

Introduction

Consider the equations

$$y''(x) + y(x) = e^x \quad (1)$$

$$y^{(17)}(x) + \sin(y(x)) = (x^x)^x \quad (2)$$

Before we want to solve these equations, we need to understand what these equations *are*.

(1) This is a second order, inhomogeneous, linear ordinary differential equation.

(2) This is a 17th order, inhomogeneous, nonlinear ordinary differential equation.

Generally, when we have a nonlinear equation, we convert it (using the Jacobian) to the “nearest” corresponding linear equation using Taylor approximations. In this case, converting equation (2), we have

$$y^{(17)}(x) + y(x) = (x^x)^x. \quad (2')$$

Now, equation (2') is linear, so it is able to be solved. It may not be pretty,¹ but it can be solved, using Laplace Transforms or other methods.

Ordinary Differential Equations

Returning to our equation (1),

$$y''(x) + y(x) = e^x, \quad (1)$$

there is one more fact that we can see — this is an equation with constant coefficients. The most general form of a n th order linear ordinary differential equation is of the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x). \quad (\dagger)$$

Specifically, we also require $a_k(x) \in C(I)$, where I is some interval (specifics will be detailed later).

Theorem (Existence and Uniqueness Theorem): Any ordinary differential equation of the form (\dagger) has unique solutions in the interval I .

There are n linearly independent solutions for $g(x) = 0$.

The corresponding homogeneous equation for (1) is

$$y''(x) + y(x) = 0. \quad (1')$$

The equations (1) and (1') are related by the linearity principle. In particular, if $y_0(x)$ is a solution to (1'), then we can add $\alpha y_0(x)$ to any solution $y_p(x)$ of (1), then we have all the solutions for (1). In particular, the solutions to (1') are

$$\begin{aligned} y_1(x) &= \sin(x) \\ y_2(x) &= \cos(x). \end{aligned}$$

To evaluate that these solutions are linearly independent, we consider the differential operator L from (\dagger) defined by

$$L[y] = \sum_{k=0}^n a_k(x)y^{(k)}(x).$$

We rewrite (\dagger) as

$$L[y] = g(x).$$

The operator L is linear, so L has the following properties:

¹Citation needed.

- $L[y_1 + y_2]$;
- $L[cy] = cL[y]$.

Now, in (1) and (1'), if we set $L[y] = y''(x) + y(x)$, then evaluating our solutions y_1 and y_2 to (1'), we get

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= c_1 L[y_1] + c_2 L[y_2] \\ &= 0. \end{aligned}$$

Now, we get

$$y_0(x) = c_1 \sin(x) + c_2 \sin(x)$$

as our general solution to (1'). By the linearity principle, all we need is one solution to $L[y] = e^x$ to find all solutions to (1).

Evaluating (†) in the most general form, we have the general solution

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{\text{homogeneous solution}} + y_p(x),$$

where $y_p(x)$ is the particular solution. In other words, our general solution is

$$y(x) = \text{span}(y_1(x), y_2(x), \dots, y_n(x)) + y_p(x).$$

For this to work, we need the set $\{y_1, \dots, y_n\}$ to be linearly independent. To do this, we evaluate the Wronskian:

$$W(x) = \det \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}.$$

Specifically, the set $\{y_1, \dots, y_n\}$ is linearly independent if $W(x) \neq 0$ for all $x \in I$.

Example. Consider the equation

$$y''(x) - y(x) = e^x \tag{1}$$

We want to find the general solution to this constant coefficient equation.

We start by finding two linearly independent homogeneous solutions to the equation, take their span, then add a particular solution.

The characteristic equation of the homogeneous equation for (1) is

$$r^2 - 1 = 0$$

We get $r = \pm 1$, which by the definition of the characteristic equation yields $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. To verify that this solution set is linearly independent

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2 \end{aligned}$$

$$\neq 0.$$

Thus, our solutions are linearly independent. We get the general form of

$$y(x) = c_1 e^x + c_2 e^{-x} + y_p(x).$$

Now, we only have to find a particular solution. This is, unfortunately, the hard part.

