Problem (Problem 1)**:** Let X and Y be simplicial complexes homeomorphic to the 2-sphere, S^2, and the torus $S^1 \times S^1$. Compute the real simplicial homology and cohomology of X and Y.

Solution: We fix the order $(v_0, v_1, v_2, v_3, v_4, v_5)$ in the simplicial complex for X. We see that the k-chains are as follows:

- $C_k(X, \mathbb{R}) = 0$ for all $k \geqslant 3$;
- $C_2(X, \mathbb{R}) = \mathbb{R}\langle v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_5, v_1v_3v_5, v_2v_3v_5\rangle \cong \mathbb{R}^6$;
- $C_1(X, \mathbb{R}) = \mathbb{R}\langle v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3, v_1v_5, v_2v_5, v_3v_5\rangle \cong \mathbb{R}^9$;
- $C_0(X, \mathbb{R}) = \mathbb{R}\langle v_0, v_1, v_2, v_3, v_4, v_5\rangle \cong \mathbb{R}^6$.

We start by applying the boundary map to $C_1(X, \mathbb{R})$, yielding

$$\begin{aligned}
v_0v_1 &\mapsto v_1 - v_0\\
v_0v_2 &\mapsto v_2 - v_0\\
v_0v_3 &\mapsto v_3 - v_0\\
v_1v_2 &\mapsto v_2 - v_1\\
v_1v_3 &\mapsto v_3 - v_1\\
v_2v_3 &\mapsto v_3 - v_2\\
v_1v_5 &\mapsto v_5 - v_1\\
v_2v_5 &\mapsto v_5 - v_2\\
v_3v_5 &\mapsto v_5 - v_3.
\end{aligned}$$

Since this forms a basis for the kernel of the linear functional given by mapping all of the v_i to 1, it follows that $B_0(X, \mathbb{R}) \cong \mathbb{R}^5$, while $Z_0(X, \mathbb{R}) \cong \mathbb{R}^6$, yielding $H_0(X, \mathbb{R}) \cong \mathbb{R}$.

Similarly, since we may find the boundary map $\partial\colon C_2(X, \mathbb{R}) \to C_1(X, \mathbb{R})$ that yields a subspace that is the kernel of a linear functional on $\mathbb{R}^9$ with codimension 4, it follows that $H_2(X, \mathbb{R}) \cong \mathbb{R}$ as well.

Finally, we see that the image of the basis for $C_2(X, \mathbb{R})$ yields a basis with six linearly independent vectors, while the kernel of ∂ on $C_1(X, \mathbb{R})$ yields another basis with six linearly independent vectors, so that $H_1(X, \mathbb{R}) \cong 0$.

Problem (Problem 2)**:** Use the definition of de Rham cohomology to prove that $H^0_{DR}(\mathbb{R}) \cong \mathbb{R}$ and all higher de Rham cohomology vector spaces are zero.

Solution: Evaluating H^0_{DR}, we see that the functions whose derivatives are zero are the constants on $\mathbb{R}$, meaning the cochains $Z^0(\mathbb{R}) \cong \mathbb{R}$, while the coboundaries $B^0(\mathbb{R}) \cong 0$.

Since $\mathbb{R}$ has dimension 1, it follows that $\Lambda^k(\mathbb{R}) \cong 0$ for all $k \geqslant 2$, so we only need to verify that $Z^1(\mathbb{R}) \cong B^1(\mathbb{R})$. This follows from the fact that every 1-form can be integrated to yield a C^∞ function on $\mathbb{R}$, while every 1-form evaluates to zero under the exterior derivative.

Problem (Problem 3)**:** Use the definition of de Rham cohomology to prove that $H^*_{DR}(S^1) \cong \mathbb{R}$ in dimensions 0 and 1 and vanishes in all higher dimensions.

Solution: Since S^1 is a 1-dimensional manifold, it follows that $H^k_{DR}(S^1) \cong 0$ for all $k \geqslant 2$ since all 2-forms vanish.

Similarly, since only the constants S^1 vanish, we have $H^0_{DR}(S^1) \cong \mathbb{R}$. Finally, to understand $H^1_{DR}(S^1)$, we observe that any exact form $d\omega$ maps to $\mathbb{R}$ by integrating,

$$f(\theta) = \int_0^\theta d\omega,$$

and such non-closed exact forms exist on S^1, so that $H^1_{DR}(S^1) \cong \mathbb{R}$.

Problem (Problem 4)**:** Prove that if M is a closed, connected manifold of dimension n that is not orientable, then the nth simplicial homology satisfies $H_n(M, \mathbb{R}) = 0$.

