

Contents

Set Theory	7
Naive Set Theory	7
Function Examples	7
Function Operations	7
Injective, Surjective, Bijective	7
Invertibility	8
Definition of Invertibility	8
Injection and Surjection Invertibility	8
Cardinality and Countability	8
Introduction to Cardinality	8
Equivalent Cardinality	9
Equivalent Cardinalities of Intervals	9
Intervals and Real Numbers	9
Finitude and Infinitude	9
Inequality of Finite Sets	9
Infinitude of the Naturals	9
Infinitude of a Set	10
Integers and Power Sets	10
Cardinality of Integers and Natural Numbers	10
Power Set and 2^X	10
Cantor's Theorem	11
Comparing Cardinality	11
Cardinality of the Power Set	11
Equivalent Cardinality Comparisons	12
Cardinality Rules	12
Cardinality of Canonical Sets	13
Countability and the Continuum Hypothesis	13
Corollary to Cantor-Schröder-Bernstein	13
Countability under Union	14
Continuum Hypothesis	14
Field Ordering	15
Ordering Relations	15
Examples of Orderings	16
Total and Directed Orderings	16
Upper and Lower Bounds	16
Examples	17
Ordering of \mathbb{Z} , \mathbb{Q} , and \mathbb{R}	17
Properties of \mathbb{Z}^+	17
Creating the Rationals	18
Fields	18
Ordering of \mathbb{Q}	19
Properties of \mathbb{Q}^+	19
Ordered Fields and the Ordering of \mathbb{R}	19
Rational Orderings	21

Important Inequalities	21
Arithmetic and Geometric Means	21
Arithmetic Mean-Geometric Mean Inequality	21
Bernoulli's Inequality	22
Cauchy's Inequality	22
Triangle Inequality	23
Metrics, Norms, and Bounds	23
Metrics and Norms on \mathbb{R}^n	23
Bounded Sets	24
Bounded Functions	24
Distance Metrics	24
Properties of Norms	25
Relating Distance Metrics and Norms	25
Metric Spaces	26
Examples of Metric Spaces	26
Open and Closed Sets in Metric Spaces	27
Supremum, Infimum, and Completeness	27
Finding a Supremum	27
Supremum Example	28
Finding an Infimum	28
Infimum Example	28
Properties of Supremum and Infimum	28
Completeness Axiom	29
Archimedean Property	29
Corollary to the Archimedean Property	29
Corollary to the Corollary: Powers of 2	29
Corollary to the Corollary: In Between Integers	29
Density	29
Density of the Rationals	29
Density of the Irrationals	30
Uniqueness of $\sqrt{2}$	30
Intervals in \mathbb{R}	31
Characterization of Intervals	31
Nested Intervals	31
Measure	32
Nested Intervals Theorem	32
Corollary to the Nested Intervals Theorem	33
Sequences and Convergence	33
Sequences in Metric Spaces	33
Finding a Sequence	34
Bounded Sequences	34
Monotonicity	35
Monotonicity Example	35
Convergence of Sequences	36
Definition of Convergence	36
Convergence Proof 1	37
Convergence Proof 2	37
Convergence and Distance	38

Comparison Proposition	38
Comparison Example 1	39
Comparison Example 2	39
Sequence Divergence	39
Sequence Divergence 1	39
Sequence Divergence 2	40
Alternating Sequence	41
Uniqueness of Limits	41
Useful Lemmas for Convergence	41
Absolutely Convergent Sequences	41
Convergence to Zero	41
Geometric Sequence	42
n th Root Convergence	43
Positive Sequence Convergence	44
n th Root of n Convergence	44
Multiplication by Geometric Sequence	45
Boundedness and Convergence	45
Algebraic Operations on Sequences	46
Ordering of Limits	47
Squeeze Theorem	47
Squeeze Theorem Examples	47
Ratio Test	47
Monotone Convergence Theorem	48
Applications of the Monotone Convergence Theorem	48
Monotone Convergence Example 1	48
Monotone Convergence Example 2	49
Alternative Proof of the Nested Intervals Theorem	50
Calculating Square Roots	50
Euler's Number	51
Monotone Divergence	52
Monotone Divergence Example	52
Subsequences and Bolzano-Weierstrass	53
Natural Sequences	53
Subsequences	53
Convergence of Subsequences	54
Corollary to Convergence of Subsequences	54
Convergence of Subsequences Example	54
Divergence and Subsequences	54
Bolzano-Weierstrass Theorem	55
Limit Superior and Limit Inferior	56
Limit Points	56
Finding the Limit Points	56
Defining Limit Superior and Limit Inferior	56
Fundamental Results in Limit Superior and Limit Inferior	57
Applying Limit Superior and Limit Inferior	57
Ratio Test and Root Test: Equivalent Convergence	57
Properties of \overline{X}	58

Cauchy and Contractive Sequences	59
Cauchy Sequences	59
Boundedness of Cauchy Sequences	59
Convergent Subsequences and Cauchy Sequences	60
Cauchy Sequence Convergence in the Reals	60
Complete Metric Spaces	60
Finding Cauchy Sequences and Convergence in \mathbb{R}	61
Cauchy Sequences and Convergence 1	61
Cauchy Sequences and Convergence 2	61
Contractive Sequences	62
Contractive and Cauchy	62
Applying Contractive Sequences 1	62
Applying Contractive Sequences 2	63
Sequence Divergence	64
Properly Divergent Sequences	64
Divergence of the Geometric Sequence	64
Monotone Divergence	65
Sequence Comparison Test	65
Applying the Sequence Comparison Test	65
Series Convergence and Divergence	66
Introduction to Infinite Series	66
Convergence of a Series of Positive Terms	66
Corollary to Convergence of a Series of Positive Terms	66
Applying Convergence of a Series of Positive Terms 1	66
Applying Convergence of a Series of Positive Terms 2	67
Applying Convergence of a Series of Positive Terms 3	67
Series Comparison Test	67
Limit Comparison Test	68
Applying the Limit Comparison Test	68
n th Term Divergence Test	69
Cauchy Condensation Test	69
p -Series	70
Sequences and Series of Functions	70
Pointwise Convergence	70
Applying Pointwise Convergence	70
Uniform Convergence	71
Applying Uniform Convergence	72
Negating Uniform Convergence	72
Negating Uniform Convergence 1	73
Changing Domain and Uniform Convergence	73
Negating Uniform Convergence 2	73
Uniform Norm	74
Applying Uniform Norm 1	74
Root Test and Series Convergence	75
Absolute Convergence	75
Series of Functions	76
Applying Pointwise Convergence of Series of Functions	76
Applying Uniform Convergence of Series of Functions	76
Weierstrass M -test	77

