Problem (Problem 1)**:** Determine and classify each of the singularities in $\mathbb{C}$ of the following functions:

(a) $\dfrac{z+1}{\sin^2(\pi/z)}$;

(b) $\dfrac{1}{z^2-1}\cos\left(\dfrac{\pi z}{z+1}\right)$;

(c) $\cot(z) - \dfrac{1}{z}$.

Solution:

(a) We observe that

$$\lim_{z\to -1} \sin^2(\pi/z) = 0,$$

and

$$\begin{aligned} \lim_{z\to -1}(z+1)\frac{z+1}{\sin^2(\pi/z)} &= \lim_{z\to -1}\frac{(z+1)^2}{\sin^2(\pi/z)}\\ &= \lim_{w\to 0}\frac{w^2}{\sin^2(w)}\\ &= 1, \end{aligned}$$

meaning that $f(z) = \frac{z+1}{\sin^2(\pi/z)}$ has a pole of order 1 at $z = -1$. Next, we observe that

$$\begin{aligned} f(z) &= \frac{z+1}{\sin^2(\pi/z)}\\ &= (z+1)\csc^2(\pi/z) \end{aligned}$$

has singularities at every other $z = \frac{1}{n}$ for all $n \in \mathbb{Z}$ with $n \neq 0, -1$; for any such satisfactory $z = \frac{1}{n}$, we have

$$\lim_{z\to\frac{1}{n}}\left(z - \frac{1}{n}\right)^2 \csc^2(\pi/z)(z+1) = 1 + \frac{1}{n},$$

meaning that at each $z = \frac{1}{n}$ with $n \in \mathbb{Z}$ and $n \neq 0, -1$, we have a pole of order 2.

Finally, there is no isolated singularity at 0 because 0 is an accumulation point of the sequence $\left(\frac{1}{n}\right)_{n\geqslant 1}$.

(b) Simplifying, we have

$$\begin{aligned} \frac{1}{z^2-1}\cos\left(\frac{\pi z}{z+1}\right) &= \frac{1}{(z-1)(z+1)}\cos\left(\pi - \frac{\pi}{z+1}\right)\\ &= \frac{1}{(z-1)(z+1)}\left(-\cos\left(\frac{\pi}{z+1}\right)\right)\\ &= -\frac{1}{(z-1)(z+1)}\cos\left(\frac{\pi}{z+1}\right). \end{aligned}$$

We observe that

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{-\cos\left(\frac{\pi}{z+1}\right)}{(z+1)}$$

$$= 0,$$

meaning that the singularity at $z = 1$ is removable. Additionally, we observe that the Laurent expansion about -1 for the function is

$$-\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos\left(\frac{\pi}{(z+1)}\right) = -\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)\pi^{2k+1}}{(2k+1)!}\frac{1}{(z+1)^{2k+1}}\right),$$

meaning that there are infinitely many negative-power terms in this Laurent expansion, so that the singularity at -1 is essential.

(c) We observe that

$$\begin{aligned}\lim_{z\to 0} z\left(\frac{\cos(z)}{\sin(z)} - \frac{1}{z}\right) &= \lim_{z\to 0}\frac{z\cos(z)}{\sin(z)} - 1\\ &= 0,\end{aligned}$$

meaning that the singularity at 0 is removable. Additionally, we see that for any $n \in \mathbb{Z}$ with $n \neq 0$,

$$\begin{aligned}\lim_{z\to n\pi}(z - n\pi)\frac{\cos(z)}{\sin(z)} &= (-1)^n \lim_{z\to n\pi}\frac{(z-n\pi)}{\sin(z)}\\ &= (-1)^n,\end{aligned}$$

whence the function has poles of order 1 at $n\pi$ when $n \neq 0$.

Problem (Problem 2)**:** Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire.

(a) Suppose there is a bounded set $U \subseteq \mathbb{C}$ such that $f(\mathbb{C} \setminus U) \subseteq \mathbb{C}$ is not dense. Show that f is a polynomial.

(b) Suppose that f is injective. Show that $f(z) = az + b$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

Solution:

(a) Since $U \subseteq \mathbb{C}$ is bounded, there is some $R > 0$ such that $U \subseteq B(0, R)$. In particular, this means that $f(\mathbb{C} \setminus B(0, R)) \subseteq \mathbb{C}$ is not dense. Consider now the set

$$\begin{aligned}V &= \left\{\frac{1}{z} \;\middle|\; z \in \mathbb{C} \setminus B(0, R)\right\}\\ &\subseteq \mathbb{C} \setminus \{0\}.\end{aligned}$$

