Complex Numbers

A complex number is an ordered pair of real numbers, $(a, b) = a + bi$. A vector in $\mathbb{R}^2$ is also an ordered pair, (a, b) of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with $\mathbb{R}^2$. However, unlike vectors in $\mathbb{R}^2$, there is also an operation $\cdot$. We desire for $(0, 1) \cdot (0, 1) = (-1, 0)$; essentially, $i^2 = -1$. We say that i is a square foot of -1; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di)\\ &:= a(c) + adi + bci + bd(i^2)\\ &:= (ac - bd) + (ad + bc)i\\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus, $\mathbb{R}^2$ with the operations $+$ and the above defined complex multiplication is known as $\mathbb{C}$. We write as $a+bi$ instead of (a, b).

Given $z = (a + bi) \in \mathbb{C}$, we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$. If $\text{Im}(z) = 0$, then $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$. However, many people say that $\mathbb{R} \subseteq \mathbb{C}$, even if $\mathbb{C}$ isn't defined as such.

Reciprocals of Complex Numbers

Let $z \in \mathbb{C}$, where $z \neq 0$. Then, $\exists w \in \mathbb{C}$ such that $zw = 1$.

Let $w = c + di$. We want to show that $zw = 1$.

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1\\ ad + bc &= 0.\end{aligned}$$

Thus, let $w = c + di$, with $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2}\\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every $z \neq 0$, with $z = a + bi$, the *reciprocal* of z is defined as $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Then, for $w \in \mathbb{C}$, we define

$$\frac{w}{z} := w\left(\frac{1}{z}\right).$$

Properties of Complex Numbers

Let $z = a + bi \in C$. Then, the (Euclidean) norm (or absolute value) of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

(i) $z\overline{z} = |z|^2$

(ii) $\overline{(\overline{z})} = z$

(iii) $\overline{(z+w)} = \overline{z} + \overline{w}$

(iv) $\overline{zw} = \overline{z} \cdot \overline{w}$

(v) $z + \overline{z} = 2\text{Re}(z)$, so $\text{Re}(z) = \frac{z+\overline{z}}{2}$

(vi) $z - \overline{z} = 2\text{Im}(z)i$, so $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

Polar Representation

Let $z = a+bi$ (or $z = (a, b)$). Then, $|z| = \sqrt{a^2 + b^2}$ is the *radius*, and the *argument* is found by $\theta = \arctan(b/a)$ for $a \neq 0$. Therefore, the full polar representation is as follows:

$$z = |z|\left(\cos\theta + i\sin\theta\right). \qquad \theta \in [0, 2\pi)$$

If $z = 0$, then $|z| = 0$, and $\arg z$ is undefined.

For example, we can find $\arg i$ in $[\pi, 3\pi)$ as $\frac{5\pi}{2}$.

For z_1 and z_2 in polar form, we have:

$$|z_1z_2| = |z_1||z_2| \tag{1}$$
$$\arg(z_1z_2) = \arg z_1 + \arg z_2 \mod 2\pi \tag{2}$$

Proof of (1):

$$\begin{aligned}
|z_1z_2|^2 &= (z_1z_2)\overline{(z_1z_2)} \\
&= z_1z_2\overline{z_1}\,\overline{z_2} \\
&= z_1\overline{z_1}z_2\overline{z_2} \\
&= |z_1|^2|z_2|^2
\end{aligned}$$

Since $|z| \geq 0$, we get $|z_1z_2| = |z_1||z_2|$.

Let $z = 2(\cos\pi/6 + i\sin\pi/6)$, and let $f : \mathbb{C} \to \mathbb{C}$ defined as $f(w) = zw$. Then, f rotates w by $\pi/6$ and scales w by 2.

