Normed Vector Spaces

Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **vector space** over $\mathbb{F}$ is a nonempty set V equipped with two operations: vector addition and scalar multiplication.

$$V \times V \xrightarrow{+} V$$
$$(v, w) \mapsto v + w \qquad\qquad \text{Vector Addition}$$
$$F \times V \to V$$
$$(\alpha, v) \mapsto \alpha v \qquad\qquad \text{Scalar Multiplication}$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

(i) $u + (v + w) = (u + v) + w$

(ii) $\exists 0_v \in V$ with $\forall v \in V$, $0_v + v = v + 0_v = v$

(iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$

(iv) $\forall v, w \in V$, $v + w = w + v$

(v) $\alpha(v + w) = \alpha v + \alpha w$, $(\alpha + \beta)v = \alpha v + \beta v$

(vi) $\alpha(\beta w) = (\alpha \beta)w$

(vii) $1 \cdot v = v$

Remarks:

(a) 0_v is unique and known as the zero vector.

(b) The vector w in (iii) is unique, and denoted $-v$.

(c) $0 \cdot v = 0_v$

(d) $(-1) \cdot v = -v$

(e) Property (iv) follows from all the other axioms.

(f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

Subspaces

Let V be a vector space over $\mathbb{F}$. A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

(i) $w \in W, \alpha \in \mathbb{F} \to \alpha w \in W$.

(ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

Remark: 0_v is always a member of any subspace; a subspace is also a vector space.

Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of V, then, $\bigcap W_i$ is a subspace of V.

Proposition: Union of Subspaces

It is not the case that the union of subspaces of V also a subspace. For example, consider $\mathbb{R}^2$ with the traditional vector space operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$$
$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

If $W_1, W_2 \in V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Generated Subspaces

Let $S \subseteq V$ be any subset of a vector space V. Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}, v_1, \ldots, v_n \in S \right\}$$

Remarks:

- $\text{span}(S) \subseteq V$ is a subspace.
- $\text{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\text{span}(S)$ is the "smallest" subspace containing S, or the subspace generated by S.

Proposition: Quotient Group on Vector Space

Let V be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

(1) $\sim_W$ is an equivalence relation.

(2) If $[v]_W$ denotes the equivalence class of v, then $[v]_W = v + W = \{v + w \mid w \in W\}$.

(3) $V/W := \{[v]_W \mid v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

Proof of (1):

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

Proof of (2):

$$
\begin{aligned}
[v]_W &= \{u \in V \mid u \sim_W v\} \\
&= \{u \in V \mid u - v \in W\} \\
&= \{u \in V \mid u = v + w \text{ some } w \in W\} \\
&= \{v + w \mid w \in W\} \\
&= v + W
\end{aligned}
$$

Proof of (3): Prove that the operation is well-defined.

Bases

Let V be a vector space and $S \subseteq V$ be a subset.

(1) S is said to be spanning for V if $\text{span}(S) = V$.

(2) S is linearly independent if, for $\sum_{j=1}^{n} \alpha_j v_j = 0_V$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, $v_1, \ldots, v_n \in S$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

(3) S is a basis for V if S is linearly independent and spanning for V.

Proposition: Existence of Basis

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that B is a basis and $B \supseteq B_0$.

Background: A relation on a set X is a subset $R \subseteq X \times X$. If R is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \to x \sim z$), and antisymmetric ($x \sim y, y \sim x \to x = y$), then R is an ordering, and we write $x \leq y$.

If $\leq$ is an ordering of X such that $\forall x, y \in X$, $x \leq y$ or $y \leq x$, then $\leq$ is a total (or linear) ordering.

Let $\leq$ be an ordering of X, let $Y \subseteq X$. An upper bound for Y is an element $u \in X$ such that $y \leq u$ $\forall y \in Y$. A maximal element in X is an element $m \in X$ such that $x \in X$, $x \geq m \to x = m$.

Example: $\mathbb{N}$ under the division ordering defines $a \leq b \leftrightarrow a \mid b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of A can be divided by 9 and 12). Meanwhile, $\mathbb{N}$ itself has no maximal elements.

This leads us to ask: given an ordered set, $(X, \leq)$, does X admit maximal elements.

Zorn's Lemma (or Axiom): Let $(X, \leq)$ be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in X. Then, X admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

Proof: Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with D linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that X has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i \in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \ldots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \ldots, v_n \in D_j$ because Y is totally ordered. However, by definition, D_j is a linearly independent set — therefore, $\alpha_k = 0$. Thus, D is linearly independent.

Since D is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. D is also an upper bound for Y. So, by Zorn's Lemma, X has a maximal element, B.

So, $B_0 \subseteq B \subseteq V$, B is independent, and B is maximal in X. We claim that B is a basis for V. Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and B' is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \ldots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.
- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \text{span}(B)$. $\perp$

Thus, we have a linearly independent set, B', with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of B. Therefore, $\text{span}(B) = V$, and B is a basis for V.

Examples: Vector Spaces

(1) n-Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y+n \end{pmatrix} = \begin{pmatrix} x_1 + y + 1 \\ \vdots \\ x_n + y + n \end{pmatrix}$$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$B = \{e_1, \ldots, e_n\}$$

where e_i denotes the unit vector at position i.

(2) $m \times n$ Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$\alpha(a_{ij}) = (\alpha a_{ij})$$

$$B = \{e_{ij}\}$$

where e_{ij} denotes a matrix of 0 everywhere except column i and row j.

(3) Functions with domain Ω:

$$\mathcal{F}(\Omega, \mathbb{F}) = \{f \mid f : \Omega \to \mathbb{F}\}$$
$$(f + g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

(4) Bounded functions with domain Ω:

$$\ell_\infty(\Omega, \mathbb{F}) = \{f \in \mathcal{F}(\Omega, \mathbb{F}) \mid \|f\|_u \le \infty\}$$
$$\|f\|_u = \sup_{x \in \Omega} |f(x)|$$

Exercises:

- Triangle Inequality: $\|f + g\|_u \le \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha| \|f\|_u$
- Positive Definite: $\|f\|_u = 0 \Rightarrow f = \mathbb{0}$

Proof of Triangle Inequality: Given $x \in \Omega$,

$$\begin{aligned}
|(f + g)(x)| &= |f(x) + g(x)| \\
&\le |f(x)| + |g(x)| \\
&\le \|f\|_u + \|g\|_u
\end{aligned}$$

Therefore,

$$\sup |(f + g)(x)| \le \|f\|_u + \|g\|_u$$
$$\|f + g\|_u \le \|f\|_u + \|g\|_u$$

(5) Continuous functions on closed and bounded intervals:

$$C([a, b], \mathbb{F}) = \{f : [a, b] \to \mathbb{F} \mid f \text{ continuous}\}$$

Check that $C([a, b], \mathbb{F}) \subseteq \ell_\infty([a, b], \mathbb{F})$ is a subspace.

