Problem (Problem 1)**:** Show that if $1 < \lambda < \infty$, then the equation

$$ze^{\lambda - z} = 1$$

has precisely one solution in $\mathbb{D}$.

Solution: Write $f(z) = ze^{\lambda-z} - 1$. Our task is to show that $f(z)$ has exactly one solution in $\mathbb{D}$. Consider the function

$$g(z) = ze^{\lambda - z}.$$

We observe that $g(0) = 0$, and for any $z \neq 0$, $g(z) \neq 0$. Furthermore, since $e^{\lambda - z} \neq 0$ for all $z \in \mathbb{D}$, we observe that g has exactly one zero at $z = 0 \in \mathbb{D}$.

Let $\Gamma = S^1 = \partial\mathbb{D}$. We then observe that g and f are never zero on S^1, and that

$$\begin{aligned} |f(z) - g(z)| &= 1 \\ &< e^{\lambda - 1} \\ &< e^{\lambda - \operatorname{Re}(z)} \\ &= \left|ze^{\lambda - z}\right| \\ &= |g(z)|, \end{aligned}$$

whence f and g have the same number of zeros in $\mathbb{D}$.

Problem (Problem 2)**:**

(a) Prove that for any constants $a_0, a_1, a_2 \in \mathbb{C}$, the following inequality holds:

$$\max_{|z|=1}\left|z^7 + a_2z^2 + a_1z + a_0\right| \geqslant 1.$$

(b) Let $U \subseteq \mathbb{C}$ be open with $B(0,1) \subseteq U$, and let $f\colon U \to \mathbb{C}$ be holomorphic. Suppose that

$$\max_{|z|=1}\left|f(z) - \frac{1}{z^2}\right| < 1.$$

Show that f is not a polynomial.

Solution:

(a) Suppose toward contradiction that this were not true. That is, there are $a_0, a_1, a_2 \in \mathbb{C}$ such that for all $z \in S^1$,

$$\left|z^7 + a_2z^2 + a_1z + a_0\right| < l.$$

Write $g(z) = -z^7$ and $f(z) = a_2z^2 + a_1z + a_0$. We observe immediately that this condition implies that f cannot have any zeros on S^1, as $|g(z)| = 1$ on S^1 always. Therefore, writing

$$\begin{aligned} \left|z^7 + a_2z^2 + a_1z + a_0\right| &= |f(z) - g(z)| \\ &< |g(z)|, \end{aligned}$$

we observe that the conditions of Rouché's theorem are satisfied, so f and g have the same number of zeros in $\mathbb{D}$. Yet, this is absurd, since f can have at most two zeros when counted with multiplicity, while g has 7 zeros in $\mathbb{D}$ when counted with multiplicity.

(b) We observe that on S^1, $\left|\frac{1}{z^2}\right| = 1$, whence the condition equals

$$\left|f(z) - \frac{1}{z^2}\right| < \left|\frac{1}{z^2}\right|.$$

We claim that the conditions of Rouché's theorem are satisfied. Toward this end, observe that, from the reverse triangle inequality,

$$|f(z)| < 2.$$

If f were to have a pole at $z_0 \in S^1$, there would be some $r > 0$ such that $|f(z)| > 2$ for all $z \in \dot{U}(z_0, r)$, which would contain some other element of S^1. Additionally, we observe that f has no zeros on S^1, as we would then have

$$\left|-\frac{1}{z^2}\right| < \left|\frac{1}{z^2}\right|$$

on S^1, another contradiction. Therefore, the sum of all orders in $\mathbb{D}$ for f and $\frac{1}{z^2}$ is the same; since $\operatorname{ord}_0\left(\frac{1}{z^2}\right) = -2$, it follows that f has two poles (counted with multiplicity) inside $\mathbb{D}$, whence f is not a polynomial (as it is not even holomorphic on U).

Problem (Problem 3)**:** Let $U \subseteq \mathbb{C}$ be open containing $B(0,1)$, and let $f, g \colon U \to \mathbb{C}$ be holomorphic such that $\operatorname{ord}_0(f) = 1$ and $\operatorname{ord}_z(f) = 0$ for all $z \in B(0,1) \setminus \{0\}$. For $w \in \mathbb{C}$, define $f_w(z) = f(z) + wg(z)$.

(a) Show that there exists some $r > 0$ dependent on g such that if $w \in U(0, r)$, then f_w has a unique zero in $B(0,1)$, which we call z_w.

(b) Show that $\lim_{w\to 0} z_w = 0$.

