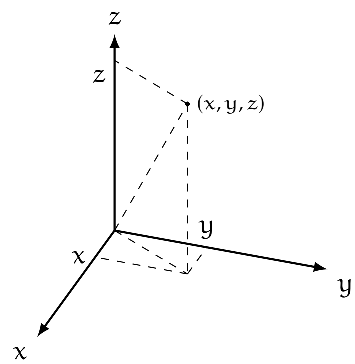
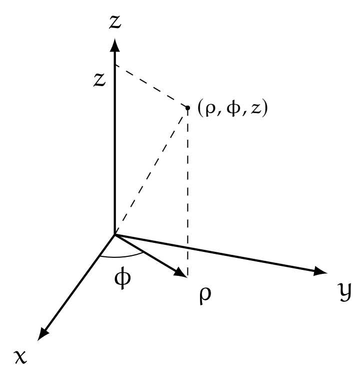
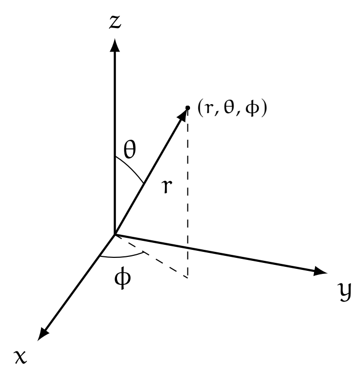
Things You Just Gotta Know

Coordinate Systems





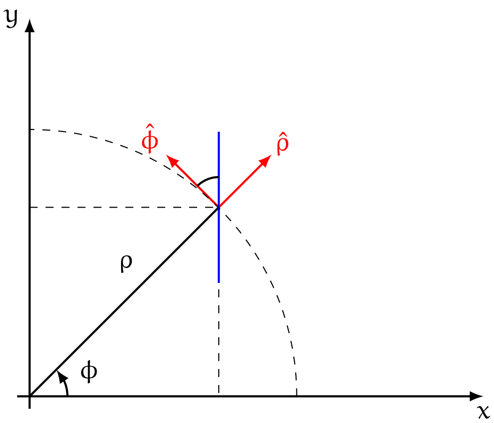


We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x,y)\hat{\imath} + V_y(x,y)\hat{\jmath}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

Polar Coordinates



We can also express the inputs to $\vec{V}$ in polar coordinates, (ρ, ϕ).

$$\vec{V}(\mathbf{r}) = V_\rho(\rho, \phi)\,\hat{\imath} + V_\phi(\rho, \phi)\,\hat{\jmath}.$$

To extract the input functions, we take

$$\begin{aligned} V_x &= \hat{\imath} \cdot \vec{V} \\ V_y &= \hat{\jmath} \cdot \vec{V}. \end{aligned}$$

Alternatively, we can project $\vec{V}$ onto the $\hat{\rho}, \hat{\phi}$ axis:

$$\vec{V}(\mathbf{r}) = V_\rho(\rho, \phi)\,\hat{\rho} + V_\phi(\rho, \phi)\,\hat{\phi},$$

and we extract

$$\begin{aligned} V_\rho &= \hat{\rho} \cdot \vec{V} \\ V_\phi &= \hat{\phi} \cdot \vec{V}. \end{aligned}$$

Notice that $\mathbf{r}$ is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned} \mathbf{s} &= x\hat{\imath} + y\hat{\jmath} \\ &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} \\ &= \rho\hat{\rho} \\ &= \sqrt{x^2 + y^2}\hat{\rho} \end{aligned}$$

The main reason we avoided using the $\hat{\rho}, \hat{\phi}$ axis up until this point is that ρ and ϕ are *position-dependent*, while the $\hat{\imath}, \hat{\jmath}$ axis is position-independent.

