Classical Mechanics

Motion in $\mathbb{R}^1$

Let $x(t)$ denote position. Then, $v(t) = \frac{dx}{dt} = \dot{x}(t)$ is velocity (where the $\cdot$ denotes derivative with respect to time), $a(t) = \dot{v}(t) = \ddot{x}(t)$, etc.

Considering Newton's second law, $F(x(t)) = m\ddot{x}(t)$, every exact solution requires initial conditions of $x(t_0)$ and $v(t_0)$. Solutions to Newton's second law are known as trajectories.

Considering a spring of constant k, $F(x) = -kx$ yields the differential equation $m\ddot{x} + kx = 0$. The general solution is

$$x(t) = a\cos(\omega t) + b\cos(\omega t),$$

with $\omega = \sqrt{k/m}$ denoting the frequency. The spring is an example of a simple harmonic oscillator.

Conservation of Energy

For a general force function $F(x)$, the kinetic energy is $\frac{1}{2}mv^2$, and the potential energy is

$$V(x) = -\int F(x)dx,$$

meaning $F(x) = -\frac{dV}{dx}$. The total energy is thus found as

$$E(x, v) = \frac{1}{2}mv^2 + V(x).$$

Theorem: Conservation of Energy

If a particle with trajectory $x(t)$ satisfies $m\ddot{x} = F(x)$, then the energy E is conserved.

Proof:

$$\begin{aligned}
\frac{d}{dt}E(x(t), \dot{x}(t)) &= \frac{d}{dt}\left(\frac{1}{2}m(\dot{x}(t))^2 + V(x(t))\right) \\
&= m\dot{x}(t)\ddot{x}(t) + \frac{dV}{dx}\dot{x}(t) \\
&= \dot{x}(t)\left(m\ddot{x}(t) - F(x(t))\right).
\end{aligned}$$

By using the conservation of energy, we can reduce the second order differential equation $F(x) = m\ddot{x}$ to a system of first order differential equations in $x(t)$ and $v(t)$ respectively:

$$\begin{aligned}
\frac{dx}{dt} &= v(t) \\
\frac{dv}{dt} &= \frac{1}{m}F(x(t)).
\end{aligned}$$

If $(x(t), v(t))$ satisfies this set of equations, then $x(t)$ satisfies Newton's second law. We say the set of all possible (x, v) forms the phase space for the particle in $\mathbb{R}^1$.

In phase space, conservation of energy implies that the set of all (x, v) must lie on the level curve of the energy function: $\{\,(x, v) \mid E(x, v) = E(x_0, v_0)\,\}$.

Using the conservation of energy, we find that, though Newton's second law is a second order differential equation in time, it is actually a first order differential equation:

$$\frac{m}{2}\left(\dot{x}(t)\right)^2 + V(x(t) = E(x(t_0), v(t_0))$$
$$\dot{x}(t) = \sqrt{\frac{2(E_0 - V(x(t)))}{m}}$$

Damping

Suppose we also introduce a force that depends on velocity — in the case of a damped simple harmonic oscillator, the equation for force changes from $F = -kx$ to $F = -kx - \gamma\dot{x}$, with $\gamma > 0$. The damping force acts in the opposite direction of velocity, meaning the particle slows down.

The equation of motion is then

$$m\ddot{x} + \gamma\dot{x} + kx = 0.$$

For γ small, the solutions are a sum sines and cosines multiplied by some exponential decay factor, but for γ large, the solutions are only the exponential decay.

Energy Conservation (or lack thereof) in Damped System

Suppose a particle moves along $x(t)$ that satisfies $F(x,\dot{x}) = F_1(x) - \gamma\dot{x}$, with $\frac{dV}{dx} = -F_1(x)$ and $\gamma > 0$. Then,

$$\frac{d}{dt}E(x(t),\dot{x}(t)) = -\gamma\dot{x}(t)^2.$$

Proof:

$$\begin{aligned}\frac{d}{dt}E(x(t),\dot{x}(t)) &= \dot{x}(t)\left(m\ddot{x}(t) - F_1(x(t))\right)\\ &= \dot{x}(t)\left(m\ddot{x}(t) - (m\ddot{x}(t) + \gamma\dot{x}(t))\right)\\ &= -\gamma\dot{x}(t)^2\end{aligned}$$