(6) Let $f : [a, b] \to \mathbb{R}$ be any function. Let $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

$$\text{var}(f; \mathcal{P}) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$
$$\text{var}(f) = \sup_{\mathcal{P}} \text{var}(f; \mathcal{P})$$
$$\text{BV}([a, b]) = \{f : [a, b] \to \mathbb{R} \mid \text{var}(f) < \infty\}$$
$$\|f\|_{\text{BV}} = |f(a)| + \text{var}(f)$$

$\text{BV}([a, b])$ is a vector space.

Question: Is $\mathbb{1}_\mathbb{Q} \in \text{BV}([0, 1])$?

(7) Suppose $K \subseteq V$ is a *convex* subset of a vector space: $v, w \in K, t \in [0, 1] \Rightarrow (1 - t)v + tw \in K$. Let $\text{Aff}(K) = \{f : K \to \mathbb{R} \mid f$ is affine$\}$, where f is affine if $\forall v, w \in K, t \in [0, 1], f((1 - t)v + tw) = (1 - t)f(v) + tf(w)$.

Exercise: Show that $\text{Aff}(K) \subseteq \mathcal{F}(K, \mathbb{R})$ is a subspace.

(8) Let S be defined as

$$S = \{(a_k)_{k=1}^\infty \mid a_k \in \mathbb{F}\}.$$

Under pointwise operations, S is a vector space.

$$(a_k)_k + (b_k)_k = (a_k + b_k)_k$$
$$\alpha(a_k)_k = (\alpha a_k)_k$$

Note 1: $S = \mathcal{F}(\mathbb{N}, \mathbb{F})$.

Note 2: $c_{00} \subseteq \ell_1 \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq S$.

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k \ne 0\}$
- $c_0 = \{(a_k)k \mid (a_k)_k \to 0\}$

- $c = \{(a_k)_k \mid (a_k)_k \to a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_u < \infty\}$
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| = a < \infty\}$

(9) $C_C(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \to \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ continuous}, \lim_{x \to \pm\infty} f(x) = 0\}$

(10) Let S be any non-empty set.

$$\mathbb{F}(S) := \{f : S \to \mathbb{F} \mid f \text{ finitely supported}\}$$
$$\operatorname{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \to \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t \in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\operatorname{supp}(f) = \{t_1, \ldots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^n f(t_k) e_{t_k} \in \operatorname{span}(\xi)$. Therefore, ξ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^n \alpha_{t_k} e_{t_k} = \mathbb{0}$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\left(\sum_{k=1}^{\alpha_{t_k}} e_{t_k} \right) = \mathbb{0}(t_1)$$

$$\alpha_{t_1} = 0.$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \ldots, n$. Therefore, ξ is linearly independent. Since ξ is linearly independent and spanning, ξ forms a basis for $\mathbb{F}(S)$.

Note: The free vector space, $\mathbb{F}(S)$, displays the universal property.

There are functions $\iota : S \to \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \to V$ for V a vector space over $\mathbb{F}$, $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \to V$ such that $\iota \circ T_\varphi = \varphi$.

$$S \xrightarrow{\iota} \mathbb{F}(S)$$
$$\varphi \searrow \quad \downarrow T_\varphi$$
$$V$$

Proof: Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^n f(t_k) e_{t_k}$, where $\operatorname{supp}(f) = \{t_1, \ldots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^n f(t_k) \varphi(t_k)$$

Exercise: Show T_φ is linear and unique.

Exercise 2: Suppose V is a vector space over $\mathbb{F}$ with basis B. Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists\, T : V \to W$ such that T is bijective and linear.

Normed Spaces

To every vector $v \in V$, we want to assign a length to v, $\|v\|$.

A **norm** on a vector space V is a map

$$\| \cdot \| : V \to \mathbb{R}^+$$
$$v \mapsto \|v\| \geq 0$$

such that

(i) Homogeneity: $\|\alpha v\| = |\alpha| \|v\|$

(ii) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

(iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = \mathbb{0}_V$.

If $p : V \to \mathbb{R}^+$ satisfies (i) and (ii), then p is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\|v\| \leq c_1 \|v\|'$$
$$\|v\|' \leq c_2 \|v\|$$

Note: On $\mathbb{R}^n$, all norms are equivalent.

Exercise: If p is any seminorm on V, then $|p(v) - p(w)| \leq p(v - w)$.

Notation: If V is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.

Examples of Normed Spaces

(1) Given $V = \mathbb{F}^n$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have different norms:

$$\|x\|_1 = \sum_{j=1}^n |x_j|$$

$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$$

$$\|x\|_2 = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.$$

In general, for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p}.$$

Exercise: Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Show that $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$.

We want to show that $\|\cdot\|_p$ defines a norm for $1 \leq p < \infty$. If $1 \leq p < \infty$, its conjugate index $q \in [1, \infty]$ whereby $\frac{1}{p} + \frac{1}{q} = 1$. For example, if $p = 1$, then $q = \infty$, and if $p = \infty$, then $q = 1$.

Lemma 1: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $f : [0, \infty) \to \mathbb{R}$, $f(t) = \frac{1}{p} t^p - t + \frac{1}{q}$. Then, $f(t) \geq 0$ for all $t \geq 0$.

Proof 1: We can see that $f'(t) = t^{p-1} - 1$. Then, $f'(t) = 0$ at $t = 1$; $f'(t) > 0$ for $t > 1$ and $f'(t) < 0$ for $t \in [0, 1)$.

So, since $f(t) \geq f(1)$ for all $t \geq 0$, and $f(1) = 0$, $f(t) \geq 0$ for all $t \geq 0$.

Lemma 2: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $z, y \geq 0$, $xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q$.

Proof 2: We know from Lemma 1, $t \leq \frac{1}{p} t^p + \frac{1}{q}$. Multiply by y^q to get

$$ty^q \leq \frac{1}{p} t^p y^q + \frac{1}{q} y^q.$$

Set $t = xy^{1-q}$. Then,

$$xy^{1-q} y^q \leq \frac{1}{p} x^p y^{p-pq} y^q + \frac{1}{q} y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $p - pq = -q$, so

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q.$$

With these two lemmas in mind, we get two important inequalities.

Hölder's Inequality: For $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. Then, for $x, y \in \mathbb{F}^n$,

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leq \|x\|_p \|y\|_q.$$

Proof of Hölder's Inequality: For $p = 1$, the solution is as follows:

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leq \sum_{j=1}^{n} |x_j| |y_j|$$
$$\leq \sum_{j=1}^{n} |x_j| \|y\|_\infty$$
$$= \|x\|_q \|y\|_\infty,$$

and similarly for $p = \infty$, $q = 1$.

For $1 < p < \infty$, assume $\|x\|_p = \|y\|_q = 1$.

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leq \sum_{j=1}^{\infty} |x_j| |y_j|$$
$$\leq \sum_{j=1}^{n} \left(\frac{1}{p} |x_j|^p + \frac{1}{q} |y_j|^q \right)$$
$$= \frac{1}{p} \left(\sum_{j=1}^{n} |x_j|^p \right) + \frac{1}{q} \left(\sum_{j=1}^{n} |y_j|^q \right)$$
$$= \frac{1}{p} + \frac{1}{q}$$
$$= 1.$$

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = \mathbb{0}_\mathbb{F}$ or $y = \mathbb{0}_\mathbb{F}$, the inequality still holds.

