

## Representations

**Definition:** If  $A$  is a  $C^*$ -algebra, a representation of  $A$  is a pair  $(\pi, H)$  where  $H$  is a Hilbert space and  $\pi: A \rightarrow B(H)$  is a  $*$ -homomorphism. If  $A$  is unital, then we require  $\pi(1) = I$ .

Note that if  $A$  does not have an identity, we can extend to the unitization  $A_1 = A \oplus \mathbb{C}$  and define  $\tilde{\pi}(a, \lambda) = \pi(a) + \lambda I$  for any  $a \in A$  and  $\lambda \in \mathbb{C}$ .

Note that every representation is contractive and the range of any representation is closed.

**Example:**

- (a) If  $A$  is a  $C^*$ -subalgebra of  $B(H)$ , then the inclusion map  $A \hookrightarrow B(H)$  is a representation.
- (b) If  $(X, \Omega, \mu)$  is a  $\sigma$ -finite measure space, then  $\pi: L_\infty(\mu) \rightarrow B(L_2(\mu))$ , where  $\pi(\phi) = M_\phi$ , is a representation.
- (c) If  $X$  is compact, and  $\mu$  is a positive Borel measure on  $X$ , then  $\pi_\mu: C(X) \rightarrow B(L_2(\mu))$  defined by  $\pi_\mu(f) = M_f$  is a representation of  $C(X)$ .

**Definition:** Let  $A$  be a  $C^*$ -algebra.

- (i) If  $d$  is a cardinal number,  $H$  a Hilbert space, we let  $H^{(d)}$  denote the direct sum of  $H$  with itself over  $d$ . If  $T \in B(H)$ , we let  $T^{(d)}$  be the direct sum of  $T$  with itself over  $d$ , which is known as the  $d$ -fold inflation of  $T$ .

Given a representation  $\pi: A \rightarrow B(H)$ , we have  $\pi^{(d)}: A \rightarrow B(H^{(d)})$ , defined by  $\pi^{(d)}(a) = \pi(a)^{(d)}$  is a representation, which is known as the inflation of  $\pi$ .

If  $d = \aleph_0$ , we will denote their respective inflations as  $H^{(\infty)}$  and  $\pi^{(\infty)}$ .

- (ii) If  $\{(\pi_i, H_i)\}_{i \in I}$  is a collection of representations of  $A$ , the direct sum of these representations is the representation

$$\bigoplus_{i \in I} \pi_i: A \rightarrow B\left(\bigoplus_{i \in I} H_i\right)$$

$$a \mapsto \bigoplus_{i \in I} \pi_i(a).$$

Note that since all representations are contractive, the direct sum is in fact a bounded operator. Furthermore, if  $\pi$  is isometric (hence injective), then so too is its inflation.

**Example:** If  $X$  is a compact topological space, and  $(\mu_n)_n$  is a sequence of positive Borel measures for  $X$ , with corresponding representations  $\pi_n: C(X) \rightarrow B(L_2(\mu_n))$  taking  $f \mapsto M_f$ , then  $\bigoplus_{n \geq 1} \pi_n$  is also a representation.

**Definition:** Two representations  $(\pi_1, H_1)$  and  $(\pi_2, H_2)$  are called equivalent if there is a unitary  $U: H_1 \rightarrow H_2$  such that  $\pi_2(a) = U\pi_1(a)U^{-1}$ .

**Definition:** If  $A$  is a  $C^*$ -algebra, then a representation  $\rho: A \rightarrow B(H)$  is *non-degenerate* if

$$[\rho(A)H] = \overline{\{\rho(a)h \mid a \in A, h \in H\}}$$

$$= H.$$

That is, the reducing subspace  $[\rho(A)H]$  for  $\rho(A)$  “lives” on the entire Hilbert space, or that the only  $g \in H$  for which  $\rho(a)g = 0$  for all  $a \in A$  is 0.

**Definition:** A representation  $\rho: A \rightarrow B(H)$  is called *cyclic* if there is some  $v \in H$  such that

$$H = [\rho(A)v].$$

We call the vector  $v$  a *cyclic vector* for  $\rho$ .

