

**Problem (Problem 1):** In this exercise, we prove another fundamental result in differential topology, called the tubular neighborhood theorem. Let  $M$  be a compact smooth manifold with orientable boundary  $N$ . For simplicity, assume that  $N$  is connected. The tubular neighborhood theorem asserts that  $N$  admits a neighborhood in  $M$  which is diffeomorphic to  $N \times [0, 1)$ .

- (a) Choose a Riemannian metric on  $M$ , and show that  $N$  admits a nonvanishing vector field that is everywhere orthogonal to the tangent space of  $N$ . That is, a vector field  $X$  such that for all  $p \in N$ ,  $g(X_p, T_p N) = 0$ .
- (b) Use the flow generated by  $X$  to find the desired neighborhood.

**Solution:**

- (a) If  $p \in N$ , then we observe that  $T_p N \subset T_p M$  is a proper subspace with codimension 1. Letting  $\{e_1, \dots, e_{n-1}\}$  be an orthonormal basis for  $T_p N$ , then we may extend to a basis for  $T_p M$  by taking a representative for a basis for  $T_p M / T_p N$ , and observing that such a vector necessarily has

$$g_p(e_n, e_k) = 0$$

for all  $k = 1, \dots, n-1$ . By smoothly varying over all points  $p \in N$ , we get our desired everywhere nonvanishing vector field normal to  $T_p N$ .

- (b) Let  $\varphi_t$  be the one-parameter diffeomorphism group generated by  $X$ , where  $\varphi_t: M \rightarrow M$  is such that  $\varphi_0(p) = p$  for all  $p \in N$ . Then,  $\varphi: (-\varepsilon, \varepsilon) \rightarrow \text{diff}(M)$  restricted to  $[0, \varepsilon)$  gives our desired neighborhood in  $M$  diffeomorphic to  $N \times [0, 1)$ .

**Problem (Problem Set 7, Problem 5):** Suppose  $G$  is a finite group acting freely on a manifold  $M$  by diffeomorphisms.

- (a) Show that  $M/G$  is a manifold.
- (b) Show that the de Rham cohomology of  $M/G$  is isomorphic to the  $G$ -invariant cohomology of  $M$ .

**Solution:**

- (a) Let  $p \in M$ , and let  $U$  be a neighborhood of  $p$ . By shrinking  $U$  if necessary, the fact that  $G$  acts freely on  $M$  implies that for all  $1 \neq g$ , we have  $g \cdot U \cap U = \emptyset$ . In particular, this gives that the projection map  $q: M \rightarrow M/G$  is a covering map. Thus, to find a chart about  $[p] \in M/G$ , we consider the image  $\bar{U} := q(U) \subseteq M/G$ . If  $\varphi$  is the coordinate map for  $U$ , define  $\bar{\varphi}: \bar{U} \rightarrow \mathbb{R}^n$  by taking  $\bar{\varphi}(q(U)) = \varphi \circ q^{-1}(\bar{U})$ .

Let  $[w] \in \bar{U} \cap \bar{V}$ . We have some  $w_1 \in U$  and  $w_2 \in V$  such that  $q(w_1) = q(w_2) = [w]$ ; in particular, there is  $g \in G$  such that  $g \cdot w_1 = w_2$ . We may define  $U' = U \cap g^{-1} \cdot V$  and  $V' = V \cap (g \cdot U)$ , where  $\bar{U}' \cap \bar{V}' \subseteq \bar{U} \cap \bar{V}$  and  $[w] \in \bar{U}' \cap \bar{V}'$ . Furthermore, we see that  $q(U' \cap V') = \bar{U}' \cap \bar{V}'$ , as any element in the latter is given by  $[x]$ , where  $x_1 \in U'$  and  $x_2 \in V'$  have  $q(x_i) = [x]$ , meaning that the element  $k \cdot x_1 = x_2$  is uniquely determined as the action is free.