Motion in $\mathbb{R}^n$

The position of a particle $\mathbf{x} = (x_1, \dots, x_n)$ lends itself to velocity $\mathbf{v} = (v_1, \dots, v_n) = (\dot{x}_1, \dots, \dot{x}_n)$, and $\mathbf{a} = (\ddot{x}_1, \dots, \ddot{x}_n)$. Similar to in $\mathbb{R}^1$, Newton's second law is denoted

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}(t), \dot{\mathbf{x}}(t)).$$

Conservation of Energy in n Dimensions

The energy function

$$E(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\left\|\dot{\mathbf{x}}\right\|^2 + V(\mathbf{x})$$

is only satisfied where $\mathbf{F} = -\nabla V$.

Proof:

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}m\left\|\dot{\mathbf{x}}\right\|^2 + V(\mathbf{x})\right) &= m\sum_{j=1}^{n}\dot{x}_j\ddot{x}_j + \sum_{j=1}^{n}\frac{\partial V}{\partial x_j}\dot{x}_j(t)\\ &= \dot{\mathbf{x}}(t)\left(m\ddot{\mathbf{x}}(t) + \nabla V\right)\\ &= \dot{x}(t)\left(\mathbf{F}(x) + \nabla V(\mathbf{x})\right),\end{aligned}$$

which is equal to zero only if $-\nabla V = \mathbf{F}$.

If $\mathbf{F}$ is a smooth $\mathbb{R}^n$ valued function on $U \subset \mathbb{R}^n$, then $\mathbf{F}$ is conservative if there exists a smooth real-valued function V such that $\mathbf{F} = -\nabla V$.

In other words, $\mathbf{F}$ is conservative if $\mathbf{F}$ is a gradient field, implying that $\nabla \times \mathbf{F} = 0$.

If $\mathbf{F}(\mathbf{x}, \mathbf{y}) = -\nabla V(\mathbf{x}) + \mathbf{F}_2(\mathbf{x}, \mathbf{y})$, with $\mathbf{v} \cdot \mathbf{F}_2 = 0$ for all $\mathbf{x}$ and $\mathbf{v}$, then energy is conserved along a given trajectory.

Systems of Particles

Let $\mathbf{x}^j = \left(x_1^j, x_2^j, \dots, x_n^j\right)$ denote the jth particle of a system of N particles. Newton's second law is thus reformulated as

$$m_j \ddot{\mathbf{x}}^j = \mathbf{F}^j\left(\mathbf{x}^1, \dots, \mathbf{x}^N, \dot{\mathbf{x}}^1, \dots, \dot{\mathbf{x}}^N\right).$$

The total energy is determined by

$$E(\mathbf{x}^1, \dots \mathbf{x}^N, \mathbf{v}^1, \dots, \mathbf{v}^N) = \left(\sum_{j=1}^{N} \frac{1}{2} m_j \left\|\mathbf{v}^j\right\|^2\right) + V(\mathbf{x}^1, \dots, \mathbf{x}^N).$$

Conservation of Energy in a System of Particles

The energy function is constant along each trajectory if $\nabla^j V = -\mathbf{F}^j$, where ∇^j denotes the gradient with respect to $\mathbf{x}^j$.

The force function along a simply connected domain U in $\mathbb{R}^{nN}$ satisfies $\nabla^j V = -\mathbf{F}^j$ if and only if

$$\frac{\partial F_k^j}{\partial x_m^l} = \frac{\partial F_m^l}{\partial x_k^j}$$

for all j, k, l, m.

Proof:

$$\begin{aligned}
\frac{dE}{dt} &= \sum_{j=1}^{N} \left(m_j \dot{\mathbf{x}}^j \cdot \ddot{\mathbf{x}}^j + \nabla^j V \cdot \mathbf{x}^j\right) \\
&= \sum_{j=1}^{N} \dot{\mathbf{x}}^j \left(m_j \ddot{\mathbf{x}}^j + \nabla^j V\right) \\
&= \sum_{j=1}^{N} \dot{\mathbf{x}} \left(\mathbf{F}^j + \nabla^j V\right),
\end{aligned}$$

which is equal to zero if $\nabla^j V = -\mathbf{F}^j$.

