

# Alternating Series and Conditional Convergence

Avinash Iyer

Occidental College

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Recap

Consider the following series:

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Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

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Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1$$

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Recap

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2}$$

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Recap

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3}$$



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Recap

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

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Recap

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

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Recap

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

This series appears to be related to the harmonic series, but also very different:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Harmonic Series

# Divergence of the Harmonic Series

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Recap

We can show that the harmonic series is divergent using the series comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

# Divergence of the Harmonic Series

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Recap

We can show that the harmonic series is divergent using the series comparison test:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots\end{aligned}$$

# Divergence of the Harmonic Series

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Recap

We can show that the harmonic series is divergent using the series comparison test:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\end{aligned}$$

# Divergence of the Harmonic Series

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Recap

We can show that the harmonic series is divergent using the series comparison test:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\ &= \infty\end{aligned}$$

# Differences

However, our alternating harmonic series is different. Let's look at partial sums.

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Recap

However, our alternating harmonic series is different. Let's look at partial sums.

$$s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$$

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Recap

However, our alternating harmonic series is different. Let's look at partial sums.

$$s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$$

$$s_1 = 1$$

# Differences

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Recap

However, our alternating harmonic series is different. Let's look at partial sums.

$$s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$$

$$s_1 = 1$$

$$s_2 = \frac{1}{2}$$

# Differences

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Recap

However, our alternating harmonic series is different. Let's look at partial sums.

$$s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$$

$$s_1 = 1$$

$$s_2 = \frac{1}{2}$$

$$s_3 = \frac{5}{6}$$

# Differences

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Recap

However, our alternating harmonic series is different. Let's look at partial sums.

$$s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$$

$$s_1 = 1$$

$$s_2 = \frac{1}{2}$$

$$s_3 = \frac{5}{6}$$

$$s_4 = \frac{7}{12}$$

$\vdots$

# Convergence?

Clearly, this sequence does not grow without bound — it is bounded above by 1, and doesn't seem to dip below  $\frac{1}{2}$ .

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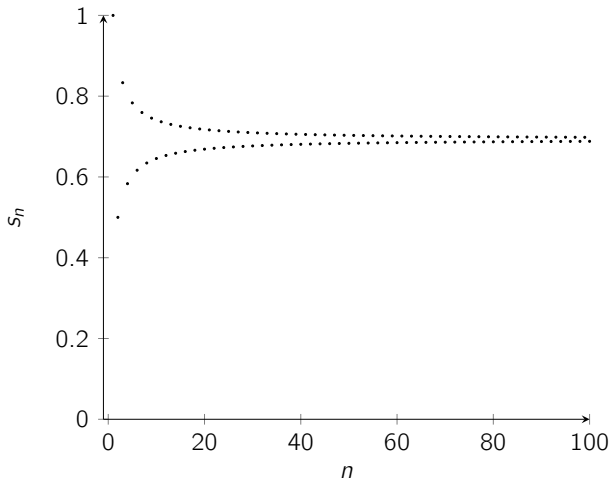
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# Convergence?

Clearly, this sequence does not grow without bound — it is bounded above by 1, and doesn't seem to dip below  $\frac{1}{2}$ .



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# Convergence? cont'd

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Recap

The alternating harmonic does converge. Courtesy of Wolfram MathWorld, we know that the series converges to the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$



# Convergence? cont'd

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Recap

The alternating harmonic does converge. Courtesy of Wolfram MathWorld, we know that the series converges to the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

...or does it?

# Rearranging the Alternating Harmonic Series

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Recap

Rearrange the series as follows:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \dots$$

# Rearranging the Alternating Harmonic Series

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Recap

Rearrange the series as follows:

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \dots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots \end{aligned}$$

# Rearranging the Alternating Harmonic Series

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Recap

Rearrange the series as follows:

$$\begin{aligned}1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \dots \\&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots \\&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \\&= \frac{1}{2} \ln 2\end{aligned}$$

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Recap

- We saw that our alternating harmonic series converges to  $\ln 2$ , but should it not converge to  $\ln 2$  all the time?