Applying the Weierstrass M -test	77
Power Series	78
Cauchy-Hadamard Theorem	78
Limits	79
Cluster Points	79
Sequential Criterion of Cluster Points	80
Definition of a Limit	80
Applying the Limit Definition: Linear Function	80
Applying the Limit Definition: Quadratic Function	81
Applying the Limit Definition: Rational Function	81
Uniqueness of Limits	82
Sequential Criterion for Limits	82
Limit Divergence and Non-Existence	82
Applying Limit Divergence using Sequences	83
Bounded Functions and Cluster Points	83
Operations with Limits	83
Squeeze Theorem	84
Trigonometric Limits	85
Strictly Positive Limits	86
One-Sided Limits	86
Limit Equality	86
Infinite Limits	87
Applying Infinite Limits	87
Limits at Infinity	87
Applying Limits at Infinity 1	87
Applying limits at Infinity: Polynomials	88
Continuity and Uniform Continuity	88
Continuity	88
Continuity and Limits	88
Sequential Criterion of Continuity	89
Left and Right Continuity	89
Continuity on Sets	89
Applying Continuity on Sets	89
Discontinuity	89
Discontinuity of the Sign Function	89
Discontinuity of Thomae's Function	90
Extension of a Function	90
Jump Discontinuities	91
Lipschitz Functions	91
Properties of Continuous Functions	91
Equality over Dense Subsets	91
Boundedness over a Dense Subset	91
Bounding Away From 0	92
Continuity over Operations	92
Fundamental Theorem of Continuous Functions on $[a, b]$	92
Uniform Continuity	94
Illustrating Non-Uniform Continuity	94
Proving Uniform Continuity 1	94
Proving Uniform Continuity 2	95
Lipschitz and Uniform Continuity	95

Uniform Continuity and Continuity	95
Negating Uniform Continuity	95
Applying Non-Uniform Continuity 1	96
Applying Non-Uniform Continuity 2	96
Uniform Continuity Theorem	96
Uniform Continuity and Lipschitz	97
Lemma: Uniform Continuity and Cauchy Sequences	97
Continuous Extension Theorem	97
Applying the Continuous Extension Theorem	98
Approximation by Step Function	98
Approximation by Piecewise Linear Function	99
Monotone Functions	99
Limits and Continuity with Monotone Functions	100
Jump of a Function	100
Countability of Monotone Function Discontinuities	101
Continuous Inverse Theorem	103
The n th Root Function	103
Derivatives	104
Definition of Differentiation	104
Applying Differentiation 1	105
Applying Differentiation 2	105
Applying Differentiation 3	105
Applying Differentiation 4	105
Applying Differentiation 5	105
Applying Differentiation 6	106
Differentiability and Continuity	106
Operations with Differentiation	106
Power Rule	107
Carathéodory's Theorem	107
Chain Rule	107
Inverse Functions	108
Applying Inverse Functions 1	108
Applying Inverse Functions 2	109
Fermat's Theorem	109
Rolle's Theorem	110
Applying Rolle's Theorem	110

Set Theory

Naive Set Theory

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Z}_+ = \{0, 1, 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

$$\mathbb{C}_q = \{a + bi \mid a, b \in \mathbb{Q}\}$$

Recall: given sets X and Y , a relation from X to Y is a subset of $X \times Y$, where \times denotes the cartesian product of X and Y .

A relation $f \subseteq X \times Y$ is a function from X to Y such that $\forall x \in X, \exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote f as $f : X \rightarrow Y$.

X is the **domain** of f and Y is the **codomain**. The range $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Function Examples

Identity Function:

$$\text{id}_X : X \rightarrow X, \text{id}_X(x) = x$$

The Characteristic Function: If $A \subseteq X$

$$\mathbb{1}_A : X \rightarrow \mathbb{R}, \mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Function Operations

Let X be any set, and $(X; \mathbb{R}) = \{f : X \rightarrow \mathbb{R}\}$ represent the function space of X with codomain \mathbb{R} .

Addition: Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

Scalar Multiplication: If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$.

Function Multiplication: If $g(x) \neq 0 \forall x \in X$, then $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$.

Composition: If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions, then $g \circ f(x) = g(f(x))$.

Injective, Surjective, Bijective

A function $f : X \rightarrow Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. For example, the shift map $S : \mathbb{N} \rightarrow \mathbb{N}$, $S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \rightarrow \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \rightarrow Y$ be a function. f is **left-invertible** if $\exists g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$. f is **right-invertible** if $\exists h : Y \rightarrow X$ such that $f \circ h = \text{id}_Y$.

f is **invertible** if $\exists k : Y \rightarrow X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \rightarrow [-\pi/2, \pi/2]$.

Definition of Invertibility

Statement: f is invertible if and only if f is left and right invertible.

Proof:

(\Rightarrow) This is via the definition of invertibility.

(\Leftarrow) Suppose g is a left-inverse of f , and h is a right-inverse of f . Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

Injection and Surjection Invertibility

Statement: If $f : X \rightarrow Y$ is a function:

- (1) f is injective $\Leftrightarrow f$ is left-invertible.
- (2) f is surjective $\Leftrightarrow f$ is right-invertible.
- (3) f is bijective $\Leftrightarrow f$ is invertible.

Proof: (1), (\Rightarrow) — suppose f is injective. Given $y \in \text{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = y$, by the definition of injective.

Let $g : Y \rightarrow X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X . We can see that $g \circ f = \text{id}_X$.

Cardinality and Countability

Introduction to Cardinality

Which set is “larger,” $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? \mathbb{N} or \mathbb{N}_0 ? \mathbb{Z} or \mathbb{Q} ?

In order to prove that one set is “the same size” as the other, we can create pairs. For two sets A and B , we can show that A is the same size as B by creating a function. For example, to show that \mathbb{N} and \mathbb{N}_0 have the same size, we create $s : \mathbb{N} \rightarrow \mathbb{N}_0$, $s(n) = n + 1$.

Equivalent Cardinality

Sets A and B have the same **cardinality** if \exists bijection $f : A \rightarrow B$. We write $\text{card}(A) = \text{card}(B)$.

Equivalent Cardinalities of Intervals

Statement: Given $a < b$ and $c < d$, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

Proof: We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}([a, b]) = \text{card}([c, d])$.

