Prelude

My REU mentor recently bought me the book *Banach Algebra Techniques in Operator Theory*, so I'm going to be reading through it here. Astute readers may already know that I am also reading through the book *Quantum Theory for Mathematicians*, and may be wondering if this is going to crowd out that book. The answer is yes — but I don't really care that much. If I come out of the summer knowing more things than I knew entering, then I will have succeeded.

Prerequisite Notes

Since Douglas's book is very advanced, I'm going to end up going back and reading other important material in order to contextualize the parts of the book I don't fully understand.

Tychonoff's Theorem

I'm drawing information for this section from Volker Runde's book *A Taste of Topology*, specifically from Chapter 3.

Definition (Product Topology)**.** Let $\{(X_i, \tau_i)\}_i$ be a family of topological spaces, and $X = \prod_{i\in I} X_i$.

The product topology on X is the coarsest topology τ on X such that

$$\prod_i : X \to X_i;\ \ f \mapsto f(i)$$

is continuous.

The product topology's open sets are of the form

$$\bigcap_{j=1}^{n} \pi_{i_j}\left(U_j\right),$$

where $i_j \in I$. The product topology is the topology of coordinatewise convergence.

Theorem (Tychonoff)**.** *Let $\{(K_i, \tau_i)\}_{i\in I}$ be a nonempty family of compact topological spaces. Then, the product space $K = \prod_{i\in I} K_i$ is compact in the product topology.*

Proof. Let $\{f_\alpha\}_{\alpha\in A}$ be a net[i] in K. Let $J \subseteq I$ be nonempty, and let $f \in K$.

[i]See future definition of nets.

We call (J, f) a partial accumulation point of $\{f_\alpha\}_{\alpha \in A}$ if $f|_J$ is a accumulation point of $\{f_\alpha|_J\}_{\alpha\in A}$ in $\prod_{j\in J} K_j$. A partial accumulation point of $\{f_\alpha\}_{\alpha\in A}$ is a accumulation point of $\{f_\alpha\}_{\alpha\in A}$ if and only if $J = I$.

Let $\mathcal{P}$ be the set of partial accumulation points of $\{f_\alpha\}_{\alpha\in A}$ For any two $(J_f, f), (J_g, g) \in \mathcal{P}$, define the order $(J_f, f) \leqslant (J_g, g)$ if and only if $J_f \subseteq J_g$ and $g|_{J_f} = f$.

Since K_i is compact for each $i \in I$, the net $\{f_\alpha\}_\alpha$ has partial accumulation points $(\{i\}, f_i)$ for each $i \in I$ (since each K_i is compact, the net analogue to sequential compactness holds); in particular, $\mathcal{P}$ is nonempty.

Let $\mathcal{Q}$ be a totally ordered subset of $\mathcal{P}$, and $J_g = \bigcup \{J_f \mid (J_f, f) \in \mathbb{Q}\}$. Define g by letting $g(j) = f(j)$ for each $j \in J_f$ with $(J_f, f) \in \mathcal{Q}$, and arbitrarily on $I \setminus J_g$.

Since $\mathcal{Q}$ is totally ordered, g is well-defined. We claim that (J_g, g) is a partial accumulation point of $\{f_\alpha\}_\alpha$.

Let $N \subseteq \prod_{j\in J_g} K_j$ be a neighborhood of $g|_{J_g}$. We may suppose that

$$N = \pi_{j_1}^{-1}\left(U_{j_1}\right) \cap \cdots \cap \pi_{j_n}\left(U_{j_n}\right),$$

where $j_1, \dots, j_n \in J_g$, and $U_{j_i} \subseteq K_{j_i}$ are open.

Let $(J_h, h) \in \mathcal{Q}$ be such that $\{j_1, \dots, j_n\} \subseteq J_h$, which is possible since $\mathcal{Q}$ is totally ordered. Since (J_h, h) is a partial accumulation point of $\{f_\alpha\}_\alpha$, there is an index α and a $\beta \geqslant \alpha$, where

$$f_\beta(j_k) = \pi_{j_k}(f_\beta) U_{j_k},$$

so $f_\beta \in N$. Thus, (J_g, g) is a partial accumulation point of $\{f_\alpha\}_\alpha$, and is an element of $\mathcal{P}$.

By Zorn's lemma,[ii] $\mathcal{P}$ has a maximal element, (J_{max}, f_{max}).

Suppose toward contradiction that $J_{max} \subset I$, meaning there is an $i_0 \in I \setminus J_{max}$. Since (J_{max}, f_{max}) is a partial accumulation point of $\{f_\alpha\}_\alpha$, there is a subnet $\{f_{\alpha_\beta}\}_\beta$ such that $\pi_j\left(f_{\alpha_\beta}\right) \to \pi_j(f_{max})$ for each $j \in J_{max}$.

Since K_{i_0} is compact, we find a subnet $\{f_{\alpha_{\beta_\gamma}}\}_\gamma$ such that $\pi_{i_0}\left(f_{\alpha_{\beta_\gamma}}\right)_\gamma$ converges to x_{i_0} in K_{i_0}.

Define $\tilde{f} \in K$ by setting $\tilde{f}|_{J_{max}} = f_{max}$, and $\tilde{f}(i_0) = x_{i_0}$. Thus, $(J_{max} \cup \{i_0\}, \tilde{f})$ is a partial accumulation point, which contradicts the maximality of (J_{max}, f_{max}). $\square$

[ii] In a partially ordered set, if every totally ordered subset contains an upper bound, the set contains a maximal element

Complex Measures and the Radon–Nikodym Theorem

I am going to be drawing much of this information from Gerald B. Folland's text on Real Analysis.

Definition (Signed Measure)**.** For (X, Ω) a measurable space, a signed measure is a function $\nu : \Omega \to [-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$
- ν assumes at most one of $\pm\infty$
- For $\{E_j\}$ a sequence of disjoint sets in Ω,

$$\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu\left(E_j\right),$$

  with the latter sum converging if $\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right)$ is finite.

Traditional measures will be referred to as positive measures.

If μ_1 and μ_2 are positive measures on Ω with at least one a finite measure, then $\nu = \mu_1 - \mu_2$ is a signed measure.

For μ a measure on Ω, if $f : X \to [-\infty, \infty]$ such that at least one of $\int f^+ \, d\mu$ or $\int f^- \, d\mu$ is finite, we call f an extended μ-integrable function, with $\nu(E) \int_E f \, d\mu$ a signed measure.

In fact, we shall soon see that every signed measure is represented in these forms.

Theorem (Hahn Decomposition)**.** *If ν is a signed measure on (X, Ω), then there exist a positive set P and a negative set N for ν such that $P \cup N = X$, and $P \cap N = 0$. If P' and N' are another set, then $P \triangle P'$ and $N \triangle N'$ are ν-null.*

Proof. We assume that ν does not assume the value of negative infinity. Let m be the supremum of $\nu(E)$ as E ranges over all positive sets; let $\{P_j\}$ be the sequence of positive sets such that $\nu(P_j) \to m$.

We set $P = \bigcup_{j=1}^{\infty} P_j$; by continuity and the property that the union of a countable family of positive sets is positive, we see that P is positive and $\nu(P) = m < \infty$. We claim that $N = X \setminus P$ is negative.

Suppose toward contradiction that it is not the case. First, we can see that N does not contain any nonnull positive sets, as for $E \subseteq N$ positive, then $E \cup P$ is positive and $\nu(E \cup P) = \nu(E) + \nu(P) > m$. Alternatively, we can see that for any $A \subseteq N$ with $\nu(A) > 0$, we find $C \subseteq A$ with $\nu(C) < 0$ (as A cannot be positive), so $B = A \setminus C$ has measure $\nu(A) - \nu(C) > \nu(A)$.

If N is nonnegative, we can find subsets $\{A_j\}$ in N and define n_j as follows. We select n_1 to be the smallest integer for which there exists a set $B \subseteq N$ with $\nu(B) > \frac{1}{n_1}$; A_1 is the given set. Inductively, select n_j the smallest integer where $B \subseteq A_{j-1}$ has measure $\nu(B) > \nu(A_{j-1}) + \frac{1}{n_j}$, with A_j as the set.

Let $A = \bigcap_{j=1}^{\infty} A_j$. Then,

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{1}{n_j} &< \lim_{j\to\infty} \nu\left(A_j\right)\\ &< \infty,\end{aligned}$$

meaning that $n_j \to \infty$ as $j \to \infty$. However, we still have $B \subseteq A$ with $\nu(B) > \nu(A) + \frac{1}{n}$ for some n; for j sufficiently large, we have $n < n_j$ with $B \subseteq A_{j-1}$, which contradicts the construction of n_j.

If P' and N' are another pair of sets, then $P \setminus P' \subseteq P$ and $P \setminus P' \subseteq N'$, meaning $P \setminus P'$ is measure zero. □

The decomposition $X = P \sqcup N$ is known as the Hahn decomposition for ν (non-unique, generally speaking).

We say two measures, μ and ν on (X, Ω) are mutually singular if there exist disjoint $E, F \in \Omega$ with $E \sqcup F = X$, where $\mu(E) = 0$ and $\nu(F) = 0$. Informally speaking, μ and ν exist on disjoint sets; we denote mutual singularity as $\mu \perp \nu$.

Theorem (Jordan Decomposition). *If ν is a signed measure, then there exist unique positive measure ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$, and $\nu^+ \perp \nu^-$.*

Proof. Let $X = P \sqcup N$ be a Hahn decomposition for ν, and define $\nu^+(E) = \nu(E \cap P)$, $\nu^-(E) = -\nu(E \cap N)$. Then, we can obviously see that $\nu = \nu^+ - \nu^-$, with $\nu^+ \perp \nu^-$.

Suppose $\nu = \mu^+ - \mu^-$ with $\nu^+ \perp \nu^-$. Let $E, F \in \Omega$ be such that $E \cap F = \emptyset$, $E \cup F = X$, and $\mu^+(F) = \mu^-(E) = 0$. Then, $X = E \sqcup F$ is another Hahn decomposition for ν, meaning $P \triangle E$ is ν-null, meaning that for any $A \in \Omega$, $\mu^+(A) = \nu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$, and similarly, $\nu^- = \mu^-$. □

Definition (Variation of Signed Measure)**.** We define ν^+ to be the positive variation of ν, ν^- to be the negative variation of ν, with the total variation of ν being defined by

$$|\nu| = \nu^+ + \nu^-.$$

If ν does not admit the value ∞, then $\nu^+(X) = \nu(P) < \infty$, meaning ν^+ is a signed measure, and ν is bounded above by ν^+.

We say ν is (σ-)finite if $|\nu|$ is (σ-)finite.

Definition (Integration with respect to a Signed Measure)**.** To integrate with respect to the signed measure ν, we take $L^1(\nu) = L^1(\nu^+) - L^1(\nu^-)$, and

$$\int_X f\,d\nu = \int_X f\,d\nu^+ - \int_X f\,d\nu^-.$$

Definition (Absolute Continuity)**.** Let ν be a signed measure, and μ a positive measure on (X, Ω). We say ν is absolutely continuous with respect to μ if $\nu(E) = 0$ for every $E \in \Omega$ where $\mu(E) = 0$. We write $\nu \ll \mu$.

We can verify that $\nu \ll \mu$ if and only if $|\nu| \ll \mu$, which is true if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Theorem (Definition of Absolute Continuity)**.** *Let ν be a signed measure and μ a positive measure on (X, Ω). Then, $\nu \ll \mu$ if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ when $\mu(E) < \delta$.*

Proof. Since $\nu \ll \mu$ if and only if $|\nu| \ll \mu$, and $|\nu(E)| \leqslant |\nu|(E)$, we only need assume ν is positive.

We can see easily that the ε-δ condition implies $\nu \ll \mu$.

In the opposite direction, suppose toward contradiction that there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, we can find $E_n \in \Omega$ with $\mu(E_n) < 2^{-n}$ with $\nu(E_n) \geqslant \varepsilon_0$.

Let $F_k = \bigcup_{n=k}^{\infty} E_n$, and $F = \bigcap_{k=1}^{\infty} F_k$. Then, $\mu(F_k) < 2^{1-k}$, meaning $\mu(F) = 0$, but $\nu(F_k) \geqslant \varepsilon_0$ for all k, meaning ν is finite and $\nu(F) = \lim_{k\to\infty} \nu(F_k) \geqslant \varepsilon$, meaning $\nu \not\ll \mu$. ◻

Corollary. *Let $\nu(E)$ be defined by $\nu(E) = \int_E F\,d\mu$. Then, if $f \in L^1(\mu)$, $\nu \ll \mu$.*

Lemma (Prelude to Radon–Nikodym)**.** *Suppose that ν and μ are finite measures on (X, Ω). Either $\nu \perp \mu$, or there exists $\varepsilon_0 > 0$ and $E \in \Omega$ such that $\mu(E) > 0$ and $\nu \geqslant \varepsilon_0\mu$ on E.*

Proof. Let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$. Let $P = \bigcup_{n=1}^{\infty} P_n$, $N = \bigcap_{n=1}^{\infty} N_n = P^c$.

Then, N is a negative set for $\nu - \frac{1}{n}\mu$ for all n, meaning $0 \leqslant \nu(N) \leqslant \frac{1}{n}\mu(N)$ for all n, so $\nu(N) = 0$.

If $\mu(P) = 0$, then $\nu \perp \mu$. If $\mu(P) > 0$, then $\mu(P_n) > 0$ for some n, and P_n is a positive set for $\nu - \frac{1}{n}\mu$. ◻

Theorem (Radon–Nikodym)**.** *Let ν be a σ-finite signed measure, μ a σ-finite positive measure on (X, Ω). Then, there exist unique σ-finite signed measures λ, ρ on (X, Ω) such that $\lambda \perp \mu$, $\rho \ll \mu$, and $\nu = \lambda + \rho$.*

Moreover, there exists an extended μ-integrable function $f : X \to \mathbb{R}$ such that $\rho(E) = \int_E f\,d\mu$. The derived measure ρ will be referred to by the shorthand, $d\rho = f\,d\mu$.

Proof.

Case 1: Suppose ν and μ are finite positive measures. Let

$$\mathcal{F} = \left\{ f : X \to [0, \infty] \mid \int_E f\, d\mu \leqslant \nu(E)\ \forall E \in \Omega \right\}.$$

Then, $\mathcal{F}$ is nonempty, since $0 \in \mathcal{F}$. If $f, g \in \mathcal{F}$, then $h = \max(f, g) \in \mathcal{F}$, since, for $A = \{x \mid f(x) > g(x)\}$,

$$\begin{aligned}\int_E h\, d\mu &= \int_{E\cap A} f\, d\mu + \int_{E\setminus A} g\, d\mu \\ &\leqslant \nu(E \cap A) + \nu(E \setminus A) \\ &= \nu(E).\end{aligned}$$

Let $a = \sup\left\{\int_X f\, d\mu \mid f \in \mathcal{F}\right\}$. Noting that $a \leqslant \nu(X) < \infty$, choose a sequence $\{f_n\} \in \mathcal{F}$ such that $\int_X f_n\, d\mu \to a$.

Let $g_n = \max(f_1, \dots, f_n)$, and $f = \sup_n f_n$. Then, $g_n \in \mathcal{F}$, increasing pointwise to f, and $\int_X g_n\, d\mu \geqslant \int_X f_n\, d\mu$. Thus, $\lim_{n\to\infty} \int_X g_n\, d\mu = a$, meaning that by monotone convergence, $f \in \mathcal{F}$ with $\int_X f\, d\mu = a$.

We claim that the measure $d\lambda = d\mu - f\, d\mu$ is singular with respect to μ. If it isn't, then there exists $E \in \Omega$ and $\varepsilon_0 > 0$ such that $\mu(E) > 0$ and $\lambda \geqslant \varepsilon_0\mu$ on E. However, $\varepsilon_0\chi_E\, d\mu \leqslant d\lambda = d\nu - f\, d\mu$, meaning $(f + \varepsilon_0\chi_E)\, d\mu \leqslant d\nu$, meaning $f + \varepsilon_0\chi_E \in \mathcal{F}$, and

$$\begin{aligned}\int_X (f + \varepsilon_0\chi_E)\, d\mu &= a + \varepsilon_0\mu(E) \\ &> a,\end{aligned}$$

contradicting the definition of a.

Thus, existence of λ, f, and $d\rho = f\, d\mu$ is proved. To show uniqueness, if it is also the case that $d\nu = d\lambda' + f' d\mu$, we have $d\lambda - d\lambda' = (f' - f)\, d\mu$. However, $\lambda - \lambda' \perp \mu$, while $(f' - f)\, d\mu \ll d\mu$, meaning $d\lambda - d\lambda' = (f' - f)\, d\mu = 0$, meaning $\lambda = \lambda'$ and $f = f'$ μ-almost everywhere.

Case 2: If μ and ν are σ-finite measures, then X is a countable disjoint union of μ-finite sets and a countable disjoint union of ν-finite sets.

Taking intersections, we obtain a sequence of disjoint sets $\{A_i\} \subseteq \Omega$ such that $\mu(A_j)$ and $\nu(A_j)$ are finite for all j, and $X = \bigcup_{j=1}^{\infty} A_j$.

Define $\mu_j(E) = \mu(E \cap A_j)$, and $\nu_j(E) = \nu(E \cap A_j)$. For each j, we have $d\nu_j = d\lambda_j + f_j\, d\mu_j$, where $\lambda_j \perp \mu_j$ (applying Case 1 to each finite measure).

