

Contents

Introduction	1
Normed Vector Spaces	2
Vector Spaces, Norms, and Basic Properties	2
Examples	3
Series Convergence and Completeness	3
Quotient Spaces	4
Bounded Linear Operators	7
Quotient Maps	11
Pillars of Functional Analysis	15
Baire Category Theorem	15
Open Mapping Theorem	17
Complemented Subspaces and Direct Sums	19
Closed Graph Theorem	20
Uniform Boundedness Principle	23
Hahn–Banach Theorems	26
Duality	34
Completions and the Double Dual	34
Dual Spaces	38
The Banach Space Adjoint	55
Properties of Reflexive Spaces	61
Tensor Products in Vector Spaces	62
Tensor Products of Banach Spaces	64
Injective Tensor Product	64
Projective Tensor Product	68
Finite-Rank and Compact Operators	71
Locally Convex Topological Vector Spaces	80
Topological Vector Spaces	80
Local Convexity	86
Structure of Locally Convex Topological Vector Spaces	87
Examples of Locally Convex Topological Vector Spaces	92
Norms and Metrics on Locally Convex Spaces	97
Hahn–Banach Theorems for Locally Convex Topological Vector Spaces	100
Hahn–Banach Extension Theorems	100
Separation Results	102
Weak Topologies	104

Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and C^* -algebras. This section of notes will be focused on the essential results in functional analysis with regard to normed spaces and topological vector spaces.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

Normed Vector Spaces

Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over \mathbb{C} . Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

Definition (Vector Space). A vector space V is a set closed under two operations

$$\begin{aligned} \alpha : V \times V &\rightarrow V, (v_1, v_2) \mapsto v_1 + v_2 \\ m : \mathbb{C} \times V &\rightarrow V, (\lambda, v) \mapsto \lambda v. \end{aligned}$$

We refer to α as addition, and m as scalar multiplication; $(V, +)$ is an abelian ring.

Definition (Norm). A norm is a function

$$\|\cdot\| : V \rightarrow \mathbb{R}^+, x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness: $\|v\| = 0$ if and only if $v = 0_V$.
- Triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.
- Absolute Homogeneity: $\|\lambda v\| = |\lambda| \|v\|$, for $\lambda \in \mathbb{C}$.

If a function $p : V \rightarrow \mathbb{R}^+$ satisfies the triangle inequality and absolute homogeneity, we say p is a semi-norm.

We say the pair $(V, \|\cdot\|)$ is a normed vector space.

Definition (Balls and Spheres). Let X be a normed vector space, $x \in X$, and $\delta > 0$. Then,

$$\begin{aligned} U(x, \delta) &= \{y \in X \mid d(x, y) < \delta\} \\ B(x, \delta) &= \{y \in X \mid d(x, y) \leq \delta\} \\ S(x, \delta) &= \{y \in X \mid d(x, y) = \delta\}. \end{aligned}$$

For a normed vector space, we will use the following conventions for common sets:

$$\begin{aligned} U_X &= U(0, 1) \\ B_X &= B(0, 1) \\ S_X &= S(0, 1) \\ \mathbb{D} &= U_{\mathbb{C}} \\ \mathbb{T} &= S_{\mathbb{C}}. \end{aligned}$$

Definition (Equivalent Norms). Two norms on V , $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there are two constants C_1 and C_2 such that

$$\begin{aligned} \|v\|_a &\leq C_1 \|v\|_b \\ \|v\|_b &\leq C_2 \|v\|_a \end{aligned}$$

for all $v \in V$. We say $\|\cdot\|_a \sim \|\cdot\|_b$.

Examples

Example (Finite-Dimensional Vector Spaces). The vector space \mathbb{C}^n with the p -norm is denoted ℓ_p^n , where for $p \in [1, \infty]$, the p -norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

In the case with $p = 2$, this gives the traditional Euclidean norm, and with $p = \infty$, this gives the sup norm:

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Example (A Sequence Space). We let $\ell_p = \{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\}$ be the collection of sequences in \mathbb{C} with finite p -norm. Here,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

In the case with $p = \infty$, this gives the sequence space ℓ_∞ , which has norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Example (A Function Space). We let $\ell^\infty(\Omega)$ denote the set of all bounded functions $f : \Omega \rightarrow \mathbb{C}$, equipped with the norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|.$$

If $\Omega = (\Omega, \mathcal{M}, \mu)$ is a measure space, then we let $L^\infty(\Omega)$ be the space of μ -a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_\infty = \text{ess sup}_{x \in \Omega} |f(x)|.$$

Series Convergence and Completeness

Proposition (Criteria for Banach Spaces): Let X be a normed vector space. The following are equivalent:

- (i) X is a Banach space.¹
- (ii) If $(x_k)_k$ is a sequence of vectors such that $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.
- (iii) If $(x_k)_k$ is a sequence in X such that $\|x_k\| < 2^{-k}$, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. To show (i) implies (ii), for $n > m > N$, we have

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^n x_k \right\| \\ &\leq \sum_{k=m+1}^n \|x_k\| \\ &< \epsilon, \end{aligned}$$

implying that s_n is Cauchy, and thus converges since X is complete.

¹Complete normed vector space.

Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let $(x_n)_n$ be a Cauchy sequence in X . We only need construct a convergent subsequence in order to show that $(x_n)_n$ converges.

Chose $n_1 \in \mathbb{N}$ such that for $n, m \geq n_1$, $\|x_m - x_n\| < \frac{1}{2^2}$, and inductively define $n_j > n_{j-1}$ such that $n, m \geq n_j$ implies $\|x_m - x_n\| < \frac{1}{2^{j+1}}$.

Let $y_1 = x_{n_1}$, $y_j = x_{n_j} - x_{n_{j-1}}$. Then,

$$\begin{aligned} \|y_j\| &= \|x_{n_j} - x_{n_{j-1}}\| \\ &< \frac{1}{2^j}, \end{aligned}$$

so $\sum_{j=1}^{\infty} y_j$ converges by our assumption. By telescoping, we see that $\sum_{j=1}^k y_j = x_{n_k}$, so $(x_{n_k})_k$ converges. \square

Quotient Spaces

Let X be a normed vector space. Then, for $E \subseteq X$ a subspace, there is a quotient space X/E with the projection map $\pi : X \rightarrow X/E$, $x \mapsto x + E$. We want to make X/E into a normed space — in order to do this, we use the distance function:

$$\text{dist}_E(x) = \inf_{y \in E} d(x, y),$$

which is uniformly continuous. For E closed, then $\text{dist}_E(x) = 0$ if and only if $x \in E$.

Proposition (Quotient Space Norm): Let X be a normed vector space, and $E \subseteq X$ a subspace. Set

$$\|x + E\|_{X/E} = \text{dist}_E(x).$$

Then,

- (1) $\|\cdot\|_{X/E}$ is a well-defined seminorm on X/E .
- (2) If E is closed, then $\|\cdot\|_{X/E}$ is a norm on X/E .
- (3) $\|x + E\|_{X/E} \leq \|x\|$ for all $x \in X$.
- (4) If E is closed, then $\pi : X \rightarrow X/E$ is Lipschitz.
- (5) If X is a Banach space and E is closed, then X/E is also a Banach space.

Proof.

- (1) We will show that $\|\cdot\|_{X/E}$ is well-defined. If $x + E = x' + E$, $x' - x \in E$, so for every $y \in E$, $x' - x + y \in E$. Thus,

$$\begin{aligned} \|x - y\| &= \|x' - (x' - x + y)\| \\ &\geq \inf_{z \in E} \|x' - z\| \\ &= \|x' + E\|_{X/E}. \end{aligned}$$

Thus, $\|x + E\|_{X/E} \geq \|x' + E\|_{X/E}$, and vice versa.

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and $x \in X$. Then,

$$\|\lambda(x + E)\|_{X/E} = \|\lambda x + E\|_{X/E}$$

$$\begin{aligned}
&= \inf_{y \in E} \|\lambda x - y\| \\
&= |\lambda| \inf_{y \in E} \|x - \lambda^{-1}y\| \\
&= |\lambda| \inf_{y' \in E} \|x - y'\| \\
&= |\lambda| \|x + E\|_{X/E}
\end{aligned}$$

Given $x, x' \in X$ and a fixed $\varepsilon > 0$, we have

$$\|x + E\| + \frac{\varepsilon}{2} > \|x - y\|$$

for some $y \in E$, and

$$\|x' + E\| + \frac{\varepsilon}{2} > \|x' - y'\|$$

for some $y' \in E$. Thus,

$$\begin{aligned}
\|(x + x') - (y + y')\| &\leq \|x - y\| + \|x' - y'\| \\
&< \varepsilon + \|x + E\| + \|x' + E\|.
\end{aligned}$$

Since $y + y' \in E$, we have

$$\begin{aligned}
\|(x + E) + (x' + E)\|_{X/E} &= \|x + x' + E\|_{X/E} \\
&\leq \|(x + x') - (y + y')\| \\
&< \varepsilon + \|x + E\|_{X/E} + \|x' + E\|_{X/E},
\end{aligned}$$

meaning

$$\|(x + E) + (x' + E)\| \leq \|x + E\| + \|x' + E\|.$$

(2) If E is closed, and $\|x + E\| = 0$, then $x \in E$ so $x + E = 0_{X/E}$.

(3) For $x \in X$,

$$\begin{aligned}
\|x + E\|_{X/E} &= \inf_{y \in E} \|x - y\| \\
&\leq \|x\|.
\end{aligned}$$

(4) We have

$$\begin{aligned}
\|(x + E) - (x' + E)\|_{X/E} &= \|x - x' + E\|_{X/E} \\
&\leq \|x - x'\|.
\end{aligned}$$

(5) Let X be complete and $E \subseteq X$ be closed. Let $(x_k + E)_k$ be a sequence in X/E with $\|x_k + E\| < 2^{-k}$. We want to show that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

For each k , since $\|x_k + E\| < 2^{-k}$, there exists $y_k \in E$ such that $\|x_k - y_k\| < 2^{-k}$. Since X is complete, $\sum_{k=1}^{\infty} x_k - y_k$ converges.

Let $(\sum_{k=1}^n x_k - y_k)_n \rightarrow x$ in X . Applying the canonical projection map, π , to both sides, we get

$$\sum_{k=1}^n (x_k + E) = \sum_{k=1}^n \pi(x_k)$$

$$\begin{aligned}
&= \pi \left(\sum_{k=1}^n (x_k - y_k) \right) \\
&\rightarrow \pi(x),
\end{aligned}$$

implying that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

□

Exercise: Consider ℓ_{∞} and its closed subspace c_0 . If $\pi : \ell_{\infty} \rightarrow \ell_{\infty}/c_0$ denotes the canonical quotient map, with $(z_k)_k \in \ell_{\infty}$, show that

$$\|(z_k)_k + c_0\| = \limsup_{k \rightarrow \infty} |z_k|$$

Solution: Let $z = (z_k)_k \in \ell_{\infty}$. We define the distance

$$\text{dist}_{c_0}(z) = \inf_{t \in c_0} |z_k - t_k|.$$

Let $w \in c_c$ be defined by

$$w = (z_1, z_2, \dots, z_{n-1}, 0, 0, \dots).$$

Then,

$$\begin{aligned}
\|z - w\|_{\infty} &= \sup_{k \in \mathbb{N}} |z_k - w_k| \\
&= \sup_{k \geq n} |z_k - w_k|,
\end{aligned}$$

meaning that

$$\text{dist}_{c_c}(z) \leq \sup_{k \geq n} |z_k|.$$

Since $c_0 \supseteq c_c$, we have

$$\begin{aligned}
\text{dist}_{c_0}(z) &\leq \text{dist}_{c_c}(z) \\
&\leq \inf_{n \geq 1} \left(\sup_{k \geq n} |z_k| \right) \\
&= \limsup_{k \rightarrow \infty} |z_k|.
\end{aligned}$$

Now, we show that $\limsup_{k \rightarrow \infty} |z_k| \leq \text{dist}_{c_c}(z)$. Given $\varepsilon > 0$, there exists $w \in c_c$ such that

$$\|z - w\| < \text{dist}_{c_c}(z) + \varepsilon.$$

Additionally, for w that terminates at $n - 1$ (i.e., is equal to 0 for all $k \geq n$), we have

$$\sup_{k \geq n} |z_k - w_k| \leq \sup_{k \in \mathbb{N}} |z_k - w_k|,$$

meaning

$$\begin{aligned}
\limsup_{k \rightarrow \infty} |z_k| &= \inf_{n \geq 1} \left(\sup_{k \geq n} |z_k| \right) \\
&\leq \sup_{k \geq n} |z_k - w_k| \\
&\leq \sup_{k \in \mathbb{N}} |z_k - w_k| \\
&= \|z - w\| \\
&< \text{dist}_{c_c}(z) + \varepsilon,
\end{aligned}$$

implying that

$$\limsup_{k \rightarrow \infty} |z_k| = \text{dist}_{c_c}(z).$$

For $\varepsilon > 0$, let $w \in c_0$ be such that

$$\|z - w\| < \text{dist}_{c_0}(z) + \varepsilon/2.$$

Additionally, let $\lambda \in c_c$ such that $\|\lambda - w\| < \varepsilon/2$. Then, we have

$$\begin{aligned} \text{dist}_{c_0}(z) + \varepsilon &> \|z - \lambda\| + \|\lambda - w\| \\ &\geq \text{dist}_{c_c}(z) + \varepsilon/2 \\ &\geq \limsup_{k \rightarrow \infty} |z_k|. \end{aligned}$$

Thus, $\limsup_{k \rightarrow \infty} |z_k| \leq \text{dist}_{c_0}(z)$, meaning $\limsup_{k \rightarrow \infty} |z_k| = \text{dist}_{c_0}(z)$.

Bounded Linear Operators

Definition (Continuous Functions). A function $f : (X, d_X) \rightarrow (Y, d_Y)$ is called Lipschitz if there is a constant $C > 0$ such that

$$d_Y(f(x), f(x')) \leq C d_X(x, x')$$

for all $x, x' \in X$.

If $C \leq 1$, a Lipschitz map is known as a contraction.

If

$$d_Y(f(x), f(x')) = d_X(x, x')$$

for all $x, x' \in X$, then f is known as an isometry.

Proposition (Categorization of Continuous Linear Maps): Let X and Y be normed vector spaces, and let $T : X \rightarrow Y$ be a linear map. The following are equivalent:

- (i) T is continuous at 0.
- (ii) T is continuous.
- (iii) T is uniformly continuous.
- (iv) T is Lipschitz.
- (v) There exists a constant $C > 0$ such that $\|T(x)\| \leq C \|x\|$ for all $x \in X$.

Definition (Bounded Linear Operator). Let X and Y be normed vector spaces, and let $T : X \rightarrow Y$ be a linear map.

- (1) T is bounded if $T(B_X)$ is bounded in Y . Equivalently, T is bounded if and only if

$$\sup_{x \in B_X} \|T(x)\| < \infty,$$

or that $\exists r > 0$ such that $T(B_X) \subseteq B_Y(0, r)$.

- (2) The operator norm of T is the value

$$\|T\|_{\text{op}} = \sup_{x \in B_X} \|T(x)\|.$$

Lemma: Let $T : X \rightarrow Y$ be a linear map between normed vector spaces. Then,

$$\|T\|_{\text{op}} = \sup_{x \in S_X} \|T(x)\|$$

and for all $x \in X$,

$$\|T(x)\| \leq \|T\|_{\text{op}} \|x\|.$$

Lemma: Let $T : X \rightarrow Y$ be a bounded linear map between normed vector spaces. Then, for any $x \in X$ and $r > 0$,

$$r \|T\|_{\text{op}} \leq \sup_{y \in B(x, r)} \|T(y)\|$$

Proof. Let $C = \sup_{y \in B(x, r)} \|T(y)\|$. If $z \in B(0, r)$, then $z + x, z - x \in B(x, r)$, meaning

$$2T(z) = T(z + x) + T(z - x),$$

so by the triangle inequality, we get

$$\begin{aligned} 2 \|T(z)\| &\leq \|T(z + x)\| + \|T(z - x)\| \\ &\leq 2 \max \{ \|T(z + x)\|, \|T(z - x)\| \} \\ &\leq 2C. \end{aligned}$$

Thus,

$$\|T(z)\| \leq \sup_{y \in B(x, r)} \|T(y)\|,$$

meaning

$$r \|T\|_{\text{op}} \leq \sup_{y \in B(x, r)} \|T(y)\|.$$

□

Remark: For a linear map $T : X \rightarrow Y$, the following are equivalent:

- (1) T is continuous.
- (2) T is bounded.
- (3) $\|T\|_{\text{op}} < \infty$.

Definition. Let X and Y be normed spaces, $T : X \rightarrow Y$ a linear map.

- (1) T is bounded below if there exists C_2 such that $\|T(x)\| \geq C_2 \|x\|$ for all $x \in X$.
- (2) T is bicontinuous if T is bounded and bounded below.

$$C_2 \|x\| \leq \|T(x)\| \leq C_1 \|x\|$$

- (3) T is a bicontinuous isomorphism if T is bijective, linear, and bicontinuous. We say X and Y are bicontinuously isomorphic.
- (4) We say T is an isometric isomorphism if T is bijective, linear, and an isometry.

Example. Let ρ be the continuous surjective wrapping function $\rho : [0, 2\pi] \rightarrow \mathbb{T}$, $\rho(t) = e^{it}$. There is an induced isometry

$$T_\rho : C(\mathbb{T}) \rightarrow C([0, 2\pi]),$$

defined by $T_\rho(f)(t) = f \circ \rho(t) = f(e^{it})$.

The range of T_ρ is $C = \{G \in C([0, 2\pi]) \mid g(0) = g(2\pi)\}$, which means that $C(\mathbb{T})$ and C are isometrically isomorphic Banach spaces.

Proposition: Let X and Y be normed spaces, and $T : X \rightarrow Y$ be a linear map. The following are equivalent.

- (i) T is bicontinuous.
- (ii) $T : X \rightarrow \text{Ran}(T)$ is a linear isomorphism and homeomorphism.

Proof. Let T be bicontinuous. Then, T is linear, injective, and surjective onto $\text{Ran}(T)$. Since T is continuous, T is bounded. Let $S : \text{Ran}(T) \rightarrow X$ be defined by $S(T(x)) = x$. We can see that S is well-defined, since $T : X \rightarrow \text{Ran}(T)$ is surjective, and so has a left inverse. Similarly, since $\|S(T(x))\| = \|x\| \leq \frac{1}{C_2} \|T(x)\|$, S is continuous.

Let $S : \text{Ran}(T) \rightarrow X$ be defined by $S(T(x)) = x$. Since T is continuous, it is bounded, so

$$\|T(x)\| \leq \|T\|_{\text{op}} \|x\|.$$

Since S is bounded,

$$\begin{aligned} \|x\| &= \|S(T(x))\| \\ &= \|S\|_{\text{op}} \|T(x)\|, \end{aligned}$$

$$\text{so } \frac{1}{\|S\|_{\text{op}}} \|x\| \leq \|T(x)\|.$$

□

Corollary: Let X be a vector space with $\|\cdot\|$ and $\|\cdot\|'$ two norms. The following are equivalent:

- (i) The norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.
- (ii) The map $\text{id}_X : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$.

Proposition (Properties of Bounded Linear Operators): Let X, Y, Z be normed spaces, $T : X \rightarrow Y$, $S : X \rightarrow Y$, and $R : Y \rightarrow Z$ be linear maps.

- (1) $\|\alpha T\|_{\text{op}} = |\alpha| \|T\|_{\text{op}}$
- (2) $\|T + S\|_{\text{op}} \leq \|T\|_{\text{op}} + \|S\|_{\text{op}}$
- (3) $\|T\|_{\text{op}} = 0$ if and only if $T = 0$
- (4) $\|R \circ T\|_{\text{op}} \leq \|R\|_{\text{op}} \|T\|_{\text{op}}$
- (5) $\|\text{id}_X\|_{\text{op}} = 1$
- (6) If $E \subseteq X$ is a subspace, then $\|T|_E\|_{\text{op}} \leq \|T\|_{\text{op}}$

Proof. We will prove (4) here. For $x \in B_X$, we have

$$\begin{aligned} \|R \circ T(x)\| &= \|R(T(x))\| \\ &\leq \|R\|_{\text{op}} \|T(x)\| \\ &\leq \|R\|_{\text{op}} \|T\|_{\text{op}}. \end{aligned}$$

Taking the supremum, we obtain $\|R \circ T\|_{\text{op}} \leq \|R\|_{\text{op}} \|T\|_{\text{op}}$.

□

Recall: $\mathcal{L}(X, Y)$ is the set of all linear operators with domain X and codomain Y .

Proposition: Let X and Y be normed spaces.

- (1) The collection $\mathcal{B}(X, Y) = \{T \in \mathcal{L}(X, Y) \mid \|T\|_{\text{op}} < \infty\}$ equipped with the operator norm is a normed space known as the space of bounded linear operators between X and Y .
- (2) If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

(3) The continuous dual space, $X^* = \mathcal{B}(X, \mathbb{C})$ is a Banach space.

Proof. We will prove (2). Let $(T_n)_n$ be Cauchy under $\|\cdot\|_{\text{op}}$. Since Cauchy sequences are bounded, there is some $C > 0$ such that $\|T_n\|_{\text{op}} \leq C$ for all $n \geq 1$. For $x \in X$,

$$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|_{\text{op}} \|x\|,$$

meaning $(T_n(x))_n$ is Cauchy in Y . Since Y is complete, we define

$$T(x) = \lim_{n \rightarrow \infty} T_n(x)$$

in Y . If $x \in B_X$, we have

$$\begin{aligned} \|T(x)\| &= \left\| \lim_{n \rightarrow \infty} T_n(x) \right\| \\ &= \lim_{n \rightarrow \infty} \|T_n(x)\| \\ &\leq \limsup_{n \rightarrow \infty} \|T_n(x)\| \\ &\leq C \|x\|, \end{aligned}$$

meaning $\|T\|_{\text{op}} \leq C$.

Let $\varepsilon > 0$, and $N \in \mathbb{N}$ large such that $n, m \geq N$, $\|T_n - T_m\|_{\text{op}} \leq \varepsilon$. For $x \in B_X$,

$$\begin{aligned} \|T_n(x) - T(x)\| &= \lim_{m \rightarrow \infty} \|T_n(x) - T_m(x)\| \\ &\leq \limsup_{m \rightarrow \infty} \|T_n - T_m\|_{\text{op}} \|x\| \\ &< \varepsilon. \end{aligned}$$

Thus, $\|T - T_n\|_{\text{op}} < \varepsilon$ for all $n \geq N$. □

Definition (Algebras). Let A be an algebra over \mathbb{C} .

- (1) If A admits a norm $\|\cdot\|$ satisfying $\|ab\| \leq \|a\| \|b\|$, then A is a normed algebra. If A is unital, then $\|1_A\| = 1$.
- (2) If A is complete with respect to its norm, then A is called a Banach algebra, and if A is unital, then A is a unital Banach algebra.

Lemma: In a normed algebra A , the map $\cdot : A \times A \rightarrow A$, $(a, b) \mapsto ab$ is continuous.

Proposition: Let X be a normed space. The set of bounded operators $\mathcal{B}(X, X) = \mathcal{B}(X)$ is a unital normed algebra. Moreover, if X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

Proposition: Let A be a unital Banach algebra, $a \in A$. The series

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges absolutely in A . We call $\exp(a)$ the exponential of a .

- (1) $\exp(0) = 1_A$
- (2) If A is commutative, then $\exp(a + b) = \exp(a)\exp(b)$.
- (3) We have $\exp(a) \in GL(A)$ with $\exp(a)^{-1} = \exp(-a)$.
- (4) $\|\exp(a)\| \leq \exp(\|a\|)$.

Quotient Maps

Definition. A map $f : X \rightarrow Y$ is called open if $U \subseteq X$ is open implies $f(U) \subseteq Y$ is open.

Proposition: Let X and Y be normed spaces, $T : X \rightarrow Y$ a linear map. The following are equivalent:

- (i) T is surjective and open.
- (ii) $T(U_X) \subseteq Y$ is open.
- (iii) There exists $\delta > 0$ such that $\delta U_Y \subseteq T(U_X)$.
- (iv) There exists δ such that $\delta B_Y \subseteq T(B_X)$.
- (v) There exists $M > 0$ such that for all $y \in Y$, there exists $x \in X$ with $T(x) = y$ and $\|x\| \leq M \|y\|$.

Proof. To see (i) implies (ii), if T is surjective and open, then it is clear that $T(U_X)$, which is the image of an open set, is open.

To see (ii) implies (iii), if $T(U_X)$ is open, we have $0_Y \in T(U_X)$, so there is some δ such that $U(0, \delta) \subseteq T(U_X)$, meaning $\delta U_Y \subseteq T(U_X)$.

Assuming (iii), we see that $\frac{\delta}{2} B_Y \subseteq \delta U_Y \subseteq T(U_X) \subseteq T(B_X)$.

To see (iv) implies (v), let δ be such that $\delta B_Y \subseteq T(B_X)$, and set $M = \frac{1}{\delta}$. Note that for $y \in Y, y \neq 0$, $\frac{\delta}{\|y\|} y \in \delta B_Y$, meaning $\frac{\delta}{\|y\|} y = T(x)$ for some $x \in B_X$, implying that $T\left(\frac{\|y\|}{\delta} x\right) = y$. Finally, since $x \in B_X$, $\frac{\|y\|}{\delta} \|x\| \leq \frac{1}{\delta} \|y\| = M \|y\|$.

To see (v) implies (i), we can see that T is surjective by the assumption. Let $U \subseteq X$ be open, $y_0 \in T(U)$. Then, there exists x_0 such that $T(x_0) = y_0$, and $\delta > 0$ such that $U(x_0, \delta) \subseteq U$. Note that $U(x_0, \delta) = x_0 + \delta U_X$, so $x_0 + \delta U_X \subseteq U$. Applying T , we get $T(x_0 + \delta U_X) \subseteq T(U)$, or $y_0 + \delta T(U_X) \subseteq T(U)$. By assumption, since given $y \in U_Y$, there exists $x \in X$ such that $\|x\| \leq M \|y\|$, meaning $\|x\| \leq M$, we have $U_Y \subseteq T(M U_X)$. Thus, $\frac{1}{M} U_Y \subseteq T(U_X)$, meaning $y_0 + \frac{\delta}{M} U_Y \subseteq y_0 + \delta T(U_X) \subseteq T(U)$, so $U_Y(y_0, \frac{\delta}{M}) \subseteq T(U)$. \square

Definition. Let X and Y be normed vector spaces.

- (1) A bounded linear map $T : X \rightarrow Y$ that is surjective and open is known as a quotient map.
- (2) If $T(U_X) = U_Y$, then T is called a 1-quotient map.

Exercise: If $T(B_X) = B_Y$, show that $T(U_X) = U_Y$.

Solution: Since $T(B_X) = B_Y$, it is the case that $(T(B_X))^\circ = B_Y^\circ$. Since T is an open map, T is continuous, meaning $(T(B_X))^\circ \subseteq T(B_Y^\circ)$. Thus, $T(U_X) = U_Y$.

Proposition: Let X and Y be normed vector spaces with $T : X \rightarrow Y$ a quotient map. If X is a Banach space, then Y is a Banach space.

Proof. We will show that Y is complete by showing that an absolutely convergent series converges.

Let $(y_k)_k$ be a sequence in Y with $\sum_{k=1}^{\infty} \|y_k\| < \infty$. Since T is a quotient map, there is a universal $M > 0$ such that for all k , there is $x_k \in X$ such that $T(x_k) = y_k$ and $\|x_k\| \leq M \|y_k\|$. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} \|x_k\| &\leq M \sum_{k=1}^{\infty} \|y_k\| \\ &< \infty. \end{aligned}$$

Since X is complete, $\sum_{k=1}^{\infty} x_k$ converges. Let $\sum_{k=1}^{\infty} x_k = x$. Then, $(T(\sum_{k=1}^n x_k))_n \xrightarrow{n \rightarrow \infty} T(x)$, meaning $\sum_{k=1}^{\infty} y_k = T(x)$. Thus, $\sum_{k=1}^{\infty} y_k$ converges in Y , so Y is a Banach space. \square

Proposition: Let X be a normed vector space, $E \subseteq X$ a closed subspace. The canonical quotient map, $\pi : X \rightarrow X/E$ is a 1-quotient map.

Proof. We know that $\|\pi(x)\| \leq \|x\|$, meaning $\pi(U_X) \subseteq U_{X/E}$.

Let $\pi(x) = x + E \subseteq U_{X/E}$. Then, $\inf_{y \in E} \|x - y\| \leq 1$, meaning there exists some y such that $\|x - y\| < 1$, meaning $\pi(x - y) = \pi(x)$. \square

Corollary: If X is a Banach space, $E \subseteq X$ a closed subspace, then X/E is a Banach space.

Corollary: Let X be a normed vector space and $E \subseteq X$ be closed. If two of $X, E, X/E$ are complete, the third is also complete.

Proof. We have shown that if X is complete, then E is necessarily complete (since E is closed) and X/E is complete as shown above.

Let E and X/E be complete. We now want to show that X is complete. Let $(x_k)_k$ be Cauchy in X .

For each k , let $x_k = s_k + y_k$, where $y_k \in E$ and $s_k + E = \pi(x_k)$. Notice that, since x_k is Cauchy, so too is s_k , as $\|s_k\| \leq \|x_k\|$ for all k . Additionally, for $m, n \geq N$, we have

$$\begin{aligned} \|x_m - x_n\| &= \|s_m + y_m - (s_n + y_n)\| \\ &\leq \|s_m - s_n\| + \|y_m - y_n\| \\ &< \varepsilon, \end{aligned}$$

implying that $(y_k)_k$ is Cauchy in E . Since X/E and E are complete, we define $x = \lim_{k \rightarrow \infty} s_k + \lim_{k \rightarrow \infty} y_k$. Finally, for $m, n \geq N$, we have

$$\begin{aligned} \|x - x_n\| &= \lim_{m \rightarrow \infty} \|x_m - x_n\| \\ &\leq \varepsilon, \end{aligned}$$

meaning $(x_k)_k \xrightarrow{k \rightarrow \infty} x$, so X is complete. \square

Proposition: Let X and Y be normed spaces, $E \subseteq X$ a closed subspace, and $T : X \rightarrow Y$ bounded linear with $E \subseteq \ker(T)$. Then, there exists a unique bounded linear map $\bar{T} : X/E \rightarrow Y$ such that $\bar{T} \circ \pi = T$. Moreover, \bar{T} is injective if and only if $E = \ker(T)$ and $\|\bar{T}\| = \|T\|$.

Proof. The existence and uniqueness of $\bar{T} : X/E \rightarrow Y$ such that $\bar{T} \circ \pi = T$ follows from the First Isomorphism Theorem for vector spaces, as does the fact that \bar{T} is injective and only if $\ker(T) = E$.

Let $x + E \in X/E$. For $y \in E$, we have

$$\begin{aligned} \|\bar{T}(x + E)\| &= \|\bar{T}(x - y + E)\| \\ &= \|T(x - y)\| \\ &\leq \|T\| \|x - y\|. \end{aligned}$$

Taking infimum over all $y \in E$, we get $\|\bar{T}(x + E)\| \leq \|T\| \|x + E\|$, meaning $\|\bar{T}\| \leq \|T\|$. Additionally,

$$\begin{aligned} \|T\| &= \|\bar{T} \circ \pi\| \\ &\leq \|\bar{T}\| \|\pi\| \\ &= \|\bar{T}\|. \end{aligned}$$

\square

Theorem (First Isomorphism Theorem for Normed Vector Spaces): Let X and Y be normed vector spaces, $T \in \mathcal{B}(X, Y)$.

(1) T is a quotient map if and only if $\bar{T} : X/\ker(T) \rightarrow Y$ is a bicontinuous isomorphism.

(2) T is a 1-quotient map if and only if $\bar{T} : X/\ker(T) \rightarrow Y$ is an isometric isomorphism.

Proof.

(1) Let $\bar{T} : X/\ker(T) \rightarrow Y$ be a bicontinuous isomorphism. Since \bar{T} is bicontinuous, it is a homeomorphism, meaning it is open and surjective. Since π is a quotient map, so too is $T : \bar{T} \circ \pi$.

Suppose T is a quotient map. Then, T is surjective, meaning \bar{T} is an isomorphism. Since T is bounded below, \bar{T} is also bounded. Let $\pi(x) = x + \ker(T) \in X/\ker(T)$, with $T(x) = y$. Let M be such that $\|x\| \leq M \|y\|$. There is an $x' \in X$ with $T(x') = y$, and $\|x'\| \leq M \|y\|$. Thus, $x - x' \in \ker(T)$, so $\pi(x) = \pi(x')$, meaning

$$\begin{aligned} \|\bar{T} \circ \pi(x)\| &= \|T \circ \pi(x')\| \\ &= \|y\| \\ &\geq M^{-1} \|x'\| \\ &\geq M^{-1} \|\pi(x')\| \\ &= M^{-1} \|\pi(x)\|, \end{aligned}$$

meaning T is bounded below.

(2) Suppose $\bar{T} : X/\ker(T) \rightarrow Y$ is an isometric isomorphism. Then, \bar{T} is a 1-quotient map, and since π is a 1-quotient map, so too is $T = \bar{T} \circ \pi$.

Suppose T is a 1-quotient map. Since T is surjective, \bar{T} is an isomorphism. Since T is a 1-quotient map, $\|T\| = \sup_{x \in U_X} \|T(x)\| \leq 1$, meaning $\|\bar{T}\| \leq \|T\| \leq 1$. Consider $S = (\bar{T})^{-1} : Y \rightarrow X/\ker(T)$; S is also an isomorphism, so $S \circ \bar{T} = \text{id}_{X/\ker(T)}$. We will now show S is a contraction, meaning \bar{T} is an isometry.

Let $y \in U_Y$. Since T is a 1-quotient map, there exists $x \in U_X$ such that $T(x) = y$. Then, $\bar{T}(x + \ker(T)) = T(x) = y$, meaning $S(y) = x + \ker(T)$, and

$$\begin{aligned} \|S(y)\| &= \|x + \ker(T)\| \\ &\leq \|x\| \\ &\leq 1, \end{aligned}$$

meaning $\|S\| \leq 1$.

□

Proposition: Every separable Banach space is isometrically isomorphic to a quotient of ℓ_1 .

Proof. Let X be a separable Banach space. Since X is separable, so too is S_X . Let $(z_n)_n$ be norm-dense in S_X , and define

$$\begin{aligned} T : \ell_1 &\rightarrow X \\ (\lambda_n)_n &\rightarrow \sum_{n=1}^{\infty} \lambda_n z_n. \end{aligned}$$

This series converges absolutely:

$$\sum_{n=1}^{\infty} \|\lambda_n z_n\| = \sum_{n=1}^{\infty} |\lambda_n| < \infty,$$

so this series converges in X . We can also see that T is linear; additionally, T is a contraction:

$$\begin{aligned} \|T((\lambda_n)_n)\| &= \left\| \sum_{n=1}^{\infty} \lambda_n z_n \right\| \\ &= \lim_{N \rightarrow \infty} \left\| \sum_{n=1}^N \lambda_n z_n \right\| \\ &\leq \lim_{N \rightarrow \infty} \sum_{n=1}^N \|\lambda_n z_n\| \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N |\lambda_n| \\ &= \|(\lambda_n)_n\|. \end{aligned}$$

Thus, $T(U_{\ell_1}) \subseteq U_X$. To show that $T(U_{\ell_1}) = U_X$, we will use the following fact (which follows from the density of z_n).

Fact. For $\delta > 0$ and $x \neq 0$ in X , and $k \in \mathbb{N}$, there exists $n > k$ such that

$$\begin{aligned} \left\| \frac{x}{\|x\|} - z_n \right\| &< \frac{\delta}{\|x\|} \\ \|x - (\|x\|) z_n\| &< \delta \end{aligned}$$

Let $x \in U_X$ with $x \neq 0$, and let $\varepsilon > 0$. Find n_1 such that

$$\|x - (\|x\|) z_{n_1}\| < \frac{\varepsilon}{2},$$

and set $\lambda_{n_1} = \|x\|$.

We find n_2 with $n_2 > n_1$ and

$$\|(x - \lambda_{n_1} z_{n_1}) - (\|x - \lambda_{n_1} z_{n_1}\|) z_{n_2}\| < \frac{\varepsilon}{2^2},$$

and set $\lambda_{n_2} = \|x - \lambda_{n_1} z_{n_1}\|$. We have

$$\|x - (\lambda_{n_1} z_{n_1} + \lambda_{n_2} z_{n_2})\| < \frac{\varepsilon}{2^2},$$

and $\lambda_{n_2} < \frac{\varepsilon}{2}$.

Inductively, we obtain the subsequence $(z_{n_k})_k$ in z_n and a sequence of scalars $(\lambda_{n_k})_k$ such that

$$\left\| x - \sum_{j=1}^k \lambda_{n_j} z_{n_j} \right\| < \frac{\varepsilon}{2^k}$$

and

$$\|\lambda_{n_k}\| < \frac{\varepsilon}{2^{k-1}}.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i = 0$ for $i \notin \{n_1, n_2, \dots\}$. We can see that

$$\begin{aligned}\|\lambda_{n_1}\| &= \left\| \lambda_{n_1} + \sum_{k=2}^{\infty} \lambda_{n_k} \right\| \\ &\leq \|x\| + \sum_{k=2}^{\infty} \frac{\varepsilon}{2^{k-1}} \\ &= \|x\| + \varepsilon.\end{aligned}$$

We choose ε such that $\|x\| + \varepsilon < 1$, meaning $\lambda \in U_{\ell_1}$.

We can also see that $\sum_{j=1}^{\infty} \lambda_{n_j} z_{n_j} = x$, meaning T is a 1-quotient map. □

Pillars of Functional Analysis

The five main theorems of functional analysis are:

- Baire Category Theorem;
- Open Mapping Theorem (and Bounded Inverse Theorem);
- Closed Graph Theorem;
- Uniform Boundedness Principle;
- and the Hahn Banach Theorems:
 - Hahn–Banach–Minkowski Theorem;
 - Hahn–Banach Extension Theorem;
 - Hahn–Banach Separation Theorem.

These theorems will appear time and again as we work through the fundamentals of functional analysis.

Baire Category Theorem

Definition (Baire Space). Let $\{A_n\}_{n \geq 1}$ be a countable collection of open, dense subsets of a topological space X . We say X is a Baire space if

$$\bigcap_{n \geq 1} A_n$$

is dense for every such collection.

Definition (Meager Set). If $X = \bigcup_{n \geq 1} F_n$, where $(\overline{F_n})^\circ = \emptyset$ for each n , then we say X is meager.^{II}

Proposition (Meager Spaces): If X is a Baire space, then X is nonmeager.

Proof. Suppose toward contradiction that $X = \bigcup_{n \geq 1} F_n$, with F_n all nowhere dense. Then,

$$X = \bigcup_{n \geq 1} C_n,$$

where $C_n = \overline{F_n}$ are closed with $C_n^\circ = \emptyset$.

^{II}In other words, X is meager if X is a countable union of nowhere dense subsets.

Let $A_n = C_n^c$. Then, A_n is open for all n , and $\overline{A_n} = \overline{C_n^c} = (C_n^c)^\circ = X$, meaning A_n are all open and dense.

Since X is a Baire space, we know that $\bigcap_{n \geq 1} A_n$ is dense. However, we also have

$$\begin{aligned} \emptyset &= X^c \\ &= \left(\bigcup_{n \geq 1} C_n \right)^c \\ &= \bigcap_{n \geq 1} C_n^c \\ &= \bigcap_{n \geq 1} A_n. \end{aligned}$$

□

Theorem (Baire Category Theorem): If (X, d) is a complete metric space, then X is a Baire space.

Proof. Let $\{A_n\}_{n \geq 1}$ be a collection of open dense subsets of X . Let U_0 be any ball of radius $r > 0$, and set $B_0 = \overline{U_0}$. Since $A_1 \cap U_0$ is open and nonempty, it contains a closed ball B_1 with radius less than $r/2$.

Set $U_1 = B_1^\circ$. Similarly, we find a closed ball B_2 with radius less than $r/4$ such that $B_2 \subseteq A_2 \cap U_1$, and set $U_2 = B_2^\circ$.

Continuing in this manner, we find a closed ball B_n with radius less than $r/2^n$ with $B_n \subseteq A_n \cap U_{n-1}$, and the chain

$$B_0 \supseteq U_0 \supseteq B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \supseteq \dots$$

Letting $(x_n)_n$ be the center of B_n , we can see that x_n forms a Cauchy sequence in X , as the distance between x_m and x_n with $n > m$ is no more than $\frac{r}{2^{m-1}}$.

Since X is complete, $(x_n)_n \rightarrow x \in X$. We claim that x belongs to $\bigcap_{n \geq 1} B_n$.

Suppose toward contradiction that $x \notin B_N$ for some $N \in \mathbb{N}$. For $n \geq N$, we have $x \notin B_n$, so $d(x_n, x) \geq \text{dist}_{B_n}(x) > 0$, which contradicts the fact that $(x_n)_n \rightarrow x$.

Thus, $x \in \bigcap_{n \geq 1} B_n \subseteq \bigcap_{n \geq 1} A_n$. Since $\bigcap_{n \geq 1} B_n \subseteq U_0$, we have $(\bigcap_{n \geq 1} A_n) \cap U_0 \neq \emptyset$, meaning $\bigcap_{n \geq 1} A_n$ is dense in X . □

Corollary: Let X be an infinite-dimensional Banach space. The cardinality of the Hamel basis of X is uncountable.

Proof. Suppose toward contradiction that $\{b_k\}_{k \in \mathbb{N}}$ is a Hamel basis for X . For each n , set $E_n = \text{span}\{b_1, \dots, b_n\}$. Each E_n is closed, meaning $\overline{E_n} = E_n \neq X$ since X is infinite-dimensional.

Additionally, $E_n^\circ = \emptyset$ for each n , meaning the E_n are nowhere dense.

Since $\{b_k\}_{k \in \mathbb{N}}$ is a spanning set,

$$X = \bigcup_{n \geq 1} E_n,$$

implying that X is meager. □

Exercise: Let X be a Banach space, and $Z \subseteq X$ a subspace. Is it true that $\dim(Z) = \dim(\overline{Z})$?

Solution: It is not the case that $\dim(Z) = \dim(\overline{Z})$. For example, consider the subspace $c_c \subseteq \ell_\infty$. Then, the Hamel basis of c_c consists of e_n , which consists of 1 at index n and zero elsewhere, implying that $\dim(c_c) = \aleph_0$. However, $\overline{c_c} = c_0$, and c_0 is an infinite-dimensional Banach space, meaning that $\dim(\overline{c_c}) = 2^{\aleph_0} \neq \aleph_0$.

Open Mapping Theorem

A surjective continuous map between topological spaces is not necessarily an open map — however, if X and Y are Banach spaces, and $f : X \rightarrow Y$ is a surjective linear map. This is the Open Mapping theorem, which yields the result that a continuous linear bijection between Banach spaces always admits a bounded inverse.

Lemma: Let X and Y be Banach spaces, and suppose $T \in \mathcal{B}(X, Y)$.

- (1) If $U_Y \subseteq \overline{T(\delta U_X)}$ for some $\delta > 0$, then $U_Y \subseteq T(2\delta U_X)$.
- (2) If $\delta U_Y \subseteq \overline{T(U_X)}$ for some $\delta > 0$, then $\frac{\delta}{2} U_Y \subseteq T(U_X)$.

Proof.

- (1) Let $y \in U_Y$. By our assumption, there exists $x_1 \in \delta U_X$ such that $\|y - T(x_1)\| < 1/2$. Additionally,

$$\begin{aligned} y - T(x_1) &\in \frac{1}{2} U_Y \\ &\subseteq \frac{1}{2} \overline{T(\delta U_X)} \\ &= \overline{T\left(\frac{\delta}{2} U_X\right)}. \end{aligned}$$

Thus, there exists $x_2 \in \frac{\delta}{2} U_X$ such that $\|(y - T(x_1)) - T(x_2)\| < \frac{1}{4}$, implying that

$$\begin{aligned} y - T(x_1) - T(x_2) &\in \frac{1}{4} U_Y \\ &\subseteq \overline{T\left(\frac{\delta}{4} U_X\right)}. \end{aligned}$$

Inductively, we have a sequence $(x_k)_k \in \frac{\delta}{2^{k-1}} U_X$ for each k , and

$$\left\| y - \sum_{j=1}^k T(x_j) \right\| < 2^{-k}.$$

We consider $\sum_{j=1}^{\infty} x_j$. Since

$$\begin{aligned} \sum_{j=1}^{\infty} \|x_j\| &\leq \sum_{j=1}^{\infty} \frac{\delta}{2^{j-1}} \\ &= 2\delta \\ &< \infty, \end{aligned}$$

the series converges to $x \in X$ since X is complete.

Additionally, since $\|x\| \leq \sum_{j=1}^{\infty} \|x_j\| \leq 2\delta$, we have $x \in 2\delta U_X$, and $T(x) = y$ by the continuity of T .

- (2) If $\delta U_Y \subseteq \overline{T(U_X)}$, then $U_Y \subseteq \frac{1}{\delta} \overline{T(U_X)}$, so $U_Y \subseteq \overline{T\left(\frac{1}{\delta} U_X\right)}$, meaning $U_Y \subseteq T\left(\frac{2}{\delta} U_X\right)$, or $\frac{\delta}{2} U_Y \subseteq T(U_X)$.

□

Theorem (Open Mapping Theorem): Let X and Y be Banach spaces, $T \in \mathcal{B}(X, Y)$ surjective. Then, T is open and thus a quotient mapping.

Proof. We will show that $\delta U_Y \subseteq T(U_X)$ for some $\delta > 0$. This is enough to show that T is a quotient mapping.

