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## Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and  $C^*$ -algebras. This section of notes will be focused on the essential results in functional analysis, starting from normed vector spaces, working our way up through  $C^*$ -algebras.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

## Normed Vector Spaces

### Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over  $\mathbb{C}$ . Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

**Definition** (Vector Space). A vector space  $V$  is a set closed under two operations

$$\begin{aligned} \alpha : V \times V &\rightarrow V, (v_1, v_2) \mapsto v_1 + v_2 \\ m : \mathbb{C} \times V &\rightarrow V, (\lambda, v) \mapsto \lambda v. \end{aligned}$$

We refer to  $\alpha$  as addition, and  $m$  as scalar multiplication;  $(V, +)$  is an abelian ring.

**Definition** (Norm). A norm is a function

$$\|\cdot\| : V \rightarrow \mathbb{R}^+, x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness:  $\|v\| = 0$  if and only if  $v = 0_V$ .
- Triangle inequality:  $\|v + w\| \leq \|v\| + \|w\|$ .
- Absolute Homogeneity:  $\|\lambda v\| = |\lambda| \|v\|$ , for  $\lambda \in \mathbb{C}$ .

If a function  $p : V \rightarrow \mathbb{R}^+$  satisfies the triangle inequality and absolute homogeneity, we say  $p$  is a semi-norm.

We say the pair  $(V, \|\cdot\|)$  is a normed vector space.

**Definition** (Balls and Spheres). Let  $X$  be a normed vector space,  $x \in X$ , and  $\delta > 0$ . Then,

$$\begin{aligned} U(x, \delta) &= \{y \in X \mid d(x, y) < \delta\} \\ B(x, \delta) &= \{y \in X \mid d(x, y) \leq \delta\} \\ S(x, \delta) &= \{y \in X \mid d(x, y) = \delta\}. \end{aligned}$$

For a normed vector space, we will use the following conventions for common sets:

$$\begin{aligned} U_X &= U(0, 1) \\ B_X &= B(0, 1) \\ S_X &= S(0, 1) \\ \mathbb{D} &= U_{\mathbb{C}} \\ \mathbb{T} &= S_{\mathbb{C}}. \end{aligned}$$

**Definition** (Equivalent Norms). Two norms on  $V$ ,  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are said to be equivalent if there are two constants  $C_1$  and  $C_2$  such that

$$\begin{aligned} \|v\|_a &\leq C_1 \|v\|_b \\ \|v\|_b &\leq C_2 \|v\|_a \end{aligned}$$

for all  $v \in V$ . We say  $\|\cdot\|_a \sim \|\cdot\|_b$ .

## Examples

**Example** (Finite-Dimensional Vector Spaces). The vector space  $\mathbb{C}^n$  with the  $p$ -norm is denoted  $\ell_p^n$ , where for  $p \in [1, \infty]$ , the  $p$ -norm is defined by

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

In the case with  $p = 2$ , this gives the traditional Euclidean norm, and with  $p = \infty$ , this gives the sup norm:

$$\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i|.$$

**Example** (A Sequence Space). We let  $\ell_p = \{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\}$  be the collection of sequences in  $\mathbb{C}$  with finite  $p$ -norm. Here,

$$\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

In the case with  $p = \infty$ , this gives the sequence space  $\ell_{\infty}$ , which has norm

$$\|x\|_{\infty} = \sup_{n \in \mathbb{N}} |x_n|.$$

**Example** (A Function Space). We let  $\ell^{\infty}(\Omega)$  denote the set of all bounded functions  $f : \Omega \rightarrow \mathbb{C}$ , equipped with the norm

$$\|f\|_{\infty} = \sup_{x \in \Omega} |f(x)|.$$

If  $\Omega = (\Omega, \mathcal{M}, \mu)$  is a measure space, then we let  $L^{\infty}(\Omega)$  be the space of  $\mu$ -a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_{\infty} = \text{ess sup}_{x \in \Omega} |f(x)|.$$