Since $\mu_j\left(A_j^c\right) = \nu_j\left(A_j^c\right) = 0$, we have $\lambda_j\left(A_j^c\right) = \nu_j\left(A_j^c\right) - \int_{A_j^c} f\, d\mu = 0$, meaning $f_j = 0$ on A_j^c.[iii]

Let $\lambda = \sum_{j=1}^{\infty} \lambda_j$ and $f = \sum_{j=1}^{\infty} f_j$. Then, $d\nu = d\lambda + f\, d\mu$, $\lambda \perp \mu$, and $d\lambda$, $f\, d\mu$ are σ-finite.

The uniqueness follows from the uniqueness of each λ_j and $f\, d\mu_j$.

In the general case, we apply each argument to ν^+ and ν^-, then subtract. □

Definition (Radon–Nikodym Derivative)**.** The composition $\nu = \lambda + \rho$, where $\lambda \perp \mu$ and $\rho \ll \mu$ is known as the Lebesgue decomposition of ν with respect to μ.

When $\nu \ll \mu$, then $d\nu = f\, d\mu$ for some f. We call f the Radon–Nikodym derivative of ν with respect to μ. We write

$$d\nu = \frac{d\nu}{d\mu} d\mu.$$

Proposition (Chain Rule)**.** *Let ν be a σ-finite signed measure, where μ, λ are σ-finite measures on (X, Ω) with $\nu \ll \mu$, $\mu \ll \lambda$.*

(a) If $g \in L^1(\nu)$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu}\, d\mu.$$

(b) We have $\nu \ll \lambda$, and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

λ-almost everywhere.

Proof. We may assume $\nu \geqslant 0$. By the definition of $\frac{d\nu}{d\mu}$, it is the case that

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu}\, d\mu$$

whenever $g = \chi_E$. Thus, by linearity, the equation is true for simple functions, and then for nonnegative measurable functions by monotone convergence, then for $L^1(\nu)$ functions by linearity.

[iii] μ-almost everywhere, of course.

Replacing ν and μ with μ and λ, setting $g = \chi_E \frac{d\nu}{d\mu}$, we have

$$\begin{aligned}\nu(E) &= \int_E \frac{d\nu}{d\mu}\,d\mu\\ &= \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\,d\lambda\end{aligned}$$

for all $E \in \Omega$, meaning

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

λ-almost everywhere. $\square$

Corollary. *If* $\mu \ll \lambda$ *and* $\lambda \ll \mu$, *then* $\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1$.

Example (Dirac δ Distribution)**.** Let μ be the Lebesgue measure on $\mathbb{R}$, and ν the point mass at 0 on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. We can see that $\nu \perp \mu$.

The "Radon–Nikodym derivative" $\frac{d\nu}{d\mu}$ is the Dirac δ distribution.

Essentials of Abstract Harmonic Analysis

In order to go further into theories of Banach algebras, we need a better understanding of the theory of topological groups and other essentials of abstract harmonic analysis. As a result, I'm going to be pulling information from Hewitt and Ross's *Abstract Harmonic Analysis, Volume I*.

Basic Definitions

Definition (Topological Group)**.** Let G be a set that is a group with a topological structure. If

(i) $\cdot : G \times G \to G, (x, y) \mapsto xy$;

(ii) and $^{-1} : G \to G, x \mapsto x^{-1}$

are continuous, then G is a topological group.

The continuity of group multiplication implies that for every neighborhood U of xy, there exists a neighborhood V of x and W of y such that $VW \subseteq U$.

Theorem (Homeomorphisms)**.** *Let* G *be a topological group. For* $a \in G$, *the maps* $\{a\} \times G \to aG$, $G \times \{a\} \to Ga$, *and inversion are homeomorphisms.*

Theorem (Translation of Bases)**.** *Let* G *be a topological group, and let* $\mathcal{U}$ *be a basis for* G *at* e. *Then, the families* $\mathcal{U}x$ *and* $x\mathcal{U}$ *for every* $x \in G$ *are bases for* G.

Proof. Let W be a nonempty open subset of G with $a \in W$. The map $x \mapsto a^{-1}x$ takes W to the set $a^{-1}W$; notice that $e \in a^{-1}W$.

Since $\mathcal{U}$ is a basis at e, there is a neighborhood $U \in \mathcal{U}$ such that $U \subseteq a^{-1}W$, meaning $aU \subseteq W$.

Thus, W is a union of the sets aU, meaning $\{xU \mid x \in G, U \in \mathcal{U}\}$ is an basis for G. □

Theorem (Product Sets)**.** *Let G be a topological group, with $A, B \subseteq G$.*

- *If A is open, then AB and BA are open.*
- *If A and B are compact, then AB is compact.*
- *If A is closed and B is compact, then AB and BA are closed.*
- *If A and B are closed, then AB need not be closed.*

Proof. To prove the first item, we have $AB = \bigcup_{b \in B} Ab$; since open sets are translation-invariant, this means AB is a union of open sets, and thus open.

Suppose A and B are compact; then, by Tychonoff's theorem, $A \times B$ is compact in $G \times G$. Since group multiplication is continuous, AB is compact.

Suppose A is closed and B is compact. Let $\{x_\alpha\}_{\alpha \in D}$ be a net in AB converging to x_0 in G. We only need show that $x_0 \in AB$. For each x_α, we have $x_\alpha = a_\alpha b_\alpha$, where $a_\alpha \in A$ and $b_\alpha \in B$.

Since B is compact, there is a subnet such that $\lim_{\beta \in E} b_\beta \to b_0$. It must be the case that $x_\beta \to x_0$, and we can see that $\left(x_\beta, b_\beta\right) \to (x_0, b_0)$. Therefore, $a_\beta = x_\beta b_\beta$ converges to $x_0 b_0^{-1}$, as it is the composition of $\left(x_\beta, y_\beta\right)$ with $(x, y) \mapsto xy^{-1}$. Since A is closed, and each $a_\beta \in A$, $a_\beta \to a_0 \in A$, meaning

$$\begin{aligned} x_0 &= \left(x_0 b_0^{-1}\right) b_0 \\ &= (a_0)\, b_0 \\ &\in AB. \end{aligned}$$

Similarly, we can see that BA is closed. □

Theorem (Characterization of Topological Groups)**.** *Let G be a topological group with $\mathcal{U}$ a basis centered at e. Then,*

(i) for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^2 \subseteq U$;

(ii) for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$;

(iii) For every $U \in \mathcal{U}$ and every $x \in U$, there is a $V \in \mathcal{U}$ such that $xV \subseteq U$;

(iv) for every $U \in \mathcal{U}$ *and every* $x \in G$*, there is a* $V \in \mathcal{U}$ *such that* $xVx^{-1} \subseteq U$.

Conversely, if G *is a group and* $\mathcal{U}$ *is a family of subsets of* G *with the finite intersection property and (i)–(iv), then the family of subsets* $\{xU \mid U \in \mathcal{U}, x \in G\}$ *is a subbasis*[iv] *for a topology on* G.

If $\mathcal{U}$ *also satisfies* $U, V \in \mathcal{U} \Rightarrow \exists W \in \mathcal{U}$ *such that* $W \subseteq U \cap V$*, then* $\{xU \mid U \in \mathcal{U}, x \in G\}$ *and* $\{Ux \mid U \in \mathcal{U}, x \in G\}$ *are open bases for the topology on* G.

Proof. In the forward direction, we see that (i) implies that $(x, y) \mapsto xy$ is continuous, (ii) implies that $x \mapsto x^{-1}$ is continuous, and (iii) implies that U is open. Finally, (iv) follows from the fact that $x \mapsto ax \mapsto axa^{-1}$ is a homeomorphism.

In the reverse direction, let $\mathcal{U}$ satisfy conditions (i)–(iv) and have the finite intersection property. Then, for $U \in \mathcal{U}$, there are $V, W \in \mathcal{U}$ such that $V^2 \subseteq U$ and $W^{-1} \subseteq V$. Since $V \cap W \neq \emptyset$, we have $e \in VW^{-1} \subseteq V^2 \subseteq U$. Thus, all elements of $\mathcal{U}$ must contain e.

Let $\tilde{\mathcal{U}}$ be the family of all sets $\bigcap_{k=1}^{n} U_k$ for $U_1, \dots, U_n \in \mathcal{U}$. For each U_k, there exist V_k such that $V_k^2 \subseteq U_k$. Thus,

$$\begin{aligned}\left(\bigcap_{k=1}^{n} V_k\right)^2 &\subseteq \bigcap_{k=1}^{n} (V_k)^2\\ &\subseteq \bigcap_{k=1}^{n} U_k.\end{aligned}$$

Thus, condition (i) holds for $\tilde{\mathcal{U}}$. Additionally, since $\left(\bigcap_{k=1}^{n} V_k\right)^{-1} = \bigcap_{k=1}^{n} V_k^{-1}$, (ii) holds for $\tilde{\mathcal{U}}$. Finally, since $x\left(\bigcap_{k=1}^{n} V_k\right) = \bigcap_{k=1}^{n} xV_k$ and $x\left(\bigcap_{k=1}^{n} V_k\right)x^{-1} = \bigcap_{k=1}^{n} \left(xV_kx^{-1}\right)$, properties (iii) and (iv) hold for $\tilde{\mathcal{U}}$.

Thus, the nonempty sets $\bigcap_{k=1}^{n} (x_kU_k)$ with $x_k \in G$ and $U_k \in \mathcal{U}$ form an open basis for the topology of G. Let $y \in \bigcap_{k=1}^{n} (x_kU_k)$, and let V_k be such that $x_kyV_k \subseteq U_k$ for each k. Then,

$$\begin{aligned}y\left(\bigcap_{k=1}^{n} V_k\right) &= \bigcap_{k=1}^{n} (yV_k)\\ &\subseteq \bigcap_{k=1}^{n} (x_kU_k),\end{aligned}$$

meaning $y\tilde{U}$ as $\tilde{U}$ runs through $\tilde{\mathcal{U}}$ forms an open basis at y for each $y \in G$.

To show that G is a topological group, let $a, b \in G$ and $\tilde{U} \in \tilde{\mathcal{U}}$. By (i) and (iv) on $\tilde{\mathcal{U}}$, there exist $\tilde{V}, \tilde{W} \in \tilde{\mathcal{U}}$ such that $\left(b^{-1}\tilde{W}b\right)\tilde{V} \subseteq \tilde{U}$, meaning $\left(a\tilde{W}\right)\left(b\tilde{V}\right) \subseteq ab\tilde{U}$, meaning group multiplication is continuous. Similarly, from (ii) and (iv), we can see that $a^{-1}\tilde{V} \subseteq \tilde{U}$. □

[iv] The topology on G is the smallest topology containing $\{xU \mid U \in \mathcal{U}, x \in G\}$.

Theorem (Symmetric Neighborhoods). *Every topological group G has a basis at e consisting of neighborhoods U such that $U = U^{-1}$.*

Proof. For an arbitrary neighborhood U of e, we can see that for $V = U \cap U^{-1}$, $V = V^{-1}$ and V is a neighborhood of V with $V \subseteq U$. □

Corollary (Regularity of Topological Groups at Identity). *Let G be a topological group. For every (open) neighborhood U of e, there is a neighborhood V of e such that $\overline{V} \subseteq U$.*

Proof. Let V be a symmetric neighborhood of e such that $V^2 \subseteq U$. Then, for $x \in \overline{V}$, we have $xV \cap V \neq \emptyset$, meaning $xv_1 = v_2$ for some $v_1, v_2 \in V$, so $x = v_2 v_1^1 \in VV^{-1} = V^2 \subseteq U$. □

Theorem (T_3 Property of Topological Groups). *Let G be a T_0 topological group.*[v] *Then, G is regular at every point, thus it is Hausdorff.*

Proof. From the corollary, we know that G is regular at e, and since left translation is a homeomorphism, this means G is regular at every element. Thus, G is Hausdorff. □

Theorem (Conjugate Neighborhoods in Compact Subsets). *Let G be a topological group, with U a neighborhood of e, $F \subseteq G$ compact. Then, there is a neighborhood e of V such that $xVx^{-1} \subseteq U$ for all $x \in F$.*

Proof. Let $\mathcal{U}$ denote the family of symmetric neighborhoods of e. We will first show that for $y \in G$, there is a $V \in \mathcal{U}$ such that $x \subseteq Vy$ implies $xVx^{-1} \subseteq U$.

Let $V_1 \in \mathcal{U}$ be such that $V_1^3 \subseteq U$, and $V_2 \in \mathcal{U}$ such that $yV_2y^{-1} \subseteq V_1$. Let $V = V_1 \cap V_2$. Then, for $x \in Vy$, we have $xy^{-1} \in V \subseteq V_1$, and $yx^{-1} \in V_1^{-1} = V_1$, meaning $xVx^{-1} \subseteq xV_2x^{-1} = \left(xy^{-1}\right) yV_2y^{-1} \left(yx^{-1}\right) \subseteq V_1^3 \subseteq U$.

For each $y \in F$, there is a $V_y \in \mathcal{U}$ such that $xV_y y \Rightarrow xV_y x^{-1} \subseteq U$. Since $F \subseteq \bigcup_{y \in F} V_y y$, and F is compact, there exist $y_1, \dots, y_n$ such that $F \subseteq \bigcup_{k=1}^{n} V_{y_k} y_k$.

Let $V = \bigcap_{k=1}^{n} V_{y_k}$. Then, for $x \in F$, $x \in V_{y_k} y_k$ for some k, meaning $xVx^{-1} \subseteq xV_{y_k}x^{-1} \subseteq U$. □

Theorem (Neighborhoods with Compact Closure). *Let G be a topological group and U an open subset of G such that for compact F, $F \subseteq U$. Then, there is a neighborhood V of e such that $(FV) \cup (VF) \subseteq U$. If G is locally compact, then V can be chosen such that $\overline{(FV) \cup (VF)}$.*

Proof. For each $x \in F$, there is a neighborhood W_x of e such that $xW_x \subseteq U$, and a neighborhood V_x of e such that $V_x^2 \subseteq W_x$.

[v] T_0 is the Kolmogorov property, implying that for two points $x \neq y$, there exists an open set O such that $x \in O \wedge y \notin O$ or vice versa.

Since $F \subseteq \bigcup_{x\in F} xV_x$, there exist $x_1,\dots,x_n \in F$ such that $F \subseteq \bigcup_{k=1}^{n} x_kV_{x_k}$. Let $V_1 = \bigcap_{k=1}^{n} V_{x_k}$. Then,

$$\begin{aligned} FV_1 &\subseteq \bigcup_{k=1}^{n} x_kV_{x_k}V_1\\ &\subseteq \bigcup_{k=1}^{n} x_kV_{x_k}^2\\ &\subseteq \bigcup_{k=1}^{n} x_kW_{x_k}\\ &\subseteq U. \end{aligned}$$

Similarly, there is a neighborhood V_2 of e such that $V_2F \subseteq U$. Letting $V = V_1 \cap V_2$, we get that $(FV) \cup (VF) \subseteq U$.

If G is locally compact, then V can be chosen such that $\overline{V}$ is compact. It follows that $F\overline{V}$ is compact; since $FV \subseteq F\overline{V}$, and $F\overline{V}$ is closed, $\overline{FV} \subseteq F\overline{V}$, meaning $\overline{FV}$ is compact. Similarly, $\overline{VF}$ is compact, meaning $\overline{(FV)\cup(VF)}$ is compact. □

As a result of the fact that translation is a homeomorphism, we can introduce a notion of "uniform nearness" of points, as well as uniform continuity for real- and complex-valued functions on G.

Considering uniform nearness, left translating x and y in G by x^{-1}, we find that $x \mapsto e$ and $y \mapsto x^{-1}y$. If $x^{-1}y$ is in a symmetric neighborhood U of e, we can say that x and y are U-near in the sense of left translation. Similarly, if $yx^{-1} \in U$, we can say that x and y are U-near in the sense of right translation.

Both of these are uniform concepts, in that they can be applied to any x and y in G. If φ is a complex-valued function on G, we can say that φ is left (right) uniformly continuous if for every $\varepsilon > 0$, there exists a neighborhood U of e such that $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever x and y are U-near in the sense of left (right) translation.

Thus, for left uniform continuity, we must have

$$|\varphi(x) - \varphi(xu)| < \varepsilon$$

for all $x \in G$ and all $u \in U$, and for right uniform continuity, we must have

$$|\varphi(x) - \varphi(ux)| < \varepsilon$$

for all $x \in G$ and all $u \in U$.

The notions of left and right uniform continuity are natural extensions of uniform continuity for a complex-valued function defined on $\mathbb{R}$; however, instead of a single $\delta > 0$ that works for all $x, y \in \mathbb{R}$, we have a single neighborhood U that works for every $x \in G$.

Definition (Uniform Structure)**.** Let G be a topological group. For every neighborhood U of e in G, let L_U be the set of points $(x, y) \in G \times G$ such that $x^{-1}y \in U$, and let R_U be the set of points $(x, y) \in G \times G$ such that $yx^{-1} \in U$. The family of sets L_U (or R_U) as U runs through all neighborhoods of e is written as $\mathcal{I}_l(G)$ (or $\mathcal{I}_r(G)$), and is called hte left (or right) uniform structure on G.

Definition (Uniformly Continuous Mapping)**.** Let G and H be topological groups, with $\varphi : G \to H$ a map. Let $\mathcal{U}$ and $\mathcal{V}$ denote the bases at the identities of G and H respectively.

