Chapter 2 Problems

2.3

Cylindrical Coordinates

Starting with our expression of $\mathbf{r}$, we have

$$\begin{aligned}\mathbf{r} &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}\\ d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \rho}d\rho + \frac{\partial \mathbf{r}}{\partial \phi}d\phi + \frac{\partial \mathbf{r}}{\partial z}dz.\end{aligned}$$

Calculating each partial derivative,

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \rho} &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\\ \hat{\rho} &= \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|}\\ &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath},\\ \frac{\partial r}{\partial \phi} &= \rho\left(-\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\right)\\ \hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\|\frac{\partial \mathbf{r}}{\partial \phi}\right\|}\\ &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\end{aligned}$$

implying

$$\frac{\partial \mathbf{r}}{\partial \phi} = \rho\hat{\phi},$$

and finally, we have

$$\frac{\partial \mathbf{r}}{\partial z} = \hat{k}.$$

The above calculations yield

$$d\mathbf{r} = (d\rho)\,\hat{\rho} + (\rho\, d\phi)\,\hat{\phi} + (dz)\,\hat{k}.$$

Spherical Coordinates

Starting with our expression of $\mathbf{x}$[1]

$$\begin{aligned}\mathbf{x} &= r\sin\phi\sin\theta\hat{\imath} + r\cos\phi\sin\theta\hat{\jmath} + r\cos\theta\hat{k}\\ d\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial r}dr + \frac{\partial \mathbf{x}}{\partial \phi}d\phi + \frac{\partial \mathbf{x}}{\partial \theta}d\theta,\end{aligned}$$

Evaluating each partial derivative, we have

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial r} &= \sin\phi\sin\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath} + \cos\theta\hat{k}\\ \hat{r} &= \frac{\frac{\partial \mathbf{x}}{\partial r}}{\left\|\frac{\partial \mathbf{x}}{\partial r}\right\|}\end{aligned}$$

[1] I am using $\mathbf{x}$ instead of $\mathbf{r}$ because r is already used in the expression of the spherical coordinates.

$$\begin{aligned}
&= \sin\phi\sin\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath} + \cos\theta\hat{k},\\
\frac{\partial \mathbf{x}}{\partial \phi} &= -r\sin\phi\sin\theta\hat{\imath} + r\cos\phi\sin\theta\hat{\jmath}\\
\hat{\phi} &= \frac{\frac{\partial \mathbf{x}}{\partial \phi}}{\left\|\frac{\partial \mathbf{x}}{\partial \phi}\right\|}\\
&= -\sin\phi\sin\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath}
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \phi} = r\sin\theta\hat{\phi},$$

and finally, we have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \theta} &= r\cos\phi\cos\theta\hat{\imath} + r\sin\phi\cos\theta\hat{\jmath} - r\sin\theta\hat{k}\\
\hat{\theta} &= \frac{\frac{\partial \mathbf{x}}{\partial \theta}}{\left\|\frac{\partial \mathbf{x}}{\partial \theta}\right\|}\\
&= \cos\phi\cos\theta\hat{\imath} + \sin\phi\cos\theta\hat{\jmath} - \sin\theta\hat{k},
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \theta} = r\hat{\theta}.$$

The above calculations yield

$$d\mathbf{x} = (dr)\,\hat{r} + (r\sin\theta d\phi)\,\hat{\phi} + (rd\theta)\,\hat{\theta}.$$

2.8

Let

$$\begin{aligned}
\vec{a} &= r_a\cos\phi_a\sin\theta_a\hat{\imath} + r_a\sin\phi_a\sin\theta_a\hat{\jmath} + r_a\cos\theta_a\hat{k}\\
\vec{b} &= r_b\cos\phi_b\sin\theta_b\hat{\imath} + r_b\sin\phi_b\sin\theta_b\hat{\jmath} + r_b\cos\theta_b\hat{k}.
\end{aligned}$$

