Normed Vector Spaces

Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **vector space** over $\mathbb{F}$ is a nonempty set V equipped with two operations: vector addition and scalar multiplication.

$$\begin{aligned} V \times V &\xrightarrow{+} V \\ (v, w) &\mapsto v + w && \text{Vector Addition} \\ F \times V &\to V \\ (\alpha, v) &\mapsto \alpha v && \text{Scalar Multiplication} \end{aligned}$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

(i) $u + (v + w) = (u + v) + w$

(ii) $\exists 0_v \in V$ with $\forall v \in V,\ 0_v + v = v + 0_v = v$

(iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$

(iv) $\forall v, w \in V,\ v + w = w + v$

(v) $\alpha(v + w) = \alpha v + \alpha w,\ (\alpha + \beta)v = \alpha v + \beta v$

(vi) $\alpha(\beta w) = (\alpha\beta)w$

(vii) $1 \cdot v = v$

Remarks:

(a) 0_v is unique and known as the zero vector.

(b) The vector w in (iii) is unique, and denoted $-v$.

(c) $0 \cdot v = 0_v$

(d) $(-1) \cdot v = -v$

(e) Property (iv) follows from all the other axioms.

(f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

Subspaces

Let V be a vector space over $\mathbb{F}$. A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

(i) $w \in W, \alpha \in \mathbb{F} \to \alpha w \in W$.

(ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

Remark: 0_v is always a member of any subspace; a subspace is also a vector space.

Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of V, then, $\bigcap W_i$ is a subspace of V.

Proposition: Union of Subspaces

It is not the case that the union of subspaces of V also a subspace. For example, consider $\mathbb{R}^2$ with the traditional vector space operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$$
$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

If $W_1, W_2 \in V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Generated Subspaces

Let $S \subseteq V$ be any subset of a vector space V. Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

Remarks:

- $\text{span}(S) \subseteq V$ is a subspace.
- $\text{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\text{span}(S)$ is the "smallest" subspace containing S, or the subspace generated by S.

Proposition: Quotient Group on Vector Space

Let V be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

(1) $\sim_W$ is an equivalence relation.

(2) If $[v]_W$ denotes the equivalence class of v, then $[v]_W = v + W = \{v + w | w \in W\}$.

(3) $V/W := \{[v]_W | v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

Proof of (1):

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

Proof of (2):

$$\begin{aligned}
[v]_W &= \{u \in V \mid u \sim_W v\} \\
&= \{u \in V \mid u - v \in W\} \\
&= \{u \in V \mid u = v + w \text{ some } w \in W\} \\
&= \{v + w \mid w \in W\} \\
&= v + W
\end{aligned}$$

Proof of (3): Prove that the operation is well-defined.

Bases

Let V be a vector space and $S \subseteq V$ be a subset.

(1) S is said to be spanning for V if $\text{span}(S) = V$.

(2) S is linearly independent if, for $\sum_{j=1}^{n} \alpha_j v_j = 0_v$ with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

(3) S is a basis for V if S is linearly independent and spanning for V.

Proposition: Existence of Basis

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that B is a basis and $B \supseteq B_0$.

Background: A relation on a set X is a subset $R \subseteq X \times X$. If R is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \to x \sim z$), and antisymmetric ($x \sim y, y \sim x \to x = y$), then R is an ordering, and we write $x \leq y$.

If $\leq$ is an ordering of X such that $\forall x, y \in X,\ x \leq y$ or $y \leq x$, then $\leq$ is a total (or linear) ordering.

Let $\leq$ be an ordering of X, let $Y \subseteq X$. An upper bound for Y is an element $u \in X$ such that $y \leq u\ \forall y \in Y$. A maximal element in X is an element $m \in X$ such that $x \in X,\ x \geq m \to x = m$.

Example: $\mathbb{N}$ under the division ordering defines $a \leq b \Leftrightarrow a | b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of A can be divided by 9 and 12). Meanwhile, $\mathbb{N}$ itself has no maximal elements.

This leads us to ask: given an ordered set, $(X, \leq)$, does X admit maximal elements.

Zorn's Lemma (or Axiom): Let $(X, \leq)$ be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in X. Then, X admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

Proof: Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with D linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that X has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i\in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \dots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \dots, v_n \in D_j$ because Y is totally ordered. However, by definition, D_j is a linearly independent set — therefore, $\alpha_k = 0$. Thus, D is linearly independent.

Since D is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. D is also an upper bound for Y. So, by Zorn's Lemma, X has a maximal element, B.

So, $B_0 \subseteq B \subseteq V$, B is independent, and B is maximal in X. We claim that B is a basis for V. Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and B' is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \dots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.
- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \text{span}(B)$. $\perp$

Thus, we have a linearly independent set, B', with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of B. Therefore, $\text{span}(B) = V$, and B is a basis for V.

