Chapter 8 Problems

8.1

(a)

$$\begin{aligned}
\int_0^1 2^x \, dx &= \int_0^1 e^{x(\ln 2)} \, dx \\
&= \frac{1}{\ln 2}\left(e^{x(\ln 2)}\Big|_0^1\right) && u = x(\ln 2) \\
&= \frac{1}{\ln 2}\left(2^x\Big|_0^1\right) \\
&= \frac{1}{\ln 2}(2-1) \\
&= \frac{1}{\ln 2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^x \, dx &= \int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2}+x-\frac{1}{2}\right)+\frac{1}{2}} \, dx && \text{Completing the square.} \\
&= e^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2} \, dx \\
&= \sqrt{2\pi e} && \text{Gaussian Integral}
\end{aligned}$$

(c)

(d)

$$\int_{-a}^{a} \sin x e^{-\alpha x^2} \, dx = 0 \qquad \text{Even/odd.}$$

(e)

$$\begin{aligned}
\int_0^1 e^{\sqrt{x}} \, dx &= xe^{\sqrt{x}}\Big|_0^1 - \frac{1}{2}\int_0^1 xe^{\sqrt{x}} \, dx && \text{Integration by Parts} \\
&= e - \int_0^1 u^3 e^u \, du && u = \sqrt{x} \\
&= e - \left(u^3e^u\Big|_0^1 - 3u^2e^u\Big|_0^1 + 6u\Big|_0^1 - 6e^u\Big|_0^1\right) && \text{Repeated integration by parts.} \\
&= 3e - 6.
\end{aligned}$$

To evaluate $\int_0^1 u^3 e^u \, du$, we used tabular integration as follows:

Sign	Differentiate	Integrate
+	u^3	e^u
-	$3u^2$	e^u
+	$6u$	e^u
-	6	e^u
+	0	e^u

Taking the boundary integrals, we obtain

$$u^3e^u\Big|_0^1 - 3u^2e^u\Big|_0^1 + 6ue^u\Big|_0^1 - 6e^u\Big|_0^1 = 6 - 2e$$

(f)

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\cosh(u)}\cosh(u)\,du && x=\sinh(u)\\ &= u + C\\ &= \sinh^{-1}(x) + C.\end{aligned}$$

(g)

$$\begin{aligned}\int \tanh x\,dx &= \int \frac{\sinh x}{\cosh x}\,dx\\ &= \int \frac{1}{u}\,du && u=\cosh x\\ &= \ln|u| + C\\ &= \ln|\cosh x| + C.\end{aligned}$$

(h)

$$\begin{aligned}\int \tan^{-1} x\,dx &= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx && \text{integration by parts}\\ &= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C. && \text{u-substitution implicit}\end{aligned}$$

(i)

$$\begin{aligned}\int_S z^2\,d\mathbf{a} &= \int_0^{\pi/2}\int_0^{\pi/2} \cos^2\theta\sin\theta\,d\phi d\theta\\ &= \frac{\pi}{2}\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta\\ &= -\frac{\pi}{2}\int_0^{-1} t^2\,dt && t=\cos\theta\\ &= \frac{\pi}{2}\left(\frac{t^3}{3}\Big|_{-1}^{0}\right)\\ &= \frac{\pi}{6}\end{aligned}$$

8.8

(a)

$$\begin{aligned}\int_0^\infty \frac{x}{e^x-1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx\\ &= \int_0^\infty xe^{-x}\left(\sum_{k=0}^{\infty} e^{-kx}\right)dx\\ &= \sum_{k=0}^{\infty}\int_0^\infty xe^{-(k+1)x}\,dx\\ &= \sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\int_0^\infty ue^{-u}\,du && u=(k+1)x\\ &= \frac{\pi^2}{6}. && \text{Basel Problem}\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty \frac{x}{e^x+1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx \\
&= \int_0^\infty xe^{-x}\sum_{k=0}^\infty (-1)^k e^{-kx}\,dx \\
&= \sum_{k=0}^\infty (-1)^k \int_0^\infty xe^{-(k+1)x}\,dx \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2}\int_0^\infty ue^{-u}\,dx && u=(k+1)x \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2}.
\end{aligned}$$

