Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

$$d\mathbf{r} = \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}.$$

We let $\hat{e}_1 = \hat{\rho}$, $\hat{e}_2 = \hat{\phi}$, and $\hat{e}_3 = \hat{z}$, with $u_1 = \rho$, $u_2 = \phi$, and $u_3 = z$. Thus, we get

- Line element:

$$\begin{aligned}
(ds)^2 &= \sum_{i,j} \frac{\partial \mathbf{r}}{\partial u_i}\frac{\partial \mathbf{r}}{\partial u_j}\left(\hat{e}_i \cdot \hat{e}_j\right) du_i\, du_j \\
&= \sum_{i=1} \left(\frac{\partial \mathbf{r}}{\partial u_i}\right)(du_i)^2 && \text{The } \hat{\rho}, \hat{\phi}, \hat{z} \text{ basis is orthogonal} \\
&= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.
\end{aligned}$$

- Area element:

$$d\mathbf{a} = \left(\sum_k \epsilon_{ijk}\hat{e}_k\right)\frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} du_i\, du_j$$

4.9

Without loss of generality, we have

$$\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ij\ell} = \epsilon_{mn1}\epsilon_{ij1},$$

where $m, n, i, j = 2, 3$. If we have $m = i, n = j$, then $\epsilon_{mn1}\epsilon_{ij1} = 1$; if $m = j, n = i$, then $\epsilon_{mn1}\epsilon_{ij1} = -1$; else, $\epsilon_{mn1}\epsilon_{ij1} = 0$.

4.11

(a)

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k \\
&= -\sum_{i,j,k} \epsilon_{jik} B_j A_i \hat{e}_k \\
&= -\left(\mathbf{B} \times \mathbf{A}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= \sum_{i,j,k} \left(\epsilon_{ijk} A_i B_j \hat{e}_k\right) \cdot A_i \hat{e}_i \\
&= \sum_{i,j,k} \delta_{ik}\left(\epsilon_{ijk} A_i^2 B_j\right) \\
&= 0.
\end{aligned}$$

(c)

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) =$$

(d)

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_\ell B_i C_j \\ &= \sum_{i,j,\ell} \left(\epsilon_{\ell ij} A_\ell B_i\right) C_j \\ &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})\end{aligned}$$

and

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_\ell B_i C_j \\ &= \sum_{i,j,\ell} \left(\epsilon_{j\ell i} C_j A_i\right) B_i \\ &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}).\end{aligned}$$

(e)

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j} \epsilon_{ijk} A_i \left(\sum_{\alpha,\beta} \epsilon_{\alpha\beta j} B_\alpha C_\beta\right) \\ &= \sum_{i,j,\alpha,\beta} \epsilon_{ijk}\epsilon_{\alpha\beta j} A_i B_\alpha C_\beta \\ &= -\left(\sum_{i,j,\alpha,\beta} \epsilon_{ikj}\epsilon_{\alpha\beta j} A_i B_\alpha C_\beta\right) \\ &= -\left(\sum_{i,j,\alpha,\beta} \left(\delta_{i\alpha}\delta_{k\beta} - \delta_{i\beta}\delta_{k\alpha}\right) A_i B_\alpha C_\beta\right) \\ &= \sum_{i,j,\alpha,\beta} \left(\delta_{k\alpha}\delta_{i\beta} - \delta_{i\alpha}\delta_{k\beta}\right) A_i B_\alpha C_\beta \\ &= \sum_{i,j,\alpha,\beta} \left(B_\alpha \delta_{k\alpha}\right)\left(A_i C_\beta \delta_{i\beta}\right) - \left(C_\beta \delta_{k\beta}\right)\left(A_i B_\alpha \delta_{i\alpha}\right) \\ &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).\end{aligned}$$

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right| \\ &= 1,\end{aligned}$$

which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \qquad \forall n \geqslant 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(b)
- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\left(\frac{1}{2}\right)\right| \\ &= \frac{1}{2} \\ &< 1,\end{aligned}$$

meaning the series converges by the ratio test.

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$$\frac{1}{n2^n} < \frac{1}{2^n} \qquad \text{for all } n \geqslant 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.