Spectrum, Resolvent, and Functional Calculus for Banach Algebras

Some Basic Background

Definition: Let A be a unital Banach algebra. For $a \in A$, the *spectrum* of a is given by

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda e - a \text{ is not invertible in } A\}.$$

The resolvent of a, denoted by $\rho(a)$, is $\mathbb{C} \setminus \sigma(a)$. The resolvent function on $\lambda \in \rho(a)$ is given by $R(a, \lambda) := (\lambda - a)^{-1}$.

Proposition: Let A be a unital Banach algebra. The group (under multiplication)

$$\mathrm{GL}(A) := \{a \in A \mid a \text{ is invertible}\}$$

is open.

Proof. The Carl Neumann series gives that if $\|a\| < 1$, then

$$(e - a)^{-1} = \sum_{n=0}^{\infty} a^n,$$

where $a^0 = 1$. If $b \in \mathrm{GL}(A)$, and $\|a\| < \frac{1}{\|b^{-1}\|}$, then $b - a = b\left(e - b^{-1}a\right)$, and since $\left\|b^{-1}a\right\| \leq \left\|b^{-1}\right\|\|a\| < 1$, we have that $e - b^{-1}a$ is invertible, so $b - a$ is invertible.

In particular, this means that $U\left(b, 1/\left\|b^{-1}\right\|\right) \subseteq \mathrm{GL}(A)$, so $\mathrm{GL}(A)$ is open. □

Corollary: If A is a unital Banach algebra, $\sigma(a)$ is a compact subset of $B(0, \|a\|)$.

Proof. Since $\mathrm{GL}(A)$ is open, so is $\rho(a)$, meaning that if $|\lambda| > \|a\|$, we have $\lambda - a = \lambda\left(e - \lambda^{-1}a\right)$, and since $\left\|\lambda^{-1}a\right\| < 1$, this is invertible, so $\sigma(a)$ is a closed subset of $B(0, \|a\|)$, and thus is compact. □

One thing to note about the spectrum is that it most certainly depends on the ambient algebra. To see this, consider the algebra $A(\mathbb{D})$, which is the set of all holomorphic functions on the unit disk that extend continuously to the unit circle S^1. For an element $f \in A(\mathbb{D})$ to have an inverse, it needs to be nonzero on $\overline{\mathbb{D}}$.

In particular, this means that $\sigma_{A(\mathbb{D})}(f) = \sigma_{C(\overline{\mathbb{D}})}(f) = \mathrm{im}(f)$. Meanwhile, we may consider the restriction map $f \mapsto f|_{S^1}$, which maps from $A(\mathbb{D})$ to $C\left(S^1\right)$. From the maximum modulus principle, this map is in fact isometric, but the spectra of elements in $C\left(S^1\right) \cap A(\mathbb{D})$ are not necessarily equivalent. For instance, the spectrum of $f(z) = z$ has $\sigma_{A(\mathbb{D})}(f) = \overline{\mathbb{D}}$, while $\sigma_{C(S^1)}(f) = S^1$.

Yet, since all holomorphic functions can be determined from their boundary values, in order to determine the spectrum from the boundary values, we may use the argument principle for this purpose. If f does not vanish on S^1, then the number of zeros of f is given by

$$n = \frac{1}{2\pi i} \oint_{S^1} \frac{f'(z)}{f(z)}\, dz,$$

meaning that we have

$$\sigma_{A(\mathbb{D})}(f) = f\left(S^1\right) \cup \left\{\lambda \in \mathbb{C} \mid n\left(\lambda; f\left(S^1\right)\right) \neq 0\right\}.$$

Proposition: If A is a unital Banach algebra, and $z \in \rho(a)$, then

$$\mathrm{dist}_{\sigma(a)}(z) \geq \frac{1}{\left\|(ze - a)^{-1}\right\|}.$$

Proof. Suppose $\lambda \in \sigma(a)$. Then, since $U\left(ze - a, 1/\left\|(ze-a)^{-1}\right\|\right) \subseteq \mathrm{GL}(A)$, this set does not include $\lambda e - a$. Thus,

$$\begin{aligned}|z - \lambda| &= \|(ze - a) - (\lambda e - a)\| \\ &\geq \frac{1}{\left\|(z-a)^{-1}\right\|}.\end{aligned}$$

□

Proposition: Let A be a unital Banach subalgebra of a unital Banach algebra B. Then, for any $a \in A$,

$$\partial\sigma_A(a) \subseteq \sigma_B(a) \subseteq \sigma_A(a).$$

Proof. If a is invertible in A, then a is invertible in B with the same inverse element, so $\sigma_B(a) \subseteq \sigma_A(a)$.

