Abstract

We discuss extremal structure in locally convex topological vector spaces, as well as a fundamental result in the theory of topological vector spaces: the Krein–Milman theorem. We also use extremal structure to prove the Stone–Weierstrass Theorem and the Banach–Stone theorem.

Extremal Structure

We need to recall some basic ideas related to convexity and compactness in topological vector spaces.

Theorem: If $A_1, \dots, A_n$ are compact convex sets in a topological vector space X, then $\operatorname{conv}(A_1 \cup \cdots \cup A_n)$ is compact.

Proof. Let $\Delta_n = \operatorname{conv}(e_1, \dots, e_n)$ be the basic simplex in $\mathbb{R}^n$, where elements look like

$$\Delta_n = \left\{ (s_1, \dots, s_n) \;\middle|\; s_i \geq 0, \sum_{i=1}^{n} s_i = 1 \right\}.$$

Define $A = A_1 \times \cdots \times A_n$, and set $f \colon \Delta_n \times A \to X$ to be defined by $f(s, a) = \sum_i s_i a_i$. We set $K = f(S \times A)$.

Note that since f is continuous (as addition and scalar multiplication are continuous), Δ_n is compact, and A is compact, we have that K is compact. Furthermore, $K \subseteq \operatorname{conv}(A_1 \cup \cdots \cup A_n)$. We will now show that the inclusion goes in the opposite direction.

We will do this by showing that K is convex. Let $(s, a), (t, b) \in S \times A$, and let $0 \leq q \leq 1$. Then, defining

$$\begin{aligned} u &= qs + (1 - q)t \\ c_i &= \frac{q s_i a_i + (1 - q) t_i b_i}{q s_i + (1 - q) t_i}, \end{aligned}$$

we have

$$\begin{aligned} qf(s, a) + (1 - q) f(t, b) &= f(u, c) \\ &\in K, \end{aligned}$$

meaning K is convex, so $\operatorname{conv}(A_1 \cup \cdots \cup A_n) \subseteq K$. □

Definition. Let K be a subset of a vector space X. A nonempty $S \subseteq K$ is called a *face* for K if the interior of any line in K that is contained in S contains its endpoints. Analytically, this means that if $x, y \in K$ are such that, for all $t \in (0, 1)$, $tx + (1 - t)y \in S$, then $x, y \in S$.

An *extreme point* of K is an extreme set of K that consists of one point. We write $\operatorname{ext}(K)$ for the extreme points of K.

Example. Let Ω be a LCH space. The extreme points of the regular Borel probability measures on Ω are the Dirac measures. That is,

$$\operatorname{ext}(\mathcal{P}_r(\Omega)) = \{\delta_x \mid x \in \Omega\}.$$

In one direction, we see that if $x \in \Omega$, and $\delta_x = \frac{1}{2}(\mu + \nu)$, then for a Borel set $E \subseteq \Omega$ with $x \in E$, we have $1 = \frac{1}{2}(\mu(E) + \nu(E))$. Therefore, $\mu(E) = \nu(E) = 1$. If $x \notin E$, then $0 = \frac{1}{2}(\mu(E) + \nu(E))$, so $\mu(E) = \nu(E) = 0$. Thus, $\mu = \nu = \delta_x$, so every δ_x is extreme.

In the opposite direction, if $\mu \in \operatorname{ext}(\mathcal{P}_r(\Omega))$, we claim that there is $x_0 \in \Omega$ with $\operatorname{supp}(\mu) = \{x_0\}$. Now, since $\mu(\Omega) = 1$, we know that $\operatorname{supp}(\mu) \neq \varnothing$.

Suppose there exist $x, y \in \operatorname{supp}(\mu)$ with $x \neq y$. Since Ω is Hausdorff, we can separate $x, y \in \operatorname{supp}(\mu)$ with disjoint open sets U and V, where $0 < \mu(U) < 1$ and $0 < \mu(V) < 1$. Set $t = \mu(U)$, and define

$$\begin{aligned}\mu_1(E) &= \frac{\mu(E \cap U)}{\mu(U)}\\ \mu_2(E) &= \frac{\mu(E^c)}{\mu(U^c)}.\end{aligned}$$

Then, μ_1, μ_2 are regular Borel probability measures with $\mu_1 \neq \mu_2$ and $t\mu_1 + (1-t)\mu_2 = \mu$, which contradicts μ being extreme. Therefore, $\operatorname{supp}(\mu) = \{x_0\}$, so $\mu = \delta_{x_0}$.

Example. The picture of a face in a convex compact set is relatively simple. If $u \colon X \to \mathbb{R}$ is an $\mathbb{R}$-linear continuous functional, and $P \subseteq X$ is compact and convex, the infimum $\inf_{x \in P} u(x) =: s$ is attained. The subset

$$P_u = \{x \in P \mid u(x) = s\}$$

is a closed face in P.

