

Problem (Problem 1):

- (a) Determine every holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ satisfying $\operatorname{Re}(f(z)) = \operatorname{Re}(z)^2 - \operatorname{Im}(z)^2$.
- (b) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be given by

$$f(z) := \sqrt{|\operatorname{Re}(z)\operatorname{Im}(z)|}.$$

Show that the Cauchy–Riemann equations are satisfied for f at $z = 0$, but f is not differentiable at $z = 0$.

Solution:

- (a) We want to determine $f: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

satisfies

$$u(x, y) = x^2 - y^2,$$

and the Cauchy–Riemann equations:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

First, we must verify that u is indeed harmonic. This follows from the fact that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= 2 \\ \frac{\partial^2 u}{\partial y^2} &= -2.\end{aligned}$$

Furthermore, we see that u is C^3 , as all of the third partial derivatives are equal to zero. Therefore, a harmonic conjugate of u exists and ensures that f is holomorphic on \mathbb{C} . By evaluating the Cauchy–Riemann equations separately, we find that

$$\frac{\partial v}{\partial y} = 2x,$$

or $v = 2xy + K(x)$, and

$$-\frac{\partial v}{\partial x} = -2y,$$

or $v = 2xy + L(y)$. These are only in harmony when $v = 2xy + c$, where $c \in \mathbb{C}$ is a constant. Thus, we find that

$$f(x + iy) = (x^2 - y^2) + i(2xy) + c$$

is necessarily (up to a constant) unique.

- (b) We write f as

$$f(x + iy) = \sqrt{|xy|}.$$

Problem (Problem 2): Let $U \subseteq \mathbb{C}$ be a region, and let $f: U \rightarrow \mathbb{C}$ be a function.

- (a) Suppose that f and \bar{f} are both holomorphic. Show that f is constant.
- (b) Suppose that f is holomorphic and $\operatorname{Re}(f)$ is constant. Show that f is constant.

Solution:

- (a) Write $f(x + iy) = u(x, y) + iv(x, y)$. Since f is holomorphic, we thus get

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Now, since \bar{f} is also holomorphic, we have

$$\overline{f(x + iy)} = u(x, y) - iv(x, y),$$

meaning that

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}\end{aligned}$$

or that

$$\begin{aligned}\frac{\partial u}{\partial x} &= \pm \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= \pm \frac{\partial v}{\partial x}.\end{aligned}$$

Considering the first equation, we then get that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, or that

$$\begin{aligned}u &= c_1(y) \\ v &= d_1(x),\end{aligned}$$

while in the second equation, we get that $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$, meaning that u and v are thus constant. Therefore, f is constant.

- (b) If f is holomorphic and $\operatorname{Re}(f)$ is constant, then $i \operatorname{Im}(f) = f - \operatorname{Re}(f)$ is holomorphic as it is the difference of two holomorphic functions, so $-i \operatorname{Im}(f)$ is holomorphic as it is a constant multiple of a holomorphic function, and thus $\operatorname{Re}(f) - i \operatorname{Im}(f)$ is holomorphic as it is the sum of two holomorphic functions. This gives \bar{f} is holomorphic, so f is constant.

Problem (Problem 3): Let $U, V \subseteq \mathbb{C}$ be open sets, $f: V \rightarrow U$ holomorphic for which $\operatorname{Re}(f), \operatorname{Im}(f) \in C^2(V)$, and $u: U \rightarrow \mathbb{R}$ harmonic. Show that $u \circ f: V \rightarrow \mathbb{R}$ is a harmonic function.

Solution: We write $f(x + iy) = k(x, y) + i\ell(x, y)$, so that $u \circ f(x + iy) = u(k(x, y), \ell(x, y))$. Observe then that this means $u \circ f$ is in $C^2(V)$, and that u is harmonic as a function of k and ℓ .

