Chapter 2 Problems

2.3

Cylindrical Coordinates

Starting with our expression of $\mathbf{r}$, we have

$$\begin{aligned}\mathbf{r} &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}\\ d\mathbf{r} &= \frac{\partial\mathbf{r}}{\partial\rho}d\rho + \frac{\partial\mathbf{r}}{\partial\phi}d\phi + \frac{\partial\mathbf{r}}{\partial z}dz.\end{aligned}$$

Calculating each partial derivative,

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\rho} &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\\ \hat{\rho} &= \frac{\frac{\partial\mathbf{r}}{\partial\rho}}{\left\|\frac{\partial\mathbf{r}}{\partial\rho}\right\|}\\ &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath},\\ \frac{\partial r}{\partial\phi} &= \rho\left(-\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\right)\\ \hat{\phi} &= \frac{\frac{\partial r}{\partial\phi}}{\left\|\frac{\partial r}{\partial\phi}\right\|}\\ &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\end{aligned}$$

implying

$$\frac{\partial\mathbf{r}}{\partial\phi} = \rho\hat{\phi},$$

and finally, we have

$$\frac{\partial\mathbf{r}}{\partial z} = \hat{k}.$$

The above calculations yield

$$d\mathbf{r} = (d\rho)\,\hat{\rho} + (\rho\,d\phi)\,\hat{\phi} + (dz)\,\hat{k}.$$

Spherical Coordinates

Starting with our expression of $\mathbf{x}$[1]

$$\begin{aligned}\mathbf{x} &= r\sin\phi\sin\theta\hat{\imath} + r\cos\phi\sin\theta\hat{\jmath} + r\cos\theta\hat{k}\\ d\mathbf{x} &= \frac{\partial\mathbf{x}}{\partial r}dr + \frac{\partial\mathbf{x}}{\partial\phi}d\phi + \frac{\partial\mathbf{x}}{\partial\theta}d\theta,\end{aligned}$$

[1] I am using $\mathbf{x}$ instead of $\mathbf{r}$ because r is already used in the expression of the spherical coordinates.

Evaluating each partial derivative, we have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial r} &= \sin\phi\sin\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath} + \cos\theta\hat{k}\\
\hat{r} &= \frac{\frac{\partial \mathbf{x}}{\partial r}}{\left\|\frac{\partial \mathbf{x}}{\partial r}\right\|}\\
&= \sin\phi\sin\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath} + \cos\theta\hat{k},\\
\frac{\partial \mathbf{x}}{\partial \phi} &= -r\sin\phi\sin\theta\hat{\imath} + r\cos\phi\sin\theta\hat{\jmath}\\
\hat{\phi} &= \frac{\frac{\partial \mathbf{x}}{\partial \phi}}{\left\|\frac{\partial \mathbf{x}}{\partial \phi}\right\|}\\
&= -\sin\phi\sin\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath}
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \phi} = r\sin\theta\hat{\phi},$$

and finally, we have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \theta} &= r\cos\phi\cos\theta\hat{\imath} + r\sin\phi\cos\theta\hat{\jmath} - r\sin\theta\hat{k}\\
\hat{\theta} &= \frac{\frac{\partial \mathbf{x}}{\partial \theta}}{\left\|\frac{\partial \mathbf{x}}{\partial \theta}\right\|}\\
&= \cos\phi\cos\theta\hat{\imath} + \sin\phi\cos\theta\hat{\jmath} - \sin\theta\hat{k},
\end{aligned}$$

implying

$$\frac{\partial \mathbf{x}}{\partial \theta} = r\hat{\theta}.$$

The above calculations yield

$$d\mathbf{x} = (dr)\,\hat{r} + (r\sin\theta d\phi)\,\hat{\phi} + (rd\theta)\,d\theta.$$

2.8

Let

$$\vec{a} = r_a\cos\phi_a\sin\theta_a\hat{\imath} + r_a\sin\phi_a\sin\theta_a\hat{\jmath} + r_a\cos\theta_a\hat{k}\vec{b} \quad = r_b\cos\phi_b\sin\theta_b\hat{\imath} + r_b\sin\phi_b\sin\theta_b\hat{\jmath} + r_b\cos\theta_b\hat{k}.$$

