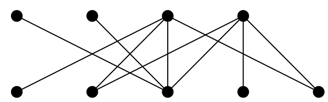
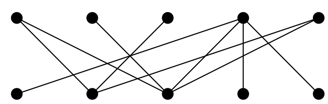
3.1.1

Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.

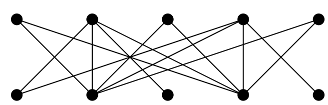
Graph 1:



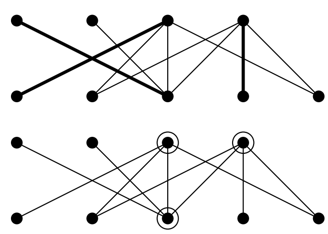
Graph 2:



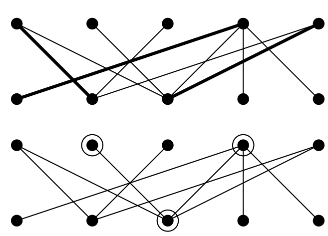
Graph 3:



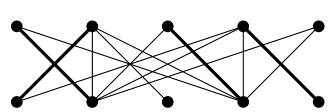
Graph 1:

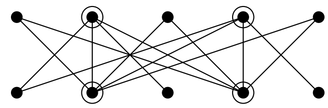


Graph 2:



Graph 3:





In all of these cases, we have solved the dual problem and found that the size of the minimum vertex cover is equal to the size of the maximum matching. Because all of these graphs are bipartite, we know that $\alpha'(G) = \beta(G)$.

3.1.2

Determine the minimum size of a maximal matching in C_n.

There are two cases of n where we will construct the minimum maximal matching.

n is odd: If n is odd, then we create M by selecting one edge, then selecting an edge traversing along C_n by skipping one vertex, then selecting the edge starting from the vertex (if it is not in the matching), then so on and so forth until M cannot be added to. In the case of any odd cycle, this means $|M| = \left\lfloor \frac{n}{2} \right\rfloor$.

n is even If n is even, then we create M by selecting an edge and traversing along the cycle by a similar technique (skip one vertex, select an edge starting at the following vertex, etc.) until M cannot be added to any further. This yields a minimum size of $|M| = \frac{n}{2} - 1$, as we can select two vertices to be skipped in any graph.

3.1.5

Prove that $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ for every graph G.

$$\begin{aligned}
\alpha(G) + \beta(G) &= n(G) \\
\alpha(G) &= n(G) - \beta(G) \\
\alpha(G) &\geq n(G) - \alpha(G)\Delta(G) && \text{since } \alpha(G)\Delta(G) \text{ must cover every edge} \\
\alpha(G) &\geq \frac{n(G)}{\Delta(G) + 1}
\end{aligned}$$

3.1.9

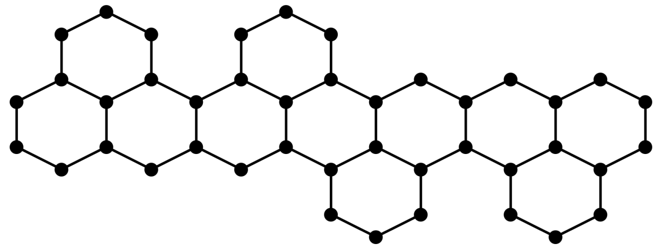
Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Let M be a maximal matching with fewer than $\alpha'(G)/2$ edges. Since M is maximal, we are assuming that every vertex in G is either in M or is adjacent to a vertex in M. Since M is a maximal matching that is *not* a maximum matching, then there must be an M-augmenting path.

Let v be the start of this M-augmenting path, and let w be its final vertex. Because v and w cannot have any other edges incident on them, we must assume that $\alpha'(G) = |M| + 1$, as we can exchange the matchings within the M-augmenting path to get a maximum matching. However, because $|M| < \alpha'(G)/2$, we must get that $\alpha'(G) < 1 + \alpha'(G)/2$, or that $\alpha'(G) < 2$. However, $\alpha'(G)/2$ must be an integer, meaning that the original assumption (that M is a maximal matching) must be false.

3.1.28

Exhibit a perfect matching in the graph below or give a short proof that it has none.



The minimum edge cover is as follows, and is of size 22, meaning that the maximum matching is of size 20, which is not a perfect matching.