Remark: As a general rule, I use the following conventions:

$$U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$
$$B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leqslant r\}$$
$$S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$$

Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a bounded region, $f \colon \overline{U} \to \mathbb{C}$ continuous such that $f|_U$ is holomorphic. Suppose f is nonvanishing in U, and that there exists $c > 0$ such that $|f(z)| = c$ for all $z \in \partial U$. Prove that there exists some $\theta \in \mathbb{R}$ such that $f(z) = ce^{i\theta}$ for all $z \in \overline{U}$.

Solution: Since f is holomorphic on the connected, bounded, open set U, it follows from the maximum modulus principle that for all $z \in U$, we have $|f(z)| \leqslant |f(w)|$ for all $w \in \partial U$. In particular, we must have $|f(z)| \leqslant c$ for all $z \in U$. Since $|f(z)| \neq 0$ for all $z \in U$, it follows that $\frac{1}{|f(z)|} \geqslant \frac{1}{c}$ for all $z \in U$. Yet, at the same time, since $\frac{1}{f(z)}$ is holomorphic, we must have $\frac{1}{|f(z)|} \leqslant \frac{1}{|f(w)|}$ for all $w \in \partial U$, meaning that $\frac{1}{|f(z)|} \leqslant \frac{1}{c}$, so that $\left|\frac{1}{f}\right| = \frac{1}{c}$, or that $|f(z)| = c$ for all $z \in U$.

In particular, for all $z \in U$, we have $|f(z)| \geqslant |f(w)|$ for all $z \in U$, the maximum modulus principle gives that f is constant. Since $|f(z)| = c$, we thus have $f(z) = ce^{i\theta}$ for some $\theta \in \mathbb{R}$.

Problem (Problem 2)**:** For $0 < r < R$, let $A(z_0, r, R) = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. Suppose that there exists a continuous $f \colon \overline{A(z_0, r, R)} \to \mathbb{C}$ such that $f|_{A(z_0,r,R)}$ is holomorphic, and that there exist constants C_r and C_R in $\mathbb{R}$ such that $\operatorname{Re}(f(z)) = C_r$ on $S(z_0, r)$, and $\operatorname{Re}(f(z)) = C_R$ on $S(z_0, R)$> Show that $C_r = C_R$, and that f is constant for all $z \in \overline{A(z_0, r, R)}$.

Solution: Without loss of generality, since we may take $g(z) = f(z - z_0)$, we may assume that $z_0 = 0$, so that we let $u(x, y) \colon \overline{A(0, r, R)} \to \mathbb{R}$ be given by $u(x, y) = \operatorname{Re}(f(x - x_0 + i(y - y_0)))$. Since u is the real part of a holomorphic function, u is necessarily harmonic, so by the extended maximum modulus principle, u takes on its maximum on either $S(0, r)$ or $S(0, R)$. In other words, it is the case that the maximum for u is either C_r or C_R.

Now, consider the function

$$w(x, y) = u(x, y) - C_r - (C_R - C_r)\frac{\ln\left(x^2 + y^2\right) - \ln\left(r^2\right)}{\ln(R^2) - \ln(r^2)}.$$

We start by verifying that w is harmonic. Towards this end, since Laplace's equation is linear, we only need to evaluate the expression of ln, as we already know that u satisfies Laplace's equation. This gives

$$\begin{aligned}
\frac{\partial w}{\partial x} &= -\frac{C_R - C_r}{\ln(R^2) - \ln(r^2)}\frac{2x}{x^2 + y^2}\\
\frac{\partial^2 w}{\partial x^2} &= -\frac{C_R - C_r}{\ln(R^2) - \ln(r^2)}\left(\frac{2}{x^2 + y^2} - 2x\left(\frac{2x}{(x^2 + y^2)^2}\right)\right)\\
&= -\frac{C_R - C_r}{\ln(R^2) - \ln(r^2)}\frac{2y^2 - 2x^2}{(x^2 + y^2)^2}\\
\frac{\partial^2 w}{\partial y^2} &= -\frac{C_R - C_r}{\ln(R^2) - \ln(r^2)}\frac{2x^2 - 2y^2}{(x^2 + y^2)^2},
\end{aligned}$$

which means that the sum is zero, and thus w is harmonic. In particular, it also satisfies the extended maximum modulus principle, meaning that w attains its maxima and minima on the boundary of the annulus. Yet, since w equals 0 on both the outer circle and inner circle of the annulus, it follows that w is identically zero.