## Series Convergence and Completeness

**Proposition** (Criteria for Banach Spaces): Let  $X$  be a normed vector space. The following are equivalent:

- (i)  $X$  is a Banach space.<sup>1</sup>
- (ii) If  $(x_k)_k$  is a sequence of vectors such that  $\sum_{k=1}^{\infty} \|x_k\|$  converges, then  $\sum_{k=1}^{\infty} x_k$  converges.
- (iii) If  $(x_k)_k$  is a sequence in  $X$  such that  $\|x_k\| < 2^{-k}$ , then  $\sum_{k=1}^{\infty} x_k$  converges.

*Proof.* To show (i) implies (ii), for  $n > m > N$ , we have

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^n x_k \right\| \\ &\leq \sum_{k=m+1}^n \|x_k\| \\ &< \epsilon, \end{aligned}$$

implying that  $s_n$  is Cauchy, and thus converges since  $X$  is complete.

Since  $\sum_{k=1}^{\infty} 2^{-k}$  converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let  $(x_n)_n$  be a Cauchy sequence in  $X$ . We only need construct a convergent subsequence in order to show that  $(x_n)_n$  converges.

Chose  $n_1 \in \mathbb{N}$  such that for  $n, m \geq n_1$ ,  $\|x_m - x_n\| < \frac{1}{2^2}$ , and inductively define  $n_j > n_{j-1}$  such that  $n, m \geq n_j$  implies  $\|x_m - x_n\| < \frac{1}{2^{j+1}}$ .

Let  $y_1 = x_{n_1}$ ,  $y_j = x_{n_j} - x_{n_{j-1}}$ . Then,

$$\begin{aligned} \|y_j\| &= \|x_{n_j} - x_{n_{j-1}}\| \\ &< \frac{1}{2^j}, \end{aligned}$$

so  $\sum_{j=1}^{\infty} y_j$  converges by our assumption. By telescoping, we see that  $\sum_{j=1}^k y_j = x_{n_k}$ , so  $(x_{n_k})_k$  converges.  $\square$

## Quotient Spaces

Let  $X$  be a normed vector space. Then, for  $E \subseteq X$  a subspace, there is a quotient space  $X/E$  with the projection map  $\pi : X \rightarrow X/E$ ,  $x \mapsto x + E$ . We want to make  $X/E$  into a normed space — in order to do this, we use the distance function:

$$\text{dist}_E(x) = \inf_{y \in E} d(x, y),$$

which is uniformly continuous. For  $E$  closed, then  $\text{dist}_E(x) = 0$  if and only if  $x \in E$ .

**Proposition** (Quotient Space Norm): Let  $X$  be a normed vector space, and  $E \subseteq X$  a subspace. Set

$$\|x + E\|_{X/E} = \text{dist}_E(x).$$

Then,

- (1)  $\|\cdot\|_{X/E}$  is a well-defined seminorm on  $X/E$ .

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<sup>1</sup>Complete normed vector space.

- (2) If  $E$  is closed, then  $\|\cdot\|_{X/E}$  is a norm on  $X/E$ .
- (3)  $\|x + E\|_{X/E} \leq \|x\|$  for all  $x \in X$ .
- (4) If  $E$  is closed, then  $\pi : X \rightarrow X/E$  is Lipschitz.
- (5) If  $X$  is a Banach space and  $E$  is closed, then  $X/E$  is also a Banach space.

*Proof.*

- (1) We will show that  $\|\cdot\|_{X/E}$  is well-defined. If  $x + E = x' + E$ ,  $x' - x \in E$ , so for every  $y \in E$ ,  $x' - x + y \in E$ . Thus,

$$\begin{aligned} \|x - y\| &= \|x' - (x' - x + y)\| \\ &\geq \inf_{z \in E} \|x' - z\| \\ &= \|x' + E\|_{X/E}. \end{aligned}$$

Thus,  $\|x + E\|_{X/E} \geq \|x' + E\|_{X/E}$ , and vice versa.

