Introduction

Consider the equations

$$\frac{d^2y}{dx^2} + y(x) = e^x \tag{1}$$

$$\frac{d^{17}y}{dx^{17}}(x) + \sin(y(x)) = (x^x)^x \tag{2}$$

Before we want to solve these equations, we need to understand what these equations *are*.

(1) This is a second order, inhomogeneous, linear ordinary differential equation.

(2) This is a 17th order, inhomogeneous, nonlinear ordinary differential equation.

Generally, when we have a nonlinear equation, we convert it (using the Jacobian) to the "nearest" corresponding linear equation using Taylor approximations. In this case, converting equation (2), we have

$$\frac{d^{17}y}{dx^{1y}}(x) + y(x) = (x^x)^x. \tag{2'}$$

Now, equation (2′) is linear, so it is able to be solved. It may not be pretty,[I] but it can be solved, using Laplace Transforms or other methods.

Ordinary Differential Equations

Returning to our equation (1),

$$\frac{d^2y}{dx^2} + y(x) = e^x, \tag{1}$$

there is one more fact that we can see — this is an equation with constant coefficients. The most general form of a nth order linear ordinary differential equation is of the form

$$a_n(x)\frac{d^ny}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}(x) + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = g(x). \tag{†}$$

Specifically, we also require $a_k(x) \in C(I)$, where I is some interval (specifics will be detailed later).

Theorem (Existence and Uniqueness Theorem)**:** Any ordinary differential equation of the form (†) has unique solutions in the interval I.

There are n linearly independent solutions for $g(x) = 0$.

The corresponding homogeneous equation for (1) is

$$\frac{d^2y}{dx^2} + y(x) = 0. \tag{1'}$$

The equations (1) and (1′) are related by the linearity principle. In particular, if $y_0(x)$ is a solution to (1′), then we can add $\alpha y_0(x)$ to any solution $y_p(x)$ of (1), then we have all the solutions for (1). In particular, the solutions to (1′) are

$$y_1(x) = \sin(x)$$
$$y_2(x) = \cos(x).$$

[I]Citation needed.

To evaluate that these solutions are linearly independent, we consider the differential operator L from (†) defined by

$$L[y] = \sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k}.$$

We rewrite (†) as

$$L[y] = g(x).$$

The operator L is linear, so L has the following properties:

- $L[y_1 + y_2]$;
- $L[cy] = cL[y]$.

Now, in (1) and (1′), if we set $L[y] = \frac{d^2y}{dx^2} + y(x)$, then evaluating our solutions y_1 and y_2 to (1′), we get

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= c_1L[y_1] + c_2L[y_2] \\ &= 0. \end{aligned}$$

Now, we get

$$y_0(x) = c_1 \sin(x) + c_2 \sin(x)$$

as our general solution to (1′). By the linearity principle, all we need is one solution to $L[y] = e^x$ to find all solutions to (1).

Evaluating (†) in the most general form, we have the general solution

$$y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{\text{homogeneous solution}} + y_p(x),$$

where $y_p(x)$ is the particular solution. In other words, our general solution is

$$y(x) = \operatorname{span}(y_1(x), y_2(x), \dots, y_n(x)) + y_p(x).$$

For this to work, we need the set $\{y_1, \dots, y_n\}$ to be linearly independent. To do this, we evaluate the Wronskian:

$$W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{pmatrix}.$$

Specifically, the set $\{y_1, \dots, y_n\}$ is linearly independent if $W(x) \neq 0$ for all $x \in I$.

Example. Consider the equation

$$\frac{d^2y}{dx^2} - y(x) = e^x \tag{1}$$

We want to find the general solution to this constant coefficient equation.

We start by finding two linearly independent homogeneous solutions to the equation, take their span, then add a particular solution.

The characteristic equation of the homogeneous equation for (1) is

$$r^2 - 1 = 0$$

We get $r = \pm 1$, which by the definition of the characteristic equation yields $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. To verify that this this solution set is linearly independent

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2 \\ &\neq 0. \end{aligned}$$

Thus, our solutions are linearly independent. We get the general form of

$$y(x) = c_1 e^x + c_2 e^{-x} + y_p(x).$$

Now, we only have to find a particular solution. This is, unfortunately, the hard part.

We begin by guessing. But, in a way that doesn't suck. Specifically, we let $y_p(x) = Axe^x$. Evaluating, we get

$$\begin{aligned} \frac{dy_p}{dx} &= A(x+1)e^x \\ \frac{d^2y_p}{dx^2} &= A(x+2)e^x \\ \frac{d^2y_p}{dx^2} - y_p(x) &= A(x+2)e^x - Axe^x \\ &= 2Ae^x, \end{aligned}$$

so $2A = 1$, and $A = \frac{1}{2}$. Thus, we have the end result of

$$y(x) = c_1 e^x + c_2 e^x + \frac{1}{2} x e^x.$$

Evaluating in Mathematica, we take

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DSolve[y''[x] - y[x] == Exp[x], y[x], x]
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and we get

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{4}(2x-1)e^x,$$

corroborating our solution.[II]

Example. Consider the equation

$$\frac{d^3y}{dx^3} - y(x) = 0.$$

The particular solution to this equation is $y(x) = 0$. The characteristic equation for this equation is

$$r^3 - 1 = 0.$$

[II]Only slightly different, but they're the same solution.