Thus, we have

$$u(x, y) = C_r + (C_R - C_r)\frac{\ln\left(x^2 + y^2\right) - \ln\left(r^2\right)}{\ln(R^2) - \ln(r^2)}.$$

Yet, this implies that

$$\mathrm{Re}(f(z)) = C_r + \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)}\left(\ln\left(|z|^2\right) - \ln\left(r^2\right)\right).$$

> **Aside:** In the solution, we use the Wirtinger derivative $\frac{\partial}{\partial \overline{z}}$. To capture exactly what this means, we recall that the Cauchy–Riemann equations say that if $f(x + iyi) = u(x, y) + iv(x, y)$, then
>
> $$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. &= 0. \end{aligned}$$
>
> Now, if $z = x + iy$, then $\overline{z} = x - iy$, meaning that
>
> $$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x} \\ &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial y} \\ &= i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \overline{z}} \end{aligned}$$
>
> meaning
>
> $$i\frac{\partial}{\partial y} = -\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}.$$
>
> By solving for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$, we get
>
> $$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned}$$
>
> Then, by applying $\frac{\partial}{\partial \overline{z}}$ to $f(x + iy) = u(x, y) + iv(x, y)$, we get
>
> $$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x, y) + iv(x, y)) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right). \end{aligned}$$
>
> Therefore, this expression equals zero precisely when f satisfies the Cauchy–Riemann equations, hence when f is holomorphic.

Since f is holomorphic, we must have

$$\begin{aligned} 0 &= \frac{\partial f}{\partial \overline{z}} \\ &= \frac{\partial\,\mathrm{Re}(f)}{\partial \overline{z}} + i\frac{\partial\,\mathrm{Im}(f)}{\partial \overline{z}} \\ &= \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)}\left(\frac{z}{|z|}\right)^2 + i\frac{\partial\,\mathrm{Im}(f)}{\partial \overline{z}} \end{aligned}$$

for all $z \in A(0, r, R)$. In particular, this must also hold for $z = \mathrm{Re}(z)$, so that

$$0 = \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)} + i\frac{\partial\,\mathrm{Im}(f)}{\partial \overline{z}}.$$

Now, since $\overline{z} = \operatorname{Re}(z)$, it follows that

$$0 = \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)} + i\frac{\partial v}{\partial x},$$

where $f(x+iy) = u(x,y) + iv(x,y)$. Yet, since the first term in this equation is purely real, and $i\frac{\partial v}{\partial x}$ is purely imaginary, it follows that both terms must be equal to zero, so that $C_R = C_r$.

This means we may take $u(x,y) = C$ for some C such that $f(z) = C + iv(x,y)$. Thus, by Cauchy–Riemann, we must have

$$\begin{aligned} \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial v}{\partial y} &= 0, \end{aligned}$$

so that $v(x,y)$ is constant, and thus f is constant.

Problem (Problem 3)**:** Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$\sup_{M_1,M_2 \geqslant 0} \int_{-M_2}^{M_2} \int_{-M_1}^{M_1} |f(x+iy)|\, dx\, dy$$

is finite. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

Solution: Letting $(x_0, y_0) \in \mathbb{R}^2$, we observe that for any $r > 0$, we have

$$\begin{aligned} |f(x_0 + iy_0)| &\leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(x_0 + r\cos(\theta), y_0 + r\sin(\theta))|\, d\theta \\ &= \frac{1}{2\pi r} \int_0^{r} \int_0^{2\pi} |f(x_0 + r\cos(\theta), y_0 + r\sin(\theta))|\, d\theta\, dr. \end{aligned}$$