Intervals and Real Numbers

$$\text{card}((0, 1)) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ is a bijection:
 - \tan is strictly increasing (and thus injective)
 - $\lim_{x \rightarrow \infty} \tan(x) = \infty$ and $\lim_{x \rightarrow -\infty} \tan(x) = -\infty$, and by intermediate value theorem, \tan is surjective
- $\ell : (0, 1) \rightarrow (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0, 1) \rightarrow \mathbb{R}$.

Finitude and Infinitude

A set F is **finite** if F is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}(\{1, 2, \dots, n\})$. A non-finite set is called infinite.

We can *enumerate* F by creating a function $\sigma : \{1, 2, \dots, n\} \rightarrow F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

Inequality of Finite Sets

Statement: If $m \neq n$, then $\text{card}\{1, 2, \dots, m\} \neq \text{card}\{1, 2, \dots, n\}$.

Proof: WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$ is our bijection. This means there are m “pigeons” and n “holes.”

One hole, j , must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since f is assumed to be injective, this is a contradiction.

Infinitude of the Naturals

Statement: \mathbb{N} is infinite.

Proof: Suppose toward contradiction that \mathbb{N} is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \rightarrow \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m+1\} \rightarrow \mathbb{N}$. i is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \rightarrow \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

Infinitude of a Set

Statement: If A is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

Proof:

$$\begin{array}{ll} \exists a_1 \in A & A \neq \emptyset \\ \exists a_2 \in A \setminus \{a_1\} & A \setminus \{a_1\} \neq \emptyset \\ \exists a_3 \in A \setminus \{a_1, a_2\} & A \setminus \{a_1, a_2\} \neq \emptyset \\ \vdots & \end{array}$$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of A .

Consider $f : \mathbb{N} \rightarrow A$, $f(n) = a_n$. f is injective as a_n are distinct.

Integers and Power Sets

Cardinality of Integers and Natural Numbers

Statement:

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

Proof:

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{N} \\ f(m) &= \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases} \end{aligned}$$

f is a bijection as $g : \mathbb{N} \rightarrow \mathbb{Z}$, $g(n) = (-1)^{n+1} \lfloor \frac{n}{2} \rfloor$ is the inverse of f .

Power Set and 2^X

Given any set X , $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of X .

$$2^X := \{f \mid f : X \rightarrow \{0, 1\}\}.$$

Statement:

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Proof: Let $\varphi : \mathcal{P}(X) \rightarrow 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbb{1}_A$.

Consider $\psi : 2^X \rightarrow \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbb{1}_A) = \mathbb{1}_A^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbb{1}_{f^{-1}(\{1\})} = f$.

Cantor's Theorem

Statement:

$$\nexists \text{ surjection } \mathbb{N} \rightarrow (0, 1)$$

Proof: From calculus we know $\forall \sigma \in (0, 1)$, σ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \dots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \rightarrow (0, 1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0, 1, \dots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \rightarrow \{0, 1, \dots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since r is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

Comparing Cardinality

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

Since the composition of two injective functions is injective, if $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

Cardinality of the Power Set

Statement: For any set A , $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Proof: Let us construct a function: $f : A \rightarrow \mathcal{P}(A)$, where $a \mapsto \{a\}$.

f is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

Claim: $\nexists g : A \rightarrow \mathcal{P}(A)$, g is surjective.

Suppose toward contradiction that such a g exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.

Since g is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. \perp

Equivalent Cardinality Comparisons

- (i) $\text{card}(A) \leq \text{card}(B)$
- (ii) $\exists f : A \hookrightarrow B$
- (iii) $\exists g : B \rightarrow A$, g surjection.

Proof:

(ii) \Rightarrow (iii) If $f : A \hookrightarrow B$, f is left-invertible, and thus $\exists g : B \rightarrow A$ with $g \circ f = id_A$. So, g is right-invertible, so g is surjective.

(iii) \Rightarrow (ii) If $g : B \rightarrow A$ is surjective, then g is right-invertible, so $\exists f : A \rightarrow B$ such that $g \circ f = id_B$. So, f is left-invertible, so f is injective.

From the above, we can see that, if $f : A \rightarrow B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$, by considering $g : A \rightarrow f(A)$ defined as $g(a) = f(a)$, which is onto, meaning \exists an injection $f(A) \hookrightarrow A$.

Cardinality Rules

- (i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)
- (ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)
- (iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)
- (iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

Proof of (iii): We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

Claim: We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, g and f are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1 = \emptyset$. \perp

We let $A_n = g \circ f(A_{n-1})$ for arbitrary n , and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \rightarrow B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where y_x is the unique element in B with $g(y_x) = x$.

Claim: We claim h is the desired bijection.

Proof of Injection: Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of H , $f(x_1) = f(x_2)$, f is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y_{x_2}) = x_2 \notin A_\infty$. This case is not possible.

Thus, h is injective.

Proof of Surjection: Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where y_x is the unique element in B with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since g is injective, $y = f(z)$, $z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$, $q(m, n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, defined as $H(m, n) = (h(m), n)$.

Claim: H is a bijection.

Proof of Injection: If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since h is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

Proof of Surjection: Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m, n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m, n) = (k, \ell)$. Set $n = \ell$, and since h is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m, n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, φ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned} \text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\ &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \text{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \text{card}(\mathbb{N}) \end{aligned}$$

Countability and the Continuum Hypothesis

A set X is *countable* if $\exists f : X \hookrightarrow \mathbb{N}$ ($\text{card}(X) \leq \text{card}(\mathbb{N})$). $\text{card}(\mathbb{N}) = \aleph_0$. If X is countable and infinite, X is *denumerable*.

Corollary to Cantor-Schröder-Bernstein

Statement: If X is denumerable, then $\text{card}(X) = \aleph_0$.

Proof: Since X is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since X is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

Countability under Union

Statement: The countable union of countable sets is countable. If I is a countable indexing set and for each $i \in I$, A_i is countable, then $\bigcup_{i \in I} A_i$ is countable.

Proof: Since each A_i is countable, $\exists \pi_i : \mathbb{N} \rightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \rightarrow \bigcup_{i \in I} A_i$$

defined as $\pi(i, j) = \pi_i(j)$.

Claim 1: π is a surjection.

Proof 1: Let $x \in \bigcup_{i \in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since π_{i_0} is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, π is surjective.

Claim 2: $I \times \mathbb{N}$ is countable.

