Problem (Problem 1)**:**

(a) Let G be a finite group. Show that for any subgroup $H \leqslant G$, we have $n_p(H) \leqslant n_p(G)$.

(b) Let $f\colon G \to G'$ be a surjective homomorphism of finite groups, and let p be a prime. Show that every p-Sylow subgroup P' of G' is the image of some p-Sylow subgroup P of G.

Solution:

(a) Suppose $|G| = p^r m$ and $|H| = p^s \ell$, with $p \nmid m, \ell$.

First, we observe that if $s = r$, then any p-Sylow subgroup of H is a p-Sylow subgroup of G that is contained in H, whence $n_p(H) \leqslant n_p(G)$.

Now, let $s < r$. We observe that if $P \leqslant H \leqslant G$ is a p-Sylow subgroup of H, then by the second Sylow theorem, P is contained in some p-Sylow subgroup, $P' \leqslant G$. We claim that any two distinct p-Sylow subgroups of H must be contained in distinct p-Sylow subgroups of G. This follows from the fact that, if $P_1, P_2 \leqslant H$ are two distinct p-Sylow subgroups, and $P_1, P_2 \leqslant P'$, then the subgroup $\langle P_1, P_2 \rangle$ generated in H is contained in both H and P', but has strictly larger order than either P_1 or P_2, which contradicts the maximality of the orders of P_1 and P_2 respectively. Thus, any p-Sylow subgroup of H is of the form $P' \cap H$ for some p-Sylow subgroup of G, whence $n_p(H) \leqslant n_p(G)$.

(b) Let $N = \ker(f)$, and let P_0 be a p-Sylow subgroup of N. By the second Sylow theorem, there is a p-Sylow subgroup of G, P_1, such that $P_0 \subseteq P_1$. From the first isomorphism theorem, we know that f induces an isomorphism $\hat{f}\colon G/N \to G'$, so we will establish the result in G/N.

First, observe that $\pi\colon G \to G/N$ induces the subgroup P_1N/N in G/N. By the second isomorphism theorem, it then follows that

$$P_1N/N \cong P_1/P_1 \cap N,$$

whence

$$\begin{aligned}|P_1N/N| &= |P_1/P_1 \cap N| \\ &= \frac{|P_1|}{|P_1 \cap N|} \\ &= \frac{|P-1|}{|P_0|},\end{aligned}$$

meaning that $f(P_1)$ is a p-group. Furthermore, since P_1 is a maximal p-group, it follows that $p \nmid [G : P_1]$, and since $P_1 \subseteq P_1N$, we have that $p \nmid [G : P_1N]$. Yet, this means that p does not divide

$$\begin{aligned}\frac{|G|}{|P_1N|} &= \frac{|G/N|}{|P_1N/N|} \\ &= [G/N : P_1N/N],\end{aligned}$$

meaning that P_1N/N is in fact a maximal p-subgroup of G/N.

Now, we show that any p-Sylow subgroup of G/N arises in this fashion. In particular, suppose $P' \leqslant G' \cong G/N$ is a p-Sylow subgroup. Then, by the fourth isomorphism theorem, it follows that P' corresponds to a subgroup $P_2 \in G$ with $N \leqslant P_2$. Such a P_2 must be a p-subgroup by Lagrange's theorem, as P_2 surjects onto P'. Our goal is to show that $p \nmid [G : P_2]$, which as we showed before, is equivalent to showing that $p \nmid [G/N : P_2N/N]$.

Now, since P_2 is a p-subgroup, there is some conjugate gP_2g^{-1} such that gP_2g^{-1} is a subgroup of P_1 by the second Sylow theorem.

Yet, this means that $(gN)(P_2N/N)(gN)^{-1} \leqslant P_1N/N$, meaning that $|P_2N/N| = |P_1N/N|$ as we have stated that $P' \cong P_2N/N$ is a p-Sylow subgroup in and of itself. Therefore, it follows that $[G/N : P_2N/N] = [G/N : P_1N/N]$, so that $p \nmid [G : P_2]$, thus P_2 is a p-Sylow subgroup.

Problem (Problem 2)**:** Let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Show that the group of upper unitriangular matrices,

$$U = \left\{ (a_{ij})_{i,j} \in GL_n(F) \,\middle|\, a_{ij} = 0 \text{ for } i > j \text{ and } a_{ii} = 1 \right\}$$

is a p-Sylow subgroup of $G = GL_n(F)$.

Solution: To start, we observe that the group of upper unitriangular matrices has an order of $p^{n(n-1)/2}$, as follows from the fact that all elements in the strict upper triangle of any given matrix can be pulled from $\mathbb{Z}/p\mathbb{Z}$.

