Here, we overview and discuss some of the most important results related to projections in von Neumann algebras.

Comparison of Projections

Recall that if H is a Hilbert space, an element $w \in B(H)$ is called a partial isometry if, for any $h \in \ker(w)^{\perp}$, we have $\|Wh\| = \|h\|$. We call $\ker(w)^{\perp}$ the initial space of W and $\operatorname{im}(w)$ the final space of W.

There are a variety of equivalent definitions for partial isometries.

Proposition: If $w \in B(H)$, then the following are equivalent:

(i) w is a partial isometry;

(ii) w^* is a partial isometry;

(iii) w^*w is a projection onto the initial space of w;

(iv) ww^* is a projection onto the final space of w;

(v) $ww^*w = w$;

(vi) $w^*ww^* = w^*$.

Theorem (Polar Decomposition)**:** Let $a \in B(H)$. Then, there is a partial isometry $w \in B(H)$ with initial space $\ker(a)^{\perp}$ and final space $\overline{\operatorname{im}(a)}$ such that $a = w|a|$.

If $a \in M \subseteq B(H)$, where M is a von Neumann algebra, then both $|a|$ and w are in M.

Equivalence of Projections

If $M \subseteq B(H)$ is a von Neumann algebra, then we say two projections $p, q \in P(M)$, where $P(M)$ denotes the space of projections of M, are (Murray–von Neumann) *equivalent* in M if there is a partial isometry $v \in P(M)$ such that $v^*v = p$ and $vv^* = q$. We will write $p \sim q$.

Note that projections have an ordering by saying that $p \leq q$ if $pq = qp = p$, or $\operatorname{im}(p) \subseteq \operatorname{im}(q)$. This allows us to say that p is *sub-equivalent* to q (in M), written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$.[I]

The sub-equivalence relation in fact forms a partial order, and equivalence as projections forms an equivalence relation. We will first show that it is a preorder.

Proposition: In a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\preceq$ is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry follows from the fact that if v is a partial isometry, then so is v^*.

Now, we will show transitivity for $\preceq$, from which we will see that $\sim$ is transitive. Letting $p, q, r \in P(M)$ be such that $p \preceq q$ and $q \preceq r$, we have partial isometries $u, v \in M$ with

[I] We will say that the projection q majorizes p if $p \preceq q$, and we will say that q dominates p if $p \leq q$.

$u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. Then, we have

$$\begin{aligned} qu &= quu^*u \\ &= (quu^*)u \\ &= uu^*u \\ &= u, \end{aligned}$$

so that

$$\begin{aligned} (vu)^*(vu) &= u^*v^*vu \\ &= u^*qu \\ &= u^*u \\ &= p \\ (vu)(vu)^* &= vuu^*v^* \\ &\leq vqv^* \\ &= vv^*vv^* \\ &= vv^* \\ &\leq r. \end{aligned}$$

Therefore, $p \preceq r$, so $\preceq$ is a transitive relation. □

To see that $\preceq$ is a partial order, we need an analogue of the Cantor–Schröder–Bernstein theorem for projections. In fact, it can be proven in a similar manner. First, we discuss a simple lemma.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ are pairwise orthogonal families of projections with $p_i \preceq q_i$, then $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$.

Proof. Let u_i be the partial isometries with $u_i^*u_i = p_i$ and $r_i \coloneqq u_iu_i^* \leq q_i$. Then, the r_i are pairwise orthogonal since the q_i are pairwise orthogonal, and for any $i \neq j$,

$$\begin{aligned} u_i^*u_j &= u_i^*u_iu_i^*u_ju_j^*u_j \\ &= u_ir_ir_ju_j \\ &= 0 \\ u_iu_j^* &= u_iu_i^*u_iu_j^*u_ju_j^* \\ &= u_ip_ip_ju_j^* \\ &= 0. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \left(\sum_{i\in I} u_i^*\right)\left(\sum_{j\in I} u_j\right) &= \sum_{i\in I} u_i^*u_i \\ &= \sum_{i\in I} p_i \end{aligned}$$

$$
\begin{aligned}
\left(\sum_{i\in I} u_i\right)\left(\sum_{j\in I} u_j^*\right) &= \sum_{i\in I} u_iu_i^* \\
&\leq \sum_{i\in I} q_i.
\end{aligned}
$$

This gives $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$. □

Theorem: If $e \preceq f$ and $f \preceq e$, then $e \sim f$.

