# Basic Properties

**Definition:** A topological space $M$ is called a *manifold* if it satisfies the following:

- $M$ is Hausdorff (points can be separated by open sets);
- $M$ is second countable (the basis for the topology of $M$ is countable);
- $M$ is locally Euclidean (every point in $M$ has a neighborhood homeomorphic to $\mathbb{R}^n$ for some $n$).

In particular, the third condition says that for every $p \in M$, there is $U \in \mathcal{O}_p$ and a homeomorphism $\varphi\colon U \to \mathbb{R}^n$. The value of $n$ is called the *dimension* of the manifold $M$.

**Definition:** Let $M$ be an $n$-manifold. A *chart* on $M$ is a pair $(U, \varphi)$ such that $U \subseteq M$ is open, $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism.

A family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is known as an *atlas* if

$$M = \bigcup_{i \in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point $p \in M$ and two charts $(U, \varphi_U)$ and $(V, \varphi_V)$ such that $p \in U$ and $p \in V$. The functions $\varphi_U\colon U \to \mathbb{R}^n$ and $\varphi_V\colon V \to \mathbb{R}^n$ are homeomorphism, meaning that $\varphi_V \circ \varphi_U^{-1}\colon \varphi_U(U \cap V)^n \to \mathbb{R}^n$ defined on the (nonempty) $U \cap V$ is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs $\varphi_V \circ \varphi_U^{-1}$ are *diffeomorphisms*. To do this, we need to first develop the derivative in $\mathbb{R}^n$.

**Definition:** Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say $f$ is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear map $L \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\frac{\|f(p+h) - f(p) - Lh\|}{\|h\|} \to 0$$

as $h \to 0$.

The *derivative* of $f$ is the association $f \mapsto L$ for each $p \in \mathbb{R}^n$.

A function $f$ is called a *diffeomorphism* if it is continuously differentiable and has a continuously differentiable inverse.

**Definition:** If $(U, \varphi_U)$ and $(V, \varphi_V)$ are charts such that $U \cap V \neq \emptyset$, the function $\varphi_V \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is known as the *transition map* between $\varphi_U$ and $\varphi_V$.

A *smooth structure* for $M$ is an atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that for all $i, j \in I$, the transition maps $\varphi_j \circ \varphi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$ are diffeomorphisms where defined (if not defined, then it is vacuously so). If $M$ admits a smooth structure, then we call $M$ a smooth manifold.

**Note:** From now on, we use "manifold" to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

**Definition:** A map $f\colon M \to N$ between manifolds is called *smooth* if for any chart $(U, \varphi_U)$ in $M$ and corresponding chart $(V, \varphi_V)$ in $N$, the map $\varphi_V \circ f \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^k$ is continuously differentiable.

The function $f$ is a *diffeomorphism* if $f$ is a smooth bijection with smooth inverse, and we say the manifolds $M$ and $N$ are diffeomorphic if they admit a diffeomorphism.