

## Math 395: Homework 1

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### Problem 3

**Problem.** Let  $V$  be an  $\mathbb{F}$ -vector space.

- (a) Prove that an arbitrary intersection of subspaces of  $V$  is again a subspace of  $V$ .
- (b) Prove that the union of two subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces is contained in the other.

**Solution.**

- (a) Let  $U, W \subseteq V$  be subspaces. Since  $U$  and  $W$  are subspaces,  $0_V \in U$  and  $0_V \in W$ , meaning  $U \cap W$  is nonempty.

Let  $u, w \in U \cap W$ , and let  $\alpha \in \mathbb{F}$ . Then, since  $u \in U$  and  $w \in U$ , it is the case that  $u + \alpha w \in U$ . Similarly, since  $u \in W$  and  $w \in W$ , it is the case that  $u + \alpha w \in W$ . Thus,  $u + \alpha w \in U \cap W$ , meaning  $U \cap W$  is a subspace.

Having shown the base case, we let  $\bigcap_{k=1}^N U_k$  be an intersection of subspaces  $U_k$ . By the inductive hypothesis, we have  $W = \bigcap_{k=1}^N U_k$ , where  $W$  is a subspace.

- (b) Let  $U, W \subseteq V$  be subspaces.

In the reverse direction, if, without loss of generality,  $U \subseteq W$ , then it is the case that  $U \cup W = W$ , meaning that  $U \cup W$  is a subspace of  $V$ .

In the forward direction, suppose toward contradiction that there exist subspaces  $U, W \subseteq V$  such that  $U \not\subseteq W$  and  $W \not\subseteq U$ , but  $U \cup W$  is a subspace of  $V$ . Since  $U \not\subseteq W$  and  $W \not\subseteq U$ , there exist non-trivial vectors  $w \in W \setminus U$  and  $u \in U \setminus W$ . Since  $w + u \in W \cup U$ , it is the case that  $w + u$  is contained either in  $U$  or in  $W$ . If  $w + u \in U$ , then  $(w + u) - u \in U$  (as  $u \in U$  and  $U$  is a subspace), meaning  $w \in U$ , which is a contradiction. Similarly, if  $w + u \in W$ , then  $(w + u) - w \in W$ , or  $u \in W$ , which is yet again a contradiction.

Thus, it must be the case that  $W \subseteq U$  or  $U \subseteq W$ .

### Problem 4

**Problem.** Let  $T \in \text{Hom}_{\mathbb{F}}(\mathbb{F}, \mathbb{F})$ . Prove there exists  $\alpha \in \mathbb{F}$  such that  $T(v) = \alpha v$  for all  $v \in \mathbb{F}$ .

**Solution.** Since  $\dim_{\mathbb{F}}(\mathbb{F}) = 1$ , we know that the basis of  $\mathbb{F}$  is  $\{\beta\}$  for some  $\beta \in \mathbb{F}$ . For  $v \in \mathbb{F}$ , it is then the case that  $v$  is a linear combination of the basis of  $\mathbb{F}$  over  $\mathbb{F}$ , meaning  $v = v_0\beta$  for some  $v_0 \in \mathbb{F}$ , implying  $\beta = (v_0^{-1})v$ .

Considering a linear transformation  $T(v)$ , we have

$$T(v) = T(v_0\beta).$$

Substituting  $\beta = v_0^{-1}v$ , and using the commutativity and associativity of multiplication under  $\mathbb{F}$ , we have

$$T(v) = T\left(v\left(v_0^{-1}v\right)\right).$$

Using the fact that  $T$  is linear and  $v \in \mathbb{F}$ , we have

$$\begin{aligned} &= vT\left(v_0^{-1}v_0\right) \\ &= vT(1). \end{aligned}$$

Thus,  $\alpha = T(1)$ .

## Problem 6

**Problem.** Let  $V$  be an  $\mathbb{F}$ -vector space. Prove that if  $\{v_1, \dots, v_n\}$  is linearly independent, then so is the set  $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$ .

**Solution.** To prove that  $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$  is linearly independent, we consider the sum

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \dots + a_{n-1}(v_{n-1} - v_n) + a_nv_n,$$

and show that this sum equals zero if and only if  $a_i = 0$  for each  $i$ . Rearranging the sum, we have

$$a_1v_1 + (a_2 - a_1)v_2 + \dots + (a_{n-1} - a_{n-2})v_{n-1} + (a_n - a_{n-1})v_n.$$

Since the set  $\{v_1, \dots, v_n\}$  are linearly independent, this linear combination equals  $0_V$  if and only if  $a_1 = (a_2 - a_1) = \dots = a_n - a_{n-1} = 0$ . In particular, since  $a_1 = 0$ , it must be the case that  $a_2 = 0$ ,  $a_3 = 0$ , and so on.

Thus,  $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$  are linearly independent.

## Problem 13

**Problem.** Let  $p$  be a prime and  $V$  a dimension  $n$  vector space over  $\mathbb{F}_p$ . Show there are

$$(p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})$$

distinct bases of  $V$ .

**Solution.** We begin by constructing our basis by selecting  $v_1 \in V \setminus \{0_V\}$ . Since  $V$  is a dimension  $n$  vector space over  $\mathbb{F}_p$ , it is the case that there are  $p^n - 1$  options to select  $v_1$ .

To select  $v_2$ , we find  $v_2 \in V \setminus \text{span}\{v_1\}$ ; since  $|\text{span}\{v_1\}| = p$ , there are  $p^n - p$  vectors that are linearly independent of  $v_1$ .

To select  $v_3$ , we find  $v_3 \in V \setminus \text{span}\{v_1, v_2\}$ ; since  $|\text{span}\{v_1, v_2\}| = p^2$ , there are  $p^n - p^2$  vectors that are linearly independent of  $\{v_1, v_2\}$ .

Continuing down the chain, we find that to select  $v_i$ , one can select from  $p^n - p^{i-1}$  vectors that are linearly independent of  $\{v_1, \dots, v_{i-1}\}$ .

Thus, the number of distinct bases of  $V$  is

$$\prod_{i=0}^{n-1} (p^n - p^i).$$