1.3.17

Let G be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

Solution

(a)

Assume toward contradiction that deleting a vertex of degree $\Delta(G)$ increases the average degree.

$$\begin{aligned}
d'_{\text{avg}} &> d_{\text{avg}} \\
\frac{2e(G) - 2\Delta(G)}{n(G)-1} &> \frac{2e(G)}{n(G)} \\
\frac{2e(G) - 2\Delta(G)}{2e(G)} &> \frac{n(G)-1}{n(G)} \\
1 - \frac{\Delta(G)}{e(G)} &> 1 - \frac{1}{n(G)} \\
\frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} &> 0 \\
\frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{\text{avg}}} &> 0 \\
\frac{d_{\text{avg}} - 2\Delta(G)}{n(G)} &> 0 \\
d_{\text{avg}} - 2\Delta(G) &> 0 \\
d_{\text{avg}} &> 2\Delta(G)
\end{aligned}$$

However, we have reached a contradiction — by definition, $\Delta(G) \geq d_{\text{avg}}$, meaning that $d_{\text{avg}} \not> \Delta(G)$, let alone $2\Delta(G)$.

(b)

Deleting a vertex of the graph $K_{1,1}$ yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in $K_{1,1}$.

1.3.25

Prove that every cycle of length $2r$ in a hypercube is contained within a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r.

Solution

Let C be a cycle of length $2r$ in Q_n. Then, C contains $2r$ n-tuples. For every tuple in C, there exists a "switched" tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since C is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are $2r$ switches (corresponding to the $2r$ edges in C), this means there are at most r coordinates that are switched, then switched back sometime along the cycle's path. This means the other $n-r$ coordinates are fixed, implying that $C \subseteq Q_r$, the r-dimensional subcube of Q_k.

There is a cycle of length 8 in Q_3, defined as follows: $000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000$.

1.3.31

Using complete graphs and counting arguments, prove the following:

(a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ for $0 \leq k \leq n$.

(b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \leq \binom{n}{2}$.

Solution

(a)

We can consider a decomposition of the edges of K_n into the edge set of K_k and K_{n-k}, and some connector edges.

The edge set of K_n has cardinality $\binom{n}{2}$, the edge set of K_k has cardinality $\binom{k}{2}$, and the edge set of K_{n-k} has cardinality $\binom{n-k}{2}$. In order for this set of edges to be a full decomposition, we need a graph that connects all the vertices in K_k with all the vertices in K_{n-k}, which takes $k(n-k)$ edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph G, where $|V(G)| = n$ with maximal clique components $H_1, \dots, H_k$. Each of these components has $e(H_i) = \binom{|V(H_i)|}{2}$, with total $\sum_{i=1}^{k} \binom{|V(H_i)|}{2}$. In comparison, if we consider $e(K_G)$, where K_G is defined as the complete graph on the vertices of G, then that value is $\binom{n}{2}$, and $n = \sum_{i=1}^{k} |V(H_i)|$. Therefore, the size of the edge set of G is less than or equal to the sum of the sizes of the edge sets of maximal clique components H_i (because the maximal clique components of G could just be G itself).

1.3.41

Prove or disprove: if G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

Solution

Let $u, v \in V(G)$ and let $d(u) = \lceil \frac{n}{2} \rceil$. Then, u is adjacent to $\lceil \frac{n}{2} \rceil$ vertices and nonadjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Let $u \not\leftrightarrow v$.

We want to show that there exists some other vertex such that there exists a u, v path through that vertex. We know that $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$ and $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$.

Since $u \not\leftrightarrow v$, $N(u), N(v) \subseteq V(G) - \{u, v\}$. So, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq \left(\lceil \frac{n}{2} \rceil\right) + \left(\lfloor \frac{n}{2} \rfloor - 1\right) - (n-2) = 1$.

Therefore, there must be at least one element in $N(u) \cap N(v)$, meaning G is connected.