Factoring, we get

$$\begin{aligned}(r-1)\left(r^2+r+1\right) &= 0\\ (r-1)(r-\zeta_3)\left(r-\zeta_3^2\right) &= 0.\end{aligned}$$

Thus, we get

$$r = \left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}.$$

Thus, our solutions are of the form

$$y(x) = c_1e^x + c_2e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Recall that the most general second order constant-coefficient linear differential equation is

$$y'' + ay' + by = 0,$$

with characteristic equation

$$r^2 + ar + b = 0.$$

The solutions to the characteristic equation are

$$r = -\frac{a}{2} \pm \frac{\sqrt{a^2-4b}}{2}.$$

There are a few cases:

(1) $r_1 \neq r_2$ with $r_1, r_2 \in \mathbb{R}$;

(2) $r_1 = r_2$ with $r_1, r_2 \in \mathbb{R}$;

(3) $r_1 = c + id$, $r_2 = c - id$, where $c, d \in \mathbb{R}$.

The solutions are $y_1 = c_1e^{r_1x}$ and $y_2 = c_2e^{r_2x}$.

Example (Solving Second-Order Equations)**.**

(1) Let

$$y'' - 3y' + 2y = 0.$$

The characteristic equation is $r^2 - 3r + 2 = 0$, whose solutions are $r = 1, r = 2$. The general solution is, thus,

$$y(x) = c_1e^x + c_2e^{2x} \tag{†}$$

The Wronskian is

$$\begin{aligned}W(x) &= \det\begin{pmatrix} e^x & e^{2x}\\ e^x & 2e^{2x}\end{pmatrix}\\ &= 2e^{3x} - e^{3x}\\ &= e^{3x}\\ &\neq 0.\end{aligned}$$

Thus, the solution is indeed (†).

(2) Let

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with solution $r = -3, -3$. Currently, we only have the solution $y_1(x) = c_1 e^{-3x}$.

Note that in an nth order linear ordinary differential equation, we always have n linearly independent solutions. Let's guess. Consider the equation $y_2(x) = c_2 x e^{-3x}$.

We can see that $y_2(x)$ is also a solution to this equation,[III] but we need to verify linear independence. Taking the Wronskian, we get

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{pmatrix} \\ &= e^{-6x}\begin{pmatrix} 1 & x \\ -3 & -3x+1 \end{pmatrix} \\ &= e^{-6x}(-3x + 1 + 3x) \\ &= e^{-6x} \\ &\neq 0. \end{aligned}$$

Thus, we have two linearly independent solutions, with the general solution of

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

(3) Let

$$y'' + 4y' + 5 = 0.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with solutions of $r = -2 \pm i$. We then have the solutions

$$\begin{aligned} y_1(x) &= e^{(-2+i)x} \\ y_2(x) &= e^{(-2-i)x}. \end{aligned}$$

Unfortunately, we cannot just let these equations stand on their own, because we want *real* solutions. Let's use Euler's theorem, $e^{ix} = \cos x + i \sin x$. Then, we get

$$\begin{aligned} y(x) &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\ &= e^{-2x}\left(c_1 e^{ix} + c_2 e^{-ix}\right). \end{aligned}$$

Let $f(x) = c_1 e^{ix} + c_2 e^{-ix}$. Using the even/odd decomposition, we get

$$\begin{aligned} f(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\ &= (c_1 + c_2)\cos(x) + i(c_1 - c_2)\sin(x). \end{aligned}$$

We "real"-ize our solution by just dropping the value of i in $f(x)$. Thus, we get the full general solution

$$y(x) = e^{-2x}(d_1 \cos(x) + d_2 \sin(x)).$$

[III] Exercise left for the reader.

(4) If we have the equation

$$y^{(4)} - 25y'',$$

then using a similar process, we get the solution

$$y(x) = c_1 + c_2x + c_3e^{5x} + c_4e^{-5x}.$$

(5) Considering the equation

$$y^{(5)} + 4y''' + 4y' = 0,$$

we take the characteristic equation $r^5 + 4r^3 + 4r = 0$. Factoring, we get solutions of $r = 0, r = \pm i\sqrt{2}$. Thus, we get the solution of

$$y(x) = c_1 + c_2\cos\left(\sqrt{2}x\right) + c_3\sin\left(\sqrt{2}x\right) + c_4x\cos\left(\sqrt{2}x\right) + c_5x\sin\left(\sqrt{2}x\right).$$

Reducing our Orders

Let

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0.$$

Suppose we know $y_1(x)$. Can we find $y_2(x)$? The answer is yes. We presume

$$y_2(x) = v(x)y_1(x).$$

Now, we have

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'', \end{aligned}$$

and inserting into the equation, we get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{=0} \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Now, we have

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p. \tag{$*$}$$

Integrating, we get

$$\ln(v') = -2\ln(y_1) - \int p(x)\,dx.$$

Taking powers, we get

$$\begin{aligned} v' &= e^{-2\ln(y_1) - \int p(x)\,dx} \\ &= y_1^{-2}e^{-\int p(x)\,dx} \\ &= \frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \\ v &= \int \frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\,dx \end{aligned}$$

Example. Consider the equation

$$\cos^2(x)\frac{d^2y}{dx^2} - \sin(x)\cos(x)y' - y(x) = 0.$$

Putting our equation into standard form, we may be able to find another solution.

$$y'' - \tan(x)y' - \sec^2(x)y = 0.$$

Guessing $y(x) = \tan(x)$, we get $y' = \sec^2(x)$ and $y'' = 2\sec^2(x)\tan(x)$. This is also another solution, $y_2(x) = \tan(x)$.