Proof. We will let $e_0 \coloneqq e$ and $f_0 \coloneqq f$. Let v and w be partial isometries with $v^*v = e$, $vv^* = f_1 \leq f$, $w^*w = f$, $ww^* = e_1 \leq e$. Inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of some projection dominated by f_1, it follows that we may write $f_2 \coloneqq ve_1(ve_1)^*$ with $f_2 \leq f_1$. Since w maps the range of f_1 onto the range of some projection dominated by e_1, it follows that we may write $wf_1(wf_1)^* \eqqcolon e_2$. Observe also that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

Inductively, we obtain decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1}, and w maps the range of f_n isometrically onto that of e_{n+1}. Defining $e_\infty \coloneqq \inf_n e_n$ and $f_\infty = \inf_n f_n$, we have that v maps the range of e_∞ onto that of f_∞, and w that of f_∞ onto the range of e_∞. Note that we have $e_\infty \sim f_\infty$.

As discussed earlier, we have that $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$, so since sums of pairwise orthogonal families of projections respects equivalence, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) &\sim \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2}) \\
\sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) &\sim \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
e &= e_\infty + \sum_{n=0}^{\infty}(e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty}(e_{2n+1} - e_{2n+2}) \\
&\sim f_\infty + \sum_{n=0}^{\infty}(f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty}(f_{2n} - f_{2n+1}) \\
&= f.
\end{aligned}
$$

□

Central Projections and the Comparison Theorem

The projections in a von Neumann algebra form a complete lattice, as the collection of closed subspaces of H form a complete lattice under the operations

$$\bigvee_{i\in I} X_i := \overline{\sum_{i\in I} X_i}$$
$$\bigwedge_{i\in I} X_i := \bigcap_{i\in I} X_i.$$

If $S \subseteq H$ is any subset, then we will define the range projection of S by

$$[S] := P_{\overline{\operatorname{span}}(S)}.$$

Proposition: If $M \subseteq B(H)$ is a von Neumann algebra, and $x \in M$, then $[xH]$ and $[x^*H]$ are in M, with $[xH] \sim [x^*H]$ in M.

Proof. Let $x = v|x|$ be the polar decomposition. Note that $v \in M$. Now, vv^* is the projection onto $\overline{xH}$ and v^*v is the projection onto $\ker(x)^\perp = \overline{x^*H}$. Thus, these projections are equivalent in M. □

Definition: Let $x \in M$. We define the *central support* to be the projection

$$z(x) = \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say p and q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $M \subseteq B(H)$ is a von Neumann algebra, then the central support of any $p \in P(M)$ is given by

$$z(p) = [MpH].$$

Let $w = [MpH]$. Since M is unital, it follows that $p \leq w$, and since $\overline{MpH}$ is a reducing subspace for both M and M', we have $w \in M \cap M'$, so $z(p) \leq w$.

Conversely, if $x \in M$, then

$$\begin{aligned} xpH &= xz(p)pH \\ &= z(p)xpH, \end{aligned}$$

meaning that $[xpH] \leq z(p)$, so $w \leq z(p)$ as x was arbitrary.

Proposition: Let M be a von Neumann algebra, and let $p, q \in P(M)$ be projections. The following are equivalent:

(i) p and q are centrally orthogonal;

(ii) $pMq = \{0\}$;

(iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ with $p_0 \sim q_0$.

Proof. Let p and q be centrally orthogonal. Then, for any $x \in M$, we have

$$pxq = pz(p)xz(q)q$$

$$\begin{aligned} &= pxz(p)z(q)q \\ &= 0. \end{aligned}$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, so $p \leq 1 - z(q)$. Since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, meaning that $z(p)z(q) = 0$. Therefore, (i) and (ii) are equivalent.

Suppose (ii) is not the case. Let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$. Defining

$$\begin{aligned} p_0 &= [pxqH] \\ q_0 &= [qx^*pH], \end{aligned}$$

we have that $p_0 \leq p$, $q_0 \leq q$, and since $(pxq)^* = qx^*p$, we have $p_0 \sim q_0$.

Now, if there are nonzero projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim q_0$, then if v is a partial isometry with $v^*v = p_0$, $vv^* = q_0$, then $v^* = p_0 v^* q_0$, meaning

$$\begin{aligned} pv^*q &= pp_0v^*q_0q \\ &= p_0v^*q_0 \\ &= v^* \\ &\neq 0, \end{aligned}$$

meaning $pMq \neq \{0\}$. □

Theorem (Comparison Theorem)**:** Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there is a central projection $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ of pairwise orthogonal projections with $p_i \sim q_i$ and, setting

$$\begin{aligned} p_0 &= \sum_{i\in I} p_i \\ q_0 &= \sum_{i\in I} q_i, \end{aligned}$$

we have $p_0 \preceq q_0$. From above, we have that $p_0 \sim q_0$.

Let $w := z(q - q_0)$. Since $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ are maximal, it follows that $z(q - q_0)$ and $z(p - p_0)$ are centrally orthogonal, yielding $(p - p_0)w = 0$, meaning $pw = p_0w$.