Assume $\|x\|_p \neq 0$, $\|y\|_p \neq 0$. Set

$$x' = \frac{x}{\|x\|_p}$$
$$y' = \frac{y}{\|y\|_p}.$$

It can be verified that $\|x'\|_p = 1 = \|y'\|_q$. Therefore,

$$\left| \sum_{j=1}^{n} x_j' y_j' \right| \leq 1$$
$$\left| \sum_{j=1}^{n} \frac{x_j}{\|x\|_p} \frac{y_j}{\|y\|_q} \right| \leq 1$$
$$\left| \sum_{j=1}^{n} x_j y_j \right| \leq \|x\|_p \|y\|_q$$

Minkowski's Inequality: Given $x, y \in \mathbb{F}^n$, $1 \leq p \leq \infty$, $\frac{1}{p} = \frac{1}{q} = 1$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Proof of Minkowski's Inequality: We can verify for $p = 1$, $q = \infty$, and vice versa.

Assume $1 < p < \infty$. Then,

$$\|x + y\|_p^p = \sum_{j=1}^{n} |x_j + y_j|^p$$

$$= \sum_{j=1}^{\infty} |x_j + y_j||x_j + y_j|^{p-1}$$

$$\leq \sum_{j=1}^{\infty} |x_j||x_j + y_j|^{p-1} + \sum_{j=1}^{n} |y_j||x_j + y_j|^{p-1}$$

$$\leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} \qquad \text{Hölder's Inequality}$$

$$= \|x\|_p\|x + y\|_p^{p/q} + \|y\|_p\|x + y\|_p^{p/q}$$

$$= (\|x\|_p + \|y\|_p)\|x + y\|_p^{p-1}$$

Divide by $\|x + y\|_p^{p-1}$ to get desired inequality.

(2) $\ell_\infty(\Omega, \mathbb{F})$ with $\|\cdot\|_u$. This includes subspaces that inherit the norm, such as

$$C([a, b]) \subseteq \ell_\infty(\Omega)$$
$$\ell_\infty(\mathbb{R}) \supseteq C_0(\mathbb{R}) \supseteq C_C(\mathbb{R})$$

Exercise: Show that $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ is a subspace.

(3) $\Omega = \mathbb{N}$, $\ell_\infty = \ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$. Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \leq \ell_\infty.$$

(4) ℓ_1 with $\|\cdot\|_1$,

$$\|(a_k)_k\|_1 = \sum_{k=1}^{n} |a_k|.$$

(5) $C([a, b])$ with

$$\|f\|_1 = \int_a^b |f(x)|\,dx.$$

(6) Let $1 \leq p < \infty$.

$$\ell_p = \left\{ (a_k)_{k=1}^\infty \mid \sum_{k=1}^\infty |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/p} = \left\|\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}\right\|_{\ell_p^n}$$

$$= \left\|\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right\|_{\ell_p^n}$$

$$\leq \left\|\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right\| + \left\|\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right\|_{\ell_p^\infty}$$

$$\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p.$$

Taking the limit as $n \to \infty$ (by the definition of an infinite series), we find that $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$.

(7) $\text{BV}([a, b]) = \{f : [a, b] \to \mathbb{R} \mid \text{Var}(f) < \infty\}$ with the norm $\|f\|_{\text{BV}} = |f(a)| + \text{Var}(f)$ is a normed space:

$$\|f\|_{\text{BV}} = 0$$
$$|f(a)| = 0$$
$$\text{Var}(f) = 0$$

given $t \in (a, b)$, look at the partition $a < t \leq b$. Then,

$$\text{Var}(f) \geq |f(t) - f(a)| + |f(b) - f(t)|$$
$$f(t) = 0$$
$$f = \mathbb{0}_f.$$

(8) $\mathbb{M}_{m,n}(\mathbb{F})$ with

$$\|a\|_{\text{op}} = \sup_{\|\xi\|_{\ell_2^n} \leq 1} \|a\xi\|_{\ell_2^m}$$

is a normed vector space. If $\|a\|_{\text{op}} = 0$, then

$$ae_j = 0 \hspace{6cm} \forall j \in \{1, \ldots, n\}.$$

take the dot product with $i \neq j$

$$ae_j \cdot e_i = a_{ij}$$
$$= 0$$

so $a_{ij} = 0$ for all a_{ij}, so a is the $\mathbb{0}$ matrix.

(9) Let V, W be vector spaces over $\mathbb{F}$. Then, $\mathcal{L}(V, W) = \{T \mid T : V \to W \text{ linear}\}$, where $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

$\mathcal{L}(V, W)$ is a vector space with operations

$$(T + S)(v) = T(v) + S(v)$$
$$(\alpha T)(v) = \alpha T(v).$$

Notation: $\mathcal{L}(V) := \mathcal{L}(V, V)$ is all linear operators on V. $\mathcal{L}(V, \mathbb{F}) = V'$ is all linear functionals.

Suppose V and W are normed vector spaces. If $T : V \to W$, set

$$\|T\|_{\text{op}} := \sup_{\|v\|_V \leq 1} \|T(v)\|_W,$$
$$\mathbb{B}(V, W) = \{T \in \mathcal{L}(V, W) \mid \|T\|_{op} \leq \infty\},$$

where $\mathbb{B}(V, W)$ is referred to as the set of all bounded linear maps from V to W. $\mathbb{B}(V, W)$ with $\|\cdot\|_{\text{op}}$ is a normed space.

- Homogeneity:

$$\|\alpha T\|_{[op]} = \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W$$
$$= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W$$
$$= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W$$
$$= |\alpha| \|T\|_{\text{op}}.$$

- Triangle Inequality: for $\|v\|_V \leq 1$,

$$\|(T + S)(v)\|_W = \|T(v) + S(v)\|_W$$
$$\leq \|T(v)\|_W + \|S(v)\|_W$$
$$\leq \|T\|_{\text{op}} + \|S\|_{\text{op}}$$

so

$$\|T + S\|_{\text{op}} = \sup_{\|v\| \leq 1} \|T + S(v)\|$$
$$\leq \|T\|_{\text{op}} + \|S\|_{\text{op}}$$

- Positive Definite: If $\|T\|_{\mathrm{op}} = 0$, then $T(v) = 0$ for all $v \in V$, $\|v\| \leq 1$.

   Let $v \in V$, $v \neq 0$. Then, $\frac{v}{\|v\|} \in B_V$.

$$T\left(\frac{v}{\|v\|}\right) = 0$$
$$\frac{1}{\|v\|}T(v) = 0$$
$$T(v) = 0$$

Special Cases: $\mathbb{B}(V) = \mathbb{B}(V, V)$, $V^* = \mathbb{B}(V, \mathbb{F})$.

Exercise: $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$.

(10) Inner Product Spaces (expanded upon below).

Inner Product Spaces

An inner product on a vector space V is a pairing

$$V \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}$$

that satisfies

(i) $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$, $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$.