Let  $\lambda \in \mathbb{C} \setminus \{0\}$ , and  $x \in X$ . Then,

$$\begin{aligned} \|\lambda(x + E)\|_{X/E} &= \|\lambda x + E\|_{X/E} \\ &= \inf_{y \in E} \|\lambda x - y\| \\ &= |\lambda| \inf_{y \in E} \|x - \lambda^{-1}y\| \\ &= |\lambda| \inf_{y' \in E} \|x - y'\| \\ &= |\lambda| \|x + E\|_{X/E} \end{aligned}$$

Given  $x, x' \in X$  and a fixed  $\varepsilon > 0$ , we have

$$\|x + E\| + \frac{\varepsilon}{2} > \|x - y\|$$

for some  $y \in E$ , and

$$\|x' + E\| + \frac{\varepsilon}{2} > \|x' - y'\|$$

for some  $y' \in E$ . Thus,

$$\begin{aligned} \|(x + x') - (y + y')\| &\leq \|x - y\| + \|x' - y'\| \\ &< \varepsilon + \|x + E\| + \|x' + E\|. \end{aligned}$$

Since  $y + y' \in E$ , we have

$$\begin{aligned} \|(x + E) + (x' + E)\|_{X/E} &= \|x + x' + E\|_{X/E} \\ &\leq \|(x + x') - (y + y')\| \\ &< \varepsilon + \|x + E\|_{X/E} + \|x' + E\|_{X/E}, \end{aligned}$$

meaning

$$\|(x + E) + (x' + E)\| \leq \|x + E\| + \|x' + E\|.$$

- (2) If  $E$  is closed, and  $\|x + E\| = 0$ , then  $x \in E$  so  $x + E = 0_{X/E}$ .

(3) For  $x \in X$ ,

$$\begin{aligned}\|x + E\|_{X/E} &= \inf_{y \in E} \|x - y\| \\ &\leq \|x\|.\end{aligned}$$

(4) We have

$$\begin{aligned}\|(x + E) - (x' + E)\|_{X/E} &= \|x - x' + E\|_{X/E} \\ &\leq \|x - x'\|.\end{aligned}$$

(5) Let  $X$  be complete and  $E \subseteq X$  be closed. Let  $(x_k + E)_k$  be a sequence in  $X/E$  with  $\|x_k + E\| < 2^{-k}$ . We want to show that  $\sum_{k=1}^{\infty} (x_k + E)$  converges.

For each  $k$ , since  $\|x_k + E\| < 2^{-k}$ , there exists  $y_k \in E$  such that  $\|x_k - y_k\| < 2^{-k}$ . Since  $X$  is complete,  $\sum_{k=1}^{\infty} x_k - y_k$  converges.

Let  $(\sum_{k=1}^n x_k - y_k)_n \rightarrow x$  in  $X$ . Applying the canonical projection map,  $\pi$ , to both sides, we get

$$\begin{aligned}\sum_{k=1}^n (x_k + E) &= \sum_{k=1}^n \pi(x_k) \\ &= \pi\left(\sum_{k=1}^n (x_k - y_k)\right) \\ &\rightarrow \pi(x),\end{aligned}$$

implying that  $\sum_{k=1}^{\infty} (x_k + E)$  converges.

□

**Exercise:** Consider  $\ell_{\infty}$  and its closed subspace  $c_0$ . If  $\pi : \ell_{\infty} \rightarrow \ell_{\infty}/c_0$  denotes the canonical quotient map, with  $(z_k)_k \in \ell_{\infty}$ , show that

$$\|(z_k)_k + c_0\| = \limsup_{k \rightarrow \infty} |z_k|$$

**Solution.** By the definition of the quotient norm, we have

$$\begin{aligned}\|(z_k)_k + c_0\|_{\ell_{\infty}/c_0} &= \inf_{(a_k)_k \in c_0} \|(z_k)_k - (a_k)_k\| \\ &= \inf_{(a_k)_k \in c_0} \sup_{k \in \mathbb{N}} |z_k - a_k| \\ &= \limsup_{k \rightarrow \infty} |z_k|.\end{aligned}$$

