

Solution (12.4, Problem 6): Upon separation of variables, we get

$$\frac{1}{a^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} Ae^{akt} & k^2 \\ At + B & 0 \\ A \cos(akt) + B \sin(akt) & -k^2 \end{cases}$$

$$X(x) = \begin{cases} Ce^{kx} & k^2 \\ Cx + D & 0 \\ C \cos(kx) + D \sin(kx) & -k^2 \end{cases}.$$

By plugging in the boundary conditions of $u(0, t) = u(1, t) = 0$, we quickly remove the former two cases, we are of the form

$$T(t) = A \cos(akt) + B \sin(akt)$$

$$X(x) = C \cos(kx) + D \sin(kx).$$

Since $X(0) = 0$, we must have $C = 0$, and since $X(1) = 0$, we have $k = n\pi$, $n \in \mathbb{Z}$. Thus, we have functions of the form

$$u_n(x, t) = (A_n \cos(n\pi t) + B_n \sin(n\pi t)) \sin(n\pi x),$$

and the general solution of

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi t) + B_n \sin(n\pi t)) \sin(n\pi x).$$

Plugging in the initial condition, we have

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$$= \frac{1}{100} \sin(3\pi x),$$

so that $A_n = \frac{1}{100}$ at $x = 3$ and 0 elsewhere. Writing our amended solution, we have

$$u(x, 0) = \left(\frac{1}{100} \cos(3\pi t) + B_3 \sin(3\pi t) \right) \sin(3\pi x).$$

Taking derivatives, we have

$$\left. \frac{\partial u}{\partial t} \right|_{(x,0)} = B_3 \sin(3\pi x)$$

$$= 0,$$

so $B_3 = 0$, and we arrive at the solution

$$u(x, t) = \frac{1}{100} \cos(3\pi t) \sin(3\pi x).$$

Solution (12.4, Problem 8): Upon separation of variables, we get

$$\frac{1}{a^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} Ae^{akt} & k^2 \\ At + B & 0 \\ A \cos(akt) + B \sin(akt) & -k^2 \end{cases}$$

$$X(x) = \begin{cases} Ce^{kx} & k^2 \\ Cx + D & 0 \\ C \cos(kx) + D \sin(kx) & -k^2 \end{cases}.$$

We plug in the boundary conditions of $\frac{\partial u}{\partial x} \Big|_{x=0} = \frac{\partial u}{\partial x} \Big|_{x=L} = 0$ to obtain

$$X_n(x) = \begin{cases} C_n \cos\left(\frac{n\pi}{L}x\right) & -k^2 \\ Cx + D & 0 \end{cases}$$

$$T_n(t) = \begin{cases} B_n \cos\left(\frac{n\pi a}{L}t\right) & -k^2 \\ At + B & 0 \end{cases}$$

We may evaluate the solution

$$u(x, t) = X_0(x)T_0(t) + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right).$$

To do this, we start with the initial condition, giving $T_0(t) = 1$ and $X_0(x) = x$. Taking the partial derivative with respect to t , we get

$$\frac{\partial u}{\partial t} = X_0(x) \frac{dT_0}{dt} - \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi}{L}x\right) \left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi a}{L}t\right).$$

Therefore,

$$u(x, t) = x$$

Solution (12.5, Problem 2): Separating variables, we have

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

Thus, we have

$$X_n = A_n \cos(\lambda x) + B_n \sin(\lambda x).$$

Using the boundary conditions of $X_n(a) = X_n(0) = 0$, we simplify to

$$X_n = B_n \sin\left(\frac{n\pi}{a}x\right).$$

Similarly, we have

$$Y_n(y) = C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right).$$

Applying the boundary condition of $\frac{\partial u}{\partial y}\bigg|_{(x,0)} = 0$, we have $D_n = 0$, and

$$u(x, y) = \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right).$$

We have

$$\begin{aligned} f(x) &= u(x, b) \\ &= \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a}x\right). \end{aligned}$$

Using the expansion of Fourier coefficients, we have

$$K_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx.$$

Solution (12.5, Problem 4): Separating variables, we get

$$\begin{aligned} \frac{1}{X} \frac{d^2 X}{dx^2} &= -\frac{1}{Y} \frac{d^2 Y}{dy^2} \\ &= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}. \end{aligned}$$

This evaluates to

$$\begin{aligned} X &= A \cos(\lambda x) + B \sin(\lambda x) \\ Y &= C \cosh(\lambda y) + D \sinh(\lambda y). \end{aligned}$$

Using the Neumann boundary condition, we get

$$\begin{aligned} X_n &= A_n \cos\left(\frac{n\pi}{a}x\right) \\ Y_n &= C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right). \end{aligned}$$

Therefore, $u(x, y)$ is of the form

$$u(x, y) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{a}x\right) \cosh\left(\frac{n\pi}{a}y\right) + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right).$$

