Problem (Problem 1)**:** Use de Rham cohomology to prove that if B^n is a closed ball in $\mathbb{R}^n$, and $f\colon B^n \to B^n$ is smooth, then f has a fixed point.

Solution: Suppose $f\colon B^n \to B^n$ is a fixed-point free self-map of the ball. It follows then that by drawing a line between v and $f(v)$, we may define a smooth retraction of the ball to the sphere S^{n-1}. Call this retraction r.

We observe then that r induces a map in cohomology $r^*\colon H^*_{DR}(S^{n-1}) \to H^*_{DR}(B^n)$. In particular, since r is a retraction to S^{n-1}, it follows that r is homotopic to the identity map when restricted to S^{n-1}, meaning r^* is an isomorphism in de Rham cohomology of $H^*_{DR}(S^{n-1})$ and $H^*_{DR}(B^n)$.

Yet, we recognize that $H^{n-1}_{DR}(S^{n-1}) \cong \mathbb{R}$, while $H^{n-1}_{DR}(B^n) \cong 0$, the latter emerging from the fact that B^n is contractible via the straight-line homotopy and the Poincaré lemma. Thus, no such r exists, whence f cannot have a fixed point.

Problem (Problem 2)**:** Suppose M is a compact smooth manifold with a smooth triangulation, and let $f\colon M \to M$ be a smooth map preserving the triangulation. Write f^*_k for the induced map on $H^k_{DR}(M)$. Prove that if

$$\begin{aligned} L(f) &= \sum_{k=0}^{n} (-1)^k \operatorname{tr}\left(f^*_k\right) \\ &\neq 0, \end{aligned}$$

then f has a fixed point.

Solution: By abuse of notation, we treat $f^*\colon H^*(M;\mathbb{R}) \to H^*(M;\mathbb{R})$ to be the corresponding map on the simplicial cohomology rather than the de Rham cohomology, which follows from de Rham's theorem and the isomorphism between singular and simplicial cohomology.

Suppose f has no fixed points. Let $\Delta \subseteq M$ be a simplex. Then, by the definition of f, we observe that $f(\Delta) \subseteq M$ is also a simplex, which we call Λ. Suppose toward contradiction that $\Lambda = \Delta$. Then, restricting the map f to Δ, we observe that $f\colon \Delta \to \Delta$ is a smooth self-map of the k-simplex Δ. Yet, since $\Delta \cong B^n$ are diffeomorphic (when considering a small neighborhood of Δ), this implies that we have a smooth self-map on Δ, whence f has a fixed point by the result of Problem (1).

From the de Rham isomorphism and the fact that M is triangulated, an arbitrary cochain on M, I_ω, can be defined by

$$I_\omega(\Delta) = \int_\Delta \omega,$$

which induces the isomorphism $H^*_{DR}(M) \cong H^*(M;\mathbb{R})$. We observe that f^* yields a map on cochains by taking

$$\begin{aligned} f^*(I_\omega)(\sigma) &= \int_\sigma f^*\omega \\ &= I_{f^*\omega}(\sigma) \end{aligned}$$

for a k-simplex σ.

We start by showing that on $C^k(M;\mathbb{R})$, we have

$$\sum_{k=0}^{n} (-1)^k \operatorname{tr}\left(f^*_k\right) = 0$$

Now, we observe that for any k-simplex $\sigma \subseteq M$, that $f(\sigma) \not\subseteq \sigma$; by selecting a k-form supported on σ, we observe then that $I_{f^*\omega}(\sigma) = 0$, whence the map $f^*_k\colon C^k(M;\mathbb{R}) \to C^k(M;\mathbb{R})$ induced by the pullback has no eigenvectors. Thus, it follows that $\operatorname{tr}(f_k^*) = 0$, so

$$\sum_{k=0}^{n} (-1)^k \operatorname{tr}(f_k^*) = 0$$

necessarily.