Now, let $\lambda \in \partial\sigma_A(a)$. Let $(z_n)_n \subseteq \rho(a) \to \lambda$. By the above proposition,

$$\begin{aligned}\left\|(z_n e - a)^{-1}\right\| &\geq \frac{1}{\mathrm{dist}_{z_n,\sigma(a)}} \\ &\geq \frac{1}{|z_n - \lambda|} \\ &\to \infty,\end{aligned}$$

so if there is an element $b \in B$ with $b(\lambda e - a) = e = (\lambda e - a)b$, then we have $b = \lim_{n\to\infty}(z_n e - a)^{-1}$, which contradicts the fact that the norms of $(z_n - a)^{-1}$ are bounded. □

Theorem: Let A be a unital Banach algebra, and let $a \in A$. Then, the map $\lambda \mapsto R(a,\lambda)$ is strongly analytic, in the sense that there is $r > 0$ and $a_n \in A$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $|z - z_0| < r$. Furthermore, $\lim_{|\lambda|\to\infty} R(a,\lambda) = 0$.

Proof. Let $z_0 \in \rho(a)$, and let $|w| < |R(a,z_0)|^{-1}$. Then,

$$\begin{aligned}((z_0 - w)e - a)^{-1} &= ((z_0 e - a) - we)^{-1} \\ &= ((z_0 e - a)(e - wR(a,z_0)))^{-1} \\ &= R(a,z_0)(e - wR(a,z_0))^{-1} \\ &= R(a,z_0)\sum_{n=0}^{\infty} R(a,z_0)^n w^n \\ &= \sum_{n=0}^{\infty} R(a,z_0)^{n+1} w^n.\end{aligned}$$

Furthermore, whenever $|\lambda| > \|a\|$, we have

$$\begin{aligned}\|R(a,\lambda)\| &= \left\|\sum_{n=0}^{\infty} \lambda^{-n-1} a^n\right\| \\ &\leq \frac{1}{|\lambda| - \|a\|},\end{aligned}$$

which converges to 0 as $|\lambda| \to \infty$. □

Theorem: If A is a unital Banach algebra, and $a \in A$, then $\sigma(a)$ is nonempty.

Proof. Suppose $\sigma(a)$ were empty. Then, $\rho(a) = \mathbb{C}$, whence we would have $\lambda \mapsto R(a, \lambda)$ is an entire function that converges to 0 as $|\lambda| \to \infty$. Since every C_0 function is bounded, it follows that we have a bounded entire strongly analytic function. Therefore, for any $\varphi \in A^*$, we have $\varphi \circ R(a, \lambda)$ is a bounded entire function, hence constant. Thus, $\varphi \circ R(a, \lambda) = 0$, meaning that $R(a, \lambda) = 0$ for all $\lambda \in \mathbb{C}$, which cannot happen. Thus, $\sigma(a)$ is nonempty. □

Definition: Let A be a Banach algebra, $a \in A$. The spectral radius of a, is given by

$$r(a) = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Theorem (Spectral Radius Formula)**:** If A is a unital Banach algebra, and $a \in A$, then

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Integration in Banach Spaces

It is relatively simple to extend Riemann integration from real (or complex) space to any Banach space.

