

Tensor Products, Bilinear Maps, and Linear Maps

First, we review some definitions of algebraic tensor products.

Definition of Tensor Products

From linear algebra, we know that if X and Y are vector spaces, then the *tensor product* of X and Y , denoted $X \otimes Y$, is the universal object such that if $t: X \times Y \rightarrow Z$ is a bilinear map, then there is a unique linear map $T: X \otimes Y \rightarrow Z$, alongside an injection $(x, y) \mapsto x \otimes y$ such that $T \circ \iota = t$.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\iota} & X \otimes Y \\ & \searrow t & \downarrow T \\ & & Z \end{array}$$

Elements of the tensor product are linear combinations of the form

$$u = \sum_{i=1}^n x_i \otimes y_i.$$

Identifying Tensor Products as Spaces of Maps

We observe that if t is a bilinear *form* — i.e., a bilinear map $t: X \times Y \rightarrow \mathbb{F}$ for some field \mathbb{F} — then there is a unique linear map $T: X \otimes Y \rightarrow \mathbb{F}$, meaning that the space of bilinear forms, $\text{hom}(X \times Y, \mathbb{F})$, is in one-to-one correspondence with elements of the algebraic dual of the tensor product, $(X \otimes Y)'$.

We can in fact view tensors in and of themselves as bilinear forms. For any $x \in X$ and $y \in Y$, we may define a bilinear form $B_{x,y}: X' \times Y' \rightarrow \mathbb{F}$ by $B_{x,y}(\varphi, \psi) = \varphi(x)\psi(y)$. Thus, we have a unique linear map $X \otimes Y \rightarrow \text{hom}(X' \times Y', \mathbb{F})$ taking $x \otimes y \mapsto B_{x,y}$.

The map is injective, since if

$$\sum_{i=1}^n B_{x_i, y_i} = 0,$$

then for any $\varphi \in X'$ and $\psi \in Y'$, we have

$$\sum_{i=1}^n \varphi(x_i)\psi(y_i) = 0.$$

Since X' separates the points of X and similarly for Y' and the points of Y , we must have that

$$\sum_{i=1}^n x_i \otimes y_i = 0.$$

Thus, we have an embedding $X \otimes Y \hookrightarrow \text{hom}(X' \times Y', \mathbb{F})$; if X and Y are dual spaces, then there is a corresponding embedding $X' \otimes Y' \hookrightarrow \text{hom}(X \times Y, \mathbb{F})$ by identifying

$$\sum_{i=1}^n \varphi_i \otimes \psi_i \mapsto \left((x, y) \mapsto \sum_{i=1}^n \varphi_i(x)\psi_i(y) \right).$$

For any bilinear form B , there are two associated linear maps, $L_B: X \rightarrow Y'$ and $R_B: Y \rightarrow X'$, given by

$$\begin{aligned} B(x, y) &= \langle y, L_B(x) \rangle \\ &= \langle x, R_B(y) \rangle, \end{aligned}$$

where we let $\langle \cdot, \cdot \rangle : X \times X' \rightarrow \mathbb{F}$ denote the canonical duality given by

$$\langle x, \varphi \rangle = \varphi(x).$$

Thus, we see that every element of the tensor product,

$$u = \sum_{i=1}^n x_i \otimes y_i,$$

gives us two linear maps

$$\begin{aligned} L_u(\varphi) &= \sum_{i=1}^n \varphi(x_i) y_i \\ R_u(\psi) &= \sum_{i=1}^n \psi(y_i) x_i, \end{aligned}$$

giving two more identifications $X \otimes Y \hookrightarrow \text{hom}(X', Y)$ and $X \otimes Y \hookrightarrow \text{hom}(Y', X)$.

If one of X or Y is a dual space, we get $X' \otimes Y \hookrightarrow \text{hom}(X, Y)$ and $X \otimes Y' \hookrightarrow \text{hom}(Y, X)$.

Specifically, the elements of $\text{hom}(X, Y)$ that correspond to elements of $X' \otimes Y$ are the finite-rank linear maps.

