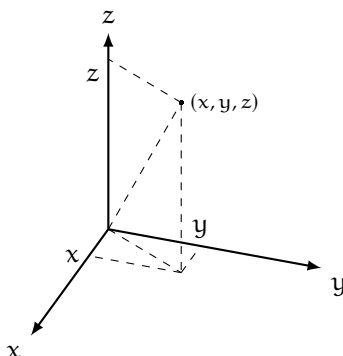


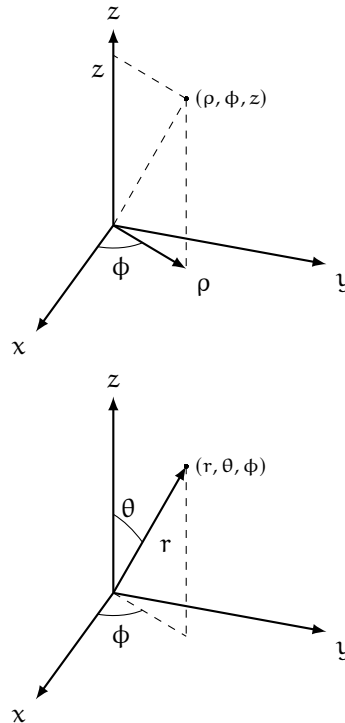
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## Things You Just Gotta Know

### Coordinate Systems



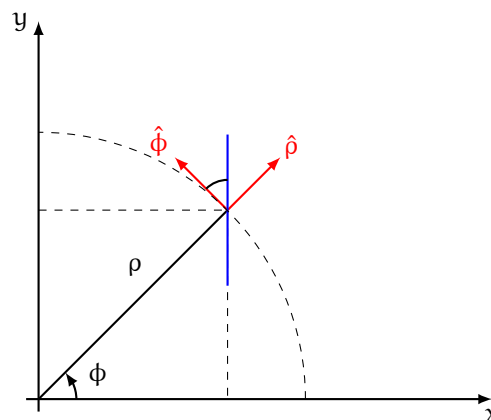


We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x, y)\hat{i} + V_y(x, y)\hat{j}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

### Polar Coordinates



We can also express the inputs to  $\vec{V}$  in polar coordinates,  $(\rho, \phi)$ .

$$\vec{V}(\mathbf{r}) = V_\rho(\rho, \phi)\hat{i} + V_\phi(\rho, \phi)\hat{j}.$$

To extract the input functions, we take

$$V_x = \hat{i} \cdot \vec{V}$$

$$V_y = \hat{j} \cdot \vec{V}.$$

Alternatively, we can project  $\vec{V}$  onto the  $\hat{\rho}, \hat{\phi}$  axis:

$$\vec{V}(\mathbf{r}) = V_{\rho}(\rho, \phi) \hat{\rho} + V_{\phi}(\rho, \phi) \hat{\phi},$$

and we extract

$$\begin{aligned} V_{\rho} &= \hat{\rho} \cdot \vec{V} \\ V_{\phi} &= \hat{\phi} \cdot \vec{V}. \end{aligned}$$

Notice that  $\mathbf{r}$  is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned} \mathbf{s} &= x\hat{i} + y\hat{j} \\ &= \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} \\ &= \rho \hat{\rho} \\ &= \sqrt{x^2 + y^2} \hat{\rho} \end{aligned}$$

The main reason we avoided using the  $\hat{\rho}, \hat{\phi}$  axis up until this point is that  $\rho$  and  $\phi$  are *position-dependent*, while the  $\hat{i}, \hat{j}$  axis is position-independent.

Now, we must figure out the position-dependence of  $\hat{\rho}$  and  $\hat{\phi}$ :

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi.$$

If we hold  $\phi$  constant, it must be the case that any change in  $\rho$  is in the  $\hat{\rho}$  direction. Therefore,

$$\begin{aligned} \hat{\rho} &= \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|} \\ &= \frac{\cos \phi \hat{i} + \sin \phi \hat{j}}{|\cos \phi \hat{i} + \sin \phi \hat{j}|} \\ &= \cos \phi \hat{i} + \sin \phi \hat{j}. \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\|} \\ &= \frac{-\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j}}{\|-\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j}\|} \\ &= -\sin \phi \hat{i} + \cos \phi \hat{j}. \end{aligned}$$

Thus, we can see that the  $\hat{\rho}, \hat{\phi}$  axis is orthogonal.

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial \phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j} \\ &= \hat{\phi}, \\ \frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\rho}, \end{aligned}$$

$$\frac{\partial \hat{\phi}}{\partial \rho} = 0,$$

and

$$\frac{\partial \hat{\rho}}{\partial \rho} = 1$$

**Example (Velocity).**

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{s}}{dt} \\ &= \frac{d}{dt} (x\hat{i}) + \frac{d}{dt} (y\hat{j}). \end{aligned}$$

In the case of cartesian coordinates,  $\hat{i}$  and  $\hat{j}$  are constants.

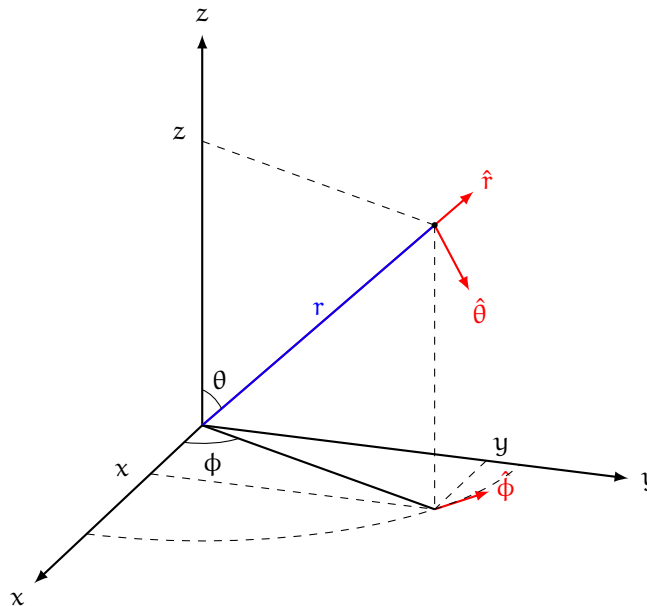
$$= v_x \hat{i} + v_y \hat{j}$$

When we examine polar coordinates, since  $\hat{\rho}$  and  $\hat{\phi}$  are position-dependent, we must use the chain rule.<sup>1</sup>

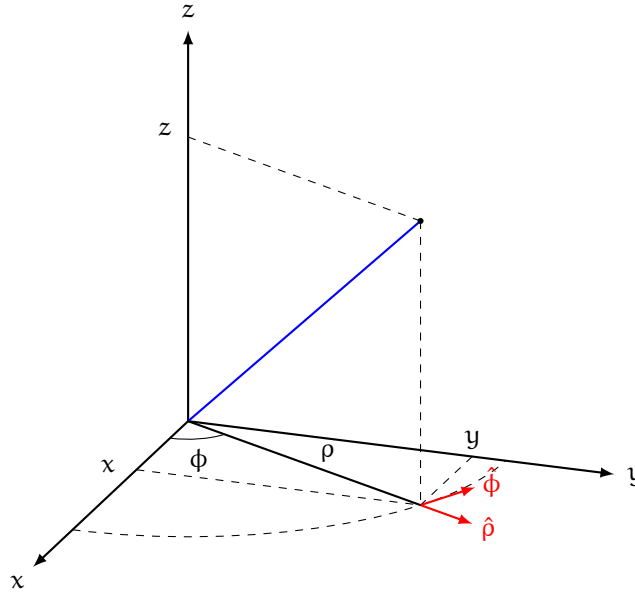
$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{s}}{dt} \\ &= \frac{d\rho}{dt} \hat{\rho} + \rho \frac{d\hat{\rho}}{dt} \\ &= \frac{d\rho}{dt} \hat{\rho} + \rho \left( \frac{\partial \hat{\rho}}{\partial \rho} \frac{d\rho}{dt} + \underbrace{\frac{\partial \hat{\rho}}{\partial \phi} \frac{d\phi}{dt}}_{=\dot{\phi}} \right) \\ &= \frac{d\rho}{dt} \hat{\rho} + \rho \frac{d\phi}{dt} \hat{\phi} \\ &= \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi}. \end{aligned}$$

Notice that  $\dot{\rho}$  is the radial velocity and  $\dot{\phi} = \omega$  is the angular velocity.

### Spherical and Cylindrical Coordinates



<sup>1</sup>Note that  $\hat{\rho} = \hat{\rho}(\rho, \phi)$  and  $\hat{\phi} = \hat{\phi}(\rho, \phi)$ .



Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho, \phi)$	$\mathbf{s} = s(\rho, \phi, z)$	$\mathbf{s} = s(r, \phi, \theta)$
$\mathbf{s} = \begin{pmatrix} \rho \cos \phi \\ \rho \sin \phi \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} \rho \cos \phi \\ \rho \sin \phi \\ z \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} r \cos \phi \sin \theta \\ r \sin \phi \sin \theta \\ r \cos \theta \end{pmatrix}$

Here,<sup>11</sup>  $\phi$  denotes the polar angle and  $\theta$  denotes the azimuthal angle. Notice that  $\phi \in [0, 2\pi)$  and  $\theta \in [0, \pi]$ .

We can see that  $\hat{r}$ ,  $\hat{\phi}$ , and  $\hat{\theta}$  in spherical coordinates are also position-dependent.

$$\begin{aligned}
 \hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\| \frac{\partial \mathbf{s}}{\partial r} \right\|} \\
 &= \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \\
 \hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\| \frac{\partial \mathbf{s}}{\partial \phi} \right\|} \\
 &= -\sin \phi \hat{i} + \cos \phi \hat{j} \\
 \hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\| \frac{\partial \mathbf{s}}{\partial \theta} \right\|} \\
 &= \cos \phi \cos \theta \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}
 \end{aligned}$$

### Scale Factors and Jacobians

Coordinate System	Line Element	Area Element	Volume Element
Polar	$d\mathbf{s} = \hat{\rho} d\rho + \rho \hat{\phi} d\phi$	$d\mathbf{a} = r dr d\phi$	—
Cylindrical	$d\mathbf{s} = \hat{\rho} d\rho + \rho \hat{\phi} d\phi + \hat{k} dz$	—	$d\tau = r dr d\phi dz$
Spherical	$d\mathbf{s} = \hat{r} dr + r \sin \theta \hat{\phi} d\phi + r \hat{\theta} d\theta$	$d\mathbf{a} = r^2 \sin \theta d\phi d\theta$	$d\tau = r^2 \sin \theta dr d\phi d\theta$

<sup>11</sup>Physicists amirite?

In cylindrical coordinates, we can use the chain rule to find the value of  $d\mathbf{r}$ :

$$d\mathbf{r} = \hat{\rho} d\rho + \rho \hat{\phi} d\phi + \hat{k} dz.$$

The extra factor of  $\rho$  in the expression of  $\rho \hat{\phi} d\phi$  is the *scale factor* on  $\phi$ .

Similarly, in spherical coordinates, we have

$$d\mathbf{r} = \hat{r} dr + r \sin \theta \hat{\phi} d\phi + r \hat{\theta} d\theta,$$

with scale factors of  $r \sin \theta$  on  $\hat{\phi} d\phi$  and  $r$  on  $\hat{\theta} d\theta$ .

When we go from line elements (of the form  $d\mathbf{r}$ ) to area elements (of the form  $d\mathbf{a}$ ), we can see that the area element in polar coordinates is  $d\mathbf{a} = \rho d\rho d\phi$  — we need the extra factor of  $\rho$  to account for the fact that the magnitude of the area element scales with the radius.

Similarly, the volume element in cylindrical coordinates is  $d\tau = r dr d\phi dz$  and the volume element in spherical coordinates is  $r^2 \sin \theta dr d\phi d\theta$ .

Recall that the definition of an angle  $\phi$  that subtends an arc length  $s$  is  $\phi = \frac{s}{r}$ , where  $r$  is the radius of a circle. We can imagine a similar concept on a sphere — a *solid angle* measured in steradians is of the form  $\Omega = \frac{A}{r^2}$ , where  $A$  denotes the surface area subtended by the angle  $\Omega$ . In particular, since  $d\Omega = \frac{dA}{r^2}$ , we find that  $d\Omega = \sin \theta d\phi d\theta$ .

When we are dealing with products of scale factors, we need to use the Jacobian to determine the proper scale factor on any given element:

$$\begin{aligned} d\mathbf{a} &= dx dy \\ &= |J| du dv, \end{aligned}$$

where  $|J|$  denotes the determinant of the Jacobian matrix. We write the Jacobian as follows:

$$\begin{aligned} J &= \frac{\partial (x, y)}{\partial (u, v)} \\ &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}. \end{aligned}$$

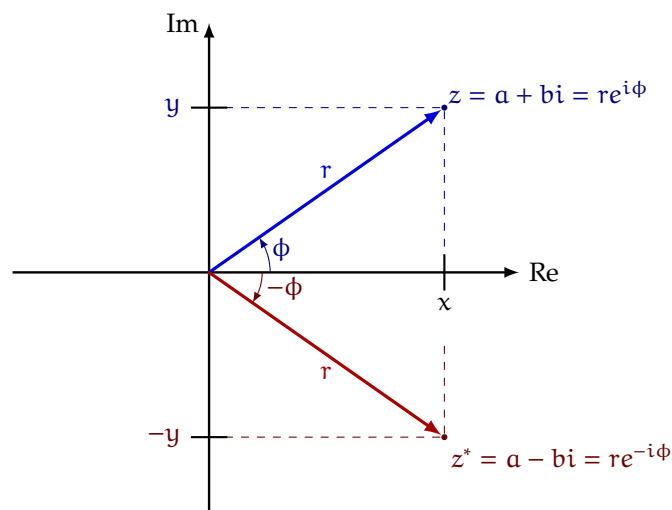
We specifically desire the determinant:

$$|J| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}.$$

## Complex Numbers

Quantity	Expression and/or Criterion
Cartesian form	$z = a + bi$
Polar form	$z = re^{i\phi}$
$r$	$\sqrt{a^2 + b^2}$
$\phi$	$\arg z = \arctan\left(\frac{b}{a}\right)$
Cartesian $z^*$	$z^* = a - bi$
Polar $z^*$	$z = re^{-i\phi}$
$ z $	$\sqrt{zz^*}$
$\operatorname{Re}(z)$	$\operatorname{Re}(z) = \frac{z+z^*}{2}$
$\operatorname{Im}(z)$	$\operatorname{Im}(z) = \frac{z-z^*}{2i}$
$\cos \phi$	$\frac{e^{i\phi} + e^{-i\phi}}{2}$
$\sin \phi$	$\frac{e^{i\phi} - e^{-i\phi}}{2i}$
$e^{i\phi}$	$\cos \phi + i \sin \phi$
$e^{in\phi}$	$\cos(n\phi) + i \sin(n\phi)$

### Introduction



A complex number is denoted

$$z = a + bi$$

where  $i^2 = -1$  and  $a, b \in \mathbb{R}$ . This is known as the cartesian representation. However, we can also imagine  $z$  as the polar representation:

$$z = re^{i\phi},$$

where  $\phi = \arg z$  is known as the argument, and  $r = |z|$  is the modulus. We can see the relation between the cartesian and polar representations through Euler's identity:<sup>III</sup>

$$r(\cos \phi + i \sin \phi) = re^{i\phi}.$$

We denote the conjugate of  $z$  as  $z^*$ <sup>IV</sup>, found by  $z^* = a - bi = re^{-i\phi}$ .

We find  $\text{Re}(z)$  and  $\text{Im}(z)$ , the real and imaginary parts of  $z$ , by

$$\begin{aligned}\text{Re}(z) &= \frac{z + z^*}{2} \\ \text{Im}(z) &= \frac{z - z^*}{2i}.\end{aligned}$$

We say that a complex number of the form  $e^{i\phi}$  is a *pure phase*, as  $|e^{i\phi}| = 1$ .

To find if some complex number  $z$  is purely real or purely imaginary, we can use the following criterion:

$$\begin{aligned}z \in \mathbb{R} &\Leftrightarrow z = z^* \\ z \in i\mathbb{R} &\Leftrightarrow z = -z^*.\end{aligned}$$

**Example** (Real, Imaginary, or Complex?). Consider

$$z_1 = i^i.$$

To find if this is purely real or complex, we take

$$\begin{aligned}z_1^* &= (-i)^{-i} \\ &= \left(\frac{1}{-i}\right)^i \\ &= i^i.\end{aligned}$$

Thus,  $z_1 \in \mathbb{R}$ . In order to determine the value of  $i^i$ , we substitute the polar form:

$$\begin{aligned}z_1 &= \left(e^{i\frac{\pi}{2}}\right)^i \\ &= e^{-\frac{\pi}{2}}.\end{aligned}$$

### Some Trigonometry with Complex Exponentials

Consider  $z = \cos \phi + i \sin \phi$ . We can see that

$$\begin{aligned}\text{Re}(z) &= \cos \phi \\ &= \frac{(\cos \phi + i \sin \phi) + (\cos \phi - i \sin \phi)}{2} \\ &= \frac{e^{i\phi} + e^{-i\phi}}{2} \\ \text{Im}(z) &= \sin \phi \\ &= \frac{(\cos \phi + i \sin \phi) - (\cos \phi - i \sin \phi)}{2i} \\ &= \frac{e^{i\phi} - e^{-i\phi}}{2i}.\end{aligned}$$

We can actually define  $\sin \phi$  and  $\cos \phi$  with the above derivation.

