Math 395

Homework 4

Due: 2/27/2024

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Collaborators:

Problem 1

Let F be a field, with $F[x]$ denoting the ring of polynomials with coefficients in F. Let $f(x) \in F[x]$ be a monic polynomial. Let $g(x) \in F[x]$ be a nonzero polynomial. We will show that there exist unique $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Consider the ideal generated by $g(x)$, $\langle g(x) \rangle \subseteq F[x]$.

Problem 4

Let $p \in \mathbb{Z}$ be a prime. Let $\mathfrak{m} = \{(pa, b) \mid a, b \in \mathbb{Z}\}$. We will prove that $\mathfrak{m}$ is a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$.

We will do so by showing that $\mathbb{Z} \times \mathbb{Z}/\mathfrak{m}$ is isomorphic to the field $\mathbb{Z}/p\mathbb{Z}$. Let $\varphi : \mathbb{Z} \times \mathbb{Z}/\mathfrak{m} \to \mathbb{Z}/p\mathbb{Z}$ be defined by $\varphi(\mathfrak{m} + (i, j)) = [i]_p$. We will show that φ is a well-defined bijective homomorphism.

Problem 5

Let p be a prime, and let J be the collection of polynomials in $\mathbb{Z}[x]$ whose constant term is divisible by p. We will show that J is a maximal ideal in $\mathbb{Z}[x]$.