Cardinality and Countability

Section 1.1: Countable Sets

Definition (Denumerable Set). A set S is denumerable if there exists a function $f : S \to \mathbb{N}$ with f a bijection. We also say S is countably infinite.

Definition (Countable Set). We say S is countable if S is either finite or denumerable.

Theorem (Countability of Unions). *If A and B are countable sets, then $A \cup B$ is countable.*

Theorem (Countability of Subsets). *If $A \subseteq B$, then if B is countable, then A is countable.*

Theorem (Union of Finite Sets). *If A and B are finite, then $A \cup B$ is finite.*

Proof. If A is finite and B has one element, then we show that $A \cup B$ is finite (with two cases).

Afterward, for $|B| > 1$, we use induction on $|B|$. □

Definition (Finite Set). A set A is finite if there exists a bijection $f : S \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N} = \{0, 1, \dots\}$.

We write $|A| = n$.

Theorem (Disjoint Union of Countable Sets). *If A is denumerable, B is finite, and $A \cap B = \emptyset$, then $A \cup B$ is denumerable.*

Proof. There exists a bijection $f : A \to \mathbb{N}$ (since A is denumerable), and a bijection $g : B \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ (since B is finite).

We create a new bijection $h : A \cup B \to \mathbb{N}$ by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since $A \cap B = \emptyset$, we know that h is well-defined.

Now, we must show that h is a bijection.

Suppose $h(x) = h(y)$.

Case 1: If $x, y \in B$, then $h(x) = g(x) - 1$, and $h(y) = g(y) - 1$, meaning $g(x) - 1 = g(y) - 1$, meaning $g(x) = g(y)$. Since g is a bijection, $x = y$.

Case 2: If $x, y \in A$, a similar argument yields that $x = y$

Case 3: Without loss of generality, let $x \in A$ and $y \in B$. If $x \in A$, then $h(x) = f(x) + n$ and $h(y) = g(y) - 1$. Thus, $f(x) + n = g(y) - 1$. However, since $f(x) + n \geqslant n$ and $0 \leqslant g(y) - 1 \leqslant n - 1$. Thus, we get that $0 \leqslant n \leqslant n - 1$, which is a contradiction.

Thus, we have shown that h is injective. □

Theorem (Cartesian Product of Natural Numbers). *$\mathbb{N} \times \mathbb{N}$ is denumerable.*

Proof. We consider $\mathbb{N} \times \mathbb{N}$ as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \cdots,$$

$$\begin{array}{lccccc}
\mathbb{N}\times\{0\}: & (0,0) & (1,0) & (2,0) & (3,0) & \cdots \\
\mathbb{N}\times\{1\}: & (0,1) & (1,1) & (2,1) & (3,1) & \cdots \\
\mathbb{N}\times\{2\}: & (0,2) & (1,2) & (2,2) & (3,2) & \cdots \\
\mathbb{N}\times\{3\}: & (0,3) & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{lccccc}
\mathbb{N}\times\{0\}: & \cancel{(0,0)}^{0} & \cancel{(1,0)}^{2} & \cancel{(2,0)}^{5} & \cancel{(3,0)}^{9} & \cdots \\
\mathbb{N}\times\{1\}: & \cancel{(0,1)}^{1} & \cancel{(1,1)}^{4} & \cancel{(2,1)}^{8} & (3,1) & \cdots \\
\mathbb{N}\times\{2\}: & \cancel{(0,2)}^{3} & \cancel{(1,2)}^{7} & (2,2) & (3,2) & \cdots \\
\mathbb{N}\times\{3\}: & \cancel{(0,3)}^{6} & (1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can also find a bijection $P : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, with

$$P(x,y) = \frac{(x+y)(x+y+1)}{2} + x$$

A fun challenge is to prove that P is a bijection. □

Theorem (Countability of the Rationals)**.** $\mathbb{Q}$ *is denumerable.*

Theorem (Countability of the Integers)**.** *The set* $\mathbb{Z}$ *is denumerable.*

Proof. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 2x & x \geqslant 0 \\ -2x-1 & x < 0 \end{cases}$$

□

Definition (Cardinality)**.** We say two sets, A and B, have the same cardinality if there exists a bijection $f : A \to B$.