We can write

$$\begin{aligned} X &= \bigcup_{n \geq 1} nU_X \\ Y &= T(X) \\ &= \bigcup_{n \geq 1} T(nU_X) \end{aligned}$$

since T is onto. Since Y is nonmeager, there is an $m \geq 1$ such that $\overline{T(mU_X)}^\circ \neq \emptyset$. There exists $y_0 \in Y$ and $\varepsilon > 0$ such that $U_Y(y_0, \varepsilon) \subseteq \overline{T(mU_X)}$. We claim that

$$\begin{aligned} \varepsilon U_Y &= U_Y(0, \varepsilon) \\ &\subseteq T(mU_X). \end{aligned}$$

Let $z \in \varepsilon U_Y$. Note that $y_0 + z$ and $y_0 - z$ are in $U_Y(y_0, \varepsilon)$, and

$$\begin{aligned} 2z &= (y_0 + z) - (y_0 - z) \\ &\in \overline{T(mU_X)} - \overline{T(mU_X)}. \end{aligned}$$

We write $2z = z_1 - z_2$, with $z_1, z_2 \in \overline{T(mU_X)}$. We can find sequences $(T(x_k))_k$ and $(T(x'_k))_k$ with $(T(x_k))_k \rightarrow z_1$ and $(T(x'_k))_k \rightarrow z_2$. Thus, we have

$$\begin{aligned} 2z &= \lim_{k \rightarrow \infty} (T(x_k) - T(x'_k)) \\ &= \lim_{k \rightarrow \infty} T(x_k - x'_k), \end{aligned}$$

where $\|x_k - x'_k\| \leq 2m$. Thus, $2z \in \overline{T(mU_X)} = 2\overline{T(mU_X)}$, so $z \in \overline{T(mU_X)}$.

We now have

$$\frac{\varepsilon}{m} U_Y \subseteq \overline{T(U_X)},$$

so

$$\frac{\varepsilon}{2m} U_Y \subseteq T(U_X).$$

Setting $\delta = \frac{\varepsilon}{2m}$, we finish the proof. □

If $T : X \rightarrow Y$ is bijective linear, then $T^{-1} : Y \rightarrow X$ is linear. If $X = Y$, we say T is invertible in the unital algebra $\mathcal{L}(X)$. However, if X and Y are normed vector spaces, we also have to be concerned with the continuity of T^{-1} .

Corollary (Bounded Inverse Theorem): Let X and Y be Banach spaces, $T : X \rightarrow Y$ is linear, bounded, and bijective. Then, $T^{-1} : Y \rightarrow X$ is also bounded.

Proof. Since T is surjective, T is open, so T^{-1} is continuous. □

Example. Consider the normed space $Y = (C([0, 1]), \|\cdot\|_1)$. To show that Y is not complete, we let $X = (C([0, 1]), \|\cdot\|_\infty)$, which we know is complete.

The identity function from X to Y is bijective and bounded linear since $\|\cdot\|_1 \leq \|\cdot\|_\infty$. If Y were to be complete, then it would imply that the inverse map is bounded. However, since there is no C such that $\|\cdot\|_\infty \leq C \|\cdot\|_1$, it is not the case that Y is complete.

Definition. Let X and Y be normed spaces. A bounded linear map $T \in \mathcal{B}(X, Y)$ is called invertible if there is a bounded linear map $S \in \mathcal{B}(Y, X)$ with $T \circ S = \text{id}_Y$ and $S \circ T = \text{id}_X$. We write $T^{-1} = S$.

Corollary: Let $T \in \mathcal{B}(X, Y)$ with X and Y Banach spaces. The following are equivalent.

- (i) T is bounded below.
- (ii) T is injective and $\text{Ran}(T) \subseteq Y$ is closed.
- (iii) $T : X \rightarrow \text{Ran}(T)$ is a bicontinuous isomorphism.

Proof. For (i) to (ii), if T is bounded below, then $\ker T = \{0\}$, so T is injective. Additionally, since T is bounded below, if $(T(x_n))_n$ is a Cauchy sequence in $\text{Ran}(T)$, then

$$\begin{aligned} C \|x_n - x_m\| &\leq \|T(x_n - x_m)\| \\ &= \|T(x_n) - T(x_m)\|, \end{aligned}$$

meaning $(x_n)_n$ is a Cauchy sequence in X . Since T is continuous, $(T(x_n))_n \rightarrow T(x) \in \text{Ran}(T)$.

For (ii) to (i), since Y is complete and $\text{Ran}(T) \subseteq Y$ is closed, $\text{Ran}(T)$ is a Banach space, so $T^{-1} : \text{Ran}(T) \rightarrow X$ is bounded. Thus,

$$\begin{aligned} \|x\| &= \|T^{-1}(T(x))\| \\ &\leq \|T^{-1}\|_{\text{op}} \|T(x)\|, \end{aligned}$$

meaning $\|T(x)\| \geq \|T^{-1}\|_{\text{op}}^{-1} \|x\|$ for all $x \in X$.

To show that (ii) is true if and only if (iii) is true, we can see that since T is bounded and T is bounded below, it is the case that T is a bicontinuous isomorphism. \square

Corollary: Let X and Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then, T is invertible if and only if T is bounded below and surjective.

Complemented Subspaces and Direct Sums

For any normed vector spaces X and Y , we can form the product $X \oplus_p Y$ by defining $\|(x, y)\| = (\|x\|^p + \|y\|^p)^{1/p}$ for all $p \in [1, \infty)$.

A vector space Z with subspaces X and Y is called the direct sum of X and Y if

- (a) for all $z \in Z$, there exist $x \in X$ and $y \in Y$ such that $z = x + y$;
- (b) $X \cap Y = \{0\}$.

We write $Z = X \oplus Y$ for the internal direct sum.

Proposition: Let $(Z, \|\cdot\|_Z)$ be a Banach space, and suppose X and Y are closed subspaces of Z with $Z = X \oplus Y$. Then, $Z \cong X \oplus_p Y$ for all $p \in [1, \infty]$.

Proof. Let $p = 1$. Set $\phi : X \oplus_1 Y \rightarrow Z$ by taking $\phi((x, y)) = x + y$. Since $Z = X \oplus Y$, this is a bijection, hence an isomorphism. Additionally,

$$\begin{aligned}\|\phi((x, y))\|_Z &= \|x + y\|_Z \\ &\leq \|x\|_Z + \|y\|_Z \\ &= \|(x, y)\|_1,\end{aligned}$$

meaning ϕ is bounded. Thus, ϕ^{-1} is also bounded, meaning ϕ is bicontinuous. The proof is similar for all other $p \in (1, \infty]$. \square

Definition. If Z is a normed space, X and Y are closed subspaces of Z such that $Z = X \oplus Y$, we say Z is the topological internal direct sum of X and Y .

Definition. Let Z be a normed space, and suppose X is a closed subspace of Z . We say X is complemented in Z if there is a closed $Y \subseteq Z$ with $X \oplus Y = Z$.

Not all closed subspaces are complemented.

Proposition: Let $T : X \rightarrow Y$ be a bounded linear map between Banach spaces. If $Z \subseteq Y$ is a closed subspace such that $Y = \text{Ran}(T) \oplus Z$, then $\text{Ran}(T)$ is closed (meaning the internal direct sum is topological).

Proof. Passing to the quotient

$$X/\ker(T) \rightarrow Y, \quad x + \ker(T) \mapsto T(x),$$

we may assume that T is injective. The map $S : X \oplus_\infty Z \rightarrow Y$, $S(x, z) = T(x) + z$ is bounded and bijective. Thus, S is bounded below, so for some $C > 0$, we have

$$\begin{aligned}\|T(x)\| &= \|S(x, 0)\| \\ &\geq C \|(x, 0)\|_\infty \\ &= C \|x\|,\end{aligned}$$

meaning T is bounded below, and thus has closed range. \square

Corollary: If X and Y are Banach spaces, and $T : X \rightarrow Y$ is bounded Fredholm,^{III} then T has closed range.

Proof. There is a subspace $C \subseteq Y$ with C linearly isomorphic to $\text{coker}(T)$, and $Y = \text{Ran}(T) \oplus C$. Since T is Fredholm, $\dim(C)$ is finite, meaning C is closed. Thus, $\text{Ran}(T)$ is closed. \square

Closed Graph Theorem

Definition. If $f : A \rightarrow B$ is a map between arbitrary sets, then the graph of f is

$$\begin{aligned}\text{graph}(f) &= \{(a, f(a)) \mid a \in A\} \\ &\subseteq A \times B.\end{aligned}$$

Proposition: If (X, d) and (Y, ρ) are metric spaces, and $f : (X, d) \rightarrow (Y, \rho)$ is continuous, then $\text{graph}(f) \subseteq X \times Y$ is closed under the product topology.^{IV}

Proof. Let $(x_n, f(x_n))_n$ be a sequence in $\text{graph}(f)$ such that $(x_n, f(x_n))_n \rightarrow (x, y)$ in $X \times Y$. Then, $(x_n)_n \rightarrow x$ in X and $(f(x_n))_n \rightarrow y$ in Y .

By the continuity of f , we have $(f(x_n))_n \rightarrow f(x)$, and since limits are unique, we have $f(x) = y$. Thus,

$$\begin{aligned}(x, y) &= (x, f(x)) \\ &\in \text{graph}(f).\end{aligned}$$

\square

^{III}A linear map is Fredholm if both $\ker(T)$ and $\text{coker}(T)$ are finite. Here, $\text{coker}(T) = Y/\text{Ran}(T)$.

^{IV}The product topology is the coarsest topology on $X \times Y$ such that the projection maps π_X and π_Y are continuous.

Thus, we can see that the graph of any continuous function is closed in the product topology. However, the converse fails in the general case. For instance,

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a closed graph, but f is not continuous.

However, with linear maps between Banach spaces, the converse is actually true.

Theorem (Closed Graph Theorem): Let X and Y be Banach spaces, and let $T : X \rightarrow Y$ be a linear map. Then, T is continuous if and only if $\text{graph}(T) \subseteq X \times Y$ is closed with respect to the product topology on $X \times Y$.

Proof. The forward direction follows from the previous proposition.

Suppose $\text{graph}(T) \subseteq X \times Y$ is closed in the product topology. Note that the product topology coincides with the $\|\cdot\|_1$ topology, with $\|(x, y)\|_1 = \|x\| + \|y\|$. Thus, $(\text{graph}(T), \|\cdot\|_1)$ is a Banach space.

Consider the projection map $P : \text{graph}(T) \rightarrow X$ defined by $P((x, T(x))) = x$, which is bijective. We also have

$$\begin{aligned} \|P((x, T(x)))\| &= \|x\| \\ &\leq \|x\| + \|T(x)\| \\ &= \|(x, T(x))\|_1, \end{aligned}$$

meaning P is bounded. Thus, P is bicontinuous, meaning it is bounded below, so for some constant C , we have

$$\begin{aligned} \|x\| &= \|P((x, T(x)))\| \\ &\geq C \|(x, T(x))\|_1 \\ &\geq C \|T(x)\|, \end{aligned}$$

meaning $\|T(x)\| \leq \frac{1}{C} \|x\|$, so T is bounded. □

Example. Consider the collection of sequences

$$X = \left\{ f : \mathbb{N} \rightarrow \mathbb{C} \mid \sum_{k \geq 1} k |f(k)| < \infty \right\}.$$

Note that $X \subseteq \ell_1$, and X is a linear subspace. Let

$$\begin{aligned} T : (X, \|\cdot\|_1) &\rightarrow \ell_1 \\ T(f)(k) &= (kf(k))_1, \end{aligned}$$

which is well-defined and linear. We will show that T is unbounded.

Let f_n denote the element of X defined by

$$f_n = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, 0, 0, \dots).$$

For each f_n , the norm is $\|f_n\|_1 = n$. We also see that

$$T(f_n) = (1, 2, \dots, n, 0, 0, \dots),$$

with norm

$$\|T(f_n)\|_1 = \frac{n(n+1)}{2}.$$

If T were bounded, we would have a constant C such that

$$\frac{n(n+1)}{2} \leq Cn$$

for all n . This is not possible, meaning T is unbounded.

However, at the same time, $\text{graph}(T)$ is closed, as for $(f_n)_n \rightarrow f$ in X and $(T(f_n))_n \rightarrow g$ in ℓ_1 , we have

$$(f_n(k))_n \rightarrow f(k)$$

and

$$(kf(k))_n \rightarrow g(k)$$

for all $k \in \mathbb{N}$, implying that $g(k) = kf(k)$, or that $T(f) = g$.

The closed graph theorem thus implies that $X \subseteq \ell_1$ must not be closed.

We can use the closed graph theorem to obtain insights about the orthogonal projection operators.^v

Proposition: Let $Z = X \oplus Y$ be a topological internal direct sum of a Banach space Z . Then, the projection operators P_X and P_Y onto the closed subspaces of X and Y respectively are bounded.

Proof. Let $(z_n)_n$ and $(P_X(z_n))_n$ be sequences in z with $(z_n)_n \rightarrow z$ and $(P_X(z_n))_n \rightarrow u$ for some $z, u \in Z$.

For each $n \in \mathbb{N}$, we may write $z_n = x_n + y_n$, with $x_n \in X$ and $y_n \in Y$. Since X is closed, and $x_n = P_X(z_n)$, it is the case that $u \in X$. Then, $y_n = z_n - x_n$ converges to $z - u$; since Y is closed, $z - u \in Y$. Setting $y = z - u$, we have

$$\begin{aligned} P_X(z) &= P_X(u + y) \\ &= P_X(u) + P_X(y) \\ &= u. \end{aligned}$$

Thus, the graph of P_X is closed. □

Proposition: Let Z be a Banach space, and let $P \in \mathcal{B}(Z)$ be a projection. Then, $\ker(P)$ and $\text{Ran}(P)$ are closed subspaces, and

$$Z = \text{Ran}(P) \oplus \ker(P).$$

Proof. For every $z \in Z$, we may express $z = P(z) + (z - P(z))$. Since $P(z - P(z)) = P(z) - P^2(z) = P(z) - P(z) = 0$, we can see that $Z = \text{Ran}(P) \oplus \ker(P)$.

We now need to show that $\text{Ran}(P)$ is closed. Let $(x_n)_n$ be a sequence in $\text{Ran}(p)$ with $(x_n)_n \rightarrow z$. Then, for each $n \in \mathbb{N}$, we may write $x_n = P(z_n)$. Since P is continuous, we can see that $(P(x_n)) \rightarrow P(z)$. However, at the same time, we may write $x_n = P(z_n)$, so

$$\begin{aligned} P(x_n) &= P(P(z_n)) \\ &= P(z_n) \\ &= x_n \\ &\rightarrow z, \end{aligned}$$

Which implies that $z = P(z) \in \text{Ran}(P)$ by the uniqueness of limits. □

Corollary: Let Z be a Banach space, and let $X \subseteq Z$ be a closed subspace. Then, X is complemented in Z if and only if there is a projection $P : Z \rightarrow Z$ with $\text{Ran}(P) = X$.

^vIdempotent operator, where $P^2 = P$, are also known as projections. I will refer to them as projections.

Uniform Boundedness Principle

The uniform boundedness principle is often useful when dealing with a family of bounded operators. It says that a pointwise bounded family of operators on a Banach space is bounded.

Theorem (Uniform Boundedness Principle): Let X and Y be normed vector spaces, and consider a family of bounded operators $\{T_i\}_{i \in I} \subseteq \mathcal{B}(X, Y)$.

- (1) If $\sup_{T_i(x)} < \infty$ for all $x \in A \subseteq X$ with A nonmeager, then

$$\sup_{i \in I} \|T_i\|_{\text{op}} < \infty.$$

- (2) If X is a Banach space with $\sup_{i \in I} \|T_i(x)\| < \infty$ for all $x \in X$, then

$$\sup_{i \in I} \|T_i\|_{\text{op}} < \infty.$$

Proof. We only need to prove (1), as (2) follows.

For each $i \in I$ and $n \in \mathbb{N}$, define

$$E_{n,i} = \{x \in X \mid \|T_i(x)\| \leq n\}.$$

Notice that

$$E_{n,i} = T_i^{-1} \circ \|\cdot\|^{-1}([0, n]).$$

Since both the norm and T_i are continuous, each $E_{n,i}$ is closed. Therefore,

$$\begin{aligned} E_n &= \bigcap_{i \in I} E_{n,i} \\ &= \left\{ x \in X \mid \sup_{i \in I} \|T_i(x)\| \leq n \right\} \end{aligned}$$

is also closed, meaning $\overline{E_n} = E_n$.

By our assumption,

$$A = \bigcup_{n \geq 1} E_n.$$

Since A is nonmeager, there is $m \in \mathbb{N}$, $x_0 \in X$, and $r > 0$ such that $B(x_0, r) \subseteq E_m$.^{vi}

We will now prove that $B(0, r) = rB_X \subseteq E_{2m}$. Let x be such that $\|x\| < r$. Then, $x + x_0 \in B(x_0, r) \subseteq E_m$. For each i , we also have

$$\begin{aligned} \|T_i(x)\| &= \|T_i(x) - T_i(x_0) + T_i(x_0)\| \\ &\leq \|T_i(x) + T_i(x_0)\| + \|T_i(x_0)\| \\ &= \|T_i(x + x_0)\| + \|T_i(x_0)\| \\ &\leq 2m, \end{aligned}$$

meaning $\sup_{i \in I} \|T_i(x)\| \leq 2m$, meaning $x \in E_{2m}$.

Given $x \in B_X \setminus \{0\}$, we have $rx \in E_{2m}$, so $T_i(rx) \leq 2m$ for all $i \in I$, meaning

$$\|T_i(x)\| \leq \frac{2m}{r}$$

for all $i \in I$, so $\sup_{x \in B_X} \|T_i(x)\| \leq \frac{2m}{r}$. Thus, $\sup_{i \in I} \|T_i\|_{\text{op}} \leq \frac{2m}{r}$. □

^{vi}Select δ such that $U(x_0, \delta) \subseteq E_m$, and then select $r = \delta/2$.

Alternative Proof of (2). Suppose toward contradiction that $\sup_{i \in I} \|T_i\|_{\text{op}} = \infty$. We can choose a sequence of operators $(T_{i_n})_n$ such that $\|T_{i_n}\|_{\text{op}} \geq 4^n$. We write $T_{i_n} := T_n$. We will construct some $x \in X$ such that $(\|T_n(x)\|)_n$ is unbounded, yielding a contradiction.

For any $x \in X$ and $r > 0$, we have

$$\begin{aligned} \sup_{y \in B(x, r)} &\geq r \|T\|_{\text{op}} \\ &> \frac{2}{3} r \|T\|_{\text{op}}. \end{aligned}$$

Setting $x_0 = 0$, $T = T_1$, and $r_0 = T^0$, we can find x_1 with

$$\begin{aligned} \|x_1 - x_0\| &\leq r_0 \\ \|T_1(x_1)\| &\geq \frac{2r}{3} \|T_1\|_{\text{op}}. \end{aligned}$$

We find x_2 given T_2 such that for $r_1 = 3^{-1}$

$$\|x_2 - x_1\| \leq r_1 \|T_2(x_2)\| \geq \frac{2r_1}{3} \|T_2\|_{\text{op}}.$$

Inductively, we find x_{n+1} such that for $r_n = 3^{-n}$,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq r_n \\ \|T_{n+1}(x_{n+1})\| &\geq \frac{2r_n}{3} \|T_{n+1}\|_{\text{op}}. \end{aligned}$$

Thus, we have a sequence $(x_n)_n$ with

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq 3^{-n} \\ \|T_n(x_n)\| &\geq \frac{2}{3^n} \|T_n\|_{\text{op}}. \end{aligned}$$

This means $(x_n)_n$ is Cauchy, and thus converges to $x \in X$ with $\|x - x_n\| \leq \frac{1}{(2)(3^{n-1})}$. Thus yields

$$\begin{aligned} \frac{1}{(2)(3^{n-1})} \|T_n\|_{\text{op}} &\geq \|T_n\|_{\text{op}} \|x_n - x\| \\ &\geq \|T_n(x_n - x)\| \\ &= \|T_n(x_n) - T_n(x)\| \\ &\geq \|T_n(x_n)\| - \|T_n(x)\| \\ &\geq \frac{2}{3^n} \|T_n\|_{\text{op}} - \|T_n(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|T_n(x)\| &\geq \frac{2}{3^n} \|T_n\|_{\text{op}} - \frac{1}{(2)(3^{n-1})} \|T_n\|_{\text{op}} \\ &= \frac{1}{(2)3^n} \|T_n\|_{\text{op}} \\ &\geq \frac{1}{2} \left(\frac{4}{3}\right)^n \end{aligned}$$

□

Corollary (Banach–Steinhaus Theorem): Let X, Y be Banach spaces, and let $(T_n)_n$ be a sequence of bounded linear operators in $\mathcal{B}(X, Y)$ with $\lim_{n \rightarrow \infty} (T_n(x))$ exists for every $x \in X$. Then,

$$T(x) = \lim_{n \rightarrow \infty} (T_n(x))$$

defines a bounded linear map from X to Y .

Proof. We can see that T is linear by its definition. Since $(T_n(x))_n$ converges in Y for all $x \in X$, we have $(T_n(x))$ is bounded in Y for all $x \in X$. Thus, $\sup_{n \geq 1} \|T_n(x)\| < \infty$ for all $x \in X$, so $\sup_{n \geq 1} \|T_n\|_{\text{op}} = C$ for some $C > 0$.

Given $x \in B_X$, we can see that

$$\begin{aligned} \|T(x)\| &= \left\| \lim_{n \rightarrow \infty} (T_n(x)) \right\| \\ &= \lim_{n \rightarrow \infty} \|T_n(x)\| \\ &\leq \limsup_{n \rightarrow \infty} \|T_n\|_{\text{op}} \|x\| \\ &\leq \limsup_{n \rightarrow \infty} \|T_n\|_{\text{op}} \\ &\leq C, \end{aligned}$$

meaning $\|T\|_{\text{op}} \leq C$. □

Example. Let $(z_k)_k$ be a sequence such that $\sum_{k=1}^{\infty} z_k y_k$ converges for every $(y_k)_k \in c_0$. We claim that $(z_k)_k \in \ell_1$.

Consider the map

$$\begin{aligned} T_n : c_0 &\rightarrow \mathbb{C} \\ T_n(y) &= \sum_{k=1}^n z_k y_k. \end{aligned}$$

We can see that

$$\begin{aligned} |T_n(y)| &= \left| \sum_{k=1}^n z_k y_k \right| \\ &\leq \sum_{k=1}^n |z_k y_k| \\ &\leq \|y\|_{\infty} \left(\sum_{k=1}^n |z_k| \right). \end{aligned}$$

Thus, each T_n is bounded linear with $\|T_n\|_{\text{op}} \leq \sum_{k=1}^n |z_k|$.^{vii}

By assumption, we have that $(T_n(y))_n$ converges, meaning $\sup_{n \geq 1} |T_n(y)| < \infty$ for all $y \in c_0$. The Uniform Boundedness Principle gives $\sup_{n \geq 1} \|T_n\|_{\text{op}} < \infty$. Thus,

$$\begin{aligned} \|z\|_1 &= \sum_{k=1}^{\infty} |z_k| \\ &= \sup_{n \geq 1} \left(\sum_{k=1}^n |z_k| \right) \end{aligned}$$

^{vii}It is an exercise to show that this is an inequality. I'm not sure how to do it though.

$$= \sup_{n \geq 1} \|T_n\|_{\text{op}} < \infty.$$

Example (Product Maps). Let $\{X_i\}_{i \in I}$ be a family of Banach spaces, and consider the product

$$\prod_{i \in I} X_i = \left\{ (x_i)_{i \in I} \mid x_i \in X_i, \sup_{i \in I} \|x_i\| < \infty \right\}.$$

The product space is a Banach space with pointwise operations and the norm $\|(x_i)_{i \in I}\|_\infty = \sup_{i \in I} \|x_i\|$.

Consider a Banach space Z and a linear map $Z \rightarrow \prod_{i \in I} X_i$.

We can see that if T is continuous, then $\pi_j \circ T : Z \rightarrow X_j$, where π_j is the canonical projection map on $\prod_{i \in I} X_i$, is also continuous for each j . However, it is also the case that this is a sufficient condition — that is, if $\pi_j \circ T$ is continuous for each j , then T is continuous.

Proof. Let $\iota_j : X_j \hookrightarrow \prod_{i \in I} X_i$ be the inclusion map, and $T_j : Z \rightarrow \prod_{i \in I} X_i$ be given by

$$T_j = \iota_j \circ \pi_j \circ T.$$

Then,

$$\begin{aligned} \sup_{j \in I} \|T_j(z)\| &= \sup_{j \in I} \|\pi_j \circ T(z)\| \\ &= \|T(z)\| \\ &< \infty \end{aligned}$$

for all $z \in Z$. Thus, $\sup_{j \in I} \|T_j\|_{\text{op}} < \infty$, so $\|T\|_{\text{op}}$ is bounded. \square

Hahn–Banach Theorems

The Hahn–Banach extension theorems allow us to extend continuous linear functionals from subspaces to the full vector space, while the separation theorems allow us to separate points from closed subspaces.

Definition (Algebraic Dual). Let X be a vector space over \mathbb{F} . Then, $X' = \mathcal{L}(X, \mathbb{F})$, which consists of *all* linear functionals from X into \mathbb{F} , is known as the algebraic dual.

Definition (Continuous Dual). Let X be a normed vector space over \mathbb{F} . Then, $X^* = \mathcal{B}(X, \mathbb{F})$, which consists of all *continuous* linear functionals from X into \mathbb{F} , is known as the continuous dual.^{viii}

Definition (Hyperplane). A hyperplane is a subspace $E \subseteq X$ such that $\dim(X/E) = 1$. An affine hyperplane is of the form $x_0 + E$ for some fixed $x_0 \in X$.

Proposition: Let X be a normed vector space over \mathbb{F} .

- (1) If $\varphi \in X'$ is not zero, then φ is bounded if and only if $\ker(\varphi) \subseteq X$ is closed. In this case, $X/\ker(\varphi) \cong \mathbb{F}$ are bicontinuously isomorphic.
- (2) Given a closed hyperplane $M \subseteq X$, there is a $\varphi \in X^*$ with $\ker(\varphi) = M$.
- (3) If $\varphi \in X'$ is unbounded, then $\ker(\varphi) \subseteq X$ is norm-dense.

Proof.

^{viii}Since this is functional analysis, any mention of the “dual space” will refer to the continuous dual.

- (1) If φ is continuous, then $\ker(\varphi) = \varphi^{-1}(0)$ is closed. Thus, from the definition of a quotient map, $\tilde{\varphi} : X/\ker(\varphi) \rightarrow \mathbb{F}$ defined by $x + \ker(\varphi) \mapsto \varphi(x)$, is a bicontinuous isomorphism from $X/\ker(\varphi)$ to \mathbb{F} .

Conversely, if $\ker(\varphi)$ is closed in X , then $X/\ker(\varphi)$ is a normed space with dimension 1, meaning $\tilde{\varphi}$ is continuous (as it is a linear map of dimension 1). Thus, $\varphi = \tilde{\varphi} \circ \pi$ is continuous.

- (2) Since X/M is a normed space with $\dim(X/M) = 1$, there is a bicontinuous isomorphism $\psi : X/M \rightarrow \mathbb{F}$. Set $\varphi = \psi \circ \pi$, where π denotes the canonical projection map. Then, φ is continuous with $\ker(\varphi) = M$.
- (3) Let $M = \ker(\varphi)$. Since φ is unbounded, then $M \subset \overline{M}$, meaning $p : X/M \rightarrow X/\overline{M}$ defined by $x + M \mapsto x + \overline{M}$ is well-defined with nontrivial kernel.

However, since $\dim(X/M) = 1$, we must have $\ker(p) = X/M$, implying that $x + \overline{M} = 0_{X/\overline{M}}$ for all $x \in X$, meaning $x \in \overline{M}$ for all $x \in X$, so $X \subseteq \overline{M}$.

□

For a vector space X over \mathbb{C} , we can consider X as a \mathbb{R} -vector space by simply “forgetting” the imaginary scalars. We can also consider the respective dual spaces over \mathbb{R} and \mathbb{C} , which we write as follows.

$$\begin{aligned}\mathcal{L}_{\mathbb{C}}(X, \mathbb{C}) &:= \{ \varphi : X \rightarrow \mathbb{C} \mid \varphi \text{ is } \mathbb{C}\text{-linear} \} \\ \mathcal{L}_{\mathbb{R}}(X, \mathbb{R}) &:= \{ \varphi : X \rightarrow \mathbb{R} \mid \varphi \text{ is } \mathbb{R}\text{-linear} \}.\end{aligned}$$

For $\varphi \in \mathcal{L}_{\mathbb{C}}(X, \mathbb{C})$, we define $\operatorname{Re}(\varphi) : X \rightarrow \mathbb{R}$ and $\operatorname{Im}(\varphi) : X \rightarrow \mathbb{R}$ to denote the real and imaginary parts of φ .

Proposition: Let X be a \mathbb{F} -vector space.

- (1) For $\varphi \in \mathcal{L}_{\mathbb{C}}(X, \mathbb{C})$, then $u := \operatorname{Re}(\varphi)$ and $v := \operatorname{Im}(\varphi)$, and $v(x) = -u(ix)$, implying $\varphi(x) = u(x) - iu(ix)$.
- (2) For $u \in \mathcal{L}_{\mathbb{R}}(X, \mathbb{R})$, the map $v : X \rightarrow \mathbb{R}$ defined by $v(x) := -u(ix)$ belongs to $\mathcal{L}_{\mathbb{R}}(X, \mathbb{R})$, and the map $\varphi : X \rightarrow \mathbb{C}$ defined by $\varphi(x) = u(x) + iv(x)$ belongs to $\mathcal{L}_{\mathbb{C}}(X, \mathbb{C})$.
- (3) If $(X, \|\cdot\|)$ is a normed space, with φ and u as in (1) or (2), then $\|\varphi\| = \|u\|$.

Proof.

- (1) For $x, y \in X$ and $t \in \mathbb{R}$, we find

$$\begin{aligned}u(x + ty) &= \operatorname{Re}(\varphi(x + ty)) \\ &= \operatorname{Re}(\varphi(x) + t\varphi(y)) \\ &= \operatorname{Re}(\varphi(x)) + \operatorname{Re}(t\varphi(y)) \\ &= \operatorname{Re}(\varphi(x)) + t\operatorname{Re}(\varphi(y)) \\ &= u(x) + tu(y).\end{aligned}$$

For any $z \in \mathbb{C}$, we have $\operatorname{Re}(iz) = -\operatorname{Im}(z)$, meaning

$$\begin{aligned}v(x) &= \operatorname{Im}(\varphi(x)) \\ &= -\operatorname{Re}(i\varphi(x)) \\ &= -\operatorname{Re}(\varphi(ix)) \\ &= -u(ix).\end{aligned}$$

(2) Since u is \mathbb{R} -linear, so too are v and φ . Thus,

$$\begin{aligned}\varphi(ix) &= u(ix) - iu(-x) \\ &= u(ix) + iu(x) \\ &= i\varphi(x).\end{aligned}$$

Thus, φ is \mathbb{C} -linear.

(3) Note that $|u(x)| = |\operatorname{Re}(\varphi(x))| \leq |\varphi(x)|$ for all $x \in X$, meaning $\|u\| \leq \|\varphi\|$.

For $\varphi(x) \neq 0$, we set $\alpha = \frac{|\varphi(x)|}{\varphi(x)}$, meaning $|\alpha| = 1$ and $\alpha\varphi(x) = |\varphi(x)|$. Thus, we have

$$\begin{aligned}|\varphi(x)| &= \alpha\varphi(x) \\ &= \varphi(\alpha x) \\ &= \operatorname{Re}(\varphi(\alpha x)) \\ &= u(\alpha x) \\ &\leq \|u\| \|\alpha x\| \\ &= \|u\| \|x\|,\end{aligned}$$

meaning $\|\varphi\| \leq \|u\|$.

□

Definition (Minkowski Functional). Let X be an \mathbb{R} -vector space. A Minkowski functional on X is a map $m : X \rightarrow \mathbb{R}$ such that the following are satisfied for every $x, y \in X$ and $t \geq 0$:

- (i) $m(x + y) \leq m(x) + m(y)$;
- (ii) $m(tx) = tm(x)$.

Before we go into the Hahn–Banach theorems, we want to consider the problem of whether, for an \mathbb{F} -vector space X and a subspace $E \subseteq X$, if there is a linear functional $\varphi : E \rightarrow \mathbb{F}$, is there an extension to $\psi : X \rightarrow \mathbb{F}$ such that $\psi|_E = \varphi$?

The answer is yes. If we select a basis for E , \mathcal{B}_0 , we extend it to a basis for X , \mathcal{B} , and define $\psi_0 : X \rightarrow \mathbb{F}$ by sending $b \mapsto \varphi(b)$ for $b \in \mathcal{B}_0$, and 0 for elements in $\mathcal{B} \setminus \mathcal{B}_0$.

We may be interested in extending *bounded* linear functionals as well, and if we can control the norms of these various extensions. This is where the Hahn–Banach theorems start to play a major role.

Theorem (Hahn–Banach–Minkowski): Let X be a real vector space with a Minkowski functional $m : X \rightarrow \mathbb{R}$, and let $E \subseteq X$ be a subspace. Suppose $\varphi : E \rightarrow \mathbb{R}$ is a linear functional with $\varphi(y) \leq m(y)$ for all $y \in E$.

Then, there exists a linear functional $\psi : X \rightarrow \mathbb{R}$ such that $\psi|_E = \varphi$ and $\psi(x) \leq m(x)$ for all $x \in X$.

Proof. Consider the collection $\mathcal{Z} = \{(Z, \phi)\}$ of all linear functionals that extend φ and are dominated by m . That is, $E \subseteq Z \subseteq X$ are subspaces, $\phi \in \mathcal{Z}'$ with $\phi|_E = \varphi$, and $\phi(z) \leq m(z)$ for all $z \in Z$.

Note that (E, φ) is in \mathcal{Z} , meaning \mathcal{Z} is nonempty. We define an ordering on \mathcal{Z} by taking

$$(Z_1, \phi_1) \leq (Z_2, \phi_2)$$

if and only if $Z_1 \subseteq Z_2$ and $\phi_2|_{Z_1} = \phi_1$.

Consider a chain in \mathcal{Z} , $\mathcal{C} = (Z_i, \phi_i)_{i \in I}$. Let $Y = \bigcup_{i \in I} Z_i$. Since Z_i are totally ordered by inclusion, it is the case that Y is a subspace of X . We define $\eta : Y \rightarrow \mathbb{R}$ by $\eta(x) = \phi_i(x)$ for $x \in Z_i$. Since Z_i are totally ordered

by inclusion, for $Z_j \subseteq Z_i$, it is the case that $\phi_i|_{Z_j} = \phi_j$, meaning (Y, η) is an upper bound for C .

By Zorn's lemma,^{ix} there is a maximal element (Z_0, ϕ_0) . We will now show that $Z_0 = X$.

Suppose there exists $x_1 \in X \setminus Z_0$. Consider the subspace

$$Z_1 = \{x + \lambda x_1 \mid \lambda \in \mathbb{R}, x \in Z_0\}.$$

Note that a vector in Z_1 has a unique expression $x + \lambda x_1$, since $x_1 \notin Z_0$. Thus, the linear functional $\phi_1 : Z_1 \rightarrow \mathbb{R}$ defined by $\phi_1(x + \lambda x_1) = \phi_0(x) + \lambda \alpha$ is well-defined. Note that $E \subseteq Z_1$ and $\phi_1|_E$ extends ϕ , since ϕ_0 extends ϕ . We need to find some α such that $\phi_1(z) \leq m(z)$ for all $z \in Z_1$, which will contradict the maximality of (Z_0, ϕ_0) .

For any $u, v \in Z_0$, we have

$$\begin{aligned} \phi_0(u) + \phi_0(v) &= \phi_0(u+v) \\ &\leq m(u+v) \\ &= m(u - x_1 + v + x_1) \\ &\leq m(u - x_1) + m(v + x_1), \end{aligned}$$

meaning that for all $u, v \in Z_0$,

$$\phi_0(u) - m(u - x_1) \leq m(v + x_1) - \phi_0(v).$$

We define

$$\alpha = \sup_{u \in Z_0} (\phi_0(u) - m(u - x_1)).$$

Then, for all $u \in Z_0$,

$$m(u - x_1) \geq \phi_0(u) - \alpha,$$

meaning $\alpha \leq m(v + x_1) - \phi_0(v)$ for all $v \in Z_0$, so

$$\phi_0(v) + \alpha \leq m(v + x_1).$$

Thus, for $t = \pm 1$, and $z \in Z_0$, we have

$$\phi_0(z) + t\alpha \leq m(z + tx_1)$$

Scaling by $c > 0$, we find

$$\phi_0(cz) + ct\alpha \leq m(cz + ct\alpha).$$

Setting $c = |\lambda|$, $z = \frac{1}{c}x$, and $t = \text{sgn}(\lambda)$, we find

$$\phi_1(x) + \lambda\alpha \leq m(x + \lambda x_1),$$

thus contradicting maximality. □

Theorem (Hahn–Banach Extension): Let X be a normed vector space, $E \subseteq X$ a subspace with a bounded linear functional $\phi \in E^*$. Then, there exists a continuous $\psi \in X^*$ such that $\|\phi\| = \|\psi\|$ and $\psi|_E = \phi$.

^{ix}If P is a partially ordered set, and every totally ordered $C \subseteq P$ admits an upper bound, then P has a maximal element.

Proof. Let $m(x) = \|\varphi\| \|x\|$ be our desired Minkowski functional. Restricting our view to \mathbb{R} , we find

$$\begin{aligned}\varphi(x) &\leq |\varphi(x)| \\ &\leq \|\varphi\| \|x\| \\ &= m(x).\end{aligned}$$

Thus, there exists $\psi \in X'$ such that $\psi|_E = \varphi$ and ψ is dominated by m on X .

We must now show that $\|\psi\| = \|\varphi\|$. We have

$$\begin{aligned}\psi(x) &\leq m(x) \\ &\leq \|\varphi\|\end{aligned}$$

for all $x \in B_X$, and we also have

$$\begin{aligned}-\psi(x) &= \psi(-x) \\ &\leq \|\varphi\| \|x\| \\ &\leq \|\varphi\|.\end{aligned}$$

Thus, $|\psi(x)| \leq \|\varphi\|$ for all $x \in B_X$. Additionally, since $\|\varphi\| \leq \|\psi\|$ necessarily (as ψ is defined on a larger space than φ), it is the case that $\|\varphi\| = \|\psi\|$.

Now, turning our attention to \mathbb{C} , we decompose $\varphi = u + iv$, where $u = \operatorname{Re}(\varphi)$ and $v(y) = -u(iy)$ for all $y \in E$. Then, $u : E \rightarrow \mathbb{R}$ is an \mathbb{R} -linear functional with $\|u\| = \|\varphi\|$. We extend u to $U \in X^*$ such that $\|U\| = \|u\| = \|\varphi\|$ with $U|_E = u$. Defining $V(x) = -U(ix)$ for all $x \in X$, we find $\psi = U + iV$ is a \mathbb{C} -linear functional with $\|\psi\| = \|U\| = \|\varphi\|$, and $\psi|_E = \varphi$. \square

Thanks to the Hahn–Banach extension result, we can see that there are “enough” linear functionals in X^* for any normed vector space X .

Theorem (Hahn–Banach Separation): Let X be a normed vector space.

- (1) Given a nonzero $x_0 \in X$, there is a $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$. We call φ a norming functional.
- (2) Given a proper closed subspace $E \subseteq X$ and $x_0 \in X \setminus E$, there is a $\varphi \in X^*$ such that $\varphi|_E = 0$, $\|\varphi\| \leq 1$, and $\varphi(x_0) = \operatorname{dist}_E(x_0)$.
- (3) The unit sphere in X^* , S_{X^*} , separates the points of X .

Proof.

- (1) Let $E = \operatorname{span}(x_0)$, and $\varphi : E \rightarrow \mathbb{F}$ defined by $\varphi(\lambda x) = \lambda \|x\|$. This is a linear functional, with

$$\begin{aligned}|\varphi(\lambda x)| &= |\lambda| \|x\| \\ &= \|\lambda x\|,\end{aligned}$$

meaning $\|\varphi\| = 1$. We can extend φ to ψ on X such that $\psi|_E = \varphi$, meaning $\psi(x_0) = \varphi(x_0) = \|x_0\|$.

- (2) There is a $\psi \in (X/E)^*$ with $\|\psi\| = 1$ and $\psi(x_0 + E) = \|x_0 + E\| = \operatorname{dist}_E(x_0)$. For the canonical quotient map $\pi : X \rightarrow X/E$, we set $\varphi = \psi \circ \pi$. Since π sends E to $0 + E$, φ must send any element of E to 0.

Additionally,

$$\begin{aligned}\|\varphi\| &= \|\psi \circ \pi\| \\ &\leq \|\psi\| \|\pi\| \\ &= 1.\end{aligned}$$

Thus, φ is such that $\varphi(x_0) = \operatorname{dist}_E(x_0)$.

- (3) If $x \neq y$ with $x, y \in X$, then $x - y \neq 0$, meaning there is $\varphi \in S_{X^*}$ with $\varphi(x - y) = \|x - y\| \neq 0$, meaning $\varphi(x) \neq \varphi(y)$.

□

Corollary: Let X be a normed space. For every $x \in X$, we have

$$\begin{aligned} \sup_{\varphi \in B_{X^*}} |\varphi(x)| &= \sup_{\varphi \in S_{X^*}} |\varphi(x)| \\ &= \|x\|. \end{aligned}$$

Proposition: Let X be a normed vector space with separable dual X^* . Then, there is an isometric embedding $X \hookrightarrow \ell_\infty$.

Proof. Let $(\varphi_n)_{n \geq 1}$ be a norm-dense subset of B_{X^*} . Consider the map

$$\mu : X \rightarrow \ell_\infty$$

with $\mu(x) = (\varphi_n(x))_{n \geq 1}$ for $x \in X$. It is pretty clear that μ is linear. Since $\|\varphi_n\| \leq 1$ for each $n \geq 1$, we have

$$\begin{aligned} \|\mu(x)\|_\infty &= \sup_{n \geq 1} |\varphi_n(x)| \\ &\leq \sup_{n \geq 1} \|\varphi_n\| \|x\| \\ &\leq \|x\|. \end{aligned}$$

Suppose $\|\mu(x)\|_\infty < \|x\|$ for some $x \in B_X$. Then, since $x \neq 0$, we set $\delta = \|x\| - \|\mu(x)\|_\infty$.

We can find $\psi \in B_{X^*}$ with $\psi(x) = \|x\|$. Since $(\varphi_n)_{n \geq 1}$ is norm-dense, there is $m \in \mathbb{N}$ with

$$\|\psi - \varphi_m\| < \delta,$$

meaning

$$\begin{aligned} \|x\| &= \psi(x) \\ &\leq |\psi(x) - \varphi_m(x)| + |\varphi_m(x)| \\ &\leq \|\psi - \varphi_m\| \|x\| + \|\mu(x)\| \\ &< \delta + \|\mu(x)\|_\infty \\ &= \|x\|, \end{aligned}$$

which is a contradiction. Thus, μ is isometric on the unit ball. □

Definition. Let X be a normed vector space with $S \subseteq X$ and $T \subseteq X^*$ subsets. We define the annihilator of S by

$$S^\perp = \{ \varphi \in X^* \mid \varphi(x) = 0 \ \forall x \in S \},$$

and the pre-annihilator of T by

$${}^\perp T = \{ x \in X \mid \varphi(x) = 0 \ \forall \varphi \in T \}.$$

Exercise: If X is a normed space and $E \subset X$ is a proper subspace, show that E^\perp is nonzero. Also, show that $S^\perp \subseteq X^*$ and ${}^\perp T \subseteq X$ are norm-closed subspaces.

Solution:

- (a) For $E \subset X$ a proper subspace, we know there exists some $\phi \in X^*$ such that $\|\phi\| \leq 1$ and $\|x_0\| = \text{dist}_E(x_0)$ for some $x_0 \in X$, meaning $E^\perp \neq \emptyset$.

(b) For $\varphi, \psi \in S^\perp$, we have $\varphi(x) = 0$ and $\psi(x) = 0$ for all $x \in S$, meaning

$$\begin{aligned} (\varphi + \alpha\psi)(x) &= \varphi(x) + \alpha\psi(x) \\ &= 0, \end{aligned}$$

so $\varphi + \alpha\psi \in S^\perp$, so S is a subspace.

Additionally, for a sequence $(\varphi_n)_{n \geq 1}$ in S^\perp such that $(\varphi_n)_n \rightarrow \varphi \in X^*$, we have $\varphi_n(x) = 0$ for all $x \in S$, so $\varphi(x) = 0$ for all $x \in S$.

(c) Similarly, for $x, y \in {}^\perp T$, we have

$$\begin{aligned} \varphi(x + \alpha y) &= \varphi(x) + \alpha\varphi(y) \\ &= 0, \end{aligned}$$

meaning ${}^\perp T$ is a subspace. Similarly, for any sequence $(x_n)_{n \geq 1}$ in ${}^\perp T$ such that $(x_n)_n \rightarrow x \in X$, we have $\varphi(x_n) = 0$ for all $\varphi \in T$, so $\varphi(x) = 0$.

Exercise: Let $T : X \rightarrow Y$ be a bounded linear map between normed vector spaces. Show

$$(1) \text{ Ran}(T)^\perp = \ker(T^*)$$

$$(2) {}^\perp \text{Ran}(T) = \ker(T).$$

Solution: Note that $T^* : Y^* \rightarrow X^*$ is defined by $T^*(\varphi)(x) = \varphi(T(x))$.

(1)

$$\begin{aligned} \text{Ran}(T)^\perp &= \{\varphi \in Y^* \mid \varphi(y) = 0, \forall y \in \text{Ran}(T)\} \\ &= \{\varphi \in Y^* \mid \varphi(T(x)) = 0, \forall x \in X\} \\ &= \{\varphi \in Y^* \mid T^*(\varphi)(x) = 0, \forall x \in X\} \\ &= \{\varphi \in Y^* \mid T^*(\varphi) = 0\} \\ &= \ker(T^*). \end{aligned}$$

(2)

$$\begin{aligned} {}^\perp \text{Ran}(T^*) &= \{x \in X \mid \psi(x) = 0, \forall \psi \in \text{Ran}(T^*)\} \\ &= \{x \in X \mid \varphi(T(x)) = 0, \forall \varphi \in Y^*\} \\ &= \{x \in X \mid T(x) = 0\} \\ &= \ker(T). \end{aligned}$$

Corollary: Let X be a normed vector space, and suppose $S \subseteq X$ is a subset. Then,

$${}^\perp S^\perp = \overline{\text{span}}(S).$$

Proof. Since $S \subseteq {}^\perp S^\perp$, and the latter is a norm-closed subspace, it is the case that $Z = \overline{\text{span}}(S) \subseteq {}^\perp S^\perp$.