## Bounded Linear Operators

**Definition** (Continuous Functions). A function  $f : (X, d_X) \rightarrow (Y, d_Y)$  is called Lipschitz if there is a constant  $C > 0$  such that

$$d_Y(f(x), f(x')) \leq C d_X(x, x')$$

for all  $x, x' \in X$ .

If  $C \leq 1$ , a Lipschitz map is known as a contraction.

If

$$d_Y(f(x), f(x')) = d_X(x, x')$$

for all  $x, x' \in X$ , then  $f$  is known as an isometry.

**Proposition** (Categorization of Continuous Linear Maps): Let  $X$  and  $Y$  be normed vector spaces, and let  $T : X \rightarrow Y$  be a linear map. The following are equivalent:

- (i)  $T$  is continuous at 0.
- (ii)  $T$  is continuous.
- (iii)  $T$  is uniformly continuous.
- (iv)  $T$  is Lipschitz.
- (v) There exists a constant  $C > 0$  such that  $\|T(x)\| \leq C \|x\|$  for all  $x \in X$ .

**Definition** (Bounded Linear Operator). Let  $X$  and  $Y$  be normed vector spaces, and let  $T : X \rightarrow Y$  be a linear map.

- (1)  $T$  is bounded if  $T(B_X)$  is bounded in  $Y$ . Equivalently,  $T$  is bounded if and only if

$$\sup_{x \in B_X} \|T(x)\| < \infty,$$

or that  $\exists r > 0$  such that  $T(B_X) \subseteq B_Y(0, r)$ .

- (2) The operator norm of  $T$  is the value

$$\|T\|_{\text{op}} = \sup_{x \in B_X} \|T(x)\|.$$

**Lemma:** Let  $T : X \rightarrow Y$  be a linear map between normed vector spaces. Then,

$$\|T\|_{\text{op}} = \sup_{x \in S_X} \|T(x)\|$$

and for all  $x \in X$ ,

$$\|T(x)\| \leq \|T\|_{\text{op}} \|x\|.$$

**Lemma:** Let  $T : X \rightarrow Y$  be a bounded linear map between normed vector spaces. Then, for any  $x \in X$  and  $r > 0$ ,

$$r \|T\|_{\text{op}} \leq \sup_{y \in B(x, r)} \|T(y)\|$$

*Proof.* Let  $C = \sup_{y \in B(x, r)} \|T(y)\|$ . If  $z \in B(0, r)$ , then  $z + x, z - x \in B(x, r)$ , meaning

$$2T(z) = T(z + x) + T(z - x),$$

so by the triangle inequality, we get

$$\begin{aligned} 2 \|T(z)\| &\leq \|T(z + x)\| + \|T(z - x)\| \\ &\leq 2 \max \{ \|T(z + x)\|, \|T(z - x)\| \} \\ &\leq 2C. \end{aligned}$$

Thus,

$$\|T(z)\| \leq \sup_{y \in B(x, r)} \|T(y)\|,$$

meaning

$$r \|T\|_{\text{op}} \leq \sup_{y \in B(x, r)} \|T(y)\|.$$

□

**Remark:** For a linear map  $T : X \rightarrow Y$ , the following are equivalent:

- (1)  $T$  is continuous.
- (2)  $T$  is bounded.
- (3)  $\|T\|_{\text{op}} < \infty$ .