Suppose that for every $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ such that $(\varphi(x), \varphi(y)) \in L_V$ for all $(x, y) \in L_U$. Then, φ is said to be a uniformly continuous mapping for the pair of uniform structures $(\mathcal{I}_l(G), \mathcal{I}_l(H))$.

Uniform continuity for the pairs of uniform structures $(\mathcal{I}_l(G), \mathcal{I}_r(H))$, $(\mathcal{I}_r(G), \mathcal{I}_l(H))$, and $(\mathcal{I}_r(G), \mathcal{I}_r(H))$ are defined similarly.

Banach Spaces

Let X be a compact Hausdorff space, and let $C(X)$ denote the set of continuous functions $f : X \to \mathbb{C}$. For $f_1, f_2 \in C(X)$ and $\lambda \in \mathbb{C}$, we define

(1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

(2) $(\lambda f_1)(x) = \lambda f_1(x)$

(3) $(f_1 f_2)(x) = f_1(x) f_2(x)$

With these operations, $C(X)$ is a commutative algebra[vi] with identity over the field $\mathbb{C}$.

For each $f \in C(X)$, f is bounded (since X is compact and f is continuous); thus, $\sup |f| < \infty$. We call this the norm of f, and denote it

$$\|f\|_\infty = \sup \{|f(x)| \mid x \in X\}.$$

Proposition (Properties of the Norm on $C(X)$)**.**

(1) Positive Definiteness: $\|f\|_\infty = 0 \Leftrightarrow f = 0$

(2) Absolute Homogeneity: $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$

(3) Subadditivity (Triangle Inequality): $\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty$

(4) Submultiplicativity: $\|fg\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$

We define a metric ρ on $C(X)$ by $\rho(f, g) = \|f - g\|_\infty$.

[vi] A vector space with multiplication.

Proposition (Properties of the Induced Metric on $C(X)$)**.**

(1) $\rho(f,g) = 0 \Leftrightarrow f = g$

(2) $\rho(f,g) = \rho(g,f)$

(3) $\rho(f,h) \leqslant \rho(f,g) + \rho(g,h)$

Proposition (Completeness of $C(X)$)**.** *If X is a compact Hausdorff space, then $C(X)$ is a complete metric space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be Cauchy. Then,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leqslant \|f_n - f_m\|_\infty \\ &= \rho(f_n, f_m) \end{aligned}$$

for each $x \in X$. Thus, $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy for each $x \in X$. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We will need to show that this implies $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

Let $\varepsilon > 0$; choose N such that $n, m \geqslant N$ implies $\|f_n - f_m\|_\infty < \varepsilon$. For $x_0 \in X$, there exists a neighborhood U such that $|f_N(x_0) - f_N(x)| < \varepsilon$ for $x \in U$.[vii] Thus,

$$\begin{aligned} |f(x_0) - f(x)| &= |f_n(x_0) - f_N(x_0) + f_N(x_0) - f_N(x) + f_N(x) - f_n(x)| \\ &\leqslant |f_n(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f_n(x)| \\ &\leqslant 3\varepsilon. \end{aligned}$$

Thus, f is continuous. Additionally, for $n \geqslant N$ and $x \in X$, we have

$$\begin{aligned} |f_n(x) - f(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\leqslant \lim_{m\to\infty} \|f_n - f_m\|_\infty \\ &\leqslant \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, meaning $C(X)$ is complete. □

Definition (Banach Space)**.** A Banach space is a vector space over $\mathbb{C}$ with a norm $\|\cdot\|$ is complete with respect to the induced metric.

Proposition (Properties of the Banach Space Operations)**.** *Let $\mathcal{X}$ be a Banach space. The functions*

- $a : \mathcal{X} \times \mathcal{X} \to \mathcal{X};\ a(f,g) = f + g,$
- $s : \mathbb{C} \times \mathcal{X} \to \mathcal{X};\ s(\lambda, f) = \lambda f,$
- $n : \mathcal{X} \to \mathbb{R}^+;\ n(f) = \|f\|$

are continuous.

[vii] This is by the continuity of $\{f_n\}_n$.

Definition (Directed Sets and Nets)**.** Let A be a partially ordered set with ordering $\leqslant$. We say A is directed if for each $\alpha, \beta \in A$, there exists a γ such that $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$.

A net is a map $\alpha \mapsto \lambda_\alpha$, where $\alpha \in A$ for some directed set A.

Definition (Convergence of Nets)**.** Let $\{\lambda_\alpha\}$ be a net in X. We say the net converges to $\lambda \in X$ if for every neighborhood U of λ, there exists α_U such that for $\alpha \geqslant \alpha_U$, every λ_α is contained in U.[viii]

Definition (Cauchy Nets in Banach Spaces)**.** A net $\{f_\alpha\}_\alpha$ in a Banach space $\mathcal{X}$ is said to be a Cauchy net if for every $\varepsilon > 0$, there exists α_0 in A such that $\alpha_1, \alpha_2 \geqslant \alpha_0$ implies $\|f_{\alpha_1} - f_{\alpha_2}\| < \varepsilon$.

Proposition (Convergence of Cauchy Nets in Banach Spaces)**.** *In a Banach space, every Cauchy net is convergent.*

Proof. Let $\{f_\alpha\}_\alpha$ be a Cauchy net in $\mathcal{X}$. Choose α_1 such that $\alpha \geqslant \alpha_1$ implies $\|f_\alpha - f_{\alpha_1}\| < 1$.

We iterate this process by choosing $\alpha_{n+1} \geqslant \alpha_n$ such that $\alpha \geqslant \alpha_{n+1}$ implies $\|f_\alpha - f_{\alpha_{n+1}}\| < \frac{1}{n+1}$.

The sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ is Cauchy, and since $\mathcal{X}$ is complete, there exists $f \in \mathcal{X}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.

We must now prove that $\lim_{\alpha\in A} f_\alpha = f$. Let $\varepsilon > 0$. Choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$, and $\|f_{\alpha_n} - f_\alpha\| < \frac{\varepsilon}{2}$. Then, for $\alpha \geqslant \alpha_n$, we have

$$\begin{aligned}\|f_\alpha - f\| &\leqslant \|f_\alpha - f_{\alpha_n}\| + \|f_{\alpha_n} - f\| \\ &< \frac{1}{n} + \frac{\varepsilon}{2} \\ &< \varepsilon.\end{aligned}$$

□

Definition (Convergence of Infinite Series)**.** Let $\{f_\alpha\}_\alpha$ be a set of vectors in $\mathcal{X}$. Let $\mathcal{F} = \{F \subseteq A \mid F \text{ finite}\}$.

Define the ordering $F_1 \leqslant F_2 \Leftrightarrow F_1 \subseteq F_2$.[ix] For each F, define

$$g_F = \sum_{\alpha\in F} f_\alpha.$$

[viii]The net convergence generalizes sequence convergence in a metric space to the case where X does not have a metric.

[ix]the inclusion ordering

If $\{g_F\}_{F\in\mathcal{F}}$ converges to some $g \in \mathcal{X}$, then

$$\sum_{\alpha\in A} f_\alpha$$

converges, and we write

$$g = \sum_{\alpha\in A} f_\alpha.$$

Proposition (Absolute Convergence of Series in Banach Space)**.** *Let $\{f_\alpha\}_\alpha$ be a set of vectors in the Banach space $\mathcal{X}$. Suppose $\sum_{\alpha\in A} \|f_\alpha\|$ converges in $\mathbb{R}$. Then, $\sum_{\alpha\in A} f_\alpha$ converges in $\mathcal{X}$.*

Proof. All we need show is $\{g_F\}_{F\in\mathcal{F}}$ is Cauchy. Since $\sum_{\alpha\in A} \|f_\alpha\|$ converges, there exists $F_0 \in \mathcal{F}$ such that $F \geqslant F_0$ implies

$$\sum_{\alpha\in F} \|f_\alpha\| - \sum_{\alpha\in F_0} \|f_\alpha\| < \varepsilon.$$

Thus, for $F_1, F_2 \geqslant F_0$, we have

$$\begin{aligned}
\|g_{F_1} - g_{F_2}\| &= \left\|\sum_{\alpha\in F_1} f_\alpha - \sum_{\alpha\in F_2} f_\alpha\right\| \\
&= \left\|\sum_{\alpha\in F_1\setminus F_2} f_\alpha - \sum_{\alpha\in F_2\setminus F_1}\right\| \\
&\leqslant \sum_{\alpha\in F_1\setminus F_2} \|f_\alpha\| + \sum_{\alpha\in F_2\setminus F_1} \|f_\alpha\| \\
&\leqslant \sum_{\alpha\in F_1\cup F_2} \|f_\alpha\| - \sum_{\alpha\in F_0} \|f_\alpha\| \\
&< \varepsilon.
\end{aligned}$$

Thus, $\{g_F\}_{F\in\mathcal{F}}$ is Cauchy, and thus the series is convergent. □

Theorem (Absolute Convergence Criterion for Banach Spaces)**.** *Let $\mathcal{X}$ be a normed vector space. Then, $\mathcal{X}$ is a Banach space if and only if for every sequence $\{f_n\}_{n=1}^{\infty}$ of vectors in $\mathcal{X}$,*

$$\sum_{n=1}^{\infty} \|f_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} f_n \textit{ convergent.}$$

Proof. The forward direction follows from the previous proposition.

Let $\{g_n\}_{n=1}^{\infty}$ be a Cauchy sequence in a normed vector space where

$$\sum_{n=1}^{\infty} \|f_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} f_n \text{ convergent.}$$

We select a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ as follows. Choose n_1 such that $i, j \geqslant n_1$ implies $\left\|g_i - g_j\right\| < 1$; recursively, we select n_{N+1} such that $\|g_{N+1} - g_N\| < 2^{-N}$. Then,

$$\sum_{k=1}^{\infty} \|g_{k+1} - g_k\| < \infty.$$

Set $f_k = g_{n_k} - g_{n_{k-1}}$ for $k > 1$, with $f_1 = g_{n_1}$. Then,

$$\sum_{k=1}^{\infty} \|f_k\| < \infty,$$

meaning $\displaystyle\sum_{k=1}^{\infty} f_k$ converges. Thus, $\{g_{n_k}\}_{k=1}^{\infty}$ converges, meaning $\{g_n\}_{n=1}^{\infty}$ converges in $\mathcal{X}$. □

Definition (Bounded Linear Functional)**.** Let $\mathcal{X}$ be a Banach space. A function $\varphi : \mathcal{X} \to \mathbb{C}$ is known as a bounded linear functional if

(1) $\varphi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1\varphi(f_1) + \lambda_2\varphi(f_2)$ for each $\lambda_1, \lambda_2 \in \mathbb{C}$ and $f_1, f_2 \in \mathcal{X}$.

(2) There exists M such that $|\varphi(f)| \leqslant M\|f\|$ for each $f \in \mathcal{X}$.

Proposition (Equivalent Criteria for Bounded Linear Functionals)**.** *Let φ be a linear functional on $\mathcal{X}$. Then, the following conditions are equivalent:*

(1) φ is bounded;

(2) φ is continuous;

(3) φ is continuous at 0.

Proof. (1) $\Rightarrow$ (2): If $\{f_\alpha\}_{\alpha\in A}$ is a net in $\mathcal{X}$ converging to f, then $\lim_{\alpha\in A}\|f_\alpha - f\| = 0$. Thus,

$$\begin{aligned}\lim_{\alpha\in A} |\varphi(f_\alpha) - \varphi(f)| &= \lim_{\alpha\in A} |\varphi(f_\alpha - f)| \\ &\leqslant \lim_{\alpha\in F} M\|f_\alpha - f\| \\ &= 0\end{aligned}$$

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): If φ is continuous at 0, then there exists $\delta > 0$ such that $\|f\| < \delta \Rightarrow |\varphi(f)| < 1$. Thus, for any $g \in X$ nonzero, we have

$$\begin{aligned} |\varphi(g)| &= \frac{2\|g\|}{\delta}\left|\varphi\left(\frac{\delta}{2\|g\|}g\right)\right| \\ &< \frac{2}{\delta}\|g\|, \end{aligned}$$

meaning φ is bounded.

□

Definition (Dual Space)**.** Let $\mathcal{X}^*$ be the set of bounded linear functionals on $\mathcal{X}$. For each $\varphi \in \mathcal{X}^*$, define

$$\|\varphi\| = \sup_{\|f\|=1} |\varphi(f)|.$$

We say $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Proposition (Completeness of the Dual Space)**.** *For $\mathcal{X}$ a Banach space, $\mathcal{X}^*$ is a Banach space.*

Proof. Both positive definiteness and absolute homogeneity are apparent from the definition of the norm. We will now show the triangle inequality as follows. Let $\varphi_1, \varphi_2 \in \mathcal{X}^*$. Then,

$$\begin{aligned} \|\varphi_1 + \varphi_2\| &= \sup_{\|f\|=1} |\varphi_1(f) + \varphi_2(f)| \\ &\leqslant \sup_{\|f\|=1} |\varphi_1(f)| + \sup_{\|f\|=1} |\varphi_2(f)| \\ &= \|\varphi_1\| + \|\varphi_2\|. \end{aligned}$$

We must now show completeness. Let $\{\varphi_n\}_n$ be a sequence in $\mathcal{X}^*$. Then, for every $f \in \mathcal{X}$, it is the case that

$$|\varphi_n(f) - \varphi_m(f)| \leqslant \|\varphi_n - \varphi_m\|\,\|f\|,$$

meaning $\{\varphi_n(f)\}_n$ is Cauchy for each f. Define $\varphi(f) = \lim_{n\to\infty}\varphi_n(f)$. It is clear that $\varphi(f)$ is linear, and for N such that $n, m \geqslant N \Rightarrow \|\varphi_n - \varphi_m\| < 1$,

$$\begin{aligned} |\varphi(f)| &\leqslant |\varphi(f) - \varphi_N(f)| + |\varphi_N(f)| \\ &\leqslant \lim_{n\to\infty} |\varphi_n(f) - \varphi_N(f)| + |\varphi_N(f)| \\ &\leqslant \left(\lim_{n\to\infty} \|\varphi_n - \varphi_N\| + \|\varphi_N\|\right)\|f\| \\ &\leqslant (1 + \|\varphi_N\|)\,\|f\|, \end{aligned}$$

so φ is bounded. Thus, we must show that $\lim_{n\to\infty}\|\varphi_n - \varphi\| = 0$. Let $\varepsilon > 0$. Set N such that $n, m \geqslant N \Rightarrow \|\varphi_n - \varphi_m\| < \varepsilon$. Then, for $f \in \mathcal{X}$,

$$\begin{aligned} |\varphi(f) - \varphi_n(f)| &\leqslant |\varphi(f) - \varphi_m(f)| + |\varphi_m(f) - \varphi_n(f)| \\ &\leqslant |(\varphi - \varphi_m)(f)| + \varepsilon\,\|f\|. \end{aligned}$$

Since $\lim_{m\to\infty} |(\varphi - \varphi_m)(f)| = 0$, we have $\|\varphi - \varphi_m\| < \varepsilon$. □

Proposition (Banach Spaces and their Duals)**.**

(1) The space ℓ^∞ consists of the set of bounded sequences. For $f \in \ell^\infty$, the norm on f is computed as $\|f\|_\infty = \sup_n |f(n)|$.

(2) The subspace $c_0 \subseteq \ell^\infty$ consists of all sequences that vanish at ∞. The norm on c_0 is inherited from the norm on ℓ_∞.

(3) The space ℓ^1 consists of the set of all absolutely summable sequences. For $f \in \ell^1$, the norm on f is computed as $\|f\| = \sum_{n=1}^{\infty} |f(n)|$.

We claim that these are all Banach spaces.

We also claim that $c_0^* = \ell^1$, and $\left(\ell^1\right)^* = \ell^\infty$.

Proofs of Banach Space.

ℓ^∞:

Proof of Normed Vector Space: Let $a, b \in \ell^\infty$, and $\lambda \in \mathbb{C}$. Then,

$$\sup_n |a(n)| = 0$$

if and only if a is the zero sequence. Additionally, we have that

$$\begin{aligned} \|\lambda a\|_\infty &= \sup_n |\lambda a(n)| \\ &= |\lambda| \sup_n |a(n)| \\ &= |\lambda| \, \|a\|_\infty, \end{aligned}$$

meaning $\|\cdot\|_\infty$ is absolutely homogeneous. Finally,

$$\begin{aligned} \|a + b\|_\infty &= \sup_n |a(n) + b(n)| \\ &\leqslant \sup_n |a(n)| + \sup_n |b(n)| \\ &= \|a\|_\infty + \|b\|_\infty. \end{aligned}$$

Proof of Completeness: Let $\{a_n\}_{n=1}^\infty$ be a Cauchy sequence of elements of ℓ^∞. Let $\varepsilon > 0$, and let N be such that $\|a_n - a_m\|_\infty < \varepsilon$ for $n, m \geqslant N$. Then, for each k,

$$\begin{aligned} |a_n(k) - a_m(k)| &= |(a_n - a_m)(k)| \\ &\leqslant \|a_n - a_m\| \\ &< \varepsilon, \end{aligned}$$

meaning that $a_n(k)$ is Cauchy in $\mathbb{C}$ for each k.