If we let v be a partial isometry implementing the equivalence $p_0 \sim q_0$, then we have that vw is a partial isometry implementing the equivalence $p_0w \sim q_0w$. Therefore, we have

$$\begin{aligned} pw &= p_0w \\ &\sim q_0w \\ &\leq q. \end{aligned}$$

Similarly, $p_0(1-w) \sim q_0(1-w)$, so since $q - q_0 \leq w$, we have $q(1-w) \preceq p(1-w)$. □

Recall that a factor is a von Neumann algebra M such that $Z(M) = \mathbb{C}1$.

Corollary: If M is a factor, then any two projections in M can be compared.

The Type Decomposition

Definition: Let M be a von Neumann algebra, and $p \in B(H)$ a projection not necessarily in M. The algebra pMp is known as a corner (or compression) of M.

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra, and let $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, and $(pMp)' = M'p$, $(M'p)' = pMp$.

Corollary: If M is a factor and $p \in P(M)$, then pMp and $M'p$ are both factors.

Definition: Let $M \subseteq B(H)$ be a von Neumann algebra. We say a projection $p \in P(M)$ is

- minimal if $p \neq 0$ and the only subprojections of p are 0 and p;
- abelian if pMp is abelian;
- finite if $q \leq p$ and $q \sim p$ implies $q = p$;
- semifinite if there are pairwise orthogonal finite projections $p_i \in P(M)$ such that $p = \sum_{i\in I} p_i$;
- purely infinite if $p \neq 0$ and there is no nonzero finite projection $q \leq p$;
- properly infinite if $p \neq 0$ and zp is not finite for any nonzero central projection $z \in Z(P(M))$.

We say that the von Neumann algebra M is finite/semifinite/purely infinite/properly infinite if the projection 1 satisfies its respective condition. Additionally, if M has no minimal projections, we say it is diffuse.

What we will be working towards is known as the type decomposition, which forms the basis for the dimension theory of von Neumann algebras. Eventually, we will show that every von Neumann algebra M can be decomposed as

$$M = M_{\text{sf}} \oplus M_{\text{III}},$$

where M_{sf} is a semifinite von Neumann algebra and M_{III} is a type III von Neumann algebra, a definition we will discuss shortly. First, we must expand on some of the ways that comparison of projections interacts with these properties of those projections.

Lemma: Let $\{p_i\}_{i\in I}$ be a family of centrally orthogonal projections in a von Neumann algebra $M \subseteq B(H)$. If each p_i is abelian (finite), then the sum $\sum_{i\in I} p_i$ is also abelian (finite).

Proof. If each p_i is abelian, then since they are centrally orthogonal for any $i \neq j$, then for any $x, y \in M$, we have $p_i x p y p_j = 0$. Therefore, we have

$$\begin{aligned}(pxp)(pyp) &= \sum_{i\in I} p_i x p_i y p_i \\ &= (pyp)(pxp),\end{aligned}$$

so p is abelian.

Now, if each p_i is finite, and $u \in M$ is such that $uu^* \leq u^*u = p$, then for all i we have $z(p_i)u^*uz(p_i) = p$, and $uz(p_i)u^* = z(p_i)uu^* \leq p_i$, meaning $uz(p_i)u^* = p_i$, and

$$\begin{aligned} uu^* &= uz(p)u^* \\ &= \sum_{i\in I} uz(p_i)u^* \\ &= p. \end{aligned}$$

□

Proposition: Let p, q be nonzero projections in a von Neumann algebra with $p \preceq q$. If q is finite (purely infinite), then p is also finite (purely infinite).

Proof. Suppose q is finite, and $p \sim q$, with $v \in M$ implementing the equivalence. Let $u \in M$ be such that $u^*u = p$ and $uu^* \leq p$. Then,

$$\begin{aligned} (vuv^*)^*(vuv^*) &= q \\ (vuv^*)(vuv^*)^* &\leq q, \end{aligned}$$

so since q is finite, $(vuv^*)(vuv^*)^* = q$, so $uu^* = p$.

Now, if $p \leq q$, then if $u^*u = p$ with $uu^* \leq p$, then by setting $w = u + (q - p)$, we have that $w^*w = q$ and $ww^* \leq q$, so $u^*u + (q-p) = ww^* = q$, meaning $uu^* = p$.

In the general case, we have some $q_0 \leq q$ such that $p \sim q_0 \leq q$.

Since projections are purely infinite when they have no nonzero finite subprojections, the purely infinite case follows from the finite case. □

Proposition: A projection $p \in P(M)$ is semifinite if and only if p is the supremum of finite projections. In particular, a supremum of semifinite projections is also semifinite.

Proof. If p is semifinite, then p is the sum (and hence the supremum) of a family of pairwise orthogonal finite projections.