To start, P_u is nonempty because the infimum is attained. Since u is continuous, P_u is closed. Furthermore, if $t \in [0,1]$ and $x, y \in P_u$, then $(1-t)x + ty \in P_u$, as

$$\begin{aligned}u((1-t)x + ty) &= (1-t)u(x) + tu(y)\\ &= (1-t)s = ts\\ &= s.\end{aligned}$$

Now, if $t \in (0,1)$ and $x, y \in P$ with $(1-t)x + ty \in P_u$, then

$$s = (1-t)u(x) + tu(y).$$

Since $u(x) \geq s$ and $u(y) \geq s$, we must have $u(x) = u(y) = s$, meaning $x, y \in P_u$.

The Krein–Milman Theorem

One of the most important results in extremal structure is the fact that every compact convex set of a topological vector space (with some relatively weak conditions) has an extreme point — moreover, there are a lot of extreme points.

Theorem (Krein–Milman)**:** Let X be a topological vector space where X^* separates points. If K is a nonempty compact convex set in X, then

$$K = \overline{\operatorname{conv}}(\operatorname{ext}(K)).$$

Proof. We start with a lemma.

Lemma: If F is a face of K and G is a face of F, then G is a face of K.

Proof. Let $x, y \in K$ be such that for all $t \in (0,1)$, $(1-t)x + ty \in G$. Then, since G is a face of F, we have $(1-t)x + ty \in F$, so since F is a face, $x, y \in F$. However, since G is a face, $x, y \in G$, so G is a face of K. □

We start by showing that $\operatorname{ext}(K) \neq \varnothing$. Let $F \subseteq K$ be a closed face. The family

$$\mathcal{G} = \{G \subseteq F \mid G \text{ is a closed face in } F\}$$

is nonempty, as $F \in \mathcal{G}$. Ordering $\mathcal{G}$ by containment, we will show that $\mathcal{G}$ satisfies the conditions of Zorn's lemma. If $\mathcal{C} \subseteq \mathcal{G}$ is a chain, then we claim that

$$I = \bigcap_{G \in \mathcal{C}} G$$

is an element of $\mathcal{G}$ that is an upper bound for $\mathcal{C}$. First, since I is an arbitrary intersection of convex sets, I is convex.

Furthermore, for any $G_1, \dots, G_n \in \mathcal{C}$, then since $\mathcal{C}$ is a chain, there is j such that $G_i \preceq G_j$ For all $i = 1, \dots, n$, meaning $\bigcap_{i=1}^{n} G_i = G_j \neq \varnothing$. Since K is compact, the finite intersection property gives $I \neq \varnothing$. Finally, let $t \in (0,1)$ with $x, y \in F$ and $(1-t)x + ty \in I$. Then, $(1-t)x + ty \in G$ for all $G \in \mathcal{C}$, so $x, y \in G$ for all $G \in \mathcal{C}$, so $x, y \in I$, meaning I is a face. Notice that for all $G \in \mathcal{C}$, we have $G \preceq I$, so the conditions of Zorn's lemma are satisfied.

By Zorn's lemma, there is a maximal $P \in \mathcal{G}$. We claim that P is a singleton.

Note that P is compact since it is closed. Let $\varphi \in X^*$ and set $u = \operatorname{Re}(\varphi)$. Since P is compact, the set

$$P_u = \left\{ p \in P \;\middle|\; u(p) = \inf_{x \in P} u(x) \right\},$$

and by maximality, we must have $P_u = P$. Since $\varphi(x) = u(x) - iu(ix)$, we must have that φ is constant on P, so $P = \{z\}$ as X^* separates points.

Since F is a face, and $P \subseteq F$ is a face, P is a face, so $z \in \operatorname{ext}(K)$.

Now, note that $C = \overline{\operatorname{conv}}(\operatorname{ext}(K)) \subseteq K$ as K is closed and convex. Suppose that this inclusion is strict. Let $x_0 \in K \setminus C$.

Then, by the Hahn–Banach separation, there is $\varphi \in X^*$ such that for all $y \in C$,

$$u(x_0) < t \leq u(y),$$

where $u = \operatorname{Re}(\varphi)$. Let $s = \inf_{k \in K} u(k)$, so that $K_u = \{x \in K \mid u(x) = s\}$. This is a closed face in K, so it has an extreme point $z \in K$, with $z \in C$. Then, $u(z) \geq t > s$, but $z \in K_u$, so $u(z) = s$. Therefore, the inclusion is not strict. □

Other Uses of Extremal Structure

The Stone–Weierstrass Theorem

The Banach–Stone Theorem