Complex Numbers

A complex number is an ordered pair of real numbers, $(a, b) = a + bi$. A vector in $\mathbb{R}^2$ is also an ordered pair, (a, b) of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with $\mathbb{R}^2$. However, unlike vectors in $\mathbb{R}^2$, there is also an operation $\cdot$. We desire for $(0, 1) \cdot (0, 1) = (-1, 0)$; essentially, $i^2 = -1$. We say that i is a square foot of -1; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di)\\ &:= a(c) + adi + bci + bd(i^2)\\ &:= (ac - bd) + (ad + bc)i\\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus, $\mathbb{R}^2$ with the operations $+$ and the above defined complex multiplication is known as $\mathbb{C}$. We write as $a+bi$ instead of (a, b).

Given $z = (a + bi) \in \mathbb{C}$, we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$. If $\text{Im}(z) = 0$, then $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$. However, many people say that $\mathbb{R} \subseteq \mathbb{C}$, even if $\mathbb{C}$ isn't defined as such.

Reciprocals of Complex Numbers

Let $z \in \mathbb{C}$, where $z \neq 0$. Then, $\exists w \in \mathbb{C}$ such that $zw = 1$.

Let $w = c + di$. We want to show that $zw = 1$.

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1\\ ad + bc &= 0.\end{aligned}$$

Thus, let $w = c + di$, with $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2}\\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every $z \neq 0$, with $z = a + bi$, the *reciprocal* of z is defined as $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Then, for $w \in \mathbb{C}$, we define

$$\frac{w}{z} := w\left(\frac{1}{z}\right).$$

Properties of Complex Numbers

Let $z = a + bi \in C$. Then, the (Euclidean) norm (or absolute value) of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

(i) $z\overline{z} = |z|^2$

(ii) $\overline{(\overline{z})} = z$

(iii) $\overline{(z+w)} = \overline{z} + \overline{w}$

(iv) $\overline{zw} = \overline{z} \cdot \overline{w}$

(v) $z + \overline{z} = 2\text{Re}(z)$, so $\text{Re}(z) = \frac{z+\overline{z}}{2}$

(vi) $z - \overline{z} = 2\text{Im}(z)i$, so $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

Polar Representation

Let $z = a+bi$ (or $z = (a, b)$). Then, $|z| = \sqrt{a^2 + b^2}$ is the *radius*, and the *argument* is found by $\theta = \arctan(b/a)$ for $a \neq 0$. Therefore, the full polar representation is as follows:

$$z = |z|\left(\cos\theta + i\sin\theta\right). \qquad \theta \in [0, 2\pi)$$

If $z = 0$, then $|z| = 0$, and $\arg z$ is undefined.

For example, we can find $\arg i$ in $[\pi, 3\pi)$ as $\frac{5\pi}{2}$.

For z_1 and z_2 in polar form, we have:

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| && (1) \\ \arg(z_1 z_2) &= \arg z_1 + \arg z_2 \mod 2\pi && (2) \end{aligned}$$

Proof of (1):

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)\overline{(z_1 z_2)} \\ &= z_1 z_2 \overline{z_1}\,\overline{z_2} \\ &= z_1 \overline{z_1} z_2 \overline{z_2} \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

Since $|z| \geq 0$, we get $|z_1 z_2| = |z_1||z_2|$.

Let $z = 2(\cos\pi/6 + i\sin\pi/6)$, and let $f : \mathbb{C} \to \mathbb{C}$ defined as $f(w) = zw$. Then, f rotates w by $\pi/6$ and scales w by 2.

Theorem: For $n \in \mathbb{N}$, if $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Proof: Induct on n. For the base case, we know that $n = 1$ satisfies this property. For $n > 1$, we have:

$$\begin{aligned} z^{n+1} &= (z^n)(z) \\ &= \left(r^n(\cos(n\theta) + i\sin(n\theta))\right) r(\cos\theta + i\sin\theta) \\ &= (r^n)(r)\left(\cos(n\theta+\theta) + i\sin(n\theta+\theta)\right) && \text{Polar Representation Definition} \\ &= r^{n+1}(\cos\left((n+1)\theta\right) + i\sin((n+1)\theta)) \end{aligned}$$

We can use this technique to find the "roots of unity." For example, to find all z such that $z^3 = 1$, we use our technique:

$$\begin{aligned}
z^3 &= 1\\
|z| &= 1\\
\arg z^3 &= 0\\
3\arg z &= 0 \mod 2\pi\\
\arg z &= \frac{k2\pi}{3}\\
&= 0, \frac{2\pi}{3}, \frac{4\pi}{3}\\
z_1 &= 1\\
z_2 &= (\cos 2\pi/3 + i\sin 2\pi/3)\\
z_3 &= (\cos 4\pi/3 + i\sin 4\pi/3)
\end{aligned}$$

We can see that $z_2^2 = z_3$.

For the n case, we find $z_2 = \cos(2\pi/n) + i\sin(2\pi/n)$, and $z_k = z_2^{k-1}$.

Exponential, Logarithm, and Trigonometric Functions in $\mathbb{C}$

Exponential

Let $z = a + bi$. We define e^{a+bi} as follows:

$$e^{a+bi} = e^a\left(\cos b + i\sin b\right)$$

Recall that for every nonzero complex number, $z = |z|\left(\cos\theta + i\sin\theta\right)$, where $\theta = \arg z$. Thus,

$$\begin{aligned}
z &= |z|e^{i\theta}\\
&= |z|e^{i\arg z}.
\end{aligned}$$

The function e^z has some properties similar to the function e^x in real numbers, and some properties varying with the real numbers.

$$\begin{aligned}
e^z e^w &= e^{z+w}\\
e^z &\neq 0
\end{aligned}$$

However, there are some differences:

$$\begin{aligned}
|e^{i\theta}| &= 1 && \forall\theta\\
e^{a+bi} &= e^a
\end{aligned}$$

From these properties, we find Euler's equation:

$$e^{i\pi} + 1 = 0$$

Additionally, e^z is periodic, while $f(x) = e^x$ is injective:

$$\begin{aligned}
e^{z+2n\pi} &= e^z\left(\cos(2n\pi) + i\sin 2n\pi\right)\\
&= e^z
\end{aligned}$$

When examining the function $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, $z \mapsto e^z$, we find that the following happen:

- $f(\mathbb{R}) = (0, \infty)$ — we apply $f(x) = e^x$.
- $f(a + bi) = e^a e^{bi}$ — e^a is rotated by b.
- $f(\mathbb{R} + bi)$ is expressed as the line along b radians through the origin.
- Therefore, $f(A_0) = \mathbb{C} \setminus \{0\}$, where $A_0 = \{a + bi \mid a \in \mathbb{R}, b \in [0, 2\pi)\}$.

Logarithm

Recall that for a function $f : A \to B$, f^{-1} is a function if f is injective. However, for any f, it is the case that $f^{-1}(b)$ does exist, defined as follows:

$$f^{-1}(b) = \{a \mid f(a) = b\}.$$

For the function $f(z) = e^z$, f is not one to one, so for $w = e^z$, $f^{-1}(w) = \{z' \in \mathbb{C} \mid e^{z'} = w\}$. We can find this as $f^{-1}(w) = \{z + 2n\pi i \mid n \in \mathbb{Z}\}$.

We define $\log(w) := \{z \in \mathbb{C} \mid e^z = w\}$. For a fixed $\theta \in \mathbb{R}$, we define $\log_{A_\theta}(w) := \{z \mid e^z = w, z \in A_\theta\}$.

Let $z = 1 + \frac{5\pi}{2} i$. Then,

$$\log_{A_{-\pi}} e^z = 1 + \frac{\pi}{2} i$$

Let $w \in \mathbb{C} \setminus \{0\}$. To find $\log w$ (all values), then

$$\begin{aligned} z &\in \log w \\ e^z &= w \\ &= |w| e^{i \arg w} \\ e^{a+bi} &= |w| e^{i \arg w} \\ e^a e^{ib} &= |w| e^{i \arg w}. \end{aligned}$$

Therefore, $a = \ln |w|$ and $b = \arg w$. Additionally, the following hold, for $z_1, z_2 \in \mathbb{C}$:

$$\log_{A_\theta}(z_1 z_2) = \log_{A_\theta}(z_1) + \log_{A_\theta}(z_2) + 2n\pi i$$

Cosine and Sine

$$\begin{aligned} e^{ib} &= \cos b + i \sin b \\ e^{-ib} &= \cos b - i \sin b \\ \cos z &:= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &:= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Complex Powers

Recall that for $s, t \in \mathbb{R}$, $s^t = e^{t \ln s}$, where $s > 0$. For $z, w \in \mathbb{C}$, $z^w = e^{w \log z}$., where $z \neq 0$.

$$\begin{aligned} (-2)^i &= e^{i \log(-2)} \\ &= e^{i(\ln(2) + i\pi)} \\ &= e^{i \ln 2 - (\pi + 2\pi n)} \\ &= e^{-\pi + 2\pi n + i \ln 2} \end{aligned}$$

This has *infinitely* many values.

Let $\alpha = u + vi$. Then,

$$\begin{aligned} z^{\alpha} &= e^{\alpha \log z} \\ &= e^{(u+vi)(\ln|z| + i \arg z)} \\ &= e^{(u \ln|z| - v \arg z)} e^{i(v \ln|z| + u \arg z)} \end{aligned}$$

Since $\arg z = \theta + 2\pi n$ for some real $\theta \in [0, 2\pi)$,

$$= e^{u \ln z} e^{-v(\theta + 2\pi n)} e^{iv \ln|z|} e^{iu(\theta + 2\pi n)}$$

Therefore, complex exponentiation is single-valued if $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Z}$, then z^{α} has only one value; if $\alpha \in \mathbb{Q}$, where $\alpha = \frac{p}{q}$ and $\gcd(p, q) = 1$, then z^{α} takes q distinct values, which are the qth-roots.

Continuous Functions with Complex Domains

Let $z \in \mathbb{C}$, let $r > 0$.

