**Problem** (Problem 1)**:** For all $n \in \mathbb{N}$, find the residue of $f(z) = (1 - e^{-z})^n$ at $z = 0$ via Cauchy's residue theorem.

**Solution:** Choose a square contour $\gamma$ defined by

$$\begin{aligned} \gamma &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\ \gamma_1 &= 1 + iy \\ \gamma_2 &= i - x \\ \gamma_3 &= -1 - iy \\ \gamma_4 &= -i + x \end{aligned}$$

with $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$. Then,

$$\begin{aligned} 2\pi i \operatorname{Res}(f; 0) &= \oint_{\gamma} f(z)\, dz \\ &= \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \int_{\gamma_3} f(z)\, dz + \int_{\gamma_4} f(z)\, dz. \end{aligned}$$

We compute

$$\int_{\gamma_1} f(z)\, dz = \int_{-1}^{1} \frac{i}{(1 - e^{-1-iy})^n}\, dy.$$

Taking $u = e^{-1-iy}$, we get

$$\begin{aligned} &= -\int_{u(-1)}^{u(1)} \frac{1}{u(1-u)^n}\, du \\ &= -\int_{e^{-1+i}}^{e^{-1-i}} \frac{1}{e^{-1-iy}} + \frac{p\left(e^{-1-iy}\right)}{(1 - e^{-1-iy})^n}\, dy, \end{aligned}$$

where $p(u) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} u^{k-1}$.

$$\int_{\gamma_2} f(z)\, dz = \int_{-1}^{1} \frac{-1}{(1 - e^{-i+x})^n}\, dx.$$

Taking $v = e^{-i+x}$

$$\begin{aligned} &= -\int_{v(-1)}^{v(1)} \frac{1}{v} + \frac{p(v)}{(1-v)^n}\, dv \\ &= -\int_{e^{-1-i}}^{e^{1-i}} \frac{1}{e^{-i+x}} + \frac{p\left(e^{-i+x}\right)}{(1 - e^{-i+x})^n}\, dx \end{aligned}$$

**Problem** (Problem 2)**:** Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2 + 1}\, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{\sin^2(x)}{x^2 + 1}\, dx.$$

**Solution:** We compute

$$\int_{-R}^{R} \frac{\sin^2(x)}{x^2 + 1}\, dx = \frac{1}{2} \int_{-R}^{R} \frac{1}{x^2 + 1}\, dx - \frac{1}{2} \int_{-R}^{R} \frac{\cos(2x)}{x^2 + 1}\, dx.$$

Calling the latter integral I, we take

$$f(z) = \frac{e^{2iz}}{z^2+1},$$

close the contour $\gamma$ in the upper half-plane with the half-circle $C_R = \left\{Re^{i\theta} \mid 0 \leqslant \theta \leqslant \pi\right\}$. This gives

$$\begin{aligned}\oint_\gamma f(z)\,dz &= I + \int_{C_R} f(z)\,dz\\ &= I + \int_0^\pi \frac{e^{2iRe^{i\theta}}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta.\end{aligned}$$

Estimating the integrand on the second integral, we see that for $R > 1$,

$$\left|\frac{iRe^{i\theta}e^{2iRe^{i\theta}}}{R^2e^{2i\theta}+1}\right| \leqslant \frac{R}{R^2-1},$$

whence the integral over $C_R$ approaches 0 as $R \to \infty$. Therefore, by Cauchy's residue theorem,

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos(2x)}{x^2+1}\,dx &= 2\pi i\,\mathrm{Res}(f;i)\\ &= 2\pi i \lim_{z\to i} \frac{e^{2iz}}{(z-i)(z+i)}\\ &= \frac{\pi}{e^2}.\end{aligned}$$

Thus, we find that

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2+1}\,dx = \frac{\pi}{2} - \frac{\pi}{2e^2}.$$