Problem (Problem 1)**:** For all $n \in \mathbb{N}$, find the residue of $f(z) = (1 - e^{-z})^n$ at $z = 0$ via Cauchy's residue theorem.

Solution: Choose a rectangular contour γ_R given by the rectangle with side length $2R$ and height going from $-\pi i$ to πi. This yields

$$\begin{aligned}
2\pi i \operatorname{Res}(f;0) &= \oint_{\gamma_R} f(z)\,dz \\
&= \int_{-R}^{R} f(x - i\pi)\,d(x - i\pi) + \int_{R}^{-R} f(x + i\pi)\,d(x + i\pi) \\
&\quad + \int_{-\pi}^{\pi} f(R + iy)\,d(R + iy) + \int_{\pi}^{-\pi} f(-R + iy)\,d(-R + iy).
\end{aligned}$$

Expanding each of these integrals, we get

$$\begin{aligned}
\int_{-R}^{R} f(x - i\pi)\,d(x - i\pi) &= \int_{-R}^{R} \frac{1}{(1 + e^x)^n}\,dx \\
\int_{R}^{-R} f(x + i\pi)\,d(x + i\pi) &= \int_{R}^{-R} \frac{1}{(1 + e^x)^n}\,dx \\
\int_{\pi}^{-\pi} f(-R + iy)\,d(-R + iy) &= i\int_{\pi}^{-\pi} \frac{1}{(1 - e^{R-iy})^n}\,dy \\
\int_{-\pi}^{\pi} f(R + iy)\,d(R + iy) &= i\int_{-\pi}^{\pi} \frac{1}{(1 - e^{-R-iy})^n}\,dy.
\end{aligned}$$

The first two integrals sum to zero. As for the third integral, we may compute

$$\begin{aligned}
\left| i\int_{-\pi}^{\pi} \frac{1}{(1 - e^{R+iy})}\,dy \right| &\leqslant \frac{2\pi}{|1 - e^{R+iy}|^n} \\
&\leqslant \frac{2\pi}{(e^R - 1)^n} \\
&\to 0
\end{aligned}$$

as $R \to \infty$. Finally, for the fourth integral, we observe that

$$\lim_{R\to\infty} \frac{1}{(1 - e^{-R-iy})^n} = 1,$$

so our goal is to show that we can exchange limit and integral by showing that this convergence is uniform, in the sense that the sequence

$$\left(\frac{1}{(1 - e^{-k-iy})^n}\right)_{k=1}^{\infty} \to 1$$

uniformly. Towards this end, we observe that

$$\begin{aligned}
\left|1 - \frac{1}{(1 - e^{-k-iy})^n}\right| &= \left|\frac{\left(1 - e^{-k-iy}\right)^n - 1}{(1 - e^{-k-iy})^n}\right| \\
&\leqslant \frac{\left|\left(1 - e^{-k-iy}\right)^n - 1\right|}{\inf_{-\pi\leqslant y\leqslant\pi}|1 - e^{-k-iy}|^n} \\
&= \frac{\left|\left(1 - e^{-k-iy}\right)^n - 1\right|}{|1 - e^{-k}|^n}.
\end{aligned}$$

Observe that the numerator converges to 0 in k as $\left(1-e^{-k-iy}\right)^n \xrightarrow{k\to\infty} 1$, whence there is some K such that the above quantity is less than ε for arbitrary $\varepsilon > 0$. In particular, for $k \geqslant K$, we have

$$\sup_{-\pi\leqslant y\leqslant\pi}\left|1-\frac{1}{\left(1-e^{-k-iy}\right)^n}\right| < \varepsilon,$$

so that the convergence is uniform. In particular, this gives

$$\begin{aligned}
2\pi i \operatorname{Res}(f;0) &= \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz\\
&= \lim_{R\to\infty} i\int_{-\pi}^{\pi}\frac{1}{\left(1-e^{-R-iy}\right)^n}\,dy\\
&= i\int_{-\pi}^{\pi}\lim_{R\to\infty}\frac{1}{\left(1-e^{-R-iy}\right)^n}\,dy\\
&= i\int_{-\pi}^{\pi}dy\\
&= 2\pi i,
\end{aligned}$$

so $\operatorname{Res}(f;0) = 1$.

