Problem 1

Problem: Determine whether each of the following statements is true or false. Prove your answers.

(1) If A is a limit ordinal, then $A + B$ is a limit ordinal.

(2) If B is a limit ordinal, then $A + B$ is a limit ordinal.

(3) If $A + B$ is a limit ordinal, then A is a limit ordinal.

(4) If $A + B$ is a limit ordinal, then B is a limit ordinal.

Solution:

(1) False — the ordinal $\omega + 1$ is a successor ordinal to ω, but ω is a limit ordinal.

(2) True — by the lexicographical ordering on $A + B$, we must have that any element of $\{1\} \times B$ is greater than any element of $\{0\} \times A$. Since B is a limit ordinal, it does not have a maximal element (or else it would be a successor ordinal), so $\{1\} \times B$ has no maximal element, so $A + B$ has no maximal element. Thus, $A + B$ is a limit ordinal.

(3) False — the limit ordinal ω is equal to $2 + \omega$, but 2 is not a limit ordinal.

(4) True — by similar reasoning to (2), we see that there is no maximal element in $A + B$, and by the lexicographical ordering, this means there is no maximal element in $\{1\} \times B\times$, so there is no maximal element in B. Thus, B is a limit ordinal.

Problem 2

Problem: Let A, B, and C be nonzero ordinals. Determine whether each of the following is true or false. Prove your answers.

(1) $A < A + B$;

(2) $B < A + B$;

(3) if $A < B$, then $A + C < B + C$;

(4) if $A < B$, then $C + A < C + B$.

Solution:

(1) Since $A \cong \{0\} \times A$ are order isomorphic, and $\{0\} \times A \subsetneq \{0\} \times A \cup \{1\} \times B \cong A + B$, we have $A < A + B$.

(2) Since $B \cong \{1\} \times B$ are order isomorphic, and $\{1\} \times B \subsetneq \{0\} \times A \cup \{1\} \times B \cong A + B$, we have $B < A + B$.

(3) If $A < B$, then $A \subsetneq B$, so $\{0\} \times A \subsetneq \{0\} \times B$, so $\{0\} \times A \cup \{1\} \times C \subsetneq \{0\} \times B \cup \{1\} \times C$, so $A + C < B + C$.

(4) By a similar reasoning, we have $\{1\} \times A \subsetneq \{1\} \times B$, so $\{0\} \times C \cup \{1\} \times A \subsetneq \{0\} \times C \cup \{1\} \times A$, so $C + A < C + B$.