Introduction

This is going to be the notes from my Honors Thesis project on amenability. We will be covering different results that are used to show that a topological group has a translation-invariant finitely additive probability measure (i.e., a mean).

The primary source texts to inform this independent study will be Volker Runde's *Lectures on Amenability* and Timothy Rainone's *Functional Analysis-en route to Operator Algebras*, as well as various notes compiled by my professor.

Group Actions, Paradoxical Decompositions, and the Banach–Tarski Paradox

In order to introduce Tarski's theorem, which is where our first condition about the amenability of groups appears, we begin by discussing paradoxical decompositions, with the goal of this section being a proof of the Banach–Tarski Paradox. The Banach–Tarski paradox says the following:

> If A and B are any bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Basics of Group Actions

The information for these essentials about group actions will be drawn from Dummit and Foote's *Abstract Algebra*.

Definition (Group Action)**.** A (left) group action of G onto a set A is a map from $G \times A$ to A that satisfies:

- $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$;
- $e \cdot a = a$[1] for all $a \in A$.

Definition (Permutation Representation)**.** For each g, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ (the group element g acts on a) is a permutation of A. There is a homomorphism associated to these actions:

$$\varphi : G \to S_A,$$

where $\varphi(g) = \sigma_g$. Recall that S_A is the symmetric group (group of permutations) on the elements of A.

This is the permutation representation for the action.

In particular, given any nonempty set A and a homomorphism G into S_A, we can define an action of G on A by taking $g \cdot a = \varphi(g)(a)$.

Definition (Kernel)**.** The kernel of the action of G is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}$$

Note: The kernel of the action is the kernel of the permutation representation $\varphi : G \to S_A$.

Definition (Stabilizer)**.** For each $a \in A$, the stabilizer of a under G is the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

[1] The identity element is usually written as 1, but I will write it as e out of familiarity.

Note: The kernel of the group action is the intersection of the stabilizers of every element of G:

$$\text{kernel} = \bigcap_{a \in A} G_a.$$

Note: For each $a \in A$, G_a is a subgroup of G.

Definition (Faithful Action)**.** An action is faithful if the kernel of the action is e.

Definition (Free Action)**.** For a set X with G acting on X, the action of G on X is free if, for every x, $g \cdot x = x$ if and only if $g = e_G$.

If the action of G on X is a free action, we say G acts freely on X.

Proposition (Equivalence Relation on A)**.** *Let* G *be a group that acts on a nonempty set* A*. We define a relation* $a \sim b$ *if and only if* $a = g \cdot b$ *for some* $g \in G$*. This is an equivalence relation, with the number of elements in* $[a]_\sim$ *found by taking* $|G : G_a|$*, which is the index of the stabilizer of* a*.*

Proof. We can see that $a \sim a$, since $e \cdot a = a$. Similarly, we can see that if $a \sim b$, then $b = g^{-1} \cdot a$, meaning $b \sim a$. Finally, let $a \sim b$ and $b \sim c$. Then, we have $a = g \cdot b$ for some $g \in G$, and $b = h \cdot c$ for some $h \in G$. Thus, we have

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$.

We say there is a bijection between the left cosets of G_a and the elements of the equivalence class of a.

Define $\mathcal{C}_a$ to be the set $\{g \cdot a \mid g \in G\}$, and let $b = g \cdot a$. Define a map $g \cdot a \mapsto gG_a$. This map is surjective since $g \cdot a$ is always an element of $\mathcal{C}_a$. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, meaning $h^{-1}g \in G_a$, and $h^{-1}g \in G_a$ if and only if $gG_a = hG_a$, this map is injective.

Since there is a one-to-one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we now know that the number of equivalence classes of a under the action of G is $|G : G_a|$. □

Definition (Orbit)**.** For any $a \in A$, we define the orbit under G of a by

$$G \cdot a = \{b \in A \mid \forall g \in G,\ b = g \cdot a\}$$

In particular, if $c \in G \cdot a$ for some $a \in A$, then $G \cdot c = G \cdot a$.

Paradoxical Decompositions

Most of the information from this section will be drawn from Volker Runde's *Lectures on Amenability*.

Definition (Paradoxical Sets and Decompositions)**.** Let G be a group that acts on a set X. Let $E \subseteq X$.

If there exist pairwise disjoint $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j,$$

then we say that E is G-paradoxical.

In particular, a paradoxical group is one where G acts on itself by left-multiplication.