- The set $D(z; r) := \{w \mid w \in \mathbb{C}, |z - w| < r\}$ is the r-neighborhood of z.
- A subset $A \subseteq \mathbb{C}$ is open if $(\forall z \in A)\,(\exists r > 0) \ni D(z; r) \subseteq A$.

For example, if $A = \{z \mid \mathrm{Re}(z) > 0\}$, we can find r equal to half the magnitude of the real component of z for any $z \in A$, meaning A is open.

Meanwhile, if $A = \{z \mid \mathrm{Re}(z) \geq 0\}$, this is not the case. If $z = 0$, then $\nexists r > 0$ such that $D(z; r) \subseteq A$, as any open ball of radius r will have some element in $\overline{A}$.

- A subset $B \subseteq \mathbb{C}$ is closed if $\overline{B} \subseteq \mathbb{C}$ is open.

For example, $A = \emptyset$ is open, by vacuous truth, so $\overline{A} = \mathbb{C}$ is closed. Similarly, since $\mathbb{C}$ is open, $\emptyset$ is closed.

Meanwhile, $A = \{x + iy \mid -1 \leq x < 1\}$ is neither open nor closed.

Limits

Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, $z_0 \in \mathbb{C}$. Then,

$$\lim_{z \to z_0} f(z) = \ell$$

means both of the following hold:

(i) for some $r > 0$, $D(z_0; r) \setminus \{z_0\} \subseteq \mathrm{dom}(f)$,

(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $f(D(z_0; \delta) \setminus \{z_0\}) \subseteq D(\ell; \varepsilon)$.

For example, if

$$f(z) = \begin{cases} z & z \in \mathbb{C} \setminus \mathbb{R} \\ 3i & z \in \mathbb{R} \end{cases}$$

Then, $\lim_{z \to 0} f(z)$ does not exist, as there is no ℓ that satisfies both conditions. Specifically, if $\ell = 3i$, and we set $\varepsilon = 1$, then a disc of any radius around 0 has some $z \in \mathbb{C} \setminus \mathbb{R}$ that maps to itself. Similarly, if we set $\ell = 0$, then there is a real number in a disc of any radius around 0.

Note: f does not have to be defined at z_0 for the limit to be defined at z_0.

Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$, and $z_0 \in A$. We say f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$. We say f is continuous on A if $\forall z_0 \in A$, f is continuous at z_0.

We will show that $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto 3z$ is continuous.

Scratch Work: We want δ such that $f(D(z_0; \delta)) \subseteq D(3z_0; \varepsilon)$. Let $z \in D(z_0; \delta)$, meaning $f(z) = 3z$. We want $3z \in D(3z_0; \varepsilon)$, meaning we want $|3z - 3z_0| < \varepsilon$, or $|z - z_0| < \frac{\varepsilon}{3}$.

Proof: Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{3}$. We show $f(D(z_0; \delta)) \subseteq D(f(z_0); \varepsilon)$. Let $z \in D(z_0; \delta)$. Then, $|z - z_0| < \delta = \varepsilon/3$, meaning $3|z - z_0| < \varepsilon$, meaning $|3z - 3z_0| < \varepsilon$, so $|f(z) - f(z_0)| < \varepsilon$. Therefore, $f(z) \in D(f(z_0); \varepsilon)$. Since f is continuous at arbitrary z_0, f is continuous on $\mathbb{C}$.

Sequences

A sequence $z_1, z_2, \dots \in \mathbb{C}$. A sequence converges to $z_0 \in \mathbb{C}$ if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M},\ |z_n - z_0| < \varepsilon$$

In words, for any radius around z_0, we can find z_n arbitrarily close to z_0 for sufficiently large n. We write $z_n \to z_0$ if this is the case.

Let $f : \mathbb{C} \to \mathbb{C}$. Then, f is continuous on $\mathbb{C}$ if and only if the following equivalent conditions are met:

(i) the inverse image of every open set is open ($f^{-1}(B) := \{a \in \mathbb{C} \mid f(a) \in B\}$);

(ii) the inverse image of every closed set is closed;

(iii) for every sequence $(z_n)_n$ such that $(z_n)_n \to z_0$, $f(z_n) \to f(z_0)$.

Let

$$f(z) = \begin{cases} 0 & z = 0 \\ 1 & z \neq 0 \end{cases}.$$

This function is not continuous. We will check that (i)–(iii) fail.

(i) Let $B = D(0; 1)$. Then, $f^{-1}(B) = \{0\}$, which is not open set.

(ii) Let $B = \text{cl}(D(1; 0.5))$. Then, $f^{-1}(B) = \mathbb{C} \setminus \{0\}$, which is not closed.

(iii) Let $z_n = \frac{1}{n}$. Then, $(z_n)_n \to 0$, but $f(z_n) = 1$ for all n, meaning $f(z_n) \to 1 \neq f(0)$.

To show limit divergence, recall the definition of limit convergence:

$$\lim_{n \to \infty} z_n = z_0 \Leftrightarrow (\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M},\ |z_n - z_0| < \varepsilon.$$

Let $z_1, \dots, \in \mathbb{C}$ be a sequence. Then, $\lim_{n \to \infty} = \infty$ means

$$(\forall M > 0)(\exists N \in \mathbb{N}) \ni \forall n > N,\ |z_n| > M.$$

In words, $|z_n|$ is arbitrarily large for sufficiently large n.

Connected Sets

Let $a, b \in \mathbb{C}$. A path from a to b is a continuous function $p : [0, 1] \to \mathbb{C}$ such that $p(0) = a$ and $p(1) = b$. Let $S \subseteq \mathbb{C}$. If $p([0, 1]) \subseteq S$, then p is a path in S.

We say S is *path-connected* if for any $s, t \in S$, there is a path in S from s to t.

Every set that is path-connected is connected, but not necessarily the other way around — if A is open and path connected, then A is connected.

An open, path-connected subset of $\mathbb{C}$ is known as a region, or a domain.

Let $A = \mathbb{R} \times \{0\}$ (or the x axis in $\mathbb{C}$). A is not a region, as A is not an open set, even if A is path-connected.

$A \subseteq \mathbb{C}$ is bounded if there exists $r > 0$ such that $A \subseteq D(0; r)$. $A = \mathbb{R} \times \{0\}$ is not bounded.

If $A \subseteq \mathbb{C}$, then A is compact if A is closed and bounded. There are various properties of compact sets that make them particularly amenable towards analysis.

Extreme Value Theorem: Every real-valued continuous function on a compact domain attains its maximum and minimum values.

Uniform Continuity Theorem: Elaborated below.

Uniform Continuity

Recall that if $f : A \to \mathbb{C}$, f is continuous if $\forall a \in A$, $\lim_{z \to a} f(z) = f(a)$.

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta_a > 0) \ni f(D(a; \delta_a)) \subseteq D(f(a); \varepsilon) \qquad \delta \text{ depends on } a$$

When f is uniformly continuous, there is one value of δ, dependent on ε, that applies for every value of a.

$$(\forall \varepsilon > 0)(\exists \delta_\varepsilon > 0) \ni (\forall a \in A), f(D(a; \delta_\varepsilon)) \subseteq D(f(a); \varepsilon)$$

Riemann Sphere

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2\}$. Let $N = (0, 0, 1)$ denote the north pole. Then, there is a continuous bijection from $S^2 \setminus \{N\} \to \mathbb{C}$.

We can visualize this by picking a random point on the sphere and drawing a line from the north pole through the sphere to this point, and finding the point that intersects the plane.

Consider the sequence $z_n = n^2 i$ for $n = 1, 2, \dots$. We can see that, on the projection from z_n to the sphere, all the values of p converge to N. Therefore, we write $\lim_{n \to \infty} z_n = \infty$, where ∞ corresponds to N on S^2.

We can define $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to be the complex plane that includes the "point at infinity" (from the projection on S^2 that corresponds to the north pole).

Analytic Functions

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ where A is open. Let $z_0 \in A$. We say f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Rules of Differentiation

- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$
- $(f \circ g)' = g'(f' \circ g)$
- For $n \in \mathbb{Z}$, $(z^n)' = nz^{n-1}$

Let $f(z) = \overline{z}$. We will find this value by directly applying the definition of the derivative.

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\ &= \lim_{z\to z_0} \frac{\overline{z - z_0}}{z - z_0}. \end{aligned}$$

Let's approach z_0 from the horizontal direction. Suppose $z = z_0 + t$ for some $t \in \mathbb{R}$. Then,

$$\lim_{z\to z_0} \frac{\overline{z_0 + t} - \overline{z_0}}{z_0 + t - z_0} = 1.$$

Let's approach z_0 from the horizontal direction. Suppose $z = z_0 + ti$ for some $t \in \mathbb{R}$. Then,

$$\begin{aligned} \lim_{z\to z_0} \frac{\overline{z_0 + ti} - \overline{z_0}}{z_0 + ti - z_0} &= \frac{-ti}{ti} \\ &= -1. \end{aligned}$$

Since $1 \neq -1$, we find that the limit does not exist.

We see that complex-differentiability is a strong condition.

Suppose that $f'(z_0) = 2i$, meaning

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 2i.$$

If z is close to z_0, then $f(z) - f(z_0) \approx 2i(z - z_0)$. Pictorially, we can visualize this as, for z_0 sufficiently close to z, the vector $z_0 - z$ is akin to a counterclockwise rotation and a scaling by 2. This is applicable for *all* z in sufficient proximity to z_0.

Specifically, we can see that the complex differentiable function is *angle-preserving*. The technical name for f is that f is *conformal*.

Analytic Function

Let $f : A \subseteq C \to \mathbb{C}$. If f is differentiable at every $z_0 \in A$, we say f is *analytic* on A.

If f is analytic on A, then f is infinitely differentiable on A.

If f is analytic on A and $f'(z_0) \neq 0$ for some $z_0 \in A$, then f is conformal at $z_0 \in A$.

Cauchy-Riemann Theorem

Given a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. Recall that we can take partial derivatives, $\frac{\partial f}{\partial x}$, and directional derivative $\frac{\partial f}{\partial u}$ for some unit vector u.