Proof 2: We know $\exists f : I \hookrightarrow \mathbb{N}$ since I is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i, n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m, n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since π is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i \in I} A_i$ is countable.

Continuum Hypothesis

Statement:

$$\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}}),$$

where I is any non-degenerate interval.

Proof:

Lemma 1: $\text{card}([0, 1]) \leq \text{card}(2^{\mathbb{N}})$.

Proof 1: Every $t \in [0, 1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0, 1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\phi} [0, 1]$, defined as $\phi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Set $f : \mathbb{N} \rightarrow \{0, 1\}$, $f(k) = \sigma_k$.

Therefore, ϕ is surjective, so $\exists \{0, 1\} \hookrightarrow 2^{\mathbb{N}}$, so $\text{card}([0, 1]) \leq 2^{\mathbb{N}}$

Lemma 2: $\text{card}([0, 1]) = \text{card}(\mathbb{R})$.

Proof 2: We have $[0, 1] \xrightarrow{i} \mathbb{R}$ via inclusion, so $\text{card}([0, 1]) \leq \text{card}(\mathbb{R})$.

Also, $\text{card}(\mathbb{R}) = \text{card}((0, 1)) \leq \text{card}([0, 1])$, so by Cantor-Schröder-Bernstein, $\text{card}(\mathbb{R}) = \text{card}([0, 1])$.

Lemma 3: Any two non-degenerate intervals I and J have the same cardinality.

Proof 3: We can create injections $I \hookrightarrow J$ and vice-versa.

Lemma 4: $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0, 1])$.

Proof 4: $\psi : 2^{\mathbb{N}} \rightarrow [0, 1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

ψ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim ψ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$, $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} \frac{1}{3^k} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$\begin{aligned} |t_f - t_g| &= \left| \sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k} \right| \\ &\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\ &\leq \sum_{k>k_0} \frac{1}{3^k} \\ &= \frac{(1/3)^{k_0+1}}{1 - (1/3)} \\ &= \frac{1}{2} \cdot \frac{1}{3^{k_0}} \end{aligned}$$

\perp

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0, 1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than \aleph_0 and strictly less than 2^{\aleph_0} is an axiom (i.e., it can be an assumption or not).

Field Ordering

Ordering Relations

Let X be a non-empty set. A relation on X is a subset of $X \times X$.

- R is *reflexive* if $\forall x \in X, (x, x) \in R$.
- R is *transitive* if $(x, y), (y, z) \in R \rightarrow (x, z) \in R$.
- If R is *antisymmetric* $(x, y), (y, x) \in R \rightarrow x = y$.

If R is reflexive, transitive, and antisymmetric, then R is an *ordering* of X .

If R is an ordering of X , then we write:

$$(x, y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \quad \forall x \in X$
- $x \leq_R y, y \leq_R z \rightarrow x \leq_R z$
- $x \leq_R y, y \leq_R x \rightarrow x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

Examples of Orderings

Algebraic Ordering of \mathbb{N}_0 : $n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$ such that $n + k = m$

\mathbb{N} ordered via division: $n \leq_D m \Leftrightarrow n|m$; under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

Inclusion: Let S be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_i B \Leftrightarrow A \subseteq B$.

Containment: With X defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

Functions: For $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \rightarrow \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in X$.

Total and Directed Orderings

- An ordering \leq of X is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.
- An ordering is *directed* if $\forall x, y \in X \exists z \in X$ such that $x \leq z$ and $y \leq z$.

If X is a totally ordered set, X is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D), (\mathcal{P}(S), \leq_i), (\mathcal{P}(S), \leq_c)$$

Upper and Lower Bounds

- (i) Let (X, \leq) be an ordered set, $A \subseteq X$. A is bounded above if $\exists v \in X$ with $a \leq v \quad \forall a \in A$. Such a v is an upper bound.
- (ii) A is bounded below if $\exists \ell \in X$ such that $a \geq \ell \quad \forall a \in A$. Such a w is a lower bound.
- (iii) If v is an upper bound of A and $v \in A$, then v is the greatest element of A , or $\max(A) = v$.
- (iv) If ℓ is a lower bound for A and $\ell \in A$, then ℓ is the least element of A , or $\min(A) = \ell$.
- (v) If u is an upper bound for A , and $u \leq v$ for all other upper bounds v of A , then u is the *least upper bound* of A , or $\sup(A) = u$ (for *supremum*).
- (vi) If ℓ is a lower bound for A , and $\ell \leq g$ for all other lower bounds g of A , then ℓ is the *greatest lower bound* of A , or $\inf(A) = \ell$ (for *infimum*).
- (vii) If A is bounded above and below, then A is bounded.

An ordered set (X, \leq) is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, \mathbb{Q} is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

Well-Ordering Principle: With (\mathbb{N}, \leq_a) , every nonempty $A \subseteq \mathbb{N}$ has a least element.

Examples

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \dots, 12\}$, we have the following:

Bounded Above? Yes.

Upper Bounds 12, 13, 14, ...

Greatest Element 12

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \dots, 10\}$

Bounded Above? Yes.

Upper Bounds 10!

Greatest Element? No.

Supremum $2^3 \cdot 3^2 \cdot 5 \cdot 7$

Bounded Below? Yes.

Lower Bound 1

Least Element? No.

Infimum 1

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$.

Supremum $\bigcup_{i \in I} A_i$

Infimum $\bigcap_{i \in I} A_i$

Ordering of \mathbb{Z} , \mathbb{Q} , and \mathbb{R}

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0, n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

Properties of \mathbb{Z}^+

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

Statement:

(1) $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$

(2) $m \leq_a n$ and $p \leq_a q \Rightarrow m + p \leq_a n + q$

(3) $m \leq_a n$ and $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$

(4) $m \leq_a n \Rightarrow -m \geq_a n$

(5) \leq_a is total.

(6) If $a_a > 0$, and $ab_a \geq 0$, then $b_a \geq 0$

(7) If $a > 0$ and $ab_a \geq ac$, then $b \geq c$.

Proof of (3):

$$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0 \text{ with } m + k = n.$$

$$\Rightarrow pm + pk = pn$$

$$pk \in \mathbb{N}_0 \text{ by the properties of } \mathbb{Z}^+. \text{ So, } pm \leq_a pn$$

Proof of (5):

Let $m, n \in \mathbb{Z}$. Consider $m - n$.

By (ii), $m - n \in \mathbb{Z}^+$ or $-(m - n) \in \mathbb{Z}^+$. Thus, $m - n = k$ for some $k \in \mathbb{Z}^+$, or $-(m - n) = \ell$ for some $\ell \in \mathbb{Z}^+$.