Now, suppose $p = \bigvee_{\alpha} p_\alpha$, where each p_α is finite. Let $\{q_\beta\}_\beta$ be a maximal family of pairwise orthogonal finite subprojections of p. If we set

$$q_0 = p - \sum_{\beta} q_\beta,$$

and suppose that $q_0 \neq 0$, then there exists some p_α such that p_α and q_0 are not orthogonal (else it would contradict maximality), so they are not centrally orthogonal. Therefore, we have a nonzero subprojection $q_1 \leq q_0$ such that $q_1 \preceq p_\alpha$, so it is finite by what we showed previously; this contradicts the maximality of the set $\{q_\beta\}_\beta$. □

Corollary: Let p be a projection in a von Neumann algebra M. If p is semifinite (purely infinite), then the central support $z(p)$ is also semifinite (purely infinite).

Proof. The central support is the supremum over all equivalent projections, so since the supremum of semifinite projections is again semifinite, it follows from the previous proposition.

Furthermore, a nonzero projection is purely infinite if and only if it is centrally orthogonal to every semifinite projection, so we obtain the corollary in this case. □

Corollary: Let p, q be nonzero projections in a von Neumann algebra such that $p \preceq q$. If q is semifinite, then so is p.

Proof. It is enough to consider the case when q is central, in which case we may take $p \leq q$. Let p_0 be the maximal semifinite subprojection of p. Since q is semifinite, it is the supremum of its finite subprojections. Since $z(p - p_0) \leq q = z(q)$, it follows that if $p - p_0 \neq 0$, then there would exist a nonzero finite subprojection that would be equivalent to a subprojection of $p - p_0$, which contradicts the definition of p_0. Thus, p is the supremum of its finite subprojections, so it is semifinite. □

Lemma: Let M be a properly infinite von Neumann algebra. Then, there exists a projection $p \in P(M)$ such that $p \sim 1 - p \sim 1$.

Proof. Since M is properly infinite, there exists $u \in M$ with $uu^* \leq u^*u = 1$. Set $p_0 = 1 - uu^*$. Then, $p_n = u^n p_0 (u^n)^*$ is a pairwise orthogonal family of equivalent projections. Let $\{q_i\}_{i\in I}$ be a maximal family of pairwise orthogonal equivalent projections in M extending $\{p_n\}_{n\in\mathbb{N}}$, and set $q_0 = 1 - \sum_{i\in I} q_i$.

By the comparison theorem, there is $z \in Z(P(M))$ such that $q_0 z \leq q_{i_0} z$ and $q_{i_0}(1 - z) \leq q_0(1 - z)$. If it were the case that $z = 0$, then we would have $q_{i_0} \leq q_0$, contradicting maximality of $\{q_i\}_{i\in I}$, so $z \neq 0$, and we have

$$\begin{aligned} z &= q_0 z + \sum_{i\in I} q_i z \\ &\preceq q_{i_0} z + \sum_{i\neq i_0} q_i z \\ &= \sum_{i\in I} q_i z \leq z, \end{aligned}$$

so $z \sim \sum_{i\in i} q_i z$ by Cantor–Schröder–Bernstein for projections. Decomposing $\{q_i\}_{i\in I}$ into infinite subsets, we may construct two projections p and $z - p$ such that $p \sim z - p \sim z$.

Now, let $\{r_j\}_{j\in J}$ be a maximal family of centrally orthogonal projections with $r_j \sim z(r_j) - r_j \sim z(r_j)$. Then, the argument above shows that $\sum_{j\in J} z(r_j) = 1$, so by setting $p = \sum_{j\in J} r_j$, we obtain our desired result. □

Proposition: Let p and q be finite projections in a von Neumann algebra M. Then, $p \vee q$ is finite.

Proof. We use Kaplansky's formula for this, which gives $p \vee q - p \sim q - p \wedge q$. This follows from observing that $x = (1 - p)q$ has $\ker(x) = \ker(q) \oplus (qH \cap pH)$, meaning that $[x^* H] = 1 - ((1 - q) + q \wedge p) = q - q \wedge p$. Symmetrically, this gives

$$\begin{aligned} [xH] &= (1 - p) - (1 - p) \wedge (1 - q) \\ &= p \vee q - p. \end{aligned}$$

In particular, since $q - p \wedge q \leq q$, we may assume that p and q are orthogonal, replacing q with $p \vee q - p$. We may also assume that $p + q = 1$ by passing to $(p+q)M(p+q)$.

Let z_0 be the supremum of all finite central projections. It follows that z_0 is finite. If $z_0 = 1$, then we are done. Else, we may use $(1 - z_0)p$ and $(1 - z_0 q)$, wherein $z_0 = 0$ and thus we assume that M is properly infinite.

