Math 395

Homework 7

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Name: Avinash Iyer

Collaborators: Antonio Cabello, Timothy Rainone, Nate Hall, Nora Manukyan, Jamie Perez-Schere

Problem 1

We say a field K/F is normal if K is the splitting field of a collection of polynomials. Equivalently, every polynomial in $F[x]$ that has a root in K splits into linear factors over K. Let $\alpha \in \mathbb{R}$ such that $\alpha^4 = 5$. We will show that $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$, but $\mathbb{Q}(\alpha + i\alpha)$ is not normal over $\mathbb{Q}$.

Note that $(\alpha + i\alpha)^2 = 2i\alpha^2$. Thus, $\mathbb{Q}(\alpha + i\alpha) = \text{Spl}_{\mathbb{Q}(i\alpha^2)}(x^2 - 2i\alpha^2)$, so $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$.

Suppose toward contradiction that $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}$. Notice that $(\alpha + i\alpha)^4 = -20$, as is $(\alpha - i\alpha)^4$. Thus, $\alpha + i\alpha$ and $\alpha - i\alpha$ are roots of $x^4 + 20$. Since $\alpha, i, i\alpha \in \mathbb{Q}(\alpha + i\alpha)$, it is the case that $\mathbb{Q}(\alpha, i) \subseteq \mathbb{Q}(\alpha + i\alpha)$. However, we have

$$\begin{aligned}
[\mathbb{Q}(\alpha, i) : \mathbb{Q}] &= [\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] \\
&= (2)(4) \\
&= 8,
\end{aligned}$$

and $[\mathbb{Q}(\alpha + i\alpha) : \mathbb{Q}] = 4$, as $m_{\alpha+i\alpha,\mathbb{Q}}(x) = x^4 + 20$. $\perp$

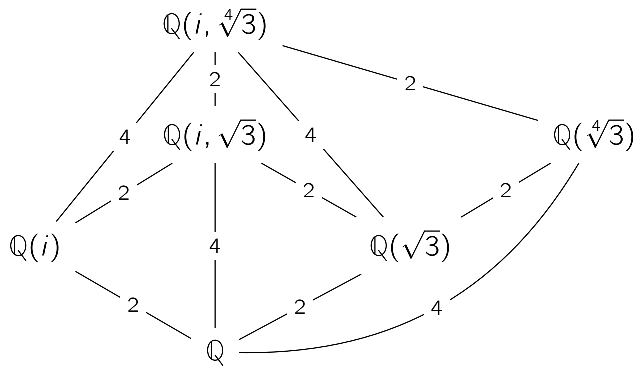
Problem 2

The roots of $f(x) = (x^5 - 2)(x^2 - 2)$ are $\pm\sqrt{2}, \zeta_5^k \sqrt[5]{2}$ for $k = 0, 1, 2, 3, 4$. We can see that $\mathbb{Q}(\zeta_5, \sqrt{2}, \sqrt[5]{2})$ contains the roots of $(x^5 - 2)(x^2 - 2)$, so $\text{Spl}_{\mathbb{Q}}(f(x)) \subseteq \mathbb{Q}(\zeta_5, \sqrt{2}, \sqrt[5]{2})$. Additionally, we see that $\sqrt[5]{2} \in \text{Spl}_{\mathbb{Q}}(f(x))$, $\zeta_5 = \frac{\zeta_5 \sqrt[5]{2}}{\sqrt[5]{2}} \in \text{Spl}_{\mathbb{Q}}(f(x))$, and $\sqrt{2} \in \text{Spl}_{\mathbb{Q}}(f(x))$. Thus, $\mathbb{Q}(\zeta_5, \sqrt[5]{2}, \sqrt{2}) = \text{Spl}_{\mathbb{Q}}(f(x))$.

For $x^6 + x^3 + 1$, we have that $x^6 + x^3 + 1 = \frac{x^9 - 1}{x^3 - 1}$. Therefore, the roots of $x^6 + x^3 + 1$ are ζ_9^d, where $\gcd(d, 9) = 1$ (since $9 = 3^2$, every $n \neq 0, 3, 6$ is a root of $x^6 + x^3 + 1$). Therefore, we can see that $x^6 + x^3 + 1 = \Phi_9(x)$, meaning $\text{Spl}_{\mathbb{Q}}(x^6 + x^3 + 1) = \mathbb{Q}(\zeta_9)$.

Problem 6

To find the subfields of $\mathbb{Q}(i, \sqrt[4]{3})$, we see that the basis of $\mathbb{Q}(i, \sqrt[4]{3})$ over $\mathbb{Q}$ is $\{1, \sqrt[4]{3}, \sqrt{3}, \sqrt[4]{27}, i, i\sqrt[4]{3}, i\sqrt{3}, i\sqrt[4]{27}\}$, meaning $[\mathbb{Q}(i, \sqrt[4]{3}) : \mathbb{Q}] = 8$. Finding subspaces of $\mathbb{Q}(i, \sqrt[4]{3})$, we arrive at the following diagram.



For any subfield $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(i, \sqrt[4]{3})$, it must be the case that $[F : \mathbb{Q}] = 2^k$ for some $k = 0, 1, 2, 3$. Therefore, it must be the case that all subfields are of degree $1, 2, 4, 8$.

Suppose there is any subfield $\mathbb{Q} \subseteq E \subseteq \mathbb{Q}(i)$. Then, it must be the case that $[E : \mathbb{Q}] = 1$ or $[E : \mathbb{Q}] = 2$, meaning $E = \mathbb{Q}$ or $E = \mathbb{Q}(i)$. The same argument applies for all degree 2 extensions in the above diagram.

Problem 7

Let $n = p^k m$ with m relatively prime to prime p. We will show that there are m distinct nth roots of unity over a field with characteristic p.

Let ζ_n be an nth root of unity. Then, $\zeta_n^n - 1 = 0$, meaning

$$\begin{aligned} \zeta_n^{p^k m} - 1 &= 0 \\ (\zeta_n^m)^{p^k} - 1 &= 0 \\ (\zeta_n^m)^{p^k} - 1^{p^k} &= 0 \\ (\zeta_n^m - 1)^{p^k} &= 0. \end{aligned}$$

Since $m \neq p^\ell \alpha$, as m and p are relatively prime, it must be the case that, the m roots of unity are distinct, and each nth root of unity is an mth root of unity, meaning there are m distinct nth roots of unity.