Problem (Problem 1)**:**

(a) Determine every holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $\operatorname{Re}(f(z)) = \operatorname{Re}(z)^2 - \operatorname{Im}(z)^2$.

(b) Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) := \sqrt{|\operatorname{Re}(z)\operatorname{Im}(z)|}.$$

Show that the Cauchy–Riemann equations are satisfied for f at $z = 0$, but f is not differentiable at $z = 0$.

Solution:

(a) We want to determine $f\colon \mathbb{C} \to \mathbb{C}$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

satisfies

$$u(x, y) = x^2 - y^2,$$

and the Cauchy–Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

First, we must verify that u is indeed harmonic. This follows from the fact that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= 2 \\ \frac{\partial^2 u}{\partial y^2} &= -2. \end{aligned}$$

Furthermore, we see that u is C^3, as all of the third partial derivatives are equal to zero. Therefore, a harmonic conjugate of u exists and ensures that f is holomorphic on $\mathbb{C}$. By evaluating the Cauchy–Riemann equations separately, we find that

$$\frac{\partial v}{\partial y} = 2x,$$

or $v = 2xy + K(x)$, and

$$-\frac{\partial v}{\partial x} = -2y,$$

or $v = 2xy + L(y)$. These are only in harmony when $v = 2xy + c$, where $c \in \mathbb{C}$ is a constant. Thus, we find that

$$f(x + iy) = \left(x^2 - y^2\right) + i(2xy) + c$$

is necessarily (up to a constant) unique.

(b) We write f as

$$f(x + iy) = \sqrt{|xy|}.$$

Problem (Problem 2)**:** Let $U \subseteq \mathbb{C}$ be a region, and let $f\colon U \to \mathbb{C}$ be a function.

(a) Suppose that f and $\overline{f}$ are both holomorphic. Show that f is constant.

(b) Suppose that f is holomorphic and $\operatorname{Re}(f)$ is constant. Show that f is constant.

Solution:

(a) Write $f(x+iy) = u(x,y) + iv(x,y)$. Since f is holomorphic, we thus get

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Now, since $\overline{f}$ is also holomorphic, we have

$$\overline{f(x+iy)} = u(x,y) - iv(x,y),$$

meaning that

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}\end{aligned}$$

or that

$$\begin{aligned}\frac{\partial u}{\partial x} &= \pm\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= \pm\frac{\partial v}{\partial x}.\end{aligned}$$

Considering the first equation, we then get that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, or that

$$\begin{aligned}u &= c_1(y)\\ v &= d_1(x),\end{aligned}$$

while in the second equation, we get that $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$, meaning that u and v are thus constant. Therefore, f is constant.

(b) If f is holomorphic and $\operatorname{Re}(f)$ is constant, then $i\operatorname{Im}(f) = f - \operatorname{Re}(f)$ is holomorphic as it is the difference of two holomorphic functions, so $-i\operatorname{Im}(f)$ is holomorphic as it is a constant multiple of a holomorphic function, and thus $\operatorname{Re}(f) - i\operatorname{Im}(f)$ is holomorphic as it is the sum of two holomorphic functions. This gives $\overline{f}$ is holomorphic, so f is constant.

Problem (Problem 3)**:** Let $U, V \subseteq \mathbb{C}$ be open sets, $f\colon V \to U$ holomorphic for which $\operatorname{Re}(f), \operatorname{Im}(f) \in C^2(V)$, and $u\colon U \to \mathbb{R}$ harmonic. Show that $u \circ f\colon V \to \mathbb{R}$ is a harmonic function.

Solution: We write $f(x+iy) = k(x,y) + \ell(x,y)$, so that $u \circ f(x+iy) = u(k(x,y), \ell(x,y))$. Observe then that this means $u \circ f$ is in $C^2(V)$, and that u is harmonic as a function of k and ℓ.

