Contents

Introduction

This is going to be the notes from my Honors Thesis project on amenability. We will be covering different results that are used to show that a topological group has a translation-invariant finitely additive probability measure (i.e., a mean).

The primary source texts to inform this independent study will be Volker Runde's *Lectures on Amenability* and Timothy Rainone's *Functional Analysis-en route to Operator Algebras*, as well as various notes compiled by my professor.

Group Actions, Paradoxical Decompositions, and the Banach–Tarski Paradox

In order to introduce Tarski's theorem, which is where our first condition about the amenability of groups appears, we begin by discussing paradoxical decompositions, with the goal of this section being a proof of the Banach–Tarski Paradox. The Banach–Tarski paradox says the following:

> If A and B are any bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Basics of Group Actions

The information for these essentials about group actions will be drawn from Dummit and Foote's *Abstract Algebra*.

Definition (Group Action)**.** A (left) group action of G onto a set A is a map from $G \times A$ to A that satisfies:

- $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$;
- $e \cdot a = a$[1] for all $a \in A$.

Definition (Permutation Representation)**.** For each g, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ (the group element g acts on a) is a permutation of A. There is a homomorphism associated to these actions:

$$\varphi : G \to S_A,$$

where $\varphi(g) = \sigma_g$. Recall that S_A is the symmetric group (group of permutations) on the elements of A.

This is the permutation representation for the action.

In particular, given any nonempty set A and a homomorphism G into S_A, we can define an action of G on A by taking $g \cdot a = \varphi(g)(a)$.

[1] The identity element is usually written as 1, but I will write it as e out of familiarity.

Definition (Kernel)**.** The kernel of the action of G is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}$$

Note: The kernel of the action is the kernel of the permutation representation $\varphi : G \to S_A$.

Definition (Stabilizer)**.** For each $a \in A$, the stabilizer of a under G is the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

Note: The kernel of the group action is the intersection of the stabilizers of every element of G:

$$\text{kernel} = \bigcap_{a \in A} G_a.$$

Note: For each $a \in A$, G_a is a subgroup of G.

Definition (Faithful Action)**.** An action is faithful if the kernel of the action is e.

Definition (Free Action)**.** For a set X with G acting on X, the action of G on X is free if, for every x, $g \cdot x = x$ if and only if $g = e_G$.

If the action of G on X is a free action, we say G acts freely on X.

Proposition (Equivalence Relation on A)**.** Let G be a group that acts on a nonempty set A. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking $|G : G_a|$, which is the index of the stabilizer of a.

Proof. We can see that $a \sim a$, since $e \cdot a = a$. Similarly, we can see that if $a \sim b$, then $b = g^{-1} \cdot a$, meaning $b \sim a$. Finally, let $a \sim b$ and $b \sim c$. Then, we have $a = g \cdot b$ for some $g \in G$, and $b = h \cdot c$ for some $h \in G$. Thus, we have

$$\begin{aligned} a &= g \cdot (h \cdot c) \\ &= (gh) \cdot c, \end{aligned}$$

meaning $a \sim c$.

We say there is a bijection between the left cosets of G_a and the elements of the equivalence class of a.

Define $\mathcal{C}_a$ to be the set $\{g \cdot a \mid g \in G\}$, and let $b = g \cdot a$. Define a map $g \cdot a \mapsto gG_a$. This map is surjective since $g \cdot a$ is always an element of $\mathcal{C}_a$. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, meaning $h^{-1}g \in G_a$, and $h^{-1}g \in G_a$ if and only if $gG_a = hG_a$, this map is injective.

Since there is a one-to-one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we now know that the number of equivalence classes of a under the action of G is $|G : G_a|$. □

Definition (Orbit)**.** For any $a \in A$, we define the orbit under G of a by

$$G \cdot a = \{b \in A \mid \forall g \in G,\ b = g \cdot a\}$$

In particular, if $c \in G \cdot a$ for some $a \in A$, then $G \cdot c = G \cdot a$.

Paradoxical Decompositions

Most of the information from this section will be drawn from Volker Runde's *Lectures on Amenability*, as well as *Amenable Banach Algebras — A Panorama*.

Definition (Paradoxical Sets and Decompositions). Let G be a group that acts on a set X. Let $E \subseteq X$.

If there exist pairwise disjoint $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j,$$

then we say that E is G-paradoxical.

In particular, a paradoxical group is one where G acts on itself by left-multiplication.