However, for $\mathbb{C}$, there is only one derivative, $f'(z_0)$, meaning that regardless of direction, $f'(z_0)$ exists and has one value. We can contextualize $f(z) = f(x + yi) = u(x, y) + iv(x, y)$, where $u(x, y) \in \mathbb{R}$ and $v(x, y) \in \mathbb{R}$. Then,

$$\frac{\partial u}{\partial x} \neq \frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial x} \neq \frac{\partial v}{\partial y}$$

but

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

We can see this by first letting $z = z_0 + \delta x$.

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z_0 + \delta x) - f(z_0)}{z_0 + \delta x - z_0} \\ &= \lim_{z \to z_0} \frac{u(x_0 + \delta x, y_0) + iv(x_0 + \delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

and in the y direction,

$$\begin{aligned} f'(z_0) &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

We set these two values equal to find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned}$$

which are the Cauchy-Riemann equations. The corresponding theorem states that if $f'(z_0)$ exists, then the Cauchy-Riemann equations must hold.

For example, if $f(z) = \overline{z}$, with $f(x + yi) = x - yi$, we have $u(x, y) = x$ and $v(x, y) = -y$. Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial v}{\partial v} &= -1, \end{aligned}$$

meaning f is not complex-differentiable.

If $f : A \to \mathbb{C}$ satisfies the Cauchy-Riemann equations at every $z_0 \in A$, then f is analytic on A.

If $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is analytic on A, then we know f' and f'' are continuous. From multivariable calculus, we know that $u_{xy} = u_{yx}$ if both are continuous. So,

$$\begin{aligned} u_{xy} &= \frac{\partial}{\partial y}(u_x) \\ &= \frac{\partial}{\partial y}(v_y) \\ &= v_{yy} \\ u_{yx} &= \frac{\partial}{\partial x}(u_y) \\ &= \frac{\partial}{\partial x}(-v_x) \\ &= -v_{xx} \end{aligned}$$

Therefore, $v_{xx} + v_{yy} = 0$. Similarly, $u_{xx} + u_{yy} = 0$.

If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ If $\varphi_{xx} + \varphi_{yy} = 0$, then we say φ is a harmonic function. Therefore, if f is an analytic function, then both the real and imaginary parts of f are harmonic.

Let $A \subseteq \mathbb{R}^2$. If $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$. Then, u and v are harmonic conjugates if $u + iv$ is an analytic function. Additionally, u and v are harmonic conjugates if and only if they satisfy the Cauchy-Riemann equations.

We may ask if there exists an analytic function f such that $\text{Re}(f) = x^3 - 3xy^2 + y$. Then,

$$\begin{aligned} v_y &= u_x = 3x^2 - 3y^2 \\ -v_x &= u_y = 1 - 6xy. \end{aligned}$$

Therefore, we find $v = -x + 3x^2y - y^3 + c$ through integration. Therefore, we have

$$\begin{aligned} f(z) &= (x^3 - 3xy^2 + y) + i(3x^2y - y^3 - x + c) \\ &= (x - iy)^3 + y - ix + ic \\ &= z^3 + i(-iy + x) + ic \\ &= \overline{z}^3 + i\left(\overline{z} + c\right) \end{aligned}$$

Recall from from multivariable calculus that $\nabla u \perp$ contour lines of u. Similarly, $\nabla v \perp$ contour lines of v. Then, using the Cauchy-Riemann equations, we find

$$\begin{aligned} \nabla u \cdot \nabla v &= (-u_x u_y) + u_x u_y \\ &= 0, \end{aligned}$$

meaning the gradients are orthogonal to each other, meaning the contours of u are perpendicular to the contours of v.

Inverse Functions

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$. Let $z_0 \in A$. If f is analytic on A and $f'(z_0) \neq 0$, then f is one to one on some neighborhood of z_0. Then, $f^{-1} : f(N) \to N$ is analytic on $f(N)$, and

$$(f^{-1})'(f(z_0)) = \frac{1}{f'(z_0)}.$$

Derivatives of Elementary Functions

Specifically, we will be working with complex exponentiation, complex trigonometric functions, and complex logarithms.

Complex Exponential

$$\frac{d}{dz}e^z = e^z,$$

since, letting $z = x + iy$,

$$\begin{aligned}
e^z &= e^x e^{iy}\\
&= e^x(\cos(y) + i\sin(y)).\\
\frac{d}{dz}e^z &= \frac{\partial}{\partial x}e^z && \text{treating } y \text{ as constant}\\
&= e^x(\cos(y) + i\sin(y))\\
&= e^{x+iy}\\
&= e^z.
\end{aligned}$$

We know that e^z is continuous on $\mathbb{C}$, but this doesn't imply differentiability at every $z_0 \in \mathbb{C}$. We can verify by checking the Cauchy-Riemann equations, where $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$. Then,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= e^x\cos(y)\\
&= \frac{\partial v}{\partial y}\\
\frac{\partial v}{\partial y} &= -e^x\sin(y)\\
&= -\frac{\partial v}{\partial x}.
\end{aligned}$$

If a function is analytic on $\mathbb{C}$, then f is known as entire.

Complex Logarithm

We might ask where $\log z$ is analytic. Let $f(z) = e^z$. Then, $\log z = f^{-1}(z)$; since f is not one to one, we restrict the domain of f to $A_\theta = \{z \mid \operatorname{Im}(z) \in [\theta, \theta + 2\pi)\}$ for any θ.

Since $f\big|_{A_\theta}$ is one to one, then

$$\left(f\big|_{A_\theta}\right)^{-1} = \log_{A_\theta}.$$

Fixing θ, set $g = f\big|_{A_\theta}$. Then,

$$g^{-1}(g(z)) = z.$$

Because g is analytic on A_θ, g^{-1} is analytic on A_θ. By chain rule, we have

$$\begin{aligned}
\frac{d}{dz}(g^{-1}(g(z))) &= \frac{d}{dz}z\\
g^{-1'}(g(z)) &= \frac{1}{g'(z)} && g'(z) \neq 0\\
g^{-1}(w) &= \frac{1}{g'(z)} && w = e^z\\
&= \frac{1}{e^z}\\
&= \frac{1}{w}.
\end{aligned}$$

Therefore, $\frac{d}{dw}\log_{A_\theta}(z) = \frac{1}{z}$. Therefore, $\operatorname{dom}(\log_{A_\theta}) = \operatorname{ran}(e^z_{A_\theta}) = \mathbb{C} \setminus \{0\}$. However, $\log_{A_0}$ (setting $\theta = 0$) is not even continuous on $\mathbb{C} \setminus \{0\}$!

Specifically, at $z = 0$, $e^z = 1$. Travelling around the unit circle counterclockwise in the image, we see that the preimage of these points travels along the imaginary axis. Approaching 1 "from the bottom," we find that the preimage of the points approaches 2π in the domain. However, they ought to be approaching 0. Therefore, the limit doesn't exist.

However, notice that the domain is not open! To fix this, we will let $B_\theta = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \in (\theta, \theta + 2\pi)\}$.

Our log function *is* when e^z is restricted to B_θ. Then, $\log_{B_\theta}$ is analytic on $\mathbb{C} \setminus \{re^{i\theta} \mid r \geq 0\}$. When $\theta = -\pi$, then $\log_{B_{-\pi}}$ is the principle branch of $\log z$.

Then, the domain is $C \setminus \{z \mid z = x + 0i, x < 0\}$ and the range is $B_{-\pi}$.

Powers

Let $\alpha \in \mathbb{C}$. We might ask

$$\frac{d}{dz}\alpha^z$$
$$\frac{d}{dz}z^\alpha.$$

Recall that $a^b = e^{b \log a}$. Specifically, $a^b = e^{b(\ln|a| + i \arg a)}$.

$$\frac{d}{dz}\alpha^z = \frac{d}{dz}e^{z\log\alpha}$$

Fix θ. Then,

$$\begin{aligned} &= \frac{d}{dz}e^{z\log_{A_\theta}\alpha} \\ &= \log_{A_\theta}\alpha e^{z\log_{A_\theta}\alpha} && \text{assuming analytic domain} \\ &= \alpha^z \log_{A_\theta}\alpha. \end{aligned}$$

Specifically, as long as $\alpha \notin \{re^{i\theta} \mid r \geq 0\}$, $z\log_{A_\theta}\alpha$ is analytic, meaning $e^{z\log_{A_\theta}\alpha}$ is analytic (composition of analytic functions).

$$\begin{aligned} z^\alpha &= e^{\alpha \log z} \\ &= e^{\alpha \log_{B_\theta} z} \\ &= e^{\alpha \log_{B_\theta} z}\frac{\alpha}{z} \\ &= \alpha z^{\alpha - 1}. \end{aligned}$$

Specifically, this holds for $z \notin \{re^{i\theta} \mid r \geq 0\}$.

We know that $\frac{d}{dz}\log_{B_{-\pi}(z)} = \frac{1}{z}$. The domain of $\log_{B_{-\pi}}$ is $\mathbb{C} \setminus (-\infty, 0]$.

Contour Integrals

Recall from multivariable that $\gamma : [a, b] \to \mathbb{R}^n$ is called a curve.

For example, $\gamma : [0, \pi] \to \mathbb{R}^2$, defined as $\gamma(\theta) = (\cos\theta, \sin\theta)$. The image of the given curve is a half circle.

We want to have γ be continuous and differentiable. Then,

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$$

is continuous/differentiable if and only if every γ_i is continuous/differentiable.

$$\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$$

If γ' is continuous, we say γ is smooth. For us, $\gamma \in C^1$ is enough, $\gamma \in C^\infty$ is not necessary.

For $\gamma : [a, b] \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, we define

$$\int_\gamma f := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

as the line integral of f over γ.

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ for A open, where $\gamma : [a, b] \to A$. Then,

$$\begin{aligned}\int_\gamma f &:= \int_a^b f(\gamma(t))\gamma'(t) dt \\ &= \lim_{n\to\infty} \sum_{k=1}^n f(z_k)\Delta z\end{aligned}$$

Rather than the dot product, we use complex multiplication.

