Solution (19.1)**:**

(a) There is a simple pole at $z = 0$. The residue at this pole is 0.

(b) There is a pole of order 4 at $z = 0$. The residue at this pole is 0.

(c) There is a pole of order 4 at $z = 0$. The residue at this pole is $\frac{1}{120}$.

(d) There is an essential singularity at $z = 0$.

(e) There is a removable singularity at $z = 0$.

Solution (19.2)**:** The poles of $\frac{e^z}{\sin z}$ occur when $\sin z = 0$, which happens when $z = n\pi$.

Solution (19.4)**:** There are no residues within $|z| < 1$.

For $1 < |z| < 2$, evaluating the a_{-1} term, we have the residue of $\frac{1}{3}$.

For $|z| > 2$, evaluating the a_{-1} term, we have a residue of $\frac{1}{3}$.

Solution (19.5)**:**

(a) There is a pole of order 2 at $z = 1$ and a pole of order 1 at $z = 0$.

(b) Around $z = 0$, we have the expansion

$$\begin{aligned}\frac{1}{z(z-1)^2} &= \frac{1}{z(1-z)^2}\\ &= \frac{1}{z}\left(\sum_{k=1}^{\infty} kz^{k-1}\right)\\ &= \sum_{k=1}^{\infty} kz^{k-2},\end{aligned}$$

which converges for all $0|z| < 1$. Around $z = 1$, we have the expansion

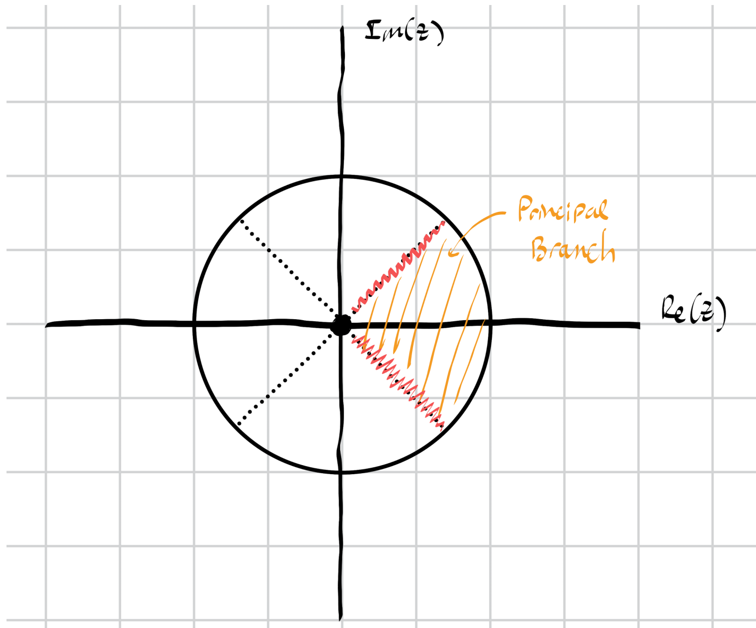
$$\begin{aligned}\frac{1}{(z-1)^2 z} &= \frac{1}{(z-1)^2(1+z-1)}\\ &= \frac{1}{(z-1)^2}\left(\sum_{k=0}^{\infty}(-1)^k(z-1)^k\right)\\ &= \sum_{k=0}^{\infty}(-1)^k(z-1)^{k-2}.\end{aligned}$$

This series converges for all $0 < |z-1| < 1$.

(c) The residue at $z = 0$ is 1, and the residue at $z = 1$ is -1.

Solution (19.9)**:** If a is not a singularity of $w(z)$, the Laurent expansion collapses into the Taylor expansion.

Solution (19.11)**:**



Solution (19.13)**:** Writing

$$\sqrt{z^2+1} = \sqrt{(z-i)(z+i)},$$

we look at the contours $\pm i$. Define

$$\begin{aligned} z_{\pm} &= (z \mp i) \\ &= r_{\pm}e^{i\varphi_{\pm}}. \end{aligned}$$

Plugging into our expression, we get

$$w(z) = \sqrt{r_+r_-}e^{i(\varphi_+ + \varphi_-)/2}.$$

If we go around the contour at $z = i$, then φ_+ will rotate around by 2π, while φ_- will not rotate around by 2π. Similarly, if we go around the contour at $z = -i$, then φ_- will rotate around by 2π while φ_+ will not rotate around by 2π.

Meanwhile, if we have the contour around both $z = i$ and $z = -i$, then both φ_+ and φ_- rotate around by 2π, meaning we do not pick up a sign change.

We take the branch cut between $z = i$ and $z = -i$ to allow contours that circle both $z = \pm i$ but disallow contours that only circle one of $z = \pm i$.

Solution (19.18)**:**

(a) Consider

$$\begin{aligned} w(1/\zeta) &= \sqrt{\left(\frac{1}{\zeta} - a_1\right)\cdots\left(\frac{1}{\zeta} - a_n\right)} \\ &= \frac{1}{\zeta^{n/2}}\sqrt{(1-a_1\zeta)\cdots(1-a_n\zeta)}. \end{aligned}$$

We have a branch point at $\zeta = 0$ whenever $n/2 \notin \mathbb{Z}$, as then it is the case that the square root has multivalued behavior.

(b) Considering

$$w(1/\zeta) = \sqrt{1/\zeta - a_1} + \cdots + \sqrt{1/\zeta - a_n}$$

$$= \frac{1}{\zeta^{1/2}}\sqrt{1-a_1\zeta} + \cdots + \frac{1}{\zeta^{1/2}}\sqrt{1-a_n\zeta},$$

we see that each $\zeta^{1/2}$ has branching behavior as $\zeta \to 0$, so w has a branch point at ∞.

Solution (19.24)**:** We must move $e^{2\pi i}$ back into the principal branch to evaluate the square root.

Solution (19.28)**:**

(a) We have

$$\begin{aligned} e^{iz} &= \cos(z) + i\sin(z) \\ &= \left(1-\sin^2(z)\right)^{1/2} + i\sin(z). \end{aligned}$$

Thus, defining $w = \sin(z)$, we have

$$\begin{aligned} iz &= \ln\left(iw + \left(1-w^2\right)^{1/2}\right) \\ z &= -i\ln\left(iw + \left(1-w^2\right)^{1/2}\right). \end{aligned}$$

Similarly, defining $w = \cos(z)$, we have

$$\begin{aligned} e^{iz} &= \cos(z) + i\left(1-\cos^2(z)\right)^{1/2} \\ iz &= \ln\left(w + i\left(1-w^2\right)^{1/2}\right) \\ &= \ln\left(w + i\left((-1)\left(w^2-1\right)\right)^{1/2}\right) \\ &= \ln\left(w + i(-i)\left(w^2-1\right)^{1/2}\right) \\ &= \ln\left(w + \left(w^2-1\right)^{1/2}\right). \end{aligned}$$

(b) The principal branch of ln gives outputs in the range $(-\pi, \pi)$.