Problem (Problem 2)**:** Evaluate

$$\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2+1}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin^2(x)}{x^2+1}\,dx.$$

Solution: We compute

$$\int_{-R}^{R}\frac{\sin^2(x)}{x^2+1}\,dx = \frac{1}{2}\int_{-R}^{R}\frac{1}{x^2+1}\,dx - \frac{1}{2}\int_{-R}^{R}\frac{\cos(2x)}{x^2+1}\,dx.$$

Calling the latter integral I, we take

$$f(z) = \frac{e^{2iz}}{z^2+1},$$

close the contour γ in the upper half-plane with the half-circle $C_R = \left\{Re^{i\theta} \,\middle|\, 0\leqslant\theta\leqslant\pi\right\}$. This gives

$$\begin{aligned}
\operatorname{Re}\oint_{\gamma} f(z)\,dz &= \operatorname{Re}(I) + \operatorname{Re}\int_{C_R} f(z)\,dz\\
&= \operatorname{Re}(I) + \operatorname{Re}\int_0^{\pi}\frac{e^{2iRe^{i\theta}}}{R^2e^{2i\theta}+1}iRe^{i\theta}\,d\theta.
\end{aligned}$$

Estimating the integrand on the second integral, we see that for $R > 1$,

$$\begin{aligned}
\left|\frac{iRe^{i\theta}e^{2iRe^{i\theta}}}{R^2e^{2i\theta}+1}\right| &\leqslant \frac{R}{R^2-1}\left|e^{2iR(\cos(\theta)+i\sin(\theta))}\right|\\
&\leqslant \frac{R}{(R^2-1)\left(e^{2R\sin(\theta)}\right)}\\
&\leqslant \frac{R}{R^2-1}
\end{aligned}$$

whence

$$\left|\int_{C_R} f(z)\,dz\right| \leqslant \pi\frac{R}{R^2-1}$$

$$\to 0.$$

Therefore, by Cauchy's residue theorem,

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos(2x)}{x^2+1}\,dx &= \operatorname{Re}(2\pi i\operatorname{Res}(f;i))\\ &= \operatorname{Re}\left(2\pi i\lim_{z\to i}\frac{(z-i)e^{2iz}}{(z-i)(z+i)}\right)\\ &= \frac{\pi}{e^2}.\end{aligned}$$

Thus, we find that

$$\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2+1}\,dx = \frac{\pi}{2}-\frac{\pi}{2e^2}.$$

Problem (Problem 3)**:** For $\xi\in\mathbb{R}$, evaluate

$$\int_{-\infty}^{\infty}\frac{\cos(\xi x)}{x^2+4x+5}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{\cos(\xi x)}{x^2+4x+5}\,dx.$$

Solution: First, if $\xi = 0$, then

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{x^2+4x+5}\,dx &= \int_{-\infty}^{\infty}\frac{1}{(x+2)^2+1}\,dx\\ &= \pi\end{aligned}$$

upon a u-substitution.

Now, let $\xi > 0$. Using $f(z) = \frac{e^{i\xi z}}{z^2+4z+5}$ and closing the contour

$$\gamma_R = [-R,R]+\left\{Re^{i\theta}\;\middle|\;0\leqslant\theta\leqslant\pi\right\}$$

in the upper half plane, we find that we get

$$\oint_{\gamma_R} f(z)\,dz = \underbrace{\int_{-R}^{R} f(x)\,dx}_{=:I}+\int_{C_R} f(z)\,dz.$$

Parametrizing the integral over C_R by $z = Re^{i\theta}$, we get

$$= I+\int_0^{\pi}\frac{e^{i\xi Re^{i\theta}}}{(Re^{i\theta}+2)^2+1}iRe^{i\theta}\,d\theta.$$

