Banach Limits

Theorem (Hahn–Banach–Minkowski)**:** Let X be a real vector space, and let $p\colon X \to \mathbb{R}$ be such that $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ and $p(tx) = tp(x)$ for all $t \geq 0$. If $f\colon Y \to \mathbb{R}$ is a linear functional defined on a subspace Y such that $f(x) \leq p(x)$ for all $x \in Y$, then there is an extension $F\colon X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for all $x \in X$ and $F|_Y = f$.

Furthermore, if $v \in X \setminus Y$, the value of $F(v)$ can be designated to be in the closed interval defined by

$$m = \sup_{w \in Y} (-p(-w-v) - f(w))$$

at the left endpoint, and

$$M = \inf_{w \in Y} (p(w+v) - f(w))$$

at the right endpoint.

Corollary: If X is a complex normed vector space with subspace $E \subseteq X$ and $\varphi \in E^*$, then there is $\Phi\colon X \to \mathbb{C}$ such that $\Phi|_E = \varphi$ and $\|\Phi\| = \|\varphi\|$.

Additionally, if there is $x_0 \in X \setminus E$, then there is $f \in X^*$ such that $f(x_0) = \operatorname{dist}_E(x_0)$ and $f|_E = 0$.

One of the most important vector spaces is the space ℓ_∞ of bounded sequences $x\colon \mathbb{N} \to \mathbb{C}$, which admits a subspace of convergent subspaces, often denoted c.

Proposition: There exists a linear functional $L\colon \ell_\infty \to \mathbb{C}$ with

(i) $\|L\| = 1$;

(ii) for any $x \in c$, $L(x) = \lim_{n\to\infty} x_n$;

(iii) for any $x \in \ell_\infty$ with $x_n \geq 0$ for each n, we have $L(x) \geq 0$;

(iv) for any $x \in \ell_\infty$, with $(S(x))_n \coloneqq x_{n+1}$, we have $L(S(x)) = L(x)$.

We will construct this linear functional using the Hahn–Banach theorem(s) by following the construction in Conway's book. We consider the real vector space $\operatorname{Re}(\ell_\infty)$, which we will write as ℓ_∞ for now.

To start we consider the subspace M of ℓ_∞ given by

$$M = \{x - S(x) \mid x \in \ell_\infty\}.$$

If $\mathbb{1}$ denotes the sequence of 1s in ℓ_∞, then we see that $\operatorname{dist}_M(\mathbb{1}) = 1$. First, $0 \in M$, so that $\operatorname{dist}_M(\mathbb{1}) \leq 1$. If there is n such that $(x - S(x))_n \leq 0$, then we see that

$$\begin{aligned} \|\mathbb{1} - (x - S(x))\| &\geq |\mathbb{1} - (x_n - (S(x))_n)| \\ &\geq 1. \end{aligned}$$

Else, if for all n, $0 \leq (x - S(x))_n = x_n - x_{n+1}$, then $x_{n+1} \leq x_n$ for all n. Since $x \in \ell_\infty$, there is $\alpha = \lim_{n\to\infty} x_n$. Therefore, $\lim_{n\to\infty}(x_n - x_{n+1}) = 0$, so $\|\mathbb{1} - (x - S(x))\| \geq 1$.

Thus, there is some linear functional $L\colon \ell_\infty \to \mathbb{R}$ such that $\|L\| = 1$ and $L(x) = L(S(x))$. This satisfies (i) and (iv) in the proposition.

Next, we show that $c_0 \subseteq \ker(L)$. Since (in our current focus) $c = c_0 + \mathbb{R}\mathbb{1}$, we would then obtain (ii). To see this, let $x \in c_0$. Observe that $S^n(x) - x$ is contained in M, meaning that $L(x) = L(S^n(x))$ for each n. If $\varepsilon > 0$, there is some N such that $|x_m| < \varepsilon$ for all $m > N$. Therefore,

$$\begin{aligned} |L(x)| &= |L(S^n(x))| \\ &\leq \|S^n(x)\| \\ &< \varepsilon. \end{aligned}$$

Since ε is arbitrary, we thus have $x \in \ker(L)$.

