A smooth map between manifolds $f\colon M \to N$ includes a certain family of local information; for instance, the derivative $D_pf\colon T_pM \to T_{f(p)}N$, which is a linear map between tangent spaces at p and q, is defined on a coordinate chart $U \subseteq M$ for p and a corresponding coordinate chart $V \subseteq N$ for $f(p)$. Yet, the properties of this linear map can give us information about the underlying map f.

To understand this, we need to dive into the world of regular and critical values.

Much of this document is based on the book *Topology from the Differentiable Viewpoint* and assorted notes from my Differential Topology class.

Sard's Theorem and the Regular Value Theorem

Definition: Let $f\colon M \to N$ be a smooth map, and let $p \in M$. We say p is a *critical point* for f if D_pf does not have the same rank as the dimension of $T_{f(p)}N$.

If D_pf has the same rank as the dimension of $T_{f(p)}N$, then we say that p is a *regular point* of f.

We say $q \in N$ is a *critical value* for f if $f^{-1}(\{q\})$ contains a critical point for f. Else,we say that q is a *regular value*.

We start with the case of Sard's Theorem on $\mathbb{R}^n$. Then, we will expand this to the case of any arbitrary manifold by means of a technical lemma.

Theorem (Sard's Theorem)**:** Let $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be a smooth map. Then, if C is the set of critical points for f, we have $f(C) \subseteq \mathbb{R}^m$ has measure zero.

The proof of Sard's Theorem is very technical, so we will not be showing the full proof. A proof can be found at this link.

A useful result used in conjunction with Sard's Theorem is the Regular Value Theorem. We will show some important results using these two theorems.

Theorem (Regular Value Theorem)**:** Let $f\colon M \to N$ be a smooth map of manifolds with dimensions $m \geq n$. If $q \subseteq N$ is a regular value, then $f^{-1}(\{q\}) \subseteq M$ is a submanifold of dimension $m - n$.

Proof. Let $p \in f^{-1}(\{q\})$, and let (U, φ) be a chart about p where $\varphi(U) \cong \mathbb{R}^m \cong T_pM$ are identified together. Since D_pf is full rank, we have that $K = \ker(D_pf)$ is of codimension n, meaning that $K \cong \mathbb{R}^{m-n}$.

Let $L\colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ be a projection, and define $F\colon U \to N \times \mathbb{R}^{m-n}$ by $x \mapsto (f(x), L(x))$. Then, since L is a linear map and the matrix representation for D_pF is block-diagonal, we have that $D_pF = (D_pf, L)$. In particular, $D_pF\colon \mathbb{R}^n \to \mathbb{R}^n$ is full rank, so by the inverse function theorem, F is invertible on a neighborhood $V \times W \subseteq N \times \mathbb{R}^{m-n}$, where W is a neighborhood of 0. We may thus identify $U \cong V \times W$.

By composing with the projection $\pi\colon N \times \mathbb{R}^{m-n} \to N$ given by $(q, W) \mapsto q$, we have that $f = \pi \circ F$, meaning $f^{-1}(\{q\}) = F^{-1}\left(\pi^{-1}(\{q\})\right)$, so that $f^{-1}(\{q\}) \cong \mathbb{R}^{m-n}$. □

Remark: If M is compact and N has the same dimension as M, $f^{-1}(\{q\})$ is discrete. Additionally, the cardinality $\left|f^{-1}(\{q\})\right|$ is a locally constant function of q.

To see this, let $p_1, \dots, p_k$ be the elements of $f^{-1}(\{q\})$ with corresponding disjoint open neighborhoods $U_1, \dots, U_k$. These neighborhoods are necessarily mapped diffeomorphically onto neighborhoods $V_1, \dots, V_k$ in N. If we let

$$V = (V_1 \cap \cdots \cap V_k) \setminus f(M \setminus (U_1 \cup \cdots \cup U_k)),$$

then for any $w \in V$, we have $\left|f^{-1}(\{w\})\right|$ is equal to $\left|f^{-1}(\{q\})\right|$.