Dijkstra's Algorithm Modification

In class, we gave a version of Dijkstra's algorithm for finding the distance between a vertex u and all the other vertices in a weighted graph. Appropriately modify that algorithm such that it not only outputs $d(u, v)$, but also outputs the shortest u, v path.

We will commence the algorithm as follows:

Input A weighted graph, G, and $u \in V(G)$

Output For each $z \in V(G)$, the distance $d(u, z)$, and the u, v-geodesic.

Initialization Extend the weight function such that if $xy \notin E(G)$, then $w(xy) = \infty$. Create S that contains all vertices whose distances from u are known. Let $S := \{u\}$. Let $t : V \to \mathbb{R}^+ \cup \{0, \infty\}$ which will keep track of the tentative distance between u and z. Let $t(z) := w(uz)$ for all $z \neq u$, and $t(u) := 0$. Additionally, create a path P_z for z, starting at u and ending at z. We will let P'_z be the tentative path.

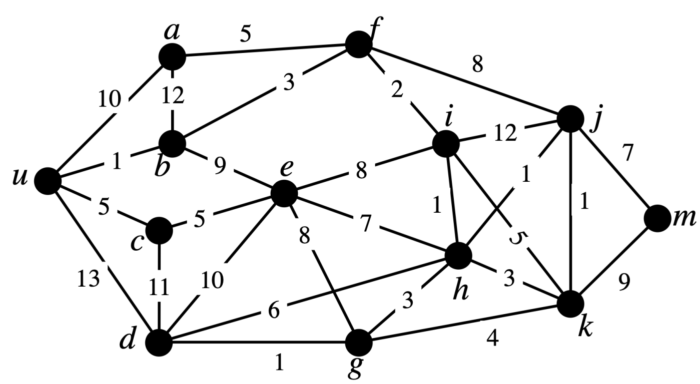
Condition to Terminate Loop If $t(z) = \infty$ for all $z \notin S$ or $S = V$, then go to end.

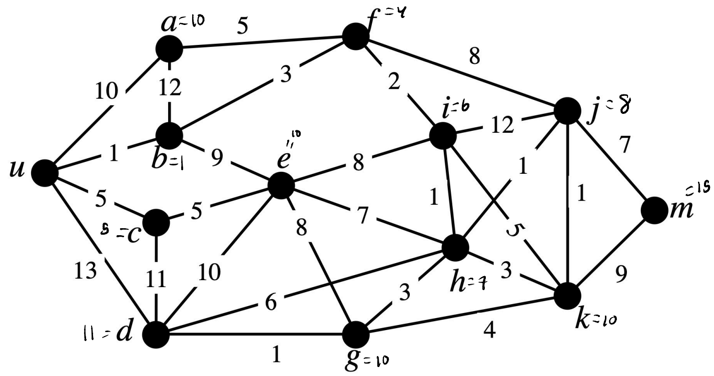
Loop Else, pick $v \in V - S$ such that $t(v) = \min_{z \notin S} t(z)$. Add v to S. Explore the edges from v to update tentative distances; for each edge vz with $z \notin S$, $t(z) := \min\{t(z), t(v) + w(vz)\}$. Add z to P'_v.

End Set $d(u, v) = t(v)\ \forall v \in V$, and set P_v to P'_v for all $v \in V$.

Modified Dijkstra's Algorithm, Example

Perform the modified Dijkstra's algorithm on the following graph:





Paths:

- $P_a = u, a$
- $P_b = u, b$
- $P_c = u, c$
- $P_d = u, b, f, i, h, g, d$
- $P_e = u, b, e$
- $P_f = u, b, f$
- $P_g = u, b, f, i, h, g$
- $P_h = u, b, f, i, h$
- $P_i = u, b, f, i$
- $P_j = u, b, f, j$
- $P_k = u, b, f, j, k$
- $P_m = u, b, f, j, m$

2.3.1

Assign integer weights to the edges of K_n. Prove that the total weight of every cycle in K_n is even if and only if the total weight of each triangle is even.

($\Rightarrow$) Suppose there exists a triangle in K_n whose weight is odd. Then, this triangle is a cycle in K_n whose weight is odd, meaning that it is not the case that the total weight of every cycle in K_n is even.

($\Leftarrow$) Suppose toward contradiction that K_n has a cycle of odd weight. Then, we will show that there exists a triangle of odd weight in K_n.

Base Case: If $n = 3$, then K_n containing a cycle of odd weight means K_3, the triangle graph, is of odd weight, so K_3 contains a triangle of odd weight. Note that this cycle contains either 1 or 3 edges of odd weight.