Theorem (Finite Subset Cardinality)**.** *If* $m, n \in \mathbb{N}$ *and* $m \neq n$*, then* $\{1,2,\dots,m\}$ *and* $\{1,2,\dots,n\}$ *do not have the same cardinality.*

Theorem (Infinitude of the Natural Numbers)**.** $\mathbb{N}$ *is not finite.*

Example. If $A \subsetneq B$ and $|A| = |B|$, then both A and B are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.

Section 1.2

Definition (Uncountable Set)**.** A set is uncountable if it is not countable.

Theorem (Uncountability of $\mathbb{R}$)**.** $\mathbb{R}$ *is uncountable.*

Proof. For all $x \in \mathbb{R}$, and for all $j \in \mathbb{N}$, we define $[x]_j$ to denote the $j+1$-th digit after the decimal point in the decimal expansion of x.

For example, $[\pi]_0 = 1$, $[\pi]_1 = 4$, etc.

Let $f : \mathbb{N} \to \mathbb{R}$. We will show that f is not surjective.

Let $y \in [0,1) \subseteq \mathbb{R}$ defined by $\forall j \in \mathbb{N}$,

$$[y]_j = \begin{cases} 0 & [f(j)]_j = 1 \\ 1 & [f(j)]_j \neq 1 \end{cases}.$$

We claim that $y \notin f(\mathbb{N})$. We will show that $\forall j \in \mathbb{N}$, $f(j) \neq y$.

We can see that if $[f(j)]_j = 1$, then $[y]_j = 0$. Similarly, if $[f(j)]_j \neq 1$, then $[y]_j = 1$. Either way, $[f(j)]_j \neq [y]_j$ for all $j \in \mathbb{N}$. □

Remark: The above proof is an example of a diagonalization proof. It can be imagined as

$$\begin{array}{c|c} f(0) & *.\cancel{a_1}\, a_2 a_3 \dots \\ f(1) & *.b_1 \cancel{b_2}\, b_3 \dots \\ f(2) & *.c_1 c_2 \cancel{c_3} \dots \\ \vdots & \vdots \end{array}$$

Note: A substantial problem that we might need to deal with is that a real number does not necessarily have a unique decimal representation. For instance, $3.999\cdots = 4.000\dots$.

In order to resolve this issue, we can default to the option with trailing 0 over trailing 9.

Definition (Power Set)**.** The power set of a set S is

$$P(S) = \{A \mid A \subseteq S\}.$$

Theorem (Power Set Surjection)**.** *Let* $f : S \to P(S)$. *Then,* f *is not surjective.*

Proof. Let $T = \{x \in S \mid x \notin f(x)\}$. Then, $T \notin f(S)$.

Let $y \in S$. We want to show that $f(y) \neq T$. Suppose toward contradiction that $f(y) = T$. Then, if $y \in T$, then $y \in f(y)$, which implies that $y \notin T$.

If $y \notin T$, then $y \notin f(y)$, which implies that $y \in T$.

Thus, it cannot be the case that $f(y) = T$. □

Definition (Cardinality Comparison)**.** Let A and B be sets. Then, we write $\text{card}(A) \leqslant \text{card}(B)$ if there exists an injective map $f : A \hookrightarrow B$.

We write $\text{card}(A) < \text{card}(B)$ if there exists an injection $f : A \hookrightarrow B$ but no bijection.