Then,

$$\begin{aligned}
\cos\gamma &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\left\|\vec{b}\right\|}\\
&= \frac{1}{r_a r_b}\left(r_a r_b\left(\sin\theta_a\sin\theta_b\left(\cos\phi_a\cos\phi_b + \sin\phi_a\sin\phi_b\right) + \cos\theta_a\cos\theta_b\right)\right)\\
&= \cos\theta_a\cos\theta_b + \sin\theta_a\sin\theta_b\cos\left(\phi_a - \phi_b\right).
\end{aligned}$$

2.9

$$
\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{d}{dt}\left(\dot{\rho}\hat{\rho}\right) + \frac{d}{dt}\left(\rho\dot{\phi}\hat{\phi}\right)\\
&= \hat{\rho}\ddot{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt}\\
&= \hat{\rho}\ddot{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \left(\frac{\partial\hat{\phi}}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\hat{\phi}}{\partial\phi}\frac{d\phi}{dt}\right)\\
&= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi}.
\end{aligned}
$$

Chapter 3 Problems

For all problems involving $\arg z$ (or equivalents), I will be using the principle branch, $\arg z \in (-\pi, \pi]$.

3.5

(a)

$$
\begin{aligned}
\sqrt{3} + i &= 2e^{i\frac{\pi}{3}}\\
-\sqrt{3} + i &= 2e^{i\frac{2\pi}{3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sqrt{2i} &= \sqrt{2}e^{i\frac{\pi}{4}+n\pi}\\
\sqrt{2+2\sqrt{3}i} &= 2e^{i\frac{\pi}{6}}
\end{aligned}
$$

3.6

(a) Real:

$$
\begin{aligned}
(-1)^{1/i} &= \left(e^{i\pi}\right)^{-i}\\
&= e^{\pi}.
\end{aligned}
$$

(b) Real:

$$
\begin{aligned}
\left(\frac{z}{z^*}\right)^i &= \left(e^{2i\arg z}\right)^i\\
&= e^{-2\arg z}.
\end{aligned}
$$

(c) Imaginary:

$$
\begin{aligned}
\left(z_1z_2^* - z_1^*z_2\right)^* &= z_1^*z_2 - z_1z_2^*\\
&= -\left(z_1z_2^* - z_1^*z_2\right).
\end{aligned}
$$

(d) Complex:

$$
\sum_{n=0}^{N} e^{in\theta} = \frac{1-e^{iN\theta}}{1-e^{i\theta}}.
$$

(e) Real: for each $a \in \{1, 2, \ldots, N\}$, $e^{ia\theta} + e^{-ia\theta} \in \mathbb{R}$.

3.9

(a)

$$\begin{aligned}\cos(a+b)+\cos(a-b) &= \frac{1}{2}\left(e^{i(a+b)}+e^{-i(a+b)}\right)+\frac{1}{2}\left(e^{i(a-b)}+e^{-i(a-b)}\right)\\ &= \frac{1}{2}\left(e^{ia}\left(e^{ib}+e^{-ib}\right)+e^{-ia}\left(e^{ib}+e^{-ib}\right)\right)\\ &= \frac{1}{2}\left(e^{ia}+e^{-ia}\right)\left(e^{ib}+e^{-ib}\right)\\ &= 2\cos a\cos b.\end{aligned}$$

(b)

$$\begin{aligned}\sin(a+b)+\sin(a-b) &= \frac{1}{2i}\left(e^{i(a+b)}-e^{-i(a+b)}\right)+\frac{1}{2}\left(e^{i(a-b)}-e^{-i(a-b)}\right)\\ &= \frac{1}{2i}\left(e^{ia}\left(e^{ib}+e^{-ib}\right)-e^{-ia}\left(e^{ib}+e^{-ib}\right)\right)\\ &= \frac{1}{2i}\left(e^{ia}-e^{-ia}\right)\left(e^{ib}+e^{-ib}\right)\\ &= 2\sin a\cos b.\end{aligned}$$

