

**Solution (11.2, Problem 2):** We evaluate

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) \, dx \\ &= \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin(nx) \, dx \\ &= \frac{1}{\pi} \left( \frac{1}{n} \cos(nx) \right) \Big|_0^{\pi} \\ &= \frac{1}{n\pi} ((-1)^n - 1). \end{aligned}$$

Therefore, our Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi} \sin(nx).$$

**Solution (11.2, Problem 8):** We evaluate

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)(3 - 2x) \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \cos(nx) - 2x \cos(nx) \, dx \\ &= \begin{cases} 3 & n = 0 \\ 0 & \text{else.} \end{cases} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx)(3 - 2x) \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \sin(nx) - 2x \sin(nx) \, dx \\ &= \frac{1}{\pi} \left( \frac{3}{n} \cos(nx) \right) \Big|_{-\pi}^{\pi} - 2 \left( \frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right) \Big|_{-\pi}^{\pi} \\ &= \frac{4(-1)^n}{n}. \end{aligned}$$

Thus, our Fourier series is

$$f(x) = 3 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx).$$

**Solution (11.2, Problem 10):** Using integration by parts, we evaluate

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi/2} \cos\left(\frac{n}{2}x\right) \cos(x) \, dx \\ &= \begin{cases} \frac{8}{\pi(4-n^2)} \cos\left(\frac{n\pi}{4}\right) & n \neq 2 \\ \frac{1}{2} & n = 2 \end{cases}. \\ b_n &= \frac{2}{\pi} \int_0^{\pi/2} \sin\left(\frac{n}{2}x\right) \cos(x) \, dx \\ &= \begin{cases} \frac{8}{\pi(4-n^2)} \left( \sin\left(\frac{n\pi}{4}\right) - \frac{n}{2} \cos\left(\frac{n\pi}{4}\right) \right) & n \neq 2 \\ \frac{1}{\pi} & n = 2 \end{cases}. \end{aligned}$$

Thus, with  $a_0 = \frac{2}{\pi}$ , we have the Fourier series

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{2}x\right) + b_n \sin\left(\frac{n}{2}x\right).$$

**Solution** (11.2, Problem 17): We first start by finding the series expansion. Evaluating, we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{\pi} x^2 \cos(nx) \, dx \\ &= \begin{cases} \frac{2(-1)^n}{n^2} & n > 0 \\ \frac{\pi^2}{3} & n = 0 \end{cases} \\ b_n &= \frac{1}{\pi} \int_0^{\pi} x^2 \sin(nx) \, dx \\ &= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}((-1)^n - 1). \end{aligned}$$

Thus, we have the Fourier series

$$x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) + \left( \frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}((-1)^n - 1) \right) \sin(nx).$$

Using the input  $x = 0$ , we get

$$\begin{aligned} 0 &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}. \end{aligned}$$

Meanwhile, using the input of  $-\pi$ , we have  $f(-\pi) = 0$ , and

$$\begin{aligned} 0 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} \\ \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

**Solution** (11.2, Problem 18): Adding, we get

$$\frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}.$$

**Solution** (11.3, Problem 6): The function

$$f(x) = e^x - e^{-x}$$

is odd.

**Solution** (11.3, Problem 10): The function

$$f(x) = |x^5|$$

is even.

**Solution** (11.3, Problem 12): This function is even, so we expand in the cosine series. This gives

$$a_n = \int_0^2 f(x) \cos\left(\frac{n\pi}{2}x\right) dx$$

$$\begin{aligned}
 &= \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx \\
 &= \begin{cases} \frac{2}{n\pi} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.
 \end{aligned}$$

**Solution (11.3, Problem 18):** This function is odd, so we expand in the sine series. This gives

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^\pi x^3 \sin(nx) dx \\
 &= \frac{(-1)^{n+1} \pi^3}{n} + \frac{6\pi(-1)^n}{n^3}.
 \end{aligned}$$

**Solution (11.3, Problem 20):** This function is odd, so we expand in a sine series.

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^\pi (x+1) \sin(nx) dx \\
 &= \frac{2}{\pi} \int_0^\pi x \sin(nx) + \sin(nx) dx \\
 &= \frac{2(1 + (-1)^{n+1} + (-1)^{n+1}\pi)}{n\pi}.
 \end{aligned}$$