Let X be a Banach space, and let $\gamma\colon [0, L] \to \mathbb{C}$ be a rectifiable oriented curve, and call $\Gamma = \operatorname{im}(\gamma)$, with $f\colon \Gamma \to X$ some continuous function. We may parametrize Γ by arc-length, giving $\frac{d|\gamma|}{dt} = 1$. Recall that if

$$\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = L\}$$

is a partition, the *norm* of the partition is given by $\|\mathcal{P}\|$, which is the maximum size of the distance between s_i and s_{i-1}. Letting $\xi_i \in \gamma([s_{i-1} - s_i])$ be a collection of tags, with $\Sigma = \{\xi_i \mid i = 1, \dots, n\}$, we define the Riemann sum with respect to $\mathcal{P}$ and Σ by

$$R(\mathcal{P}, \Sigma) = \sum_{i=1}^{n} f(\xi_i)(\gamma(s_i) - \gamma(s_{i-1})).$$

We say the Riemann integral

$$\int_\Gamma f(z)\, dz = w$$

exists if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$, and any collection of tags Σ, we have

$$\|R(\mathcal{P}, \Sigma) - w\| < \varepsilon.$$

Theorem: Let X be a Banach space, Γ an oriented rectifiable curve in $\mathbb{C}$, and $f\colon \Gamma \to X$ a continuous function. Then,

$$\int_C f(z)\, dz$$

exists as a Riemann integral.

Proof. Let $\delta > 0$ be such that $|s - t| < \delta$ implies $\|f(\gamma(s)) - f(\gamma(t))\| < \varepsilon/2L$. Let $(\mathcal{P}_1, \Sigma_1), (\mathcal{P}_2, \Sigma_2)$ be tagged partitions with $\|\mathcal{P}_i\| < \delta$. Take their common refinement, $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$, and choose a tag set $\{\omega_i\}_{i=1}^n$ for $\mathcal{P}$.

Write

$$\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = L\}$$
$$\mathcal{P}_1 = \{0 = s_0 < s_{i_1} < \cdots < s_{i_m} = s_n = L\}.$$

For each $i_{k-1} < i \leq i_k$, the sample points ω_i and ξ_k lie in $\omega([s_{i-k-1}, s_{i_k}])$, so that $\|f(\omega_i) - f(\xi_k)\| <$

$\varepsilon/2L$. We thus have

$$\begin{aligned}\|R(\mathcal{P},\Omega)-R(\mathcal{P}_1,\Sigma_1)\| &= \left\|\sum_{i=1}^{n} f(\omega_i)(\gamma(s_i)-\gamma(s_{i-1})) - \sum_{k=1}^{m} f(\xi_k)\big(\gamma(s_{i_k})-\gamma(s_{i_{k-1}})\big)\right\| \\ &= \left\|\sum_{i=1}^{n} f(\omega_i)(\gamma(s_i)-\gamma(s_{i-1})) - \sum_{k=1}^{m} f(\xi_k)\sum_{i=i_{k-1}+1}^{i_k}(\gamma(s_i)-\gamma(s_{i-1}))\right\| \\ &\leq \sum_{i=1}^{n}\|f(\omega_i)-f(\xi_k)\||\gamma(s_i)-\gamma(s_{i-1})| \\ &< \frac{\varepsilon}{2L}\sum_{i=1}^{n}|\gamma(s_i)-\gamma(s_{i-1})| \\ &\leq \frac{\varepsilon}{2}.\end{aligned}$$

Similarly, we have $\|R(\mathcal{P},\Omega)-R(\mathcal{P}_2,\Sigma_2)\| < \varepsilon/2$, meaning that

$$\|R(\mathcal{P}_2,\Sigma_2)-R(\mathcal{P}_1,\Sigma_1)\| < \varepsilon.$$

Thus, the net $R(\mathcal{P},\Sigma)$ directed by partition norm is Cauchy in X, whence it is convergent.

Thus, f is Riemann-integrable. □

Holomorphic Functional Calculus

We start our discussion of functional calculus by evaluating $f(a)$ for a holomorphic function $f(z) \in \mathcal{O}(U)$ defined on a neighborhood $\sigma(a)$. We can actually use Cauchy's Integral Formula for this purpose.

