1.2.20

Let v be a cut-vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

Solution

Let $x, y, v \in V(\overline{G})$, where v is a cut-vertex of G.

Suppose x and y belong to distinct components of $G - v$. Then, $xy \notin E(G)$, meaning that $xy \in E(\overline{G})$, meaning there is an x, y path in $\overline{G}$, so there is an x, y path in $\overline{G} - v$.

Suppose x and y are in the same component of $G - v$. Since v is a cut-vertex, this means there must be at least two components in $G - v$. Let H_1 be the component that x, y are in, while $\exists w \in H_2$ is a vertex in H_2 disjoint from H_1. Since H_1 and H_2 are disjoint, this means the components do not contain any edges between them, so $x \not\leftrightarrow w$ and $y \not\leftrightarrow w$ in $G - v$ — however, this means that $x \leftrightarrow w$ and $y \leftrightarrow w$ in $\overline{G}$, meaning that $\exists x, y$ path in $\overline{G} - v$.

1.2.22

Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Solution

Let G be a graph where there exists a partition of its vertices into two non-empty sets such that there is no edge with endpoints in both sets. Call these sets A and B. By our assumptions, $\forall u \in A$ and $\forall v \in B$, $\nexists e$ such that $e = uv$. Therefore, we cannot create a path between any $u \in A$ and any $v \in B$ as there is no edge to connect any element in A and any element in B. Therefore, G is disconnected.

Suppose G is a disconnected graph. Then, G contains more than one component — we can create a partition of $V(G)$ by letting $H_1, H_2, \dots, H_k$ refer to the k components of G. Each of these components is necessarily disjoint from every other component. By taking $H = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ as one set and H_k as our other set, we know that $H_1, \dots, H_k$ are all disjoint, meaning that H and H_k are disjoint, meaning that there is no edge connecting any vertex H with any vertex in H_k, meaning we have created a partition of G such that there exists no edge between any vertex in one set and any vertex in the other set.

1.2.26

Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

Solution

Suppose G is bipartite. Then, there exists a partition of the vertices $V = X \sqcup Y$ such that X and Y are independent sets. Let H be a subgraph of G, and let $H_X = X \cap V(H)$ and $H_Y = Y \cap H$. Because H is a subgraph of G, each vertex of H must be an element of either H_X or H_Y, or that $V(H) = H_X \sqcup H_Y$. WLOG, let $|H_X| > |H_Y|$. Since $H_X \subseteq X$ and X is an independent set, H_X is an independent subset consisting of at least half of $V(H)$.

Suppose every subgraph of G has an independent set consisting of at least half of $V(H)$. We will suppose toward contradiction that G is not bipartite. Then, G must contain an odd cycle, H_1. However, an independent set of H_1 consists of at most $\lfloor \frac{|V(H_1)|}{2} \rfloor < \frac{|V(H_1)|}{2}$, otherwise two vertices would be adjacent. Because H_1 is an independent set with less than half of the elements of $V(H)$, we have reached a contradiction. Therefore, G must be bipartite.

1.2.38

Prove that every n-vertex graph with at least n edges contains a cycle.

Solution

We proceed via induction as follows:

For the base case where $|V(G)| = 1$, we know that there is a cycle with one edge that connects back on the vertex.

For the case where $|V(G)| > 1$, if $v \in V(G)$ has degree at most 1, then $G - v$ has $n-1$ vertices and at least $n-1$ edges, so by our inductive hypothesis, we know that $G - v$ contains a cycle. Meanwhile, if $\forall v \in V(G), d(v) \geq 2$, we know by Lemma 1.2.25 that G contains a cycle.