

Solution (20.1): We know that $\sin(z)$ is conformal when $\frac{d}{dz}(\sin(z)) \neq 0$, meaning that we verify when $\cos(z) \neq 0$. This occurs at $z = n\pi$, where $n \in \mathbb{Z}$.

We know that $\sin(z) = 0$ when $z = \pi$, with $\cos(z) = -1 = e^{i\pi}$. Therefore, the image of $z = \pi$ is not stretched, and is rotated by an angle of π .

We know that the image of $z = i\pi$ is stretched by a factor of $|\cos(i\pi)| = |\cosh(\pi)|$. Since $\cosh(\pi) = |\cosh(\pi)|$, the image is rotated by an angle of 0.

Evaluating $\cos(\pi/2 + i\pi)$, we get that it is equal to $-\sin(\pi/2)\sin(i)$, or $-i\sinh(1) = \sinh(1)e^{-i\pi/2}$. Therefore, the image of $z = \pi/2 + i$ is stretched by a factor of $\sinh(1)$ and rotated by an angle of $-\pi/2$.

Solution (20.9): From Table 20.1, we find that

$$w = \frac{z+1}{1-z}$$

maps the *unit* circle to the right half plane. Therefore, scaling everything by $\sqrt{2}$, we have

$$w = \frac{\sqrt{2}z+1}{1-\sqrt{2}z}.$$

Solution (20.10): The first map of e^z has it such that $\text{Re}(w)$ ranges from $e^{\text{Re}(z_1)}$ to $e^{\text{Re}(z_2)}$, while $\arg(w)$ ranges from 0 to π , which agrees with the map showing an annular strip in the w -plane.

The second map of e^z maps z_1, z_2 , and z_3 to $e^{\text{Re}(z_1)}, 1$, and $e^{\text{Re}(z_3)}$, eventually converging to 0 as z_3 becomes more and more negative. Similarly, e^z maps z_4, z_5 , and z_6 to $e^{i\pi \text{Re}(z_4)}, -1$, and $e^{i\pi \text{Re}(z_6)}$, similarly converging to 0 as z_4 becomes more and more negative.

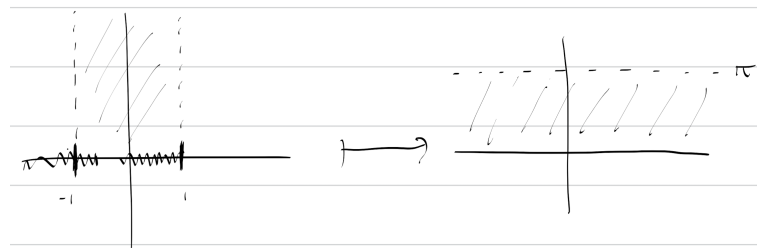
Solution (20.11):

- Since w is a composition of conformal maps (a Möbius transformation and the principal branch \ln function), $w(z)$ is conformal.
- The cut line occurs when the argument, $\frac{z-1}{z+1}$, is less than or equal to zero, meaning that the cut line is along the real axis with $z \leq 1$.
- To start, we know that in the UHP, $\arg(w)$ ranges from 0 to π . Now, the Möbius transformation $\frac{z-1}{z+1}$ maps $\infty \rightarrow 1, 0 \rightarrow -1$, and $1 \rightarrow 0$. Now, we have

$$\ln\left(\frac{(x-1)+iy}{(x+1)+iy}\right) = \ln(r) + i \arctan\left(\frac{2y}{(x^2-1)+y^2}\right),$$

where r is somewhat immaterial.

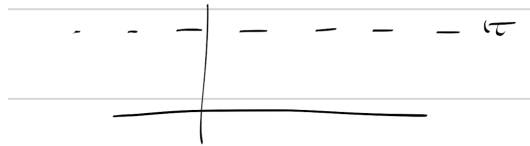
I'm not sure how exactly to show it from here, but this is sort of what I feel this is getting at.



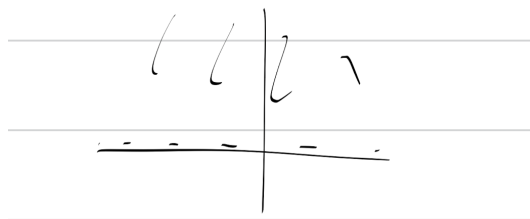
Solution (20.12): We know that the strip $0 < y < \pi$ maps to the UHP under $w_1 = e^z$. Then, using either the cross ratio or the result in Example 20.3, we use the ratio $\frac{z-i}{z+i}$ to map the UHP into the unit disk. Thus, we have the final

conformal map of

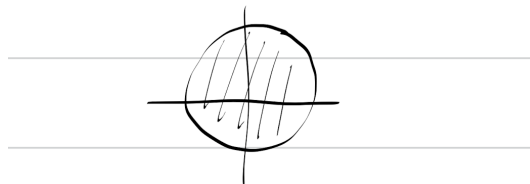
$$w(z) = \frac{e^z - i}{e^z + i}.$$



$$e^z \mapsto$$



$$\begin{aligned} \infty &\mapsto 1 \\ i &\mapsto 0 \Rightarrow \frac{e^z - i}{e^z + i} \\ -i &\mapsto \infty \end{aligned}$$



Solution (20.14): Note that if $z = e^{i\varphi}$ for $0 \leq \varphi \leq \pi$, then

$$z + \frac{1}{z} = e^{i\varphi} + e^{-i\varphi},$$

which ranges from -2 to 2 . Now, if $|x| > 1$, then

$$w(z) = x + \frac{x}{x^2 + y^2} + i \left(y - \frac{y}{x^2 + y^2} \right),$$

and, setting $y = 0$, we have

$$w(x) = x + \frac{1}{x}.$$

This gives a map from the x axis to the x axis.

Solution (20.15): Taking the derivative of Ω , we get

$$\frac{d\Omega}{dz} = -iV_0,$$

with magnitude $\left| \frac{d\Omega}{dz} \right| = V_0$.

I don't know where to go from here (a lack of physics background here is really hurting me here).

Solution (20.16):

(a) Substituting $z = x + iy$, we get

$$\begin{aligned} w(z) &= (x + e^x \cos(y)) + i(y + e^x \sin(y)) \\ &= u + iv. \end{aligned}$$

Now, for $y = \pm\pi$, we have that $u < -1$, and that $v = \pm\pi$. The line $y = 0$ maps to $y = 0$.

(b) I'm not sure I exactly understand how to solve for the equipotentials or the electric field lines (or if I've already solved for them?).

Solution (20.17): I presume the idea goes something along the lines of the following:

- (1) find a conformal map from the scenario from this scenario to the scenario with two parallel plates (i.e., one conformal map converting the right angles into semi-infinite plates, then the inverse of 20.16);
- (2) solving for the electric field/equipotentials in the simpler scenario;
- (3) converting back to this scenario.

Unfortunately, an issue I am facing here is that I do not know how to do step (1) (or step (3) for that matter).