Set $a(k) = \lim_{n\to\infty} a_n(k)$. We must now show that $\lim_{n\to\infty} \|a - a_n\| = 0$. Let $\varepsilon > 0$, and set N such that for $n, m \geqslant N$, $\|a_m - a_n\| < \varepsilon$. Then,

$$\begin{aligned} |a(k) - a_n(k)| &\leqslant |a(k) - a_m(k)| + |a_m(k) - a_n(k)| \\ &\leqslant |a(k) - a_m(k)| + \|a_m - a_n\| \\ &< |a(k) - a_m(k)| + \varepsilon. \end{aligned}$$

Since $\lim_{m\to\infty} |a(k) - a_m(k)| = 0$, we have $\|a - a_n\| < \varepsilon$.[x]

c_0:

Proof of Subspace: Let $a, b \in c_0$, and $\lambda \in \mathbb{C} \setminus \{0\}$. Let $\varepsilon > 0$. Set N_1 such that $|a(n)| < \frac{\varepsilon}{2|\lambda|}$ for all $n \geqslant N_1$, and set N_2 such that $|b(n)| < \frac{\varepsilon}{2}$ for all $n \geqslant N_2$.

Then, for all $n \geqslant \max\{N_1, N_2\}$,

$$\begin{aligned} |\lambda a(n) + b(n)| &\leqslant |\lambda||a(n)| + |b(n)| \\ &< |\lambda|\frac{\varepsilon}{2|\lambda|} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Proof of Completeness: In order to show completeness, we must show that c_0 is closed in ℓ^∞. Let $\{a_k\}_{k=1}^\infty$ be a sequence in c_0, with $a_k \to a$.

We will need to show that $a \in c_0$.[xi] Let $\varepsilon > 0$, and set K such that for all $k \geqslant K$, $\|a_k - a\| < \varepsilon/2$. For each k, choose N such that $|a_k(n)| < \varepsilon/2$ for all $n \geqslant N$. Then, for all $n \geqslant N$,

$$\begin{aligned} |a(n)| &\leqslant |a(n) - a_k(n)| + |a_k(n)| \\ &< \|a - a_k\| + |a_k(n)| \\ &< \varepsilon. \end{aligned}$$

Since c_0 is closed in ℓ^∞, it is thus complete.

ℓ^1:

[x]The reason we had to go about it like this was that we defined the sequence a pointwise; however, we need to show convergence *in norm*.

[xi]Sequential criterion for closure.

Proof of Normed Vector Space: Let $a, b \in \ell^1$, and $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned} \|\lambda a + b\| &= \sum_{k=1}^{\infty} |\lambda a(k) + b(k)| \\ &\leqslant \sum_{k=1}^{\infty} |\lambda a(k)| + \sum_{k=1}^{\infty} |b(k)| \\ &= |\lambda| \sum_{k=1}^{\infty} |a(k)| + \sum_{k=1}^{\infty} |b(k)| \\ &= |\lambda| \, \|a\| + \|b\| \,. \end{aligned}$$

Thus, $\lambda a + b \in \ell^1$. We have also shown both the triangle inequality and absolute homogeneity. We can also see that, if $\|a\| = 0$,

$$\begin{aligned} \|a\| &= \sum_{k=1}^{\infty} |a(k)| \\ &= 0, \end{aligned}$$

which is only true if $a(k) = 0$ for all k.

□

Example (Pointwise Convergence and Convergence in Norm)**.** Consider a sequence $\{\varphi_n\}_n$ in $\mathcal{X}^*$. If the sequence converges in norm to φ, then it must also converge pointwise. However, the converse isn't true.

For each k, define $L_k(f) = f(k)$, where $f \in \ell^1$. We can see that $L_k \in \left(\ell^1\right)^*$, and $\lim_{k\to\infty} L_k(f) = 0$ for each $f \in \ell^1$. The sequence of L_k thus converges to the zero functional pointwise, but since $\|L_k\| = 1$ always, it isn't the case that L_k converges to the zero functional in norm.

Definition (Weak Topology and w^*-Topology)**.** Let X be a set, Y a topological space, and $\mathcal{F}$ be a family of functions from X to Y. The weak topology on X is the topology for which all functions in $\mathcal{F}$ are continuous.

For each f in $\mathcal{X}$, let $\hat{f} : \mathcal{X}^* \to \mathbb{C}$ be defined by $\hat{f}(\varphi) = \varphi(f)$. The w^*-topology on $\mathcal{X}^*$ is the weak topology on $\mathcal{X}^*$ defined by the family of functions $\left\{\hat{f} \mid f \in \mathcal{X}\right\}$.

If Y is Hausdorff and $\mathcal{F}$ separates the points of X, then the weak topology is Hausdorff.[xii]

Proposition (Hausdorff Property of w^*-Topology)**.** *The w^*-topology on $\mathcal{X}^*$ is Hausdorff.*

Proof. If $\varphi_1 \neq \varphi_2$, then there exists at least one f such that $\varphi_1(f) \neq \varphi_2(f)$, meaning $\left\{\hat{f} \mid f \in \mathcal{X}\right\}$ separates the points of $\mathcal{X}^*$, so the w^*-topology is Hausdorff. □

[xii] I am trying to find a source to prove this, will include the proof of this implicit proposition hopefully.

Proposition (Convergence in the w^*-Topology). *A net $\{\varphi_\alpha\}_\alpha$ converges to $\varphi \in \mathcal{X}^*$ in the w^* topology if and only if $\lim_{\alpha\in A} \varphi_\alpha = \varphi$.*[xiii]

Proposition (Determination of the w^*-Topology). *Let $\mathcal{M}$ be a dense subset of $\mathcal{X}$, and let $\{\varphi_\alpha\}_{\alpha\in A}$ be a uniformly bounded net in $\mathcal{X}^*$, where $\lim_{\alpha\in A} \varphi_\alpha(f) = \varphi(f)$ for each $f \in \mathcal{M}$. Then, the net $\{\varphi_\alpha\}_{\alpha\in A}$ converges to φ in the w^* topology.*

Proof. Let $M = \sup_{\alpha\in A} \max\{\|\varphi_\alpha\|, \|\varphi\|\}$, and let $\varepsilon > 0$.

Given $g \in \mathcal{X}$, choose $f \in \mathcal{M}$ such that $\|f - g\| < \frac{\varepsilon}{3M}$. Let $\alpha_0 \in A$ such that $\alpha \geqslant \alpha_0$ implies $|\varphi_\alpha(f) - \varphi(f)| < \frac{\varepsilon}{3}$. Then, for all $\alpha \geqslant \alpha_0$,

$$\begin{aligned}|\varphi_\alpha(g) - \varphi(g)| &\leqslant |\varphi_\alpha(g) - \varphi_\alpha(f)| + |\varphi_\alpha(f) - \varphi(f)| + |\varphi(f) - \varphi(g)| \\ &\leqslant \|\varphi_\alpha\| \|f - g\| + \frac{\varepsilon}{3} + \|\varphi\| \|f - g\| \\ &< \varepsilon.\end{aligned}$$

□

Definition (Unit Ball). For $\mathcal{X}$ a Banach space, we denote the unit ball as $B_{\mathcal{X}} = \{f \in \mathcal{X} \mid \|f\| \leqslant 1\}$.[xiv]

Theorem (Banach–Alaoglu). *The set $B_{\mathcal{X}^*}$ is compact in the w^*-topology.*

Proof. Let $f \in B_{\mathcal{X}}$. Let $\overline{\mathbb{D}}^f$ denote the f-labeled copy of the closed unit disc in $\mathbb{C}$. Set

$$P = \prod_{f\in B_{\mathcal{X}}} \overline{\mathbb{D}}^f.$$

Then, P is compact by Tychonoff's theorem.

Define $\Lambda : B_{\mathcal{X}^*} \to P$ by $\Lambda(\varphi) = \varphi|_{B_{\mathcal{X}}}$. Notice that $\Lambda(\varphi_1) = \Lambda(\varphi_2)$ implies that $\varphi_1 = \varphi_2$ on $B_{\mathcal{X}}$, meaning $\varphi_1 = \varphi_2$. Therefore, Λ is injective.

Let $\{\varphi_\alpha\}_{\alpha\in A}$ be a net in $\mathcal{X}^*$ converging to φ in the w^*-topology. Then,

$$\begin{aligned}\lim_{\alpha\in A} \varphi_\alpha(f) &= \varphi(f) \\ \lim_{\alpha\in A} (\Lambda(\varphi_\alpha))(f) &= \lim_{\alpha\in A} (\Lambda(\varphi))(f),\end{aligned}$$

meaning

$$\lim_{\alpha\in A} \Lambda(\varphi_\alpha) = \Lambda(\varphi)$$

in P. Since Λ is one-to-one, we can see that $\Lambda : \mathcal{B}_{\mathcal{X}^*} \to \Lambda(B_{\mathcal{X}^*}) \subseteq P$ is a linear homeomorphism.

[xiii] In the special case of Hilbert space $\mathcal{H}$, we know from the Riesz Representation Theorem that each $\varphi \in \mathcal{H}^*$ is represented by ψ such that $\varphi(f) = \langle f, \psi\rangle$.

[xiv] The book uses a different notation, but I don't like that notation.

Let $\{\Lambda(\varphi_\alpha)\}_{\alpha\in A}$ be a net in $\Lambda(B_{\mathcal{X}^*})$ converging in the product topology to ψ. Let $f, g \in B_{\mathcal{X}^*}$ and $\xi \in \mathbb{C}$ with $f+g \in B_{\mathcal{X}^*}$ and $\xi f \in B_{\mathcal{X}^*}$. Then,

$$\begin{aligned}\psi(f+g) &= \lim_{\alpha\in A} (\Lambda(\varphi_\alpha))(f+g)\\ &= \lim_{\alpha\in A} (\Lambda(\varphi_\alpha))(f) + \lim_{\alpha\in A} (\Lambda(\varphi_\alpha))(g)\\ &= \psi(f)+\psi(g)\end{aligned}$$

and

$$\begin{aligned}\psi(\xi f) &= \lim_{\alpha\in A} (\Lambda(\varphi_\alpha))(\xi f)\\ &= \lim_{\alpha\in A} \varphi_\alpha(\xi f)\\ &= \varphi(\xi f)\\ &= \xi\varphi(f)\\ &= \xi(\Lambda(\varphi))(f)\\ &= \xi\psi(f).\end{aligned}$$

Thus, $\psi(f)$ determines $\tilde{\psi}(f) = \frac{1}{\|f\|}\psi(f)$ in $B_{\mathcal{X}^*}$ for all $f \in \mathcal{X}\setminus\{0\}$. If $f \in B_{\mathcal{X}}$, then $\tilde{\psi} \in \mathcal{B}_{\mathcal{X}^*}$ and $\Lambda(\tilde{\psi}) = \psi$.

Thus, $\Lambda(B_{\mathcal{X}^*})$ is closed in P, meaning $B_{\mathcal{X}^*}$ is compact in the w^*-topology. □

We will be able to use the Banach–Alaoglu theorem to prove that every Banach space is isomorphic to a subspace of $C(X)$ for some compact Hausdorff space X. However, we will need some theorems and machinery to prove that

Definition (Sublinear Functionals)**.** Let $\mathcal{E}$ be a real linear space, and let p be a real-valued functional on $\mathcal{E}$. We say p is a sublinear functional if $p(f+g) \leqslant p(f)+p(g)$ for all $f, g \in \mathcal{E}$, and $p(\lambda f) = \lambda p(f)$.

Theorem (Hahn–Banach Dominated Extension)**.** *Let $\mathcal{E}$ be a real linear space, and p a (real-valued) sublinear functional on $\mathcal{E}$. Let $\mathcal{F} \subseteq \mathcal{E}$ be a subspace, and φ a real linear functional on $\mathcal{F}$ such that $\varphi(f) \leqslant p(f)$ for all $f \in \mathcal{F}$.*

Then, there exists a real linear functional Φ on $\mathcal{E}$ such that $\Phi(f) = \varphi(f)$ for $f \in \mathcal{F}$, and $\Phi(g) \leqslant p(g)$ for all $g \in \mathcal{E}$.

Proof. Let $\mathcal{F} \subseteq \mathcal{E}$ be a nonempty subspace, and let $f \notin \mathcal{F}$. Select $\mathcal{G} = \{g+\lambda f \mid g \in \mathcal{F},\ \lambda \in \mathbb{R}\}$.

We will extend φ to $\Phi_{\mathcal{G}}$ by taking $\Phi(g+\lambda f) \leqslant p(g+\lambda f)$. Dividing by $|\lambda|$, we find that, for all $h \in \mathcal{F}$

$$\Phi(f-h) \leqslant p(f-h)$$

and

$$-p\left(h-f\right)\leqslant \Phi\left(h-f\right).$$

Thus, recalling that $\Phi(h) = \varphi(h)$ for $h \in \mathcal{F}$,

$$-p\left(h-f\right)+\varphi(h)\leqslant \Phi\left(f\right)\leqslant p(f-h)+\varphi(h).$$

The desired Φ only has this property if

$$\sup_{h\in\mathcal{F}}\left\{\varphi(h)-p(h-f)\right\}\leqslant \inf_{k\in\mathcal{F}}\left\{\varphi(k)+p(f-k)\right\}.$$

However, we also have

$$\begin{aligned}\varphi(h)-\varphi(k) &= \varphi(h-k)\\ &\leqslant p(h-k)\\ &\leqslant p(f-k)+p(h-f),\end{aligned}$$

meaning

$$\varphi(h)-p(h-f)\leqslant \varphi(k)+p(f-k).$$

Therefore, we can thus extend φ on $\mathcal{F}$ to Φ on $\mathcal{G}$, where $\Phi(h) \leqslant p(h)$. We label this as $\Phi_{\mathcal{G}}$.

Let $\mathcal{P} = \left\{(\mathcal{G}_\delta, \Phi_{\mathcal{G}_\delta})\right\}_{\delta\in D}$ denote the class of extensions of φ such that $\Phi_{\mathcal{G}_\delta}(h) \leqslant p(h)$ for all $h \in \mathcal{G}_\delta$.

An element of $\mathcal{P}$ contains $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{E}$, where $\Phi_{\mathcal{G}}$ extends φ, meaning $\mathcal{P}$ is nonempty.

The partial order on $\mathcal{P}$ can be set by $\left(\mathcal{G}_1, \Phi_{\mathcal{G}_2}\right) \leqslant \left(\mathcal{G}_2, \Phi_{\mathcal{G}_2}\right)$ if $G_1 \subseteq G_2$ and $\Phi_{\mathcal{G}_1}(f) = \Phi_{\mathcal{G}_2}(f)$ for all $f \in \mathcal{G}_1$.

Consider a chain[xv] $\left\{\left(\mathcal{G}_\alpha, \Phi_{\mathcal{G}_\alpha}\right)\right\}_{\alpha\in A}$. To find an upper bound, consider

$$\mathcal{G} = \bigcup_{\alpha\in A}\mathcal{G}_\alpha,$$

where $\Phi_{\mathcal{G}}(f) = \Phi_{\mathcal{G}_\alpha}(f)$ for every $f \in \mathcal{G}_\alpha$. Then, $\Phi_{\mathcal{G}}$ is a linear functional that satisfies the given properties,[xvi] and $\left(\mathcal{G}, \Phi_{\mathcal{G}}\right)$ is an upper bound for $\left\{\left(\mathcal{G}_\alpha, \Phi_{\mathcal{G}_\alpha}\right)\right\}$.

Thus, by Zorn's Lemma, there is a maximal element of $\mathcal{P}$, $\left(\mathcal{G}_{\max}, \Phi_{\mathcal{G}_{\max}}\right)$. If $\mathcal{G}_0 \neq \mathcal{E}$, then we can find a $f \notin \mathcal{G}_0$ and repeat the process performed at the beginning of the proof, which would contradict maximality.

Thus, we have constructed a linear functional Φ such that $\Phi(f) \leqslant p(f)$ for all $f \in \mathcal{E}$ that extends φ. □

[xv] totally ordered subset

[xvi] I am too lazy to prove this.

Theorem (Hahn–Banach Continuous Extension)**.** *Let $\mathcal{M}$ be a subspace of the Banach space $\mathcal{X}$. If φ is a bounded linear functional on $\mathcal{M}$, then there exists Φ on $\mathcal{X}^*$ such that $\Phi(f) = \varphi(f)$ for all $f \in \mathcal{M}$ and $\|\Phi\| = \|\varphi\|$.*

Proof. Consider $\tilde{\mathcal{X}}$ as the real linear space on which $\|\cdot\|$ is the sublinear functional. Set $\psi = \operatorname{Re}(\varphi)$ on $\mathcal{M}$.

We can see that, since $\operatorname{Re}(\varphi(f)) \leqslant |\varphi(f)|$, $\|\psi\| \leqslant \|\varphi\|$.

Set $p(f) = \|\varphi\|\,\|f\|$. Since $\psi(f) \leqslant p(f)$ for all $f \in \mathcal{X}$, by the dominated extension theorem, there exists Ψ defined on $\tilde{\mathcal{X}}$ that extends ψ. In particular, we can see that $\Psi(f) \leqslant \|\varphi\|\,\|f\|$.