Theorem: For $n \in \mathbb{N}$, if $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Proof: Induct on n. For the base case, we know that $n = 1$ satisfies this property. For $n > 1$, we have:

$$\begin{aligned}
z^{n+1} &= (z^n)(z) \\
&= \left(r^n(\cos(n\theta) + i\sin(n\theta))\right) r(\cos\theta + i\sin\theta) \\
&= (r^n)(r)\left(\cos(n\theta + \theta) + i\sin(n\theta + \theta)\right) && \text{Polar Representation Definition} \\
&= r^{n+1}(\cos\left((n+1)\theta\right) + i\sin((n+1)\theta))
\end{aligned}$$

We can use this technique to find the "roots of unity." For example, to find all z such that $z^3 = 1$, we use our technique:

$$\begin{aligned}
z^3 &= 1\\
|z| &= 1\\
\arg z^3 &= 0\\
3\arg z &= 0 \mod 2\pi\\
\arg z &= \frac{k2\pi}{3}\\
&= 0, \frac{2\pi}{3}, \frac{4\pi}{3}\\
z_1 &= 1\\
z_2 &= (\cos 2\pi/3 + i\sin 2\pi/3)\\
z_3 &= (\cos 4\pi/3 + i\sin 4\pi/3)
\end{aligned}$$

We can see that $z_2^2 = z_3$.

For the n case, we find $z_2 = \cos(2\pi/n) + i\sin(2\pi/n)$, and $z_k = z_2^{k-1}$.

Exponential, Logarithm, and Trigonometric Functions in $\mathbb{C}$

Exponential

Let $z = a + bi$. We define e^{a+bi} as follows:

$$e^{a+bi} = e^a(\cos b + i\sin b)$$

Recall that for every nonzero complex number, $z = |z|(\cos\theta + i\sin\theta)$, where $\theta = \arg z$. Thus,

$$\begin{aligned}
z &= |z|e^{i\theta}\\
&= |z|e^{i\arg z}.
\end{aligned}$$

The function e^z has some properties similar to the function e^x in real numbers, and some properties varying with the real numbers.

$$\begin{aligned}
e^z e^w &= e^{z+w}\\
e^z &\neq 0
\end{aligned}$$

However, there are some differences:

$$\begin{aligned}
|e^{i\theta}| &= 1 && \forall\theta\\
e^{a+bi} &= e^a
\end{aligned}$$

From these properties, we find Euler's equation:

$$e^{i\pi} + 1 = 0$$

Additionally, e^z is periodic, while $f(x) = e^x$ is injective:

$$\begin{aligned}
e^{z+2n\pi} &= e^z(\cos(2n\pi) + i\sin 2n\pi)\\
&= e^z
\end{aligned}$$

When examining the function $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, $z \mapsto e^z$, we find that the following happen:

- $f(\mathbb{R}) = (0, \infty)$ — we apply $f(x) = e^x$.
- $f(a + bi) = e^a e^{bi}$ — e^a is rotated by b.
- $f(\mathbb{R} + bi)$ is expressed as the line along b radians through the origin.
- Therefore, $f(A_0) = \mathbb{C} \setminus \{0\}$, where $A_0 = \{a + bi \mid a \in \mathbb{R}, b \in [0, 2\pi)\}$.

Logarithm

Recall that for a function $f : A \to B$, f^{-1} is a function if f is injective. However, for any f, it is the case that $f^{-1}(b)$ does exist, defined as follows:

$$f^{-1}(b) = \{a \mid f(a) = b\}.$$

For the function $f(z) = e^z$, f is not one to one, so for $w = e^z$, $f^{-1}(w) = \{z' \in \mathbb{C} \mid e^{z'} = w\}$. We can find this as $f^{-1}(w) = \{z + 2n\pi i \mid n \in \mathbb{Z}\}$.

We define $\log(w) := \{z \in \mathbb{C} \mid e^z = w\}$. For a fixed $\theta \in \mathbb{R}$, we define $\log_{A_\theta}(w) := \{z \mid e^z = w, z \in A_\theta\}$.

