

# Cardinality and Countability

## Section 1.1: Countable Sets

**Definition** (Denumerable Set). A set  $S$  is denumerable if there exists a function  $f : S \rightarrow \mathbb{N}$  with  $f$  a bijection. We also say  $S$  is countably infinite.

**Definition** (Countable Set). We say  $S$  is countable if  $S$  is either finite or denumerable.

**Theorem** (Countability of Unions). If  $A$  and  $B$  are countable sets, then  $A \cup B$  is countable.

**Theorem** (Countability of Subsets). If  $A \subseteq B$ , then if  $B$  is countable, then  $A$  is countable.

**Theorem** (Union of Finite Sets). If  $A$  and  $B$  are finite, then  $A \cup B$  is finite.

*Proof.* If  $A$  is finite and  $|B|$  has one element, then we show that  $A \cup B$  is finite (with two cases).

Afterward, for  $|B| > 1$ , we use induction on  $|B|$ . □

**Definition** (Finite Set). A set  $A$  is finite if there exists a bijection  $f : S \rightarrow \{1, 2, \dots, n\}$  for some  $n \in \mathbb{N} = \{0, 1, \dots\}$ .

We write  $|A| = n$ .

**Theorem** (Disjoint Union of Countable Sets). If  $A$  is denumerable,  $B$  is finite, and  $A \cap B = \emptyset$ , then  $A \cup B$  is denumerable.

*Proof.* There exists a bijection  $f : A \rightarrow \mathbb{N}$  (since  $A$  is denumerable), and a bijection  $g : B \rightarrow \{1, 2, \dots, n\}$  for some  $n \in \mathbb{N}$  (since  $B$  is finite).

We create a new bijection  $h : A \cup B \rightarrow \mathbb{N}$  by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since  $A \cap B = \emptyset$ , we know that  $h$  is well-defined.

Now, we must show that  $h$  is a bijection.

Suppose  $h(x) = h(y)$ .

**Case 1:** If  $x, y \in B$ , then  $h(x) = g(x) - 1$ , and  $h(y) = g(y) - 1$ , meaning  $g(x) - 1 = g(y) - 1$ , meaning  $g(x) = g(y)$ . Since  $g$  is a bijection,  $x = y$ .

**Case 2:** If  $x, y \in A$ , a similar argument yields that  $x = y$ .

**Case 3:** Without loss of generality, let  $x \in A$  and  $y \in B$ . If  $x \in A$ , then  $h(x) = f(x) + n$  and  $h(y) = g(y) - 1$ . Thus,  $f(x) + n = g(y) - 1$ . However, since  $f(x) + n \geq n$  and  $0 \leq g(y) - 1 \leq n - 1$ . Thus, we get that  $0 \leq n \leq n - 1$ , which is a contradiction.

Thus, we have shown that  $h$  is injective. □

**Theorem** (Cartesian Product of Natural Numbers).  $\mathbb{N} \times \mathbb{N}$  is denumerable.

*Proof.* We consider  $\mathbb{N} \times \mathbb{N}$  as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \dots,$$

$$\begin{array}{rcll}
\mathbb{N} \times \{0\} : & (0, 0) & (1, 0) & (2, 0) & (3, 0) & \dots \\
\mathbb{N} \times \{1\} : & (0, 1) & (1, 1) & (2, 1) & (3, 1) & \dots \\
\mathbb{N} \times \{2\} : & (0, 2) & (1, 2) & (2, 2) & (3, 2) & \dots \\
\mathbb{N} \times \{3\} : & (0, 3) & (1, 3) & (2, 3) & (3, 3) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{rcll}
\mathbb{N} \times \{0\} : & \cancel{(0,0)}^0 & \cancel{(1,0)}^2 & \cancel{(2,0)}^5 & \cancel{(3,0)}^9 & \dots \\
\mathbb{N} \times \{1\} : & \cancel{(0,1)}^1 & \cancel{(1,1)}^4 & \cancel{(2,1)}^8 & (3,1) & \dots \\
\mathbb{N} \times \{2\} : & \cancel{(0,2)}^3 & \cancel{(1,2)}^7 & (2,2) & (3,2) & \dots \\
\mathbb{N} \times \{3\} : & \cancel{(0,3)}^6 & (1,3) & (2,3) & (3,3) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can also find a bijection  $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ , with

$$P(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

A fun challenge is to prove that  $P$  is a bijection. □

**Theorem** (Countability of the Rationals).  $\mathbb{Q}$  is denumerable.

**Theorem** (Countability of the Integers). The set  $\mathbb{Z}$  is denumerable.

*Proof.* Let  $f : \mathbb{Z} \rightarrow \mathbb{N}$  be defined by

$$f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

□

**Definition** (Cardinality). We say two sets,  $A$  and  $B$ , have the same cardinality if there exists a bijection  $f : A \rightarrow B$ .

