

Complex Numbers

A complex number is an ordered pair of real numbers, $(a, b) = a + bi$. A vector in \mathbb{R}^2 is also an ordered pair, (a, b) of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with \mathbb{R}^2 . However, unlike vectors in \mathbb{R}^2 , there is also an operation \cdot . We desire for $(0, 1) \cdot (0, 1) = (-1, 0)$; essentially, $i^2 = -1$. We say that i is a square root of -1 ; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di) \\ &:= a(c) + adi + bci + bd(i^2) \\ &:= (ac - bd) + (ad + bc)i \\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus, \mathbb{R}^2 with the operations $+$ and the above defined complex multiplication is known as \mathbb{C} . We write as $a + bi$ instead of (a, b) .

Given $z = (a + bi) \in \mathbb{C}$, we write $\operatorname{Re}(z) = a$ and $\operatorname{Im}(z) = b$. If $\operatorname{Im}(z) = 0$, then $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$. However, many people say that $\mathbb{R} \subseteq \mathbb{C}$, even if \mathbb{C} isn't defined as such.

Reciprocals of Complex Numbers

Let $z \in \mathbb{C}$, where $z \neq 0$. Then, $\exists w \in \mathbb{C}$ such that $zw = 1$.

Let $w = c + di$. We want to show that $zw = 1$.

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1 \\ ad + bc &= 0.\end{aligned}$$

Thus, let $w = c + di$, with $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2} \\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every $z \neq 0$, with $z = a + bi$, the *reciprocal* of z is defined as $\frac{1}{z} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$. Then, for $w \in \mathbb{C}$, we define

$$\frac{w}{z} := w \left(\frac{1}{z} \right).$$

Properties of Complex Numbers

Let $z = a + bi \in \mathbb{C}$. Then, the (Euclidean) norm (or absolute value) of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of $z = a + bi$ is $\bar{z} = a - bi$.

$$(i) \quad z\bar{z} = |z|^2$$

$$(ii) \quad \overline{(\bar{z})} = z$$

$$(iii) \overline{(z + w)} = \bar{z} + \bar{w}$$

$$(iv) \overline{zw} = \bar{z} \cdot \bar{w}$$

$$(v) z + \bar{z} = 2\operatorname{Re}(z), \text{ so } \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$(vi) z - \bar{z} = 2i\operatorname{Im}(z), \text{ so } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

Polar Representation

Let $z = a + bi$ (or $z = (a, b)$). Then, $|z| = \sqrt{a^2 + b^2}$ is the *radius*, and the *argument* is found by $\theta = \arctan(b/a)$ for $a \neq 0$. Therefore, the full polar representation is as follows:

$$z = |z| (\cos \theta + i \sin \theta). \quad \theta \in [0, 2\pi)$$

If $z = 0$, then $|z| = 0$, and $\arg z$ is undefined.

For example, we can find $\arg i$ in $[\pi, 3\pi)$ as $\frac{5\pi}{2}$.

For z_1 and z_2 in polar form, we have:

$$|z_1 z_2| = |z_1| |z_2| \quad (1)$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi} \quad (2)$$

Proof of (1):

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2) \overline{(z_1 z_2)} \\ &= z_1 z_2 \bar{z}_1 \bar{z}_2 \\ &= z_1 \bar{z}_1 z_2 \bar{z}_2 \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

Since $|z| \geq 0$, we get $|z_1 z_2| = |z_1| |z_2|$.

Let $z = 2(\cos \pi/6 + i \sin \pi/6)$, and let $f : \mathbb{C} \rightarrow \mathbb{C}$ defined as $f(w) = zw$. Then, f rotates w by $\pi/6$ and scales w by 2.

Theorem: For $n \in \mathbb{N}$, if $z = r(\cos \theta + i \sin \theta)$, then $z^n = r^n(\cos(n\theta) + i \sin(n\theta))$.

Proof: Induct on n . For the base case, we know that $n = 1$ satisfies this property. For $n > 1$, we have:

$$\begin{aligned} z^{n+1} &= (z^n)(z) \\ &= (r^n(\cos(n\theta) + i \sin(n\theta))) r(\cos \theta + i \sin \theta) \\ &= (r^n)(r) (\cos(n\theta + \theta) + i \sin(n\theta + \theta)) && \text{Polar Representation Definition} \\ &= r^{n+1}(\cos((n+1)\theta) + i \sin((n+1)\theta)) \end{aligned}$$

We can use this technique to find the “roots of unity.” For example, to find all z such that $z^3 = 1$, we use our

technique:

$$\begin{aligned}
 z^3 &= 1 \\
 |z| &= 1 \\
 \arg z^3 &= 0 \\
 3 \arg z &= 0 \pmod{2\pi} \\
 \arg z &= \frac{k2\pi}{3} \\
 &= 0, \frac{2\pi}{3}, \frac{4\pi}{3} \\
 z_1 &= 1 \\
 z_2 &= (\cos 2\pi/3 + i \sin 2\pi/3) \\
 z_3 &= (\cos 4\pi/3 + i \sin 4\pi/3)
 \end{aligned}$$

We can see that $z_2^2 = z_3$.

