Revised Problems

Problem (Homework 3, Problem 3 (b))**:** Prove that S^∞ is contractible.

Solution: We view S^∞ as a topological subspace of $\mathbb{R}^\infty$ (finitely supported real sequences) equipped with the Euclidean norm; i.e., if $(x_n) \in \mathbb{R}^\infty$, then

$$\|(x_n)\| = \left(\sum_{i=0}^{\infty} x_i^2\right)^{1/2},$$

where the sum is finite by definition. We consider $S^\infty \subseteq \mathbb{R}^\infty$ to be the space of all finitely supported sequences (x_n) such that

$$\|(x_n)\| = 1.$$

We consider the continuous 1-parameter family given by

$$f_t(x_n) = (1-t)(x_n) + tV(x_n),$$

where V denote the right unilateral shift mapping $(x_0, x_1, \dots)$ to $(0, x_0, x_1, \dots)$. To show that $f_t(x_n)$ is never zero, we start by considering $(x_0, \dots, x_k, 0)$ viewed in S^{k+1}, and observe then that f_t, restricted to S^{k+1}, yields

$$f_t((x_0, \dots, x_k, 0)) = ((1-t)x_0, (1-t)x_1 + tx_0, \dots, (1-t)x_k + tx_{k-1}, tx_k).$$

Without loss of generality, we may consider x_0 as being nonzero; then, we observe that the second coordinate $1 + t(x_0 - x_1)$ will be nonzero for all $0 \leq t \leq 1$ if $|x_0 - x_1| < 1$, and will be zero at $t = 1$ only when $|x_0 - x_1| = 1$, but that can only happen if either x_0 or x_1 is 1 and every other coordinate is 0; yet, in such a scenario, it is necessarily the case that f_t is nonzero. Therefore, $\|f_t\|$ is nonzero for all $0 \leq t \leq 1$ acting on S^∞.

In particular, when we consider the homotopy $H\colon S^\infty \times [0,1] \to S^\infty$ given by

$$H((x_n), t) = \begin{cases} (1-t)(x_n) + 2tV(x_n) & 0 \leq t \leq 1/2 \\ (2-2t)V(x_n) + (2t-1)(1, 0, \dots) & 1/2 \leq t \leq 1 \end{cases},$$

we observe that H is continuous along each of $S^\infty \times [0, 1/2]$ and $S^\infty \times [1/2, 1]$, and is equal at $t = 1/2$, so by the pasting lemma, H is continuous along $[0,1]$. Since $H(\cdot, t)/\|H(\cdot, t)\|$ is contained in S^∞ (with well-definedness following from the earlier discussion), and is a homotopy between the identity and a constant map, it follows that the identity is null-homotopic, so S^∞ is contractible.