Thus, $n \leq_a m$ in the first case, or $m \leq_a n$ in the second case.

Creating the Rationals

Recall that $Q = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}^+, b \neq 0\}$. Consider the equivalence relation:

$$(a, b) \sim (c, d) \stackrel{\text{def}}{\iff} ad = bc$$

We will let $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in Q\}$ be the set of all equivalence classes in Q . We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined: $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{a'd' + c'b'}{b'd'} = \frac{ad + bc}{bd}$.

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make \mathbb{Q} a **field**.

Fields

A ring is a nonempty set R equipped with two binary operations:

- $+: R \times R \rightarrow R, (a, b) \mapsto a + b$ ("addition")
- $\cdot: R \times R \rightarrow R, (a, b) \mapsto a \cdot b$ ("multiplication")

such that the following hold:

- (1) $(a + b) + c = a + (b + c)$
- (2) $\exists z \in R$ such that $a + z = a = z + a \forall a \in R$; there is at most one such z . Set $z = 0_R$.
- (3) $\forall a \in R, \exists b \in R$ such that $a + b = 0_R = b + a$; there is at most one such b . Set $b = -a$.
- (4) $\forall a, b \in R, a + b = b + a$.
- (5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (6) $a \cdot (b + c) = a \cdot b + a \cdot c, (a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If $(R, +, \cdot)$ satisfies $ab = ba$, R is a commutative ring.

If there exists $u \in R$ such that $ua = au = a \forall a \in R$, R is a unital ring; there is at most one unit. Set $u = 1_R$

An integral domain is a unital, commutative ring such that $ab = 0 \Rightarrow a = 0 \vee b = 0$. For example, \mathbb{Z} is an integral domain. However, $C(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ is a unital, commutative ring, but there exist two functions such that $f, g \neq \mathbf{0}$, but $f \cdot g = \mathbf{0}$.

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R, \text{ with } ab = 1_R$$

There is only one such b . Set $b = a^{-1}$.

Ordering of \mathbb{Q}

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

\leq is a well-defined total ordering of \mathbb{Q} , and $j : \mathbb{Z} \hookrightarrow \mathbb{Q}, j(n) = \frac{n}{1}$ is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

Properties of \mathbb{Q}^+

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

$$(i) \quad q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+, q_1 q_2 \in \mathbb{Q}^+$$

$$(ii) \quad q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$$

$$(iii) \quad \pm q \in \mathbb{Q}^+, q = 0$$

$$(iv) \quad x \leq y, u \leq v \Rightarrow x + u \leq y + v$$

$$(v) \quad x \leq y, 0 \leq z \Rightarrow zx \leq zy$$

Ordered Fields and the Ordering of \mathbb{R}

An **ordered field** is a field F equipped with a total ordering \leq_F such that:

$$(i) \quad \text{if } s \leq_F t, \text{ and } x \leq_F y, \text{ then } s + x \leq_F t + y$$

$$(ii) \quad \text{if } s \leq_F t \text{ and } 0 \leq_F z, \text{ then } zs \leq_F zt$$

For example, \mathbb{Q} with its ordering is an ordered field.

Statement: If (F, \leq_F) is an ordered field, we define $F^+ = \{x \in F \mid x_F \geq 0\}$ with the following properties:

$$(1) \quad x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$$

$$(2) \quad x \in F \Rightarrow x \in F^+, -x \in F^+$$

$$(3) \quad \pm x \in F^+ \Rightarrow x = 0_F$$

Proofs:

$$(1) \quad \text{Let } x, y \in F^+. \text{ Then, } x \geq 0 \text{ and } y \geq 0, \text{ so by property (i) of an ordered field, } x + y \geq 0 + 0 = 0, \text{ so } x + y \in F^+. \text{ Additionally, we have } x \cdot y \geq x \cdot 0 = 0, \text{ so } xy \in F^+.$$

- (2) Let $x \in F$. Since the ordering on F is total, $x \geq 0$ or $0 \geq x$. In the first case, $x \in F^+$. In the second case, we add $-x$ to both sides, so by (i), $-x \geq 0$, so $-x \in F^+$.
- (3) We have $x \geq 0$ and $-x \geq 0$. So $x \geq 0$ and $x + (-x) \geq x + 0$, so $x \geq 0$ and $0 \geq x$. So, $x = 0$ by antisymmetry.

Note: $x \leq_F y \Leftrightarrow y - x \in F^+$.

Statement: Let F be an ordered field. Then, the following is true:

- (1) $\forall a \in F, a^2 \in F^+$
- (2) $0, 1 \in F^+$
- (3) If $n \in \mathbb{N}$, $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$
- (4) If $x \in F^+$, and $x \neq 0$, then $x^{-1} \in F^+$
- (5) If $xy > 0$, then $x, y \in F^+$, or $-x, -y \in F^+$
- (6) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$
- (7) If $x \leq y$, then $-y \leq -x$
- (8) $x \geq 1 \Rightarrow x^2 \geq x \geq 1$, and $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$.

Proof:

- (1) Let $a \in F$. Then, $a \in F^+$ or $-a \in F^+$.
 Case 1 If $a \in F^+$, then by the previous proposition, $a^2 \in F^+$.
 Case 2 If $-a \in F^+$, then by the previous proposition, $(-a)(-a) = a^2 \in F^+$.
- (2) $0 \geq 0$, so $0 \in F^+$. $1 = 1 \cdot 1 = 1^2 \in F^+$ by the previous result.
- (3) $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}} \in F^+$ by the previous proposition.
- (4) Let $x \neq 0, x \in F^+$. Suppose toward contradiction that $x^{-1} \notin F^+$, then $-x^{-1} \in F^+$. Thus, $x \cdot (-x^{-1}) \in F^+$, so $-1 \in F^+$, but $1 \in F^+$, so $1 = 0$. \perp
- (5) Let $xy > 0$, meaning $xy \in F^+$. Suppose toward contradiction that $x > 0$ and $y < 0$. So, $x > 0$ and $-y > 0$, so $(x)(-y) > 0$, so $-(xy) \in F^+$, so $xy = 0$. \perp
- (6) Let $0 < x \leq y$. We know $x^{-1} \in F^+$, so $x^{-1}x \leq x^{-1}y$. So $1 \leq x^{-1}y$. We also know $y \in F^+$, so $y^{-1} \in F^+$. So, $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$.
- (7) Let $x \leq y$. Then, $0 \leq y - x$, so $-y \leq -x$.
- (8) Let $x \geq 1$. Then, $x \cdot x \geq 1 \cdot x \geq 1$.