For the n case, we find $z_2 = \cos(2\pi/n) + i \sin(2\pi/n)$, and $z_k = z_2^{k-1}$.

Exponential, Logarithm, and Trigonometric Functions in \mathbb{C}

Exponential

Let $z = a + bi$. We define e^{a+bi} as follows:

$$e^{a+bi} = e^a (\cos b + i \sin b)$$

Recall that for every nonzero complex number, $z = |z| (\cos \theta + i \sin \theta)$, where $\theta = \arg z$. Thus,

$$\begin{aligned}
 z &= |z| e^{i\theta} \\
 &= |z| e^{i \arg z}.
 \end{aligned}$$

The function e^z has some properties similar to the function e^x in real numbers, and some properties varying with the real numbers.

$$\begin{aligned}
 e^z e^w &= e^{z+w} \\
 e^z &\neq 0
 \end{aligned}$$

However, there are some differences:

$$\begin{aligned}
 |e^{i\theta}| &= 1 \\
 e^{a+bi} &= e^a
 \end{aligned}
 \quad \forall \theta$$

From these properties, we find Euler's equation:

$$e^{i\pi} + 1 = 0$$

Additionally, e^z is periodic, while $f(x) = e^x$ is injective:

$$\begin{aligned}
 e^{z+2n\pi} &= e^z (\cos(2n\pi) + i \sin 2n\pi) \\
 &= e^z
 \end{aligned}$$

When examining the function $f : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$, $z \mapsto e^z$, we find that the following happen:

- $f(\mathbb{R}) = (0, \infty)$ — we apply $f(x) = e^x$.
- $f(a + bi) = e^a e^{bi}$ — e^a is rotated by b .
- $f(\mathbb{R} + bi)$ is expressed as the line along b radians through the origin.
- Therefore, $f(A_0) = \mathbb{C} \setminus \{0\}$, where $A_0 = \{a + bi \mid a \in \mathbb{R}, b \in [0, 2\pi)\}$.

Logarithm

Recall that for a function $f : A \rightarrow B$, f^{-1} is a function if f is injective. However, for any f , it is the case that $f^{-1}(b)$ does exist, defined as follows:

$$f^{-1}(b) = \{a \mid f(a) = b\}.$$

For the function $f(z) = e^z$, f is not one to one, so for $w = e^z$, $f^{-1}(w) = \{z' \in \mathbb{C} \mid e^{z'} = w\}$. We can find this as $f^{-1}(w) = \{z + 2n\pi i \mid n \in \mathbb{Z}\}$.

We define $\log(w) := \{z \in \mathbb{C} \mid e^z = w\}$. For a fixed $\theta \in \mathbb{R}$, we define $\log_{A_\theta}(w) := \{z \mid e^z = w, z \in A_\theta\}$.

Let $z = 1 + \frac{5\pi}{2}i$. Then,

$$\log_{A_{-\pi}} e^z = 1 + \frac{\pi}{2}i$$

Let $w \in \mathbb{C} \setminus \{0\}$. To find $\log w$ (all values), then

$$\begin{aligned} z &\in \log w \\ e^z &= w \\ &= |w|e^{i \arg w} \\ e^{a+bi} &= |w|e^{i \arg w} \\ e^a e^{ib} &= |w|e^{i \arg w}. \end{aligned}$$

Therefore, $a = \ln |w|$ and $b = \arg w$. Additionally, the following hold, for $z_1, z_2 \in \mathbb{C}$:

$$\log_{A_\theta}(z_1 z_2) = \log_{A_\theta}(z_1) + \log_{A_\theta}(z_2) + 2n\pi i$$

Cosine and Sine

$$\begin{aligned} e^{ib} &= \cos b + i \sin b \\ e^{-ib} &= \cos b - i \sin b \\ \cos z &:= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &:= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Complex Powers

Recall that for $s, t \in \mathbb{R}$, $s^t = e^{t \ln s}$, where $s > 0$. For $z, w \in \mathbb{C}$, $z^w = e^{w \log z}$, where $z \neq 0$.

$$\begin{aligned} (-2)^i &= e^{i \log(-2)} \\ &= e^{i(\ln(2) + i\pi)} \\ &= e^{i \ln 2 - (\pi + 2\pi n)} \\ &= e^{-\pi + 2\pi n + i \ln 2} \end{aligned}$$

This has *infinitely* many values.

