12

Suppose that n teams $1, 2, \ldots, n$ are involved in a softball tournament in which every two teams play each other exactly once. For $n = 9$ and $n = 10$, set up a schedule of games that takes place during the smallest number of days so that no team plays more than one game per day.

In the $n = 10$ case, it should take 9 days for every team to play every other team (as there are 9 1-factorizations of K_{10}).

In the $n = 9$ case, we still have 9 days, as we can reconceptualize the case as $n = 10$ but team 10 did not show up.

13

Show that the graph in Figure 7.13 is not 1-factorable.

In Figure 7.13, there are 10 vertices, meaning there are 5 edges in any given 1-factor.

Every 1-factor must contain either u_1v_1 or u_4v_4, as otherwise the 1-factors would be wholly contained within the pentagon components of u or v — however, there can be at most 2 1-factors in any given pentagon component.

So, since there need to be 4 1-factors for G to be 1-factorable, G must not be 1-factorable.

14

Show that if a cubic bridgeless graph is Hamiltonian, then it is also 1-factorable.

Let G be a cubic bridgeless graph. Then, by Petersen's Theorem, G must contain a perfect matching (i.e., it is an even graph).

Because the graph is Hamiltonian, we can create a Hamiltonian C in G that contains within it two 1-factors. Upon creating those two 1-factors, we can create the third 1-factor from the remaining edges (which is possible because the graph is cubic and even).

So, G is 1-factorable.

15

Determine whether the 6-regular graph G below is Hamiltonian-factorable.

If G is Hamiltonian-factorable, then there are 3 Hamiltonian cycles. Each Hamiltonian cycle needs at least 2 of u_iv_i, where $1 \leq i \leq 4$.

However, 3 Hamiltonian cycles will require 6 edges, meaning this is not possible.

16

A committee of seven students has a luncheon meeting three times during the semester. When they meet, they sit at a round table with seven chairs. Show that it is possible for the committee to meet so that every two students sit next to each other during only one of the three meetings.

We can consider the seating arrangement as K_7. K_7 has 21 edges, and since K_7 is a complete graph, it is Hamiltonian-factorable, with 7 edges per Hamiltonian-factor, meaning there are 3 Hamiltonian cycles in the graph.