k charts such that $B(0,2) \subseteq \psi_{p_k}\left(V_{i_{p_k}}\right)$.

(c)

Problem (Problem 6)**:** Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Solution: Let $\{U_i\}_{i\in I}$ be a locally finite open cover of M, and let $\{(U_i, \varphi_i)\}_{i\in I}$ be the corresponding C^k atlas for M where $B(0,2) \subseteq \varphi_i(U_i)$, and M is covered by $\varphi_i^{-1}(U(0,1))$. Then, we may define

$$f_i = \begin{cases} G \circ \varphi_i & \text{on } U_i \\ 0 & \text{on } U_i^c, \end{cases}$$

where

$$G(x) = \frac{e^{\frac{1}{4-|x|^2}}}{e^{\frac{1}{4-|x|^2}} + e^{\frac{1}{|x|^2-1}}}$$

is a C^∞ function supported on $B(0,2)$ and equal to 1 on $U(0,1)$. Defining

$$f = \sum_{i\in I} f_i,$$

we see that $f \neq 0$ everywhere, as M is covered by the family $\varphi_i^{-1}(U(0,1))$, and since $\{U_i\}_{i\in I}$ is locally finite, f is also C^k as each f_i is the composition of a C^k diffeomorphism and a C^∞ function. The functions

$$g_i = \frac{f_i}{f}$$

are thus smooth, $0 \leqslant g_i \leqslant 1$, and $\sum_{i\in I} g_i = 1$.

Problem (Problem 7)**:** Let X and Y be topological spaces, and let $C(X, Y)$ be the set of continuous maps from X to Y. Equip $C(X, Y)$ with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space and let X be compact. We note that a neighborhood basis in the topology of uniform convergence on $C(X,Y)$ consists of sets of the form

$$U_{f,\varepsilon} = \left\{ g \;\middle|\; \sup_{x\in X} d(f(x),g(x)) < \varepsilon \right\}.$$

Similarly, a subbase for the compact open topology consists of sets of the form

$$U_{f,K,\varepsilon} = \left\{ g \;\middle|\; \sup_{x\in K} d(f(x),g(x)) < \varepsilon \right\};$$

the fact that Y is a metric space allows us to take this refinement of the compact-open topology.

Thus, to prove that the compact-open topology and the topology of uniform convergence are equivalent, we show that any basis element of the topology of uniform convergence is contained in a basis element of the compact-open topology, and vice versa.

First, we see that almost by definition, if $K \subseteq X$ is any compact subset, then

$$U_{f,\varepsilon} \subseteq U_{f,K,\varepsilon},$$

as any function whose supremum distance is less than ε over X must have that supremum distance hold over $K \subseteq X$.

Now, in the reverse direction, we fix f and ε. We wish to show that there is a finite family of subsets U_{K_i,V_i} with $f \in U_{K_i,V_i}$ for each i, and their intersection lies in $U_{f,\varepsilon}$. We see that every point $x \in X$ has a pre-compact open neighborhood U_x such that $f\left(\overline{U_x}\right) \subseteq U(f(x),\varepsilon/3)$, which follows from the fact that compact subsets of Y are bounded. The family $\{U_x \mid x \in X\}$ is an open cover for X, so admits a finite subcover $\{U_{x_i}\}_{i=1}^{n}$. Since each $\left\{\overline{U_{x_i}}\right\}_{i=1}^{n}$ is compact, and for each i, $f \in U_{\overline{U_{x_i}},U(f(x_i),\varepsilon/3)}$, we see that

$$V = \bigcap_{i=1}^{n} U_{\overline{U_{x_i}},U(f(x_i),\varepsilon/3)}$$

is a nonempty open subset in the compact-open topology on $C(X,Y)$ that contains f. Now, for any $g \in V$ and for any $x \in X$, we see that there is some U_{x_j} such that $x \in U_{x_j}$, and since $g \in U_{\overline{U_{x_j}},U(f(x_j),\varepsilon/3)}$, we have that

$$\begin{aligned} d(g(x),f(x)) &\leqslant d\left(g(x),f\left(x_j\right)\right) + d\left(f\left(x_j\right),f(x)\right) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon, \end{aligned}$$

so $V \subseteq U_{f,\varepsilon}$, meaning the topologies are equal.