Examples: Vector Spaces

(1) n-Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y+n \end{pmatrix} = \begin{pmatrix} x_1 + y + 1 \\ \vdots \\ x_n + y + n \end{pmatrix}$$
$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$
$$B = \{e_1, \dots, e_n\}$$

where e_i denotes the unit vector at position i.

(2) $m \times n$ Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$
$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$
$$\alpha(a_{ij}) = (\alpha a_{ij})$$
$$B = \{e_{ij}\}$$

where e_{ij} denotes a matrix of 0 everywhere except column i and row j.

(3) Functions with domain Ω:

$$\begin{aligned}\mathcal{F}(\Omega,\mathbb{F}) &= \{f \mid f:\Omega\to\mathbb{F}\}\\ (f+g)(x) &= f(x)+g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

(4) Bounded functions with domain Ω:

$$\begin{aligned}\ell_\infty(\Omega,\mathbb{F}) &= \{f\in\mathcal{F}(\Omega,\mathbb{F}) \mid \|f\|_u \leq \infty\}\\ \|f\|_u &= \sup_{x\in\Omega}|f(x)|\end{aligned}$$

Exercises:

- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha|\|f\|_u$
- Positive Definite: $\|f\|_u = 0 \Rightarrow f = \mathbb{0}$

Proof of Triangle Inequality: Given $x\in\Omega$,

$$\begin{aligned}|(f+g)(x)| &= |f(x)+g(x)|\\ &\leq |f(x)|+|g(x)|\\ &\leq \|f\|_u+\|g\|_u\end{aligned}$$

Therefore,

$$\begin{aligned}\sup|(f+g)(x)| &\leq \|f\|_u+\|g\|_u\\ \|f+g\|_u &\leq \|f\|_u+\|g\|_u\end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a,b],\mathbb{F}) = \{f:[a,b]\to\mathbb{F} \mid f \text{ continuous}\}$$

Check that $C([a,b],\mathbb{F})\subseteq\ell_\infty([a,b],\mathbb{F})$ is a subspace.

(6) Let $f:[a,b]\to\mathbb{R}$ be any function. Let $\mathcal{P}: a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

$$\begin{aligned}\operatorname{var}(f;\mathcal{P}) &:= \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\\ \operatorname{var}(f) &= \sup_{\mathcal{P}}\operatorname{var}(f;\mathcal{P})\\ \mathrm{BV}([a,b]) &= \{f:[a,b]\to\mathbb{R} \mid \operatorname{var}(f)<\infty\}\\ \|f\|_{\mathrm{BV}} &= |f(a)|+\operatorname{var}(f)\end{aligned}$$

$\mathrm{BV}([a,b])$ is a vector space.

Question: Is $\mathbb{1}_{\mathbb{Q}}\in\mathrm{BV}([0,1])$?

(7) Suppose $K\subseteq V$ is a *convex* subset of a vector space: $v,w\in K, t\in[0,1]\Rightarrow(1-t)v+tw\in K$. Let $\mathrm{Aff}(K) = \{f:K\to\mathbb{R} \mid f \text{ is affine}\}$, where f is affine if $\forall v,w\in K, t\in[0,1], f((1-t)v+tw) = (1-t)f(v)+tf(w)$.

Exercise: Show that $\mathrm{Aff}(K)\subseteq\mathcal{F}(K,\mathbb{R})$ is a subspace.

(8) Let S be defined as

$$S = \{(a_k)_{k=1}^{\infty} \mid a_k\in\mathbb{F}\}.$$

Under pointwise operations, S is a vector space.

$$\begin{aligned}(a_k)_k+(b_k)_k &= (a_k+b_k)_k\\ \alpha(a_k)_k &= (\alpha a_k)_k\end{aligned}$$

Note 1: $S = \mathcal{F}(\mathbb{N},\mathbb{F})$.

Note 2: $c_{00}\subseteq\ell_1\subseteq c_0\subseteq c\subseteq\ell_\infty\subseteq S$.

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k\neq 0\}$
- $c_0 = \{(a_k)k \mid (a_k)_k\to 0\}$

- $c = \{(a_k)_k \mid (a_k)_k \to a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_u < \infty\}$
- $\ell_1 = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k| = a < \infty\right\}$

(9) $C_C(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \to \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ continuous}, \lim_{x\to\pm\infty} f(x) = 0\}$

(10) Let S be any non-empty set.

$$\begin{aligned} \mathbb{F}(S) &:= \{f : S \to \mathbb{F} \mid f \text{ finitely supported}\} \\ \operatorname{supp}(f) &= \{x \in S \mid f(x) \neq 0\} \end{aligned}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \to \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t\in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\operatorname{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^{n} f(t_k)e_{t_k} \in \operatorname{span}(\xi)$. Therefore, ξ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^{n} \alpha_{t_k} e_{t_k} = \mathbb{0}$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\begin{aligned} \left(\sum_{k=1}^{\alpha_{t_k}} e_{t_k}\right) &= \mathbb{0}(t_1) \\ \alpha_{t_1} &= 0. \end{aligned}$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \dots, n$. Therefore, ξ is linearly independent. Since ξ is linearly independent and spanning, ξ forms a basis for $\mathbb{F}(S)$.