Therefore, we have a projection $r \in P(M)$ such that $r \sim 1 - r \sim 1$. By comparison, there is $z \in P(Z(M))$ such that

$$\begin{aligned} z(p \wedge r) &\preceq z(q \wedge (1-r)) \\ (1-z)(q \wedge (1-r)) &\preceq (1-z)(p \wedge r). \end{aligned}$$

Additionally, $zr \sim z(1-r) \sim z$, and

$$\begin{aligned} z(p \wedge r) &= zp \wedge zr \\ &\preceq z(1-r) \wedge zq, \end{aligned}$$

so by using Kaplansky's formula, we have

$$\begin{aligned} zr &= z(r - r \wedge p) + z(r \wedge p) \\ &\preceq z(r \vee p - p) + z(q \wedge (1-r)) \\ &= zq, \end{aligned}$$

Therefore, $zr = 0$ since zq is finite and M is properly infinite, so $z \sim 0$. In particular, this gives $q \wedge (1-r) \preceq p \wedge r$. Replacing p with q and r with $1 - r$, we get

$$\begin{aligned} 1 - r &= (1 - r - (1-r) \wedge q) + ((1-r) \wedge q) \\ &\preceq ((1-r) \vee q - q) + (p \wedge r) \\ &= p, \end{aligned}$$

which gives a contradiction since p is finite. □

Proposition: Let p and q be finite projections with $p \sim q$. Then, $1 - p$ and $1 - q$ are equivalent.

Proof. We have that $p \vee q$ is finite, so we may assume that M is finite. By the comparison theorem, there are projections p_1 and q_1 and a central projection $z \in P(Z(M))$ such that

$$\begin{aligned} (1-p)z \sim q_1 &\leq (1-q)z \\ (1-q)(1-z) \sim p_1 &\leq (1-p)(1-z). \end{aligned}$$

Then, we have

$$\begin{aligned} z &= (1-p)z + pz \\ &\sim q_1 + qz \\ &\leq (1-q)z + qz \end{aligned}$$

$$= z$$

and

$$\begin{aligned}(1-z) &= (1-q)(1-z) + q(1-z)\\ &\sim p_1 + p(1-z)\\ &\leq (1-z),\end{aligned}$$

so since both z and $(1-z)$ are finite, we have $q_1 = (1-q)z$ and $p_1 = (1-p)(1-z)$. Thus, $1-q \sim 1-p$. □

Definition: Let M be a von Neumann algebra.

- We say M is type I if every nonzero central projection in M majorizes a nonzero abelian projection in M.
- We say M is type II_1 if it is finite, has no nonzero abelian projections, and every nonzero central projection in M majorizes a nonzero finite projection.
- We say M is type II_∞ if every nonzero central projection majorizes a nonzero finite projection, and has no nonzero finite central projections.
- We say M is type III if it is purely infinite.

Theorem: Every von Neumann algebra M uniquely decomposes into a direct sum of those of type I, II_1, II_∞, and III. Moreover, very projection e in M can be uniquely written as the sum of centrally orthogonal projections e_1 and e_2 in M such that e_1 is finite and e_2 is properly infinite.

Proof. Let $\{e_i\}_{i\in I}$ be a maximal family of centrally orthogonal abelian projections in M, and let $e = \sum_{i\in I} e_i$. Then, e is abelian; define $z_{\text{I}} = z(e)$.

If z is a nonzero central projection majorized by z_{I}, then ze is a nonzero abelian projection, so Mz_{I} is of type I.

By construction, there is then no nonzero abelian projection in $M(1-z_{\text{I}})$, meaning that $M(1-z_{\text{I}})$ has no nontrivial direct summand of type I.

Let $\{f_j\}_{j\in J}$ be a maximal family of centrally orthogonal finite projections in $M(1-z_{\text{I}})$, and let $f = \sum_{j\in J} f_j$. Then, f is finite. Set $z_{\text{II}} = z(f)$. By construction, we have that Mz_{II} has no nonzero abelian projections, and every nonzero projection z in Mz_{II} majorizes a nonzero finite projection zf. Thus, Mz_{II} is of type II.

By the maximality of $\{f_j\}_{j\in J}$, it follows that $1 - z_{\text{II}} = z_{\text{III}}$ does not majorize any finite projection, so Mz_{III} is of type III. We have that $z_{\text{I}} + z_{\text{II}} + z_{\text{III}} = 1$.