We may plug this into an expression for $u(x, 0)$ to get

$$x = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{a}x\right),$$

meaning

$$\begin{aligned} B_0 &= a \\ B_n &= \frac{2((-1)^n - 1)a}{n^2\pi^2}, \end{aligned}$$

and plugging in the condition that $u(x, b) = 0$, we have

$$D_n = -\coth\left(\frac{n\pi b}{a}\right) B_n.$$

Solution (12.5, Problem 6): Separating variables, we have

$$-\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2} = \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

We thus have

$$Y = A \cos(\lambda y) + B \sin(\lambda y).$$

Using the Neumann boundary condition in y , we may simplify this to

$$Y_n = A_n \cos(ny).$$

This gives

$$X_n = B_n \cosh(nx) + C_n \sinh(nx).$$

We thus have

$$u(x, y) = \sum_{n=0}^{\infty} A_n \cosh(nx) \cos(ny) + \sum_{n=1}^{\infty} C_n \sinh(nx) \cos(ny).$$

Evaluating the boundary condition at $u(0, y)$, we have

$$g(y) = \sum_{n=0}^{\infty} A_n \cos(ny),$$

so that

$$A_n = \frac{2}{\pi} \int_0^{\pi} g(y) \cos(ny) dy.$$

Evaluating the derivative at $x = 1$, we have

$$\begin{aligned} \left. \frac{\partial u}{\partial x} \right|_{(1, y)} &= \sum_{n=1}^{\infty} n A_n \sinh(n) \cos(ny) + \sum_{n=1}^{\infty} n C_n \cosh(n) \cos(ny) \\ &= 0. \end{aligned}$$

Therefore, $C_n = -A_n \tanh(n)$.

Solution (12.5, Problem 8): Separating variables, we have

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}$$

Using the Dirichlet boundary condition on X , we get

$$X_n = A_n \sin(n\pi x)$$

$$Y_n = B_n \cosh(n\pi y) + C_n \sinh(n\pi y).$$

Thus, we have $u(x, y)$ of the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cosh(n\pi y) + \sum_{n=1}^{\infty} B_n \sin(n\pi x) \sinh(n\pi y).$$

Setting

$$\begin{aligned}\left. \frac{\partial u}{\partial y} \right|_{y=0} &= \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) \\ &= 0,\end{aligned}$$

we have $B_n = 0$, so

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cosh(n\pi y).$$

Solving for the Fourier coefficients, we get

$$A_n = \frac{2}{\cosh(n\pi)} \int_0^1 f(x) \sin(n\pi x) dx.$$

Solution (12.6, Problem 2): We may homogenize the boundary condition by letting $u(x, t) = v(x, t) + \psi(x)$, and solving

$$k \frac{d^2 \psi}{dx^2} = 0,$$

where $\psi(0) = u_0$ and $\psi(1) = 0$, so $\psi = -\frac{1}{u_0}x + u_0$. This gives the boundary value problem

$$\frac{\partial v}{\partial t} - k \frac{\partial^2 v}{\partial x^2} = 0$$

with Dirichlet boundary of $v(0) = v(1) = 0$. Separating variables as $v(x, t) = X(x)T(t)$, we then get

$$\begin{aligned}\frac{1}{T} \frac{dT}{dt} &= \frac{k}{X} \frac{d^2 X}{dx^2} \\ &= \begin{cases} -\alpha^2 \\ 0 \\ \alpha^2 \end{cases},\end{aligned}$$

so that

$$\begin{aligned}X_n &= A_n \sin(n\pi\sqrt{k}x) \\ T_n &= e^{-kn^2\pi^2 t},\end{aligned}$$

and

$$v(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi\sqrt{k}x) e^{-kn^2\pi^2 t}.$$

We have the initial condition $v(x, 0) = f(x) - \psi(x)$, so

$$A_n = 2 \int_0^1 \left(f(x) + \frac{1}{u_0}x - u_0 \right) \sin(n\pi x) dx,$$

and

$$u(x, t) = v(x, t) - \frac{1}{u_0}x + u_0.$$

Solution (12.6, Problem 4): Homogenizing the boundary conditions, we have

$$u(x, t) = v(x, t) + \psi(x),$$

where

$$k \frac{d^2 \psi}{dx^2} = -r.$$

This gives

$$\psi(x) = -\frac{r}{2k}x^2 + Bx + C.$$

Plugging in the boundary conditions, we get

$$\psi(x) = -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0.$$

The homogeneous heat equation is given by

$$\frac{\partial v}{\partial t} - k \frac{\partial^2 v}{\partial x^2} = 0,$$

where $v(x, 0) = f(x) - \psi(x)$, $v(0, t) = v(1, t) = 0$. By separating variables, we get

$$\begin{aligned} \frac{1}{T} \frac{d^2 T}{dt^2} &= \frac{k}{X} \frac{d^2 X}{dx^2} \\ &= \begin{cases} -\alpha^2 \\ 0 \\ \alpha^2 \end{cases} \end{aligned}$$

Using the Dirichlet boundary conditions, we get

$$\begin{aligned} X_n &= A_n \sin(n\pi\sqrt{k}x) \\ T_n &= e^{-kn^2\pi^2 t}. \end{aligned}$$

This gives

$$v(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi\sqrt{k}x) e^{-kn^2\pi^2 t}.$$

The Fourier coefficients are

$$A_n = 2 \int_0^1 (f(x) - \psi(x)) \sin(n\pi x) dx,$$

and we get the solution of $u(x, t) = v(x, t) + \psi(x)$.

