# Chapter 31 Problems

## Problem 1

Since the inner product is positive definite, and

$$k_m = \langle P_m \,|\, P_m \rangle\,,$$

we must have $k_m \geqslant 0$. Since $P_m \neq 0$, we must have $k_m \neq 0$.

## Problem 2

I don't know how to do this problem.

## Problem 3

$$\begin{aligned}\langle f+g \,|\, f+g\rangle &= \langle f \,|\, f\rangle + \langle g \,|\, g\rangle + 2\operatorname{Re}\left(\langle f \,|\, g\rangle\right)\\ &\leqslant \langle f \,|\, f\rangle + \langle g \,|\, g\rangle + 2\left|\langle f \,|\, g\rangle\right|\\ &\leqslant \langle f \,|\, f\rangle + \langle g \,|\, g\rangle + 2\sqrt{\langle f \,|\, f\rangle\langle g \,|\, g\rangle}\\ &< \infty.\end{aligned}$$

# Chapter 32 Problems

## Problem 2

(a) We have

$$\begin{aligned}|P_0\rangle &= 1\\ |P_1\rangle &= x\\ |P_2\rangle &= \frac{1}{2}\left(3x^2-1\right)\\ |P_3\rangle &= |\chi_3\rangle - \left\langle \hat{P}_0 \,\middle|\, \chi_3\right\rangle\left|\hat{P}_0\right\rangle - \left\langle \hat{P}_1 \,\middle|\, \chi_3\right\rangle\left|\hat{P}_0\right\rangle - \left\langle \hat{P}_2 \,\middle|\, \chi_3\right\rangle\left|\hat{P}_2\right\rangle\\ &= |\chi_3\rangle - \left\langle \hat{P}_1 \,\middle|\, \chi_3\right\rangle\left|\hat{P}_1\right\rangle\\ &= x^3 - \left(\sqrt{\frac{3}{2}}\int_{-1}^{1} t^4\,dt\right)\left(\sqrt{\frac{3}{2}}x\right)\\ &= \frac{1}{2}\left(5x^3-3x\right).\end{aligned}$$

(b) We have

$$\begin{aligned}|L_0\rangle &= 1\\ |L_1\rangle &= |\chi_1\rangle - \langle L_0 \,|\, \chi_1\rangle\,|L_0\rangle\\ &= x - \left(\int_0^\infty xe^{-x}\,dx\right)(1)\\ &= 1-x\\ |L_2\rangle &= |\chi_2\rangle - \langle L_1 \,|\, \chi_2\rangle\,|L_1\rangle - \langle L_0 \,|\, \chi_2\rangle\,|L_0\rangle\\ &= \frac{1}{2}\left(x^2-4x+2\right)\\ |L_3\rangle &= |\chi_3\rangle - \langle L_2 \,|\, \chi_3\rangle\,|L_2\rangle - \langle L_1 \,|\, \chi_3\rangle\,|L_1\rangle - \langle L_0 \,|\, \chi_3\rangle\,|L_0\rangle\\ &= \frac{1}{6}\left(-x^3+9x^2-18x+6\right).\end{aligned}$$

## Problem 7

$$\begin{aligned}
\langle P_0 \mid f\rangle &= \frac{1}{2}\int_{-1}^{1} P_0(x)e^{ikx}\,dx\\
&= \frac{2\sin(k)}{k}\\
\langle P_1 \mid f\rangle &= \frac{3}{2}\int_{-1}^{1} P_1(x)e^{ikx}\,dx\\
&= 2i\frac{-k\cos(k)+\sin(k)}{k^2}\\
\langle P_2 \mid f\rangle &= \frac{5}{2}\int_{-1}^{1} \frac{1}{2}\left(3x^2-1\right)e^{-ikx}\,dx\\
&= 6\frac{\cos(k)}{k^2}+2\frac{\left(-3+k^2\right)\sin(k)}{k^3}.
\end{aligned}$$

## Problem 8

$$\begin{aligned}
\langle L_0 \mid f\rangle &= \int_{0}^{\infty} e^{-x/2}\,dx\\
&= 2\\
\langle L_1 \mid f\rangle &= \int_{0}^{\infty} (1-x)\,e^{-x/2}dx\\
&= -2\\
\langle L_2 \mid f\rangle &= \int_{0}^{\infty} \frac{1}{2}\left(x^2-4x+2\right)e^{-x/2}\,dx\\
&= 2.
\end{aligned}$$

## Problem 16

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= -\frac{1}{2}\frac{d}{dx}\left(1-x^2\right)\\
&= x\\
P_2(x) &= \frac{1}{8}\frac{d^2}{dx^2}\left(x^4-2x^2+1\right)\\
&= \frac{1}{2}\left(3x^2-1\right).
\end{aligned}$$