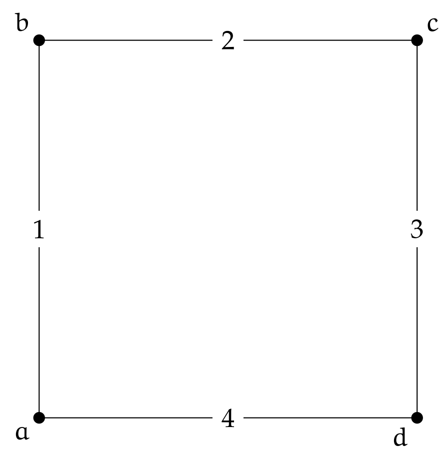
Inductive Hypothesis: If K_n contains a cycle of odd weight, then K_{n-1} for some deleted vertex contains a cycle of odd weight, meaning that K_{n-1} has a triangle of odd weight.

Proof: For any cycle with odd weight, there must be an odd number of edges in the cycle of odd weight. Thus, for any vertex with an *even* number of edges with odd weight incident on it, deletion maintains the parity of the total number of edges with odd weight. So, K_{n-1} for the deleted vertex will still contain an odd cycle, and thus by the inductive hypothesis, K_{n-1} will have a triangle of odd weight.

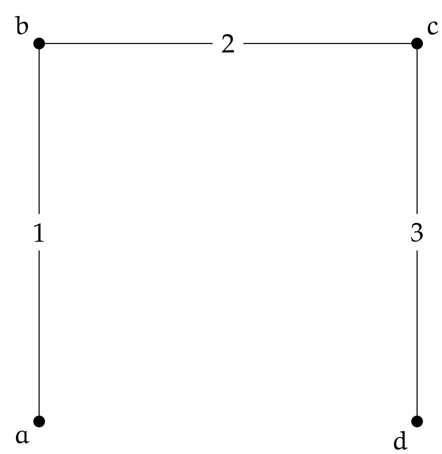
2.3.2

Prove or disprove: If T is a minimum weight spanning tree of a weighted graph G, then the u, v path in T is the minimum weight u, v path in G.

This assertion is **false**, as we can see in the following graph:



The minimum weight spanning tree is as follows:



But, we can see that the shortest weight path between a and d in G is of weight 4, while it is of weight 6 in the minimum weight spanning tree.

2.3.5

There are five cities in a network. The travel time for travelling directly from i to j is the entry $a_{i,j}$ in the matrix below. The matrix is not symmetric, and $a_{i,j} = \infty$ indicates there is no direct route. Determine the least travel time and quickest route for each pair i, j.

$$\begin{pmatrix} 0 & 10 & 20 & \infty & 17 \\ 7 & 0 & 5 & 22 & 33 \\ 14 & 13 & 0 & 15 & 27 \\ 30 & \infty & 17 & 0 & 10 \\ \infty & 15 & 12 & 8 & 0 \end{pmatrix}$$

Using Dijkstra's algorithm, we are able to find the optimal travel time as seen in the following table (where the column entry represents i and the row entry represents j).

(i, j)	1	2	3	4	5
1	0	10	15	25	17
2	7	0	5	14	24
3	14	13	0	15	25
4	30	25	17	0	17
5	22	15	12	8	0

2.3.7

Let G be a weighted connected graph with distinct edge weights. Without using Kruskal's algorithm, prove that G has only one minimum weight spanning tree.

Let $E_T = \{e_1, \dots, e_k\}$ be the edge set of the minimum weight spanning tree of $T \subseteq G$. Suppose there exists a T' such that $w(T') \leq w(T)$. Then, by the result from Exercise 2.1.37, we know that we can create T' from T by subtracting one edge in G and adding a different edge in G, or that $T' = T - e + e'$.

Since all the edges of G are of distinct weight, then $w(e) \neq w(e')$, meaning that $w(T') \neq w(T)$. Therefore, it must be the case that the inequality is sharp, or that $w(T') < w(T)$ — however, since we had assumed that T was a minimum weight spanning tree, this means T' cannot exist (or else it would be a minimum weight spanning tree), so there is only one minimum weight spanning tree in G.

2.3.13

Let T be a minimum weight spanning tree and let T' be another spanning tree in G. Prove that T' can be turned into T by a list of steps that exchange one edge of T' for one edge of T such that the edge set is always a spanning tree and the total weight never increases.

Let $E(T')$ denote the edge list in T' and let $E(T)$ denote the edge list in T. By Theorem 2.1.7, we know that for any edge $e' \in E(T') - E(T)$, there exists an edge $e \in E(T) - E(T')$ such that $T' + e - e'$ is also a spanning tree of G. In the case where T is a minimum weight spanning tree of G, we know that T' must have a weight greater than or equal to T, and that any edge e' that is exchanged for e, $w(e') \geq w(e)$. So, at the end of the program (where we have exchanged every $e' \in E(T') - E(T)$ for every $e \in E(T) - E(T')$), we know that any edge exchange must either reduce the total weight of T' or keep the total weight the same.