**Theorem:** Let  $\pi$  be a representation for the  $C^*$ -algebra  $A$ . Then, there is a family of cyclic representations  $\{\pi_i\}_{i \in I}$  for  $A$  such that

$$\pi \cong \bigoplus_{i \in I} \pi_i.$$

*Proof.* Let  $\mathcal{E}$  be the family of sets of nonzero vectors in  $H$  such that  $[\pi(A)e] \perp [\pi(a)f]$  for any  $e \neq f \in E \in \mathcal{E}$ . Ordering  $E$  by inclusion, using Zorn's Lemma gives us that  $\mathcal{E}$  has a maximal element  $E_0$ . Define

$$H_0 = \bigoplus_{e \in E_0} [\pi(A)e].$$

Let  $h \in H_0^\perp$ , so that  $0 = \langle \pi(a)e, h \rangle$  for all  $a \in A$  and  $e \in E_0$ . Therefore, if we have  $a, b \in A$  with  $e \in E_0$ , we have

$$\begin{aligned} 0 &= \langle \pi(b^*a)e, h \rangle \\ &= \langle \pi(b)^* \pi(a)e, h \rangle \\ &= \langle \pi(a)e, \pi(b)h \rangle. \end{aligned}$$

That is,  $\pi(A)e \perp \pi(A)h$  for all  $e \in E_0$ , meaning that  $E_0 \cup \{h\} \in \mathcal{E}$ , meaning that by maximality of  $E_0$ , we must have that  $h = 0$  and  $H = H_0$ .

Letting  $H_e := [\pi(A)e]$ , then for any  $a \in A$ , we have  $\pi(a)H_e \subseteq H_e$ , and  $\pi(a)^* = \pi(a^*)$ , so that  $H_e$  reduces  $\pi(a)$ . If we define  $\pi_e: A \rightarrow B(H_e)$ , we have that  $\pi_e$  is a representation of  $a$ , with

$$\pi = \bigoplus_{e \in E_0} \pi_e.$$

□

**Definition:** A representation  $\pi$  of a  $C^*$ -algebra  $A$  is called *irreducible* if the only invariant subspaces for  $\pi(A)$  are 0 and  $H$ .

The best example of an irreducible representation is a cyclic representation.

**Lemma:** A representation of a  $C^*$ -algebra  $A$  is irreducible if and only if the only operators commuting with  $\pi(A)$  are multiples of the identity.

*Proof.* If  $\pi$  has a nontrivial invariant subspace  $V$ , then  $P_V$  commutes with every  $\pi(a)$  and is not a scalar. Conversely, if there is a non-scalar operator  $T$  commuting with  $\pi(A)$ , then either the real or imaginary part of  $T$  is a non-scalar self-adjoint operator  $S$  commuting with  $\pi(A)$ , so there is some spectral projection  $P$  for  $S$  that is neither the 0 projection or the identity that commutes with  $\pi(A)$ , meaning that  $P(H)$  is an invariant subspace for  $\pi(A)$ . □

## States

For now, we will assume that  $M$  is a unital self-adjoint subspace of a  $C^*$ -algebra  $A$ . If  $\rho$  is a linear functional on  $M$ , then the equation

$$\rho^*(a) = \overline{\rho(a^*)}$$

defines another linear functional; if  $\rho = \rho^*$ , then we call  $\rho$  hermitian. Equivalently,  $\rho$  is hermitian if  $\rho(a^*) = \overline{\rho(a)}$ . If  $\rho$  is a bounded hermitian functional on  $M$ , then we claim that

$$\|\rho\| = \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\}.$$

This follows from the fact that if  $\varepsilon > 0$ , then from the Riesz lemma, we may find  $a$  in the unit ball of  $M$  with  $|\rho(a)| > \|\rho\| - \varepsilon$ . For a suitable  $\lambda$  with  $|\lambda| = 1$ , we have

$$\|\rho\| - \varepsilon < |\rho(a)|$$

$$\begin{aligned}
&= \rho(\lambda a) \\
&= \overline{\rho(\lambda a)} \\
&= \rho((\lambda a)^*).
\end{aligned}$$

If  $a_0 = \operatorname{Re}(\lambda a)$ , we have  $\|a_0\| \leq 1$  with  $\rho(a_0) > \|\rho\| - \varepsilon$ . Thus,

$$\|\rho\| \leq \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\},$$

with the reverse inequality being true by definition.

We say the linear functional  $\rho$  is *positive* if for any  $a \in M_+$ ,  $\rho(a) \geq 0$ ; if  $\rho(1) = 1$ , then we say  $\rho$  is a state. In fact, if  $\rho$  is positive, then  $\rho$  is hermitian (as we will see below).