Suppose there exists $x_0 \in {}^\perp S^\perp \setminus Z$. Then, we must have a $\varphi \in X^*$ with $\varphi|_Z = 0$ and $\varphi(x_0) \neq 0$, meaning $\varphi \in S^\perp$, so $\varphi(x_0) = 0$, which is a contradiction. \square

Having used the Hahn–Banach theorems to understand some structure results on normed vector spaces, we now turn to an application to complex analysis.

Definition. Let X be a normed vector space and consider a function $f : \Omega \rightarrow X$, where $\Omega \subseteq \mathbb{C}$ is open. We say f is differentiable at $z \in \Omega$ if

$$\begin{aligned} f'(z) &= \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z} \\ &= \lim_{\zeta \rightarrow 0} \frac{f(z + \zeta) - f(z)}{\zeta} \end{aligned}$$

exists in X . Here, the limit is defined through the norm of X .

If $f'(z)$ exists for all $z \in \Omega$, we say f is holomorphic on Ω . If $\Omega = \mathbb{C}$, then we say f is entire.

Example. Let A be a Banach algebra with $a \in A$ fixed. The exponential map

$$\begin{aligned} f : \mathbb{C} &\rightarrow A \\ z &\mapsto \exp(az) \end{aligned}$$

is entire.

For any $z \in \mathbb{C}$,

$$\begin{aligned} f'(z) &= \lim_{w \rightarrow 0} \frac{\exp(az + aw) - \exp(az)}{w} \\ &= \lim_{w \rightarrow 0} \frac{\exp(az) \exp(aw) - \exp(az)}{w} \\ &= \exp(az) \left(\lim_{w \rightarrow 0} \frac{\exp(aw) - 1}{w} \right) \\ &= a \exp(az). \end{aligned}$$

Lemma: Let X be a normed space, and suppose $f : \Omega \rightarrow X$ is differentiable at $z \in \Omega$. If $\varphi \in X^*$, then $\varphi \circ f$ is also differentiable at z . Similarly, if f is holomorphic on Ω , then $\varphi \circ f$ is holomorphic on Ω .

Proof. By continuity and linearity of φ , we have

$$\begin{aligned} (\varphi \circ f)'(z) &= \lim_{w \rightarrow z} \frac{\varphi(f(w)) - \varphi(f(z))}{w - z} \\ &= \lim_{w \rightarrow z} \varphi \left(\frac{f(w) - f(z)}{w - z} \right) \\ &= \varphi \left(\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z} \right) \\ &= \varphi(f'(z)). \end{aligned}$$

□

Liouville's theorem is a fundamental result in complex function theory which provides insight into the behavior of holomorphic functions.

Theorem (Liouville's Theorem): If $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire, and $\sup_{z \in \mathbb{C}} |f(z)| < \infty$, then there is some $z_0 \in \mathbb{C}$ such that $f(z) = z_0$ for all $z \in \mathbb{C}$.

A proof of Liouville's theorem is presented below.

Proof. Let f be holomorphic and bounded on \mathbb{C} . Then, f is analytic, meaning

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

and Cauchy's integral formula gives

$$\begin{aligned} a_k &= \frac{f^{(k)}(0)}{k!} \\ &= \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w^{k+1}} dw, \end{aligned}$$

where C_r denotes a circle of radius r about the origin. Since f is bounded on \mathbb{C} , f is bounded on C_r . Thus, we have

$$|a_k| = \frac{1}{2\pi} \left| \oint_{C_r} \frac{f(w)}{w^{k+1}} dw \right|$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \oint_{C_r} \left| \frac{f(w)}{w^{k+1}} \right| |dw| \\
&\leq \frac{1}{2\pi} \oint_{C_r} \frac{M}{r^{k+1}} |dw| \\
&= \frac{M}{2\pi r^{k+1}} \oint_{C_r} |dw| \\
&= \frac{M}{r^k}.
\end{aligned}$$

Since r was arbitrary, this means $a_k = 0$ for all $k \geq 1$, meaning $f(z) = a_0$ for all z . \square

Corollary: Let X be a normed space. If $f : \mathbb{C} \rightarrow X$ is entire and bounded, then there is $x_0 \in X$ such that $f(z) = x_0$ for all $z \in \mathbb{C}$.

Proof. Suppose this is not the case. Then, there are $z_1, z_2 \in \mathbb{C}$ such that $f(z_1) \neq f(z_2)$. Thus, there is $\varphi \in X^*$ such that $\varphi(f(z_1)) \neq \varphi(f(z_2))$. Additionally, $\varphi \circ f : \mathbb{C} \rightarrow \mathbb{C}$ is entire, and

$$\begin{aligned}
\sup_{z \in \mathbb{C}} |\varphi \circ f| &= \sup_{z \in \mathbb{C}} |\varphi(f(z))| \\
&\leq \sup_{z \in \mathbb{C}} \|\varphi\| |f(z)| \\
&< \infty,
\end{aligned}$$

so $\varphi \circ f$ is constant. However, this contradicts $\varphi(f(z_1)) \neq \varphi(f(z_2))$. \square

Duality

If X is a normed vector spaces, we know that the continuous dual space, $X^* = \mathcal{B}(X, \mathbb{C})$, is a Banach space as \mathbb{C} is complete. This section will be focused on understanding dual spaces and their operators.

Completions and the Double Dual

Definition (Double Dual). We define $X^{**} = \mathcal{B}(X^*, \mathbb{C})$ to be the double dual of X .

Remark: X^{**} is complete.

Proposition: Let X be a normed vector space. If $x \in X$, the map

$$\begin{aligned}
\hat{x} : X^* &\rightarrow \mathbb{F} \\
\hat{x}(\varphi) &= \varphi(x)
\end{aligned}$$

is linear and bounded with $\hat{x}_{\text{op}} = \|x\|$. The map $\iota_X : X \hookrightarrow X^{**}$ with $x \mapsto \hat{x}$, is a linear isometry known as the canonical embedding.

Proof. First, we show that \hat{x} is linear. Let $\varphi, \psi \in X^*$ and $\alpha \in \mathbb{F}$. Then,

$$\begin{aligned}
\hat{x}(\varphi + \alpha\psi) &= (\varphi + \alpha\psi)(x) \\
&= \varphi(x) + \alpha\psi(x) \\
&= \hat{x}(\varphi) + \alpha\hat{x}(\psi).
\end{aligned}$$

We can also see, by the Hahn–Banach Separation theorem,

$$\begin{aligned}
\|\hat{x}\|_{\text{op}} &= \sup_{\varphi \in B_{X^*}} |\hat{x}(\varphi)| \\
&= \sup_{\varphi \in B_{X^*}} |\varphi(x)|
\end{aligned}$$

$$= \|x\|.$$

Now, we will show that ι_X is linear. Let $x, y \in X, \lambda \in \mathbb{F}$. Then,

$$\begin{aligned}\iota_X(x + \lambda y) &= \widehat{x + \lambda y} \\ &= \widehat{x} + \widehat{\lambda y} \\ &= \widehat{x} + \lambda \widehat{y} \\ &= \iota_X(x) + \lambda \iota_X(y).\end{aligned}$$

□

Definition (Completion of a Metric Space). For a metric space X , the completion \widetilde{X} is a complete metric space where $X \subseteq \widetilde{X}$ and $\overline{X} = \widetilde{X}$.

Since X^{**} is complete, we can make a completion of X using the canonical embedding.

Definition (Norm Completion). Let X be a normed vector space. A norm completion of X is a pair (Z, j) , with Z is a Banach space and $j : X \hookrightarrow Z$ is a linear isometry such that $\overline{\text{Ran}(j)} = Z$.

First, we show the existence of a norm completion. Then, we will show that norm completions are unique up to isometric isomorphism.

Proposition: Let X be a normed vector space. Set $\widetilde{X} = \overline{\iota_X(X)}^{\|\cdot\|_{\text{op}}} \subseteq X^{**}$, where ι_X is the canonical embedding. The pair, (\widetilde{X}, ι_X) , is a completion of X .

Proof. Since X^{**} is a Banach space, the closed subspace $\overline{\iota_X(X)}^{\|\cdot\|_{\text{op}}}$ is complete. Since ι_X is an isometric isomorphism, and $\overline{\text{Ran}(\iota_X)} = \overline{\iota_X(X)}^{\|\cdot\|_{\text{op}}}$. □

Note: If X is a Banach space, $\text{Ran}(\iota_X)$ is already norm-closed, since X is complete, so $\widetilde{X} = \iota_X(X) \cong X$.

To show that completions are unique up to isometric isomorphism, we need to be able to extend bounded linear operators from normed spaces to their completions that preserve the operator norm.

Proposition: Let X be a normed space with $E \subseteq X$ a linear subspace.

- (1) The closure \overline{E} is a closed linear subspace.
- (2) If $T : E \rightarrow Z$ is a bounded linear map into a Banach space Z , then there is a unique bounded linear map $\widetilde{T} : \overline{E} \rightarrow Z$ such that $\widetilde{T}|_E = T$ and $\|\widetilde{T}\|_{\text{op}} = \|T\|_{\text{op}}$. If T is an isometry, then so is \widetilde{T} , and $\text{Ran}(\widetilde{T}) = \overline{\text{Ran}(T)}$.

Proof.

- (1) Let $(x_n)_n \rightarrow x$ and $(y_n)_n \rightarrow y$ in \overline{E} . Then, $x + \alpha y = \lim_{n \rightarrow \infty} (x_n)_n + \alpha \lim_{n \rightarrow \infty} (y_n)_n$, or $x + \alpha y = \lim_{n \rightarrow \infty} (x_n + \alpha y_n)_n$, meaning $x + \alpha y \in \overline{E}$.
- (2) Let $x \in \overline{E}$. There is a sequence $(x_n)_n$ in E with $(x_n)_n \rightarrow x$. Then,

$$\begin{aligned}\|T(x_n) - T(x_m)\| &= \|T(x_n - x_m)\| \\ &\leq \|T\| \|x_n - x_m\|.\end{aligned}$$

Thus, $(T(x_n))_n$ is a Cauchy sequence in Z , meaning it converges to $z \in Z$.

If $(x'_n)_n$ is a different sequence in E with $(x'_n)_n \rightarrow x$, then similarly, $(T(x'_n))_n$ is Cauchy in Z that converges to z' . We have

$$z = \lim_{n \rightarrow \infty} T(x_n)$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} (T(x_n) - T(x'_n) + T(x'_n)) \\
&= \lim_{n \rightarrow \infty} (T(x_n) - T(x'_n)) + \lim_{n \rightarrow \infty} T(x'_n) \\
&= z'.
\end{aligned}$$

We define $\tilde{T} : \bar{E} \rightarrow Z$ with $\tilde{T}(x) := \lim_{n \rightarrow \infty} T(x_n)$ for any sequence $(x_n)_n$ in E that converges to x .

Linearity of \tilde{T} follows from the linearity of limits.

We now verify that $\|\tilde{T}\|_{\text{op}} = \|T\|_{\text{op}}$.

$$\begin{aligned}
\|\tilde{T}(x)\| &= \left\| \lim_{n \rightarrow \infty} T(x_n) \right\| \\
&= \lim_{n \rightarrow \infty} \|T(x_n)\| \\
&\leq \lim_{n \rightarrow \infty} \|T\| \|x_n\| \\
&= \|T\| \|x\|,
\end{aligned}$$

implying that $\|\tilde{T}\|_{\text{op}} \leq \|T\|_{\text{op}}$. Additionally, since \tilde{T} is defined on a larger space than T , $\|T\|_{\text{op}} \leq \|\tilde{T}\|_{\text{op}}$. Uniqueness follows from the fact that two continuous functions defined on a dense subset agree on the whole set.

Additionally, if T is an isometry, then

$$\begin{aligned}
\|\tilde{T}(x)\| &= \left\| \lim_{n \rightarrow \infty} T(x_n) \right\| \\
&= \lim_{n \rightarrow \infty} \|T(x_n)\| \\
&= \lim_{n \rightarrow \infty} \|x_n\| \\
&= \|x\|.
\end{aligned}$$

By definition, we can see that $\text{Ran}(T) \subseteq \text{Ran}(\tilde{T}) \subseteq \overline{\text{Ran}(T)}$. Since \tilde{T} is an isometry, its range is closed, so we have $\text{Ran}(\tilde{T}) = \overline{\text{Ran}(T)}$.

□

Theorem: Let X be a normed vector space. There is a Banach space \tilde{X} and a linear isometry $j : X \hookrightarrow \tilde{X}$ such that $\overline{j(X)} = \tilde{X}$. If (Z, k) is any other completion of X , then $Z \cong \tilde{X}$ are isometrically isomorphic.

Proof. We have shown existence with the earlier proposition.

Let (Z_1, j_1) and (Z_2, j_2) be norm completions of X . Consider the mappings $T : \text{Ran}(j_1) \rightarrow Z_2$ and $S : \text{Ran}(j_2) \rightarrow Z_1$ given by

$$\begin{aligned}
T(j_1(x)) &= j_2(x) \\
S(j_2(x)) &= j_1(x).
\end{aligned}$$

Since j_1 and j_2 are isometries, then T and S are well-defined linear isometries. Thus, T and S extend to isometries $T : Z_1 \rightarrow Z_2$ and $S : Z_2 \rightarrow Z_1$.

We have $S \circ T(z) = z$ for all $z \in \text{Ran}(j_1)$, meaning $S \circ T = \text{id}_{Z_1}$, and similarly, $T \circ S = \text{id}_{Z_2}$. Thus, $Z_1 \cong Z_2$ are isometrically isomorphic. □

Corollary: Let Z be a Banach space with $X \subseteq Z$ a subspace. Then, $\tilde{X} \cong \bar{X}$.

Remark: For a normed space $(X, \|\cdot\|)$, we let $\bar{X}^{\|\cdot\|}$ denote the norm completion \tilde{X} , and $X \subseteq \bar{X}^{\|\cdot\|}$ is considered as a dense subspace.

We are now able to extend bounded linear operators to the completion.

Proposition: Let X and Y be normed spaces. If $T \in \mathcal{B}(X, Y)$, there is a unique $\tilde{T} \in \mathcal{B}(\tilde{X}, \tilde{Y})$ such that $\tilde{T} \circ \iota_X = \iota_Y \circ T$. The following diagram commutes.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{T}} & \tilde{Y} \\ \iota_X \uparrow & & \uparrow \iota_Y \\ X & \xrightarrow{T} & Y \end{array} \quad \text{Additionally, } \|T\|_{\text{op}} = \|\tilde{T}\|_{\text{op}}. \text{ If } T \text{ is an isometry, then so is } \tilde{T}, \text{ and if } T \text{ is an isometric}$$

isomorphism, so is \tilde{T} .

Proof. Define $T_0 : \text{Ran}(\iota_X) \rightarrow \tilde{Y}$ by $T_0(\iota_X(x)) = \iota_Y(T(x))$. Since ι_X is injective, T_0 is well-defined and linear. Additionally, since ι_X and ι_Y are isometries, for every $x \in X$, we have

$$\begin{aligned} \|T_0(\iota_X(x))\| &= \|\iota_Y(T(x))\| \\ &= \|T(x)\| \\ &\leq \|T\|_{\text{op}} \|x\| \\ &= \|T\|_{\text{op}} \|\iota_X(x)\|, \end{aligned}$$

meaning T_0 is bounded. Thus, T_0 extends to \tilde{T} defined on $\overline{\text{Ran}(\iota_X)} = \tilde{X}$, such that $\|\tilde{T}\|_{\text{op}} = \|T_0\|_{\text{op}} \leq \|T\|_{\text{op}}$.

It is also the case that $\|T\|_{\text{op}} \leq \|\tilde{T}\|_{\text{op}}$ since \tilde{T} is an extension.

Thus, the diagram commutes. Uniqueness follows from the fact that $\text{Ran}(\iota_X) \subseteq \tilde{X}$ is dense.

If T is isometric, then

$$\begin{aligned} \|T_0(\iota_X(x))\| &= \|\iota_Y(T(x))\| \\ &= \|T(x)\| \\ &= \|x\| \\ &= \|\iota_X(x)\|, \end{aligned}$$

meaning T_0 is an isometry. Thus, the extension to $\tilde{T} : \tilde{X} \rightarrow \tilde{Y}$ is also an isometry.

If T is an isometric isomorphism, then

$$\begin{aligned} \text{Ran}(\tilde{T}) &= \overline{\text{Ran}(T_0)} \\ &= \overline{\iota_Y(\text{Ran}(T))} \\ &= \overline{\iota_Y(Y)} \\ &= \tilde{Y}, \end{aligned}$$

meaning \tilde{T} is an isomorphism as well. □

Corollary: Let X be a vector space and Y a Banach space. Suppose there is an injective linear map $T : X \rightarrow Y$, and we write $\|\cdot\|_X$ to be the norm on X induced by T , defined by $\|x\|_X = \|T(x)\|_Y$. If $Z := \overline{T(X)}$ is the closure of the range of T in Y , then $\tilde{X} \cong Z$.

Proof. By the definition of the norm, $T : X \rightarrow Z$ is an isometry into the Banach space Z , and so extends to $\tilde{T} : \tilde{X} \rightarrow Z$, where $\tilde{T}|_X = T$. Since T is an isometry and \tilde{X} is complete, the range $\tilde{T}(\tilde{X})$ is closed in Y , meaning

$$\begin{aligned} T(X) &\subseteq \tilde{T}(\tilde{X}) \\ \overline{T(X)} &\subseteq \tilde{T}(\tilde{X}). \end{aligned}$$

Thus, \tilde{T} is surjective. □

Dual Spaces

Definition. A normed vector space X is called a dual space if there is a normed vector space Z such that $Z^* \cong X$ are isometrically isomorphic. The space Z is known as the pre-dual of X .

It is generally not easy to find whether or not X is a dual space. However, we can attempt to find some dual spaces that for various Banach spaces.

A linear map is uniquely defined on the Hamel basis of its domain. However, when dealing with infinite-dimensional Banach spaces, we know that they cannot have a countable Hamel basis, which makes working with them very difficult. However, we can use a different notion of basis to assist us in our study.

Definition. Let X be a Banach space over \mathbb{F} . A sequence $(e_n)_n$ in X is called a Schauder basi for X if every $x \in X$ admits a unique sequence $(\lambda_n)_n$ in \mathbb{F} such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n$$

is a norm convergent sum in X .

Proposition: The canonical coordinate vectors $(e_n)_n^*$ is a Schauder basis for c_0 and ℓ_p for $1 \leq p < \infty$.

Proof. We will show this for c_0 . Let $x \in c_0$, $x = (\lambda_n)_n$. For each $N \geq 1$, define

$$x_N = \sum_{n=1}^N \lambda_n e_n.$$

Whenever $N \geq M$, we have

$$\begin{aligned} \|x_N - x_M\|_{\infty} &= \left\| \sum_{n=M+1}^N \lambda_n e_n \right\|_{\infty} \\ &= \sup_{M+1 \leq n \leq N} |\lambda_n|. \end{aligned}$$

Since $(\lambda_n)_n \rightarrow 0$ as $x \in c_0$, we have $\|x_N - x_M\|_{\infty}$ is small for large N, M . Thus, $(x_N)_N$ is Cauchy.

By the completeness of c_0 , we know that $\sum_{n=1}^{\infty} \lambda_n e_n$ thus converges. Since

$$\begin{aligned} \|x - x_N\|_{\infty} &= \sup_{n \geq N+1} |\lambda_n| \\ &\rightarrow 0, \end{aligned}$$

we have $x = \sum_{n=1}^{\infty} \lambda_n e_n$. Thus, we have shown existence.

*The sequence with 1 at position n and 0 everywhere else.

To show uniqueness, let $\varphi_k \in c_0^*$ be given by $\varphi_k((z_n)_n) = z_k$ for each $k \in \mathbb{N}$. Suppose that there exist two sequences of coefficients $(\lambda_n)_n$ and $(\mu_n)_n$ such that

$$\begin{aligned} x &= \sum_{n=1}^{\infty} \lambda_n e_n \\ &= \sum_{n=1}^{\infty} \mu_n e_n. \end{aligned}$$

For each k , since φ_k is continuous, we have

$$\begin{aligned} \lambda_k &= \sum_{n=1}^{\infty} \lambda_n \varphi_k(e_n) \\ &= \varphi_k\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) \\ &= \varphi_k(x) \\ &= \varphi_k\left(\sum_{n=1}^{\infty} \mu_n e_n\right) \\ &= \sum_{n=1}^{\infty} \mu_n \varphi_k(e_n) \\ &= \mu_k. \end{aligned}$$

Thus, $(\lambda_n)_n = (\mu_n)_n$. □

Exercise: Every Banach space that admits a Schauder basis is separable. Conclude that the canonical sequence of coordinate vectors $(e_n)_n$ is not a Schauder basis for ℓ_∞ .

Solution: Let X admit a Schauder basis $(e_n)_n$, and let $(\lambda_n)_n$ be a sequence of coefficients in \mathbb{C} such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n.$$

Define

$$E = \left\{ \sum_{k=1}^{\infty} \alpha_k e_k \mid \alpha_k \in \mathbb{C}_Q \right\}.$$

Since the coefficients in E are defined over \mathbb{C}_Q , E is countable. We claim that $\overline{E} = X$.

Let $\varepsilon > 0$. Since $\overline{\mathbb{C}_Q} = \mathbb{C}$, we find $\alpha_1 \in \mathbb{C}_Q$ such that $\|(\alpha_1 - \lambda_1)e_1\| < \frac{\varepsilon}{2}$, and inductively find $\alpha_k \in \mathbb{C}_Q$ such that

$$\|(\alpha_k - \lambda_k)e_k\| < \frac{\varepsilon}{2^k}.$$

Define $\mu = \sum_{k=1}^{\infty} \alpha_k e_k$. Then,

$$\begin{aligned} \|\mu - x\| &= \left\| \sum_{k=1}^{\infty} (\alpha_k - \lambda_k) e_k \right\| \\ &\leq \sum_{k=1}^{\infty} \|(\alpha_k - \lambda_k) e_k\| \\ &< \varepsilon. \end{aligned}$$

Since ℓ_∞ is not separable,^{x1} it is the case that ℓ_∞ does not admit a Schauder basis.

^{x1}The set $E = \{(\alpha_k)_k \mid \alpha_k \in \{0, 1\}\}$ under $\|\cdot\|_\infty$ has the discrete metric but is uncountable (hence, not separable), but any subset of a separable set is necessarily separable.

We can now find some dual spaces.

Proposition: The dual space of c_0 is isometrically isomorphic to ℓ_1 . That is, $c_0^* \cong \ell_1$.

Proof. Let $\theta : \ell_1 \rightarrow c_0^*$ be defined by $\theta(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n z_n$, where $\lambda = (\lambda_n)_n \in \ell_1$ and $z = (z_n)_n \in c_0$. We will show that θ is an isometric isomorphism.

We start by showing that θ is well-defined. To see this, note that

$$\begin{aligned} \sum_{n=1}^{\infty} |\lambda_n z_n| &\leq \sum_{n=1}^{\infty} |\lambda_n| \|z\|_{\infty} \\ &= \|\lambda\|_1 \|z\|_{\infty}, \end{aligned}$$

which is finite since $\lambda \in \ell_1$ and $z \in \ell_{\infty}$. Thus, $\sum_{n=1}^{\infty} \lambda_n z_n$ converges absolutely, and thus is convergent.

For each $\lambda \in \ell_1$, the map $\theta(\lambda) : c_0 \rightarrow \mathbb{F}$ is linear. We now need to show that θ is bounded.

Let $z \in c_0$. Then,

$$\begin{aligned} |\theta(\lambda)(z)| &= \left| \sum_{n=1}^{\infty} \lambda_n z_n \right| \\ &\leq \sum_{n=1}^{\infty} |\lambda_n| |z_n| \\ &\leq \sum_{n=1}^{\infty} |\lambda_n| \|z\|_{\infty} \\ &= \|\lambda\|_1 \|z\|_{\infty}. \end{aligned}$$

Thus, $\|\theta(\lambda)\|_{\text{op}} \leq \|\lambda\|_1$, so $\theta(\lambda) \in c_0^*$.

The map $\lambda \mapsto \theta(\lambda)$ is linear. Now, we show that θ is an isometry. We know that $\|\theta(\lambda)\|_{\text{op}} \leq \|\lambda\|_1$. For $(\lambda_n)_n \in \ell_1$, we know that

$$z_{\lambda} = (\text{sgn}(\lambda_1), \text{sgn}(\lambda_1), \dots, \text{sgn}(\lambda_n))$$

is a member of B_{c_0} , as its nonzero terms have modulus at most 1. Here, $\text{sgn}(z)$ is the complex number of modulus 1 defined by $\text{sgn}(z) = \frac{|z|}{z}$. Thus,

$$\begin{aligned} \|\theta(\lambda)\|_{\text{op}} &\geq |\theta(\lambda)(z_1)| \\ &= \left| \sum_{i=1}^n \lambda_i \text{sgn}(\lambda_i) \right| \\ &= \sum_{i=1}^n |\lambda_i|. \end{aligned}$$

Sending $n \rightarrow \infty$, we have $\|\theta(\lambda)\|_{\text{op}} \geq \|\lambda\|_1$. Thus, θ is an isometry.

Finally, we must show that θ is onto. Let $\varphi \in c_0^*$, and let $\lambda_n = \varphi(e_n)$. We set $\omega_n = \text{sgn}(\varphi(e_n))$. Then,

$$\sum_{n=1}^N |\lambda_n| = \sum_{n=1}^N |\varphi(e_n)|$$

$$\begin{aligned}
&= \sum_{n=1}^N \varphi_n \operatorname{sgn}(\varphi(e_n)) \\
&= \sum_{n=1}^N \varphi_n(e_n \omega_n) \\
&= \varphi\left(\sum_{n=1}^N \omega_n e_n\right) \\
&\leq \|\varphi\|_{\text{op}} \left\| \sum_{n=1}^N \omega_n e_n \right\|_{\infty} \\
&\leq \|\varphi\|_{\text{op}}.
\end{aligned}$$

Sending $N \rightarrow \infty$, we find that $(\lambda_n)_n \in \ell_1$. For an arbitrary $(z_n)_n \in c_0$, we have

$$\begin{aligned}
\varphi(z) &= \varphi\left(\sum_{n=1}^{\infty} z_n e_n\right) \\
&= \sum_{n=1}^{\infty} z_n \varphi(e_n) \\
&= \sum_{n=1}^{\infty} z_n \lambda_n \\
&= \theta(\lambda)(z),
\end{aligned}$$

thus meaning φ and $\theta(\lambda)$ agree on an arbitrary member of c_0 . \square

Remark: What made this problem so much easier than it might have been is the fact that since c_0 has a Schauder basis, we could define how $\varphi \in c_0^*$ behaved entirely through its action on the basis elements. This is similar to how we can find properties of linear maps in linear algebra by finding their actions on the Hamel bases of the vector space.

Exercise: Consider the map $\phi : \ell_1 \rightarrow \ell_{\infty}^*$ defined by $\phi(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n z_n$ for $\lambda = (\lambda_n)_n \in \ell_1$ and $z = (z_n)_n \in \ell_{\infty}$. Prove that ϕ is a linear isometry, but ϕ is *not* onto.

Solution: A similar method to the proof that $c_0^* \cong \ell_1$ gives ϕ is an isometry. We will replicate it here.

First, we show that ϕ is bounded and well-defined.

$$\begin{aligned}
|\phi(\lambda)(z)| &= \left| \sum_{n=1}^{\infty} \lambda_n z_n \right| \\
&\leq \sum_{n=1}^{\infty} |\lambda_n z_n| \\
&\leq \sum_{n=1}^{\infty} |\lambda_n| \|z\|_{\infty} \\
&= \|\lambda\|_1 \|z\|_{\infty} \\
&< \infty.
\end{aligned}$$

Thus, $\|\phi(\lambda)\|_{\text{op}} \leq \|\lambda\|_1$, so ϕ is bounded. It is also clear that ϕ is linear in λ and in z . Similarly, for $\lambda = (\lambda_n)_n \in \ell_1$, we define

$$\begin{aligned}
z_{\lambda} &= (\operatorname{sgn}(\lambda_1), \dots, \operatorname{sgn}(\lambda_n), 0, 0, \dots) \\
&\in B_{c_0}.
\end{aligned}$$

Thus,

$$\|\phi(\lambda)\|_{\text{op}} \geq |\phi(\lambda)(z_{\lambda})|$$

$$= \sum_{k=1}^n |\lambda_k|,$$

meaning $\|\phi(\lambda)\|_{\text{op}} \geq \|\lambda\|_1$, so ϕ is an isometry.

In particular, ϕ is an isometric isomorphism from ℓ_1 to $c_0^* \subset \ell_\infty^*$, so ϕ cannot be onto ℓ_∞^* .

Proposition: The dual of ℓ_1 is isometrically isomorphic to ℓ_∞ .

Proof. Let $\theta : \ell_\infty \rightarrow \ell_1^*$ be defined by $\theta(z)(\lambda) = \sum_{n=1}^\infty z_n \lambda_n$ for $z = (z_n)_n \in \ell_\infty$ and $(\lambda_n)_n \in \ell_1$.

We can see that θ is well-defined with $\|\theta(z)\|_{\text{op}} \leq \|z\|_\infty$ by a similar reasoning to the case of $\theta : \ell_1 \rightarrow c_0^*$. Additionally, since

$$\begin{aligned} \|\theta(z)\|_{\text{op}} &\geq |\theta(z)(e_n)| \\ &= |z_n| \end{aligned}$$

for every $n \geq 1$, we have $\|z\|_\infty \leq \|\theta(z)\|_{\text{op}}$, meaning θ is an isometry.

Suppose $\varphi \in \ell_1^*$. Set $z_n = \varphi(e_n)$. Note that $z \in \ell_\infty$ since $|z_n| = |\varphi(e_n)| \leq \|\varphi\|_{\text{op}}$ for all $n \geq 1$. For $\lambda = (\lambda_n)_n \in \ell_1$, then $\lambda = \sum_{n=1}^\infty \lambda_n e_n$ is convergent in the 1-norm, as

$$\begin{aligned} \left\| \sum_{n=1}^\infty \lambda_n e_n \right\| &\leq \sum_{n=1}^\infty \|\lambda_n e_n\| \\ &= \sum_{n=1}^\infty |\lambda_n| \|e_n\| \\ &= \sum_{n=1}^\infty |\lambda_n| \\ &< \infty. \end{aligned}$$

Thus,

$$\begin{aligned} \varphi(\lambda) &= \varphi\left(\sum_{n=1}^\infty \lambda_n e_n\right) \\ &= \sum_{n=1}^\infty \lambda_n \varphi(e_n) \\ &= \sum_{n=1}^\infty \lambda_n z_n \\ &= \theta(z)(\lambda). \end{aligned}$$

□

This result can be generalized to L_p spaces, which are spaces of measurable functions satisfying some integration property.

Theorem: Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

- (1) If μ is a σ -finite measure, and $p, q \in (1, \infty)$ satisfying $p^{-1} + q^{-1} = 1$, then the map $L_q(\Omega, \mu) \rightarrow L_p(\Omega, \mu)^*$ given by $f \mapsto \varphi_f$, with

$$\varphi_f(g) = \int_{\Omega} g f \, d\mu,$$

is an isometric isomorphism.

(2) If μ is semi-finite, then $L_\infty \rightarrow L_1(\Omega, \mu)^*$ given by $f \mapsto \varphi_f$ is an isometric isomorphism.

We will now turn our attention to spaces of continuous functions. Let Ω be a locally compact Hausdorff space. We want to understand $C_0(\Omega)^*$, the space of bounded linear functionals on $C_0(\Omega)$.

Note that $C_0(\Omega)$ is a $*$ -algebra, and admits a cone of positive elements, $C_0(\Omega)_+$.^{xii} We start by understanding the properties of positive linear functionals.

Definition. Let Ω be a locally compact Hausdorff space.

(1) A linear functional $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ is called positive if $\varphi(f) \geq 0$ whenever $f \in C_0(\Omega)_+$.

(2) A positive linear functional $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ is called faithful if

$$\ker(\varphi) \cap C_0(\Omega)_+ = \{0\}.$$

(3) A state on $C_0(\Omega)$ is a positive linear functional $\varphi \in C_0(\Omega)'$ with $\|\varphi\|_{\text{op}} = 1$. The collection of states

$$\begin{aligned} S(C_0(\Omega)) &= \{\varphi \in C_0(\Omega)^* \mid \varphi \text{ is a state}\} \\ &\subseteq B_{C_0(\Omega)^*}. \end{aligned}$$

is called the state space of Ω , which we abbreviate $S(\Omega)$.

Lemma: Let Ω be a locally compact Hausdorff space, and suppose $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ is a positive linear functional.

(1) If $h \in C_0(\Omega, \mathbb{R})$, then $\varphi(h) \in \mathbb{R}$.

(2) If $h \leq k$ in $C_0(\Omega, \mathbb{R})$, then $\varphi(h) \leq \varphi(k)$.

(3) If $f \in C_0(\Omega)$, then $\varphi(f^*) = \overline{\varphi(f)}$.

Proof.

(1) We write $h = h_+ - h_-$, and see that $\varphi(h) = \varphi(h_+) - \varphi(h_-) \in \mathbb{R}$.

(2) If $h \leq k$ in $C_0(\Omega, \mathbb{R})$, then $k - h \geq 0$, so $\varphi(k - h) \geq 0$, so $\varphi(h) \leq \varphi(k)$.

(3) We write f as the Cartesian decomposition, $f = h + ik$, where $h, k \in C_0(\Omega, \mathbb{R})$. Thus,

$$\begin{aligned} \varphi(f^*) &= \varphi(h - ik) \\ &= \varphi(h) - i\varphi(k) \\ &= \overline{\varphi(h) + i\varphi(k)} \\ &= \overline{\varphi(f)}. \end{aligned}$$

□

Proposition: If $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ is positive and linear, then φ is bounded.

Proof. Let

$$M = \sup \{ \varphi(f) \mid f \in C_0(\Omega)_+, \|f\|_{\text{u}} \leq 1 \}.$$

We claim that M is finite. If $M = \infty$ then for each $n \in \mathbb{N}$, there exists $f_n \in C_0(\Omega)_+$ with $\|f_n\|_{\text{u}} \leq 1$ and $\varphi(f_n) \geq 2^n$. Note that

$$\sum_{n=1}^{\infty} \|2^{-n} f_n\|_{\text{u}} \leq \sum_{n=1}^{\infty} 2^{-n}$$

^{xii} $C_0(\Omega)_+ = \{f \in C_0(\Omega) \mid f(x) \geq 0 \forall x \in \Omega\}$.

$$= 1,$$

meaning the series $f = \sum_{n=1}^{\infty} 2^{-n} f_n$ converges uniformly in $C_0(\Omega)$.

For $N \in \mathbb{N}$, we can see that $f \geq \sum_{n=1}^N 2^{-n} f_n \geq 0$, as f_n are all positive. Since φ preserves the order on $C_0(\Omega)$, we have

$$\begin{aligned} \varphi(f) &\geq \varphi\left(\sum_{n=1}^N 2^{-n} f_n\right) \\ &= \sum_{n=1}^N 2^{-n} \varphi(f_n) \\ &\geq \sum_{n=1}^N 2^{-n} 2^n \\ &= N. \end{aligned}$$

This is absurd since N is arbitrary. Thus, $M < \infty$.

Let $f \in C_0(\Omega)$ with $\|f\|_u \leq 1$. Let $f = h + ik$ be the Cartesian decomposition. Then, we can write $f = (h_+ - h_-) + i(k_+ - k_-)$. Since φ is linear, $|\varphi(f)| \leq 4M$, meaning $\|\varphi\|_{op} \leq 4M$. \square

Proposition: If Ω is a compact Hausdorff space, and $\varphi : C(\Omega) \rightarrow \mathbb{F}$ is linear, then φ is positive if and only if $\|\varphi\| = \varphi(\mathbb{1}_\Omega)$.

Proof. Let $h \in C(\Omega, \mathbb{R})$. Then,

$$-\|h\| \mathbb{1}_\Omega \leq h \leq \|h\| \mathbb{1}_\Omega.$$

Thus, since φ is positive,

$$-\|h\| \varphi(\mathbb{1}_\Omega) \leq \varphi(h) \leq \|h\| \varphi(\mathbb{1}_\Omega).$$

Thus, $|\varphi(h)| \leq \|h\| \varphi(\mathbb{1}_\Omega)$.

Let $f \in C(\Omega)$, and find $\omega \in \mathbb{C}$ such that $|\omega| = 1$ and $\omega\varphi(f) = |\varphi(f)|$. Set $g = \omega f$. Note that $\|g\|_u = \|f\|_u$. Then,

$$\begin{aligned} |\varphi(f)| &= \operatorname{Re}(|\varphi(f)|) \\ &= \operatorname{Re}(\omega\varphi(f)) \\ &= \operatorname{Re}(\varphi(\omega f)) \\ &= \operatorname{Re}(\varphi(g)) \\ &= \frac{1}{2} \left(\varphi(g) + \overline{\varphi(g)} \right) \\ &= \varphi\left(\frac{g + \bar{g}}{2}\right) \\ &= \varphi(\operatorname{Re}(g)) \\ &\leq \varphi(\mathbb{1}_\Omega) \|\operatorname{Re}(g)\|_u \\ &\leq \varphi(\mathbb{1}_\Omega) \|g\| \\ &= \varphi(\mathbb{1}_\Omega) \|f\|_u, \end{aligned}$$

meaning $\|\varphi\|_{op} \leq \varphi(\mathbb{1}_\Omega)$.

Suppose $\|\varphi\| = \varphi(\mathbb{1}_\Omega)$. We start by claiming that if $h \in C(\Omega, \mathbb{R})$, then $\varphi(h) \in \mathbb{R}$. Let $\varphi(h) = a + bi$; we will show that $b = 0$. Let $t \in \mathbb{R}$ be arbitrary.

$$\begin{aligned}\|h + it\mathbb{1}_\Omega\|_u^2 &= \left(\sup_{x \in \Omega} |h(x) + it| \right)^2 \\ &= \sup_{x \in \Omega} |h(x) + it|^2 \\ &= \sup_{x \in \Omega} (h(x)^2 + t^2) \\ &= \|h\|_u^2 + t^2.\end{aligned}$$

Thus,

$$\begin{aligned}|\varphi(h + it\mathbb{1}_\Omega)|^2 &\leq (\varphi(\mathbb{1}_\Omega))^2 \|h + it\mathbb{1}_\Omega\|^2 \\ &= (\varphi(\mathbb{1}_\Omega))^2 (\|h\|^2 + t^2).\end{aligned}$$

Alternatively,

$$\begin{aligned}|\varphi(h + it\mathbb{1}_\Omega)|^2 &= |\varphi(h) + it\varphi(\mathbb{1}_\Omega)|^2 \\ &= |a + i(b + t\varphi(\mathbb{1}_\Omega))|^2 \\ &= a^2 + (b + t\varphi(\mathbb{1}_\Omega))^2 \\ &= a^2 + b^2 + t^2(\varphi(\mathbb{1}_\Omega))^2 + 2bt\varphi(\mathbb{1}_\Omega).\end{aligned}$$

Thus, we get $a^2 + b^2 + 2bt\varphi(\mathbb{1}_\Omega) \leq (\varphi(\mathbb{1}_\Omega))^2 \|h\|^2$. If $b \neq 0$, sending $t \rightarrow \infty$ yields a contradiction. Thus, $b = 0$.

Assume $f \in C(\Omega)_+$. Observe that $\text{Ran}(f - \|f\|\mathbb{1}_\Omega) \subseteq [-\|f\|, \|f\|]$, meaning $\|f - \|f\|\mathbb{1}_\Omega\| \leq \|f\|$. Thus,

$$\begin{aligned}|\varphi(f) - \|f\|\varphi(\mathbb{1}_\Omega)| &= |\varphi(f - \|f\|\mathbb{1}_\Omega)| \\ &\leq \varphi(\mathbb{1}_\Omega) \|f - \|f\|\mathbb{1}_\Omega\| \\ &\leq \varphi(\mathbb{1}_\Omega) \|f\|.\end{aligned}$$

Since $\varphi(f) \in \mathbb{R}$, we have $\varphi(f) \geq 0$, so φ is positive. □

Corollary: Let Ω be a compact Hausdorff space. The state space

$$S(\Omega) = \{\varphi \in C(\Omega)^* \mid \varphi(\mathbb{1}_\Omega) = 1 = \|\varphi\|\}$$

is convex.

Proof. Suppose $\varphi_1, \varphi_2 \in S(\Omega)$, $t \in [0, 1]$. Then, $(1 - t)\varphi_1 + t\varphi_2$ is a positive functional, meaning

$$\begin{aligned}\|(1 - t)\varphi_1 + t\varphi_2\| &= (1 - t)\varphi(\mathbb{1}_\Omega) + t\varphi(\mathbb{1}_\Omega) \\ &= (1 - t) + t \\ &= 1,\end{aligned}$$

so $S(\Omega)$ is convex. □

Now, we can associate bounded linear functionals with regular measures.

Theorem (Riesz Representation Theorem): Let Ω be a locally compact Hausdorff space. If $\varphi : C_c(\Omega) \rightarrow \mathbb{C}$ is a positive linear functional, then there exists a unique Radon measure μ such that

$$\varphi(f) = \int_{\Omega} f \, d\mu,$$

with $|\varphi(f)| \leq \|f\| \mu(\text{supp}(f))$. The Radon measure μ also satisfies the following.

(1) For every open $U \subseteq \Omega$, we have

$$\mu(U) = \sup \{ \varphi(f) \mid f \in C_c(\Omega, [0, 1]), \text{supp}(f) \subseteq U \}.$$

(2) For every compact $K \subseteq \Omega$,

$$\mu(K) = \inf \{ \varphi(f) \mid f \geq \mathbb{1}_K \}.$$

The following proof is adapted from Folland's *Real Analysis*. In order to provide a complete proof of the Riesz Representation theorem, we need some previously established theorems and lemmas.

Lemma: Let A and B be disjoint subsets of a normal space X .^{xiii} Let $\Delta = \{k2^{-n} \mid n \geq 1, 0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$.

There is a family $\{U_r \mid r \in \Delta\}$ of open sets in X such that $A \subseteq U_r \subseteq B^c$ for all $r \in \Delta$, and $\overline{U_r} \subseteq U_s$ for $r < s$.

Proof. Since X is normal, there exist disjoint open sets V and W such that $A \subset V$ and $B \subset W$.

Let $U_{1/2} = V$. Since W^c is closed, it is the case that

$$A \subset U_{1/2} \subset \overline{U_{1/2}} \subset W^c \subset B^c.$$

We select U_r for $r = k2^{-n}$ by inducting on n .

Suppose we have chosen U_r for $r = k2^{-n}$ for $0 < k < 2^n$ and $n \leq N - 1$. To find U_r for $r = (2j + 1)2^{-N}$ (where $0 \leq j \leq 2^{N-1}$), we see that

$$\overline{U_{j2^{1-N}}} \cap (U_{(j+1)2^{1-N}})^c = \emptyset,$$

where we have defined $\overline{U_0} = A$ and $U_1^c = B$. So, we can select U_r such that

$$A \subset \overline{U_{j2^{1-N}}} \subset U_r \subset \overline{U_r} \subset U_{(j+1)2^{1-N}} \subset B^c.$$

□

Theorem (Urysohn's Lemma): Let X be a normal space. Then, if A and B are disjoint closed sets in X , there exists $f \in C(X, [0, 1])$ such that $f(A) = 0$ and $f(B) = 1$.

Proof. Let U_r be defined by the previous lemma for $r \in \Delta$. Set $U_1 = X$.

For $x \in X$, define $f(x) = \inf \{r \mid x \in U_r\}$. Since $A \subset U_r \subseteq B^c$ for $0 < r < 1$, we have $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$, and $0 \leq f(x) \leq 1$ for all $x \in X$.

Now, we need to show that f is continuous.

We can see that $f(x) < \alpha$ if and only if $x \in U_r$ for some $r < \alpha$, which happens if and only if $x \in \bigcup_{r < \alpha} U_r$. Thus, $f^{-1}((-\infty, \alpha)) = \bigcup_{r < \alpha} U_r$.

Additionally, $f(x) > \alpha$ if and only if $x \notin U_r$ for some $r > \alpha$, which happens if and only if $x \notin \overline{U_s}$ for some $s > \alpha$, thus $x \in \bigcup_{s > \alpha} (\overline{U_s})^c$, meaning $f^{-1}((\alpha, \infty)) = \bigcup_{s > \alpha} (\overline{U_s})^c$ is open.

Since the open half lines generate the topology on \mathbb{R} , we have f is continuous. □

Proposition: Let X be a locally compact Hausdorff space, and $K \subset U \subset X$, where K is compact, and U is open. Then, there exists an open set V with compact closure such that $K \subset V \subset \overline{V} \subset U$.

^{xiii}A normal space is a space where, if A and B are disjoint closed subsets, then there exist disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.

Proof. Since X is locally compact, we can choose a compact neighborhood N_x for each $x \in K$ with $N_x \subset U$. Then

$$K \subset \bigcup_{x \in K} N_x^\circ$$

is an open cover of K . Let $\{N_{x_i}\}_{i=1}^n$ be a finite subcover, and let $V = \bigcup_{i=1}^n N_{x_i}^\circ$. Then, $K \subset V$, and $\bar{V} = \bigcup_{i=1}^n \bar{N}_{x_i}$ is compact and contained in U . \square

Theorem (Urysohn's Lemma for Locally Compact Hausdorff Spaces): Let X be a locally compact Hausdorff space, and $K \subset U \subset X$ with K compact and U open. Then, there exists $f \in C_c(X, [0, 1])$ such that $f = 1$ on K and $\text{supp}(f) \subset U$.

Proof. Let V be open such that $K \subset V \subset \bar{V} \subset U$. Then, \bar{V} is normal, so there exists $f \in C(\bar{V}, [0, 1])$ such that $f = 1$ on K and $f = 0$ on ∂V , as $\partial V \subset \bar{V}$ is closed and $\partial V \cap K = \emptyset$. We extend f to X by setting $f = 0$ on \bar{V}^c .