We begin by guessing. But, in a way that doesn't suck. Specifically, we let $y_p(x) = A x e^x$. Evaluating, we get

$$\begin{aligned} y_p'(x) &= A(x+1)e^x \\ y_p''(x) &= A(x+2)e^x \\ y_p''(x) - y_p(x) &= A(x+2)e^x - A x e^x \\ &= 2A e^x, \end{aligned}$$

so $2A = 1$, and $A = \frac{1}{2}$. Thus, we have the end result of

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x.$$

Evaluating in Mathematica, we take

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DSolve[y''[x] - y[x] == Exp[x], y[x], x]
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and we get

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{4}(2x - 1)e^x,$$

corroborating our solution.¹¹

Example. Consider the equation

$$y'''(x) - y(x) = 0.$$

The particular solution to this equation is $y(x) = 0$. The characteristic equation for this equation is

$$r^3 - 1 = 0.$$

Factoring, we get

$$\begin{aligned} (r-1)(r^2 + r + 1) &= 0 \\ (r-1)(r - \zeta_3)(r - \zeta_3^2) &= 0. \end{aligned}$$

Thus, we get

$$r = \left\{ 1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} \right\}.$$

Thus, our solutions are of the form

$$y(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

¹¹Only slightly different, but they're the same solution.

Recall that the most general second order constant-coefficient linear differential equation is

$$y'' + ay' + by = 0,$$

with characteristic equation

$$r^2 + ar + b = 0.$$

The solutions to the characteristic equation are

$$r = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}.$$

There are a few cases:

- (1) $r_1 \neq r_2$ with $r_1, r_2 \in \mathbb{R}$;
- (2) $r_1 = r_2$ with $r_1, r_2 \in \mathbb{R}$;
- (3) $r_1 = c + id, r_2 = c - id$, where $c, d \in \mathbb{R}$.

The solutions are $y_1 = c_1 e^{r_1 x}$ and $y_2 = c_2 e^{r_2 x}$.

Example (Solving Second-Order Equations).

- (1) Let

$$y'' - 3y' + 2y = 0.$$

The characteristic equation is $r^2 - 3r + 2 = 0$, whose solutions are $r = 1, r = 2$. The general solution is, thus,

$$y(x) = c_1 e^x + c_2 e^{2x} \tag{†}$$

The Wronskian is

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \\ &= 2e^{3x} - e^{3x} \\ &= e^{3x} \\ &\neq 0. \end{aligned}$$

Thus, the solution is indeed (†).

- (2) Let

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with solution $r = -3, -3$. Currently, we only have the solution $y_1(x) = c_1 e^{-3x}$.

Note that in an n th order linear ordinary differential equation, we always have n linearly independent solutions. Let's guess. Consider the equation $y_2(x) = c_2 x e^{-3x}$.

We can see that $y_2(x)$ is also a solution to this equation,^{III} but we need to verify linear independence. Taking the Wronskian, we get

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{pmatrix} \\ &= e^{-6x} \begin{pmatrix} 1 & x \\ -3 & -3x + 1 \end{pmatrix} \\ &= e^{-6x}(-3x + 1 + 3x) \\ &= e^{-6x} \\ &\neq 0. \end{aligned}$$

Thus, we have two linearly independent solutions, with the general solution of

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

(3) Let

$$y'' + 4y' + 5 = 0.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with solutions of $r = -2 \pm i$. We then have the solutions

$$\begin{aligned} y_1(x) &= e^{(-2+i)x} \\ y_2(x) &= e^{(-2-i)x}. \end{aligned}$$

Unfortunately, we cannot just let these equations stand on their own, because we want *real* solutions. Let's use Euler's theorem, $e^{ix} = \cos x + i \sin x$. Then, we get

$$\begin{aligned} y(x) &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\ &= e^{-2x} (c_1 e^{ix} + c_2 e^{-ix}). \end{aligned}$$

Let $f(x) = c_1 e^{ix} + c_2 e^{-ix}$. Using the even/odd decomposition, we get

$$\begin{aligned} f(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\ &= (c_1 + c_2) \cos(x) + i(c_1 - c_2) \sin(x). \end{aligned}$$

We "real"-ize our solution by just dropping the value of i in $f(x)$. Thus, we get the full general solution

$$y(x) = e^{-2x} (d_1 \cos(x) + d_2 \sin(x)).$$

(4) If we have the equation

$$y^{(4)} - 25y'' = 0,$$

then using a similar process, we get the solution

$$y(x) = c_1 + c_2 x + c_3 e^{5x} + c_4 e^{-5x}.$$

(5) Considering the equation

$$y^{(5)} + 4y''' + 4y' = 0,$$

we take the characteristic equation $r^5 + 4r^3 + 4r = 0$. Factoring, we get solutions of $r = 0, r = \pm i\sqrt{2}$. Thus, we get the solution of

$$y(x) = c_1 + c_2 \cos(\sqrt{2}x) + c_3 \sin(\sqrt{2}x) + c_4 x \cos(\sqrt{2}x) + c_5 x \sin(\sqrt{2}x).$$

^{III}Exercise left for the reader.