We start by considering a version of the Cauchy–Schwarz inequality for states.

**Proposition:** If  $\rho$  is a positive linear functional on a  $C^*$ -algebra  $A$ , then

$$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b).$$

*Proof.* With  $a \in A$ , we have  $a^*a \in A_+$ , so  $\rho(a^*a) \geq 0$ . Then, since  $\rho$  is hermitian, we have that

$$\langle a, b \rangle = \rho(b^*a)$$

defines a positive sesquilinear form on  $A$ , so the traditional Cauchy–Schwarz inequality gives the desired result.  $\square$

**Proposition:** Let  $\rho$  be a bounded linear functional on a  $C^*$ -algebra  $A$ . The following are equivalent:

- (i)  $\rho$  is positive;
- (ii) for every approximate unit  $(e_i)_{i \in I}$ ,  $\|\rho\| = \lim_i \tau(e_i)$ ;
- (iii) for some approximate unit  $(e_i)_{i \in I}$ ,  $\|\rho\| = \lim_i \tau(e_i)$ .

*Proof.* We may assume that  $\|\rho\| = 1$ . To see that (i) implies (ii), we assume  $\rho$  is positive, and let  $(e_i)_{i \in I}$  be an approximate unit for  $A$ . Then,  $(\rho(e_i))_{i \in I}$  is an increasing net in  $\mathbb{R}$  that converges to its supremum, which is not greater than 1, so  $\lim_i \rho(e_i) \leq 1$ .

Now, we let  $a \in A$  be such that  $\|a\| \leq 1$ . We have

$$\begin{aligned}
|\rho(e_i a)|^2 &\leq \rho(e_i^2) \rho(a^* a) \\
&\leq \rho(e_i) \rho(a^* a) \\
&\leq \lim_i \rho(e_i),
\end{aligned}$$

so  $|\rho(a)|^2 \leq \lim_i \rho(e_i)$ , meaning  $1 \leq \lim_i \rho(e_i)$ .

Showing that (ii) implies (iii) is pretty much by definition. For (iii) implies (i), let  $(e_i)_i$  be an approximate unit with  $1 = \lim_i \rho(e_i)$ . Let  $a \in A_{\text{s.a.}}$  with  $\|a\| \leq 1$ , and write  $\rho(a) = \alpha + i\beta$  for  $\alpha, \beta \in \mathbb{R}$ . We may assume that  $\beta \leq 0$ , and we will show that  $\beta = 0$ . Letting  $n$  be any positive integer, we have

$$\begin{aligned}
\|a - ine_i\|^2 &= \|(a + ine_i)(a - ine_i)\| \\
&= \|a^2 + n^2 e_i^2 - in(ae_i - e_i a)\| \\
&\leq 1 + n^2 + n\|ae_i - e_i a\|,
\end{aligned}$$

so that

$$|\rho(a - ine_i)|^2 \leq 1 + n^2 + n\|ae_i - e_i a\|.$$

Yet, since  $\lim_i \rho(a - ine_i) = \rho(a) - in$ , with  $\lim_i \|ae_i - e_i a\| = 0$ , in the limit, we get

$$|\alpha + i\beta - in|^2 \leq 1 + n^2,$$

so by expanding, we have

$$-2n\beta \leq 1 - \beta^2 - \alpha^2.$$

Since  $\beta \leq 0$ , and this inequality holds for all positive  $n$ , it follows that  $\beta = 0$ . If  $a$  is positive with  $\|a\| \leq 1$ , we have  $e_i - a$  is hermitian with  $\|e_i - a\| \leq 1$ , so  $\rho(e_i - a) \leq 1$ . Then,  $1 - \rho(a) = \lim_i \rho(e_i - a) \leq 1$ , meaning  $\rho(a) \geq 0$ . Thus,  $\rho$  is positive.  $\square$

**Corollary:** If  $\rho$  is a bounded linear functional on a unital  $C^*$ -algebra  $A$ , then  $\rho$  is positive if and only if  $\rho(1) = \|\rho\|$ .

*Proof.* The sequence consisting exclusively of 1 is an approximate unit for  $A$ .  $\square$

The best-known example of a state is that of the *vector state* on  $B(H)$ , given by  $\rho_v: B(H) \rightarrow \mathbb{C}$ ,

$$\rho_v(T) = \langle Tv, v \rangle.$$

for a unit vector  $v \in H$ .