Examples

Example (Matrices): If we let \mathbb{F}^n and \mathbb{F}^m be endowed with the standard bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$, we may identify $\mathbb{F}^n \otimes \mathbb{F}^m \cong \mathbb{M}_{m,n}(\mathbb{F})$.

We may identify $e_i \otimes f_j$ with the matrix unit e_{ij} .

Example (Vector-Valued Functions): Let $\mathcal{F}(S)$ denote the vector space of functions from a set S into the field \mathbb{F} ; if X is a vector space, then $\mathcal{F}(S, X)$ may denote the vector space of all functions from S to X with pointwise operations. Given $f \in \mathcal{F}(S)$ and any $x \in X$, we may define a function from S to X by taking $s \mapsto f(s)x$. We may write this as $f \cdot x$

This defines a bilinear map $\mathcal{F}(S) \times X \rightarrow \mathcal{F}(S, X)$ taking $(f, x) \mapsto f \cdot x$.

We may thus find a linear map

$$\sum_{i=1}^n f_i \otimes x_i \mapsto \sum_{i=1}^n f_i \cdot x_i.$$

We show that this map is injective. Suppose we have

$$\sum_{i=1}^n f_i \cdot x_i = 0,$$

so that

$$\sum_{i=1}^n f_i(s) x_i = 0.$$

Yet, since the evaluation functionals $f \mapsto f(s)$ are a separating subset of $\mathcal{F}(S)$, we have

$$\sum_{i=1}^n f_i \otimes x_i = 0.$$

This gives an embedding of $\mathcal{F}(S) \otimes X \hookrightarrow \mathcal{F}(S, X)$.

Example (Vector-Valued Measures): Let \mathcal{A} be an algebra of subsets of S . The vector space $M_{f.a.}(S)$ denotes the space of all finitely additive scalar-valued measures on \mathcal{A} . An element of $M_{f.a.}(S)$ is a function from \mathcal{A} to \mathbb{F} such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigsqcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$$

for every finite collection of pairwise disjoint sets in \mathcal{A} .

For every vector space X , we can similarly define $M_{f.a.}(S, X)$ to be the space of finitely additive X -valued measures on \mathcal{A} , and obtain an embedding of the tensor product $M_{f.a.}(S) \otimes X \hookrightarrow M_{s.a.}(S, X)$.

Summary

- If X and Y are vector spaces and $t: X \times Y \rightarrow Z$ is a bilinear map, then there is a unique linear map $T: X \otimes Y \rightarrow Z$ such that $T \circ \iota = t$.
- Letting $\text{hom}(X \times Y, \mathbb{F})$ denote the space of bilinear forms on $X \times Y$ to an underlying field \mathbb{F} , there are identifications

$$\begin{aligned} (X \otimes Y)' &\leftrightarrow \text{hom}(X \times Y, \mathbb{F}) \\ X \otimes Y &\hookrightarrow \text{hom}(X' \times Y', \mathbb{F}) \\ X' \otimes Y &\hookrightarrow \text{hom}(X, Y) \\ X \otimes Y' &\hookrightarrow \text{hom}(Y, X). \end{aligned}$$

- We may identify vector-valued functions and measures as

$$\begin{aligned} \mathcal{F}(S) \otimes X &\hookrightarrow \mathcal{F}(S, X) \\ M_{f.a.}(S) \otimes X &\hookrightarrow M_{f.a.}(S, X). \end{aligned}$$

Projective Tensor Products

If X and Y are Banach spaces, there are a variety of ways we may seek to norm the tensor product $X \otimes Y$. The basic requirement we have is that we want

$$\|x \otimes y\| \leq \|x\| \|y\|,$$

and for any representation

$$u = \sum_{i=1}^n x_i \otimes y_i,$$

we want

$$\|u\| \leq \sum_{i=1}^n \|x_i\| \|y_i\|.$$

Therefore, we must have

$$\|u\| \leq \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

The latter value is thus the largest possible candidate for a norm on $X \otimes Y$ that has these desired qualities. We thus define

$$\|u\|_{\wedge} = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}$$

to be the *projective norm* on $X \otimes Y$.