To define $\gamma'(t)$, we can imagine it as

$$\begin{aligned}\gamma(t) &= \gamma_1(t) + i\gamma_2(t) \\ \gamma'(t_0) &= \lim_{t\to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \\ &= \gamma_1'(t_0) + i\gamma_2'(t_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_\gamma f &= \int_\gamma \underbrace{f(\gamma(t))\gamma'(t)}_{u(t)+iv(t)} dt \\ &= \int_a^b u(t)dt + i\int_a^b v(t)dt\end{aligned}$$

Let γ be the line from i to 2, and f as $\text{Im}(z)$. Find $\int_\gamma f$.

To solve, we need a formula for $\gamma : [0, 1] \to \mathbb{C}$. We can consider $\gamma(t) = i(1 - t) + 2t$. For any straight line, we can define $\gamma : [0, 1] \to \mathbb{C}$ as $\gamma(t) = p(1 - t) + qt$, or $p + t(q - p)$.

So,

$$\begin{aligned}\int_\gamma f &= \int_0^1 f(\gamma(t))\gamma'(t)dt\\ &= \int_0^1 \operatorname{Im}(2t+i(1-t))(2-i)dt\\ &= (2-i)\int_0^1 (1-t)dt\\ &= (2-i)\left(t-\frac{t^2}{2}\right)\Big|_0^1\\ &= \frac{1}{2}(2-i)\end{aligned}$$

We could also have $\tilde{\gamma} : [0,1] \to \mathbb{C}$, $\tilde{\gamma}(t) = 2t^2 + i(1-t^2)$. The image of $\tilde{\gamma}$ is the same as the image of γ, and (not coincidentally), so is its line integral.

Theorem: Reparametrization

Let $f : A \to \mathbb{C}$ be analytic, $\gamma : [a,b] \to A$ and $\tilde{\gamma} : [\tilde{a},\tilde{b}] \to A$ smooth curves such that $\tilde{\gamma}$ is a reparametrization of γ. Then,

$$\int_\gamma f = \int_{\tilde{\gamma}} f.$$

If $\gamma : [a,b] \to A$, then $\tilde{\gamma}[\tilde{a},\tilde{b}] \to A$ is a reparametrization if $\exists r : [a,b] \to [\tilde{a},\tilde{b}]$ such that $r(a) = \tilde{a}$ and $r(b) = \tilde{b}$, and $\tilde{\gamma} \circ r = \gamma$.

For a quick proof, we look at

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)dt\\ &= \int_a^b f(\tilde{\gamma}\circ r(t))(\tilde{\gamma}\circ r)(t)dt\\ &= \int_a^b f(\tilde{\gamma}\circ r(t))\tilde{\gamma}'(r(t))r'(t)dt\end{aligned}$$

$u = r(t), du = r'(t)dt$

$$\begin{aligned}&= \int_{r(a)}^{r(b)} f(\tilde{\gamma}(u))\tilde{\gamma}'(u)du\\ &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(u))\tilde{\gamma}(t)du\end{aligned}$$

Cauchy's Theorem: A Generalization

Note: I was out of class the previous week so we jumped to this location

So far, we know that if γ is a simple closed curve and f is analytic on and inside γ, then $\int_\gamma f = 0$. However, the theorem is much stronger.

If γ is a closed curve, and f is analytic on $A \subseteq \mathbb{C}$, with γ contained in A, and γ is homotopic to a point in A, then $\int_\gamma f = 0$.

Let $A \subseteq \mathbb{C}$, with $j = 0,1$, and $\gamma_j : [0,1] \to A$ closed curves. We say γ_0 is homotopic in A to γ_1 if there exists continuous $H : [0,1] \times [0,1] \to A$ such that

- $H_t : [0, 1] \to A$ defined by $x \mapsto H(x, t)$ is a closed curve
- $H_0 = \gamma_0$ and $H_1 = \gamma_1$.

If such H exists, we write $\gamma_0 \sim \gamma_1$.

For example, if $\gamma_0(\theta) = e^{2\pi i\theta}$ and $\gamma_3(\theta) = 3e^{2\pi i\theta}$, we can show they are homotopic by using a linear homotopy:

$$H_t(\theta) = (1 - t)e^{2\pi i\theta} + t\left(3e^{2\pi i\theta}\right),$$

which is both continuous and satisfies the given requirements.

In general, for two arbitrary closed curves γ_0 and γ_1, we can't go wrong by trying the linear homotopy $H_t(\theta) := (1 - t)\gamma_0 + t\gamma_1$.

If a closed curve γ is homotopic in A to a point in A (i.e., the curve is homotopic to a constant map), we say γ is null-homotopic.

A set in $\mathbb{C}$ is simply connected if it is path-connected and every closed curve in the set is null-homotopic in the set. A set $A \subseteq \mathbb{C}$ is convex if $\forall z_0, z_1 \in A, t \in [0, 1]$, $tz_1 + (1 - t)z_0 \in A$.

Let $f : A \to \mathbb{C}$, where f is analytic on A. If γ_0 and γ_1 are curves in A such that $\gamma_0 \sim \gamma_1$ in A, then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Consider ρ, a path connecting some point in γ_0 to some point in γ_1 (if they are closed), which exists by the homotopy. Then, $\Gamma := \gamma_0 + \rho - \gamma_1 - \rho$ (where we traverse along γ_0, then ρ, then γ_1, then reverse ρ.) is null-homotopic. So, Cauchy's Theorem implies that

$$\begin{aligned}
\int_{\Gamma} f &= \int_{\gamma_0} + \int_{\rho} f - \int_{\gamma_1} f - \int_{\rho} f \\
&= 0 \\
\int_{\gamma_0} f &= \int_{\gamma_1} f.
\end{aligned}$$

Cauchy's Integral Formula

We know that

$$\int_{\gamma} f(z)dz = 0$$

occurs if one of these conditions is satisfied.

(i) If γ is a simple closed curve and γ is analytic on and inside γ.

(ii) If γ is homotopic in a region R to a point, where f is analytic on R.

(iii) If f has an antiderivative in the region, and γ is a closed curve.

(iv) If γ is closed and contained in a simply connected region R that f is analytic on.

We can also show that

$$\int_{\gamma} \frac{1}{z - z_0} dz = 2\pi i$$

where γ is a simple closed curve and z_0 is contained within the region with boundary γ.

Let f be analytic on a simply connected open set $D \subseteq \mathbb{C}$. Then, for every piecewise smooth closed curve $\gamma \in D$ and every point $z_0 \in D \setminus \operatorname{im}(\gamma)$,

$$\begin{aligned}\int_{\Gamma} \frac{f(z)}{z-z_0}\,dz &= 2\pi i f(z_0)\\ f(z_0) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-z_0}\,dz\end{aligned}$$

For every z_0 inside Γ, $f(z_0)$ is determined by the values of f on Γ.

For an outline of the proof, consider C, a circle of radius $\varepsilon > 0$ centered at z_0. Since $\Gamma \sim C$ in $D \setminus \{z_0\}$, we know that

$$\int_{\Gamma}\frac{f(z)}{z-z_0}\,dz = \int_{C}\frac{f(z)}{z-z_0}\,dz$$

Therefore, on C, $f(z) \approx f(z_0)$ if ε is small. So,

$$\begin{aligned}&\approx f(z_0)\int_C \frac{1}{z-z_0}\,dz\\ &= 2\pi i f(z_0)\end{aligned}$$

For example, we can find

$$\begin{aligned}\int_{|z|=4}\frac{\cos(z)}{(z-\pi)(z-5)}\,dz &= \int_{|z|=4}\left(\frac{\cos z}{z-5}\right)\frac{1}{z-\pi}\,dz\\ &= 2\pi i\frac{\cos(\pi)}{\pi-5}\\ &= \frac{2\pi i}{5-\pi}\end{aligned}$$

Suppose $f(z)$ is continuous on a contour Γ (not necessarily closed). Let

$$g(w) = \int_{\Gamma}\frac{f(z)}{z-w}\,dz.$$

Then, g is defined for all $w \notin \operatorname{im}(\Gamma)$, and g is differentiable at every $w \notin \operatorname{im}(\Gamma)$. In other words, g is analytic on $\mathbb{C}\setminus\operatorname{im}(\Gamma)$. Additionally, g' is also analytic on $\mathbb{C}\setminus\operatorname{im}(\Gamma)$, with

$$\begin{aligned}g'(w) &= \frac{d}{dw}\int_{\Gamma}\frac{f(z)}{z-w}\,dz\\ &= \int_{\Gamma}\frac{d}{dw}\frac{f(z)}{z-w}\,dz\\ &= \int_{\Gamma}\frac{f(z)}{(z-w)^2}\,dz\end{aligned}$$

This is what we use to prove that any complex-differentiable function is infinitely complex-differentiable.

If f is analytic on D, then f' is analytic on D. Since f is analytic, then

$$f(w) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-w}\,dz$$

where Γ is a circle centered at w. So,

$$\begin{aligned} f'(w) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-w)^2}dz\\ &= \frac{1}{2\pi i}\int_{\Gamma}\frac{(f(z)/(z-w))}{z-w}dz \end{aligned}$$

The numerator $\frac{f(z)}{z-w}$ is continuous on Γ because $w \notin \Gamma$, so by the previous theorem, the integral is analytic on $D \setminus \operatorname{im}(\Gamma)$. Therefore, f' is differentiable at w, so f' is analytic on D.

If Γ is a simple closed curve, w is inside Γ, and f is analytic on D with $\Gamma \subseteq D$. Then,

$$\begin{aligned} f'(w) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-w)^2}dz\\ f''(w) &= \frac{2}{2\pi i}2!\int_{\Gamma}\frac{f(z)}{(z-w)^3}dz\\ f^{(n)}(w) &= \frac{n!}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-w)^{n+1}}dz \end{aligned}$$

For example,

$$\int_{|z|=2}\frac{e^{-z}}{(z+1)^3}dz = e^{-2}\pi i$$

If f is continuous on a domain D and $\int_{\Gamma} f = 0$ for every closed Γ in D, then f is analytic on D.

By the path independence theorem, f has an antiderivative F on D. So, F is analytic on D as $F' = f$. Thus, $F^{(n)}$ is analytic for all n, so F' is analytic, meaning f is analytic. The converse does not hold.