<sup>III</sup>This can be proven relatively easily through substitution into the Taylor series, which is allowed because  $e^z$  is entire.

<sup>IV</sup>Physicists amirite?



**Theorem** (De Moivre):

$$\begin{aligned} e^{inx} &= \cos(nx) + i \sin(nx) \\ &= \left(e^{ix}\right)^n \\ &= (\cos x + i \sin x)^n. \end{aligned}$$

**Example** (Finding  $\cos(2x)$  and  $\sin(2x)$ ).

$$\begin{aligned} \cos(2x) + i \sin(2x) &= (\cos x + i \sin x)^2 \\ &= (\cos^2 x - \sin^2 x) + i(2 \sin x \cos x). \end{aligned}$$

Since the real parts and imaginary parts have to be equal, this means

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin^2 x &= 2 \sin x \cos x. \end{aligned}$$

In particular, we can see that  $e^{in\pi} = (-1)^n$  and  $e^{in\frac{\pi}{2}} = i^n$ .<sup>v</sup>

Additionally, we can see that for  $z = re^{i\phi}$ ,

$$\begin{aligned} z^{1/m} &= \left(re^{i\phi+2\pi n}\right)^{1/m} \\ &= r^{1/m} e^{i\frac{1}{m}(\phi+2\pi n)}, \end{aligned}$$

where  $n \in \mathbb{N}$  and  $m$  is fixed. For  $r = 1$ , we call these values the  $m$  roots of unity.

**Example** (Waves and Oscillations). Recall that for a wave with spatial frequency  $k$ , angular frequency  $\omega$ , and amplitude  $A$ , the wave is represented by

$$f(x, t) = A \cos(kx - \omega t).$$

The speed of a wave  $v$  is equal to  $\frac{\omega}{k}$ .

Simple harmonic motion is characterized by the solution to the differential equation  $\ddot{x} = -\omega^2 x$ , where  $x$  denotes position. In simple harmonic motion, there is no spatial motion, meaning our function is only of time:

$$\begin{aligned} f(t) &= A \cos \omega t \\ &= \operatorname{Re} \left( A e^{i\omega t} \right). \end{aligned}$$

As a result of the representation of complex numbers in polar form, we can do math entirely in exponentials, then take the real part of our solution to find  $f(t)$ .

Unfortunately, in the real world, there is friction; as a result, our oscillation is damped by an exponential factor.

**Example** (Hyperbolic Sine and Hyperbolic Cosine). We wish to calculate  $\cos ix$  and  $\sin ix$ .

$$\begin{aligned} \cos ix &= \frac{1}{2} \left( e^{i(ix)} + e^{-i(ix)} \right) \\ &= \frac{e^{-x} + e^x}{2} \end{aligned}$$

---

<sup>v</sup>This will be especially useful when we get to Fourier series.

We define  $\cosh x = \cos(ix)$ . Additionally,

$$\begin{aligned} -i \sin ix &= -i \frac{1}{2i} (e^{i(ix)} - e^{-i(ix)}) \\ &= i \frac{e^x - e^{-x}}{2i} \\ &= \frac{e^x - e^{-x}}{2}. \end{aligned}$$

We define  $\sinh x = -i \sin(ix)$ .

Similar to how  $\cos^2 x + \sin^2 x = 1$ , we can find that  $\cosh^2 x - \sinh^2 x = 1$ .

## Index Algebra

We usually denote vectors by either  $\vec{A}$ ,  $\mathbf{A}$ , or

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is defined by a basis.

If we imagine we are in  $n$ -dimensional space, we can let  $A_i$  where  $i = 1, 2, \dots, n$  denote both

- the  $i$ th component of  $\vec{A}$ ;
- the entire vector  $\vec{A}$  (since  $i$  can be arbitrary).

## Contractions and Dummy Indices

Consider  $C = AB$ , where  $A, B$  are  $n \times m$  and  $m \times p$  matrices respectively.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}.$$

**Definition** (Matrix Multiplication in Index Notation). For matrices  $A$  and  $B$ , where  $A$  is an  $m \times n$  and  $B$  is a  $n \times p$  matrix, we write

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

We say that  $k$  is a dummy index, since  $k$  takes values from 1 to  $n$ . Note that the value we calculate is  $C_{ij}$ ; in other words, in the sum  $\sum_k A_{ik} B_{kj}$ , the indices of the form  $ij$  are the “net indices” from the multiplication.

Note that if  $C = BA$ , then

$$\begin{aligned} C_{ij} &= \sum_{k=1}^n B_{ik} A_{kj} \\ &= \sum_{k=1}^n A_{kj} B_{ik} \end{aligned}$$

$$\neq \sum_{k=1}^n A_{ik} B_{kj}.$$

The corresponding fact is that  $AB \neq BA$  necessarily.

Note that the index that is summed over always appears exactly twice.

**Definition** (Symmetric Matrix). Let  $C$  be a matrix. Then, we say  $C$  is symmetric if

$$C_{ij} = C_{ji}$$

**Definition** (Antisymmetric Matrix). Let  $C$  be a matrix. We say  $C$  is antisymmetric if

$$C_{ij} = -C_{ji}.$$

We can always decompose a random matrix into the sum of a symmetric matrix and an antisymmetric matrix.

## Two Special Tensors

Name	Notation	Definition
Kronecker Delta	$\delta_{ij}$	$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
Levi-Civita Symbol	$\epsilon_{ijk}$	$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = (1, 2, 3) \text{ cyclically} \\ -1 & (i, j, k) = (2, 1, 3) \text{ cyclically} \\ 0 & \text{else} \end{cases}$

Order of (i, j, k)	Value of $\epsilon_{ijk}$
1, 2, 3	1
3, 1, 2	1
2, 3, 1	1
1, 3, 2	-1
2, 1, 3	-1
3, 2, 1	-1
else	0

Value	Index Notation
$\mathbf{A} \times \mathbf{B}$	$\sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k$
$(\mathbf{A} \times \mathbf{B})_\ell$	$\sum_{i,j} \epsilon_{ij\ell} A_i B_j$
$(\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k$	$\epsilon_{ijk}$
$B_i$	$\sum_{\alpha} B_{\alpha} \delta_{\alpha i}$
$\mathbf{A} \cdot \mathbf{B}$	$\sum_{i,j} A_i B_j \delta_{ij}$
$\sum_{j,k} \epsilon_{mjk} \epsilon_{njk}$	$2\delta_{mn}$
$\sum_{\ell} \epsilon_{mnl} \epsilon_{ijl}$	$\delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}$

**Definition** (Kronecker Delta). The Kronecker Delta,  $\delta_{ij}$ , is the tensor that denotes the identity matrix.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

**Example** (Extracting an Index). Consider  $A$  as vector. Then,

$$\sum_i A_i \delta_{ij} = A_j.$$

In other words, the Kronecker Delta collapses the sum to the  $j$ th index.

**Example** (Orthonormal Basis from Kronecker Delta). Let  $\{\hat{e}_i\}_{i=1}^n$  be a basis for some vector space  $V$ . If

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

for every  $i, j$ , then  $\{\hat{e}_i\}_{i=1}^n$  is an orthonormal basis for  $V$ .

**Definition** (Levi-Civita Symbol). In two dimensions, as a matrix, we write

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

meaning

$$\epsilon_{ij} = \begin{cases} 1 & i = 1, j = 2 \\ -1 & i = 2, j = 1 \\ 0 & \text{else} \end{cases}.$$

The Levi-Civita Symbol is antisymmetric, just as the Kronecker Delta is symmetric.

In three dimensions, we define

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = (1, 2, 3) \text{ cyclically} \\ -1 & (i, j, k) = (2, 1, 3) \text{ cyclically} \\ 0 & \text{else} \end{cases}.$$

In other words,  $\epsilon_{ijk} = -\epsilon_{jik}$ .