Applying a higher dimension version of $\nabla \times \mathbf{F}$ to each coordinate pair (a, b), we find the identity that shows $\mathbf{F}$ is a gradient field.

Momentum of a System of Particles

The momentum of a particle $\mathbf{p}^j$ is defined by

$$\mathbf{p}^j = m_j \dot{\mathbf{x}}^j.$$

Observe that $\frac{d\mathbf{p}^j}{dt} = m_j\ddot{\mathbf{x}}^j = \mathbf{F}^j$. The total momentum is then

$$\mathbf{p} = \sum_{j=1}^{N} \mathbf{p}^j.$$

Newton's third law, which states "for every action there is an equal and opposite reaction" applies if

- $\mathbf{F}^j = \sum_{k\neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{y}^j)$;
- $\mathbf{F}^{j,k}(\mathbf{x}_j, \mathbf{x}_k) = -\mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j)$.

If each $\mathbf{F}^j$ is also a conservative force, then satisfying these conditions yields potential energy in the form of

$$V(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N) = \sum_{j<k} V^{j,k}(\mathbf{x}^j - \mathbf{x}^k).$$

Newton's Third Law and Conservation of Momentum

If the system of particles satisfies the conditions of

- $\mathbf{F}^j = \sum_{k\neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{y}^j)$
- and $\mathbf{F}^{j,k}(\mathbf{x}_j, \mathbf{x}_k) = -\mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j)$,

then total momentum is conserved.

Proof:

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= \sum_{j=1}^{N} \mathbf{F}^{\mathbf{j}}\\ &= \sum_{j=1}^{N}\sum_{k\neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k),\end{aligned}$$

and since $F^{j,k}(\mathbf{x}^j, \mathbf{x}^k) + \mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j) = 0$, we find $\frac{d\mathbf{p}}{dt} = 0$.

Translation Invariance of Potential and Momentum Conservation

Let V denote the potential for a conservative force. Then, momentum is conserved if and only if V is translation invariant, meaning that for all $\mathbf{a} \in \mathbb{R}^n$,

$$V(\mathbf{x}^1 + \mathbf{a}, \mathbf{x}^2 + \mathbf{a}, \dots, \mathbf{x}^N + \mathbf{a}) = V(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N).$$

Proof: Let $\mathbf{a} = t\mathbf{e}_k$. Then, differentiating at $t = 0$ with respect to t, we find

$$\begin{aligned}0 &= \sum_{j=1}^{N} \frac{\partial V}{\partial x_k^j}\\ &= -\sum_{j=1}^{N} F_k^j\\ &= -\sum_{j=1}^{N} \frac{dp_k^j}{dt}\\ &= -\frac{dp_k}{dt},\end{aligned}$$

with p_k denoting the kth component of $\mathbf{p}$. Therefore, $\mathbf{p}$ is constant in time.

If $\mathbf{p}$ is conserved, then the sum of all forces is 0 at each point for all t, meaning that for all t,

$$\begin{aligned}\frac{d}{dt}V(\mathbf{x}^1+t\mathbf{a},\mathbf{x}^2+t\mathbf{a},\dots,\mathbf{x}^N+t\mathbf{a}) &= \sum_{j=1}^{N}\nabla^j V(\mathbf{x}^1+t\mathbf{a},\mathbf{x}^2+t\mathbf{a},\dots,\mathbf{x}^n+t\mathbf{a})\cdot\mathbf{a}\\ &= -\left(\sum_{j=1}^{N}\mathbf{F}^j(\mathbf{x}^1+t\mathbf{a},\mathbf{x}^2+t\mathbf{a},\dots,\mathbf{x}^N+t\mathbf{a})\right)\cdot\mathbf{a}\\ &= 0\end{aligned}$$

meaning V is equal at $t=0$ and $t=1$.