Let $z = 1 + \frac{5\pi}{2} i$. Then,

$$\log_{A_{-\pi}} e^z = 1 + \frac{\pi}{2} i$$

Let $w \in \mathbb{C} \setminus \{0\}$. To find $\log w$ (all values), then

$$\begin{aligned} z &\in \log w \\ e^z &= w \\ &= |w| e^{i \arg w} \\ e^{a+bi} &= |w| e^{i \arg w} \\ e^a e^{ib} &= |w| e^{i \arg w}. \end{aligned}$$

Therefore, $a = \ln |w|$ and $b = \arg w$. Additionally, the following hold, for $z_1, z_2 \in \mathbb{C}$:

$$\log_{A_\theta}(z_1 z_2) = \log_{A_\theta}(z_1) + \log_{A_\theta}(z_2) + 2n\pi i$$

Cosine and Sine

$$\begin{aligned} e^{ib} &= \cos b + i \sin b \\ e^{-ib} &= \cos b - i \sin b \\ \cos z &:= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &:= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Complex Powers

Recall that for $s, t \in \mathbb{R}$, $s^t = e^{t \ln s}$, where $s > 0$. For $z, w \in \mathbb{C}$, $z^w = e^{w \log z}$., where $z \neq 0$.

$$\begin{aligned} (-2)^i &= e^{i \log(-2)} \\ &= e^{i(\ln(2) + i\pi)} \\ &= e^{i \ln 2 - (\pi + 2\pi n)} \\ &= e^{-\pi + 2\pi n + i \ln 2} \end{aligned}$$

This has *infinitely* many values.

Let $\alpha = u + vi$. Then,

$$\begin{aligned} z^{\alpha} &= e^{\alpha \log z} \\ &= e^{(u+vi)(\ln|z| + i \arg z)} \\ &= e^{(u \ln|z| - v \arg z)} e^{i(v \ln|z| + u \arg z)} \end{aligned}$$

Since $\arg z = \theta + 2\pi n$ for some real $\theta \in [0, 2\pi)$,

$$= e^{u \ln z} e^{-v(\theta + 2\pi n)} e^{iv \ln|z|} e^{iu(\theta + 2\pi n)}$$

Therefore, complex exponentiation is single-valued if $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Z}$, then z^{α} has only one value; if $\alpha \in \mathbb{Q}$, where $\alpha = \frac{p}{q}$ and $\gcd(p, q) = 1$, then z^{α} takes q distinct values, which are the qth-roots.

Continuous Functions with Complex Domains

Let $z \in \mathbb{C}$, let $r > 0$.

- The set $D(z; r) := \{w \mid w \in \mathbb{C}, |z - w| < r\}$ is the r-neighborhood of z.
- A subset $A \subseteq \mathbb{C}$ is open if $(\forall z \in A)\,(\exists r > 0) \ni D(z; r) \subseteq A$.

For example, if $A = \{z \mid \text{Re}(z) > 0\}$, we can find r equal to half the magnitude of the real component of z for any $z \in A$, meaning A is open.

Meanwhile, if $A = \{z \mid \text{Re}(z) \geq 0\}$, this is not the case. If $z = 0$, then $\nexists r > 0$ such that $D(z; r) \subseteq A$, as any open ball of radius r will have some element in $\overline{A}$.

- A subset $B \subseteq \mathbb{C}$ is closed if $\overline{B} \subseteq \mathbb{C}$ is open.

For example, $A = \emptyset$ is open, by vacuous truth, so $\overline{A} = \mathbb{C}$ is closed. Similarly, since $\mathbb{C}$ is open, $\emptyset$ is closed.

Meanwhile, $A = \{x + iy \mid -1 \leq x < 1\}$ is neither open nor closed.

Limits

Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, $z_0 \in \mathbb{C}$. Then,

$$\lim_{z \to z_0} f(z) = \ell$$

means both of the following hold:

(i) for some $r > 0$, $D(z_0; r) \setminus \{z_0\} \subseteq \text{dom}(f)$,

(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $f(D(z_0; \delta) \setminus \{z_0\}) \subseteq D(\ell; \varepsilon)$.

For example, if

$$f(z) = \begin{cases} z & z \in \mathbb{C} \setminus \mathbb{R} \\ 3i & z \in \mathbb{R} \end{cases}$$

Then, $\lim_{z \to 0} f(z)$ does not exist, as there is no ℓ that satisfies both conditions. Specifically, if $\ell = 3i$, and we set $\varepsilon = 1$, then a disc of any radius around 0 has some $z \in \mathbb{C} \setminus \mathbb{R}$ that maps to itself. Similarly, if we set $\ell = 0$, then there is a real number in a disc of any radius around 0.