Order Axiom: \mathbb{R} is an ordered field. The injection $\mathbb{Q} \hookrightarrow \mathbb{R}$, $i(q) = q$ is an order embedding.

Rational Orderings

Statement: If $a \leq b$, then $a \leq \frac{1}{2}(a+b) \leq b$.

Proof: $2a = a + a \leq a + b \leq b + b$, all by property (i) of an ordered field.

Therefore, $2a \leq a + b \leq 2b$. Since $2 = 1 + 1 \in \mathbb{R}^+$, $2^{-1} \in \mathbb{R}^+$, so $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$, so $a \leq \frac{1}{2}(a+b) \leq b$.

Statement: If $a \geq 0$ and $(\forall \epsilon > 0), a \leq \epsilon$, then $a = 0$.

Proof: Suppose toward contradiction that $a \geq 0$ and $a \neq 0$, so $a > 0$. So, we have that $\frac{1}{2}a < a$. Let $\epsilon = \frac{1}{2}a$. We also have that $a \leq \epsilon = \frac{1}{2}a < a$, so $a < a$. \perp

Important Inequalities

Arithmetic and Geometric Means

Given $a_1, a_2, \dots, a_n \in \mathbb{R}^+$:

Arithmetic Mean

$$= \frac{\sum_{i=1}^n a_i}{n}$$

Geometric Mean

$$= \sqrt[n]{a_1 a_2 \cdots a_n}$$

Arithmetic Mean-Geometric Mean Inequality

Statement: Let $a, b \geq 0$.

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

Proof: If $x, y \geq 0$, $x \leq y \Leftrightarrow x^2 \leq y^2$.

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y$$

by property (ii) of ordered fields

Therefore,

$$\begin{aligned} (ab)^{1/2} &\leq \frac{1}{2}(a+b) \\ ab &\leq \frac{1}{4}(a^2 + 2ab + b^2) \\ 4ab &\leq a^2 + 2ab + b^2 \\ 0 &\leq a^2 - 2ab + b^2 \\ 0 &\leq (a-b)^2 \end{aligned}$$

by definition

Challenge: Prove for m .

Remark: The harmonic mean is defined as:

$$\frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$$

Bernoulli's Inequality

Statement: If $x \geq -1$, then $(1+x)^n \geq 1+nx$, for any $n \in \mathbb{N}_0$.

Proof: By induction, we know that for $n=0$ and $n=1$, this holds.

Assume the inequality holds for some $m \geq 1$.

$$\begin{aligned}
 (1+x)^{m+1} &= (1+x)^m(1+x) \\
 &\geq (1+mx)(1+x) && \text{by the inductive hypothesis} \\
 &= 1+x+mx+mx^2 \\
 &= 1+(m+1)x+mx^2 \\
 &\geq 1+(m+1)x
 \end{aligned}$$

Cauchy's Inequality

Statement: Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. Then

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left(\sum_{j=1}^n a_j^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra language, this is equivalent to $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$.

Proof: Consider $f: \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^n (a_i - b_i x)^2$$

We know that $f(x) \geq 0$ for all $x \in \mathbb{R}$

$$\begin{aligned}
 &= \sum_{i=1}^n (a_i^2 - 2a_i b_i x + b_i^2 x^2) \\
 &= \left(\sum_{j=1}^n b_j^2 \right) x^2 + \left(\sum_{j=1}^n -2a_j b_j \right) x + \sum_{j=1}^n a_j^2 \\
 &= Ax^2 + Bx + C
 \end{aligned}$$

Therefore, $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned}
 \left(-2 \sum_{j=1}^n a_j b_j \right)^2 &\leq 4 \left(\sum_{j=1}^n a_j^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \\
 \left| \sum_{j=1}^n a_j b_j \right| &= \left(\sum_{j=1}^n a_j^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2}
 \end{aligned}$$

As we know from linear algebra, the way we get equality is when $\vec{v} = c\vec{w}$, or that $a_j = cb_j \forall j$ for some $c \in \mathbb{R}$.

Triangle Inequality

Statement: Given $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left(\sum_{j=1}^n (a_j + b_j)^2 \right)^{1/2} \leq \left(\sum_{j=1}^n a_j^2 \right)^{1/2} + \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra, this is equivalent to $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

Proof:

$$\begin{aligned} \sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\ &\leq \sum a_j^2 + 2 \left(\sum a_j^2 \right)^{1/2} \left(\sum b_j^2 \right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\ &= \left(\left(\sum a_j^2 \right)^{1/2} + \left(\sum b_j^2 \right)^{1/2} \right)^2 \end{aligned}$$

we take square roots to get our end result

Metrics, Norms, and Bounds

Metrics and Norms on \mathbb{R}^n

Consider $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$, defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

Theorems about Absolute Value:

- (i) $|ab| = |a||b|$
- (ii) $|a^2| = |a|^2$
- (iii) $|-a| = |a|$
- (iv) $|a| \in \mathbb{R}^+$
- (v) $-|a| \leq a \leq |a|$
- (vi) $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$ for $\delta > 0$
- (vii) $|a + b| \leq |a| + |b|$, $|a - b| \leq |a| + |b|$, $||a| - |b|| \leq |a - b|$

Proof of (i):

Case 1: If $a, b \in \mathbb{R}^+$, then $|a| = a$, and $|b| = b$, and $ab \in \mathbb{R}^+$, so $|ab| = ab$

Case 2: If $a, b \notin \mathbb{R}^+$, then $|a| = -a$, and $|b| = -b$. Additionally, $(-a)(-b) = ab \in \mathbb{R}^+$, so $|ab| = ab$. The LHS = ab , and the RHS = ab .

Case 3: $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$. Then, $|a||b| = (a)(-b) = -ab$. Then, since $a(-b) \in \mathbb{R}^+$, $-ab \in \mathbb{R}^+$, so $|ab| = -ab$. Therefore, the LHS and RHS are equal.

Proof of (vii): Having established that $|a + b| \leq |a| + |b|$, we will show that $||a| - |b|| \leq |a - b|$.