**Definition.** Let  $X$  and  $Y$  be normed spaces,  $T : X \rightarrow Y$  a linear map.

- (1)  $T$  is bounded below if there exists  $C_2$  such that  $\|T(x)\| \geq C_2 \|x\|$  for all  $x \in X$ .
- (2)  $T$  is bicontinuous if  $T$  is bounded and bounded below.

$$C_2 \|x\| \leq \|T(x)\| \leq C_1 \|x\|$$

- (3)  $T$  is a bicontinuous isomorphism if  $T$  is bijective, linear, and bicontinuous. We say  $X$  and  $Y$  are bicontinuously isomorphic.
- (4) We say  $T$  is an isometric isomorphism if  $T$  is bijective, linear, and an isometry.

**Example.** Let  $\rho$  be the continuous surjective wrapping function  $\rho : [0, 2\pi] \rightarrow \mathbb{T}$ ,  $\rho(t) = e^{it}$ . There is an induced isometry

$$T_\rho : C(\mathbb{T}) \rightarrow C([0, 2\pi]),$$

defined by  $T_\rho(f)(t) = f \circ \rho(t) = f(e^{it})$ .

The range of  $T_\rho$  is  $C = \{g \in C([0, 2\pi]) \mid g(0) = g(2\pi)\}$ , which means that  $C(\mathbb{T})$  and  $C$  are isometrically isomorphic Banach spaces.

**Proposition:** Let  $X$  and  $Y$  be normed spaces, and  $T : X \rightarrow Y$  be a linear map. The following are equivalent.

- (i)  $T$  is bicontinuous.
- (ii)  $T : X \rightarrow \text{Ran}(T)$  is a linear isomorphism and homeomorphism.

*Proof.* Let  $T$  be bicontinuous. Then,  $T$  is linear, injective, and surjective onto  $\text{Ran}(T)$ . Since  $T$  is continuous,  $T$  is bounded. Let  $S : \text{Ran}(T) \rightarrow X$  be defined by  $S(T(x)) = x$ . We can see that  $S$  is well-defined, since  $T : X \rightarrow \text{Ran}(T)$  is surjective, and so has a left inverse. Similarly, since  $\|S(T(x))\| = \|x\| \leq \frac{1}{C_2} \|T(x)\|$ ,  $S$  is continuous.

Let  $S : \text{Ran}(T) \rightarrow X$  be defined by  $S(T(x)) = x$ . Since  $T$  is continuous, it is bounded, so

$$\|T(x)\| \leq \|T\|_{\text{op}} \|x\|.$$

Since  $S$  is bounded,

$$\begin{aligned} \|x\| &= \|S(T(x))\| \\ &= \|S\|_{\text{op}} \|T(x)\|, \end{aligned}$$

so  $\frac{1}{\|S\|_{\text{op}}} \|x\| \leq \|T(x)\|$ . □

**Corollary:** Let  $X$  be a vector space with  $\|\cdot\|$  and  $\|\cdot\|'$  two norms. The following are equivalent:

- (i) The norms  $\|\cdot\|$  and  $\|\cdot\|'$  are equivalent.
- (ii) The map  $\text{id}_X : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$ .

**Proposition (Properties of Bounded Linear Operators):** Let  $X, Y, Z$  be normed spaces,  $T : X \rightarrow Y$ ,  $S : X \rightarrow Y$ , and  $R : Y \rightarrow Z$  be linear maps.

- (1)  $\|\alpha T\|_{\text{op}} = |\alpha| \|T\|_{\text{op}}$

- (2)  $\|T + S\|_{\text{op}} \leq \|T\|_{\text{op}} + \|S\|_{\text{op}}$
- (3)  $\|T\|_{\text{op}} = 0$  if and only if  $T = 0$
- (4)  $\|R \circ T\|_{\text{op}} \leq \|R\|_{\text{op}} \|T\|_{\text{op}}$
- (5)  $\|\text{id}_X\|_{\text{op}} = 1$
- (6) If  $E \subseteq X$  is a subspace, then  $\|T|_E\|_{\text{op}} \leq \|T\|_{\text{op}}$