(c) Show that

$$z_w = \frac{1}{2\pi i}\oint_{S(0,1)} \frac{f'_w(\xi)}{f_w(\xi)}\xi \, d\xi.$$

Solution:

(a) To start, we ignore the case where $g(z)$ is identically zero, as it then holds for all $w \in \mathbb{C}$. We observe that if we let $\Gamma = S^1$, we desire to find w such that the conditions for Rouché's Theorem hold:

$$\begin{aligned} |f_w(z) - f(z)| &< |f(z)| \\ |wg(z)| &< |f(z)| \\ |w||g(z)| &< |f(z)| \end{aligned}$$

Now, we observe a few things. First, since S^1 is compact, f is holomorphic, and we assume that $\operatorname{ord}_z(f) = 0$ for all $z \in B(0,1) \setminus \{0\}$, we see that there is some $K > 0$ such that $|f(z)| \geqslant K$ for all $z \in S^1$. Furthermore, since g is holomorphic, there is some $M > 0$ such that $|g(z)| \leqslant M$ for all $z \in S^1$. Therefore, we let $r = \frac{K}{M}$.

Then, if $|w| < r$, we have

$$\begin{aligned} |f_w(z) - f(z)| &= |w||g(z)| \\ &< r|g(z)| \\ &\leqslant rM \\ &= K \\ &\leqslant |f(z)| \end{aligned}$$

on S^1. Additionally, to verify that the conditions of Rouche's Theorem still hold for $r = \frac{K}{M}$, we see that

$$|f(z) + wg(z)| \geqslant |f(z)| - |w||g(z)|$$

$$\begin{aligned} &\geqslant K - |w|M \\ &> K - rM \\ &= 0, \end{aligned}$$

whence $f_w(z)$ has no zeros on S^1. Thus, given these conditions, $f_w(z)$ and $f(z)$ have the same number of zeros in $\mathbb{D}$, whence $f_w(z)$ has a unique zero.

(b) Let $(w_n)_n \to 0$ be a sequence in $U(0,r)$ with corresponding sequence $(z_n)_n \subseteq \mathbb{D}$. We observe then that

$$\lim_{n\to\infty} (f_{w_n}(z_n)) = \lim_{n\to\infty} (f(z_n) + w_n g(z_n)).$$

Since g is bounded above by M, we observe that

$$\begin{aligned} \lim_{n\to\infty} |w_n g(z_n)| &\leqslant \lim_{n\to\infty} |w_n|M \\ &= 0, \end{aligned}$$

whence

$$\begin{aligned} \lim_{n\to\infty} |f(z_n)| &= \lim_{n\to\infty} |(f_{w_n}(z_n) - w_n g(z_n))| \\ &\leqslant \lim_{n\to\infty} |w_n||g(z_n)| \\ &= 0, \end{aligned}$$

so that $\lim_{n\to\infty} f(z_n) = 0$. Since $(z_n)_n \subseteq \overline{\mathbb{D}}$, $(z_n)_n$ admits a subsequence $(z_{n_k})_k \to z \in \overline{\mathbb{D}}$. Yet, this means $f(z) = 0$, meaning $z = 0$ as f has exactly one zero in $\overline{\mathbb{D}}$. Since $(w_n)_n \to 0$, it follows that $(w_{n_k})_k \to 0$, whence $\lim_{w\to 0} z_w = 0$.

(c) We write

$$f_w(z) = (z - z_w)h(z),$$

where $h(z)$ is holomorphic and has $h(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$. We see then that

$$\frac{f_w'(z)}{f_w(z)} = \frac{1}{z - z_w} + \frac{h'(z)}{h(z)},$$

meaning that the expression $\frac{h'(z)}{h(z)}$ is holomorphic. Therefore,

$$\oint_{S^1} \frac{f_w'(\xi)}{f_w(\xi)}\xi \, d\xi = \oint_{S^1} \frac{\xi}{\xi - z_w} \, d\xi + \oint_{S^1} \xi \frac{h'(\xi)}{h(\xi)} \, d\xi.$$

Since $h(\xi) \neq 0$ on $\mathbb{D}$, it follows that the second integral vanishes by Cauchy's Integral Theorem, whence by Cauchy's Integral Formula, we have

$$= z_w$$

Problem (Problem 4)**:** For all $n \in \mathbb{N}$ find the residue at $z = 0$ for each of the following functions.

(a) $\dfrac{e^{z^2}}{z^{2n+1}}$;

(b) $z^{-n}e^{\alpha z}$ for $\alpha \in \mathbb{Z}$;

(c) $\dfrac{z^{n-1}}{\sin^n(z)}$.

Solution:

(a) Using the Taylor expansion for e^{z^2}, we find that

$$\begin{aligned}\frac{1}{z^{2n+1}}e^{z^2} &= \frac{1}{z^{2n+1}}\sum_{k=0}^{\infty}\frac{z^{2k}}{k!}\\ &= \sum_{k=0}^{\infty}\frac{z^{2k-2n-1}}{k!},\end{aligned}$$

meaning that the coefficient at a_{-1} is $\frac{1}{n!}$.