To resolve

$$\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

we take

$$= \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) - \frac{1}{4}\underbrace{\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)}_{\frac{\pi^2}{6}},$$

meaning

$$\int_0^\infty \frac{x}{e^x+1}\,dx = \frac{\pi^2}{12}.$$

8.14

$$\begin{aligned}
I_0(a) &= \int_0^\infty x^0 e^{-ax^2}\,dx \\
&= \frac{1}{2}\sqrt{\frac{\pi}{a}}
\end{aligned}$$

$$\begin{aligned}
I_1(a) &= \int_0^\infty xe^{-ax^2}\,dx \\
&= \frac{1}{2}\left(\frac{1}{-a}e^{-ax^2}\Big|_0^\infty\right) \\
&= \frac{1}{2a}
\end{aligned}$$

$$\begin{aligned}
I_2(a) &= \int_0^\infty x^2 e^{-ax^2}\,dx \\
&= -\frac{1}{2a}\left(xe^{-ax^2}\Big|_0^\infty\right) + \frac{1}{2a}\int_0^\infty e^{-ax^2}\,dx
\end{aligned}$$

$$= \frac{1}{4a}\sqrt{\frac{\pi}{a}}.$$

$$\begin{aligned}
I_3(a) &= \int_0^\infty x^3 e^{-ax^2}\,dx\\
&= -\frac{1}{2a}\left(x^2e^{-ax^2}\Big|_0^\infty\right) + \frac{1}{a}\int_0^\infty xe^{-ax^2}\,dx\\
&= \frac{1}{2a^2}
\end{aligned}$$

$$\begin{aligned}
I_4(a) &= \int_0^\infty x^4 e^{-ax^2}\,dx\\
&= -\frac{1}{2a}x^3e^{-ax^2}\Big|_0^\infty + \frac{3}{2a}\int_0^\infty x^2e^{-ax^2}\,dx\\
&= \frac{3}{2a}I_2\\
&= \frac{3}{8a^3}\sqrt{\frac{\pi}{a}}.
\end{aligned}$$

8.24

$$\begin{aligned}
J(a) &= \lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 e^{-a\frac{\sum_{i=1}^n x_i}{n}}\frac{n}{\sum_{i=1}^n x_i}\,dx_1dx_2\cdots dx_n\\
J'(a) &= -\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 e^{-a}\,dx_1dx_2\cdots dx_n\\
&= -e^{-a}
\end{aligned}$$

meaning

$$\begin{aligned}
J(a) &= e^{-a}\\
J(0) &= 1.
\end{aligned}$$

8.26

(a)