Note: The free vector space, $\mathbb{F}(S)$, displays the universal property.

> There are functions $\iota : S \to \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \to V$ for V a vector space over $\mathbb{F}$, $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \to V$ such that $\iota \circ T_\varphi = \varphi$.

Proof: Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^{n} f(t_k)e_{t_k}$, where $\operatorname{supp}(f) = \{t_1, \dots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^{n} f(t_k)\varphi(t_k)$$

Exercise: Show T_φ is linear and unique.

Exercise 2: Suppose V is a vector space over $\mathbb{F}$ with basis B. Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists\, T : V \to W$ such that T is bijective and linear.

Normed Spaces

To every vector $v \in V$, we want to assign a length to v, $\|v\|$.

A **norm** on a vector space V is a map

$$\begin{aligned} \|\cdot\| &: V \to \mathbb{R}^+ \\ v &\mapsto \|v\| \geq 0 \end{aligned}$$

such that

(i) Homogeneity: $\|\alpha v\| = |\alpha|\|v\|$

(ii) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

(iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = \mathbb{0}_V$.

If $p : V \to \mathbb{R}^+$ satisfies (i) and (ii), then p is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\begin{aligned}\|v\| &\leq c_1\|v\|'\\ \|v\|' &\leq c_2\|v\|\end{aligned}$$

Note: On $\mathbb{R}^n$, all norms are equivalent.

Exercise: If p is any seminorm on V, then $|p(v) - p(w)| \leq p(v - w)$.

Notation: If V is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.

Examples of Normed Spaces

(1) Given $V = \mathbb{F}^n$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have different norms:

$$\begin{aligned}\|x\|_1 &= \sum_{j=1}^{n} |x_j|\\ \|x\|_\infty &= \max_{1\leq j\leq n} |x_j|\\ \|x\|_2 &= \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}.\end{aligned}$$

In general, for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}.$$

Exercise: Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

We want to show that $\|\cdot\|_p$ defines a norm for $1 \leq p < \infty$. If $1 \leq p < \infty$, its conjugate index $q \in [1, \infty]$ whereby $\frac{1}{p} + \frac{1}{q} = 1$. For example, if $p = 1$, then $q = \infty$, and if $p = \infty$, then $q = 1$.

Lemma 1: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $f : [0, \infty) \to \mathbb{R}$, $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then, $f(t) \geq 0$ for all $t \geq 0$.

Proof 1: We can see that $f'(t) = t^{p-1} - 1$. Then, $f'(t) = 0$ at $t = 1$; $f'(t) > 0$ for $t > 1$ and $f'(t) < 0$ for $t \in [0, 1)$.

So, since $f(t) \geq f(1)$ for all $t \geq 0$, and $f(1) = 0$, $f(t) \geq 0$ for all $t \geq 0$.

Lemma 2: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $z, y \geq 0$, $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

Proof 2: We know from Lemma 1, $t \leq \frac{1}{p}t^p + \frac{1}{q}$. Multiply by y^q to get

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Set $t = xy^{1-q}$. Then,

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq} y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $p - pq = -q$, so

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

With these two lemmas in mind, we get two important inequalities.

Hölder's Inequality: For $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. Then, for $x, y \in \mathbb{F}^n$,

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \|x\|_p \|y\|_q.$$

Proof of Hölder's Inequality: For $p = 1$, the solution is as follows:

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j y_j\right| &\leq \sum_{j=1}^{n} |x_j||y_j| \\ &\leq \sum_{j=1}^{n} |x_j| \|y\|_\infty \\ &= \|x\|_q \|y\|_\infty,\end{aligned}$$

and similarly for $p = \infty, q = 1$.

For $1 < p < \infty$, assume $\|x\|_p = \|y\|_q = 1$.

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j y_j\right| &\leq \sum_{j=1}^{\infty} |x_j||y_j| \\ &\leq \sum_{j=1}^{n} \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q\right) \\ &= \frac{1}{p}\left(\sum_{j=1}^{n} |x_j|^p\right) + \frac{1}{q}\left(\sum_{j=1}^{n} |y_j|^q\right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1.\end{aligned}$$

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = \mathbb{0}_{\mathbb{F}}$ or $y = \mathbb{0}_{\mathbb{F}}$, the inequality still holds.

