Problem 1

Show that a discrete metric space is compact if and only if it is finite.

Proof: Let (X, d) be a discrete metric space. Suppose (X, d) is not finite. Then, we can create an open cover of X defined by

$$X = \bigcup_{x \in X} \{x\}.$$

Since every subset of X is open, this is an open cover, but this does not contain a finite subcover as X is infinite.

Suppose (X, d) is not compact. Then, there is an open cover of X

$$X \subseteq \bigcup_{i \in I} U_i$$

with no finite subcover. Specifically this means that for each $i \in I$, there is some $x_i \in U_i$ such that $x_i \notin \bigcup U_{-i}$. Therefore, we have $\{x_i\}_{i=1}^{\infty} \subseteq X$, so X is infinite.

Problem 2

Let X be a metric space and suppose $Y \subseteq X$. Show that $K \subseteq Y$ is compact in Y with the relative topology if and only if K is compact in X.

Proof: Let $K \subseteq Y$ be compact in Y. For $\mathcal{V}$ an open cover of K, we have that

$$K \subseteq \bigcup_{i \in I} (Y \cap V_i),$$

where each of the V_i are open in X. Since K is compact, there exists a finite F such that

$$K \subseteq \bigcup_{i \in F} (Y \cap V_i).$$

Thus, $K \subseteq Y \cap \bigcup_{i \in F} V_i$, meaning $K \subseteq \bigcup_{i \in F} V_i$, meaning the open cover $\bigcup V_i$ in X has a finite subcover for K in X.

Let $K \subseteq Y \subseteq X$ be compact in X. Then, for an open cover $\mathcal{V} = \bigcup_{i \in I} V_i$, there exists a finite subcover F such that $K \subseteq \bigcup_{i \in F} V_i$. Thus,

$$\begin{aligned} K &= Y \cap K \\ &\subseteq Y \cap \left(\bigcup_{i \in F} V_i \right) \\ &= \bigcup_{i \in F} (Y \cap V_i) \end{aligned}$$

is an open cover with finite subcover for K in Y, meaning K is compact in Y.

Problem 3

Let X be a metric space. Let $(x_n)_n$ be a sequence in X which converges to a point $x_0 \in X$. Show that $\{x_0, x_1, \dots\}$ is compact.

Proof: Since $(x_n)_n \to x_0 \in \{x_0, x_1, x_2, \dots\} = A$ is a bounded sequence, the set A is bounded. Thus, all sequences in A are bounded; since we can extract a convergent subsequence in A by selecting a natural sequence by recursively selecting the smallest following index that contains x_i, i greater than the index of the current point. If no such i exists, then the sequence converges necessarily nonetheless.

Since every sequence in $\{x_0, x_1, \dots\}$ admits a convergent subsequence, $\{x_0, x_1, \dots\}$ is sequentially compact, hence compact in X.

Problem 4

Let (X, d) be a metric space. If $C, K \subseteq X$, we define $\text{dist}(C, K) := \inf_{x \in C, y \in K} d(x, y)$.

(i) If K is compact and C is closed, show that

$$K \cap C = \emptyset \Leftrightarrow \text{dist}(C, K) > 0$$

Can we remove the requirement that K is compact and only require it to be closed?

(ii) If both K and C are compact, show that there is $x \in C$ and $y \in K$ with $\text{dist}(C, K) = d(x, y)$.

Proof:

(i) Let $K \cap C = \emptyset$. Then, by the normal property, $\exists U, V \in \tau_X$ with $K \subset U$ and $C \subset V$ and $U \cap V = \emptyset$. Choose $x \in U \setminus K$ and $y \in V \setminus C$. Then, $\exists \varepsilon_x, \varepsilon_y > 0$ with $U(x, \varepsilon_x) \subseteq U$ and $U(y, \varepsilon_y) \subseteq V$. Thus, $d(x, y) > \varepsilon_x + \varepsilon_y > 0$, meaning $\text{dist}(C, K) > \varepsilon_x + \varepsilon_y > 0$. This direction of the proof did not require compactness.

Problem 5

Let V be a finite-dimensional normed space. Show that the unit ball $B := \{v \in V \mid \|v\| \leq 1\}$ is compact.

Proof: Having shown that all norms on V are equivalent, we can create a homeomorphism $f : \ell_2^n \to V$, where $\dim(V) = n$. Consider $f^{-1}(B_V)$. Since B_V is bounded and closed, its continuous image under f^{-1} is bounded and closed. Thus, $f^{-1}(B_V)$ is compact in ℓ_2^n. So, $f(f^{-1}(B_V)) = B_V$ is a continuous image of a compact set, which is compact. Thus, B_V is compact in V.

Problem 6

Prove that the unit ball in $C([0, 1])$ is not compact.

Proof: We have shown that B_V is compact if and only if V is finite-dimensional. Since $C([0, 1])$ is infinite-dimensional, it must be the case that B_V is not compact.

Problem 7

Let V be a normed space and let $K, L \subseteq V$ be compact. Show that

$$K + L := \{x + y \mid x \in K, y \in L\}$$

is also compact.

Proof: We will show that $K+L$ is complete and totally bounded.