Note: f does not have to be defined at z_0 for the limit to be defined at z_0.

Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$, and $z_0 \in A$. We say f is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. We say f is continuous on A if $\forall z_0 \in A$, f is continuous at z_0.

We will show that $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto 3z$ is continuous.

Scratch Work: We want δ such that $f(D(z_0; \delta)) \subseteq D(3z_0; \varepsilon)$. Let $z \in D(z_0; \delta)$, meaning $f(z) = 3z$. We want $3z \in D(3z_0; \varepsilon)$, meaning we want $|3z - 3z_0| < \varepsilon$, or $|z - z_0| < \frac{\varepsilon}{3}$.

Proof: Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{3}$. We show $f(D(z_0; \delta)) \subseteq D(f(z_0); \varepsilon)$. Let $z \in D(z_0; \delta)$. Then, $|z - z_0| < \delta = \varepsilon/3$, meaning $3|z - z_0| < \varepsilon$, meaning $|3z - 3z_0| < \varepsilon$, so $|f(z) - f(z_0)| < \varepsilon$. Therefore, $f(z) \in D(f(z_0); \varepsilon)$. Since f is continuous at arbitrary z_0, f is continuous on $\mathbb{C}$.

Sequences

A sequence $z_1, z_2, \dots \in \mathbb{C}$. A sequence converges to $z_0 \in \mathbb{C}$ if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M},\ |z_n - z_0| < \varepsilon$$

In words, for any radius around z_0, we can find z_n arbitrarily close to z_0 for sufficiently large n. We write $z_n \to z_0$ if this is the case.

Let $f : \mathbb{C} \to \mathbb{C}$. Then, f is continuous on $\mathbb{C}$ if and only if the following equivalent conditions are met:

(i) the inverse image of every open set is open ($f^{-1}(B) := \{a \in \mathbb{C} \mid f(a) \in B\}$);

(ii) the inverse image of every closed set is closed;

(iii) for every sequence $(z_n)_n$ such that $(z_n)_n \to z_0$, $f(z_n) \to f(z_0)$.

Let

$$f(z) = \begin{cases} 0 & z = 0 \\ 1 & z \neq 0 \end{cases}.$$

This function is not continuous. We will check that (i)–(iii) fail.

(i) Let $B = D(0; 1)$. Then, $f^{-1}(B) = \{0\}$, which is not open set.

(ii) Let $B = \mathrm{cl}(D(1; 0.5))$. Then, $f^{-1}(B) = \mathbb{C} \setminus \{0\}$, which is not closed.

(iii) Let $z_n = \frac{1}{n}$. Then, $(z_n)_n \to 0$, but $f(z_n) = 1$ for all n, meaning $f(z_n) \to 1 \neq f(0)$.

To show limit divergence, recall the definition of limit convergence:

$$\lim_{n\to\infty} z_n = z_0 \Leftrightarrow (\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M},\ |z_n - z_0| < \varepsilon.$$

Let $z_1, \dots, \in \mathbb{C}$ be a sequence. Then, $\lim_{n\to\infty} = \infty$ means

$$(\forall M > 0)(\exists N \in \mathbb{N}) \ni \forall n > N,\ |z_n| > M.$$

In words, $|z_n|$ is arbitrarily large for sufficiently large n.

Connected Sets

Let $a, b \in \mathbb{C}$. A path from a to b is a continuous function $p : [0,1] \to \mathbb{C}$ such that $p(0) = a$ and $p(1) = b$. Let $S \subseteq \mathbb{C}$. If $p([0,1]) \subseteq S$, then p is a path in S.

We say S is *path-connected* if for any $s, t \in S$, there is a path in S from s to t.

Every set that is path-connected is connected, but not necessarily the other way around — if A is open and path connected, then A is connected.

An open, path-connected subset of $\mathbb{C}$ is known as a region, or a domain.