Now, we must figure out the position-dependence of $\hat{\rho}$ and $\hat{\phi}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi.$$

If we hold ϕ constant, it must be the case that any change in ρ is in the $\hat{\rho}$ direction. Therefore,

$$\hat{\rho} = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\|}$$

$$
\begin{aligned}
&= \frac{\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}}{\left|\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\right|} \\
&= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|} \\
&= \frac{-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}}{\left\|-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}\right\|} \\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}.
\end{aligned}
$$

Thus, we can see that the $\hat{\rho}, \hat{\phi}$ axis is orthogonal.

$$
\begin{aligned}
\frac{\partial \hat{\rho}}{\partial \phi} &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
&= \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\rho}, \\
\frac{\partial \hat{\phi}}{\partial \rho} &= 0,
\end{aligned}
$$

and

$$
\frac{\partial \hat{\rho}}{\partial \rho} = 0
$$

Example (Velocity)**.**

$$
\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{s}}{dt} \\
&= \frac{d}{dt}\left(x\hat{\imath}\right) + \frac{d}{dt}\left(y\hat{\jmath}\right).
\end{aligned}
$$

In the case of cartesian coordinates, $\hat{\imath}$ and $\hat{\jmath}$ are constants.

$$
= v_x\hat{\imath} + v_y\hat{\jmath}
$$

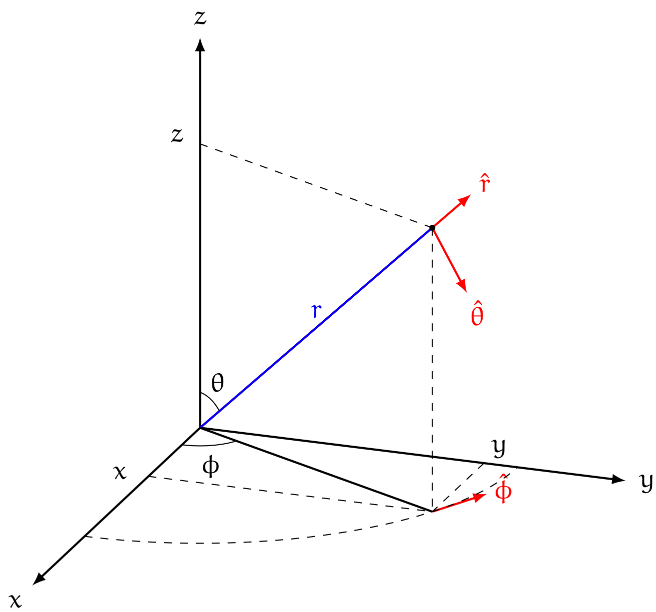
When we examine polar coordinates, since $\hat{\rho}$ and $\hat{\phi}$ are position-dependent, we must use the chain rule.[1]

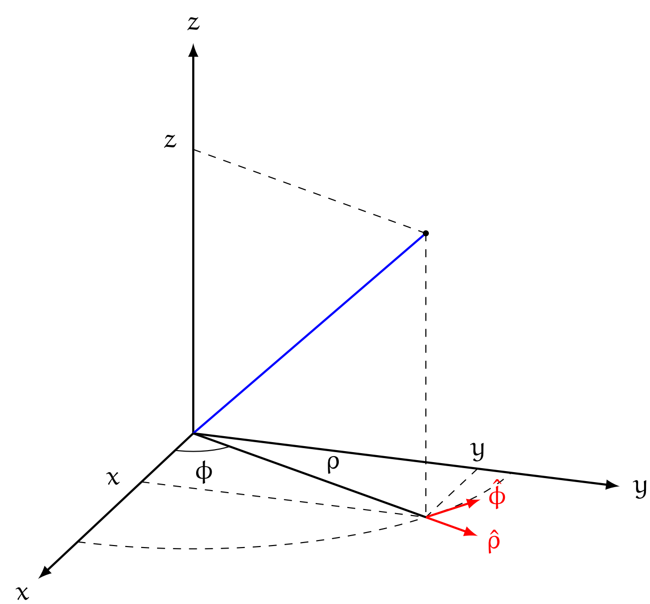
$$
\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{s}}{dt} \\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} \\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\cancelto{0}{\frac{\partial\hat{\rho}}{\partial\rho}}\frac{d\rho}{dt} + \underbrace{\frac{\partial\hat{\rho}}{\partial\phi}}_{=\hat{\phi}}\frac{d\phi}{dt}\right) \\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi} \\
&= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}.
\end{aligned}
$$

Notice that $\dot{\rho}$ is the radial velocity and $\dot{\phi} = \omega$ is the angular velocity.