Let $E \subseteq [0, 1]$ be closed. If $0 \notin E$, then $f^{-1}(E) = (f|_{\bar{V}})^{-1}(E)$, and if $0 \in E$, then $f^{-1}(E) = (f|_{\bar{V}})^{-1}(E) \cup \bar{V}^c$, as $\partial V \subseteq (f|_{\bar{V}})^{-1}(E)$. Thus, $f^{-1}(E)$ is closed, so f is continuous. \square

Proposition (Partition of Unity): If K is a compact set in a locally compact Hausdorff space Ω with an open cover $\{U_j\}_{j=1}^n$, then there is a partition of unity^{xiv} on K consisting of compactly supported functions g_j such that $\text{supp}(g_j) \subseteq U_j$.

Proof. For each $x \in K$, there is a compact neighborhood N_x such that $N_x \subseteq U_j$ for some j . The set $\{N_x^\circ \mid x \in K\}$ is an open cover of K , meaning there exist x_1, \dots, x_m such that $K \subseteq \bigcup_{k=1}^m N_{x_k}$.

Let F_j be the union of the set of N_{x_k} that are a subset of U_j . Then, $F_j \subseteq U_j$ is compact, so by Urysohn's lemma, there are $g_1, \dots, g_n \in C_c(\Omega)$ such that $g_j = 1$ on F_j and $\text{supp}(g_j) \subseteq U_j$.

Since F_j covers K , it is the case that $\sum_{k=1}^n g_k \geq 1$ on K . By Urysohn's Lemma, there $f \in C_c(\Omega, [0, 1])$ such that $f = 1$ on K and $\text{supp}(f) \subseteq \{x \mid \sum_{k=1}^n g_k(x) > 0\}$.

Define $g_{n+1} = 1 - f$, meaning $\sum_{k=1}^{n+1} g_k > 0$ everywhere. For $j = 1, \dots, n$, let $h_j = \frac{g_j}{\sum_{k=1}^{n+1} g_k}$. Then, $\text{supp}(h_j) = \text{supp}(g_j)$, $\text{supp}(h_j) \subseteq U_j$, and $\sum_{j=1}^n h_j = 1$ on K . \square

With these results in mind, we can now begin the proof of the Riesz Representation theorem.

Notation: For an open set U , we let $f < U$ denote $f \in C_c(\Omega, [0, 1])$ and $\text{supp}(f) \subset U$.

Proof. We start by establishing uniqueness.

Let μ be a Radon measure such that $\varphi(f) = \int_\Omega f \, d\mu$ for all $f \in C_c(\Omega)$. For $U \subseteq X$ open, it is the case that $\varphi(f) \leq \mu(U)$ for $f < U$.

If $K \subseteq U$ is compact, Urysohn's lemma gives $f \in C_c(\Omega)$ such that $f < U$ and $f = 1$ on K , meaning $\mu(K) \leq \int_\Omega f \, d\mu = \varphi(f)$. Since μ is inner regular on U , we satisfy condition (1). Thus, μ is determined on all open sets by φ , and so is determined on all Borel sets since μ is outer regular.

To show existence, we start by defining

$$\mu(U) = \sup \{ \varphi(f) \mid f \in C_c(\Omega), f < U \}.$$

^{xiv}A collection of functions in $C(X, [0, 1])$ such that their sum is 1 on E and every neighborhood has only finitely many nonzero functions.

For any $E \subseteq X$, we define

$$\mu^*(E) = \inf \{ \mu(U) \mid E \subseteq U, U \text{ open} \}.$$

Note that by the way we define μ , if $U \subseteq V$, then $\mu(U) \leq \mu(V)$, meaning $\mu^*(U) = \mu(U)$ if U is open.

We begin by showing that μ^* is an outer measure. It suffices to show that for a sequence of open sets $\{U_j\}_{j=1}^\infty$ such that $U = \bigcup_{j=1}^\infty U_j$, $\mu(U) \leq \sum_{j=1}^\infty \mu(U_j)$. From this, we can see that, for any $E \subseteq X$,

$$\begin{aligned} \mu^*(E) &= \inf \left\{ \sum_{j=1}^\infty \mu(U_j) \mid U_j \text{ open}, E \subseteq \bigcup_{j=1}^\infty U_j \right\} \\ &= \inf \{ \mu(U) \mid E \subseteq U, U \text{ open} \} \end{aligned}$$

defines an outer measure.

Let $U = \bigcup_{j=1}^\infty U_j$ and $f \in C_c(\Omega)$ with $f < U$. Let $K = \text{supp}(f)$. Since K is compact, $K \subseteq \bigcup_{i=1}^n U_i$ for some n .

By the partition of unity proposition, we can construct $g_i < U_i$ with $\sum_{i=1}^n g_i = 1$ on K . Then, $f = \sum_{i=1}^n f g_i$, with $f g_i < U_i$. Thus,

$$\begin{aligned} \varphi(f) &= \sum_{i=1}^n \varphi(f g_i) \\ &\leq \sum_{i=1}^n \mu(U_i) \\ &\leq \sum_{j=1}^\infty \mu(U_j). \end{aligned}$$

Thus, $\sup_{f \in C_c(\Omega)} (\varphi(f)) = \mu(U) \leq \sum_{j=1}^\infty \mu(U_j)$. Since this holds for any $f < U$, we conclude that $\mu(U) \leq \sum_{j=1}^\infty \mu(U_j)$.

We now show that for any open set $U \subseteq \Omega$, U is μ^* -measurable.

Let U be open, $E \subseteq \Omega$ with $\mu^*(E) < \infty$. Suppose E is open. Then, $E \cap U$ is open, so for $\varepsilon > 0$, we can find $f \in C_c(\Omega)$ such that $f < E \cap U$ and $\varphi(f) > \mu(E \cap U) - \varepsilon$. Additionally, $E \setminus \text{supp}(f)$ is open, so we can find $g \in C_c(\Omega)$ such that $g < E \setminus \text{supp}(f)$, and $\varphi(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon$. However, $f + g < E$, so

$$\begin{aligned} \mu(E) &\geq \varphi(f) + \varphi(g) \\ &> \mu(E \cap U) + \mu(E \setminus \text{supp}(f)) - 2\varepsilon \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon. \end{aligned}$$

We obtain the desired inequality as $\varepsilon \rightarrow 0$.

For any $E \subseteq \Omega$, we find V open with $E \subseteq V$ and $\mu(V) < \mu^*(E) + \varepsilon$ (by our definition of μ^*). Thus,

$$\begin{aligned} \mu^*(E) + \varepsilon &> \mu(V) \\ &\geq \mu^*(V \cap U) + \mu^*(V \setminus U) \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus U). \end{aligned}$$

We let $\varepsilon \rightarrow 0$ to obtain our desired inequality.

Now, we show that for any compact $K \subseteq \Omega$, $\mu(K) = \inf \{ \varphi(f) \mid f \in C_c(\Omega), f \geq 1_K \}$.

Let K be compact, $f \in C_c(\Omega)$, and $f \geq 1_K$. Let $U_\varepsilon = \{x \mid f(x) > 1 - \varepsilon\}$. Then, U_ε is open, and for $g < U_\varepsilon$, we have $(1 - \varepsilon)^{-1} f - g \geq 0$.

Thus, $\varphi(g) \leq (1 - \varepsilon)^{-1} \varphi(f)$, meaning $\mu(K) \leq \mu(U_\varepsilon) \leq (1 - \varepsilon)^{-1} \varphi(f)$. Letting $\varepsilon \rightarrow 0$, we find $\mu(K) \leq \varphi(f)$.

For any $K \subseteq U$, Urysohn's lemma gives $f \in C_c(\Omega)$ such that $f \geq 1_K$ and $f < U$, meaning $\varphi(f) \leq \mu(U)$. Since μ is outer regular, we obtain condition (2).

With our measure μ defined, we only need to show that $\varphi(f) = \int_\Omega f \, d\mu$ for $f \in C_c(\Omega, [0, 1])$, as $C_c(\Omega)$ is the linear span of $C_c(\Omega, [0, 1])$.

Set $f \in C_c(\Omega, [0, 1])$. Given $N \in \mathbb{N}$, for $1 \leq j \leq N$, let $K_j = \left\{x \mid f(x) \geq \frac{j}{N}\right\}$. Let $K_0 = \text{supp}(f)$. Define $f_1, \dots, f_N \in C_c(\Omega)$ with $f_j(x) = 0$ for $x \notin K_{j-1}$, $f_j(x) = f(x) - \frac{j-1}{N}$ for $x \in K_{j-1} \setminus K_j$, and $f_j = \frac{1}{N}$ for $x \in K_j$.

We, thus have $\frac{1_{K_j}}{N} \leq f_j \leq \frac{1_{K_{j-1}}}{N}$, meaning

$$\frac{1}{N} \mu(K_j) \leq \int_\Omega f_j \, d\mu \leq \frac{1}{N} \mu(K_{j-1}).$$

If U is an open set containing K_{j-1} , then $Nf_j < U$, meaning $\varphi(f_j) \leq \frac{\mu(U)}{N}$. Thus, by condition (2), and outer regularity,

$$\frac{1}{N} \mu(K_j) \leq \varphi(f_j) \leq \frac{1}{N} \mu(K_{j-1}).$$

Since $f = \sum_{j=1}^N f_j$, we have

$$\frac{1}{N} \sum_{j=1}^N \mu(K_j) \leq \int_\Omega f \, d\mu \leq \frac{1}{N} \sum_{j=1}^N \mu(K_{j-1}),$$

and

$$\frac{1}{N} \sum_{j=1}^N \mu(K_j) \leq \varphi(f) \leq \frac{1}{N} \sum_{j=1}^{N-1} \mu(K_j).$$

Thus,

$$\begin{aligned} \left| \varphi(f) - \int_\Omega f \, d\mu \right| &\leq \frac{\mu(K_0) - \mu(K_N)}{N} \\ &\leq \frac{\mu(\text{supp}(f))}{N}. \end{aligned}$$

Since $\mu(\text{supp}(f)) < \infty$, and N is arbitrary, we have $I(f) = \int_\Omega f \, d\mu$. □

We can state the Riesz representation theorem for functions on $C(X)$.

Theorem (Riesz Representation Theorem, $C(X)$): Let X be compact. Let $\varphi \in (C(X))^*$ be a state. Then, for $f \in C(X)$, there is a unique Borel probability measure μ such that

$$\varphi(f) = \int_X f \, d\mu.$$

Proof. We start by letting $X = \Delta$, where $\Delta = \prod_{\mathbb{N}} \{0, 1\}$ be the Cantor set. Here, $\{0, 1\}$ is endowed with the discrete topology, and Δ the product topology.

We consider $C \subseteq \mathcal{P}(\Delta)$ to be the set of all clopen subsets of Δ . Since the empty set and Δ are clopen, the complement of any clopen set is clopen, and the finite union of clopen sets is clopen, it is the case that C is an algebra of subsets of Δ .

For $E \in C$, we define $\mu_0(E) = \varphi(\mathbb{1}_E)$. We can see that

$$\begin{aligned}\mu_0(E \sqcup F) &= \varphi(\mathbb{1}_{E \sqcup F}) \\ &= \varphi(\mathbb{1}_E + \mathbb{1}_F) \\ &= \varphi(\mathbb{1}_E) + \varphi(\mathbb{1}_F) \\ &= \mu_0(E) + \mu_0(F).\end{aligned}$$

Additionally, for a pairwise disjoint countable collection $\{E_k\}_{k \geq 1} \subseteq C$ with $\bigcup_{k \geq 1} E_k \in C$, we have

$$\begin{aligned}\mu_0\left(\bigcup_{k \geq 1} E_k\right) &= \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{k=1}^n E_k\right) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu_0(E_k) \\ &= \sum_{k=1}^{\infty} \mu_0(E_k).\end{aligned}$$

Thus, we have established μ_0 as a premeasure on C .

To see that μ_0 is σ -finite, we examine $\Delta' \subseteq [0, 1]$ homeomorphic to the Cantor set, where $r : \Delta \rightarrow \Delta'$ is the homeomorphism. There, we can see that $E'_k = [0, \frac{1}{k}] \cap \Delta'$ is a clopen set, so $r^{-1}(E'_k) \in C$.

Thus, $\Delta = r^{-1}(\bigcup_{k \geq 1} E'_k)$, meaning

$$\begin{aligned}\mu_0\left(\bigcup_{k \geq 1} r^{-1}(E'_k)\right) &= \varphi\left(\mathbb{1}_{\bigcup_{k \geq 1} r^{-1}(E'_k)}\right) \\ &= \varphi(\mathbb{1}_\Delta) \\ &= 1,\end{aligned}$$

meaning μ_0 is σ -finite.

By Carathéodory's extension theorem, μ_0 extends to a unique measure μ on $\sigma(C)$.

We now show that $\sigma(C) = \mathcal{B}_\Delta$. To do this, we can see that since Δ is totally disconnected and (locally) compact, Δ is zero-dimensional, and thus admits a base of compact open (hence clopen) sets. Since the topology τ on Δ is generated by C , we have $\tau \subseteq \sigma(C)$, so

$$\begin{aligned}\mathcal{B}_\Delta &= \sigma(\tau) \\ &\subseteq \sigma(C),\end{aligned}$$

so $\mathcal{B}_\Delta = \sigma(C)$. Thus, the unique measure μ that extends μ_0 is a Borel probability measure on Δ .

Let $f \in C(\Delta, \mathbb{R})$ be positive. Then, there is a pointwise increasing sequence of simple functions, $(s_n)_n$ such that $(s_n)_n \rightarrow f$ pointwise, meaning f is measurable. Since $f \in C(\Delta)$, f is bounded, so $(s_n)_n$ can be defined such that $s_n \leq f$ for all n , meaning that the convergence $(s_n)_n \rightarrow f$ is uniform.

By dominated convergence, we have

$$\lim_{n \rightarrow \infty} \varphi(s_n) = \lim_{n \rightarrow \infty} \int_{\Delta} s_n \, d\mu$$

$$\begin{aligned}
&= \int_{\Delta} \lim_{n \rightarrow \infty} s_n \, d\mu \\
&= \int_{\Delta} f \, d\mu.
\end{aligned}$$

For the general case of $f \in C(\Delta)$, we split f into

$$f = (u_+ - u_-) + i(v_+ - v_-),$$

and apply the previous process to each of u_{\pm}, v_{\pm} . Thus,

$$\varphi(f) = \int_{\Delta} f \, d\mu.$$

For the general case of X as a compact metric space, we use the following theorem.

Theorem (Cantor Surjection Theorem): Let X be a compact metric space. There is a surjection $f : \Delta \rightarrow X$ which induces an embedding $C(X) \hookrightarrow C(\Delta)$.

To prove this theorem, we use some lemmas. The proof of the Cantor Surjection Theorem is adapted from Aliprantis and Border's *Infinite Dimensional Analysis*.

Lemma: If X is a compact metric space, X is homeomorphic to a subset of the Hilbert cube, $[0, 1]^{\mathbb{N}}$.

Proof. Let X be a compact metric space. Then, X is normal. Let \mathcal{B} be a countable topological base of nonempty subsets of X , and let $C = \{(U, V) \mid \overline{U} \subset V \text{ and } U, V \in \mathcal{B}\}$. Since X is normal, C is nonempty, and since \mathcal{B} is countable, so too is C .

For each n , we select a continuous real function f_n with values in $[0, 1]$ such that $f_n(\overline{U}) = 1$ and $f_n(V^c) = 0$; such a function must exist by Urysohn's Lemma. Since X is Hausdorff, the family $\{f_n\}_{n=1}^{\infty}$ must separate points.

Define $\varphi : X \rightarrow [0, 1]^{\mathbb{N}}$ by

$$\varphi(x) = (f_n(x))_n.$$

If C is finite, we fill the sequence with zero functions.

Since $\{f_n\}_{n=1}^{\infty}$ separates the points of X , φ is one to one, and since f_n is continuous, so too is φ . All we need show is that φ^{-1} is continuous.

Let $\varphi(x_{\alpha}) \rightarrow \varphi(x)$ be a net, and let W be a neighborhood of x . Since \mathcal{B} is a base, there is some U_n and some V_n in C such that $x \in U_n \subset \overline{U_n} \subset V_n \subset W$, meaning $f_n(x) = 1$. Thus, we must have $f_n(x_{\alpha}) \rightarrow f_n(x)$ for each n , meaning that for large enough α , $f_n(x_{\alpha}) > 0$. This means $x_{\alpha} \in V_n \subseteq W$ for large α . Thus, $x_{\alpha} \rightarrow x$. \square

Lemma: The Cantor set, Δ , is homeomorphic with $\Delta^{\mathbb{N}}$.

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be a partition of \mathbb{N} consisting of pairwise disjoint countably infinite sets. We order each A_k according to the algebraic ordering on \mathbb{N} . In particular, $A_k = \{n_{i,k}\}_{i=1}^{\infty}$.

We let $a = (a_1, a_2, \dots) \in \Delta^{\mathbb{N}}$, and take $a_k = (a_{1,k}, a_{2,k}, \dots)$ be the k th index of each a_i , and define $b_{n_{i,k}} = a_{i,k}$.

We define $\psi : \Delta^{\mathbb{N}} \rightarrow \Delta$ by $\psi(a_1, a_2, \dots) = (b_1, b_2, \dots)$. Then, it is the case that ψ is a continuous bijection, meaning ψ is a homeomorphism. \square

Lemma: For each closed $K \subseteq \Delta$, there exists a continuous surjection $f : \Delta \rightarrow K$ with $f(x) = x$ for each $x \in K$.

Proof. Let $K \subseteq \Delta$ be closed. Then, K is compact. Thus, for each $x \in \Delta$, there is a unique $y \in K$ such that $d(x, y)$ is minimized. We define $f(x) = y$.

Since $\text{dist}_K(x)$ is continuous, it is the case that f is a continuous surjection with $f(x) = x$ for all $x \in K$. \square

Lemma: For any proper map (a continuous map where the preimage of any compact set is compact) $\tau : \Omega \rightarrow \Lambda$, the induced map $T_\tau : C_0(\Lambda) \rightarrow C_0(\Omega)$ is a linear isometric embedding, where $T_\tau(f) = f \circ \tau$.

If τ is surjective, then T_τ is injective.

Proof. Since f and τ are continuous, $T_\tau(f)$ is also continuous (and linear, since composition is linear in the first argument).

Additionally, since τ is proper, it is the case that T_τ vanishes at infinity. For any $\varepsilon > 0$ and $f \in C_0(\Lambda)$, there is a compact $K \subseteq \Lambda$ such that $|f(y)| \leq \varepsilon$ for $y \notin K$. If $x \notin \tau^{-1}(K)$, then $\tau(x) \notin K$, so $|T_\tau(f)(x)| = |f(\tau(x))| \leq \varepsilon$.

If τ is surjective, then if $T_\tau(f) = 0$, then $f|_{\text{Ran}(\tau)} = 0$, so $f = 0$. We have shown already that if τ is surjective, then $\|T_\tau(f)\| = \|f\|_\infty$, meaning T_τ is an isometry. \square

With all these in mind, we can now prove the Cantor surjection theorem.

Proof of Cantor Surjection Theorem. Let X be a compact metric space. Then, there is an embedding $\varphi : X \hookrightarrow [0, 1]^N$. Let $Y = \varphi(X) \subseteq [0, 1]^N$.

There is a continuous surjection $\psi : \Delta \rightarrow [0, 1]^N$. Since Y is compact (hence closed), $\psi^{-1}(Y) \subseteq \Delta$ is closed. There exists $f : \Delta \rightarrow \psi^{-1}(Y)$ such that $f(x) = x$ for all $x \in \psi^{-1}(Y)$.

We can make a continuous surjection from Δ to X by taking

$$\Delta \xrightarrow{f} \psi^{-1}(Y) \xrightarrow{\psi} Y \xrightarrow{\varphi^{-1}} X.$$

Note that since the composition $\tau = \varphi^{-1} \circ \psi \circ f$ is surjective, it is the case that the induced map $T_\tau : C(X) \hookrightarrow C(\Delta)$ is a linear isometric embedding. \square

Now that we have proven the Cantor surjection theorem, we let X be any compact metric space, and $\tau : \Delta \rightarrow X$ be a continuous surjection. Letting μ be the given measure on $C(\Delta)$, we use the pushforward measure $\nu = \tau_*\mu$ on X by setting

$$\begin{aligned} \nu(E) &= \tau_*\mu(E) \\ &= \mu\left(\tau^{-1}(E)\right). \end{aligned}$$

Thus, we have proven the Riesz representation theorem for $C(X)$. \square

Having proven the Riesz Representation Theorem, we can begin to describe all bounded linear functionals on continuous function spaces, starting with the positive ones.

Proposition: Let Ω be a locally compact Hausdorff space.

(1) For μ a positive and finite regular measure on Ω ,

$$\begin{aligned} \varphi_\mu : C_0(\Omega) &\rightarrow \mathbb{C} \\ \varphi_\mu(f) &= \int_\Omega f \, d\mu \end{aligned}$$

defines a positive linear functional with $\|\varphi_\mu\|_{\text{op}} = \mu(\Omega)$. Moreover, φ_μ is faithful if μ has full support.

- (2) If $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ is a positive linear functional, then there exists a unique positive and finite regular measure μ_φ such that $\varphi = \varphi_{\mu_\varphi}$. That is,

$$\varphi(f) = \int_{\Omega} f \, d\mu_\varphi.$$

Moreover, $\|\varphi\|_{\text{op}} = \mu(\Omega)$, and μ has full support if φ is faithful.

Proof. The map

$$\varphi_\mu(f) = \int_{\Omega} f \, d\mu$$

is positive, linear, and well-defined, since

$$\begin{aligned} |\varphi_\mu(f)| &\leq \left| \int_{\Omega} f \, d\mu \right| \\ &\leq \int_{\Omega} |f| \, d\mu \\ &\leq \|f\|_{\infty} \mu(\Omega). \end{aligned}$$

This also shows that $\|\varphi_\mu\|_{\text{op}} \leq \mu(\Omega)$. Given $\varepsilon > 0$, the regularity of μ allows us to find $K \subseteq \Omega$ with $\mu(\Omega) - \varepsilon < \mu(K)$. Urysohn's lemma gives a continuous $f \in C_c(\Omega, [0, 1])$ with $f = 1$ on K . Thus,

$$\begin{aligned} \|\varphi_\mu\|_{\text{op}} &\geq \varphi_\mu(f) \\ &= \int_{\Omega} f \, d\mu \\ &\geq \int_{\Omega} \mathbb{1}_K \, d\mu \\ &= \mu(K) \\ &> \mu(\Omega) - \varepsilon. \end{aligned}$$

Since ε was arbitrary, $\mu(\Omega) \leq \|\varphi_\mu\|_{\text{op}}$.

Suppose $g \in C_0(\Omega)$ with $g \geq 0$ and $g \neq 0$. Then, there is a nonempty open subset $U \subseteq \Omega$ and $\delta > 0$ such that $g(x) \geq \delta$ for all $x \in U$. If μ has full support, $\mu(U) > 0$, meaning

$$\begin{aligned} \varphi(g) &= \int_{\Omega} g \, d\mu \\ &\geq \int_{\Omega} \delta \mathbb{1}_U \, d\mu \\ &= \delta \mu(U) \\ &> 0. \end{aligned}$$

Thus, we have proven (1).

If $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ is a positive linear functional, then φ is bounded. The restriction $\varphi_0 = \varphi|_{C_c(\Omega)} : C_c(\Omega) \rightarrow \mathbb{C}$ is positive and linear, meaning $\varphi_0 = I_\mu$ for some Radon measure μ by the Riesz Representation Theorem. We set $\mu_\varphi = \mu$.

By the Riesz Representation theorem, we have

$$\mu(\Omega) = \sup \{ I_\mu(f) \mid f \in C_c(\Omega, [0, 1]), \text{ supp}(f) \subseteq \Omega \}$$

$$\begin{aligned}
&\leq \sup \{ \varphi(f) \mid f \in C_0(\Omega), \|f\| \leq 1 \} \\
&= \|\varphi\|_{\text{op}} \\
&< \infty.
\end{aligned}$$

Additionally, if $f \in C_c(\Omega)$, then

$$\begin{aligned}
|\varphi(f)| &= |\varphi_0(f)| \\
&\leq \left| \int_{\Omega} f \, d\mu \right| \\
&\leq \int_{\Omega} |f| \, d\mu \\
&\leq \|f\|_{\infty} \mu(\Omega).
\end{aligned}$$

Since $C_c(\Omega) \subseteq C_0(\Omega)$ is dense, we have $|\varphi(f)| \leq \|f\|_{\infty} \mu(\Omega)$ for all $f \in C_0(\Omega)$, meaning $\|\varphi\|_{\text{op}} \leq \mu(\Omega)$.

Since μ is finite and Radon, μ is regular. We can see that φ and φ_{μ} are bounded functionals that agree on $C_c(\Omega)$, so $\varphi = \varphi_{\mu}$.

Suppose φ is faithful. Let $U \subseteq \Omega$ be open and nonempty. By Urysohn's lemma, there is $f : \Omega \rightarrow [0, 1]$ that is nonzero and compactly supported with $\text{supp}(f) \subseteq U$. Thus,

$$\begin{aligned}
\mu(U) &= \int_{\Omega} \mathbb{1}_U \, d\mu \\
&\geq \int_{\Omega} f \, d\mu \\
&= \varphi_{\mu}(f) \\
&= \varphi(f) \\
&> 0,
\end{aligned}$$

meaning μ has full support. □

We let $\mathcal{P}_r(\Omega)$ be the convex set of all regular probability measures on $(\Omega, \mathcal{B}_{\Omega})$. The following corollary follows from the previous proposition.

Corollary: Let Ω be a compact Hausdorff space. There exists a bijective affine map $T : S(\Omega) \rightarrow \mathcal{P}_r(\Omega)$ given by $T(\varphi) = \mu_{\varphi}$.

We can use the positive linear functionals to generate all bounded linear functionals on a function space.

Theorem (Jordan Decomposition): Let Ω be a locally compact Hausdorff space, and suppose $\varphi : C_0(\Omega) \rightarrow \mathbb{C}$ be bounded linear. Then, there exists positive linear functionals $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ such that

$$\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4),$$

and $\|\varphi_j\|_{\text{op}} \leq \|\varphi\|_{\text{op}}$.

Remark: The Jordan decomposition for positive linear functionals follows from the Hahn decomposition for complex measures.

If $\varphi \in C_0(\Omega)^*$, then the Jordan decomposition provides $\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4)$, which gives the positive regular measures μ_1, \dots, μ_4 such that $\varphi_j(f) = \int_{\Omega} f \, d\mu_j$ for $f \in C_0(\Omega)$.

Then, $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ has total variation $|\mu|$, meaning it is regular itself. Thus,

$$\begin{aligned}
\varphi(f) &= \varphi_1(f) - \varphi_2(f) + i(\varphi_3(f) - \varphi_4(f)) \\
&= \int_{\Omega} f \, d\mu_1 - \int_{\Omega} f \, d\mu_2 + i \left(\int_{\Omega} f \, d\mu_3 - \int_{\Omega} f \, d\mu_4 \right)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} f \, d\mu \\
&= \varphi_{\mu}(f).
\end{aligned}$$

Thus, we get the following characterization of $C_0(\Omega)^*$.

Theorem (Riesz–Markov Theorem): Let Ω be a locally compact Hausdorff space. Then, $M_r(\Omega) \cong C_0(\Omega)^*$. The map $\Phi : M_r(\Omega) \rightarrow C_0(\Omega)^*$ defined by $\Phi(\mu) = \varphi_{\mu}$, where

$$\varphi_{\mu} = \int_{\Omega} f \, d\mu,$$

is an isometric isomorphism.

The Banach Space Adjoint

Definition. Let X, Y be normed vector spaces, and let $T \in \mathcal{B}(X, Y)$. Then, the map $T^* : Y^* \rightarrow X^*$, defined by $T^*(\varphi) = \varphi \circ T$ is called the Banach space adjoint (or adjoint) of T .

We can see that T^* is well-defined since it is the composition of two bounded linear operators. Additionally,

$$\begin{aligned}
\|T^*(\varphi)\|_{\text{op}} &= \|\varphi \circ T\|_{\text{op}} \\
&\leq \|\varphi\|_{\text{op}} \|T\|_{\text{op}}.
\end{aligned}$$

Proposition: If X and Y are normed vector spaces, $T \in \mathcal{B}(X, Y)$, and T^* be the adjoint of T . Then, $T^* \in \mathcal{B}(Y^*, X^*)$, and $\|T^*\|_{\text{op}} = \|T\|_{\text{op}}$.

Proof. Let $\varphi_1, \varphi_2 \in Y^*$, $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned}
T^*(\varphi_1 + \alpha\varphi_2) &= (\varphi_1 + \alpha\varphi_2) \circ T \\
&= \varphi_1 \circ T + \alpha\varphi_2 \circ T \\
&= T^*(\varphi_1) + \alpha T^*(\varphi_2),
\end{aligned}$$

meaning T^* is linear. Recall that for any normed space Z and $z \in Z$, we have $\|z\| = \sup_{\varphi \in B_{Z^*}} |\varphi(z)|$. Thus, for $x \in X$, we have

$$\begin{aligned}
\|T(x)\| &= \sup_{\varphi \in B_{Y^*}} |\varphi(T(x))| \\
&= \sup_{\varphi \in B_{Y^*}} |T^*(\varphi)(x)| \\
&= \sup_{\varphi \in B_{Y^*}} |T^*(\varphi)| \|x\| \\
&\leq \|T^*\|_{\text{op}} \|x\|.
\end{aligned}$$

Thus, we have $\|T\|_{\text{op}} \leq \|T^*\|_{\text{op}}$. □

We can now understand some of the analytic and algebraic properties of the adjoint.

Theorem: Let X, Y, Z be normed spaces, and let $T, S : X \rightarrow Y$, $R : Y \rightarrow Z$ be bounded linear operators. The following are true.

- (1) $(T + S)^* = T^* + S^*$
- (2) $(\alpha T)^* = \alpha T^*$
- (3) $(R \circ T)^* = T^* \circ R^*$

(4) $(\text{id}_X)^* = \text{id}_{X^*}$

(5) The following diagram commutes.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ \uparrow \iota_X & & \uparrow \iota_Y \\ X & \xrightarrow{T} & Y \end{array}$$

Proof of (5).

$$\begin{aligned} T^{**}(\iota_X(x))(\varphi) &= T^{**}(\hat{x})(\varphi) \\ &= \hat{x} \circ T^*(\varphi) \\ &= \hat{x}(\varphi \circ T) \\ &= \varphi \circ T(x) \\ &= \widehat{T(x)}(\varphi) \\ &= \iota_Y(T(x))(\varphi). \end{aligned}$$

Thus, $T^{**}(\iota_X(x)) = \iota_Y(T(x))$, meaning $T^{**} \circ \iota_X = \iota_Y \circ T$. □

Theorem: Let X and Y be normed spaces.

- (1) If $T : X \rightarrow Y$ is an isometric isomorphism, then $T^* : Y^* \rightarrow X^*$ is an isometric isomorphism.
- (2) If $T : X \rightarrow Y$ is a bicontinuous isomorphism, then $(T^{-1})^* = (T^*)^{-1}$, so T^* is a bicontinuous isomorphism.
- (3) If $T : X \rightarrow Y$ is an isometry, then $T^* : Y^* \rightarrow X^*$ is a 1-quotient map. We have $T^*(B_{Y^*}) = B_{X^*}$.
- (4) If $T : X \rightarrow Y$ is a 1-quotient map, then $T^* : Y^* \rightarrow X^*$.
- (5) $T : X \rightarrow Y$ is an isometry if and only if $T^{**} : X^{**} \rightarrow Y^{**}$ is an isometry.
- (6) If $T \in \mathcal{B}(X, Y)$, and T^* is a 1-quotient map, then T is an isometry.
- (7) If X, Y are Banach spaces, and $T \in \mathcal{B}(X, Y)$, and T^* is an isometry, then T is a 1-quotient map.

Proof. Starting by proving (2), we have

$$\begin{aligned} \text{id}_{X^*} &= (\text{id}_X)^* \\ &= (T^{-1} \circ T)^* \\ &= T^* \circ (T^{-1})^* \end{aligned}$$

and

$$\text{id}_{X^*} = T^* \circ (T^{-1})^*,$$

meaning $(T^*)^{-1} = (T^{-1})^*$.

Now, turning our attention to (4), if T is a 1-quotient map, then $T(U_X) = U_Y$. For $\psi \in Y^*$, we have

$$\begin{aligned} \|T^*(\psi)\|_{\text{op}} &= \|\psi \circ T\|_{\text{op}} \\ &= \sup_{x \in U_X} \|\psi \circ T(x)\| \end{aligned}$$

$$\begin{aligned}
&= \sup_{y \in Y} \|\psi(y)\| \\
&= \|\psi\|_{\text{op}}.
\end{aligned}$$

Thus, T^* is an isometry.

Focusing on (3), we have $\|T^*\|_{\text{op}} = \|T\|_{\text{op}} = 1$, so $T^*(B_{Y^*}) \subseteq B_{X^*}$. Let $\varphi \in B_{X^*}$. We want to find $\psi \in B_{Y^*}$ such that $T^*(\psi) = \varphi$. Let $\psi_0 : \text{Ran}(T) \rightarrow \mathbb{F}$ be defined by $\psi_0(T(x)) = \varphi(x)$. This is well-defined since T is injective. For $x \in B_X$, we have

$$\begin{aligned}
|\psi_0(T(x))| &= |\varphi(x)| \\
&\leq \|\varphi\|_{\text{op}} \|x\| \\
&\leq \|\varphi\|_{\text{op}}.
\end{aligned}$$

Thus, $\|\psi_0\|_{\text{op}} \leq \|\varphi\|_{\text{op}} \leq 1$. There is an extension $\psi \in Y^*$ of ψ_0 such that $\|\psi\|_{\text{op}} = \|\psi_0\|_{\text{op}}$, and $\psi \circ T = \varphi$. Thus, $T^*(B_{Y^*}) = B_{X^*}$.

The veracity of (1) follows from (2), (3), and (4).

To show (5), we let T be isometric, which implies that T^* is a 1-quotient map. Thus, we have T^{**} is isometric. Conversely, if T^{**} is isometric, then

$$\begin{aligned}
\|T(x)\| &= \|\iota_Y(T(x))\| \\
&= \|T^{**}(\iota_X(x))\| \\
&= \|\iota_X(x)\| \\
&= \|x\|.
\end{aligned}$$

To see (6), if T^* is a 1-quotient map, then T^{**} is isometric, so T is isometric.

We will show (7) later. □

Example. Let $\tau : \Omega \rightarrow \Lambda$ be a proper map^{xv} between locally compact Hausdorff spaces. The induced contractive map $T_\tau : C_0(\Lambda) \rightarrow C_0(\Omega)$ has an adjoint map

$$T_\tau^* : C_0(\Omega) \rightarrow C_0(\Lambda)^*,$$

where $T_\tau^*(\varphi) = \varphi \circ T_\tau$. By the Riesz–Markov theorem, we can identify $C_0(\Omega)^* \cong M_r(\Omega)$.

The push forward measure is an induced map

$$\tau_* : M(\Omega) \rightarrow M(\Lambda),$$

defined by

$$\tau_*\mu(E) := \mu(\tau^{-1}(E))$$

for any $E \in \mathcal{B}_\Lambda$. We can verify that $\tau_*\mu$ is a complex measure on $(\Lambda, \mathcal{B}_\Lambda)$, and that $\tau_*(\mathcal{P}(\Omega)) \subseteq \mathcal{P}(\Lambda)$.

We can also see that for all $f \in C_0(\Lambda)$,

$$\int_\Omega f \circ \tau \, d\mu = \int_\Lambda f \, d(\tau_*\mu).$$

^{xv}Inverse image of a compact subset of Λ is compact in Ω

This is because, for some $E \in \mathcal{B}_\Lambda$, we have $\mathbb{1}_E \circ \tau = \mathbb{1}_{\tau^{-1}(E)}$, so

$$\begin{aligned} \int_{\Omega} \mathbb{1}_E \circ \tau \, d\mu &= \int_{\Omega} \mathbb{1}_{\tau^{-1}(E)} \, d\mu \\ &= \mu(\tau^{-1}(E)) \\ &= \tau_*\mu(E) \\ &= \int_{\Lambda} \mathbb{1}_E \, d(\tau_*\mu). \end{aligned}$$

Thus, since this identity holds for all simple functions, and every bounded function is the uniform limit of a simple function, we get that

$$\int_{\Omega} f \circ \tau \, d\mu = \int_{\Lambda} f \, d(\tau_*\mu)$$

for all $f \in C_0(\Lambda)$ by Dominated Convergence.

Thus, we can see that for any $\tau : \Omega \rightarrow \Lambda$ that is a proper map between locally compact Hausdorff spaces, then $T_{\tau}^*(\varphi_{\mu}) = \varphi_{\tau_*\mu}$ for all $\mu \in M_r(\Omega)$.

$$\begin{aligned} T_{\tau}^*(\varphi_{\mu})(f) &= \varphi_{\mu} \circ T_{\tau}(f) \\ &= \varphi_{\mu}(f \circ \tau) \\ &= \int_{\Omega} f \circ \tau \, d\mu \\ &= \int_{\Lambda} f \, d(\tau_*\mu) \\ &= \varphi_{\tau_*\mu}(f). \end{aligned}$$

This also shows that the push forward of a regular complex Borel measure is regular, or $\tau_*(M_r(\Omega)) \subseteq M_r(\Lambda)$.

Suppose we are dealing with the special case of $\tau : \Omega \rightarrow \Lambda$ is a homeomorphism. Then, T_{τ} is an isometric isomorphism of Banach spaces, meaning

$$T_{\tau}^* : C_0(\Omega)^* \rightarrow C_0(\Lambda)^*,$$

or

$$\tau_* : M_r(\Omega) \rightarrow M_r(\Lambda),$$

is an isometric isomorphism of Banach spaces. The restriction to the state space

$$T_{\tau}^*|_{S(\Omega)} : S(\Omega) \rightarrow S(\Lambda),$$

or

$$\tau_*|_{\mathcal{P}_r(\Omega)} : \mathcal{P}_r(\Omega) \rightarrow \mathcal{P}_r(\Lambda)$$

is an affine bijection.

This allows us to consider the idea of an *invariant* measure (or, equivalently, invariant state).

Definition. Let Ω be a locally compact state, and let $\tau : \Omega \rightarrow \Omega$ be a continuous transformation. A regular Borel probability measure, $\mu \in \mathcal{P}_r(\Omega)$ is called τ -invariant if $\tau_*\mu = \mu$, or $T_{\tau}^*(\varphi_{\mu}) = \varphi_{\mu}$. We write

$$\mathcal{P}_r(\Omega, \tau) := \{\mu \in \mathcal{P}_r(\Omega) \mid \tau_*\mu = \mu\}$$

to denote the set of τ -invariant measures.

We can now use the analytic and algebraic properties of adjoints to identify duals of subspaces and quotient spaces. Recall that for a normed vector space X and $E \subseteq X$ a subset, then

$$E^\perp = \{\varphi \in X^* \mid \varphi|_E = 0\}$$

is the annihilator of E .

Proposition: Let X be a normed vector space, $E \subseteq X$ a subspace. The dual space of E is isometrically isomorphic to X^*/E^\perp .

Proof. The embedding $\iota : E \hookrightarrow X$ is an isometry, so $\iota^* : X^* \rightarrow E^*$ is a 1-quotient map. The first isomorphism theorem gives $X^*/\ker(\iota^*) \cong E^*$. Thus,

$$\begin{aligned} \ker(\iota^*) &= \{\varphi \in X^* \mid \iota^*(\varphi) = 0\} \\ &= \{\varphi \in X^* \mid \varphi \circ \iota = 0\} \\ &= \{\varphi \in X^* \mid \varphi|_E = 0\} \\ &= E^\perp. \end{aligned}$$

□

Now, we can look at the dual of a quotient space.

Proposition: Let X be a normed vector space and $E \subseteq X$ a closed subspace. Then, $(X/E)^* \cong E^\perp$.

Proof. Let $\pi : X \rightarrow X/E$ be the canonical projection map. Since π is a 1-quotient map, we have $\pi^* : (X/E)^* \rightarrow X^*$ is an isometry, so $(X/E)^* \cong \text{Ran}(\pi^*)$. We have

$$\begin{aligned} \text{Ran}(\pi^*) &= \{\pi^*(\psi) \mid \psi \in (X/E)^*\} \\ &= \{\psi \circ \pi \mid \psi \in (X/E)^*\}. \end{aligned}$$

We claim that $\text{Ran}(\pi^*) = E^\perp$. For any $\psi \in (X/E)^*$, we have $\psi \circ \pi \in X^*$, and $\psi \circ \pi|_E = 0$, meaning $\text{Ran}(\pi^*) \subseteq E^\perp$.

For the reverse direction, if $\varphi \in E^\perp$, and $\varphi : X \rightarrow \mathbb{C}$ sends E to 0, then φ factors through X/E , meaning $\varphi(x) = \tilde{\varphi}(x + E)$ for some bounded linear operator $\tilde{\varphi}$ such that $\|\tilde{\varphi}\|_{\text{op}} = \|\varphi\|_{\text{op}}$. Thus, $\tilde{\varphi} \circ \pi = \varphi$, so $\varphi \in \text{Ran}(\pi^*)$, so $\text{Ran}(\pi^*) = E^\perp$.

Thus, we have $(X/E)^* \cong \text{Ran}(\pi^*) \cong E^\perp$.

□

Proposition (Existence of Auerbach Basis): Let X be a finite dimensional normed vector space. Then, there exists a collection $\{u_1, \dots, u_n\} \subseteq S_X$ that forms a basis for X , and a collection $\{\varphi_1, \dots, \varphi_n\} \subseteq S_{X^*}$ that forms a basis for X^* , such that $\varphi_u(u_j) = \delta_{ij}$. We call the collection $\{(u_i, \varphi_i)\}_{i=1}^n$ an Auerbach basis for X .

Proof. Let $\theta : X \rightarrow \ell_\infty^n$ be a bicontinuous isomorphism. Consider the map

$$D : \underbrace{B_X \times \dots \times B_X}_{n \text{ times}} \rightarrow \mathbb{C}$$

defined by

$$D(x_1, \dots, x_n) = \det(\theta(x_1) \ \theta(x_2) \ \dots \ \theta(x_n)),$$

where $\theta(x_n)$ are column vectors. It is the case that D is continuous, as D is a polynomial over its entries, and since $B_X \times \dots \times B_X$ is compact (as B_X is compact, and the finite product of compact sets is compact), we must have

$$\max_{x_j \in B_X} |D(x_1, \dots, x_n)| = |D(u_1, \dots, u_n)|$$

$$:= M$$

for some $u_1, \dots, u_n \in B_X \times \dots \times B_X$. Define $\varphi : X \rightarrow \mathbb{C}$ by

$$\varphi_k(x) = \frac{1}{M} D(u_1, \dots, x, \dots, u_n),$$

where x is in the k th index. Since D is multilinear, φ is linear. Additionally, $|\varphi_k(x)| \leq 1$ for each B_X , so $\|\varphi_k\|_{\text{op}} \leq 1$, while $|\varphi_k(u_k)| = 1$, so $\|\varphi_k\|_{\text{op}} = 1$.

Let $\omega_k = \text{sgn}(\varphi_k(u_k))$, and set $v_k = \omega_k u_k$. Then, $\varphi_k(v_j) = \delta_{jk}$.

If $D(x_1, \dots, x_n) = 0$, then $\{\theta(x_1), \dots, \theta(x_n)\}$ is linearly dependent in \mathbb{F}^n , and since θ is a linear isomorphism, then $\{x_1, \dots, x_n\}$ is linearly dependent. Thus, $\{v_1, \dots, v_n\}$ is a basis for X .

Similarly, if

$$\sum_{i=1}^n \alpha_i \varphi_i = 0,$$

then

$$\sum_{i=1}^n \alpha_i \varphi_i(v_j) = 0,$$

so $\alpha_j = 0$ for each j .

□

Proposition: Let $E \subseteq X$ be a subspace of a normed vector space X . If E is finite-dimensional, then E is topologically complemented — i.e., there exists $F \subseteq X$ closed such that $X = E \oplus F$, or, equivalently, there exists $P \in \mathcal{B}(X)$ with $P^2 = P$ and $\text{Ran}(P) = E$.

Proof. Since $\dim(E) = n$, E admits an Auerbach basis, $\{(u_i, \varphi_i)\}_{i=1}^n$. For each $i \in \{1, \dots, n\}$, the Hahn-Banach theorem allows an extension to $\psi_i \in X^*$ with $\|\varphi_i\|_{\text{op}} = \|\psi_i\|_{\text{op}} = 1$.

We will define $P : X \rightarrow X$ by

$$P(x) = \sum_{k=1}^n \psi_k(x) u_k.$$

We can see that P is linear, and $\text{Ran}(P) \subseteq E$. Note that $P(y) = y$ for all $y \in E$, since

$$\begin{aligned} P(u_j) &= \sum_{k=1}^n \psi_k(u_j) u_k \\ &= \sum_{k=1}^n \varphi_k(u_j) u_k \\ &= \sum_{k=1}^n \delta_{kj} u_k \\ &= u_j \end{aligned}$$

for each j . It follows that, for $y = \sum_{j=1}^n \alpha_j u_j$,

$$P(y) = \sum_{j=1}^n \alpha_j P(u_j)$$

$$\begin{aligned}
&= \sum_{j=1}^n \alpha_j u_j \\
&= y.
\end{aligned}$$

Thus, $\text{Ran}(P) = E$ and $P^2 = P$. □

Exercise: If Z is a normed vector space with $z_0 \in Z$ and $\psi \in Z^*$ fixed, prove that the map $T : Z \rightarrow Z$ defined by $T(z) = \psi(z)z_0$ is continuous.

Solution:

$$\begin{aligned}
\|T(z)\| &= \|\psi(z)z_0\| \\
&= \|z_0\| \|\psi(z)\| \\
&\leq \|z_0\| \|\psi\|_{\text{op}} \|z\|,
\end{aligned}$$

meaning $\|T\|_{\text{op}} \leq \|z_0\| \|\psi\|_{\text{op}}$, so T is bounded.

Properties of Reflexive Spaces

Definition. A normed vector space X is reflexive if $\iota_X : X \hookrightarrow X^{**}$ is surjective.

Proposition: Let X be a normed vector space, $E \subseteq X$ a subspace. The following are true.

