Problem (Problem 1)**:** A topological group is a group which is also a Hausdorff topological space where the group operations are continuous.

Recall the definition of the concatenation operation on the fundamental group. Now, let G be a path-connected topological group, and let $\pi_1(G, e)$ be the fundamental group of G with base point e. Use the Hilton–Eckmann argument to prove that the concatenation operation on the fundamental group is commutative.

Problem (Problem 2)**:** Let M and N be smooth, orientable, closed manifolds of the same dimension n, and let $f\colon M \to N$ be a smooth function. Show that f induces a map $f^*\colon H^n_{DR}(N) \to H^n_{DR}(M)$ which is multiplication by an integer. This is called the degree of f and is written $\deg(f)$.

Solution: Letting $\omega \in H^n_{DR}(N)$ be a nonvanishing top-dimensional form, we observe that there are two operations we are able to do on ω. We may normalize to take

$$\int_N \omega = 1.$$

By the naturality of the de Rham isomorphism, it follows that there is some $\delta \in \mathbb{R}$ such that.

$$\int_M f^*\omega = \delta \int_N \omega$$

Our task now is to show that $\delta \in \mathbb{Z}$.

Toward this end, let q be a regular value of f. We may use a smooth bump function to restrict ω to a small open neighborhood V of q. It follows then that $f^{-1}(q) = \{p_1, \dots, p_\ell\}$ for some ℓ, with corresponding disjoint open neighborhoods $U_1, \dots, U_\ell$ locally diffeomorphic to V, whence the support of $f^*\omega$ is contained in the union of $U_1, \dots, U_\ell$. If $f^{-1}(q) = \emptyset$, then

$$\begin{aligned}\int_M f^*\omega &= \int_\emptyset f^*\omega\\ &= \delta \int_N \omega\\ &= 0,\end{aligned}$$

whence $\delta = 0$. If $f^{-1}(q) \neq \emptyset$, then we see that

$$\begin{aligned}\int_M f^*\omega &= \sum_{k=1}^{\ell} \int_{U_k} f^*\omega\\ &= r \int_N \omega,\end{aligned}$$

for some particular integer r (as, by the definition of the pullback, integration over the pullback sums over the same domain with sign changes). In particular, this means that in the general case, $\delta \in \mathbb{Z}$.

Problem (Problem 3)**:** Recall the definition of the degree of f from one of the previous problem sets, counting the sums of signs of determinants of the derivative of f over the preimage of a regular value of f. Prove that the two definitions of the degree agree.

Solution: As we have seen in the previous problem, given a regular value $q \in N$ with the same setup as above, we have

$$\int_M f^*\omega = \sum_{k=1}^{\ell} \int_{U_k} f^*\omega.$$

In the particular case of exactly one regular value, we observe that by the change of coordinates formula,

$$\int_U f^*\omega = \operatorname{sgn}(\det(D_p f)) \int_V \omega.$$

Therefore, in the general case, we have

$$\begin{aligned}\int_M f^*\omega &= \sum_{k=1}^{\ell} \operatorname{sgn}(\det(D_{p_k} f)) \int_{U_k} f^*\omega \\ &= \sum_{k=1}^{\ell} \operatorname{sgn}(\det(D_{p_k} f)) \int_V \omega \\ &= \deg(f) \int_V \omega \\ &= \deg(f) \int_N \omega.\end{aligned}$$

In particular, this gives

$$\deg(f) = \sum_{k=1}^{\ell} \operatorname{sgn}(\det(D_{p_k} f)).$$

Problem (Problem 4)**:** With the setup of the previous exercises, prove that if ω is an arbitrary n-form on N, then

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

Solution:

$$\begin{tikzcd} H^n_{DR}(N) \arrow[r, "f^*"] \arrow[d, "\int_N"'] & H^n_{DR}(M) \arrow[d, "\int_M"] \\ \mathbb{R} \arrow[r, "\Delta"] & \mathbb{R} \end{tikzcd}$$

Since the de Rham isomorphism induces a natural transformation, commutativity gives

$$\begin{aligned}\int_M f^*\omega &= \delta \int_N \omega \\ &= \deg(f) \int_N \omega.\end{aligned}$$