**Solution (11.3, Problem 34):** Since  $f(0) = 0$ , we expand in a sine series. This gives

$$b_n = \int_0^2 x(2-x) \sin\left(\frac{n\pi}{2}x\right) dx = \frac{16}{n^3\pi^3} (1 + (-1)^{n+1}).$$

**Solution (11.4, Problem 2):** We have the cases of  $\lambda = \alpha^2, 0, -\alpha^2$ , with corresponding solution forms of

$$\begin{aligned}
 y &= c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\
 y &= c_1 + c_2 x \\
 y &= c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).
 \end{aligned}$$

Substituting our boundary conditions for each of these cases, where the first listed equation is the case  $y(1) = 0$  and the second listed equation is the case  $y(0) + y'(0) = 0$ , we have

$$\begin{aligned}
 c_1 \cos(\alpha) + c_2 \sin(\alpha) &= 0 \\
 \alpha(c_2 - c_1) &= 0
 \end{aligned}$$

$$\begin{aligned}
 c_1 + c_2 &= 0 \\
 c_1 + c_2 &= 0
 \end{aligned}$$

$$\begin{aligned}
 c_1 \cosh(\alpha) + c_2 \sinh(\alpha) &= 0 \\
 \alpha(c_1 + c_2) &= 0.
 \end{aligned}$$

In the case with  $\lambda = \alpha^2$ , we have  $c_1 = c_2$  (as  $\alpha \neq 0$ ), giving  $y(1) = c_1 \cos(\alpha) + c_1 \sin(\alpha)$ , which simplifies to  $\sqrt{2}c_1 \sin\left(\alpha + \frac{\pi}{4}\right) = 0$ . This has solutions of  $\alpha = n\pi - \frac{\pi}{4}$ , where  $n \in \mathbb{Z}$ .

In the case with  $\lambda = 0$ , we have the case  $y = c_1 - c_1 x$ .

The case with  $\lambda = -\alpha^2$  simplifies to  $c_1 = -c_2$ , and  $c_1 \cosh(\alpha) + c_1 \sinh(\alpha) = 0$ , which is only true if  $c_1 = c_2 = 0$ . This is a trivial solution.

Therefore, we have eigenvalues  $\lambda = 0, \left(n\pi - \frac{\pi}{4}\right)^2$ .

$$\lambda_1 = 0$$

$$\begin{aligned}\lambda_2 &= \frac{\pi^2}{16} \\ \lambda_3 &= \frac{9\pi^2}{16} \\ \lambda_4 &= \frac{25\pi^2}{16}.\end{aligned}$$

**Solution** (11.4, Problem 4): We have the periodic Sturm–Liouville boundary condition of

$$\begin{aligned}y(-L) - y(L) &= 0 \\ y'(-L) - y'(L) &= 0.\end{aligned}$$

Letting  $\lambda = \alpha^2$ , we have solutions of the form

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Using the first boundary condition, we have

$$2c_2 \sin(\alpha L) = 0,$$

so  $\alpha = \frac{n\pi}{L}$ , where  $n \in \mathbb{Z}$ . Varying our coefficients and our values of  $n$ , we recover the family

$$y = \left\{ 1, \sin\left(\frac{n\pi}{L}\right), \cos\left(\frac{n\pi}{L}\right) \mid n \in \mathbb{Z} \right\}.$$

**Solution** (11.4, Problem 8):

(a) Using the guess  $y = e^{kt}$ , we get the characteristic equation of

$$k^2 + k + \lambda = 0,$$

which has solutions

$$k = -\frac{1}{2} \pm \sqrt{\lambda}.$$

This splits into three cases.

If  $\lambda = 0$ , then  $y = Ae^{-\frac{1}{2}t} + Bte^{-\frac{1}{2}t}$ . Plugging in the boundary conditions, we get that  $A = B = 0$ , so this cannot be a solution.

Similarly, if  $\lambda > 0$ , then  $y = A \cosh\left(\left(-\frac{1}{2} + \sqrt{\lambda}\right)t\right) + B \sinh\left(\left(-\frac{1}{2} + \sqrt{\lambda}\right)t\right)$ , which once again yields  $A = B = 0$  when plugging in the boundary conditions.