[1]Note that $\hat{\rho} = \hat{\rho}(\rho, \phi)$ and $\hat{\phi} = \hat{\phi}(\rho, \phi)$.

Spherical and Cylindrical Coordinates





Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho, \phi)$	$\mathbf{s} = s(\rho, \phi, z)$	$\mathbf{s} = s(r, \phi, \theta)$
$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$

Here,[II] ϕ denotes the polar angle and θ denotes the azimuthal angle. Notice that $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

[II]Physicists amirite?

We can see that $\hat{\rho}$, $\hat{\phi}$, and $\hat{\theta}$ in spherical coordinates are also position-dependent.

$$\begin{aligned}
\hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\|\frac{\partial \mathbf{s}}{\partial r}\right\|}\\
&= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}\\
\hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\|\frac{\partial \mathbf{s}}{\partial \phi}\right\|}\\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\\
\hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\|\frac{\partial \mathbf{s}}{\partial \theta}\right\|}\\
&= \cos\phi\cos\theta\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}
\end{aligned}$$

Scale Factors and Jacobians

Coordinate System	Line Element	Area Element	Volume Element
Polar	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi$	$d\mathbf{a} = r\,drd\phi$	—
Cylindrical	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz$	—	$d\mathbf{v} = r\,drd\phi dz$
Spherical	$d\mathbf{s} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta$	$d\mathbf{a} = r^2\sin\theta d\phi d\theta$	$d\mathbf{v} = r^2\sin\theta\,drd\phi d\theta$

In cylindrical coordinates, we can use the chain rule to find the value of $d\mathbf{r}$:

$$d\mathbf{r} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz.$$

The extra factor of ρ in the expression of $\rho\hat{\phi}d\phi$ is the *scale factor* on ϕ.

Similarly, in spherical coordinates, we have

$$d\mathbf{r} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta,$$

with scale factors of $r\sin\theta$ on $\hat{\phi}d\phi$ and r on $\hat{\theta}d\theta$.

When we go from line elements (of the form $d\mathbf{r}$) to area elements (of the form $d\mathbf{a}$), we can see that the area element in polar coordinates is $d\mathbf{a} = \rho d\rho d\phi$ — we need the extra factor of ρ to account for the fact that the magnitude of the area element scales with the radius.

Similarly, the volume element in cylindrical coordinates is $d\mathbf{v} = rdrd\phi dz$ and the volume element in spherical coordinates is $r^2\sin\theta drd\phi d\theta$.

Recall that the definition of an angle ϕ that subtends an arc length s is $\phi\frac{s}{r}$, where r is the radius of a circle. We can imagine a similar concept on a sphere — a *solid angle* measured in steradians is of the form $\Omega = \frac{A}{r^2}$, where A denotes the surface area subtended by the angle Ω. In particular, since $d\Omega = \frac{dA}{r^2}$, we find that $d\Omega = \sin\theta d\phi d\theta$.

When we are dealing with products of scale factors, we need to use the Jacobian to determine the proper scale factor on any given element:

$$\begin{aligned}
d\mathbf{a} &= dxdy\\
&= |J|\ dudv,
\end{aligned}$$

where $|J|$ denotes the determinant of the Jacobian matrix. We write the Jacobian as follows:

$$\begin{aligned} J &= \frac{\partial (x, y)}{\partial (u, v)} \\ &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}. \end{aligned}$$