(ii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$

(iii) $\langle v, v \rangle \geq 0$.

(iv) If $\langle v, v \rangle = 0$, then $v = 0$.

The pair $(V, \langle \cdot, \cdot \rangle)$ is known as an inner product space.

Remarks: $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$, $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$.

If $\langle \cdot, \cdot \rangle$ is an inner product on a linear space V, then set

$$\|v\|_2 := \langle v, v \rangle^{1/2}.$$

Exercise: $\|\alpha v\|_2 = |\alpha| \|v_2\|$, $\|v\|_2 = 0 \Rightarrow v = 0$.

$v, w \in (V, \langle \cdot, \cdot \rangle)$ are *orthogonal* if $\langle v, w \rangle = 0$.

The Pythagoran theorem states that for $v_1, \ldots, v_n \in V$ mutually orthogonal, then

$$\left\|\sum_{i=1}^{n} v_i\right\|^2 = \sum_{j=1}^{n} \|v_j\|^2.$$

For two vectors $v, w \in V$, $P_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.

Exercise: Check that $\langle P_w(v), v - P_w(v) \rangle$, meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

Cauchy-Schwarz Inequality: In any inner product space,

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|.$$

Proof of Cauchy-Schwarz: From the exercise,

$$\|v\| \geq \|P_w(v)\|$$
$$\|v\| \geq \left\|\frac{\langle v, w \rangle}{\langle w, w \rangle} w\right\|$$
$$= \frac{|\langle v, w \rangle|}{\|w\|^2}\|w\|$$

therefore,

$$\|v\|\|w\| \geq |\langle v, w \rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

Proof of Triangle Inequality:

$$\begin{aligned}
\|v + w\|_2^2 &= \langle v + w, v + w \rangle \\
&= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\
&= \|v\|^2 + \|w\|^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} \\
&= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\,\langle v, w \rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w \rangle| \\
&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \qquad\qquad\qquad\qquad\qquad \text{Cauchy-Schwarz Inequality} \\
&= (\|v\| + \|w\|)^2\,.
\end{aligned}$$

Take square roots on both sides.

(1) $\ell_2^n = \mathbb{F}^n$ with

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left| \sum_{j=1}^n x_j \overline{y_j} \right| \leq \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |y_j|^2 \right)^{1/2}.$$

(2) ℓ_2 with

$$\langle (a_j)_j, (b_j)_j \rangle = \sum_{j=1}^\infty a_j \overline{b_j}.$$

We can see that for any finite n, the Cauchy-Schwarz inequality in ℓ_2^n states

$$\left| \sum_{j=1}^n a_j \overline{b_j} \right| \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |b_j|^2 \right)^{1/2}$$

$$\leq \left(\sum_{j=1}^\infty |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^\infty |b_j|^2 \right)^{1/2}.$$

Taking the limit as $n \to \infty$, we see that $\langle (a_j)_j, (b_j)_j \rangle$ is convergent.

(3) $C([a, b])$ with

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

(4) Let $V = \mathbb{M}_n(\mathbb{C})$.

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let $\mathrm{Tr} : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$, $\mathrm{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$.

- $\mathrm{Tr}(I_n) = n$
- $\mathrm{Tr}(a + \alpha b) = \mathrm{Tr}(a) + \alpha \mathrm{Tr}(b)$
- $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$

Then, if $\text{Tr}(a^*a) = 0$, then $a = \mathbb{0}_{\mathbb{M}_n}$.

$$a^*a = (\overline{a_{ji}})_{i,j}(a_{ij})_{i,j}$$

$$= \left(\sum_{k=1}^n \overline{ki}a_{kj}\right)_{i,j}$$

$$\text{Tr}(a^*a) = \sum_{i=1}^n \sum_{k=1}^n \overline{a_{ki}}a_{ki}$$

$$= \sum_{i,k=1}^n |a_{ki}|^2$$

$$= \sum_{i,j=1}^n |a_{ij}|^2.$$

If $\text{Tr}(a^*a) = 0$, then $a_{ij} = 0$ for all i, j.

We define

$$\langle a, b \rangle_{\text{HS}} = \text{Tr}(b^*a).$$

(i) $(b_1 + b_2)^* = b_1^* + b_2^*$

(ii) $(\alpha b)^* = \overline{\alpha}b^*$

(iii) $(b_1 b_2)^* = b_2^* b_1^*$

(iv) $b^{**} = b$

The norm is defined as

$$\|a\|_{\text{HS}} = \langle a, a \rangle^{1/2}$$

$$= \text{Tr}(a^*a)^{1/2}$$

$$= \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$$

Metric Spaces

We looked at normed spaces, where we attach a length $\|v\|$ to very vector v. We can also speak of the distance between two vectors, defined as $d(v, w) = \|v - w\|$.

Notice that the following hold:

- $d(v, w) \geq 0$

-

$$d(v, w) = \|v - w\|$$
$$= \|(-1)(w - v)\|$$
$$= |-1|\,\|w - v\|$$
$$= \|w - v\|$$

-

$$d(u, w) = \|u - w\|$$
$$= \|u - v + v - w\|$$
$$\leq \|u - v\| + \|v - w\|$$
$$= d(u, v) + d(v, w).$$

- $d(v, v) = \|v - v\| = 0$. If $d(v, w) = 0$, then $\|v - w\| = 0$, so $v - w = \mathbb{0}$, so $v = w$.

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely $(\mathbb{R}, |\cdot|)$. We will expand these concepts to all metric spaces.

Definition of a Metric Space

Let X be a non-empty set. A **metric** on X is a map

$$d : X \times X \to \mathbb{R}^+$$
$$(x, y) \mapsto d(x, y) \geq 0$$

such that

  (i) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.

  (ii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

  (iii) Zero Distance: $d(x, x) = 0$

  (iv) Definite: $d(x, y) = 0 \Rightarrow x = y$

If d satisfies (i), (ii), and (iii), then d is called a semi-metric. If d satisfies (iv) as well, then d is a metric.

If d is a (semi-)metric on X, the pair (X, d) is called a (semi-)metric space.

Two metrics, d and ρ, on X, are equivalent if $\exists c_1, c_2 \geq 0$ such that $d(x, y) \leq c_1 \rho(x, y)$ and $\rho(x, y) \leq c_2 d(x, y)$ for all x, y.

Examples of Metric Spaces

  (1) Discrete Metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

     for X any set.

  (2) Hamming distance: between two bit strings of equal length. Let

$$X = \{0, 1\}^n$$
$$= \{0, 1\} \underbrace{\times \cdots \times}_{n \text{ times}} \{0, 1\}$$
$$d_H((x_j)_1^n, (y_j)_1^n) = \left| \{j \mid x_j \neq y_j\} \right|.$$

  (3) Any normed space $(V, \|\cdot\|)$ is a metric space.

$$d(v, w) = \|v - w\|.$$

    Exercise: Show that if two norms are equivalent, their induced metrics are equivalent.

  (4) Subset of Metric Space: If (X, d) is a metric space, and $Y \subseteq X$ is non-empty. Then, (Y, d) is a metric space.

