

Problem 1

If F is a finite set and $k : F \rightarrow F$ is a self-map, prove that k is injective if and only if k is surjective.

Let k be injective.

$$\begin{aligned} \text{card}(F) &= \text{card}(k(F)) && \text{definition of injection} \\ k(F) &\subseteq F && \text{definition of function} \\ k(F) &= F \end{aligned}$$

Let k be surjective.

$$\begin{aligned} k(F) &= F && \text{definition of surjection} \\ \text{card}(k(F)) &= \text{card}(F) && \text{cardinality definition} \end{aligned}$$

Therefore, k is injective, as F is finite.

Problem 2

Prove that a set A is infinite if and only if there is a non-surjective injection $f : A \hookrightarrow A$.

(\Rightarrow) Let A be infinite. Then, $\exists i : \mathbb{N} \hookrightarrow A$; $\forall n \in \mathbb{N}, a_n := i(n)$. Let $f : A \rightarrow A$, $f(a_i) = a_{i+1}$. Then, for $a_{i_1} \neq a_{i_2}$, $f(a_{i_1}) = a_{i_1+1} \neq f(a_{i_2}) = a_{i_2+1}$. Therefore, f is injective.

(\Leftarrow) Suppose A is finite. Then, by the result in Problem 1, $\forall f : A \hookrightarrow A$, f must be surjective.

Problem 3

Let A , B , and C be sets and suppose $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$. Prove that $\text{card}(A) < \text{card}(C)$.

Problem 4

If $A \subseteq B$ is an inclusion of sets with A countable and B uncountable, show that $B \setminus A$ is uncountable.

Suppose toward contradiction that $B \setminus A$ is countable.

Then, $A \cup (B \setminus A)$ must be countable, by union of countable sets.

However, $A \cup (B \setminus A) = B$, and B is uncountable, meaning that $B \setminus A$ must be uncountable.

Problem 5

Is the set $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ countable?

Let $q : \mathbb{Q} \rightarrow \mathbb{N}$ be the enumeration of the rationals; let $f : \{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ be defined as $f(x) = q(x^2)$.

$x > 0 \longrightarrow t(x) = x^2$ is a bijection; \mathbb{Q} countable $\longrightarrow q$ is a bijection; $f = q \circ t \longrightarrow f$ is a bijection, and thus $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ is countable.

Problem 6

Consider the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of \mathbb{N} . Is $\mathcal{F}(\mathbb{N})$ countable?

Let $f : \mathcal{F} \rightarrow \mathbb{N}$ be defined as follows, where p_n denotes the n th prime number.

$$f(\{a_1, a_2, \dots, a_n\}) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_n^{a_n}$$

By the fundamental theorem of arithmetic, every natural number is equal to a unique product of powers of prime numbers, meaning that f is injective, so \mathcal{F} is countable.

Problem 7

Let $k \in \mathbb{N}$.

- (i) Prove that $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ is countable.
- (ii) Show that the set $\mathbb{N}^\infty := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}\}$ consisting of all sequences of natural numbers is uncountable.
- (iii) Prove that the set of **finitely-supported** natural sequences $c_c(\mathbb{N}) := \{(n_k)_{k \geq 1} \mid n_k \in \mathbb{N}, n_k = 0 \text{ for all but finitely many } k\}$ is countable.

(i)

Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be defined as follows, where p_n denotes the n th prime number in the sequence $\{2, 3, 5, \dots\}$

$$f((a_1, a_2, \dots, a_k)) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$$

By the fundamental theorem of arithmetic, every natural number is equal to a unique product of powers of prime numbers, so $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is an injection, meaning \mathbb{N}^k is countable

(ii)

Suppose toward contradiction that the set of all sequences of natural numbers is countable: $f : A_n \rightarrow \mathbb{N}$ is surjective.

$$A_1 = \{a_{11}, a_{12}, a_{13}, \dots\}$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, \dots\}$$

$$\vdots$$

Create a new sequence N defined as follows:

$$n_k = a_{kk} + 1$$

Since f is surjective, $\exists A_m = \{a_{m1}, a_{m2}, \dots, a_{mm}, \dots\} = \{n_1, n_2, \dots, n_m\}$. However, since by definition, $n_m \neq a_{mm}$, f must not be surjective. Thus, \mathbb{N}^∞ is not countable.

(iii)

Let $f : c_c(\mathbb{N}) \rightarrow \mathbb{N}$ be defined as follows, where p_n denotes the n th prime number:

$$f(\{n_1, n_2, \dots, n_k\}) = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$$

Since every natural number is represented uniquely by a product of powers of primes by the fundamental theorem of arithmetic, f is injective, meaning $c_c(\mathbb{N})$ is countable.

Problem 8

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range $\text{ran}(f)$ cannot contain any interval.

Problem 9

Prove that the set

$$\mathcal{P} := \left\{ \sum_{k=0}^n a_k x^k \mid n \subseteq \mathbb{N}_0, a_k \in \mathbb{Q} \right\}$$

consisting of all polynomials with rational coefficients, is countable.

Let $q : \mathbb{Q} \rightarrow \mathbb{N}$ be the enumeration of the rationals, and let p_n denote the n th element in the sequence of prime numbers, where $p_1 = 2, p_2 = 3$, etc.

Let $f : \mathcal{P} \rightarrow \mathbb{N}^k$ be defined as follows:

$$f(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots) = (q(a_0), q(a_1), \dots, q(a_k), \dots)$$

Since \mathbb{Q} is countable, $\forall a \in \mathbb{Q}, q(a) \in \mathbb{N}$, so the output of f is a bijection to \mathbb{N}^k , meaning \mathcal{P} is countable.

Problem 10

A real number t is called **algebraic** if there is a nonzero polynomial p with rational coefficients such that $p(t) = 0$. If $t \in \mathbb{R}$ is not algebraic, then it is called **transcendental**. For example, $\sqrt{2}$ is algebraic, but π is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.

Because \mathcal{P} is countable, and there are k roots in a k -degree polynomial, there are a countable number of polynomial roots, so the algebraic numbers are rational.

Since \mathbb{R} is uncountable, $\mathbb{R} \setminus \mathbb{A}$ is uncountable as shown by a previous result.