We observe now that the transition map  $\bar{\psi} \circ \bar{\varphi}^{-1}: \bar{\varphi}(\bar{U}' \cap \bar{V}') \rightarrow \bar{\psi}(\bar{U}' \cap \bar{V}')$  is then given, by the definition of these maps, by the transition map between  $\varphi(U \cap (g^{-1} \cdot V))$  to  $\psi((g \cdot U) \cap V)$ . Therefore,  $M/G$  is a manifold.

- (b) Consider a closed form  $\omega \in \mathcal{A}^*(M/G)$ . The pullback  $q^*\omega \in \mathcal{A}^*(M)$  is necessarily  $G$ -invariant. This descends to a map in cohomology

$$q^*: H_{\text{DR}}^*(M/G) \rightarrow H_{\text{DR}}^*(M)^G.$$

Our goal is to show that this map is injective and surjective. First, let  $[\omega] \in \ker(q^*)$ . Then,  $\omega$  is

exact, meaning that  $q^*\omega$  is exact, so there is some  $\eta$  such that  $q^*\omega = d\eta$ . If we let

$$\xi = \frac{1}{|G|} \sum_{g \in |G|} g^*\eta,$$

then we see that  $\xi$  is  $G$ -invariant and has  $d\xi = \eta$ . By the commutativity of pullback and  $d$ , there is then some  $\bar{\xi} \in H_{\text{DR}}^*(M/G)$  such that  $q^*\bar{\xi} = \xi$ , meaning that  $d\xi$  and  $\omega$  are in the same cohomology class. Thus,  $\ker(q^*) = \{0\}$ .

Now, to see surjectivity, let  $[\omega] \in H_{\text{DR}}^*(M)$  be  $G$ -invariant. By using the same averaging process, we have a representative of the cohomology class that can be found by pullback of a closed form in  $H_{\text{DR}}^*(M/G)$ , so that  $H_{\text{DR}}^*(M/G) \cong H_{\text{DR}}^*(M)^G$ .

**Problem** (Problem Set 8, Problem 3): Compute the de Rham cohomology of  $\mathbb{RP}^n$ .

**Solution:** We observe that the antipodal map,  $x \mapsto -x$ , is a finite free action on the manifold  $S^n$ , and is such that the orbit space is  $\mathbb{RP}^n$ , given by  $\mathbb{Z}/2\mathbb{Z}$ .

We know that the cohomology for  $S^n$  is given by  $\mathbb{R}$  at  $H^0$ , and  $\mathbb{R}$  at  $H^n$ , with 0 everywhere else. The antipodal map is of degree 1 if and only if  $n$  is odd, which means that the antipodal map is thus a sign-preserving local diffeomorphism. In particular, this means that for an  $n$ -form  $\omega \in \mathcal{A}^n(\mathbb{RP}^n)$ , we have

$$\int_{S^n} q^*\omega = \int_{\mathbb{RP}^n} \omega,$$

so that the simplicial cohomology is identical. Thus, if  $n$  is odd, then  $\mathbb{RP}^n$  and  $S^n$  have the same cohomology.

Now, if  $n$  is even, we know that the degree of the antipodal map is  $-1$ . Yet, this means that there are no invariant closed  $n$ -forms, meaning that the top-dimensional cohomology of  $\mathbb{RP}^n$  is 0.

**Problem** (Problem Set 8, Problem 5): Use the Mayer–Vietoris sequence to prove the Künneth Formula: if  $M$  and  $N$  are smooth manifolds, then  $H_{\text{DR}}^*(M \times N)$  is the tensor product of  $H_{\text{DR}}^*(M)$  and  $H_{\text{DR}}^*(N)$ . Specifically, in each degree  $\ell$ , we have

$$H_{\text{DR}}^\ell(M \times N) = \bigoplus_{i+j=\ell} H_{\text{DR}}^i(M) \otimes H_{\text{DR}}^j(N).$$

**Solution:** For the sake of being able to solve this problem, we focus on the case where  $M$  and  $N$  are closed smooth manifolds.