Center of Mass

For a system of N particles, the center of mass is denoted

$$\mathbf{c} = \sum_{j=1}^{N}\frac{m_j}{\sum_{j=1}^{N}m_j}\mathbf{x}_j.$$

We denote $\sum_{j=1}^{N}m_j = M$. Differentiating $\mathbf{c}$, we get

$$\begin{aligned}\frac{d\mathbf{c}}{dt} &= \frac{1}{M}\sum_{j=1}^{N}m_j\dot{\mathbf{x}}^j\\ &= \frac{\mathbf{p}}{M}.\end{aligned}$$

Notice that if $\mathbf{p}$ is conserved, then $\mathbf{c}(t) = \mathbf{c}(t_0) + (t-t_0)\frac{\mathbf{p}}{M}$.

For a system of two particles, if $V(\mathbf{x}^1,\mathbf{x}^2)$ is invariant under translation, then $V(\mathbf{x}^1,\mathbf{x}^2) = \tilde{V}(\mathbf{x}^1-\mathbf{x}^2)$, and $\tilde{V}(\mathbf{a}) = V(\mathbf{a},0)$.

The positions $\mathbf{x}^1$ and $\mathbf{x}^2$ can be recovered from knowledge about $\mathbf{c}$ and the relative position $\mathbf{y} := \mathbf{x}^1-\mathbf{x}^2$:

$$\begin{aligned}\mathbf{x}^1 &= \frac{\mathbf{c}+m_2\mathbf{y}}{m_1+m_2}\\ \mathbf{x}^2 &= \frac{\mathbf{c}-m_1\mathbf{y}}{m_1+m_2}.\end{aligned}$$

Thus, we can calculate

$$\begin{aligned}\ddot{\mathbf{y}} &= \ddot{\mathbf{x}}^1-\ddot{\mathbf{x}}^2\\ &= -\frac{1}{m_1}\nabla\tilde{V}\left(\mathbf{x}^1-\mathbf{x}^2\right) - \frac{1}{m_2}\nabla\tilde{V}\left(\mathbf{x}^1-\mathbf{x}^2\right).\end{aligned}$$

Motion of Relative Position under Translation Invariant Potential

For a two particle system with translation invariant potential, the relative position $\mathbf{y} = \mathbf{x}^1-\mathbf{x}^2$ is a solution to the differential equation

$$\mu\ddot{\mathbf{y}} = -\nabla\tilde{V}(\mathbf{y}),$$

where

$$\mu = \frac{m_1m_2}{m_1+m_2}.$$

This implies that when momentum is conserved, the relative position of the two particle system evolves as a one-particle system with effective mass μ.

Angular Momentum

A particle moving in $\mathbb{R}^2$ with position $\mathbf{x}$, velocity $\mathbf{v}$, and momentum $\mathbf{p} = m\mathbf{v}$ has angular momentum J denoted as

$$J = x_1 p_2 - x_2 p_1,$$

or $J = \|\mathbf{x} \times \mathbf{p}\| = \|\mathbf{x}\| \|\mathbf{p}\| \sin\phi$, with ϕ measured counterclockwise. In polar coordinates, we find

$$\begin{aligned} J &= mr^2 \frac{d\theta}{dt} \\ &= 2M \frac{dA}{dt}, \end{aligned}$$

where $A = 1/2 \int r^2 d\theta$ denotes the area swept out by $\mathbf{x}(t)$.

Conservation of Angular Momentum

Suppose a particle of mass m is moving in $\mathbb{R}^2$ under the influence of a conservative force with potential $V(\mathbf{x})$. V is invariant under rotation if and only if J is conserved.

Proof:

$$\begin{aligned} \frac{dJ}{dt} &= \frac{dx_1}{dt} p_2 + x_1 \frac{dp_2}{dt} - \frac{dx_2}{dt} p_1 - x_2 \frac{dp_1}{dt} \\ &= \frac{1}{m} p_1 p_2 - x_1 \frac{\partial V}{\partial x_2} - \frac{1}{m} p_2 p_1 + x_2 \frac{\partial V}{\partial x_1} \\ &= x_2 \frac{\partial V}{\partial x_1} - x_1 \frac{\partial V}{\partial x_2}. \end{aligned}$$

Alternatively, consider $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Differentiating V along R_θ, we get

$$\begin{aligned} \left.\frac{d}{d\theta} V(R_\theta \mathbf{x})\right|_{\theta=0} &= \frac{\partial V}{\partial x}\frac{dx}{d\theta} + \frac{\partial V}{\partial y}\frac{dy}{d\theta} \\ &= -x_2 \frac{\partial V}{\partial x_1} + x_1 \frac{\partial V}{\partial x_2} \\ &= -\frac{dJ}{dt}(\mathbf{x}) \end{aligned}$$

Thus, $\frac{dJ}{dt} = 0$ if and only if the angular derivative of V is zero.