Let $(a_n)_n$ be a Cauchy sequence in $K+L$. Then, $a_n = \chi_n + \sigma_n$ for $\chi_n \in K$ and $\sigma_n \in L$, both Cauchy. For large m, n, we have

$$\begin{aligned} |a_m - a_n| &= |(\chi_m + \sigma_m) - (\chi_m + \sigma_n)| \\ &\leq |\chi_m - \chi_n| + |\sigma_m - \sigma_n| \\ &< \varepsilon, \end{aligned}$$

and since $(\chi_n)_n \to \chi \in K$ and $(\sigma_n)_n \to \sigma \in L$, it must be the case that $(a_n)_n \to \chi + \sigma \in K + L$. Therefore, $K+L$ is complete.

Let $\varepsilon > 0$. Since K is totally bounded, $\exists x_1, \dots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} U(x_i, \varepsilon/2)$. Similarly, since L is totally bounded, $\exists y_1, \dots, y_m \in L$ such that $L \subseteq \bigcup_{j=1}^{m} U(y_i, \varepsilon/2)$.

Let $x \in K + L$. Then, $x = x_K + y_L$ for $x_K \in K$ and $y_L \in L$. Since there exist $x_i \in K$ and $y_j \in L$ with $\|x_K - x_i\| < \varepsilon/2$ and $\|y_L - y_j\| < \varepsilon$, we have

$$\begin{aligned} \|x - (x_i + y_j)\| &= \|(x_K + y_L) - (x_i - y_j)\| \\ &\leq \|x_K - x_i\| + \|y_L - y_j\| \\ &< \varepsilon. \end{aligned}$$

Thus, it is the case that

$$K + L \subseteq \bigcup_{j=1}^{m} \left(\bigcup_{i=1}^{n} U(x_i + y_j, \varepsilon) \right),$$

meaning $K+L$ is totally bounded.

Since $K+L$ is complete and totally bounded, it is compact.

Problem 8

Let $(f_n : [0,1] \to \mathbb{R})_{n\geq 1}$ be a sequence of differentiable functions with $\sup \|f_n\|_u < \infty$ and $\sup \|f_n'\|_u < \infty$. Show that there is a subsequence $(f_{n_k})_k$ that converges uniformly to a continuous function $f : [0,1] \to \mathbb{R}$.

Proof: Let $(f_n)_n$ be the sequence defined as above.

Let $K = \sup_{n\geq 1} \|f_n'\|_u$. By the Mean Value Theorem, for all $x, y \in [0,1]$, we have that $|f_n(x) - f_n(y)| \leq K|x-y|$. Letting $\delta = \frac{\varepsilon}{2K}$, we have that $(f_n)_n$ is an equicontinuous family of functions.

Since $\sup_{n\geq 1} \|f_n\|_u < \infty$, the family $(f_n)_n$ is also bounded.

By Arzelà-Ascoli, $\exists n_k$ such that $(f_{n_k})_k \to f$ uniformly, as $\mathcal{F} = \{f_n\}$ is compact.

Problem 9

Let $(X_n, d_n)_n$ be a sequence of compact metric spaces. Show that the product $\prod X_n$ with the product metric is also compact.

Proof: Let (X_n, d_n) be a sequence of compact metric spaces with the distance between $x = (x_k)_k, y = (y_k)_k \in \prod X_n$ defined by $\sum_{k=1}^{\infty} 2^{-k} d_k(x_k, y_k)$.

Problem 10

Let (X, d) be a compact metric space and let $\mathcal{V}$ be an open cover of X. Show that there is a number $L(\mathcal{V})$ satisfying that given any nonempty $E \subseteq X$ with $\text{diam}(E) < L(\mathcal{V})$, there exists $V \in \mathcal{V}$ with $E \subseteq V$.

Proof: Suppose toward contradiction that no such $L(\mathcal{V})$ exists. Then, for any $E \subseteq X$ with $\text{diam}(E) < \frac{1}{n}$, there does not exist $V \in \mathcal{V}$ with $E \subseteq V$.

Let $(x_n)_n$ be a sequence in X. Since X is compact, we can extract n_k such that $(x_{n_k})_k \to x \in X$. For $\varepsilon > 0$, it must be the case that $U(x, \varepsilon) \subseteq V$ for some $V \in \mathcal{V}$ (as V is an open cover of X).

Since $(x_{n_k})_k \to x$, we have that $\exists N_k$ large such that for all $k \geq N_k$, $x_{n_k} \in U(x, \varepsilon/2)$, and $\frac{1}{N_k} < \varepsilon/2$. Letting $E \subseteq X$ be a set of diameter $\frac{1}{n_k}$, we have that for $y \in E$,

$$\begin{aligned}
d(y, x) &\leq d\left(y, x_{n_k}\right) + d\left(x_{n_k}, x\right) \\
&\leq \text{diam}(E) + \frac{\varepsilon}{2} \\
&\leq \frac{1}{n_k} + \frac{\varepsilon}{2} \\
&\leq \frac{1}{N_k} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, $E \subseteq U(x, \varepsilon) \subseteq V$. $\perp$