Assume $\|x\|_p \neq 0$, $\|y\|_p \neq 0$. Set

$$\begin{aligned}x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_p}.\end{aligned}$$

It can be verified that $\|x'\|_p = 1 = \|y'\|_q$. Therefore,

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j' y_j'\right| &\leq 1 \\ \left|\sum_{j=1}^{n} \frac{x_j}{\|x\|_p} \frac{y_j}{\|y\|_q}\right| &\leq 1 \\ \left|\sum_{j=1}^{n} x_j y_j\right| &\leq \|x\|_p \|y\|_q\end{aligned}$$

Minkowski's Inequality: Given $x, y \in \mathbb{F}^n$, $1 \leq p \leq \infty$, $\frac{1}{p} = \frac{1}{q} = 1$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Proof of Minkowski's Inequality: We can verify for $p = 1, q = \infty$, and vice versa.

Assume $1 < p < \infty$. Then,

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{j=1}^{n} |x_j + y_j|^p \\
&= \sum_{j=1}^{\infty} |x_j + y_j||x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^{\infty} |x_j||x_j + y_j|^{p-1} + \sum_{j=1}^{n} |y_j||x_j + y_j|^{p-1} \\
&\leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} && \text{Hölder's Inequality} \\
&= \|x\|_p \|x+y\|_p^{p/q} + \|y\|_p \|x+y\|_p^{p/q} \\
&= \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p-1}
\end{aligned}$$

Divide by $\|x+y\|_p^{p-1}$ to get desired inequality.

(2) $\ell_\infty(\Omega, \mathbb{F})$ with $\|\cdot\|_u$. This includes subspaces that inherit the norm, such as

$$\begin{aligned}
C([a,b]) &\subseteq \ell_\infty(\Omega) \\
\ell_\infty(\mathbb{R}) \supseteq C_0(\mathbb{R}) &\supseteq C_C(\mathbb{R})
\end{aligned}$$

Exercise: Show that $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ is a subspace.

(3) $\Omega = \mathbb{N}$, $\ell_\infty = \ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$. Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \leq \ell_\infty.$$

(4) ℓ_1 with $\|\cdot\|_1$,

$$\|(a_k)_k\|_1 = \sum_{k=1}^{n} |a_k|.$$

(5) $C([a,b])$ with

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

(6) Let $1 \leq p < \infty$.

$$\ell_p = \left\{ (a_k)_{k=1}^{\infty} \mid \sum_{k=1}^{\infty} |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\begin{aligned}
\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} &= \left\| \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \right\|_{\ell_p^n} \\
&= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
&\leq \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\| + \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^\infty} \\
&\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p.
\end{aligned}$$

Taking the limit as $n \to \infty$ (by the definition of an infinite series), we find that $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$.

(7) $\mathrm{BV}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid \mathrm{Var}(f) < \infty\}$ with the norm $\|f\|_{\mathrm{BV}} = |f(a)| + \mathrm{Var}(f)$ is a normed space:

$$\begin{aligned} \|f\|_{\mathrm{BV}} &= 0\\ |f(a)| &= 0\\ \mathrm{Var}(f) &= 0 \end{aligned}$$

given $t \in (a,b]$, look at the partition $a < t \leq b$. Then,

$$\begin{aligned} \mathrm{Var}(f) &\geq |f(t) - f(a)| + |f(b) - f(t)|\\ f(t) &= 0\\ f &= \mathbb{0}_f. \end{aligned}$$

(8) $\mathbb{M}_{m,n}(\mathbb{F})$ with

$$\|a\|_{\mathrm{op}} = \sup_{\|\xi\|_{\ell_2^n} \leq 1} \|a\xi\|_{\ell_2^m}$$

is a normed vector space. If $\|a\|_{\mathrm{op}} = 0$, then

$$ae_j = 0 \qquad \forall j \in \{1, \dots, n\}.$$

take the dot product with $i \neq j$

$$\begin{aligned} ae_j \cdot e_i &= a_{ij}\\ &= 0 \end{aligned}$$

so $a_{ij} = 0$ for all a_{ij}, so a is the $\mathbb{0}$ matrix.

(9) Let V, W be vector spaces over $\mathbb{F}$. Then, $\mathcal{L}(V,W) = \{T \mid T : V \to W \text{ linear}\}$, where $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

$\mathcal{L}(V,W)$ is a vector space with operations

$$\begin{aligned} (T+S)(v) &= T(v) + S(v)\\ (\alpha T)(v) &= \alpha T(v). \end{aligned}$$

Notation: $\mathcal{L}(V) := \mathcal{L}(V,V)$ is all linear operators on V. $\mathcal{L}(V,\mathbb{F}) = V'$ is all linear functionals.

Suppose V and W are normed vector spaces. If $T : V \to W$, set

$$\begin{aligned} \|T\|_{\mathrm{op}} &:= \sup_{\|v\|_v \leq 1} \|T(v)\|_W,\\ \mathbb{B}(V,W) &= \{T \in \mathcal{L}(V,W) \mid \|T\|_{op} \leq \infty\}, \end{aligned}$$

where $\mathbb{B}(V,W)$ is referred to as the set of all bounded linear maps from V to W. $\mathbb{B}(V,W)$ with $\|\cdot\|_{\mathrm{op}}$ is a normed space.