We don't want to guess anymore. Let $y_2(x) = v(x)y_1(x)$. We get

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx.$$

We have $-p(x) = \tan(x)$, so $-\int p(x)\,dx = \ln(\sec(x))$. Thus, $e^{-\int p(x)\,dx} = \sec(x)$. Thus, we get

$$\begin{aligned}
v(x) &= \int \frac{\sec(x)}{\tan^2(x)}\,dx \\
&= \int \frac{\cos(x)}{\sin^2(x)}\,dx \\
&= \int \frac{1}{u^2}\,du && u = \sin(x) \\
&= -\frac{1}{u} \\
&= -\csc(x).
\end{aligned}$$

Thus, we have $y_2(x) = -\csc(x)\tan(x) = -\sec(x)$.

Example. Consider the equation

$$x^2(\ln(x) - 1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{dy}{dx} = 0.$$

We can use the power of inspection to find one solution, $y_1(x) = x$. Dividing out, we have

$$y'' - \frac{1}{x(\ln(x) - 1)}y' + \frac{1}{x^2(\ln(x) - 1)}y = 0.$$

Using the reduction of order, we guess $y_2(x) = v(x)y_1(x)$, and have

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

Noting that $-p(x) = \frac{1}{x(\ln(x)-1)}$, we have $\int \frac{1}{x(\ln(x)-1)}\,dx = \ln(\ln(x) - 1)$.

Now, we have

$$\begin{aligned}
v(x) &= \int \frac{\ln(x) - 1}{x^2}\,dx \\
&= \frac{1 - \ln(x)}{x} - \int -\frac{1}{x^2}\,dx && u = \ln(x) - 1, dv = x^{-2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\ln(x)}{x} - \frac{1}{x} \\
&= -\frac{\ln(x)}{x}.
\end{aligned}
$$

Thus, we get $y_2(x) = -\ln(x)$, and the general solution of $y(x) = c_1x + c_2\ln(x)$.

Example (Cauchy–Euler Equation)**.** A second-order Cauchy–Euler equation is of the form

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy(x) = 0.$$

More generally,

$$\sum_{k=0}^{n} c_kx^ky^{(k)}(x) = 0.$$

We guess $y(x) = x^r$. Then, $\frac{dy}{dx} = rx^{r-1}$ and $\frac{d^2y}{dx^2} = r(r-1)x^{r-2}$. This yields

$$
\begin{aligned}
a(r)(r-1)x^r + brx^r + cx^r &= x^r\left(a\left(r^2 - r\right) + br + c\right) \\
&= 0.
\end{aligned}
$$

Example (Solving a Cauchy–Euler Equation)**.** Consider the equation

$$x^2y'' + xy' - y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 1 = 0,$$

so our general solution is $y(x) = c_1x + c_2/x$.

Example (Solving another Cauchy–Euler Equation)**.** Consider the equation

$$x^2y'' - 3xy' + 4y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 4r + 4 = 0,$$

so our solutions are x^2 and x^2. This is not good enough, we need another solution.

Now, we place our equation into standard form.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y' = 0.$$

Thus, we get $p(x) = -\frac{3}{x}$. Using reduction of order, we get $y_2(x) = v(x)y_1(x)$,

$$
\begin{aligned}
v(x) &= \int \frac{e^{-\int -3/x\,dx}}{x^4}\,dx \\
&= \int \frac{e^{3\ln(x)}}{x^4}\,dx \\
&= \int \frac{x^3}{x^4}\,dx \\
&= \ln(x).
\end{aligned}
$$

Thus, we have the solution $y_2(x) = \ln(x)x^2$, and the general solution of $y(x) = c_1x^2 + c_2\ln(x)x^2$.

Example. Consider the equation

$$x^2y'' + 3xy' + 5y = 0.$$

We get the characteristic equation of

$$\begin{aligned} 0 &= r^2 - 4r + 5 \\ r &= 2 \pm i. \end{aligned}$$

Now, we need to figure out what $x^{2\pm i}$ means.

To solve this part, we keep the positive exponent, so we only need to try to understand $y = x^{2+i}$. Now, we get $y = x^2x^i$. To evaluate x^i, we take $x = \left(e^{\ln x}\right)^i = e^{i\ln x}$. Using Euler's identity, we get

$$y = x^2(\cos(\ln x) + i\sin(\ln x)).$$

Since our solutions are real, get

$$y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x).$$

Example. Consider the equation

$$x^4y^{(4)} - 2x^2y'' + y = 2.$$

We have the particular solution $y_p(x) = 2$. Substituting into our method for the Cauchy–Euler equation, we have

$$r(r-1)(r-2)(r-3) - 2r(r-1) + 1 = 0.$$

Factoring, we have

$$r(r-1)^2(r-4) + 1 = 0.$$

Unfortunately, to go forward from here we need Mathematica.

