

Problem (Problem 1): Let F be a field, and for $n \geq 1$, let $\text{Mat}_n(F)$ be the set of $n \times n$ matrices with entries in F .

- (a) Show that $\text{GL}_n(F) := \{x \in \text{Mat}_n(F) \mid \det(x) \neq 0\}$ is a group under matrix multiplication.
- (b) Show that $\text{SL}_n(F) := \{x \in \text{Mat}_n(F) \mid \det(x) = 1\}$ is a normal subgroup of $\text{GL}_n(F)$, and identify the quotient $\text{GL}_n(F)/\text{SL}_n(F)$.

Solution:

- (a) We see that if $a, b \in \text{GL}_n(F)$, then since $\det(a) \neq 0$, the properties of the determinant yield $0 \neq \det(a)^{-1} = \det(a^{-1})$, meaning that $a^{-1} \in \text{GL}_n(F)$, and $0 \neq \det(a)\det(b) = \det(ab)$, meaning that $ab \in \text{GL}_n(F)$, since fields have no zero-divisors.
- (b) If $a \in \text{SL}_n(F)$, then for any $x \in \text{GL}_n(F)$, we have

$$\begin{aligned} \det(xax^{-1}) &= \det(x)\det(a)\det(x^{-1}) \\ &= \det(x)\det(a)\det(x)^{-1} \\ &= \det(a) \\ &= 1, \end{aligned}$$

meaning that $xax^{-1} \in \text{SL}_n(F)$ for any $x \in \text{GL}_n(F)$. In particular, we note that the map

$$\det: \text{GL}_n(F) \rightarrow F \setminus \{0\},$$

given by $a \mapsto \det(a)$ is a group homomorphism, as has been established by the properties of the determinant, and it is surjective, as the matrix $\text{diag}(a, 1_F, \dots, 1_F)$ has determinant a , for any $a \in F$. Finally, we see that $\det^{-1}(\{1_F\})$ is $\text{SL}_n(F)$, meaning that by the First Isomorphism Theorem, $\text{GL}_n(F)/\text{SL}_n(F) \cong F \setminus \{0\}$.

Problem (Problem 3): Let G be a group, and let $H_1, H_2 \leq G$ be subgroups.

- (a) Show that if H_1 and H_2 are finite, with $\gcd(|H_1|, |H_2|) = 1$, then $H_1 \cap H_2 = \{e\}$.
- (b) Show that if both H_1 and H_2 are normal subgroups, and $H_1 \cap H_2 = \{e\}$, then $h_1h_2 = h_2h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$.

Solution:

- (a) Let $g \in H_1 \cap H_2$. Then, we see that $\text{ord}(g) \mid |H_1|$ and $\text{ord}(g) \mid |H_2|$; yet, since $\gcd(|H_1|, |H_2|) = 1$, this means that $\text{ord}(g) = 1$, meaning $g = \{e\}$.
- (b) If H_1 and H_2 are normal subgroups, then for $h_1 \in H_1$ and $h_2 \in H_2$, we consider the commutator $c = h_1h_2h_1^{-1}h_2^{-1}$. Notice that by grouping as $(h_1h_2h_1^{-1})h_2^{-1}$, since H_2 is a normal subgroup, $c \in H_2$. Similarly, by grouping as $h_1(h_2h_1^{-1}h_2^{-1})$, since H_1 is normal, we see that $c \in H_1$. Since $H_1 \cap H_2 = \{e\}$, we see that $h_1h_2h_1^{-1}h_2^{-1} = e$, so $h_1h_2 = h_2h_1$.