Thus, we are left with  $\lambda < 0$ , which has solutions of the form

$$y = e^{-\frac{1}{2}t} \left( A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t) \right).$$

Plugging in the boundary conditions, we get that  $A = 0$ , and with the other boundary condition, we get

$$B \sin(2\sqrt{\lambda}) = 0,$$

meaning that since  $B \neq 0$ , we have  $\lambda = \frac{n^2\pi^2}{4}$ . The corresponding eigenvectors are

$$y = e^{-\frac{1}{2}t} \sin\left(\frac{n\pi}{2}t\right),$$

where  $n = 1, 2, \dots$

(b) We may put the equation in self-adjoint form by multiplying by  $e^x$ , giving

$$\frac{d}{dx} \left( e^x \frac{dy}{dx} \right) + e^x \lambda y = 0.$$

(c) We have the orthogonality relation

$$\int_0^2 y_n(x) y_m(x) e^x dx = k_n \delta_{mn}.$$

**Solution** (11.4, Problem 10): We multiply out by  $e^{-x^2}$  to get

$$\frac{d}{dx} \left( e^{-x^2} \frac{dy}{dx} \right) + 2ne^{-x^2} y = 0,$$

with orthogonality relation

$$\int_{-\infty}^{\infty} y_n(x) y_m(x) e^{-x^2} dx = \delta_{mn}.$$

**Solution** (12.3, Problem 2): Separating variables with the guess  $u(x, t) = X(x)T(t)$ , we get the equation of

$$\begin{aligned} \frac{1}{T} \frac{dT}{dt} &= \frac{k}{X} \frac{d^2X}{dx^2} \\ &= \begin{cases} \alpha^2 \\ 0 \\ -\alpha^2 \end{cases}. \end{aligned}$$

We see that these cases yield

$$u(x, t) = e^{\alpha^2 t} \left( A \cosh\left(\frac{\alpha}{\sqrt{k}} x\right) + B \sinh\left(\frac{\alpha}{\sqrt{k}} x\right) \right)$$

$$u(x, t) = Ax + B$$

$$u(x, t) = e^{-\alpha^2 t} \left( A \cos\left(\frac{\alpha}{\sqrt{k}} x\right) + B \sin\left(\frac{\alpha}{\sqrt{k}} x\right) \right).$$

Inspecting the first two cases, we see that they only satisfy the boundary conditions if  $A = B = 0$ , so we turn our attention to the third case. Evaluating the boundary conditions, we get that  $A = 0$ , and

$$B \sin\left(\frac{\alpha}{\sqrt{k}} L\right) = 0,$$

so

$$\begin{aligned} \alpha &= \frac{n\pi\sqrt{k}}{L} \\ X_n &= B_n \sin\left(\frac{n\pi}{L} x\right). \end{aligned}$$

Solving for the initial condition entails finding  $\{B_n\}_{n \in \mathbb{N}}$  such that

$$\begin{aligned} u(x, 0) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) \\ &= x(L - x). \end{aligned}$$

Finding the Fourier sine expansion, we get

$$B_n = \frac{4 + 4(-1)^{n+1} L^2}{n^3 \pi^3}.$$

This gives the solution

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4 + 4(-1)^{n+1} L^2}{n^3 \pi^3} e^{-\frac{k n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} x\right).$$

**Solution** (12.3, Problem 4): Taking the general solution

$$u(x) = \frac{1}{L} \int_0^L f(x) dx + \frac{2}{L} \sum_{n=1}^{\infty} \left( \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx \right) e^{-\frac{k n^2 \pi^2}{L^2} t} \cos\left(\frac{n\pi}{L} x\right),$$

we substitute in our value of  $L = 2$  and

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases},$$

to give

$$\begin{aligned} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx &= \int_0^1 x \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \begin{cases} \frac{-4+4(-1)^{(n/2)}}{n^2\pi^2} & n \in 2\mathbb{Z} \\ \frac{-4+2n\pi(-1)^{(n-1)/2}}{n^2\pi^2} & n \in 2\mathbb{Z} + 1 \end{cases} \\ &=: B_n \end{aligned}$$

Thus, we have

$$u(x, t) = \frac{1}{4} + \sum_{n=1}^{\infty} B_n e^{-\frac{k n^2 \pi^2}{L^2} t} \cos\left(\frac{n\pi}{L}x\right).$$