- Homogeneity:

$$\begin{aligned} \|\alpha T\|_{[op]} &= \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W\\ &= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W\\ &= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W\\ &= |\alpha| \|T\|_{\mathrm{op}}. \end{aligned}$$

- Triangle Inequality: for $\|v\|_V \leq 1$,

$$\begin{aligned} \|\left(T+S\right)(v)\|_W &= \|T(v) + S(v)\|_W\\ &\leq \|T(v)\|_W + \|S(v)\|_W\\ &\leq \|T\|_{\mathrm{op}} + \|S\|_{\mathrm{op}} \end{aligned}$$

so

$$\begin{aligned} \|T+S\|_{\mathrm{op}} &= \sup_{\|v\| \leq 1} \|T + S(v)\|\\ &\leq \|T\|_{\mathrm{op}} + \|S\|_{\mathrm{op}} \end{aligned}$$

- Positive Definite: If $\|T\|_{\text{op}} = 0$, then $T(v) = 0$ for all $v \in V$, $\|v\| \leq 1$.

  Let $v \in V$, $v \neq 0$. Then, $\frac{v}{\|v\|} \in B_V$.

$$\begin{aligned} T\left(\frac{v}{\|v\|}\right) &= 0 \\ \frac{1}{\|v\|}T(v) &= 0 \\ T(v) &= 0 \end{aligned}$$

Special Cases: $\mathbb{B}(V) = \mathbb{B}(V, V)$, $V^* = \mathbb{B}(V, \mathbb{F})$.

Exercise: $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$.

(10) Inner Product Spaces (expanded upon below).

Inner Product Spaces

An inner product on a vector space V is a pairing

$$V \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}$$

that satisfies

(i) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$, $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$.

(ii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

(iii) $\langle v, v\rangle \geq 0$.

(iv) If $\langle v, v\rangle = 0$, then $v = 0$.

The pair $(V, \langle \cdot, \cdot\rangle)$ is known as an inner product space.

Remarks: $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$, $\langle v, \alpha w\rangle = \overline{\alpha}\langle v, w\rangle$.

If $\langle \cdot, \cdot\rangle$ is an inner product on a linear space V, then set

$$\|v\|_2 := \langle v, v\rangle^{1/2}.$$

Exercise: $\|\alpha v\|_2 = |\alpha| \|v_2\|$, $\|v\|_2 = 0 \Rightarrow v = 0$.

$v, w \in (V, \langle, \cdot, \cdot\rangle)$ are *orthogonal* if $\langle v, w\rangle = 0$.

The Pythagoran theorem states that for $v_1, \dots, v_n \in V$ mutually orthogonal, then

$$\left\|\sum_{i=1}^{n} v_i\right\|^2 = \sum_{j=1}^{n} \|v_j\|^2.$$

For two vectors $v, w \in V$, $P_w(v) = \frac{\langle v, w\rangle}{\langle w, w\rangle} w$.

Exercise: Check that $\langle P_w(v), v - P_w(v)\rangle$, meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

Cauchy-Schwarz Inequality: In any inner product space,

$$|\langle v, w\rangle| \leq \|v\| \cdot \|w\|.$$

Proof of Cauchy-Schwarz: From the exercise,

$$\begin{aligned} \|v\| &\geq \|P_w(v)\| \\ \|v\| &\geq \left\|\frac{\langle v, w\rangle}{\langle w, w\rangle} w\right\| \\ &= \frac{|\langle v, w\rangle|}{\|w\|^2}\|w\| \end{aligned}$$

therefore,

$$\|v\|\|w\| \geq |\langle v, w\rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

Proof of Triangle Inequality:

$$\begin{aligned}
\|v+w\|_2^2 &= \langle v+w, v+w\rangle \\
&= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\
&= \|v\|^2 + \|w\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} \\
&= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v,w\rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2|\langle v,w\rangle| \\
&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| && \text{Cauchy-Schwarz Inequality} \\
&= (\|v\|+\|w\|)^2.
\end{aligned}$$

Take square roots on both sides.

(1) $\ell_2^n = \mathbb{F}^n$ with

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i\overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left|\sum_{j=1}^n x_j\overline{y_j}\right| \leq \left(\sum_{j=1}^n |x_j|^2\right)^{1/2}\left(\sum_{j=1}^n |y_j|^2\right)^{1/2}.$$

(2) ℓ_2 with

$$\left\langle (a_j)_j, (b_j)_j\right\rangle = \sum_{j=1}^\infty a_j\overline{b}_j.$$

We can see that for any finite n, the Cauchy-Schwarz inequality in ℓ_2^n states

$$\begin{aligned}
\left|\sum_{j=1}^n a_j\overline{b_j}\right| &\leq \left(\sum_{j=1}^n |a_j|^2\right)^{1/2}\left(\sum_{j=1}^n |b_j|^2\right)^{1/2} \\
&\leq \left(\sum_{j=1}^\infty |a_j|^2\right)^{1/2}\left(\sum_{j=1}^\infty |b_j|^2\right)^{1/2}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we see that $\left\langle (a_j)_j, (b_j)_j\right\rangle$ is convergent.

