

1.15

Problem. Define $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by $f(a, b) = 2^a 3^b$. Show that f is injective. Use the Cantor–Schröder–Bernstein theorem to deduce that $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Solution. Suppose $2^{a_1} 3^{b_1} = 2^{a_2} 3^{b_2}$. By the fundamental theorem of arithmetic, it must be the case that $a_1 = a_2$ and $b_1 = b_2$, meaning f is injective.

Since we have an injection $g : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ with $g(n) = (n, 0)$, it is the case that, by the Cantor–Schröder–Bernstein theorem, there exists some bijection between $\mathbb{N} \times \mathbb{N}$ and \mathbb{N} , meaning they have the same cardinality.

1.16

Problem. Let A be the set of all finite subsets of \mathbb{N} . Find injective functions from \mathbb{N} to A and vice versa. Use the Cantor–Schröder–Bernstein theorem to deduce that A is countably infinite. Then, prove that the number of infinite subsets of \mathbb{N} is uncountable.

Solution. There is a simple injection from \mathbb{N} to $A = \mathcal{F}(\mathbb{N})$ by taking $f(n) = \{n\}$. In the reverse direction, for $B \in A$, define $g_B : B \rightarrow \{1, 2, \dots, n\}$ to be the bijection that defines the cardinality of B . We let b_i be the element of B such that $g(b_i) = b_i$. Taking $h : A \rightarrow \mathbb{N}$ by $h(B) = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$, where p_n denotes the n th prime number, we have an injection (by the fundamental theorem of arithmetic) between A and \mathbb{N} . Thus, by the Cantor–Schröder–Bernstein theorem, there is a bijection between A and \mathbb{N} , so A is countably infinite.

Since the set of all subsets of \mathbb{N} , $\mathcal{P}(\mathbb{N})$, is uncountable, and $A = \mathcal{F}(\mathbb{N})$ is countable, it is the case that the set of infinite subsets of \mathbb{N} , $\mathcal{P}(\mathbb{N}) \setminus \mathcal{F}(\mathbb{N})$, is uncountable. To show this, suppose toward contradiction that $\mathcal{P}(\mathbb{N}) \setminus \mathcal{F}(\mathbb{N})$ is countable. Then, we would have $\mathcal{F}(\mathbb{N}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{F}(\mathbb{N}))$ is a countable union of countable sets, implying $\mathcal{P}(\mathbb{N})$ is countable, which is a contradiction.

1.17

Problem. Let \mathbb{R}^\times denote the set of nonzero real numbers. Use the Cantor–Schröder–Bernstein theorem to deduce that $|\mathbb{R}^\times| = |\mathbb{R}|$. Now, try to explicitly define a bijection between the sets.

Solution. The inclusion map $\iota : \mathbb{R}^\times \rightarrow \mathbb{R}$ is an injection, implying that $|\mathbb{R}^\times| \leq |\mathbb{R}|$. Additionally, the map $f : \mathbb{R} \rightarrow \mathbb{R}^\times$ defined by $f(x) = \arctan(x) + \pi/2$ is an injection from \mathbb{R} into \mathbb{R}^\times , meaning $|\mathbb{R}| \leq |\mathbb{R}^\times|$. Thus, by Cantor–Schröder–Bernstein, there is a bijection from \mathbb{R} to \mathbb{R}^\times .

The function

$$f : \mathbb{R} \rightarrow \mathbb{R}^\times$$

defined by

$$f(x) = \begin{cases} x + 1 & x \in \mathbb{N} \\ x & x \notin \mathbb{N} \end{cases}$$

is a bijection from \mathbb{R} to \mathbb{R}^\times .

1.18

Problem. Let $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $B = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Find injective functions $f : A \rightarrow B$ and $g : B \rightarrow A$, and deduce that $|A| = |B|$. Try to define an explicit bijection between A and B .

Solution. The inclusion map $\iota : A \hookrightarrow B$ is an injection between $(0, 1)$ and $[0, 1]$. Additionally, $g : [0, 1] \rightarrow (0, 1)$ defined by $g(x) = \frac{1}{2}x$ is also an injection between $[0, 1]$ and $(0, 1)$. Thus, by Cantor–Schröder–Bernstein, there is a bijection between A and B .

For an attempt at a bijection between $[0, 1]$ and $(0, 1)$, let

$$f_n : [0, 1] \rightarrow \left[\frac{1}{n}, 1 - \frac{1}{n} \right]$$

be a linear function. Notice that $\lim_{n \rightarrow \infty} \left[\frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1)$. We define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. I don't know exactly how to prove that this is a bijection, but it should be one.

Alternatively, we take

$$\left\{ \frac{1}{n} \mid n \geq 2 \right\},$$

and map $\frac{1}{2}$ to 0, $\frac{1}{3} \mapsto 1$, and $\frac{1}{n+2} \mapsto \frac{1}{n}$ for $n \geq 2$. For $x \notin \left\{ \frac{1}{n} \mid n \geq 2 \right\}$, we map $x \mapsto x$. This yields a bijection from $(0, 1)$ to $[0, 1]$.

1.19

Problem. Let $S = \{s_1, \dots, s_n\}$ be a nonempty set of finitely many symbols. Show that the number of finite strings consisting of elements of S is countably infinite. What happens if S is countably infinite?

Solution. We let S_i be the set of strings of length i ; there are n^i elements of S_i , which is finite. The set of all finite strings in S is

$$\bigcup_{i=1}^{\infty} S_i.$$

Since the set S_i are disjoint, it is the case that the set of all finite strings in S is a countably infinite union of finite disjoint sets, which is countably infinite.

If S is countably infinite, then by ordering the finite subsets of S by length and lexicographical order, we find that the set of finite subsets of S is finite.

1.20

Problem. The two questions below refer to Hilbert's Hotel, discussed at the end of the chapter.

- A fleet of countably infinite busses arrives with countably infinite passengers. Describe a way to assign rooms to everyone, including those currently in the hotel, such that no rooms are left empty.
- There are now a countably infinite number of fleets of countably infinite buses with a countably infinite number of people. Find a way for the desk attendant to accommodate all guests.

Solution.

- Move every current resident of the hotel to 2 multiplied by their current room number. Use the Cantor pairing function to map $\mathbb{N} \times \mathbb{N}$ to map each of the countably infinite busses' countably infinite members to \mathbb{N} . Then, for each new resident, multiply their room number by 2 and add 1.
- Proceeding in a similar manner, we can compose the Cantor pairing function with itself to create a bijection from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to \mathbb{N} , then multiply by 2 and add 1 to map every new resident to an odd room, while mapping every current resident to an even room.