As a result of the conservation of angular momentum, we thus get Kepler's Second Law: if $\mathbf{x}(t)$ is the trajectory of a particle under the influence of a force with rotationally invariant potential, then the area swept out by $\mathbf{x}(t)$ between $t = a$ and $t = b$ is $\frac{b-a}{2m} J$.

In $\mathbb{R}^3$, $\mathbf{J}$ is a vector given by $\mathbf{x} \times \mathbf{p}$. Meanwhile, in $\mathbb{R}^n$, the angular momentum is a skew-symmetric matrix defined by

$$J_{jk} = x_j p_k - x_k p_j.$$

The total angular momentum of a system of N particles in $\mathbb{R}^n$ is given by $\mathbf{J}$ with entries

$$J_{jk} = \sum_{l=1}^{N} \left(x_j^l p_k^l - x_k^l - p_j^l \right).$$

Similar to the case of linear momentum, angular momentum is constant in the presence of a conservative force if and only if the potential function V is rotationally invariant. That is,

$$V(R\mathbf{x}^1, R\mathbf{x}^2, \dots, R\mathbf{x}^N) = V(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N)$$

for all rotation matrices R.

Hamiltonian Mechanics

The Hamiltonian is the total energy function, but formulated in terms of position and momentum rather than position and velocity. If a particle in $\mathbb{R}^n$ has the usual energy function, we write

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\sum_{j=1}^{n} p_j^2 + V(\mathbf{x}),$$

where $p_j = m_j\dot{x}_j$. Observe that the equations of motion can be written as

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial H}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial x_j}. \end{aligned}$$

In the basic formulation, we can see that the first equation is just $\dot{x}_j = p_j/m$, and $\dot{p}_j = F_j$.The equations of motion written with Hamiltonians are known as Hamilton's equations.

Poisson Bracket

Let f and g be two smooth functions on $\mathbb{R}^{2n}$, with each element of $\mathbb{R}^{2n}$ being denoted by $(\mathbf{x}, \mathbf{p})$. The Poisson bracket of f and g is equal to

$$\{f, g\}(\mathbf{x}, \mathbf{p}) = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}\right).$$

The Poisson bracket satisfies the following properties:

- Linearity: $\{f, g + ch\} = \{f, g\} + c\{f, h\}$
- Antisymmetry: $\{g, f\} = -\{f, g\}$
- Product Rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$
- Jacobi Identity: $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$.

It can be easily verified that the following Poisson bracket relations hold:

$$\begin{aligned} \{x_j, x_k\} &= 0 \\ \{p_j, p_k\} &= 0 \\ \{x_j, p_k\} &= \delta_{jk}, \end{aligned}$$

where δ_{jk} denotes the Kronecker delta function.

Functions of Solutions to Hamilton's Equations

If $(\mathbf{x}(t), \mathbf{p}(t))$ is a solution to Hamilton's Equations, then for any smooth f on $\mathbb{R}^{2n}$, we have

$$\frac{df}{dt} = \{f, h\}.$$

Proof:

$$\begin{aligned} \frac{df}{dt} &= \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{dx_j}{dt} + \frac{\partial f}{\partial p_j}\frac{dp_j}{dt}\right) \\ &= \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{\partial H}{\partial p_j} + \frac{\partial f}{\partial p_j}\left(-\frac{\partial H}{\partial x_j}\right)\right) \\ &= \{f, H\}. \end{aligned}$$

Conserved Quantities

Let $f \in C^1(\mathbb{R}^{2n})$ be called conserved if $f(\mathbf{x}(t), \mathbf{p}(t))$ is independent of t for each solution to Hamilton's equation. Then, f is a conserved quantity if and only if

$$\{f, H\} = 0.$$

Note that H is also a conserved quantity.