*Proof.* We will prove (4) here. For  $x \in B_X$ , we have

$$\begin{aligned} \|R \circ T(x)\| &= \|R(T(x))\| \\ &\leq \|R\|_{\text{op}} \|T(x)\| \\ &\leq \|R\|_{\text{op}} \|T\|_{\text{op}}. \end{aligned}$$

Taking the supremum, we obtain  $\|R \circ T\|_{\text{op}} \leq \|R\|_{\text{op}} \|T\|_{\text{op}}$ . □

**Recall:**  $\mathcal{L}(X, Y)$  is the set of all linear operators with domain  $X$  and codomain  $Y$ .

**Proposition:** Let  $X$  and  $Y$  be normed spaces.

- (1) The collection  $\mathcal{B}(X, Y) = \{T \in \mathcal{L}(X, Y) \mid \|T\|_{\text{op}} < \infty\}$  equipped with the operator norm is a normed space known as the space of bounded linear operators between  $X$  and  $Y$ .
- (2) If  $Y$  is a Banach space, then  $\mathcal{B}(X, Y)$  is a Banach space.
- (3) The continuous dual space,  $X^* = \mathcal{B}(X, \mathbb{C})$  is a Banach space.

*Proof.* We will prove (2). Let  $(T_n)_n$  be Cauchy under  $\|\cdot\|_{\text{op}}$ . Since Cauchy sequences are bounded, there is some  $C > 0$  such that  $\|T_n\|_{\text{op}} \leq C$  for all  $n \geq 1$ . For  $x \in X$ ,

$$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|_{\text{op}} \|x\|,$$

meaning  $(T_n(x))_n$  is Cauchy in  $Y$ . Since  $Y$  is complete, we define

$$T(x) = \lim_{n \rightarrow \infty} T_n(x)$$

in  $Y$ . If  $x \in B_X$ , we have

$$\begin{aligned} \|T(x)\| &= \left\| \lim_{n \rightarrow \infty} T_n(x) \right\| \\ &= \lim_{n \rightarrow \infty} \|T_n(x)\| \\ &\leq \limsup_{n \rightarrow \infty} \|T_n(x)\| \\ &\leq C \|x\|, \end{aligned}$$

meaning  $\|T\|_{\text{op}} \leq C$ .

Let  $\varepsilon > 0$ , and  $N \in \mathbb{N}$  large such that  $n, m \geq N$ ,  $\|T_n - T_m\|_{\text{op}} \leq \varepsilon$ . For  $x \in B_X$ ,

$$\begin{aligned} \|T_n(x) - T(x)\| &= \lim_{m \rightarrow \infty} \|T_n(x) - T_m(x)\| \\ &\leq \limsup_{m \rightarrow \infty} \|T_n - T_m\|_{\text{op}} \|x\| \\ &< \varepsilon. \end{aligned}$$

Thus,  $\|T - T_n\|_{\text{op}} < \varepsilon$  for all  $n \geq N$ . □



**Definition** (Algebras). Let  $A$  be an algebra over  $\mathbb{C}$ .

- (1) If  $A$  admits a norm  $\|\cdot\|$  satisfying  $\|ab\| \leq \|a\| \|b\|$ , then  $A$  is a normed algebra. If  $A$  is unital, then  $\|1_A\| = 1$ .
- (2) If  $A$  is complete with respect to its norm, then  $A$  is called a Banach algebra, and if  $A$  is unital, then  $A$  is a unital Banach algebra.

**Lemma:** In a normed algebra  $A$ , the map  $\cdot : A \times A \rightarrow A, (a, b) \mapsto ab$  is continuous.

**Proposition:** Let  $X$  be a normed space. The set of bounded operators  $\mathcal{B}(X, X) = \mathcal{B}(X)$  is a unital normed algebra. Moreover, if  $X$  is a Banach space, then  $\mathcal{B}(X)$  is a Banach algebra.

**Proposition:** Let  $A$  be a unital Banach algebra,  $a \in A$ . The series

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges absolutely in  $A$ . We call  $\exp(a)$  the exponential of  $a$ .