The order of $GL_n(F)$ can be seen to be

$$|GL_n(F)| = (p^n - 1)(p^n - p)\cdots\left(p^n - p^{n-1}\right).$$

Taking out powers of p from each of the factors that divides p, we find that we get

$$\begin{aligned} |GL_n(F)| &= (p^n - 1)(p)\left(p^{n-1} - 1\right)\cdots\left(p^{n-1}\right)(p-1) \\ &= p \cdot p^2 \cdots p^{n-1}(p^n - 1)\left(p^{n-1} - 1\right)\cdots(p-1) \\ &= p^{n(n-1)/2}(p^n - 1)\left(p^{n-1} - 1\right)\cdots(p-1). \end{aligned}$$

Now, we observe that each of the factors of the form $p^k - 1$ are coprime to p^k, meaning that p necessarily cannot divide $p^k - 1$ for each k, so that any p-Sylow subgroup of $GL_n(F)$ has the order $p^{n(n-1)/2}$.

Problem (Problem 4)**:** Let G be a finite group of order p^n with $n \geqslant 1$.

Show that for every $m = 0, 1, \dots, n$, the group G has a subgroup of order p^m.

Solution: We prove using induction. If $n = 1$, then G is a group of the form $\mathbb{Z}/p\mathbb{Z}$, meaning that G has a subgroup of order 1, which is $\{e\}$, and a group of order p, which is the group itself.

Now, let $|G| = p^n$, and suppose that for any group with order p^k with $k < n$, we have that said group contains subgroups of all prime orders from 0 to k. Letting G act on itself via conjugation, we obtain from the class equation that

$$|G| = |Z(G)| + \sum_{a \in A} [G : Z_G(a)].$$

Taking residues modulo p, we observe that $|Z(G)| \equiv |G|$ modulo p, and since $|Z(G)| \geqslant 1$ as $\{e\} \in Z(G)$, it follows that $|Z(G)| \geqslant p$.

Now, if $Z(G) = G$, then G is abelian, so by the structure of finite abelian groups, we have

$$G \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k}\mathbb{Z},$$

where we may use powers of p as the sole factors by virtue of the fact that G is a p group. Taking residue classes modulo $\mathbb{Z}/p\mathbb{Z} \times \{0\} \times \cdots \times \{0\}$, we observe then that the quotient group $G/(\mathbb{Z}/p\mathbb{Z})$ has subgroups of orders up to $n - 1$ all of which contain $\mathbb{Z}/p\mathbb{Z}$ by the fourth isomorphism theorem, whence G has subgroups of all orders up to n.

Meanwhile, if $Z(G) \neq G$, then $Z(G)$ is a p-group of order less than p^n, and the quotient group $G/Z(G)$ has order strictly less than p^n as well, meaning that the former contains p-subgroups up to the order of $Z(G)$, while the latter contains p-subgroups up to the order of $G/Z(G)$, each of which contains $Z(G)$, so that G contains p-subgroups of all orders up to p^n.

Problem (Problem 5)**:** Show that a group of order 351 always has a normal p-Sylow subgroup for some prime p dividing the order.

Solution: The prime factorization of 351 yields $3^3 \cdot 13$. We observe that the number of 13-Sylow subgroups is congruent to 1 modulo 13 and divides 27; in particular, we have the cases of 1 and 27. If there is one 13-Sylow subgroup, this subgroup is normal, and we are done. Else, if there are 27 13-Sylow subgroups, these subgroups intersect at the identity (as each is isomorphic to $\mathbb{Z}/13\mathbb{Z}$), and thus there are 324 elements of order 13, giving 27 elements with order not equal to 13. Since, by the first Sylow theorem, there is at least one 3-Sylow subgroup, this is the 3-Sylow subgroup, which is necessarily normal.

Problem (Problem 8)**:** Let G be a group of order $3 \cdot 5^2 \cdot 17$.

(a) Show that $n_{17}(G) = 1$. That is, a 17-Sylow subgroup H is normal.

(b) The conjugation action of G on H defines a group homomorphism $G \to \operatorname{aut}(H)$. Show that this homomorphism is trivial, and conclude that $H \subseteq Z(G)$.

Solution:

(a) By the third Sylow theorem, we know that $n_{17}(G)|75$ and $n_{17}(G) \equiv 1$ modulo 17. Writing out the possibilities for n_{17} under the second condition explicitly gives

$$n_{17}(G) = 1, 18, 35, 52, 69, 86, \dots$$

of which only 1 divides 75. Thus, there is only one 17-Sylow subgroup.

(b) Let H be the 17-Sylow subgroup of G. Since $gHg^{-1} = H$ for all $g \in G$, it follows that the map $g \mapsto \iota_g$ defines a group homomorphism $f\colon G \to \operatorname{aut}(H)$. Since H is abelian, $H \leqslant \ker(f)$, so by the first isomorphism theorem, there is an induced homomorphism $\overline{f}\colon G/H \to \operatorname{aut}(H)$.

Now, we observe that, since H has prime order, $H \cong \mathbb{Z}/17\mathbb{Z}$, meaning that $\operatorname{aut}(H) \cong (\mathbb{Z}/17\mathbb{Z})^{\times}$, which is a group of order 16, while $|G/H| = 75$. Yet, since $16 \nmid 75$, it follows that $\overline{f}$ must in fact be the trivial homomorphism, meaning that $ghg^{-1} = g$ for each $g \in G$ (as it is already true for all $g \in H$ and any representative for $gH \in G/H$). Therefore, $H \subseteq Z(G)$.