We recall from linear algebra that a linear operator $T\colon V \to V$ is called diagonalizable if there is an orthonormal basis $\{e_j\}_{j\in J}$ and a bounded collection of elements $\{\lambda_j\}_{j\in J}$ such that for every $x \in V$, we have

$$Tx = \sum_{j\in J} \lambda_j \langle x, e_j\rangle e_j.$$

When V is a Hilbert space, there are a variety of generalizations. It will be useful to review the basic properties of compact and Fredholm operators.

Spectral Theory for Compact Normal Operators

The first, most basic version of the spectral theorem is the one for compact normal operators. We recall the different types of spectra.

Definition: Let $T \in B(X)$, where X is a Banach space.

(i) The *point spectrum* of T is the set

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \ker(T - \lambda I) \neq \{0\}\},$$

which are the eigenvalues of T.

(ii) The *approximate point spectrum* of T is the set

$$\pi(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not bounded below}\}.$$

(iii) The *compression spectrum* of T is

$$\gamma(T) = \{\lambda \in \mathbb{C} \mid \operatorname{im}(T - \lambda I) \text{ is not dense in } X\}.$$

There is a useful characterization of compact operators as follows.

Lemma: The following for $T \in B(H)$ are equivalent:

(i) T is compact;

(ii) $T|_{B_H}$ is a weak–norm continuous function from B_H into H.

Proof. Suppose T is compact. Then, if $(x_i)_{i\in I}$ is a weakly convergent net in B_H with limit x, and $\varepsilon > 0$, there is some finite-rank $S \in F(H)$ with $\|S - T\|_{\text{op}} < \varepsilon/3$. We have

$$\begin{aligned}\|Tx_i - Tx\| &\leq \|Tx_i - Sx_i\| + \|Sx_i - Sx\| + \|Sx - Tx\| \\ &\leq 2\|T - S\|_{\text{op}} + \|Sx_i - Sx\|.\end{aligned}$$

Every operator in $B(H)$ is weak–weak continuous, and since $\operatorname{im}(S)$ is finite-dimensional, all norms coincide, so that $Sx_i \to Sx$ in norm, giving that $\|Tx_i - Tx\| < \varepsilon/3$ for sufficiently large i. Thus, T is weak–norm continuous.

If T is weak–norm continuous, then since B_H is weakly compact, it follows that $T(B_H)$ is compact by continuity. □

Lemma: A diagonalizable operator T in $B(H)$ is compact if and only if its eigenvalues $\{\lambda_j \mid j \in J\}$ corresponding to an orthonormal basis $\{e_j \mid j \in J\}$ belongs to $c_0(J)$.

Proof. Since T is diagonalizable, we have

$$Tx = \sum_{j\in J} \lambda_j \langle x, e_j\rangle e_j.$$

If $T \in K(H)$, and $\varepsilon > 0$, then we set

$$J_\varepsilon = \{j \in J \mid |\lambda_j| \geq \varepsilon\}.$$

If J_ε is infinite, then since $\langle x, e_j\rangle \to 0$ by Parseval's identity, we have that the net $(e_j)_{j\in J_\varepsilon}$ converges weakly to zero. Yet, since $\|Te_j\| = |\lambda_j| \geq \varepsilon$ for any $j \in J_\varepsilon$, this contradicts the fact that T is weak–norm continuous. Thus, J_ε is finite for any $\varepsilon > 0$, so $(\lambda_j)_{j\in J}$ vanishes at infinity.

Now, if J_ε is finite for every $\varepsilon > 0$, we may define $T_\varepsilon \in F(H)$ by

$$T_\varepsilon = \sum_{j\in J_\varepsilon} \lambda_j \langle \cdot, e_j\rangle e_j,$$

and

$$\begin{aligned}\|(T - T_\varepsilon)x\|^2 &= \left\|\sum_{j\notin J_\varepsilon} \lambda_j \langle x, e_j\rangle e_j\right\|^2 \\ &= \sum_{j\in J_\varepsilon} |\lambda_j|^2 |\langle x, e_j\rangle|^2 \\ &\leq \varepsilon^2 \|x\|^2,\end{aligned}$$

so $\|T - T_\varepsilon\| \leq \varepsilon$, meaning that $T \in \overline{F(H)} = K(H)$. □

Note that by some basic computations, if T is diagonalizable, then we have

$$\begin{aligned}T^* &= \sum_{j\in J} \overline{\lambda_j} \langle \cdot, e_j\rangle e_j \\ T^*T &= \sum_{j\in J} |\lambda_j|^2 \langle \cdot, e_j\rangle e_j \\ &= TT^*.\end{aligned}$$

Thus, in particular, we have that every diagonalizable operator is normal.