- (1) If X is finite-dimensional, then X is reflexive.
- (2) If X is reflexive, then E is reflexive.
- (3) If X is reflexive, then X/E is reflexive.
- (4) X is reflexive if and only if X^* is reflexive.

Proof. To prove (1), let $\mathcal{B} = \{u_1, \dots, u_n\}$ be a basis for X . Since X is finite-dimensional, all linear operators on X are continuous. In particular, $X^* = X' = \mathcal{L}(X, \mathbb{F})$. Consider the linear functional $\varepsilon_i : X \rightarrow \mathbb{F}$ defined by

$$\varepsilon_i \left(\sum_{j=1}^n \alpha_j u_j \right) = \alpha_i.$$

Then $\mathcal{B}' = \{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for X^* , so $\dim(X) = \dim(X^*)$. Similarly, $\dim(X^*) = \dim(X^{**})$. Since $\iota_X : X \rightarrow X^{**}$ is injective, the rank-nullity theorem has ι_X is also surjective. Thus, X is reflexive.

To prove (2), we can see that if $\iota_X : X \rightarrow X^{**}$ is a bijection, then $\iota_X|_E : E \rightarrow E^{**}$ is also a bijection, so E is reflexive.

To see the forward direction of (4), we let X be reflexive, and consider $\iota_{X^*} : X^* \hookrightarrow X^{***}$. We will show that ι_{X^*} is surjective.

Let $h \in X^{***}$, so $h : X^{**} \rightarrow \mathbb{F}$ is bounded linear. Let $\varphi = h \circ \iota_X \in X^*$.

We claim that $\iota_{X^*}(\varphi) = h$. In order to do this, we will show that $\iota_{X^*}(\varphi)(\xi) = h(\xi)$ for all $\xi \in X^{**}$. For each $\xi \in X^{**}$, we know that $\xi = \iota_X(x) = \hat{x}$ for some $x \in X$ since X is reflexive. Thus,

$$\begin{aligned}
\iota_{X^*}(\varphi)(\xi) &= \hat{\varphi}(\xi) \\
&= \xi(\varphi) \\
&= \hat{x}(\varphi) \\
&= \varphi(x).
\end{aligned}$$

Alternatively, we have

$$\begin{aligned} h(\xi) &= h(\hat{x}) \\ &= h(\iota_X(x)) \\ &= h \circ \iota_X(x) \\ &= \varphi(x). \end{aligned}$$

For the converse, assuming X^* is reflexive, we have shown that X^{**} is reflexive. However, since $\iota_X(X) \subseteq X^{**}$, it is the case that X is reflexive. \square

Tensor Products in Vector Spaces

In order to understand tensor products and their norms in the case of Banach spaces, we begin by examining tensor products and linear functionals over general vector spaces.

Definition (Bilinear Map). Let X, Y, Z be vector spaces.

A map $b : X \times Y \rightarrow Z$ is bilinear if for all $x \in X$ and $y \in Y$, $b(x, \cdot) : Y \rightarrow Z$ and $b(\cdot, y) : X \rightarrow Z$ are linear.

We write $\text{Bil}(X, Y; Z)$ to denote the collection of all bilinear maps with domain $X \times Y$ and codomain Z .

Definition (Tensor Product). Let X and Y be \mathbb{F} -vector spaces. A tensor product of X and Y is a pair (V, b) where V is a \mathbb{F} -vector space and $b : X \times Y \rightarrow V$ is a bilinear map satisfying the universal property where, for any vector space Z and any bilinear map $\beta : X \times Y \rightarrow Z$, there is a unique linear map T_β such that $T_\beta \circ b = \beta$.

The following diagram commutes.

$$\begin{array}{ccc} X \times Y & \xrightarrow{b} & V \\ & \searrow \beta & \downarrow \exists! T_\beta \\ & & Z \end{array}$$

Fact. Let X and Y be vector spaces, and suppose (V, b) and (V', b') are tensor products of X and Y . Then, V and V' are linearly isomorphic.

Proof. Consider the following diagram depicting the universal property on V and V' .

$$\begin{array}{ccccc} & & V' & & \\ & \nearrow b' & & \searrow T'_\beta & \\ X \times Y & & & & Z \\ & \searrow b & & \nearrow T_\beta & \\ & & V & & \end{array}$$

Since $T_\beta = T'_\beta$, it is the case that T_β and T'_β agree on the bases of V and V' respectively. Thus, V and V' are linearly isomorphic. \square

We can now construct tensor products.

Let X and Y be vector spaces over \mathbb{F} . We consider the set $X \times Y$, and consider the free vector space, $\mathbb{F}(X \times Y)$. The space $\mathbb{F}(X \times Y)$ has basis $\{\delta_{(x,y)} \mid x \in X, y \in Y\}$, where $\delta_{(x,y)}$ are defined by

$$\delta_{(x,y)}(x', y') = \begin{cases} 1 & x = x', y = y', \\ 0 & \text{otherwise} \end{cases}.$$

Let $N = \text{span}(E_1 \cup E_2 \cup E_3 \cup E_4)$, where we define

$$\begin{aligned} E_1 &:= \{ \delta_{(x_1+x_2, y)} - \delta_{(x_1, y)} - \delta_{(x_2, y)} \mid x_1, x_2 \in X, y \in Y \} \\ E_2 &:= \{ \delta_{(x, y_1+y_2)} - \delta_{(x, y_1)} - \delta_{(x, y_2)} \mid x \in X, y_1, y_2 \in Y \} \\ E_3 &:= \{ \delta_{(\alpha x, y)} - \alpha \delta_{(x, y)} \mid x \in X, y \in Y, \alpha \in \mathbb{F} \} \\ E_4 &:= \{ \delta_{(x, \alpha y)} - \alpha \delta_{(x, y)} \mid x \in X, y \in Y, \alpha \in \mathbb{F} \}. \end{aligned}$$

We define the tensor product of X and Y as $X \otimes Y = \mathbb{F}(X \times Y) / N$.^{xvi} Elementary tensors are of the form $x \otimes y = \delta_{(x, y)} + N$. The following are some equivalences over \otimes that follow our definition of the tensor product.

Lemma: Let X, Y be \mathbb{F} -vector spaces. Then, for $x, x_1, x_2 \in X, y, y_1, y_2 \in Y$, and $\alpha \in \mathbb{F}$, we have

- $\alpha(x \otimes y) = \alpha x \otimes y = x \otimes \alpha y$;
- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$;
- $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$.

The natural map $b : X \times Y \rightarrow X \otimes Y$ defined by $b(x, y) = x \otimes y$ is, as a result, bilinear. Additionally, $0_{X \otimes Y} = 0_X \otimes y = x \otimes 0_Y$ for any x, y .

Proposition (Existence of Tensor Product): Let X, Y be \mathbb{F} -vector spaces, and $X \otimes Y$ be defined as above. Let $b : X \times Y \rightarrow X \otimes Y$ also be defined as above. Then, the pair $(X \otimes Y, b)$ is a tensor product.

Proof. We need to show the universal property. Let Z be a space, and suppose $\beta : X \times Y \rightarrow Z$ is a bilinear map.

By the universal property of the free vector space, there is a unique linear map $\bar{\beta} : \mathbb{F}(X \times Y) \rightarrow Z$ satisfying $\bar{\beta}(\delta_{(x, y)}) = \beta(x, y)$.

Note that, for $x_1, x_2 \in X$ and $y \in Y$, we have

$$\begin{aligned} \bar{\beta}(\delta_{(x_1+x_2, y)} - \delta_{(x_1, y)} - \delta_{(x_2, y)}) &= \bar{\beta}(\delta_{(x_1+x_2, y)}) - \bar{\beta}(\delta_{(x_1, y)}) - \bar{\beta}(\delta_{(x_2, y)}) \\ &= \beta(x_1 + x_2, y) - \beta(x_1, y) - \beta(x_2, y) \\ &= \beta(x_1, y) + \beta(x_2, y) - \beta(x_1, y) - \beta(x_2, y) \\ &= 0 \end{aligned}$$

Similarly, this means $\bar{\beta}$ sends $E_1 \cup E_2 \cup E_3 \cup E_4$ to 0, meaning $\bar{\beta}$ send N to 0.

By the First Isomorphism Theorem, there is a unique linear map $T_\beta : \mathbb{F}(X \times Y) / N \rightarrow Z$ satisfying $T_\beta \circ \pi = \bar{\beta}$, where π is the canonical quotient map.

Thus, we have

$$\begin{aligned} T_\beta \circ b(x, y) &= T_\beta(x \otimes y) \\ &= T_\beta \circ \pi(\delta_{(x, y)}) \\ &= \bar{\beta}(\delta_{(x, y)}) \\ &= \beta(x, y). \end{aligned}$$

The uniqueness of T_β follows from the fact that $X \otimes Y$ is spanned by elementary tensors. □

We can now turn our attention to tensor products of linear operators and linear functionals.

^{xvi}The construction of N essentially “mods out” all the expected relations between inputs of a bilinear map. For instance, $b(x_1 + x_2, y) = b(x_1, y) + b(x_2, y)$.

Proposition: Let X, Y, V, W be vector spaces over \mathbb{F} .

- (1) If $T \in \mathcal{L}(X, V)$ and $S \in \mathcal{L}(Y, W)$, there is a unique linear map

$$T \otimes S : X \otimes Y \rightarrow V \otimes W$$

satisfying $T \otimes S(x \otimes y) = T(x) \otimes S(y)$ for all $x \in X$ and $y \in Y$.

- (2) For every $\varphi \in X'$ and $\psi \in Y'$, there is a linear map $\varphi \times \psi \in (X \otimes Y)'$ satisfying $(\varphi \times \psi)(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in X$ and $y \in Y$.

Proof.

- (1) The map $X \times Y \rightarrow V \otimes W$ given by $(x, y) \mapsto T(x) \otimes S(y)$ is a bilinear map. Thus, by the universal property, there is a unique linear map $T \otimes S : X \otimes Y \rightarrow V \otimes W$ satisfying the required condition.
- (2) The map $X \times Y \rightarrow \mathbb{F}$ given by $(x, y) \mapsto \varphi(x)\psi(y)$ is bilinear, meaning the universal property gives a map $\varphi \times \psi : X \otimes Y \rightarrow \mathbb{F}$.

□

Proposition: Let X and Y be vector spaces over \mathbb{F} . If $\{x_1, \dots, x_n\}$ is linearly independent and

$$\sum_{j=1}^n x_j \otimes y_j = 0,$$

then $y_j = 0$ for all j .

Proof. For every $\varphi \in X'$ and $\psi \in Y'$, we have

$$\begin{aligned} 0 &= (\varphi \times \psi) \left(\sum_{j=1}^n x_j \otimes y_j \right) \\ &= \sum_{j=1}^n \varphi(x_j) \psi(y_j) \\ &= \sum_{j=1}^n \varphi(\psi(y_j) x_j) \\ &= \varphi \left(\sum_{j=1}^n \psi(y_j) x_j \right), \end{aligned}$$

meaning $\sum_{j=1}^n \psi(y_j) x_j = 0$. Since x_j are linearly independent, $\psi(y_j) = 0$ for all j and all ψ , so $y_j = 0$. □

Tensor Products of Banach Spaces

Now that we have a better understanding of tensor products (especially the dual space of a tensor product), we can start examining norms on tensor products of Banach spaces.

Injective Tensor Product

We start by defining the injective norm. We know that for $\varphi \in X'$ and $\psi \in Y'$, there is a linear functional $\varphi \times \psi \in (X \otimes Y)'$ that satisfies $(\varphi \times \psi)(x \otimes y) = \varphi(x)\psi(y)$.

We can translate this into a norm on any normed spaces X and Y , since $X^* \subseteq X'$ and $Y^* \subseteq Y'$.

Proposition: Let X and Y be normed vector spaces. Given $t \in X \otimes Y$, we define

$$\|t\|_V = \sup_{\varphi \in B_{X^*}, \psi \in B_{Y^*}} |(\varphi \times \psi)(t)|$$

to be the injective norm on $X \otimes Y$. The injective norm has the cross property — that is,

$$\|x \otimes y\|_V = \|x\| \|y\|$$

for all $x \in X$ and $y \in Y$.

Proof. We start by showing the supremum is finite. For any representation

$$t = \sum_{k=1}^n x_k \otimes y_k,$$

and any $\varphi \in B_{X^*}, \psi \in B_{Y^*}$, we have

$$\begin{aligned} \|(\varphi \times \psi)(t)\| &= \left| \sum_{k=1}^n \varphi(x_k) \psi(y_k) \right| \\ &\leq \sum_{k=1}^n |\varphi(x_k)| |\psi(y_k)| \\ &\leq \sum_{k=1}^n \|x_k\| \|y_k\|, \end{aligned}$$

recalling that $\|x\| = \sup_{\varphi \in B_{X^*}} |\varphi(x)|$.

The triangle inequality and homogeneity of the injective norm are clear from its definition. We will now prove that the norm is positive definite.

Let $\|t\|_V = 0$. We express t as

$$t = \sum_{k=1}^n x_k \otimes y_k,$$

where $\{y_1, \dots, y_n\}$ are linearly independent.

We have that for all $\varphi \in B_{X^*}$ and $\psi \in B_{Y^*}$,

$$\begin{aligned} 0 &= (\varphi \times \psi)(t) \\ &= \sum_{k=1}^n \varphi(x_k) \psi(y_k) \\ &= \psi \left(\sum_{k=1}^n \varphi(x_k) y_k \right). \end{aligned}$$

Since this is true for all $\psi \in B_{Y^*}$, the Hahn–Banach separation result holds that

$$\sum_{k=1}^n \varphi(x_k) y_k = 0$$

in Y . Since $\{y_1, \dots, y_n\}$ are linearly independent, we also get that $\varphi(x_k) = 0$ for all k and all $\varphi \in B_{X^*}$, so $x_k = 0$ for all k , and $t = 0$.

To see that the injective norm is cross, we have $\sup_{\varphi \in B_{Z^*}} |\varphi(z)| = \|z\|$. Thus,

$$\begin{aligned} \|x \otimes y\|_V &= \sup_{\varphi \in B_{X^*}, \psi \in B_{Y^*}} |\varphi(x)\psi(y)| \\ &= \sup_{\varphi \in B_{X^*}} |\varphi(x)| \sup_{\psi \in B_{Y^*}} |\psi(y)| \\ &= \|x\| \|y\|. \end{aligned}$$

□

Definition. If X and Y are normed spaces, the completion of $X \otimes Y$ with respect to the injective norm $\|\cdot\|_V$ is called the injective tensor product of X and Y , denoted $X \check{\otimes} Y$.

Exercise: Let X and Y be normed spaces, and let $\varphi \in X^*$, $\psi \in Y^*$. Show that $\varphi \times \psi \in (X \check{\otimes} Y)^*$, and $\|\varphi \times \psi\| = \|\varphi\| \|\psi\|$.

Solution: Let $\varphi \times \psi \in (X \check{\otimes} Y)^*$. Then,

$$\begin{aligned} \|\varphi \times \psi\| &= \sup_{\|x \otimes y\|_V = 1} |(\varphi \times \psi)(x \otimes y)| \\ &= \sup_{\|x \otimes y\|_V = 1} |\varphi(x)| |\psi(y)| \\ &= \sup_{\|x\| \|y\| = 1} |\varphi(x)| |\psi(y)| \\ &= \|\varphi\| \|\psi\|. \end{aligned}$$

Thanks to the injective tensor product, we are able to continuously extend bounded linear operators in a manner that preserves operator norm.

Proposition: Let X, Y, V, W be normed vector spaces, and let $T \in \mathcal{B}(X, V)$, $S \in \mathcal{B}(Y, W)$.

- (1) The operator $T \otimes S : X \otimes Y \rightarrow V \otimes W$ is continuous with respect to the injective norm, and extends continuously to a bounded linear operator $T \check{\otimes} S \in \mathcal{B}(X \check{\otimes} Y, V \check{\otimes} W)$ with $\|T \check{\otimes} S\|_{\text{op}} = \|T\|_{\text{op}} \|S\|_{\text{op}}$.
- (2) If T and S are isometric, so is $T \check{\otimes} S$.

Proof. Set $T_1 = \frac{1}{\|T\|_{\text{op}}} T$, and $S_1 = \frac{1}{\|S\|_{\text{op}}} S$. Let $t = \sum_{k=1}^n x_k \otimes y_k$ in $X \otimes Y$. Then,

$$\begin{aligned} \|T \otimes S(t)\|_V &= \left\| \sum_{k=1}^n T(x_k) \otimes S(y_k) \right\|_V \\ &= \sup_{\varphi \in B_{V^*}, \psi \in B_{W^*}} \left| \sum_{k=1}^n \varphi(T(x_k)) \psi(S(y_k)) \right| \\ &= \|T\| \|S\| \sup_{\varphi \in B_{V^*}, \psi \in B_{W^*}} \left| \sum_{k=1}^n \varphi(T_1(x_k)) \psi(S_1(y_k)) \right| \\ &= \|T\| \|S\| \sup_{\varphi \in B_{V^*}, \psi \in B_{W^*}} \left| \sum_{k=1}^n T_1^*(\varphi)(x_k) S_1^*(\psi)(y_k) \right| \\ &\leq \|T\| \|S\| \sup_{\omega \in B_{X^*}, \tau \in B_{Y^*}} \left| \sum_{k=1}^n \omega(x_k) \tau(y_k) \right| \\ &= \|T\| \|S\| \|t\|_V. \end{aligned}$$

The inequality in the final line follows from the fact that T_1 and S_1 are contractions, so $T_1^*(B_{V^*}) \subseteq B_{X^*}$ and $S_1^*(B_{W^*}) \subseteq B_{Y^*}$. Thus, we have $\|T \otimes S\| \leq \|T\| \|S\|$.

If $x \in B_X$ and $y \in B_Y$, then $\|x \otimes y\|_V = \|x\| \|y\| \leq 1$. Thus,

$$\begin{aligned} \|T \otimes S\| &\geq \|T(x) \otimes S(y)\|_V \\ &= \|T(x)\| \|S(y)\|. \end{aligned}$$

Taking the supremum, we get

$$\|T \otimes S\| \geq \|T\| \|S\|.$$

The norm-preserving continuous extension to $T \otimes S$ follows from the norm-preserving continuous extension to the completion in the case of an ordinary Banach space.

If T and S are isometries, then $T_1 = T$ and $S_1 = S$, so $T_1^*(B_{V^*}) = B_{X^*}$ and $S_1^*(B_{W^*}) = B_{Y^*}$. Thus, $T \otimes S$ is an isometry. \square

Proposition: Let X and Y be normed spaces. There is a canonical isometric embedding $X \otimes Y \hookrightarrow \mathcal{B}(Y^*, X)$.

Proof. For $x \in X$ and $y \in Y$, we consider the linear map $\theta_{x,y} : Y^* \rightarrow X$ given by $\theta_{x,y}(\varphi) = \varphi(y)x$. By the Hahn–Banach theorems, $\|\theta_{x,y}\| = \|x\| \|y\|$.

Next, the map $X \times Y \rightarrow \mathcal{B}(Y^*, X)$ given by $(x, y) \mapsto \theta_{x,y}$ is bilinear. Thus, there is a unique linear map $u : X \otimes Y \rightarrow \mathcal{B}(Y^*, X)$ with $x \otimes y \mapsto \theta_{x,y}$.

Let $t = \sum_{k=1}^n x_k \otimes y_k$ in $X \otimes Y$. Then,

$$\begin{aligned} \|u(t)\| &= \left\| \sum_{k=1}^n \theta_{x_k, y_k} \right\| \\ &= \sup_{\psi \in B_{Y^*}} \left\| \left(\sum_{k=1}^n \theta_{x_k, y_k} \right) (\psi) \right\| \\ &= \sup_{\psi \in B_{Y^*}} \left\| \sum_{k=1}^n \psi(y_k) x_k \right\| \\ &= \sup_{\psi \in B_{Y^*}, \varphi \in B_{X^*}} \left| \varphi \left(\sum_{k=1}^n \psi(y_k) x_k \right) \right| \\ &= \sup_{\psi \in B_{Y^*}, \varphi \in B_{X^*}} \left| \sum_{k=1}^n \psi(y_k) \varphi(x_k) \right| \\ &= \|t\|_V. \end{aligned}$$

Note that we use the Hahn–Banach norming results to convert from $\|x\|$ to $\varphi(x)$ and vice versa.

Thus, u is an isometry, so u extends to an isometry $U : X \otimes Y^* \hookrightarrow \mathcal{B}(Y^*, X)$. \square

Remark: We can construct similar isometric embeddings $X \otimes Y \hookrightarrow \mathcal{B}(X^*, Y)$. These embeddings yield other injective norms.

$$\begin{aligned} \left\| \sum_{k=1}^n x_k \otimes y_k \right\|_V &= \sup_{\psi \in B_{Y^*}} \left\| \sum_{k=1}^n \psi(y_k) x_k \right\| \\ &= \sup_{\varphi \in B_{X^*}} \left\| \sum_{k=1}^n \varphi(x_k) y_k \right\|. \end{aligned}$$

Example. Let Ω and Λ be compact Hausdorff spaces. We claim that

$$C(\Omega) \otimes C(\Lambda) \cong C(\Omega \times \Lambda).$$

For $f \in C(\Omega)$ and $g \in C(\Lambda)$, there is a continuous function $f \times g : \Omega \times \Lambda \rightarrow \mathbb{F}$ given by $(f \times g)(x, y) = f(x)g(y)$.

Since $(f, g) \mapsto f \times g$ is a bilinear map from $C(\Omega) \times C(\Lambda)$ to $C(\Omega \times \Lambda)$, there is a unique linear map $u : C(\Omega) \otimes C(\Lambda) \rightarrow C(\Omega \times \Lambda)$ satisfying $u(f \otimes g) = f \times g$. We will show that u is isometric under the injective norm.

If $t = \sum_{k=1}^n f_k \otimes g_k$ in $C(\Omega) \otimes C(\Lambda)$, then

$$\begin{aligned}
 \|t\|_v &= \sup_{\psi \in B_{C(\Lambda)^*}} \left\| \sum_{k=1}^n \psi(g_k) f_k \right\| \\
 &= \sup_{\psi \in B_{C(\Lambda)^*}} \sup_{x \in \Omega} \left| \sum_{k=1}^n \psi(g_k) f_k(x) \right| \\
 &= \sup_{\psi \in B_{C(\Lambda)^*}} \sup_{x \in \Omega} \left| \psi \left(\sum_{k=1}^n g_k f_k(x) \right) \right| \\
 &= \sup_{x \in \Omega} \left\| \sum_{k=1}^n f_k(x) g_k \right\| \\
 &= \sup_{x \in \Omega, y \in \Lambda} \left| \sum_{k=1}^n f_k(x) g_k(y) \right| \\
 &= \sup_{(x, y) \in \Omega \times \Lambda} \left| \sum_{k=1}^n (f_k \times g_k)((x, y)) \right| \\
 &= \left\| \sum_{k=1}^n (f_k \times g_k) \right\| \\
 &= \|u(t)\|.
 \end{aligned}$$

Thus, u extends to an isometry, $C(\Omega) \otimes C(\Lambda) \rightarrow C(\Omega \times \Lambda)$.

We will now show that u is onto. Since

$$\begin{aligned}
 (f_1 \times g_1)(f_2 \times g_2) &= f_1 f_2 \times g_1 g_2 \\
 \overline{(f \times g)} &= \bar{f} \times \bar{g} \\
 \mathbb{1}_\Omega \times \mathbb{1}_\Lambda &= \mathbb{1}_{\Omega \times \Lambda},
 \end{aligned}$$

we see that $\text{span}\{f \times g \mid f \in C(\Omega), g \in C(\Lambda)\}$ is a self-adjoint, unital, separating subalgebra of $C(\Omega \times \Lambda)$. Thus, by the Stone–Weierstrass theorem, $\overline{\text{Ran}(u)} = C(\Omega \times \Lambda)$. However, since u is isometric, its range is closed, meaning u is surjective.

Projective Tensor Product

Loosely speaking, the projective tensor product preserves the space of bilinear forms, $\text{Bil}(X, Y; Z)$.

Definition. Let X, Y, Z be normed spaces, and let $b : X \times Y \rightarrow Z$ be a bilinear map. We define the bilinear operator norm as

$$\|b\|_{\text{op}} = \sup_{x \in B_X, y \in B_Y} \|b(x, y)\|.$$

If $\|b\|_{\text{op}} < \infty$, then we say b is bounded bilinear.

Proposition: Let X, Y, Z be normed spaces, and suppose $b : X \times Y \rightarrow Z$ be bilinear. Then, b is continuous if and only if b is bounded bilinear.

Proposition: Let X, Y, Z be Banach spaces. The space of all bounded bilinear maps under pointwise addition and scalar multiplication with the norm $\|\cdot\|_{\text{op}}$ is a Banach space.

Proposition: Let X and Y be normed spaces. The formula

$$\|t\|_{\wedge} = \inf \left\{ \sum_{k=1}^n \|x_k\| \|y_k\| \mid t = \sum_{k=1}^n x_k \otimes y_k \right\}$$

defines a norm on $X \otimes Y$ called the projective norm. The projective norm is a cross norm, and satisfies $\|t\|_{\vee} \leq \|t\|_{\wedge}$.

Proof. The norm is clearly homogeneous.

If $t, t' \in X \otimes Y$ and have representations $t = \sum_{k=1}^n x_k \otimes y_k$ and $t' = \sum_{j=1}^m u_j \otimes v_j$, then

$$\begin{aligned} t + t' &= \sum_{k=1}^n x_k \otimes y_k + \sum_{j=1}^m u_j \otimes v_j \\ \|t + t'\|_{\wedge} &\leq \sum_{k=1}^n \|x_k\| \|y_k\| + \sum_{j=1}^m \|u_j\| \|v_j\|. \end{aligned}$$

Taking the infimum over representations of t , we have

$$\|t + t'\|_{\wedge} \leq \|t\|_{\wedge} + \sum_{j=1}^m \|u_j\| \|v_j\|,$$

and taking the infimum over representations of t' ,

$$\|t + t'\| \leq \|t\|_{\wedge} + \|t'\|_{\wedge}.$$

Note that for $t = \sum_{k=1}^n x_k \otimes y_k$, we have

$$\begin{aligned} \|t\|_{\vee} &= \sup_{\varphi \in B_{X^*}, \psi \in B_{Y^*}} \left| \sum_{k=1}^n \varphi(x_k) \psi(y_k) \right| \\ &\leq \sum_{k=1}^n |\varphi(x_k)| |\psi(y_k)| \\ &\leq \sum_{k=1}^n \|x_k\| \|y_k\|. \end{aligned}$$

Taking the infimum over all representations, we have $\|t\|_{\vee} \leq \|t\|_{\wedge}$.

If $\|t\|_{\wedge} = 0$, then $\|t\|_{\vee} = 0$, so $t = 0$ since $\|t\|_{\vee}$ is a norm.

For $x \in X$ and $y \in Y$, we have

$$\begin{aligned} \|x\| \|y\| &= \|x \otimes y\|_{\vee} \\ &\leq \|x \otimes y\|_{\wedge} \\ &\leq \|x\| \|y\|. \end{aligned}$$

□

Definition. If X and Y are normed spaces, the Banach space completion of $X \otimes Y$ with respect to the projective norm $\|\cdot\|_\wedge$ is called the projective tensor product of X and Y , denoted $X \hat{\otimes} Y$.

Proposition: Let X, Y, Z be normed spaces.

- (1) If b is bounded bilinear, $b \in \text{Bil}(X, Y; Z)$. Then, there is a unique $T_b \in \mathcal{B}(X \hat{\otimes} Y, Z)$ such that

$$T_b(x \otimes y) = b(x, y),$$

$$\text{and } \|T_b\|_{\text{op}} = \|b\|_{\text{op}}.$$

- (2) If $T \in \mathcal{B}(X \hat{\otimes} Y, Z)$, there is a $b \in \text{Bil}(X, Y; Z)$ with $T_b = T$, where T_b is given as above.
 (3) The map $b \mapsto T_b$ from $\text{Bil}(X, Y; Z)$ to $\mathcal{B}(X \hat{\otimes} Y, Z)$ is a linear isometric isomorphism.

Proof.

- (1) Since b is bilinear, the universal property for tensor products gives a unique linear $T_b : X \otimes Y \rightarrow Z$ such that $T_b(x \otimes y) = b(x, y)$. For $t = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$, we have

$$\begin{aligned} \|T_b(t)\| &= \left\| \sum_{k=1}^n b(x_k, y_k) \right\| \\ &\leq \sum_{k=1}^n \|b(x_k, y_k)\| \\ &\leq \sum_{k=1}^n \|b\|_{\text{op}} \|x_k\| \|y_k\|. \end{aligned}$$

Taking the infimum, we have

$$\|T_b(t)\| \leq \|b\|_{\text{op}} \|t\|_\wedge,$$

so

$$\|T_b\|_{\text{op}} \leq \|b\|_{\text{op}}.$$

For $x \in B_X, y \in B_Y$ we have $\|x \otimes y\|_\wedge \leq 1$, meaning

$$\begin{aligned} \|b(x, y)\| &= \|T_b(x \otimes y)\| \\ &\leq \|T_b\|_{\text{op}}. \end{aligned}$$

Taking the supremum, we get $\|b\|_{\text{op}} \leq \|T_b\|_{\text{op}}$. It is the case that T_b has a norm-preserving continuous extension to $X \hat{\otimes} Y \rightarrow Z$.

- (2) Define $b(x, y) = T(x \otimes y)$. It is the case that b is bilinear and bounded, since

$$\begin{aligned} \|b(x, y)\| &= \|T(x \otimes y)\| \\ &\leq \|T\|_{\text{op}} \|x \otimes y\|_\wedge \\ &= \|T\|_{\text{op}} \|x\| \|y\|. \end{aligned}$$

Since $T_b(x \otimes y) = b(x, y) = T(x \otimes y)$, T agrees with T_b on $X \otimes Y$. Since both T and T_b are continuous with respect to the projective norm, they agree on $X \hat{\otimes} Y$.

- (3) Since $b \mapsto T_b$ is linear, and we have shown this to be an isometric isomorphism, it is the case that this is an isometric isomorphism.

□

We can now identify the dual of the projective tensor product.

Proposition: Let X and Y be normed spaces. Then,

$$(X \hat{\otimes} Y)^* \cong \mathcal{B}(X, Y^*)$$

are isometrically isomorphic.

Proof. Consider the map $T : (X \hat{\otimes} Y)^* \rightarrow \mathcal{B}(X, Y^*)$ defined by $\psi \mapsto T_\psi$, where $T_\psi(x)(y) = \psi(x \otimes y)$. It is the case that $T_\psi(x)$ is a linear operator on Y , and

$$\begin{aligned} |T_\psi(x)(y)| &= |\psi(x \otimes y)| \\ &\leq \|\psi\| \|x \otimes y\|_\wedge \\ &= \|\psi\| \|x\| \|y\|, \end{aligned}$$

so $\|T_\psi(x)\|_{\text{op}} \leq \|\psi\| \|x\|$, so $T_\psi \in Y^*$. It is also clear that T_ψ is linear on x , so $\|T_\psi\|_{\text{op}} \leq \|\psi\|$, meaning T is a contraction.

We now define $\eta : \mathcal{B}(X, Y^*) \rightarrow (X \hat{\otimes} Y)^*$. Fixing $T \in \mathcal{B}(X, Y^*)$, we have a map $X \times Y \rightarrow \mathbb{F}$ defined by $T(x)(y)$ that is both bilinear and bounded:

$$\begin{aligned} \sup_{x \in B_X, y \in B_Y} |T(x)(y)| &= \sup_{x \in B_X} \|T(x)\| \\ &= \|T\|. \end{aligned}$$

Thus, there is some $\eta(T) \in (X \hat{\otimes} Y)^*$ with $\eta(T)(x \otimes y) = T(x)(y)$, and $\|\eta(T)\|_{\text{op}} = \|T\|$. We have η is linear, and

$$\begin{aligned} \eta(T_\psi)(x \otimes y) &= T_\psi(x)(y) \\ &= \psi(x \otimes y). \end{aligned}$$

Since both $\eta(T_\psi)$ and ψ are continuous and linear, we get $\eta(T_\psi) = \psi$. Since both T and η are contractions, we must have T is an isometric isomorphism. \square

Finite-Rank and Compact Operators

Definition. A linear map $T : X \rightarrow Y$ between vector spaces is said to be finite-rank if $\dim(\text{Ran}(T)) < \infty$.

The collection of finite-rank operators, $F(X, Y)$, forms an ideal in $\mathcal{L}(X, Y)$.

Not all finite rank operators are bounded, but we can look at ones that are.

Definition. Let X and Y be normed spaces. We write

$$\mathbb{F}(X, Y) = \mathcal{B}(X, Y) \cap F(X, Y)$$

to denote the set of all bounded finite-rank operators between X and Y . We also write $\mathbb{F}(X) = \mathbb{F}(X, X)$.

Proposition: Let X and Y be normed spaces. For each $x \in X$ and $\psi \in Y^*$, there is a rank-one operator

$$L_{x, \psi} : Y \rightarrow X$$

defined by

$$L_{x, \psi}(y) = \psi(y)x.$$

The following are true.

- (1) Each $L_{x,\psi}$ is bounded with $\|L_{x,\psi}\|_{\text{op}} = \|\psi\| \|x\|$.
- (2) The map $X \times Y^* \rightarrow \mathcal{B}(Y, X)$ mapping $(x, \psi) \mapsto L_{x,\psi}$ is bilinear.
- (3) Each $T \in \mathcal{F}(Y, X)$ can be written as

$$T = \sum_{k=1}^n L_{x_k, \psi_k},$$

where $\{x_1, \dots, x_n\}$ is a Hamel basis for $\text{Ran}(T)$ and $\psi_1, \dots, \psi_n \in Y^*$. Moreover, every operator of this form is bounded and of finite rank.

Proof.

- (1) We compute

$$\begin{aligned} \|L_{x,\psi}\|_{\text{op}} &= \sup_{y \in B_Y} \|L_{x,\psi}(y)\| \\ &= \sup_{y \in B_Y} \|\psi(y)x\| \\ &= \sup_{y \in B_Y} |\psi(y)| \|x\| \\ &= \|\psi\|_{\text{op}} \|x\|. \end{aligned}$$

- (2) This is clear from the definition of $L_{x,\psi}$.
- (3) Let $\{x_1, \dots, x_n\}$ be a Hamel basis for $\text{Ran}(T)$. There is then a bicontinuous isomorphism $\varphi : \ell_\infty^n \rightarrow \text{Ran}(T)$ with $\varphi(e_k) = x_k$ for each k .

We set $\psi_k = \pi_k \circ \varphi^{-1} \circ T$, where $\pi_k : \ell_\infty^n \rightarrow \mathbb{F}$ is the projection to the k th coordinate.

Since T is bounded, $\psi_k \in Y^*$, so for every $y \in Y$, we have

$$\begin{aligned} \sum_{k=1}^n L_{x_k, \psi_k}(y) &= \sum_{k=1}^n \psi_k(y) x_k \\ &= \sum_{k=1}^n \pi_k \circ \varphi^{-1} \circ T(y) x_k \\ &= \sum_{k=1}^n \pi_k \circ \varphi^{-1} \circ T(y) \varphi(e_k) \\ &= \varphi \left(\sum_{k=1}^n \pi_k \circ \varphi^{-1} \circ T(y) e_k \right) \\ &= \varphi \left(\varphi^{-1}(T(y)) \right) \\ &= T(y). \end{aligned}$$

Here, we use the fact that $z = \sum_{k=1}^n \pi_k(z) e_k$ for each $z \in \ell_\infty^n$.

Since $\text{Ran}(T) \subseteq \text{span}\{x_1, \dots, x_n\}$, $\text{Ran}(T)$ is finite-rank, and is bounded by its definition.

□

Exercise: Let $E = E^2$ be a bounded finite-rank idempotent operator on a Banach space X . Show that we can write

$$E = \sum_{k=1}^n L_{x_k, \varphi_k},$$

where $\{x_1, \dots, x_n\}$ is a Hamel basis for $\text{Ran}(E)$ and $\varphi_k \in X^*$ with $\varphi_k(x_j) = \delta_{kj}$.

Solution: Since $\text{Ran}(E)$ is finite, $\text{Ran}(E)$ has an Auerbach basis, $\{(x_i, \varphi_k)\}_{i=1}^n$ such that $\varphi_i(x_j) = \delta_{ij}$. We can necessarily write

$$E(x) = \sum_{k=1}^n \varphi_k(x) x_k$$

for each $x \in X$, which means

$$E = \sum_{k=1}^n L_{x_k, \varphi_k}.$$

Proposition: Let X and Y be normed spaces. There is an isometric isomorphism $L : X \otimes Y^* \rightarrow \mathbb{F}(Y, X)$ with $X \otimes Y$ equipped with the injective norm. Additionally, L extends to an isometric isomorphism $\tilde{L} : X \tilde{\otimes} Y^* \rightarrow \overline{\mathbb{F}(Y, X)}^{\|\cdot\|_{\text{op}}}$.

Proof. For every $x \in X$ and $y \in Y^*$, we consider $L_{x, \psi} : Y \rightarrow X$ as defined above. By the universal property of tensor products, there is a linear map $L : X \otimes Y^* \rightarrow \mathcal{B}(Y, X)$ satisfying $L(x \otimes \psi) = L_{x, \psi}$.

We will show this map is isometric with $X \otimes Y^*$ equipped with the injective norm. If $t = \sum_{k=1}^n x_k \otimes \psi_k$ is in $X \otimes Y^*$, then

$$\begin{aligned} \|L(t)\| &= \left\| \sum_{k=1}^n L_{x_k, \psi_k} \right\| \\ &= \sup_{y \in Y} \left\| \left(\sum_{k=1}^n L_{x_k, \psi_k} \right)(y) \right\| \\ &= \sup_{y \in Y} \left\| \sum_{k=1}^n \psi_k(y) x_k \right\| \\ &= \sup_{y \in B_Y, \varphi \in B_{X^*}} \left| \varphi \left(\sum_{k=1}^n \psi_k(y) x_k \right) \right| \\ &= \sup_{y \in B_Y, \varphi \in B_{X^*}} \left| \sum_{k=1}^n \psi_k(y) \varphi(x_k) \right| \\ &= \sup_{y \in B_Y, \varphi \in B_{X^*}} \left| \left(\sum_{k=1}^n \varphi(x_k) \psi_k \right)(y) \right| \\ &= \sup_{\varphi \in B_{X^*}} \left\| \sum_{k=1}^n \varphi(x_k) \psi_k \right\| \\ &= \|t\|_V. \end{aligned}$$

Since L is surjective, and L is an isometry, it is the case that L is an isometric isomorphism. We can extend L to get an isometric isomorphism $X \tilde{\otimes} Y^* = \overline{\mathbb{F}(Y, X)}^{\|\cdot\|_{\text{op}}}$. \square

Remark: We can view $X \otimes Y^*$ as being contained in $\mathcal{B}(Y, X)$, since every $t \in X \otimes Y^*$ represents a finite rank operator from Y to X with $\|t\|_\wedge = \|t\|_{\text{op}}$.

Definition. Let X and Y be normed spaces. A linear map $T : X \rightarrow Y$ is said to be compact if $\overline{T(B_X)} \subseteq Y$ is compact.

We write $\mathbb{K}(X, Y)$ to denote the set of all compact operators from X to Y .

Example. Let $k : [0, 1]^2 \rightarrow \mathbb{C}$ be continuous. Given $f \in C([0, 1])$, we define

$$K(f) : [0, 1] \rightarrow \mathbb{C}$$

by

$$K(f)(s) = \int_0^1 k(s, t) f(t) dt.$$

We will show that $K(f)$ is continuous and that $K : C([0, 1]) \rightarrow C([0, 1])$ is a compact operator.

Fix $f \in C([0, 1])$, and let $\varepsilon > 0$. Since k is uniformly continuous,^{xvii} there is $\delta > 0$ such that if $(s, t), (s', t') \in [0, 1]^2$ and $d_2((s, t), (s', t')) < \delta$, then

$$|k(s, t) - k(s', t')| < \frac{\varepsilon}{\|f\|_u}.$$

If $|s - s'| < \delta$, then $d_2((s, t), (s', t')) < \delta$, so

$$\begin{aligned} |K(f)(s) - K(f)(s')| &\leq \int_0^1 |k(s, t) - k(s', t')| |f(t)| dt \\ &\leq \int_0^1 \frac{\varepsilon}{\|f\|_u} \|f\| dt \\ &= \varepsilon. \end{aligned}$$

Thus, $K(f)$ is continuous, so K is well-defined.

It is the case that K is linear, and $\|K\|_{op} \leq \|k\|_u$.

We will show that K is a compact operator by using the Arzela–Ascoli theorem. Consider the set

$$\mathcal{F} = \{K(f) \mid f \in C([0, 1]), \|f\|_u \leq 1\}.$$

If $s \in [0, 1]$, then $|K(f)(s)| \leq \|k\|_u$ for all $f \in B_{C([0, 1])}$, meaning \mathcal{F} is pointwise bounded. Similarly, for ε and δ as in the proof of the well-definedness of K , then if $|s - s'| < \delta$, $|K(f)(s) - K(f)(s')| \leq \varepsilon$ for all $f \in B_{C([0, 1])}$, meaning \mathcal{F} is equicontinuous.

By the Arzela–Ascoli theorem, $\overline{K(B_{C([0, 1])})}$ is compact.

We can characterize compact operators in a variety of ways.

Proposition: Let X and Y be normed spaces, $T : X \rightarrow Y$ is a linear map. Then, we have the following.

- (1) If $T \in \mathbb{F}(X, Y)$, then T is compact.
- (2) If $T \in \mathbb{K}(X, Y)$, then $T(B_X)$ is totally bounded in Y , which implies that $T \in \mathcal{B}(X, Y)$. If Y is a Banach space, then T is compact if and only if $T(B_X)$ is totally bounded.
- (3) T is compact if and only if for every bounded sequence $(x_n)_n$ in X , there is a subsequence $(x_{n_k})_k$ such that $(T(x_{n_k}))_k$ converges in Y .

^{xvii}Continuous and defined over compact domain.

(4) In the following statements:

- (i) $T(B_X)$ is compact in Y ;
- (ii) T is a compact operator;
- (iii) $T|_{B_X} : (B_X, w) \rightarrow (Y, \|\cdot\|)$ is continuous.

We have (i) \Rightarrow (ii) \Rightarrow (iii). If X is reflexive, then (iii) \Rightarrow (i).

Proof.

- (1) Let T be bounded and of finite rank. Let $Z = \text{Ran}(T)$. Since $\dim(Z) < \infty$, B_Z is compact, so $\|T\| B_Z$ is compact. For $x \in B_X$, we have $\frac{T(x)}{\|T\|} \in B_Z$, so $T(x) \in \|T\| B_Z$.
- (2) In metric spaces, compact sets are totally bounded, so if T is compact, then $\overline{T(B_X)}$, hence $T(B_X)$, is totally bounded in Y .

If $T(B_X)$ is totally bounded, then there exist y_1, \dots, y_m such that

$$T(B_X) \subseteq \bigcup_{j=1}^m U(y_j, 1).$$

We set $C = \max_{j=1}^m \{\|y_j\|\} + 1$, meaning $\|T\|_{\text{op}} \leq C$.

If Y is complete, then total boundedness is equivalent to having compact closure (i.e., precompact).

- (3) Let $(x_n)_n$ be a sequence in X . Then, we have $(\frac{1}{c}x_n)_n$ is a sequence in B_X for some $c > 0$, meaning $(T(\frac{1}{c}x_n))_n$ is a sequence in $T(B_X) \subseteq \overline{T(B_X)}$. Since T is compact, $\overline{T(B_X)}$ is compact, so there is a subsequence $(\frac{1}{c}x_{n_k})_k \rightarrow y$, so $(T(x_{n_k}))_k \rightarrow cy$.

Conversely, let $(y_n)_n$ be a sequence in $\overline{T(B_X)}$ with a convergent subsequence. We approximate $\|y_n - z_n\| < 1/n$, where $T(x_n) = z_n$. By our assumption, there is a subsequence $(x_{n_k})_k$ such that $(T(x_{n_k}))_k = z_{n_k} \rightarrow y$ for some $y \in Y$. Then,

$$\begin{aligned} \|y_{n_k} - y\| &\leq \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - y\| \\ &\leq \frac{1}{n_k} + \|z_{n_k} - y\| \\ &< \varepsilon, \end{aligned}$$

meaning $\overline{T(B_X)}$ is sequentially compact, hence compact.

- (4) We can see that (i) implies (ii).

Let T be compact, and let $(x_i)_i \rightarrow x$ be a weakly convergent net in B_X . Consider any subnet $(T(x_{i_j}))_j$. Since $\overline{T(B_X)}$ is compact, there exists a norm-convergent subnet, $(T(x_{i_{j_k}}))_k \rightarrow y$. Let $\varphi \in Y^*$. Then,

$$\varphi(T(x_{i_{j_k}}))_k \xrightarrow{k \rightarrow \infty} \varphi(y).$$

Note that $\varphi \circ T \in X^*$, and $(x_{i_{j_k}})_k \rightarrow x$, meaning

$$\varphi \circ T(x_{i_{j_k}})_k \rightarrow \varphi \circ T(x).$$

Thus, $\varphi \circ T(x) = \varphi(y)$. Since this is true for all $\varphi \in Y^*$, we have $T(x) = y$.

We have X is reflexive if and only if (B_X, w) is compact. Thus, if $T|_{B_X} : (B_X, w) \rightarrow (Y, \|\cdot\|)$ is continuous, and B_X is weakly compact, then $T(B_X)$ is norm-compact in Y . This shows (iii) implying (i).

□

Lemma: Let $T : X \rightarrow Y$ be a compact operator between normed spaces. Then, $\overline{\text{Ran}}(T)$ is separable.

Proof. Since $\overline{T(B_X)}$ is compact, it is separable, so there is a set $S = \{y_n\}_{n \geq 1}$ in $\overline{T(B_X)}$ with $\overline{S} = \overline{T(B_X)}$.

We claim S is total^{xviii} in $\overline{\text{Ran}}(T)$.

Since $T(B_X) \subseteq \text{Ran}(T)$, we can take closures to find

$$\begin{aligned} S &\subseteq \overline{T(B_X)} \\ &\subseteq \overline{\text{Ran}}(T). \end{aligned}$$

Since $\overline{\text{Ran}}(T)$ is a subspace, we have $\text{span}(S) \subseteq \overline{\text{Ran}}(T)$. Taking closures yet again, we get $\overline{\text{span}}(S) \subseteq \overline{\text{Ran}}(T)$.