This set is open as $\frac{1}{z}$ is holomorphic on the open set $\mathbb{C} \setminus B(0, R)$. Furthermore, for $\varepsilon > 0$, we can see that $\dot{U}(0, \varepsilon) \subseteq V$, as if $\varepsilon < \frac{1}{R}$, then for

$$\dot{U}(0, \varepsilon) = \left\{re^{i\theta} \;\middle|\; 0 < r < \varepsilon, 0 \leqslant \theta < 2\pi\right\}$$

we see that

$$\begin{aligned}\frac{1}{\dot{U}(0, \varepsilon)} &= \left\{se^{-i\theta} \;\middle|\; s > \frac{1}{\varepsilon}, -2\pi < \theta \leqslant 0\right\}\\ &\subseteq \mathbb{C} \setminus B(0, R).\end{aligned}$$

In particular, since $f(\mathbb{C} \setminus B(0, R))$ is not dense in $\mathbb{C}$, if we define

$$g(z) = f\left(\frac{1}{z}\right),$$

then we observe that

$$g\big(\dot{U}(0,\varepsilon)\big) \subseteq \mathbb{C}$$

is not dense, meaning that 0 is not an essential singularity of g by the contrapositive to the Casorati–Weierstrass Theorem. Thus, we may write g in the form

$$g(z) = \sum_{k=0}^{n} a_k z^{-k}$$

for some $n \geqslant 0$. Thus, since $f(z) = g\left(\frac{1}{z}\right)$, it follows that

$$f(z) = \sum_{k=0}^{n} a_k z^k,$$

or that f is a polynomial (constant if $n = 0$).

(b) Let f be an injective entire function. First, we note that f cannot be a constant function by definition.

Since f is injective, it follows that $f(\mathbb{C} \setminus B(0,1)) \cap f(U(0,1)) = \emptyset$, while since $\mathbb{C} \setminus B(0,1)$ and $U(0,1)$ are both open, and f is nonconstant, it follows from the open mapping principle that there is some $z_0 \in f(\mathbb{C} \setminus B(0,1))$ and some $r > 0$ such that $U(z_0, r) \subseteq \mathbb{C} \setminus B(0,1)$ and $U(z_0, r) \cap f(U(0,1)) = \emptyset$. Thus, in particular, f is a polynomial, as follows from part (a).

Finally, we observe that f cannot have degree greater than 1, since f can only have one value map to zero, and any functions of the form

$$f(z) = a_n (z - \alpha)^n$$

would have $\alpha + 1$ and $\alpha + e^{2i\pi/n}$ map to the same value, once again violating injectivity. Thus, f is a nonconstant polynomial with degree at most 1, whence $f(z) = az + b$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

Problem (Problem 3)**:** We say a function $h\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic if the following hold:

(i) for every $z_0 \in \mathbb{C}$ with $h(z_0) \neq \infty$, there is $r > 0$ such that $h(U(z_0, r)) \subseteq \mathbb{C}$ and h is holomorphic on $U(z_0, r)$;

(ii) for every $z_0 \in \mathbb{C}$ with $h(z_0) = \infty$, there exists some $r > 0$ such that $\widetilde{h}(z) = \frac{1}{h(z)}$ has $\widetilde{h}\big(\dot{U}(z_0, r)\big) \subseteq \mathbb{C}$, $\widetilde{h}$ is holomorphic on $\dot{U}(z_0, r)$, and z_0 is removable for $\widetilde{h}$;

(iii) if $h(\infty) \neq \infty$, then there exists some $r > 0$ such that $\widetilde{h}(z) = h\left(\frac{1}{z}\right)$ has $\widetilde{h}\big(\dot{U}(0, r)\big) \subseteq \mathbb{C}$, $\widetilde{h}$ is holomorphic on $\dot{U}(0, r)$, and 0 is removable for $\widetilde{h}$;

(iv) if $h(\infty) = \infty$, then there exists some $r > 0$ such that $\widetilde{h} = \frac{1}{h\left(\frac{1}{z}\right)}$ is such that $\widetilde{h}\big(\dot{U}(0, r)\big) \subseteq \mathbb{C}$, $\widetilde{h}$ is holomorphic on $\dot{U}(0, r)$, and 0 is removable for $\widetilde{h}$.

Show that if $h\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is injective and holomorphic, then h is a linear fractional transformation.

Solution: Using this definition, we claim that a function $h\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic when restricted to $\mathbb{C}$, and that if $h(\infty) = \infty$, then h is meromorphic on $\mathbb{C}$ with a pole at ∞.