Recall that if Γ is a piecewise C^1 closed curve, and $n(\Gamma;w) \neq 0$, with $w \in \mathbb{C}\setminus\Gamma$, then

$$f(w)n(\Gamma;w) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{z-w}\,dz.$$

Now, given a neighborhood U of $\sigma(a)$, and Γ such a curve, we may define

$$f(a) = \frac{1}{2\pi i}\oint_{\Gamma} f(z)R(a,z)\,dz.$$

This is a good start, but we can go much further in the special case of C^*-algebras, which we will do in the following section.

Characters and Continuous Functional Calculus for C^*-Algebras

Definition: If A is a Banach algebra, then a character is a nonzero (unital if necessary) algebra homomorphism $\varphi\colon A \to \mathbb{C}$.

Proposition: If A is a Banach algebra, and φ is a character, then φ is continuous with $\|\varphi\| \leq 1$. Furthermore, if A is unital, then $\|\varphi\| = 1$.

Proof. Suppose toward contradiction that $\|\varphi\| > 1$. Then, there is $x \in A$ with $\|x\| < |\varphi(x)|$. Set $y = x/\varphi(x)$, and $z = \sum_{n=1}^{\infty} y^n$, which converges as $\|y\| < 1$. Note that $z = y + yz$, so by applying φ, we get

$$\begin{aligned}\varphi(z) &= \varphi(y) + \varphi(y)\varphi(z) \\ &= 1 + \varphi(z).\end{aligned}$$

This yields a contradiction, so $\|\varphi\| \leq 1$. □

Theorem: Let A be a unital commutative Banach algebra. There is a bijective correspondence between the set of characters on A and the set of maximal ideals in A, given by $\varphi \leftrightarrow \ker(\varphi)$.

Proof. We know that if φ is a character, then $M = \ker(\varphi)$ is a closed ideal with codimension 1, so it is maximal.

Meanwhile, suppose M is a maximal ideal of A. Since M is maximal, $M = \overline{M}$, and $\overline{M} \cap U(e,1) = \emptyset$ as $U(e,1) \subseteq \mathrm{GL}(A)$. In particular, A/M is a unital commutative Banach algebra that is necessarily a field, hence equal to $\mathbb{C}$. Thus, $\varphi\colon A \to A/M \cong \mathbb{C}$ is a character with $\ker(\varphi) = M$, and since $\varphi(e) = 1$, it follows that $A/M = \mathbb{C}\overline{e}$. □

Definition: If A is a unital commutative Banach algebra, the character space (or maximal ideal space) of A, denoted $\hat{A}$, is the subspace of the unit ball B_{A^*} endowed with the weak* topology.

The *Gelfand transform* is the map $\Gamma\colon A \to C(\hat{A})$, $a \mapsto \hat{a}$, where

$$\hat{a}(\varphi) = \varphi(a).$$

Theorem: Let A be a unital commutative Banach algebra. Then, $\hat{A}$ is nonempty and compact, with Γ a contractive algebra homomorphism. The subalgebra $\Gamma(A) \subseteq C(\hat{A})$ separates points of $\hat{A}$.

Proof. Since there is a maximal ideal in A, there is a corresponding character.

Now, we show that $\hat{A}$ is w^*-closed. Let $(\varphi_i)_{i\in I} \xrightarrow{w^*} \varphi \in A^*$. Then, for any $a, b \in A$, we have

$$\begin{aligned}\varphi(ab) &= \lim_{i\in I} \varphi_i(ab)\\ &= \lim_{i\in I} \varphi_i(a)\varphi_i(b)\\ &= \varphi(a)\varphi(b),\end{aligned}$$

meaning that φ is indeed a character. Since the unit ball of A^* is w^*-compact by the Banach–Alaoglu theorem, it follows that $\hat{A}$ is a compact Hausdorff space.

We know that each $\hat{a}$ is a continuous function. Since every $\varphi \in \hat{A}$ has norm 1, it follows that Γ is necessarily contractive. Furthermore, since

$$\begin{aligned}\widehat{ab}(\varphi) &= \varphi(ab)\\ &= \varphi(a)\varphi(b)\\ &= \hat{a}(\varphi)\hat{b}(\varphi),\end{aligned}$$

it follows that Γ is an algebra homomorphism.