Proposition: Let X and Y be Banach spaces. Then, $\|\cdot\|_{\wedge}$ is a norm on $X \otimes Y$ with $\|x \otimes y\|_{\wedge} = \|x\|\|y\|$.

Proof. We start by showing homogeneity. Assume $\lambda \neq 0$. Then, if

$$u = \sum_{i=1}^n x_i \otimes y_i,$$

we have

$$\lambda u = \sum_{i=1}^n (\lambda x_i) \otimes y_i,$$

we have

$$\begin{aligned} \|\lambda u\|_{\wedge} &\leq \sum_{i=1}^n \|\lambda x_i\| \|y_i\| \\ &= |\lambda| \sum_{i=1}^n \|x_i\| \|y_i\| \\ &= |\lambda| \|u\|_{\wedge}. \end{aligned}$$

Similarly,

$$\|u\|_{\wedge} \leq |\lambda|^{-1} \|\lambda u\|_{\wedge},$$

whence $\|\lambda u\|_{\wedge} = |\lambda| \|u\|_{\wedge}$.

Now, let $\varepsilon > 0$ and let $u, v \in X \otimes Y$ have representations

$$\begin{aligned} u &= \sum_{i=1}^n x_i \otimes y_i \\ v &= \sum_{j=1}^m w_j \otimes z_j \end{aligned}$$

such that

$$\begin{aligned} \sum_{i=1}^n \|x_i\| \|y_i\| &\leq \|u\|_{\wedge} + \varepsilon/2 \\ \sum_{j=1}^m \|w_j\| \|z_j\| &\leq \|v\|_{\wedge} + \varepsilon/2. \end{aligned}$$

Then, we have a representation

$$u + v = \sum_{i=1}^n x_i \otimes y_i + \sum_{j=1}^m w_j \otimes z_j,$$

so that

$$\begin{aligned} \|u + v\|_{\wedge} &\leq \sum_{i=1}^n \|x_i\| \|y_i\| + \sum_{j=1}^m \|w_j\| \|z_j\| \\ &\leq \|u\|_{\wedge} + \|v\|_{\wedge} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we obtain the triangle inequality.

Now, we show that the norm is definite. Let $\|u\|_{\wedge} = 0$. Then, for any $\varepsilon > 0$, there is a representation

$$u = \sum_{i=1}^n x_i \otimes y_i$$

such that

$$\sum_{i=1}^n \|x_i\| \|y_i\| \leq \varepsilon.$$

In particular, for any $\varphi \in X^*$ and $\psi \in Y^*$, we have

$$\left| \sum_{i=1}^n \varphi(x_i) \psi(y_i) \right| \leq \varepsilon \|\varphi\| \|\psi\|.$$

Since the quantity $\sum_{i=1}^n \varphi(x_i) \psi(y_i)$ is independent of the representation of u , it follows that this sum equals zero. Yet, since X^* and Y^* separate the points of X and Y , it follows that $u = 0$.

Finally, we know that $\|x \otimes y\|_{\wedge} \leq \|x\| \|y\|$, so we let $\varphi \in B_{X^*}$ and $\psi \in B_{Y^*}$ such that $\varphi(x) = \|x\|$ and $\psi(y) = \|y\|$. We let $b: X \times Y \rightarrow \mathbb{F}$ be given by $B(w, z) = \varphi(w) \psi(z)$. The linearization B is a linear functional on $X \otimes Y$ with

$$\begin{aligned} \left| B\left(\sum_{i=1}^n x_i \otimes y_i\right) \right| &\leq \sum_{i=1}^n |B(x_i \otimes y_i)| \\ &= \sum_{i=1}^n |\varphi(x_i) \psi(y_i)| \\ &\leq \sum_{i=1}^n \|x_i\| \|y_i\|, \end{aligned}$$

so that $|B(u)| \leq \|u\|_{\wedge}$ for every $u \in X \otimes Y$. In particular, this means that B is a bounded linear functional on the normed space $(X \otimes Y, \|\cdot\|_{\wedge})$ with norm at most 1, whence $\|x\| \|y\| = B(x \otimes y) \leq \|x \otimes y\|_{\wedge}$. \square

We may thus complete $(X \otimes Y, \|\cdot\|_{\wedge})$ with respect to the projective norm to obtain the *projective tensor product* of the Banach spaces X and Y , which we denote $X \hat{\otimes} Y$.