In fact, we will show in the next section that this is, to an extent, “every” state on a  $C^*$ -algebra.

## The GNS Construction

The most important fact about states is that they allow us to represent any  $C^*$ -algebra as a subalgebra of  $B(H)$ .

**Theorem (GNS Construction):** Let  $A$  be a  $C^*$ -algebra, and let  $\rho: A \rightarrow \mathbb{C}$  be a state. Then, there is a representation  $\pi_\rho: A \rightarrow B(H_\rho)$  with unit cyclic vector  $\xi_\rho$  such that

$$\langle \pi_\rho(a) \xi_\rho, \xi_\rho \rangle = \rho(a)$$

for all  $a \in A$ .

Furthermore, if  $\xi$  is a unit cyclic vector for a representation  $\pi: A \rightarrow B(H_\pi)$ , then the vector state

$$\begin{aligned} \tau: A &\rightarrow \mathbb{C} \\ a &\mapsto \langle \pi(a) \xi, \xi \rangle \end{aligned}$$

induces a unitary isomorphism of  $H_\rho$  onto  $H_\pi$  such that  $\pi(a) = U\pi_\tau(a)U^{-1}$  for all  $a \in A$ .

*Proof.* To start, we let  $\rho$  be a state on a  $C^*$ -algebra  $A$ , and define the subspace

$$N_\rho = \{a \in A \mid \rho(a^*a) = 0\}.$$

We see that  $\rho(b^*a) = 0$  if either  $a$  or  $b$  are in  $N_\rho$ , meaning there is a well-defined inner product on  $A/N_\rho$  given by

$$\langle a + N_\rho, b + N_\rho \rangle = \rho(b^*a).$$

We may define  $H_\rho$  to be the Hilbert space completion of  $A/N_\rho$ . We will show that  $N_\rho$  is a left ideal, by taking, for  $a \in A$  and  $x \in N_\rho$ , and using the identity  $x^*a^*ax \leq \|a\|^2 x^*x$ , to find

$$\begin{aligned} \langle ax, ax \rangle &= \phi((ax)^*ax) \\ &\leq \phi(\|a\|^2 x^*x) \\ &= 0, \end{aligned}$$

meaning that  $ax \in N_\rho$ . Furthermore, we see that

$$\begin{aligned} \|a(b + N_\rho)\|^2 &= \rho(b^*a^*ab) \\ &\leq \|a\|^2 \rho(b^*b) \end{aligned}$$

$$= \|a\|^2 \|b + N_\rho\|^2$$

Therefore, we may uniquely extend elements of  $a$  to bounded operators on  $H_\rho$ , which defines a representation  $\pi_\rho: A \rightarrow B(H_\rho)$ .

Now, if  $A$  is unital, we observe that  $1 + N_\rho$  is cyclic for  $\pi_\rho$ , as

$$\begin{aligned} [\pi_\rho(A)\xi_\rho] &= \overline{\{\pi_\rho(a)\xi_\rho \mid a \in A\}} \\ &= \overline{\{a(1 + N_\rho) \mid a \in A\}} \\ &= \overline{A/N_\rho} \\ &= H_\rho, \end{aligned}$$

and we observe that

$$\begin{aligned} \langle \pi_\rho(a)\xi_\rho, \xi_\rho \rangle &= \langle a(1 + N_\rho), 1 + N_\rho \rangle \\ &= \langle a + N_\rho, 1 + N_\rho \rangle \\ &= \rho(a). \end{aligned}$$

Meanwhile, if  $A$  is not unital, then we may extend  $\rho$  to a state  $\tau$  on the unitization  $\tilde{A}$ , which induces an isometry  $V$  of  $H_\rho$  to  $H_\tau$  mapping  $a + N_\rho$  to  $a + N_\tau$  that intertwines  $\pi_\rho$  and  $\pi_\tau$ , in the sense that  $V\pi_\rho(a) = \pi_\tau(a)V$ . We may identify  $H_\rho$  with the subspace  $VH_\rho \subseteq H_\tau$ . This gives that  $\pi_\tau|_A = \pi_\rho \oplus 0$  in  $H_\rho \oplus H_\rho^\perp$ . The projection of  $1 + N_\tau$  onto  $H_\rho$ , which we may denote  $h_\rho$ , then satisfies

$$\begin{aligned} \pi_\rho(a)h_\rho &= \pi_\tau(a)(1 + N_\tau) \\ &= a + N_\tau, \end{aligned}$$

meaning that  $h_\rho$  is cyclic for  $\pi_\rho$ , with

$$\begin{aligned} \langle \pi_\rho(a)h_\rho, h_\rho \rangle &= \langle \pi_\tau(a)(1 + N_\tau), 1 + N_\tau \rangle \\ &= \tau(a) \\ &= \rho(a). \end{aligned}$$