**Exercise** (Relations between  $\delta_{ij}$  and  $\epsilon_{ijk}$ ).

$$\sum_{j,k} \epsilon_{mjk} \epsilon_{nj k} = 2\delta_{mn}$$

$$\sum_{\ell} \epsilon_{mn\ell} \epsilon_{ij\ell} = \delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}$$

**Definition** (Dot Product). Let  $\{\hat{e}_i\}_{i=1}^n$  be an orthonormal basis for  $V$ . Let  $\mathbf{A} = \sum_i A_i \hat{e}_i$  and  $\mathbf{B} = \sum_i B_i \hat{e}_i$ . Then,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \sum_{i,j} (A_i \hat{e}_i) \cdot (B_j \hat{e}_j) \\ &= \sum_{i,j} A_i B_j (\hat{e}_i \cdot \hat{e}_j) \\ &= \sum_{i,j} A_i B_j \delta_{ij} \\ &= \sum_i A_i B_i \end{aligned}$$

**Definition** (Cross Product). Let  $\{\hat{e}_i\}_{i=1}^3$  be the standard basis over  $\mathbb{R}^3$ . Let  $\mathbf{A} = \sum_i A_i \hat{e}_i$  and  $\mathbf{B} = \sum_i B_i \hat{e}_i$ . Then,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \sum_{i,j} (A_i \hat{e}_i) \times (B_j \hat{e}_j) \\ &= \sum_{i,j} A_i B_j (\hat{e}_i \times \hat{e}_j) \\ &= \sum_{i,j,k} A_i B_j (\epsilon_{ijk} \hat{e}_k).\end{aligned}$$

Instead of asking about  $\mathbf{A} \times \mathbf{B}$ , we ask about  $(\mathbf{A} \times \mathbf{B})_\ell$ , yielding

$$\begin{aligned}(\mathbf{A} \times \mathbf{B})_\ell &= (\mathbf{A} \times \mathbf{B}) \cdot \hat{e}_\ell \\ &= \left( \sum_{i,j,k} A_i B_j (\epsilon_{ijk} \hat{e}_k) \right) \cdot \hat{e}_\ell \\ &= \sum_{i,j} \epsilon_{ij\ell} A_i B_j.\end{aligned}$$

**Remark:** This notation for  $\mathbf{A} \times \mathbf{B}$  automatically shows us that

$$\begin{aligned}(\mathbf{B} \times \mathbf{A})_\ell &= \sum_{i,j} \epsilon_{ij\ell} B_i A_j \\ &= - \sum_{i,j} \epsilon_{ji\ell} B_i A_j \\ &= - \sum_{i,j} \epsilon_{ji\ell} A_j B_i \\ &= - \sum_{i,j} \epsilon_{ij\ell} A_i B_j \quad \text{ } i, j \text{ are dummy indices} \\ &= -(\mathbf{A} \times \mathbf{B})_\ell.\end{aligned}$$

**Example** (Central Force and Angular Momentum). A central force is defined by

$$\mathbf{F} = f(r) \hat{\mathbf{r}},$$

where  $\hat{\mathbf{r}}$  is a radial vector.

Angular momentum is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where  $\mathbf{r}$  denotes position and  $\mathbf{p}$  denotes momentum. Then,

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) \\ &= \left( \frac{d}{dt} \mathbf{r} \times \mathbf{p} \right) + \mathbf{r} \times \left( \frac{d\mathbf{p}}{dt} \right) \\ &= m \left( \frac{d}{dt} \mathbf{r} \times \frac{d}{dt} \mathbf{r} \right) + \mathbf{r} \times (f(r) \hat{\mathbf{r}}) \\ &= f(r) (\mathbf{r} \times \hat{\mathbf{r}}).\end{aligned}$$

This implies that  $\frac{d\mathbf{L}}{dt} = 0$  under a central force.

**Example (Determinant).** Let  $\mathbf{M} = M_{ij}$  be square. We denote  $\mathbf{M}_i$  to be the vector denoting the  $i$ th-row. Then,

$$\begin{aligned} m &= |\mathbf{M}| \\ &= \mathbf{M}_1 \cdot (\mathbf{M}_2 \times \mathbf{M}_3) \\ &= \mathbf{M}_3 \cdot (\mathbf{M}_1 \times \mathbf{M}_2) \\ &= \mathbf{M}_2 \cdot (\mathbf{M}_3 \times \mathbf{M}_1). \end{aligned}$$

**Example (Trace).** Let  $\mathbf{M} = M_{ij}$  be a square matrix. We define  $\text{tr}(\mathbf{M}) = \sum_i M_{ii}$ . Equivalently,

$$\begin{aligned} \text{tr}(\mathbf{M}) &= \sum_{ij} M_{ij} \delta_{ij} \\ &= \sum_i M_{ii}. \end{aligned}$$

Note that

$$\begin{aligned} \text{tr}(\mathbf{I}_n) &= \sum_i \delta_{ii} \\ &= n. \end{aligned}$$

When we upgrade to 3 matrices, we take

$$\begin{aligned} \text{tr}(ABC) &= \sum_{i,j} \left( \sum_{k,\ell} A_{ik} B_{k\ell} C_{\ell j} \right) \delta_{ij} \\ &= \sum_{i,k,\ell} A_{ik} B_{k\ell} C_{\ell i} \\ &= \sum_{i,k,\ell} C_{\ell i} A_{ik} B_{k\ell} \\ &= \text{tr}(CAB). \end{aligned}$$

In other words, the trace is invariant under cyclic permutations.

**Example (Moment of Inertia Tensor).**

Recall that

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p}, \\ &= I \boldsymbol{\omega}. \end{aligned}$$

where  $\mathbf{p} = m\dot{\mathbf{x}}$ , and  $I$  denotes the moment of inertia. Note that  $I \sim m r^2$ . On a more fundamental level, it is the case that the first equation,  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , is the “true” definition of  $\mathbf{L}$ .

Consider a small portion  $m_\alpha$  about some axis at radius  $\mathbf{r}_\alpha$  and momentum  $\mathbf{p}_\alpha$ . Then, we have

$$\begin{aligned} \mathbf{L}_\alpha &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\ &= \sum_\alpha m_\alpha (\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)). \end{aligned}$$

In the infinitesimal case (i.e., as  $\alpha \rightarrow 0$ ), we get

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \rho \, d\tau,$$

where  $\rho$  denotes volume density. Applying the identity  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ , we find

$$\mathbf{L} = \int (\boldsymbol{\omega} (\mathbf{r} \cdot \mathbf{r}) - \mathbf{r} (\mathbf{r} \cdot \boldsymbol{\omega})) \rho \, d\tau.$$

Switching to index notation, we have

$$\begin{aligned}
 L_i &= \int \left( \omega_i r^2 - r_i \sum_j r_j \omega_j \right) \rho \, d\tau \\
 &= \sum_j \int \omega_j \left( \delta_{ij} r^2 - r_i r_j \right) \rho \, d\tau \\
 &= \sum_j \omega_j \underbrace{\left( \int \left( \delta_{ij} r^2 - r_i r_j \right) \rho \, d\tau \right)}_{\text{moment of inertia tensor}} \\
 &= \sum_j I_{ij} \omega_j.
 \end{aligned}$$

## Binomial Theorem

The binomial theorem allows us to calculate the expansion

$$(x + y)^n = \sum_{m=0}^n \binom{n}{m} x^{n-m} y^m.$$

In the case of  $(x + y)^2 = x^2 y^0 + 2x^1 y^1 + x^0 y^2 = x^2 + 2xy + y^2$ . Recall that

$$\binom{n}{m} = \frac{n!}{m! (n - m)!}.$$

Recall that  $0! = 1$ .

## Infinite Series

Let

$$S = \sum_{k=0}^{\infty} a_k$$

be an infinite series. We are often curious as to the convergence of this sum (for a variety of reasons). Formally, we have to invoke partial sums

$$S_N = \sum_{k=0}^N a_k,$$

and see if the sequence of partial sums is convergent. However, we will prefer to use series convergence tests.

**Example (Geometric Series).** Let

$$\begin{aligned}
 S &= \sum_{k=0}^{\infty} r^k \\
 &= 1 + r + r^2 + \dots
 \end{aligned}$$

Then, we have

$$S_N = \sum_{k=0}^N r^k$$

$$rS_N = \sum_{k=0}^N r^k.$$

Subtracting, we get

$$(1-r)S_N = 1 - r^{N+1}$$

$$S_N = \frac{1 - r^{N+1}}{1 - r}.$$

In the limit, we expect that if  $r \rightarrow \infty$ , and  $r < 1$ , then  $r^{N+1} \rightarrow 0$ . In the infinite case, we have

$$S = \sum_{k=0}^{\infty} r^k$$

$$= \frac{1}{1-r},$$

if  $r < 1$ .

There are a few prerequisites for series convergence:

- there exists some  $K$  for which for all  $k \geq K$ ,  $a_{k+1} \leq a_k$ ;
- $\lim_{k \rightarrow \infty} a_k < \infty$ ;
- we need the series to reduce “quickly” enough.