Define Φ on $\mathcal{X}$ by $\Phi(f) = \Psi(f) - i\Psi(if)$ for any $f \in \mathcal{X}$. We will show that Φ is a complex bounded linear functional that extends φ and has norm $\|\varphi\|$. We can see that

$$\begin{aligned}\Phi(f+g) &= \Psi(f+g) - i\Psi\left(i\left(f+g\right)\right)\\ &= \Psi(f) - i\Psi(if) + \Psi(g) - i\Psi(ig)\\ &= \Phi(f) + \Phi(g),\end{aligned}$$

and for $\lambda_1, \lambda_2 \in \mathbb{R}$,[xvii]

$$\Phi\left(\left(\lambda_1 + i\lambda_2\right)f\right) = \Phi\left(\lambda_1 f\right) + \Phi\left(i\lambda_2 f\right) \qquad = \left(\lambda_1 + i\lambda_2\right)\Phi(f).$$

To verify that $\Phi(f)$ extends $\varphi(f)$, let $f \in \mathcal{M}$, and we can see that

$$\begin{aligned}\Phi(f) &= \Psi(f) - i\Psi(if)\\ &= \psi(f) - i\psi(if)\\ &= \operatorname{Re}\left(\varphi(f)\right) - i\operatorname{Re}\left(\varphi(if)\right)\\ &= \operatorname{Re}\left(\varphi(f)\right) - i\left(-\operatorname{Im}\left(\varphi(f)\right)\right)\\ &= \varphi(f).\end{aligned}$$

Finally, to verify that $\|\Phi\| = \|\varphi\|$, all we need show is that $\|\Phi\| \leqslant \|\Psi\|$. Let $\Phi(f) = re^{i\theta}$. Then,

$$\begin{aligned}|\Phi(f)| &= r\\ &= e^{-i\theta}\Phi(f)\\ &= \Phi\left(e^{-i\theta}f\right)\\ &= \Psi\left(e^{-i\theta}f\right)\\ &\leqslant \left|\Psi\left(e^{-i\theta}f\right)\right|\\ &\leqslant \|\Psi\|\,\|f\|,\end{aligned}$$

[xvii] Notice that $\Phi(if) = \Psi(if) - i\Psi(-f) = i\Psi(f) + \Psi(if) = i\Phi(f)$

meaning

$$\|\Phi\| \|f\| \leqslant \|\Psi\| \|f\|.$$

□

Corollary (Norming Functional)**.** *If* $f \in \mathcal{X}$*, then there exists* $\varphi \in \mathcal{X}^*$ *such that* $\|\varphi\| = 1$ *and* $\varphi(f) = \|f\|$.

Proof. Assume $f \neq 0$. Let $\mathcal{M} = \{\lambda f \mid \lambda \in \mathbb{C}\}$, and define ψ on $\mathcal{M}$ by $\psi(\lambda f) = \lambda \|f\|$. Then, $\|\psi\| = 1$ and an extension of ψ to $\mathcal{X}$ has the desired properties. □

Theorem (Banach)**.** *Let* $\mathcal{X}$ *be any Banach space. Then,* $\mathcal{X}$ *is isometrically isomorphic to some closed subspace of* $C(X)$ *for compact Hausdorff* X.

Proof. Set $X = B_{\mathcal{X}^*}$ in the w^*-topology, which by Banach–Alaoglu, is compact.

Set $\beta : \mathcal{X} \to C(X)$ by $\beta(f)(\varphi) = \varphi(f)$. Then, for $\lambda_1, \lambda_2 \in \mathbb{C}$, $f_1, f_2 \in \mathcal{X}$,

$$\begin{aligned} \beta(\lambda_1 f_1 + \lambda_2 f_2)(\varphi) &= \varphi(\lambda_1 f_1 + \lambda_2 f_2) \\ &= \lambda_1 \varphi(f_1) + \lambda_2 \varphi(f_2) \\ &= (\lambda_1 \beta(f_1) + \lambda_2 \beta(f_2))(\varphi). \end{aligned}$$

Let $f \in \mathcal{X}$. Then,

$$\begin{aligned} \|\beta(f)\|_\infty &= \sup_{\varphi \in B_{\mathcal{X}^*}} |\beta(f)(\varphi)| \\ &= \sup_{\varphi \in B_{\mathcal{X}^*}} |\varphi(f)| \\ &\leqslant \sup_{\varphi \in B_{\mathcal{X}^*}} \|\varphi\| \|f\| \\ &\leqslant \|f\|. \end{aligned}$$

Additionally, since there exists a norming functional in $B_{\mathcal{X}^*}$, we have that $\|\beta(f)\|_\infty = \|f\|$, meaning β is an isometric isomorphism. □

Note: The preceding construction cannot yield an isometric isomorphism to $C(B_{\mathcal{X}^*})$ itself, even if $\mathcal{X} = C(Y)$ for some Y.

It can be shown via topological arguments that if $\mathcal{X}$ is separable, we can take X to be the interval $[0, 1]$.

Now, we turn to finding the dual space of $C([0, 1])$. In particular, we will soon find out that $C([0, 1]) = BV([0, 1])$, which is the space of all functions of bounded variation.

Definition (Bounded Variation)**.** If φ is a complex function with domain $[0, 1]$, φ is said to be of bounded variation if for every partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, it is the case that

$$\sum_{i=0}^{n} |\varphi(t_{n+1}) - \varphi(t_n)| \leqslant M.$$

The infimum of all such values of M is denoted $\|\varphi\|_{BV}$.[xviii] Henceforth, all functions of

[xviii]The book uses $\|\varphi\|_\nu$, but I think that's more confusing than BV.

bounded variation will be referred to as BV functions.

Proposition (Limits of BV Functions)**.** *A BV function possesses a limit from the left and right at each endpoint.*

Proof. Let $\varphi : [0,1] \to \mathbb{C}$ not have a limit from the left at some point $t \in (0,1]$.

Then, for any $\delta > 0$, there exist s_1, s_2 such that $t - \delta < s_1 < s_2 < t$ and $|\varphi(s_2) - \varphi(s_1)| \geqslant \varepsilon$. Selecting $\delta_2 = t - s_2$, we inductively create a sequence $\{s_n\}_{n=1}^{\infty}$ where $0 < s_1 < s_2 < \cdots < s_n < \cdots < t$.

Consider a partition $t_0 = 0$, and $t_k = s_k$ for $k = 1, 2, \ldots, N$, and $t_{N+1} = 1$, we have

$$\begin{aligned}\sum_{k=0}^{N} |\varphi(t_{k+1}) - \varphi(t_k)| &\geqslant \sum_{k=1}^{N} |\varphi(s_{k+1}) - \varphi(s_k)| \\ &\geqslant N\varepsilon.\end{aligned}$$

Thus, φ is not a BV function. □

Corollary (Discontinuities of a BV Function)**.** *Let $\varphi : [0,1] \to \mathbb{C}$ be a BV function. Then, φ has countably many discontinuities.*

Proof. Notice that φ is discontinuous at a point t if and only if $\varphi(t) \neq \varphi(t^+)$ or $\varphi(t) \neq \varphi(t^-)$.

If $t_0, t_1, \ldots, t_n$ are distinct points of $[0,1]$, then

$$\sum_{i=0}^{N} \left|\varphi(t) - \varphi(t^+)\right| + \sum_{i=0}^{N} |\varphi(t) - \varphi(t^-)| \leqslant \|\varphi\|_{BV}.$$

Thus, for every $\varepsilon > 0$, there exist at most finitely many t such that $|\varphi(t) - \varphi(t^+)| + |\varphi(t) - \varphi(t^-)| \geqslant \varepsilon$, meaning there can be at most countably many discontinuities. □

Definition (Riemann–Stieltjes Integral)**.** Let $f \in C([0,1])$, and let $\varphi \in BV([0,1])$. Then, we denote the Riemann–Stieltjes integral

$$\int_0^1 f\, d\varphi = \sum_{i=0}^{n} f\left(t_i'\right)\left[\varphi\left(t_{i+1}\right) - \varphi\left(t_i\right)\right],$$

where $\{t_i\}$ is a partition and $t_i' \in [t_i, t_{i+1}]$.

Proposition (Essential properties of the Riemann–Stieltjes Integral)**.** *If $f \in C([0,1])$ and $\varphi \in BV([0,1])$, then*

(1) $\int_0^1 f\, d\varphi$ *exists;*

(2) $\int_0^1 (\lambda_1 f_1 + \lambda_2 f_2)\ d\varphi = \lambda_1 \int_0^1 f_1\ d\varphi + \lambda_2 \int_0^1 f_2\ d\varphi$ *for* $\lambda_1, \lambda_2 \in \mathbb{C}$ *and* $f_1, f_2 \in C([0,1])$;

(3) $\int_0^1 f\ d(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 \int_0^1 f_1\ d\varphi_1 + \lambda_2 \int_0^1 f_2\ d\varphi_2$ *for* $\lambda_1, \lambda_2 \in \mathbb{C}$ *and* $\varphi_1, \varphi_2 \in BV([0,1])$;

(4) $\left|\int_0^1 f\ d\varphi\right| \leqslant \|f\|_\infty \|\varphi\|_{BV}$ *for* $f \in C([0,1])$ *and* $\varphi \in BV([0,1])$.

Proposition (BV Function Limits and Riemann–Stieltjes Integrals). *Let* $\varphi \in BV([0,1])$ *and* ψ *be defined by* $\psi(t) = \varphi(t^-)$ *for* $t \in (0,1)$, *where* $\psi(0) = \varphi(0)$ *and* $\psi(1) = \varphi(1)$.

Then, $\psi \in BV([0,1])$, $\|\psi\|_{BV} \leqslant \|\varphi\|_{BV}$, *and*

$$\int_0^1 f\ d\varphi = \int_0^1 f\ d\psi$$

for $f \in C([0,1])$.

Proof. We list the set $\{s_i\}_{i\geqslant 1}$ the points where φ is discontinuous from the left. By the definition of ψ, we have $\psi(t) = \varphi(t)$ for $t \notin \{s_i\}_{i\geqslant 1}$.

Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ be a partition where if $t_i \in \{s_i\}_{i\geqslant 1}$, then neither t_{i-1} nor t_{i+1} is. To show that ψ is BV, then we must show

$$\sum_{i=0}^{n} |\psi(t_{i+1}) - \psi(t_i)| \leqslant \|\varphi\|_{\mathrm{BV}}.$$

Set $\varepsilon > 0$. If $t_i \notin \{s_i\}_{i\geqslant 1}$, $i = 0$, or $i = n+1$, then set $t_i' = t_i$. If $t_i \in \{s_i\}_{i\geqslant 1}$ and $i \neq 0, n+1$, choose $t_i' \in (t_{i-1}, t_i)$ such that $\left|\varphi(t_i^-) - \varphi(t_i')\right| < \frac{\varepsilon}{2n}$. Then, $0 = t_0' < t_1' < \cdots < t_n' < t_{n+1}' = 1$ is a partition of $0, 1$ with

$$\begin{aligned}
\sum_{i=0}^{n} |\psi(t_{i+1}) - \psi(t_i)| &= \sum_{i=0}^{n} \left|\varphi\left(t_{i+1}^-\right) - \varphi_{t_i^-}\right| \\
&\leqslant \sum_{i=0}^{n} \left|\varphi\left(t_{i+1}^-\right) - \varphi\left(t_{i+1}'\right)\right| + \sum_{i=0}^{n} \left|\varphi\left(t_{i+1}'\right) - \varphi(t_i')\right| + \sum_{i=0}^{n} \left|\varphi\left(t_i'\right) - \varphi\left(t_i^-\right)\right| \\
&\leqslant \frac{\varepsilon}{2} + \|\varphi\|_{\mathrm{BV}} + \frac{\varepsilon}{2}
\end{aligned}$$

Since ε was arbitrary, $\psi \in \mathrm{BV}([0,1])$, with $\|\psi\|_{\mathrm{BV}} \leqslant \|\varphi\|_{\mathrm{BV}}$.

For N any arbitrary integer, define $\eta_N(t) = 0$ for t not in $\{s_1, s_2, \dots, s_N\}$, and $\eta_N(s_i) = \varphi(s_i) - \psi(s_i)$. Then, we can see that $\|\varphi - (\psi + \eta_N)\|_{BV} = 0$, with $\int_0^1 f\, d\eta_N = 0$. Thus,

$$\begin{aligned}\int_0^1 f\, d\varphi &= \int_0^1 f\, d\psi + \lim_{N\to\infty}\int_0^1 f\, d\eta_N\\ &= \int_0^1 f\, d\psi.\end{aligned}$$

□

We let BV ([0, 1]) be the space of all BV functions with pointwise addition and scalar multiplication, with norm $\|\cdot\|_{BV}$.[xix]

Theorem. *BV* ([0, 1]) *is a Banach space.*

Proof. Suppose $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence in BV ([0, 1]) such that

$$\sum_{n=1}^{\infty}\|\varphi_n\|_{BV} < \infty.$$

Additionally,

$$\begin{aligned}|\varphi_n(t)| &\leqslant |\varphi_n(t) - \varphi_n(0)| + |\varphi_n(1) - \varphi_n(t)|\\ &\leqslant \|\varphi_n\|_{BV}\end{aligned}$$

for $t \in [0, 1]$, meaning

$$\sum_{n=1}^{\infty}\varphi_n(t)$$

converges uniformly and absolutely to a function φ defined on [0, 1]. We can see that $\varphi(0) = 0$ and φ is continuous from the left on (0, 1). We must now show that φ is of bounded variation and

$$\lim_{N\to\infty}\left\|\varphi - \sum_{n=1}^{N}\varphi_n\right\| = 0.$$

To start, let $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ be a partition of [0, 1]. Then,

$$\begin{aligned}\sum_{i=0}^{k}|\varphi(t_{i+1}) - \varphi(t_i)| &= \sum_{i=0}^{k}\left|\sum_{n=1}^{\infty}\varphi_n(t_{i+1}) - \sum_{n=1}^{\infty}\varphi_n(t_i)\right|\\ &\leqslant \sum_{n=1}^{\infty}\left(\sum_{i=0}^{k}|\varphi_n(t_{i+1}) - \varphi_n(t_i)|\right)\\ &\leqslant \sum_{n=1}^{\infty}\|\varphi_n\|_{BV}.\end{aligned}$$

[xix] Yes, technically before now I was engaging in a gross abuse of notation.

Thus, $\varphi \in BV([0,1])$. Additionally,

$$\begin{aligned}\sum_{i=0}^{k}\left|\left(\varphi - \sum_{n=1}^{N}\varphi_n\right)(t_{i+1}) - \left(\varphi - \sum_{n=1}^{N}\varphi_n\right)(t_i)\right| &= \sum_{i=0}^{k}\left|\sum_{n=N+1}^{\infty}\varphi_n(t_{i+1}) - \sum_{n=N+1}^{\infty}\varphi_n(t_i)\right|\\ &\leqslant \sum_{i=0}^{k}\sum_{n=N+1}^{\infty}\left|\varphi_n(t_{i+1}) - \varphi_n(t_i)\right|\\ &\leqslant \sum_{n=N+1}^{\infty}\|\varphi_n\|_{BV},\end{aligned}$$

meaning $\varphi = \sum_{n=1}^{\infty}\varphi_n$ in the BV norm. □

Theorem (Riesz). *Let* $\hat{\varphi}(f) = \int_0^1 f\,d\varphi$. *Then,* $\varphi \to \hat{\varphi}$ *is an isometric isomorphism between* $(C([0,1]))^*$ *and BV* $([0,1])$.

Proof. We must show that the map $\varphi \mapsto \hat{\varphi}$ is an isometric isomorphism.

We can see that, to start, $\hat{\varphi} \in (C([0,1]))^*$, with $\|\hat{\varphi}\| \leqslant \|\varphi\|_{BV}$.

We must now show that for $L \in (C([0,1]))^*$, there exists $\psi \in BV([0,1])$ such that $\hat{\psi} = L$, $\left\|\hat{\psi}\right\|_{BV} \leqslant \|L\|$, and ψ is unique.

Let $B([0,1])$ be the space of all *bounded* functions on $[0,1]$. It is readily apparent that $C([0,1]) \subseteq B([0,1])$,[xx] and we can see $B([0,1])$ is a Banach space with pointwise addition and scalar multiplication under the norm $\|f\|_u = \sup_{t\in[0,1]}|f(t)|$.[xxi] For $E \subseteq [0,1]$, define I_E to be the indicator function on E. The indicator function is always bounded.[xxii]

Since L is a bounded linear functional on $C([0,1])$, the Hahn–Banach continuous extension theorem allows us to create a (not necessarily unique) bounded linear functional L' on $B([0,1])$ with $\|L'\| = \|L\|$.

In particular, we can choose L' such that $L'\left(I_{\{0\}}\right) = 0$, by extending L to the linear span of $I_{\{0\}}$ and $C([0,1])$:

$$\begin{aligned}\left|L'\left(f + \lambda I_{\{0\}}\right)\right| &= |L(f)|\\ &\leqslant \|L\|\,\|f\|_\infty\\ &\leqslant \|L\|\left\|f + \lambda I_{\{0\}}\right\|_u\end{aligned}$$

[xx] Extreme Value Theorem

[xxi] Obviously $B([0,1])$ is a normed vector space. For a Cauchy sequence of functions $(f_n)_n \in B([0,1])$, completeness has pointwise convergence to f. Boundedness and convergence follows from the properties of the supremum.

[xxii] I am using I_E instead of $\mathbb{1}_E$ since it's easier for me to type that faster.

for all $f \in C([0,1])$ and $\lambda \in \mathbb{C}$.

FOr $0 < t \leqslant 1$, let $\varphi(t) = L\left(I_{(t,t+1]}\right)$, with $\varphi(0) = 0$. We aim to show that $\varphi \in BV([0,1])$ and $\|\varphi\|_{BV} \leqslant \|L\|$.

