

## Problem 1

Show that  $C_0(\mathbb{R})$  is a Banach space.

**Proof:** We know that  $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$ , meaning we need show  $C_0(\mathbb{R})$  is closed under the uniform norm.

Let  $(f_n)_n \rightarrow f$ , with  $(f_n)_n \in C_0(\mathbb{R})$  for all  $n \in \mathbb{N}$ . Let  $\varepsilon > 0$ . Then,

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f_n(x) - f(x)| + |f_n(x)| \\ &\leq \|f_n - f\|_u + |f_n(x)| \end{aligned}$$

By the definition of uniform convergence, for all  $n \geq N_\varepsilon$ ,  $\|f_n - f\| < \varepsilon/2$  and by the definition of vanishing at  $\pm\infty$ , for all  $|x| > M$ ,  $|f_n(x)| < \varepsilon/2$ . Thus,

$$< \varepsilon,$$

meaning  $f(x) \in C_0(\mathbb{R})$ , so  $C_0(\mathbb{R})$  is closed, so it is complete.

## Problem 2

Show that  $\ell_2$  is a Hilbert space.

**Proof:** Let  $(x_n)_n$  be a Cauchy sequence in  $\ell_2$ . Let  $x_n(k)$  denote the index  $k$  of the sequence  $x_n \in \ell_2$ . Then, by the equivalence of norms,  $\exists c \in \mathbb{R}$  such that

$$\begin{aligned} |x_n(k) - x_m(k)| &\leq c \|x_m(k) - x_n(k)\|_2 \\ &\rightarrow 0 \end{aligned} \quad \text{since } (x_n)_n \text{ is Cauchy in } \ell_2.$$

Since  $\mathbb{R}$  (or  $\mathbb{C}$ ) is complete,  $x_n(k) \rightarrow x(k)$  as  $k \rightarrow \infty$ . We set  $(x(k))_k$  to be the sequence such that  $x_n(k) \rightarrow x(k)$  for each  $n$ .

We must show that  $\|x_n - x\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ . This is equivalent to

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{k=1}^N \lim_{m \rightarrow \infty} |x_n(k) - x_m(k)|^2 &\leq \lim_{m \rightarrow \infty} \sup_{m \geq M} \|x_n - x_m\|^2 \\ &\leq \varepsilon^2 \end{aligned} \quad \text{since } (x_n)_n \text{ is Cauchy.}$$

Thus,  $\|x_n - x_m\| \rightarrow 0$  as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ , so  $\|x_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ .

## Problem 3

Suppose  $(X, d)$  is a complete metric space and  $(x_n)_n$  is a sequence in  $X$  such that there is a  $\theta \in (0, 1)$  with  $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$ . Show that  $(x_n)_n$  is convergent.

**Proof:** We will show that  $(x_n)_n$  is convergent by showing that  $(x_n)_n$  Cauchy. Let  $m, n$  be such that  $m \geq n$ .

Notice that  $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$ . Thus,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq d(x_2, x_1) (\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}) \\ &= d(x_2, x_1) \theta^{n-1} (1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}) \\ &\leq d(x_2, x_1) \frac{\theta^{n-1}}{1 - \theta}. \end{aligned}$$

Notice that the sequence  $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \rightarrow 0$  in  $\mathbb{R}$ , meaning  $(x_n)_n$  is Cauchy. Since  $X$  is complete,  $(x_n)_n$  is convergent.

## Problem 4

Let  $(X, d)$  be a complete metric space, and suppose  $f : X \rightarrow X$  is a contractive map — i.e., there is a  $\theta \in (0, 1)$  with

$$d(f(x), f(y)) \leq \theta d(x, y).$$

Prove that  $f$  has a unique fixed point.