Now, for the converse, we see that the linear functional  $\tau$  defined by

$$\tau(a) = \langle \pi(a)\xi, \xi \rangle$$

is positive with norm at most 1; in fact, since for any approximate identity we have  $\pi(e_i)\xi \rightarrow \xi$ , it follows that  $\tau$  in fact has norm 1. We have that

$$\begin{aligned} N_\tau &= \{a \in A \mid \langle \pi(a^*a)\xi, \xi \rangle\} \\ &= \{a \in A \mid \pi(a)\xi = 0\}, \end{aligned}$$

so there is a well-defined linear map  $U_0: A/N_\tau \rightarrow H_\pi$  defined by  $U_0(a + N_\tau) = \pi(a)\xi$ , which is isometric, since

$$\begin{aligned} \langle U_0(a + N_\tau), U_0(b + N_\tau) \rangle &= \langle \pi(b^*a)\xi, \xi \rangle \\ &= \tau(b^*a) \\ &= \langle a + N_\tau, b + N_\tau \rangle, \end{aligned}$$

meaning that  $U_0$  extends to an isometric linear map  $U$  on  $H_\tau$  which surjects onto  $[\pi(A)\xi] = H_\pi$  since  $\xi$  is cyclic. Therefore,  $U$  is unitary, and we have

$$\begin{aligned} U\pi_\tau(a)(b + N_\tau) &= U(ab + N_\tau) \\ &= \pi(ab)\xi \\ &= \pi(a)\pi(b)\xi \\ &= \pi(a)U(b + N_\tau). \end{aligned}$$

Therefore,  $U\pi_\tau(a)U^{-1} = \pi(a)$ . □

**Corollary:** If  $\pi$  is an irreducible representation of a  $C^*$ -algebra  $A$ , and  $\xi \in H_\pi$  is any unit vector, then  $\pi$  is unitarily equivalent to the GNS representation  $\pi_\tau$ , where  $\tau$  is the vector state  $a \mapsto \langle \pi(a)\xi, \xi \rangle$ .

*Proof.* We observe that  $K = [\pi(A)\xi]$  is invariant under  $\pi$ , so by irreducibility we have that  $K$  is either 0 or  $H_\pi$ . Since  $\pi$  is non-degenerate, it follows that  $\pi(e_i)\xi \rightarrow \xi$  for some approximate unit  $(e_i)_i$ , so  $\xi$  is a nonzero vector in  $K$  and  $K$  is all of  $H_\pi$ .  $\square$

## Representations and the Extremal Structure of the State Space

The state space,  $S(A)$ , can be seen to be convex, as if  $A$  is unital with  $\phi, \psi \in S(A)$ , then

$$\begin{aligned} ((1-t)\phi + t\psi)(1) &= (1-t)\phi(1) + t\psi(1) \\ &= 1. \end{aligned}$$

Furthermore, by taking a net  $(\phi_i)_{i \in I} \subseteq S(A)$ , we see that the state space is  $w^*$ -closed. Thus, from the [Krein–Milman Theorem](#), it follows that the state space is equal to the  $w^*$ -closure of the convex hull of the extreme points of  $S(A)$ . The extreme points are known as *pure states*, and they have a relationship with representations of  $A$ .

**Theorem:** Let  $\rho \in S(A)$ . Then, the GNS representation  $\pi_\rho$  is irreducible if and only if  $\rho$  is a pure state.

*Proof.* Let  $\pi := \pi_\rho$  be non-irreducible. That is, there is an invariant subspace  $K \subseteq H_\rho$  with corresponding projection  $P$  such that  $P, I - P \neq 0$ . We will write  $\rho$  as a nontrivial convex combination of states.