Let $A = \mathbb{R} \times \{0\}$ (or the x axis in $\mathbb{C}$). A is not a region, as A is not an open set, even if A is path-connected.

$A \subseteq \mathbb{C}$ is bounded if there exists $r > 0$ such that $A \subseteq D(0; r)$. $A = \mathbb{R} \times \{0\}$ is not bounded.

If $A \subseteq \mathbb{C}$, then A is compact if A is closed and bounded. There are various properties of compact sets that make them particularly amenable towards analysis.

Extreme Value Theorem: Every real-valued continuous function on a compact domain attains its maximum and minimum values.

Uniform Continuity Theorem: Elaborated below.

Uniform Continuity

Recall that if $f : A \to \mathbb{C}$, f is continuous if $\forall a \in A$, $\lim_{z \to a} f(z) = f(a)$.

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta_a > 0) \ni f(D(a; \delta_a)) \subseteq D(f(a); \varepsilon) \qquad \delta \text{ depends on } a$$

When f is uniformly continuous, there is one value of δ, dependent on ε, that applies for every value of a.

$$(\forall \varepsilon > 0)(\exists \delta_\varepsilon > 0) \ni (\forall a \in A), f(D(a; \delta_\varepsilon)) \subseteq D(f(a); \varepsilon)$$

Riemann Sphere

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2\}$. Let $N = (0, 0, 1)$ denote the north pole. Then, there is a continuous bijection from $S^2 \setminus \{N\} \to \mathbb{C}$.

We can visualize this by picking a random point on the sphere and drawing a line from the north pole through the sphere to this point, and finding the point that intersects the plane.

Consider the sequence $z_n = n^2 i$ for $n = 1, 2, \ldots$. We can see that, on the projection from z_n to the sphere, all the values of p converge to N. Therefore, we write $\lim_{n \to \infty} z_n = \infty$, where ∞ corresponds to N on S^2.

We can define $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to be the complex plane that includes the "point at infinity" (from the projection on S^2 that corresponds to the north pole).

Analytic Functions

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ where A is open. Let $z_0 \in A$. We say f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Rules of Differentiation

- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$
- $(f \circ g)' = g'(f' \circ g)$
- For $n \in \mathbb{Z}$, $(z^n)' = nz^{n-1}$

Let $f(z) = \overline{z}$. We will find this value by directly applying the definition of the derivative.

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\ &= \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}. \end{aligned}$$

Let's approach z_0 from the horizontal direction. Suppose $z = z_0 + t$ for some $t \in \mathbb{R}$. Then,

$$\lim_{z \to z_0} \frac{\overline{z_0 + t} - \overline{z_0}}{z_0 + t - z_0} = 1.$$

Let's approach z_0 from the horizontal direction. Suppose $z = z_0 + ti$ for some $t \in \mathbb{R}$. Then,

$$\begin{aligned} \lim_{z \to z_0} \frac{\overline{z_0 + ti} - \overline{z_0}}{z_0 + ti - z_0} &= \frac{-ti}{ti} \\ &= -1. \end{aligned}$$

Since $1 \neq -1$, we find that the limit does not exist.

We see that complex-differentiability is a strong condition.

Suppose that $f'(z_0) = 2i$, meaning

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 2i.$$

If z is close to z_0, then $f(z) - f(z_0) \approx 2i(z - z_0)$. Pictorially, we can visualize this as, for z_0 sufficiently close to z, the vector $z_0 - z$ is akin to a counterclockwise rotation and a scaling by 2. This is applicable for *all* z in sufficient proximity to z_0.

Specifically, we can see that the complex differentiable function is *angle-preserving*. The technical name for f is that f is *conformal*.

Analytic Function

Let $f : A \subseteq C \to \mathbb{C}$. If f is differentiable at every $z_0 \in A$, we say f is *analytic* on A.

If f is analytic on A, then f is infinitely differentiable on A.

If f is analytic on A and $f'(z_0) \neq 0$ for some $z_0 \in A$, then f is conformal at $z_0 \in A$.

Cauchy-Riemann Theorem

Given a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. Recall that we can take partial derivatives, $\frac{\partial f}{\partial x}$, and directional derivative $\frac{\partial f}{\partial u}$ for some unit vector u.