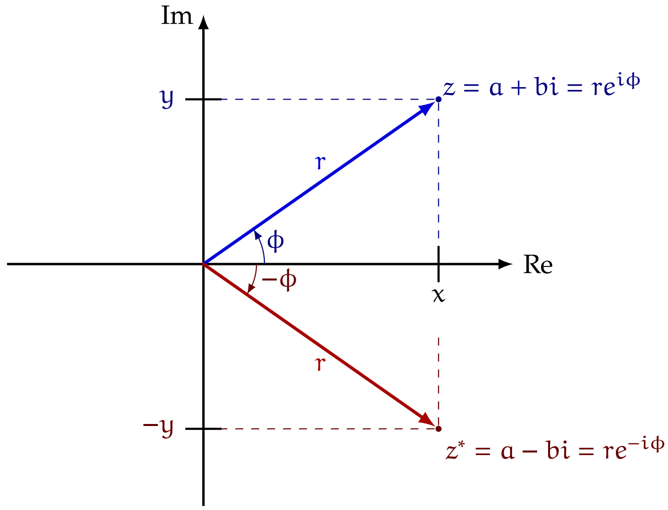
We specifically desire the determinant:

$$|J| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Complex Numbers

Quantity	Expression and/or Criterion
Cartesian form	$z = a + bi$
Polar form	$z = re^{i\phi}$
r	$\sqrt{a^2 + b^2}$
ϕ	$\arg z = \arctan\left(\frac{b}{a}\right)$
Cartesian z^*	$z^* = a - bi$
Polar z^*	$z = re^{-i\phi}$
$\|z\|$	$\sqrt{zz^*}$
$\mathrm{Re}(z)$	$\mathrm{Re}(z) = \frac{z+z^*}{2}$
$\mathrm{Im}(z)$	$\mathrm{Im}(z) = \frac{z-z^*}{2i}$
$\cos\phi$	$\frac{e^{i\phi}+e^{-i\phi}}{2}$
$\sin\phi$	$\frac{e^{i\phi}-e^{-i\phi}}{2i}$
$e^{i\phi}$	$\cos\phi + i\sin\phi$
$e^{in\phi}$	$\cos(n\phi) + i\sin(n\phi)$

Introduction



A complex number is denoted

$$z = a + bi$$

where $i^2 = -1$ and $a, b \in \mathbb{R}$. This is known as the cartesian representation. However, we can also imagine z as the polar representation:

$$z = re^{i\phi},$$

where $\phi = \arg z$ is known as the argument, and $r = |z|$ is the modulus. We can see the relation between the cartesian and polar representations through Euler's identity:[III]

$$r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

We denote the conjugate of z as z^*[IV], found by $z^* = a - bi = re^{-i\phi}$.

We find $\text{Re}(z)$ and $\text{Im}(z)$, the real and imaginary parts of z, by

$$\begin{aligned} \text{Re}(z) &= \frac{z + z^*}{2} \\ \text{Im}(z) &= \frac{z - z^*}{2i}. \end{aligned}$$

We say that a complex number of the form $e^{i\phi}$ is a *pure phase*, as $\left|e^{i\phi}\right| = 1$.

To find if some complex number z is purely real or purely imaginary, we can use the following criterion:

$$\begin{aligned} z \in \mathbb{R} &\Leftrightarrow z = z^* \\ z \in i\mathbb{R} &\Leftrightarrow z = -z^*. \end{aligned}$$

Example (Real, Imaginary, or Complex?)**.** Consider

$$z_1 = i^i.$$

To find if this is purely real or complex, we take

$$z_1^* = (-i)^{-i}$$

[III]This can be proven relatively easily through substitution into the Taylor series, which is allowed because e^z is entire.
[IV]Physicists amirite?

$$\begin{aligned} &= \left(\frac{1}{-i}\right)^{i} \\ &= i^{i}. \end{aligned}$$

Thus, $z_1 \in \mathbb{R}$. In order to determine the value of i^i, we substitute the polar form:

$$\begin{aligned} z_1 &= \left(e^{i\frac{\pi}{2}}\right)^{i} \\ &= e^{-\frac{\pi}{2}}. \end{aligned}$$

