Problem (Problem 2)**:** Prove the claim from class that the open star cover of a simplicial complex is good.

Problem (Problem 4)**:** Compute the de Rham cohomology of $\mathbb{R}^2 \setminus \{0\}$, and find representatives of all nontrivial classes.

Solution: We observe that $\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R}$, so by the Poincaré lemma, we have

$$H^*_{DR}\left(\mathbb{R}^2 \setminus \{0\}\right) \cong H^*_{DR}\left(S^1\right)$$

or

$$\begin{aligned} H^0_{DR}\left(\mathbb{R}^2 \setminus \{0\}\right) &\cong \mathbb{R} \\ H^1_{DR}\left(\mathbb{R}^2 \setminus \{0\}\right) &\cong \mathbb{R} \\ H^k_{DR}\left(\mathbb{R}^2 \setminus \{0\}\right) &\cong 0 \text{ for } k \geqslant 2. \end{aligned}$$

We know that a complete set of representatives for cohomology classes of S^1 are 1 for H^0 and $d\theta$ for H^1. We know from the lemma that then, $d\theta$ corresponds to $\pi^*(d\theta)$, where $\pi\colon S^1 \times \mathbb{R} \to S^1$ is the projection. Thus, we observe that $\{1, \pi^*(d\theta)\}$ is the complete set of representatives of cohomology classes for $H^*_{DR}\left(\mathbb{R}^2 \setminus \{0\}\right)$.

Problem (Problem 6)**:** Let U and V be open subsets of a smooth manifold M, and let $W = U \cup V$. Write i_U, i_V for the inclusions of U and V into W respectively, and write j_U, j_V for the inclusions of $U \cap V$ into U and V respectively. Show that the sequence

$$0 \longrightarrow \mathcal{A}^k(W) \xrightarrow{\left(i_U^*, i_V^*\right)} \mathcal{A}^k(U) \oplus \mathcal{A}^k(V) \xrightarrow{j_U^* - j_V^*} \mathcal{A}^k(U \cap V) \longrightarrow 0$$

is exact.

Solution: Exactness at $\mathcal{A}^k(W)$ follows from the fact that $\left(i_U^*, i_V^*\right)$ is an inclusion map, hence has kernel 0.

To verify that the sequence is exact at $\mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$, we observe that if $\omega \in \mathcal{A}^k(W)$, then $(\omega|_U, \omega|_V)$ yields zero when subjected to $j_U^* - j_V^*$ as ω when restricted to $U \cap V$ is equal to itself. Therefore, the sequence is exact at $\mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$.

Finally, we let $\{f_U, f_V\}$ be a partition of unity for W subordinate to $\{U, V\}$. If $\omega \in \mathcal{A}^k(U \cap V)$, we observe that $f_U\omega$ extends to 0 on $V \setminus (U \cap V)$, whence $f_U\omega \in \mathcal{A}^k(V)$, and similarly for $f_V\omega \in \mathcal{A}^k(U)$. Therefore, $(f_V\omega, -f_U\omega) \in \mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$ maps to $\omega \in \mathcal{A}^k(U \cap V)$, meaning $j_U^* - j_V^*$ is surjective, so the sequence is exact at $\mathcal{A}^k(U \cap V)$.