To start, if $z_0 \in \mathbb{C}$ is such that $h(z_0) \neq \infty$, then by condition, h there is some $r > 0$ such that h is holomorphic on $U(z_0, r)$. Now, if $z_0 \in \mathbb{C}$ is such that $h(z_0) = \infty$, then we observe that, on $\dot{U}(z_0, r)$, we have the function

$$\widetilde{h}(z) = \frac{1}{h(z)}$$

is such that

$$\lim_{z\to z_0}(z - z_0)\widetilde{h}(z) = 0,$$

and also that

$$\begin{aligned}\widetilde{h}(z_0) &= \frac{1}{h(z_0)}\\ &= 0,\end{aligned}$$

whence $\widetilde{h}$ has a holomorphic extension g to $\dot{U}(z_0, r)$ with $g(z_0) = 0$. Thus, on $\dot{U}(z_0, r)$, we have

$$\widetilde{h}(z) = (z - z_0)^k \widetilde{g}(z)$$

with $\widetilde{g}(z_0) \neq 0$, as all zeros of $\widetilde{h}$ are isolated. Thus, from real analysis, there is some $0 < s < r$ such that $\widetilde{g} \neq 0$ on $U(z_0, s)$, meaning that on $\dot{U}(z_0, s)$, we have

$$h(z) = (z - z_0)^{-k}\frac{1}{\widetilde{g}(z)},$$

whence h has a pole of order k at z_0. In particular, this means that the non-removable singularities of h are exclusively poles, so h is meromorphic. Additionally, this also means that a meromorphic function is holomorphic on $\hat{\mathbb{C}}$ upon defining the function to equal ∞ at its poles.

Now, if h is such that $h(\infty) = \infty$, then we observe from the definition that the function

$$\widetilde{h}(z) = h\left(\frac{1}{z}\right)$$

has $\widetilde{h}(0) = \infty$, meaning that $\widetilde{h}$ has a pole at 0 from earlier, so that $h(\infty) = \infty$ implies that h has a pole at ∞.

Finally, if $h(\infty) \neq \infty$, we start by showing that either there is some z_0 with $h(z_0) = \infty$ or h is constant. If there is no such z_0, then $h\colon \hat{\mathbb{C}} \to \mathbb{C}$ is holomorphic and bounded (following from the extreme value theorem, as $\hat{\mathbb{C}}$ is compact), meaning that $h|_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, so that h is constant. Thus, there is some z_0 such that $h(z_0) = \infty$, whence h is meromorphic when restricted to $\mathbb{C}$.

Now, suppose h is an injective holomorphic function on $\hat{\mathbb{C}}$. Suppose $h(\infty) = \infty$. Then, for all $z_0 \in \mathbb{C}$, it follows that $h(z_0) \neq \infty$, whence h is holomorphic on $\mathbb{C}$ by condition (i). We have already shown thus far that $h(\infty) = \infty$ implies that h has a pole at ∞, meaning that h is an entire function that has a pole at infinity, hence h is a polynomial. Finally, since h is injective, we know from Problem 2 (b) that this means $h(z) = az + b$, hence h is a fractional linear transformation.

Now, if h is an injective holomorphic function on $\hat{\mathbb{C}}$ that does not have $h(\infty) = \infty$, then $h(\infty) = k$ for some $k \in \mathbb{C}$, and h is still a meromorphic function as seen above. Then, we observe that

$$p(z) = \frac{1}{z - k}$$

is meromorphic on $\mathbb{C}$, hence

$$(p \circ h)(z) = \frac{1}{h(z) - k}$$

is also meromorphic on $\mathbb{C}$ (hence holomorphic on $\hat{\mathbb{C}}$), injective, and has $(p \circ h)(\infty) = \infty$, so that $(p \circ h)(z) = az + b$, whence

$$\begin{aligned} \frac{1}{h(z) - k} &= az + b \\ h(z) &= \frac{1}{az + b} + k, \end{aligned}$$

which is yet again a fractional linear transformation.

Problem (Problem 4)**:** Let $P \colon \mathbb{C} \to \mathbb{C}$ be a polynomial not uniformly zero.

(a) Show that $\sum_{n=0}^{\infty} P(n) z^n$ has radius of convergence exactly 1.

(b) Show that if $f = \sum_{n=0}^{\infty} P(n) z^n$ then there exists a meromorphic function $g \colon \mathbb{C} \to \mathbb{C}$ that is rational and satisfies $g(z) = f(z)$ on $\mathbb{D}$.

(c) Show that at least one of the poles of g satisfies $|z_0| = 1$.