Some Trigonometry with Complex Exponentials

Consider $z = \cos\phi + i\sin\phi$. We can see that

$$\begin{aligned} \operatorname{Re}(z) &= \cos\phi \\ &= \frac{(\cos\phi + i\sin\phi) + (\cos\phi - i\sin\phi)}{2} \\ &= \frac{e^{i\phi} + e^{-i\phi}}{2} \\ \operatorname{Im}(z) &= \sin\phi \\ &= \frac{(\cos\phi + i\sin\phi) - (\cos\phi - i\sin\phi)}{2i} \\ &= \frac{e^{i\phi} - e^{-i\phi}}{2i}. \end{aligned}$$

We can actually define $\sin\phi$ and $\cos\phi$ with the above derivation.

Theorem (De Moivre)**.**

$$\begin{aligned} e^{inx} &= \cos(nx) + i\sin(nx) \\ &= \left(e^{ix}\right)^{n} \\ &= (\cos x + i\sin x)^{n}. \end{aligned}$$

Example (Finding $\cos(2x)$ and $\sin(2x)$)**.**

$$\begin{aligned} \cos(2x) + i\sin(2x) &= (\cos x + i\sin x)^2 \\ &= \left(\cos^2 x - \sin^2 x\right) + i\,(2\sin x\cos x). \end{aligned}$$

Since the real parts and imaginary parts have to be equal, this means

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin^2 x &= 2\sin x\cos x. \end{aligned}$$

In particular, we can see that $e^{in\phi} = (-1)^n$ and $e^{in\frac{\pi}{2}} = i^n$.[v]

Additionally, we can see that for $z = re^{i\phi}$,

$$\begin{aligned} z^{1/m} &= \left(re^{i\phi + 2\pi n}\right)^{1/m} \\ &= r^{1/m} e^{i\frac{1}{m}(\phi + 2\pi n)}, \end{aligned}$$

where $n \in \mathbb{N}$ and m is fixed. For $r = 1$, we call these values the m roots of unity.

[v] This will be especially useful when we get to Fourier series.

Example (Waves and Oscillations)**.** Recall that for a wave with spatial frequency k, angular frequency ω, and amplitude A, the wave is represented by

$$f(x,t) = A\cos(kx - \omega t).$$

The speed of a wave v is equal to $\frac{\omega}{k}$.

Simple harmonic motion is characterized by the solution to the differential equation $\ddot{\mathbf{x}} = -\omega^2\mathbf{x}$, where $\mathbf{x}$ denotes position. In simple harmonic motion, there is no spatial motion, meaning our function is only of time:

$$\begin{aligned} f(t) &= A\cos\omega t \\ &= \operatorname{Re}\left(Ae^{i\omega t}\right). \end{aligned}$$

As a result of the representation of complex numbers in polar form, we can do math entirely in exponentials, then take the real part of our solution to find $f(t)$.

Unfortunately, in the real world, there is friction; as a result, our oscillation is damped by an exponential factor.

Example (Hyperbolic Sine and Hyperbolic Cosine)**.** We wish to calculate $\cos ix$ and $\sin ix$.

$$\begin{aligned} \cos ix &= \frac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right) \\ &= \frac{e^{-x} + e^{x}}{2} \end{aligned}$$

We define $\cosh x = \cos(ix)$. Additionally,

$$\begin{aligned} -i\sin ix &= -i\frac{1}{2i}\left(e^{i(ix)} - e^{-i(ix)}\right) \\ &= i\frac{e^{ix} - e^{-ix}}{2i}. \\ &= \frac{e^{x} - e^{-x}}{2}. \end{aligned}$$

We define $\sinh x = -i\sin(ix)$.

Similar to how $\cos^2 x + \sin^2 x = 1$, we can find that $\cosh^2 x - \sinh^2 x = 1$.

Index Algebra

We usually denote vectors by either $\vec{A}$, $\mathbf{A}$, or

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is defined by a basis.

If we imagine we are in n-dimensional space, we can let A_i where $i = 1, 2, \dots, n$ denote both

- the ith component of $\vec{A}$;
- the entire vector $\vec{A}$ (since i can be arbitrary).