This has the solution set of of

$$\begin{aligned} r_1 &= \frac{3}{2} - \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\ r_2 &= \frac{3}{2} - \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad + \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\ r_3 &= \frac{3}{2} + \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \end{aligned}$$

$$r_4 = \frac{3}{2} + \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} + \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}}$$

Varying our Parameters

Given a set of n linearly independent homogeneous solutions, we want to find a particular solution.

To find this, we start with the general second-order inhomogeneous equation in standard form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = g(x).$$

Given y_1, y_2, we find $y_p(x)$ by taking

$$y_p = v_1y_1 + v_2y_2.$$

Finding the derivatives, we have

$$\begin{aligned} y_p' &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2 \\ y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2. \end{aligned}$$

Substituting, we have

$$\begin{aligned} y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2 \\ py_p' &= pv_1y_1' + pv_1'y_1 + pv_2y_2' + pv_2'y_2 \\ qy_p &= qv_1y_1 + qv_2y_2 \end{aligned}$$

$$g(x) = v_1\overbrace{\left(y_1'' + py_1' + qy_1\right)}^{=0} + v_2\overbrace{\left(y_2'' + py_2' + qy_2\right)}^{=0} + v_1'\left(2y_1' + py_1\right) + v_1''y_1 + v_2\left(2y_2' + py_2\right) + v_2''y_2$$

$$g(x) = v_1'\left(2y_1' + py_1\right) + v_1''y_1 + v_2\left(2y_2' + py_2\right) + v_2''y_2.$$

We suppose that $v_1'y_1 + v_2'y_2 = 0$. Then,

$$\begin{aligned} \frac{d}{dx}\left(v_1'y_1 + v_2'y_2\right) &= 0 \\ v_1''y_1 + v_1'y_1' + v_2''y_2 + v_2'y_2 &= 0. \end{aligned}$$

Plugging into our earlier expression, we get the expression of

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_2'y2' + v_2'y_2' &= g(x). \end{aligned}$$

Plugging into matrix form, we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

Since the Wronskian is nonzero, we have

$$\begin{pmatrix} \frac{dv_1}{dx} \\ \frac{dv_2}{dx} \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$
$$= \frac{1}{y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx}} \begin{pmatrix} -y_2(x)g(x) \\ y_1(x)g(x) \end{pmatrix} \qquad (\ddagger)$$

Example. Let

$$y'' - 2y' + y = e^x.$$

Solving the homogeneous solution, we have the characteristic equation of $r^2 - 2r + 1 = 0$. Thus, $y_1(x) = e^x$ and $y_2(x) = xe^x$.

To find $y_p(x)$, we guess $y_p(x) = x^2 e^x$. Using the power of computation in Sage, we get the answer of

Avoiding Variation of Parameters

```
de = diff(y,x,2) - 2*diff(y,x) + y - e^(x)
g = desolve(de,y)
latex(expand(g))
```

$$y_p(x) = K_2 x e^x + K_1 e^x + \frac{1}{2}x^2 e^x.$$

However, this is a very unsatisfying method.

Using (‡), we can find a different solution. We find

$$\frac{dv_1}{dx} = \frac{1}{e^{2x}}((-1)(xe^x)(e^x))$$
$$= -x,$$

yielding

$$v_1(x) = -\frac{x^2}{2} + c_2.$$

Similarly, we get

$$\frac{dv_2}{dx} = \frac{1}{e^{2x}}(e^x)(e^x)$$
$$v_2(x) = x + c_2.$$

This gives

$$y_p(x) = \frac{1}{2}x^2 e^x.$$

Example. Let

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = x + e^x.$$

Using the characteristic equation, we have $y_1(x) = 1$, $y_2(x) = e^x$, and $y_3(x) = e^{-x}$.

Now, using the Wronskian, we get

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ x + e^x \end{pmatrix}.$$

This would suck, but we would be able to find a solution nonetheless.

In the general form, with linearly independent homogeneous solutions $y_1, \dots, y_n$, we have the solution of

$$\begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ g(x) \end{pmatrix}$$
$$y(x) = \sum_{i=1}^{n} c_i y_i(x) + \sum_{i=1}^{n} v_i(x) y_i(x).$$

Example (Solving a Coupled System)**.** Before we can start using variation of parameters for systems, we need to recall how to solve constant-coefficient systems.

$$x'(t) = 3x(t) + y(t)$$
$$y'(t) = x(t) + 3y(t).$$

Here, setting

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

we get system of linear equations

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Remark: In the matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

the eigenvalues are

$$\lambda_1 = a + b$$
$$\lambda_2 = a - b$$

with eigenvectors of

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example (General n-dimensional System of Differential Equations)**.** Consider the system of equations defined by

$$\begin{aligned}x_1'(t) &= g_1(t, x_1(t), \dots, x_n(t))\\ &\vdots\\ x_n'(t) &= g_n(t, x_1(t), \dots, x_n(t)).\end{aligned}$$

We will refine this slightly so as to be a system of *linear* equations. Let

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{pmatrix}\\ \frac{d\mathbf{x}}{dt} &= \begin{pmatrix}x_1'(t)\\ \vdots\\ x_n'(t)\end{pmatrix}\\ \mathbf{F} &= \begin{pmatrix}f_1(t)\\ \vdots\\ f_n(t)\end{pmatrix}\\ \mathbf{x}_{t_0} &= \begin{pmatrix}x_1(t_0)\\ \vdots\\ x_n(t_0)\end{pmatrix}.\end{aligned}$$

Now, we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where $\mathbf{x}(t_0) = \mathbf{x}_{t_0}$ and A is some matrix that represents some linear transformation.