Recall that $\varphi(x, y)$ is harmonic on D if $\varphi_{xx} + \varphi_{yy} = 0$. If $f(z) = u(x, y) + iv(x, y)$, then $f' = u_x + iv_x$, or $f' = v_y - iu_y$. If f is analytic, then both u and v are harmonic. Similarly, u_x, v_x are harmonic, and u_y, v_y are harmonic (since the analyticity of f implies that f' is also analytic).

Bounds for Analytic Functions and the Fundamental Theorem of Algebra

Liouville's Theorem: every non-constant entire function is unbounded.

Recall that

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-w)^{n+1}}dz.$$

Suppose that f is analytic on $C_R(z_0) = \{z \mid |z - z_0| = R\}$ and f is bounded on C_R. Then, $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$.

$$\begin{aligned} |f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-w)^{n+1}}dz\right|\\ &= \frac{n!}{2\pi}\left|\int_{C_R}\frac{f(z)}{(z-w)^{n+1}}dz\right| \end{aligned}$$

given $|f(z)| \leq M$,

$$\begin{aligned} \left|\frac{f(z)}{(z-z_0)^{n+1}}\right| &= \frac{|f(z)|}{R^{n+1}}\\ &\leq \frac{M}{R^{n+1}} \end{aligned}$$

So

$$\begin{aligned}|f^{(n)}(z_0)| &\leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi R\\ &= \frac{n!M}{R^n}\end{aligned}$$

To show Liouville's Theorem, by the above result, $|f'(z_0)| \leq \frac{M}{R}$. Since f is entire and M is fixed, we can make R arbitrarily large. So, $|f'(z_0)| = 0$, with z_0 arbitrary. Thus, f is constant.

Fundamental Theorem of Algebra

Every non-constant polynomial has at least one root in the complex plane.

To prove this, suppose $P(z) = a_n z^n + \cdots + a_1 z + a_0$ has no root. Then, $\frac{1}{P(z)}$ is also entire. We have that

$$\begin{aligned}\lim_{|z|\to\infty}\left|\frac{1}{P(z)}\right| &= \lim_{|z|\to\infty}\left|\frac{1}{z^n(a_0/z^n + \cdots + a_n)}\right|\\ &= \lim \frac{1}{|z^n|}\left|\frac{1}{a_0/z^n + \cdots + a_n}\right|\\ &\to 0.\end{aligned}$$

Therefore, there exists M such that for $|z| > M$, $|P(z)| < 1$. Examining $D_M := \{z \mid |z| \leq M\}$. Since $\left|\frac{1}{P(z)}\right|$ is a real-valued continuous function, it attains a maximum value A on D_M since D_M is compact. Thus, $|1/P(z)| \leq \max\{1, A\}$ for all $z \in \mathbb{C}$. Thus, $1/P(z)$ is bounded and entire, meaning $P(z)$ is constant. $\perp$

Extrema of Non-Constant Analytic Functions

Let f be analytic on $A \subseteq \mathbb{C}$ open. Then, $|f|$ admits a local maximum at $z_0 \in A$ only if f is constant.

f has a local maximum at z_0 if $\exists \varepsilon > 0$ such that for all $z \in D_\varepsilon(z_0) := \{z \mid |z - z_0| < \varepsilon\}$, $|f(z)| \leq |f(z_0)|$.

Maximum modulus principle: If f is analytic on a bounded domain D and continuous on ∂D, then f attains its maximum on ∂D.

For example, if $f(z) = z^2 - 1$, then to find the absolute extrema of f on $D_2(0)$ (the closed disk of radius 2 about 0), we know that f attains its absolute extrema on the boundary of $D_2(0)$.

$$\begin{aligned}|f(z)| &= |z^2 - 1|\\ &\leq |z^2| + |1| = 5\\ &\geq |z^2| - |1| = 3\end{aligned}$$

We have that $|f(2)| = 3$ and $|f(2i)| = 5$.

If f is a non-constant, non-zero analytic function on a bounded domain D, f has no local minimum.

Proof: Let $g(z) = \frac{1}{f(z)}$. Since $f(z)$ is non-zero on D, and f is analytic on D, so too is g. Therefore, $|g|$ admits its maximum on ∂D. Since $\max|g| = \min|f|$, $|f|$ attains its minimum on ∂D.

To prove the maximum modulus principle, we use the following lemma:

Lemma: If f is analytic, and $|f|$ is non-constant on a disk $|z - z_0| < r$, then $|f(z_0)|$ is not maximal on D.

Proof of Lemma: Suppose toward contradiction that $|f(z_0)|$ is the maximum of $|f(z)|$. By the hypothesis, there exists $z_1 \in D$ with $|f(z_1)| < |f(z_0)|$. Let Γ be the circle $|z - z_0| = |z_1 - z_0|$. Since f is analytic,

$$2\pi i f(z_0) = \int_{\Gamma} \frac{f(z)}{z - z_0}\,dz.$$

On Γ, $|z - z_0| = |z_1 - z_0|$, so

$$\begin{aligned}
\left|\int_{\Gamma} \frac{f(z)}{z - z_0}\,dz\right| &\leq \int_{\Gamma} \left|\frac{f(z)}{z - z_0}\right| dz\\
&= \int_{\Gamma} \frac{|f(z)|}{|z - z_0|}\,dz\\
&= \frac{1}{|z_1 - z_0|}\int_{\Gamma} |f(z)|\,dz\\
&< \frac{1}{|z_1 - z_0|}\int_{\Gamma} |f(z_0)|\,dz && (*)\\
&= \frac{\ell(\Gamma)|f(z_0)|}{|z_1 - z_0|}\\
&= \frac{2\pi|z_1 - z_0||f(z_0)|}{|z_1 - z_0|}\\
&= 2\pi|f(z_0)|
\end{aligned}$$

$(*)$: There must exist $\varepsilon > 0$ such that for $|z - z_1| < \varepsilon$, $|f(z)| < |f(z_0)|$, since f is continuous and $|f(z_1)| < |f(z_0)|$. Let $\Gamma_1 = \Gamma \cap D(z_1, \varepsilon)$, and $\Gamma_2 = \Gamma \setminus \Gamma_1$. Then,

$$\begin{aligned}
\left|\int_{\Gamma}\right| &= \left|\int_{\Gamma_1} + \int_{\Gamma_2}\right|\\
&\leq \left|\int_{\Gamma_1}\right| + \left|\int_{\Gamma_2}\right|\\
&< \left|\int_{\Gamma_1} |f(z_0)|\right| + \left|\int_{\Gamma_2} |f(z_0)|\right|.
\end{aligned}$$

However, this means $|f(z_0)| < |f(z_0)|$, which is a contradiction.

Alternatively, if $f'(z_0) \neq 0$, then f approximately rotates and stretches or contracts a small disk around z_0. If we draw a line from 0 to $f(z_0)$ through the disk, then there is some point in $\operatorname{im}(f)$ in the disk that has a larger modulus than $f(z_0)$.

Winding Number

Recall the Cauchy Integral Formula: if f is analytic on a simply connected domain D, and Γ is a simple closed curve in D, with z_0 inside Γ, then

$$f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_0}\,dz.$$

There is a generalized version: if f is analytic on any domain D, and Γ is any closed curve that is null-homotopic in D. If $z_0 \notin \Gamma$, then

$$f(z_0) I(\Gamma, z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_0}\,dz,$$

where $I(\Gamma, z_0)$ denotes the winding number of Γ about z_0.

We define

$$I(\Gamma, z_0)\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{z-z_0}dz$$

for $z_0 \notin \Gamma$. We assert that $I(\Gamma, z_0)$ is always an integer.

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-z_0}dz &= f(z_0)I(\Gamma, z_0)\\ \int_{\Gamma}\frac{f(z)}{z-z_0}dz &= \int_{\Gamma}\frac{f(z_0)}{z-z_0}dz\end{aligned}$$

Series and Sequences

A sequence in $\mathbb{C}$ is a function $a : \mathbb{N} \to \mathbb{C}$. We denote $a(n) = a_n$.

A sequence $(a_n)_n$ converges to $L \in \mathbb{C}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$. In other words, $(a_n)_n$ converges to L if a_n is arbitrarily close to L for all sufficiently large n.

A series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to some S if the sequence of partial sums converges to S, where $s_n := \displaystyle\sum_{k=1}^{n} a_n$.

Tests for Convergence and Divergence

Divergence Test: In real numbers, if $\lim_{n\to\infty} x_n \neq 0$, then $\sum x_n$ diverges.

Similarly, in complex numbers, if $\lim_{n\to\infty} |a_n| \to 0$, then $\sum a_n$ diverges.

Ratio Test: Let

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If $L < 1$, then $\sum a_n$ converges, and if $L > 1$, then $\sum a_n$ diverges. If $L = 1$, then the test is inconclusive.

Comparison Test: Given $\sum a_n$ and $\sum b_n$ series. If $|a_n| \leq |b_n|$ for sufficiently large n, and $\sum b_n$ converges, then $\sum |a_n|$ converges (so $\sum a_n$ converges).

Geometric Series: If $a_{n+1}/a_n = c$ for all n, then $\sum a_n = \sum a_0 c^n$, and we say a_n is a geometric series. If $|c| < 1$, then a_n converges, and if $|c| > 1$, then $\sum a_n$ converges.

The partial sums

$$\begin{aligned}s_n &= a_0 + \cdots + a_0 c^n\\ cs_n &= a_0 c + \cdots + a_0 c^{n+1}\\ s_n(1-c) &= a_0 - a_0 c^{n+1}\\ s_n &= \frac{a_0 - a_0 c^{n+1}}{1-c}\\ &= a_0\frac{1-c^{n+1}}{1-c}\\ \lim_{n\to\infty} &= a_0\lim_{n\to\infty}\frac{1-c^{n+1}}{1-c}\\ &= a_0\frac{1}{1-c} && \text{since } |c| < 1\end{aligned}$$

Convergence of Functions

To find for which $z \in \mathbb{C}$ does $\sum \frac{1}{z^n}$ converge, we use the geometric series, meaning $\left|\frac{1}{z}\right| < 1$, meaning $|z| > 1$ is necessary for the series to converge. When $|z| > 1$, the series converges to $\frac{1}{1-(1/z)}$.