Now, let $\{z_k\}_{k\in K}$ be a maximal orthogonal family of finite central projections in Mz_{II}, and set $z_{\text{II}_1} = \sum_{k\in K} z_k$, and $z_{\text{II}_\infty} = z_{\text{II}} - z_{\text{II}_1}$. It follows that Mz_{II_1} is of type II_1 and Mz_{II_∞} is of type II_∞. Therefore, we get the direct sum decomposition

$$M = Mz_{\text{I}} \oplus Mz_{\text{II}_1} \oplus Mz_{\text{II}_\infty} \oplus Mz_{\text{III}}.$$

As for uniqueness, we suppose that there is another orthogonal decomposition $1 = w_{\text{I}} +$

$w_{\mathrm{II}_1} + w_{\mathrm{II}_\infty} + w_{\mathrm{III}}$. Then, we must have that $w_{\mathrm{I}}(1 - z_{\mathrm{I}}) = 0$ since $1 - z_{\mathrm{I}}$ does not majorize any nonzero abelian projection, while $w_{\mathrm{I}}(1 - z_{\mathrm{I}})$ is a central projection in Mw_{I}. Therefore, $w_{\mathrm{I}} \leq z_{\mathrm{I}}$. Similarly, $z_{\mathrm{I}}(1 - w_{\mathrm{I}}) = 0$ for the same reason, meaning $z_{\mathrm{I}} \leq w_{\mathrm{I}}$, so they are equal. By similar arguments, all the other summands are equal to each other.

Finally, we let e be a nonzero projection in M. By considering $M_e \coloneqq eMe$, we may assume that $e = 1$. We let e_1 be the sum of a maximal orthogonal family of finite central projections. Then, e_1 is finite and central, with $1 - e_1$ properly infinite. The uniqueness of this decomposition has the same flavor as the arguments for $\{z_{\mathrm{I}}, \dots, z_{\mathrm{III}}\}$. $\square$

Definition: We say a factor is atomic if it contains a minimal projection, and otherwise we say it is diffuse.

Structure of Type I and II von Neumann Algebras

Now, we will discuss some structural results related to type I and II von Neumann algebras. Before we can do this, we must discuss tensor products.

Tensor Products of Hilbert Spaces and Operators

Definition: Let H and K be Hilbert spaces. There is an inner product on the algebraic tensor product $H \odot K$ given by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle,$$

whenever $\xi_1, \eta_1 \in H$ and $\xi_2, \eta_2 \in K$. The Hilbert space tensor product of H and K, denoted $H \otimes K$, is the completion of $H \odot K$ with respect to the norm induced by this inner product.

Proposition: Let H and K be Hilbert spaces with orthonormal bases $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$. Then,

(i) $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal basis for $H \otimes K$;

(ii) if $|J| = \alpha$ for some cardinal α, then $H \otimes K \cong H^{(\alpha)} \cong \bigoplus_{j \in J} H$;

(iii) if $H = L_2(X, \mu)$ for some σ-finite regular Borel measure space (X, μ), and K is separable, then $H \otimes K \cong L_2(X, \mu, K)$, where the latter denotes the space of square-integrable Borel functions with respect to the norm on K.

Proof.

(i) We observe that the set $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal set. The spans of these elementary tensors are all the vectors of the form $x \otimes y$, provided that x is a finite linear combination of the e_i and y is a finite linear combination of the f_j. Therefore, the completion is equal to the completion of $H \otimes K$, so the set is an orthonormal basis.

(ii) We find that

$$H \otimes K \cong \bigoplus_{j \in J} H \otimes \mathbb{C} f_j$$

is an ℓ_2 direct sum of $|J|$ copies of H.

(iii) Define maps $Y_j\colon L_2(X,\mu,K) \to L_2(X,\mu)$ by taking

$$(Y_j f)(x) = \langle f(x), f_j\rangle,$$

a representative of an equivalence class modulo μ. Then, we have that

$$f \mapsto \sum_{j\in J} Y_j f$$

defines a map from $L_2(X,\mu,K) \to \bigoplus_{j\in J} H \otimes \mathbb{C} f_j$. By Tonelli's theorem and Parseval's identity, we get

$$\begin{aligned}\|Yf\|^2 &= \sum_{j\in J}\|Y_j f\|^2\\ &= \sum_{j\in J}\int |\langle f(x), f_j\rangle|^2\,d\mu\\ &= \int \sum_{j\in J}|\langle f(x), f_j\rangle|^2\,d\mu\\ &= \int \|f(x)\|^2\,d\mu\\ &= \|f\|^2.\end{aligned}$$

Therefore, Y is an isometry. The range is dense since, if $h \in L_2(\mu)$, then $f \coloneqq hf_j$ is mapped to the vector with h in position j and 0 elsewhere. Thus, Y is a unitary map from $L_2(X,\mu,K)$ onto $H \otimes K$.

□

Proposition: Let $A \in B(H)$ and $B \in B(K)$. Then, there is a unique $A \otimes B \in B(H \otimes K)$ such that $(A\otimes B)(\xi \otimes \eta) = A\xi \otimes B\eta$. Furthermore, $\|A \otimes B\| = \|A\|\|B\|$.