Select a partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, and set

$$\lambda_k = \frac{\varphi(t_{k+1}) - \varphi(t_k)}{|\varphi(t_{k+1}) - \varphi(t_k)|}$$

for $\varphi(t_{k+1}) \neq \varphi(t_k)$, and $\lambda_k = 0$ otherwise. Then,

$$f = \sum_{k=0}^{n} \lambda_k I_{(t_k,t_{k+1}]} \in B([0,1])$$

with $\|f\|_u \leqslant 1$, and

$$\begin{aligned}
\sum_{k=0}^{n} |\varphi(t_{k+1}) - \varphi(t_k)| &= \sum_{k=0}^{n} \lambda_k (\varphi(t_{k+1} - t_k)) \\
&= \sum_{k=0}^{n} L'\left(I_{(t_k - t_{k+1}]}\right) \\
&= L'(f) \\
&\leqslant \|L'\| = \|L\|.
\end{aligned}$$

Thus, $\|\varphi\|_{BV} \leqslant \|L\|$.

Now, we need to show that $L(g) = \int_0^1 g \, d\varphi$ for every $g \in C([0,1])$.

Let $g \in C([0,1])$. For $\varepsilon > 0$, set $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ a partition such that

$$|g(s) - g(s')| < \frac{\varepsilon}{2\|L'\|}$$

for every $s, s' \in (t_k, t_{k+1}]$, and

$$\left| \int_0^1 g \, d\varphi - \sum_{k=0}^{n} g(t_k)(\varphi(t_{k+1}) - \varphi(t_k)) \right| < \frac{\varepsilon}{2}.$$

Thus, for $f = \sum_{k=0}^{n} g(t_k) I_{(t_k,t_{k+1}]} + g(0) I_{\{0\}}$, we have

$$\begin{aligned}
\left| L(g) - \int_0^1 g \, d\varphi \right| &\leqslant |L(g) - L'(f)| + \left| L'(f) - \int_0^1 g \, d\varphi \right| \\
&\leqslant \|L'\| \|g - f\|_u + \left| \sum_{k=0}^{n} g(t_k)(\varphi(t_{k+1}) - \varphi(t_k)) - \int_0^1 g \, d\varphi \right| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, $L(g) = \int_0^1 g \, d\varphi$.

We obtain $\psi \in BV([0,1])$ with $\|\psi\|_{BV} \leqslant \|\varphi\|_{BV} \leqslant \|L\|$ (see function limits), and

$$\begin{aligned}\hat{\psi}(g) &= \int_0^1 g \, d\psi \\ &= \int_0^1 g \, d\varphi \\ &= L(g).\end{aligned}$$

Now, we must show that the mapping $\varphi \mapsto \hat{\varphi}$ is injective.

Let $\varphi \in BV([0,1])$. Fix $0 < t_0 \leqslant 1$, and let f_n be a sequence of functions in $C([0,1])$ defined by

$$f_n(t) = \begin{cases} 1 & 0 \leqslant t \leqslant \frac{n-1}{n} t_0 \\ n\left(1 - \frac{t}{t_0}\right) & \frac{n-1}{n} t_0 < t \leqslant t_0 \\ 0 & t_0 < t \leqslant 1 \end{cases}.$$

The function $I_{(0,t_0]} - f_n$ is zero outside the open interval $\left(\frac{n-1}{n} t_0, t_0\right)$. If we define

$$\varphi_n(t) = \begin{cases} \varphi\left(\frac{n-1}{n} t_0\right) & 0 \leqslant t \leqslant \frac{n-1}{n} t_0 \\ \varphi(t) & \frac{n-1}{n} t_0 < t \leqslant t_0 \\ \varphi(t_0) & t_0 < t \leqslant 1 \end{cases},$$

then

$$\begin{aligned}\left|\int_0^1 \left(I_{(0,t_0]} - f_n\right) \, d\varphi\right| &= \left|\int_0^1 \left(I_{(0,t_0]} - f_n\right) \, d\varphi_n\right| \\ &\leqslant \|\varphi_n\|_{BV}.\end{aligned}$$

We claim that $\lim_{n\to\infty} \|\varphi\|_{BV} = 0$.

Since φ is left continuous at t_0, there exists $\delta > 0$ such that $0 < t_0 - t < \delta$ implies $|\varphi(t_0 - t)| < \frac{\varepsilon}{2}$. Let $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$ be a partition of $[0,1]$, where

$$\left| \|\varphi\|_[BV] - \left(\sum_{i=0}^{k} |\varphi(t_{i+1}) - \varphi(t_i)|\right)\right| < \frac{\varepsilon}{2}.$$

Let $t_0 = t_{i_0}$ for some i_0, where $t_{i_0} - t_{i_0 - 1} < \delta$. Then,

$$|\varphi(t_{i_0}) - \varphi(t_{i_0 - 1})| < \frac{\varepsilon}{2},$$

and $\text{Var}(\varphi)_{[t_{i_0-1},t_{i_0}]} < \varepsilon$. Therefore,

$$\begin{aligned}\varphi(t_0) &= \int_0^1 I_{(0,t_0]}\,d\varphi\\ &= \lim_{n\to\infty}\int_0^1 f\,d\varphi,\end{aligned}$$

with $\hat{\varphi} = 0$ implying $\varphi = 0$. Thus, $(C([0,1]))^* = BV([0,1])$. □

Example (Conjugate Space of $C(X)$)**.** If X is any compact Hausdorff space, rather than merely $[0,1]$, it makes no sense to talk about bounded variation (since X may not have an ordering on it), so to find $(C(X))^*$ would require some extra work.

Every countably additive measure on $\mathcal{B}_X$ gives rise to a bounded linear functional on $C(X)$. Using the Hahn–Banach continuous extension theorem, we can extend this to the Banach space of bounded Borel functions, and obtain a Borel measure by evaluating the extended linear functional on the indicator functions of Borel subsets of X.

If we restrict our attention to regular measures[xxiii], the extended functional *is* unique, and we can identify $(C(X))^*$ to be $M(X)$, which is the set of complex regular Borel measures on X.

This result is known as the Riesz–Markov–Kakutani Representation Theorem.

Example (Quotient Spaces of Banach Spaces)**.** Let $\mathcal{X}$ be a Banach space, and $\mathcal{M}$ be a closed subspace of $\mathcal{X}$. We will try to find a norm on $\mathcal{X}/\mathcal{M}$.

The space $\mathcal{X}/\mathcal{M}$ is the set of equivalence classes of $f \in \mathcal{X}$ where $[f] = \{f + g \mid g \in \mathcal{M}\}$. The norm can be defined by

$$\|[f]\| = \inf_{g\in\mathcal{M}} \|f - g\|.$$

If $\|[f]\| = 0$, then there is a sequence g_n such that $\lim_{n\to\infty}\|f - g_n\| = 0$,[xxiv] meaning $g_n \to f$; since $\mathcal{M}$ is closed, this implies that $[f] = [0]$. In the converse direction, if $[f] = [0]$, then $0 \leqslant \|[f]\| \leqslant \|f - f\| = 0$. Thus, $\|[f]\|$ is positive definite.

[xxiii] Inner regular means the measure of a set can be approximated by compact subsets, outer regular means the measure of a set can be approximated by open supersets, and regular means both inner and outer regular.

[xxiv] I am using $\|[f]\| = \inf_{g\in\mathcal{M}}\|f - g\|$ instead since that is what my professor uses.

To show homogeneity, let $f \in \mathcal{X}$ and $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned} \|\lambda[f]\| &= \inf_{g\in\mathcal{M}} \|\lambda f - g\| \\ &= \inf_{h\in\mathcal{M}} \|\lambda(f-h)\| \\ &= |\lambda| \inf_{h\in\mathcal{M}} \|f-h\| \\ &= |\lambda|\,\|[f]\|. \end{aligned}$$

Finally, to show the triangle inequality, let $f_1, f_2 \in \mathcal{X}$. Then,

$$\begin{aligned} \|[f_1]+[f_2]\| &= \|[f_1+f_2]\| \\ &= \inf_{g\in\mathcal{M}} \|(f_1+f_2)-g\| \\ &= \inf_{g_1,g_2\in\mathcal{M}} \|(f_1-g_1)+(f_2-g_2)\| \\ &\leqslant \inf_{g_1\in\mathcal{M}} \|f_1-g_1\| + \inf_{g_2\in\mathcal{M}} \|f_2-g_2\| \\ &= \|[f_1]\| + \|[f_2]\|. \end{aligned}$$

Finally, to show completeness, we let $\{[f_n]\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{X}/\mathcal{M}$. Then, there exists a subsequence $\{[f_{n_k}]\}_{k=1}^{\infty}$ such that $\|[f_{n_{k+1}}]-[f_{n_k}]\| < \frac{1}{2^k}$.

Select $h_k \in [f_{n_{k+1}} - f_{n_k}]$ such that $\|h_k\| < \frac{1}{2^k}$. Then, $\sum_{k=1}^{\infty}\|h_k\| < 1 < \infty$, meaning $\sum_{k=1}^{\infty} h_k = h$ for some h.

Since

$$\begin{aligned} [f_{n_k} - f_{n_1}] &= \sum_{i=1}^{k-1} [f_{n_{i+1}} - f_{n_i}] \\ &= \sum_{i=1}^{k-1} [h_i], \end{aligned}$$

we must have $\lim_{k\to\infty}[f_{n_k} - f_{n_1}] = [h]$, meaning $\lim_{k\to\infty}[f_{n_k}] = [h + f_{n_1}]$.

We can see that there is a natural (projection) map $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$, defined by $\pi(f) = [f]$. This is a contraction and a surjective (which we will later see to be the same as open) map.

Definition (Bounded Linear Transformation)**.** Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. The linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is bounded if

$$\begin{aligned} \|T\|_{op} &= \sup_{\|f\|=1} \|T(f)\| \\ &< \infty \end{aligned}$$

The set of all bounded linear transformations from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. We have proven earlier that a linear transformation is bounded if and only if it is continuous.[xxv]

[xxv]This holds in all normed vector spaces, not just Banach spaces.

Proposition (Properties of $\mathcal{L}(\mathcal{X},\mathcal{Y})$)**.** *The space* $\mathcal{L}(\mathcal{X},\mathcal{Y})$ *is a Banach space.*

Proof. It is readily apparent that $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is a normed vector space under pointwise addition and scalar multiplication. All we need to show now is completeness.

Let $(T_n)_n$ be a Cauchy sequence of elements of $\mathcal{L}(\mathcal{X},\mathcal{Y})$. Then, for $\varepsilon > 0$, there exists N such that for $m, n > N$,

$$\|T_m - T_n\|_{op} < \varepsilon.$$

This means that for any $f \in \mathcal{X}$, there exists N_f such that for $m, n > N_f$,

$$\begin{aligned} \|(T_m - T_n)(f)\| &\leqslant \|f\| \, \|T_m - T_n\|_{op} \\ &< \frac{\varepsilon}{\|f\|} \|f\| \\ &= \varepsilon. \end{aligned}$$

Since for each f, $(T_n(f))_n$ is Cauchy, and $\mathcal{Y}$ is complete, we define T to be the pointwise limit of $(T_n)_n$.

Thus, since

$$\begin{aligned} \lim_{m\to\infty} \|T_m - T_n\|_{op} &= \|T - T_n\|_{op} \\ &< \varepsilon, \end{aligned}$$

we have that $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is complete. □

Theorem (Open Mapping)**.** *Let* $\mathcal{X}$, $\mathcal{Y}$ *be Banach spaces, and let* $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ *be surjective. Then,* T *is an open map.*

Note: I don't like order that Douglas's book introduces the Open Mapping/Bounded Inverse/Uniform Boundedness principle as well as the proofs, so I'm going to be drawing the following proofs mostly from Stein and Shakarchi's Functional Analysis text.

Proof. We see

$$\mathcal{X} = \bigcup_{n=1}^{\infty} U_{\mathcal{X}}(0, n),$$

Since T is surjective, we have

$$\mathcal{Y} = \bigcup_{n=1}^{\infty} T\left(U_{\mathcal{X}}(0, n)\right).$$

Since Y is complete, the Baire Category Theorem states that there must be at least one value of n such that $\overline{T\left(U_{\mathcal{X}}(0, n)\right)}^{\circ}$ is nonempty. Since T is linear, in particular we can see that $\overline{T(U_{\mathcal{X}}(0,1))}$ has a nonempty interior.

We let $U_{\mathcal{Y}}(y_0,\varepsilon) \subseteq \overline{T(U_{\mathcal{X}}(0,1))}$. By the definition of closure, we may select $y_1 = Tx_1$ for $x_1 \in T(U_{\mathcal{X}}(0,1))$ such that $\|y_1 - y_0\| < \frac{\varepsilon}{2}$.

Inductively, we can select $y_2 = Tx_2$ for $x_2 \in T(U_{\mathcal{X}}(0,1/2))$ such that $\|y_0 - y_1 - y_2\| < \frac{\varepsilon}{4}$, and so on, selecting $x_n \in T\left(U_{\mathcal{X}}\left(0, \frac{1}{2^{n-1}}\right)\right)$ such that $\left\|y_0 - \sum_{j=1}^{n} Tx_j\right\| < \frac{\varepsilon}{2^n}$.

Since $\left\|x_j\right\| < \frac{1}{2^{j-1}}$ for $j \in \mathbb{N}$, it is clear that $\sum_{j=1}^{\infty}\left\|x_j\right\|$ converges — thus, since X is a Banach space, there exists x such that $x = \sum_{j=1}^{n} x_j$. Moreover, since $\left\|y_0 - \sum_{j=1}^{n} Tx_j\right\| < \frac{\varepsilon}{2^n}$, and T is continuous, the limit of $\left\{x_j\right\}_{j=1}^{n}$ must be such that $T(x) = y_0$.

Therefore, we must have that $U_{\mathcal{Y}}\left(0, \frac{1}{2}\right) \subseteq T(U_{\mathcal{X}}(0,1))$. □

Theorem (Bounded Inverse)**.** *Let* $T : \mathcal{X} \to \mathcal{Y}$ *be a bounded bijective linear transformation. Then,* $T^{-1} : \mathcal{Y} \to \mathcal{X}$ *is also bounded.*

Proof. Since T is bijective, T is an open map, meaning T^{-1} must be continuous. □

Theorem (Uniform Boundedness Principle)**.** *Let* $\mathcal{L}$ *be a collection of continuous linear functionals on a Banach space* $\mathcal{X}$*. Then, if* $\sup_{\varphi\in\mathcal{L}} |\varphi(f)| < \infty$ *for all* f *in a residual subset* $A \subseteq \mathcal{X}$*, then* $\sup_{\varphi\in\mathcal{L}} \|\varphi\| < \infty$*.*

Proof. Suppose $\sup_{\varphi\in\mathcal{L}} |\varphi(f)| < \infty$ for all $f \in A$, where A is residual. For every M, define $A_{M,\varphi} = \{f \in \mathcal{X} \mid |\varphi(f)| \leqslant M\}$. Each of $A_{M,\varphi}$ is closed since φ is continuous. Define $A_M = \bigcap_{\varphi\in\mathcal{L}} A_{M,\varphi}$; each A_M is closed.

We can see that

$$A = \bigcup_{M=1}^{\infty} \bigcap_{\varphi\in\mathcal{L}} A_{M,\varphi}.$$

Since A is residual, there must be some M_0 such that A_{M_0} has nonempty interior, so there exists $f_0 \in \mathcal{X}$ and $r > 0$ such that $U_{\mathcal{X}}(f_0, r) \subseteq A_{M_0}$.

Thus, for every $\varphi \in \mathcal{L}$, we have $|\varphi(f)| \leqslant M_0$ for all f where $\|f - f_0\| < r$. Thus, for all $\|g\| < r$ and $\varphi \in \mathcal{L}$, we have

$$\begin{aligned} |\varphi(g)| &\leqslant |\varphi(g + f_0)| + |\varphi(-f_0)| \\ &\leqslant 2M_0, \end{aligned}$$

meaning $\|\varphi\| < \infty$ for all $\varphi \in \mathcal{L}$. □

Definition (Lebesgue Spaces)**.** Let μ be a probability measure on a σ-algebra Ω of the subsets of a set X.

We let $\mathcal{L}^1$ denote the vector space of all integrable complex-valued functions, with $\mathcal{N} \subseteq \mathcal{L}^1$ denoting the subspace of all $f \in \mathcal{L}^1$ where

$$\int_X |f| \, d\mu = 0.$$

Then, $L^1 = \mathcal{L}^1/\mathcal{N}$ is the space of equivalence classes $[f]$, where $\|[f]\|_1 = \int_X |f| d\mu$.

For each $1 \leqslant p < \infty$, we set $\mathcal{L}^p$ to be the functions in $\mathcal{L}^1$ where $\int_X |f|^p \, d\mu < \infty$; then, defining $\mathcal{N}^p = \mathcal{N} \cap \mathcal{L}^p$, the quotient space $L^p = \mathcal{L}^p/\mathcal{N}^p$ is the space of equivalence classes $[f]$ with norm

$$\|[f]\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

To construct L^∞, we start by constructing $\mathcal{L}^\infty$, which is the set of all essentially bounded functions, where $\mu \{x \in X \mid |f(x)| > M\} = 0$ for some M; we say $\|f\|_\infty$ is the infimum of all such M. Equivalently, $\|f\|_\infty = \operatorname{ess\,sup} |f|$. The set $\mathcal{N}^\infty = \mathcal{N} \cap \mathcal{L}^\infty$, and $L^\infty = \mathcal{L}^\infty/\mathcal{N}^\infty$ is the set of the equivalence classes $[f]$ where $\|[f]\|_\infty = \|f\|_\infty < \infty$ for f a representative of $[f]$.

We can see that all the L^p spaces are normed vector spaces; to show completeness will take more work, but we will show completeness for both L^1 and L^∞.