**Theorem** (Finite Subset Cardinality). If  $m, n \in \mathbb{N}$  and  $m \neq n$ , then  $\{1, 2, \dots, m\}$  and  $\{1, 2, \dots, n\}$  do not have the same cardinality.

**Theorem** (Infinitude of the Natural Numbers).  $\mathbb{N}$  is not finite.

**Example.** If  $A \subsetneq B$  and  $|A| = |B|$ , then both  $A$  and  $B$  are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.

## Section 1.2

**Definition** (Uncountable Set). A set is uncountable if it is not countable.

**Theorem** (Uncountability of  $\mathbb{R}$ ).  $\mathbb{R}$  is uncountable.

*Proof.* For all  $x \in \mathbb{R}$ , and for all  $j \in \mathbb{N}$ , we define  $[x]_j$  to denote the  $j + 1$ -th digit after the decimal point in the decimal expansion of  $x$ .

For example,  $[\pi]_0 = 1$ ,  $[\pi]_1 = 4$ , etc.

Let  $f : \mathbb{N} \rightarrow \mathbb{R}$ . We will show that  $f$  is not surjective.

Let  $y \in [0, 1) \subseteq \mathbb{R}$  defined by  $\forall j \in \mathbb{N}$ ,

$$[y]_j = \begin{cases} 0 & [f(j)]_j = 1 \\ 1 & [f(j)]_j \neq 1 \end{cases}.$$

We claim that  $y \notin f(\mathbb{N})$ . We will show that  $\forall j \in \mathbb{N}$ ,  $f(j) \neq y$ .

We can see that if  $[f(j)]_j = 1$ , then  $[y]_j = 0$ . Similarly, if  $[f(j)]_j \neq 1$ , then  $[y]_j = 1$ . Either way,  $[f(j)]_j \neq [y]_j$  for all  $j \in \mathbb{N}$ .  $\square$

**Remark:** The above proof is an example of a diagonalization proof. It can be imagined as

$$\begin{array}{c|l} f(0) & *.a_1 \overline{a_2} a_3 \dots \\ f(1) & *.b_1 \overline{b_2} b_3 \dots \\ f(2) & *.c_1 c_2 \overline{c_3} \dots \\ \vdots & \vdots \end{array}$$

**Note:** A substantial problem that we might need to deal with is that a real number does not necessarily have a unique decimal representation. For instance,  $3.999\dots = 4.000\dots$ .

In order to resolve this issue, we can default to the option with trailing 0 over trailing 9.

**Definition** (Power Set). The power set of a set  $S$  is

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

**Theorem** (Power Set Surjection). Let  $f : S \rightarrow \mathcal{P}(S)$ . Then,  $f$  is not surjective.

*Proof.* Let  $T = \{x \in S \mid x \notin f(x)\}$ . Then,  $T \notin f(S)$ .

Let  $y \in S$ . We want to show that  $f(y) \neq T$ . Suppose toward contradiction that  $f(y) = T$ . Then, if  $y \in T$ , then  $y \in f(y)$ , which implies that  $y \notin T$ .

If  $y \notin T$ , then  $y \notin f(y)$ , which implies that  $y \in T$ .

Thus, it cannot be the case that  $f(y) = T$ .  $\square$

**Definition** (Cardinality Comparison). Let  $A$  and  $B$  be sets. Then, we write  $\text{card}(A) \leq \text{card}(B)$  if there exists an injective map  $f : A \hookrightarrow B$ .

We write  $\text{card}(A) < \text{card}(B)$  if there exists an injection  $f : A \hookrightarrow B$  but no bijection.

**Example** (Cardinality of the Power Set). For every set,

$$\text{card}(S) < \text{card}(\mathcal{P}(S)).$$

- (1) We know that  $\text{card}(S) \leq \text{card}(\mathcal{P}(S))$ , defining  $f : S \hookrightarrow \mathcal{P}(S)$ ,  $f(a) = \{a\}$ , since if  $f(x) = f(y)$ , then  $\{x\} = \{y\}$ , meaning  $x \in \{y\}$ , so  $x = y$ .

In the case of  $f : \emptyset \rightarrow \{\emptyset\}$ , we define  $\emptyset = f \subseteq \emptyset \times \{\emptyset\}$ .

- (2) Since there exists no bijection  $f : S \rightarrow \mathcal{P}(S)$ , it is the case that  $\text{card}(S) \neq \text{card}(\mathcal{P}(S))$ .

**Example** (Decimal Expansion). We know that for some decimal expansion

$$\begin{aligned} 3.14159 \dots &= 3 + \frac{1}{10} + \frac{4}{100} + \dots \\ &= \sum_{i=0}^{\infty} \frac{n_i}{10^i}, \end{aligned}$$

with  $0 \leq n_i \leq 9$  for  $i \geq 1$ .

However, we can also write any real number as

$$\sum_{i=0}^{\infty} \frac{n_i}{3^i}$$

with  $0 \leq n_i \leq 2$  for all  $i \geq 1$ .