3.10

(a)

$$\begin{aligned}e^{i\alpha}+e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)}+e^{i\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)}\\ &= e^{i\frac{\alpha+\beta}{2}}\left(e^{i\frac{\alpha-\beta}{2}}+e^{-i\frac{\alpha-\beta}{2}}\right)\\ &= 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{i\frac{\alpha+\beta}{2}}.\end{aligned}$$

(b)

$$\begin{aligned}e^{i\alpha}-e^{i\beta} &= e^{i\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)}-e^{i\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)}\\ &= e^{i\frac{\alpha+\beta}{2}}\left(e^{i\frac{\alpha-\beta}{2}}-e^{-i\frac{\alpha-\beta}{2}}\right)\\ &= 2i\sin\left(\frac{\alpha-\beta}{2}\right)e^{i\frac{\alpha+\beta}{2}}\end{aligned}$$

3.12

$$\begin{aligned}\frac{1}{2i}\ln\left(\frac{a+ib}{a-ib}\right) &= \frac{1}{2i}\left(\ln(a+ib)-\ln(a-ib)\right)\\ &= \frac{1}{2i}\left(\ln|a+ib|+i\arctan\left(\frac{b}{a}\right)-\left(\ln|a+ib|+i\arctan\left(-\frac{b}{a}\right)\right)\right)\\ &= \arctan\left(\frac{b}{a}\right).\end{aligned}$$

3.13

$$\begin{aligned}
\frac{d^n}{dt^n}\left(e^{at}\sin bt\right) &= \frac{1}{2i}\frac{d^n}{dt^n}\left(e^{(a+ib)t} - e^{(a-ib)t}\right)\\
&= \frac{1}{2i}\left((a+ib)^n e^{(a+ib)t} - (a-ib)^n e^{(a-ib)t}\right)\\
&= e^{at}\frac{1}{2i}\left(a^2+b^2\right)^{n/2}\left(e^{i\left(b+n\arctan\left(\frac{b}{a}\right)\right)t} - e^{i\left(b+n\arctan\left(\frac{b}{a}\right)\right)}\right)\\
&= e^{at}\left(a^2+b^2\right)^{n/2}\sin\left(bt + n\arctan\left(\frac{b}{a}\right)\right)
\end{aligned}$$

3.20

Showing the equivalence between $C_1\cos kx + C_2\sin kx$ and $A\cos(kx+\alpha)$ and $B\sin(kx+\beta)$, we have

$$\begin{aligned}
A\cos(kx+\alpha) &= A\cos kx\cos\alpha - A\sin kx\sin\alpha\\
B\sin(kx+\beta) &= B\cos kx\sin\beta + B\sin kx\cos\beta
\end{aligned}$$

meaning (assuming $\alpha,\beta \neq \pi n, \pi/2 + \pi n$)

$$\begin{aligned}
A &= \frac{C_1}{\cos\alpha}\\
&= -\frac{C_2}{\sin\alpha}\\
B &= \frac{C_1}{\sin\beta}\\
&= \frac{C_2}{\cos\beta}.
\end{aligned}$$

Now, we show the equivalence between $C_1\cos kx + C_2\sin kx$ and $D_1e^{ikx} + D_2e^{-ikx}$.

$$\begin{aligned}
D_1e^{ikx} + D_2e^{-ikx} &= \frac{D_1+D_2}{2}\left(e^{ikx}+e^{-ikx}\right) + \frac{D_1+D_2}{2}\left(e^{ikx}-e^{-ikx}\right)\\
&= (D_1+D_2)\cos kx + i(D_1-D_2)\sin kx.
\end{aligned}$$

meaning

$$\begin{aligned}
C_1 &= D_1 + D_2\\
C_2 &= i(D_1 - D_2).
\end{aligned}$$

Finally, we show the equivalence between $\operatorname{Re}\left(Fe^{ikx}\right)$ and $C_1\cos kx + C_2\sin kx$.

$$\begin{aligned}
\operatorname{Re}\left((a+ib)e^{ikx}\right) &= \operatorname{Re}(a\cos kx + ia\sin kx + ib\cos kx - b\sin kx)\\
&= a\cos kx - b\sin kx,
\end{aligned}$$

meaning

$$\begin{aligned}
C_1 &= a\\
C_2 &= -b.
\end{aligned}$$