Theorem (Completeness of L^1). *The space L^1 is complete with respect to the norm $\|[f]\|_1 = \int_X |f| \, d\mu$.*

Proof. Let $\{[f_n]\}_{n=1}^\infty$ be a sequence in L^1 where $\sum_{n=1}^\infty \|[f_n]\|_1 \leqslant M < \infty$.

Select representatives f_n from each equivalence class. The sequence $\left\{\sum_{n=1}^N f_n\right\}_{N=1}^\infty$ is increasing for every $x \in X$ and non-negative, meaning

$$\begin{aligned} \int_X \left(\sum_{n=1}^N |f_n| \right) d\mu &= \sum_{n=1}^N \|[f_n]\|_1 \\ &\leqslant M, \end{aligned}$$

so by the dominated convergence theorem[xxvi] (with $g = M$, whose integral is finite because μ is a probability measure), we have that $\left\{\sum_{n=1}^N |f_n|\right\}_{N=1}^\infty$ is integrable and converges μ-almost everywhere to $[k] \in \mathcal{L}^1$.

[xxvi] The book states that they use Fatou's Lemma but I couldn't really understand where it comes into use so I decided to use the dominated convergence theorem and provide an explanation.

Finally,

$$\begin{aligned}\left\|[k] - \int_{n=1}^{N}\right\|_1 &= \int_X \left|\sum_{n=1}^{\infty} f_n - \sum_{n=1}^{N} f_n\right| \,d\mu \\ &\leqslant \sum_{n=N+1}^{\infty} \int_X |f_n| \,d\mu \\ &\leqslant \sum_{n=N+1}^{\infty} \|[f_n]\|_1 .\end{aligned}$$

Thus, $\sum_{n=1}^{\infty}[f_n] = [k]$. □

Theorem (Completeness of L^{∞})**.** *The space* L^{∞} *is complete with respect to the norm* $\|[f]\|_{\infty} = \operatorname{ess\,sup}|f|$.[xxvii]

Proof. Let $\{[f_n]\}_{n=1}^{\infty}$ be a sequence of elements of L^{∞} with $\sum_{n=1}^{\infty}\|[f_n]\|_{\infty} \leqslant M < \infty$. Choose representatives f_n from $[f_n]$, such that $|f_n|$ is bounded everywhere by $\|[f_n]\|_{\infty}$.

For $x \in X$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} |f_n(x)| &\leqslant \sum_{n=1}^{\infty} \|[f_n]\| \\ &\leqslant M.\end{aligned}$$

Thus, by dominated convergence, the sequence $\sum_{n=1}^{\infty} f_n = \lim_{N\to\infty}\sum_{n=1}^{N} f_n$ converges to a measurable bounded function h, where

$$\begin{aligned}|h(x)| &= \left|\sum_{n=1}^{\infty} f_n(x)\right| \\ &\leqslant \sum_{n=1}^{\infty} |f_n(x)| \\ &\leqslant M.\end{aligned}$$

Thus, $h \in \mathcal{L}^{\infty}$. Finally, we can see that

$$\begin{aligned}\left|[h] - \sum_{n=1}^{N}\right| &= \left|\sum_{n=1}^{\infty} f_n - \sum_{n=1}^{N} f_n\right| \\ &\leqslant \sum_{n=N+1}^{\infty} |f_n| \\ &\leqslant \sum_{n=N+1}^{\infty} \|f_n\|_{\infty} ,\end{aligned}$$

[xxvii] I had a proof of this in my Real Analysis II notes with Cauchy sequences. Here, I'll be going off the book's proof, which uses the absolute convergence determination criterion for Banach spaces.

meaning $\left\|[h] = \sum_{n=1}^{N} f_n\right\|_\infty$ converges to 0. □

The traditional abuse of notation for elements of L^p spaces is to refer to $f \in L^1$ to mean the equivalence class of μ-almost everywhere functions equal to $f \in \mathcal{L}^1$.

Now, we turn our attention to the dual of L^1, $\left(L^1\right)^*$.

Theorem (Dual of L^1). *Let $\hat{\varphi}$ be the linear functional defined by*

$$\hat{\varphi}(f) = \int_X f\varphi \, d\mu$$

for $f \in L^1$ and $\varphi \in L^\infty$. Then, the map $\varphi \mapsto \hat{\varphi}$ is an isometric isomorphism of L^∞ onto $\left(L^1\right)^*$.

Proof. If $\varphi \in L^\infty$, then for $f \in L^1$, it is the case that $|(\varphi f)(x)| \leqslant \|\varphi\|_\infty |f(x)|$ almost everywhere. Thus, φf is integrable, meaning $\hat{\varphi}$ is well-defined and linear, with

$$\begin{aligned}
|\hat{\varphi}(f)| &= \left|\int_X f\varphi \, d\mu\right| \\
&\leqslant \|\varphi\|_\infty \int_X |f| \, d\mu \\
&\leqslant \|\varphi\|_\infty \|f\|_1,
\end{aligned}$$

meaning $\hat{\varphi} \in \left(L^1\right)^*$ and $\|\hat{\varphi}\| \leqslant \|\varphi\|_\infty$.

Let $L \in \left(L^1\right)^*$. For E a measurable subset of X, I_E, the indicator function on E, is L^1, with $\|I_E\|_1 = \mu(E)$.

If we set $\lambda(E) = L(I_E)$, we can see that λ is a finitely additive complex-valued measure, with $|\lambda(E)| \leqslant \mu(E)\|L\|$. Moreover, for $\{E_n\}_{n=1}^\infty$ a nested sequence of measurable sets with $\bigcap_{n=1}^\infty E_n = \emptyset$, we have

$$\begin{aligned}
\left|\lim_{n\to\infty} \lambda(E_n)\right| &\leqslant \lim_{n\to\infty} |\lambda(E_n)| \\
&\leqslant \|L\| \lim_{n\to\infty} \mu(E_n) \\
&= 0.
\end{aligned}$$

Thus, λ is dominated by μ, meaning that by the Radon–Nikodym theorem,[xxviii] there exists an integrable function φ on X such that $\lambda(E) = \int_X I_E \varphi \, d\mu$ for all measurable sets E. What we need to show now is that φ is essentially bounded, and $L(f) = \int_X f\varphi \, d\mu$ for all $f \in L^1$.

[xxviii] Someday I will actually learn this theorem for real.

Set

$$E_N = \left\{ x \in X \middle| \|L\| + \frac{1}{N} \leqslant |\varphi(x)| \leqslant N \right\}.$$

Then, E_N is measurable, and $I_{E_N}\varphi$ is bounded.

If $f = \sum_{i=1}^{k} c_i I_{E_i}$, then we can see that $L(f) = \int_X f\varphi \, d\mu$. We can also see that for any $f \in L_1$ with $\operatorname{supp}(f) = E_N$, $L(f) = \int_X f\varphi \, d\mu$.

Let $g = \frac{\overline{\varphi(x)}}{|\varphi(x)|}$ if $x \in E_N$ and $\varphi(x) \neq 0$; otherwise, $g = 0$. Then, $g \in L^1$ with $\operatorname{supp}(g) = E_n$ and $\|g\|_1 = \mu(E_N)$. Thus, we have

$$\begin{aligned}
\mu(E_N)\|L\| &\geqslant |L(g)| \\
&= \left| \int_X g\varphi \, d\mu \right| \\
&= \int_X |\varphi| \, I_{E_N} \, d\mu \\
&\geqslant \left(\|L\| + \frac{1}{N} \right) \mu\left(E_N\right),
\end{aligned}$$

meaning $\mu(E_N) = 0$. Thus, $\mu\left(\bigcup_{N=1}^{\infty} E_N\right) = 0$, meaning φ is essentially bounded and $\|\varphi\|_\infty \leqslant \|L\|$. □

Definition (Hardy Spaces)**.** Let $\mathbb{T}$ denote the unit circle in the complex plane, and μ the Lebesgue measure normalized such that $\mu(\mathbb{T}) = 1$. We define $L^p(\mathbb{T})$ with respect to μ as the Lebesgue space on $\mathbb{T}$.

The Hardy space, H^p is a closed subspace of $L^p(\mathbb{T})$.

For $n \in \mathbb{Z}$, we define χ_n on $\mathbb{T}$ such that $\chi_n(z) = z^n$. We define

$$H^1 = \left\{ f \in L^1(\mathbb{T}) \mid \frac{1}{2\pi} \int_0^{2\pi} f\chi_n \, dt = 0 \right\}.$$

We can see that H^1 is a linear subspace, and is a kernel of a bounded linear functional on $L^1(\mathbb{T})$, meaning it is closed.

For similar reasons,

$$H^\infty = \left\{ \varphi \in L^\infty(\mathbb{T}) \mid \frac{1}{2\pi} \int_0^{2\pi} \varphi\chi_n \, dt = 0 \right\}$$

is also a closed subspace of $L^\infty(\mathbb{T})$. In particular, this is the kernel of the w^*-continuous function

$$\hat{\chi}_n(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi\chi_n \, dt,$$

meaning H^∞ is also w^*-closed.

Banach Algebras

Earlier, we showed that $C(X)$, where X is a compact Hausdorff space, is a Banach space; additionally, every Banach space is isomorphic to some subspace of $C(X)$. We can also see that $C(X)$ is an algebra[xxix] with multiplication continuous in the norm topology.

Definition (Multiplicative Linear Functional)**.** A linear functional $\varphi : C(X) \to \mathbb{C}$ is multiplicative if $\varphi(fg) = \varphi(f)\varphi(g)$, meaning $\varphi(1) = 1$.

For each $x \in X$, we define $\varphi_x(f) = f(x)$.

The space of multiplicative linear functionals on $C(X)$ is denoted $M_{C(X)}$.

Proposition. *Let $\psi : X \to M_{C(X)}$ be defined by $\psi(x) = \varphi_x$.*

Then, ψ is a homeomorphism from X onto $M_{C(X)}$, where $M_{C(X)}$ is given the w^*-topology on $(C(X))^*$.

Proof. Let $\varphi \in M_{C(X)}$, and set

$$\begin{aligned}\mathfrak{R} &= \ker \varphi \\ &= \{f \in C(X) \mid \varphi(f) = 0\}.\end{aligned}$$

We show that there exists x_0 in X such that $f(x_0) = 0$ for all $f \in \mathfrak{R}$.

If this were not the case, then for each $x \in X$, there would exist $f_x \in \mathfrak{R}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there exists a neighborhood U_x of x where $f_x \neq 0$. Since X is compact, and $\{U_x\}_{x\in X}$ is an open cover of X, there exist $U_{x_1}, \dots, U_{x_N}$ with $X = \bigcup_{n=1}^{N} U_{x_n}$.

If we set $g = \sum_{n=1}^{N} \overline{f_{x_n}} f_{x_n}$, then

$$\begin{aligned}\varphi(g) &= \sum_{n=1}^{n} \varphi\left(\overline{f_{x_n}}\right)\varphi\left(f_{x_n}\right) \\ &= 0,\end{aligned}$$

implying $g \in \mathfrak{R}$. However, $g \neq 0$ on $C(X)$, meaning g is invertible, implying $\varphi(1) = \varphi(g)\varphi(1/g) = 0$. Thus, there must exist $x_0 \in X$ such that $f(x_0) = 0$.

If $f \in C(X)$, then $f - (1)(\varphi(f))$ is in $\mathfrak{R}$, since $\varphi\left(f - (1)\varphi(f)\right) = \varphi(f) - \varphi(f) = 0$, meaning $f(x_0) - \varphi(f) = 0$, and $\varphi = \varphi_{x_0}$.

Since each $\varphi \in M_{C(X)}$ is bounded, we can give $M_{C(X)}$ the subspace topology of the w^*-topology on $(C(X))^*$.

[xxix] Vector space with multiplication.

Consider $\psi : X \to M_{C(X)}$. Since X is compact and Haursdorff, it is normal, meaning that by Urysohn's lemma, there exists $f \in C(X)$ such that $f(x) \neq f(y)$, meaning

$$\begin{aligned}\psi(x)(f) &= \varphi_x(f)\\ &= f(x)\\ &\neq f(y)\\ &= \varphi_y(f)\\ &= \psi(y)(f),\end{aligned}$$

implying ψ is injective.

To show ψ is continuous, let $\{x_\alpha\}_{\alpha\in A}$ be a net in X converging to x. Then, for every $f \in C(X)$, $\lim_{\alpha\in A} f(x_\alpha) = f(x)$, or $\lim_{\alpha\in A} \psi(x_\alpha)(f) = \psi(x)(f)$.

Thus, $\{\psi(x_\alpha)\}_{\alpha\in A}$ converges in the w^*-topology to $\psi(x)$, meaning ψ is continuous.

Since ψ is injective and continuous mapping a compact space onto a Hausdorff space, ψ is a homeomorphism. □

Definition (Banach Algebra)**.** A Banach algebra $\mathfrak{B}$ is an algebra over $\mathbb{C}$ with identity e which has a norm that makes it a Banach space, where $\|e\| = 1$ and $\|fg\| \leqslant \|f\|\,\|g\|$.

Proposition (Invertible Elements)**.** *If* $f \in \mathfrak{B}$ *with* $\|e - f\| < 1$, *then* f *is invertible and*

$$\left\|f^{-1}\right\| \leqslant \frac{1}{1 - \|e - f\|}.$$

Proof. If we set $\eta = \|e - f\| < 1$, then for $N \geqslant M$, we have

$$\begin{aligned}\left\|\sum_{n=0}^{N}(e-f)^n - \sum_{n=0}^{M}(e-f)^n\right\| &= \left\|\sum_{n=M+1}^{N}(e-f)^n\right\|\\ &\leqslant \sum_{n=M+1}^{N}\|e-f\|^n\\ &= \sum_{n=M+1}^{N}\eta^n\\ &\leqslant \frac{\eta^{M+1}}{1-\eta},\end{aligned}$$

meaning the sequence of partial sums $\left\{\sum_{n=0}^{N}(1-f)^n\right\}_{N=0}^{\infty}$ is Cauchy.

If $g = \sum_{n=0}^{\infty} (e - f)^n$, then

$$\begin{aligned}
fg &= (e - (e - f)) \left(\sum_{n=0}^{\infty} (e - f)^n\right) \\
&= \lim_{N\to\infty} \left((1 - (e - f)) \sum_{n=0}^{N} (e - f)^n\right) \\
&= \lim_{N\to\infty} \left(1 - (e - f)^{N+1}\right) \\
&= 1.
\end{aligned}$$

Similarly, $gf = 1$, meaning f is invertible with $f^{-1} = g$. We can also see

$$\begin{aligned}
\|g\| &= \lim_{N\to\infty} \left\|\sum_{n=0}^{N} (e - f)^n\right\| \\
&\leqslant \lim_{N\to\infty} \sum_{n=0}^{N} \|e - f\|^n \\
&= \frac{1}{1 - \|e - f\|}.
\end{aligned}$$

□

Definition (Set of Invertible Elements). For a Banach algebra $\mathfrak{B}$, we denote the collection of invertible elements as $\mathcal{G}$, with $\mathcal{G}_l$ denoting the left-invertible elements that are not invertible, and $\mathcal{G}_r$ the collection of right-invertible elements that are not invertible.

Proposition (Openness of Sets of Invertible Elements). *For $\mathfrak{B}$ a Banach algebra, the sets $\mathcal{G}$, $\mathcal{G}_l$, and $\mathcal{G}_r$ are open.*

Proof. Let $f \in \mathcal{G}$. Then, if $\|f - g\| \leqslant \frac{1}{\|f^{-1}\|}$, then $1 > \left\|f^{-1}\right\| \|f - g\| \geqslant \left\|e - f^{-1}g\right\|$, implying that $f^{-1}g \in \mathcal{G}$, and $g \in \mathcal{G}$, meaning $\mathcal{G}$ contains the open ball of radius $\frac{1}{\|f^{-1}\|}$ about each element.

If $f \in \mathcal{G}_l$, then there exists $h \in \mathfrak{B}$ such that $hf = 1$; if $\|f - g\| < \frac{1}{\|h\|}$, then $1 > \|h\| \|f - g\| \geqslant \|1 - hg\|$, implying hg is invertible and g is left invertible. Thus, $\mathcal{G}_l$ has the open ball of radius $\frac{1}{\|h\|}$ about every element of f, meaning $\mathcal{G}_l$ is open.

A similar argument holds for $\mathcal{G}_r$. □

Corollary (Topological Group of Invertible Elements). *If $\mathfrak{B}$ is a Banach algebra, then $f \mapsto f^{-1}$ defined on $\mathcal{G}$ is continuous.*

Proof. If $f \in \mathcal{G}$, then $\|f - g\| < \frac{1}{2}\left\|f^{-1}\right\|$ implies $\left\|e - f^{-1}g\right\| < \frac{1}{2}$. Thus,

$$\begin{aligned}\left\|g^{-1}\right\| &\leqslant \left\|g^{-1}f\right\|\left\|f^{-1}\right\| \\ &= \left\|\left(f^{-1}g\right)^{-1}\right\|\left\|f^{-1}\right\| \\ &\leqslant 2\left\|f^{-1}\right\|.\end{aligned}$$

Thus,

$$\begin{aligned}\left\|f^{-1} - g^{-1}\right\| &= \left\|f^{-1}(f-g)g^{-1}\right\| \\ &\leqslant 2\left\|f^{-1}\right\|^2\|f-g\|,\end{aligned}$$

meaning $f \mapsto f^{-1}$ is Lipschitz. □

Proposition (Connected Component with Identity)**.** *Let $\mathfrak{B}$ be a Banach algebra, with $\mathcal{G}$ the group of invertible elements. Let $\mathcal{G}_0$ be the connected component in $\mathcal{G}$ that contains the identity.*

Then, $\mathcal{G}_0$ is a clopen normal subgroup of $\mathcal{G}$, the cosets of $\mathcal{G}_0$ are the components of $\mathcal{G}$, and $\mathcal{G}/\mathcal{G}_0$ is a discrete group[xxx]

Proof. Since $\mathcal{G}$ is an open subset of a locally connected space, its components are clopen subsets of $\mathcal{G}$.