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a - b| + |b| \\ |a| - |b| &\leq |a - b| \end{aligned}$$

Similarly, by exchanging a for b

$$|b| - |a| \leq |b - a|$$

$$|b| - |a| \leq |a - b|$$

Let $t = |a| - |b|$. We have shown that

$$\pm t \leq |a - b|$$

$$-|a - b| \leq t \leq |a - b|$$

$$|t| \leq |a - b|$$

Bounded Sets

A subset $A \subseteq \mathbb{R}$ is **bounded** $\Leftrightarrow \exists c \geq 0$ such that $\forall x \in A, |x| \leq c$.

(\Rightarrow) Suppose $A \subseteq \mathbb{R}$ is bounded. Then, $\exists \ell, u \in \mathbb{R}$ such that $\ell \leq x \leq u \forall x \in A$. Let $c := \max\{|\ell|, |u|\}$.

Since $|u| \leq c$, we have that $x \leq c$.

Since $|\ell| \leq c$, and $-\ell \leq x$, we get that $-x \leq |\ell| \leq c$.

Since $x \leq c$ and $-x \leq c$, $|x| \leq c$.

(\Leftarrow) If such a c exists, then $|x| \leq c$ if and only if $-c \leq x \leq c$. Thus, $-c$ is the lower bound and c is the upper bound.

Bounded Functions

Let D be any set. A function $f : D \rightarrow \mathbb{R}$ is bounded if $\text{Ran}(D) \subseteq \mathbb{R}$ is bounded.

For example, let $f : [3, 7] \rightarrow \mathbb{R}$, $f(x) = \frac{x^2+2x+1}{x-1}$. We will show that f is bounded.

$$3 \leq x \leq 7 \Rightarrow 2 \leq x - 1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}.$$

$$\text{Also, } 4 \leq x + 1 \leq 8 \Rightarrow 16 \leq x^2 + 2x + 1 \leq 64 \Rightarrow |x^2 + 2x + 1| \leq 64.$$

So, $|f(x)| \leq 32$.

Distance Metrics

For $s, t \in \mathbb{R}$, we will define $d(s, t) = |s - t|$ to be the **distance** between s and t .

Properties:

(i)

$$d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$$

$$(s, t) \mapsto d(s, t) \geq 0$$

(ii) $d(s, t) = d(t, s)$

(iii) $d(s, r) \leq d(s, t) + d(t, r)$

(iv) $d(s, s) = 0$

(v) If $d(s, t) = 0$, then $s = t$.

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^n |x_j|$$

- ∞ -norm:

$$\|v\|_\infty = \max_{j=1}^n |x_j|$$

- 2-norm:

$$\|v\|_2 = \left(\sum_{j=1}^n x_j^2 \right)^{1/2}$$

Properties of Norms

Statement: With v, w above, let $p = 1, 2, \infty$. The following are true:

- (1) $\|v\|_p \geq 0$
- (2) $\|v + w\|_p \leq \|v\|_p + \|w\|_p$
- (3) $\|\vec{0}\|_p = 0$
- (4) $\|v\|_p = 0 \Rightarrow v = \vec{0}$
- (5) $\forall t \in \mathbb{R}, \|tv\|_p = |t|\|v\|_p$

Proofs: Let $p = \infty$. We will prove (2).

Say $\|v\|_{\infty} = |x_i|$ and $\|w\|_\infty = |y_k|$. We want to show that $\|v + w\|_\infty = \max_{j=1}^n |x_j + y_j| \leq |x_i| + |y_k|$.

Note that $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| && \text{Triangle Inequality} \\ &\leq |x_i| + |y_k| \\ &= \|v\|_\infty + \|w\|_\infty \end{aligned}$$

Therefore, $\|v + w\|_\infty \leq \|v\|_\infty + \|w\|_\infty$.

Relating Distance Metrics and Norms

A **norm** on \mathbb{R}^n is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$, $v \mapsto \|v\|$, satisfying the following properties for $v \in \mathbb{R}^n$:

- (1) $\|v\| \geq 0$
- (2) $\|v + w\| \leq \|v\| + \|w\|$
- (3) $\|\vec{0}\| = 0$
- (4) $\|v\| = 0 \Rightarrow v = \vec{0}$

$$(5) \forall t \in \mathbb{R}, \|tv\| = |t|\|v\|$$

If $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a norm, we define $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$, defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for $v, w \in \mathbb{R}^n$.

The properties of distance in \mathbb{R} still hold for distance in \mathbb{R}^n :

- (1) $d(v, w) = d(w, v)$
- (2) $d(u, w) \leq d(u, v) + d(v, w)$
- (3) $d(v, v) = 0$
- (4) $d(v, w) = 0 \Rightarrow v = w$

Metric Spaces

A **metric space** is a nonempty set X equipped with a function $d : X \times X \rightarrow \mathbb{R}^+$, $(x, y) \mapsto d(x, y) \geq 0$. The metric has the following properties:

- (1) $d(x, y) = d(y, x) \forall x, y \in X$
- (2) $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X$
- (3) $d(x, x) = 0$
- (4) $d(x, y) = 0 \Leftrightarrow x = y$

The map d is called a *metric* on X .

Examples of Metric Spaces

- \mathbb{R} with $d(x, y) = |x - y|$.
- \mathbb{R}^n with the *Euclidean metric*:

$$\begin{aligned} d_2(v, w) &= \|v - w\|_2 \\ &= \left(\sum_{j=1}^n (x_j - y_j)^2 \right)^{1/2} \end{aligned}$$

- \mathbb{R}^n with the 1-norm:

$$\begin{aligned} d_1(v, w) &= \|v - w\|_1 \\ &= \sum_{j=1}^n |x_j - y_j| \end{aligned}$$

- \mathbb{R}^n with the ∞ -norm:

$$\begin{aligned} d_\infty(v, w) &= \|v - w\|_\infty \\ &= \max_{j=1}^n |x_j - y_j| \end{aligned}$$

Open and Closed Sets in Metric Spaces

Let (X, d) be a metric space.

- (1) The **open ball** centered at $x_0 \in X$ with radius δ is:

$$V(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

- (2) The **closed ball** centered at $x_0 \in X$ with radius δ is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

- (3) A set $U \subseteq X$ is **open** if $\forall x \in U, \exists \delta > 0$ such that $V(x, \delta) \subseteq U$.

- (4) A set $C \subseteq X$ is **closed** if $\overline{C} = X - C \subseteq X$ is open.

For example,

In \mathbb{R} with $d(s, t) = |s - t|$:

$$\begin{aligned} V(x_0, \delta) &= \{y \in \mathbb{R} \mid d(y, x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0, \delta) &= [x_0 - \delta, x_0 + \delta] \end{aligned}$$

The interval $A = [1, \infty)$ is not open, as $\forall \delta > 0, U(1, \delta) \not\subseteq [1, \infty)$.