Example (Our First Paradoxical Group). The free group on two generators, $\mathbb{F}(a,b)$,[II] is paradoxical. To see this, we let

$$W(x) = \{w \in \mathbb{F}(a,b) \mid w \text{ starts with } x\}.$$

Here, "starts with" refers to the left-most element. For instance, $ba^2ba^{-1} \in W(b)$.

In particular, we can see that

$$\mathbb{F}(a,b) = \left\{e_{\mathbb{F}(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For any $w \in \mathbb{F}(a,b) \setminus W(a)$, we can see that $a^{-1}w \in W\left(a^{-1}\right)$, meaning $w \in aW\left(a^{-1}\right)$. Therefore, $\mathbb{F}(a,b) = W(a) \sqcup aW\left(a^{-1}\right)$.

Similarly, for any $v \in \mathbb{F}(a,b) \setminus W(b)$, $b^{-1}v \in W\left(b^{-1}\right)$, so $v \in bW\left(b^{-1}\right)$. Therefore, $\mathbb{F}(a,b) = W(b) \sqcup bW\left(b^{-1}\right)$.

Proposition (Free Action of a Paradoxical Group). Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$ be pairwise disjoint, with $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

We let $M \subseteq X$ contain exactly one element from every orbit of G.

The set $\{g \cdot M \mid g \in G\}$ is a partition of X. Since M contains exactly one element from every orbit of G, it is then the case that $\bigcup_{g \in G} g \cdot M = X$, since $G \cdot M = X$.

Additionally, if $x, y \in M$ with $g \cdot x = h \cdot y$, then $\left(h^{-1}g\right) \cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Thus, we must have $h^{-1}g = e_G$, as we assume G acts freely.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$ if $g_1 \neq g_2$, meaning $\{g \cdot M \mid g \in G\}$ is a partition.

[II]The set of all reduced words over $\left\{a, b, a^{-1}, b^{-1}, e_{\mathbb{F}(a,b)}\right\}$. In particular, a word is reduced when the pairs aa^{-1} and bb^{-1} are replaced with the identity $e_{\mathbb{F}(a,b)}$.

Define A_j^* to be the subset of X that is the result of the elements of A_j acting on M. In other words,

$$A_j^* = \bigcup_{g \in A_j} g \cdot M.$$

As a useful shorthand, we can say $A_j^* = A_j \cdot M$.[III] Similarly, we define

$$\begin{aligned} B_j^* &= \bigcup_{h \in B_j} h \cdot M \\ &= B_j \cdot M. \end{aligned}$$

We can see that $A_1^*, A_2^*, \dots, A_n^*, B_1^*, B_2^*, \dots, B_m^* \subseteq X$ are disjoint, since $\{g \cdot M \mid g \in G\}$ is a partition, and $A_1, \dots, A_n, B_1, \dots, B_m$ are disjoint in G.

Thus, we have

$$\begin{aligned} \bigcup_{j=1}^{n} g_j \cdot A_j^* &= \bigcup_{j=1}^{n} \left(g_j A_j\right) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Similarly,

$$\begin{aligned} \bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} \left(h_j B_j\right) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Thus, we see that X has a paradoxical decomposition, meaning X is G-paradoxical. □

Note: We invoked the axiom of choice when we defined M to contain exactly one element from each orbit in X.

Paradoxical Decompositions of the Unit Sphere and Unit Ball

We are aware of $\mathbb{F}(a, b)$ being a paradoxical group — in particular, we hope to use the properties of $\mathbb{F}(a, b)$ to yield paradoxical decompositions of the unit sphere in $\mathbb{R}^3$, denoted S^2.

Definition (Special Orthogonal Group)**.** For $n \in \mathbb{N}$, we define the special orthogonal group to consist of all real $n \times n$ matrices A such that

$$A^T A = AA^T = I,$$

with $\det(A) = 1$.

In particular, SO(3) denotes the set of all rotations about some line that runs through the origin. An important fact about SO(3) is that it contains a paradoxical subgroup.

Theorem. There are rotations A and B about lines through the origin in $\mathbb{R}^3$ which generate a subgroup of SO(3) isomorphic to $\mathbb{F}(a, b)$.

[III] Yes, I know that A_j is not technically a group acting on M, but this will help illuminate the final conclusion.

Proof. We set

$$A^{\pm} = \begin{bmatrix} 1/3 & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$B^{\pm} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

Here, A^{+} denotes A, and A^{-} denotes A^{-1}, and similarly with B.