If $A \subseteq X$ and $B \subseteq Y$ are subsets, then we will let

$$A \otimes B := \{x \otimes y \mid x \in A, y \in B\}.$$

Proposition: The closed unit ball of $X \hat{\otimes} Y$ is the closed convex hull of $B_X \otimes B_Y$.

Proof. Since the closed unit ball is the closure of the unit ball in $X \otimes Y$, it suffices to prove the proposition for the space $(X \otimes Y, \|\cdot\|_{\wedge})$. Let u be an element of the open unit ball of $(X \otimes Y, \|\cdot\|_{\wedge})$.

By the definition of the projective norm, there is a representation

$$u = \sum_{i=1}^n x_i \otimes y_i$$

such that

$$\sum_{i=1}^n \|x_i\| \|y_i\| < 1.$$

Let

$$w_i = \|x_i\|^{-1} x_i$$

$$z_i = \|y_i\|^{-1} y_i$$

$$\lambda_i = \|x_i\| \|y_i\|.$$

Then, we have

$$u = \sum_{i=1}^n \lambda_i w_i \otimes z_i$$

with $w_i \in B_X$, $z_i \in B_Y$, $\lambda_i \geq 0$, and $\sum_{i=1}^n \lambda_i < 1$. Thus, $u \in \text{conv}(B_X \otimes B_Y)$, meaning that the closed unit ball of $X \otimes Y$ is contained in $\overline{\text{conv}}(B_X \otimes B_Y)$. Yet, we must also have $B_X \otimes B_Y$ contained in the closed unit ball of $(X \otimes Y, \|\cdot\|_\wedge)$, so it holds for $\overline{\text{conv}}(B_X \otimes B_Y)$. \square

Tensor Products of Linear Operators on Banach Spaces

In general, if we have two linear maps $S: X \rightarrow E$ and $T: Y \rightarrow F$, we have a bilinear map $X \times Y \rightarrow E \otimes F$ given by $(x, y) \mapsto (Sx) \otimes (Ty)$. From the universal property, we get a linear map $S \otimes T: X \otimes Y \rightarrow E \otimes F$ such that $(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$.

When X and Y have norms, and we are concerned with continuity, we observe that if $u \in X \otimes Y$ has representation

$$u = \sum_{i=1}^n x_i \otimes y_i,$$

we have

$$\begin{aligned} \|(S \otimes T)(u)\|_\wedge &= \left\| \sum_{i=1}^n (Sx_i) \otimes (Ty_i) \right\| \\ &\leq \|S\|_{\text{op}} \|Y\|_{\text{op}} \sum_{i=1}^n \|x_i\| \|y_i\| <> \end{aligned}$$

whence

$$\|(S \otimes T)(u)\|_\wedge \leq \|S\|_{\text{op}} \|T\|_{\text{op}} \|u\|_\wedge.$$

This gives that $\|S \otimes T\|_{\text{op}} \leq \|S\|_{\text{op}} \|T\|_{\text{op}}$. Meanwhile, since $\|x \otimes y\|_\wedge = \|x\| \|y\|$, we have $\|S \otimes T\|_{\text{op}} \geq \|S\|_{\text{op}} \|T\|_{\text{op}}$, so that $\|S \otimes T\|_{\text{op}} = \|S\|_{\text{op}} \|T\|_{\text{op}}$.

Finally, we may extend $S \otimes T$ to the completions $X \hat{\otimes} Y$ and $E \hat{\otimes} F$. This gives the following proposition.