Letting $f_n(z) = s_n(z)$, we have that f_n is itself a sequence of functions. Letting $g(z) = \frac{1}{1-(1/z)}$. Then, for each fixed z with $|z| > 1$, we see that $\lim f_n(z) = g(z)$. So, on the set $|z| > 1$, the sequence f_n converges *pointwise* to g.

Let $(f_n)_n$ be a sequence of functions with $f_n : A \to \mathbb{C}$, $A \subseteq \mathbb{C}$. We say $(f_n)_n$ converges pointwise to g on A if $\forall z \in A$, $\forall \varepsilon > 0$, $\exists M \in \mathbb{N}$ such that for all $n \geq M$, $|f_n(z) - g(z)| < \varepsilon$.

We say f_n converges uniformly to g on A if $\forall \varepsilon > 0$, $\exists M$ such that for all $z \in A$ and $\forall n \geq M$, $|f_n(z) - g(z)| < \varepsilon$.

Let $f_n = \sum_{k=0}^{n} \frac{1}{z}$. Does f_n converge to $g(z) = \frac{1}{1-(1/z)}$ uniformly on $|z| > 1$?

We want to show that for some $\varepsilon_0 > 0$, there does not exist M such that $\forall z \in A, \forall n > M, |f_n(z) - g(z)| < \varepsilon_0$. Let $\varepsilon_0 = 1$. Fix $M \in \mathbb{N}$. We will show $\exists z$ with $|z| > 1$ such that for some $n > M$, $|f_n(z) - g(z)| \geq 1$. We have

$$\begin{aligned} |f_n(z) - g(z)| &= \left| \frac{1 - \frac{1}{z^{n+1}}}{1 - \frac{1}{z}} - \frac{1}{1 - \frac{1}{z}} \right| \\ &= \frac{1}{z^{n+1}\left(1 - \frac{1}{z}\right)} \end{aligned}$$

Let $z = 1 + \delta$ for $\delta > 0$ sufficiently small. Then,

$$\geq 1$$

Taylor Series

Recall from Calc II that for $f : \mathbb{R} \to \mathbb{R}$, a Taylor series for f centered at x_0 is

$$T_{x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

If f is infinitely differentiable at x_0, we have that $T_{x_0}(x)$ will converge to f in an interval of convergence about x_0. For a finite-degree polynomial, we have that

$$P_k(x) := \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

approximates f. Specifically, we can see that $P_k^{(j)}(x_0) = f^{(j)}(x_0)$ for $j \leq k$.

We say that $f(z)$ is analytic on z_0 if $f(z)$ is analytic on $D(z_0; \delta)$ for some $\delta > 0$. If $f(z)$ is analytic at z_0, then the Taylor series for $f(z)$ around z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0).$$

If $f(z)$ is analytic on an open disk $D(z_0; r)$, then the Taylor series for $f(z)$ around z_0 converges to $f(z)$ on $D(z_0; r)$, and converges uniformly on $D(z_0; r') \subset D(z_0; r)$.

For example, if $f(z) = (c-z)^{-1}$, we can find a Taylor series for f about 0, and find the disk of convergence.

$$\begin{aligned}
f'(z_0) &= (c-z)^{-2}\\
f''(z_0) &= 2(c-z)^{-3}\\
f^{(3)}(z_0) &= 3!(c-z)^{-4}\\
&\vdots\\
f^{(n)}(z_0) &= n!(c-z)^{-(n+1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
T(f,z_0) &= \sum_{n=0}^{\infty} \frac{n!(c-z_0)^{-(n+1)}}{n!}(z-z_0)^n\\
&= \sum_{n=0}^{\infty} c^{-(n+1)}z^n.
\end{aligned}$$

To find the radius of convergence, we find that f is analytic on $\mathbb{C}\setminus\{c\}$. Thus, $T(f,z_0)$ is convergent about $D(0;|c|)$.

Considering $f(z) = (c-z)^{-1}$ again, we find

$$\begin{aligned}
f(z) &= \frac{1}{c}\frac{1}{1-\frac{z}{c}}\\
&= \frac{1}{c}\sum_{n=0}^{\infty} c^{-n}z^n && \text{true iff } |z/c| < 1\\
&= \sum_{n=0}^{\infty} c^{-(n+1)}z^n.
\end{aligned}$$

To find a Taylor series for $g(z) = (c-z)^{-2}$, we have that $g(z) = f'(z)$, so we can take the Taylor series for f and differentiate it. Since f is analytic on $|z| < |c|$, and g is equal to f', we have that g is convergent on the same disk that f is convergent on.

If f is analytic at z_0, and $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ on some disk $D(z_0;r)$, then $f'(z) = \sum_{n=1}^{\infty} c_n n(z-z_0)^{n-1}$, and this series converges on $D(z_0;r)$. We can also do integration term-by-term.

For example, to find the Taylor series for $f(z) = \operatorname{Log}(z)$ around $z_0 \in \mathbb{C}\setminus(\infty,0]$, we take integrals term-by-term on $g(z) = 1/z$.

$$\begin{aligned}
g(z) &= \frac{1}{z}\\
&= \frac{1}{z_0-(z_0-z)}\\
&= \frac{1}{z_0}\frac{1}{1-\left(1-\frac{z}{z_0}\right)}\\
&= \frac{1}{z_0}\sum_{n=0}^{\infty}\left(1-\frac{z}{z_0}\right)^n.
\end{aligned}$$

We have that the series converges if $|1 - z/z_0| < 1$.

$$\begin{aligned} &= \frac{1}{z_0}\sum_{n=0}^{\infty}\left(\frac{z_0 - z}{z_0}\right) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{z_0^{n+1}}(z - z_0)^n \end{aligned}$$

so,

$$\begin{aligned} f(z) &= \int f'(z)dz \\ &= \int \sum_{n=0}^{\infty}\frac{(-1)^n}{z_0^{n+1}}(z - z_0)^n\ dz \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{z_0^{n+1}}\int (z - z_0)^n\ dz \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{z_0^{n+1}}\frac{(z - z_0)^{n+1}}{n+1} + C \end{aligned}$$

Specifically, $C = \operatorname{Log}(z_0)$. Thus,

$$f(z) = \operatorname{Log}(z_0) + \sum_{n=0}^{\infty}\frac{(-1)^n}{z_0^{n+1}}\frac{(z - z_0)^{n+1}}{n+1}.$$

We have that the radius of convergence in $\mathbb{C}$ is equal to $\operatorname{dist}_{(-\infty,0]}(z_0) = |\operatorname{Im}(z_0)|$.

For f and g with respective Taylor series, we can find their sum relatively easily (coefficient-wise addition), but for fg, we require convolution.

$$\begin{aligned} \left(a_0 + a_1 z + a_2 z^2 + \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + \cdots\right) &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + \cdots \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \end{aligned}$$

Power Series

Recall that the Taylor series for $f(z)$ about z_0 is

$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z - z_0)^j \\ &= \sum_{j=0}^{\infty} c_j (z - z_0)^j \\ c_j &= \frac{f^{(j)}(z_0)}{j!}. \end{aligned}$$

Suppose instead that we start with the sequence $(c_n)_n$. For example, let $c_j = \frac{j^2+1+i}{(2i)^j}$. We may ask if $\sum c_j (z - z_0)^j$ is convergent (and thus the Taylor series for some analytic function about z_0).

If $\sum c_j (z - z_0)^j$ converges for some $z \neq z_0$, then it indeed is. A series of the form $f(z) = \sum c_j (z - z_0)^j$ is known as a power series. However, the function that the power series converges to may not be an elementary function.

For every power series $\sum c_j(z-z_0)^j$, there exists a single value $R \in [0,\infty]$ such that the power series converges on $|z-z_0| < R$ and diverges on $|z-z_0| \geq R$. For every $r < R$, the power series converges *uniformly* on $|z-z_0| < r$. If R is finite, $|z-z_0| = R$ is called the circle of convergence. The power series may converge at some, all, or no points on $|z-z_0| = R$. R is known as the radius of convergence.

Let $c_k = \frac{k^2+1+i}{(2i)^k}$, with $\sum c_k(z-5i)^k$ the series we must find the radius of convergence for. Using the ratio test, we find

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{\frac{(k+1)^2+1+i}{(2i)^{k+1}}(z-5i)^{k+1}}{\frac{k^2+1+i}{(2i)^k}(z-5i)^k}\right| &= \lim_{k\to\infty}\left|(z-5i)\frac{(k+1)^2+1+i}{(k^2+1+i)(2i)}\right| \\
&= \lim_{k\to\infty}\left|\frac{(z-5i)}{2i}\right| \\
&= \frac{|z-5i|}{2}.
\end{aligned}$$

If $\frac{|z-5i|}{2} < 1$, then the power series converges, so we have $R = 2$.

We care about the uniform convergence of the power series since if $(f_n)_n$ is a sequence of continuous functions that converges uniformly to f on D, then f is continuous on D. If $(f_n)_n$ are analytic under the same condition, then f is analytic.

Notice that $f_n(z) = \sum_{j=0}^{n} c_j(z-z_0)^j$ is a polynomial. Since the $(f_n)_n$ are analytic, if it is the case that the power series converges uniformly on D, then $f(z)$ is analytic.

(i) Every power series with nonzero radius of convergence is an analytic function inside its circle of convergence.

(ii) The Taylor series for the function $\sum_{j=0}^{\infty} c_j(z-z_0)^j$ is itself.

For example, let $g(z) = \sum_{k=0}^{\infty}\frac{k^2+1+i}{(2i)^k}(z-5i)^k$. Then, g is analytic on $|z-5i| < 2$, and $g^{(9)}(5i) = \left(\frac{82+i}{(2i)^9}\right)(9!)$

To prove (ii), consider $\sum_{j=0}^{\infty} a_j(z-z_0)^j = \sum_{j=0}^{\infty} b_j(z-z_0)^j$. We then ask if $a_j = b_j$ for all j. The constant term of the nth-derivative of the left-hand side is $a_n n!$, and the constant term of the nth derivative of the right-hand side is $b_n n!$. Plugging in z_0 to the respective nth derivatives, we find that $a_n = b_n$.

