3

The graph G in Figure 50 is connected and contains no bridges. Find a strong orientation of G.

4

Suppose that D is an orientation of a connected graph G such that for each vertex v of G, some edge is directed toward v and some edge is directed away from v. Is D a strong orientation on G.

Since G is a connected graph, there must be a path P between any two vertices v_1 and v_2. Call this path P, travelling along the orientation D. Then, the path "exits" v_1 and "enters" v_2. Suppose that there is no path from v_2 back to v_1.

Then, within $G - P$ it must be the case that either v_1 or v_2 are of degree 0, or there is a point in $G - P$ wherein the interior vertices have no edges directed "out" both of which would contradict the assumptions. Additionally, since there is at least one edge directed "out" from v_2 and directed "in" v_1.

Extra Problem 1

Determine whether the following statements are true, and prove if so.

(a) A graph G has a strong orientation if and only if G is connected and has an orientation such that every pair of distinct vertices in G is in a directed cycle.

(b) A graph G has a strong orientation if and only if G is connected and has an orientation such that every pair of distinct vertices in G is in a directed circuit.

(c) A graph G has a strong orientation if and only if G is connected and has an orientation such that every pair of distinct vertices in G is in a directed closed walk.

(a)

($\Rightarrow$) If G has a strong orientation, then for $u, v \in V(G)$ distinct, $\exists P = u, \dots, v$, and $P' = v, \dots, u \in G - P$ paths. Therefore, by appending P and P' together, we find that u and v are in a directed cycle.

($\Leftarrow$) Let $u, v \in V(G)$ distinct such that $u, v \in C$, where C is a directed cycle. Thus, there must be no bridge between u and v, as deletion of any edge must allow a path in $C - e$ — so, by the condition of Robbin's Theorem, it must be the case that there is a strong orientation on G.

(b)

($\Rightarrow$) Suppose G has a strong orientation. Then, every pair of distinct vertices $u, v \in V(G)$ must have a path $P = u, \dots, v$ and a path $P' = v, \dots, u \in G - P$. By appending these paths together, we get a directed cycle, which is also a directed circuit.

$\Leftarrow$ Suppose G is connected and has an orientation such that for every $u, v \in V(G)$ distinct, u, v are in a directed circuit C'. This means there is a directed u, v trail in G — and thus, a directed u, v path P in G. Similarly, in $G - P$, there must be a directed v, u trail, and thus a directed v, u path. So, by the conditions of Robbin's Theorem, it must be the case that G has a strong orientation.

(c)

Since a closed walk is able to repeat edges, it is not necessarily the case that G is a bridgeless graph, and thus has a strong orientation.

Extra Problem 2

Let K_n be a strong tournament with $n \geq 3$.

(a) Prove that for every j in $\{2, \dots, n-2\}$, K_n has a directed cycle of length $1+j$ or $1+n-j$.

(b) Prove that for every j in $\{2, \dots, n-2\}$, K_n has n distinct directed cycles $C_1, \dots, C_n$ such that each C_i has length $1+j$ or $1+n-j$.

(a)

Let $v_1, v_2, \dots, v_n, v_1$ be a Hamiltonian cycle in the strong tournament K_n, which we know exists by Theorem 9.5. Then, there is an edge connecting v_1 and v_{j+1} for each $j \in \{2, \dots, n-2\}$.

If $e = v_1 \to v_{j+1}$, then we trace $v_1 \to v_{j+1} \to \cdots \to v_n \to v_1$ with length $n-j+1$. Otherwise, we have $v_1 \to v_2 \to \cdots \to v_j \to v_1$, with length $j+1$.