(3) $C([a,b])$ with

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

(4) Let $V = \mathbb{M}_n(\mathbb{C})$.

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let $\mathrm{Tr} : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$, $\mathrm{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$.

- $\mathrm{Tr}(I_n) = n$
- $\mathrm{Tr}(a+\alpha b) = \mathrm{Tr}(a) + \alpha\mathrm{Tr}(b)$
- $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$

Then, if $\text{Tr}(a^*a) = 0$, then $a = \mathbb{0}_{\mathbb{M}_n}$.

$$\begin{aligned}
a^*a &= (\overline{a_{ji}})_{i,j}(a_{ij})_{i,j}\\
&= \left(\sum_{k=1}^{n}\overline{ki}a_{kj}\right)_{i,j}\\
\text{Tr}(a^*a) &= \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{a_{ki}}a_{ki}\\
&= \sum_{i,k=1}^{n}|a_{ki}|^2\\
&= \sum_{i,j=1}^{n}|a_{ij}|^2.
\end{aligned}$$

If $\text{Tr}(a^*a) = 0$, then $a_{ij} = 0$ for all i, j.

We define

$$\langle a, b\rangle_{\text{HS}} = \text{Tr}(b^*a).$$

(i) $(b_1 + b_2)^* = b_1^* + b_2^*$
(ii) $(\alpha b)^* = \overline{\alpha}b^*$
(iii) $(b_1b_2)^* = b_2^*b_1^*$
(iv) $b^{**} = b$

The norm is defined as

$$\begin{aligned}
\|a\|_{\text{HS}} &= \langle a, a\rangle^{1/2}\\
&= \text{Tr}(a^*a)^{1/2}\\
&= \left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{1/2}
\end{aligned}$$

Metric Spaces

We looked at normed spaces, where we attach a length $\|v\|$ to very vector v. We can also speak of the distance between two vectors, defined as $d(v, w) = \|v - w\|$.

Notice that the following hold:

- $d(v, w) \geq 0$
- $$\begin{aligned}
d(v, w) &= \|v - w\|\\
&= \|(-1)(w - v)\|\\
&= |-1|\,\|w - v\|\\
&= \|w - v\|
\end{aligned}$$
- $$\begin{aligned}
d(u, w) &= \|u - w\|\\
&= \|u - v + v - w\|\\
&\leq \|u - v\| + \|v - w\|\\
&= d(u, v) + d(v, w).
\end{aligned}$$
- $d(v, v) = \|v - v\| = 0$. If $d(v, w) = 0$, then $\|v - w\| = 0$, so $v - w = \mathbb{0}$, so $v = w$.

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely $(\mathbb{R}, |\cdot|)$. We will expand these concepts to all metric spaces.

Definition of a Metric Space

Let X be a non-empty set. A **metric** on X is a map

$$\begin{aligned} d &: X \times X \to \mathbb{R}^+ \\ (x, y) &\mapsto d(x, y) \geq 0 \end{aligned}$$

such that

(i) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.

(ii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(iii) Zero Distance: $d(x, x) = 0$

(iv) Definite: $d(x, y) = 0 \Rightarrow x = y$

If d satisfies (i), (ii), and (iii), then d is called a semi-metric. If d satisfies (iv) as well, then d is a metric.

If d is a (semi-)metric on X, the pair (X, d) is called a (semi-)metric space.

Two metrics, d and ρ, on X, are equivalent if $\exists c_1, c_2 \geq 0$ such that $d(x, y) \leq c_1\rho(x, y)$ and $\rho(x, y) \leq c_2 d(x, y)$ for all x, y.

Examples of Metric Spaces

(1) Discrete Metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for X any set.

(2) Hamming distance: between two bit strings of equal length. Let

$$\begin{aligned} X &= \{0, 1\}^n \\ &= \{0, 1\} \underbrace{\times \cdots \times}_{n \text{ times}} \{0, 1\} \\ d_H((x_j)_1^n, (y_j)_1^n) &= \left|\{j \mid x_j \neq y_j\}\right|. \end{aligned}$$

(3) Any normed space $(V, \|\cdot\|)$ is a metric space.

$$d(v, w) = \|v - w\|.$$

Exercise: Show that if two norms are equivalent, their induced metrics are equivalent.

(4) Subset of Metric Space: If (X, d) is a metric space, and $Y \subseteq X$ is non-empty. Then, (Y, d) is a metric space.

(5) Paris metric: let (X, ρ) be a metric space. Let $p \in X$ be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

(6) Bounded metric: Let ρ be a (semi-)metric on X. Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We claim that d is a (semi-)metric. Notice that $0 \leq d(x, y) \leq 1$.