However, for $\mathbb{C}$, there is only one derivative, $f'(z_0)$, meaning that regardless of direction, $f'(z_0)$ exists and has one value. We can contextualize $f(z) = f(x + yi) = u(x, y) + iv(x, y)$, where $u(x, y) \in \mathbb{R}$ and $v(x, y) \in \mathbb{R}$. Then,

$$\frac{\partial u}{\partial x} \neq \frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial x} \neq \frac{\partial v}{\partial y}$$

but

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

We can see this by first letting $z = z_0 + \delta x$.

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z_0 + \delta x) - f(z_0)}{z_0 + \delta x - z_0} \\ &= \lim_{z \to z_0} \frac{u(x_0 + \delta x, y_0) + iv(x_0 + \delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

and in the y direction,

$$\begin{aligned} f'(z_0) &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

We set these two values equal to find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned}$$

which are the Cauchy-Riemann equations. The corresponding theorem states that if $f'(z_0)$ exists, then the Cauchy-Riemann equations must hold.

For example, if $f(z) = \overline{z}$, with $f(x + yi) = x - yi$, we have $u(x, y) = x$ and $v(x, y) = -y$. Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial v}{\partial v} &= -1, \end{aligned}$$

meaning f is not complex-differentiable.

If $f : A \to \mathbb{C}$ satisfies the Cauchy-Riemann equations at every $z_0 \in A$, then f is analytic on A.

If $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is analytic on A, then we know f' and f'' are continuous. From multivariable calculus, we know that $u_{xy} = u_{yx}$ if both are continuous. So,

$$\begin{aligned} u_{xy} &= \frac{\partial}{\partial y}(u_x) \\ &= \frac{\partial}{\partial y}(v_y) \\ &= v_{yy} \\ u_{yx} &= \frac{\partial}{\partial x}(u_y) \\ &= \frac{\partial}{\partial x}(-v_x) \\ &= -v_{xx} \end{aligned}$$

Therefore, $v_{xx} + v_{yy} = 0$. Similarly, $u_{xx} + u_{yy} = 0$.

If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ If $\varphi_{xx} + \varphi_{yy} = 0$, then we say φ is a harmonic function. Therefore, if f is an analytic function, then both the real and imaginary parts of f are harmonic.

Let $A \subseteq \mathbb{R}^2$. If $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$. Then, u and v are harmonic conjugates if $u + iv$ is an analytic function. Additionally, u and v are harmonic conjugates if and only if they satisfy the Cauchy-Riemann equations.

We may ask if there exists an analytic function f such that $\text{Re}(f) = x^3 - 3xy^2 + y$. Then,

$$\begin{aligned} v_y &= u_x = 3x^2 - 3y^2 \\ -v_x &= u_y = 1 - 6xy. \end{aligned}$$

Therefore, we find $v = -x + 3x^2y - y^3 + c$ through integration. Therefore, we have

$$\begin{aligned} f(z) &= (x^3 - 3xy^2 + y) + i(3x^2y - y^3 - x + c) \\ &= (x - iy)^3 + y - ix + ic \\ &= z^3 + i(-iy + x) + ic \\ &= \overline{z}^3 + i\left(\overline{z} + c\right) \end{aligned}$$

Recall from from multivariable calculus that $\nabla u \perp$ contour lines of u. Similarly, $\nabla v \perp$ contour lines of v. Then, using the Cauchy-Riemann equations, we find

$$\begin{aligned} \nabla u \cdot \nabla v &= (-u_x u_y) + u_x u_y \\ &= 0, \end{aligned}$$

meaning the gradients are orthogonal to each other, meaning the contours of u are perpendicular to the contours of v.

Inverse Functions

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$. Let $z_0 \in A$. If f is analytic on A and $f'(z_0) \neq 0$, then f is one to one on some neighborhood of z_0. Then, $f^{-1} : f(N) \to N$ is analytic on $f(N)$, and

$$(f^{-1})'(f(z_0)) = \frac{1}{f'(z_0)}.$$