Since  $K$  is invariant,  $\pi(a)P = P\pi(a) = P\pi(a)P$  for all  $a \in A$ . Since  $\xi_\rho$  is cyclic, it follows that  $\|P\xi_\rho\| \neq 0$ , as otherwise we would have

$$\begin{aligned} \pi(A)P\xi_\rho &= P\pi(A)\xi_\rho \\ &= PH_\rho \\ &= 0. \end{aligned}$$

Similarly, we have  $\|(1-P)\xi_\rho\| \neq 0$ , meaning that

$$\begin{aligned} \phi(a) &:= \frac{1}{\|P\xi_\rho\|^2} \langle \pi(a)P\xi_\rho, P\xi_\rho \rangle \\ \psi(a) &:= \frac{1}{\|(1-P)\xi_\rho\|^2} \langle \pi(a)(1-P)\xi_\rho, (1-P)\xi_\rho \rangle \end{aligned}$$

define states for  $A$  with  $\rho(a) = (\lambda)\phi(a) + (1-\lambda)\psi(a)$ , where  $\lambda = \|P\xi_\rho\|^2$ , and  $0 < \lambda < 1$ . Now, if we had  $\rho = \phi$ , then  $P\pi(a)P = \pi(a)P$  would imply that

$$\langle \pi(a)\xi_\rho, \xi_\rho \rangle = \frac{1}{\|P\xi_\rho\|^2} \langle \pi(a)\xi_\rho, P\xi_\rho \rangle$$

for all  $a \in A$ , but this is only possible if  $\|P\xi_\rho\|^2 \xi_\rho = P\xi_\rho$ , implying that  $\|P\xi_\rho\|^2 P\xi_\rho = P\xi_\rho$ , meaning  $\|P\xi_\rho\|^2 = 1$ , contradicting  $0 < \lambda < 1$ . Similarly,  $\rho \neq \psi$ ,  $\rho$  thus admits a nontrivial convex decomposition, meaning  $\rho$  is not an extreme point of  $S(A)$ .

Now, suppose  $\pi = \pi_\rho$  is irreducible, with  $\rho(a) = \langle \pi(a)\xi, \xi \rangle$  for some  $\xi \in H_\pi$ . Suppose  $\rho = \lambda\phi + (1-\lambda)\psi$  for some states  $\phi, \psi$  and some  $0 < \lambda < 1$ . Since  $\psi$  and  $\phi$  are positive, we have that

$$\begin{aligned} N_\rho &= \{a \in A \mid \rho(a^*a) = 0\} \\ &\subseteq N_\phi \\ &= \{a \in A \mid \phi(a^*a) = 0\}. \end{aligned}$$

Since  $\pi(a)h = \pi(b)h$  whenever  $a - b \in N_\rho$ , the Cauchy–Schwarz inequality implies that the sesquilinear form

$$(\pi(a)h, \pi(b)h) = \lambda\phi(b^*a)$$

is well-defined on the dense subspace  $\pi(A)h \subseteq H_\pi$ . By the polarization identity,  $(\cdot, \cdot)$  is bounded on  $\pi(A)h$ , so it can be extended to a bounded sesquilinear form  $q$  on  $H_\pi$ , meaning there is some bounded operator  $T \in B(H_\pi)$  such that  $q(h, k) = \langle h, Tk \rangle$ . In particular, we have

$$\begin{aligned}\langle \pi(a)h, T\pi(b)h \rangle &= (\pi(a)h, \pi(b)h) \\ &= \lambda\phi(b^*a).\end{aligned}$$

Since  $\phi$  is positive and  $\pi(A)h$  is dense, it follows that  $T$  is a positive operator with norm at most 1. Now, we claim that  $T$  commutes with  $\pi(A)$ . If  $a, b, c \in A$ , then

$$\begin{aligned}\langle \pi(a)h, T\pi(c)\pi(b)h \rangle &= \lambda\phi((cb)^*a) \\ &= \lambda\phi(b^*(c^*a)) \\ &= \langle \pi(c^*a)h, T\pi(b)h \rangle \\ &= \langle \pi(a)h, \pi(c)T\pi(b)h \rangle,\end{aligned}$$

so since  $\pi(A)h$  is dense, it follows that  $\pi(c)T = T\pi(c)$ . Since  $\pi$  is irreducible and  $T$  is positive, it follows that there is some  $z \geq 0$  such that  $T = zI$ . For any approximate identity  $(e_i)_{i \in I}$  for  $A$  and all  $a \in A$ , it then follows that