Then,

$$\begin{aligned}
\cos\gamma &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\\
&= \frac{1}{r_ar_b}\left(r_ar_b\left(\sin\theta_a\sin\theta_b\left(\cos\phi_a\cos\phi_b + \sin\phi_a\sin\phi_b\right) + \cos\theta_a\cos\theta_b\right)\right)\\
&= \cos\theta_a\cos\theta_b + \sin\theta_a\sin\theta_b\cos\left(\phi_a - \phi_b\right).
\end{aligned}$$

2.9

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{d}{dt}\left(\dot{\rho}\hat{\rho}\right) + \frac{d}{dt}\left(\rho\dot{\phi}\hat{\phi}\right)\\
&= \hat{\rho}\ddot{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt}\\
&= \hat{\rho}\ddot{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\dot{\phi}\left(\frac{\partial\hat{\phi}}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\hat{\phi}}{\partial\phi}\frac{d\phi}{dt}\right)\\
&= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi}.
\end{aligned}$$

2.12

(a)

(i) $d\mathbf{a} = \rho \, d\phi dz$

(ii) $d\mathbf{a} = d\rho dz$

(iii) $d\mathbf{a} = \rho \, d\rho d\phi$

(b)

(i) $d\mathbf{a} = r^2 \sin\theta \, d\theta d\phi$

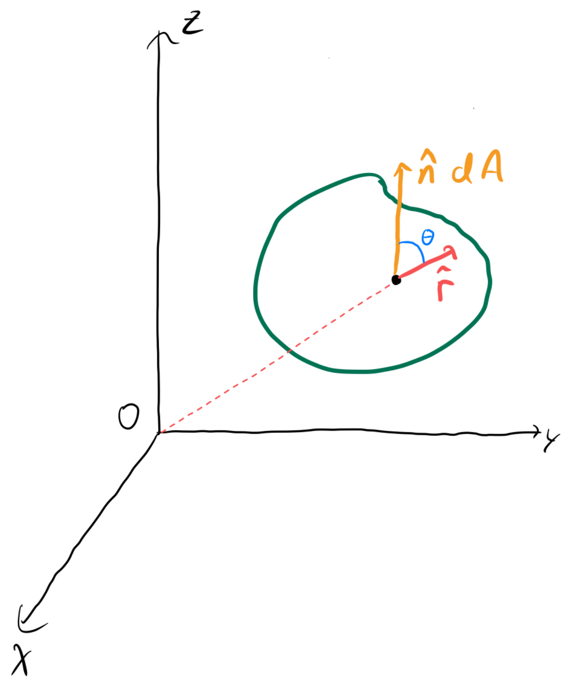
(ii) $d\mathbf{a} = r \sin\theta \, dr d\phi$

(iii) $d\mathbf{a} = r \, dr d\theta$

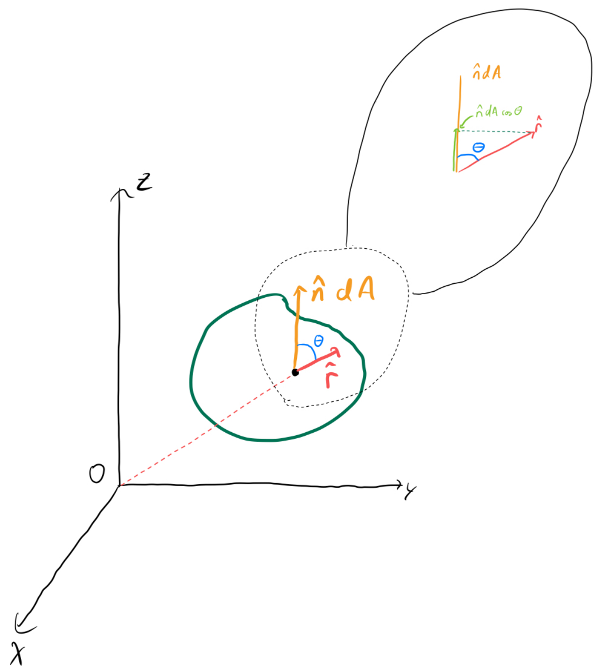
2.14

(a)

$$\begin{aligned} d\Phi &= \mathbf{E} \cdot \hat{n} \, dA \\ &= \|\mathbf{E}\| \, \|\hat{n}\| \cos\theta dA \\ &= \frac{q}{4\pi\epsilon_0 r^2} \cos\theta dA. \end{aligned}$$