Flow and Liouville's Theorem

Solving Hamilton's equations on $\mathbb{R}^{2n}$ yields a flow Φ_t[1] with $\Phi_t(\mathbf{x}, \mathbf{p})$ equal to the solution at time t with initial condition $(\mathbf{x}, \mathbf{p})$.

The Φ_t aren't necessarily defined on all of $\mathbb{R}^{2n}$, but if Φ_t is defined on $\mathbb{R}^{2n}$ for all t, then we say Φ_t is complete.

Liouville's Theorem[2] states that the flow preserves the $2n$-dimensional measure

$$dx_1 dx_2 \cdots dx_n dp_1 dp_2 \cdots dp_n.$$

More specifically, if E is a measurable subset of the domain of Φ_t, then $\mu\left(\Phi_t(E)\right) = \mu(E)$.

Proof: Hamilton's equations can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial x_1} \\ \vdots \\ -\frac{\partial H}{\partial x_n} \end{bmatrix}.$$

Hamilton's equations describe the flow along the vector field appearing on the right side — by a result in vector calculus,[3] the flow preserves the $2n$-dimensional area measure if and only if the divergence of the vector field is zero.

$$\begin{aligned} \nabla \cdot \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial x_1} \\ \vdots \\ -\frac{\partial H}{\partial x_n} \end{bmatrix} &= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_k}\frac{\partial H}{\partial x_k} \\ &= \sum_{k=1}^{n} \frac{\partial^2 H}{\partial x_k \partial p_k} - \frac{\partial^2 H}{\partial p_k \partial x_k} \\ &= 0 \end{aligned}$$

The condition of zero divergence is equivalent to Φ_t preserving a particular symplectic form ω defined by

$$\omega\left((\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}')\right) = \mathbf{x} \cdot p' - \mathbf{p} \cdot x',$$

meaning that for any t and any $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n}$, the partial derivatives of Φ_t preserves ω.

[1] the Φ_t are diffeomorphisms, or differentiable isomorphisms with differentiable inverses
[2] not the one from complex analysis
[3] Author's Note: I do not know this result yet, but hopefully I will soon!

Alternatively, this is equivalent to Φ_t preserving Poisson brackets:

$$\{ f \circ \Phi_t, g \circ \Phi_t \} = \{ f, g \} \circ \Phi_t.$$

Thus, Φ_t is an example of a symplectomorphism.

Hamiltonian Flow and Hamiltonian Generators

We say $f \in C^1(\mathbb{R}^{2n})$ is the Hamiltonian generator of the flow that results from solving Hamilton's equations with f substituted for H:

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial f}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial f}{\partial x_j}. \end{aligned}$$

It is possible to see that

$$f_{\mathbf{a}}(\mathbf{x}, \mathbf{p}) = \mathbf{a} \cdot \mathbf{p}$$

yields the flow

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_0 + t\mathbf{a} \\ \mathbf{p}(t) &= \mathbf{p}_0, \end{aligned}$$

and

$$g_{\mathbf{b}}(\mathbf{x}, \mathbf{p}) = \mathbf{b} \cdot \mathbf{x}$$

yields the flow

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_0 \\ \mathbf{p}(t) &= \mathbf{p}_0 - t\mathbf{b}. \end{aligned}$$

Thus, the Hamiltonian flow generated by momentum yields translation in position, and the Hamiltonian flow generated by position yields translation in momentum.

In this light, we can think of *the* Hamiltonian as the Hamiltonian generator that yields time evolution.Other Hamiltonian generators represent some other family of symmetries of the system.

Hamiltonian Flow generated by Angular Momentum

For a particle moving in $\mathbb{R}^2$, the Hamiltonian flow generated by

$$J(\mathbf{x}, \mathbf{p}) = x_1 p_2 - x_2 p_1$$

consists of simultaneous rotations of $\mathbf{x}$ and $\mathbf{p}$.