- (1)  $\exp(0) = 1_A$
- (2) If  $A$  is commutative, then  $\exp(a + b) = \exp(a)\exp(b)$ .
- (3) We have  $\exp(a) \in GL(A)$  with  $\exp(a)^{-1} = \exp(-a)$ .
- (4)  $\|\exp(a)\| \leq \exp(\|a\|)$ .

## Quotient Maps

**Definition.** A map  $f : X \rightarrow Y$  is called open if  $U \subseteq X$  is open implies  $f(U) \subseteq Y$  is open.

**Proposition:** Let  $X$  and  $Y$  be normed spaces,  $T : X \rightarrow Y$  a linear map. The following are equivalent:

- (i)  $T$  is surjective and open.
- (ii)  $T(U_X) \subseteq Y$  is open.
- (iii) There exists  $\delta > 0$  such that  $\delta U_Y \subseteq T(U_X)$ .
- (iv) There exists  $\delta$  such that  $\delta B_Y \subseteq T(B_X)$ .
- (v) There exists  $M > 0$  such that for all  $y \in Y$ , there exists  $x \in X$  with  $T(x) = y$  and  $\|x\| \leq M \|y\|$ .

*Proof.* To see (i) implies (ii), if  $T$  is surjective and open, then it is clear that  $T(U_X)$ , which is the image of an open set, is open.

To see (ii) implies (iii), if  $T(U_X)$  is open, we have  $0_Y \in T(U_X)$ , so there is some  $\delta$  such that  $U(0, \delta) \subseteq T(U_X)$ , meaning  $\delta U_Y \subseteq T(U_X)$ .

Assuming (iii), we see that  $\frac{\delta}{2} B_Y \subseteq \delta U_Y \subseteq T(U_X) \subseteq T(B_X)$ .

To see (iv) implies (v), let  $\delta$  be such that  $\delta B_Y \subseteq T(B_X)$ , and set  $M = \frac{1}{\delta}$ . Note that for  $y \in Y, y \neq 0$ ,  $\frac{\delta}{\|y\|} y \in \delta B_Y$ , meaning  $\frac{\delta}{\|y\|} y = T(x)$  for some  $x \in B_X$ , implying that  $T\left(\frac{\|y\|}{\delta} x\right) = y$ . Finally, since  $x \in B_X$ ,  $\frac{\|y\|}{\delta} \|x\| \leq \frac{1}{\delta} \|y\| = M \|y\|$ .

To see (v) implies (i), we can see that  $T$  is surjective by the assumption. Let  $U \subseteq X$  be open,  $y_0 \in T(U)$ . Then, there exists  $x_0$  such that  $T(x_0) = y_0$ , and  $\delta > 0$  such that  $U(x_0, \delta) \subseteq U$ . Note that  $U(x_0, \delta) = x_0 + \delta U_X$ , so  $x_0 + \delta U_X \subseteq U$ . Applying  $T$ , we get  $T(x_0 + \delta U_X) \subseteq T(U)$ , or  $y_0 + \delta T(U_X) \subseteq T(U)$ . By assumption, since given  $y \in U_Y$ , there exists  $x \in X$  such that  $\|x\| \leq M \|y\|$ , meaning  $\|x\| \leq M$ , we have  $U_Y \subseteq T(M U_X)$ . Thus,  $\frac{1}{M} U_Y \subseteq T(U_X)$ , meaning  $y_0 + \frac{\delta}{M} U_Y \subseteq y_0 + \delta T(U_X) \subseteq T(U)$ , so  $U_Y(y_0, \frac{\delta}{M}) \subseteq T(U)$ .  $\square$

**Definition.** Let  $X$  and  $Y$  be normed vector spaces.

- (1) A bounded linear map  $T : X \rightarrow Y$  that is surjective and open is known as a quotient map.
- (2) If  $T(U_X) = U_Y$ , then  $T$  is called a 1-quotient map.