Let  $V = M \times N$  be the product manifold for  $M$  and  $N$ . If  $\pi_1: V \rightarrow M$  and  $\pi_2: V \rightarrow N$  are the projection maps on  $M$  and  $N$  respectively, we get the composed maps

$$\mathcal{A}^k(M) \times \mathcal{A}^\ell(N) \rightarrow \mathcal{A}^{k+\ell}(V)$$

given by  $(\omega, \eta) \mapsto \pi_1^*\omega \wedge \pi_2^*\eta$ . If  $\omega$  and  $\eta$  are closed forms, then we observe that

$$\begin{aligned} d(\pi_1^*\omega \wedge \pi_2^*\eta) &= d\pi_1^*\omega \wedge \pi_2^*\eta + (-1)^k \pi_1^*\omega \wedge d\pi_2^*\eta \\ &= \pi_1^*(d\omega) \wedge \pi_2^*\eta + (-1)^k \pi_1^*\omega \wedge \pi_2^*(d\eta) \\ &= 0. \end{aligned}$$

Furthermore, if we let  $\omega' = \omega + d\tau$  and  $\eta' = \eta + d\rho$ , then we know from earlier work that  $\pi_1^*\omega' \wedge \pi_2^*\eta'$  can be expressed as  $\pi_1^*\omega \wedge \pi_2^*\eta + d\sigma$  for some form  $\sigma$  by using the fact that  $d$  and the pullback commute. Thus, it follows that the map descends to a map in cohomology, given by

$$H_{\text{DR}}^k(M) \times H_{\text{DR}}^\ell(N) \rightarrow H^{k+\ell}(M \times N)$$

$$([\omega], [\eta]) \mapsto [\pi_1^* \omega \wedge \pi_2^* \eta],$$

whence via the universal property of tensor products and direct sums, we get the map

$$\psi: H_{\text{DR}}^*(M) \otimes H_{\text{DR}}^*(N) \rightarrow H^*(M \times N).$$

Our goal now is to show that  $\psi$  is indeed an isomorphism.

Toward this end, suppose we have two open sets in the good cover for  $M$ , given by  $U_1$  and  $U_2$ . From the Mayer–Vietoris sequence, this yields the following exact sequence in cohomology for a fixed  $k$ , where  $D_k$  denote the connecting homomorphisms from  $H^k(U_1 \cap U_2)$  to  $H^{k+1}(M)$ .

$$\dots \xrightarrow{D_{k-1}} H_{\text{DR}}^k(M) \xrightarrow{i} H_{\text{DR}}^k(U_1) \oplus H_{\text{DR}}^k(U_2) \xrightarrow{j} H_{\text{DR}}^k(U_1 \cap U_2) \xrightarrow{D_k} \dots$$

Since the tensor product preserves exact sequences, we observe that by taking the tensor product with  $H_{\text{DR}}^\ell(N)$ , giving the following.

$$\dots \xrightarrow{D_{k-1}} H_{\text{DR}}^k(M) \otimes H_{\text{DR}}^\ell(N) \xrightarrow{i} H_{\text{DR}}^k(U_1) \otimes H_{\text{DR}}^\ell(N) \oplus H_{\text{DR}}^k(U_2) \otimes H_{\text{DR}}^\ell(N) \xrightarrow{j} H^k(U_1 \cap U_2) \otimes H^\ell(N) \xrightarrow{D_k} \dots$$

Taking direct sums with the same dimension, we obtain the following diagram.