Furthermore, we may make an inhomogeneous equation by

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{F}.$$

Example. Going back to our example of

$$\frac{d\mathbf{x}}{dt} = \underbrace{\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}}_{A}\mathbf{x}.$$

We find eigenvalues of $\lambda_1 = 4, \lambda_2 = 2$ and eigenvectors $\mathbf{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}1\\ -1\end{pmatrix}$. This gives

$$\mathbf{x}_1 = e^{4t}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\mathbf{x}_2 = e^{2t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

In general, if we have two distinct eigenvalues, then our solutions are

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

Define

$$\begin{aligned}\Phi_A(t) &= \begin{pmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{pmatrix}\\ &= \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}.\end{aligned}$$

We call Φ_A a fundamental matrix for A.

The general solution to the system is given by

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\\ &= c_1\begin{pmatrix}e^{4t}\\ e^{4t}\end{pmatrix} + c_2\begin{pmatrix}e^{2t}\\ -e^{2t}\end{pmatrix}\\ &= \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}.\end{aligned}$$

Example. Consider the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$A = \begin{pmatrix}4 & 2 & 1\\ 0 & 4 & 2\\ 0 & 0 & 4\end{pmatrix} \tag{A}$$

Notice that we have a triple-repeated eigenvalue,

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= 4\\ \lambda_3 &= 4.\end{aligned}$$

Unfortunately, to find the eigenvectors, this will be a bit harder.

$$\begin{aligned}(A - 4I)\mathbf{v} &= 0\\ \begin{pmatrix}0 & 2 & 1\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} &= \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.\end{aligned}$$

This gives

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so $b = c = 0$, and our eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We may need some more eigenvectors. Currently, our solution is

$$\mathbf{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We need to go into the realm of generalized eigenvectors. If λ is repeated, we need to do the following.

(1) Find all the eigenvectors for which $(A - \lambda I)\mathbf{v} = 0$. If we come up short, then we have a defective system.

(2) For the remaining eigenvectors, we solve the system

$$(A - \lambda I)\mathbf{v}_j = \mathbf{v}_k,$$

where $\mathbf{v}_k$ is known, and we desire $\mathbf{v}_j$. The $\mathbf{v}_j$ are known as generalized eigenvectors.

(3) Continue this process until we are done.

Now, in this case, we get

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and a generalized eigenvector of

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}.$$

Going at it again, we have

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix},$$

giving the equation

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix},$$

giving

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1/8 \\ 1/4 \end{pmatrix}.$$

Note that when we take generalized eigenvectors, we "integrate" with respect to t before adding. For instance

$$\begin{aligned} \mathbf{x}_1 &= e^{\lambda t}\mathbf{v}_1 \\ \mathbf{x}_2 &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) \\ \mathbf{x}_3 &= e^{\lambda t}\left(\frac{t^2}{t}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right). \end{aligned}$$

Now, our linearly independent solutions to the system in (A) is of the form

$$\begin{aligned} \mathbf{x}_1(t) &= e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \mathbf{x}_2(t) &= e^{4t}\left(t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}\right) \\ \mathbf{x}_3(t) &= e^{4t}\left(\frac{t^2}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/8 \\ 1/4 \end{pmatrix}\right). \end{aligned}$$

This gives the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{4t} & te^{4t} & \frac{t^2}{2}e^{4t} \\ 0 & \frac{1}{2}e^{4t} & e^{4t}\left(\frac{t}{2} - \frac{1}{8}\right) \\ 0 & 0 & \frac{1}{4}e^{4t} \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}.$$

The general solution is, then,

$$\mathbf{x}(t) = e^{At}\mathbf{c},$$

where $\mathbf{c}$ is a constant vector, and e^{At} is the matrix exponential of A.

Example. Consider A as the matrix with eigenvalue λ and eigenvector $\mathbf{v}_1$ and generalized eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$. Then, the solution set

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v}_1\\
\mathbf{x}_2(t) &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)\\
\mathbf{x}_3(t) &= e^{\lambda t}\left(\frac{t^2}{2}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \lambda e^{\lambda t}\mathbf{v}_1\\
A\mathbf{x}_1(t) &= Ae^{\lambda t}\mathbf{v}_1\\
&= e^{\lambda t}A\mathbf{v}_1\\
&= \lambda e^{\lambda t}\mathbf{v}_1.
\end{aligned}$$

Now, recalling that $A\mathbf{v}_1 = \lambda\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$, we have

$$\begin{aligned}
\frac{d\mathbf{x}_2}{dt} &= \lambda e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) + e^{\lambda t}\mathbf{v}_1\\
A\mathbf{x}_2(t) &= Ae^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)\\
&= e^{\lambda t}(tA\mathbf{v}_1 + A\mathbf{v}_2)\\
&= e^{\lambda t}(t\lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_1)\\
&= \lambda e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) + e^{\lambda t}\mathbf{v}_1.
\end{aligned}$$

Finally, we have $A\mathbf{v}_3 = \lambda\mathbf{v}_3 + \mathbf{v}_2$.