To prove that a sequence of continuous functions $(f_n)_n \to f$ uniformly to a continuous function f, we pick $z_0 \in D$ to show that f is continuous at z_0.

Let $\varepsilon > 0$. We want to show that there exists $\delta > 0$ such that $|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$.

Since $(f_n)_n \to f$ uniformly on D, there exists M such that $\forall n \geq M$ and $\forall z \in D$, $|f_n(z)-f(z)| < \varepsilon$. Since f_M is continuous, we have that $\exists\delta > 0$ such that $|f_M(z)-f_M(z_0)| < \varepsilon$ for $|z-z_0| < \delta$. Then,

$$\begin{aligned}
|f(z)-f(z_0)| &= |f(z)-f_M(z)+f_M(z)-f_M(z_0)+f_M(z_0)-f(z_0)| \\
&\leq |f(z)-f_M(z)| + |f_M(z)-f_M(z_0)| + |f_M(z_0)-f(z_0)| \\
&< 3\varepsilon
\end{aligned}$$

Laurent Series

Suppose $\sum_{j=1}^{\infty} a_j(z-z_0)^j$ converges on $|z-z_0| < R_1$. Then, $\sum_{j=1}^{\infty} a_j w^j$, where $w = z-z_0$ converges on $|w| < R_1$. Then, $\sum_{j=1}^{\infty} a_j\left(\frac{1}{z-z_0}\right)^j$ converges where $\left|\frac{1}{z-z_0}\right| < R_1$, so it converges with $|z-z_0| > \frac{1}{R_1}$.

We write it as $\sum_{j=1}^{\infty} a_j(z-z_0)^{-j}$. Let $c_{-1} = a_1$, $c_{-2} = a_2$, etc.; then, we write the series as $\sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j}$. Suppose also that $\sum_{j=0}^{\infty} b_j(z-z_0)^j$ converges on $|z-z_0| < R_2$ such that $\frac{1}{R_1} < R_2$. Then, both series converge on the annulus defined by $\frac{1}{R_1} < |z-z_0| < R_2$. Let $c_j = b_j$ for $j \geq 0$. Then,

$$\sum_{j=-\infty}^{\infty} c_j(z-z_0)^j = \sum_{j=0}^{\infty} b_j(z-z_0)^j + \sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j}$$

converges on the annulus.

Suppose f is analytic on the annulus $r < |z-z_0| < R$, with $r, R \in [0, \infty]$. Then, for all z in the annulus, the series $= \sum_{j=-\infty}^{\infty} c_j(z-z_0)^j$ converges to $f(z)$, where c_j is given by

$$c_j = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_0)^{j+1}}\,dz,$$

where γ is any counterclockwise simple closed curve in the annulus.

When j is positive, we have that

$$\begin{aligned} c_j &= \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_0)^{j+1}} \\ &= \frac{1}{n!} f^{(j)}(z_0). \end{aligned}$$

To find the Laurent series for $f(z) = \frac{e^z}{z-i}$ in $\mathbb{C} \setminus \{i\}$, we do the following.

We need $\sum_{j=-\infty}^{\infty} c_j(z-i)^j$. We can write $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$ about 0.

$$\begin{aligned} e^z &= e^{z-i+i} \\ &= e^i e^{z-i} \\ &= e^i \sum_{j=0}^{\infty} \frac{(z-i)^j}{j!}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{e^z}{z-i} &= \frac{1}{z-i} e^i \sum_{j=0}^{\infty} \frac{(z-i)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{e^i}{j!}(z-i)^{j-1}, \end{aligned}$$

meaning it converges on $0 < |z-i| < \infty$.

To try to find the Laurent series for $\frac{1}{z^2(z-i)}$, we may consider on different annuli. For $0 < |z| < 1$, we first have to find the Laurent series for $\frac{1}{z-i}$.

$$\begin{aligned} \frac{1}{z-i}\frac{i}{i} &= \frac{i}{1-(iz)} \\ &= \sum_{j=0}^{\infty} i^{j+1} z^j. \end{aligned}$$

This Taylor series converges on $|z| < 1$. Thus,

$$\begin{aligned}\frac{1}{z^2(z-i)} &= \frac{1}{z^2}\sum_{j=0}^{\infty} i^{j+1}z^j \\ &= \sum_{j=0}^{\infty} i^{j+1}z^{j-2}.\end{aligned}$$

For $1 < |z| < \infty$, we can do

$$\begin{aligned}\frac{1}{z-i} &= \frac{1}{z\left(1-\frac{i}{z}\right)} \\ &= \frac{1}{z}\sum_{j=0}^{\infty} i^j z^{-j} \\ \frac{1}{z^2(z-i)} &= \sum_{j=0}^{\infty} i^j z^{-j-3}.\end{aligned}$$

To find the Laurent series for $f(z) = \frac{1}{(z-2)(z-3)}$ on $|z| < 2$, we do the following.

$$\begin{aligned}\frac{1}{(z-2)(z-3)} &= \frac{1}{z-3} - \frac{1}{z-2} \\ \frac{1}{z-3} &= \frac{1}{-3\left(1-(z/3)\right)} \\ &= -\frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{z}{3}\right)^j \\ \frac{1}{z-2} &= -\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j\end{aligned}$$

Both of these series converge on $|z| < 2$, so

$$\begin{aligned}\frac{1}{z-3} - \frac{1}{z-2} &= \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}} - \frac{z^j}{3^{j+1}} \\ &= \sum_{j=0}^{\infty} \frac{3^{j+1}-2^{j+1}}{6^{j+1}} z^j.\end{aligned}$$

Cauchy Criterion and Convergence

Let $(a_n)_n \in \mathbb{C}$ be such that $\forall \varepsilon > 0$, $\exists N$ large such that for $m, n \geq N$, $|a_m - a_n| < \varepsilon$. A sequence is convergent if and only if it is Cauchy.

Let $(a_n)_n \to \ell \in \mathbb{C}$. Let $\varepsilon > 0$. Then, $\exists N$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$. Let $m, n \geq N$. Then,

$$\begin{aligned}|a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |a_m - L| \\ &< 2\varepsilon\end{aligned}$$

The other direction requires the axiom of choice.

Recall the comparison test: if $\sum b_j$ converges, and $|a_j| < b_j$ for all j, then $\sum a_j$ converges. To prove this, we require the Monotone Convergence Theorem — every bounded monotone sequence of real numbers converges.

Let $(a_j)_j$ be nondecreasing and bounded above by B. Since $(a_j)_j$ is bounded above, it has a least upper bound L. Let $\varepsilon > 0$. Since L is the least upper bound, $L - \varepsilon$ is not an upper bound for $(a_j)_j$, meaning $\exists j$ such that $a_j > L - \varepsilon$. Thus, $L - \varepsilon < a_j \leq a_{j+1} \leq \cdots < L$. So, for all $k > j$, $|a_k - L| < \varepsilon$.

To show the comparison test, let $S_n = \sum_{j=0}^{n} |a_j|$. We have that S_n is monotone increasing. Additionally, S_n is bounded above since $S_n \leq \sum_{j=0}^{n} b_j \leq \sum b_j$, which converges. Let $T_n = \sum_{j=0}^{n} a_j$. Pick m, n with $m < n$. Given $\varepsilon > 0$, we have that for $m, n \geq N$,

$$\begin{aligned}
|T_m - T_n| &= \left| \sum_{j=m+1}^{n} a_j \right| \\
&\leq \sum_{j=m+1}^{n} |a_j| \\
&= S_n - S_m \\
&< \varepsilon,
\end{aligned}$$

so T_n is Cauchy, and thus convergent.

Let $A \subseteq \mathbb{R}$. Then, $\sup A$ is the least upper bound of A — if A is not bounded above, then $\sup A = \infty$.

For a sequence $(a_n)_n \in \mathbb{R}$, define $(x_n)_n$ as $x_n = \sup(a_j)_{j\geq n}$. Then, $x_n = \sup\{a_n, x_{n+1}\}$. Thus, we have $x_n \geq x_{n+1} \geq \cdots$, so $(x_n)_n$ may converge to some L, or $x_n \to \pm\infty$. We define $\lim_{n\to\infty} x_n$

Let $(a_n)_n = (-1)^n$. Then, $\limsup a_n = 1$. However, $\limsup(-2)^n = \infty$.

Cauchy-Hadamard Theorem

Given any power series $\sum_{j=0}^{\infty} a_j(z - z_0)^j$, $\exists R \in [0, \infty]$ such that the series converges uniformly for $|z - z_0| < R$, diverges on $|z - z_0| > R$, and converges uniformly on $|z - z_0| < r < R$.

Case 1: Let $\ell = \limsup \sqrt[n]{|a_n|}$. Let $\ell \in (0, \infty)$. Then, $R = \frac{1}{\ell}$.

Let $z \in \mathbb{C}$ such that $|z - z_0| < \frac{1}{\ell}$. Then, $\exists \ell'$ such that $|z - z_0| < \frac{1}{\ell'} < \frac{1}{\ell}$. Then, $c = \ell'|z - z_0| < 1$.

We have $\limsup \sqrt[n]{|a_n|} = \ell$, let $x_n = \sup\{\sqrt[n]{|a_n|}, \{\sqrt[n+1]{|a_{n+1}|}, \dots, \}$. Then, $\lim_{n\to\infty} x_n = \ell$. Then, $\exists p$ such that $\forall n \geq p$, $|x_n - \ell| < \ell' - \ell$. So, $\sup\{\sqrt[p]{|a_p|}, \dots\} < \ell'$, meaning that $\forall n \geq p$,

$$\begin{aligned}
&\sqrt[n]{a_n}|z - z_0| < \ell'|z - z_0| = c \\
&|a_n||z - z_0|^n < c^n,
\end{aligned}$$

so $\sum a_j(z - z_0)^j$ converges by comparison with $\sum c^j$, since $c < 1$.