**Example (Ratio Test).** A series  $S = \sum_k a_k$  converges if the ratio of consecutive terms is (eventually) less than 1:

$$r = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1.$$

**Example (Applying the Ratio Test).** Consider  $S = \sum_k \frac{1}{k!}$ . Then,

$$r = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{k+1}$$

$$= 0 < 1$$

**Example (Riemann Zeta Function).** We write

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

In order to evaluate the convergence of the Riemann zeta function. We have

$$r = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)^s}}{\frac{1}{k^s}}$$

$$= \lim_{k \rightarrow \infty} \left( \frac{k}{k+1} \right)^s$$

$$= 1.$$

Unfortunately, this means the ratio test is inconclusive.



For examples of evaluations of the zeta function, we have

$$\begin{aligned}\zeta(1) &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ \zeta(2) &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots \\ &= \frac{\pi^2}{6}.\end{aligned}$$

**Example** (Absolute Convergence). In our original ratio test, we had assumed that  $a_k$  are real and positive. However, if the  $a_k \in \mathbb{C}$ , we have to look at the convergence in modulus:

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

If  $\sum_k |a_k|$  converges, this is known as absolute convergence.

**Example** (Alternating Series Test). If the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

has the following conditions:

- $a_{k+1} < a_k$  for  $k > K$ ;
- $\lim_{k \rightarrow \infty} a_k = 0$ ;

then  $\sum_k (-1)^k a_k$  converges.

For instance, the alternating harmonic series converges

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2.$$

## Power Series

Consider the function

$$S(x) = \sum_{k=0}^{\infty} a_k x^k.$$

This is a series both in  $a_k$  and in  $x$ . In order to determine convergence, we use the ratio test as follows:

$$\begin{aligned}\lim_{k \rightarrow \infty} \left| \frac{a_{k+1} x^{k+1}}{a_k x^k} \right| &= |x| \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| \\ &\equiv |x| r.\end{aligned}$$

In particular, for convergence, it must be the case that

$$|x| r < 1.$$

We define

$$R = \begin{cases} \frac{1}{r} & 0 < r < \infty \\ 0 & r = \infty \\ \infty & r = 0 \end{cases}.$$

In particular, this means

$$|x| < R.$$

**Definition** (Radius of Convergence). For a power series  $\sum_k a_k x^k$ , the series converges for  $|x| < R$ ,<sup>vi</sup> where

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$R = \begin{cases} \frac{1}{r} & 0 < r < \infty \\ 0 & r = \infty \\ \infty & r = 0 \end{cases}.$$

Note that convergence for  $|x| < R$  does not provide information regarding convergence at the boundary.

**Example** (Geometric Series). We have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

has  $R = 1$ , meaning the power series converges for  $|x| < 1$ .

**Example** (Exponential Function). We have

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

with  $R = \infty$ .

**Example** (Natural Log). We have

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$$

In particular, since  $R = 1$ , we know that the radius of convergence is  $|x| < 1$ . However, the series does converge on the boundary when  $x = 2$ , but not when  $x = 0$  (for obvious reasons).

**Example** (Why Radius of Convergence?). Consider two series

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

We can see that the first series converges for  $|x| < 1$ . However, even though  $\frac{1}{1+x^2}$  has a domain across the entire real numbers, it is still the case that the *series* converges for  $|x| < 1$ .

The primary reason that the radius of convergence is defined as such is because, over the complex numbers, it is the case that  $x^2 + 1 = 0$  at  $x = \pm i$ , meaning  $\frac{1}{1+z^2}$  has singularities at those values of  $z$ .

The main reason power series are useful is that, when truncated, they are simply polynomials. In particular, with power series, we can reverse the order of sum and derivative.

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<sup>vi</sup>The definition is not the true radius of convergence; it is actually that  $r = \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|}$ . It just happens to be the case that the ratio test and root test return the same value when they're regular limits (rather than limits superior).

## Taylor Series

Function	Taylor Series
$f(x)$	$\sum_{k=0}^{\infty} \frac{(x-x_0)^n}{n!} \left( \left. \frac{d^n f}{dx^n} \right _{x=x_0} \right)$
$e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$
$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$ <sup>vii</sup>

**Definition.** The Taylor series of a function  $f(x)$  about  $x_0$  is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left( \left. \frac{d^n f}{dx^n} \right|_{x=x_0} \right).$$

**Remark:** The reason we write  $\frac{d^n f}{dx^n}$  is because  $\frac{d^n}{dx^n}$  is an operator in and of itself.

**Example** (The Most Important Taylor Series).

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \end{aligned}$$

**Example** (Equilibrium Points). Let  $U(x)$  denote a potential over  $x$ . Then,  $F = -\nabla U$ . We have

$$U(x) = U(x_0) + (x-x_0) U'(x_0) + \frac{1}{2!} (x-x_0)^2 U''(x_0) + \frac{1}{3!} (x-x_0)^3 U'''(x_0) + \dots$$

When we analyze an equilibrium point, we disregard the  $U(x_0)$  term, and see that the derivative of  $U$  is zero; thus, we can truncate our series at the second derivative close to  $x = x_0$ :

$$\begin{aligned} U(x) &\approx \frac{1}{2} U''(x_0) (x-x_0)^2 \\ &= \frac{1}{2} m\omega^2 (x-x_0)^2. \end{aligned}$$

In other words, when we are very close to equilibrium, we have simple harmonic motion.

**Example** (Faster Taylor Series). Consider the function

$$\exp\left(\frac{x}{1-x}\right).$$

In order to create its Taylor series, we can create this Taylor series piecewise:

$$\exp\left(\frac{x}{1-x}\right) = 1 + \left(\frac{x}{1-x}\right) + \frac{1}{2!} \left(\frac{x}{1-x}\right)^2 + \frac{1}{3!} \left(\frac{x}{1-x}\right)^3 + \dots$$

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<sup>vii</sup>We define  $(\alpha)_n = \prod_{k=0}^{n-1} (\alpha - k)$

Now, we expand the denominators as geometric series:

$$= 1 + x \left( \sum_{k=0}^{\infty} x^k \right) + \frac{x^2}{2!} \left( \sum_{k=0}^{\infty} x^k \right)^2 + \frac{x^3}{3!} \left( \sum_{k=0}^{\infty} x^k \right)^3 + \dots$$

If we want to expand through  $x^3$ , we have to expand by keeping track of *every* term:

$$= 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + O(x^4).$$

We say we have expanded the series through the third order; the lowest order correction, denoted  $O(x^n)$ , is the fourth order (in this case).

**Example** (Exponentiated Operator). Consider a (square) matrix  $M$ . Then, we define

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!},$$

where  $M^k = \prod_{i=1}^k M$ ; we define  $M^0 = I$ . Similarly,

$$e^{\frac{d}{dx}} = \sum_{k=0}^{\infty} \frac{d^k}{dx^k} \frac{1}{k!}.$$

In particular,  $e^{\frac{d}{dx}}$  is the Taylor series operator.

**Remark:** In quantum mechanics, the momentum operator is

$$P = -i\hbar \frac{d}{dx}.$$

**Example** (Binomial Expansion). For any  $\alpha \in \mathbb{C}$  and  $|x| < 1$ , we have

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

Note that if  $\alpha \in \mathbb{Z}^+$ , then the series truncates (and we recover the binomial theorem again).

The main use of the binomial expansion is with very small quantities. For instance,

$$\begin{aligned} E &\sim \frac{1}{(x^2 + a^2)^{3/2}} \\ &= \frac{1}{x^3 \left(1 + \frac{a^2}{x^2}\right)^{3/2}} \\ &\approx \frac{1}{x^3} \left(1 - \frac{3}{2} \frac{a^2}{x^2}\right) \end{aligned} \quad \text{For } x \gg a$$

**Remark:** The binomial expansion only applies to the form  $(1+x)^\alpha$ . If we are dealing with an expression of the form  $(a+x)^\alpha$ , we need to factor out  $a$ , making the expression  $a^\alpha (1+x/a)^\alpha$ .

**Example** (Special Relativity with the Binomial Expansion). In the theory of special relativity, Einstein came up with the equations

$$\begin{aligned} E &= \gamma mc^2 \\ \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \end{aligned}$$

We can use the binomial expansion to find more information about  $\gamma$ .

$$\begin{aligned}
 E &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} mc^2 \\
 &= \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} \left(-\frac{v^2}{c^2}\right)^2 + \dots\right) mc^2 \\
 &= mc^2 + \underbrace{\frac{1}{2} mv^2 \left(1 + \frac{3}{4} \frac{v^2}{c^2} + \frac{5}{8} \left(\frac{v^2}{c^2}\right)^2 + \dots\right)}_{\text{Kinetic Energy}}
 \end{aligned}$$

As we take  $v \ll c$ , we only need to keep the first order term in the expansion, meaning we have  $E = mc^2 + \frac{1}{2}mv^2$ .