Proof. Let $A \otimes I_K$ be the amplification to $H \otimes K$. Fixing an orthonormal basis $\{\eta_j\}_{j\in J}$ for K, then any vector $\omega \in H \otimes K$ (from part (ii) above) can be expressed as

$$\omega = (\zeta_j \otimes \eta_j)_j$$

for a collection of $\zeta_j \in H$, and we may define

$$\begin{aligned}\|\omega\|^2 &= \sum_{j\in J}\|\zeta_j\|^2\\ &< \infty\end{aligned}$$

Therefore, we may rewrite

$$(A \otimes I)\omega = (A\zeta_j \otimes \eta_j)_j,$$

giving

$$\begin{aligned}\|(A\otimes I)\omega\|^2 &= \left\|\left(A\zeta_j\otimes\eta_j\right)_j\right\|^2\\ &= \sum_{j\in J}\|A\zeta_j\|^2\\ &\leq \|A\|^2 + \sum_{j\in J}\|\zeta_j\|^2\\ &= (\|A\|\|\omega\|)^2.\end{aligned}$$

It follows that $\|A\otimes I\| = \|A\|$. Similarly, we may define $I\otimes B$, and set $A\otimes B = (A\otimes I)(I\otimes B)$. Thus, we have

$$\begin{aligned}\|A\otimes B\|(\xi\otimes\eta) &= (A\otimes I)(I\otimes B)(\xi\otimes\eta)\\ &= (A\otimes I)(\xi\otimes B\eta)\\ &= A\xi\otimes B\eta.\end{aligned}$$

We have that $\|A\otimes B\| \leq \|A\otimes I\|\|I\otimes B\| = \|A\|\|B\|$. Since $\|(A\otimes B)(\xi\otimes\eta)\| = \|A\xi\|\|B\eta\|$, we may choose unit vectors ξ and η appropriately to approximate $\|A\|\|B\|$. Finally, since $H\odot K$ is dense in $H\otimes K$, we have that $A\otimes B$ is well-defined. $\square$

Definition: Let M and N be von Neumann algebras. Then, the von Neumann algebra tensor product $M\bar{\otimes}N$ is the WOT-closure of $M\odot N$ in $B(H\otimes K)$. If $N = \mathbb{C}1$, then $M\bar{\otimes}\mathbb{C}1 = M^{(\alpha)}$ is an amplification of M.

Proposition: Let M be a von Neumann algebra. If $\{e_i\}_{i\in I}$ is an orthogonal family of mutually equivalent projections in M, then $\sum_{i\in I} e_i = 1$, then

$$M \cong e_1Me_1\bar{\otimes}B(\ell_2(I)).$$

Type I von Neumann Algebras

We start by considering the structure of a von Neumann algebra of type I. For starters, we know that every projection dominates a rank one projection (which is minimal), so since any minimal projection is abelian, it follows that $B(H)$ is a factor of type I. Furthermore, we can show that any atomic von Neumann algebra is type I.

Proposition:

(i) If A is an abelian von Neumann algebra, then $A\bar{\otimes}B(H)$ is type I for any Hilbert space H.

(ii) If A is an abelian von Neumann algebra, then the matrix algebra $\mathbb{M}_n(A)$ is a finite von Neumann algebra of type I, and the A-valued map

$$\Phi(x) = \frac{1}{n}\sum_{i=1}^{n} x_{ii}$$

is

- center-valued: $\Phi\left((\delta_{ij}a)_{i,j}\right) = a$ whenever $a \in A$;
- A-modular: $\Phi(axb) = a\Phi(x)b$ whenever $a, b \in A$;
- positive definiteness: $\Phi(x^*x) \geq 0$, with equality only when $x = 0$;
- traciality: $\Phi(x^*x) = \Phi(xx^*)$.

Proof.

(i) Let p be a minimal projection in $B(H)$, and let $e = 1 \otimes p$. Then,

$$\begin{aligned} e(A\bar{\otimes}B(H))e &= A\bar{\otimes}pB(H)p \\ &= A \otimes \mathbb{C}p, \end{aligned}$$

so e is an abelian projection. Since the center of $A\bar{\otimes}B(H)$ is $A \otimes \mathbb{C}$, it follows that the central support $z(e)$ of e is the identity. Thus, $A\bar{\otimes}B(H)$ is type I.

(ii) The first two properties follow from the definition, while the latter two properties follow from taking

$$\begin{aligned} \Phi(x^*x) &= \frac{1}{n}\sum_{i,j=1}^{n} x_{ij}^* x_{ij} \\ &= \frac{1}{n}\sum_{i,j=1}^{n} x_{ij} x_{ij}^* \\ &= \Phi(xx^*). \end{aligned}$$

□

Definition: An A-valued map satisfying the properties in (ii), except for possibly the faithfulness condition, is known as an A-valued trace.

