Preliminary Statements

Theorem (Definition of Countability). *A set S is countable if and only if there exists an injection $f : S \hookrightarrow \mathbb{N}$.*

Proof. Let S be countable.

Case 1: We have S is finite if and only if there is a map $f : S \to \{1, 2, \dots, n\}$, where f is a bijection. Letting $\mathrm{id} : \{1, 2, \dots, n\} \to \mathbb{N}$ be defined by $\mathrm{id}(n) = n$, it is clear that id is an injection.

Considering the map $\mathrm{id} \circ f : S \to \mathbb{N}$, since id is injective and f is injective, so too is $\mathrm{id} \circ f$, meaning our desired injection is $\mathrm{id} \circ f$.

Case 2: By definition, a set S is countably infinite if and only if there exists a bijection $g : S \to \mathbb{N}$, which is our desired injection.

□

Theorem (Injection into a Finite Set). *Let S be a nonempty set. If there exists an injection $S \hookrightarrow \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, then S is finite.*
Then, S is finite if and only if there exists an injection $S \hookrightarrow \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

Proof. We begin by showing the reverse direction.

Let $\sigma : S \hookrightarrow \{1, 2, \dots, n\}$ be an injection for some $n \in \mathbb{N}$. Define s_i by $\sigma(s_i) = i$ for $i \in \mathrm{im}(\sigma)$.

Notice that $\sigma' : S \to \sigma(S)$ is a bijection, since σ is injective and any map of the form $f : A \to f(A)$ is surjective by definition.

We define $r : \sigma(S) \hookrightarrow \mathbb{N}$ selecting i_1 to be the least element in $\sigma(S)$ (which exists by the well-ordering principle since $\{1, 2, \dots, n\} \subseteq \mathbb{N}$ is nonempty), and mapping $r(i_1) = 1$. Similarly, we inductively select i_k to be the least element in $\sigma(S) \setminus \{i_1, i_2, \dots, i_{k-1}\}$, and map $r(i_k) = k$. From this construction, it is clear that r is injective.

Then, defining $r' : \sigma(S) \to r(\sigma(S))$, we can see that r' is a bijection, with $r(\sigma(S)) = \{1, 2, \dots, j\}$ for some $j \leqslant n$ (since, by definition, σ is an injection, meaning $\sigma(s_i) \leqslant n$ for all n).

Taking $r' \circ \sigma' : S \to \{1, 2, \dots, j\}$, we see that this is a composition of bijections, meaning it is a bijection. Thus, S is finite.

In the forward direction, we can see that if S is finite, then the bijection $h : S \to \{1, 2, \dots, n\}$ is an injection, and we are done. □

1.1

Problem. Show that the function $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = (-1)^{n+1} \left\lfloor \frac{n+1}{2} \right\rfloor$$

is a bijection.

Solution. We begin by showing that f is injective. Let $f(n_1) = f(n_2)$. Then, we have two cases: one if $f(n_1)$ and $f(n_2)$ are positive, and one if $f(n_1)$ and $f(n_2)$ are negative. In the either case, we have

$$f(n_1) = (-1)^{n_1+1} \left\lfloor \frac{n_1+1}{2} \right\rfloor, f(n_2) \qquad = (-1)^{n_2+1} \left\lfloor \frac{n_2+1}{2} \right\rfloor,$$

meaning

$$\left\lfloor \frac{n_1+1}{2} \right\rfloor = \left\lfloor \frac{n_2+1}{2} \right\rfloor.$$

If $f(n_1)$ and $f(n_2)$ are positive, this implies that n_1 and n_2 are odd (so that n_1+1, n_2+1 are even). Since n_1+1 and n_2+1 are even, this implies

$$\begin{aligned}\left\lfloor \frac{n_1+1}{2} \right\rfloor &= \frac{n_1+1}{2}\\ \left\lfloor \frac{n_2+1}{2} \right\rfloor &= \frac{n_2+1}{2},\end{aligned}$$

meaning $n_1 = n_2$.

If $f(n_1)$ and $f(n_2)$ are odd, this implies that n_1 and n_2 are even, so

$$\begin{aligned}\left\lfloor \frac{n_1+1}{2} \right\rfloor &= \frac{n_1}{2}\\ \left\lfloor \frac{n_2+1}{2} \right\rfloor &= \frac{n_2}{2},\end{aligned}$$

once again implying that $n_1 = n_2$.

