Revised Problem

Problem (Homework 4, Problem 2)**:** Prove that, if $f\colon X \to Y$ is a map, then Y is a deformation retract of the mapping cylinder M_f.

Solution: We consider the set $W = (X \times [0,1]) \sqcup Y$, where we let $q\colon W \to M_f$ be the quotient map that identifies $(x,1) \sim f(x)$. We will show that W admits a deformation retract to $X \times \{1\} \sqcup Y$, as by composing with the quotient map and using the universal property, we then obtain a deformation retract from M_f to Y.

Now, define the homotopy

$$H\colon W \times [0,1] \to W$$

on each component separately, taking

$$H(w,t) = \begin{cases} (p, \max(s,t)), & w = (p,s) \in X \times [0,1]; \\ y, & w = y \in Y. \end{cases}$$

Since H is defined on a disjoint union, we only need to show that H is continuous on each component, as then H is continuous. First, since the maximum function is continuous, it follows that H is continuous on $(X \times [0,1]) \times [0,1]$. Furthermore, since H is constant in t along $Y \times [0,1]$ and is equal to the identity on each $Y \times \{t\}$, it follows that H is continuous on the disjoint union $W \times [0,1]$.

Now, when $t = 0$, we have that $H(w,0) = (p,s)$ whenever $(p,s) \in X \times [0,1]$, and $H(w,0) = y$ whenever $w = y \in Y$. In particular, this means that $H(\cdot, 0)$ is the identity on $X \times [0,1] \sqcup Y$. Similarly, if $t = 1$, then we have $H(w,1) = (p,1)$ whenever $w = (p,s) \in X \times [0,1]$, and $H(w,1) = y$ whenever $w = y \in Y$. Additionally, since $(p,1) \in X \times [0,1]$ is fixed as t ranges from 0 to 1, it follows that H is a homotopy relative to $X \times \{1\} \sqcup Y$ with its image equal to the piecewise-defined map

$$\begin{aligned} v\colon X \times [0,1] \sqcup Y &\to X \times [0,1] \sqcup Y \\ v(w) &= \begin{cases} (p,1), & w = (p,s) \in X \times [0,1]; \\ y, & w = y \in Y. \end{cases} \end{aligned}$$

Thus, H is a deformation retraction onto $X \times \{1\}$, so since q maps $X \times \{1\}$ to $f(X) \subseteq Y$, it follows that upon composition, we have that $q \circ H$ is a deformation retraction onto $q(X \times \{1\} \sqcup Y) \subseteq Y$.

Current Problems

Problem (Problem 1)**:** Consider the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$, an n-fold Cartesian product. Prove that if T^n is homeomorphic to T^m, then $n = m$.

Solution: Suppose T^n is homeomorphic to T^m. Then, T^n and T^m are homotopy equivalent, meaning that they admit an isomorphism of fundamental groups $\varphi\colon \pi_1(T^n) \to \pi_1(T^m)$. Since taking fundamental groups commutes with direct products, it follows that we have $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is an isomorphism of $\mathbb{Z}$-modules. By taking the tensor product with the fraction field $\mathbb{Q}$, we see that we have a linear isomorphism of $\mathbb{Q}$-vector spaces $\mathrm{id} \otimes \varphi\colon \mathbb{Q}^n \to \mathbb{Q}^m$, meaning that $n = m$ by the invariance of dimension.

Problem (Problem 2)**:** Give a presentation of $\mathbb{Z} \times \mathbb{Z}$ by generators and relations.

Solution: Consider the surjection from the free group F_2 onto $\mathbb{Z} \times \mathbb{Z}$ given by $a \mapsto (1,0)$ and $b \mapsto (0,1)$. Then, since we must have $ab = ba$ as $\mathbb{Z} \times \mathbb{Z}$ is abelian, it follows that a presentation for $\mathbb{Z} \times \mathbb{Z}$ is given by $\langle a, b \mid aba^{-1}b^{-1} = e \rangle$.