Proof: Clearly, $d(x, y) = d(y, x)$. Additionally, $d(x, x) = 0$. If $d(x, y) = 0$ and ρ is a metric, then $\rho(x, y) = 0$, so $x = y$.

To show the triangle inequality, we examine the function

$$\begin{aligned} f(t) &= \frac{t}{1 + t} \\ f'(t) &= \frac{1}{(1 + t)^2} > 0. \end{aligned}$$

Since ρ satisfies the triangle inequality, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Apply f on both sides. Then,

$$\begin{aligned}
\underbrace{\frac{\rho(x,z)}{1+\rho(x,z)}}_{d(x,z)} &\leq \frac{\rho(x,y)+\rho(y,z)}{1+(\rho(x,y)+\rho(y,z))} \\
&= \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\
&\leq \underbrace{\frac{\rho(x,y)}{1+\rho(x,y)}}_{d(x,y)} + \underbrace{\frac{\rho(y,z)}{1+\rho(y,z)}}_{d(y,z)}.
\end{aligned}$$

(7) If $d_1, \dots, d_n$ are metrics on X, $c_1, \dots, c_n \geq 0$. Then,

$$d(x,y) = \sum_{k=1}^{n} c_k d_k(x,y)$$

is a metric.

(8) Let $\{\rho_k\}_{k=1}^{\infty}$ be a family of semi-metrics. Assume the family is separating — for all $x \neq y$, there exists k such that $\rho_k(x,y) \neq 0$.

Let d_k be defined as

$$d_k(x,y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}.$$

Note that $\{d_k\}_{k=1}^{\infty}$ is also separating.

Then,

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x,y)$$

is a metric.

We will now define the Frechet Metric using this method. Let $X = C(\mathbb{R})$. For each $k = 1,2,3,\dots$, set $p_k(f) = \sup_{x\in[-k,k]} |f(x)|$.

We can verify that p_k defines a seminorm. We can then check $\rho_k(f,g) = p_k(f-g)$ is a semi-metric.

We claim that $\{\rho_k\}$ is separating: if $f \neq g$, then there exists $x_0 \in \mathbb{R}$ with $f(x_0) \neq g(x_0)$. Since f and g are continuous, there is a neighborhood $[x_0-\delta, x_0+\delta]$ such that $f(x) \neq g(x)$ for all $x \in [x_0-\delta, x_0+\delta]$. Find k such that $[x_0-\delta, x_0+\delta] \subseteq [-k,k]$. Then, $\rho_k(f-g) > 0$.

Construct d_k as above, and then d as follows:

$$d_{\text{F}} = \sum \frac{2^{-k} p_k(f-g)}{1+p_k(f-g)}$$

(9) Product of metric spaces: let $(X_k, \rho_k)_{k=1}^{\infty}$ be a countable family of metric spaces. For each k, let

$$d_k(x,y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}.$$

Remark: If the ρ_k are already uniformly bounded, let $d_k = \rho_k$.

Let

$$\begin{aligned}
X &= \prod_{k=1}^{\infty} X_k \\
&= \{(x_k)_k \mid x_k \in X_k\} \\
&= \left\{ f : \mathbb{N} \to \coprod_{k=1}^{\infty} X_k \mid f(k) \in X_k \right\}.
\end{aligned}$$

Define $D : X \times X \to [0, \infty)$ as

$$\begin{aligned} D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} \rho_k(x_k, y_k), \\ D(f, g) &= \sum_{k=1}^{\infty} 2^{-k} \rho(f(k), g(k)). \end{aligned}$$

For example, for each k, let $X_k = \{0, 1\}$ with the discrete metric. Let

$$\begin{aligned} \Delta &= \prod_{k \in \mathbb{N}} \{0, 1\} \\ &= \{(x_k)_k \mid x_k \in \{0, 1\}\} \\ D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k| && (x_k)_k, (y_k)_k \in \Delta. \end{aligned}$$

Δ is known as the abstract Cantor set; every compact metric space is a surjective image of the abstract Cantor set.

(10) Geodesic Distance: let $\langle \cdot, \cdot \rangle$ be the standard dot product on $\mathbb{R}^3 (\mathbb{R}^n)$, then

$$\begin{aligned} S^2 &= \left\{ x \in \mathbb{R}^3 \mid \|x\|_2 = 1 \right\} \\ S^{n-1} &= \{ x \in \mathbb{R}^n \mid \|x\|_2 = 1 \}. \end{aligned}$$

To find the geodesic distance, we take $d(x, y) = \arccos(\langle x, y \rangle)$. We claim d is a metric.