Now, to show that this passes to cohomology, we make use of a lemma related to short exact sequences of vector spaces. Specifically, in the following diagram, we claim that $\operatorname{tr}(f) = \operatorname{tr}(g) + \operatorname{tr}(h)$, where A, B, C are dimensional vector spaces.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & A \oplus B \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus B \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

This follows from putting the matrix representations $[g]_\alpha$ and $[h]_\beta$ to yield, for $\gamma = (\alpha \times \{0\}) \cup (\{0\} \times \beta)$,

$$[f]_\gamma = \begin{pmatrix} [g]_\alpha & K \\ 0 & [h]_\beta \end{pmatrix},$$

for some matrix K, whence $\operatorname{tr}(f) = \operatorname{tr}(g) + \operatorname{tr}(h)$.

Thus, we consider the following short exact sequences, where we relabel g_k^* for the f_k^* acting on the k-cochains, and use f_k^* for the map in homology.

$$\begin{tikzcd}
0 \arrow[r] & Z^k \arrow[r] \arrow[d, "q_k"] & C^k \arrow[r] \arrow[d, "g_k^*"] & B^{k+1} \arrow[r] \arrow[d, "h_{k+1}"] & 0 \\
0 \arrow[r] & Z^k \arrow[r] & C^k \arrow[r] & B^{k+1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & B^k \arrow[r] \arrow[d, "h_k"] & Z^k \arrow[r] \arrow[d, "q_k"] & H^k \arrow[r] \arrow[d, "f_k^*"] & 0 \\
0 \arrow[r] & B^k \arrow[r] & Z^k \arrow[r] & H^k \arrow[r] & 0
\end{tikzcd}$$

This yields

$$\begin{aligned}
\operatorname{tr}(g_k^*) &= \operatorname{tr}(q_k) + \operatorname{tr}(h_{k+1}) \\
\operatorname{tr}(q_k) &= \operatorname{tr}(f_k^*) + \operatorname{tr}(h_k).
\end{aligned}$$

In particular, we have $\operatorname{tr}(g_k^*) = \operatorname{tr}(f_k^*) + \operatorname{tr}(h_k) + \operatorname{tr}(h_{k+1})$. Therefore, we get

$$\begin{aligned}
0 &= \sum_{k=0}^{n} (-1)^k \operatorname{tr}(g_k^*) \\
&= \sum_{k=0}^{n} (-1)^k \left(\operatorname{tr}(f_k^*) + \operatorname{tr}(h_k) + \operatorname{tr}(h_{k+1})\right) \\
&= \sum_{k=0}^{n} (-1)^k \operatorname{tr}(f_k^*).
\end{aligned}$$

Problem (Problem 3)**:** Compute the de Rham cohomology of $\mathbb{RP}^n$.

Solution: To start, we observe that $\mathbb{RP}^1 \cong S^1$, meaning that the de Rham cohomology of $\mathbb{RP}^1$ is

$$H^*_{DR}(\mathbb{RP}^1) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = 1 \\ 0 & \text{else} \end{cases}.$$

In higher dimensions, we consider the family of charts defined by

$$U_k = \{[x_0 : \cdots : x_k : \cdots : x_n] \mid x_{i\neq k} \in \mathbb{R}, x_k \neq 0\}.$$

We seek to understand the picture of

$$\begin{aligned} U_{k\neq 0} &= \bigcup_{k=1}^{n} U_k \\ &= \bigcup_{k=1}^{n} \{[x_0 : \cdots : x_n] \mid x_k \neq 0\}. \end{aligned}$$

In particular, the only elements of U_0 that are not in $U_{k\neq 0}$ are the ones of the form $[1 : 0 : \cdots : 0]$, whence $U_{k\neq 0} \cong \mathbb{R}^n \setminus \{0\}$.

Next, we observe that

$$\begin{aligned} U_0 \cap U_{k\neq 0} &= \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\} \cap \bigcup_{k=1}^{n} \{[x_0 : \cdots : x_n] \mid x_k \neq 0\} \\ &= \{[x_0 : \cdots : x_n] \mid x_0 \neq 0, x_k \neq 0 \text{ for at least one } 1 \leqslant k \leqslant n\} \\ &= U_0 \setminus \{[1 : 0 : \cdots : 0]\} \\ &\cong \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

Thus, by Mayer–Vietoris, we obtain the following short exact sequence.

