

## Problem 1

Let  $X$  be a metric space. Show that  $X$  is second countable if and only if  $X$  is separable. Conclude that if  $X$  is a separable metric space, then every open set is the union of countably many open balls.

**Proof:**

( $\Rightarrow$ ): Let  $X$  be second countable. Then,  $X$  contains base  $U_1, U_2, \dots \in \mathcal{B}$  such that each  $U_i$  is nonempty. Let  $x_1 \in U_1, x_2 \in U_2, \dots$

The set  $\{x_i\}_{i \geq 1}$  is countable, as each  $x_i \in U_i$ . For any  $U \in \tau_X$  where  $U \neq \emptyset$ ,  $U = \bigcup_{i \in I} U_i$ , meaning that  $U \cap \{x_i\}_{i \geq 1} \neq \emptyset$ . Thus,  $\{x_i\}_{i \geq 1}$  is dense in  $X$ , meaning  $X$  is separable.

( $\Leftarrow$ ): Let  $X$  be separable, with countable dense subset  $\{x_i\}_{i \geq 1}$ . Let

$$\mathcal{B} = \{U(x_i, 1/n) \mid x_i \in \{x_i\}_{i \geq 1}, n \in \mathbb{N}\}.$$

Then, for every  $U \in \tau_X$ , since  $U \cap \{x_i\}_{i \geq 1} \neq \emptyset$ , and  $\exists n$  such that  $U(x_k, 1/n) \subseteq U$ , it must be the case that  $\mathcal{B}$  is a base for  $\tau_X$ . Thus,  $X$  is second countable.

If  $X$  is a separable metric space, then it admits a countable base, and any element of  $\tau_X$  is a union of the elements of the base, implying that any element of  $\tau_X$  is a union of countably many open balls.

## Problem 2

Let  $(X, d)$  be a metric space,  $(x_n)_n$  a sequence in  $x$ , and  $x \in X$ . The following are equivalent:

- (i)  $(x_n)_n \rightarrow x$  in  $X$ ;
- (ii)  $(d(x_n, x))_n \rightarrow 0$  in  $\mathbb{R}$ ;
- (iii) For every neighborhood  $V \in \mathcal{N}_x$ , there is an  $N \in \mathbb{N}$  with  $n \geq N \Rightarrow x_n \in V$ .

**Proof:** Let  $(x_n)_n \rightarrow x$  in  $X$ . Then, for any  $\varepsilon > 0$ ,  $\exists N$  large such that  $n \geq N \Rightarrow d(x_n, x) < \varepsilon$ . However, this is precisely the same as  $|d(x_n, x) - 0| < \varepsilon$ , which is true if and only if  $(d(x_n, x))_n \rightarrow 0$ .

## Problem 3

Let  $X$  be a metric space. Show that a sequence  $(x_n)_n$  converges in  $X$  if and only if every subsequence  $(x_{n_k})_k$  admits a convergent subsequence  $(x_{n_{k_j}})_{j \geq 1}$ .

**Proof:** I don't know how to do this.

## Problem 4

Let  $\{(X_k, d_k)\}$  be a family of metric spaces. Assume that for every  $k \geq 1$ , we have  $d_k(x, y) \leq 1$  for all  $x, y \in X_k$ . Let

$$X := \prod_{k \geq 1} X_k$$

denote the product with metric

$$d(f, g) := \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k)).$$

Show that a sequence  $(f_n)_n$  converges to  $f$  in  $X$  if and only if  $(f_n(k))_n \rightarrow f(k)$  for every  $k \geq 1$ .

**Proof:** Let  $(f_n)_n \rightarrow f$ . Then,  $(d(f_n, f))_n \rightarrow 0$ . Therefore, for  $\varepsilon > 0$ , there exists an  $N$  large such that

$$\sum_{k=1}^{\infty} 2^{-k} d_k(f_n(k), f(k)) < \varepsilon$$

for  $n \geq N$ .

## Problem 5

Let  $V$  be a normed space. Show that the operations

$$\begin{aligned} a : V \times V &\rightarrow V; \\ a(v, w) &= v + w \end{aligned}$$

and

$$\begin{aligned} \mu : \mathbb{F} \times V &\rightarrow V; \\ \mu(\alpha, v) &= \alpha v \end{aligned}$$

are continuous.