Thus, we can find kinetic energy as  $KE = (\gamma - 1) mc^2$ . Notice that this means that *most* energy is internal energy emergent as mass.

## Ten Integration Techniques

While Mathematica may exist,<sup>viii</sup> it is still valuable to know how to take various integrals. More importantly, knowing how to take integrals provides valuable insights into *what* exactly integrals are.

### Integration by Parts

**Definition** (Integration by Parts). Using the product rule, we have

$$\begin{aligned}
 \int \frac{d}{dx} (uv) dx &= \int \frac{du}{dx} v - \frac{dv}{dx} u dx \\
 &= \int \frac{du}{dx} v dx - \int \frac{dv}{dx} u dx.
 \end{aligned}$$

Thus, we get

$$\int u dv = uv - \int v du.$$

In the case where our integrals are definite, we have

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

We say  $uv \Big|_a^b$  is the boundary term (or surface term).<sup>ix</sup>

**Example.**

$$\begin{aligned}
 \int x e^{ax} dx &= \frac{1}{a} x e^{ax} - \int \frac{1}{a} e^{ax} dx & u = x, dv = e^{ax} dx \\
 &= \frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax} \\
 &= \frac{1}{a^2} e^{ax} (ax - 1).
 \end{aligned}$$

The +C is implicit.

<sup>viii</sup>Citation needed.

<sup>ix</sup>We can also use integration by parts to define the (weak) derivative, assuming the boundary term is zero.

**Example.**

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \left( \frac{1}{x} \right) dx & u = \ln x, \, dv = dx \\ &= x \ln x - x.\end{aligned}$$

### Change of Variables

**Definition** (u-Substitution). Let  $x = x(u)$ , meaning  $dx = \frac{dx}{du} du$ . Thus, we get

$$\int_{x_1}^{x_2} f(x) \, du = \int_{u(x_1)}^{u(x_2)} f(x(u)) \frac{dx}{du} \, du.$$

**Example.**

$$\begin{aligned}I_1 &= \int_0^\infty x e^{-ax^2} \, dx \\ &= \frac{1}{2} \int_0^\infty e^{-au} \, du & u = x^2 \\ &= \frac{1}{2a}\end{aligned}$$

**Example.**

$$\begin{aligned}\int_0^\pi \sin \theta \, d\theta &= \int_{-1}^1 du & u = \cos \theta \\ &= 2.\end{aligned}$$

More generally, we have, for  $f(\theta) = f(\cos \theta)$ ,

$$\int_0^\pi f(\theta) \sin \theta \, d\theta = \int_{-1}^1 f(u) \, du.$$

**Example** (Trig Substitution).

$$\begin{aligned}\int_0^a \frac{x}{x^2 + a^2} \, dx &= \int_0^{\pi/4} \frac{a^2 \tan \theta \sec^2 \theta}{a^2 (1 + \tan^2 \theta)} \, d\theta & x = a \tan \theta \\ &= \int_0^{\pi/4} \tan \theta \, d\theta \\ &= -\ln(\cos \theta) \Big|_0^{\pi/4} \\ &= \ln(\sqrt{2}) \\ &= \frac{1}{2} \ln(2).\end{aligned}$$

**Example** (Trig Substitution 2.0). For rational functions of  $\sin \theta$  and  $\cos \theta$ , we can use the half-angle trig substitution  $u = \tan(\theta/2)$ .<sup>x</sup> This yields

$$\begin{aligned}d\theta &= \frac{2du}{1+u^2} \\ \sin \theta &= \frac{2u}{1+u^2}\end{aligned}$$

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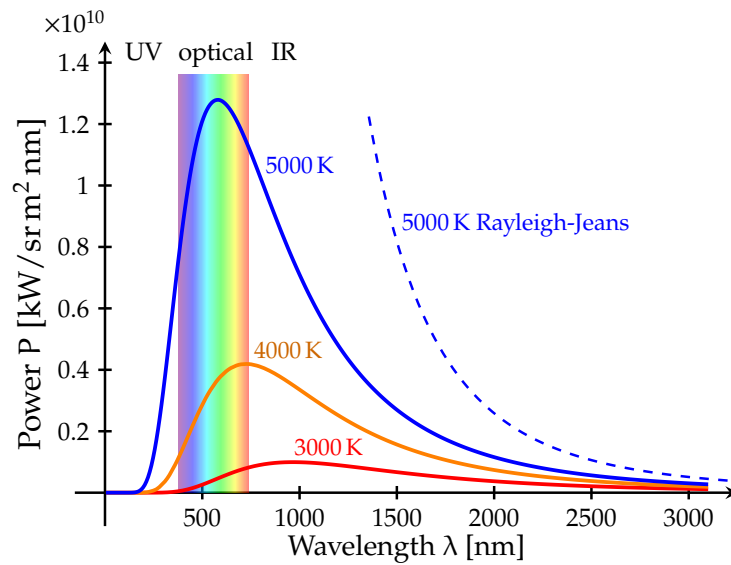
<sup>x</sup> $\tan(\theta/2) = \frac{\sin \theta}{1 + \cos \theta}$

$$\cos \theta = \frac{1 - u^2}{1 + u^2}.$$

For instance,

$$\begin{aligned} \int \frac{1}{1 + \cos \theta} d\theta &= \int \frac{1}{1 + \frac{1-u^2}{1+u^2}} \frac{2}{1+u^2} du \\ &= \int du \\ &= \tan(\theta/2) \\ &= \frac{\sin \theta}{1 + \cos \theta}. \end{aligned}$$

**Example** (Dimensionless Integrals).



Anything that has a nonzero absolute temperature radiates some energy. In particular, we want to know how this radiation is distributed among various wavelengths.

For a box of photons in equilibrium at temperature  $T$ , the energy per volume per wavelength  $\lambda^x$  is

$$u(\lambda) = \frac{8\pi hc}{\lambda^5 (e^{hc/\lambda kT} - 1)}.$$

Here,  $h$  denotes Planck's constant,  $c$  is the speed of light, and  $k$  is Boltzmann's constant.

In order to find the total energy density, we have to integrate  $u(\lambda)$  over all possible values of  $\lambda$ :

$$\begin{aligned} U &= \int_0^\infty u(\lambda) d\lambda \\ &= 8\pi hc \int_0^\infty \frac{1}{\lambda^5 (e^{hc/\lambda kT} - 1)} d\lambda \end{aligned}$$

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<sup>x</sup>read as (energy per volume) per wavelength

This integral is, for lack of a better word, hard. However, if we remove the dimensions of  $\lambda$  by substituting  $x = \frac{hc}{\lambda kT}$ , we can verify that the value of  $U$  now becomes

$$U = 8\pi hc \left( \frac{kT}{hc} \right)^4 \underbrace{\int_0^\infty \frac{x^3}{e^x - 1} dx}_{\text{scalar}}.$$

Thus, all the physics<sup>xii</sup> is captured as a coefficient on the integral; namely, this integral captures the Stefan-Boltzmann law, which has that energy density scales by  $T^4$ .

Using some fancy techniques we will learn later, we can evaluate

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}.$$

### Even/Odd

**Definition** (Even and Odd Functions). A function  $f(x)$  is

- even if  $f(-x) = f(x)$ ;
- odd if  $f(-x) = -f(x)$ .

Just as a matrix can be decomposed into a sum of a symmetric and antisymmetric matrix, we can decompose a function into a sum of an even function and an odd function.

Integrals over symmetric intervals on functions with definite parity are very simple:

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx & f \text{ odd} \\ 0 & f \text{ even} \end{cases}.$$

For the case of a function  $g(x) = g(|x|)$ , we have

$$\int_{-a}^b g(|x|) dx = \int_{-a}^0 g(-x) dx + \int_0^b g(x) dx.$$

### Products and Powers of Sines and Cosines

Value	Expression
$\sin(\alpha \pm \beta)$	$\sin \alpha \cos \beta \pm \sin \beta \cos \alpha$
$\cos(\alpha \pm \beta)$	$\cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
$\sin \alpha \cos \beta$	$\frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))$
$\cos \alpha \cos \beta$	$\frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta))$
$\sin \alpha \sin \beta$	$\frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$

**Example.** If we have an integral

$$\begin{aligned} \int \sin(3x) \cos(2x) dx &= \frac{1}{2} \int \sin(5x) + \sin(x) dx \\ &= \frac{1}{2} \left( -\frac{1}{5} \cos(5x) - \cos(x) \right). \end{aligned}$$

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<sup>xii</sup>Who cares about that stuff?