  (5) Paris metric: let (X, ρ) be a metric space. Let $p \in X$ be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

  (6) Bounded metric: Let ρ be a (semi-)metric on X. Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

    We claim that d is a (semi-)metric. Notice that $0 \leq d(x, y) \leq 1$.

    Proof: Clearly, $d(x, y) = d(y, x)$. Additionally, $d(x, x) = 0$. If $d(x, y) = 0$ and ρ is a metric, then $\rho(x, y) = 0$, so $x = y$.

    To show the triangle inequality, we examine the function

$$f(t) = \frac{t}{1 + t}$$
$$f'(t) = \frac{1}{(1 + t)^2} > 0.$$

Since ρ satisfies the triangle inequality, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Apply f on both sides. Then,

$$\underbrace{\frac{\rho(x, z)}{1 + \rho(x, z)}}_{d(x,z)} \leq \frac{\rho(x, y) + \rho(y, z)}{1 + (\rho(x, y) + \rho(y, z))}$$

$$= \frac{\rho(x, y)}{1 + \rho(x, y) + \rho(y, z)} + \frac{\rho(y, z)}{1 + \rho(x, y) + \rho(y, z)}$$

$$\leq \underbrace{\frac{\rho(x, y)}{1 + \rho(x, y)}}_{d(x,y)} + \underbrace{\frac{\rho(y, z)}{1 + \rho(y, z)}}_{d(y,z)}.$$

(7) If $d_1, \ldots, d_n$ are metrics on X, $c_1, \ldots, c_n \geq 0$. Then,

$$d(x, y) = \sum_{k=1}^{n} c_k d_k(x, y)$$

is a metric.

(8) Let $\{\rho_k\}_{k=1}^{\infty}$ be a family of semi-metrics. Assume the family is separating — for all $x \neq y$, there exists k such that $\rho_k(x, y) \neq 0$.

Let d_k be defined as

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Note that $\{d_k\}_{k=1}^{\infty}$ is also separating.

Then,

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric.

We will now define the Frechet Metric using this method. Let $X = C(\mathbb{R})$. For each $k = 1, 2, 3, \ldots$, set $p_k(f) = \sup_{x \in [-k, k]} |f(x)|$.

We can verify that p_k defines a seminorm. We can then check $\rho_k(f, g) = p_k(f - g)$ is a semi-metric.

We claim that $\{\rho_k\}$ is separating: if $f \neq g$, then there exists $x_0 \in \mathbb{R}$ with $f(x_0) \neq g(x_0)$. Since f and g are continuous, there is a neighborhood $[x_0 - \delta, x_0 + \delta]$ such that $f(x) \neq g(x)$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Find k such that $[x_0 - \delta, x_0 + \delta] \subseteq [-k, k]$. Then, $\rho_k(f - g) > 0$.

Construct d_k as above, and then d as follows:

$$d_F = \sum \frac{2^{-k} p_k(f - g)}{1 + p_k(f - g)}$$

(9) Product of metric spaces: let $(X_k, \rho_k)_{k=1}^{\infty}$ be a countable family of metric spaces. For each k, let

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Remark: If the ρ_k are already uniformly bounded, let $d_k = \rho_k$.

Let

$$X = \prod_{k=1}^{\infty} X_k$$

$$= \{(x_k)_k \mid x_k \in X_k\}$$

$$= \left\{ f : \mathbb{N} \to \bigsqcup_{k=1}^{\infty} X_k \mid f(k) \in X_k \right\}.$$

Define $D : X \times X \to [0, \infty)$ as

$$D(x, y) = \sum_{k=1}^{\infty} 2^{-k} \rho_k(x_k, y_k),$$

$$D(f, g) = \sum_{k=1}^{\infty} 2^{-k} \rho(f(k), g(k)).$$

For example, for each k, let $X_k = \{0, 1\}$ with the discrete metric. Let

$$\Delta = \prod_{k \in \mathbb{N}} \{0, 1\}$$

$$= \{(x_k)_k \mid x_k \in \{0, 1\}\}$$

$$D(x, y) = \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k| \qquad\qquad (x_k)_k, (y_k)_k \in \Delta.$$

Δ is known as the abstract Cantor set; every compact metric space is a surjective image of the abstract Cantor set.

(10) Geodesic Distance: let $\langle \cdot, \cdot \rangle$ be the standard dot product on $\mathbb{R}^3 (\mathbb{R}^n)$, then

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$$

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}.$$

To find the geodesic distance, we take $d(x, y) = \arccos(\langle x, y \rangle)$. We claim d is a metric.

- Symmetry: self-evident.
- $d(x, x) = \arccos(1) = 0$. Suppose $d(x, y) = 0$. Then, $\langle x, y \rangle = 1$, meaning $\|x - y\|^2 = 0$, so $x = y$.
- Let $\theta = \arccos(\langle x, y \rangle)$, $\varphi = \arccos(\langle y, z \rangle)$, where $\theta, \varphi \in [0, \pi]$.

$$p_x = \frac{\langle x, y \rangle}{\langle y, y \rangle} y$$

$$= \cos(\theta) y$$

$$x = \cos(\theta) y + \sin(\theta) u$$

where

$$u = \frac{x - p_x}{\|x - p_x\|}.$$

Similarly, we can take

$$z = \cos(\varphi) y + \sin(\varphi) v$$

where

$$v = \frac{z - p_z}{\|z - p_z\|}.$$

So,

$$\langle x, z \rangle = \cos(\theta) \cos(\varphi) + \sin(\theta) \sin(\varphi) \langle u, v \rangle$$

$$\geq \cos(\theta) \cos(\varphi) - \sin(\theta) \sin(\varphi) \qquad\qquad \langle u, v \rangle \geq -1$$

$$= \cos(\theta + \varphi).$$

Since arccos is decreasing,

$$\arccos(\langle x, z \rangle) \leq \arccos(\cos(\theta + \varphi))$$

$$= \theta + \varphi$$

$$= \arccos(\langle x, y \rangle) + \arccos(\langle y, z \rangle).$$

Therefore, $d(x, y) \leq d(x, y) + d(y, z)$.

- Let $\Gamma = (V, E)$ be a simple connected graph. We define $d : V \times V \to [0, \infty)$ to be the length of the shortest path between vertices u and v.

  Exercise: Show this is a metric.

(11) Let (X, d) be any metric space. If $E \subseteq X$, define $\operatorname{diam}(E) = \sup_{x, y \in E} d(x, y)$. E is bounded if $\operatorname{diam}(E) < \infty$.

Exercise: If $(V, \|\cdot\|)$ is a normed space, $E \subseteq V$ is a subset, show the following are equivalent:

(i) E is bounded (in the metric sense)

(ii) $\sup_{v \in E} \|v\| < \infty$

(iii) $\exists r > 0$ such that $E \subseteq r B_V$.

Let Ω be any set. The function $f : \Omega \to X$ is bounded if $f(\Omega) \subseteq X$ is bounded. We let $\mathrm{Bd}(\Omega, X) = \{ f : \Omega \to X \mid f$ is bounded$\}$.