Example (Cardinality of the Power Set)**.** For every set,

$$\text{card}(S) < \text{card}(P(S)).$$

(1) We know that $\text{card}(S) \leqslant \text{card}(P(S))$, defining $f : S \hookrightarrow P(S)$, $f(a) = \{a\}$, since if $f(x) = f(y)$, then $\{x\} = \{y\}$, meaning $x \in \{y\}$, so $x = y$.

    In the case of $f : \emptyset \to \{\emptyset\}$, we define $\emptyset = f \subseteq \emptyset \times \{\emptyset\}$.

(2) Since there exists no bijection $f : S \to P(S)$, it is the case that $\text{card}(S) \neq \text{card}(P(S))$.

Example (Decimal Expansion)**.** We know that for some decimal expansion

$$\begin{aligned} 3.14159\ldots &= 3 + \frac{1}{10} + \frac{4}{100} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{n_i}{10^i}, \end{aligned}$$

with $0 \leqslant n_i \leqslant 9$ for $i \geqslant 1$.

However, we can also write any real number as

$$\sum_{i=0}^{\infty} \frac{n_i}{3^i}$$

with $0 \leqslant n_i \leqslant 2$ for all $i \geqslant 1$.

Example (Finite Strings)**.** Let S be the set of all finite strings of 0 and 1. S is countable.

Proof 1: We define $f : S \to \mathbb{N}$ by, for a string $x \in S$, x starts with n_1 zeroes, then has n_2 ones, then n_3 zeroes, etc. We define $f(x) := 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times 7^{n_4} \times 11^{n_5} \cdots$, or

$$f(x) = \prod_{i}^{\infty} p_i^{n_i},$$

where p_i denotes the ith prime number. We can see that f is an injection.

Since S is infinite (proof omitted), we can see that $f(S)$ is also infinite.[1] Since $f(S)$ is an infinite subset of $\mathbb{N}$, $f(S)$ is denumerable, meaning there exists a bijection $q : f(S) \to \mathbb{N}$. Therefore, we have $q \circ f : S \to \mathbb{N}$ is a bijection, meaning S is denumerable.

Proof 2: List the elements of S by length and lexicographic order: short strings come before long strings, and 0s come before 1s.

Rank	String
0	0
1	1
2	00
3	01
4	10
5	11
$\vdots$	$\vdots$

This pattern yields a systematic way to map S to the natural numbers.

Proof 3: We can see that

$$S = \bigcup_{i=1}^{\infty} S_i,$$

where S_i is the set of all strings of length i, each of which contains 2^i elements.

Since each S_i is finite, and $S_i \cap S_j = \emptyset$ (by definition). Thus, S is a countable union of pairwise disjoint countable sets, so S is countable.

[1] If $f(S)$ is finite, then there exists a bijection $g : f(S) \to \{1, \ldots, n\}$. Composing g and f, we find S is finite as $g \circ f|_S$ is a bijection.

Example (All Possible Writings)**.** Let W be the set of all possible writings in English. We let W_n denote the writing with n characters. Then,

$$W = \bigcup_{n=1}^{\infty} W_n,$$

which is a countable union of disjoint finite sets, which is countable.

Similarly, we can list all the writings by length and lexicographic order.

This result implies that "almost all" real numbers, in a sense, are unable to be described.

Section 1.3: Cantor–Schröder–Bernstein Theorem

Example. If we have $|A| \leqslant |B|$ and $|B| \leqslant |A|$, it does not necessarily imply $|A| = |B|$.

This is because the $\leqslant$ in the cardinality comparison implies there exist injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, not that the cardinalities are necessarily "less than or equal to" each other.

However, at the same time, this fact is true — this is what is known as the Cantor–Schröder–Bernstein Theorem.

Theorem (Cantor–Schröder–Bernstein)**.** *Let* $f : C \hookrightarrow D$ *and* $g : D \hookrightarrow C$ *be injective maps. Then,* $|C| = |D|$.

An Informal Proof Sketch. Consider C to be a set of cats and D to be a set of dogs. Every cat chases a dog, and every dog chases a cat, with different cats chasing different dogs and vice versa.