To show surjectivity, let $z \in \mathbb{Z}$. Suppose $z < 0$. Then, we find $n \in \mathbb{N}$ by taking $n = -2z$. If $z > 0$, we take $n = 2z - 1$, and if $z = 0$, we take $n = 0$.

1.2

Problem. Given bijections $f : \mathbb{N} \to \mathbb{Z}$ and $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, show that the function $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ defined by $h(x, y) = P\left(f^{-1}(x), f^{-1}(y)\right)$ is bijective.

Solution. We begin by showing injectivity. Since f is bijective, so too is f^{-1}, meaning that for

$$h(x, y) = h(x', y'),$$

we have

$$\begin{aligned}P\left(f^{-1}(x), f^{-1}(y)\right) &= P\left(f^{-1}(x'), f^{-1}(y')\right)\\ f^{-1}(x) &= f^{-1}(x')\\ f^{-1}(y) &= f^{-1}(y') && \text{since } P \text{ is bijective}\end{aligned}$$

meaning

$$\begin{aligned}x &= x'\\ y &= y' && \text{since } f^{-1} \text{ is bijective.}\end{aligned}$$

Thus, h is injective.

Let $n \in \mathbb{N}$. Since P is surjective, there exist a, b such that $P(a, b) = n$. Since f^{-1} is surjective, there exists $x, y \in \mathbb{Z}$ such that $f^{-1}(x) = a$ and $f^{-1}(y) = b$. Thus, there exist $x, y \in \mathbb{Z}$ such that $h(x, y) = n$.

1.3

Problem. If A and B are countably infinite, show that $A \times B$ is countably infinite.

Solution. By the definition of countably infinite sets, there exist bijections $\alpha : A \to \mathbb{N}$ and $\beta : B \to \mathbb{N}$. Additionally, we know that there exists a bijection $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Define $h : A \times B \to \mathbb{N}$ by $h(a, b) = P(\alpha(a), \beta(b))$. Then, since h is a composition of bijections, h is a bijection between $A \times B$ and $\mathbb{N}$.

1.5

Problem. If $A_1, A_2, \dots$ is an infinite sequence of (pairwise) disjoint finite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

Solution. Let $\chi_n : A_n \to \{1, 2, \dots, \alpha_n\}$ be the bijection that defines the cardinality of each A_n. We define $a_{i,n} \in A_n$ to be the unique element of a_n such that $\chi_n(a_{i,n}) = i$. Let p_n denote the nth prime number.

The function $h : \bigcup_{n=1}^{\infty} A_n \to \mathbb{N}$ defined by $h(a_{i,k}) = p_k^{\chi_k(a_{i,k})}$, where $a_{i,k} \in A_k$, is an injection (since prime numbers do not divide each other). Thus, we know that $\bigcup_{n=1}^{\infty} A_n$ is countable (by the theorem above).

Solution (Proposed). For all $i \in \mathbb{N}$, there exists $f_i : A_i \to \{1, 2, \dots, n_i\}$ such that f_i is a bijection (by the definition of finitude).

Let $x \in \bigcup_{i=1}^{\infty} A_i$.

Then, $x \in A_i$ for some i, and only one i, because the sets A_i are pairwise disjoint.

Define

$$p(x) = f_i(x) - 1 + \sum_{j=1}^{i-1} n_j.$$

Thus, $\bigcup_{i=1}^{\infty} A_i$ is denumerable.

1.6

Problem. If $A_1, A_2, \dots$ is an infinite sequence of disjoint countably infinite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

Solution. Proceeding in a similar manner to 1.5, we define $\chi_n : A_n \to \mathbb{N}$ to be bijections that define the cardinality of A_n, and let $a_{i,n} \in A_n$ be defined by $\chi_n(a_{i,n}) = i$. We let p_n denote the nth prime number.

The function $h : \bigcup_{n=1}^{\infty} A_n \to \mathbb{N}$ defined by $h(a_{i,k}) = p_k^{\chi_k(a_{i,k})}$ is an injection, as each A_k is disjoint and prime numbers do not divide each other. Thus, we know that $\bigcup_{n=1}^{\infty} A_n$ is countable.

Solution. Since A_i is denumerable for all $i \in \mathbb{Z}_{>0}$, there exists a bijection $f_i : A_i \to \mathbb{N}$.

