Introduction: naive set theory

$$\begin{aligned}\mathbb{N} &= \{1, 2, 3, \dots,\} \\ \mathbb{Z} &= \{0, \pm 1, \pm 2, \dots,\} \\ \mathbb{Z}_+ &= \{0, 1, 2, \dots,\} \\ \mathbb{Q} &= \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\} \\ \mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} \\ \mathbb{C}_q &= \{a + bi \mid a, b \in \mathbb{Q}\}\end{aligned}$$

Recall: given sets X and Y, a relation from X to Y is a subset of $X \times Y$, where $\times$ denotes the cartesian product of X and Y.

A relation $f \subseteq X \times Y$ is a function from X to Y such that $\forall x \in X$, $\exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote f as $f : X \to Y$.

X is the **domain** of f and Y is the **codomain**. The range $\operatorname{ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\operatorname{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Examples

$$\mathrm{id}_x : X \to X, \mathrm{id}_X(x) = x$$

This is the identity function.

The Characteristic Function: If $A \subseteq X$

$$\mathbf{1}_A : X \to \mathbb{R},\ \mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Algebra of Functions

Let X be any set, and $(X; \mathbb{R}) = \{f : X \to \mathbb{R}\}$ represent the function space of X with codomain $\mathbb{R}$.

Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$ (scalar multiplication). If $g(x) \neq 0 \forall x \in X$, then $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$.

Finally, we have composition. If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f(x) = g(f(x))$.

Injective, Subjective, and Bijective

A function $f : X \to Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.For example, the shift map $S : \mathbb{N} \to \mathbb{N},\ S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \to \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\operatorname{ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \to Y$ be a function. f is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. f is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$.

f is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \mathrm{id}_Y$ and $k \circ f = \mathrm{id}_X$.

Proposition

f is invertible if and only if f is left and right invertible.

Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose g is a left-inverse of f, and h is a right-inverse of f. Therefore, $g \circ f = \mathrm{id}_X$, and $f \circ h = \mathrm{id}_Y$. Observe that $g = g \circ \mathrm{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$.

Theorem

If $f : X \to Y$ is a function:

1. f is injective $\Leftrightarrow$ f is left-invertible.
2. f is surjective $\Leftrightarrow$ f is right-invertible.
3. f is bijective $\Leftrightarrow$ f is invertible.

We will prove the first proposition in the forward direction. Suppose f is injective. Given $y \in \mathrm{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \mathrm{ran}(f) \\ x_0 & y \notin \mathrm{ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X. We can see that $g \circ f = \mathrm{id}_X$.

For example, the function $\mathrm{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

Cardinality and Finitude

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets A and B, we can show that A is the same size as B by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

Definition

Sets A and B have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\text{card}(A) = \text{card}(B)$.

Example

Given $a < b$ and $c < d$, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

Example 2

$$\text{card}((0, 1)) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - tan is strictly increasing (and thus injective)
  - $\lim_{x\to\infty} \tan(x) = \infty$ and $\lim_{x\to-\infty} \tan(x) = -\infty$, and by intermediate value theorem, tan is surjective
- $\ell : (0, 1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0, 1) \to \mathbb{R}$.

Definition

A set F is **finite** if F is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}(\{1, 2, \dots, n\})$. A non-finite set is called infinite.

We can *enumerate* F by creating a function $\sigma : \{1, 2, \dots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

Proposition

If $m \neq n$, then $\text{card}\{1, 2, \dots, m\} = \text{card}\{1, 2, \dots, n\}$.

WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is our bijection. This means there are m "pigeons" and n "holes."

One hole, j, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since f is assumed to be injective, this is a contradiction.

Proposition

$\mathbb{N}$ is infinite.

Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m+1\} \to \mathbb{N}$. i is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \to \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

Proposition

If A is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

$\exists a_1 \in A$, as $A \neq \emptyset$.

$A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$.

$A \setminus \{a_1, a_2\} \neq \emptyset$, so $\exists a_3 \in A \setminus \{a_1, a_2\}$.

$\vdots$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of A.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. f is injective as a_n are distinct.

Example

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

f is a bijection as $g : \mathbb{N} \to \mathbb{Z},\ g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of f.

Definition

Given any set X, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of X.

$2^X := \{f \mid f : X \to \{0, 1\}\}$.

Proposition

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbf{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$.

Cantor's theorem

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

Fact from calculus: $\forall \sigma \in (0,1)$, σ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \dots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0, 1, \dots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0, 1, \dots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since r is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

Comparing Cardinalities

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

Transitive Property

If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

The composition of two injective functions is injective.

Canonical Set Comparisons

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

Cantor-Schröder-Bernstein

For any set A, $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Let us construct a function: $f : A \to \mathcal{P}(A)$, where $a \mapsto \{a\}$.

f is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

Claim $\nexists g : A \to \mathcal{P}(A)$, g is surjective.

Suppose toward contradiction that such a g exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since g is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. $\perp$

Equivalent Propositions

(i) $\text{card}(A) \leq \text{card}(B)$

(ii) $\exists f : A \hookrightarrow B$

(iii) $\exists g : B \to A$, g surjection.

By definition, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) If $f : A \hookrightarrow B$, f is left-invertible, and thus $\exists g : B \to A$ with $g \circ f = id_A$. So, g is right-invertible, so g is surjective.

(iii) $\Rightarrow$ (ii) If $g : B \to A$ is surjective, then g is right-invertible, so $\exists f : A \to B$ such that $g \circ f = id_B$. So, f is left-invertible, so f is injective.

Corollary

If $f : A \to B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$.

Consider $g : A \to f(A)$, where $g(a) = f(a)$. So, g is onto, so $\exists$ an injection $f(A) \hookrightarrow A$.

More Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, q(m, n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $H(m, n) = (h(m), n)$.

Claim H is a bijection.

Proof of Injection If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since h is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

Proof of Surjection Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m, n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m, n) = (k, \ell)$. Set $n = \ell$, and since h is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m, n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, φ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned} \text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\ &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \text{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \text{card}(\mathbb{N}) \end{aligned}$$

Cardinality Rules

(i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)

(ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)

(iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)

(iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

Proof of (iii) We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

Claim We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, g and f are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1$. $\perp$

We let $A_n = g \circ f(A_{n-1})$ for arbitrary n, and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \to B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where y_x is the unique element in B with $g(y_x) = x$.

Claim We claim h is the desired bijection.

Proof of Injection Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of H, $f(x_1) = f(x_2)$, f is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y(x_2)) = x_2 \notin A_\infty$. This case is not possible.

Thus, h is injective.

Proof of Surjection Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where y_x is the unique element in B with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since g is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Therefore, we have $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$.