These are some definitions and ideas I will be using regularly throughout this presentation.

**Definition** (Groups). Let $A$ be a set, and let $\star\colon A \times A \to A$ be such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists $e_A \in A$ such that $e_A \star a = a \star e_A$ for all $a \in A$;
- for each $a \in A$, there is $a^{-1}$ such that $a \star a^{-1} = a^{-1} \star a = e_A$.

Then, we say the pair $(A, \star)$ is a *group*. We abbreviate $a \star b = ab$.

**Definition** (Subgroups and Quotient Groups). Let $G$ be a group.

- If $H \subseteq G$ is a subset such that for all $a, b \in H$, $ab^{-1} \in H$, then we say $H$ is a *subgroup*.
- If $N \subseteq G$ is a subgroup such that for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say $N$ is a *normal subgroup*.
- If $N$ is a normal subgroup, we may define the equivalence relation $g \sim_N g'$ if $g^{-1}g' \in N$; the equivalence classes $gN \coloneqq [g]_{\sim_N}$ form the *quotient group*, $G/N$.
- If $H \subseteq G$ is a subgroup, then the *index* of $H$ is the number of cosets, $gH \coloneqq \{gh \mid h \in H\}$, written $[G : H]$.

**Definition** (Group Actions). If $G$ is a group, and $X$ is a set, then $\rho\colon G \times X \to X$ is called an *action* of $G$ onto $X$ if $\rho$ satisfies

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

We write $\rho(g, x) = g \cdot x$.

Every group is equipped with a family of canonical actions, $\sigma\colon G \times G \to G$, given by $(a, x) \mapsto ax$, known as *left-multiplication*.

**Definition** (Algebras, $\sigma$-Algebras of Subsets). If $X$ is a set, then a collection $\mathcal{A} = \{A_i\}_{i \in I} \subseteq P(X)$ is known as an *algebra* of subsets of $X$ if

(1) $\varnothing, X \in \mathcal{A}$;

(2) for all $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;

(3) for all $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

If condition (3) holds for any countable subcollection $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, then we say $\mathcal{A}$ is a *$\sigma$-algebra* of subsets.

**Definition** (Measures). If $X$ is a set and $\mathcal{A}$ is a $\sigma$-algebra, then a map $\mu\colon \mathcal{A} \to [0, \infty]$ that satisfies

- $\mu(\varnothing) = 0$;
- for disjoint $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say $\mu$ is a *finitely additive measure*. If, for any countable collection of disjoint sets, $\{A_n\}_{n \geq 1}$, we have

$$\mu\left(\bigsqcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n),$$

then we say $\mu$ is a *measure*. If $\mu(X) = 1$, then we say $\mu$ is a probability measure.