Amenability: A (Somewhat) Brief Introduction

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Outline

Contents

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

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We abbreviate $a \star b$ as ab.

Subgroups, Quotient Groups

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- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* G/N.

Some Groups

- The integers $\mathbb{Z}$ are a group under addition.
- The group of invertible $n \times n$ matrices over $\mathbb{C}$, $\mathrm{GL}_n(\mathbb{C})$, is a group under matrix multiplication.
- The subgroup $\mathrm{SO}(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under multiplication.

Group Actions

Let G be a group, and X a set. Let $\rho\colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X. We write $\rho(g, x) = g \cdot x$.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i\in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a *σ-algebra* of subsets.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

σ-Algebras and Measures, Cont'd

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If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say μ is a measure. If $\mu(X) = 1$, then we say μ is a probability measure.

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Questions?

- If G is a group, is it possible to reconstruct G by using some subset of G?
- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We define $F(a,b)$ to be the set of all "words" in the alphabet $\left\{a,b,a^{-1},b^{-1}\right\}$, subject to the condition that, for $w,w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

A Curiosity

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A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

These decompositions seem to be downright paradoxical — we take a part of the group, translate some of it, and get the whole group back!

Defining Paradoxical Decompositions

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$;

such that

$$\begin{aligned} G &= \bigcup_{i=1}^{n} g_i A_i \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Actions

If G acts on a set X, then a subset $A \subseteq X$ is *G-paradoxical* if there exist

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq A$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$

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such that

$$A = \bigcup_{i=1}^{n} g_i \cdot A_i = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

A paradoxical group is a paradoxical set under the action of left-multiplication.

Examples

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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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This is proven using the Ping-Pong lemma.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

- In other words, not all subsets of $\mathbb{R}^3$ have a definite "volume" invariant under isometry.

Equidecomposability

Let G be a group that acts on a set X, and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \ldots, A_n \subseteq A$, $B_1, \ldots, B_n \subseteq B$
- group elements $g_1, \ldots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say A and B are *G-equidecomposable*.

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Effectively, A and B are "equal" to each other up to the group action.

If A is G-paradoxical, then so too is B.

The Banach–Tarski Paradox: Proof Outline I

1. We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a, b)$.

The Banach–Tarski Paradox: Proof Outline II

2. We use the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right) \end{aligned}$$

   to duplicate the unit sphere in $\mathbb{R}^3$, S^2, except for a countable subset D. (The *Hausdorff Paradox*.)
3. We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable — there is thus a paradoxical decomposition of S^2.
4. We show that the unit ball, $B(0,1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group E(3).

The Banach–Tarski Paradox: Proof Outline III

5. Define a relation $A \preceq B$ if A is G-equidecomposable with a subset of B, and show that if $A \preceq B$ and $B \preceq A$, then A and B are G-equidecomposable.
6. Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

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Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped "create" the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu\colon F(a,b) \to [0,1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\ &= \nu(F(a,b)) + \nu(F(a,b)) \\ &= 2. \end{aligned}$$

Ill-Behaved Groups, Cont'd

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Huh.

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu\colon G \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

If G admits a mean, we say G is *amenable*.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

  Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

  is a mean.
- Abelian (commutative) groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

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Some Recent Developments