Problem (Problem 8)**:** Let $C^k(M,N)$ be the set of C^k maps from M to N. The compact-open topology on $C^k(M,N)$ is defined similarly. Let $f \in C^k(M,N)$, (U,φ) and (V,ψ) charts on M and N, let $K \subseteq U$ be compact such that $f(K) \subseteq V$, and let $\varepsilon > 0$. We obtain a basic neighborhood $N(f,U,\varphi,V,\psi,K,\varepsilon)$ by looking at all the maps $g \in C^k(M,N)$ such that $g(K) \subseteq V$, and

$$\left\|D^r\left(\psi f \varphi^{-1}\right)(x) - D^r\left(\psi g \varphi^{-1}\right)(x)\right\|_{op} \leqslant \varepsilon \tag{$*$}$$

for all integers $0 \leqslant r \leqslant k$.

The Whitney topology is slightly different. Let $\Phi = \{(U_i,\varphi_i)\}_{i\in I}$ be a locally finite atlas on M, let $K_i \subseteq U_i$ be compact for all i, let Ψ be an atlas on N, and let $\{\varepsilon_i\}_{i\in I}$ be a family of positive numbers. A basic neighborhood of $f \in C^k(M,N)$ in this topology is given by all g such that $g(K_i) \subseteq V_i$ for all i, and

$$\left\|D^r\left(\psi_i f \varphi_i^{-1}\right)(x) - D^r\left(\psi_i g \varphi_i^{-1}\right)(x)\right\|_{op} \leqslant \varepsilon_i \tag{**}$$

for all $x \in \varphi_i(K_i)$ and all integers $0 \leqslant r \leqslant k$.

For infinite values of k, we take the compact-open and Whitney topologies on $C^\infty(M,N)$ to be the union of these topologies via the inclusion $C^\infty(M,N) \subseteq C^k(M,N)$. Show the following:

(a) these basic neighborhoods actually give a basis for a topology in both cases;

(b) if M is compact, these two topologies coincide;

(c) if M is compact and has no boundary, then the C^k diffeomorphisms from M to N are open in $C^k(M,N)$ in the Whitney topology.

Solution:

(a) Clearly, in both the compact open topology and the Whitney topology, the respective neighborhoods cover $C^k(M,N)$, so we only need to verify the condition that if $X_1, X_2 \subseteq C^k(M,N)$ are open subsets such that $f \in X_1 \cap X_2$, then there is $X_3 \subseteq C^k(M,N)$ open such that $X_3 \subseteq X_1 \cap X_2$.

We start with the case of the compact-open topology. Let $f \in X_1 \cap X_2$, where X_1 and X_2 are open in the compact-open topology. Since $f \in X_1$, there is a chart (U_1, φ_1) of M, a chart (V_1, ψ_1) of N, $K_1 \subseteq U_1$ compact such that $f(K_1) \subseteq V_1$, and $\varepsilon_1 > 0$ such that $(*)$ holds and $N(f, U_1, \varphi_1, V_1, \psi_1, \varepsilon_1) \subseteq X_1$. Similarly, since $f \in X_2$, there are charts (U_2, φ_2) and (V_2, ψ_2) of M and N respectively, $K_2 \subseteq U_2$ compact with $f(K_2) \subseteq V_2$, and $\varepsilon_2 > 0$ such that $(*)$ holds, and $N(f, U_2, \varphi_2, V_2, \psi_2, \varepsilon_2) \subseteq X_2$. Note that by the characterization, $(*)$ holds for the supremum over all $x \in \varphi_j\left(K_j\right)$ for $j = 1, 2$.