**Proof:**

- $a : V \times V \rightarrow V, a(v, w) = v + w:$

$$\begin{aligned} \|a(v, w) - a(v', w')\| &= \|v + w - (v' + w')\| \\ &= \|v - v' + w - w'\| \\ &\leq \|v - v'\| + \|w - w'\| \\ &= d(v, v') + d(w, w') \\ &= d_1((v, w), (v', w')), \end{aligned}$$

meaning  $a$  is Lipschitz.

- $\mu : \mathbb{F} \times V \rightarrow V, \mu(\alpha, v) = \alpha v:$

$$\begin{aligned} \|\mu(\alpha, v) - \mu(\beta, w)\| &= \|\alpha v - \beta w\| \\ &= \|\alpha v - \alpha w + \alpha w - \beta w\| \\ &\leq |\alpha| \|v - w\| + |\alpha - \beta| \|w\| \end{aligned}$$

If  $(\alpha_n)_n \rightarrow \beta$  and  $(v_n)_n \rightarrow w$ , then

$$\begin{aligned} \|\alpha_n v_n - \beta w\| &\leq |\alpha_n| \|v_n - w\| + |\alpha_n - \beta| \|w\| \\ &\rightarrow 0. \end{aligned}$$

## Problem 6

Let  $(X, d)$  be a metric space,  $f, g : X \rightarrow \mathbb{F}$  continuous maps, and  $\alpha \in \mathbb{F}$ . Show that  $f + g$ ,  $fg$ , and  $\alpha f$  are continuous.

**Proof:** Let  $(x_n)_n \rightarrow x \in X$ . Then, we know that  $|f(x_n) - f(x)| \rightarrow 0$  and  $|g(x_n) - g(x)| \rightarrow 0$  (where  $|\cdot|$  denotes absolute value in  $\mathbb{F}$ ). Let  $\varepsilon > 0$ . Therefore, for  $N$  large, we know that

$$\begin{aligned} |f(x_n) + g(x_n) - (f(x) + g(x))| &\leq |f(x_n) - f(x)| + |g(x_n) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

meaning  $|f(x_n) + g(x_n) - (f(x) + g(x))| \rightarrow 0$ , so  $(f(x_n) + g(x_n))_n \rightarrow f(x) + g(x)$ . Thus,  $f + g$  is continuous.

Similarly,

$$\begin{aligned} |f(x_n)g(x_n) - f(x)g(x)| &= |f(x_n)g(x_n) - f(x_n)g(x) + f(x_n)g(x) - f(x)g(x)| \\ &= |f(x_n)(g(x_n) - g(x)) + g(x)(f(x_n) - f(x))| \\ &\leq |f(x_n)||g(x_n) - g(x)| + |g(x)||f(x_n) - f(x)| \\ &\leq c|g(x_n) - g(x)| + |g(x)||f(x_n) - f(x)| \quad \text{convergent sequences are bounded} \\ &< \varepsilon \end{aligned}$$

so  $(f(x_n)g(x_n))_n \rightarrow f(x)g(x)$ .

## Problem 8

Let  $h : X \rightarrow Y$  be a homeomorphism of metric spaces. Show that the map

$$T_h : (C(X), \|\cdot\|_u) \rightarrow (C(Y), \|\cdot\|_u)$$

$$T_h(f) = f \circ h$$

is an isometric isomorphism of normed spaces.

**Proof:**

## Problem 9

Suppose  $T : V \rightarrow W$  is a bijective linear map between normed spaces with  $\|T\|_{\text{op}} \leq 1$  and  $\|T^{-1}\|_{\text{op}} \leq 1$ . Show that  $T$  is an isometry.

**Proof:** Since the operator norm for  $T$  is less than or equal to 1, we know that for  $v, w \in V$ ,

$$\|T(v) - T(w)\|_W \leq \|v - w\|_V$$

and

$$\|T^{-1}(T(v)) - T^{-1}(T(w))\|_V \leq \|T(v) - T(w)\|_W$$

so, since  $T$  is bijective,

$$\|v - w\|_V \leq \|T(v) - T(w)\|_W$$

meaning

$$\|T(v) - T(w)\|_W = \|v - w\|_V$$

so  $T$  is an isometry.