Estimating the second integral, we see that for $R > 5$,

$$\begin{aligned}\left|\frac{iRe^{i\theta}e^{i\xi Re^{i\theta}}}{(Re^{i\theta}+2)^2+1}\right| &\leqslant \frac{R}{R^2-4R-5}\left|e^{i\xi R(\cos(\theta)+i\sin(\theta))}\right|\\ &\leqslant \frac{R}{(R^2-4R-5)\left(e^{\xi R\sin(\theta)}\right)}\\ &\leqslant \frac{R}{R^2-4R-5}\end{aligned}$$

meaning that

$$\left|\int_{C_R} f(z)\,dz\right| \leqslant \pi\frac{R}{R^2-4R-5}$$

$$\to 0.$$

Therefore, we find that

$$\begin{aligned}
2\pi i\operatorname{Res}(-2+i) &= \lim_{R\to\infty}\oint_{\gamma_R} f(z)\,dz\\
&= \int_{-\infty}^{\infty}\frac{e^{i\xi x}}{x^2+4x+5}\,dx\\
&= 2\pi i\lim_{z\to -2+i}\frac{(z-(-2+i))e^{i\xi z}}{(z-(-2+i))(z-(-2-i))}\\
&= 2\pi i\frac{e^{i\xi(-2+i)}}{2i}\\
&= \frac{\pi}{e^{\xi}}e^{-2i\xi}\\
&= \frac{\pi}{e^{\xi}}(\cos(2\xi)-i\sin(2\xi))\\
&= \frac{\pi}{e^{\xi}}\cos(2\xi)-i\frac{\pi}{e^{\xi}}\sin(2\xi).
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{\cos(\xi x)}{x^2+4x+5}\,dx &= \operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{x^2+4x+5}\,dx\\
&= \frac{\pi}{e^{\xi}}\cos(2\xi).
\end{aligned}$$

Now, let $\xi < 0$. We take η_R to be the contour

$$\eta_R = [-R,R] + \left\{Re^{-i\theta}\;\middle|\;0\leqslant\theta\leqslant\pi\right\}.$$

We find that

$$\begin{aligned}
\oint_{\eta_R} f(z)\,dz &= \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz\\
&= I + \int_0^{\pi}\frac{e^{i\xi\left(Re^{-i\theta}\right)}}{\left(Re^{-i\theta}+2\right)^2+1}\left(-iRe^{-i\theta}\right)\,d\theta.
\end{aligned}$$

Estimating the second integrand, we have for $R > 5$

$$\begin{aligned}
\left|\frac{-iRe^{i\theta}e^{i\xi\left(Re^{-i\theta}\right)}}{\left(Re^{-i\theta}+2\right)^2+1}\right| &\leqslant \frac{R}{R^2-4R-5}\left|e^{i\xi R(\cos(\theta)-i\sin(\theta))}\right|\\
&\leqslant \frac{R}{R^2-4R-5}e^{\xi R\sin(\theta)}\\
&\leqslant \frac{R}{R^2-4R-5}.
\end{aligned}$$

Thus,

$$\left|\int_{C_R} f(z)\,dz\right| \leqslant \pi\frac{R}{R^2-4R-5},$$

whence the integral over C_R goes to zero as $R\to\infty$. Therefore, we have

$$-2\pi i\operatorname{Res}(f;-2-i) = \lim_{R\to\infty}\int_{\eta_R} f(z)\,dz$$

$$
\begin{aligned}
&= I + \lim_{R\to\infty}\int_{C_R} f(z)\,dz \\
&= I \\
&= -2\pi i \lim_{z\to -2-i} \frac{(z-(-2-i))e^{i\xi z}}{(z-(-2-i))(z-(-2+i))} \\
&= -2\pi i \frac{e^{i\xi(-2-i)}}{-2i} \\
&= \pi e^{i\xi(-2-i)} \\
&= \pi e^{\xi}(\cos(2\xi) - i\sin(2\xi)) \\
&= \pi e^{\xi}\cos(2\xi) - i\pi e^{\xi}\sin(2\xi).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+4x+5}\,dx &= \operatorname{Re}(I) \\
&= \pi e^{\xi}\cos(2\xi).
\end{aligned}
$$

In closed form, this yields the solution

$$
\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+4x+5}\,dx = \pi e^{-|\xi|}\cos(2\xi).
$$

Problem (Problem 4)**:** Evaluate

$$
\int_0^{\infty} \frac{(\log x)^2}{x^2+1}\,dx.
$$

Solution: Select the branch of the logarithm that ignores $[0,\infty)$, so that $\arg(z) \in (0, 2\pi)$ for all $z \in \mathbb{C}\setminus[0,\infty)$. Draw a keyhole contour $\gamma_{\delta,\varepsilon,R}$ with an inner *semi*circle of radius δ, an outer semicircle of radius R, and returning along the negative real axis to the start of the semicircle of radius δ.