Define $f : \bigcup_{i=1}^{\infty} A_i \to \mathbb{N} \times \mathbb{N}$ by

$$f(x) = (i - 1, f_i(x))$$

for $x \in A_i$.

We know f is well-defined, since A_i are pairwise disjoint, so $x \in A_i$ for exactly one value of i.

To define $g : \bigcup_{i=1}^{\infty} A_i \to \mathbb{N}$, we take $g = P \circ f$. We can see that g is a bijection since P and f are bijections.

1.7

Problem. Construct an explicit polynomial bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Solution. Let $Q : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $Q(x, y, z) = P(P(x, y), z)$, where $P(x, y) = \frac{(x+y)(x+y+1)}{2} + x$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

We know that Q is a bijection since it is a composition of bijections. I do not want to expand this expression.

Extra Problem 1

Problem. Prove that if A and B are finite sets, then $A \cup B$ is finite.

Solution. We have $A \cup B = A \setminus B \cup B \setminus A \cup A \cap B$. Since $A \setminus B \subseteq A$, $B \setminus A \subseteq B$, and $A \cap B \subseteq A$, with all three disjoint, this is a finite disjoint union of finite sets, meaning it is finite.[1]

Solution (Proposed)**.** We know $A \setminus B \subseteq A$; since A is finite, so too is $A \setminus B$ (by Extra Problem 3).

Since $A \cup B = (A \setminus B) \cup B$ is a disjoint union of finite sets, $A \cup B$ is finite.

Remark (Disjoint Union of Finite Sets is Finite)**:** Let A, B be disjoint finite sets. Then, $A \cup B$ is finite.

To prove this, by the definition of finitude, there exist $\alpha : A \to \{1, 2, \ldots, m\}$ and $\beta : B \to \{1, 2, \ldots, n\}$ bijections for some $m, n \in \mathbb{N}$.

We can create a new function $f : A \cup B \to \{1, 2, \ldots, m + n\}$ by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) + m & x \in B \end{cases}.$$

We can see that h is a well-defined bijection since $A \cap B = \emptyset$.

Extra Problem 2

Problem. Prove that for every $n \in \mathbb{N}$, every subset of $\{0, 1, \ldots, n\}$ is finite.

Solution. For any subset $P \subseteq \{0, 1, \ldots, n\}$, the identity map is an injection into $\{0, 1, \ldots, n\}$; composing the identity map with the bijection $a : \{0, 1, \ldots, n\} \to \{1, 2, \ldots, n + 1\}$ defined by $a(m) = m + 1$, we see that there is an injection $a \circ \mathrm{id} : P \hookrightarrow \{1, 2, \ldots, n + 1\}$, meaning P is finite by the theorem above.

[1] In the order of my completing homework, I proved the injection to finite sets, then the subset of a finite set, then this problem.

Extra Problem 3

Problem. Prove that every subset of a finite set is finite.

Solution. Since every empty set is finite, so too is every subset of the empty set. Similarly, any empty subset of a given finite set is also finite.

Let A be a nonempty finite set. Then, there exists a bijection $\alpha : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

Let $B \subseteq A$ be nonempty. The inclusion map $\iota : B \hookrightarrow A$ defined by $\iota(x) = x$ is an injection.

Thus, $\alpha \circ \mathrm{id} : B \hookrightarrow \{1, 2, \dots, n\}$ is an injection, as it is a composition of injections. By the established theorem above, this means B is finite.

Extra Problem 4

Problem. Prove that every infinite subset of $\mathbb{N}$ is denumerable.

Solution. Let $A \subseteq \mathbb{N}$ be infinite.

Since A is nonempty, by the well-ordering principle, there must exist a least element of A, which we label as a_0.

Consider $A \setminus \{a_0\}$. Since A is infinite, $A \setminus \{a_0\}$ must also be infinite, meaning there is a least element of $A \setminus \{a_0\}$ by the well-ordering principle. We label this element as $\{a_1\}$.

Now, we consider $A \setminus \{a_0, a_1\}$, and use the well-ordering principle to extract a_2, and inductively extract a_i by using the well ordering principle on $A \setminus \{a_0, a_1, \dots, a_{i-1}\}$.

The function $f : A \to \mathbb{N}$ defined by $f(a_i) = i$ is a bijection, since $f(a_i) = f\left(a_j\right)$ if and only if $i = j$.

Thus, f is a denumeration of A.