Contractions and Dummy Indices

Consider $C = AB$, where A, B are $n \times m$ and $m \times p$ matrices respectively.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}.$$

Definition (Matrix Multiplication in Index Notation)**.** For matrices A and B, where A is an $m \times n$ and B is a $n \times p$ matrix, we write

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

We say that k is a dummy index, since k takes values from 1 to n. Note that the value we calculate is C_{ij}; in other words, in the sum $\sum_k A_{ik}B_{kj}$, the indices of the form ij are the "net indices" from the multiplication.

Note that if $C = BA$, then

$$\begin{aligned} C_{ij} &= \sum_{k=1}^{n} B_{ik} A_{kj} \\ &= \sum_{k=1}^{n} A_{kj} B_{ik} \\ &\neq \sum_{k=1}^{n} A_{ik} B_{kj}. \end{aligned}$$

The corresponding fact is that $AB \neq BA$ necessarily.

Note that the index that is summed over always appears exactly twice.

Definition (Symmetric Matrix)**.** Let C be a matrix. Then, we say C is symmetric if

$$C_{ij} = C_{ji}$$

Definition (Antisymmetric Matrix)**.** Let C be a matrix. We say C is antisymmetric if

$$C_{ij} = -C_{ji}.$$

We can always decompose a random matrix into the sum of a symmetric matrix and an antisymmetric matrix.

Two Special Tensors

Name	Notation	Definition
Kronecker Delta	δ_{ij}	$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
Levi–Civita Symbol	ϵ_{ijk}	$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3) \text{ cyclically} \\ -1 & (i,j,k) = (2,1,3) \text{ cyclically} \\ 0 & \text{else} \end{cases}$

Order of (i, j, k)	Value of ϵ_{ijk}
$1, 2, 3$	1
$3, 1, 2$	1
$2, 3, 1$	1
$1, 3, 2$	-1
$2, 1, 3$	-1
$3, 2, 1$	-1
else	0

Definition (Kronecker Delta)**.** The Kronecker Delta, δ_{ij}, is the tensor that denotes the identity matrix.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example (Extracting an Index)**.** Consider A as vector. Then,

$$\sum_{i} A_i \delta_{ij} = A_j.$$

In other words, the Kronecker Delta collapses the sum to the jth index.

Example (Orthonormal Basis from Kronecker Delta)**.** Let $\{\hat{e}_i\}_{i=1}^{n}$ be a basis for some vector space V. If

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

for every i, j, then $\{\hat{e}_i\}_{i=1}^{n}$ is an orthonormal basis for V.

Definition (Levi–Civita Symbol)**.** In two dimensions, as a matrix, we write

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

meaning

$$\epsilon_{ij} = \begin{cases} 1 & i = 1, j = 2 \\ -1 & i = 2, j = 1 \\ 0 & \text{else} \end{cases}.$$

The Levi–Civita Symbol is antisymmetric, just as the Kronecker Delta is symmetric.

In three dimensions, we define

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = (1, 2, 3) \text{ cyclically} \\ -1 & (i, j, k) = (2, 1, 3) \text{ cyclically} \\ 0 & \text{else} \end{cases}.$$

In other words, $\epsilon_{ijk} = -\epsilon_{jik}$.

Exercise (Relations between δ_{ij} and ϵ_{ijk})**.**

$$\sum_{j,k} \epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn}$$
$$\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ijl} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$$

Definition (Dot Product)**.** Let $\{\hat{e}_i\}_{i=1}^{n}$ be an orthonormal basis for V. Let $\mathbf{A} = \sum_i A_i\hat{e}_i$ and $\mathbf{B} = \sum_i B_i\hat{e}_i$. Then,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \sum_{i,j}(A_i\hat{e}_i)\cdot\left(B_j\hat{e}_j\right)\\ &= \sum_{i,j}A_iB_j\left(\hat{e}_i\cdot\hat{e}_j\right)\\ &= \sum_{i,j}A_iB_j\delta_{ij}\\ &= \sum_i A_iB_i\end{aligned}$$