If $x \in X$, we have

$$\begin{aligned} T(x) &= \|x\| T\left(\frac{x}{\|x\|}\right) \\ &\in \|x\| T(B_X) \\ &\subseteq \|x\| \overline{T(B_X)} \\ &= \|x\| \overline{S} \\ &= \overline{\|x\| S} \\ &\subseteq \overline{\text{span}}(S), \end{aligned}$$

so $\text{Ran}(T) \subseteq \overline{\text{span}}(S)$, meaning $\overline{\text{Ran}}(T) \subseteq \overline{\text{span}}(S)$. Thus, S is total in $\overline{\text{Ran}}(T)$, so $\text{Ran}(T)$ is separable. □

Exercise: If $T : X \rightarrow Y$ is a compact operator between normed spaces, and $E \subseteq X$ is a subspace, then $T|_E : E \rightarrow Y$ is also compact.

Solution: Since $E \subseteq X$ is linear, we have $B_E \subseteq B_X$, so $T|_E(B_E) = T(B_E) \subseteq T(B_X)$, meaning

$$\overline{T|_E(B_E)} \subseteq \overline{T(B_X)}.$$

Since $\overline{T(B_X)}$ is compact, so too is $\overline{T|_E(B_E)}$.

Proposition: Let X and Y be Banach spaces, and let $T : X \rightarrow Y$ be compact. Then, $T^* : Y^* \rightarrow X^*$ is also compact.

Proof. We only need show that $T^*(B_{Y^*})$ is totally bounded.

Let $\varepsilon > 0$. We know that $T(B_X)$ is totally bounded in Y , so there are x_1, \dots, x_n such that

$$T(B_X) \subseteq \bigcup_{i=1}^n U(x_i, \varepsilon/3).$$

^{xviii}Linear span is dense.

Consider the finite rank operator $F : Y^* \rightarrow \ell_\infty^n$ defined by

$$\begin{aligned} F(\varphi) &= (\varphi(T(x_i)))_{i=1}^n \\ &= (T^*(\varphi)(x_i))_{i=1}^n. \end{aligned}$$

We have F is bounded (as it is a composition of bounded linear maps), so F is compact, meaning $F(B_{Y^*})$ is totally bounded in ℓ_∞^n . Thus, there are functionals $\varphi_1, \dots, \varphi_n$ such that

$$F(B_{Y^*}) \subseteq \bigcup_{i=1}^n U(\varphi_i, \varepsilon/3).$$

Given $\psi \in B_{Y^*}$ and $x \in B_X$, we select i and j such that

$$\begin{aligned} \|T(x) - T(x_i)\| &< \varepsilon/3 \\ \|F(\psi) - F(\varphi_j)\|_\infty &< \varepsilon/3. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|T^*(\psi)(x) - T^*(\varphi_j)(x)\| &\leq \|T^*(\psi)(x) - T^*(\psi)(x_i)\| + \|T^*(\psi)(x_i) - T^*(\varphi_j)(x_i)\| + \|T^*(\varphi_j)(x_i) - T^*(\varphi_j)(x)\| \\ &\leq \|\psi(T(x) - T(x_i))\| + \|F(\psi) - F(\varphi_j)\| + \|\varphi_j(T(x_i) - T(x))\| \\ &< \varepsilon. \end{aligned}$$

Since $x \in B_X$ was arbitrary, we have $\|T^*(\psi) - T^*(\varphi_j)\|_{\text{op}} \leq \varepsilon$, and since $\psi \in B_{Y^*}$ was arbitrary, we also have

$$T^*(B_{Y^*}) \subseteq \bigcup_{i=1}^n U(\varphi_i, \varepsilon).$$

□

Exercise: Let X and Y be Banach spaces, and suppose $T : X \rightarrow Y$ is compact.

- (1) If T is surjective, show that $\dim(Y) < \infty$.
- (2) If T has closed range, then T is finite-rank.
- (3) If $X = Y$ and T is idempotent, then T is finite-rank.
- (4) If X is infinite-dimensional and T is injective, then $\text{Ran}(T)$ is not closed.
- (5) If X is infinite-dimensional, then T cannot be isometric.

Solution:

- (1) Let T be surjective. Then, by the Open Mapping Theorem, T is a quotient map, so there exists $\delta > 0$ such that $B_Y \subseteq T\left(\frac{1}{\delta}B_X\right)$.

Let $(y_n)_n$ be a sequence in B_Y . Since T is surjective, there exists $(x_n)_n$ such that $(T(x_n))_n = y_n$. $B_Y \subseteq T\left(\frac{1}{\delta}B_X\right)$, we must have $(x_n)_n \subseteq \frac{1}{\delta}B_X$, so $(x_n)_n$ is bounded. Hence, $y_n = (T(x_n))_n$ has a convergent subsequence as T is compact. Thus, B_Y is compact, hence $\dim(Y) < \infty$.

- (2) Let $Z = T(X)$. Then, $Z \subseteq Y$ is a Banach space as Z is closed and Y is a Banach space. Therefore, $T : X \rightarrow Z$ is surjective, hence an open map, meaning that Z is finite-dimensional by (1).
- (3) Let T be idempotent. Then, we have $X = \ker(T) \oplus \text{Ran}(T)$, and $\text{Ran}(T)$ is closed, meaning that by (2), $\text{Ran}(T)$ is finite-dimensional.
- (4) Let T be injective. Then, $\ker(T) = \{0\}$, and since $X = X/\ker(T) \cong \text{Ran}(T)$ by the First Isomorphism Theorem, $\text{Ran}(T)$ is infinite-dimensional, implying that T is not finite rank. Thus, $\text{Ran}(T)$ is not closed by the contrapositive of (2).
- (5) Let T be isometric. Then, $T^* : Y^* \rightarrow X^*$ is a 1-quotient map, hence surjective. Since T^* is compact, it must be the case that $\dim(X^*) < \infty$, so X is also finite-dimensional.

Compact operators have special properties related to their eigenspaces, including (but not limited to) the fact that the eigenspaces corresponding to nonzero eigenvalues are finite-dimensional and invariant.

Proposition: Let X be a Banach space, $T : X \rightarrow Y$ be a compact operator, and $0 \neq \lambda \in \mathbb{F}$.

(1) The eigenspace

$$E_\lambda(T) = \ker(T - \lambda I)$$

is T -invariant and finite-dimensional.

(2) The range $\text{Ran}(T - \lambda I)$ is closed.

(3) The cokernel $\text{coker}(T - \lambda I)$ is finite-dimensional.

(4) The operator $T - \lambda I$ is Fredholm.

Proof.

(1) We know that E_λ is T -invariant by the properties of eigenspaces, and since E_λ is the kernel of a continuous operator, it is evident that it is closed.

Let $T|_{E_\lambda} : E_\lambda \rightarrow E_\lambda$ be the restriction of T to the invariant subspace. Then, S is compact (as T is compact), and by the definition of the eigenspace, $S = \lambda \text{id}_{E_\lambda}$. Thus,

$$\begin{aligned} \overline{S(B_{E_\lambda})} &= \overline{\lambda B_{E_\lambda}} \\ &= \lambda \overline{B_{E_\lambda}} \\ &= \lambda B_{E_\lambda}, \end{aligned}$$

is compact. Since $\lambda \neq 0$, B_{E_λ} is compact, so $\dim(E_\lambda) < \infty$.

(2) Since $E_\lambda(T)$ is finite-dimensional, there is a closed subspace $F \subseteq X$ with $X = E_\lambda(T) \oplus F$.

Let $S = (T - \lambda I)|_F : F \rightarrow X$ be the restriction. Then, $\text{Ran}(T - \lambda I) = \text{Ran}(S)$, and S is injective. We will show that S is bounded below.

Suppose there is a sequence of unit vectors $(f_n)_n$ with $(\|S(f_n)\|)_n \rightarrow 0$. Since T is compact, we may assume $(T(f_n))_n \rightarrow z$ for some $z \in X$. Thus, $(f_n)_n \rightarrow \lambda^{-1}z$; since $\|z\| = \lambda \neq 0$, and F is closed, $z \in F$. Thus, $(T(f_n))_n \rightarrow \lambda^{-1}T(z)$, so $T(z) = \lambda z$, or $S(z) = 0$.

(3) Since $\text{Ran}(T - \lambda I)$ is closed, we find

$$\begin{aligned} \text{coker}((T - \lambda I)^*) &= (X/\text{Ran}(T - \lambda I))^* \\ &\cong (\text{Ran}(T - \lambda I))^\perp \\ &= \ker(T^* - \lambda I). \end{aligned}$$

Since T is compact, T^* is also compact, meaning $\ker(T^* - \lambda I)$ is finite-dimensional, hence $\text{coker}(T - \lambda I)$ is finite dimensional.

(4) This follows from both $\ker(T - \lambda I)$ and $\text{coker}(T - \lambda I)$ being finite-dimensional.

□

Corollary: If X is a Banach space, and $P \in \mathcal{K}(X)$ is idempotent, then P is finite-rank.

Proof. Note that $E_1(P) = \text{Ran}(P)$, so by the previous proposition, P is finite-rank.

□

Theorem (Algebraic Properties of the Space of Compact Operators): Let X, Y, Z be Banach spaces.

- (1) $\mathbb{K}(X, Y) \subseteq \mathcal{B}(X, Y)$ is a closed linear subspace.
- (2) If $S \in \mathcal{B}(Y, Z)$ and $T \in \mathbb{K}(X, Y)$, then $S \circ T \in \mathbb{K}(X, Z)$. Additionally, if $R \in \mathcal{B}(Z, X)$, then $T \circ R \in \mathbb{K}(Z, Y)$.
- (3) $\mathbb{K}(X) \subseteq \mathcal{B}(X)$ is a closed ideal.
- (4) $I_X \in \mathbb{K}(X)$ if and only if $\dim(X) < \infty$, in which case $\mathbb{K}(X) = \mathcal{B}(X)$.

Proof.

- (1) Let $T, S \in \mathbb{K}(X, Y)$ and $\alpha \in \mathbb{C}$. Let $(x_n)_n$ be a bounded sequence in X . Since T is compact, there exists a subsequence such that $(T(x_{n_k}))_k$ converges in Y . Since S is convergent, there exists a subsequence such that $(S(x_{n_{k_j}}))_j$ converges in Y . Thus, $(S + \alpha T)(x_{n_{k_j}})_j$ converges in Y , so $S + \alpha T$ is compact.

Let $(\|T_n - T\|_{\text{op}})_n \rightarrow 0$, where $(T_n)_n$ is a sequence in $\mathbb{K}(X, Y)$. We will show that $T(B_X)$ is totally bounded.

Let $\varepsilon > 0$. There is N such that $\|T_N - T\| < \varepsilon$, and since $T_N(B_X)$ is totally bounded, there exist y_1, \dots, y_m such that $T_N(B_X) \subseteq \bigcup_{j=1}^m \mathcal{U}(y_j, \varepsilon/2)$.

For $x \in B_X$, we can find j such that $\|T_N(x) - y_j\| < \varepsilon/2$, meaning

$$\begin{aligned} \|T(x) - y_j\| &\leq \|T(x) - T_N(x)\| + \|T_N(x) - y_j\| \\ &\leq \|T - T_N\| \|x\| + \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

meaning $T(B_X) \subseteq \bigcup_{j=1}^m \mathcal{U}(y_j, \varepsilon)$, so $T(B_X)$ is totally bounded, meaning T is compact.

- (2) Let $(x_n)_n$ be bounded in X . Then, there exists a subsequence such that $(T(x_{n_k}))_k \rightarrow y$ in Y . Since S is continuous, $(S(T(x_{n_k})))_k \rightarrow S(y)$ in Z , so $S \circ T \in \mathbb{K}(X, Z)$.

Since $R(B_Z) \subseteq \|R\|_{\text{op}} B_X$, we have

$$\begin{aligned} T \circ R(B_Z) &\subseteq T(\|R\|_{\text{op}} B_X) \\ &= \|R\|_{\text{op}} T(B_X) \\ &\subseteq \|R\|_{\text{op}} \overline{T(B_X)}. \end{aligned}$$

Since T is compact, $\overline{T(B_X)}$ is compact, meaning $\overline{T \circ R(B_Z)}$ is compact, so $T \circ R \in \mathbb{K}(Z, Y)$.

- (3) This fact follows from (1) and (2).
- (4) We have X is finite-dimensional if and only if B_X is compact if and only if $I(B_X)$ is compact, if and only if I is compact. Since any ideal that contains the multiplicative identity is the whole algebra, we know $\mathbb{K}(X) = \mathcal{B}(X)$.

□

Remark: Since the compact operators form a closed subspace and every bounded finite-rank operator is compact, we have

$$\overline{\mathbb{F}(X)}^{\|\cdot\|_{\text{op}}} \subseteq \mathbb{K}(X),$$

where X is a Banach space. This inclusion is sometimes strict. Banach spaces for which $\overline{\mathbb{F}(X)}^{\|\cdot\|_{\text{op}}} = \mathbb{K}(X)$ are said to have the approximation property.

Hilbert spaces have the approximation property.

Locally Convex Topological Vector Spaces

Now that we have a proper understanding of normed vector spaces and operators on them, we can investigate further into a more general structure that normed vector spaces fall into — topological vector spaces.

Topological Vector Spaces

Definition. Let X be a vector space over \mathbb{F} , and let τ be a topology on X . We say τ is compatible with the vector space structure of X if

- (1) for every $x \in X$, $\{x\}$ is closed;
- (2) scalar multiplication:

$$\begin{aligned} m : \mathbb{F} \times X &\rightarrow X \\ m(\lambda, x) &= \lambda x \end{aligned}$$

is continuous (where $\mathbb{F} \times X$ is endowed with the product topology);

- (3) vector addition

$$\begin{aligned} a : X \times X &\rightarrow X \\ a(x, y) &= x + y \end{aligned}$$

is continuous (where $X \times X$ is endowed with the product topology).

A topological vector space is a pair (X, τ) where X is a vector space and τ is a topology on X that is compatible with the vector space structure of X .

Remark: Since the topologies we are dealing with are not necessarily first countable,^{xxx} we have to use nets instead of sequences to understand ideas such as convergence.

Proposition: Let X be a topological vector space, and fix $x_0 \in X$, $\alpha \neq 0$. Then,

- (1) $T_{x_0} : X \rightarrow X$ defined by $T_{x_0}(x) = x + x_0$ is a homeomorphism;
- (2) $m_\alpha : X \rightarrow X$ defined by $m_\alpha(x) = \alpha x$ is a homeomorphism.

Corollary: Let X be a TVS, with $U \subseteq X$ open, $C \subseteq X$ closed, and $K \subseteq X$ compact. Let $x_0 \in X$ and $\alpha \neq 0$.

Then, $x_0 + U$ and αU are open, $x_0 + C$ and αC are closed, and $x_0 + K$ and αK are compact.

To show that the topology on a topological vector space is Hausdorff, we begin by showing a useful fact.

Lemma: Let X be a topological vector space. Suppose $V \subseteq X$ is a neighborhood of 0_X . There is a symmetric^{xx} open neighborhood of 0_X such that $W + W \subseteq V$.

Proof. We assume V is open. Since $a : X \times X \rightarrow X$ is continuous, the preimage $a^{-1}(V) \subseteq X \times X$ is an open neighborhood of $(0_X, 0_X)$ in the product topology of $X \times X$, meaning there are neighborhoods $W_1 \times W_2$ of $(0_X, 0_X)$ with W_1, W_2 open. Note that $W_1 + W_2 \subseteq V$. If $W_3 = W_1 \cap W_2$, then W_3 is an open neighborhood of 0_X with $W_3 + W_3 \subseteq V$. Setting $W = W_3 \cap (-W_3)$, we have W is symmetric with $W + W \subseteq V$. \square

Proposition: If (X, τ) is a topological vector space, then τ is Hausdorff.

^{xxx}Every point has a countable neighborhood system on it.

^{xx}A subset W of X such that $W = -W$.

Proof. Let $x, y \in X$ with $x \neq y$. Since $\{y\}$ is closed, there is $U \subseteq X$ with $x \in U \subseteq \{y\}^c$. Translating $V = U - X$, V is thus an open neighborhood of 0_X . There is then a symmetric open neighborhood W with $W + W \subseteq V$.

We have $x + W$ and $y + W$ are open neighborhoods of x and y respectively. We claim $(x + W) \cap (y + W) = \emptyset$.

Suppose toward contradiction that this is not the case. Then, there is some $w, w' \in W$ such that $x + w = y + w'$. Thus,

$$\begin{aligned} y &= x + w - w' \\ &\in x + W + (-W) \\ &= x + W + W \\ &\subseteq x + V \\ &= x + (U - X) \\ &= U, \end{aligned}$$

meaning $y \in \{y\}^c$, which is a contradiction.

Thus, τ is Hausdorff. □

We can now look at subspaces and quotient spaces of topological vector spaces.

Proposition: Let X be a topological vector space, and let $E \subseteq X$ be a linear subspace.

- (1) E is a topological vector space under the subspace topology.
- (2) \bar{E} is also a subspace.
- (3) Assuming E is closed, X/E endowed with the quotient topology is a topological vector space, and $\pi : X \rightarrow X/E$ is an open map.

Proof.

- (1) Since every element of E is an element of X , we must have $\{x\}$ closed in E as $\{x\}$ is closed in X . Similarly, since scalar multiplication is continuous in X , and E is a subspace, scalar multiplication is closed in X , and similarly is vector addition.
- (2) We know that the subspace topology on E is compatible with the vector space structure of E . Let $(x_\alpha)_\alpha \rightarrow x$ and $(y_\alpha)_\alpha \rightarrow y$ be nets in E , and let $\lambda \in \mathbb{F}$. Then,

$$(x_\alpha + \lambda y_\alpha)_\alpha \rightarrow x + \lambda y,$$

so \bar{E} is also a subspace.

- (3) We will show that π is an open map. If $U \subseteq X$ is open, then

$$\begin{aligned} \pi^{-1}(\pi(U)) &= U + E \\ &= \bigcup_{x \in E} x + U, \end{aligned}$$

meaning $\pi^{-1}(\pi(U))$ is open, so $\pi(U)$ is open by the definition of the quotient topology.

We need to show that the quotient topology is compatible with the vector space structure.

Let $x + E \neq y + E$ in X/E . Then, $(x - y) \notin E$, so there exists U with $x - y \in U \subseteq E^c$. Then, $y \in x + U$ and $x + U$ is open, meaning

$$y + E = \pi(y)$$

$$\begin{aligned} &\in \pi(x + U) \\ &\subseteq X/E. \end{aligned}$$

Since π is open, $x + U$ is open. We claim that $x + E \notin x + U$.

If $x + E = \pi(x + u) = x + u + E$, then $u \in E$, which is a contradiction as $U \subseteq E^c$. Thus $\{x + E\}^c$ is open, so X/E is a T1 topology.

Let $\alpha : X \times X \rightarrow X$ and $\bar{\alpha} : X/E \times X/E \rightarrow X/E$ denote the addition maps. The following diagram commutes.

$$\begin{array}{ccccc} X \times X & \xrightarrow{\pi \times \pi} & X/E \times X/E & \xrightarrow{\bar{\alpha}} & X/E \\ & \searrow \alpha & & \nearrow \pi & \\ & & X & & \end{array}$$

We can see that for open $V \subseteq X/E$, $(\bar{\alpha})^{-1}(V) = (\pi \times \pi)(\alpha^{-1}(\pi^{-1}(V)))$, which is open as π is open and α is continuous.

Similarly, we can see that $\bar{m} : F \times X/E \rightarrow X/E$ is continuous.

□

Proposition: Let (X, τ) be a topological vector space. Then, the collection of all closed subspaces,

$$\mathcal{M} = \{M \mid M \subseteq X \text{ closed subspace}\}$$

is a lattice ordered by inclusion.

Proof. If $M, N \subseteq X$, then

$$M \wedge N = M \cap N$$

is the largest closed subspace contained in both M and N , while

$$M \vee N = \overline{M + N}^\tau$$

is the smallest closed subspace containing both M and N .

□

Definition. Let (X, τ) be a topological vector space. A local base for X is a neighborhood base of 0_X . The family $\mathcal{O}_0 \subseteq \tau$ is defined such that

- $0_X \in V$ for all $V \in \mathcal{O}_0$
- \mathcal{O}_0 is directed by reverse inclusion — that is, if $U \in \tau$ with $0_X \in U$, then there exists $B \in \mathcal{O}_0$ with $B \subseteq U$.

Note that for any $x \in X$, we can find a neighborhood base for x by taking $x + B$ for all $B \in \mathcal{O}_0$.

Definition. A map $T : X \rightarrow Y$ between topological vector spaces is uniformly continuous if, for all $V \in \mathcal{O}_{0_Y}$, there exists $U \in \mathcal{O}_{0_X}$ such that if $x_1, x_2 \in X$ with $x_1 - x_2 \in U$, then $T(x_1) - T(x_2) \in V$.

Proposition: Let X and Y be topological vector spaces, and let $T : X \rightarrow Y$ be linear. Then, the following are equivalent.

- (1) T is continuous at 0_X ;
- (2) T is continuous;

(3) T is uniformly continuous.

Proof. To see (i) implies (ii), we let $(x_\alpha)_\alpha \rightarrow x$ be a net convergent in X , so $(x_\alpha - x)_\alpha \rightarrow 0_X$ by continuity of addition. Since T is continuous at the origin, $(T(x_\alpha - x))_\alpha \rightarrow T(0_X) = 0_Y$. Since T is linear, we have $(T(x_\alpha) - T(x))_\alpha \rightarrow 0_Y$, and since addition is continuous, we have $(T(x_\alpha))_\alpha \rightarrow T(x)$.

It is clear that (ii) implies (i).

To see that (iii) implies (i), we let $V \in \mathcal{O}_{0_Y}$, meaning there exists $U \in \mathcal{O}_{0_X}$ such that $x_1 - x_2 \in U$ implies $T(x_1) - T(x_2) \in V$. Taking $x_2 = 0$, we have $T(U) \subseteq V$.

To see that (i) implies (iii), let $V \in \mathcal{O}_{0_Y}$. By continuity, there is $U \in \mathcal{O}_{0_X}$ such that $T(U) \subseteq V$. If $x_1, x_2 \in X$ with $x_1 - x_2 \in U$, then $T(x_1 - x_2) \in V$. Since T is linear, we have $T(x_1) - T(x_2) \in V$. \square

Definition. We say that two topological vector spaces X and Y are isomorphic if there is a linear isomorphism $T : X \rightarrow Y$ that is a homeomorphism. We write $X \cong Y$ as topological vector spaces.

Definition. A subset of a vector space $A \subseteq X$ is said to be balanced if, for every $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$, we have $\alpha A \subseteq A$.

Lemma: Let X be a topological vector space, and suppose U is a neighborhood of the origin 0_X . Then, there exists a balanced open set W_U with $0 \in W_U \subseteq U$.

Proof. Since $m : \mathbb{F} \times X \rightarrow X$ is continuous, $m^{-1}(U)$ is an open neighborhood of $(0, 0_X)$ in the product topology. There must exist $V \in \mathcal{O}_{0_X}$ and $\delta > 0$ such that $(-\delta, \delta) \times V \subseteq m^{-1}(U)$. We have $m((-\delta, \delta) \times V) \subseteq U$, so $\alpha V \subseteq U$ whenever $|\alpha| < \delta$. Thus, we have

$$W_U = \bigcup_{|\alpha| < \delta} \alpha V$$

is the set we are looking for. \square

Proposition: Every topological vector space admits a local base consisting of balanced sets.

Proof. Let \mathcal{O}_{0_X} be a local base for the topological vector space X . For each $U \in \mathcal{O}_{0_X}$, there is $W_U \subseteq U$ with W_U balanced. Thus, $\{W_U\}_{U \in \mathcal{O}_{0_X}}$ is a local base consisting of balanced sets. \square

Proposition: Let X be a topological vector space of dimension n . Then, $X \cong \ell_2^n$ as topological vector spaces.

Proof. Let $\{b_1, \dots, b_n\}$ be a basis for X , and consider the map $\varphi : \ell_2^n \rightarrow X$ given by

$$\varphi(\lambda) = \sum_{j=1}^n \lambda(j) b_j,$$

where

$$\lambda = \sum_{j=1}^n \lambda(j) e_j.$$

It is clear that φ is a linear isomorphism. Additionally, we claim that φ is continuous.

If $(\lambda_k)_k \xrightarrow{k \rightarrow \infty} \lambda$ in ℓ_2^n , then $\lambda_k(j) \xrightarrow{k \rightarrow \infty} \lambda(j)$ for each j . Thus, $\lambda_k(j) b_j \xrightarrow{k \rightarrow \infty} \lambda(j) b_j$ for each j , since multiplication is continuous. Using the fact that addition is continuous, we get

$$(\varphi(\lambda_k))_k \xrightarrow{k \rightarrow \infty} \sum_{j=1}^n \lambda(j) b_j.$$

To show that the inverse is continuous, we need a compactness argument.

We claim that, for $B = B_{\ell_2^n}$, $\varphi(B)$ contains an open neighborhood of 0_X .

The unit sphere, $S = S_{\ell_2^n}$, is compact since it is closed and bounded, so $\varphi(S) \subseteq X$ is compact. Note that $0_X \notin \varphi(S)$ since φ is injective. So, for every $x \in \varphi(S)$, we choose U_x, V_x such that $U_x \cap V_x = \emptyset$, $0_X \in U_x$, and $x \in V_x$.

Now, we have an open cover

$$\varphi(S) \subseteq \bigcup_{x \in \varphi(S)} V_x,$$

which admits a finite subcover

$$\varphi(S) \subseteq \bigcup_{j=1}^n V_{x_j}.$$

Set

$$U = \bigcap_{j=1}^n U_{x_j},$$

noting that $U \cap \varphi(S) = \emptyset$.

There is an open and balanced W with $0_X \in W \subseteq U$. We will show that $W \subseteq \varphi(B)$.

Suppose toward contradiction that there is $w \in W \setminus \varphi(B)$. Since φ is onto, we can write $w = \varphi(z)$ for some $z \in B^c$. We have

$$\begin{aligned} \varphi\left(\frac{1}{\|z\|}z\right) &= \frac{1}{\|z\|}\varphi(z) \\ &= \frac{1}{\|z\|}w, \end{aligned}$$

which belongs to W since $\|z\| > 1$ and W is balanced. However, $\frac{1}{\|z\|}z \in S$, and $W \cap \varphi(S) = \emptyset$, which is our contradiction.

If $V \subseteq \ell_2^n$ is an open neighborhood of the origin $0_{\ell_2^n}$, then there is $\delta > 0$ such that $\delta B \subseteq V$. Set $W' = \delta W$, which is yet again an open neighborhood of 0_X . We have

$$\begin{aligned} W' &= \delta W \\ &\subseteq \delta \varphi(B) \\ &= \varphi(\delta B) \\ &\subseteq \varphi(V), \end{aligned}$$

implying that $\varphi^{-1}(W) \subseteq \varphi^{-1}(\varphi(V)) = V$, so φ^{-1} is continuous at 0_X , hence continuous. \square

Corollary: Let X be a topological vector space with finite dimension, and let Y be an arbitrary topological vector space. Then, any linear map $T : X \rightarrow Y$ is continuous.

Proof. We have an isomorphism of topological vector spaces $\varphi : \ell_2^n \rightarrow X$. Since $\text{Ran}(T)$ is finite dimensional and inherits the subspace topology on Y , there is also an isomorphism of topological vector spaces $\psi : \text{Ran}(T) \rightarrow \ell_2^m$ for some $m \leq n$.

We know that $S = \psi \circ T \circ \varphi$ is continuous since any map between finite-dimensional normed spaces is continuous, so $T = \psi^{-1} \circ S \circ \varphi^{-1}$ is continuous. \square

We can extend notions of completeness to the case of a TVS using generalized versions of Cauchy sequences.

Definition. Let X be a topological vector spaces.

- (1) A sequence $(x_n)_n$ in X is called Cauchy if $U \in \mathcal{O}_X$, there exists N such that for all $m, n \geq N$, $x_m - x_n \in U$.
- (2) A net $(x_\alpha)_{\alpha \in A}$ is called Cauchy if, for every $U \in \mathcal{O}_X$, there exists $\gamma \in A$ such that for all $\alpha, \beta \geq \gamma$, $x_\alpha - x_\beta \in U$.
- (3) We say X is complete if every Cauchy net converges in X .

Proposition: A convergent net $(x_\alpha)_\alpha \rightarrow x$ in a topological vector space is always Cauchy.

Proof. Let $V \in \mathcal{O}_X$. We find symmetric $W \in \mathcal{O}_X$ with $W + W \subseteq V$. Note that $x + W \in \mathcal{O}_x$, so for all large α , we have $x_\alpha \in x + W$. If α and β are sufficiently large, we have

$$\begin{aligned} x_\alpha - x_\beta &= (x_\alpha - x) + (x - x_\beta) \\ &\in W + (-W) \\ &= W + W \\ &\subseteq V. \end{aligned}$$

□

Exercise: Suppose $(x_\alpha)_\alpha$ is a Cauchy net in a topological vector space X that admits a convergent subnet $(x_{\alpha_\beta})_\beta \rightarrow x$. Show that $(x_\alpha)_\alpha \rightarrow x$.

Solution: Let $V \in \mathcal{O}_X$ be such that $V + V = V$. Then, there is α_0 such that for all $\alpha, \beta \geq \alpha_0$, $x_\alpha - x_\beta \in V$. We also have that, for $V + x \in \mathcal{O}_x$, there is β_0 such that for $\beta \geq \beta_0$, $x_{\alpha_\beta} \in V + x$.

Thus, we have, for γ such that $\gamma \geq \beta_0$ and $\gamma \geq \alpha_0$, if $\alpha, \beta \geq \gamma$, we have

$$\begin{aligned} (x_\alpha - x_{\alpha_\beta}) + (x_{\alpha_\beta}) &\in V + (V + x) \\ &= V + x \\ &\in \mathcal{O}_x, \end{aligned}$$

meaning $(x_\alpha)_\alpha \rightarrow x$.

Proposition: If X is a first countable topological vector space, then X is complete if and only if every Cauchy sequence converges.

Proof. We only need to show sufficiency. Assume that every Cauchy sequence converges. Let $(x_\alpha)_\alpha$ be a Cauchy net in X . Since X is first countable, there is a countable local base $(V_n)_n$. Without loss of generality, we may take $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$. We inductively construct $(\alpha_n)_n \subseteq A$ with $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$, with, for $\alpha, \beta \geq \alpha_n$, $x_\alpha - x_\beta \in V_n$.

Then, $(x_{\alpha_n})_n$ is a Cauchy sequence in X , and thus converges. Therefore, $(x_\alpha)_\alpha$ converges. □

Corollary: If X is a normed spaces, then X is complete as a topological vector space if and only if X is a Banach space.

Proposition: Let X be a topological vector space, and suppose $E \subseteq X$ is a subspace.

- (1) If E is complete as a topological vector space, then $E \subseteq X$ is closed.
- (2) If $E \subseteq X$ is closed and X is complete, then E is complete.

Proposition: Let X and Y be topological vector spaces, and suppose $\varphi : X \rightarrow Y$ is an isomorphism of topological vector spaces. Then, X is complete if and only if Y is complete.

Proof. Let $(x_\alpha)_\alpha$ be a Cauchy net in X . Then, for $V \in \mathcal{O}_{0_Y}$, for large α, β , $x_\alpha - x_\beta \in V$. We then have $\varphi(x_\alpha) - \varphi(x_\beta) \in U$ for some $U \in \mathcal{O}_{0_Y}$, as φ is an isomorphism (hence linear).

We thus have $(x_\alpha)_\alpha$ is Cauchy if and only if $(\varphi(x_\alpha))_\alpha$ (as φ is a homeomorphism) is Cauchy, hence X is complete if and only if Y is complete. \square

Corollary: Let X be a topological vector space, and suppose $E \subseteq X$ is a subspace. If $\dim(E) < \infty$, then E is closed.

Proof. Endowing E with the subspace topology, we see that E is a finite-dimensional topological vector space, so $E \cong \ell_2^n$ for some n . Since ℓ_2^n is a Banach space, it is complete as a topological vector space, meaning E is complete as a topological vector space, meaning E is closed. \square

Local Convexity

Topological vector spaces that arise in functional analysis are generally constructed as weak topologies. These spaces always admit a local base of convex sets.

Definition. Let X be a vector space.

- (1) A subset $C \subseteq X$ is called convex if, for any elements $x_1, x_2 \in C$, and $t \in [0, 1]$, we have $(1 - t)x_1 + tx_2 \in C$.
- (2) Given a subset $A \subseteq X$, the convex hull of A is defined as

$$\text{conv}(A) = \bigcap \{C \mid A \subseteq C \subseteq X, C \text{ convex}\}.$$

Equivalently,

$$\text{conv}(A) = \left\{ \sum_{j=1}^n t_j x_j \mid n \in \mathbb{N}, t_j \in [0, 1], x_j \in A, \sum_{j=1}^n t_j = 1 \right\}.$$

Definition. A topological vector space is called locally convex if it admits a local base of convex sets. Such a space is called a locally convex topological vector space, abbreviated LCTVS.

Exercise: Let U be an open and balanced set in a topological vector space X . Prove that $\text{conv}(U)$ is open and balanced.

Solution: We will show that the interior of any convex set is convex. Let S be convex, $U = S^\circ$. Since $tS + (1 - t)S = S$ by convexity, we have $tU + (1 - t)U \subseteq S$, so $tU + (1 - t)U \subseteq S^\circ$. Since S° is the smallest open set that contains $tU + (1 - t)U$, it is the case that $tU + (1 - t)U = U$.

In particular, it must be the case that for any convex C with $U \subseteq C \subseteq X$, we must have $U \subseteq C^\circ \subseteq X$ with C° convex. Thus, $\text{conv}(U)$ is open.

Since U is balanced, we know that $\alpha U \subseteq U$ for all $|\alpha| \leq 1$. Since $U \subseteq \text{conv}(U)$, we have $\alpha U \subseteq \text{conv}(U)$, so $\text{conv}(\alpha U) \subseteq \text{conv} U$, so $\alpha \text{conv}(U) \subseteq \text{conv}(U)$, so $\text{conv}(U)$ is balanced.

Proposition: Every locally convex topological vector space has a local base of balanced convex sets.

Proof. Let \mathcal{O}_{0_X} be a local base for X consisting of convex sets. We construct the family $\{W_U\}_{U \in \mathcal{O}_{0_X}}$, which is a local base consisting of balanced sets with $W_U \subseteq U$ for each $U \in \mathcal{O}_{0_X}$. For each U , we set $V_U = \text{conv}(W_U)$. Since $V_U \subseteq U$ for each U , as U is convex, and V_U is open, convex, and balanced, we have $\{V_U\}_{U \in \mathcal{O}_{0_X}}$ as our desired local base. \square

Proposition (Preservation of Local Convexity): Let X be a locally convex topological vector space, and suppose $E \subseteq X$ is a subspace.

- (1) Equipped with the subspace topology, E is a locally convex topological vector space.

(2) If E is closed, X/E with the quotient topology is a locally convex topological vector space.

Proof. We have shown that E and X/E are topological vector spaces earlier. We now only need to show that E and X/E admit local bases of convex sets. Let \mathcal{O}_{0_X} be a local base for X consisting of convex sets.

(1) We can see that $\{U \cap E\}_{U \in \mathcal{O}_{0_X}}$ is a local base of E consisting of convex sets.

(2) The projection $\pi : X \rightarrow X/E$ is an open map, so the family $\{\pi(U) \mid U \in \mathcal{O}_{0_X}\}$ is a local base for X/E .

If $y_1 = \pi(x_1)$ and $y_2 = \pi(x_2)$ are in $\pi(U)$ for some $x_1, x_2 \in U \in \mathcal{O}_{0_X}$, and $t \in [0, 1]$, we have

$$\begin{aligned} (1-t)y_1 + ty_2 &= (1-t)\pi(x_1) + t\pi(x_2) \\ &= \pi((1-t)x_1 + tx_2) \\ &\in \pi(U), \end{aligned}$$

since U is convex. Thus, $\{\pi(U) \mid U \in \mathcal{O}_{0_X}\}$.

□

Structure of Locally Convex Topological Vector Spaces

So far, we have mostly been discussing the properties of the topologies on locally convex topological vector spaces, but it is valuable to understand just exactly *from where* this topology comes from. In particular, we will construct the topology on a given vector space as an initial topology for a separating family of seminorms.

Definition. Let X be a vector space, and let \mathcal{P} be a family of seminorms on X . For each $x \in X$ and $p \in \mathcal{P}$, we consider the maps

$$f_{p,z} : X \rightarrow [0, \infty)$$

defined by $f_{p,z}(x) = p(x - z)$.^{xxi} We write $\tau_{\mathcal{P}}$ to be the topology induced by the family

$$\mathcal{F}_{\mathcal{P}} = \{f_{p,z} \mid p \in \mathcal{P}, z \in X\}.$$

Specifically, $\tau_{\mathcal{P}}$ is the topology induced by the family of open sets

$$\xi = \{f_{p,z}^{-1}(U) \mid p \in \mathcal{P}, z \in X, U \subseteq [0, \infty) \text{ open}\}.$$

Remark: Note that $\tau_{\mathcal{P}}$ is the initial topology induced by the family of semimetrics, $\mathcal{F}_{\mathcal{P}}$, not the family of seminorms \mathcal{P} . The convergence of nets is defined by $(x_{\alpha})_{\alpha} \rightarrow x$ if and only if $(f_{p,z}(x_{\alpha}))_{\alpha} \rightarrow f_{p,z}(x)$ for all $p \in \mathcal{P}$ and $z \in X$.

Equivalently, $(x_{\alpha})_{\alpha} \rightarrow x$ if and only if $(p(x_{\alpha} - z))_{\alpha} \rightarrow p(x - z)$ for all $p \in \mathcal{P}$ and $z \in X$.

Definition (Separating Family). A family of seminorms \mathcal{P} on a vector space X is called separating if, for all nonzero $x \in X$, there is $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Note that, by this definition, $\mathcal{F}_{\mathcal{P}}$ separates the points of X , as for $x \neq x'$, there is $p \in \mathcal{P}$ with $p(x - x') \neq 0$, so $f_{p,x'}(x) \neq f_{p,x'}(x')$.

We will start by showing that the topology $\tau_{\mathcal{P}}$ on X admits a local base of balanced, convex sets.

Theorem: Let $X, \mathcal{P}, \mathcal{F}_{\mathcal{P}}$, and $\tau_{\mathcal{P}}$ be defined as above.

(1) Given $p \in \mathcal{P}, z \in X$, and $\varepsilon > 0$, the set

$$U_p(z, \varepsilon) = \{x \in X \mid p(x - z) < \varepsilon\}$$

is a convex open neighborhood of z . Additionally, $U_p(z, \varepsilon)$ is balanced.

^{xxi}This is effectively equivalent to defining $f_{p,z}$ as a semimetric induced by p in relation to a fixed z .

- (2) The convergence of a net $(x_\alpha)_\alpha \rightarrow x$ holds if and only if $p(x_\alpha - x) \rightarrow 0$ for all $p \in \mathcal{P}$.
- (3) If \mathcal{P} is a separating family, then $(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space. A neighborhood base at $z \in X$ is given by

$$\mathcal{O}_z = \left\{ \bigcap_{j=1}^n U_{p_j}(z, \varepsilon) \mid p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0 \right\}.$$

Proof.

- (1) Since $[0, \varepsilon] \subseteq [0, \infty)$ is open, and $f_{p,z}$ is continuous, we have $U_p(z, \varepsilon) = f_{p,z}^{-1}([0, \varepsilon))$ is open. We also know $z \in U_p(z, \varepsilon)$.

For $x_1, x_2 \in U_p(z, \varepsilon)$ and $t \in [0, 1]$, we have

$$\begin{aligned} p((1-t)x_1 + tx_2 - z) &= p((1-t)x_1 - (1-t)z + tx_2 - tz) \\ &\leq p((1-t)(x_1 - z)) + p(t(x_2 - z)) \\ &= (1-t)p(x_1 - z) + tp(x_2 - z) \\ &< (1-t)\varepsilon + t\varepsilon \\ &= \varepsilon, \end{aligned}$$

meaning $U_p(z, \varepsilon)$ is convex.

If $x \in U_p(0, \varepsilon)$, then for $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$, we have $p(\alpha x) = |\alpha|p(x) \leq p(x) < \varepsilon$, so $\alpha x \in U_p(0, \varepsilon)$.

- (2) Taking $z = x$ in the net convergence criterion that $(x_\alpha)_\alpha \rightarrow x$ if and only if $(p(x_\alpha - z))_\alpha \rightarrow p(x - z)$ for all $p \in \mathcal{P}$ and $z \in X$ gives one direction.

In the reverse direction, let $z \in X$. Then, by the reverse triangle inequality, since $p(x_\alpha - x) = p(x_\alpha - z + z - x)$, we have

$$|p(x_\alpha - z) + p(x - z)| \leq p(x_\alpha - x).$$

Since, by assumption $p(x_\alpha - x) \rightarrow 0$, we must have $p(x_\alpha - z) \rightarrow p(x - z)$, meaning $(x_\alpha)_\alpha \rightarrow x$ by the net convergence criterion.

- (3) Since \mathcal{P} is a separating family, $\mathcal{F}_{\mathcal{P}}$ separates the points of X , so $\tau_{\mathcal{P}}$ is Hausdorff.

We must now show that $\tau_{\mathcal{P}}$ is compatible with the vector space structure.

- (i) Let $(x_\alpha)_\alpha \rightarrow x$ and $(y_\alpha)_\alpha \rightarrow y$. Then, for arbitrary $p \in \mathcal{P}$, we have

$$\begin{aligned} 0 &\leq p((x_\alpha + y_\alpha) - (x + y)) \\ &\leq p(x_\alpha - x) + p(y_\alpha - y) \\ &\rightarrow 0, \end{aligned}$$

meaning $(x_\alpha + y_\alpha) \rightarrow (x + y)$.

- (ii) Let $(x_\alpha)_\alpha \rightarrow x$ and $(\lambda_\alpha)_\alpha \rightarrow \lambda$ in \mathbb{F} . We can find α_0 such that for $\alpha \geq \alpha_0$, $|\lambda_\alpha| \leq |\lambda| + 1$. Thus, for $\alpha \geq \alpha_0$, we have

$$\begin{aligned} p(\lambda_\alpha x_\alpha - \lambda x) &\leq p(\lambda_\alpha x_\alpha - \lambda_\alpha x) + p(\lambda_\alpha x - \lambda x) \\ &\leq |\lambda_\alpha| p(x_\alpha - x) + |\lambda_\alpha - \lambda| p(x) \\ &\leq (|\lambda| + 1) p(x_\alpha - x) + |\lambda_\alpha - \lambda| p(x) \\ &\rightarrow 0, \end{aligned}$$

meaning $(\lambda_\alpha x_\alpha) \rightarrow \lambda x$.

We define

$$\xi = \{f_{p,z}^{-1}(U) \mid p \in \mathcal{P}, z \in X, U \subseteq [0, \infty) \text{ open}\}.$$

Now, we must show that

$$\tau_{\mathcal{P}} = \tau(\xi).$$

Consider the family

$$\xi' = \{f_{p,z}^{-1}([0, \varepsilon)) \mid p \in \mathcal{P}, z \in X, \varepsilon > 0\}.$$

We claim that $\tau_{\mathcal{P}} = \tau(\xi') = \tau(\xi)$.

Since $\xi' \subseteq \xi$, it is the case that $\tau(\xi') \subseteq \tau(\xi)$. To show the reverse inclusion, it suffices to show that $\xi \subseteq \tau(\xi')$.

Let $U \subseteq [0, \infty)$ be open. We write $U = \bigcup_{j=1}^{\infty} I_j$, where I_j are open intervals in $[0, \infty)$. Then, $f_{p,z}^{-1}(U) = \bigcup_{j=1}^{\infty} f_{p,z}^{-1}(I_j)$. We will now show that $f_{p,z}^{-1}(I) \in \tau(\xi')$, for I an open interval.

If $I = (a, b)$, then $f_{p,z}^{-1}(I) = \{x \mid a < p(x - z) < b\}$. For $x \in f_{p,z}^{-1}(I)$, set $\delta = \min\{b - p(x - z), p(x - z) - a\}$, meaning $x \in U_p(x, \delta) \subseteq f_{p,z}^{-1}(I)$. Note that $U_p(0, \delta) = f_{p,z}^{-1}([0, \delta)) \in \xi'$. Thus, $f_{p,z}^{-1}(I) \in \tau(\xi')$.

We fix $z \in X$. Each set of the form $\bigcap_{j=1}^n U_{p_j}(z, \varepsilon_j)$ is open, convex, and contains z . Let $z \in W \in \tau_{\mathcal{P}}$. Since ξ' generates the topology $\tau_{\mathcal{P}}$, we can find $p_1, \dots, p_n \in \mathcal{P}$ and $z_1, \dots, z_n \in X$ such that

$$z \in \bigcap_{j=1}^n U_{p_j}(z_j, \varepsilon_j) \in W.$$

We choose $\varepsilon > 0$ such that $\varepsilon < \min\{\varepsilon_j - p_j(z - z_j)\}$. For $y \in \bigcap_{j=1}^n U_{p_j}(z, \varepsilon)$ and $j \in \{1, \dots, n\}$, we have

$$\begin{aligned} p_j(y - z_j) &\leq p(y - z) + p_j(z - z_j) \\ &< \varepsilon + p_j(z - z_j) \\ &< \varepsilon_j - p_j(z - z_j) + p_j(z - z_j) \\ &= \varepsilon_j, \end{aligned}$$

meaning

$$\left\{ \bigcap_{j=1}^n U_{p_j}(z, \varepsilon) \mid p_j \in \mathcal{P}, \varepsilon > 0 \right\}$$

is a neighborhood system at $z \in X$ consisting of convex sets.

□

Having shown that a separating family of seminorms generates a topology compatible with the vector space X , we can move in the opposite. Specifically, we can realize every open, convex, balanced neighborhood of the origin as the open unit ball for some seminorm.