Proposition: Let $S: X \rightarrow E$ and $T: Y \rightarrow F$ be bounded linear operators. Then, there is a unique operator

$$\begin{aligned} S \hat{\otimes} T: X \hat{\otimes} Y &\rightarrow E \hat{\otimes} F \\ (x \otimes y) &\mapsto (Sx) \otimes (Ty). \end{aligned}$$

Furthermore, $\|S \hat{\otimes} T\|_{\text{op}} = \|S\|_{\text{op}} \|T\|_{\text{op}}$.

Inheritance of the Projective Norm

In general, the projective tensor product does not respect subspaces, in the sense that if $W \leq X$ is a subspace, so that $W \otimes Y \leq X \otimes Y$ is an algebraic subspace, the norm on $W \otimes Y$ induced by $(X \otimes Y, \|\cdot\|_\wedge)$ is not necessarily the same as the projective norm on $W \otimes Y$.

This follows from the fact that the definition of the norm $(W \otimes Y, \|\cdot\|_\wedge)$ is restricted to all representations in $W \otimes Y$, and since there are more representations for u in $X \otimes Y$, it follows that the norm of u in $(X \otimes Y, \|\cdot\|_\wedge)$ is lesser than or equal to the norm of u in $(W \otimes Y, \|\cdot\|_\wedge)$.

We start by discussing the special case of complemented subspace. Recall that a closed subspace $E \leq X$ is called *complemented* if there is another closed subspace W such that $X = E \oplus W$. An equivalent characterization of a complemented subspace is that there is a continuous projection $P_E: X \rightarrow E$ such that $X = E \oplus \ker(P_E)$.

Proposition: Let E and F be complemented subspaces of X and Y respectively. Then, $E \otimes F$ is complemented in $X \otimes Y$, and the norm on $E \otimes F$ induced by the projective norm on $X \otimes Y$ is equivalent to the projective norm (in the sense of norm equivalence) on $E \otimes F$.

If E and F are complemented by projections of norm 1, then $E \otimes F$ is a subspace of $X \otimes Y$ that is also complemented by a projection of norm 1.

Proof. Let P and Q be projections from X, Y onto E, F respectively. Then, $P \otimes Q$ is a projection of $X \otimes Y$ onto $E \otimes F$.

Let $u \in E \otimes F$. We have that $\|u\|_{\wedge, X \otimes Y} \leq \|u\|_{\wedge, E \otimes F}$, so we let

$$u = \sum_{i=1}^n x_i \otimes y_i$$

be a representation of u in $X \otimes Y$. Then,

$$\begin{aligned} u &= P \otimes Q(u) \\ &= \sum_{i=1}^n (Px_i) \otimes (Qy_i) \end{aligned}$$

is a representation of u in $E \otimes F$, whence

$$\begin{aligned} \|u\|_{\wedge, E \otimes F} &\leq \sum_{i=1}^n \|Px_i\| \|Qy_i\| \\ &\leq \|P\|_{\text{op}} \|Q\|_{\text{op}} \sum_{i=1}^n \|x_i\| \|y_i\|, \end{aligned}$$

so it follows that for every representation of u in $X \otimes Y$, we have

$$\begin{aligned} \|u\|_{\wedge, X \otimes Y} &\leq \|u\|_{\wedge, E \otimes F} \\ &\leq \|P\|_{\text{op}} \|Q\|_{\text{op}} \|u\|_{\wedge, X \otimes Y}. \end{aligned}$$

If E and F are complemented by projections of norm 1, we have $\|u\|_{\wedge, X \otimes Y} = \|u\|_{\wedge, E \otimes F}$ for every $u \in E \otimes F$, as $\|P \otimes Q\|_{\text{op}} = \|P\|_{\text{op}} \|Q\|_{\text{op}}$. \square

Now, we may explain why exactly the norm is known as the projective norm. Recall that a linear map $Q: X \rightarrow Y$ is known as a 1-quotient map if Q is surjective and $Q(B_X) = B_Y$, meaning that Y is isometrically isomorphic to $X/\ker(Q)$. An equivalent condition to this is

$$\|y\| = \inf\{\|x\| \mid x \in X, Qx = y\}$$

for every $y \in Y$.