$$0 \longrightarrow H^*(\mathbb{RP}^n) \longrightarrow H^*(\mathbb{R}^n) \oplus H^*(\mathbb{R}^n \setminus \{0\}) \longrightarrow H^*(\mathbb{R}^n \setminus \{0\}) \longrightarrow 0$$

Focusing on the case of H^0, this yields the following exact sequence, whence $H^0(\mathbb{RP}^n) \cong \mathbb{R}$.

$$0 \longrightarrow H^0(\mathbb{RP}^n) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \cdots$$

Since the $H^k(\mathbb{R}^n)$ are zero for all $k \geqslant 1$, it follows that we have $H^k(\mathbb{RP}^n) \cong 0$ for $1 \leqslant k < n$.

Finally, concerning ourselves with $H^n(\mathbb{RP}^n)$, we concern ourselves with orientability; specifically, $H^n(\mathbb{RP}^n) \cong \mathbb{R}$ if n is odd and $H^n(\mathbb{RP}^n) \cong 0$ if n is even, as $\mathbb{RP}^n$ is orientable if and only if n is odd.

Problem (Problem 4)**:** Prove the Five Lemma. Namely, consider the following commutative diagram of vector spaces, where the horizontal sequences are exact. Show that if f_1, f_2, f_4, f_5 are isomorphisms, that f_3 is also an isomorphism.

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5 \end{tikzcd}$$

Solution: We start by showing that f_3 is injective. Let $x \in \ker(f_3)$.

- By commutativity, we have
$$\begin{aligned} 0 &= \beta_3 \circ f_3(x) \\ &= f_4 \circ \alpha_3(x), \end{aligned}$$
so it follows that $\alpha_3(x) = 0$ as f_4 is injective, so $x \in \ker(\alpha_3)$. By exactness, we let $a_2 \in A_2$ be such that $\alpha_2(a_2) = x$, and define $f_2(a_2) = b_2$.

- By commutativity,
$$\begin{aligned} \beta_2(b_2) &= \beta_2(f_2(a_2)) \\ &= f_3(\alpha_2(a_2)) \\ &= f_3(x) \\ &= 0, \end{aligned}$$
so $b_2 \in \ker(\beta_2)$, meaning that by exactness, there is $b_1 \in B_1$ such that $\beta_1(B_1) = b_2$. Since f_1 is surjective, we let $a_1 \in A_1$ be such that $f_1(a_1) = b_1$.

- Finally, by commutativity, we have
$$\begin{aligned} f_2(\alpha_1(a_1)) &= \beta_2(f_1(a_1)) \\ &= \beta_1(b_1) \\ &= b_2 \\ &= f_2(a_2), \end{aligned}$$
and since f_2 is injective, we have $a_2 = \alpha_1(a_1)$.

- Thus, since $x = \alpha_2(a_2)$, we have
$$\begin{aligned} x &= \alpha_2(\alpha_1(a_1)) \\ &= (\alpha_2 \circ \alpha_1)(a_1) \\ &= 0, \end{aligned}$$
so f is injective.

Now, we show that f is surjective. Let $b \in B_3$.

- Since f_4 is surjective, there is some $a_4 \in A_4$ such that $f_4(a_4) = \beta_3(b)$.

- By commutativity, we have
$$\begin{aligned} f_5(\alpha_4(a_4)) &= \beta_4(f_4(a_4)) \\ &= \beta_4(\beta_3(b)) \\ &= 0, \end{aligned}$$
whence $\alpha_4(a_4) = 0$ since f_5 is an isomorphism, so $a_4 \in \ker(\alpha_4)$. By exactness, there is then $a_3 \in A_3$ such that $\alpha_3(a_3) = a_4$.

- By commutativity, we have
$$\begin{aligned} \beta_3(f_3(a_3)) &= f_4(\alpha_3(a_3)) \\ &= f_4(a_4) \\ &= \beta_3(b). \end{aligned}$$
Thus, $b - f_3(a_3) \in \ker(\beta_3)$. Therefore, by exactness, there is some $b_2 \in B_2$ such that $\beta_2(b_2) = b - f_3(a_3)$. Since f_2 is an isomorphism, there is $a_2 \in A_2$ such that $f_2(a_2) = b_2$.

