Problem (Problem 2)**:** Prove the claim from class that the open star cover of a simplicial complex is good.

Solution: Let X be the simplicial complex for M. We start by observing that an open n-simplex contained in $\mathbb{R}^n$ is itself contractible, as it is convex, hence there is a straight-line homotopy to any point in its interior.

Now, if $v \in X$ is a vertex, then the open star for v consists of finitely many open simplices that contain v, whence it is possible to contract via a straight-line homotopy to v. If v is not a vertex of X, then v is contained in some open n-simplex, so it is once again possible to straight-line homotopy to v.

Finally, we observe that any non-empty $(k+1)$-fold intersection of open stars in $\mathcal{U}$ defines an open k-simplex of X, whence every point on the interior of the k-simplex can be contracted to the given point via the straight-line homotopy once again.

Thus, the open star cover of M is good.

Problem (Problem 3)**:** Let ω be a closed k-form on a closed manifold M of dimension n, and let η be a closed $(n-k)$-form on M. Prove that if $\omega \wedge \eta$ is nonzero at every point of M, then ω is nonvanishing in $H^k_{\mathrm{DR}}(M)$.

Solution: Since $\omega \wedge \eta$ is nonvanishing, it follows from the definitions that $\Lambda^n T^*M$ admits a smooth nonvanishing section, meaning that M admits an orientation. In particular, integration on M is well-defined.

Thus, we may show that $\omega \wedge \eta$ is not exact by seeing that if there were some $(n-1)$-dimensional form ξ, such that $d\xi = \omega \wedge \eta$, then

$$\begin{aligned}\int_M d\xi &= \int_{\partial M} \xi \\ &= 0,\end{aligned}$$

which would be a contradiction as $\omega \wedge \eta$ is nonvanishing. We observe then that, if $\omega = d\tau$ for some $\tau \in \mathcal{A}^{k-1}(M)$, then

$$\begin{aligned}\omega \wedge \eta &= d\tau \wedge \eta \\ &= d\tau \wedge \eta + (-1)^{k-1}\tau \wedge d\eta \\ &= d(\tau \wedge \eta),\end{aligned}$$

which would be a contradiction.

Problem (Problem 4)**:** Compute the de Rham cohomology of $\mathbb{R}^2 \setminus \{0\}$, and find representatives of all nontrivial classes.

Solution: We observe that $\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R}$, so by the Poincaré lemma, we have

$$H^*_{\mathrm{DR}}\left(\mathbb{R}^2 \setminus \{0\}\right) \cong H^*_{\mathrm{DR}}\left(S^1\right)$$

or

$$\begin{aligned}H^0_{\mathrm{DR}}\left(\mathbb{R}^2 \setminus \{0\}\right) &\cong \mathbb{R} \\ H^1_{\mathrm{DR}}\left(\mathbb{R}^2 \setminus \{0\}\right) &\cong \mathbb{R} \\ H^k_{\mathrm{DR}}\left(\mathbb{R}^2 \setminus \{0\}\right) &\cong 0 \text{ for } k \geqslant 2.\end{aligned}$$

We know that a complete set of representatives for cohomology classes of S^1 are 1 for H^0 and $d\theta$ for H^1. We know from the lemma that then, $d\theta$ corresponds to $\pi^*(d\theta)$, where $\pi\colon S^1 \times \mathbb{R} \to S^1$ is the projection. Thus, we observe that $\{1, \pi^*(d\theta)\}$ is the complete set of representatives of cohomology classes for $H^*_{\mathrm{DR}}\left(\mathbb{R}^2 \setminus \{0\}\right)$.

Problem (Problem 5)**:** Let G be a finite group acting freely on a manifold M by diffeomorphisms. Show that:

- M/G is a manifold;
- the de Rham cohomology of M/G is isomorphic to the G-invariant cohomology of M.

Solution:

(i) We observe that the quotient map $\pi\colon M \to M/G$, taking $p \mapsto [p]$ is a covering map. This follows from the fact that for any $p \in M$, there is a sufficiently small $U \subseteq M$ such that $g \cdot U \cap U = \emptyset$ for all $g \in G$ with $g \neq e$, as G is finite and the action of G on M is free.

Let $\varphi\colon U \to \mathbb{R}^n$ be a coordinate map for $p \in U \subseteq M$, where U is as above (where $g \cdot U \cap U = \emptyset$). An open neighborhood of $[p] \in M/G$, where M/G is endowed with the quotient topology, thus admits $\varphi^*\colon U^* \to \mathbb{R}^n$ by taking $\varphi^*(U^*) = \varphi\left(\pi^{-1}(U^*) \cap U\right)$. Therefore, M/G admits a manifold structure.

Problem (Problem 6)**:** Let U and V be open subsets of a smooth manifold M, and let $W = U \cup V$. Write i_U, i_V for the inclusions of U and V into W respectively, and write j_U, j_V for the inclusions of $U \cap V$ into U and V respectively. Show that the sequence

$$0 \longrightarrow \mathcal{A}^k(W) \xrightarrow{(i_U^*, i_V^*)} \mathcal{A}^k(U) \oplus \mathcal{A}^k(V) \xrightarrow{j_U^* - j_V^*} \mathcal{A}^k(U \cap V) \longrightarrow 0$$

is exact.

Solution: Exactness at $\mathcal{A}^k(W)$ follows from the fact that $\left(i_U^*, i_V^*\right)$ is an inclusion map, hence has kernel 0.

To verify that the sequence is exact at $\mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$, we observe that if $\omega \in \mathcal{A}^k(W)$, then $(\omega|_U, \omega|_V)$ yields zero when subjected to $j_U^* - j_V^*$ as ω when restricted to $U \cap V$ is equal to itself. Therefore, the sequence is exact at $\mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$.

Finally, we let $\{f_U, f_V\}$ be a partition of unity for W subordinate to $\{U, V\}$. If $\omega \in \mathcal{A}^k(U \cap V)$, we observe that $f_U\omega$ extends to 0 on $V \setminus (U \cap V)$, whence $f_U\omega \in \mathcal{A}^k(V)$, and similarly for $f_V\omega \in \mathcal{A}^k(U)$. Therefore, $(f_V\omega, -f_U\omega) \in \mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$ maps to $\omega \in \mathcal{A}^k(U \cap V)$, meaning $j_U^* - j_V^*$ is surjective, so the sequence is exact at $\mathcal{A}^k(U \cap V)$.