- Symmetry: self-evident.
- $d(x, x) = \arccos(1) = 0$. Suppose $d(x, y) = 0$. Then, $\langle x, y \rangle = 1$, meaning $\|x - y\|^2 = 0$, so $x = y$.
- Let $\theta = \arccos(\langle x, y \rangle)$, $\varphi = \arccos(\langle y, z \rangle)$, where $\theta, \varphi \in [0, \pi]$.

$$\begin{aligned} p_x &= \frac{\langle x, y \rangle}{\langle y, y \rangle} y \\ &= \cos(\theta) y \\ x &= \cos(\theta) y + \sin(\theta) u \end{aligned}$$

where

$$u = \frac{x - p_x}{\|x - p_x\|}.$$

Similarly, we can take

$$z = \cos(\varphi) y + \sin(\varphi) v$$

where

$$v = \frac{z - p_z}{\|z - p_z\|}.$$

So,

$$\begin{aligned} \langle x, z \rangle &= \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) \langle u, v \rangle \\ &\geq \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) && \langle u, v \rangle \geq -1 \\ &= \cos(\theta + \varphi). \end{aligned}$$

Since arccos is decreasing,

$$\begin{aligned} \arccos(\langle x, z \rangle) &\leq \arccos(\cos(\theta + \varphi)) \\ &= \theta + \varphi \\ &= \arccos(\langle x, y \rangle) + \arccos(\langle y, z \rangle). \end{aligned}$$

Therefore, $d(x, y) \leq d(x, y) + d(y, z)$.

- Let $\Gamma = (V, E)$ be a simple connected graph. We define $d : V \times V \to [0, \infty)$ to be the length of the shortest path between vertices u and v.

  Exercise: Show this is a metric.

(11) Let (X, d) be any metric space. If $E \subseteq X$, define $\operatorname{diam}(E) = \sup_{x, y \in E} d(x, y)$. E is bounded if $\operatorname{diam}(E) < \infty$.

Exercise: If $(V, \|\cdot\|)$ is a normed space, $E \subseteq V$ is a subset, show the following are equivalent:

(i) E is bounded (in the metric sense)
(ii) $\sup_{v \in E} \|v\| < \infty$
(iii) $\exists r > 0$ such that $E \subseteq rB_V$.

Let Ω be any set. The function $f : \Omega \to X$ is bounded if $f(\Omega) \subseteq X$ is bounded. We let $\text{Bd}(\Omega, X) = \{f : \Omega \to X \mid f \text{ is bounded}\}$.

Remark: $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty(\Omega, \mathbb{F})$.

(12) $\text{Bd}(\Omega, X)$ with

$$D_u(f, g) = \sup_{x \in \Omega} d(f(x), g(x)).$$

Exercise: Show that D_u defines a metric.

Consider $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty$. Look at the subset

$$E = \{f \in \text{Bd}(\Omega, \mathbb{F}) \mid f(x) \in \{0, 1\}\}.$$

Then,

$$\begin{aligned} D_u(f, g) &= \sup_{x \in \Omega} |f(x) - g(x)|. \\ &= \begin{cases} 1 & f \neq g \\ 0 & f = g \end{cases}. \end{aligned}$$

When we take a particular subset of $D_u(f, g)$, we find that we get the discrete metric.

Taking an overview of the concepts we have learned so far, we see

Inner Product Spaces $\subseteq$ Normed Vector Spaces $\subseteq$ Metric Spaces

Topology of Metric Spaces

Throughout this section, let (X, d) be a metric space.

(1) Let $x_0 \in X$, $\delta > 0$.

(i) We say

$$U(x_0, \delta) = \{x \in X \mid d(x, x_0) < \delta\}$$

is the open ball centered at x_0 with radius δ.

(ii) We say

$$B(x_0, \delta) = \{x \in X \mid d(x, x_0) \leq \delta\}$$

is the closed ball.

(iii) We say

$$S(x_0, \delta) = \{x \in X \mid d(x, x_0) = \delta\}$$

is the sphere.

(2) $U \subseteq X$ is open if

$$(\forall x \in U)(\exists \delta > 0) \ni U(x, \delta) \subseteq U.$$

Let

$$\begin{aligned} \tau_X &= \{U \subseteq X \mid U \text{ open}\} \\ &\subseteq \mathcal{P}(X). \end{aligned}$$

(3) $D \subseteq X$ is closed if D^c is open.

(4) If $x \in U \in \tau_X$, then U is called an open neighborhood of x. If $x \in U \subseteq N$, where $U \in \tau_X$, then N is a neighborhood of x.

$$\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$$

(5) Let $A \subseteq X$. The interior of A is

$$A^0 = \bigcup \{V \mid V \subseteq A, V \text{ open}\}.$$

The closure of A is

$$\overline{A} = \bigcap \{D \mid A \subseteq D, D \text{ closed}\}.$$

The boundary of A is

$$\partial A = \overline{A} \setminus A^0.$$

Exercise: $\overline{A^c} = (A^0)^c$, $(\overline{A})^c = (A^c)^0$.