

Spectral Theory

Problem ([Con90, Exercise IX.1.2]): Show that the unit ball of $B(H)$ is WOT-compact.

Solution: Consider the set

$$K = \prod_{x,y \in B_H} \overline{\mathbb{D}},$$

where \mathbb{D} represents the complex unit disk and B_H denotes the closed unit ball of H . The space K is compact by Tychonoff's theorem. Let $\phi: B_{B(H)} \rightarrow K$ be defined by

$$\phi(T) = (\langle Tx, y \rangle)_{x,y}.$$

Observe that by Cauchy–Schwarz, we have that

$$\begin{aligned} |\langle Tx, y \rangle| &\leq \|T\|_{\text{op}} \|x\| \|y\| \\ &\leq 1, \end{aligned}$$

so ϕ is indeed well-defined. Furthermore, ϕ is injective since for any two operators T and S , we have that $T = S$ if and only if $\langle Tx, y \rangle = \langle Sx, y \rangle$ for all $x, y \in B_H$, and ϕ is continuous by the definition of the weak operator topology. Therefore, we only need to show that ϕ has a closed range.

Let $(T_i)_i$ be a net of operators in $B_{B(H)}$ such that

$$\lim_{i \in I} (\langle T_i x, y \rangle)_{x,y} = (z_{x,y})_{x,y}.$$

Then, from the Cauchy–Schwarz inequality, it follows that $(z_{x,y})_{x,y} \in K$, and by the definition of convergence in the product topology, we have, for each x, y ,

$$\lim_{i \in I} \langle T_i x, y \rangle = z_{x,y}.$$

Therefore, we may define a semidefinite sesquilinear form $F: H \times H \rightarrow \mathbb{C}$ given by

$$F(x, y) = \lim_{i \in I} \langle T_i x, y \rangle$$

for each $x, y \in H$. From the structure of sesquilinear forms, it then follows that there is some $T \in B_{B(H)}$ such that $F(x, y) = \langle Tx, y \rangle$, and thus $(T_i)_{i \in I} \rightarrow T$ in WOT.

Problem ([Con90, Exercise IX.1.13]): A representation $\rho: A \rightarrow B(H)$ is *irreducible* if the only projections in $B(H)$ that commute with every $\rho(a)$ for $a \in A$ are 0 and 1. Prove that if A is abelian and ρ is an irreducible representation, then $\dim(H) = 1$. Find the corresponding spectral measure.

Solution: Since A is abelian, so too is $\rho(A)$, meaning that $\rho(A) \subseteq \rho(A)'$. Since $\rho(A)' = \mathbb{C}1$ by the assumption of irreducibility, it follows then that $\rho(A) = \mathbb{C}1$, whence $H = [\rho(A)v] = \mathbb{C}v$.

Without loss of generality, we may assume that $A = C_0(X)$ for some locally compact Hausdorff space X , and $\rho: C_0(X) \rightarrow \mathbb{C}$ is a character. The characters of $C_0(X)$ are given by evaluation at $x_0 \in X$, meaning that their corresponding spectral measure is the Dirac mass δ_{x_0} .

Problem ([Con90, Exercise IX.2.1]): Show that $\lambda \in \sigma_p(N)$ if and only if $E(\{\lambda\}) \neq 0$. Moreover, if $\lambda \in \sigma_P(N)$, then $E(\{\lambda\})$ is the orthogonal projection onto $\ker(N - \lambda I)$.

Solution: Let $\lambda \in \sigma_p(N)$. We start by assuming that $\lambda \neq 0$. We have $\ker(N - \lambda I) \neq 0$, and let P_λ be the projection onto this subspace. We will show that $E(\{\lambda\}) = P_\lambda$. To see that $E(\{\lambda\})H \subseteq P_\lambda H$, we see that for all $x, y \in H$,

$$\langle (N - \lambda I)E(\{\lambda\})x, y \rangle = \int_{\sigma(N)} (z - \lambda) \mathbb{1}_{\{\lambda\}} dE_{x,y}$$

$$= 0,$$

so that $E(\{\lambda\})H \subseteq P_\lambda H$. Meanwhile, since we assume $\lambda \neq 0$, we have, for any $x \in \ker(T - \lambda I)$,

$$\begin{aligned} Tx &= \lambda x \\ &= \left(\int_{\sigma(N)} \lambda dE \right) x \\ &= \left(\int_{\sigma(N)} \mathbb{1}_{\{\lambda\}} z dE \right) x \\ &= E(\{\lambda\})Tx \\ &= E(\{\lambda\})(\lambda x) \\ &= \lambda x, \end{aligned}$$

so that $E(\{\lambda\})x = x$, whence $E(\{\lambda\}) = P_\lambda$. If $\lambda \neq 0$, we shift the spectrum of N by subtracting a multiple of I , and shift it back to obtain the same result.

In particular, this means that $E(\{\lambda\})$ is nonzero if and only if P_λ is nonzero, meaning that $\lambda \in \sigma_p(N)$ if and only if $E(\{\lambda\}) \neq 0$.

Problem ([Con90, Exercise IX.2.10]): Let A be a hermitian operator with spectral measure E on a separable space. For each real number t , define a projection $P(t) = E(-\infty, t)$. Show:

- (a) $P(s) \leq P(t)$ for $s \leq t$;
- (b) if $t_n \leq t_{n+1}$ and $(t_n)_n \rightarrow t$, then $P(t_n) \rightarrow P(t)$ in SOT;
- (c) for all but a countable number of points t , $P(t_n) \rightarrow P(t)$ in SOT if $(t_n)_n \rightarrow t$;
- (d) for any $f \in C(\sigma(A))$, we have

$$f(A) = \int_{-\infty}^{\infty} f(t) dP(t),$$

where the integral is defined in the Riemann–Stieltjes sense.

Solution: Let $x \in H$. Then, we observe that

$$\begin{aligned} \langle P(s)x, x \rangle &= \int_{\sigma(A)} \mathbb{1}_{(-\infty, s)} dE_{x,x} \\ &= \int_{\sigma(A) \cap (-\infty, s)} dE_{x,x} \\ &\leq \int_{\sigma(A) \cap (-\infty, t)} dE_{x,x} \\ &= \langle P(t)x, x \rangle. \end{aligned}$$

Since x is arbitrary, and the measure $E_{x,x}$ is real by the assumption that A is hermitian, it follows that $P(s) \leq P(t)$ whenever $s \leq t$. This shows (a).

To show (b), we recall that a sequence of projections $(P_n)_n \rightarrow P$ in SOT if and only if it converges in WOT, as

$$\begin{aligned} \langle (P_n - P)x, x \rangle &= \langle (P_n - P)^2 x, x \rangle \\ &= \langle (P_n - P)x, (P_n - P)x \rangle \\ &= \|(P_n - P)x\|^2. \end{aligned}$$

References

- [Con90] John B. Conway. *A Course in Functional Analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.