Set $f(z) = \frac{(\log z)^2}{z^2+1}$, and observe that for $0 < \varepsilon < \delta < 1 < R$, we have

$$
\begin{aligned}
\oint_{\gamma_{\delta,\varepsilon,R}} f(z)\,dz &= 2\pi i(\operatorname{Res}(f; i)) \\
&= 2\pi i\left(\lim_{z\to i}(z-i)\frac{(\log(z))^2}{(z-i)(z+i)}\right) \\
&= -\frac{\pi^3}{4}.
\end{aligned}
$$

Meanwhile, we observe that in the limit as $\varepsilon \to 0$, we are left with a few integrals

$$
\begin{aligned}
\oint_{\gamma_{\delta,\varepsilon,R}} f(z)\,dz &= \int_{\delta}^{R} \frac{(\log(x))^2}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{(\log(x))^2}{x^2+1}\,dx && (*) \\
&\quad + \int_0^{\pi} \frac{\log\left(\delta e^{-i\theta}\right)^2}{\delta^2 e^{-2i\theta}+1}\left(-i\delta e^{-i\theta}\right)d\theta + \int_0^{\pi} \frac{\log\left(Re^{i\theta}\right)^2}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta && (**)
\end{aligned}
$$

We start by estimating the integrals in $(**)$ by the circles $\delta e^{-i\theta}$ and $Re^{i\theta}$. Towards this end, we observe that

$$
\left|\frac{-i\delta e^{-i\theta}(\ln(\delta) - i\theta)^2}{\delta^2 e^{-2i\theta}+1}\right| \leqslant \frac{\delta|\ln(\delta)|^2 + 2\theta\delta|\ln(\delta)| + \theta^2\delta}{1-\delta^2}
$$

$$\leqslant \frac{\delta|\ln(\delta)|^2 + 4\pi\delta|\ln(\delta)| + 4\pi^2\delta}{1-\delta^2}$$
$$\to 0$$

as $\delta \to 0$. Thus,

$$\left|\int_0^{2\pi} \frac{-i\delta e^{-i\theta}(\ln(\delta) - i\theta)}{\delta^2 e^{2i\theta} + 1}\,d\theta\right| \leqslant \pi\frac{\delta|\ln(\delta)|^2 + 4\pi\delta|\ln(\delta)| + 4\pi^2\delta}{1-\delta^2}$$
$$\to 0.$$

Similarly,

$$\begin{aligned}\left|\frac{Re^{i\theta}(\ln(R) + i\theta)^2}{R^2e^{2i\theta} + 1}\right| &\leqslant \frac{R|\ln(R)|^2 + 2\theta R|\ln(R)| + \theta^2 R}{R^2 - 1}\\ &\leqslant \frac{R|\ln(R)|^2}{R^2-1} + \frac{2\pi R}{R^2-1} + \frac{4\pi^2}{R^2-1}\\ &= \frac{|\ln(R)|^2}{R - \frac{1}{R}}\frac{2\pi R}{R^2-1} + \frac{4\pi^2}{R^2-1}\\ &\to 0\end{aligned}$$

as $R \to \infty$, so the corresponding integral also goes to zero.

Now, we turn our attention to $(*)$. We observe that by the coordinate change $x \mapsto -x$, we get

$$\int_\delta^R \frac{\ln(x)}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{(\ln(x))^2}{x^2+1}\,dx = 2\int_\delta^R \frac{(\ln(x))^2}{x^2+1}\,dx + 2\pi i\int_\delta^R \frac{\ln(x)}{x^2+1}\,dx - \pi^2\int_\delta^R \frac{1}{x^2+1}\,dx.$$

As we take the limit as $\delta \to 0$ and $R \to \infty$, we observe that we get the equation

$$\frac{\pi^3}{4} = 2\underbrace{\int_0^\infty \frac{(\ln(x))^2}{x^2+1}\,dx}_{=:I_1} + 2\pi i\underbrace{\int_0^\infty \frac{\ln(x)}{x^2+1}\,dx}_{=:I_0}$$