(b)



(c)

$$\oiint_S d\Phi = \oiint_S \frac{q}{4\pi\epsilon_0 r^2}\, d\mathbf{a}$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{q}{4\pi\epsilon_0 r^2} r^2 \sin\theta \, d\theta d\phi$$
$$= (2\pi)\left(\frac{q}{4\pi\epsilon_0}\right)\left(-\cos\theta\Big|_0^{\pi}\right)$$
$$= \frac{q}{\epsilon_0}.$$

Chapter 3 Problems

For all problems involving $\arg z$ (or equivalents), I will be using the principle branch, $\arg z \in (-\pi, \pi]$.

3.5

(a)

$$\sqrt{3} + i = 2e^{i\frac{\pi}{3}}$$
$$-\sqrt{3} + i = 2e^{i\frac{2\pi}{3}}$$

(b)

$$\sqrt{2i} = \sqrt{2}e^{i\frac{\pi}{4}}$$
$$\sqrt{2 + 2\sqrt{3}i} = 2e^{i\frac{\pi}{6}}$$

3.6

(a) Real:

$$\begin{aligned}(-1)^{1/i} &= \left(e^{i\pi}\right)^{-i}\\ &= e^{\pi}.\end{aligned}$$

(b) Real:

$$\begin{aligned}\left(\frac{z}{z^*}\right)^{i} &= \left(e^{2i\arg z}\right)^{i}\\ &= e^{-2\arg z}.\end{aligned}$$

(c) Imaginary:

$$\begin{aligned}\left(z_1 z_2^* - z_1^* z_2\right)^* &= z_1^* z_2 - z_1 z_2^*\\ &= -\left(z_1 z_2^* - z_1^* z_2\right).\end{aligned}$$

(d) Complex:

$$\sum_{n=0}^{N} e^{in\theta} = \frac{1-e^{iN\theta}}{1-e^{i\theta}}.$$

(e) Real: for each $a \in \{1, 2, \ldots, N\}$, $e^{ia\theta} + e^{-ia\theta} \in \mathbb{R}$.

3.9

(a)

$$\begin{aligned}\cos(a+b) + \cos(a-b) &= \frac{1}{2}\left(e^{i(a+b)} + e^{-i(a+b)}\right) + \frac{1}{2}\left(e^{i(a-b)} + e^{-i(a-b)}\right)\\ &= \frac{1}{2}\left(e^{ia}\left(e^{ib} + e^{-ib}\right) + e^{-ia}\left(e^{ib} + e^{-ib}\right)\right)\\ &= \frac{1}{2}\left(e^{ia} + e^{-ia}\right)\left(e^{ib} + e^{-ib}\right)\\ &= 2\cos a\cos b.\end{aligned}$$

(b)

$$\begin{aligned}\sin(a+b) + \sin(a-b) &= \frac{1}{2i}\left(e^{i(a+b)} - e^{-i(a+b)}\right) + \frac{1}{2}\left(e^{i(a-b)} - e^{-i(a-b)}\right)\\ &= \frac{1}{2i}\left(e^{ia}\left(e^{ib} + e^{-ib}\right) - e^{-ia}\left(e^{ib} + e^{-ib}\right)\right)\\ &= \frac{1}{2i}\left(e^{ia} - e^{-ia}\right)\left(e^{ib} + e^{-ib}\right)\\ &= 2\sin a\cos b.\end{aligned}$$

3.10

(a)

$$e^{i\alpha} + e^{i\beta} = e^{i\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)} + e^{i\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)}$$

$$\begin{aligned}
&= e^{i\frac{\alpha+\beta}{2}}\left(e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}}\right)\\
&= 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{i\frac{\alpha+\beta}{2}}.
\end{aligned}$$