- By commutativity, we have

$$\begin{aligned} f_3(\alpha_2(a_2)) &= \beta_2(f_2(a_2)) \\ &= \beta_2(b_2) \\ &= b - f_3(a_3), \end{aligned}$$

  so

$$f_3(\alpha_2(a_2) - a_3) = b.$$

Thus, f_3 is an isomorphism.

Problem (Problem 5)**:** Use the Mayer–Vietoris sequence to prove the Künneth Formula: if M and N are smooth manifolds, then $H^*_{DR}(M \times N)$ is the tensor product of $H^*_{DR}(M)$ and $H^*_{DR}(N)$.

Specifically, in each degree ℓ, we have

$$H^{\ell}_{DR}(M \times N) = \bigoplus_{i+j=\ell} H^i_{DR}(M) \otimes H^j_{DR}(N).$$

Solution: Let $\{U_i\}_{i\in I} = \mathcal{U}$ be a good open cover of M. We observe then that $\{U_i \times N\}_{i\in I}$ is an open cover of $M \times N$; we observe that for a specific $U_i \in \mathcal{U}$, that U_i is contractible, whence by the Poincaré Lemma,

$$H^*_{DR}(U_i \times N) \cong H^*_{DR}(N).$$

Now, we observe that the projection maps $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ induce maps in cohomology

$$\begin{aligned} \pi_M^* &\colon H^*_{DR}(M) \to H^*_{DR}(M \times N) \\ \pi_N^* &\colon H^*_{DR}(N) \to H^*_{DR}(M \times N). \end{aligned}$$

Problem (Problem 6)**:** Compute the de Rham cohomology of the n-torus $(S^1)^n$ and of $(S^2)^n$.

Solution: By using the Künneth Formula, we see that

$$\begin{aligned} H^0(S^1 \times S^1) &= H^0(S^1) \otimes H^0(S^1) \\ &= \mathbb{R} \\ H^1(S^1 \times S^1) &= (H^1(S^1) \otimes H^0(S^1)) \oplus (H^0(S^1) \otimes H^1(S^1)) \\ &= \mathbb{R}^2 \\ H^2(S^1 \times S^1) &= H^1(S^1) \otimes H^1(S^1) \\ &= \mathbb{R}. \end{aligned}$$

Analogously, we have

$$H^*\left((S^1)^n\right) = H^*\left((S^1)^{n-1}\right) \otimes H^*(S^1).$$

We use the induction hypothesis of

$$H^k\left((S^1)^{n-1}\right) = \mathbb{R}^{\binom{n-1}{k}}$$

for $0 \leqslant k \leqslant n-1$. We observe then that for $0 \leqslant k \leqslant n-1$,

$$H^k\left((S^1)^n\right) = H^k\left((S^1)^{n-1}\right) \otimes \mathbb{R} \oplus H^{k-1}\left((S^1)^{n-1}\right) \otimes \mathbb{R}$$

$$
\begin{aligned}
&= \mathbb{R}^{\binom{n-1}{k}} \oplus \mathbb{R}^{\binom{n-1}{k-1}} \\
&= \mathbb{R}^{\binom{n}{k}},
\end{aligned}
$$

while $H^0\left(\left(S^1\right)^n\right) = \mathbb{R}$ and $H^1\left(\left(S^1\right)^n\right) = \mathbb{R}$ by connectedness/orientability.

Meanwhile, upon computing a number of homology group for $H^k\left(\left(S^2\right)^n\right)$, we find the following recurrence relation by using the Künneth formula and the fact that the $H^k\left(S^2\right) = \mathbb{R}$ if $k = 0$ or $k = 2$.

$$
\begin{aligned}
H^0\left(\left(S^2\right)^n\right) &= \mathbb{R} \\
H^k\left(\left(S^2\right)^n\right) &= \begin{cases} 0 & k \text{ odd} \\ H^k\left(\left(S^2\right)^{n-1}\right) \oplus H^{k-2}\left(\left(S^2\right)^{n-1}\right) & k \text{ even}, 2 \leqslant k < 2n \end{cases} \\
H^{2n}\left(\left(S^2\right)^n\right) &= \mathbb{R}.
\end{aligned}
$$