Now, to evaluate I_0, we use the same contour for $g(z) = \frac{\ln(z)}{z^2+1}$, giving

$$\begin{aligned}\int_{\gamma_{\delta,\varepsilon,R}} g(z)\,dz &= \int_\delta^R \frac{\ln(x)}{x^2+1}\,dx + \int_{-R}^{-\delta} \frac{\ln(x)}{x^2+1}\,dx\\ &\quad + \int_0^\pi \frac{\ln\left(Re^{i\theta}\right)}{R^2e^{2i\theta}+1}iRe^{i\theta}\,d\theta + \int_0^\pi \frac{\ln\left(\delta e^{-i\theta}\right)}{\delta^2e^{2i\theta}+1}\left(-i\delta e^{-i\theta}\right)\,d\theta.\end{aligned}$$

The circle integrands may be estimated by

$$\begin{aligned}\left|\frac{iRe^{i\theta}|\ln(R) + i\theta|}{R^2e^{2i\theta}}\right| &\leqslant \frac{R\ln(R) + R\theta}{R^2-1}\\ &\leqslant \frac{R\ln(R) + \pi R}{R^2-1}\\ &\to 0\end{aligned}$$

as $R \to \infty$, so that

$$\left|\int_0^\pi \frac{\ln\left(Re^{i\theta}\right)}{R^2e^{2i\theta}+1}iRe^{i\theta}\,d\theta\right| \leqslant \pi\frac{R\ln(R) + \pi R}{R^2-1}$$

$$\to 0.$$

Similarly,

$$\begin{aligned}\left|\frac{-i\delta e^{-i\theta}(\ln(\delta) - i\theta)}{\delta^2 e^{2i\theta} + 1}\right| &\leqslant \frac{\delta|\ln(\delta)| + \pi\delta}{1 - \delta^2}\\ &\to 0\end{aligned}$$

as $\delta \to \infty$, so that

$$\begin{aligned}\left|\int_0^{\pi} \frac{\ln\left(\delta e^{-i\theta}\right)}{\delta^2 e^{2i\theta} + 1}\left(-i\delta e^{-i\theta}\right)\, d\theta\right| &\leqslant \pi\frac{\delta|\ln(\delta)| + \pi\delta}{1 - \delta^2}\\ &\to 0.\end{aligned}$$

Thus, we must evaluate the first two integrals. Yet, by using the substitution $x \mapsto -x$, we see that

$$\int_\delta^R \frac{\ln(x)}{x^2 + 1}\, dx + \int_{-R}^{-\delta} \frac{\ln(x)}{x^2 + 1}\, dx = 2\int_\delta^R \frac{\ln(x)}{x^2 + 1}\, dx + i\pi\int_\delta^R \frac{1}{x^2 + 1}\, dx.$$

Taking limits and evaluating residues gives

$$\begin{aligned}2\pi i\operatorname{Res}(g; i) &= 2\pi i\left(\frac{i\pi/2}{2i}\right)\\ &= i\frac{\pi^2}{2}\\ &= 2\int_0^\infty \frac{\ln(x)}{x^2 + 1}\, dx + i\pi\int_0^\infty \frac{1}{x^2 + 1}\, dx\\ &= 2\int_0^\infty \frac{\ln(x)}{x^2 + 1}\, dx + i\frac{\pi^2}{2},\end{aligned}$$

whence the integral for $g(z)$ is zero.

Thus, we find that

$$\int_0^\infty \frac{(\ln(x))^2}{x^2 + 1}\, dx = \frac{\pi^3}{8}$$

Problem (Problem 5)**:** For $\xi \in \mathbb{R}$, evaluate

$$\text{p. v.} \int_{-\infty}^{\infty} \frac{x^3}{\left(x^2 + 1\right)^2} e^{-2\pi i x\xi}\, dx.$$

Solution: We write

$$\int_{-\infty}^{\infty} \frac{x^3}{\left(x^2 + 1\right)^2} e^{-2\pi i x\xi}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3}{(x - i)^2 (x + i)^2} e^{-2\pi i x\xi}\, dx.$$

Write $f(z) = \frac{z^3}{\left(z^2+1\right)^2} e^{-2\pi i z\xi}$.

Suppose $\xi = 0$. Then, we seek to compute

$$\int_{-\infty}^{\infty} \frac{x^3}{\left(x^2 + 1\right)^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3}{\left(x^2 + 1\right)^2}\, dx.$$

Since the integral in R is an odd integrand over a symmetric interval, it follows that the principal value is 0.