Using the fact that $u \circ f$ is in $C^2(V)$, we use the chain rule by taking

$$\frac{\partial^2(u \circ f)}{\partial x^2} + \frac{\partial^2(u \circ f)}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial(u \circ f)}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial(u \circ f)}{\partial y} \right)$$

$$\begin{aligned}
&= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial k} \frac{\partial k}{\partial x} + \frac{\partial u}{\partial \ell} \frac{\partial \ell}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial k} \frac{\partial k}{\partial y} + \frac{\partial u}{\partial \ell} \frac{\partial \ell}{\partial y} \right) \\
&= \frac{\partial u}{\partial k} \frac{\partial^2 k}{\partial x^2} + \frac{\partial u}{\partial \ell} \frac{\partial^2 \ell}{\partial x^2} + \frac{\partial u}{\partial k} \frac{\partial^2 k}{\partial y^2} + \frac{\partial u}{\partial \ell} \frac{\partial^2 \ell}{\partial y^2} \\
&\quad + \frac{\partial k}{\partial x} \left(\frac{\partial k}{\partial x} \frac{\partial}{\partial k} + \frac{\partial \ell}{\partial x} \frac{\partial}{\partial \ell} \right) \left(\frac{\partial u}{\partial k} \right) + \frac{\partial \ell}{\partial x} \left(\frac{\partial k}{\partial x} \frac{\partial}{\partial k} + \frac{\partial \ell}{\partial x} \frac{\partial}{\partial \ell} \right) \left(\frac{\partial u}{\partial \ell} \right) \\
&\quad + \frac{\partial k}{\partial y} \left(\frac{\partial k}{\partial y} \frac{\partial}{\partial k} + \frac{\partial \ell}{\partial y} \frac{\partial}{\partial \ell} \right) \left(\frac{\partial u}{\partial k} \right) + \frac{\partial \ell}{\partial y} \left(\frac{\partial k}{\partial y} \frac{\partial}{\partial k} + \frac{\partial \ell}{\partial y} \frac{\partial}{\partial \ell} \right) \left(\frac{\partial u}{\partial \ell} \right) \\
&= \frac{\partial u}{\partial k} \frac{\partial^2 k}{\partial x^2} + \frac{\partial u}{\partial \ell} \frac{\partial^2 \ell}{\partial x^2} + 2 \frac{\partial^2 u}{\partial k \partial \ell} \frac{\partial k}{\partial x} \frac{\partial \ell}{\partial x} + \frac{\partial u}{\partial k} \frac{\partial^2 k}{\partial y^2} + \frac{\partial u}{\partial \ell} \frac{\partial^2 \ell}{\partial y^2} + 2 \frac{\partial^2 u}{\partial k \partial \ell} \frac{\partial k}{\partial y} \frac{\partial \ell}{\partial y} \\
&\quad + \frac{\partial^2 u}{\partial k^2} \left(\frac{\partial k}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial \ell^2} \left(\frac{\partial \ell}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial k^2} \left(\frac{\partial k}{\partial y} \right)^2 + \frac{\partial^2 u}{\partial \ell^2} \left(\frac{\partial \ell}{\partial y} \right)^2,
\end{aligned}$$

where we first used the fact that the mixed partials of u are continuous as u is harmonic. Since k and ℓ are C^2 real/imaginary components of a holomorphic function, they are harmonic, so by reducing via the Cauchy–Riemann equations, we find

$$\begin{aligned}
&= \frac{\partial u}{\partial k} \left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2} \right) + \frac{\partial u}{\partial \ell} \left(\frac{\partial^2 \ell}{\partial x^2} + \frac{\partial^2 \ell}{\partial y^2} \right) \\
&\quad + \frac{\partial^2 u}{\partial k \partial \ell} \left(\frac{\partial \ell}{\partial y} \right) \frac{\partial \ell}{\partial x} + \frac{\partial^2 u}{\partial k \partial \ell} \left(-\frac{\partial \ell}{\partial x} \right) \frac{\partial \ell}{\partial y} \\
&\quad + \left(\frac{\partial k}{\partial x} \right)^2 \left(\frac{\partial^2 u}{\partial k^2} + \frac{\partial^2 u}{\partial \ell^2} \right) + \left(\frac{\partial k}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial k^2} + \frac{\partial^2 u}{\partial \ell^2} \right) \\
&= 0,
\end{aligned}$$

so $u \circ f$ is harmonic.