$$\begin{aligned}
\int_0^\infty e^{-ax}\sin kx\,dx &= -\frac{1}{k}e^{-ax}\cos x\Big|_0^\infty - \frac{a}{k}e^{-ax}\sin x\Big|_0^\infty - \frac{a^2}{k^2}\int_0^\infty e^{-ax}\sin x\,dx\\
\left(1+a^2\right)\int_0^\infty e^{-ax}\sin x\,dx &= -e^{-ax}\cos x\Big|_0^\infty - ae^{-ax}\sin x\Big|_0^\infty\\
\left(1+a^2\right)\int_0^\infty e^{-ax}\sin x\,dx &= -\frac{1}{k}\\
\int_0^\infty e^{-ax}\sin x\,dx &= -\frac{1}{k\left(1+\frac{a^2}{k^2}\right)}\\
&= -\frac{k}{k^2+a^2}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty e^{-ax}\sin kx\,dx &= \frac{1}{2i}\int_0^\infty e^{-ax}\left(e^{ikx}-e^{-ikx}\right)\,dx\\
&= \frac{1}{2i}\left(\frac{1}{-a+ik}e^{-x(a-ik)}-\frac{1}{-a-ik}e^{-x(a+ik)}\right)\Big|_0^\infty\\
&= \frac{1}{2i}\left(\frac{1}{-a+ik}\left(e^{a-ik}\right)^{-x}-\frac{1}{-a-ik}\left(e^{a+ik}\right)^{-x}\right)\Big|_0^\infty\\
&= \frac{1}{2i}\left(\frac{1}{-a+ik}-\frac{1}{-a-ik}\right)\\
&= \frac{1}{2}\left(\frac{1}{-k-ia}-\frac{1}{k-ia}\right)\\
&= \frac{1}{2}\left(\frac{(k-ia)-(-k-ia)}{(-k-ia)(k-ia)}\right)\\
&= \frac{1}{2}\left(\frac{2k}{-k^2-a^2}\right)\\
&= -\frac{k}{k^2+a^2}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^\infty e^{-ax}\sin kx\,dx &= \int_0^\infty e^{-ax}\,\mathrm{Im}\left(e^{ikx}\right)\,dx\\
&= \mathrm{Im}\left(\int_0^\infty e^{-ax}e^{ikx}\,dx\right)\\
&= \mathrm{Im}\left(\frac{1}{-a+ik}e^{(-a+ik)x}\Big|_0^\infty\right)\\
&= \mathrm{Im}\left(\frac{1}{-a+ik}\left(e^{a-ik}\right)^{-x}\Big|_0^\infty\right)\\
&= \mathrm{Im}\left(\frac{1}{-a+ik}\right)\\
&= \mathrm{Im}\left(\frac{-a-ik}{a^2+k^2}\right)\\
&= \mathrm{Im}\left(\frac{-a}{a^2+k^2}-\frac{k}{k^2+a^2}i\right)\\
&= -\frac{k}{k^2+a^2}.
\end{aligned}$$

Chapter 9 Problems

9.1

(a)

$$\int_{-2}^{3}\left(x^3-(2x+5)^2\right)\delta(x-1)\,dx = -48.$$

(b)

$$\int_{0}^{3}\left(5x^2-3x+4\right)\delta(x+2)\,dx = 0.$$

(c)

$$\int_0^1 \cos x \delta(x - \pi/6)\ dx = \frac{\sqrt{3}}{2}.$$

(d)

$$\int_{-\pi}^{\pi} \ln(\sin x + 2)\, \delta(x + \pi/2)\ dx = \ln(3).$$

(e)

$$\int_{-1}^{1} \left(x^3 - 3x^2 + 2\right) \delta(x/7)\ dx = 14.$$

(f)

$$\int_{-1}^{1} (x-1)e^{x^2}\delta(-3x)\ dx = -\frac{1}{3}.$$

(g)

$$\int_{-\pi}^{\pi} 4x^2 \arccos(x)\delta(2x - 1)\ dx = \frac{\pi}{6}.$$

(h)

$$\int_p^{\infty} \delta(x + q)\ dx = \begin{cases} 1 & p < q \\ 0 & p > q \end{cases}.$$

(i)

$$\begin{aligned} \int_0^{2b} x\delta\left(\left(x^2 - b^2\right)\left(x - \frac{b}{2}\right)\right)\ dx &= \int_0^{2b} x\left(b^2\delta(x-b) + 3b^2\delta(x+b) - \frac{3}{4}b^2\delta(x - b/2)\right)\ dx \\ &= b^3 - \frac{3}{8}b^3 \\ &= \frac{5}{8}b^3. \end{aligned}$$

(j)

$$\int_{-\pi/2}^{\pi/2} e^x \delta(\tan(x))\ dx = 1.$$

9.2

We can see that $\frac{d\Theta}{dx} = 0$ for $x < 0$ and $x > 0$, and $\frac{d\Theta}{dx}$ is undefined for $x = 0$.

Additionally, for any f, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\frac{d\Theta}{dx}\ dx &= \int_{-\infty}^{\infty} f(x)\ d\Theta \\ &= f(0). \end{aligned}$$

Thus, since $\frac{d\Theta}{dx}$ has the two necessary conditions to be a delta distribution. $\frac{d\Theta}{dx} = \delta(x)$.