Proposition: Let $Q: W \rightarrow X$ and $R: Z \rightarrow Y$ be 1-quotient maps. Then, $Q \otimes R$ is a quotient operator mapping $W \hat{\otimes} Z \rightarrow X \hat{\otimes} Y$.

Proof. It suffices to show that

$$Q \otimes R: W \otimes Z \rightarrow X \otimes Y$$

is a 1-quotient map. To see that $Q \otimes R$ is surjective, let $\sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$. There exist $w_i \in W$ and

$z_i \in Z$ with $Qw_i = x_i$ and $Rz_i = y_i$, so that

$$Q \otimes R \left(\sum_{i=1}^n w_i \otimes z_i \right) = \sum_{i=1}^n x_i \otimes y_i.$$

Thus, $Q \otimes R$ is surjective.

Now, let $u \in X \otimes Y$. Letting $(Q \otimes R)v = u$, so that

$$\begin{aligned} \|u\|_{\wedge} &\leq \|Q\|_{\text{op}} \|R\|_{\text{op}} \|v\|_{\wedge} \\ &= \|v\|_{\wedge}. \end{aligned}$$

Given $\varepsilon > 0$, pick a representation

$$u = \sum_{i=1}^n x_i \otimes y_i$$

such that

$$\sum_{i=1}^n \|x_i\| \|y_i\| \leq \|u\|_{\wedge} + \varepsilon.$$

For each i , select $w_i \in W$ and $z_i \in Z$ with $Qw_i = x_i$, $Rz_i = y_i$, and

$$\begin{aligned} \|w_i\| &\leq (1 + 2^{-n}\varepsilon) \|x_i\| \\ \|y_i\| &\leq (1 + 2^{-n}\varepsilon) \|y_i\|. \end{aligned}$$

Then,

$$(Q \otimes R) \left(\sum_{i=1}^n w_i \otimes z_i \right) = u,$$

and by using the estimate

$$\prod_{i=1}^n (1 + a_i) \leq e^{\sum_{i=1}^n a_i},$$

we have

$$\begin{aligned} \left\| \sum_{i=1}^n w_i \otimes z_i \right\|_{\wedge} &\leq e^{4\varepsilon} \left(\sum_{i=1}^n \|x_i\| \|y_i\| \right) \\ &\leq e^{4\varepsilon} (\|u\|_{\wedge} + \varepsilon). \end{aligned}$$

Since this holds for every $\varepsilon > 0$, it follows that

$$\|u\|_{\wedge} = \inf \{ \|v\|_{\wedge} \mid v \in W \otimes Z, (Q \otimes R)v = u \}.$$

□

The Dual Space of $X \hat{\otimes} Y$

Recall that a bilinear map $B: X \times Y \rightarrow Z$ is called bounded if there exists a constant C such that

$$\|B(x, y)\| \leq C \|x\| \|y\|$$

for every $x \in X$ and $y \in Y$.

Theorem: Let $b: X \rightarrow Y$ be a bounded bilinear map. Then, there exists a unique linear map $B: X \hat{\otimes} Y \rightarrow Z$ such that $B(x \otimes y) = b(x, y)$ for every x, y . The correspondence $b \leftrightarrow B$ is an isometric isomorphism between $B(X \times Y, Z)$ and $B(X \hat{\otimes} Y, Z)$.

Proof. The existence of the linear map follows from the universal property of tensor products. We start by showing that B is bounded on the projective norm of $X \otimes Y$. For $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, we have

$$\begin{aligned} \|Bu\| &= \left\| \sum_{i=1}^n b(x_i, y_i) \right\| \\ &\leq \|b\|_{\text{op}} \sum_{i=1}^n \|x_i\| \|y_i\|. \end{aligned}$$

This holds for every representation of u , meaning that $\|Bu\| \leq \|b\|_{\text{op}} \|u\|_{\wedge}$, so B is bounded and satisfies $\|B\|_{\text{op}} \leq \|b\|_{\text{op}}$. Yet, since

$$\begin{aligned} \|b(x, y)\| &= \|B(x \otimes y)\| \\ &\leq \|B\|_{\text{op}} \|x\| \|y\|, \end{aligned}$$

so that $\|b\|_{\text{op}} = \|B\|_{\text{op}}$. The operator B has a unique extension to $B: X \hat{\otimes} Y \rightarrow Z$ with the same norm. We only need to show now that this is surjective.