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Recap

- We saw that our alternating harmonic series converges to  $\ln 2$ , but should it not converge to  $\ln 2$  all the time?
- For example, no matter how we arrange

$$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

The sum should always equal 1.

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Recap

- We saw that our alternating harmonic series converges to  $\ln 2$ , but should it not converge to  $\ln 2$  all the time?
- For example, no matter how we arrange

$$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

The sum should always equal 1.

- Maybe we should redefine convergence?



# Alternating Series

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Recap

- The answer is that the alternating harmonic series is *conditionally* convergent.

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Recap

- The answer is that the alternating harmonic series is *conditionally* convergent.
- We can always rearrange the terms of the alternating harmonic series to form whatever sum we want.

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Recap

- The answer is that the alternating harmonic series is *conditionally* convergent.
- We can always rearrange the terms of the alternating harmonic series to form whatever sum we want.
- In general, alternating series, of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

can be convergent, while at the same time

$$\sum_{n=1}^{\infty} a_n$$

is divergent.

# Alternating Series Test

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Recap

- In general, we can find if an alternating series is *conditionally* convergent as follows:

# Alternating Series Test

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Recap

- In general, we can find if an alternating series is *conditionally* convergent as follows:
  - The (absolute value) series terms are strictly positive and decreasing.

$$0 < a_{n+1} < a_n$$

# Alternating Series Test

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Recap

- In general, we can find if an alternating series is *conditionally* convergent as follows:
  - The (absolute value) series terms are strictly positive and decreasing.

$$0 < a_{n+1} < a_n$$

- The series terms tend to zero:

$$\lim_{n \rightarrow \infty} a_n = 0$$

# Applying the Alternating Series Test

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Recap

In the alternating harmonic series, we see that

# Applying the Alternating Series Test

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Recap

In the alternating harmonic series, we see that

$$0 < \frac{1}{n+1} < \frac{1}{n},$$



# Applying the Alternating Series Test

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Recap

In the alternating harmonic series, we see that

$$0 < \frac{1}{n+1} < \frac{1}{n},$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

# Applying the Alternating Series Test

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Recap

In the alternating harmonic series, we see that

$$0 < \frac{1}{n+1} < \frac{1}{n},$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

So the series is *conditionally* convergent.

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# What is Absolute Convergence?

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We know two facts:

# What is Absolute Convergence?

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Recap

We know two facts:

- The alternating harmonic series converges conditionally
- The harmonic series diverges

# What is Absolute Convergence?

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Recap

We know two facts:

- The alternating harmonic series converges conditionally
- The harmonic series diverges

We need a stronger term for series convergence — *absolute* convergence — when a series converges to a single value.

# Finding Absolute Convergence

If the absolute value of the terms in the series converges, then the series converges absolutely.

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Recap

# Finding Absolute Convergence

If the absolute value of the terms in the series converges, then the series converges absolutely.

## Absolutely Convergent Alternating Series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely. Why?



# Finding Absolute Convergence

If the absolute value of the terms in the series converges, then the series converges absolutely.

## Absolutely Convergent Alternating Series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely. Why?

By the geometric series,

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2^n} \right| = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

converges.

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# What We Have Learned

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- The same series can converge to different values depending on the arrangement of terms — known as *conditional convergence*

# What We Have Learned

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Recap

- The same series can converge to different values depending on the arrangement of terms — known as *conditional convergence*
- We can use the *alternating series test* to find if a series converges conditionally.

# What We Have Learned

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Recap

- The same series can converge to different values depending on the arrangement of terms — known as *conditional convergence*
- We can use the *alternating series test* to find if a series converges conditionally.
- However, we would need to use other tools to find if a series is absolutely convergent.

# Questions?

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Thank you for listening. Any questions?