$$\begin{aligned}\lambda\phi(a) &= \lim_{i \in I} \lambda\phi(e_i a) \\ &= \lim_i \langle \pi(a)h, T\pi(e_i)h \rangle \\ &= z\langle \pi(a)h, h \rangle \\ &= z\rho(a),\end{aligned}$$

meaning that  $\lambda = z$ , so  $\phi = \rho$ . □

## The Universal Representation

To progress further, we must ensure that the state space separates the points of a  $C^*$ -algebra, so that we may use the states of a  $C^*$ -algebra to construct the universal representation.

**Lemma:** Let  $A$  be a  $C^*$ -algebra, and let  $a \in A_{s.a.}$ . Then, there is a state  $\rho$  of  $A$  such that  $|\rho(a)| = \|a\|$ . In particular, for any  $a \in A$ , there is a state  $\rho$  such that  $\rho(a^*a) = \|a\|^2$ .

*Proof.* We may assume that  $A$  has a unit. If we let  $B = C^*(a)$ , then there is an isometric isomorphism from  $C^*(a) \cong C(\hat{B})$ , where  $\hat{B}$  is the character space of  $B$ . In particular, since  $\hat{a}$  is a continuous map on a compact space, there is some  $\phi \in \hat{B}$  such that

$$\begin{aligned}|\phi(a)| &= |\hat{a}(\phi)| \\ &= \|\hat{a}\| \\ &= \|a\|.\end{aligned}$$

Therefore, there is some  $\rho \in A^*$  such that  $\rho|_B = \phi$  and  $\|\rho\| = \|\phi\| = 1$ . Since  $\phi(1) = 1$ , we have  $\rho(1) = 1$ , so  $\rho$  is a state with  $|\rho(a)| = \|a\|$ . Thus,  $\rho$  is a state on  $A$  with the desired property. □

**Proposition:** Every  $C^*$ -algebra  $A$  has a faithful non-degenerate representation.

*Proof.* For each  $a \neq 0$  in  $A$ , let  $\rho_a$  be a state such that  $\rho_a(a^*a) = \|a\|^2$ . Let  $\pi_{\rho_a}$  be the corresponding GNS representation with cyclic vector  $\xi_{\rho_a}$ . Then,  $\pi_{\rho_a}(a) \neq 0$ , as

$$\begin{aligned}0 &< \|a\|^2 \\ &= \rho_a(a^*a) \\ &= \langle \pi_{\rho_a}(a^*a)\xi_{\rho_a}, \xi_{\rho_a} \rangle \\ &= \|\pi_{\rho_a}(a)\xi_{\rho_a}\|^2.\end{aligned}$$

Therefore, the representation

$$\pi := \bigoplus_{\substack{a \in A \\ a \neq 0}} \pi_{\rho_a}$$

is a nondegenerate faithful representation. □

**Definition:** The *universal representation* for  $A$  is the pair  $(H_u, \pi_u)$ , where

$$H_u = \bigoplus_{\rho \in S(A)} H_\rho$$

$$\pi_u = \bigoplus_{\rho \in S(A)} \pi_\rho.$$

We observe that, as we showed in the lemma, the state space of  $A$  separates the points of  $A$ , so this is in fact a faithful representation.

## References

- [Mur90] Gerard J. Murphy.  *$C^*$ -algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990, pp. x+286. ISBN: 0-12-511360-9.
- [KR97] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*. Vol. 15. Graduate Studies in Mathematics. Elementary theory, Reprint of the 1983 original. American Mathematical Society, Providence, RI, 1997, pp. xvi+398. ISBN: 0-8218-0819-2. DOI: [10.1090/gsm/015](https://doi.org/10.1090/gsm/015). URL: <https://doi.org/10.1090/gsm/015>.
- [RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace  $C^*$ -algebras*. Vol. 60. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. xiv+327. ISBN: 0-8218-0860-5. DOI: [10.1090/surv/060](https://doi.org/10.1090/surv/060). URL: <https://doi.org/10.1090/surv/060>.
- [Con00] John B. Conway. *A course in operator theory*. Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: [10.1090/gsm/021](https://doi.org/10.1090/gsm/021). URL: <https://doi.org/10.1090/gsm/021>.