Finally, to show (c), we let $x \in \ell_\infty$ be such that $x_n \geq 0$ for all n, and assume toward contradiction that $L(x) < 0$. Dividing out by $\|x\|$, we have that $L(x) < 0$ and $0 \leq x_n \leq 1$ for all n. Yet, this would imply that $\|\mathbb{1} - x\| \leq 1$ and $L(\mathbb{1} - x) = 1 - L(x) > 1$, contradicting (a).

To extend to $\mathbb{C}$, and rewriting the functional on $\mathbb{R}$ as L_1, we may write any element $x \in \ell_\infty$ as $x = x_1 + ix_2$. Observe then that

$$L(x) = L_1(x_1) + iL_1(x_2)$$

is a linear functional on ℓ_∞. Now, observe that $\|L\| \geq 1$ almost by design. Since L is a nonzero linear functional, we let x be such that $L(x) \neq 0$, and set

$$\alpha = \frac{|L(x)|}{L(x)}.$$

We have that $|\alpha| = 1$ and $\alpha L(x) = |L(x)|$. We may then compute

$$\begin{aligned}
|L(x)| &= L(\alpha x) \\
&= \operatorname{Re}(L(\alpha x)) \\
&= L_1(\alpha x) \\
&\leq \|L_1\|\|\alpha x\| \\
&= \|x\|.
\end{aligned}$$

In particular, this means $\|L\| \leq 1$, so $\|L\| = 1$.

We call such a shift-invariant extension of the limit to all of ℓ_∞ a *Banach limit.* A quick observation gives that this limit functional cannot in fact be an algebra homomorphism. Considering the case of $a_n = (-1)^n$, we have that $a_{n+1} = -a_n$, so that

$$\begin{aligned}
L(S(a)) &= L(a) \\
&= L(-a) \\
&= -L(a),
\end{aligned}$$

or that the Banach limit must be equal to 0. However, if L were instead an algebra homomorphism (where the multiplication operation on ℓ_∞ is given pointwise), we would have

$$\begin{aligned}
L\left(a^2\right) &= L(\mathbb{1}) \\
&= 1 \\
&= (L(a))^2,
\end{aligned}$$

meaning that we would have $L(a) = \pm 1$, This means that such a limit that is an algebra homomorphism cannot be shift-invariant in the general case.

Regarding the case of $(-1)^n$, we observe that from our work above that 0 is the only Banach limit for this sequence. The sequences where every Banach limit assigns the same value for them are known as the *almost convergent* sequences. Lorentz (1948) showed that an equivalent criterion for almost convergence is that, for all $\varepsilon > 0$, there exists p_0 such that for all $p > p_0$ and all $n \in \mathbb{N}$, we have

$$\left|\frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} - L\right| < \varepsilon.$$

Every convergent sequence is almost convergent by definition.

Generalized Limits beyond Banach Limits

We have obtained one limit. However, there are lots of other extensions of the limit, each of which has norm 1. In fact, if we consider the restriction of ℓ_∞ to the reals, we know from undergrad real analysis that

$$p(x) = \limsup_{n\to\infty}((x_n)_n)$$

is in fact a sublinear functional. Furthermore, since

$$\liminf((x_n)_n) = -\limsup((-x_n)_n),$$

we can specify an extension to the limit functional to $L\colon \ell_\infty \to \mathbb{R}$ with $\|L\| = 1$ such that for any $(x_n)_n \in \ell_\infty$, we have

$$\liminf_{n\to\infty}((x_n)_n) \leq L(x) \leq \limsup_{n\to\infty}((x_n)_n)$$

However, these generalized limits may not align to the same requirement of shift-invariance as the aforementioned Banach limit. We can in fact construct a Banach limit from any generalized limit by using Césaro-type means: letting $x = (x_n)_n \in \ell_\infty$, we may define

$$c_n = \frac{1}{n}\sum_{k=1}^{n} a_k,$$

and then, for a generalized limit L, defining L' by

$$L'(a) = L(c).$$

From undergrad real analysis, we know that if $(a_n)_n \to a$ is convergent, then we have

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)_n \to a$$

as well. Thus, we see that

$$\begin{aligned}
L'(S(a)) - L'(a) &= L'((a_{n+1})_n) - L((a_n)_n)\\
&= L\left(\left(\frac{1}{n}\sum_{k=1}^{n} a_{k+1} - \frac{1}{n}\sum_{k=1}^{n} a_k\right)_n\right)\\
&= L\left(\left(\frac{1}{n}(a_{n+1} - a_1)\right)_n\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{n}(a_{n+1} - a_1)\right)\\
&= 0,
\end{aligned}$$

with the second-to-last equality emerging from the fact that $a \in \ell_\infty$, so the sequence $(a_{n+1} - a_1)_n$ is necessarily bounded.