(b) We have a pole of order n at $z = 0$, as $e^{\alpha z} \neq 0$ for all z. Thus, computing the residue directly, we find

$$\begin{aligned}\operatorname{Res}(f;0) &= \frac{1}{(n-1)!}\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(e^{\alpha z}\right)\\ &= \frac{\alpha^{n-1}}{(n-1)!}.\end{aligned}$$

(c) We observe that the order of the numerator at $z = 0$ is $n - 1$, while the order in the denominator at $z = 0$ is n, meaning that we have a simple pole at $z = 0$. Therefore, we compute

$$\begin{aligned}\operatorname{Res}(f;0) &= \lim_{z\to 0}\frac{z^n}{\sin^n(z)}\\ &= \left(\lim_{z\to 0}\frac{z}{\sin(z)}\right)^n\\ &= 1.\end{aligned}$$

Problem (Problem 5)**:** For each positive $n \in \mathbb{N}$, let γ_N be the loop consisting of the square with vertices at $\left(N + \frac{1}{2}\right)(-1 - i)$, $\left(N + \frac{1}{2}\right)(1 - i)$, $\left(N + \frac{1}{2}\right)(1 + i)$, and $\left(N + \frac{1}{2}\right)(-1 + i)$.

Let $f(z) = \frac{\pi\cot(\pi z)}{z^4}$. By evaluating $\oint_{\gamma_N} f(z)\,dz$, determine

$$\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Solution: We observe that the poles of $f(z)$ are at $-N, -N+1, \dots, 0, \dots, N-1, N$. To compute the residue at each of these poles, we separate into the case of $z = 0$ and of $z \neq 0$. For the case with $z \neq 0$, we find that f has a simple pole at $z = k$ for each such k, whence

$$\begin{aligned}\operatorname{Res}(f;k) &= \lim_{z\to k}\frac{\pi\cos(\pi z)}{z^4\left.\frac{d}{dz}\right|_{z=n}\pi\sin(\pi z)}\\ &= \frac{1}{k^4}.\end{aligned}$$

Since $z^4\sin(\pi z)$ has a zero of order 5 at 0, and $\cos(\pi z)$ does not have a zero at $z = 0$, it follows that

$$f(z) = \frac{\pi\cos(\pi z)}{z^4\sin(\pi z)}$$

has a pole of order 5 at 0. We compute

$$\operatorname{Res}(f;0) = \frac{1}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\left(z^5 f(z)\right)$$

$$= \frac{1}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}(\pi z\cot(\pi z)).$$

Upon tedious computation, we find that

$$\operatorname{Res}(f;0) = -\frac{\pi^4}{45}.$$

Therefore, we find that

$$\frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz = 2\sum_{k=1}^{N}\frac{1}{k^4} - \frac{\pi^4}{45}.$$

Now, we want to evaluate

$$\lim_{N\to\infty}\frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz.$$

We observe that

$$\begin{aligned}|f(z)| &= \frac{|\pi\cot(\pi z)|}{|z^4|}\\ &\leqslant \frac{|\pi\cot(\pi z)|}{\left(N+\frac{1}{2}\right)^4}.\end{aligned}$$

Now, we want to establish an upper bound for $|\pi\cot(\pi z)|$ on the square γ_N. We observe that cot is an odd function, so we only need to concern ourselves with establishing this bound on the upper half-plane. Toward this end, we see that

$$\pi\cot\left(\pi\left(N+\frac{1}{2}\right)\right) = 0.$$

Computing, for $0 < k \leqslant N + \frac{1}{2}$,

$$\begin{aligned}\pi\cot\left(\pi\left(N+\frac{1}{2}\right)+\pi k i\right) &= \pi\frac{\cos\left(\pi\left(N+\frac{1}{2}\right)+\pi k i\right)}{\sin\left(\pi\left(N+\frac{1}{2}\right)+ki\right)}\\ &= \pi\frac{\pm i\sinh(\pi k)}{\pm\cosh(\pi k)}\\ &= \pm\pi i\tanh(\pi k),\end{aligned}$$

whence

$$|\pi\coth(\pi z)| \leqslant 1$$

for all $z \in \gamma_N$. In particular, this yields

$$\left|\oint_{\gamma_N} f(z)\,dz\right| \leqslant 4(2N+1)\frac{1}{\left(N+\frac{1}{2}\right)^4},$$

so that

$$\lim_{N\to\infty}\oint_{\gamma_N} f(z)\,dz = 0.$$

In particular, this yields

$$0 = \lim_{N\to\infty}\oint_{\gamma_N} f(z)\,dz$$

$$= \lim_{N\to\infty}\left(2\sum_{k=1}^{N}\frac{1}{k^4} - \frac{\pi^4}{45}\right),$$

so that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}.$$