If $T \in B(X \hat{\otimes} Y, Z)$, then the bounded bilinear map $b: X \times Y \rightarrow Z$ may be defined by $b(x, y) = T(x \otimes y)$. \square

This gives a canonical identification

$$B(X \times Y, Z) \cong B(X \hat{\otimes} Y, Z).$$

In particular, if $Z = \mathbb{F}$, then

$$(X \hat{\otimes} Y)^* \cong B(X \times Y, \mathbb{F}),$$

with the action given by

$$\left\langle \sum_{i=1}^n x_i \otimes y_i, b \right\rangle = \sum_{i=1}^n b(x_i, y_i),$$

where $\langle \cdot, \cdot \rangle$ is the canonical duality between a vector space and its dual. In particular, from the Hahn–Banach Theorems, we get the expression for the projective norm

$$\|u\|_{\wedge} = \sup \left\{ |u, b| \mid b \in B(X \times Y, \mathbb{F}), \|b\|_{\text{op}} \leq 1 \right\}.$$

This can be expanded further by using the identification

$$B(X \times Y, \mathbb{F}) \cong B(X, Y^*),$$

whence

$$(X \hat{\otimes} Y)^* \cong B(X, Y^*),$$

with the action of $S \in B(X, Y^*)$ given by

$$\left\langle \sum_{i=1}^n x_i \otimes y_i, S \right\rangle = \sum_{i=1}^n \langle y_i, Sx_i \rangle.$$

This gives two more variations on the duality for a projective norm, given by

$$\begin{aligned} \|u\|_{\wedge} &= \sup \left\{ |\langle u, S \rangle| \mid S \in B(X, Y^*), \|S\|_{\text{op}} \leq 1 \right\} \\ &= \sup \left\{ |\langle u, T \rangle| \mid T \in B(Y, X^*), \|T\|_{\text{op}} \leq 1 \right\}. \end{aligned}$$

Corollary: Let W be a subspace of X . Then, $W \hat{\otimes} Y$ is a subspace of $X \hat{\otimes} Y$ if and only if every operator $S: W \rightarrow Y^*$ extends to an operator with the same norm from X into Y^* .

Recall that a Banach space Z is called *injective* if it has the property that, for every Banach space X and every subspace W of X , every operator $S: W \rightarrow Z$ extends to an operator $\tilde{S}: X \rightarrow Z$ with the same norm.

Equivalently, Z is injective if and only if Z is complemented by a norm-one projection in any Banach space that contains Z as a subspace.

Theorem: Every bounded bilinear form on $X \times Y$ admits an extension to a bounded bilinear form on $X^{**} \times Y^{**}$ with the same norm.

Proof. Let A be a bounded bilinear form on $X \times Y$, and let S be the operator from X into Y^* given such that $A(x, y) = \langle y, Sx \rangle$ for every $x \in X$ and $y \in Y$.

Consider the bounded bilinear form B on $X^{**} \times Y^{**}$ given by

$$B(x^{**}, y^{**}) = \langle S^* y^{**}, x^{**} \rangle,$$

where $S^*: Y^{**} \rightarrow X^*$ is the adjoint of S . For any $x \in \iota(X)$ and $y \in \iota(Y)$, where ι is the canonical injection into the double dual, we have

$$\begin{aligned} B(\hat{x}, \hat{y}) &= \langle S^* \hat{y}, \hat{x} \rangle \\ &= \langle y, Sx \rangle \\ &= A(x, y), \end{aligned}$$

so that B is an extension of A satisfying $\|B\|_{\text{op}} = \|A\|_{\text{op}}$. □

References

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