Example. We assume A is a $n \times n$ real matrix. Then, all complex eigenvalues of A come in conjugate pairs, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$.

Then, our eigenvectors are of the form $\mathbf{v}_1 = \mathbf{u} + i\mathbf{w}$ and $\mathbf{v}_2 = \mathbf{u} - i\mathbf{w}$.

Note that if we find the solution for λ_1 and $\mathbf{v}_2$. This gives

$$\begin{aligned}
e^{\lambda t}\mathbf{v} &= e^{(a+ib)}(\mathbf{u} + i\mathbf{w})\\
&= e^{at}(\cos(bt) + i\sin(bt))(\mathbf{u} + i\mathbf{w})\\
&= e^{at}((\cos(bt)\mathbf{u} - \sin(bt)\mathbf{w}) + i(\cos(bt)\mathbf{w} + \sin(bt)\mathbf{u})).
\end{aligned}$$

Example. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -4\\ 0 & 3 & 0\\ 2 & 0 & 5 \end{pmatrix}$$

for the system of equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Using the power of computation, we have

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= 3+2i\\ \lambda_3 &= 3-2i,\end{aligned}$$

and eigenvectors of

$$\begin{aligned}\mathbf{v}_1 &= \begin{pmatrix}0\\1\\0\end{pmatrix}\\ \mathbf{v}_2 &= \begin{pmatrix}-4\\0\\2+2i\end{pmatrix}\\ \mathbf{v}_3 &= \begin{pmatrix}-4\\0\\2-2i\end{pmatrix}.\end{aligned}$$

Now, we see that

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\lambda_1 t}\mathbf{v}_1\\ &= \begin{pmatrix}0\\e^{3t}\\0\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_2(t) &= e^{3t}\left(\cos(2t)\begin{pmatrix}-4\\0\\2\end{pmatrix} - \sin(2t)\begin{pmatrix}0\\0\\2\end{pmatrix}\right)\\ \mathbf{x}_3(t) &= e^{3t}\left(\cos(2t)\begin{pmatrix}0\\0\\2\end{pmatrix} + \sin(2t)\begin{pmatrix}-4\\0\\2\end{pmatrix}\right).\end{aligned}$$

This gives the matrix

$$\begin{aligned}\Phi(t) &= \begin{pmatrix}0 & -4e^{3t}\cos(2t) & -4e^{3t}\sin(2t)\\ e^{3t} & 0 & 0\\ 0 & 2e^{3t}(\cos(2t)-\sin(2t)) & 2e^{3t}(\sin(2t)+\cos(2t))\end{pmatrix}\\ W(t) &= \det(\Phi(t))\\ &= -e^{3t}\left(-8e^{6t}\left(\cos(2t)\sin(2t)+\cos^2(2t)\right) + 8e^{6t}\left(\sin(2t)\cos(2t)-\sin^2(2t)\right)\right)\\ &= 8e^{9t}\\ &\neq 0.\end{aligned}$$

Example. We wish to solve $\frac{dx}{dt} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

To find our eigenvalues and eigenvectors, we begin by finding

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)^2 \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 5 \\ 0 & -2 & -1-\lambda \end{pmatrix} \\ &= (2-\lambda)^3 \det\begin{pmatrix} 1-\lambda & 5 \\ -2 & -1-\lambda \end{pmatrix} \\ &= (2-\lambda)^3((1-\lambda)(-1-\lambda)+10). \end{aligned}$$

We have five eigenvalues,

$$\lambda = \pm 3i, 2, 2, 2.$$

For $\lambda_{1,2} = \pm 3i$, then

$$\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now for $\lambda_3 = 2$, we have

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now, we have

$$(A - 2I)\mathbf{v}_3 = 0$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ 3 \end{pmatrix} = 0.$$

From this equation, we have

$$\begin{aligned} c &= 0 \\ -b + 5d &= 0 \\ -2b - 3d &= 0. \end{aligned}$$

Now, we have independent a and e. This gives

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that both $\mathbf{v}_3$ and $\mathbf{v}_4$ are regular eigenvectors. Now, we wish to find one generalized eigenvector. We find this generalized eigenvector, $\mathbf{w}$, by observing that the 1 in entry $A_{1,3}$ effectively ties our vector $\mathbf{v}_4$ to vector $\mathbf{v}_{1,2}$. Thus, we get

$$(A - 2I)\mathbf{w} = \mathbf{v}_4$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, solving this, we get $c = 1$, giving the generalized eigenvector of

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we have

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{v}_4 \longrightarrow \mathbf{w}$ is a chain of length 2. This gives the JCF of

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1+3i & 1-3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3i & 0 & 0 & 0 & 0 \\ 0 & -3i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1+3i & 1-3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^{-1}.$$

We get the solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ -\cos(3t) - 3\sin(3t) \\ 0 \\ 2\cos(3t) \\ 0 \end{pmatrix}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} 0 \\ 3\cos(3t) - \sin(3t) \\ 0 \\ 2\sin(3t) \\ 0 \end{pmatrix}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{2t} \end{pmatrix}$$

$$\mathbf{x}_4(t) = \begin{pmatrix} e^{2t} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_5(t) = \begin{pmatrix} te^{2t} \\ 0 \\ e^{2t} \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{x}_5(t) = e^{2t}(t\mathbf{v}_4 + \mathbf{v}_5)$.

The fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} 0 & 0 & 0 & e^{2t} & te^{2t} \\ -\cos(3t) + 3\sin(3t) & 3\cos(3t) - \sin(3t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2t} \\ 2\cos(3t) & 2\sin(3t) & 0 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \end{pmatrix}.$$

Now, we want to find $\Phi(0)$, or $\Phi(t_0)$. Furthermore, we need to find $\Phi^{-1}(0)$, or $\Phi^{-1}(t_0)$.

Example (Implementing Initial Conditions)**.** Looking back at our equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

we may apply the initial condition of

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

We use the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

with the initial condition

$$\mathbf{x}_0 = \begin{pmatrix} 4 \\ 15 \end{pmatrix}.$$

Generally our approach to solving this kind of problem, we take the eigenvectors and eigenvalues, giving

$$\begin{aligned}\lambda_1 &= 4\\ \mathbf{v}_1 &= \begin{pmatrix}1\\1\end{pmatrix}\\ \lambda_2 &= 2\\ \mathbf{v}_2 &= \begin{pmatrix}1\\-1\end{pmatrix},\end{aligned}$$

and associated solutions of

$$\begin{aligned}\mathbf{x}_1 &= \begin{pmatrix}e^{4t}\\e^{4t}\end{pmatrix}\\ \mathbf{x}_2 &= \begin{pmatrix}e^{2t}\\-e^{2t}\end{pmatrix}.\end{aligned}$$

Then, we form a fundamental matrix of solutions:

$$\Phi(t) = \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}.$$

Note that, for any vector of constants $\mathbf{c}$, we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}$$

is a solution of $\frac{dx}{dt} = A\mathbf{x}$.

To find $\mathbf{c}$, we see that

$$\mathbf{x}_0 = \Phi(0)\mathbf{c},$$

so that

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0$$

is the solution to our initial value problem.

Calculating

$$\Phi(0) = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},$$

we find

$$\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Thus, we get the solutions of

$$\mathbf{x}(t) = \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}4\\15\end{pmatrix}$$

$$= \begin{pmatrix} \frac{19}{2}e^{4t} - \frac{11}{2}e^{2t} \\ \frac{19}{2}e^{4t} + \frac{11}{2}e^{2t} \end{pmatrix}.$$

Note that we may define

$$\Psi(t) = \Phi(t)\Phi^{-1}(0),$$

giving

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0.$$

We may calculate

$$\begin{aligned}
\Psi(t) &= \Phi(t)\Phi^{-1}(0) \\
&= \frac{1}{2}\begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{4t} + e^{2t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{4t} + e^{2t} \end{pmatrix}.
\end{aligned}$$

Example (The Matrix Exponential)**.** When we have a single first-order equation, such as

$$\frac{dy}{dt} = 3y,$$

with initial condition $y(0)$, we solve it by taking $y(t) = \pi e^{3t}$.

Similarly, if we're given

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

we may want to know if there is an analogous e^{At}.

In fact, there is. Using the Taylor expansion, we have

$$\begin{aligned}
e^{At} &= I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots \\
&= \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}.
\end{aligned}$$

Note that we may take P to be the matrix of unit eigenvectors of A, and D to be the matrix of eigenvalues corresponding to column eigenvectors

$$A = PDP^{-1}.$$

This is assuming A can be diagonalized. This gives $D = P^{-1}AP$.

Now, if A can be diagonalized, we can take

$$\begin{aligned}
e^{At} &= I + \left(PDP^{-1}\right)t + \left(PDP^{-1}\right)^2\frac{t^2}{2} + \cdots \\
&= \sum_{k=0}^{\infty}\left(PDP^{-1}\right)^k\frac{t^k}{k!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} PD^kP^{-1}\frac{t^k}{k!}\\
&= P\left(\sum_{k=0}^{\infty} D^k\frac{t^k}{k!}\right)P^{-1}.
\end{aligned}$$

We can find the power on any diagonal matrix much more easily than we can on a general matrix. In particular, this gives

$$e^{At} = P\begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t}\end{pmatrix}P^{-1}$$

Given A in the system $\frac{dx}{dt} = A\mathbf{x}$, we wish to find e^{At}, and show that $e^{At} = \Psi(t)$, where $\Psi(t) = \Phi(t)\Phi^{-1}(0)$.

Example. Let

$$A = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}.$$

We see that A has repeated eigenvalues of 4 and 4.

Our first eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now, evaluating

$$\begin{aligned}
(A - 4I)\mathbf{v} &= 0\\
\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

giving $2b = 0$.

