

## Normed Vector Spaces

### Vector Spaces

Throughout,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . A **vector space** over  $\mathbb{F}$  is a nonempty set  $V$  equipped with two operations: vector addition and scalar multiplication.

$$\begin{aligned} V \times V &\xrightarrow{+} V \\ (v, w) &\mapsto v + w \\ F \times V &\rightarrow V \\ (\alpha, v) &\mapsto \alpha v \end{aligned}$$

The vector space is an Abelian group, where  $u, v, w \in V$  and  $\alpha, \beta \in \mathbb{F}$ , we have:

- (i)  $u + (v + w) = (u + v) + w$
- (ii)  $\exists 0_v \in V$  with  $\forall v \in V, 0_v + v = v + 0_v = v$
- (iii)  $(\forall v \in V)(\exists w \in V)$  with  $v + w = 0_v$  —
- (iv)  $\forall v, w \in V, v + w = w + v$
- (v)  $\alpha(v + w) = \alpha v + \alpha w, (\alpha + \beta)v = \alpha v + \beta v$
- (vi)  $\alpha(\beta w) = (\alpha\beta)w$
- (vii)  $1 \cdot v = v$

**Remarks:**

- (a)  $0_v$  is unique and known as the zero vector.
- (b) The vector  $w$  in (iii) is unique, and denoted  $-v$ .
- (c)  $0 \cdot v = 0_v$
- (d)  $(-1) \cdot v = -v$
- (e) Property (iv) follows from all the other axioms.
- (f) For  $n \in \mathbb{N}$ ,  $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

### Subspaces

Let  $V$  be a vector space over  $\mathbb{F}$ . A **subspace** is a nonempty subset  $W \subseteq V$  satisfying the following:

- (i)  $w \in W, \alpha \in \mathbb{F} \rightarrow \alpha w \in W$ .
- (ii)  $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$ .

**Remark:**  $0_v$  is always a member of any subspace; a subspace is also a vector space.

#### Proposition: Intersection of Subspaces

If  $\{W_i\}_{i \in I}$  is a family of subspaces of  $V$ , then,  $\bigcap W_i$  is a subspace of  $V$ .

#### Proposition: Union of Subspaces

It is not the case that the union of subspaces of  $V$  also a subspace. For example, consider  $\mathbb{R}^2$  with the traditional vector space operations:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} x + x' \\ y + y' \end{pmatrix} \\ \alpha \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \end{aligned}$$

If  $W_1, W_2 \subseteq V$  are subspaces such that  $W_1 \cup W_2$  is a subspace, then  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

**Generated Subspaces**

Let  $S \subseteq V$  be any subset of a vector space  $V$ . Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^n \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

**Remarks:**

- $\text{span}(S) \subseteq V$  is a subspace.
- $\text{span}(S) = \bigcap W$ , where  $S \subseteq W$  and  $W \subseteq V$  is a subspace. Thus,  $\text{span}(S)$  is the "smallest" subspace containing  $S$ , or the subspace generated by  $S$ .

**Proposition: Quotient Group on Vector Space**

Let  $V$  be a vector space, and let  $W \subseteq V$  is a subspace. Define  $u \sim_W v \leftrightarrow u - v \in W$ .

- (1)  $\sim_W$  is an equivalence relation.
- (2) If  $[v]_W$  denotes the equivalence class of  $v$ , then  $[v]_W = v + W = \{v + w \mid w \in W\}$ .
- (3)  $V/W := \{[v]_W \mid v \in V\}$  is a vector space with  $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$  and  $\alpha[v]_W = [\alpha v]_W$ .

**Proof of (1):**

- Reflexive:  $u \sim_W u$ , since  $u - u = 0 \in W$ .
- Transitive: Suppose  $u \sim_W v$ , and  $v \sim_W z$ . Then,  $u - v \in W$ , and  $v - z \in W$ . So,  $(u - v) + (v - z) \in W$ , so  $u - z \in W$ . Whence,  $u \sim_W z$ .
- Symmetric: If  $u \sim_W v$ , then  $u - v \in W$ , so  $-1 \cdot (u - v) \in W$ , so  $v - u \in W$ . Whence,  $v \sim_W u$ .

**Proof of (2):**

$$\begin{aligned} [v]_W &= \{u \in V \mid u \sim_W v\} \\ &= \{u \in V \mid u - v \in W\} \\ &= \{u \in V \mid u = v + w \text{ some } w \in W\} \\ &= \{v + w \mid w \in W\} \\ &= v + W \end{aligned}$$

**Proof of (3):** Prove that the operation is well-defined.

**Bases**

Let  $V$  be a vector space and  $S \subseteq V$  be a subset.

- (1)  $S$  is said to be spanning for  $V$  if  $\text{span}(S) = V$ .
- (2)  $S$  is linearly independent if, for  $\sum_{j=1}^n \alpha_j v_j = 0_v$  with  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ ,  $v_1, \dots, v_n \in S$ , then  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .
- (3)  $S$  is a basis for  $V$  if  $S$  is linearly independent and spanning for  $V$ .

**Proposition: Existence of Basis**

Every vector space admits a basis. If  $S \subseteq V$  is linearly independent,  $\exists B \subseteq V$  such that  $B$  is a basis and  $S \subseteq B$ .

**Zorn's Lemma:**