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \bigoplus_{k+\ell=i-1} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^{i-1}((U_1 \cap U_2) \times N) \\ \downarrow D_{i-1} & & \downarrow D_{i-1} \\ \bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i(M \times N) \\ \downarrow i & & \downarrow i \\ \bigoplus_{k+\ell=i} (H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) & \xrightarrow{\psi} & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \\ \downarrow j & & \downarrow j \\ \bigoplus_{k+\ell=i} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i((U_1 \cap U_2) \times N) \\ \downarrow D_i & & \downarrow D_i \\ \vdots & & \vdots \end{array}$$

Since  $U_1$ ,  $U_2$ , and  $U_1 \cap U_2$  are contractible, under the good cover assumption, it follows from the Poincaré Lemma that the following subsection of the diagram is commutative, with  $\psi$  necessarily an isomorphism in each of the columns.

$$\begin{array}{ccc} \bigoplus_{k+\ell=i} (H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) & \xrightarrow{\psi} & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \\ \downarrow j & & \downarrow j \\ \bigoplus_{k+\ell=i} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i((U_1 \cap U_2) \times N) \end{array}$$

Similarly, we have that the following diagram is commutative, following from the Mayer–Vietoris sequence.

$$\begin{array}{ccc} \bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i(M \times N) \\ \downarrow i & & \downarrow j \\ \bigoplus_{k+\ell=i} (H^k(U_1) \otimes H^\ell(N) \oplus H^k(U_2) \otimes H^\ell(N)) & \xrightarrow{\psi} & H^i(U_1 \times N) \oplus H^i(U_2 \times N) \end{array}$$

Therefore, we only need to verify commutativity for the following square.

$$\begin{array}{ccc} \bigoplus_{k+\ell=i-1} H^k(U_1 \cap U_2) \otimes H^\ell(N) & \xrightarrow{\psi} & H^{i-1}((U_1 \cap U_2) \times N) \\ D_{i-1} \downarrow & & \downarrow D_{i-1} \\ \bigoplus_{k+\ell=i} H^k(M) \otimes H^\ell(N) & \xrightarrow{\psi} & H^i(M \times N) \end{array}$$

First, from the Mayer–Vietoris sequence and the fact that the coboundary map in de Rham cohomology emerges from the exterior derivative, we have that the map  $D_i$  is given by

$$D_{i-1}([\omega]) = \begin{cases} [d(-f_U \omega)] \\ [d(f_V \omega)] \end{cases}$$

for any cohomology class representative  $\omega$ . Now, we observe that

$$\begin{aligned} \psi(D_{i-1}([\omega], [\eta])) &= [\pi_1^*(D_{i-1}(\omega)) \wedge \pi_2^* \eta] \\ D_{i-1}(\psi([\omega], [\eta])) &= [D_{i-1}(\pi_1^* \omega \wedge \pi_2^* \eta)] \end{aligned}$$

In particular, since  $\pi_1^* f_U$  and  $\pi_1^* f_V$  form a partition of unity for  $M \times F$ , we have

$$\begin{aligned} \pi_1^*(D_{i-1}(\omega)) \wedge \pi_2^* \eta &= \pi_1^*(d(f_V \omega)) \wedge \pi_2^* \eta \\ &= d(\pi_1^*(f_V \omega)) \wedge \pi_2^* \eta \\ D_{i-1}(\pi_1^* \omega \wedge \pi_2^* \eta) &= d(\pi_1^* f_V \pi_1^* \omega \wedge \pi_2^* \eta) \\ &= d(\pi^*(f_V \omega)) \wedge \pi_2^* \eta. \end{aligned}$$

Since  $\psi$  at each of  $U$ ,  $V$ , and  $U \cap V$  is an isomorphism, and the diagram commutes, the Five Lemma gives that  $\psi$  at  $M$  is an isomorphism.

For any finite good cover with more than 2 elements, induction gives the desired result.

**Problem** (Problem Set 10, Problem 7): Show that the invariant cohomology  $H_L^*(G)$  is isomorphic to the de Rham cohomology of  $G$ . Conclude that the de Rham cohomology of an  $n$ -torus is isomorphic to the exterior algebra on  $\mathbb{R}^n$ .