Problem (Problem 1)**:** Let R be a commutative ring. An R-module M is called torsion if for any $m \in M$, there is a nonzero $r \in R$ such that $rm = 0$. An R-module M is called divisible if for any nonzero $r \in R$, we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$, there is $x \in M$ such that $rx = m$.

(a) Suppose M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Solution:

(a) It is enough to show that any simple tensor $m \otimes n \in M \otimes_R N$ is the zero tensor. To see this, we let $r \in R$ be such that $rm = 0$, and observe that there is some $x \in N$ such that $rx = n$. By using property (R3) of tensor products, we observe then that

$$\begin{aligned} m \otimes n &= m \otimes (rx) \\ &= (rm) \otimes x \\ &= 0 \otimes x \\ &= 0. \end{aligned}$$

Thus, $M \otimes_R N = \{0\}$.

(b) It is enough to show that $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible, as we may then apply (a). To see that $\mathbb{Q}/\mathbb{Z}$ is torsion, we have that

$$\begin{aligned} b\left[\frac{a}{b}\right] &= [a] \\ &= [0] \end{aligned}$$

for any element $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$. Additionally, for any $n \in \mathbb{Z}$, we have

$$\left[\frac{a}{b}\right] = n\left[\frac{a}{nb}\right],$$

so $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible.

Problem (Problem 2)**:** Let R be a commutative ring, $\{N_\alpha\}_{\alpha \in A}$ a collection of R-modules, and M another R-module.

(a) Prove that $M \otimes \left(\bigoplus_\alpha N_\alpha\right) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) Show by example that $M \otimes \left(\prod_\alpha N_\alpha\right)$ need not be isomorphic to $\prod_\alpha (M \otimes N_\alpha)$.

Solution:

(a) Consider the map on elementary tensors

$$f \colon M \times \left(\bigoplus_\alpha N_\alpha\right) \to \bigoplus_\alpha (M \otimes N_\alpha)$$

that takes

$$(m, (n_\alpha)_\alpha) \to (m \otimes n_\alpha)_\alpha.$$

We observe that, since the $(n_\alpha)_\alpha$ are nonzero for all but finitely many indices α, and that the map is R-bilinear, we have a well-defined and unique R-linear map $\overline{f} \colon M \otimes \left(\bigoplus_\alpha N_\alpha\right) \to \bigoplus_\alpha (M \otimes N_\alpha)$ that maps $m \otimes (n_\alpha)_\alpha \mapsto (m \otimes n_\alpha))_\alpha$.

We observe that for each index i, we have an inclusion homomorphism

$$M \times N_i \hookrightarrow M \otimes \left(\bigoplus_\alpha N_\alpha\right)$$

that takes $(m, n_\alpha) \mapsto m \otimes (n_\alpha)_\alpha$, where $(n_\alpha)_\alpha$ is zero everywhere except for index i. This is an R-bilinear map, so it induces a unique linear map $M \otimes N_i \hookrightarrow M \otimes (\bigoplus_\alpha N_\alpha)$ by the universal property for tensor products. By the universal property of the direct sum, this induces a different unique homomorphism $g\colon \bigoplus_\alpha (M \otimes N_\alpha) \to M \otimes (\bigoplus_\alpha N_\alpha)$ given by taking

$$(m_\alpha \otimes n_\alpha)_\alpha \mapsto \sum_\alpha m_\alpha \otimes (n_\alpha)_\alpha,$$

where the summand $(n_\alpha)_\alpha$ is defined as above, and the sum is finite by the definition of the direct sum. Since g and f are inverses of each other via their action on simple tensors, it follows that $M \otimes (\bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha (M \otimes N_\alpha)$.

(b) We consider the direct product

$$M = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z},$$

regarded as a $\mathbb{Z}$-module. Notice that M is not torsion, as the element $m = (1, 1, \dots)$ is such that there is no $z \in \mathbb{Z}$ with $zm = 0$. Therefore, considering the extension of scalars

$$\mathbb{Q} \otimes M = \mathbb{Q} \otimes \left(\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}\right),$$

we have that this is not a zero module, since by using the same element $(1, 1, \dots)$, the Archimedean property implies that for any $\frac{a}{b}$ in lowest terms in $\mathbb{Q}$, there is some k such that $2^k > a$. Therefore, this element is not torsion, so from Problem 8(c) on p. 376 in Dummit and Foote, it follows that this element is not zero. Yet, since each of the individual $\mathbb{Z}/2^i\mathbb{Z}$ has torsion, it would follow that

$$\prod_{i=1}^{\infty} (\mathbb{Q} \otimes \mathbb{Z}/2^i\mathbb{Z}) = 0,$$

so we have that tensor products do not commute with direct sums.