Reducing our Orders

Let

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Suppose we know $y_1(x)$. Can we find $y_2(x)$? The answer is yes. We presume

$$y_2(x) = v(x)y_1(x).$$

Now, we have

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'', \end{aligned}$$

and inserting into the equation, we get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v \underbrace{(y_1'' + py_1' + qy_1)}_{=0} \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Now, we have

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p. \quad (*)$$

Integrating, we get

$$\ln(v') = -2\ln(y_1) - \int p(x) dx.$$

Taking powers, we get

$$\begin{aligned} v' &= e^{-2\ln(y_1) - \int p(x) dx} \\ &= y_1^{-2} e^{-\int p(x) dx} \\ &= \frac{e^{-\int p(x) dx}}{y_1(x)^2} \\ v &= \int \frac{e^{-\int p(x) dx}}{y_1(x)^2} dx \end{aligned}$$

Example. Consider the equation

$$\cos^2(x)y''(x) - \sin(x)\cos(x)y' - y(x) = 0.$$

Putting our equation into standard form, we may be able to find another solution.

$$y'' - \tan(x)y' - \sec^2(x)y = 0.$$

Guessing $y(x) = \tan(x)$, we get $y' = \sec^2(x)$ and $y'' = 2\sec^2(x)\tan(x)$. This is also another solution, $y_2(x) = \tan(x)$.

We don't want to guess anymore. Let $y_2(x) = v(x)y_1(x)$. We get

$$v(x) = \int \frac{e^{-\int p(x) dx}}{y_1^2(x)} dx.$$

We have $-p(x) = \tan(x)$, so $-\int p(x) dx = \ln(\sec(x))$. Thus, $e^{-\int p(x) dx} = \sec(x)$. Thus, we get

$$\begin{aligned} v(x) &= \int \frac{\sec(x)}{\tan^2(x)} dx \\ &= \int \frac{\cos(x)}{\sin^2(x)} dx \\ &= \int \frac{1}{u^2} du & u = \sin(x) \\ &= -\frac{1}{u} \\ &= -\csc(x). \end{aligned}$$

Thus, we have $y_2(x) = -\csc(x) \tan(x) = -\sec(x)$.

Example. Consider the equation

$$x^2(\ln(x) - 1)y''(x) - xy'(x) + y'(x) = 0.$$

We can use the power of inspection to find one solution, $y_1(x) = x$. Dividing out, we have

$$y'' - \frac{1}{x(\ln(x) - 1)}y' + \frac{1}{x^2(\ln(x) - 1)}y = 0.$$

Using the reduction of order, we guess $y_2(x) = v(x)y_1(x)$, and have

$$v(x) = \int \frac{e^{-\int p(x) dx}}{y_1^2} dx.$$

Noting that $-p(x) = \frac{1}{x(\ln(x)-1)}$, we have $\int \frac{1}{x(\ln(x)-1)} dx = \ln(\ln(x) - 1)$.

Now, we have

$$\begin{aligned} v(x) &= \int \frac{\ln(x) - 1}{x^2} dx \\ &= \frac{1 - \ln(x)}{x} - \int -\frac{1}{x^2} dx & u = \ln(x) - 1, dv = x^{-2} \\ &= \frac{-\ln(x)}{x} - \frac{1}{x} \\ &= -\frac{\ln(x)}{x}. \end{aligned}$$

Thus, we get $y_2(x) = -\ln(x)$, and the general solution of $y(x) = c_1x + c_2 \ln(x)$.

Example (Cauchy–Euler Equation). A second-order Cauchy–Euler equation is of the form

$$ax^2y''(x) + bxy'(x) + cy(x) = 0.$$

More generally,

$$\sum_{k=0}^n c_k x^k y^{(k)}(x) = 0.$$

We guess $y(x) = x^r$. Then, $y'(x) = rx^{r-1}$ and $y''(x) = r(r-1)x^{r-2}$. This yields

$$a(r)(r-1)x^r + brx^r + cx^r = x^r \left(a(r^2 - r) + br + c \right) = 0.$$

Example (Solving a Cauchy–Euler Equation). Consider the equation

$$x^2 y'' + xy' - y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 1 = 0,$$

so our general solution is $y(x) = c_1 x + c_2/x$.

