Avoid looking at these solutions until you have genuinely given the practice problems an honest attempt. Each part will have solutions on a separate section, given in the table of contents, so even if you do want to see a solution, please only look at the solution for the part you are stuck on.

Contents

Problem 1

Problem:

(a) Consider the solid defined by rotating the region bounded by $x = 0$, $x = \pi/2$, $y = 0$, and $y = \cos(x)$. Set up integrals I_1 and I_2 for the volume and surface area of this solid respectively.

(b) Find the volume of the solid by resolving the integral.

(c) Similarly, find an expression for the surface area of this solid. To find the antiderivative, you may find the following steps useful.

(i) Use a substitution to express the integral entirely in terms of square roots and polynomial expressions.

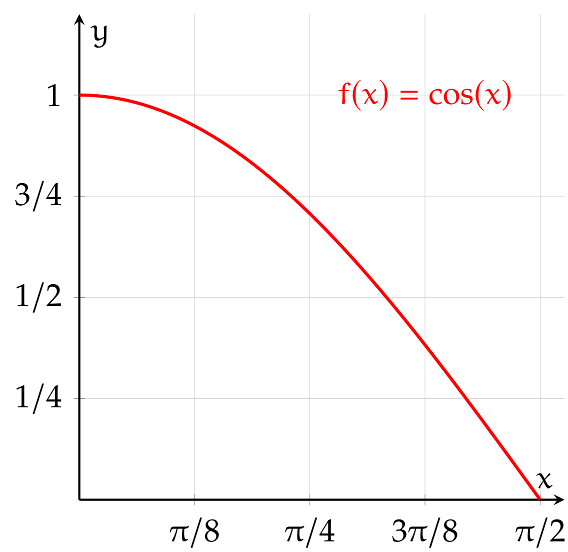
(ii) Take $u = \tan(\theta)$ and use trigonometric identities to express the integral solely in terms of $\sec(\theta)$.

(iii) Extract a factor of $\sec^2(\theta)$ and use integration by parts and a trigonometric identity to reduce this integral to that of $\sec(\theta)$.

(iv) To evaluate the integral of $\sec(\theta)$, multiply top and bottom by $\sec(\theta) + \tan(\theta)$, then use a substitution.

Solution:

(a) We start by considering the following graph.



In order to write the volume of the solid, we see that at any x, the solid looks similar to a cylinder of height dx and radius $\cos(x)$. In the limit, this gives that the volume can be written as

$$A = \int_0^{\pi/2} \pi \cos^2(x)\, dx.$$

Similarly, the arc length is given by $\sqrt{dx^2 + dy^2}$, or $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$, meaning that, by using the surface area expression, we get

$$S = \int_0^{\pi/2} 2\pi \cos(x)\sqrt{1 + \sin^2(x)}\,dx.$$

(b) To evaluate the volume, we use the identity that $\cos(2x) = 2\cos^2(x) - 1$. By rearranging, this gives that $\cos^2(x) = \frac{1+\cos(2x)}{2}$. Therefore, we get

$$\int_0^{\pi/2} \pi \cos^2(x)\,dx = \frac{\pi}{2}\int_0^{\pi/2} 1\,dx + \frac{\pi}{2}\int_0^{\pi/2} \cos(2x)\,dx.$$

By substituting with $u = 2x$ on the second integral and evaluating the first integral, we get

$$\begin{aligned} &= \frac{\pi^2}{4} + \frac{\pi}{4}\int_0^{\pi} \cos(u)\,du. && = \frac{\pi^2}{4} + \frac{\pi}{4}\left(\sin(u)\big|_0^{\pi}\right) \\ &= \frac{\pi^2}{4} + \frac{\pi}{4}(\sin(\pi) - \sin(0)) \\ &= \frac{\pi^2}{4}. \end{aligned}$$

(c) Evaluating the surface area integral will be a bit more involved. For now, we will disregard the bounds, and only evaluate them at the very end.

First, define $t = \sin(x)$, meaning $dt = \cos(x)dx$, and the substitution gives

$$\int_0^{\pi/2} \cos(x)\sqrt{1 + \sin^2(x)}\,dx = \int_0^1 \sqrt{1 + t^2}\,dt.$$

We then define $t = \tan(\theta)$, or that $\theta = \arctan(t)$, meaning that we get $dt = \sec^2(\theta)\,d\theta$. This gives

$$\begin{aligned} \int_0^1 \sqrt{1 + t^2}\,dt &= \int_0^{\pi/4} \sqrt{1 + \tan^2(\theta)}\sec^2(\theta)\,d\theta \\ &= \int_0^{\pi/4} \sec^3(\theta)\,d\theta. \end{aligned}$$

To evaluate this particular integral, we take out a factor of $\sec^2\theta$, and use integration by parts. Since the derivative of $\tan(\theta)$ is $\sec^2\theta$, we let $dv = \sec^2\theta d\theta$, $u = \sec\theta$, and get

$$\int_0^{\pi/4} \sec^3(\theta)\,d\theta = \sec(\theta)\tan(\theta)\big|_0^{\pi/4} - \int_0^{\pi/4} \sec(\theta)\tan^2\theta\,d\theta.$$

Substituting $\tan^2\theta = \sec^2\theta - 1$, we have

$$= \sec(\theta)\tan(\theta)\big|_0^{\pi/4} - \int_0^{\pi/4} \sec^3\theta \, d\theta + \int_0^{\pi/4} \sec(\theta)\, d\theta.$$

Therefore, we get that

$$\int_0^{\pi/4} \sec^3(\theta)\, d\theta = \left(\frac{1}{2}\sec(\theta)\tan(\theta)\right)\Bigg|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec(\theta)\, d\theta.$$

Finally, to evaluate the integral of $\sec(\theta)$, we multiply both numerator and denominator by $\sec(\theta) + \tan(\theta)$. All in all, this gives

$$\left(\frac{1}{2}\sec(\theta)\tan(\theta)\right)\Bigg|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec(\theta)\, d\theta = \frac{1}{2}\sqrt{2} + \int_0^{\pi/4} \frac{\sec^2\theta + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\, d\theta.$$

Using one final substitution, this time taking the dummy variable $q = \sec(\theta) + \tan(\theta)$, we get

$$\begin{aligned} &= \frac{1}{2}\sqrt{2} + \frac{1}{2}\int_1^{\sqrt{2}+1} \frac{1}{q}\, dq \\ &= \frac{1}{2}\sqrt{2} + \left(\frac{1}{2}\ln|q|\right)\Bigg|_1^{\sqrt{2}+1} \\ &= \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1 + \sqrt{2}\right) \end{aligned}$$

This is our final surface area.

Remark: It is possible to use a different evaluation technique, where instead of substituting tangent, one instead substitutes a hyperbolic trigonometric function. The hyperbolic sine and cosine are given by

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2}. \end{aligned}$$

Similar to how sin and cos are defined on the circle $x^2 + y^2 = 1$, with sin being the y-value and cos being the x-value, sinh and cosh emerge from the hyperbola $x^2 - y^2 = 1$, with sinh being the y-value and cosh being the x-value. The corresponding “Pythagorean” identity for sinh and cosh is then

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Do you think you can rework the solution by using this identity for your substitution?