We observe that there is a closed square containing the closed disk $B(z_0, r)$ given by the set of all $z \in \mathbb{C}$ such that $|\operatorname{Re}(z) - \operatorname{Re}(z_0)| \leqslant r$ and $|\operatorname{Im}(z) - \operatorname{Im}(z_0)| \leqslant r$. Since the double integral is evaluating over a positive function, the integral over this square is larger than the integral over the corresponding disk, so that we have

$$\begin{aligned} \frac{1}{2\pi r} \int_0^{r} \int_0^{2\pi} |f(x_0 + r\cos(\theta), y_0 + r\sin(\theta))|\, d\theta\, dr &\leqslant \frac{1}{2\pi r} \int_{y_0 - r}^{y_0 + r} \int_{x_0 - r}^{x_0 + r} |f(x,y)|\, dx\, dy \\ &\leqslant \frac{1}{2\pi r} \sup_{M_1,M_2 \geqslant 0} \int_{-M_2}^{M_2} \int_{-M_1}^{M_1} |f(x,y)|\, dx\, dy. \end{aligned}$$

Since the quantity in the supremum is finite, f is entire, and r was arbitrary, it follows that we may take the limit as $r \to \infty$, so that $f(x_0 + iy_0) = 0$. Since x_0 and y_0 are arbitrary, this thus holds for all $z \in \mathbb{C}$, so $f \equiv 0$.

Problem (Problem 4)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Show that if $u(x,y) = |f(x+iy)|$ is a harmonic function, then f is constant.

Solution: Let $z_0 \in U$ and $r > 0$ be such that $B(z_0, r) \subseteq U$. Since $|f|$ is harmonic, the mean value property holds that

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + re^{i\theta}\right) \right| d\theta.$$

Observe that since $u(x,y)$ is bounded below, then if $u(x,y) = 0$ at z_0, then $u(x,y)$ on $S(z_0, r)$, so that $u = 0$ on U.

On $B(z_0, r)$, by introducing a factor of r, we find that

$$|f(z_0)| = \frac{1}{2\pi r}\int_0^r \int_0^{2\pi} \left|f\left(z_0 + te^{i\theta}\right)\right| \, d\theta \, dt,$$

so by Fubini's theorem, this gives

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{r}\int_0^r \left|f\left(z_0 + te^{i\theta}\right)\right| \, dt\right) d\theta.$$

Therefore, $|f|$ maintains this mean value property on the whole ball $B(z_0, r)$, and in particular, $|f|$ maintains it along any ray originating from z_0 and terminating on $S(z_0, r)$. Now, since $|f|$ is harmonic on $U(z_0, r)$, it follows that $|f|$ attains its infimum over $B(z_0, r)$ on the boundary at some $z_1 \in S(z_0, r)$, and that $|f| - |f(z_0)| \geqslant 0$ on $B(z_0, r)$ as a result.

In particular, along the ray connecting z_0 to z_1, $v(x, y) = u(x, y) - |f(z_0)|$ still maintains the mean value of $|f(z_0)| - |f(z_1)|$. At the same time, $v(x_1, y_1) = 0$, and $v(x, y) \geqslant 0$ on $B(z_0, r)$, meaning that if v is to maintain the mean value that is found at its center, it follows that $v \equiv 0$ on $B(z_0, r)$, so $|f| = |f(z_0)|$ on $B(z_0, r)$.

Since $|f(z_0)|$ is a local maximum on $B(z_0, r)$, it follows by the maximum modulus principle that f is constant on $B(z_0, r)$, so by the identity theorem, f is constant on U.

Problem (Problem 5)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Suppose there exist positive integers $m, n \in \mathbb{N}$ such that $f(z)^m = \overline{f(z)}^n$ for all $z \in U$. Show that f is constant.

Solution: Multiplying both sides by $f(z)^n$, we get

$$f(z)^{m+n} = |f(z)|^{2n}.$$

Therefore, $f(z)^{m+n}$ is a holomorphic function mapping U to a subset of $\mathbb{R}$. By the open mapping theorem, $f(z)^{m+n}$ is necessarily constant, implying that f is constant.