Solution: Let $p \in M$; since M is orientable, if we select an n-simplex with a vertex at p, we find that both $v_0 v_1 \cdots v_n$ and $v_1 v_0 \cdots v_n$ yield valid orientations for $T_p M$. Taking a boundary of two of these n-simplices, we find that if σ_i and σ_j are two such simplices in M, we may orient σ_i such that ∂ yields a positive value on this boundary, so that $B_n(M, \mathbb{R}) \cong \mathbb{R}$. Thus, we find that $H_n(M, \mathbb{R}) \cong 0$.

Problem (Problem 5)**:** A smooth map $f\colon M \to n$ is called a submersion if it induces surjections on tangent spaces. Prove that if M and N are smooth manifolds and $A \subseteq N$ is a smooth submanifold, then f is transverse to A.

Solution: Let $p \in f^{-1}(A)$. By the definition of the submersion, we have $T_{F(p)}N = D_p F(T_p M)$, meaning that $D_p F(T_p M) + T_{F(p)}A = T_{F(p)}N$.

Problem (Problem 6)**:** In this exercise, we will prove a version of the Transversality Theorem. Let M and N be smooth manifolds. The transversality theorem asserts that for all $1 \leqslant r \leqslant \infty$, the set of C^r maps $M \to N$ that are transverse to A is dense in any of the natural topologies $C^r(M, N)$.

We will restrict our attention to manifolds embedded in Euclidean space and prove a slightly weaker version of the transversality theorem.

(a) Let M, N, and A be as above, and let Y be an arbitrary smooth manifold. Let $F\colon Y \times M \to N$ be a smooth map transverse to A. For each $y \in Y$, let $f_y : M \to N$ be defined by $F(y, \cdot)$, and let $\pi\colon Y \times M \to Y$ be the projection.

Prove that for every regular value $y \in Y$ of π, the map f_y is transverse to A.

(b) Let $f\colon M \to \mathbb{R}^n$ be a smooth map, and let $A \subseteq \mathbb{R}^n$ be a smooth submanifold. Show that the set of $p \in \mathbb{R}^n$ for which $f_p(x) := f(x) + p$ is not transverse to A has measure zero.

(c) Prove that if M and N are smooth submanifolds of $\mathbb{R}^n$, then for all $p \in \mathbb{R}^n$ outside a set of measure zero, the manifolds $M + p$ and N intersect transversely.

(d) Prove that if $f\colon M \to N$ is smooth, and $A \subseteq N$ is a smooth submanifold, then f is smoothly homotopic to a map that is transverse to A.

Solution:

(a) Let $p \in A$, and let y be a regular value for π. Observe that, by the regular value theorem, we have that $\pi^{-1}(y) = \{y\} \times M$ is a smooth submanifold of $Y \times M$. It follows from the definition of the f_y that $F \circ \pi^{-1}(y) \equiv f_y$.

Since F is transverse to A, it follows that for any $(z, q) \in F^{-1}(p)$, we have

$$D_{(z,q)}F\big(T_{(z,q)}(Y \times M)\big) + T_p A = T_p N.$$

We have, by chain rule and the inverse function theorem (seeing as y is a regular value of π),

$$\begin{aligned} D_q f_y &= D_q\left(F \circ \pi^{-1}(y)\right) \\ &= D_{(y,q)}F \circ \left(D_{\pi^{-1}(y)}\pi\right)^{-1}(y) \\ &= D_{(y,q)}F, \end{aligned}$$

so that

$$\begin{aligned} D_q f_y(T_q M) + T_p A &= D_{(y,q)}F\big(T_{(y,q)}(Y \times M)\big) + T_p A \\ &= T_p N, \end{aligned}$$

meaning f_y is transverse to A for any regular value $y \in Y$ of π.

(b) If we let $Y \equiv \mathbb{R}^n$ in part (a), and let $F\colon \mathbb{R}^n \times M \to \mathbb{R}^n$ be defined by $F(p, x) = f(x) + p$, then we observe that for every regular value p of π, that $f(x) + p$ is transverse to A. In particular, since the set of critical values has measure zero in $\mathbb{R}^n$, it follows that for almost every p, $f(x) + p$ is transverse to A.

(c) Since $N \subseteq \mathbb{R}^n$ is a smooth submanifold, then we may apply part (b) to $\iota\colon M \hookrightarrow \mathbb{R}^n \supseteq N$, whence $M + p$ and N intersect transversely for almost every $p \in \mathbb{R}^n$.

(d) Since we treat $A \subseteq N \subseteq \mathbb{R}^n$ as a smooth submanifold, we know that the set of all p for which $f_p(x) = f(x) + p$ is not transverse to A is a set of measure zero; in particular, we may find a smooth homotopy from f to f_p where f_p is a translate of f that intersects A and is transverse to A (which exists by the fact that the set of all points where this does not hold is of measure zero). Thus, f is smoothly homotopic to a map that is transverse to A.