Remark: $\mathrm{Bd}(\Omega, \mathbb{F}) = \ell_\infty(\Omega, \mathbb{F})$.

(12) $\mathrm{Bd}(\Omega, X)$ with

$$D_u(f, g) = \sup_{x \in \Omega} d(f(x), g(x)).$$

Exercise: Show that D_u defines a metric.

Consider $\mathrm{Bd}(\Omega, \mathbb{F}) = \ell_\infty$. Look at the subset

$$E = \{ f \in \mathrm{Bd}(\Omega, \mathbb{F}) \mid f(x) \in \{0, 1\} \}.$$

Then,

$$D_u(f, g) = \sup_{x \in \Omega} |f(x) - g(x)|.$$
$$= \begin{cases} 1 & f \neq g \\ 0 & f = g \end{cases}.$$

When we take a particular subset of $D_u(f, g)$, we find that we get the discrete metric.

Taking an overview of the concepts we have learned so far, we see

$$\text{Inner Product Spaces} \subseteq \text{Normed Vector Spaces} \subseteq \text{Metric Spaces}$$

Topology of Metric Spaces

Throughout this section, let (X, d) be a metric space.

(1) Let $x_0 \in X$, $\delta > 0$.

(i) We say

$$U(x_0, \delta) = \{ x \in X \mid d(x, x_0) < \delta \}$$

is the open ball centered at x_0 with radius δ.

(ii) We say

$$B(x_0, \delta) = \{ x \in X \mid d(x, x_0) \leq \delta \}$$

is the closed ball.

(iii) We say

$$S(x_0, \delta) = \{ x \in X \mid d(x, x_0) = \delta \}$$

is the sphere.

(2) $U \subseteq X$ is open if

$$(\forall x \in U)(\exists \delta > 0) \ni U(x, \delta) \subseteq U.$$

Let

$$\tau_X = \{ U \subseteq X \mid U \text{ open} \}$$
$$\subseteq \mathcal{P}(X).$$

(3) $D \subseteq X$ is closed if D^c is open.

(4) If $x \in U \in \tau_X$, then U is called an open neighborhood of x. If $x \in U \subseteq N$, where $U \in \tau_X$, then N is a neighborhood of x.

$$\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$$

(5) Let $A \subseteq X$. The interior of A is

$$A^0 = \bigcup \{V \mid V \subseteq A, V \text{ open}\}.$$

The closure of A is

$$\overline{A} = \bigcap \{D \mid A \subseteq D, D \text{ closed}\}.$$

The boundary of A is

$$\partial A = \overline{A} \setminus A^0.$$

Exercise: $\overline{A^c} = (A^0)^c$, $(\overline{A})^c = (A^c)^0$.

Remarks: A^0 is the largest open set contained in A. So, if V is open and $V \subseteq A$, then $V \subseteq A^0$. Similarly, $\overline{D}$ is the smallest closed set containing D. If C is closed and $D \subseteq C$, then $\overline{D} \subseteq C$.

- For example, $(a, b]^0 = (a, b)$. This is because (a, b) is open and contained in $(a, b]$, so $(a, b) \subseteq (a, b]^0$.

- We will show that $\overline{A^c} \subseteq (A^0)^c$.

$$A^0 \subseteq A$$
$$(A^0)^c \supseteq A^c$$

The union of open sets is open, so A^0 is open, so $(A^0)^c$ is closed by definition. Therefore,

$$(A^0)^c \supseteq \overline{A^c}.$$

Topology of Open Sets in a Metric Space

The open sets τ_X form a topology:

(i) $\emptyset, X \in \tau_X$.

(ii) If $\{V_i\}_{i \in I} \subseteq \tau_X$, then

$$\bigcup_{i \in I} V_i \in \tau_X.$$

(iii) If $V_i, \ldots, V_n \in \tau_X$, then

$$\bigcap_{i=1}^{n} V_i \in \tau_X.$$

Remark: This is only true of finite intersections. For a counterexample, if $V_n = (-1/n, 1/n) \subseteq \mathbb{R}$ with the Euclidean metric, then the infinite intersection yields $\{0\}$, which is closed in $\mathbb{R}$ with the Euclidean metric.

Proof:

(1) Clearly, $\emptyset$ (by vacuous truth) and X are open.

(2) Let $x \in \bigcup_{i \in I} V_i$. Then, $\exists i_0 \in I$ with $x \in V_{i_0}$. Since V_{i_0} is open, $\exists \varepsilon > 0$ such that $U(x, \varepsilon) \subseteq V_{i_0} \subseteq \bigcup V_i$.

(3) Let $x \in \bigcap_{i=1}^{n} V_i$. Then, $x \in V_i$ for all $i \in 1, \ldots, n$. Since each V_i is open, $\exists \varepsilon_1, \ldots, \varepsilon_n$ with $U(x, \varepsilon_i) \subseteq V_i$ for each $i = 1, \ldots, n$. Set $\varepsilon = \min\{\varepsilon_i\}_{i=1}^{n}$. Then, $U(x, \varepsilon) \subseteq U(x, \varepsilon_i) \subseteq V_i$ for all i. Therefore, $U(x, \varepsilon) \subseteq \bigcap_{i=1}^{n} V_i$.

Exercise: Show all open balls are open. In particular, show all open intervals are open.

Exercise: Show the following:

(1) $X, \emptyset$ are closed.

(2) If $\{C_i\}_{i \in I}$ is a family of closed sets, then $\bigcap_{i \in I} C_i$ is closed.

(3) For $C_1, \ldots, C_n$ closed, then $\bigcup_{i=1}^{n} C_i$ is closed.

(4) Closed balls are closed. Spheres are closed.

Let $x \in X$. Recall that $\mathcal{N}_x$ is the set of all neighborhoods of x.

(i) $N \in \mathcal{N}_x \Leftrightarrow \exists \delta > 0 : U(x, \delta) \in N$

(ii) $N \in \mathcal{N}_x, N \subseteq M \Rightarrow M \in \mathcal{N}_x$

(iii) $N_1, N_2 \in \mathcal{N}_x \Rightarrow N_1 \cap N_2 \in \mathcal{N}_x$

In this sense, $\mathcal{N}_x$ is a directed set with reverse inclusion.

Pointwise Characterization of Subsets

Let $A \subseteq X$.

(i) $x \in A^0 \Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A$.

(ii) $x \in \overline{A} \Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset$.

(iii) $x \in \partial A \Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset$ and $U(x, \delta) \cap A^c \neq \emptyset$.

Proof: Let $A \subseteq X$

(i)

$$x \in A^0 \Leftrightarrow x \in \bigcup_{\substack{V \in \tau_X \\ V \subseteq A}} V$$
$$\Leftrightarrow \exists V \in \tau_X, V \subseteq A, x \in V$$
$$\Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A.$$

(ii)

$$x \notin \overline{A} \Leftrightarrow x \in (\overline{A})^c$$
$$\Leftrightarrow x \in (A^c)^0$$
$$\Leftrightarrow \exists \delta > 0 : U(x, \delta) \subseteq A^c$$
$$\Leftrightarrow \exists \delta > 0 : U(x, \delta) \cap A = \emptyset.$$

We negate both sides.