Lemma: Suppose e is an abelian projection. Then, for any projection f with $e \leq z(f)$, we have $e \preceq f$.

Proof. We will show that if $e \leq z(f)$ and $e \succeq f$, then $e \sim f$. In this case, we have $z(e) = z(f)$.

Let $Z := Z(M)$; observe that we have $eMe = Ze$. Choose a partial isometry $u \in M$ with $u^*u = f$ and $uu^* = e_1 \leq e$. By the comparison theorem, there is a central projection z such that $e_1 = ez$. Equivalence of projections preserves central support, so that $z(f) = z(e_1) \leq z(e)$, meaning $z(e_1) = z(e)$, so that $z(e) \leq z$ and $e = e_1$. Therefore, $f \sim e_1 = e$. □

Lemma: Let z be a central projection in M. If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthogonal families of abelian projections with $z(e_i) = z(f_j) = z$ for any $i \in I$ and $j \in J$, and if $\sum_{i\in I} e_i = z = \sum_{j\in J}$, then the cardinalities of I and J are equal.

Proof. By passing to zMz, we may assume that $z = 1$. By the above lemma, we have that $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$, with $e_i \sim f_j$. If we let α and β are the cardinalities of I and J respectively, then we have

$$\begin{aligned} M &\cong e_1 M e_1 \bar{\otimes} B(\ell_2(I)) \\ &\cong f_1 M f_1 \bar{\otimes} B(\ell_2(J)). \end{aligned}$$

Since e_1 and f_1 are abelian projections with $z(e_1) = z(f_1) = 1$, we must have $e_1 M e_1 \cong f_1 M f_1 \cong Z$, where Z is the center of M.

If α is finite, then we may consider the Z-valued trace defined by

$$\Phi(x) = \frac{1}{n} \sum_{i=1}^{n} x_{ii},$$

based on the tensor product decomposition $M \cong e_1 M e_1 \otimes B(\ell_2(I)) \cong Z \otimes B(\ell_2(I))$. Since we have $e_i \sim f_j$, we have $\Phi(e_i) = \Phi(f_j)$ for all $i \in I$ and $j \in J$.

Therefore, we have

$$\begin{aligned} \alpha \Phi(e_1) &= \sum_{i \in I} \Phi(e_i) \\ &= \Phi(1) \\ &= \sum_{j \in J} \Phi(f_j) \\ &= \beta \Phi(f_1), \end{aligned}$$

so that $\alpha = \beta$.

Now, suppose α is infinite. By symmetry, β is infinite. If φ is a normal (σ-WOT continuous) state on Z, we have that the map $\sigma_i \colon x \mapsto x e_i$ taking $Z \to e_i M e_i$ is an isomorphism, so that we may define a normal state φ_i on M by $\varphi_i(x) = \varphi \circ \sigma_i^{-1}(e_i x e_i)$. If z is the support of φ in Z, then $z e_i$ is the support of φ_i for each i.

We let $J_i = \{j \in J \mid \varphi_i(f_j) \neq 0\}$. Since $\{f_j\}_{j \in J}$ is orthogonal, it follows that J_i is countable. If $\varphi_i(f_j) = 0$, then $z e_i f_j z e_i = 0$, meaning that $z e_i f_j e_i = 0$, meaning $z f_j e_i = 0$. Since

$$\begin{aligned} \sum_{i \in I} z f_j e_i &= z f_j \\ &\neq 0, \end{aligned}$$

it follows that $\varphi_i(f_j) \neq 0$ for some $i \in I$. Therefore, we have $J = \bigcup_{i \in I} J_i$, meaning $\beta \leq \alpha \aleph_0$, meaning $\beta \leq \alpha$, and thus $\alpha = \beta$ by symmetry. $\square$

Theorem: Let M be a type I von Neumann algebra. Then, there exists a unique family of orthogonal central projections $\{z_\alpha\}_\alpha$ indexed by cardinal numbers such that $\sum_\alpha z_\alpha = 1$, and $z_\alpha M z_\alpha$ is isomorphic to the tensor product of an abelian von Neumann algebra A_α and

$B(H_\alpha)$, where $\dim(H_\alpha) = \alpha$. In particular, this gives

$$M \cong \bigoplus_{\alpha} A_\alpha \bar{\otimes} B(H_\alpha).$$

Additionally, if M is finite, then each of the α is finite.

Proof. Call a central projection $z \in M$ α-homogeneous if z is the sum of α orthogonal abelian projections with central support equal to z.

Letting $\{z_i\}_{i\in I}$ be an orthogonal family of α-homogeneous central projections, then we have that $\sum_{i\in I} z_i$ is α-homogeneous. □

Traces on Finite von Neumann Algebras

Traces on Semifinite von Neumann Algebras

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