(b)

$$\begin{aligned}
e^{i\alpha} - e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)} - e^{i\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)}\\
&= e^{i\frac{\alpha+\beta}{2}}\left(e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}}\right)\\
&= 2i\sin\left(\frac{\alpha-\beta}{2}\right)e^{i\frac{\alpha+\beta}{2}}
\end{aligned}$$

3.12

$$\begin{aligned}
\frac{1}{2i}\ln\left(\frac{a+ib}{a-ib}\right) &= \frac{1}{2i}\left(\ln(a+ib) - \ln(a-ib)\right)\\
&= \frac{1}{2i}\left(\ln|a+ib| + i\arctan\left(\frac{b}{a}\right) - \left(\ln|a+ib| + i\arctan\left(-\frac{b}{a}\right)\right)\right)\\
&= \arctan\left(\frac{b}{a}\right).
\end{aligned}$$

3.13

$$\begin{aligned}
\frac{d^n}{dt^n}\left(e^{at}\sin bt\right) &= \frac{1}{2i}\frac{d^n}{dt^n}\left(e^{(a+ib)t} - e^{(a-ib)t}\right)\\
&= \frac{1}{2i}\left((a+ib)^n e^{(a+ib)t} - (a-ib)^n e^{(a-ib)t}\right)\\
&= \frac{1}{2i}e^{at}\left(\left(\left(a^2+b^2\right)^{n/2} e^{in\arctan\left(\frac{b}{a}\right)}\right)e^{ibt} - \left(\left(a^2+b^2\right)^{n/2} e^{-in\arctan\left(\frac{b}{a}\right)}\right)e^{-ibt}\right)\\
&= e^{at}\frac{1}{2i}\left(a^2+b^2\right)^{n/2}\left(e^{i\left(b+n\arctan\left(\frac{b}{a}\right)\right)t} - e^{i\left(b+n\arctan\left(\frac{b}{a}\right)\right)}\right)\\
&= e^{at}\left(a^2+b^2\right)^{n/2}\sin\left(bt + n\arctan\left(\frac{b}{a}\right)\right)
\end{aligned}$$

3.20

Showing the equivalence between $C_1\cos kx + C_2\sin kx$ and $A\cos(kx+\alpha)$ and $B\sin(kx+\beta)$, we have

$$\begin{aligned}
A\cos(kx+\alpha) &= A\cos kx\cos\alpha - A\sin kx\sin\alpha\\
B\sin(kx+\beta) &= B\cos kx\sin\beta + B\sin kx\cos\beta
\end{aligned}$$

meaning (assuming $\alpha, \beta \neq \pi n, \pi/2 + \pi n$)

$$\begin{aligned}
A &= \frac{C_1}{\cos\alpha}\\
&= -\frac{C_2}{\sin\alpha}\\
B &= \frac{C_1}{\sin\beta}
\end{aligned}$$

$$= \frac{C_2}{\cos\beta}.$$

Now, we show the equivalence between $C_1 \cos kx + C_2 \sin kx$ and $D_1 e^{ikx} + D_2 e^{-ikx}$.

$$\begin{aligned} D_1 e^{ikx} + D_2 e^{-ikx} &= \frac{D_1 + D_2}{2}\left(e^{ikx} + e^{-ikx}\right) + \frac{D_1 + D_2}{2}\left(e^{ikx} - e^{-ikx}\right) \\ &= (D_1 + D_2)\cos kx + i\,(D_1 - D_2)\sin kx. \end{aligned}$$

meaning

$$\begin{aligned} C_1 &= D_1 + D_2 \\ C_2 &= i\,(D_1 - D_2). \end{aligned}$$

Finally, we show the equivalence between $\operatorname{Re}\left(Fe^{ikx}\right)$ and $C_1 \cos kx + C_2 \sin kx$.

$$\begin{aligned} \operatorname{Re}\left((a + ib)\, e^{ikx}\right) &= \operatorname{Re}\left(a\cos kx + ia\sin kx + ib\cos kx - b\sin kx\right) \\ &= a\cos kx - b\sin kx, \end{aligned}$$

meaning

$$\begin{aligned} C_1 &= a \\ C_2 &= -b. \end{aligned}$$