Example (Solving another Cauchy–Euler Equation). Consider the equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 4r + 4 = 0,$$

so our solutions are x^2 and x^2 . This is not good enough, we need another solution.

Now, we place our equation into standard form.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0.$$

Thus, we get $p(x) = -\frac{3}{x}$. Using reduction of order, we get $y_2(x) = v(x)y_1(x)$,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int -3/x \, dx}}{x^4} \, dx \\ &= \int \frac{e^{3 \ln(x)}}{x^4} \, dx \\ &= \int \frac{x^3}{x^4} \, dx \\ &= \ln(x). \end{aligned}$$

Thus, we have the solution $y_2(x) = \ln(x)x^2$, and the general solution of $y(x) = c_1 x^2 + c_2 \ln(x)x^2$.

Example. Consider the equation

$$x^2 y'' + 3xy' + 5y = 0.$$

We get the characteristic equation of

$$\begin{aligned} 0 &= r^2 - 4r + 5 \\ r &= 2 \pm i. \end{aligned}$$

Now, we need to figure out what $x^{2 \pm i}$ means.

To solve this part, we keep the positive exponent, so we only need to try to understand $y = x^{2+i}$. Now, we get $y = x^2 x^i$. To evaluate x^i , we take $x = (e^{\ln x})^i = e^{i \ln x}$. Using Euler's identity, we get

$$y = x^2 (\cos(\ln x) + i \sin(\ln x)).$$

Since our solutions are real, get

$$y = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x).$$

Example. Consider the equation

$$x^4 y^{(4)} - 2x^2 y'' + y = 2.$$

We have the particular solution $y_p(x) = 2$. Substituting into our method for the Cauchy–Euler equation, we have

$$r(r-1)(r-2)(r-3) - 2r(r-1) + 1 = 0.$$

Factoring, we have

$$r(r-1)^2(r-4) + 1 = 0.$$

Unfortunately, to go forward from here we need Mathematica.

This has the solution set of of

$$\begin{aligned} r_1 &= \frac{3}{2} - \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \\ r_2 &= \frac{3}{2} - \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad + \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \\ r_3 &= \frac{3}{2} + \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \\ r_4 &= \frac{3}{2} + \frac{1}{2} \sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad + \frac{1}{2} \sqrt{6 - \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3} \sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}}}} \end{aligned}$$

Varying our Parameters

Given a set of n linearly independent homogeneous solutions, we want to find a particular solution.

To find this, we start with the general second-order inhomogeneous equation in standard form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x).$$

Given y_1, y_2 , we find $y_p(x)$ by taking

$$y_p = v_1 y_1 + v_2 y_2.$$

Finding the derivatives, we have

$$\begin{aligned} y_p' &= v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2 \\ y_p'' &= v_1 y_1'' + 2v_1' y_1' + v_1'' y_1 + v_2 y_2'' + 2v_2' y_2' + v_2'' y_2. \end{aligned}$$

Substituting, we have

$$\begin{aligned} y_p'' &= v_1 y_1'' + 2v_1' y_1' + v_1'' y_1 + v_2 y_2'' + 2v_2' y_2' + v_2'' y_2 \\ p y_p' &= p v_1 y_1' + p v_1' y_1 + p v_2 y_2' + p v_2' y_2 \\ q y_p &= q v_1 y_1 + q v_2 y_2 \end{aligned}$$

$$g(x) = v_1 \overbrace{(y_1'' + p y_1' + q y_1)}^{=0} + v_2 \overbrace{(y_2'' + p y_2' + q y_2)}^{=0} + v_1' (2y_1' + p y_1) + v_1'' y_1 + v_2' (2y_2' + p y_2) + v_2'' y_2$$

$$g(x) = v_1' (2y_1' + p y_1) + v_1'' y_1 + v_2' (2y_2' + p y_2) + v_2'' y_2.$$

We suppose that $v_1' y_1 + v_2' y_2 = 0$. Then,

$$\begin{aligned} \frac{d}{dx} (v_1' y_1 + v_2' y_2) &= 0 \\ v_1'' y_1 + v_1' y_1' + v_2'' y_2 + v_2' y_2' &= 0. \end{aligned}$$

Plugging into our earlier expression, we get the expression of

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_2' y_2' + v_2'' y_2 &= g(x). \end{aligned}$$

Plugging into matrix form, we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

Since the Wronskian is nonzero, we have

$$v_1' = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$