Case 2: Let $\ell = 0$. We let $R = \infty$. Let $z \in \mathbb{C}$. Then, $\limsup \sqrt[n]{|a_n|} = 0$. If $z = z_0$, we have convergence. If not, let $c = |z - z_0| > 0$. Then, there exists p such that for all $n \geq p$, $|x_n - 0| < \frac{1}{2c}$, where x_n denotes the tail sequence.

Then, for all $j > p$, $\sqrt[j]{|a_j|} < \frac{1}{2c}$, so $|a_j| < \left(\frac{1}{2c}\right)^j$, so, $|a_j(z - z_0)| < \left(\frac{1}{2c}\right)^j c^j$, meaning $\sum a_j(z - z_0)^j$ converges by comparison with $\sum \frac{1}{2^j}$.

Case 3: Let $\ell = \infty$. We set $R = 0$. Let $z \neq z_0$. We will show that $\lim |a_j(z - z_0)^j| \nrightarrow 0$.

With this, we have shown absolute convergence as well. To show uniform convergence, we need the M-test.

Weierstrass M-Test: Let $f_j : A \subseteq \mathbb{C} \to \mathbb{C}$, $j \in \mathbb{Z}_{\geq 0}$, with f_j not necessarily continuous. Suppose $|f_j| \leq M_j$ on A. If $\sum M_j$ converges, then $\sum f_j(z)$ converges absolutely and uniformly on A.

For each $z \in A$, we know $\sum f_j(z)$ converges by the comparison test with $\sum M_j$. Let $g(z) = \sum f_j(z)$.

Let $g_n(z) = \sum_{j=0}^{n} f_j(z)$. We want to show $g_n \to g$ uniformly. Given $\varepsilon > 0$, $\exists p$ such that for all $n \geq p$ and for all $z \in A$, $|g_n(z) - g(z)| < \varepsilon$.

$$\begin{aligned}|g(z) - g_n(z)| &= \left|\sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^{n} f_j(z)\right| \\ &= \left|\sum_{j=n+1}^{\infty} f_j(z)\right| \\ &\leq \sum_{j=n+1}^{\infty} M_j.\end{aligned}$$

So, for sufficiently large n, $|S_n - S_\infty| < \varepsilon$ for sufficiently large n, as $\sum_{j=0}^{\infty} M_j$ converges.

To show uniform convergence within $|z - z_0| < r < R$, let $z_1 = z_0 + r$. So, $|z - z_0| = r < R$, meaning $\sum a_j(z_1 - z_0)^j$ converges. So, $|f_j(z)| \leq |a_j||z - z_0|^j$. By above, we have $\sum_{j=0}^{\infty} M_j$ converges, where $M_j = |a_j(z_1 - z_0)^j|$.

Then, for all z with $|z - z_0| < |z_1 - z_0|$, we have $|a_j(z - z_0)^j| \leq M_j$, meaning $\sum a_j(z - z_0)^j$ converges uniformly on $|z - z_0| < r$ by the M-test.

Zeros and Singularities

We want to use the features of Laurent series to study behavior of functions.

(i) If f is analytic at z_0,, $f(z_0) = 0$, and f is not constant, then $f(z) = (z - z_0)^m g(z)$ for some $m \geq 1$ and some g analytic at z_0 such that $g(z_0) \neq 0$.

(ii) If f is analytic on $0 < |z - z_0| < R$, and $\lim_{z \to z_0} |f(z)| = \infty$, then $f(z) = \frac{g(z)}{(z - z_0)^m}$ for some $m \geq 1$ and g analytic such that $g(z_0) \neq 0$.

For example, $\sin(\pi) = 0$, meaning $\sin(z) = (z - \pi)^m g(z)$ for some $m \geq 1$ and analytic g.

Suppose f is analytic at z_0 and for some $m \geq 1$, and $f'(z_0) = 0, \dots, f^{(m-1)}(z_0) = 0$, and $f^{(m)}(z_0) \neq 0$, then we say z_0 is a zero of f of order m.

For example, the order of the zero of $\sin z$ at $z_0 = \pi$ is 1, meaning π is a zero of order 1 for $\sin z$.

If f has a zero of order m at z_0, then $f(z) = (z - z_0)^m g(z)$ for some analytic $g(z)$ with $g(z_0) \neq 0$.

To show this, notice that f is analytic at z_0 by the definition of a zero, so

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

for some $|z - z_0| < R$. Since $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, we have

$$\begin{aligned} f(z) &= \sum_{j=m}^{\infty} \frac{f^{(j)}(z_0)}{j'}(z - z_0)^j \\ &= (z - z_0)^m \underbrace{\sum_{j=m}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^{j-m}}_{g(z)}. \end{aligned}$$

We have that g is analytic since the series converges on $|z - z_0| < R$. We have that $g(z_0) = f^{(m)}(z_0) \neq 0$ by definition of order m.

If f is analytic on some punctured disk $0 < |z - z_0| < R$, but f is not analytic at z_0, then we say f has an isolated singularity at z_0. For example, $f(z) = \frac{1}{z(z-2)}$ has 0 and 2 as isolated singularity.

Let $f(z) = \sum_{j=-\infty}^{\infty} c_j(z - z_0)^j$ be the Laurent series for f on $0 < |z - z_0| < R$, where f has an isolated singularity at z_0.

- Removable singularity: if $0 = c_{-1} = c_{-2} = \cdots$, then z_0 is a removable singularity;
- Pole singularity: if $\exists k$ such that $0 \neq c_{-1}, c_{-2}, \dots, c_{-(m-1)}$, and $c_{-k} = 0$ for all $k \geq m$, then we say f has a pole of order m at z_0;
- Essential singularity: if there are infinitely many $j < 0$ with $c_j \neq 0$.

In a more concise way, let $m = \inf\{j \mid c_j \neq 0\}$. If $m = 0$, then z_0 is a removable singularity, if $m = -\infty$, then z_0 is an essential singularity, and if $m \in (0, \infty)$, then z_0 is a pole singularity.

For example, we can classify singularities in the following functions:

- $\frac{\sin z}{z}$: We see that $z = 0$ is an isolated singularity since $\sin z$ is entire. We see that $\sin z = \displaystyle\sum_{j=0}^{\infty}(-1)^j \frac{z^{2j+1}}{(2j+1)!}$, meaning $\displaystyle\frac{\sin z}{z} = \sum_{j=0}^{\infty}(-1)^j \frac{z^{2j}}{(2j+1)!}$. Since this Laurent series doesn't have any non-zero negative index coefficients, we see that $z = 0$ is a removable singularity.
- $\frac{\sin z}{z^2}$: by the same logic, we find that $\displaystyle\frac{\sin z}{z^2} = \sum_{j=0}^{\infty}(-1)^j \frac{z^{2j-1}}{(2j+1)!}$, which means we have a pole of order 1.
- $\sin(1/z)$: since $\sin(1/z) = \displaystyle\sum_{j=0}^{\infty}(-1)^j \frac{1}{z^{2j+1}(2j+1)!}$, the singularity at $z = 0$ is essential.

Type of Singularity	Removable	Pole	Essential
Behavior	$\lim\limits_{z\to z_0} f(z)$	$\lim\limits_{z\to z_0} \lvert f(z)\rvert = \infty$	$\lvert f\rvert$ unbounded near z_0, and $\lim\limits_{z\to z_0} \lvert f(z)\rvert \neq \infty$

We can then find the following equivalent statements:

(i) f has a removable singularity at z_0

(ii) $\lim_{z\to z_0} f(z)$ exists

(iii) $|f(z)|$ is bounded on some punctured disk $0 < |z - z_0| < R$

(iv) $\exists R > 0$ such that $f|_{0<|z-z_0|<R}$ can be extended to an analytic function on $|z - z_0| < R$.

We can verify this for $f(z) = \frac{\sin z}{z-\pi}$. For (i), we see that for $z_0 = \pi$, we find the Taylor Series for $\sin z$ about π, which is

$$\begin{aligned}\sin z &= -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots \\ &= \sum_{j=0}^{\infty}(-1)^{j+1}\frac{(z-\pi)^{2j+1}}{(2j+1)!} \\ \frac{\sin z}{z-\pi} &= \sum_{j=0}^{\infty}(-1)^{j+1}\frac{(z-\pi)^{2j}}{(2j+1)!},\end{aligned}$$

meaning $z = \pi$ is a removable singularity.

To show (ii), we see

$$\begin{aligned}\lim_{z\to\pi}\frac{\sin z}{z-\pi} &= \lim_{z\to\pi}\left((-1) + \frac{(z-\pi)^2}{3!} + \cdots\right) \\ &= -1.\end{aligned}$$

We *could* use L'Hôpital's rule, but we would have to prove it.

To show (iv), we see that $\frac{\sin z}{z-\pi}$ extends to the analytic function

$$f_{\text{new}}(z) = \begin{cases}\frac{\sin z}{z-\pi} & z \neq \pi \\ -1 & z = \pi\end{cases},$$

where $R = \infty$, as f_{new} is equal to the Taylor series, which is itself entire.

We can also use Picard's theorem: if $f(z)$ has an essential singularity at z_0, then f assumes every value except possibly one on every neighborhood of z_0. In other words, there exists w_0 such that $\forall w \in \mathbb{C} \setminus \{w_0\}$ and $\forall \varepsilon > 0$, $\exists z$ such that $|z - z_0| < \varepsilon$ and $f(z) = w$.

Let z_0 be an isolated singularity of f. Then, the following are equivalent:

(i) z_0 is a pole of order m

(ii) $\lim_{z\to z_0}|f(z)| = \infty$

(iii) $f(z) = \frac{g(z_0)}{(z-z_0)^m}$, where $g(z_0) \neq 0$

Additionally, the following are equivalent:

(i) z_0 is an essential singularity

(ii) $|f(z)|$ is unbounded on every neighborhood of z_0, and $\lim_{z\to z_0}|f(z)| \neq \infty$

(iii) f assumes every value except possibly one on every neighborhood of z_0.

The following are equivalent:

(i) z_0 is a zero of order m for f

(ii) z_0 is a pole of order m for $\frac{1}{f}$

and

(i) z_0 is a pole of order m for f

(ii) Defining $(1/f)(z_0) = 0$, z_0 is a zero of order m for $\frac{1}{f}$.