Solution (12.6, Problem 10): We start with $u(x, t) = v(x, t) + \psi(x)$. Then,

$$a \frac{d^2 \psi}{dx^2} - g = 0.$$

Thus,

$$\psi = \frac{g}{2a}x^2 + Bx + C,$$

giving $C = 0$ and $B = -\frac{g}{2a}$.

We then have the homogeneous heat equation of

$$a \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}$$

$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases},$$

with $v(x, 0) = -\psi(x)$. Using the Dirichlet boundary conditions, this gives

$$\begin{aligned} X_n &= A_n \sin(n\pi x) \\ T_n &= B_n \cos(n\pi\sqrt{a}t) + C_n \sin(n\pi\sqrt{a}t). \end{aligned}$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cos(n\pi\sqrt{a}t) + \sum_{n=1}^{\infty} B_n \sin(n\pi x) \sin(n\pi\sqrt{a}t).$$

We have

$$-\frac{g}{2a}x(x-1) = \sum_{n=1}^{\infty} A_n \sin(n\pi x),$$

and

$$A_n = 2 \int_0^1 -\frac{g}{2a}x(x-1) \sin(n\pi x) dx.$$

I do not know how to deal with the B_n .

Solution (Extra Problems):

(i) In standard form, we have

$$\frac{\partial u}{\partial t} + \frac{x+1}{3} \frac{\partial u}{\partial x} = 0,$$

so that

$$\frac{dx}{dt} = \frac{x+1}{3},$$

and

$$x = ke^{t/3} - 1.$$

With $x(0) = k - 1$, we calculate

$$\begin{aligned} k &= (x+1)e^{-t/3} \\ k-1 &= (x+1)e^{-t/3} - 1. \end{aligned}$$

Thus, via the method of characteristics, we have

$$u(x, t) = \left((x+1)e^{-t/3} - 1 \right)^2.$$

(ii) In standard form, we have

$$\frac{\partial u}{\partial t} + 2(t+1) \frac{\partial u}{\partial x} = 0,$$

so

$$\begin{aligned} \frac{dx}{dt} &= 2(t+1) \\ x &= (t+1)^2 + C. \end{aligned}$$

Finding $x(0) = C + 1$, we get

$$C + 1 = x - (t+1)^2 + 1,$$

and

$$\begin{aligned} u(x, t) &= u_0(x_0) \\ &= \left(x - (t + 1)^2 + 1 \right)^3. \end{aligned}$$

(iii) In standard form, we have

$$\frac{\partial u}{\partial x} + \frac{4}{x} \frac{\partial u}{\partial t} = 0.$$

Thus, solving

$$\frac{dx}{dt} = \frac{4}{x},$$

we have

$$\frac{x^2}{8} = t + C,$$

so that

$$x(0) = 2\sqrt{2C}.$$

Thus,

$$x_0 = \sqrt{x^2 - 8t},$$

and

$$\begin{aligned} u(x, t) &= u_0(x_0) \\ &= 2\sqrt{x^2 - 8t} - 5. \end{aligned}$$

(iv) Evaluating the homogeneous equation, we have

$$\frac{\partial u}{\partial t} + 7 \frac{\partial u}{\partial x} = 0,$$

with solution

$$u_h(x, t) = (x - 7t)^2.$$

Solving

$$\frac{du}{dt} = u^2,$$

we have

$$u_p(x, t) = -\frac{1}{t}.$$

Thus,

$$u(x, t) = -\frac{1}{t} + (x - 7t)^2.$$

I am aware that this is probably wrong.

(v) We have

$$\frac{\partial u}{\partial t} + \frac{3}{x} \frac{\partial u}{\partial x} = 9t,$$

so evaluating the homogeneous equation, we have

$$\frac{dx}{dt} = \frac{3}{x}.$$

Thus, we have

$$\frac{x^2}{2} = 3t + C$$

$$x = \sqrt{6t + K}$$

and

$$x(0) = \sqrt{K}$$

$$= \sqrt{x^2 - 6t}.$$

Now, we have

$$\int_{t_0}^t \frac{d}{dx}(u(x(s), s)) ds = u(x(t), t) - u_0(x_0)$$

$$= \int_0^t 9t dt$$

$$= \frac{9}{2}t^2.$$

Thus,

$$u(x, t) = \frac{9}{2}t^2 + \cos(\sqrt{x^2 - 6t}).$$

(vi) We have

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + t \frac{\partial u}{\partial y} = 0,$$

so

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = t,$$

and

$$x(t) = t + c_1$$

$$y(t) = \frac{t^2}{2} + c_2,$$

with $x(0) = c_1$, $y(0) = c_2$, and

$$c_1 = x - t$$

$$c_2 = y - \frac{t^2}{2}.$$

Thus,

$$u(x, y, t) = u_0(x_0, y_0)$$

$$= (x - t) + \left(y - \frac{t^2}{2}\right).$$