Lemma: Let X be a topological vector space, and suppose $C \subseteq X$ is an open and convex set containing 0_X . The map

$$\begin{aligned} \mu_C : X &\rightarrow [0, \infty) \\ \mu_C(x) &= \inf\{s > 0 \mid x \in sC\} \end{aligned}$$

is well-defined. The following properties hold.

- (1) μ_C is a Minkowski functional continuous at 0_X , with

$$C = \{x \in X \mid \mu_C(x) < 1\}.$$

- (2) If C is balanced, then μ_C is a continuous seminorm.

- (3) For any $\delta > 0$, $x \in \delta C$ if and only if $\mu_C(x) < \delta$. That is,

$$\delta C = \{x \mid \mu_C(x) < \delta\}.$$

Proof. Since scalar multiplication is continuous, we have $(\frac{1}{n}x)_n \rightarrow 0$. If C is open, then there is n_0 such that $\frac{1}{n_0}x \in C$, meaning $\mu_C(x) \leq n_0$. Thus, $\mu_C(x) < \infty$ for each x , so μ_C is well-defined.

- (1) Let $t > 0$. We must show that $\mu_C(tx) = t\mu_C(x)$. If $\varepsilon > 0$, there is s_ε such that $\mu_C(tx) + \varepsilon > s_\varepsilon$, with $tx \in s_\varepsilon C$. We thus have $x \in \frac{s_\varepsilon}{t}C$, so $\mu_C \leq \frac{s_\varepsilon}{t}$, or $t\mu_C(x) \leq s_\varepsilon \leq \varepsilon + \mu_C(tx)$. Since this holds for all $\varepsilon > 0$, we have $t\mu_C(x) \leq \mu_C(tx)$.

Let $\varepsilon > 0$, and set $\varepsilon' = \frac{\varepsilon}{t}$. We find $r_\varepsilon > 0$ such that $\mu_C(x) + \varepsilon > r_\varepsilon$ and $x \in r_\varepsilon C$. Multiplying by t , we get $t\mu_C(x) + \varepsilon > tr_\varepsilon \geq \mu_C(tx)$, as $tx \in tr_\varepsilon C$. Thus, $\mu_C(tx) \leq t\mu_C(x)$. Combined, we get $\mu_C(tx) = t\mu_C(x)$.

To show subadditivity, we take $x, y \in C$ and let $\varepsilon > 0$. There exist s_ε and r_ε such that $\frac{1}{s_\varepsilon}x \in C$ and $\frac{1}{r_\varepsilon}y \in C$, and $\mu_C(x) + \frac{\varepsilon}{2} > s_\varepsilon$, $\mu_C(y) + \frac{\varepsilon}{2} > r_\varepsilon$.

Since C is convex, we have

$$\begin{aligned} \frac{1}{s_\varepsilon + r_\varepsilon}(x + y) &= \frac{x}{s_\varepsilon + r_\varepsilon} + \frac{y}{s_\varepsilon + r_\varepsilon} \\ &= \frac{s_\varepsilon}{s_\varepsilon + r_\varepsilon} \left(\frac{x}{s_\varepsilon} \right) + \frac{r_\varepsilon}{s_\varepsilon + r_\varepsilon} \left(\frac{y}{r_\varepsilon} \right) \\ &\in C. \end{aligned}$$

Thus, we have $\mu_C(x + y) \leq s_\varepsilon + r_\varepsilon < \mu_C(x) + \mu_C(y) + \varepsilon$, so $\mu_C(x + y) \leq \mu_C(x) + \mu_C(y)$.

We show continuity at the origin by letting $(x_\alpha)_\alpha \rightarrow 0_X$ be a net, and let $\varepsilon > 0$. Since C is an open neighborhood of the origin, so is εC , meaning we can find α_0 such that $x_\alpha \in \varepsilon C$ for $\alpha \geq \alpha_0$. Thus, $\mu_C(x) \leq \varepsilon$ for $\alpha \geq \alpha_0$, meaning $(\mu_C(x_\alpha))_\alpha \rightarrow 0 = \mu_C(0_X)$.

For $\mu_C(x) < 1$, there exists $0 < s < 1$ with $x \in sC$. Since $sC = (1 - s)0_X + sC \subseteq C$, we have $x \in C$.

Let $x \in C$, and consider the map $f : \mathbb{R} \rightarrow X$ mapping $t \mapsto tx$. It is the case that f is continuous, and $f(1) = x$. Thus, $f^{-1}(C)$ is an open neighborhood of $t = 1$, so there exists $t > 1$ such that $tx \in C$. Thus, $\mu_C(x) \leq \frac{1}{t} < 1$. Thus, $C = \{x \in X \mid \mu_C(x) < 1\}$.

- (2) Let C be balanced. Then, we have $\frac{x}{s} \in C$ if and only if $\frac{x}{s}\omega \in C$ for $\omega \in \mathbb{F}$ and $|\omega| = 1$. Thus, for such ω , we have

$$\begin{aligned} \mu_C(\omega x) &= \inf \left\{ s > 0 \mid \frac{1}{s}\omega x \in C \right\} \\ &= \inf \left\{ s > 0 \mid \frac{x}{s} \in C \right\} \\ &= \mu_C(x). \end{aligned}$$

If $\alpha \in \mathbb{F}$ with $\alpha \neq 0$, we have

$$\begin{aligned} \mu_C(\alpha x) \mu_C\left(\frac{\alpha}{|\alpha|} |\alpha| x\right) \\ = |\alpha| \mu_C\left(\frac{\alpha}{|\alpha|} x\right) \\ = |\alpha| \mu_C(x), \end{aligned}$$

meaning μ_C is a seminorm.

To show continuity, if $x, y \in X$, then $\mu_C(x) \leq \mu_C(x - y) + \mu_C(y)$, meaning $\mu_C(x) - \mu_C(y) \leq \mu_C(x - y)$. Similarly, we get $\mu_C(y) - \mu_C(x) \leq \mu_C(x - y)$, meaning

$$|\mu_C(x) - \mu_C(y)| \leq \mu_C(x - y),$$

so μ_C is Lipschitz.

(3) We have

$$\begin{aligned} x \in \delta C &\Leftrightarrow \frac{x}{\delta} \in C \\ &\Leftrightarrow \mu_C\left(\frac{x}{\delta}\right) < 1 \\ &\Leftrightarrow \frac{1}{\delta} \mu_C(x) < 1 \\ &\Leftrightarrow \mu_C(x) < \delta. \end{aligned}$$

□

Thus, we can formally characterize locally convex topological vector spaces.

Theorem (Characterization of Locally Convex Topological Vector Spaces): Let (X, τ) be a locally convex topological vector space with local base \mathcal{O}_{0_X} consisting of open, convex, and balanced sets.

Let $\mathcal{P} = \{\mu_C \mid C \in \mathcal{O}_{0_X}\}$ be the associated family of seminorms.

Then, \mathcal{P} is a separating family of seminorms and if

$$\tau_{\mathcal{P}} = \tau \left(\left\{ f_{p,z}^{-1}(U) \mid f_{p,z}(x) = p(z - x), z \in X, U \subseteq X \text{ open} \right\} \right),$$

then the map $\text{id}_X : (X, \tau) \rightarrow (X, \tau_{\mathcal{P}})$ is an isomorphism of topological vector spaces.

Proof. If $0_X \neq x \in X$, since X is Hausdorff we can find $U, V \in \tau$ with $U \cap V = \emptyset$, $0_X \in U$, and $x \in V$. Since \mathcal{O}_{0_X} is a local base, we can find $C \in \mathcal{O}_{0_X}$ with $C \subseteq U$, meaning we must have $\mu_C(x) \geq 1$. This shows \mathcal{P} is separating.

We must now show that the identity map, $\text{id}_X : (X, \tau) \rightarrow (X, \tau_{\mathcal{P}})$ is a homeomorphism. We only need show that $(x_{\alpha})_{\alpha} \xrightarrow{\tau} 0_X$ if and only if $(x_{\alpha})_{\alpha} \xrightarrow{\tau_{\mathcal{P}}} 0_X$, which is true if and only if $(\mu_C(x_{\alpha}))_{\alpha} \rightarrow 0$ for all $C \in \mathcal{O}_{0_X}$.

It is thus enough to show that $(x_{\alpha})_{\alpha} \xrightarrow{\tau} 0_X$ if and only if $(\mu_C(x_{\alpha}))_{\alpha} \rightarrow 0$ for all $C \in \mathcal{O}_{0_X}$. Since $\mu_C : (X, \tau) \rightarrow [0, \infty)$ is continuous, the forward direction is proven.

In the reverse direction, let $(\mu_C(x_{\alpha}))_{\alpha} \rightarrow 0$ for all $C \in \mathcal{O}_{0_X}$. If $V \in \tau$ with $0_X \in V$, then there is $C_0 \in \mathcal{O}_{0_X}$ with $C_0 \subseteq V$. Since $(\mu_{C_0}(x_{\alpha}))_{\alpha} \rightarrow 0$, there exists α_0 such that for $\alpha \geq \alpha_0$, $\mu_{C_0}(x_{\alpha}) < 1$, so $x_{\alpha} \in C_0 \subseteq V$ for all $\alpha \geq \alpha_0$. Therefore, we see that $(x_{\alpha})_{\alpha} \xrightarrow{\tau} 0_X$. □

Examples of Locally Convex Topological Vector Spaces

Example (Norm Topology). Let X be a vector space over \mathbb{F} with $\|\cdot\|$ the norm on X . If we set $\mathcal{P} = \{\|\cdot\|\}$, then $\tau_{\mathcal{P}}$ is the metric topology. We have

$$\begin{aligned} (x_{\alpha})_{\alpha} \xrightarrow{\tau} x &\Leftrightarrow \|x_{\alpha} - x\| \rightarrow 0 \\ &\Leftrightarrow (x_{\alpha})_{\alpha} \xrightarrow{\|\cdot\|} x. \end{aligned}$$

Example (Compact Convergence). Let Ω be a locally compact Hausdorff space. For a compact subset $K \subseteq \Omega$, we define $p_K : C(\Omega) \rightarrow [0, \infty)$ by $p_K(f) = \sup_{x \in K} |f(x)|$.

It is the case that $\mathcal{P} = \{p_K \mid K \subseteq X \text{ compact}\}$ is a separating family of seminorms. We have, for all K ,

$$\begin{aligned} (f_{\alpha})_{\alpha} \xrightarrow{\tau_{\mathcal{P}}} f &\Leftrightarrow p_K(f_{\alpha} - f) \rightarrow 0 \\ &\Leftrightarrow \sup_{x \in K} |f_{\alpha}(x) - f(x)| \rightarrow 0 \\ &\Leftrightarrow \|f_{\alpha}|_K - f|_K\|_{\infty} \rightarrow 0. \end{aligned}$$

This is known as uniform convergence on compact subsets, or compact convergence.

Example (Weak Topology). Let $(X, \|\cdot\|)$ be a normed vector space. For each $\varphi \in X^*$, we define

$$p_{\varphi} : X \rightarrow [0, \infty)$$

by $p_{\varphi}(x) = |\varphi(x)|$. By the Hahn–Banach separation result, we know that $\mathcal{P} = \{p_{\varphi} \mid \varphi \in X^*\}$ is a separating family of seminorms.

The topology $\tau_{\mathcal{P}}$ is called the weak topology on X , written $\sigma(X, X^*) = w$. Convergence in the weak topology is as follows; for all $\varphi \in X^*$,

$$\begin{aligned} (x_{\alpha}) \xrightarrow{w} x &\Leftrightarrow p_{\varphi}(x_{\alpha} - x) \rightarrow 0 \\ &\Leftrightarrow |\varphi(x_{\alpha} - x)| \rightarrow 0 \\ &\Leftrightarrow (\varphi(x_{\alpha}))_{\alpha} \rightarrow \varphi(x). \end{aligned}$$

We can also consider the initial topology on X induced by the family $\mathcal{F} = X^*$, denoted $\tau_{\mathcal{F}}$. We can see that the particular topologies coincide. We have

$$\begin{aligned} (x_{\alpha})_{\alpha} \xrightarrow{\tau_{\mathcal{F}}} x &\Leftrightarrow (\varphi(x_{\alpha}))_{\alpha} \rightarrow \varphi(x) \\ &\Leftrightarrow (x_{\alpha})_{\alpha} \xrightarrow{w} x. \end{aligned}$$

The weak topology is weaker than the norm topology, since the identity map, $\text{id}_X : (X, \|\cdot\|) \rightarrow (X, w)$ is continuous:

$$\begin{aligned} (x_n)_n \xrightarrow{\|\cdot\|} x &\Rightarrow (\varphi(x_n))_n \rightarrow \varphi(x) \\ &\Rightarrow (x_n)_n \xrightarrow{w} x. \end{aligned}$$

Thus, if $V \subseteq X$ is weakly open, then V is norm-open.

Exercise: Consider the canonical coordinate vectors $(e_n)_{n \geq 1}$ in c_0 . Prove that $(e_n)_n \xrightarrow{w} 0$.

Solution: Let $\varphi \in c_0^*$. Then, there is a unique $(\lambda_n)_n \in \ell_1$ such that

$$\varphi(z) = \sum_{n=1}^{\infty} \lambda_n z_n,$$

where $z \in c_0$. Therefore, we have

$$\varphi(e_n) = \lambda_n.$$

Note that since $(\lambda_n)_n$ is absolutely summable, we have

$$|\varphi(e_n)| \xrightarrow{n \rightarrow \infty} 0,$$

so $(e_n)_n \xrightarrow{w} 0$.

Example. Let $(X, \|\cdot\|)$ be a normed vector space. For every $x \in X$, we define

$$p_x : X^* \rightarrow [0, \infty)$$

to be $p_x(\varphi) = |\varphi(x)|$. We have $\mathcal{P}_* = \{p_x \mid x \in X\}$ is a separating family of seminorms on X^* .

The locally convex topology $\tau_{\mathcal{P}_*}$ is called the weak* topology on X^* . It is written $w^* = \sigma(X^*, X)$. Convergence in the weak topology is as follows: for each $x \in X$, we have

$$\begin{aligned} (\varphi_\alpha)_\alpha \xrightarrow{w^*} \varphi &\Leftrightarrow p_x(\varphi_\alpha - \varphi) \rightarrow 0 \\ &\Leftrightarrow |\varphi_\alpha(x) - \varphi(x)| \rightarrow 0 \\ &\Leftrightarrow (\varphi_\alpha(x))_\alpha \rightarrow \varphi(x) \\ &\Leftrightarrow (\varphi_\alpha)_\alpha \xrightarrow{\text{p.w.}} \varphi. \end{aligned}$$

We can also consider the initial topology on X^* induced by $\mathcal{F} = \{\hat{x} \mid x \in X\}$, where $\hat{x} : X^* \rightarrow \mathbb{F}$ is the canonical evaluation map, $\hat{x}(\varphi) = \varphi(x)$. We have, for all $x \in X$,

$$\begin{aligned} (\varphi_\alpha)_\alpha \xrightarrow{w^*} \varphi &\Leftrightarrow \varphi_\alpha(x) \rightarrow \varphi(x) \\ &\Leftrightarrow \hat{x}(\varphi_\alpha) \rightarrow \hat{x}(\varphi) \\ &\Leftrightarrow (\varphi_\alpha)_\alpha \xrightarrow{\tau_{\mathcal{F}}} \varphi. \end{aligned}$$

Note that the weak* topology on X^* is weaker than the norm topology, since $\text{id}_{X^*} : (X^*, \|\cdot\|_{\text{op}}) \rightarrow (X^*, w^*)$ is continuous, as for $(\varphi_n)_n \xrightarrow{\|\cdot\|_{\text{op}}} \varphi$, we have

$$\begin{aligned} |\varphi_n(x) - \varphi(x)| &= |(\varphi_n - \varphi)(x)| \\ &\leq \|\varphi_n - \varphi\|_{\text{op}} \|x\| \\ &\rightarrow 0. \end{aligned}$$

If $V \subseteq X^*$ is w^* -open, it is open in the operator norm.

Remark: If $\dim(X) = \infty$, then we will be able to show that the weak topology on X is strictly weaker than the norm topology, and the weak* topology on X^* is strictly weaker than the norm topology in the operator norm.

Exercise: Let X be a normed vector space, and suppose $A \subseteq X$ is a total subset, which is a subset such that $\overline{\text{span}}(A) = X$. If $\varphi \in X^*$ and $(\varphi_\alpha)_\alpha$ is a bounded net in X^* , show that $(\varphi_\alpha)_\alpha \rightarrow \varphi$ if and only if $(\varphi_\alpha(x))_\alpha \rightarrow \varphi(x)$.

Solution: The forward direction is clear, as pointwise convergence in X automatically implies pointwise convergence in A .

In the reverse direction, for any $x \in A$, we know that

$$(\varphi_\alpha(x))_\alpha \xrightarrow{\varphi} (x),$$

so, for any $x \in \text{span}(A)$, we have

$$(\varphi_\alpha(x))_\alpha \xrightarrow{\varphi} (x).$$

Let $v \in X$. Then, there is a sequence $(x_n)_n \in \text{span}(A)$ with $(x_n)_n \rightarrow v$. We have

$$|\varphi_\alpha(v) - \varphi(v)| \leq |\varphi_\alpha(v) - \varphi_\alpha(x_n)| + |\varphi_\alpha(x_n) - \varphi(x_n)| + |\varphi(x_n) - \varphi(v)|.$$

By the uniform boundedness principle, we are aware that $\|\varphi_\alpha\|_{\text{op}} \leq \sup_\alpha \|\varphi\|_\alpha < \infty$ as φ_α is a bounded net. Thus, we have

$$\begin{aligned} &\leq \sup_\alpha \|\varphi_\alpha\|_{\text{op}} \|x_n - v\| + |(\varphi_\alpha - \varphi)(x_n)| + \|\varphi\|_{\text{op}} \|x_n - v\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Exercise: Let Ω be a compact Hausdorff space, and suppose μ is a regular positive measure with full support. Show that

$$\overline{C(\Omega, \mu)}^{w^*} = L_\infty(\Omega, \mu).$$

Solution: We are aware that $L_\infty(\Omega, \mu) = (L_1(\Omega, \mu))^*$, so for each $f \in L_\infty$, there is a unique $\varphi_f \in L_1$ such that $\varphi_f(g) = \int_\Omega fg \, d\mu$.

Additionally, we know that, since $C_c(\Omega) \subseteq L_1(\Omega)$ is dense, there is a sequence $(h_n)_n \subseteq C_c(\Omega) = C(\Omega)$ such that $(h_n)_n \rightarrow f$.

Starting with the case of f positive, we have $(h_n)_n \rightarrow f$ is increasing, so by the dominated convergence theorem, we have

$$\begin{aligned} \varphi_{h_n}(g) &= \int_\Omega h_n g \, d\mu \\ &\rightarrow \int_\Omega fg \, d\mu \\ &= \varphi_f(g). \end{aligned}$$

For $\text{Ran}(f) \in \mathbb{R}$, we split f into $f = f_+ - f_-$, and apply the previous process to each of f_+ and f_- .

Finally, for $\text{Ran}(f) \in \mathbb{C}$, we split f into $f = (u_+ - u_-) + i(v_+ - v_-)$, and apply to each of u_\pm and v_\pm .

Therefore, we have $\overline{C(\Omega, \mu)}^{w^*} = (L_1(\Omega, \mu))^* \cong L_\infty(\Omega, \mu)$.

Example (Strong Operator Topology). Let $(X, \|\cdot\|)$ be a normed space. For each $x \in X$, consider

$$\eta_x : \mathcal{B}(X) \rightarrow [0, \infty)$$

defined by $\eta_x(T) = \|T(x)\|$.

We can see that $\eta = \{\eta_x \mid x \in X\}$ is a separating family of seminorms on $\mathcal{B}(X)$.

We let τ_η be the locally convex topology on $\mathcal{B}(X)$ generated by the family of seminorms η . This is known as the strong operator topology, or SOT.

Convergence in the SOT is as follows: for all $x \in X$, we have

$$\begin{aligned} (T_\alpha)_\alpha \xrightarrow{\tau_\eta} T &\Leftrightarrow \eta_x(T_\alpha - T) \rightarrow 0 \\ &\Leftrightarrow \|(T_\alpha - T)(x)\| \rightarrow 0 \\ &\Leftrightarrow \|T_\alpha(x) - T(x)\| \rightarrow 0 \\ &\Leftrightarrow (T_\alpha(x))_\alpha \rightarrow T(x). \end{aligned}$$

In other words, the strong operator topology is the topology of pointwise convergence. We have

$$\text{id}_{\mathcal{B}(X)} : (\mathcal{B}(X), \|\cdot\|_{\text{op}}) \rightarrow (\mathcal{B}(X), \text{SOT})$$

is continuous. This is because, we have $\|T_n(x) - T(x)\| \leq \|T_n - T\|_{\text{op}} \|x\|$. Thus, the strong operator topology is weaker than the norm topology.

We may also ask when the SOT is *strictly* weaker than the norm topology. For this, consider $X = \ell_2$ with the linear projections

$$\begin{aligned} P_n : \ell_2 &\rightarrow \ell_2 \\ (\xi_k)_k &\mapsto (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots). \end{aligned}$$

We claim that $(P_n)_n \rightarrow I$ in the strong operator topology. Note that for every $\xi \in \ell_2$, we have

$$\begin{aligned} \|I(\xi) - P_n(\xi)\|^2 &= \|(0, 0, \dots, 0, \xi_{n+1}, \xi_{n+2}, \dots)\|^2 \\ &= \sum_{k=n+1}^{\infty} |\xi_k|^2 \\ &< \varepsilon^2 \end{aligned}$$

by the definition of convergence. However, since $\|(I - P_n)(e_{n+1})\| = \|e_{n+1}\| = 1$, it is not the case that $(P_n)_n \xrightarrow{\|\cdot\|_{\text{op}}} I$.

It is thus the case that the SOT is strictly weaker than the norm topology on $\mathcal{B}(\ell_2)$.

Example (Weak Operator Topology). Let $(X, \|\cdot\|)$ be a normed space. For every pair $(x, \varphi) \in X \times X^*$, we have

$$\omega_{x, \varphi} : \mathcal{B}(X) \rightarrow [0, \infty)$$

defined by $\omega_{x, \varphi}(T) = |\varphi(T(x))|$.

The collection $\omega = \{\omega_{x, \varphi} \mid \varphi \in X^*, x \in X\}$ is a separating family of seminorms. We let τ_ω be the locally convex topology on $\mathcal{B}(X)$ generated by the family ω . This is known as the weak operator topology, or WOT.

Convergence in the weak operator topology is as follows: for all $x \in X$ and for all $\varphi \in X^*$,

$$\begin{aligned} (T_\alpha)_\alpha \xrightarrow{\text{WOT}} T &\Leftrightarrow \omega_{x, \varphi}(T_\alpha - T) \rightarrow 0 \\ &\Leftrightarrow |\varphi((T_\alpha - T)(x))| \rightarrow 0 \\ &\Leftrightarrow |\varphi(T_\alpha(x)) - \varphi(T(x))| \rightarrow 0 \\ &\Leftrightarrow \varphi(T_\alpha(x)) \rightarrow \varphi(T(x)) \\ &\Leftrightarrow (T_\alpha(x)) \xrightarrow{w} T(x). \end{aligned}$$

In other words, the WOT is equivalent to pointwise convergence in the weak topology on X .

We can verify that

$$\text{id}_{\mathcal{B}(X)} : (\mathcal{B}(X), \text{SOT}) \rightarrow (\mathcal{B}(X), \text{WOT})$$

is continuous. However, in general, the converse is not true.

Consider the operators

$$S_n : c_0 \rightarrow c_0$$

defined by $S_n(z) = z_1 e_n$, where $(e_n)_n$ are canonical coordinate vectors. Then, since $z_1 e_n \xrightarrow{w} 0$, we have $(S_n)_n \rightarrow 0$ in the WOT, but since $\|S_n(e_1)\| = \|e_n\| = 1$, which does not converge to 0 in the SOT.

Example (Strict Topology). Let Ω be a locally compact Hausdorff space. For each $\phi \in C_0(\Omega)$, define a seminorm

$$p_\phi : C_b(\Omega) \rightarrow [0, \infty)$$

by $p_\phi(f) = \|\phi f\|$.

Note that $C_0(\Omega) \subseteq C_b(\Omega)$ is not only a closed subspace, but is also an ideal — i.e., $\phi f \in C_0(\Omega)$ for any $\phi \in C_0(\Omega)$ and $f \in C_b(\Omega)$.

To verify that p_ϕ is a seminorm, we let $f \in C_b(\Omega)$ be nonzero. Then, $S = \text{supp}(f) \neq \emptyset$, so there is $\phi \in C_c(\Omega)$ with $\text{supp}(\phi) \subseteq S$. By Urysohn's Lemma, there is $K \subseteq \text{supp}(\phi)$ with $\phi = 1$ on K . We must have $\sup_{x \in K} |\phi(x)f(x)| \neq 0$, so $\|\phi f\|_u \neq 0$ necessarily.

The family $\beta = \{p_\phi \mid \phi \in C_0(\Omega)\}$ is a separating family of seminorms on $C_b(\Omega)$, so there is a locally convex topology τ_β (which we will abuse notation and refer to as β). This is known as the strict topology on $C_b(\Omega)$.

Convergence in the strict topology is as follows: for all $\phi \in C_0(\Omega)$,

$$(f_\alpha)_\alpha \xrightarrow{\beta} f \Leftrightarrow (\phi f_\alpha)_\alpha \rightarrow \phi f.$$

Exercise: Show that $(C_b(\Omega), \beta)$ is complete as a topological vector space, and $\overline{C_0(\Omega)}^\beta = C_b(\Omega)$.

Solution: Let $(f_\alpha)_\alpha$ be a Cauchy net in $(C_b(\Omega), \beta)$. Let $x \in V$ such that V is open and precompact, with $K = \bar{V}$. We have $(f_\alpha|_K)_\alpha$ is uniformly Cauchy, so there is $f_K \in C(K)$ with $\|f_\alpha - f_K\|_u \rightarrow 0$ on K . Thus, we have $|f_\alpha(x) - f_K(x)| \rightarrow 0$ for all $x \in K$.

We define $f(x) = f_K(x)$. We must now verify that f is well-defined. To this end, if $L = \bar{U}$ with $x \in U$, then it is the case that

$$\begin{aligned} |f_K(x) - f_L(x)| &\leq |f_K(x) - f_\alpha(x)| + |f_\alpha(x) - f_L(x)| \\ &\leq \varepsilon, \end{aligned}$$

so $\|f_K(x) - f_L(x)\|_u \leq \varepsilon$ for arbitrary ε , so $f_K(x) = f_L(x)$.

Seeing as $C_0(\Omega)$ is an ideal, it is the case that, for $(f_\alpha)_\alpha$ uniformly Cauchy in $(C_b(\Omega), \beta)$ and $\phi \in C_0(\Omega)$, that $(\phi f_\alpha)_\alpha$ is uniformly Cauchy in C_0 , implying that $(\phi f_\alpha)_\alpha \rightarrow H_\phi$ for some $H_\phi \in C_0(\Omega)$. In particular, for any $K \subseteq \Omega$ compact, we must have $(\phi f)_\alpha \rightarrow \phi f$, so $H_\phi = \phi f$.

We must now show that $f \in C_b(\Omega)$. Suppose it were not the case that f were bounded. Then there would exist $(x_n)_n \subseteq \Omega$ such that $|f(x_n)| \geq n$ for each n . We can find $\phi \in C_0(\Omega)$ such that $\phi(x_n) = \frac{1}{n}$, meaning that $|\phi(x_n)f(x_n)| \geq 1$ for each n , so $\|\phi(x_n)f(x_n)\|_n \geq 1$, which is a contradiction of the fact that $\phi f \in C_0(\Omega)$.

Therefore, we must have $f \in C_b(\Omega)$, so $(C_b(\Omega), \beta)$ is complete as a topological vector space.

Example (Functions of Rapid Decrease). Consider the space of all infinitely-differentiable functions,

$$C^\infty(\mathbb{R}) = \left\{ \phi : \mathbb{R} \rightarrow \mathbb{C} \mid \frac{d^k}{dx^k}(\phi) \text{ exists and is continuous for all } k \geq 0 \right\}.$$

Considering the norm

$$\begin{aligned} \|\phi\|_{n,k} &= \sup_{x \in \mathbb{R}} \left| x^n \frac{d^k}{dx^k}(\phi)(x) \right| \\ &= \left\| x^n \frac{d^k}{dx^k}(\phi) \right\|_u, \end{aligned}$$

we let

$$\mathcal{J} = \{ \phi \in C^\infty(\mathbb{R}) \mid \|\phi\|_{n,k} < \infty \text{ for all } n, k \in \mathbb{Z}_+ \}.$$

Functions in \mathcal{J} have derivatives that decay faster than the inverse of any polynomial — the family $\|\cdot\|_{n,k}$ is separating for \mathcal{J} , which yields a locally convex topological vector space known as the functions of rapid decrease.

Norms and Metrics on Locally Convex Spaces

Given a locally convex topological vector space, we may ask if the topology is compatible with a particular norm.

Definition. Let X be a vector space.

- (1) A metric d on X is called translation-invariant if

$$d(x+z, y+z) = d(x, y)$$

for all $x, y, z \in X$.

- (2) If X is a locally convex topological vector space whose topology is compatible with a translation-invariant complete metric, we say X is a Fréchet space.

Proposition: Let X be a vector space, and let $\mathcal{P} = \{p_n \mid n \in \mathbb{N}\}$ be a countable separating family of seminorms on X . We let $\tau_{\mathcal{P}}$ be the locally convex topology generated by the family of seminorms on \mathcal{P} .

The form $d : X \times X \rightarrow [0, \infty)$ defined by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

is a translation-invariant metric, with

$$\text{id}_X : (X, \tau_{\mathcal{P}}) \rightarrow (X, d)$$

a linear isomorphism and homeomorphism.

Proof. Since p_n is a family of seminorms, the collection $(p_n((\cdot) - (\cdot)))_n$ is a separating family of semimetrics on X , meaning that d is a translation-invariant metric.

Suppose $(x_\alpha)_\alpha$ is a net in X with $d(x_\alpha, x) \rightarrow 0$. We will show that $(x_\alpha)_\alpha \rightarrow x$ in $\tau_{\mathcal{P}}$.

For each $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \rho_n(x_\alpha, x) \\ &= \frac{p_n(x_\alpha - x)}{1 + p_n(x_\alpha - x)} \\ &\leq 2^n d(x_\alpha, x), \end{aligned}$$

meaning that $\rho_n(x_\alpha, x) \rightarrow 0$. Note that $p_n(x_\alpha - x) = \frac{\rho_n(x_\alpha, x)}{1 - \rho_n(x_\alpha, x)}$, so $p_n(x_\alpha, x) \rightarrow 0$ for every $n \geq 1$.

It follows that $(x_\alpha)_\alpha \xrightarrow{\tau_{\mathcal{P}}} x$ by the definition of the topology $\tau_{\mathcal{P}}$.

Let $(x_\alpha)_\alpha \rightarrow x$, and let $\varepsilon > 0$. Find N such that $\sum_{n>N} 2^{-n} < \varepsilon/2$.

For each $n = 1, \dots, N$, it is the case that $p_n(x_\alpha - x) \rightarrow 0$, so $\rho_n(x_\alpha, x) \rightarrow 0$. Thus, there is α_0 such that for all $\alpha \geq \alpha_0$, $\rho_n(x_\alpha, x) < \varepsilon/2$.

Thus, for $\alpha \geq \alpha_0$, we have

$$\begin{aligned} d(x_\alpha, x) &= \sum_{n=1}^{\infty} \rho_n(x_\alpha, x) \\ &= \sum_{n=1}^N 2^{-n} \rho_n(x_\alpha, x) + \sum_{n>N} 2^{-n} \rho_n(x_\alpha, x) \\ &\leq \sum_{n=1}^N 2^{-n} (\varepsilon/2) + \sum_{n>N} 2^{-n} \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Thus, $d(x_\alpha, x) \rightarrow 0$. □

Example. Let X be the vector space $C(\mathbb{R})$. For each $n \geq 1$, we define

$$p_n : X \rightarrow [0, \infty)$$

by

$$p_n(f) = \sup_{x \in [-n, n]} |f(x)|.$$

It is the case that the set $\mathcal{P} = \{p_n \mid n \geq 1\}$ is a separating family of seminorms. Thus, $(X, \tau_{\mathcal{P}})$ is metrizable with the Fréchet metric,

$$d_F(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f - g)}{1 + p_n(f - g)}.$$

It is the case that the Fréchet metric is complete. To see this, we let $(f_n)_n$ be a sequence of functions Cauchy in the Fréchet metric.

We start by showing that for a fixed x , $(f_n(x))_n$ is Cauchy in \mathbb{R} . Setting

$$\rho_k(f, g) = \frac{p_k(f - g)}{1 + p_k(f - g)},$$

we have that, for any $\varepsilon > 0$,

$$\begin{aligned} \frac{|f_n(x) - f_m(x)|}{1 + |f_n(x) - f_m(x)|} &\leq \rho_k(f_n, f_m) \\ &< 2^k \varepsilon, \end{aligned}$$

where k is such that $x \in [-k, k]$. Then, it is necessarily the case that the sequence $(f_n)_n$ is uniformly Cauchy in $[-k, k]$, hence $(f_n(x))_n$ is Cauchy in \mathbb{R} .

We define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for each $x \in \mathbb{R}$. We must show that $(f_n)_n \xrightarrow{d_F} f$.

For $\varepsilon > 0$, we find K such that $\sum_{k>K} 2^{-k} < \varepsilon/2$. Then, there exists N such that for all $n \geq N$, we have $\rho_k(f_n, f) < \varepsilon/2$ for each $k = 1, \dots, K$. Therefore, for $n \geq N$, we have

$$d_F(f_n, f) = \sum_{k=1}^{\infty} 2^{-k} \rho_k(f_n, f)$$

$$\begin{aligned}
&= \sum_{k=1}^K 2^{-k} \rho_k(f_n, f) + \sum_{k>K} 2^{-k} \rho_k(f_n, f) \\
&\leq \sum_{k=1}^K 2^{-k} (\varepsilon/2) + \varepsilon/2 \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Thus, $(f_n)_n$ converges to f — since, for any k , $(f_n)_n \xrightarrow{\|\cdot\|_u} f$ on $[-k, k]$, it is the case that f is continuous on any such $[-k, k]$, meaning $f \in C(\mathbb{R})$.

Example (Functions of Rapid Decrease). We will show that the space \mathcal{J} of functions of rapid decrease is a Fréchet space.

Since there are countably many seminorms, we know that there is a translation-invariant metric. We must show that this metric is complete.

Let $(\phi_j)_j$ be Cauchy. It follows that $\left(x^n \frac{d^k \phi_j}{dx^k}\right)_j$ is uniformly Cauchy for all $n, k \in \mathbb{Z}^+$, so there are $\psi_{n,k} \in C(\mathbb{R})$ such that $\left(x^n \frac{d^k \phi_j}{dx^k}\right)_j \rightarrow \psi_{n,k}$ uniformly.

Set $\psi = \psi_{0,0}$. The fundamental theorem of calculus gives

$$\phi_j(x) = \phi_j(0) + \int_0^x \phi_j'(t) dt,$$

with $(\phi_j)_j \rightarrow \psi$ uniformly and $(\phi_j')_j \rightarrow \psi_{0,1}$ uniformly. Together, we get

$$\psi(x) = \psi(0) + \int_0^x \psi_{0,1}(t) dt,$$

meaning $\psi \in C^1(\mathbb{R})$, with $\psi' = \psi_{0,1}$. Thus, we get $x^n \frac{d^k \psi}{dx^k} = \psi_{n,k}$, meaning $\psi \in \mathcal{J}$ and $(\phi_j)_j \rightarrow \psi$ in the metric.

We may next ask when we can apply a norm to a locally convex topological vector space.

Note that if $(X, \|\cdot\|)$ is a normed vector space, then $U_X = \{x \mid \|x\| < 1\}$ is open and bounded — as it turns out, this condition is sufficient.

Definition. Let X be a topological vector space. A subset $A \subseteq X$ is said to be bounded if for every neighborhood $U \in \mathcal{N}_{0_X}$, there is $t > 0$ such that $A \subseteq tU$.

Lemma: Let X be a topological vector space. If $C \subseteq X$ is an open, convex, balanced, bounded subset with $0_X \in C$, then the Minkowski functional

$$\begin{aligned}
\mu_C : X &\rightarrow [0, \infty) \\
\mu_C(x) &= \inf \{s > 0 \mid x \in sC\}
\end{aligned}$$

is a norm.

Proof. We are aware that μ_C is a seminorm. Now, we show that if $x \neq 0$, then $\mu_C(x) \neq 0$.

Since X is Hausdorff, there exist open sets $U, V \subseteq X$ with $0 \in U, x \in V, U \cap V = \emptyset$. Since C is bounded, there is $t > 0$ such that $C \subseteq tU$, meaning $\frac{1}{t}C \subseteq U$. Thus, we have $\{z \mid \mu_C(z) < \frac{1}{t}\} = \frac{1}{t}C \subseteq U$. Since $x \notin U$, $\mu_C(x) \geq \frac{1}{t} > 0$. \square

Proposition: Let (X, τ) be a locally convex topological vector space. Then, X is normable if and only if X contains a nonempty open and bounded set.

Proof. If X is equipped with a norm, then the set $C = U_X$ is nonempty, open, and bounded.

Let $\emptyset \neq U \subseteq X$ be open and bounded. Pick $x \in U$, and note that $V = U - x$ is a nonempty, open, and bounded neighborhood of 0_X . Since X is locally compact, we can find an open, balanced, and convex $C \subseteq V$ containing 0_X . Note that C is also bounded since V is. Thus, μ_C is our desired norm.

We claim that $(X, \tau) \rightarrow (X, \mu_C)$ is an isomorphism of topological vector spaces.

If $(x_\alpha)_\alpha \xrightarrow{\tau} 0$, then $(\mu_C(x_\alpha))_\alpha \rightarrow \mu_C(0) = 0$, since μ_C is continuous. Thus, $\mu_C(x_\alpha - 0) \rightarrow 0$, so $(x_\alpha) \xrightarrow{\mu_C} 0$.

If $(x_\alpha)_\alpha \xrightarrow{\mu_C} 0$, we must show that $(x_\alpha)_\alpha \rightarrow \tau 0$. Let $V \in \mathcal{N}_{0_X}$. Since C is bounded, there is $t > 0$ with $C \subseteq tV$. We find α_0 such that for $\alpha \geq \alpha_0$, $\mu_C(x_\alpha) < \frac{1}{t}$. This implies that $x_\alpha \in \frac{1}{t}C \subseteq V$. \square

Hahn–Banach Theorems for Locally Convex Topological Vector Spaces

Remembering our Hahn–Banach results on normed vector spaces (such as extension and separation results), we can expand them to locally convex spaces.

Hahn–Banach Extension Theorems

Definition. Let X be a topological vector space. The continuous dual of X is the collection

$$X^* = \{ \varphi \in X' \mid \varphi \text{ is continuous} \}.$$

In normed vector spaces, a linear functional is continuous if and only if it is bounded. We can find an analogous result for general topological vector spaces.

Proposition: Let X be a topological vector space. Suppose $\varphi \in X'$ is a linear functional. The following are equivalent

- (i) φ is uniformly continuous;
- (ii) φ is continuous;
- (iii) φ is continuous at 0;
- (iv) $\ker(\varphi) \subseteq X$ is closed
- (v) For X a locally convex topological vector space whose topology is generated by a family of separating seminorms \mathcal{P} , then there exists $K > 0$ and $p_1, \dots, p_n \in \mathcal{P}$ such that

$$|\varphi(x)| \leq K \max_{j=1}^n |p_j(x)|.$$

Proof. We have shown the first three cases. It is also the case that (ii) implies (iv), as $\ker(\varphi) = \varphi^{-1}(\{0\})$.

To show that (iv) implies (ii), we form the quotient $X/\ker(\varphi)$, which is a topological vector space. By the first isomorphism theorem, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{F} \\ \varphi \downarrow & \nearrow \tilde{\varphi} & \\ X/\ker(\varphi) & & \end{array}$$

where $\tilde{\varphi}(x + \ker(\varphi)) = \varphi(x)$ is injective. Thus, $\dim(X/\ker(\varphi)) = 1$, so $\tilde{\varphi}$ is continuous. Since π is also continuous, $\varphi = \tilde{\varphi} \circ \pi$ is also continuous.

We can now show that (v) implies (iii). If $(x_\alpha)_\alpha$ is a net in X converging to 0_X , then $(p(x_\alpha))_\alpha \rightarrow 0$ for all $p \in \mathcal{P}$, including p_1, \dots, p_n , so $(\varphi(x_\alpha))_\alpha \rightarrow 0$.

Let φ be continuous at 0_X . Since $\varphi(0_X) = 0$, $\varphi^{-1}(U_F)$ is an open neighborhood of 0_X , meaning there are $\varepsilon > 0$ and p_1, \dots, p_n such that

$$\varphi\left(\bigcap_{j=1}^n U_{p_j}(0, \varepsilon)\right) \subseteq U_F.$$

We set $K = 2/\varepsilon$.

Let $y \in X$ be arbitrary. If $p_j(y) = 0$ for all $j = 1, \dots, n$, then $p_j(ty) = 0$ for all $t > 0$, meaning $|\varphi(ty)| < 1$ for all $t > 0$, meaning $|\varphi(y)| < \frac{1}{t}$ for all $t > 0$, meaning $\varphi(y) = 0$.

If $p_i(y) \neq 0$ for some $i \in \{1, \dots, n\}$, then $z = \frac{\varepsilon y}{2 \max_{j=1}^n |p_j(y)|}$ belongs to $\bigcap_{j=1}^n U_{p_j}(0, \varepsilon)$. Thus, we have $|\varphi(z)| < 1$, so $|\varphi(y)| < K \max_{j=1}^n |p_j(y)|$. \square

Theorem (Hahn–Banach Extension for Locally Convex Spaces): Let X be a locally convex topological vector space, and suppose $E \subseteq X$ is a subspace. If $\varphi \in E^*$, there is $\psi \in X^*$ such that $\psi|_E = \varphi$.

Proof. Let \mathcal{P} be a family of seminorms on X . We can see that $\mathcal{P}_E = \{p|_E \mid p \in \mathcal{P}\}$ defines the subspace topology on E , which is also locally convex. Since $\varphi \in E^*$, there are $p_1, \dots, p_n \in \mathcal{P}_E$ and $K > 0$ such that

$$|\varphi(x)| \leq K \max_{j=1}^n |p_j(x)|$$

for all $x \in E$. We define $q : X \rightarrow [0, \infty)$ by $q(x) = K \max_{j=1}^n p_j(x)$. Then, q is a seminorm with $|\varphi(x)| \leq q(x)$ for all $x \in E$. Applying the Hahn–Banach–Minkowski extension theorem, we get a linear map $\psi : X \rightarrow \mathbb{F}$ with $|\psi(x)| \leq q(x) = K \max_{j=1}^n |p_j(x)|$ for all $x \in X$, meaning $\psi \in X^*$. \square

The following corollary is akin to the existence of the norming functional we had exhibited in the case of the Hahn–Banach extension theorems on normed spaces.

Corollary: Let X be a locally convex topological vector space. Suppose $\{x_1, \dots, x_n\} \subseteq X$ is a linearly independent subset, and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$. Then, there exists $\varphi(x_j) = \alpha_j$ for all j .

Proof. Let

$$E = \text{span}(\{x_1, \dots, x_n\}),$$

and define $\psi : E \rightarrow \mathbb{F}$ by

$$\psi\left(\sum_{j=1}^n \lambda_j x_j\right) = \sum_{j=1}^n \lambda_j \alpha_j.$$

We see ψ is continuous as $\dim(E) = n$. We apply the extension result. \square

The following corollary is a direct implication of the above theorem.

Corollary: Let X be a locally convex topological vector space. Suppose $x \in X$ is nonzero. There exists $\varphi \in X^*$ with $\varphi(x) \neq 0$.

Thus, X^* separates points.

Separation Results

It is often useful to separate points with continuous functionals — now that we have shown that X^* separates the points of X , we want to be able to separate points from closed sets using linear functionals.

Consider two disjoint, open, convex sets A, B of \mathbb{R}^2 , and suppose a line $y = mx + b$ lies “below” A and “above” B . The linear map $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\varphi(x, y) = y - mx$ is such that $A \subseteq \{x \mid \varphi(x) > b\}$, and $B \subseteq \{x \mid \varphi(x) < b\}$, meaning the hyperplane $\{(x, y) \in \mathbb{R}^2 \mid \varphi(x, y) = b\}$ separates the convex sets A and B .

Exercise: Let X be a topological vector space. Prove that an affine hyperplane is either closed or dense in X .

Solution: Let $H = x_0 + E$ be an affine hyperplane. Then, $E \subseteq X$ is a maximal proper subspace of X by the definition of a hyperplane. At the same time, we have $E \subseteq \bar{E}$, meaning either $\bar{E} = X$, in which case E is dense, or $\bar{E} = E$, in which case E is closed.

Similarly, this means H is either closed (as it is a translate of E) or is equal to X .

Definition. Let X be a topological vector space over \mathbb{R} .

(1) An open-half space is a set of the form

$$\{x \in X \mid u(x) > t\} \text{ or } \{x \in X \mid u(x) < t\}$$

for $0 \neq u \in X^*$ and $t \in \mathbb{R}$.

(2) A closed half space is a set of the form

$$\{x \in X \mid u(x) \geq t\} \text{ or } \{x \in X \mid u(x) \leq t\}$$

for $0 \neq u \in X^*$ and $t \in \mathbb{R}$.

Theorem (Hahn–Banach Separation Theorem for Topological Vector Spaces): Let X be a topological vector space over \mathbb{F} . Let A and B be convex disjoint subsets of X . If A is open, there exists $\varphi \in X^*$, with $\varphi = u + iv$, and a scalar $t \in \mathbb{R}$, such that

$$u(a) < t \leq u(b)$$

for all $a \in A$ and $b \in B$.

Proof. We start by letting $\mathbb{F} = \mathbb{R}$. Let $x_0 \in A$ and $y_0 \in B$, and set $z = y_0 - x_0$. Then, the set $A - B + z$ is open, convex, and contains 0, so $\mu_C : X \rightarrow [0, \infty)$ is a Minkowski functional with $C = \{x \mid \mu_C(x) < 1\}$.