Suppose $\xi > 0$. Let γ_R be the rectangular contour in the lower half-plane with horizontal side length $2R$ and height R sitting on the real axis that enters the lower half-plane. Then,

$$\begin{aligned}
-2\pi i \operatorname{Res}(f; -i) &= \oint_{\gamma_R} f(z)\, dz \\
&= \int_{-R}^{R} f(x)\, dx + \int_{0}^{R} f(R - iy)\, d(R - iy) \\
&+ \int_{R}^{-R} f(x - iR)\, d(x - iR) + \int_{R}^{0} f(-R - iy)\, d(-R - iy).
\end{aligned}$$

Writing each of the integrals not equal to the original integral, we get

$$\int_0^R f(R - iy)\, d(R - iy) = -i \int_0^R \frac{(R - iy)^3}{\left((R - iy)^2 + 1\right)^2} e^{-2\pi i \xi (R - iy)}\, dy \tag{1}$$

$$\int_R^{-R} f(x - iR)\, d(x - iR) = e^{-2\pi \xi R} \int_R^{-R} \frac{(x - iR)^3}{\left((x - iR)^2 + 1\right)^2} e^{-2\pi i \xi x}\, dx \tag{2}$$

$$\int_R^{0} f(-R - iy)\, d(-R - iy) = -i \int_R^{0} \frac{(-R - iy)^3}{\left((-R - iy)^2 + 1\right)^2} e^{-2\pi i \xi (-R - iy)}\, dy \tag{3}$$

Our goal now is to show that the remaining terms vanish. We start with (1), and observe that

$$\begin{aligned}
\left| \int_0^R -i \frac{(R - iy)^3}{\left((R - iy)^2 + 1\right)^2} e^{-2\pi i \xi R} e^{-2\pi \xi y}\, dy \right| &\leqslant \frac{\int_0^R e^{-2\pi \xi y}\, dy}{\inf_{0 \leqslant y \leqslant R} \left| (R - iy) + \frac{2}{(R - iy)} + \frac{1}{(R - iy)^3} \right|} \\
&\leqslant \frac{\frac{1}{2\pi \xi}\left(1 - e^{-2\pi \xi R}\right)}{R - \frac{2}{R} - \frac{1}{R^3}} \\
&\to 0.
\end{aligned}$$

Next, regarding (2), we have

$$\begin{aligned}
&\left| e^{-2\pi \xi R} \int_R^{-R} \frac{(x - iR)^3}{\left((x - iR)^2 + 1\right)^2} e^{-2\pi i \xi x}\, dx \right| \\
&\qquad\leqslant e^{-2\pi \xi R} \frac{2R}{\inf_{-R \leqslant x \leqslant R} \left| (x - iR) + \frac{2}{(x - iR)} + \frac{1}{(x - iR)^3} \right|} \\
&\qquad\leqslant e^{-2\pi \xi R} \frac{1}{1 - \frac{2}{R^2} - \frac{1}{R^4}} \\
&\qquad\to 0.
\end{aligned}$$

Finally, (3) follows from the same argument as (1). Thus, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\, dx &= \lim_{R \to \infty} \oint_{\gamma_R} f(z)\, dz \\
&= -2\pi i \operatorname{Res}(f; -i)
\end{aligned}$$

$$\begin{aligned}
&= -2\pi i\left(\frac{d}{dz}\left(\frac{z^3}{(z-i)^2}e^{-2\pi i\xi z}\right)\Bigg|_{z=i}\right)\\
&= -2\pi i\left(\frac{3z^2}{(z-i)^2}e^{-2\pi i\xi z} - 2\frac{z^3}{(z-i)^3}e^{-2\pi i\xi z} - 2\pi i\xi\frac{z^3}{(z-i)^2}e^{-2\pi i\xi z}\right)\Bigg|_{z=i}\\
&= -2\pi i\left(\frac{3(-i)^2}{(-2i)^2}e^{2\pi\xi} - 2\frac{(-i)^3}{(-2i)^3}e^{2\pi\xi} - 2\pi i\xi\frac{(-i)^3}{(-2i)^2}e^{2\pi\xi}\right)\\
&= -2\pi i\left(\frac{-3}{-4}e^{2\pi\xi} - 2\frac{i}{8i}e^{2\pi\xi} - 2\pi i\xi\frac{i}{-4}e^{2\pi\xi}\right)\\
&= -2\pi i\left(\frac{1}{2} - \frac{1}{2}\pi\xi\right)e^{2\pi\xi}\\
&= i\pi(\pi\xi - 1)e^{2\pi\xi}.
\end{aligned}$$