Theorem: An operator $T \in B(H)$ is diagonalizable with eigenvalues vanishing at infinity if and only if it is a compact normal operator.

Proof. Now we only need to show that every compact normal operator is diagonalizable. Since T is compact, we know that the spectrum of T consists of 0 and a countable set of isolated points, and since T is normal, its spectral radius is equal to the operator norm, meaning that there is some λ such that $|\lambda| = \|T\|_{\mathrm{op}}$. In particular, there is an eigenvector for T.

Let $\mathcal{Z}$ be the family of orthonormal systems of eigenvectors of T, ordered by inclusion. Since we have established that this family is nonempty, and the union provides an upper bound for any chain in $\mathcal{Z}$, there is some maximal orthonormal system $\{e_j\}_{j\in J}$ with corresponding eigenvalues $\{\lambda_j\}_{j\in J}$. We let P be the projection onto the closed subspace spanned by the e_j. For each $x \in H$, we have

$$\begin{aligned}TPx &= T\left(\sum_{j\in J} \langle x, e_j\rangle e_J\right) \\ &= \sum_{j\in J} \lambda_j \langle x, e_j\rangle e_j \\ &= \sum_{j\in J} \left\langle x, \overline{\lambda_j} e_j\right\rangle e_j \\ &= \sum_{j\in J} \langle x, T^* e_j\rangle e_j \\ &= \sum_{j\in J} \langle Tx, e_j\rangle e_j \\ &= PTx.\end{aligned}$$

Thus, the operator $(I-P)T$ is normal, and is also compact. If $P \neq I$, then either $(I-P)T = 0$, and every unit vector in $(I-P)(H)$ is an eigenvector for T (contradicting maximality), or else $(I-P)T \neq 0$, in which case there is $e_0 \in (I-P)(H)$ with $Te_0 = \lambda e_0$ and $|\lambda| = \|(I-P)T\|_{\text{op}}$, which once again contradicts maximality.

Thus, $P = I$, and we are done. □

Spectral Theory for Normal Operators

We now generalize from the special case of compact operators. Here, we cannot use the convenient properties of compact operators with respect to finite dimensionality/codimensionality.

First, we notice that if $T \in B(H)$ is a normal operator, then $C^*(T)$, the C^*-algebra generated by T, is abelian, so from the Gelfand isomorphism, we have that $C^*(T) \cong C(\sigma(T))$ are isometrically $*$-isomorphic.

We will generalize this in a moment, but first we will apply the continuous functional calculus to show an important commutation relation. In $\mathbb{M}_n(\mathbb{C})$, we know that an operator S commutes with a normal operator T if and only if all the eigenspaces for T are invariant under S; since T and T^* commute, it then follows that S commutes with T^*.

It turns out that this generalizes to infinite-dimensional spaces, but the proof requires the use of the continuous functional calculus.

Proposition (Fuglede's Theorem)**:** If S and T are operators in $B(H)$, and T is normal, then $ST = TS$ implies $ST^* = T^*S$.

Proof. Define

$$e^{\lambda T} = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}.$$

This is an element of $C^*(T)$ by the continuous functional calculus, and similarly, $e^{\lambda T^*} \in C^*(T)$, with

$$e^{\lambda T^*} = e^{\lambda T^* - \overline{\lambda} T} e^{\overline{\lambda} T}.$$

There is some self-adjoint operator R such that $\lambda T^* - \overline{\lambda} T = iR$, meaning that

$$U(\lambda) = e^{\lambda T^* - \overline{\lambda} T}$$

is a unitary operator in $C^*(T)$ with $U(\lambda)^* = U(-\lambda)$.

It follows from the expression for $e^{\lambda T}$ that S commutes with $e^{\lambda T}$ for every λ, so that

$$e^{-\lambda T^*} S e^{\lambda T^*} = U(-\lambda) S U(\lambda),$$

with the operators uniformly bounded in norm by $\|S\|$.

Fixing $x, y \in H$, define $f\colon \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) = \left\langle e^{-\lambda T^*} S e^{\lambda T^*} x, y \right\rangle.$$

It follows that f is an entire function with $|f(\lambda)| \leq \|S\|$ for all λ, so that

$$\begin{aligned}\left\langle e^{-\lambda T^*} S e^{\lambda T^*} x, y \right\rangle - \langle Sx, y\rangle &= f(\lambda) - f(0)\\ &= 0,\end{aligned}$$

so that

$$e^{-\lambda T^*} S e^{\lambda T^*} - S = 0.$$

Thus, $ST^* - T^*S = 0$. □

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