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}. \end{aligned}$$

Proof: Plugging J Hamilton's equations, we get

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{\partial J}{\partial p_1} = -x_2\\
\frac{dp_1}{dt} &= -\frac{\partial J}{\partial x_1} = -p_2\\
\frac{dx_2}{dt} &= \frac{\partial J}{\partial p_2} = x_1\\
\frac{dp_2}{dt} &= -\frac{\partial J}{\partial x_2} = p_1.
\end{aligned}$$

It's important to note that the parameter t in the Hamiltonian flow for J is the rotation, not time. That is, J is the Hamiltonian generator of rotations.

If f is any smooth function, it is the case that the time derivative of any other function g along the Hamiltonian flow generated by f is $\frac{dg}{dt} = \{g, f\}$. In particular, the derivative of H along the flow generated by f is $\{H, f\}$, meaning that f is constant along the flow generated by H if and only if $\{f, H\} = 0$, which is true if and only if H is constant along the flow generated by H.

Thus, we find that f is conserved for solutions of Hamilton's equations if and only if H is invariant under the Hamiltonian flow generated by f. Of particular note, we find that J is conserved if and only if H is invariant under rotations of $\mathbf{x}$ and $\mathbf{p}$.

Kepler's Problem

Consider an orbit, where the sun with mass M exerts a force $\mathbf{F}$ on a planet with mass m. Then, by Newton's universal law of gravitation, the force is found by

$$\mathbf{F} = -GmM\frac{\mathbf{x}}{\|\mathbf{x}\|^3},$$

with G equal to the gravitational constant. We denote $k = GmM$, and find that in Newton's second law,

$$\begin{aligned}
m\ddot{\mathbf{x}} &= -GmM\frac{\mathbf{x}}{\|\mathbf{x}\|^3}\\
\ddot{\mathbf{x}} &= -GM\frac{\mathbf{x}}{\|\mathbf{x}\|^3}.
\end{aligned}$$

The potential associated with $\mathbf{F}$ is

$$V(\mathbf{x}) = -\frac{k}{\|\mathbf{x}\|}.$$

Since V is invariant under rotations, $\mathbf{J} = \mathbf{x} \times \mathbf{p}$ will always be constant and perpendicular to $\mathbf{x}(t)$. We call the plane perpendicular to $\mathbf{J}$ the plane of motion.

Runge–Lenz Vector

The Runge–Lenz vector is yet another conserved quantity for the orbit. We define the Runge–Lenz vector on $\mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^3$ by

$$\mathbf{A}(\mathbf{x}, \mathbf{p}) = \frac{1}{mk}\mathbf{p} \times \mathbf{J} - \frac{\mathbf{x}}{\|\mathbf{x}\|},$$

where $\mathbf{x}$ represents position and $\mathbf{p}$ represents momentum. Recall that $k = GmM$.

Proof:

$$\begin{aligned}\dot{\mathbf{A}}(t) &= \frac{1}{mk}\mathbf{F}\times\mathbf{J} - \frac{1}{\|\mathbf{x}\|}\frac{\mathbf{p}}{m} + \frac{\mathbf{x}}{\|\mathbf{x}\|^2}\sum_{j=1}^{3}\frac{\partial\|\mathbf{x}\|}{\partial x_j}\frac{dx_j}{dt}\\ &= -\frac{1}{m}\|\mathbf{x}\|^3\mathbf{x}\times(\mathbf{x}\times\mathbf{p}) - \frac{1}{\|\mathbf{x}\|}\frac{\mathbf{p}}{m} + \frac{\mathbf{x}}{\|\mathbf{x}\|^2}\sum_{j=1}^{3}\frac{x_j}{\|\mathbf{x}\|}\frac{p_j}{m}\\ &= \frac{1}{m}\left(-\frac{1}{\|\mathbf{x}\|^3}\mathbf{x}(\mathbf{x}\cdot\mathbf{p}) + \frac{1}{\|\mathbf{x}\|^3}\mathbf{p}(\mathbf{x}\cdot\mathbf{x}) - \frac{\mathbf{p}}{\|\mathbf{x}\|} + \frac{(\mathbf{x}\cdot\mathbf{p})}{\|\mathbf{x}\|^3}\right)\\ &= 0\end{aligned}$$

Trajectories for the Kepler Problem

The magnitude of the Runge–Lenz vector $\mathbf{A}$ is found by

$$\|\mathbf{A}\|^2 = 1 + \frac{2\|\mathbf{J}\|^2}{mk^2}E,$$

where $E = \dfrac{\|\mathbf{p}\|^2}{2m} - \dfrac{k}{\|\mathbf{x}\|}$.