Let $\alpha = u + vi$. Then,

$$\begin{aligned} z^\alpha &= e^{\alpha \log z} \\ &= e^{(u+vi)(\ln |z| + i \arg z)} \\ &= e^{(u \ln |z| - v \arg z)} e^{i(v \ln |z| + u \arg z)} \end{aligned}$$

Since $\arg z = \theta + 2\pi n$ for some real $\theta \in [0, 2\pi)$,

$$= e^{u \ln z} e^{-v(\theta + 2\pi n)} e^{i v \ln |z|} e^{i u(\theta + 2\pi n)}$$

Therefore, complex exponentiation is single-valued if $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Z}$, then z^α has only one value; if $\alpha \in \mathbb{Q}$, where $\alpha = \frac{p}{q}$ and $\gcd(p, q) = 1$, then z^α takes q distinct values, which are the q th-roots.

Continuous Functions with Complex Domains

Let $z \in \mathbb{C}$, let $r > 0$.

- The set $D(z; r) := \{w \mid w \in \mathbb{C}, |z - w| < r\}$ is the r -neighborhood of z .
- A subset $A \subseteq \mathbb{C}$ is open if $(\forall z \in A) (\exists r > 0) \ni D(z; r) \subseteq A$.

For example, if $A = \{z \mid \operatorname{Re}(z) > 0\}$, we can find r equal to half the magnitude of the real component of z for any $z \in A$, meaning A is open.

Meanwhile, if $A = \{z \mid \operatorname{Re}(z) \geq 0\}$, this is not the case. If $z = 0$, then $\nexists r > 0$ such that $D(z; r) \subseteq A$, as any open ball of radius r will have some element in \overline{A} .

- A subset $B \subseteq \mathbb{C}$ is closed if $\overline{B} \subseteq \mathbb{C}$ is open.

For example, $A = \emptyset$ is open, by vacuous truth, so $\overline{A} = \mathbb{C}$ is closed. Similarly, since \mathbb{C} is open, \emptyset is closed.

Meanwhile, $A = \{x + iy \mid -1 \leq x < 1\}$ is neither open nor closed.

Limits

Let $A \subseteq \mathbb{C}$, $f : A \rightarrow \mathbb{C}$, $z_0 \in \mathbb{C}$. Then,

$$\lim_{z \rightarrow z_0} f(z) = \ell$$

means both of the following hold:

- for some $r > 0$, $D(z_0; r) \setminus \{z_0\} \subseteq \operatorname{dom}(f)$,
- $\forall \varepsilon > 0, \exists \delta > 0$ such that $f(D(z_0; \delta) \setminus \{z_0\}) \subseteq D(\ell; \varepsilon)$.

For example, if

$$f(z) = \begin{cases} z & z \in \mathbb{C} \setminus \mathbb{R} \\ 3i & z \in \mathbb{R} \end{cases}$$

Then, $\lim_{z \rightarrow 0} f(z)$ does not exist, as there is no ℓ that satisfies both conditions. Specifically, if $\ell = 3i$, and we set $\varepsilon = 1$, then a disc of any radius around 0 has some $z \in \mathbb{C} \setminus \mathbb{R}$ that maps to itself. Similarly, if we set $\ell = 0$, then there is a real number in a disc of any radius around 0.

Note: f does not have to be defined at z_0 for the limit to be defined at z_0 .

Let $A \subseteq \mathbb{C}$ be open, $f : A \rightarrow \mathbb{C}$, and $z_0 \in A$. We say f is continuous at z_0 if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$. We say f is continuous on A if $\forall z_0 \in A$, f is continuous at z_0 .

We will show that $f : \mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto 3z$ is continuous.

Scratch Work: We want δ such that $f(D(z_0; \delta)) \subseteq D(3z_0; \varepsilon)$. Let $z \in D(z_0; \delta)$, meaning $f(z) = 3z$. We want $3z \in D(3z_0; \varepsilon)$, meaning we want $|3z - 3z_0| < \varepsilon$, or $|z - z_0| < \frac{\varepsilon}{3}$.

Proof: Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{3}$. We show $f(D(z_0; \delta)) \subseteq D(f(z_0); \varepsilon)$. Let $z \in D(z_0; \delta)$. Then, $|z - z_0| < \delta = \varepsilon/3$, meaning $3|z - z_0| < \varepsilon$, meaning $|3z - 3z_0| < \varepsilon$, so $|f(z) - f(z_0)| < \varepsilon$. Therefore, $f(z) \in D(f(z_0); \varepsilon)$. Since f is continuous at arbitrary z_0 , f is continuous on \mathbb{C} .