If $f, g \in \mathcal{G}_0$, then $f\mathcal{G}_0$ is a connected subset of $\mathcal{G}$ which contains fg and f, meaning $\mathcal{G}_0 \cup f\mathcal{G}_0$ is connected, and so contained in $\mathcal{G}_0$, so $fg \in \mathcal{G}_0$, and thus $\mathcal{G}_0$ is a semigroup.

Similarly, $f^{-1}\mathcal{G}_0 \cup \mathcal{G}_0$ is connected, meaning it is contained in $\mathcal{G}_0$, so $\mathcal{G}_0$ is a subgroup of $\mathcal{G}$.

Finally, for $f \in \mathcal{G}$, then $f\mathcal{G}_0 f^{-1} = \mathcal{G}_0$, meaning $\mathcal{G}_0$ is normal.

Since $f\mathcal{G}_0$ is a clopen connected subset of $\mathcal{G}$ for every $f \in \mathcal{G}$, the cosets of $\mathcal{G}$ are components of $\mathcal{G}$.

Finally, $\mathcal{G}/\mathcal{G}_0$ is discrete, since $\mathcal{G}_0$ is open and closed in $\mathcal{G}$.[xxxi] □

Definition (Abstract Index Group)**.** For $\mathfrak{B}$ a Banach algebra, the abstract index group for $\mathfrak{B}$, denoted $\Lambda_{\mathfrak{B}}$, is the quotient group $\mathcal{G}/\mathcal{G}_0$. The abstract index is the natural homomorphism $\gamma : \mathcal{G} \to \Lambda_{\mathfrak{B}}$.

Definition (Exponential Map)**.** Let $\mathfrak{B}$ be a Banach algebra. Then, the exponential map on $\mathfrak{B}$, denoted exp, is defined by

$$\exp f = \sum_{n=0}^{\infty} \frac{1}{n!} f^n.$$

[xxx] Totally disconnected group.

[xxxi] A result in abstract harmonic analysis holds that a quotient group over G is discrete if and only if the normal subgroup is open in G.

Remark: The traditional properties of the exponential map, such as its absolute convergence, hold in all commutative Banach algebras, but do not necessarily hold in noncommutative Banach algebras.

Lemma (Exponential Properties). *For* $f, g \in \mathfrak{B}$,

$$\exp(f+g) = \exp(f) + \exp(g).$$

Lemma (Elements in Range of Exponential Map). *If* $f \in \mathfrak{B}$ *is such that* $\|e - f\| < 1$, *then* $f \in \exp \mathfrak{B}$.

Proof. Let $g = \sum_{n=1}^{\infty} -\frac{1}{n}(1-f)^n$. This series converges absolutely, and substituting into the expansion for $\exp g$, we find that

$$\exp g = f.$$

□

Theorem (Collection of Finite Products in $\exp \mathfrak{B}$). *Let* $\mathfrak{B}$ *be a (not necessarily commutative) Banach algebra. Then, the collection of finite products of elements of* $\exp \mathfrak{B}$ *is* $\mathcal{G}_0$.

Proof. Let $f = \exp g$. Then, $f \exp(-g) = \exp(g - g) = 1 = \exp(-g) f$, meaning $f \in \mathcal{G}$.

The map $\varphi : [0,1] \to \exp \mathfrak{B}$ defined by $\varphi(\lambda) = \exp(\lambda g)$ is a continuous map such that $\varphi(0) = e$ and $\varphi(1) = f$, meaning $f \in \mathcal{G}_0$. Thus, $\exp \mathfrak{B} \subseteq \mathcal{G}_0$.

If $\mathcal{F}$ denotes the collection of finite products of elements of $\exp \mathfrak{B}$, then $\mathcal{F}$ is a subgroup contained in $\mathcal{G}_0$, meaning $\mathcal{F}$ is open. Finally, since each of the left cosets of $\mathcal{F}$ is open, it follows that $\mathcal{F}$ is clopen in $\mathcal{G}_0$, so $\mathcal{F} = \mathcal{G}_0$. □

Corollary (Collection of Finite Prodcuts of Commutative Banach Algebra). *If* $\mathfrak{B}$ *is commutative, then* $\exp \mathfrak{B} = \mathcal{G}_0$.

Proof. If $\exp \mathfrak{B}$ is commutative, then $\exp \mathfrak{B}$ is a subgroup of $\mathcal{G}_0$. □

For a given Banach algebra $\mathfrak{B}$, the set of multiplicative linear functionals on $\mathfrak{B}$ is denoted $M_{\mathfrak{B}} = M$.[xxxii]

Proposition (Norm of Multiplicative Linear Functional). *For* $\mathfrak{B}$ *a Banach algebra, if* $\varphi \in M$, *then* $\|\varphi\| = 1$.

[xxxii] There was a small section here relating the abstract index group of $C(X)$ for a compact Hausdorff space X to $\pi^1(X)$, which is the group of homotopy classes of continuous maps of X to $\mathbb{T}$ (the circle group). I don't know any algebraic topology so I didn't really understand this part, and it doesn't seem to be particularly necessary outside of these facts.

Proof. Let $\mathfrak{N} = \ker\varphi$. Since $\varphi(f - \varphi(f)e) = 0$, we can see that every element in $\mathfrak{B}$ can be written as $\lambda e + f$ for $\lambda \in \mathbb{C}$ and $f \in \mathfrak{N}$. Thus,

$$\begin{aligned}\|\varphi\| &= \sup_{\|g\|=1} |\varphi(g)|\\ &= \sup_{\substack{f\in\mathfrak{N}\\ \|\lambda+f\|=1}} |\varphi(\lambda + f)|\\ &= \sup_{\substack{f\in\mathfrak{N}\\ \|\lambda+f\|=1}} |\varphi(\lambda)|\\ &= 1\end{aligned}$$

□

Proposition (Compactness of M in $\mathfrak{B}^*$). *If $\mathfrak{B}$ is a Banach algebra, then M is a w^*-compact subset of $B_{\mathfrak{B}^*}$.*

Proof. Let $\{\varphi_\alpha\}_{\alpha\in A}$ be a net in M that converges in the w^*-topology on $B_{\mathfrak{B}^*}$ to $\varphi \in B_{\mathfrak{B}^*}$.

All we need show is that φ is multiplicative with $\varphi(e) = 1$. First, we see that

$$\begin{aligned}\varphi(e) &= \lim_{\alpha\in A} \varphi_\alpha(e) &&= \lim_{\alpha\in A} 1\\ &= 1.\end{aligned}$$

Further, for $f, g \in \mathfrak{B}$, we have

$$\begin{aligned}\varphi(fg) &= \lim_{\alpha\in A} (\varphi_\alpha(f)\varphi_\alpha(g))\\ &= \left(\lim_{\alpha\in A} \varphi_\alpha(f)\right)\left(\lim_{\alpha\in A} \varphi_\alpha(g)\right)\\ &= \varphi(f)\varphi(g).\end{aligned}$$

□

Thus, M is compact in the subspace w^*-topology. Recall that for every $f \in \mathfrak{B}$, there is a w^*-continuous function $\hat{f} : B_{\mathfrak{B}^*} \to \mathbb{C}$ given by $\hat{f}(\varphi) = \varphi(f)$.

Since $M \subseteq B_{\mathfrak{B}^*}$, then $\hat{f}|_M$ is continuous.

Definition (Gelfand Transform). For $\mathfrak{B}$, if $M \neq \emptyset$, then the Gelfand transform $\Gamma : \mathfrak{B} \to C(M)$ is given by $\Gamma(f) = \hat{f}|_M$.

Proposition (Properties of the Gelfand Transform). *Let $\mathfrak{B}$ be a Banach algebra, and $\Gamma : \mathfrak{B} \to C(M)$ be the Gelfand transform on $\mathfrak{B}$. Then,*

(1) *Γ is an algebra homomorphism;*

(2) $\|\Gamma(f)\|_\infty \leqslant \|f\|$ *for all* $f \in \mathfrak{B}$.

Proof. To show that Γ is an algebra homomorphism, we show that for $f, g \in \mathfrak{B}$,

$$\begin{aligned}\Gamma(fg)(\varphi) &= \varphi(fg)\\ &= \varphi(f)\varphi(g)\\ &= \Gamma(f)(\varphi)\Gamma(g)(\varphi)\\ &= (\Gamma(f)\Gamma(g))(\varphi).\end{aligned}$$

Additionally, for $f \in \mathfrak{B}$,

$$\begin{aligned}\|\Gamma(f)\|_\infty &= \left\|\hat{f}|_M\right\|_\infty\\ &\leqslant \left\|\hat{f}\right\|_\infty\\ &= \|f\|.\end{aligned}$$

Thus, Γ is a contractive algebra homomorphism. □

Remark (Notes on the Gelfand Transform)**:** Note that $\Gamma(fg - gf) = 0$, meaning that if $\mathfrak{B}$ is not commutative, then the subalgebra of $C(M)$ that is $\operatorname{ran}(\Gamma)$ may not reflect the properties of $\mathfrak{B}$.

In the commutative case, though, M is not only nonempty, but sufficiently large such that the invertibility of $f \in \mathfrak{B}$ is determined by the invertibility of $\Gamma(f)$ in $C(M)$.

Definition (Spectrum of an Element)**.** Let $f \in \mathfrak{B}$ for $\mathfrak{B}$ a Banach algebra. Then,

$$\sigma_{\mathfrak{B}}(f) = \{\lambda \in \mathbb{C} \mid f - \lambda e \notin \mathcal{G}\}.$$

The resolvent of f is

$$\rho_{\mathfrak{B}}(f) = \mathbb{C} \setminus \sigma_{\mathfrak{B}}(f).$$

Finally, the spectral radius of f is

$$r_{\mathfrak{B}}(f) = \sup_{\lambda \in \sigma_{\mathfrak{B}}(f)} |\lambda|.$$

We write $\sigma(f)$, $\rho(f)$, and $r(f)$.

Proposition (Properties of the Spectrum)**.** *Let $\mathfrak{B}$ be a Banach algebra. Then, $\sigma(f)$ is compact in $\mathbb{C}$ and $r(f) \leqslant \|f\|$.*

Proof. Define $\varphi : \mathbb{C} \to \mathfrak{B}$, $\varphi(\lambda) = f - \lambda e$. Then, φ is continuous, and $\rho(f) = \varphi^{-1}(\mathcal{G})$ is open. Thus, $\sigma(f)$ is closed.

If $|\lambda| > \|f\|$, then

$$\begin{aligned}1 &> \frac{\|f\|}{|\lambda|}\\ &= \left\|\frac{f}{\lambda}\right\|\\ &= \left\|e - \left(e - \frac{f}{\lambda}\right)\right\|,\end{aligned}$$

meaning $e - \frac{f}{\lambda}$ is invertible, so $f - \lambda e$ is invertible. Thus, $\lambda \in \rho(f)$, so $\sigma(f)$ is bounded (hence compact), and $r(f) \leqslant \|f\|$. □

Theorem (Existence of Spectrum)**.** *Let* $f \in \mathfrak{B}$*. Then,* $\sigma(f)$ *is nonempty.*

Proof. Consider $F : \rho(f) \to \mathfrak{B}$ defined by $F(\lambda) = (f - \lambda e)^{-1}$. We will show that F is an analytic $\mathfrak{B}$-valued function on $\rho(f)$ that is bounded at infinity (thus, a contradiction).

Since inversion is continuous, we have that for $\lambda_0 \in \rho(f)$,

$$\begin{aligned}
\lim_{\lambda\to\lambda_0} \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0} &= \lim_{\lambda\to\lambda_0} \frac{(f - \lambda_0 e)^{-1}\left((f - \lambda - 0e) - (f - \lambda e)\right)(f - \lambda e)^{-1}}{\lambda - \lambda_0}\\
&= \lim_{\lambda\to\lambda_0} (f - \lambda_0 e)^{-1}(f - \lambda e)^{-1}\\
&= (f - \lambda e)^{-2}.
\end{aligned}$$

In particular, for $\varphi \in \mathfrak{B}^*$, the function $\varphi(F)$ is holomorphic on $\rho(f)$. Further, for $|\lambda| \geqslant \|f\|$, we have that $e - \frac{f}{\lambda}$ is invertible, and

$$\left\|\left(e - \frac{f}{\lambda}\right)^{-1}\right\| \leqslant \frac{1}{1 - \left\|\frac{f}{\lambda}\right\|},$$

meaning

$$\begin{aligned}
\lim_{\lambda\to\infty} \|F(\lambda)\| &= \lim_{\lambda\to\infty} \left\|\frac{1}{\lambda}\left(\frac{f}{\lambda} - e\right)^{-1}\right\|\\
&\leqslant \lim_{|\lambda|\to\infty} \sup \frac{1}{|\lambda|}\frac{1}{1 - \left\|\frac{f}{\lambda}\right\|}\\
&= 0.
\end{aligned}$$

Thus, for $\varphi \in \mathfrak{B}^*$, $\lim_{\lambda\to\infty} \varphi\left(F(\lambda)\right) - 0$.

If $\sigma(f)$ is empty, then $\rho(f) = \mathbb{C}$, meaning that for $\varphi \in \mathfrak{B}^*$, it follows that $\varphi(F)$ is an entire function that vanishes at infinity, meaning $\varphi(F) = 0$ by Liouville's Theorem.

In particular, for $\lambda \in \mathbb{C}$, $\varphi\left(F(\lambda)\right) = 0$, meaning that $F(\lambda) = 0$, which contradicts $F(\lambda)$ being invertible in $\mathfrak{B}$. □

Theorem (Gelfand–Mazur)**.** *If* $\mathfrak{B}$ *is a Banach algebra that is also a division algebra,*[xxxiii] *then there exists a unique isometric isomorphism of* $\mathfrak{B}$ *onto* $\mathbb{C}$*.*

Proof. Let $f \in \mathfrak{B}$. Then $\sigma(f)$ is nonempty; for $\lambda_f \in \sigma(f)$, we have that $f - \lambda_f e$ is not invertible, meaning that $f - \lambda_f e = 0$ since $\mathfrak{B}$ is a division algebra.

[xxxiii] Every nonzero element has a nonzero inverse.

Moreover, for $\lambda \neq \lambda_f$, $f - \lambda e = \lambda_f e - \lambda e$, which is invertible. Thus, $\sigma(f)$ consists of exactly one $\lambda_f \in \mathbb{C}$ for each f.

The map $\psi : \mathfrak{B} \to \mathbb{C}$ defined by $\psi(f) = \lambda_f$ is an isometric isomorphism of $\mathfrak{B}$ onto $\mathbb{C}$.

Moreover, for $\psi' : \mathfrak{B} \to \mathbb{C}$, we would have that $\psi'(f) \in \sigma(f)$, meaning $\psi'(f) = \psi(f)$. □

Definition (Quotient Algebra)**.** Let $\mathfrak{B}$ be a Banach algebra, and let $\mathfrak{M}$ be a closed two-sided ideal in $\mathfrak{B}$. Since $\mathfrak{M}$ is closed in $\mathfrak{B}$, we can define a norm on $\mathfrak{B}/\mathfrak{M}$ to make it into a Banach space, and since $\mathfrak{M}$ is a two-sided ideal in $\mathfrak{B}$, we know that $\mathfrak{B}/\mathfrak{M}$ is an algebra.

To verify that $\mathfrak{B}/\mathfrak{M}$ is a Banach algebra, we need to verify two facts.

To show that $\|[e]\| = 1$, we see that $\|[e]\| = \inf_{g\in\mathfrak{M}} \|e - g\| = 1$; if $\|e - g\| < 1$, then g is invertible.[xxxiv]

For $f, g \in \mathfrak{B}$, we have

$$\begin{aligned}
\|[f][g]\| &= \|[fg]\| \\
&= \inf_{h\in\mathfrak{M}} \|fg - h\| \\
&\leqslant \inf_{h_1,h_2\in\mathfrak{M}} \|(f - h_1)(g - h_2)\| \\
&\leqslant \inf_{h_1\in\mathfrak{M}} \|f - h_1\| \inf_{h_2\in\mathfrak{M}} \|g - h_2\| \\
&= \|[f]\| \, \|[g]\| \, .
\end{aligned}$$

Thus, we can see that $\mathfrak{B}/\mathfrak{M}$ is a Banach algebra, with the natural map $\pi : f \to [f]$ a contractive algebra homomorphism.

Proposition (Multiplicative Linear Functionals and Maximal Ideal Space)**.** *If $\mathfrak{B}$ is a commutative Banach algebra, then there is a bijection between $M_{\mathfrak{B}}$ and the set of maximal two-sided ideals in $\mathfrak{B}$.*

[xxxiv] Any proper ideal in $\mathfrak{B}$ cannot contain any invertible elements, since if $x \in \mathfrak{M}$ is invertible, then for $y \in \mathfrak{B}$, $y = \left(yx^{-1}\right) x \in \mathfrak{M}$, implying $\mathfrak{M} = \mathfrak{B}$.