

Solution (18.1):

- (a) The function $f(z) = z^n$ is analytic on \mathbb{C} .
- (b) The functions $f(z) = \sin(z)$ and $f(z) = \cos(z)$ are analytic on \mathbb{C} , while $f(z) = \tan(z)$ is analytic everywhere except for singularities at $n\pi/2$.
- (c) The function $f(z) = |z|$ is analytic nowhere.
- (d) The function $f(z) = \frac{z-i}{z+1}$ is analytic everywhere except for $z = -1$.
- (e) The function $f(z) = \frac{z^2+1}{z}$ is analytic everywhere except for $z = 0$.
- (f) The function $f(z) = \frac{p_n(z)}{q_m(z)}$ is analytic everywhere except for the roots of q .
- (g) The function $x^2 + y^2$ is analytic nowhere.
- (h) The function e^z is analytic on \mathbb{C} .
- (i) The function e^{-iy} is analytic nowhere.
- (j) The function $\ln(z)$ is analytic everywhere except for $(-\infty, 0]$.

Solution (18.2): Let $w(z) = u(x, y) + iv(x, y)$. Then,

$$\begin{aligned} i \frac{\partial}{\partial x}(w(x + iy)) - \frac{\partial}{\partial y}(w(x + iy)) &= i \frac{\partial}{\partial x}(u(x, y) + iv(x, y)) - \frac{\partial}{\partial y}(u(x, y) + iv(x, y)) \\ &= i \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \end{aligned}$$

Thus, we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

Solution (18.4):

Solution (18.5): We see that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial v}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial v}{\partial x \partial y}, \end{aligned}$$

so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Symmetrically, we also have that v satisfies Laplace's equation.

Solution (18.6):

- (a) We start by writing everything in terms of z , so we have $w(z) = u(z) + iv(z)$. Since w is complex-differentiable, by linearity we have

$$w'(z) = \frac{du}{dz} + i \frac{dv}{dz}.$$

We write $z = x + iy$, or $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$.

$$\begin{aligned} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial z} \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right). \end{aligned}$$

(b) We have that

$$\begin{aligned}\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right).\end{aligned}$$

Thus, by tedious algebraic manipulations heavily prone to error, we recover the Cauchy–Riemann equations:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Solution (18.7): We find that

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= 1 \\ &= \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \\ &= -i.\end{aligned}$$

Since these derivatives are path-dependent, we have that $w(z) = \bar{z}$ is not differentiable.

Solution (18.11):

Solution (18.14): We know that $\frac{1}{z^m}$ is defined for all $z \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$, so we only need to show that if $z \neq 0$, then $\frac{1}{z^m}$ admits a derivative. We see that

$$\frac{d}{dz} \left(\frac{1}{z^m} \right) = \frac{-m}{z^{m+1}},$$

which is yet again defined for all $z \neq 0$, so $\frac{1}{z^m}$ is analytic on its domain.

Solution (18.15):

Solution (18.18):