Integral	Shortcut
$\int \sin^m(x) \cos^{2k+1}(x) dx$	$\int u^m (1 - u^2)^k du$
$\int \sin^{2k+1}(x) \cos^n(x) dx$	$-\int (1 - u^2)^k u^n du$
$\int \sin^2(x) dx$	$\frac{x}{2} - \frac{1}{4} \sin(2x)$
$\int \cos^2(x) dx$	$\frac{x}{2} + \frac{1}{4} \sin(2x)$

**Example.** To evaluate

$$\int \sin^2(x) dx,$$

$$\int \cos^2(x) dx$$

we use the identity

$$\sin^2(x) = \frac{1}{2} (1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2} (1 + \cos(2x)),$$

and take

$$\begin{aligned} \int \sin^2(x) dx &= \frac{1}{2} \int (1 - \cos(2x)) dx \\ &= \frac{x}{2} - \frac{1}{4} \sin(2x) \\ \int \cos^2(x) dx &= \frac{1}{2} \int (1 + \cos(2x)) dx \\ &= \frac{x}{2} + \frac{1}{4} \sin(2x). \end{aligned}$$

Thus, we can see that

$$\begin{aligned} \int_0^\pi \sin^2(x) dx &= \frac{\pi}{2} \\ \int_0^\pi \cos^2(x) dx &= \frac{\pi}{2} \end{aligned}$$

### Axial and Spherical Symmetry

Consider a function of the form  $f(x, y) = x^2 + y^2$ . If we were to integrate with respect to  $dx dy$ , we would need a two dimensional integral. With polar coordinates, though, we would have  $dx dy = r dr d\phi$ . Since  $f$  is axially symmetric, we would have our  $dx dy = 2\pi r dr$ , which is a one-dimensional integral.

If we have something with spherical symmetry, then there is no dependence on either  $\theta$  or  $\phi$ , yielding a function  $f(\mathbf{r}) = f(r)$ , meaning

$$\begin{aligned} \int f(\mathbf{r}) d\tau &= \int f(r) r^2 \sin \theta dr d\theta d\phi \\ &= 4\pi \int f(r) r^2 dr. \end{aligned}$$

Note that  $\int \sin \theta d\theta d\phi$  over the sphere is  $4\pi$ .

**Example.** Consider a surface  $S$  with charge density  $\sigma(\mathbf{r})$ . Finding the total charge requires evaluating

$$Q = \int_S \sigma(\mathbf{r}) \, dA.$$

If  $S$  is hemispherical with  $z > 0$  with radius  $R$ , and  $\sigma = k \frac{x^2 + y^2}{R^2}$ , the integrand is axially symmetric.

Using spherical coordinates, we evaluate

$$\begin{aligned} Q &= \int_S \sigma(\mathbf{r}) \, dA \\ &= \frac{k}{R^2} \int x^2 + y^2 \, dA \\ &= \frac{k}{R^2} \int \left( R^2 \sin^2 \theta \cos^2 \phi + R^2 \sin^2 \theta \sin^2 \phi \right) R^2 \sin \theta \, d\theta d\phi \\ &= kR^2 \int_S \sin^3 \theta \, d\theta d\phi \\ &= 2\pi kR^2 \int_0^{\pi/2} \sin^3 \theta \, d\theta \\ &= \frac{4\pi kR^2}{3}. \end{aligned}$$

**Example.** Let

$$\Phi(\mathbf{r}) = \int \frac{e^{-i\mathbf{k} \cdot \mathbf{r}}}{(2\pi)^3 \|\mathbf{k}\|^2} \, d^3k$$

where  $k$ -space is an abstract 3-dimensional Euclidean space. In Cartesian coordinates,  $d^3k = dk_x dk_y dk_z$ , which yields the integral

$$\Phi(\mathbf{r}) = \int \frac{e^{-ik_x x} e^{-ik_y y} e^{-ik_z z}}{(2\pi)^3 (k_x^2 + k_y^2 + k_z^2)} \, dk_x dk_y dk_z.$$

This integral is very hard to evaluate (over Cartesian coordinates, anyway),<sup>xiii</sup> so we need to use some other methods.

In spherical coordinates, we have  $d^3k = k^2 dk d\Omega$ , yielding

$$\Phi(\mathbf{r}) = \frac{1}{(2\pi)^3} \int k^2 \frac{e^{-i\mathbf{k} \cdot \mathbf{r} \cos \theta}}{k^2} \, dk d(\cos \theta) d\phi.$$

Since we are summing away all our  $k$ -dependence, we can orient  $\mathbf{r}$  along the  $k_z$  axis. Thus, we can evaluate the integral as

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int k^2 \frac{e^{-i\mathbf{k} \cdot \mathbf{r} \cos \theta}}{k^2} \, dk d(\cos \theta) d\phi \\ &= \frac{1}{(2\pi)^2} \int_{-1}^1 \int_0^\infty e^{-i\mathbf{k} \cdot \mathbf{r} \cos \theta} \, dk d(\cos \theta) \\ &= \frac{1}{(2\pi)^2} \int \frac{1}{(-i\mathbf{k} \cdot \mathbf{r})} \left( e^{-i\mathbf{k} \cdot \mathbf{r}} - e^{i\mathbf{k} \cdot \mathbf{r}} \right) \, dk \end{aligned}$$

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<sup>xiii</sup>Citation needed.

$$\begin{aligned}
&= \frac{1}{(2\pi)^2} \int_0^\infty \frac{2 \sin(kr)}{kr} dk \\
&= \frac{1}{2\pi^2} \underbrace{\int_0^\infty \frac{\sin(kr)}{kr} dk}_{\text{sinc integral}}.
\end{aligned}$$

In order to evaluate the sinc integral, we have to use some different techniques.

### Differentiation with Respect to a Parameter

**Example.** We can evaluate

$$\begin{aligned}
\int x e^{ax} dx &= \frac{\partial}{\partial a} \left( \int e^{ax} dx \right) \\
&= \frac{\partial}{\partial a} \left( \frac{1}{a} e^{ax} \right) \\
&= -\frac{1}{a^2} e^{ax} + \frac{1}{a} x e^{ax} \\
&= \frac{1}{a^2} e^{ax} (ax - 1)
\end{aligned}$$

When differentiating with respect to a parameter, it is important to remember that we are often differentiating *with respect to the parameter*, not with respect to our main variable.

**Example** (Introducing a Parameter). We wish to solve the sinc integral,

$$\int_0^\infty \frac{\sin x}{x} dx.$$

In order to do this, we will introduce a parameter such that differentiation will cancel out the  $x$  in the denominator:

$$J(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\sin x}{x} dx. \quad \alpha > 0$$

In particular,  $\alpha > 0$ . We calculate

$$\begin{aligned}
\frac{dJ}{d\alpha} &= - \int_0^\infty e^{-\alpha x} \sin x dx \\
&= -\frac{1}{1 + \alpha^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J(\alpha) &= - \int \frac{1}{\alpha^2} d\alpha \\
&= -\arctan(\alpha) + C.
\end{aligned}$$

In order to determine the value of  $C$ , we need to make sure  $J(\infty) = 0$ . Therefore,  $C = \frac{\pi}{2}$ . Therefore, we have

$$J(0) = \frac{\pi}{2}.$$

## Gaussian Integral

We cannot evaluate  $I_0 = \int_0^\infty e^{-ax^2} dx$  using elementary methods, because  $e^{-ax^2}$  is not an elementary function. The reason we care a lot about  $e^{-ax^2}$  is because it is very important in quantum mechanics and statistics.<sup>xiv</sup>

It is clear that  $I_0$  converges. We can see that the dimension of  $a$  is  $x^{-2}$ , and since we are integrating with respect to  $dx$ , we can see that our integral is related to  $\frac{1}{\sqrt{a}}$ .

**Example.** We will not solve for  $I_0$ , but for  $I_0^2$ . Thus, we have

$$\begin{aligned}
 I_0^2 &= \left( \frac{1}{2} \int_{-\infty}^{\infty} e^{-ax^2} dx \right) \left( \frac{1}{2} \int_{-\infty}^{\infty} e^{-ay^2} dy \right) \\
 &= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy \\
 &= \frac{1}{4} \int_0^{2\pi} \int_0^\infty r e^{-ar^2} dr d\phi \\
 &= \frac{\pi}{2} \int_0^\infty r e^{-ar^2} dr \\
 &= \frac{\pi}{2} \left( \frac{1}{2} \int_0^\infty e^{-au} du \right) \\
 &= \frac{\pi}{4a}.
 \end{aligned}$$

Therefore,  $I_0 = \frac{1}{2} \sqrt{\frac{\pi}{a}}$ .