Finally, since A separates the points of A^*, it follows that $\Gamma(A)$ separates the points of $\hat{A}$. □

Now, we may specialize to the case of C^*-algebras.

Proposition: Let A be a non-unital C^*-algebra. Then, A embeds into a unital C^*-algebra $\widetilde{A}$ as a maximal ideal with codimension 1.

Proof. Adjoin a symbol 1 by taking

$$\widetilde{A} = A + \mathbb{C}1,$$

and define the norm by

$$\|a + \lambda 1\| = \|L_a + \lambda I\|_{\mathrm{op}},$$

where $L_a\colon A \to A$ is given by left-multiplication. Observe that for any $a \in A$, we have

$$\begin{aligned}\|L_a\| &\leq \|a\|\\ &= \frac{\|aa^*\|}{\|a^*\|}\end{aligned}$$

$$\begin{aligned} &= \left\|L_a \frac{a^*}{\|a^*\|}\right\| \\ &\leq \|L_a\|, \end{aligned}$$

whence $\|L_a\| = \|a\|$. Now, to verify that this is in fact a C^*-norm on $\widetilde{A}$, we have

$$\begin{aligned} \|a+\lambda 1\|^2 &= \sup_{\|b\|\leq 1} \|ab+\lambda b\|^2 \\ &= \sup_{\|b\|\leq 1} \left\|(ab+\lambda b)^*(ab+\lambda b)\right\| \\ &= \sup_{\|b\|\leq 1} \left\|b^*a^*ab + \lambda b^*a^*b + \overline{\lambda} b^*ab + |\lambda|^2 b^*b\right\| \\ &= \sup_{\|b\|\leq 1} \left\|b^*\left(a^*a + \lambda a^* + \overline{\lambda} a + |\lambda|^2 1\right)b\right\| \\ &\leq \left\|a^*a + \lambda a^* + \overline{\lambda} a + |\lambda|^2 1\right\| \\ &= \left\|(a+\lambda 1)^*(a+\lambda 1)\right\| \\ &\leq \left\|(a+\lambda 1)^*\right\|\|a+\lambda 1\|. \end{aligned}$$

Thus, $\|a+\lambda 1\| \leq \left\|(a+\lambda 1)^*\right\|$. Exchanging a and a^* gives $\|a+\lambda 1\| = \left\|(a+\lambda 1)^*\right\|$.

Therefore, we have

$$\begin{aligned} \|a+\lambda 1\|^2 &\leq \left\|(a+\lambda 1)^*(a+\lambda 1)\right\| \\ &\leq \|a+\lambda 1\|^2, \end{aligned}$$

which forces equality. Thus, $\widetilde{A}$ is a C^*-algebra. $\square$

Theorem: Let A be a commutative C^*-algebra. Then, the Gelfand map $\Gamma\colon A \to C_0(\hat{A})$ is an isometric $*$-isomorphism.

Proof. Assume A is unital. Let $a = a^* \in A$, and $\varphi \in \hat{A}$. We claim that $\varphi(a)$ is real. Toward this end, take

$$\begin{aligned} u_t &= \exp(ita) \\ &= \sum_{n=0}^{\infty} \frac{(ita)^n}{n!}. \end{aligned}$$

Then, $u_t^{-1} = u_{-t}$, and so

$$\begin{aligned} u_t^* &= \sum_{n=0}^{\infty} \frac{((ita)^n)^*}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-ita)^n}{n!} \\ &= u_{-t} \\ &= u_t^{-1}, \end{aligned}$$

meaning that u_t is unitary, so $\|u_t\|^2 = 1$. We thus find

$$\begin{aligned} 1 &\geq |\varphi(u_t)| \\ &= \left|\sum_{n=0}^{\infty} \frac{\varphi(ita)^n}{n!}\right| \end{aligned}$$

$$= \left|e^{i\varphi(a)t}\right|,$$

so that $\varphi(a) \in \mathbb{R}$.