Since $A \cap B = \emptyset$, we know that $z \notin C$, so $\mu_C(z) \geq 1$. Define $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ by $u_0(sz) = s$. If $s \geq 0$, then $u_0(sz) = s \leq s\mu_C(z) = \mu_C(sz)$, and if $s < 0$, then $u_0(sz) = s \leq \mu_C(sz)$. Either way, u_0 is dominated by the Minkowski functional μ_C , so by the Hahn–Banach–Minkowski extension theorem, there is an \mathbb{R} -linear $u : X \rightarrow \mathbb{R}$ such that $u|_{\mathbb{R}z} = u_0$ and $u \leq \mu_C$ on X .

We claim that u is continuous. We only need check that u is continuous at the origin. Let $\varepsilon > 0$, and let $V = \varepsilon C \cap (-\varepsilon)C$, which is an open neighborhood of 0_X , as

$$\begin{aligned} x \in V &\Rightarrow x \in \varepsilon C \\ &\Rightarrow \varepsilon^{-1}x \in C \\ &\Rightarrow u(\varepsilon^{-1}x) \leq \mu_C(\varepsilon^{-1}x) < 1 \\ &\Rightarrow u(x) < \varepsilon. \end{aligned}$$

Additionally,

$$x \in V \Rightarrow x \in -\varepsilon C$$

$$\begin{aligned}
&\Rightarrow -\varepsilon^{-1}x \in C \\
&\Rightarrow u(-\varepsilon^{-1}x) \leq \mu_C(-\varepsilon^{-1}x) < 1 \\
&\Rightarrow u(x) > -\varepsilon,
\end{aligned}$$

meaning $u(V) \subseteq (-\varepsilon, \varepsilon)$.

If $a \in A$ and $b \in B$, then $a - b + z \in C$, so $u(a - b + z) \leq \mu_C(a - b + z) < 1$. Since $u(z) = 1$, we have $u(a) - u(b) + 1 < 1$, so $u(a) < u(b)$.

Let $t = \sup_{a \in A} u(a)$. Then, $t \leq u(b)$ for all $b \in B$, meaning $u(a) \leq t \leq u(b)$ for all $a \in A$ and $b \in B$.

If $u(a_0) = t$ for some $a_0 \in A$, we consider the map $f : \mathbb{R} \rightarrow X$ defined by $r \mapsto a_0 + rz$. Since $f(0) = a_0$, there exists $\delta > 0$ such that $|r| < \delta$ implies $a_0 + rz \in A$, meaning $t + r = u(a_0) + r = u(a_0 + rz) \leq t$ for all $r \in (-\delta, \delta)$, meaning if $r = \delta/2$, we get a contradiction. Thus, $u(a) < t \leq u(b)$ for all $a \in A$ and $b \in B$.

If $\mathbb{F} = \mathbb{C}$, we construct the \mathbb{R} -linear continuous $u : X \rightarrow \mathbb{R}$ as above, and set $\varphi(x) = u(x) + iv(x)$, where $v(x) = -u(ix)$. Thus, $\varphi \in X^*$, and the inequality holds. \square

Remark: If A and B are both open, then we can find strict inequalities in the above statement.

Now, we can prove the fact that we can separate two closed convex sets with at least one compact via a double hyperplane, provided the space is locally convex.

Theorem (Hahn–Banach Separation Theorem for Locally Convex Topological Vector Spaces): Let X be a locally convex topological vector space, and suppose $C, K \subseteq X$ are convex, closed, and disjoint subsets of X . Assume K is compact. Then, there exist $\varphi \in X^*$ with $\varphi = u + iv$, $t \in \mathbb{R}$, and $\delta > 0$, such that

$$u(x) \leq t < t + \delta \leq u(y)$$

for all $x \in C$ and $y \in K$. This is to say that K and C live in disjoint closed half-spaces.

Proof. If $y \in K$, then $C - y$ is closed, and $0_X \in (C - y)^c$. Thus, for each $y \in K$, there is an open, convex, and balanced set $V_y \subseteq X$ with $0_X \in V_y \subseteq (C - y)^c$. Thus, we have

$$K \subseteq \bigcup_{y \in K} \left(y + \frac{1}{3}V_y \right),$$

so by compactness, there are $y_1, \dots, y_n \in K$ with

$$K \subseteq \bigcup_{j=1}^n \left(y_j + \frac{1}{3}V_{y_j} \right).$$

The set

$$V = \bigcap_{j=1}^n \frac{1}{3}V_{y_j}$$

is open, convex, and contains zero. The sets

$$\begin{aligned}
K + V &= \bigcup_{y \in K} y + V \\
C + V &= \bigcup_{x \in C} (x + V)
\end{aligned}$$

are both open and convex with $K \subseteq K + V$ and $C \subseteq C + V$. We claim that $(K + V) \cap (C + V) = \emptyset$. Suppose $y + v = x + v'$ for some $y \in K$, $x \in C$, and $v, v' \in V$. There is necessarily $i \in \{1, \dots, n\}$ with $y \in y_i + \frac{1}{3}V_{y_i}$, and $v, v' \in y_i + \frac{1}{3}V_{y_i}$. Thus,

$$\begin{aligned} x &= y + v - v' \\ &\in y_i + \frac{1}{3}V_{y_i} + \frac{1}{3}V_{y_i} - \frac{1}{3}V_{y_i} \\ &= y_i + \frac{1}{3}V_{y_i} + \frac{1}{3}V_{y_i} + \frac{1}{3}V_{y_i} \\ &\subseteq y_i + V_{y_i}. \end{aligned}$$

Thus, we have $x - y_i \in V_{y_i} \cap (C - y_i)$, which contradicts the fact that $V_{y_i} \subseteq (C - y_i)^c$.

We apply the separation result on topological vector spaces to the open, convex, disjoint sets $K + V$ and $C + V$, meaning we get $\varphi \in X^*$ with $\varphi = u + iv$ such that

$$u(a) < t < u(b)$$

for all $a \in C + V$ and $b \in K + V$. In particular, we have that the inequality holds true for $a = x \in C$ and $b = y \in K$. Moreover, since K is compact, $\inf_{y \in K} u(y) = u(y_0)$ for some $y_0 \in K$. Thus,

$$u(x) < t < u(y_0) \leq u(y)$$

for every $x \in C$ and $y \in K$. We set $\delta = u(y_0) - t$. □

The following result will be very useful as we go further into the weak and weak* topologies.

Corollary: Let X be a vector space admitting locally convex topologies $\tau_1 \subseteq \tau_2$. Moreover, suppose $(X, \tau_1)^* = (X, \tau_2)^*$. If $C \subseteq X$ is a convex subset, then $\overline{C}^{\tau_1} = \overline{C}^{\tau_2}$.

Proof. Since every τ_1 -closed subset is τ_2 -closed, we have

$$\begin{aligned} \overline{C}^{\tau_1} &= \bigcap \{D \mid D \supseteq C, D \text{ is } \tau_1\text{-closed}\} \\ &\supseteq \bigcap \{D \mid D \supseteq C, D \text{ is } \tau_2\text{-closed}\} \\ &= \overline{C}^{\tau_2}. \end{aligned}$$

Suppose toward contradiction that $\overline{C}^{\tau_2} \subset \overline{C}^{\tau_1}$, and let $x_0 \in \overline{C}^{\tau_1} \setminus \overline{C}^{\tau_2}$. Since \overline{C}^{τ_2} is τ_2 -closed, and since τ_2 is locally convex, there is an open and convex neighborhood U of x_0 with $U \cap \overline{C}^{\tau_2} = \emptyset$.

By the Hahn–Banach separation theorems applied to (X, τ_2) , there is a hyperplane in X that separates U from C . In particular, there exists $\alpha \in \mathbb{R}$ and $\varphi \in (X, \tau_2)^*$ with $u = \operatorname{Re}(\varphi)$, such that

$$u(x_0) < \alpha \leq u(y)$$

for all $y \in C$.

We have $x_0 \in \overline{C}^{\tau_1}$, so there is a net $(y_i)_i \in C$ with $(y_i)_i \xrightarrow{\tau_1} x_0$. By the hypothesis, we also see that $u = (X, \tau_1) \rightarrow \mathbb{R}$ is continuous, so $(u(y_i))_i \rightarrow u(x_0)$, meaning $u(x_0) \geq \alpha$, contradicting the fact that $u(x_0) < \alpha$. □

Weak Topologies

Let X be a vector space over \mathbb{F} . The algebraic dual X' is the set of all linear functionals on X .

Let $W \subseteq X'$ be a nonempty subset. For each $\varphi \in W$, we consider the map

$$p_\varphi : X \rightarrow [0, \infty)$$

with $p_\varphi(x) = |\varphi(x)|$. Note that each p_φ is a seminorm. Since each member of W is a linear map, the family W is separating if and only if, for every nonzero $x \in X$, there is $\varphi \in W$ with $\varphi(x) \neq 0$.

Moreover, for W separating, the collection

$$\mathcal{P}_W = \{p_\varphi \mid \varphi \in W\}$$

is a separating family of seminorms. We may construct the collection of maps

$$\mathcal{F}_{\mathcal{P}_W} = \{p_\varphi((\cdot) - z) \mid \varphi \in W, z \in X\},$$

and define $\tau_{\mathcal{P}_W}$ to be the initial topology induced by $\mathcal{F}_{\mathcal{P}_W}$. This is the locally convex topology generated by the family of seminorms \mathcal{P}_W .

Alternatively, we may consider $\sigma(X, W)$, which is the initial topology^{xxii} on X induced by the family W . As it turns out, these topologies coincide.

Proof. If $(x_\alpha)_\alpha$ is a net in X , then for each $\varphi \in W$, we have

$$\begin{aligned} (x_\alpha)_\alpha \xrightarrow{\tau_{\mathcal{P}_W}} x &\Leftrightarrow p_\varphi(x_\alpha - x) \rightarrow 0 \\ &\Leftrightarrow |\varphi(x_\alpha - x)| \rightarrow 0 \\ &\Leftrightarrow |\varphi(x_\alpha) - \varphi(x)| \rightarrow 0 \\ &\Leftrightarrow (\varphi(x_\alpha))_\alpha \rightarrow \varphi(x) \\ &\Leftrightarrow (x_\alpha)_\alpha \xrightarrow{\sigma(X, W)} x. \end{aligned}$$

□

It follows that for $W \subseteq X'$ a separating family, then $(X, \sigma(X, W))$ is a locally convex topological vector space.

Definition. Let X be a \mathbb{F} -vector space.

- (1) If $W \subseteq X'$ is a separating family, then the initial topology on X generated by W , denoted $\sigma(X, W)$, is called the weak topology on X induced by W .
- (2) If $(X, \|\cdot\|)$ is a normed vector space, then $w = \sigma(X, X^*)$ is the weak topology on X .
- (3) If $(X, \|\cdot\|)$ is a normed vector space, then $w^* = \sigma(X^*, \iota(X))$ is called the weak* topology on X^* . Here, $\iota : X \hookrightarrow X^{**}$ denotes the canonical embedding.

Recall that the definition of convergence in the weak topology is

$$(x_\alpha)_\alpha \xrightarrow{w} x \Leftrightarrow (\varphi(x_\alpha))_\alpha \rightarrow \varphi(x),$$

for all $\varphi \in X^*$, while convergence in the weak* topology is

$$(\varphi_\alpha)_\alpha \xrightarrow{w^*} \varphi \Leftrightarrow (\varphi_\alpha(x))_\alpha \rightarrow \varphi(x)$$

for all $x \in X$.

Exercise: Suppose $T : X \rightarrow Y$ is a linear map between Banach spaces that is w - w continuous. Prove that T is bounded.

^{xxii}The coarsest topology such that every map in W is continuous.

Solution: Let $(x_n)_n \xrightarrow{\|\cdot\|} x$, with $(T(x_n))_n \xrightarrow{\|\cdot\|} y$. Then, $(x_n)_n \xrightarrow{w} x$, so $(T(x_n))_n \xrightarrow{w} T(x)$, meaning $T(x) = y$, so $(T(x_n))_n \xrightarrow{\|\cdot\|} T(x)$.

Thus, $\text{graph}(T) \subseteq X \times Y$ is closed, so T is continuous.

When a normed vector space is infinite-dimensional, the weak topology is strictly weaker than the norm topology.

Proposition: If X is an infinite-dimensional normed space and $W \subseteq X'$ is a separating family, then any nonempty weakly open $U \subseteq X$ is unbounded.

Proof. Assume $0_X \in U$. Since U is weakly open, U contains a basic open set for the weak topology $\sigma(X, W)$. Thus, there exist $\varphi_1, \dots, \varphi_n \in W$ and $\varepsilon > 0$ such that

$$0 \in \bigcap_{i=1}^n U_{p_{\varphi_i}}(0, \varepsilon) \subseteq U,$$

where

$$\begin{aligned} U_{p_{\varphi_i}}(0, \varepsilon) &= \{x \in X \mid p_{\varphi_i}(x) < \varepsilon\} \\ &= \{x \in X \mid |\varphi_i(x)| < \varepsilon\}. \end{aligned}$$

Note that

$$0 \in V := \bigcap_{i=1}^n \ker(\varphi_i) \subseteq \bigcap_{i=1}^n U_{p_{\varphi_i}}(0, \varepsilon) \subseteq U.$$

Since $\dim(X)$ is infinite, V is a non-trivial subspace, which is unbounded, meaning U is unbounded.

If U is arbitrary, we take the translate $U' = U - x_0$ for some $x_0 \in U$, meaning U' is unbounded. \square

Corollary: If $(X, \|\cdot\|)$ is an infinite-dimensional normed vector space, then the open unit ball $U_X = \{x \in X \mid \|x\| < 1\}$ is open in the norm topology but not weakly open.

Recall that if (X, τ) is a topological vector space, then the continuous dual $(X, \tau)^*$ is the space of all continuous functionals on X . We now wish to determine the continuous dual of $(X, \sigma(X, W))^*$, where X is a vector space and $W \subseteq X'$ is a separating family of linear functionals.

Proposition: Let X be a vector space, and suppose $W \subseteq X'$ is a separating family. If X is equipped with the weak topology, $\sigma(X, W)$, then $X^* = \text{span}(W)$.

Proof. Any $\varphi \in W$ is continuous as the initial topology is, by definition, the coarsest topology for which every member of W is continuous. Similarly, we obtain that $\text{span}(W) \subseteq X^*$.

We now establish the reverse inclusion. Let $F \in X^*$. Thus, there exist $k > 0$ and $\varphi_1, \dots, \varphi_n \in W$ such that

$$|F(x)| \leq k \max_{j=1}^n |\varphi_j(x)|,$$

meaning $\bigcap_{j=1}^n \ker(\varphi_j) \subseteq \ker(F)$. Thus, we find

$$F = \sum_{j=1}^n \alpha_j \varphi_j \in \text{span}(W).$$

\square

Corollary: Let X be a normed space, and let $\varphi : X \rightarrow \mathbb{F}$ be a linear functional. Then, φ is norm-continuous if and only if φ is weakly continuous.

Proof. We have $(X, w)^* = \text{span}(X^*) = X^*$. \square

Corollary: Let X be a normed space, and suppose $\xi : X^* \rightarrow \mathbb{F}$ be a linear functional. If ξ is w^* -continuous, then $\xi = \hat{x}$ for a unique $x \in X$.

Proof. We have $(X^*, w^*)^* = \text{span}(\hat{X}) = \hat{X}$. Uniqueness follows from the fact that X^* separates points. \square

Theorem (Mazur's Theorem): Let $(X, \|\cdot\|)$ be a normed space, and suppose $C \subseteq X$ is a convex subset. The weak and norm closures of C coincide.

Proof. The weak topology is weaker than the norm topology, so we use the corollary that $\tau_1 \subseteq \tau_2 \Rightarrow \overline{C}^{\tau_1} = \overline{C}^{\tau_2}$. \square

If $T : X \rightarrow Y$ is a bounded linear map, we know that the adjoint map $T^* : Y^* \rightarrow X^*$ is norm-continuous.

Lemma: Let X and Y be Banach spaces.

- (1) If $T \in \mathcal{B}(X, Y)$, then the adjoint $T^* : Y^* \rightarrow X^*$ is w^* - w^* continuous.
- (2) If $T : X \rightarrow Y$ is an isometric isomorphism, then $T^* : Y^* \rightarrow X^*$ is an isometric isomorphism and a w^* - w^* homeomorphism.
- (3) If $\pi : X \rightarrow Y$ is a 1-quotient map, then $\pi^* : Y^* \rightarrow X^*$ is a w^* - w^* embedding.
- (4) If $S : Y^* \rightarrow X^*$ is a linear map that is w^* - w^* continuous, there is $T \in \mathcal{B}(X, Y)$ such that $T^* = S$.

Proof.

- (1) Let $(\psi_i)_i$ be a net in Y^* that is w^* convergent to ψ . Then, $(\psi_i(y))_i \rightarrow \psi(y)$ for all $y \in Y$. In particular,

$$\begin{aligned} (T^*(\psi_i(x)))_i &= (\psi_i(T(x)))_i \\ &\rightarrow \psi(T(x)) \\ &= T^*(\psi(x)), \end{aligned}$$

meaning $(T^*(\psi_i))_i \xrightarrow{w^*} T^*(\psi)$.

- (2) We have established that T^* is an isometric isomorphism. To see that T^* is w^* - w^* continuous, consider a net $(\psi_i)_i$ and ψ in Y^* . Then, for all $y \in Y$ and $x \in X$,

$$\begin{aligned} (\psi_i)_i \xrightarrow{w^*} \psi &\Leftrightarrow (\psi_i(y))_i \rightarrow \psi(y) \\ &\Leftrightarrow (\psi_i(T(x)))_i \rightarrow \psi(T(x)) \\ &\Leftrightarrow (\psi_i \circ T(x))_i \rightarrow \psi \circ T(x) \\ &\Leftrightarrow (T^*(\psi_i(x)))_i \rightarrow T^*(\psi(x)) \\ &\Leftrightarrow (T^*(\psi_i))_i \xrightarrow{w^*} T^*(\psi). \end{aligned}$$

- (3) We know that ψ^* is an isometry that is w^* - w^* continuous. Using the fact that π is onto, we have, for all $x \in X$,

$$\begin{aligned} (\pi^*(\psi_i))_i \xrightarrow{w^*} \pi^*(\psi) &\Rightarrow (\psi^*(\psi_i(x)))_i \rightarrow \pi^*(\psi(x)) \\ &\Rightarrow (\psi_i \circ \pi(x))_i \rightarrow \psi \circ \pi(x) \\ &\Rightarrow (\psi_i(y))_i \rightarrow \psi(y). \end{aligned}$$

(4) Fix $x \in X$, and consider the linear functional

$$\eta_x : Y^* \rightarrow \mathbb{C}$$

defined by $\eta_x(\psi) = S(\psi)(x)$. Since S is w^* - w^* continuous, we have η_x is w^* continuous. Thus, there is a unique $y_x \in Y$ with $\eta_x = \hat{y}_x$. For every x we continue this process, and set $T : X \rightarrow Y$ by $T(x) = y_x$. Thus,

$$S(\psi)(x) = \psi(T(x))$$

for all $\psi \in Y^*$ and $x \in X$. Thus, T is linear.

To see that T is bounded, we show that T is w - w continuous. For all $\varphi \in X^*$ and $\psi \in Y^*$, we have

$$\begin{aligned} (x_i)_i \xrightarrow{w} x &\Rightarrow (\psi(x_i))_i \rightarrow \varphi(x) \\ &\Rightarrow (S(\psi(x_i)))_i \rightarrow S(\psi(x)) \\ &\Rightarrow (\psi(T(x_i)))_i \rightarrow \psi(T(x)) \\ &\Rightarrow (T(x_i))_i \xrightarrow{w} T(x). \end{aligned}$$

Thus, we have $T^* = S$.

□

Exercise: Let X be a normed vector space, and $N \subseteq X^*$ nonempty. Show that

$$({}^\perp N)^\perp = \overline{\text{span}}^{w^*}(N).$$

Solution: Since $({}^\perp N)^\perp \subseteq X^*$ is a norm-closed subspace, it is contained in a weak*-closed subspace that also contains N , so $\overline{\text{span}}^{w^*}(N) \subseteq ({}^\perp N)^\perp$.

Let $\varphi \in ({}^\perp N)^\perp$. Thus, for all $x \in {}^\perp N$, $\varphi(x) = 0$, meaning ${}^\perp N \subseteq \ker(\varphi)$.

Since for any collection $\varphi_i \in N$, ${}^\perp N \subseteq \ker(\varphi_i)$, it is the case that

$$\bigcap_{j=1}^n \ker(\varphi_j) \subseteq \ker(\varphi),$$

so $\varphi \in \text{span}(N)$, so $\varphi \in \overline{\text{span}}^{w^*}(N)$. \perp

Exercise: Let X be a Banach space, and suppose $N \subseteq X^*$ is a subspace. We will show here that $\overline{N}^{\|\cdot\|} \subseteq \overline{N}^{w^*}$ may be a strict inclusion.

Let $X = c_0$, so $X^* = \ell_1$. Consider

$$N = \left\{ (a_k)_k \in \ell_1 \mid \sum_{k=1}^{\infty} a_k = 0 \right\}.$$

Prove that N is norm-closed and ${}^\perp N = \{0\}$.

Solution: To see that N is norm-closed, we let $(x_i)_i$ be a sequence in N converging to $x \in \ell_1$. We want to show that $\sum_{k \geq 1} x(k) = 0$. We can see that,

$$\sum_{k \geq 1} x_i(k) \rightarrow \sum_{k \geq 1} x(k),$$

and since $\sum_{k \geq 1} x_i(k) = 0$ for each i , it is the case that $\sum_{k \geq 1} x(k) = 0$, so $x \in N$.

Suppose toward contradiction there exists $(x_k)_k \in {}^\perp N$ with $(x_k)_k \neq 0$. For some j with $x(j) \neq 0$, we construct $(a_j)_j \in N$ by finding i with $0 < |x(i)| < |x(j)|$, and setting $(a_j)_j$ to be 0 everywhere except for -1 at index i and 1 at index j . Then,

$$\sum_{k \geq 1} a_k x_k \neq 0,$$

which is a contradiction. Thus, ${}^\perp N = \{0\}$.

Proposition: Let X be a Banach space, and suppose $N \subseteq X^*$ is a w^* -closed subspace. Then, N is a dual space with the induced w^* topology coinciding with the subspace w^* topology inherited from X^* .

Proof. Consider the norm-closed subspace ${}^\perp N \subseteq X$, and the quotient space $X/({}^\perp N)$. Then, the 1-quotient map $\pi : X \rightarrow X/({}^\perp N)$ induces the isometric isomorphism

$$\pi^* : (X/({}^\perp N))^* \rightarrow ({}^\perp N)^\perp.$$

Since N is a w^* -closed subspace, and so too is $({}^\perp N)^\perp = \overline{\text{span}}^{w^*}(N)$, it is the case that $N = ({}^\perp N)^\perp$.

Thus, N has a predual, and π^* is a w^* - w^* homomorphism. \square

We are aware that the closed unit ball is norm-compact if and only if the vector space is finite-dimensional. We can use the following theorem to see that the weak* topology is coarse enough such that in *any* normed vector space, the closed unit ball is compact in the weak* topology.

Theorem (Banach–Alaoglu Theorem): Let X be a normed vector space.

- (1) The closed unit ball B_{X^*} is compact in the w^* topology.
- (2) A subset $C \subseteq X$ is w^* -compact if and only if C is w^* -closed and norm bounded.

Proof.

- (1) Let $\overline{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ denote the closed unit disc in \mathbb{C} . The set

$$Z = \prod_{x \in B_X} \overline{D}$$

equipped with the product topology is compact by Tychonoff's theorem.

Let $F : (B_{X^*}, w^*) \rightarrow Z$ be defined by $F(\varphi) = (\varphi(x))_{x \in B_X}$.

Note that

$$\begin{aligned} |\varphi(x)| &\leq \|\varphi\|_{\text{op}} \|x\| \\ &\leq 1, \end{aligned}$$

so F is well-defined.

We claim that F is an embedding. By the definition of the product topology, we must have

$$\begin{aligned} (\varphi_\alpha)_\alpha &\xrightarrow{w^*} \varphi \Leftrightarrow (\varphi_\alpha(x)) \rightarrow \varphi(x) \\ &\Leftrightarrow (\varphi_\alpha(x))_{x \in B_X} \rightarrow (\varphi(x))_{x \in B_X} \\ &\Leftrightarrow F(\varphi_\alpha) \rightarrow F(\varphi). \end{aligned}$$

We must now show that F is injective. If $F(\varphi_1) = F(\varphi_2)$, then $(\varphi_1(x))_{x \in B_X} = (\varphi_2(x))_{x \in B_X}$, so $\varphi_1(x) = \varphi_2(x)$ for each $x \in B_X$, meaning $\varphi_1 = \varphi_2$ upon rescaling.

We now claim that $\text{Ran}(F) \subseteq Z$ is closed. Let $(F(\varphi_\alpha))_\alpha$ be a net in $\text{Ran}(F)$ converging to $\beta \in Z$. Let $\beta = (\beta_x)_{x \in B_X}$.

Since $F(\varphi_\alpha) = (\varphi_\alpha(x))_{x \in B_X}$, so for each $x \in B_X$, we have $(\varphi_\alpha(x)) \rightarrow \beta_x$. Note that for $x \neq 0$, we have

$$\begin{aligned}\varphi_\alpha(x) &= \|x\| \varphi_\alpha\left(\frac{x}{\|x\|}\right) \\ &\rightarrow \|x\| \beta_{\frac{x}{\|x\|}},\end{aligned}$$

and that $(\varphi_\alpha(0))_\alpha \rightarrow 0$. Thus, for any $x \in X$, $\varphi(x) = \lim_\alpha \varphi_\alpha(x)$ exists. Additionally, $\varphi : X \rightarrow \mathbb{F}$ is linear, and for each $x \in B_X$, we have

$$\begin{aligned}|\varphi(x)| &= \left| \lim_\alpha \varphi_\alpha(x) \right| \\ &= \lim_\alpha |\varphi_\alpha(x)| \\ &\leq 1,\end{aligned}$$

meaning $\|\varphi\|_{\text{op}} \leq 1$, so $\varphi \in B_{X^*}$. Thus, $\text{Ran}(F)$ is closed.

Since $\text{Ran}(F)$ is a closed subspace of a compact space, $\text{Ran}(F)$ is compact, so (B_{X^*}, w^*) is compact.

- (2) If $C \subseteq X^*$ is norm-bounded and w^* -closed, then C is a w^* -closed subset of δB_{X^*} for some $\delta > 0$. Since B_{X^*} is w^* -compact, C is w^* -compact.

If C is w^* -compact, then C is w^* -closed as the w^* topology is Hausdorff. Set $x \in X$. The map $\hat{x} : X^* \rightarrow \mathbb{F}$ is w^* -continuous, so $\hat{x}(C) \subseteq \mathbb{F}$ is compact, hence bounded.

Thus, $\sup_{\varphi \in C} |\varphi(x)| < \infty$, so $\sup_{\varphi \in C} \|\varphi\|_{\text{op}} < \infty$ by the Uniform Boundedness Principle, so C is norm-bounded.

□

Example (Banach Limits). Consider the sequence of bounded functions $(\mu_n)_n$ in ℓ_∞^* given by

$$\mu_n((z_k)_k) = \frac{1}{n} \sum_{k=1}^n z_k.$$

We can see that $\|\mu_n\|_{\text{op}} = 1$ for each n , meaning there is a subnet $(\mu_{n_j})_j \xrightarrow{w^*} L$ for some $L \in B_{\ell_\infty^*}$.

If $(z_k)_k \rightarrow z$, then $\mu_n((z_k)_k) \xrightarrow{n \rightarrow \infty} z$, meaning $L((z_k)_k) = z$. In other words, L is a bounded linear functional extending the limit. We call L the Banach limit.

We have shown earlier that $\phi(\lambda)(z) = \sum_{k=1}^\infty \lambda_k z_k$ mapping ℓ_1 to ℓ_∞^* is a linear isometry. However, if $\phi(\lambda) = L$ for some $\lambda = (\lambda_k)_k$, then $\lambda_n = L(e_n) = 0$, implying $\lambda = 0$, which is a contradiction as $L \neq 0$. This is another way to prove that ϕ is not onto.

Exercise: Let X be a separable normed space, and let $(x_n)_n$ be a dense subset of B_X . Show that

$$d(\varphi, \psi) = \sum_{n=1}^\infty 2^{-n} |\varphi(x_n) - \psi(x_n)|$$

defines a metric on B_{X^*} , and the map

$$\text{id}_{B_{X^*}} : (B_{X^*}, d) \rightarrow (B_{X^*}, w^*)$$

is a homeomorphism.

Solution: For $\varphi \in B_{X^*}$, we define

$$p_k(\varphi) = |\varphi(x_k)|.$$

By the Hahn–Banach separation results, we know that $\{p_k \mid k \in \mathbb{N}\}$ is a collection of separating seminorms. Additionally, it is the case that $p_k(\varphi) \leq 1$ for all k . Thus, the Fréchet metric, defined by

$$\begin{aligned} d(\varphi, \psi) &= \sum_{k=1}^{\infty} 2^{-k} p_k(\varphi - \psi) \\ &= \sum_{k=1}^{\infty} 2^{-k} |(\varphi - \psi)(x_k)| \end{aligned}$$

is indeed a metric.

To show that the identity map is a homeomorphism, we see that,

$$\begin{aligned} (\varphi_\alpha)_\alpha &\xrightarrow{d} \varphi \Leftrightarrow (\varphi_\alpha(x_k))_\alpha \rightarrow \varphi(x_k) & \forall x_k \in (x_k)_k \\ &\Leftrightarrow (\varphi_\alpha(x))_\alpha \rightarrow \varphi(x) & \forall x \in B_X \\ &\Leftrightarrow (\varphi_\alpha)_\alpha \xrightarrow{w^*} \varphi. \end{aligned}$$

Thus, the identity map is necessarily a continuous bijection with continuous inverse.

Corollary: If X is a normed space, there is a compact Hausdorff space Ω and a linear embedding $X \hookrightarrow C(\Omega)$. If X is separable, then Ω is metrizable.

Proof. We know that B_{X^*} with the weak* topology is compact Hausdorff. Every $x \in X$ gives rise to an evaluation function

$$e_x : B_{X^*} \rightarrow \mathbb{C}$$

defined by $e_x(\varphi) = \varphi(x)$.

By the definition of the weak* topology, e_x is continuous. We consider the map $e : X \rightarrow C(B_{X^*})$ defined by $e(x) = e_x$. Then, since φ is linear, e is linear. Additionally,

$$\begin{aligned} \|e_x\|_u &= \sup_{\varphi \in B_{X^*}} |e_x(\varphi)| \\ &= \sup_{\varphi \in B_{X^*}} |\varphi(x)| \\ &= \|x\|. \end{aligned}$$

In the exercise, we showed that if X is separable, then B_{X^*} is metrizable. □

If X is a reflexive space, we can fully understand the double dual, but if X is not reflexive, we will need Goldstine's theorem to approximate elements of the double dual.

Theorem (Goldstine's Theorem): Let X be a normed space. Set $B = \iota_X(B_X) \subseteq B_{X^{**}}$. Then, $\overline{B}^{w^{**}} = B_{X^{**}}$.

Proof. By the Banach–Alaoglu theorem, we know that $B_{X^{**}}$ is w^{**} -compact, so $C = \overline{B}^{w^{**}}$ is also w^{**} -compact. Additionally, since ι_X is linear and B_X is convex, B is also convex, so C is convex as C is the closure of a convex set.

Suppose toward contradiction there exists $\xi \in B_{X^{**}} \setminus C$. By the Hahn–Banach theorems for separation by hyperplanes, there is a w^{**} -continuous linear $F : X^{**} \rightarrow \mathbb{R}$, $\varepsilon > 0$, and $t \in \mathbb{R}$ such that

$$\operatorname{Re}(F)(\eta) < t < t + \varepsilon < \operatorname{Re}(F)(\xi)$$

for all $\xi \in C$.

We know that there is some $\varphi \in X^*$ such that $F = \iota_{X^*}(\varphi) = \hat{\varphi}$. Substituting, we have

$$\operatorname{Re}(\eta(\varphi)) < t < t + \varepsilon < \operatorname{Re}(\eta(\varphi)).$$

In particular, this also holds for $\eta = \hat{x}$, where $x \in B_X$, so

$$\operatorname{Re}(\varphi(x)) < t < t + \varepsilon < \operatorname{Re}(\xi(\varphi))$$

for all $x \in B_X$. Since $0 \in B_X$, we must have $t > 0$. Dividing by t , and setting $\psi = \frac{1}{t}\varphi$, $\delta = \frac{1}{t}\varepsilon$, we have

$$\operatorname{Re}(\psi(x)) < 1 < 1 + \delta < \operatorname{Re}(\xi(\psi)).$$

For each $x \in B_X$, we also have

$$\begin{aligned} |\psi(x)| &= \operatorname{sgn}(\psi(x)) \\ &= \psi(\operatorname{sgn}(\psi(x))x) \\ &= \operatorname{Re}(\psi(\operatorname{sgn}(\psi(x))x)) \\ &\leq 1. \end{aligned}$$

Thus, $\|\psi\| \leq 1$, so

$$\begin{aligned} 1 + \delta &< \operatorname{Re}(\xi(\psi)) \\ &\leq |\xi(\psi)| \\ &\leq \|\xi\| \|\psi\| \\ &\leq 1, \end{aligned}$$

which is a contradiction. Thus, no such ξ exists, meaning $C = B_{X^{**}}$. □

Exercise: If X is a normed space, prove that $\overline{\iota_X(S_X)}^{w^{**}} = S_{X^{**}}$.

Solution: Since S_X is convex, so too is $\iota_X(S_X)$, so $S = \overline{\iota_X(S_X)}^{w^{**}}$ is w^{**} -compact.

If it were the case that there were some $\xi \in S_{X^{**}} \setminus S$, then by a similar pattern of reasoning to the proof of Goldstine's theorem, we would see that $\|\xi\| > 1$.

Exercise: Let X be a normed space. Prove that X is reflexive if and only if (B_X, w) is compact.

Solution: Let X be reflexive. Then, convergence of nets provides, for all $\varphi \in X^*$,

$$\begin{aligned} (x_\alpha)_\alpha \xrightarrow{w} x &\Leftrightarrow \varphi(x_\alpha) \rightarrow \varphi(x) \\ &\Leftrightarrow \hat{x}_\alpha(\varphi) \rightarrow \hat{x}(\varphi) \\ &\Leftrightarrow (\hat{x}_\alpha) \xrightarrow{w^{**}} \hat{x}. \end{aligned}$$

Since $\iota : X \rightarrow X^{**}$ is bijective, and $B_{X^{**}}$ is compact, then $\iota^{-1}(B_{X^{**}}) = B_X$, which is thus weakly compact.

Let (B_X, w) be compact. Then, $\overline{\iota_X(B_X)}^{w^{**}} = B_{X^{**}}$ by Goldstine's theorem. Since ι is an embedding, it is the case that $\iota_X(B_X)$ is closed, meaning $\overline{\iota_X(B_X)}^{w^{**}} = \iota_X(B_X) = B_{X^{**}}$. Thus, $\iota : X \rightarrow X^{**}$ is surjective, hence X is reflexive.

We can use the Banach–Alaoglu theorem to obtain insights into continuous function spaces.

Note that $B_{C_0(\Omega)^*}$ is w^* -compact, but it may not be the case that $S(\Omega)$, the state space, is w^* -closed. For instance, the evaluation functionals $\delta_n : C_0(\mathbb{R}) \rightarrow \mathbb{C}$ given by $\delta_n(f) = f(n)$ are elements of $S(C_0(\mathbb{R}))$, but $(\delta_n)_n \xrightarrow{w^*} 0$ as $f(n) \rightarrow 0$ as $n \rightarrow \infty$.

However, when Ω is compact, the state space is always compact in the w^* -topology.

Proposition: Let Ω be a compact Hausdorff space. The state space $S(C(\Omega))$ is convex and w^* -compact.

Proof. We have shown that the state space is convex, so we show that $S(\Omega)$ is w^* -closed. If $(\varphi_i)_{i \in I}$ is a net in $S(\Omega)$ with $(\varphi_i)_i \xrightarrow{w^*} \varphi$, then for $f \geq 0$, we have $\varphi_i(f) \rightarrow \varphi(f)$. Since, $\varphi_i(f) \geq 0$, we must have $\varphi(f) \geq 0$.

Additionally, since $1 = \varphi_i(1_\Omega) \rightarrow \varphi(1_\Omega)$, we must have $\|\varphi\| = 1$. Thus, φ is a state. \square

Corollary: If $\tau : \Omega \rightarrow \Omega$ is a self-homeomorphism of a compact Hausdorff space, there exists an invariant regular Borel probability measure $\mu \in \mathcal{P}_r(\Omega, \tau)$. Moreover, $\mathcal{P}_r(\Omega, \tau)$ is w^* -compact.

Proof. We identify $S(\Omega) = \mathcal{P}_r(\Omega)$. Since τ is a homeomorphism, $T_{\tau^*}|_{S(\Omega)} : S(\Omega) \rightarrow S(\Omega)$ is an affine bijection.

Since $T_\tau^k = T_{\tau^k}$, we have $(T_\tau^*)^k = (T_{\tau^k})^*$. Thus, for any $\mu \in \mathcal{P}_r(\Omega)$ and $f \in C(\Omega)$, we have

$$\begin{aligned} (T_\tau^*)^k(\mu)(f) &= (T_{\tau^k})^*(\mu)(f) \\ &= \mu(T_{\tau^k}(f)) \\ &= \mu(f \circ \tau^k). \end{aligned}$$

Fix $\mu \in \mathcal{P}_r(\Omega)$. For each $n \in \mathbb{N}$, we consider the state

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} (T_\tau^*)^k(\mu).$$

Since $S(\Omega)$ is w^* -compact, there is a subnet $(\mu_{n_i})_i \xrightarrow{w^*} \mu_\infty$ for some $\mu_\infty \in \mathcal{P}_r(\Omega)$. We see that

$$\begin{aligned} |\mu_n(f) - \mu_n(f \circ \tau)| &= \left| \frac{1}{n} \sum_{k=0}^{n-1} \mu(f \circ \tau^k) - \frac{1}{n} \sum_{k=0}^{n-1} \mu(f \circ \tau^{k+1}) \right| \\ &= \frac{1}{n} \left| \sum_{k=0}^{n-1} \mu(f \circ \tau^k) - \mu(f \circ \tau^{k+1}) \right| \\ &= \frac{1}{n} |\mu(f) - \mu(f \circ \tau^n)| \\ &= \frac{1}{n} |\mu(f - f \circ \tau^n)| \\ &\leq \frac{1}{n} \|f - f \circ \tau^n\|_u \\ &\leq \frac{2}{n} \|f\|_u. \end{aligned}$$

Taking the limit along the subnet, we have $\mu_\infty(f) = \mu_\infty(f \circ \tau)$. Thus, we have

$$\begin{aligned} \int_\Omega f \, d\mu_\infty &= \int_\Omega f \circ \tau \, d\mu_\infty \\ &= \int_\Omega f \, d(\tau_*\mu_\infty). \end{aligned}$$

Thus, $\mu_\infty = \tau_*\mu_\infty$, so μ_∞ is invariant.

Since T_τ^* is w^* - w^* continuous, $\mathcal{P}_r(\Omega, \tau)$ is w^* -closed in $\mathcal{P}_r(\Omega)$, so it is w^* -compact. \square

Now that we know the state space is compact if the underlying set is compact, we may be interested to see how we can implement this for the case where Ω is only locally compact rather than compact.

By Urysohn's lemma, we can characterize the topology of a locally compact Hausdorff space by its continuous bounded functions, $C_b(\Omega)$.

Fixing a locally compact Hausdorff space Ω , for every $x \in \Omega$, there is an evaluation linear functional $\delta_x : C_b(\Omega) \rightarrow \mathbb{F}$ defined by $\delta_x(f) = f(x)$. These functionals are known as characters (or pure states) on $C_b(\Omega)$. We have

$$\begin{aligned} |\delta_x(f)| &= |f(x)| \\ &\leq \|f\|_u. \end{aligned}$$

Additionally, we can define $g_x \in C(\Omega, [0, 1])$ by $g_x(x) = 1$ by taking $x \in K$ for some compact neighborhood K , and applying Urysohn's lemma to define $g = 1$ on K and 0 outside a compact superset of K . Then, $\|g\|_u = 1$ and $\delta_x(g_x) = 1$, meaning $\|\delta_x\|_{\text{op}} = 1$.

We have the map

$$\Delta : \Omega \rightarrow B_{C_b(\Omega)^*}$$

defined by $\Delta(x) = \delta_x$. We claim that Δ is an embedding, where $B_{C_b(\Omega)^*}$ is equipped with the w^* topology.

Proof. If $\Delta(x) = \Delta(y)$, then $\delta_x = \delta_y$, so

$$\begin{aligned} f(x) &= \delta_x(f) \\ &= \delta_y(f) \\ &= f(y) \end{aligned}$$

for every $f \in C_b(\Omega)$. Since $C_b(\Omega)$ separates points, Urysohn's lemma provides $x = y$.

If $x \in \Omega$ and $(x_\alpha)_\alpha$ is a net in Ω , then

$$\begin{aligned} (\Delta(x_\alpha))_\alpha &\xrightarrow{w^*} \Delta(x) \Leftrightarrow (\delta_{x_\alpha})_\alpha \xrightarrow{w^*} \delta_x \\ &\Leftrightarrow (\delta_{x_\alpha}(f))_\alpha \rightarrow \delta_x(f) \\ &\Leftrightarrow (f(x_\alpha))_\alpha \rightarrow f(x) \\ &\Leftrightarrow (x_\alpha)_\alpha \rightarrow x. \end{aligned}$$

□

We can now construct a compactification of the locally compact Hausdorff space Ω .

Theorem (Stone-Čech Compactification): Let Ω be a locally compact Hausdorff space. There exists a compact space $\beta\Omega$, called the Stone-Čech compactification of Ω , and a map $\Delta : \Omega \rightarrow \beta\Omega$ such that the following are true.

- (1) Δ is an embedding.
- (2) $\Delta(\Omega) \subseteq \beta\Omega$ is dense.
- (3) For each $f \in C_b(\Omega)$, there exists a unique continuous extension $f^\beta : \beta\Omega \rightarrow \mathbb{F}$ such that the following diagram commutes.

$$\begin{array}{ccc} \Omega & \xhookrightarrow{\Delta} & \beta\Omega \\ & \searrow f & \downarrow f^\beta \\ & & \mathbb{F} \end{array}$$

(4) The map $C_b(\Omega) \rightarrow C(\beta\Omega)$ mapping $f \mapsto f^\beta$ is an isometric isomorphism of Banach spaces.

Proof. Letting $\Delta : \Omega \rightarrow B_{C_b(\Omega)^*}$ be the map given by $\Delta(x) = \delta_x$, and defining $\beta\Omega = \overline{\Delta(\Omega)}^{w^*}$, we verify (1) and (2).

To verify (3), if $f \in C_b(\Omega)$, we define

$$f^\beta : \beta\Omega \rightarrow \mathbb{F}$$

by

$$f^\beta(\varphi) = \varphi(f)$$

for all $\varphi \in \beta\Omega \subseteq B_{C_b(\Omega)^*}$.

If $(\varphi_\alpha)_\alpha \xrightarrow{w^*} \varphi$ in $\beta\Omega$, then for each $f \in C_b(\Omega)$, we have $\varphi_\alpha(f) \rightarrow \varphi(f)$, so $f^\beta(\varphi_\alpha) \rightarrow f^\beta(\varphi)$. Thus, f^β is continuous.

Finally, we see that for $x \in \Omega$ and $f \in C_b(\Omega)$,

$$\begin{aligned} f^\beta \circ \Delta(x) &= f^\beta(\delta_x) \\ &= \delta_x(f) \\ &= f(x). \end{aligned}$$

The uniqueness of f^β follows from the fact that f^β is densely defined in $\beta\Omega$.

To show (4), for $f, g \in C_b(\Omega)$ and $\alpha \in \mathbb{C}$, we have

$$(f + \alpha g)^\beta = f^\beta + \alpha g^\beta,$$

since both of these expressions agree on $\Delta(\Omega)$, which is a dense subset of $\beta\Omega$. If $g \in C(\beta\Omega)$, we define $f = g \circ \Delta$, meaning f is continuous and bounded, with $f^\beta = g$.

Finally, since $\beta\Omega$ is compact, $f^\beta(\beta\Omega)$ is closed, so

$$\begin{aligned} f^\beta(\beta\Omega) &= f^\beta\left(\overline{\Delta(\Omega)}\right) \\ &\subseteq \overline{f^\beta(\Delta(\Omega))} \\ &= \overline{f(\Omega)} \\ &= \overline{f^\beta(\Delta(\Omega))} \\ &\subseteq \overline{f^\beta(\beta\Omega)} \\ &= f^\beta(\beta\Omega). \end{aligned}$$

We get

$$\begin{aligned} \|f\|_u &= \sup f(\Omega) \\ &= \sup \overline{f(\Omega)} \\ &= \sup f^\beta(\beta\Omega) \\ &= \|f^\beta\|_u. \end{aligned}$$

Thus, $f \mapsto f^\beta$ is a surjective isometry. □

Remark: The Stone–Čech compactification is a very large set. For instance, $\text{card}(\beta\mathbb{N}) = \mathfrak{c}^{\mathfrak{c}}$.

Remark: With the Stone–Čech compactification, we can finally characterize $\ell_\infty(\mathbb{N})^*$, which is the space of all regular Borel measures on $\beta\mathbb{N}$.

Exercise: Let I be a set with the discrete topology. Show that βI is Stonean — that is, the closure of every open set is clopen.

Solution: We start by showing that for any partition $A \sqcup B = I$, that $\overline{A} \sqcup \overline{B} = \beta I$ is a partition. Considering the function $f = \mathbb{1}_A$, we see that f^β is continuous, and that

$$\begin{aligned} f^\beta(\overline{A}) &\subseteq \overline{f^\beta(A)} \\ &= \overline{f(A)} \\ &= \overline{\{1\}} \\ &= \{1\} \\ f^\beta(\overline{B}) &\subseteq \overline{f^\beta(B)} \\ &= \overline{f(B)} \\ &= \overline{\{0\}} \\ &= \{0\}. \end{aligned}$$