Sequences

A sequence $z_1, z_2, \dots \in \mathbb{C}$. A sequence converges to $z_0 \in \mathbb{C}$ if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M}, |z_n - z_0| < \varepsilon$$

In words, for any radius around z_0 , we can find z_n arbitrarily close to z_0 for sufficiently large n . We write $z_n \rightarrow z_0$ if this is the case.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$. Then, f is continuous on \mathbb{C} if and only if the following equivalent conditions are met:

- (i) the inverse image of every open set is open ($f^{-1}(B) := \{a \in \mathbb{C} \mid f(a) \in B\}$);
- (ii) the inverse image of every closed set is closed;
- (iii) for every sequence $(z_n)_n$ such that $(z_n)_n \rightarrow z_0$, $f(z_n) \rightarrow f(z_0)$.

Let

$$f(z) = \begin{cases} 0 & z = 0 \\ 1 & z \neq 0 \end{cases}.$$

This function is not continuous. We will check that (i)–(iii) fail.

- (i) Let $B = D(0; 1)$. Then, $f^{-1}(B) = \{0\}$, which is not open set.
- (ii) Let $B = \text{cl}(D(1; 0.5))$. Then, $f^{-1}(B) = \mathbb{C} \setminus \{0\}$, which is not closed.
- (iii) Let $z_n = \frac{1}{n}$. Then, $(z_n)_n \rightarrow 0$, but $f(z_n) = 1$ for all n , meaning $f(z_n) \rightarrow 1 \neq f(0)$.

To show limit divergence, recall the definition of limit convergence:

$$\lim_{n \rightarrow \infty} z_n = z_0 \Leftrightarrow (\forall \varepsilon > 0)(\exists M \in \mathbb{N}) \ni \forall z_{n>M}, |z_n - z_0| < \varepsilon.$$

Let $z_1, \dots \in \mathbb{C}$ be a sequence. Then, $\lim_{n \rightarrow \infty} z_n = \infty$ means

$$(\forall M > 0)(\exists N \in \mathbb{N}) \ni \forall n > N, |z_n| > M.$$

In words, $|z_n|$ is arbitrarily large for sufficiently large n .

Connected Sets

Let $a, b \in \mathbb{C}$. A path from a to b is a continuous function $p : [0, 1] \rightarrow \mathbb{C}$ such that $p(0) = a$ and $p(1) = b$. Let $S \subseteq \mathbb{C}$. If $p([0, 1]) \subseteq S$, then p is a path in S .

We say S is *path-connected* if for any $s, t \in S$, there is a path in S from s to t .

Every set that is path-connected is connected, but not necessarily the other way around — if A is open and path connected, then A is connected.

An open, path-connected subset of \mathbb{C} is known as a region, or a domain.

Let $A = \mathbb{R} \times \{0\}$ (or the x axis in \mathbb{C}). A is not a region, as A is not an open set, even if A is path-connected.

$A \subseteq \mathbb{C}$ is bounded if there exists $r > 0$ such that $A \subseteq D(0; r)$. $A = \mathbb{R} \times \{0\}$ is not bounded.

If $A \subseteq \mathbb{C}$, then A is compact if A is closed and bounded. There are various properties of compact sets that make them particularly amenable towards analysis.

Extreme Value Theorem: Every real-valued continuous function on a compact domain attains its maximum and minimum values.

Uniform Continuity Theorem: Elaborated below.

Uniform Continuity

Recall that if $f : A \rightarrow \mathbb{C}$, f is continuous if $\forall a \in A, \lim_{z \rightarrow a} f(z) = f(a)$.

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta_a > 0) \ni f(D(a; \delta_a)) \subseteq D(f(a); \varepsilon) \quad \delta \text{ depends on } a$$

When f is uniformly continuous, there is one value of δ , dependent on ε , that applies for every value of a .

$$(\forall \varepsilon > 0)(\exists \delta_\varepsilon > 0) \ni (\forall a \in A), f(D(a; \delta_\varepsilon)) \subseteq D(f(a); \varepsilon)$$

Riemann Sphere

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $N = (0, 0, 1)$ denote the north pole. Then, there is a continuous bijection from $S^2 \setminus \{N\} \rightarrow \mathbb{C}$.

We can visualize this by picking a random point on the sphere and drawing a line from the north pole through the sphere to this point, and finding the point that intersects the plane.

Consider the sequence $z_n = n^2 i$ for $n = 1, 2, \dots$. We can see that, on the projection from z_n to the sphere, all the values of p converge to N . Therefore, we write $\lim_{n \rightarrow \infty} z_n = \infty$, where ∞ corresponds to N on S^2 .

We can define $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to be the complex plane that includes the “point at infinity” (from the projection on S^2 that corresponds to the north pole).