(iii)

$$x \in \partial A \Leftrightarrow x \in \overline{A} \setminus A^0$$
$$\Leftrightarrow x \in \overline{A} \cap (A^0)^c$$
$$\Leftrightarrow x \in \overline{A} \cap \overline{A^c}$$
$$\Leftrightarrow x \in \overline{A} \text{ and } x \in \overline{A^c}$$
$$\Leftrightarrow \forall \delta > 0 : U(x, \delta) \cap A \neq \emptyset, U(x, \delta) \cap A^c \neq \emptyset$$

Remark: $\overline{U(v, \delta)} = B(v, \delta)$ in a normed space. $\partial U(v, \delta) = \partial B(v, \delta) = S(v, \delta)$ in a normed space. Also, $B(v, \delta)^0 = U(v, \delta)$.

Proof: We show that $\overline{U(v, \delta)} = B(v, \delta)$. Since $B(v, \delta)$ is closed, and $U(v, \delta) \subseteq B(v, \delta)$, we know $\overline{U(v, \delta)} \subseteq B(v, \delta)$.

Let $w \in B(v, \delta)$. If $\|w - v\| < \delta$, then $w \in U(v, \delta)$. Assume $\|w - v\| = \delta$. Let $u_t = (1 - t)v + tw$, where $t \in [0, 1]$.

$$\|w - u_t\| = \|w - (1 - t)v - tw\|$$
$$= \|(1 - t)(w - v)\|$$
$$= (1 - t)\|w - v\|$$
$$= (1 - t)\delta.$$

Let $\varepsilon > 0$. Let $t \in (0, 1)$ such that $(1 - t)\delta < \varepsilon$. Then, $u_t \in U(w, \varepsilon) \cap U(v, \delta)$. Therefore, $w \in \overline{U(v, \delta)}$.

Unions and Intersections of Closure/Interior

Let (X, d) be a metric space.

(i)

$$\left(\bigcup_{i \in I} A_i \right)^0 \supseteq \bigcup_{i \in I} A_i^0 \qquad\qquad \text{may be strict}$$

(ii)

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$$

(iii)

$$\bigcap_{k=1}^{n} A_k^0 = \left(\bigcap_{k=1}^{n} A_k\right)^0$$

(iv)

$$\overline{\bigcup_{k=1}^{n} D_k} = \bigcup_{k=1}^{n} \overline{D_k}$$

Proof:

(i)

$$A_i^0 \subseteq A_i$$
$$\bigcup_{i \in I} A_i^0 \subseteq \bigcup_{i \in I} A_i$$
$$\bigcup_{i \in I} A_i^0 \subseteq \left(\bigcup_{i \in I} A_i\right)^0$$

Remark: We claim $\overline{\mathbb{Q}} = \mathbb{R}$ under the absolute value metric. We know that $\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{R}$ is closed, meaning $\overline{\mathbb{Q}} \subseteq \mathbb{R}$. Let $t \in \mathbb{R}$, $\delta > 0$. We know that $(t - \delta, t + \delta) \cap \mathbb{Q} \neq \emptyset$. Therefore, $t \in \overline{\mathbb{Q}}$. Thus, $\overline{\mathbb{Q}} = \mathbb{R}$.

Properties of Boundary

Let $A \subseteq X$.

(1) ∂A is closed.

(2) $\partial A = \partial A^c$

(3) $\overline{A} = A \cup \partial A$

(4) $A \setminus \partial A = A^0$

Proof:

(1)

$$\partial A = \overline{A} \setminus A^0$$
$$= \overline{A} \cap (A^0)^c.$$

(2) Follows from pointwise characterization.

(3) Clearly, $A \cup \partial A \subseteq \overline{A}$. Let $x \in \overline{A}$. If $x \in A$, we're done. Otherwise, $x \in \overline{A} \setminus A \subseteq \overline{A} \setminus A^0 = \partial A$.

(4)

$$A \setminus \partial A = A \cap (\partial A)^c$$
$$= A \cap (\overline{A} \setminus A^0)^c$$
$$= A \cap \left(\overline{A} \cap (A^0)^c\right)^c$$
$$= A \cap \left(\overline{A}^c \cup A^0\right)$$
$$= \left(A \cap \overline{A}^c\right) \cup \left(A \cap A^0\right)$$
$$= A^0$$

Density and Separability

Let (X, d) be a metric space.

(1) $A \subseteq X$ is d-dense if $\overline{A} = X$.

(2) $N \subseteq X$ is nowhere dense if $(\overline{N})^0 = \emptyset$.

(3) (X, d) is separable if there is a countable dense subset.

Exercise: If $N \subseteq X$ is closed, then N is nowhere dense if and only if N^c is dense.

Exercise: The following are equivalent.

(1) $A \subseteq X$ is dense.

(2) $\forall \emptyset \neq U \in \tau_X$, $U \cap A \neq \emptyset$.

(3) $\forall x \in X, \forall \varepsilon > 0$, $U(x, \varepsilon) \cap A \neq \emptyset$.

(4) $\forall x \in X, \forall \varepsilon > 0, \exists a \in A$ such that $d(x, a) < \varepsilon$.

Let X be a metric space.

(1) A base for τ_X is a family of open subsets $\mathcal{B}$ such that:

$$(\forall U \in \tau_X)\,(\forall x \in U) \exists B \in \mathcal{B} \ni x \in B \subseteq U.$$

Equivalently,

$$\forall U \in \tau_X, U = \bigcup_{i \in I} B_i. \qquad\qquad B_i \in \mathcal{B}$$

(2) We say that (X, d) is second countable if τ_X admits a countable base.

- For any (X, d) a metric space, $\mathcal{B} = \{U(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a base. Indeed, given any $x \in U \subseteq \tau_X$, by definition, $\exists \varepsilon > 0$ such that $U(x, \varepsilon) \subseteq U$. Alternatively, $\mathcal{B}' = \{U(x, 1/n) \mid x \in X, n \geq 1\}$ is a topological base.

- Let $X = \mathbb{R}^d$ with the Euclidean metric. Then, for $\mathcal{B} = \{U(q, 1/n) \mid n \geq 1, q \in \mathbb{Q}^d\}$, we claim this is a base.

  Let $V \subseteq \mathbb{R}^d$ be open, $r \in V$. Since V is open, $\exists \delta > 0$ with $U(r, \delta) \subseteq V$. Find n large such that $1/n < \delta$. Find $q \in \mathbb{Q}^d$ with $\|r - q\| < 1/2n$. This is always possible as $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$.