Using the fact that $u \circ f$ is in $C^2(V)$, we use the chain rule by taking

$$\frac{\partial^2(u\circ f)}{\partial x^2} + \frac{\partial^2(u\circ f)}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial(u\circ f)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial(u\circ f)}{\partial y}\right)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial k}\frac{\partial k}{\partial y} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial y}\right)\\
&= \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial x^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial x^2} + \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial y^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial y^2}\\
&+ \frac{\partial k}{\partial x}\left(\frac{\partial k}{\partial x}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial x}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial k}\right) + \frac{\partial \ell}{\partial x}\left(\frac{\partial k}{\partial x}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial x}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial \ell}\right)\\
&+ \frac{\partial k}{\partial y}\left(\frac{\partial k}{\partial y}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial y}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial k}\right) + \frac{\partial \ell}{\partial y}\left(\frac{\partial k}{\partial y}\frac{\partial}{\partial k} + \frac{\partial \ell}{\partial y}\frac{\partial}{\partial \ell}\right)\left(\frac{\partial u}{\partial \ell}\right)\\
&= \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial x^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial x^2} + 2\frac{\partial^2 u}{\partial k\partial \ell}\frac{\partial k}{\partial x}\frac{\partial \ell}{\partial x} + \frac{\partial u}{\partial k}\frac{\partial^2 k}{\partial y^2} + \frac{\partial u}{\partial \ell}\frac{\partial^2 \ell}{\partial y^2} + 2\frac{\partial^2 u}{\partial k\partial \ell}\frac{\partial k}{\partial y}\frac{\partial \ell}{\partial y}\\
&+ \frac{\partial^2 u}{\partial k^2}\left(\frac{\partial k}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial \ell^2}\left(\frac{\partial \ell}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial k^2}\left(\frac{\partial k}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial \ell^2}\left(\frac{\partial \ell}{\partial y}\right)^2,
\end{aligned}
$$

where we first used the fact that the mixed partials of u are continuous as u is harmonic. Since k and ℓ are C^2 real/imaginary components of a holomorphic function, they are harmonic, so by reducing via the Cauchy–Riemann equations, we find

$$
\begin{aligned}
&= \frac{\partial u}{\partial k}\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2}\right) + \frac{\partial u}{\partial \ell}\left(\frac{\partial^2 \ell}{\partial x^2} + \frac{\partial^2 \ell}{\partial y^2}\right)\\
&+ \frac{\partial^2 u}{\partial k\partial \ell}\left(\frac{\partial \ell}{\partial y}\right)\frac{\partial \ell}{\partial x} + \frac{\partial^2 u}{\partial k\partial \ell}\left(-\frac{\partial \ell}{\partial x}\right)\frac{\partial \ell}{\partial y}\\
&+ \left(\frac{\partial k}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial k^2} + \frac{\partial^2 u}{\partial \ell^2}\right) + \left(\frac{\partial k}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial k^2} + \frac{\partial^2 u}{\partial \ell^2}\right)\\
&= 0,
\end{aligned}
$$

so $u \circ f$ is harmonic.

Problem (Problem 4)**:** Define $g\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $g(z) = \frac{z+1}{z-1}$ and $f(z) = e^{g(z)}$.

(a) Prove that f is bounded in $\mathbb{D}$.

(b) Compute $\lim_{t\searrow 0} f(t + (1-t)a)$ for all $a \in \partial\mathbb{D} \setminus \{1\}$.

(c) Compute $\lim_{\theta\searrow 0} f\left(e^{i\theta}\right)$.

(d) Compute $\lim_{\theta\nearrow 0} f\left(e^{i\theta}\right)$.

Solution:

(a) We start by observing that

$$
\begin{aligned}
|f(z)| &= \left|e^{g(z)}\right|\\
&= e^{\operatorname{Re}(g(z))}.
\end{aligned}
$$

Therefore, to establish that $f(z)$ is bounded, we must establish an upper bound on $\operatorname{Re}(g(z))$ when $z \in \mathbb{D}$. To this end, we establish that g maps $\mathbb{D}$ to the left half-plane, $\{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$.

We start with the Cayley transform,

$$
h_1(z) = \frac{z-i}{z+i},
$$

which bijectively maps the upper half plane to the unit disc. Therefore, the inverse of the Cayley transform, given by

$$\begin{aligned} h_2(z) &= \frac{iz + i}{-z+1} \\ &= \frac{i(z+1)}{-(z-1)} \\ &= -i\frac{z+1}{z-1} \end{aligned}$$

bijectively maps the unit disc to the upper half plane. Since

$$g(z) = ih_2(z),$$

it follows that $g(z)$ bijectively maps $\mathbb{D}$ to the left half-plane, meaning that $\operatorname{Re}(g(z)) < 0$ for all $z \in \mathbb{D}$, so f is bounded on $\mathbb{D}$.

(b) Since e^w is defined for all $w \in \mathbb{C}$, we may evaluate the limit in g, then apply the exponential to obtain our desired result.