Let w be a reduced word in A, B, A^{-1}, and B^{-1} which is not the empty word. We claim that w cannot be the identity. Without loss of generality, we assume w ends in A or A^{-1} — this is because w acts as the identity if and only if AwA^{-1} or $A^{-1}wA$ act as the identity.

In particular, we will show that there exist $a, b, c \in \mathbb{Z}$ with $b \not\equiv 0$ modulo 3 such that

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix},$$

where k is the length of w. The main reason we wish to show this is that, if we have $b \not\equiv 0$ modulo 3, it is the case that w necessarily cannot map $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to itself.

We start with induction on the length of w. In particular, for $w = A^{\pm}$, we have

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix},$$

proving the base case.

Suppose $k > 0$, meaning $w = A^{\pm}w'$ or $w = B^{\pm}w'$, with w' not equal to the empty word. The inductive hypothesis says that

$$w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix},$$

for some $a', b', c' \in \mathbb{Z}$ with $b \not\equiv 0$ modulo 3. In particular,

$$A^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a' \mp 4b' \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix}$$
$$B^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} 3a' \\ (b' \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix},$$

where we say

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

i.e., we set the coordinates of $w \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$ through their definition in $A^{\pm}w'$ or $B^{\pm}w'$.

In order to show that $b \not\equiv 0$ modulo 3, we must examine the following four cases.

Let w^* denote the word such that

$$w^* \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3^{k-2}}\begin{pmatrix}a''\\b''\sqrt{2}\\c''\end{pmatrix},$$

with $a'', b'', c'' \in \mathbb{Z}$ and $b'' \not\equiv 0$ modulo 3. It is important to note here that w^* may be the empty word.

Case 1: Suppose $w = A^{\pm}B^{\pm}w^*$. Then, we have $b = b' \mp 2a'$, where $a' = 3a''$. Since $b' \not\equiv 0$ modulo 3 by the inductive hypothesis assumption, it is also the case that $b \not\equiv 0$ modulo 3.

Case 2: Suppose $w = B^{\pm}A^{\pm}w^*$. Then, we have $b = b' \mp 2c'$, where $c' = 3c''$. Similarly, since $b' \not\equiv 0$ modulo 3 by the inductive hypothesis assumption, it is also the case that $b \not\equiv 0$ modulo 3.

Case 3: Suppose $w = A^{\pm}A^{\pm}w^*$. Then, we have

$$\begin{aligned} b &= b' \pm 2a' \\ &= b' \pm 2\left(a'' \mp 4b''\right) \\ &= b' + (b'' \pm 2a'') - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Since $b', b'' \not\equiv 0$ modulo 3 by the inductive hypothesis, it is also the case that $b \not\equiv 0$ modulo 3.

Case 4: Suppose $w = B^{\pm}B^{\pm}w^*$. Then, we have

$$\begin{aligned} b &= b' \mp 2c' \\ &= b' \mp 2\left(c'' \pm 4b''\right) \\ &= b' + (b'' \mp 2c'') - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Since $b', b'' \not\equiv 0$ modulo 3 by the inductive hypothesis, it is also the case that $b \not\equiv 0$ modulo 3.

Thus, we have shown that any non-empty reduced word over A, A^{-1}, B, B^{-1} does not act as the identity. The subgroup of $SO(3)$ generated by A, B, A^{-1}, and B^{-1} is thus isomorphic to $\mathbb{F}(a, b)$. □

Remark: For any element of $SO(n)$ with $n \geqslant 3$, we can write A_n (denoting the $n \times n$ matrix corresponding to A) as

$$\begin{aligned} A_n &= \begin{pmatrix} A_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\end{pmatrix} \\ B_n &= \begin{pmatrix} B_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \end{aligned}$$

where **0** denotes a block matrix consisting of 0 and **1** denotes a block matrix equal to the identity.

This means that our subgroup of $SO(3)$ isomorphic to $\mathbf{F}(a, b)$ embeds into $SO(n)$ via the above block matrices.

Theorem (Hausdorff Paradox)**.** There is a countable subset D of S^2 such that $S^2 \setminus D$ is paradoxical under the action of $SO(3)$.

Proof. Let A and B be the rotations in $SO(3)$ that serve as the generators of the subgroup isomorphic to $\mathbb{F}(a, b)$.