Now, for any $a \in A$, we may take $x = \frac{1}{2}(a + a^*)$ and $y = \frac{1}{2i}(a - a^*)$, so that $x = x^*$, $y = y^*$, and $a = x + iy$. It follows that

$$\begin{aligned}\varphi(a^*) &= \varphi(x - iy)\\ &= \varphi(x) - i\varphi(y)\\ &= \overline{\varphi(a)},\end{aligned}$$

so that $\Gamma(a^*) = \Gamma(a)^*$, meaning Γ is a $*$-homomorphism.

Now, if $a \in A_{\text{s.a.}}$, then $\|a\|^2 = \left\|a^2\right\|$, whence

$$\begin{aligned}\|\hat{a}\| &= r(a)\\ &= \lim_{n\to\infty}\left\|a^{2^n}\right\|^{2^{-n}}\\ &= \|a\|.\end{aligned}$$

Meanwhile, for general $a \in A$, we have

$$\begin{aligned}\|a\|^2 &= \|a^*a\|\\ &= \left\|\widehat{a^*a}\right\|\\ &= \|\hat{a}\|^2.\end{aligned}$$

Thus, Γ is isometric. Since $\Gamma(A)$ is a norm-closed unital self-adjoint subalgebra of $C(\hat{A})$ that separates points, it follows that $\Gamma(A) = C(\hat{A})$.

In the non-unital case, then we let $\widetilde{A}$ be the unitization of A. Then, $\Gamma(\widetilde{A}) = C(\widehat{\widetilde{A}})$, and $\hat{A} = \widehat{\widetilde{A}}\backslash\varphi_\infty$, where $\varphi_\infty(a + \lambda 1) = \lambda$. The restriction of Γ to A is the Gelfand map of A, and it is mapped into $\ker(\varphi_\infty) = C_0(\hat{A})$, which is an isometric $*$-isomorphism which separates points and does not vanish except at φ_∞. Thus, it is surjective. ☐

Since normal elements generate a commutative C^*-algebra, we get an immediate corollary.

Corollary: Let a be a normal element. Then, $C^*(a)$ is $*$-isomorphic to $C(\sigma(a))$ via $\Gamma(a) = \iota$, where $\iota\colon \sigma(a) \to \sigma(a)$ is given by $\iota(z) = z$. The not necessarily unital subalgebra $C^*(a, a^*)$ is isomorphic to $C_0(\sigma(a) \setminus \{0\})$.

Proof. The non-closed $*$-algebra generated by $1, a, a^*$ consists of all polynomials in a and a^* for any $p \in \mathbb{C}[x, y]$, with $C^*(a)$ the norm closure. Since $C^*(a)$ is a unital C^*-algebra, it is isomorphic to $C(\widehat{C^*(a)})$. Since characters in $\widehat{C^*(a)}$ are in bijection with $\sigma(a)$ via evaluation, so that $C^*(a) \cong C(\sigma(a))$.

Meanwhile, the subalgebra generated by a, a^* is isomorphic to the subalgebra of $C(\sigma(a))$ that is generated by both z and $\overline{z}$. Since this is a self-adjoint subalgebra that separates points and vanishes at zero, it is either equal to the ideal of functions vanishing at 0 if $0 \in \sigma(a)$ or all of $C(\sigma(a))$ if not. ☐

Theorem (Continuous Functional Calculus)**:** Let a be a normal element of a unital C^*-algebra A. Then:

(i) there is an isometric $*$-isomorphism from $C(\sigma(a))$ onto $C^*(a)$ such that $\iota(a) = a$;

(ii) $\sigma(f(a)) = f(\sigma(a))$ for $f \in C(\sigma(a))$;

(iii) if g is continuous on $f(\sigma(a))$, then $g(f(a)) = (g \circ f)(a)$.

Proof.

(i) By the above corollary, $C^*(a)$ is a unital commutative C^*-algebra, where the Gelfand map is an isomorphism onto $C(\sigma(a))$ with $\Gamma(a) = \iota$. We may then define $f(a) = \Gamma^{-1}(f)$.