**Proof:** Let  $x_0 \in X$ , and define  $x_n = f(x_{n-1})$ . For all  $n$ , we have

$$d(x_n, x_{n-1}) \leq \theta^n d(x_1, x_0).$$

Therefore, for  $x_m, x_n$  arbitrary in  $X$  with  $m > n$ , we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0) \\ &= d(x_1, x_0) \theta^{n+1} (1 + \theta + \cdots + \theta^{m-n-1}) \\ &\leq d(x_1, x_0) \frac{\theta^{n+1}}{1 - \theta}. \end{aligned}$$

Since  $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , it must be the case that  $d(x_m, x_n) \rightarrow 0$  as  $m, n \rightarrow \infty$ . Since  $X$  is complete, this means  $(x_n)_n \rightarrow x$  for some  $x \in X$ , meaning  $f(x) = x$ .

Suppose it were the case that there existed  $s, t$  distinct with  $f(s) = s$  and  $f(t) = t$ . Then,  $d(f(s), f(t)) = d(s, t) \leq \theta d(s, t)$ , but  $\theta < 1$ , which is a contradiction. Thus,  $x$  is unique.

## Problem 5

If  $(f_k)_k$  is an orthonormal sequence in a Hilbert space  $\mathcal{H}$ , show that the map

$$\begin{aligned} \varphi : \ell_2 &\rightarrow \mathcal{H} \\ \varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k \end{aligned}$$

is a well-defined isometry.

**Proof:** Let  $\xi, \eta \in \ell_2$ . Then,

$$\begin{aligned} d(\xi, \eta) &= \|\xi - \eta\|_2 \\ \varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k \\ \varphi(\eta) &= \sum_{k=1}^{\infty} \eta(k) f_k \\ d(\varphi(\xi), \varphi(\eta)) &= \left( \sum_{k=1}^{\infty} \langle \xi(k) - \eta(k), f_k \rangle \right)^{1/2} \\ &= \|\xi - \eta\|_2 \end{aligned}$$

Parseval's Identity.

## Problem 6

Let  $T : V \rightarrow W$  be a continuous linear map between normed spaces which is bounded below; that is, there is a  $C > 0$  such that  $\|T(v)\| \geq C \|v\|$  for all  $v \in V$ . If  $V$  is complete, show that  $\text{Ran}(T) \subseteq W$  is a closed subspace, and that  $V \cong \text{Ran}(T)$  are uniformly isomorphic.

**Proof:** Let  $(v_n)_n$  be a Cauchy sequence in  $V$ .

## Problem 7

Let  $X$  and  $Y$  be metric spaces with completions  $(\tilde{X}, \iota_X)$  and  $(\tilde{Y}, \iota_Y)$  respectively. If  $f : X \rightarrow Y$  is an isometry, show that there is a unique isometry  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$  that extends  $f$ . That is, the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \uparrow \iota_X & & \uparrow \iota_Y \\ X & \xrightarrow{f} & Y \end{array}$$

**Proof:**

## Problem 10

Let  $V$  be an infinite-dimensional normed space with linear basis  $B$ .

- (i) If  $W \subset V$  is a proper subspace, show that  $W^\circ = \emptyset$ .
- (ii) If  $V$  is a Banach space, show that  $B$  is uncountable. You may use the fact that finite-dimensional subspaces are always closed.

**Proof of (i):** Let  $W \subset V$  be proper. Suppose  $U(x, \varepsilon) \subseteq W$  for some  $x \in V$  and  $\varepsilon > 0$ . Then, for  $v \in V$ , we have that  $\frac{\varepsilon}{2} \frac{v}{\|v\|} + x \in U(x, \varepsilon)$ , meaning  $v \in W$ , so  $V \subseteq W$ , which is a contradiction. Thus,  $W^\circ = \emptyset$ .

**Proof of (ii):** Let  $\{e_n\}_{n \geq 1}$  be a countable, linearly independent set. Let  $W_1 = \text{span}\{e_1\}$ ,  $W_2 = \text{span}\{e_1, e_2\}$ , and so on. We have that each  $W_n \subseteq V$  is closed (by assumption), and  $W_1 \subseteq W_2 \subseteq \dots$ . Since each  $W_n$  has empty interior, it cannot be the case that  $V = \bigcup W_n$  by Baire's Theorem.