Problem (Problem 3)**:** Let R be a domain, and let M be a free R-module with basis $\{e_1, \dots, e_k\}$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$.

Solution: Suppose we had $m \otimes n = e_1 \otimes e_2 + e_2 \otimes e_1$. Write

$$\begin{aligned} m \otimes n &= \left(\sum_{i=1}^{k} m_i e_i\right) \otimes \left(\sum_{j=1}^{k} n_j e_j\right) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} m_i n_j (e_i \otimes e_j) \\ &= e_1 \otimes e_2 + e_2 \otimes e_1. \end{aligned}$$

In particular, this means that we must have $m_1 n_2 = m_2 n_1 = 1$ and everything else equal to zero. Yet, since $m_1 n_1 = 0$ and $m_2 n_2 = 0$, we must have one of either $m_1 = 0$ or $n_1 = 0$ since R is a domain, and similarly we must have either $m_2 = 0$ or $n_2 = 0$. In any of these scenarios, we obtain a contradiction, so there is no such representation.

Problem (Problem 4)**:** Let R be commutative, and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r + I) \otimes (r' + J)$

to $rr' + (I + J)$.

Solution:

(a) By using R-bilinearity, we observe that an arbitrary simple tensor in $R/I \otimes R/J$ can be written as

$$\begin{aligned}(r + I) \otimes (s + J) &= (r(1 + I)) \otimes (s + J)\\ &= r((1 + I) \otimes (s + J))\\ &= (1 + I) \otimes (rs + J).\end{aligned}$$

Since any element of $R/I \otimes_R R/J$ can be written as a sum of simple tensors, and each simple tensor can be written in the above form, it follows from bilinearity that every element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (r + J)$.

(b) We consider the map

$$f\colon R/I \times R/J \mapsto R/(I + J)$$

given by

$$(r + I, r' + J) \mapsto rr' + (I + J).$$

This map is R-bilinear by the distributive properties of multiplication, so it induces a homomorphism on the tensor product given by

$$(r + I) \otimes (r' + J) \mapsto rr' + (I + J).$$

As was established above, any element of $R/I \otimes R/J$ can be written as $(1 + I) \otimes (s + J)$, so we may establish an inverse from any element of $R/(I + J)$ to $R/I \otimes R/J$ by taking $t + (I + J) \mapsto (1 + I) \otimes (t + J)$. This establishes our desired isomorphism.

Problem (Problem 5)**:** Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $\mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an R-module annihilated by both 2 and x.

(a) Show that the map $\varphi\colon I \times I \to \mathbb{Z}/2\mathbb{Z}$ given by

$$\varphi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = \left[\frac{a_0}{2}b_1\right]_{\mathbb{Z}/2\mathbb{Z}}$$

is R-bilinear.

(b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2}q'(0)$, where q' denotes the usual polynomial derivative of q.

(c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Solution:

(a) By the well-definedness of addition in R/I, we have that φ is additive in each variable. Now, letting $p(x) \in R$ and $a(x), b(x) \in I$ be defined by

$$\begin{aligned}a(x) &= a_0 + a_1x + \cdots + a_nx^n\\ b(x) &= b_0 + b_1x + \cdots + b_mx^m\\ p(x) &= p_0 + p_1x + \cdots + p_\ell x^\ell,\end{aligned}$$

we note that

$$p(x) + I = [p_0]_{\mathbb{Z}/2\mathbb{Z}}.$$

Using various definitions, we see that

$$\varphi(p(x)a(x), b(x)) = \varphi\big(p_0a_0 + O(x), b_0 + b_1x + O\big(x^2\big)\big)$$

$$
\begin{aligned}
&= \left[\frac{p_0a_0}{2}b_1\right]\\
&= [p_0]\left[\frac{a_0}{2}b_1\right]\\
&= (p(x)+I)\varphi(a(x),b(x)),
\end{aligned}
$$

and since $b_0 \in I$,

$$
\begin{aligned}
\varphi(a(x),p(x)b(x)) &= \left[\frac{a_0}{2}(p_0b_1+p_1b_0)\right]\\
&= \left[\frac{a_0}{2}(p_0b_1)\right]\\
&= (p(x)+I)\varphi(a(x),b(x)).
\end{aligned}
$$

Thus, φ is R-bilinear.

