## Problem 1

Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a family of subsets satisfying

(i) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;

(ii) If $\{A_k\}_{k\geq 1}$ is a countable family of pairwise disjoint members of $\mathcal{A}$, then $\bigsqcup_{k\geq 1} A_k \in \mathcal{A}$.

Prove that $\mathcal{A}$ is a $\sigma$-algebra on $\Omega$.

## Problem 2

Consider the family $\mathcal{E} : \{(-\infty, b) \mid b \in \mathbb{R}\}$. Show that $\sigma(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$.

**Proof:** Consider the family $\mathcal{E}' := \{[a, b) \mid a, b \in \mathbb{R}\}$. We have established that $\sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

We see that for any element of $\mathcal{E}$, $(-\infty, b) = \bigcup_{n=1}^{\infty}[a-n, b)$, meaning $\mathcal{E} \in \sigma(\mathcal{E}')$, so $\sigma\left(\mathcal{E}\right) \subseteq \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

Additionally, $[a, b) = (-\infty, b) \setminus (-\infty, a)$, meaning $\mathcal{E}' \in \sigma(\mathcal{E})$, so $\sigma(\mathcal{E}') \subseteq \sigma(\mathcal{E})$, so $\sigma\left(\mathcal{E}\right) = \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

## Problem 3

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. We define the product $\sigma$-algebra on $\Omega \times \Lambda$ as

$$\mathcal{M} \otimes \mathcal{N} := \sigma\left(\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}\right).$$

Prove that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

**Proof:** For $a < b$ and $c < d$, it is the case that $(a, b) \times (c, d) \subseteq \mathbb{R}^2$ is open, meaning

$$\begin{aligned}\sigma\left(\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{R}\}\right) &= \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}\\ &\subseteq \mathcal{B}_{\mathbb{R}^2}.\end{aligned}$$

Letting $U \in \mathcal{B}_{\mathbb{R}^2}$, it is the case that $U = \bigcup_{j=1}^{\infty} U(x_j, r_j)$. For each $U(x_j, r_j)$, take $I_j = (x_{jx} - r_j, x_{jx} + r_j) \times (x_{jy} - r_j, x_{jy} + r_j)$, so $U \subseteq \bigcup_{j=1}^{\infty} I_j$. Thus, $U \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, so $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

## Problem 4

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. A map $f : \Omega \to \Lambda$ is $\mathcal{M}$-$\mathcal{N}$-measurable if $E \in \mathcal{N} \Rightarrow f^{-1}(E) \in \mathcal{M}$.

Let $(\Omega, \mathcal{M})$ be a measurable space and suppose $E \in \mathcal{M}$. Show that $\mathcal{M}_E = \{M \cap E \mid M \in \mathcal{M}\}$ is a $\sigma$-algebra on $E$ and the inclusion map $\iota : E \to \Omega$ is $\mathcal{M}_E$-$\mathcal{M}$-measurable.

**Proof:** Let $M \in \mathcal{M}$. Then, $\iota^{-1}(M) = E \cap M \in \mathcal{M}_E$. Thus, $f$ is $\mathcal{M}_E$-$\mathcal{M}$-measurable.

## Problem 5

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. Suppose $\mathcal{N}$ is generated as a $\sigma$-algebra by a family of subsets $\mathcal{E} \subseteq \mathcal{P}(\Lambda)$. Prove that a map $f : \Omega \to \Lambda$ is $\mathcal{M}$-$\mathcal{N}$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$. Conclude that a continuous function $f : X \to Y$ between metric spaces is $\mathcal{B}_X$-$\mathcal{B}_Y$-measurable.

**Proof:**

# Problem 6

Suppose $(\Omega, \mathcal{M})$ is a measurable space and $f : \Omega \to \Lambda$ is a map. Show that $\mathcal{N} := \{E \subseteq \Lambda \mid f^{-1}(E) \in \mathcal{M}\}$ is a $\sigma$-algebra on $\Lambda$ and $f$ is $\mathcal{M}$-$\mathcal{N}$-measurable. $\mathcal{N}$ is called the $\sigma$-algebra produced by $f$.

# Problem 7

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and suppose $\{E_k\}_{k\geq 1}$ is a decreasing sequence of measurable sets with $\mu(E_1) < \infty$. Show that

$$\begin{aligned} \mu\left(\bigcap_{k\geq 1} E_k\right) &= \lim_{k\to\infty} \mu(E_k) \\ &= \inf_{k\geq 1} \mu(E_k). \end{aligned}$$

# Problem 8

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces and suppose $f : \Omega \to \Lambda$ is measurable. If $\mu$ is a measure on $\mathcal{M}$, show that

$$f * \mu : \mathcal{N} \to [0, \infty]; \quad f * \mu(E) := \mu(f^{-1}(E))$$

defines a measure on $(\Lambda, \mathcal{N})$. This is called the push-forward measure.

# Problem 9

A group $G$ is paradoxical if there are pairwise disjoint subsets of $G$; $E_1, \dots, E_n, F_1, \dots, F_m$ and group elements $t_1, \dots, t_n, s_1, \dots, s_m$ such that

$$\begin{aligned} G &= \bigsqcup_{j=1}^{n} t_j E_j \\ &= \bigsqcup_{k=1}^{m} s_k F_k. \end{aligned}$$

A mean on a group $G$ is a finitely additive probability measure $\nu : \mathcal{P}(G) \to [0, 1]$ that is translation invariant; that is, $\nu(tE) = \nu(E)$ for all $E \subseteq G$ and $t \in G$. A group is said to be amenable if it admits a mean.

Show that a paradoxical group is nonamenable.

# Problem 10

Let $\Delta$ be a totally disconnected compact metric space (for example, the Cantor set). Suppose $\varphi : C(\Delta) \to \mathbb{R}$ is a state — $\varphi$ is linear, continuous, positive, and $\varphi(\mathbb{1}_\Delta) = 1$.

(i) Show that $\mathcal{C} := \{E \mid E \subseteq \Delta\}$ is an algebra of subsets on $\Delta$.

(ii) Show that

$$\mu_0 : \mathcal{C} \to [0, 1]; \quad \mu_0(E) = \varphi(\mathbb{1}_E)$$

is a well-defined finitely additive measure.

(iii) If $\{E_k\}_{k\geq 1}$ is a countable family of members of $\mathcal{C}$ such that $\bigsqcup_{k\geq 1} E_k \in \mathcal{C}$, show that

$$\mu_0\left(\bigsqcup_{k\geq 1} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k).$$