Example (Our First Paradoxical Group)**.** The free group on two generators, $\mathbb{F}(a,b)$,[II] is paradoxical. To see this, we let

$$W(x) = \{w \in \mathbb{F}(a,b) \mid w \text{ starts with } x\}.$$

Here, "starts with" refers to the left-most element. For instance, $ba^2ba^{-1} \in W(b)$.

In particular, we can see that

$$\mathbb{F}(a,b) = \left\{e_{\mathbb{F}(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For any $w \in \mathbb{F}(a,b) \setminus W(a)$, we can see that $a^{-1}w \in W\left(a^{-1}\right)$, meaning $w \in aW\left(a^{-1}\right)$. Therefore, $\mathbb{F}(a,b) = W(a) \sqcup aW\left(a^{-1}\right)$.

Similarly, for any $v \in \mathbb{F}(a,b) \setminus W(b)$, $b^{-1}v \in W\left(b^{-1}\right)$, so $v \in bW\left(b^{-1}\right)$. Therefore, $\mathbb{F}(a,b) = W(b) \sqcup bW\left(b^{-1}\right)$.

Proposition (Free Action of a Paradoxical Group)**.** *Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.*

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$ be pairwise disjoint, with $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

We let $M \subseteq X$ contain exactly one element from every orbit of G.

The set $\{g \cdot M \mid g \in G\}$ is a partition of X. Since M contains exactly one element from every orbit of G, it is then the case that $\bigcup_{g \in G} g \cdot M = X$, since $G \cdot M = X$.

Additionally, if $x, y \in M$ with $g \cdot x = h \cdot y$, then $\left(h^{-1}g\right) \cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Thus, we must have $h^{-1}g = e_G$, as we assume G acts freely.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$ if $g_1 \neq g_2$, meaning $\{g \cdot M \mid g \in G\}$ is a partition.

Define A_j^* to be the subset of X that is the result of the elements of A_j acting on M. In other words,

$$A_j^* = \bigcup_{g \in A_j} g \cdot M.$$

As a useful shorthand, we can say $A_j^* = A_j \cdot M$.[III] Similarly, we define

$$\begin{aligned} B_j^* &= \bigcup_{h \in B_j} h \cdot M \\ &= B_j \cdot M. \end{aligned}$$

[II] The set of all reduced words over $\left\{a, b, a^{-1}, b^{-1}, e_{\mathbb{F}(a,b)}\right\}$. In particular, a word is reduced when the pairs aa^{-1} and bb^{-1} are replaced with the identity $e_{\mathbb{F}(a,b)}$.

[III] Yes, I know that A_j is not technically a group acting on M, but this will help illuminate the final conclusion.

We can see that $A_1^*, A_2^*, \dots, A_n^*, B_1^*, B_2^*, \dots, B_m^* \subseteq X$ are disjoint, since $\{g \cdot M \mid g \in G\}$ is a partition, and $A_1, \dots, A_n, B_1, \dots, B_m$ are disjoint in G.

Thus, we have

$$\begin{aligned}\bigcup_{j=1}^{n} g_j \cdot A_j^* &= \bigcup_{j=1}^{n} (g_j A_j) \cdot M \\ &= G \cdot M \\ &= X.\end{aligned}$$

Similarly,

$$\begin{aligned}\bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} (h_j B_j) \cdot M \\ &= G \cdot M \\ &= X.\end{aligned}$$

Thus, we see that X has a paradoxical decomposition, meaning X is G-paradoxical. □

Note: We invoked the axiom of choice when we defined M to contain exactly one element from each orbit in X.

Paradoxical Decompositions of the Unit Sphere

We are aware of $\mathbb{F}(a, b)$ being a paradoxical group — in particular, we hope to use the properties of $\mathbb{F}(a, b)$ to yield paradoxical decompositions of the unit sphere in $\mathbb{R}^3$, denoted S^2.

Definition (Special Orthogonal Group)**.** For $n \in \mathbb{N}$, we define the special orthogonal group to consist of all real $n \times n$ matrices A such that

$$A^T A = AA^T = I,$$

with $\det(A) = 1$.

In particular, $SO(3)$ denotes the set of all rotations about some line that runs through the origin. An important fact about $SO(3)$ is that it contains a paradoxical subgroup.

Theorem. *There are rotations A and B about lines through the origin in $\mathbb{R}^3$ which generate a subgroup of $SO(3)$ isomorphic to $\mathbb{F}(a, b)$.*