(ii) We show that Γ preserves spectra by showing that a is invertible if and only if $\hat{a}$ is invertible in $C(\hat{A})$. We observe that, since Γ is an algebra homomorphism, we have $\hat{a}^{-1} = \Gamma(a^{-1})$. Conversely, if a is not invertible, then $\langle a \rangle$ is a proper ideal of A, and thus is contained in a maximal ideal M. If φ is the character that has $\ker(\varphi) = M$, then $\varphi(a) = 0$, so $\hat{a}$ has a zero on $\hat{A}$, meaning $\hat{a}$ is not invertible in $C(\hat{A})$. In particular, spectral mapping follows from the fact that Γ^{-1} also preserves spectrum.

(iii) Note that $f(a)$ is normal, so $g(f(a))$ makes sense via the functional calculus for $f(a)$. Since $g \circ f \in C(\sigma(a))$, it follows that $(g \circ f)(a)$ is defined.

If $g(z) = z^j \overline{z}^k$, we have $g \circ f(z) = f(z)^j \overline{f(z)}^k$, meaning

$$\begin{aligned} g(f(a)) &= f(a)^j (f(a)^*)^k \\ &= (g \circ f)(a). \end{aligned}$$

By linearity, this extends to polynomials $p \in \mathbb{C}[z, \overline{z}]$. By Stone–Weierstrass and continuity, it follows for all $g \in C(\sigma(f(a)))$.

□

Corollary: If a is a normal element of a non-unital C^*-algebra A, then $0 \in \sigma(a)$ and there is an isometric $*$-isomorphism from $C_0(\sigma(a) \setminus \{0\})$ onto $C^*(a, a^*)$ such that $\iota(a) = a$.

Proof. Since $C^*(a, a^*)$ is abelian and non-unital, with a not invertible, we have $0 \in \sigma(a)$ and Γ carries $C^*(a, a^*)$ onto $C_0(\sigma(a) \setminus \{0\})$. □

One of the most important properties of the continuous functional calculus is that it commutes with homomorphisms.

Corollary: Let $\pi\colon A \to B$ be a continuous $*$-homomorphism of a C^*-algebra A into a C^*-algebra B. Let a be a normal element of A, and let $f \in C(\sigma(a))$ in the unital case and $f \in C_0(\sigma(a) \setminus \{0\})$ in the non-unital case. Then, $\pi(f(a)) = f(\pi(a))$.

Proof. Let $b = \pi(a)$. Then, b is normal, so for any polynomial $p \in \mathbb{C}[z, \overline{z}]$, we have $\pi(p(a, a^*)) = p(b, b^*)$. In the non-unital case, we may use polynomials with constant term 0. These polynomials are dense in their respective spaces $C(\sigma(a))$ or $C_0(\sigma(a) \setminus \{0\})$. Continuity gives that this map extends to $\pi(f(a)) = f(b)$. □

Using the Banach–Stone theorem, this allows us to show some very powerful results regarding the automatic continuity of $*$-homomorphisms of C^*-algebras.

Proposition: Let $\varphi\colon A \to B$ be a $*$-homomorphism of C^*-algebras. Then, φ is contractive. Furthermore, φ is isometric if and only if it is injective.

Proof. We may assume without loss of generality that A and B are unital, since unitization preserves norms. If $x \in A$ is normal, then so too is $\varphi(x)$, so since $\sigma(\varphi(x)) \subseteq \sigma(x)$, we thus have that

$$\begin{aligned} \|\varphi(x)\| &= r(\varphi(x)) \\ &\leq r(x) \\ &= \|x\|. \end{aligned}$$

If a is arbitrary, then $a^* a$ is normal, as is $\varphi(a^* a)$, so by the C^*-identity,

$$\begin{aligned} \|\varphi(a)\|^2 &= \|\varphi(a^* a)\| \\ &\leq \|a^* a\| \\ &= \|a\|^2. \end{aligned}$$

If φ is injective, then since the Gelfand transform commutes with $*$-homomorphisms, we have a corresponding map $\pi\colon C(\hat{A}) \to C(\hat{B})$. □

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