**Definition** (Family of Gaussian Integrals).

$$I_n = \int_0^\infty x^n e^{-ax^2} dx.$$

Expression	Value
$I_0$	$\frac{1}{2} \sqrt{\frac{\pi}{a}}$
$I_1$	$\frac{1}{2a}$
$I_{2n}$	$(-1)^n \frac{d^n}{da^n} I_0$
$I_{2n+1}$	$(-1)^n \frac{d^n}{da^n} I_1$

It is important to note that there are different expressions for the Gaussian integral:

$$\begin{aligned}
 &\int e^{-ax^2} dx \\
 &\int e^{-a^2x^2} dx \\
 &\int e^{-a^2x^2/2} dx \\
 &\int e^{-x^2/a} dx \\
 &\int e^{-x^2/a^2} dx,
 \end{aligned}$$

meaning we have to be careful when evaluating these integrals.

<sup>xiv</sup>Who cares about that?

**Example** (Error Function). Consider the integral

$$\int_0^{53} e^{-ax^2} dx.$$

Unfortunately, there is no way to do this integral analytically. It is only able to be calculated numerically.

We define

$$\text{erf}(u) = \int_0^u e^{-ax^2} dx$$

### Completing the Square

**Example.** Consider the integral

$$\int_{-\infty}^{\infty} e^{-ax^2-bx} dx.$$

This integral is Gaussian-esque, but it isn't fully Gaussian, yet.

To do this, we will complete the square:

$$\begin{aligned} ax^2 + bx &= a \left( x^2 + \frac{b}{a}x \right) \\ &= a \left( x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} \right) \\ &= a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a}. \end{aligned}$$

In particular, this turns the integral into

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ax^2-bx} dx &= \int_{-\infty}^{\infty} e^{-a(x+b/2a)^2+b^2/4a} dx \\ &= e^{b^2/4a} \int_{-\infty}^{\infty} e^{-a(x+b/2a)} dx \\ &= e^{b^2/4a} \left( \sqrt{\frac{\pi}{a}} \right) \\ &= e^{b^2/4a} \sqrt{\frac{\pi}{a}}. \end{aligned}$$

### Series Expansion

Function	Expression
$\Gamma(s)$	$\int_0^{\infty} x^{s-1} e^{-x} dx$
$\zeta(s)$	$\sum_{k=1}^{\infty} \frac{1}{k^s}$
$\Gamma(s+1)$	$s\Gamma(s)$

Consider the integral

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

This is a very nasty integral,<sup>xv</sup> but we will need to know this value because it is useful in statistical mechanics.<sup>xvi</sup> We want to ensure this converges.

Notice that for large  $x$ , the integrand looks like  $e^{-x}x^{s-1}$ .

**Example.** To resolve the integral we take

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \int_0^{\infty} \frac{e^{-x}x^{s-1}}{1 - e^{-x}} dx$$

We will use the geometric series expansion for the denominator:

$$\begin{aligned} &= \int_0^{\infty} e^{-x}x^{s-1} \sum_{k=0}^{\infty} e^{-kx} dx \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} x^{s-1} e^{-(k+1)x} dx. \end{aligned}$$

We make the change of variables  $u = (n+1)x$ .

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} \int_0^{\infty} u^{s-1} e^{-u} du \\ &= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^s}}_{\zeta(s)} \underbrace{\int_0^{\infty} u^{s-1} e^{-u} du}_{\Gamma(s)}. \end{aligned}$$

Thus, our integral resolves to

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s)\zeta(s).$$

## Partial Fractions

**Example** (A Partial Fraction Decomposition).

$$\begin{aligned} \frac{1}{1-x^2} &= \frac{\alpha}{1-x} + \frac{\beta}{1+x} \\ &= \frac{1/2}{1-x} + \frac{1/2}{1+x}. \end{aligned}$$

**Example** (Integrating using Partial Fractions). To evaluate

$$\int \frac{4-2x}{(x^2+1)(x-1)^2} dx,$$

we do the partial fraction decomposition to find

$$\int \frac{4-2x}{(x^2+1)(x-1)^2} dx = \int \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} dx.$$

<sup>xv</sup>Citation needed.

<sup>xvi</sup>Okay actually I do kinda care about this.

**Example** (Mean Value Theorem). If we have a function  $f$  defined on  $[a, b]$ , then there is a point  $c \in (a, b)$  such that

$$f(c)(b - a) = \int_a^b f(x) dx.$$

More generally, the mean value theorem says there exists  $c \in (a, b)$  such that

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx$$

## Delta Distribution

Consider a “function”  $\delta(x)$  such that

$$\int_{-\infty}^{\infty} f(x)\delta(x - a) dx = f(a).$$

This idea seems absurd on its face — after all, singletons have measure zero, so the idea of an integral collapsing into a single point doesn’t sound normal.

The structure of the delta distribution is

$$\delta(x - a) = \begin{cases} +\infty & x = a \\ 0 & \text{else} \end{cases}.$$

In particular, we also have to define

$$\int_{-\infty}^{\infty} \delta(x - a) dx = 1.$$

This is known as the Dirac delta function (or rather, distribution). The delta distribution “weights”  $f$  to infinity at  $x = a$  and zero everywhere else.

**Example** (Delta Distribution as a Limit). Imagine a Gaussian function with area under the curve 1. In particular,

$$f_n(x) = \frac{1}{\sqrt{\pi}} n e^{-n^2 x^2}.$$

In particular, we have

$$\delta(x) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\pi}} n e^{-n^2 x^2}$$

**Example** (A Physical Example). Imagine a ball is kicked. The force is dependent on time,  $F(t)$ .

There isn’t an easy to find the force, but by Newton’s second law, we have

$$\delta p = \int F(t) dt,$$

where

$$I \equiv \int F(t) dt$$

is the impulse.

If we want to model  $F(t)$ , where we don't care about a nonzero time over which the force is occurring, we can simply state  $F(t)$  as

$$F(t) = \Delta p \delta(t - t_0).$$

Taking this integral yields I.

**Example** (Fourier Integral Representation of Delta Distribution). A different representation of  $\delta(x)$  is

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk.$$

We are superimposing all the waves  $e^{ikx}$  — in particular, for all values of  $k \neq 0$ , both  $e^{ikx}$  and  $e^{-ikx}$  are “added” together, yielding absolute destructive interference.

The factor of  $\frac{1}{2\pi}$  is necessary to normalize the integral.

### Properties of the Delta Distribution

**Normalization:**

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(x) dx &= 1 \\ \int_{x_1}^{x_2} \delta(x - a) dx &= \begin{cases} 1 & x_1 < a < x_2 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

**Sieve:**

$$\int_{x_1}^{x_2} f(x) \delta(x - a) dx = \begin{cases} f(a) & x_1 < a < x_2 \\ 0 & \text{else} \end{cases}.$$

**Example** (Delta Distribution as a Limit of Rectangles). We define the family of functions

$$\phi_k(x) = \begin{cases} k/2 & |x| < 1/k \\ 0 & |x| > 1/k \end{cases}.$$

We can see that integrating  $\phi_k$  over  $\mathbb{R}$  yields 1 for each  $k$ .

We now need to evaluate if  $\lim_{k \rightarrow \infty} \phi_k(x) = \delta(x)$ . In order to see this, we take

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \phi_k(x) dx &= \lim_{k \rightarrow \infty} \frac{k}{2} \int_{-1/k}^{1/k} f(x) dx \\ &= \lim_{k \rightarrow \infty} f(c_k) \left( \frac{k}{2} \int_{-1/k}^{1/k} dx \right) \\ &= \lim_{k \rightarrow \infty} f(c_k), \end{aligned}$$

where we define  $c_k$  from the mean value theorem. In particular, since  $c_k \in (-1/k, 1/k)$ , it is the case that  $c_k \rightarrow 0$  as  $k \rightarrow \infty$ , so

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \phi_k(x) dx &= \lim_{k \rightarrow \infty} f(c_k) \\ &= f(0). \end{aligned}$$

Thus,  $\lim_{k \rightarrow \infty} \phi_k(x) = \delta(x)$ .



We can imagine the delta distribution to be the density distribution of a single point.

The units of  $\delta(x)$  are

$$[\delta(x)] = x^{-1}.$$

**Complicated Arguments:** Imagine  $\delta(ax)$ . For instance, we want to evaluate

$$\int_{-\infty}^{\infty} f(x) \delta(ax) \, dx.$$

To do so, we use  $u$  substitution with  $u = ax$ :

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \delta(ax) \, dx &= \frac{1}{a} \int_{-\infty}^{\infty} f(u/a) \delta(u) \, du \\ &= \frac{1}{a} f(0). \end{aligned}$$

It is important to note that the integration variable  $dx$  and the argument of  $\delta(x)$  must be equal.

In general, we have

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$