  Consider $U(q, 1/2n)$. Then, $r \subseteq U(q, 1/2n) \subseteq U(r, \delta) \subseteq V$ because $\|r - q\| < 1/2n$, and if $t \in U(q, 1/2n)$, then

$$\begin{aligned}
\|t - r\| &\leq \|t - q\| + \|q - r\| \\
&< 1/2n + 1/2n \\
&= 1/n \\
&< \delta.
\end{aligned}$$

Separable, Non-Separable, Dense, and Non-Dense Sets

(1) $(R^d, \|\cdot\|_p)$ is separable for any $p \in [1, \infty]$. Indeed, $\mathbb{Q}^d \subseteq \mathbb{R}^d$ is the countable dense subset of $\mathbb{R}^d$.

Let $r = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \in \mathbb{R}^d$. Find $q = \begin{bmatrix} q_1 \\ \vdots \\ q_d \end{bmatrix} \in \mathbb{Q}^d$ with $|r_j - q_j| < \varepsilon/d$. Then,

$$\|r - q\|_1 = \sum_{j=1}^{d} |r_j - q_j|$$
$$< d.$$

We know that for any vector $r \in \mathbb{R}^d$, we can find a vector q such that

$$\|q - r\|_p \leq c \|q - r\|_1,$$

so for arbitrary p, find q such that $\|q - r\|_1 < \varepsilon/c$.

(2) Similarly, $\mathbb{C}_{\mathbb{Q}} = \{a + bi \mid a, b \in \mathbb{Q}\}$ is also countable, meaning $\mathbb{C}_{\mathbb{Q}}^d \subseteq \mathbb{C}^d$ is dense and $\mathbb{C}^d$ is dense.

Proposition: Separable Subsets

If (X, d) is separable, and $Y \subseteq X$, then (Y, d) is also separable.

Let $\{a_k\}$ be a countable dense subset in X. Let $N = \{(m, n) \mid U(a_m, 1/n) \cap Y \neq \emptyset\}$. Clearly, N is nonempty. For each $(m, n) \in N$, choose $b_{(m,n)} \in Y \cap U(a_m, 1/n)$. We claim $\{b_{(m,n)} \mid m, n \geq 1\}$ is dense in Y.

Let $y \in Y$, $\varepsilon > 0$. Find N large so that $\frac{1}{n} < \varepsilon/2$. Since $A \subseteq X$ is dense, find $U(y, 1/n) \cap A \neq \emptyset$. Suppose $d(a_m, y) < 1/n$. Then,

$$d(b_{(m,n)}, y) \leq d(b_{(m,n)}, a_m) + d(a_m, y)$$
$$< \frac{1}{n} + \frac{1}{n}$$
$$= \frac{2}{n}$$
$$< \varepsilon.$$

(1) ℓ_p^n is separable.

(2) $c_{00} = \{(a_k)_{k=1}^n \mid \text{finitely many } a_k \neq 0\}$ with $\|\cdot\|_u$ is separable.

Recall that $e_k = (0, 0, \ldots, 1, 0, 0, \ldots)$ where 1 is at position k. Consider $E = \mathbb{Q}\text{-span}\{e_k \mid k \geq 1\}$,

$$E = \left\{\sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q}, n \geq 1\right\}.$$

The set E is countable. If we fix $n \geq 1$, we have

$$E_n = \left\{\sum_{k=1}^n \alpha_k e_k \mid \alpha_k \in \mathbb{Q}\right\}.$$

Then, $E = \bigcup E_n$. Note

$$\underbrace{\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}}_{n} \to E_n$$

$$(\alpha_1, \ldots, \alpha_n) \mapsto \sum_{k=1}^n \alpha_k e_k.$$

Thus, E_n is countable, and E is a countable union of countable sets.

We claim that E is dense. Given $z \in c_{00}$, $\varepsilon > 0$, we know that $z = \sum_{k=1}^n a_k e_k$ for some n and $a_k \in \mathbb{R}$. Find $\alpha_k \in \mathbb{Q}$ such that $|\alpha_k - a_k| < \varepsilon$. Set $w = \sum_{k=1}^n \alpha_k e_k$. Then, $\|z - w\|_u = \sup |\alpha_k - a_k| < \varepsilon$.

(3) c_0 with $\|\cdot\|_u$ is separable.

(4) ℓ_∞ is not separable.

Suppose ℓ_∞ were separable. Consider $E = \{(a_k)_k \in \ell_\infty \mid a_k \in \{0, 1\}\}$. Then, E is separable. Recall that $(E, \|\cdot\|_u)$ has the discrete metric.

In the discrete metric, every subset is open, meaning every subset is closed. Therefore, if X is separable and discrete, then X is countable.

However, E is not countable by Cantor's theorem. $\text{card}(E) = 2^{\aleph_0}$.

Alternatively, we can show that

$$(a_k)_k \mapsto \sum_{k=1}^\infty 2^{-k} a_k$$

is onto.

Exercise: ℓ_p is separable for $1 \leq p < \infty$.

(5) We will show that

$$\mathbb{P}[0, 1] \left\{ \sum_{k=1}^{n} a_k x^k \mid a_k \in \mathbb{R}, n \geq 1 \right\}$$

is $\|\cdot\|_u$-dense in $C([0, 1])$. Using this, we can show that $(C([0, 1]), \|\cdot\|_u)$ is separable.

The Cantor Set

$$C_0 = [0, 1]$$
$$C_1 = [0, 1/3] \cup [2/3, 1]$$
$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$
$$C_3 = [0, 1/27] \cup [2/27, 1/9] \cup \cdots \cup [26/27, 1]$$
$$\vdots$$

In each step, we delete the middle third of each interval. This process repeated ad infinitum yields the Cantor set.

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k + 0}{3^n}, \frac{3k + 1}{3^n} \right] \cup \left[\frac{3k + 2}{3^n}, \frac{3k + 3}{3^n} \right] \right).$$

(i) $\mathcal{C}$ is closed as it is the intersection of closed sets.

(ii) length$(\mathcal{C}) = 0$. Look at the total length of the removed intervals,

$$\begin{aligned} l &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots \\ &= \sum_{k=1}^{\infty} \left(\frac{2^{k-1}}{3^k} \right) \\ &= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{2}{3} \right)^k \\ &= 1. \end{aligned}$$

Thus, length$(\mathcal{C}) = 0$.

(iii) $\mathcal{C}$ is nowhere dense — $(\overline{\mathcal{C}})^0 = \emptyset$. Since $\mathcal{C}$ is closed, $\mathcal{C}^0 = \emptyset$.

Suppose $\mathcal{C}^0 \neq \emptyset$. Then, $\exists x \in \mathcal{C}, \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}$. So, $(x - \varepsilon, x + \varepsilon) \subseteq C_n$ for all n.

Note C_n is the disjoint union of 2^n subintervals, each with length $1/3^n$. Find m so large such that $3^{-m} < \varepsilon$. We know that $(x - \varepsilon, x + \varepsilon) \subseteq C_m$.

However, $(x - \varepsilon, x + \varepsilon)$ has length $2\varepsilon > \frac{2}{3^m}$. Each subinterval in C_m has length $1/3^m$. This implies C_m contains an interval of length greater than $\frac{2}{3^m}$. $\perp$

(iv) card$(\mathcal{C}) = $ card$(\mathbb{R})$