Solution:

(a) We write $P(z) = a_k z^k + \cdots + a_1 z + a_0$ where $k > 1$ and not all terms zero. Then, if P is constant, we observe that

$$\sum_{n=0}^{\infty} a_0 z^n = a_0 \sum_{n=1}^{\infty} z^n,$$

which has a radius of convergence of exactly 1 as it is a geometric series. Therefore, if P is nonconstant, we have

$$P(n) = n^k \left(a_k + a_{k-1} n^{-1} + \cdots + a_1 n^{-k+1} + a_0 n^{-k}\right),$$

so that

$$\begin{aligned} \frac{1}{R} &= \limsup_{n\to\infty} (P(n))^{1/n} \\ &= \limsup_{n\to\infty} \left(n^k\right)^{1/n} \left(a_k + a_{k-1} n^{-1} + \cdots + a_1 n^{-k-1} + a_0 n^{-k}\right)^{1/n} \\ &= \limsup_{n\to\infty} \left(n^{1/n}\right)^k (a_k)^{1/n} \\ &= 1. \end{aligned}$$

Therefore, the radius of convergence of the power series $\sum_{n=0}^{\infty} P(n) z^n$ is exactly 1.

(b) From uniform convergence, we observe that on $\mathbb{D}$, whenever $k > 0$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^k z^n &= \frac{d}{dz}\left(\sum_{n=1}^{\infty} \frac{n^k}{n+1} z^{n+1}\right) \\ &= \frac{d}{dz}\left(\sum_{n=1}^{\infty} n^{k-1} z^{n+1}\right) - \frac{d}{dz}\left(\sum_{n=1}^{\infty} \frac{n^{k-1}}{n+1} z^{n+1}\right) \\ &= \frac{d}{dz}\left(z \sum_{n=1}^{\infty} n^{k-1} z^n\right) - \sum_{n=1}^{\infty} n^{k-1} z^n \\ &= z \frac{d}{dz}\left(\sum_{n=1}^{\infty} n^{k-1} z^n\right). \end{aligned}$$

Therefore, if we write

$$q_k(z) = \sum_{n=1}^{\infty} n^k z^n,$$

we observe that we have the recurrence relation

$$q_k = z q_{k-1}',$$

where

$$\begin{aligned} q_0(z) &= \sum_{n=1}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} z^{n+1} \\ &= \frac{z}{1-z}, \end{aligned}$$

so by solving this recurrence relation for each of the $k > 0$ terms in

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left(a_k n^k + \cdots + a_1 n + a_0\right) z^n \\ &= a_k \sum_{n=1}^{\infty} n^k z^n + \cdots + a_1 \sum_{n=1}^{\infty} n z^n + a_0 \sum_{n=0}^{\infty} z^n, \end{aligned}$$

and using the identity

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

on the term with a_0, we observe that this gives us an expression for f entirely in terms of rational functions, whence f has a meromorphic extension to $\mathbb{C}$.

(c) Suppose toward contradiction that for all poles z_0 of g, it were the case that $|z_0| > 1$. Considering the Taylor expansion of g about 0, we would have

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

on $U(0, r)$ with $r > 1$, as the Cauchy Integral Formula provides. Yet, since this new Taylor expansion agrees with $f(z) = \sum_{n=0}^{\infty} P(n) z^n$ on $\mathbb{D}$, it follows that these two expansions are equal on $U(0, r)$ by the identity theorem. Yet, we have already shown that f has radius of convergence exactly 1, which contradicts the assertion that there is any pole of g with $|z_0| > 1$.

Alternatively, we can use the construction from part (b) to show that g has exactly one pole at 1. Toward this end, observe that

$$\frac{d}{dz}\left(\frac{p(z)}{(1-z)^k}\right) = \frac{p'(z)}{(1-z)^k} + \frac{kp(z)}{(1-z)^{k+1}}.$$

In particular, taking the derivative of a rational function whose denominator is some power of $(1-z)^k$ does not alter the fact that this rational function has a pole at 1, and neither does multiplying this rational function by a polynomial that has no factors in common with $(1-z)^k$. Since we constructed g by taking a series of linear combinations of derivatives $\frac{1}{1-z}$ and multiplications by z, it follows that g has a pole exclusively at 1.

Problem (Problem 5)**:** For $m \in \mathbb{N}$, evaluate the integral

$$\oint_{S^1} \frac{z^{m-1}}{2z^m - 1}\,dz.$$

Solution: We will use the argument principle to evaluate this integral. Toward this end, we see that

$$\begin{aligned}\oint_{S^1} \frac{z^{m-1}}{2z^m - 1}\,dz &= \frac{1}{2m}\oint_{S^1} \frac{mz^{m-1}}{z^m - (1/2)}\,dz\\ &= \frac{1}{2m}\int_{S^1} \frac{f'(z)}{f(z)}\,dz\\ &= \frac{1}{2m}(2\pi i)\sum_{z_0} n\left(S^1; z_0\right)\operatorname{ord}_{z_0}(f)\\ &= \frac{\pi i}{m}(m)\\ &= \pi i.\end{aligned}$$