Since A and B are rotations, any word in the subgroup generated by A and B will also be a rotation — as a result, all such (non-empty) words contain two fixed points.

Let

$$F = \left\{x \in S^2 \mid x \text{ is a fixed point for some word } w\right\}.$$

Since the set of all words in A and B is countably infinite, so too is F. Therefore, the union of all these fixed points under the action of all such words w is also countable:

$$D = \bigcup_{w \in G} w \cdot F.$$

Since the set of words in A and B act freely on $S^2 \setminus D$, it must be the case that $S^2 \setminus D$ is paradoxical under the action of the group of all such words. □

Definition (Equidecomposable Sets)**.** Let G act on X, $A, B \subseteq X$. We say A and B are equidecomposable under G if there are $A_1, \dots, A_n \subseteq A$, $B_1, \dots, B_n \subseteq B$, and $g_1, \dots, g_n \in G$ such that

(i) $A = \bigcup_{j=1}^{n} A_j$ and $B = \bigcup_{j=1}^{n} B_j$;

(ii) the collection $\left\{A_j\right\}_{j=1}^{n}$ are pairwise disjoint and the collection $\left\{B_j\right\}_{j=1}^{n}$ are pairwise disjoint;

(iii) for each j, $g_j \cdot A_j = B_j$.

We write $A \sim_G B$ if A and B are equidecomposable under G.

Remark: The relation $\sim_G$ is an equivalence relation.

In particular, to see transitivity, we have the partitions $A_1, \dots, A_n \subseteq B$ and $B_1, \dots, B_n \subseteq B$ with $g_i \cdot A_i = B_i$, and the partitions $B_1, \dots, B_m \subseteq B$, $C_1 \cdots C_m \subseteq C$ with $h_j \cdot B_j = C_j$.

We find the partition of A by taking $A_{ij} = B_i \cap B_j$, where $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$. We then have $h_j g_i \cdot A_{ij}$ maps to a refined partition of C, yielding equidecomposability between A and C.

Remark: For equidecomposable sets A and B, there is a bijection $\phi : A \to B$ by, for each $C \subseteq A$, taking $C_i = C \cap A_i$, where $A_1, \dots, A_n$ is the partition of A, and mapping $\varphi(C_i) = g_i \cdot C_i$.

Proposition. Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \setminus D$ are equidecomposable under the action of $SO(3)$.

Proof. Let L be a line in $\mathbb{R}^3$ with the property that $L \cap D = \emptyset$. Such a L must necessarily exist as the set of all antipodes in S^2 is uncountable.

Define $\rho_\theta \in SO(3)$ to be a rotation about L by an angle of θ. For fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{x \in D \mid \rho_\theta^n \cdot x \in D\right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

Define $W_n = \{\theta \mid R_{n,\theta} \neq \emptyset\}$. The injection $\theta \mapsto \rho_\theta^n \cdot x$ into D shows that for each n, W_n is countable. Thus, defining

$$W = \bigcup_{n \in N} W_n$$

it is evident that W is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. Define ρ to be a rotation about L by ω. Then, for every $n, m \in \mathbb{N}$,

$$\rho^n \cdot D \cap \rho^m \cdot D = \emptyset.$$

We let $\tilde{D} = \bigsqcup_{n=0}^{\infty} \rho^n \cdot D$. Notice that, in particular, $\rho \cdot \tilde{D} = \bigsqcup_{n=1}^{\infty} \rho^n \cdot D$, meaning $\tilde{D}$ and $\tilde{D} \setminus D$ are equidecomposable under $SO(3)$.

Thus, we have

$$\begin{aligned} S^2 &= \tilde{D} \sqcup \left(S^2 \setminus \tilde{D}\right) \\ &\sim_{SO(3)} \rho \cdot \tilde{D} \sqcup \left(S^2 \setminus \tilde{D}\right) \\ &= (\tilde{D} \setminus D) \sqcup \left(S^2 \setminus \tilde{D}\right) \\ &= S^2 \setminus D, \end{aligned}$$

establishing the equidecomposability of S^2 and $S^2 \setminus D$. □

Proposition. Let G act on X, with E and E' subsets of X such that $E \sim_G E'$. Then, if E is paradoxical under the action of G, so too is E'.