Additionally, if $\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, then

$$\mathbf{A}\cdot\hat{\mathbf{x}} = \frac{\|\mathbf{J}\|^2}{mk\|\mathbf{x}\|} - 1$$

for all nonzero $\mathbf{x}$. Thus,

$$\|\mathbf{x}\| = \frac{\|\mathbf{J}\|^2}{mk(1+\mathbf{A}\cdot\hat{\mathbf{x}})}.$$

Introduction to Quantum Mechanics

Observable quantities such as position and momentum in quantum mechanics are represented by operators on a complex-valued Hilbert space (an inner product space that is complete with respect to the induced metric) — specifically, these quantities are *unbounded* linear operators.

In physics, the inner product is linear in the second factor and conjugate linear in the first factor:

$$\begin{aligned}\langle\phi,\lambda\psi\rangle &= \lambda\langle\phi,\psi\rangle\\ \langle\lambda\phi,\psi\rangle &= \overline{\lambda}\langle\phi,\psi\rangle.\end{aligned}$$

Alternatively, in Dirac notation:

$$\begin{aligned}\langle\phi\,|\,\lambda\psi\rangle &= \lambda\langle\phi\,|\,\psi\rangle\\ \langle\lambda\phi\,|\,\psi\rangle &= \overline{\lambda}\langle\phi\,|\,\psi\rangle.\end{aligned}$$

A Taste of Operator Theory

A linear operator $A:\mathbf{H}\to\mathbf{H}$ is bounded if it has finite operator norm:[4]

$$\sup_{\|\psi\|\leq 1}\|A\psi\| < \infty.$$

[4] I'm using more operator-theoretic language than the book uses because I'm ~~pretentious~~ a mathematician, not a physicist.

For each bounded operator A, there exists a unique bounded operator A^* such that $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$. The existence of A^* follows from the Riesz representation theorem. A bounded operator is said to be self-adjoint if $A^* = A$.

Self-adjoint operators are nice for a variety of reasons, and as a result we desire for our operators in quantum mechanics to be self-adjoint. However, this brings a significant problem — unbounded self-adjoint operators are not necessarily defined on $\mathbf{H}$.

For this case, we define unbounded operators as linear operators defined on a dense subspace of $\mathbf{H}$:

$$A : \text{Dom}(A) \subseteq \mathbf{H} \to \mathbf{H}$$

subject to

$$\overline{\text{Dom}(A)} = \mathbf{H}.$$

In addition to the domain of A not necessarily being equal to $\mathbf{H}$, the linear functional $\langle \phi, A\cdot \rangle$ is not necessarily bounded (meaning we cannot use the Riesz representation theorem to find $A^*\phi$). The adjoint of A, as a result, will be defined on a subspace of $\mathbf{H}$.

A vector $\phi \in \mathbf{H}$ is said to belong to the domain $\text{Dom}(A^*)$ if the linear functional $\langle \phi, A\cdot \rangle$ on $\text{Dom}(A)$ is bounded. Then, we define A^* to be the unique vector χ such that $\langle \chi, \psi \rangle = \langle \phi, A\psi \rangle$ for all $\psi \in \text{Dom}(A)$.

Having defined adjoints of an unbounded operator, we can now commit to defining self-adjoint operators. The operator A is symmetric if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$, and is self-adjoint if $\text{Dom}(A) = \text{Dom}(A^*)$ and $A^*\phi = A\phi$ for all $\phi \in \text{Dom}(A)$. Finally, A is essentially self-adjoint if the closure of the graph of A in $\mathbf{H} \times \mathbf{H}$ is self-adjoint.

Essentially, A is self-adjoint if A and A^* are the same operator with the same domain.