Part 1

1.7, Problem 10

For

$$\frac{dy}{dt} = e^{-y^2} + \alpha,$$

there are zero equilibrium solutions for $\alpha \geqslant 0$ and $\alpha < -1$, while there are two equilibrium solutions for $\alpha \in (-1, 0)$ and one equilibrium solution for $\alpha = -1$.

1.7, Problem 13

(a) This is a graph of (iii), as (iii) decomposes into $y(A - y^2)$, meaning 0 is always an equilibrium solution, as well as the map $A = y^2$.

(b) This is a graph of (v), as $y^2 - A = 0$ when $A = y^2$, so it yields a source when $y > 0$ and a sink when $y < 0$.

(c) This is a graph of (iv) as it is the opposite sign of (v).

(d) This is a graph of (iv), as (iv) decomposes into $(A - y)\,y$, meaning 0 is always an equilibrium solution, as well as some linear factor.

Chapter 1 Review, Problem 3

There are no equilibrium solutions for $\frac{dy}{dt} = t^2(t^2 + 1)$

Chapter 1 Review, Problem 4

One of the solutions to $\frac{dy}{dt} = -\left|\sin^5 y\right|$ is the equilibrium solution $y = 0$.

Chapter 1 Review, Problem 10

The bifurcation occurs at $a = -4$, where there is one equilibrium solution, with zero equilibrium solutions on either side of $a = -4$.

Chapter 1 Review, Problem 11

This is true. We can see that $\frac{dy}{dt} = e^{-t} = \left|-e^{-t}\right|$.

Chapter 1 Review, Problem 12

This is false. For example, the differential equation

$$\frac{dy}{dt} = (y + 5)\left(t^2 + 2\right)$$

is separable, but it is not autonomous.

Chapter 1 Review, Problem 13

This is true. We can see that

$$\frac{dy}{dt} = a(y)$$

implies

$$\int \frac{1}{a(y)}\, dy = \int dt,$$

so $\frac{dy}{dt} = a(y)$ is separable.

Chapter 1 Review, Problem 14

This is false. The differential equation

$$\frac{dy}{dt} = 3yt^2 + 2t$$

is linear but is not separable.

Chapter 1 Review, Problem 49

(a) This slope field reflects equation (iv), as the slopes are both independent of y and negative for $t > 1$.

(b) This slope field represents equation (vii), as equation (vii) is equal to $t\,(y-1)$, reflected in the fact that for $t = 0$ and $y = 1$, the slopes are zero, while the variation in the sign of the slope reflects the respective signs of t and $y - 1$.

(c) This slope field represents equation (viii), as the slopes are independent of t, the equation has equilibrium solutions at $y = \pm 1$, and at $y = 0$, and at $y = 0$, the slope is negative.

(d) This slope field represents equation (vi), as at $y = 0$, the slopes are a function of t^2, and the slopes increase with y for $t = 0$.

Chapter 1 Review, Problem 52

(a)

$$\begin{aligned} \frac{dy}{dt} &= -2ty^2 \\ \int -\frac{1}{y^2}\, dy &= \int t\, dt \\ \frac{1}{y} &= t^2 + C \\ y &= \frac{1}{t^2 + C}. \end{aligned}$$

(b)

$$y_0 = \frac{1}{1 + C},$$

We must have $C \neq -1$, and additionally that $t^2 + C \neq 0$ for all $t > -1$, meaning we must have $C > 0$, implying $0 < y_0 < 1$.