(b) Using the universal property for tensor products, there is a unique R-linear homomorphism $\overline{\varphi}\colon I\otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ such that

$$
\begin{aligned}
\overline{\varphi}(a(x)\otimes b(x)) &= \left[\frac{a_0}{2}b_1\right]\\
&= \left[\frac{p(0)}{2}q'(0)\right].
\end{aligned}
$$

(c) We observe that $\overline{\varphi}(2\otimes x) = 1$ while $\overline{\varphi}(x\otimes 2) = 0$, so they cannot be equal to each other in $I\otimes_R I$.

Problem (Problem 6)**:** Suppose R is commutative, and let I, J be ideals of R.

(a) Show that there is a surjective R-module homomorphism from $I\otimes_R J$ to the product ideal IJ mapping $i\otimes j$ to ij.

(b) Give an example to show that the map in (a) need not be injective.

Solution:

(a) We define the R-bilinear map $\varphi\colon I\times J\to IJ$ by

$$\varphi(i,j) = ij.$$

This induces a linear map $\overline{\varphi}\colon I\otimes_R J\to IJ$ such that $i\otimes j\mapsto ij$. Since every element of $I\otimes_R J$ is a finite sum of elementary tensors, this surjects onto IJ since every element of IJ is a finite sum of elements of the form ij.

(b) The map from Problem 5, given by $I\otimes I\to I^2$ applied in the case of $2\otimes x$ and $x\otimes 2$, is not injective, as $2\otimes x\neq x\otimes 2$, but R is commutative.

Problem (Problem 7)**:**

(a) Let V be a finite-dimensional vector space over $\mathbb{C}$. Note that V can be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Prove that $V\otimes_{\mathbb{C}} V$ is not isomorphic to $V\otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.

(b) Let R be an integral domain and F its field of fractions. Prove that $F\otimes_R R\cong F\otimes_F F\cong F$ as F-modules.

Solution:

(a) We may consider $V\cong\mathbb{C}^k$, so that

$$
\begin{aligned}
V\otimes_{\mathbb{C}} V &\cong \mathbb{C}^{k^2}\\
&\cong \mathbb{R}^{2k^2}\\
V\otimes_{\mathbb{R}} V &\cong \mathbb{R}^{4k^2}.
\end{aligned}
$$

(b) We will show that both of the tensor products are generated by $1 \otimes 1$ as an F-vector space. Observe that in $F \otimes_R F$, we have

$$\begin{aligned}\frac{a}{b} \otimes \frac{c}{d} &= \frac{ad}{bd} \otimes \frac{c}{d} \\ &= \frac{a}{bd} \otimes c \\ &= \frac{ac}{bd}(1 \otimes 1),\end{aligned}$$

while in $F \otimes_F F$, $\frac{a}{b} \otimes \frac{c}{d} = \frac{ac}{bd}(1 \otimes 1)$, meaning that both tensor products are generated by $1 \otimes 1$, and thus both have dimension 1 over F, so that they are all isomorphic to F.

Problem (Problem 8)**:** Let R be a subring of the commutative ring S, and let x be an indeterminate over S. Prove that $S[x]$ and $S \otimes R[x]$ are isomorphic as S-algebras.

Solution: Using the fact that $R[x]$ is a free R-algebra with basis $\{1, x, x^2, \dots\}$, we only need to define a multiplicative R-bilinear map on the basis elements. Consider the map $\phi\colon S \times R[x] \to S[x]$ given by

$$\begin{aligned}\phi(s, x^m) &= sx^m \\ \phi((s_1, x^{m_1})(s_2, x^{m_2})) &= s_1 s_2 x^{m_1+m_2}\end{aligned}$$

This is a multiplicative R-bilinear map, so it extends uniquely to a homomorphism on the tensor product

$$\begin{aligned}\varphi\colon S \otimes R[x] &\to S[x] \\ s \otimes x^m &\mapsto sx^m.\end{aligned}$$

This map is concordant with the algebra structure on $S \otimes R[x]$, as

$$\begin{aligned}(s_1 \otimes x^{m_1})(s_2 \otimes x^{m_2}) &= s_1 s_2 \otimes x^{m_1} x^{m_2} \\ &= s_1 s_2 \otimes x^{m_1+m_2} \\ &\mapsto s_1 s_2 x^{m_1+m_2}.\end{aligned}$$

Since this homomorphism has an inverse given by $\psi(sx^m) = s \otimes x^m$ that can once again be extended by linearity, it follows that $S \otimes R[x] \cong S[x]$ as R-algebras.