9.3

First, we can see that

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_n(x)\,dx &= \frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-n^2x^2}\,dx\\ &= \frac{n}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{n}\right)\\ &= 1,\end{aligned}$$

meaning that we satisfy normalization. Additionally, for $x_0 \neq 0$,

$$\begin{aligned}\lim_{n\to\infty}\phi_n(x_0) &= \lim_{n\to\infty}\frac{n}{\sqrt{\pi}}e^{-n^2x_0^2}\\ &= \frac{1}{\sqrt{\pi}}\lim_{n\to\infty} n\left(e^{x_0^2}\right)^{-n^2}\\ &= 0,\end{aligned}$$

and for $x = 0$, $\lim_{n\to}\phi_n(x)$ diverges. Finally, for $f(x)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\frac{n}{\pi}e^{-n^2x^2}\,dx &= f(c)\int_{-\infty}^{\infty}\frac{n}{\sqrt{\pi}}e^{-n^2x^2}\,dx\\ &= f(c)\end{aligned}$$

for some $c \in (-\infty,\infty)$. In particular, since $\frac{n}{\sqrt{\pi}}f(x)e^{-n^2x^2}$ tends to zero for any $x \neq 0$, it must be the case that $f(c) = f(0)$.

9.10

(a) Knowing that δ instantiates an integral at a particular value, we know that

$$\begin{aligned}M &= \int_V \rho\,d\tau\\ &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\rho r\,dr d\phi dz\\ &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} M\delta(r-R)\,\delta(z)\,r\,dr d\phi dz,\end{aligned}$$

meaning $\rho = M\delta(r-R)\,\delta(z)$.

(b) Similarly, we have

$$\begin{aligned}M &= \int_V \rho\,d\tau\\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\rho r^2\sin\theta\,dr d\phi d\theta\\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} M\delta(r-R)\,\delta\Big(\theta-\frac{\pi}{2}\Big)\,r^2\sin\theta\,dr d\phi d\theta,\end{aligned}$$

meaning $\rho = M\delta(r-R)\,\delta\big(\theta - \frac{\pi}{2}\big)$.

9.11

Notice that

$$\begin{aligned}\frac{d}{dx}\left(i\pi\operatorname{sgn}(x)\right) &= \frac{d}{dx}\left(\int_{-\infty}^{\infty}\frac{e^{ikx}}{k}\,dk\right)\\ &= i\int_{-\infty}^{\infty}e^{ikx}\,dk.\end{aligned}$$

Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dk = \frac{1}{2}\left(\frac{d}{dx}\operatorname{sgn}(x)\right),$$

meaning

$$\begin{aligned}\int_a^b f(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dk\right)\,dx &= \int_a^b f(x)\left(\frac{d}{dx}\operatorname{sgn}(x)\right)\,dx\\ &= \frac{1}{2}f(x)\operatorname{sgn}(x)\Big|_a^b - \frac{1}{2}\int_a^b f'(x)\operatorname{sgn}(x)\,dx\\ &= \frac{1}{2}\left(f(b)\operatorname{sgn}(b) - f(a)\operatorname{sgn}(a)\right) - \frac{1}{2}\left(\int_a^0 f'(x)\operatorname{sgn}(x)\,dx + \int_0^b f'(x)\operatorname{sgn}(x)\,dx\right)\\ &= \frac{1}{2}\left(f(b)\operatorname{sgn}(b) - f(a)\operatorname{sgn}(a)\right) - \frac{1}{2}\left(-\operatorname{sgn}(a)\left(f(a)-f(0)\right) + \operatorname{sgn}(b)\left(f(b)-f(0)\right)\right).\end{aligned}$$

Without loss of generality, we say $a < b$. If $\operatorname{sgn}(a) = \operatorname{sgn}(b)$, then this expression resolves to 0, and if $\operatorname{sgn}(a) \neq \operatorname{sgn}(b)$, this expression resolves to $f(0)$.