Definition (Cross Product)**.** Let $\{\hat{e}_i\}_{i=1}^{3}$ be the standard basis over $\mathbb{R}^3$. Let $\mathbf{A} = \sum_i A_i\hat{e}_i$ and $\mathbf{B} = \sum_i B_i\hat{e}_i$. Then,

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \sum_{i,j}(A_i\hat{e}_i)\times\left(B_j\hat{e}_j\right)\\ &= \sum_{i,j}A_iB_j\left(\hat{e}_i\times\hat{e}_j\right)\\ &= \sum_{i,j,k}A_iB_j\left(\epsilon_{ijk}\hat{e}_k\right).\end{aligned}$$

Instead of asking about $\mathbf{A}\times\mathbf{B}$, we ask about $(\mathbf{A}\times\mathbf{B})_\ell$, yielding

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})_\ell &= (\mathbf{A}\times\mathbf{B})\cdot\hat{e}_\ell\\ &= \left(\sum_{i,j,k}A_iB_j\left(\epsilon_{ijk}\hat{e}_k\right)\right)\cdot\hat{e}_\ell\\ &= \sum_{i,j}\epsilon_{ij\ell}A_iB_j.\end{aligned}$$

Remark: This notation for $\mathbf{A}\times\mathbf{B}$ automatically shows us that

$$\begin{aligned}(\mathbf{B}\times\mathbf{A})_\ell &= \sum_{i,j}\epsilon_{ij\ell}B_iA_j\\ &= -\sum_{i,j}\epsilon_{ji\ell}B_iA_j\\ &= -\sum_{i,j}\epsilon_{ji\ell}A_jB_i\\ &= -\sum_{i,j}\epsilon_{ij\ell}A_iB_j && i,j\text{ are dummy indices}\\ &= -(\mathbf{A}\times\mathbf{B})_\ell.\end{aligned}$$

Example (Central Force and Angular Momentum)**.** A central force is defined by

$$\mathbf{F} = f(r)\hat{r},$$

where $\hat{r}$ is a radial vector.

Angular momentum is defined by

$$\mathbf{L} = \mathbf{r}\times\mathbf{p},$$

where $\mathbf{r}$ denotes position and $\mathbf{p}$ denotes momentum. Then,

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\mathbf{r}\times\mathbf{p}\right)\\
&= \left(\frac{d}{dt}\mathbf{r}\times\mathbf{p}\right) + \mathbf{r}\times\left(\frac{d\mathbf{p}}{dt}\right)\\
&= m\left(\frac{d}{dt}\mathbf{r}\times\frac{d}{dt}\mathbf{r}\right) + \mathbf{r}\times\left(f(r)\hat{r}\right)\\
&= f(r)\left(\mathbf{r}\times\hat{r}\right).
\end{aligned}$$

This implies that $\frac{d\mathbf{L}}{dt} = 0$ under a central force.

Example (Determinant)**.** Let $\mathbf{M} = M_{ij}$ be square. We denote $\mathbf{M}_i$ to be the vector denoting the ith-row. Then,

$$\begin{aligned}
m &= |\mathbf{M}|\\
&= \mathbf{M}_1\cdot\left(\mathbf{M}_2\times\mathbf{M}_3\right)\\
&= \mathbf{M}_3\cdot\left(\mathbf{M}_1\times\mathbf{M}_2\right)\\
&= \mathbf{M}_2\cdot\left(\mathbf{M}_3\times\mathbf{M}_1\right).
\end{aligned}$$

Example (Trace)**.** Let $\mathbf{M} = M_{ij}$ be a square matrix. We define $\mathrm{tr} = \sum_i M_{ii}$. Equivalently,

$$\begin{aligned}
\mathrm{tr}\left(\mathbf{M}\right) &= \sum_{ij} M_{ij}\delta_{ij}\\
&= \sum_i M_{ii}.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{tr}\left(\mathbb{1}_n\right) &= \sum_i \delta_{ii}\\
&= n
\end{aligned}$$