In particular, this means that there are plenty of Banach limits, since we can take *any* generalized extension of the limit to ℓ_∞, then use this Césaro-type summation to yield a Banach limit.

Now, the question becomes: what are all the generalized limits? To understand this, we need to take a tour of topology. First, we recall an important structural theorem.

> **Theorem** (Riesz Representation Theorem)**:** Let X be a compact Hausdorff space. Then,
>
> $$C(X)^* \cong M_r(X),$$
>
> where $M_r(X)$ denotes the space of complex regular Borel measures on X.

Now, we consider the fact that ℓ_∞ can be identified with $C_b(\mathbb{N})$ when N is equipped with the discrete topology. In particular, there is some $R > 0$ such that, for any $x \in \ell_\infty$, we have $\operatorname{im}(x) \subseteq B(0, R)$. The universal property of the Stone–Čech compactification thus gives a unique extension $\beta x\colon \beta\mathbb{N} \to \mathbb{C}$ such that $\beta x|_{\mathbb{N}} = x$, where $\beta\mathbb{N}$ is the Stone–Čech compactification of the naturals. In particular, we may thus identify $\ell_\infty \cong C_b(\mathbb{N})$ with $C(\beta\mathbb{N})$, so $\ell_\infty^* \cong M_r(\beta\mathbb{N})$.

The next question emerges: what exactly *is* $\beta\mathbb{N}$?

Proposition: The space $\beta\mathbb{N}$ is the space of all ultrafilters on $\mathbb{N}$.

Recall that a filter $\mathcal{F}$ on a set X is a family of subsets $\mathcal{F}$ that is "closed upwards," in the sense that if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$, and is such that $X \in \mathcal{F}$ and, if $A, B \in \mathcal{F}$, then so is $A \cap B$.

An ultrafilter $\mathcal{U}$ on X is a maximal *proper* filter, and has the property that if $A \in P(X)$, then either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$. An ultrafilter on $\mathbb{N}$ is called principal if it contains a finite subset, else it is called a non-principal ultrafilter (or free ultrafilter). The principal ultrafilters in $\beta\mathbb{N}$ can be identified with $\mathbb{N}$, since they are all of the form

$$\mathcal{F}_q = \{A \subseteq N \mid q \in A\}.$$

If $\mathcal{U}$ is an ultrafilter, then we can define a limit of a sequence $(x_n)_n$ along $\mathcal{U}$ by saying that $\lim_{\mathcal{U}}(x_n)_n = L$ if and only if, for every $\varepsilon > 0$, we have

$$\{n \mid |x_n - L| \leq \varepsilon\} \in \mathcal{U}.$$

Limits along principal ultrafilters are very boring — if $\mathcal{F}_p$ is the principal ultrafilter corresponding to p, then $\lim_{\mathcal{F}_p}(x_n)_n = x_p$. However, limits along non-principal ultrafilters are where things start to get interesting.

Limits along ultrafilters are linear functionals on ℓ_∞, so they can be identified with $M_r(\beta\mathbb{N})$. Since limits along ultrafilters are generalized limits that correspond one-to-one with non-principal ultrafilters, and they each have norm 1, it follows that the limits along (non-principal) ultrafilters correspond to Dirac measures, $\delta_{\mathcal{U}}$, on the space $\beta\mathbb{N} \setminus \mathbb{N}$, where we identify $\mathbb{N}$ with the principal ultrafilters.

In fact, we can consider all the generalized limits as some some probability measure on $\beta\mathbb{N} \setminus \mathbb{N}$, and the Banach limits as the probability measures that are invariant under the action of $\mathbb{Z}$-translation (i.e., elementwise addition on the ultrafilter). In particular, since the probability measures form a compact convex subset of $M_r(\beta\mathbb{N})$, it follows that every generalized limit is in the w^*-closure of the extreme points of the probability measures, which are the limits along *all* ultrafilters, including the principal ultrafilters.