Problem (Problem 1)**:** Let G be a Lie group, which is a topological group that is also a smooth manifold and where all group operations are smooth. For convenience, we will always assume that G is path-connected. Prove that the tangent bundle TG of G is trivial — i.e., TG composes as a direct product.

Solution: From Cayley's Theorem, we know that G acts on itself transitively by left-multiplication. That is, for any $g \in G$, there is a map $L_g \colon G \to G$ that takes $h \mapsto gh$. This is a diffeomorphism of smooth manifolds since L_g is smooth and admits the smooth inverse $L_{g^{-1}}$. In particular, this means that

$$D_e(L_g) \colon T_eG \to T_gG$$

is invertible as a linear map. Letting $T_eG \cong \mathbb{R}^n$ have a local basis $\mathcal{B}_e = \left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$, we then observe that $D_e(L_g)$ then maps this basis to a basis for T_gG since $D_e(L_g)$ is a linear isomorphism, meaning that

$$\begin{aligned} TG &= \bigsqcup_{g \in G} T_gG \\ &= \bigsqcup_{g \in G} D_e(L_g)(T_eG) \\ &\cong \bigsqcup_{g \in G} \mathbb{R}^n \\ &\cong G \times \mathbb{R}^n. \end{aligned}$$

Thus, TG is trivial.

Problem (Problem 2)**:** Note that a Lie group can act on itself by left or right multiplication. A vector field on G is called *left-invariant* if it is invariant under the differential of left multiplication L_g for every $g \in G$. Prove that T_eG can be identified with left invariant vector fields on G.

Solution: We observe that by definition, a left-invariant vector field X is one where $g \cdot X = X$ for every $g \in G$. In particular, this means that for any vector field $X_e \in T_eG$, there is a corresponding left-invariant vector field on G defined at each $g \in G$ by taking $X_g = D_e(L_g)(X_e)$; that such a vector field is left-invariant follows from the fact that L_g is a diffeomorphism of G onto itself. Thus, we get the correspondence between vector fields at T_eG and left-invariant vector fields on G.

Problem (Problem 3)**:** Similar to invariant vector fields, invariant forms are ones for which $L_g^*\omega = \omega$. Prove that invariant forms are stable under taking d and under contraction by a left-invariant vector field.

Solution: Let ω be left-invariant. Then, by definition of the pullback,

$$\begin{aligned} L_g^*(d\omega) &= d(L_g^*\omega) \\ &= d\omega. \end{aligned}$$

Similarly, by definition of the contraction, if $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$ are a k-dimensional collection of vector fields, then

$$\begin{aligned} L_g^*(\iota_X(\omega))\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) &= L_g^*\left(\omega\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)\right) \\ &= \left(L_g^*\omega\right)\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \\ &= \omega\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \\ &= \iota_X(\omega)\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right). \end{aligned}$$