Thus, we're going to need a generalized eigenvector. We have

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

giving $(A - \lambda I)\mathbf{w} = \mathbf{v}_1$. Thus, we have $2b = 1$. Thus, we have

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

Now, we have

$$\mathbf{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t} \\ 0 \end{pmatrix}$$

$$\begin{aligned}\mathbf{x}_2(t) &= te^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{4t}\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \\ &= \begin{pmatrix} te^{4t} \\ \frac{1}{2}e^{4t} \end{pmatrix}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\ &= \begin{pmatrix} e^{4t} & te^{4t} \\ 0 & \frac{1}{2}e^{4t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.\end{aligned}$$

Now, we see that

$$\begin{aligned}\Phi(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\ \Phi^{-1}(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.\end{aligned}$$

Therefore, our matrix exponential is

$$\begin{aligned}\Psi(t) &= \begin{pmatrix} e^{4t} & te^{4t} \\ 0 & \frac{1}{2}e^{4t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{4t} & 2te^{4t} \\ 0 & e^{4t} \end{pmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{x}(t) &= \Psi(t)\mathbf{x}_0 \\ &= \begin{pmatrix} e^{4t} & 2te^{4t} \\ 0 & e^{4t} \end{pmatrix}\mathbf{x}_0.\end{aligned}$$

We often refer to $\Psi(t)$ as the flow matrix.

Now, because $\Psi(0) = I$, we have

$$\Psi(t)\Psi(-t) = I,$$

meaning that $\Psi(-t) = \Psi(t)^{-1}$.

Example. Consider

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

as the set of eigenvector equations, giving

$$\begin{aligned}\begin{pmatrix}A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n\end{pmatrix} &= \begin{pmatrix}\mathbf{v}_1\lambda_1 & \mathbf{v}_2\lambda_2 & \cdots & \mathbf{v}_n\lambda_n\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix},\end{aligned}$$

giving the expression $AP = PD$, where P is the set of eigenvectors.

Now, if we have generalized eigenvectors, we have a different case.

Consider the case of a chain. We know that

$$(A - \lambda I)\mathbf{v}_1 = 0$$

is the expression of an eigenvector. Now, if we have repeated eigenvalues, we get the second equation of

$$\begin{aligned}(A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1\\ (A - \lambda I)\mathbf{v}_3 &= \mathbf{v}_1\\ &\vdots\\ (A - \lambda I)\mathbf{v}_n &= \mathbf{v}_{n-1}.\end{aligned}$$

We start by changing these equations to give

$$\begin{aligned}A\mathbf{v}_1 &= \lambda\mathbf{v}_1\\ A\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1\\ A\mathbf{v}_3 &= \lambda\mathbf{v}_3 + \mathbf{v}_2\\ &\vdots\\ A\mathbf{v}_n &= \lambda\mathbf{v}_n + \mathbf{v}_{n-1}.\end{aligned}$$

We see that these are effectively the eigenvalue equations with a small perturbation. Constructing the matrix, we have

$$\begin{aligned}\begin{pmatrix}A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 & \cdots & A\mathbf{v}_n\end{pmatrix} &= \begin{pmatrix}\lambda\mathbf{v}_1 & \lambda\mathbf{v}_2 + \mathbf{v}_1 & \lambda\mathbf{v}_3 + \mathbf{v}_2 & \cdots \lambda\mathbf{v}_n + \mathbf{v}_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}\underbrace{\begin{pmatrix}\lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & 1\\ & & & & \lambda\end{pmatrix}}_{J}.\end{aligned}$$

We call the matrix J the Jordan canonical form of A, and we get the expression $AP = PJ$, where P is the matrix of generalized eigenvectors as columns.

Now, if we have multiple chains, we get multiple blocks. For instance, if we have the chains $\mathbf{v}_1 \to \mathbf{v}_2 \to \mathbf{v}_3$, $\mathbf{v}_4 \to \mathbf{v}_5 \to \mathbf{v}_6 \to \mathbf{v}_7$, and $\mathbf{v}_8$ being standalone, all for the same eigenvalue λ. This gives the Jordan canonical form of

$$J = \begin{pmatrix} \lambda & 1 & 0 & & & & & \\ 0 & \lambda & 1 & 0 & & & & \\ 0 & 0 & \lambda & 0 & & & & \\ & & & \lambda & 1 & 0 & 0 & \\ & & & 0 & \lambda & 1 & 0 & \\ & & & 0 & 0 & \lambda & 1 & \\ & & & 0 & 0 & 0 & \lambda & \\ & & & & & & & \lambda \end{pmatrix}$$

The reason block matrices are useful is that they simplify calculations massively. We may consider the block matrices as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

For instance, if we have 4×4 matrices, we convert this multiplication into 8 2×2 matrix multiplications. On first glance, this doesn't seem more efficient, but if there are a lot of zeros, it does actually become more efficient.

Example. In the general case, our flow matrix is of the form

$$\begin{aligned} \Psi(t) &= e^{At} \\ &= Pe^{Jt}P^{-1}. \end{aligned}$$

If

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

we now want to find e^{Jt}.

Now, if we had

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$

then

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{pmatrix}.$$

Note that the Jordan–Chevalley decomposition allows us to take

$$\begin{aligned} e^{Jt} &= e^{(D+N)t} \\ &= e^{Dt}e^{Nt}. \end{aligned}$$

where

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We may calculate

$$\begin{aligned} e^{Nt} &= I + Nt + \frac{N^2}{2}t^2 + \frac{N^3}{6}t^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$e^{Jt} = \begin{pmatrix} e^{\lambda t} & & \\ & e^{\lambda t} & \\ & & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$