Now, let $\xi < 0$. Then, we let γ_R be the same square with side length $2R$, but this time contained in the *upper* half-plane. This gives

$$\begin{aligned}
2\pi i\,\mathrm{Res}(f; i) &= \oint_{\gamma_R} f(z)\,dz\\
&= \int_{-R}^{R} f(x)\,dx + \int_0^R f(R+iy)\,d(R+iy)\\
&\quad + \int_R^{-R} f(x+iR)\,d(x+iR) + \int_R^0 f(-R+iy)\,d(-R+iy).
\end{aligned}$$

Similarly, we write out the three integrals not equal to our desired result, and find

$$\begin{aligned}
\int_0^R f(R+iy)\,d(R+iy) &= i\int_0^R \frac{(R+iy)^3}{\left((R+iy)^2+1\right)^2}e^{-2\pi i\xi(R+iy)}\,dy\\
\int_R^{-R} f(x+iR)\,d(x+iR) &= e^{2\pi\xi R}\int_R^{-R}\frac{(x+iR)^3}{\left((x+iR)^2+1\right)^2}e^{-2\pi i x}\,dx\\
\int_R^0 f(-R+iy)\,d(-R+iy) &= i\int_R^0 \frac{(-R+iy)^3}{\left((R+iy)^2+1\right)^2}e^{-2\pi i\xi(-R+iy)}\,dy
\end{aligned}$$

We use the same estimate process as with the case of $\xi > 0$.

$$\begin{aligned}
\left|\int_0^R i\frac{(R+iy)^3}{\left((R+iy)^2+1\right)^2}e^{-2\pi i\xi(R+iy)}\,dy\right| &\leq \frac{1}{\inf_{0\leq y\leq R}\left|(R+iy) + \frac{2}{(R+iy)} + \frac{1}{(R+iy)^3}\right|}\int_0^R e^{2\pi\xi y}\,dy\\
&\leq \frac{1}{R - \frac{2}{R} - \frac{1}{R^3}}\left(-\frac{1}{2\pi\xi} + \frac{1}{2\pi\xi}e^{2\pi\xi R}\right)\\
&\to 0\\
\left|e^{2\pi\xi R}\int_R^{-R}\frac{(x+iR)^3}{\left((x+iR)^2+1\right)^2}\,dx\right| &\leq e^{2\pi\xi R}\frac{2R}{\inf_{-R\leq x\leq R}\left|(x+iR) + \frac{2}{(x+iR)} + \frac{1}{(x+iR)^3}\right|}\\
&\leq e^{2\pi\xi R}\frac{1}{1 - \frac{2}{R^2} - \frac{1}{R^4}}
\end{aligned}$$

$$\to 0,$$

and the third integral vanishes analogously to the first integral. Thus, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz \\
&= 2\pi i\,\mathrm{Res}(f;i) \\
&= 2\pi i\left(\frac{d}{dz}\left(\frac{z^3}{(z+i)^2}e^{-2\pi i\xi z}\right)\bigg|_{z=i}\right) \\
&= 2\pi i\left(\frac{3z^2}{(z+i)^2}e^{-2\pi i\xi z} - 2\frac{z^3}{(z+i)^3}e^{-2\pi i\xi z} - 2\pi i\xi\frac{z^3}{(z+i)^2}e^{-2\pi i\xi z}\right)\bigg|_{z=i} \\
&= 2\pi i\left(\frac{3(i)^2}{(2i)^2}e^{2\pi\xi} - 2\frac{(i)^3}{(2i)^3}e^{2\pi\xi} - 2\pi i\xi\frac{(i)^3}{(2i)^2}e^{2\pi\xi}\right) \\
&= 2\pi i\left(\frac{-3}{-4}e^{2\pi\xi} - 2\frac{-i}{-8i}e^{2\pi\xi} - 2\pi i\xi\frac{-i}{-4}e^{2\pi\xi}\right) \\
&= i\pi(\pi\xi - 1)e^{2\pi\xi}.
\end{aligned}$$