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m$ be pairwise disjoint subsets of E and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} E &= \bigsqcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} h_j \cdot B_j, \end{aligned}$$

which follows from the paradoxicality of E. We let

$$\begin{aligned} A &= \bigsqcup_{i=1}^{n} A_i \\ B &= \bigsqcup_{j=1}^{m} B_j. \end{aligned}$$

It follows that $A \sim_G E$ and $B \sim_G E$; to see this, set the partition of A to be $A_1, \dots, A_n$, and set the partition of E to be $g_i \cdot A_i$ for $i \in \{1, \dots, n\}$, and similarly for G.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$, implying that there exists a paradoxical decomposition of E' in $A_1, \dots, A_n$ and $B_1, \dots, B_m$. □

Since $S^2 \setminus D$ and S^2 are equidecomposable under the action of $SO(3)$, and $S^2 \setminus D$ is paradoxical under the action of $SO(3)$, the above proposition implies the following corollary.

Corollary. S^2 is paradoxical under $SO(3)$.

Definition (Euclidean Group)**.** The Euclidean group $E(n)$ consists of all isometries of a Euclidean space. An isometry of a Euclidean space consists of translations, flips about the origin, and rotation.

In particular, $E(n) = T(n) \rtimes O(n)$, where $T(n)$ denotes all translations and $O(n)$ is the orthogonal group, which denotes all rotations or flips.

We define $E_+(n)$ to be all orientation-preserving isometries of Euclidean space. In particular, $E_+(n) = T(n) \rtimes SO(n)$, where $SO(n)$ is the special orthogonal group, which denotes all orientation-preserving rotations.

Corollary (Weak Banach–Tarski Paradox). Every closed ball in $\mathbb{R}^3$ is paradoxical under the Euclidean group $E(3)$.

Proof. We only need to show that the closed unit ball, $B(0,1)$, is paradoxical under the action of $E(3)$.

To start, we can show that $B(0,1) \setminus \{0\}$ is paradoxical. Since $SO(3)$ is paradoxical, there exist pairwise disjoint $A_1, \dots, A_n, B_1, \dots, B_m \subseteq S^2$ and $g_1, \dots, g_n, h_1, \dots, h_m \in SO(3)$ such that

$$\begin{aligned} S^2 &= \bigsqcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigsqcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

Define

$$\begin{aligned} A_i^* &= \{tx \mid t \in (0,1], x \in A_i\} \\ B_j^* &= \left\{ty \mid t \in (0,1], y \in B_j\right\}. \end{aligned}$$

Then, $A_1^*, \dots, A_n^*, B_1^*, \dots, B_m^* \subseteq B(0,1) \setminus \{0\}$ are pairwise disjoint, and

$$\begin{aligned} B(0,1) \setminus 0 &= \bigcup_{i=1}^{n} g_i \cdot A_i^* \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j^*. \end{aligned}$$

Thus, we have established that $B(0,1) \setminus \{0\}$ is paradoxical under $SO(3) \leqslant E(3)$.[IV]

Now, we want to show that $B(0,1) \setminus \{0\}$ and $B(0,1)$ are equidecomposable under $E(3)$. To do this, let $x \in B(0,1) \setminus \{0\}$, and let ρ be a rotation about x by a line that misses the origin such that $\rho^n \cdot 0 \neq \rho^m \cdot 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$.[V] Let $D = \{\rho^n \cdot 0 \mid n \in \mathbb{N}\}$. We can see that $\rho \cdot D = D \setminus \{0\}$, and that D and $\rho \cdot D$ are equidecomposable under $E(3)$.

Thus, we have

$$\begin{aligned} B(0,1) &= D \sqcup (B(0,1) \setminus D) \\ &\sim_{E(3)} (\rho \cdot D) \sqcup (B(0,1) \setminus D) \\ &= (D \setminus \{0\}) \sqcup (B(0,1) \setminus D) \\ &= B(0,1) \setminus \{0\}, \end{aligned}$$

establishing the equidecomposability of $B(0,1)$ and $B(0,1) \setminus \{0\}$.

Thus, $B(0,1)$ is paradoxical under the action of $E(3)$. □

[IV] Essentially, we take the paradoxical decomposition of S^2 under $SO(3)$, and scale by t to cover all of $B(0,1) \setminus \{0\}$.

[V] This is why we need our underlying group acting on $\mathbb{R}^3$ to be the Euclidean group rather than $SO(3)$. It is still the case that $SO(3) \leqslant E(3)$, meaning that $E(3)$ is necessarily paradoxical when acting on $\mathbb{R}^3$.