There are four potential arrangements:

(1) A set of cats and dogs are chasing each other in a circle.

(2) A chain of dogs chasing cats that starts with a dog.

(3) A chain of cats chasing dogs that starts with a cat.

(4) An endless chain of cats chasing dogs with no discernible start or end point.

These four cases create a bijection from C to D:

(1) Pair each cat with the dog that it is chasing.

(2) Pair each cat with the dog that it is chasing.

(3) Pair each cat with the dog that *is chasing it*.

(4) Pair each cat with the dog that it is chasing.

□

A More Formal Proof Sketch. For $C = \{c_i\}_{i\in I}$ and $D = \{d_i\}_i$, we have four types of sequences.

(i) Circular sequence: for some $m \in \mathbb{N}$, there exist $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$, where $c_{m+1} = c_1$.

(ii) Cat sequence: there is $c_1, c_2, \ldots$ and $d_1, d_2, \ldots$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

(iii) Dog sequence: there is $c_1, c_2, \ldots$ and $d_1, d_2, \ldots$ such that $f(c_i) = d_{i+1}$ and $g(d_i) = c_i$.

(iv) Bi-infinite sequence: $\{c_i\}_{i\in\mathbb{Z}}$ and $\{d_i\}_{i\in\mathbb{Z}}$ such that $f(c_i) = d_i$ and $g(d_i) = c_{i+1}$.

Claim 1: For every $c \in C$, c is in exactly one sequence that is either a circular sequence, a cat sequence, a dog sequence, or a bi-infinite sequence.

We define our bijection $h : C \to D$ by

$$h(c) = \begin{cases} g^{-1}(c) & c \text{ in a dog sequence} \\ f(c) & \text{else} \end{cases}.$$

Claim 2: h is well-defined.

Claim 3: h is a bijection.

□

Theorem. *For every set* A, B, *either* $|A| \leqslant |B|$ *or* $|B| \leqslant |A|$.

In order to prove this, we need the axiom of choice.

Example (Cardinality of the Reals)**.** Recall that $|\mathbb{N}| < |P(\mathbb{N})|$ and $|\mathbb{N}| < |\mathbb{R}|$. According to the previous theorem, it is the case that either $|P(\mathbb{N})| \leqslant |\mathbb{R}|$ or $|\mathbb{R}| \leqslant |P(\mathbb{N})|$.

In particular, $|P(\mathbb{N})| = |\mathbb{R}|$.

An Informal Proof. Let S be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. We will show that $|S| = |P(\mathbb{N})|$ and $|S| = |\mathbb{R}|$. This will show that $|P(\mathbb{N})| = |\mathbb{R}|$ (by composing bijections).

To show that $|S| = |P(\mathbb{N})|$, define a subset of $\mathbb{N}$ by the support[II] of some element of S. This is a bijection between $P(\mathbb{N})$ and S.

To show $|S| = |\mathbb{R}|$, we place a decimal point in front of the string, and consider it as a real number in base 2, which yields a bijection between S and $[0, 1]$.

Next, we show that $|[0, 1]| = |(0, 1)|$.

Finally, we show that $|(0, 1)| = \mathbb{R}$. Take $f : (0, 1) \to \mathbb{R}$ to be $\cot(\pi x)$ — or $\tan(\pi x - \pi/2)$. These are bijections from $(0, 1)$ to $\mathbb{R}$. □

Definition (Continuum Hypothesis)**.** We are aware that

$$|\mathbb{N}| < |\mathbb{R}| = |P(\mathbb{N})|.$$

The continuum hypothesis states that there exists no set S such that

$$|\mathbb{N}| < |\S| < |\mathbb{R}|.$$

The continuum hypothesis is independent of the ZFC axioms.[III]

[II]The elements that f does not map to 0 for some $f \in S$.
[III]Zermelo–Frankel Axioms with the Axiom of Choice.