However, A is closed, as $\overline{A} = (-\infty, 1)$ is open: given $t \in \overline{A}$, choose $\delta = 1 - t$. Let $s \in V_\delta(t)$. Then, $s \in (t - \delta, t + \delta)$, so $s \in (t - (1 - t), t + (1 - t))$, or $s \in (2t - 1, 1)$, so $s < 1$.

In (\mathbb{R}^2, d_2) , $B(0_{\mathbb{R}^2}, 1)$ is the **unit disc** centered at $(0, 0)$.

However, in (\mathbb{R}^2, d_∞) :

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

Supremum, Infimum, and Completeness

Finding a Supremum

Statement: Let $0 \neq A \subseteq \mathbb{R}$. Let $u \in \mathbb{R}$ be an upper bound for A . The following are equivalent:

- (i) $u = \sup(A)$
- (ii) If $t < u$, then $\exists a_t \in A$ such that $a_t > t$
- (iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $u - \varepsilon < a_\varepsilon$

Proof:

(i) \Rightarrow (ii): Given $t < u$, if no such $a \in A$ with $t < a$ exists, then $a \leq t \forall a \in A$. Thus t would be an upper bound. However, $t < u$ and u is the supremum of A . \perp

(ii) \Rightarrow (iii): Given $\varepsilon > 0$, set $t = u - \varepsilon < u$. So, by (ii), $\exists a_t \in A$ with $t < a_t$. Thus, $u - \varepsilon < a_t$. Set $a_\varepsilon = a_t$.

(iii) \Rightarrow (i): Let v be an upper bound for A . Suppose $v < u$. Then, set $\varepsilon = u - v > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u - \varepsilon < a_\varepsilon$. So $u - (u - v) < a_\varepsilon$, so $v < a_\varepsilon$, meaning v cannot be an upper bound. \perp

Supremum Example

$\sup[0, 1) = 1$: Certainly, 1 is an upper bound for $[0, 1)$. Let $\varepsilon > 0$.

If $\varepsilon \geq 1$, pick $t = \frac{1}{2}$. Then, $1 - \varepsilon < 0 < \frac{1}{2}$

If $0 < \varepsilon < 1$, let $t = (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$. Then, $t \in [0, 1)$, and $1 - \varepsilon < 1 - \varepsilon/2 = t$

Finding an Infimum

Let $\emptyset \neq A \subseteq \mathbb{R}$. Let $\ell \in \mathbb{R}$ be a lower bound for A . The following are equivalent:

- (i) $\ell = \inf(A)$
- (ii) If $t > \ell$, $\exists a_t$ such that $t > a_t$
- (iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $\ell + \varepsilon > a_\varepsilon$

Infimum Example

$\inf \left\{ \frac{1}{n} \mid n \geq 1 \right\}$: Clearly, $0 < \frac{1}{n} \forall n \geq 1$. Let $\varepsilon > 0$.

We need to find $a \in \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ with $\varepsilon > a$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$.
Let $a_\varepsilon = \frac{1}{m}$.

Properties of Supremum and Infimum

- If $A \subseteq \mathbb{R}$ and $\max(A) = u$, then $u = \sup(A)$: u is an upper bound of A by the definition of \max , and if $v \neq u$ is any upper bound of A , then $u < v$ since $u \in A$.
- If $\min(A) = \ell$, then $\ell = \inf(A)$ (by the same logic).
- If A is not bounded above, $\sup(A) = +\infty$, and if A is not bounded below, then $\inf(A) = -\infty$.
- If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.
- If $A \subseteq B$, then $\inf(A) \geq \inf(B)$: Let $\ell_A = \inf(A)$ and $\ell_B = \inf(B)$. By definition, $\ell_B \leq b \forall b \in B$. Since $A \subseteq B$, $\ell_B \leq a \forall a \in A$. Thus, ℓ_B is a lower bound for A . By definition of ℓ_A , $\ell_B \leq \ell_A$.

Let $A, B \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Then, the following are also sets:

- (1) $A + B = \{a + b \mid a \in A, b \in B\}$
- (2) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$
- (3) $t \cdot A = \{ta \mid a \in A\}$
- (4) $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

Completeness Axiom

If $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above, then $\sup(A)$ exists.

Well-Ordering Property: if $\emptyset \neq S \subseteq \mathbb{N}$, then $\min(S)$ exists.

Therefore, we can prove that if $F \subseteq \mathbb{Z}$ is bounded, then F has a least and greatest element.

Archimedean Property

Statement: If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x \leq n_x$.

Proof: Suppose there exists no natural number greater than x , then \mathbb{N} is bounded above by x . Let $u = \sup(\mathbb{N})$. By the Completeness Axiom, $u \in \mathbb{R}$ exists. Let $\varepsilon = 1$. Then, $\exists n \in \mathbb{N}$ with $u - 1 < n$. So, $u < n + 1$, but $n + 1 \in \mathbb{N}$, so u cannot be an upper bound.

Corollary to the Archimedean Property

$$\forall t > 0 \exists n \in \mathbb{N} \text{ such that } \frac{1}{n} < t$$

Corollary to the Corollary: Powers of 2

Statement:

$$\forall t > 0 \exists m \in \mathbb{N} \text{ such that } \frac{1}{2^m} < t$$

Proof: By the corollary to the Archimedean Property, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < t$. By Bernoulli's inequality with $x = 1$, we have $2^n \geq n$, so $2^{-n} < n^{-1} < t$.

Corollary to the Corollary: In Between Integers

Statement:

$$\forall x \in \mathbb{R} \exists n_x \in \mathbb{Z} \text{ such that } n_x - 1 \leq x < n_x$$

Proof: Assume $x \geq 0$. Let $S_x = \{n \mid n \in \mathbb{N}, x < n\}$.

$S_x \neq \emptyset$ by the Archimedean Property. By the well-ordering property, let $n_x = \min(S_x)$.

Therefore, $x < n_x$. Also, $n_x - 1 \notin S_x$. Therefore $n_x - 1 \leq x$.

Density

Let (X, d) be any metric space. A subset $D \subseteq X$ is **dense** if $\forall x \in X, \varepsilon > 0, U(x, \varepsilon) \cap D \neq \emptyset$.

In the case of $X = \mathbb{R}$ and $d(s, t) = |s - t|$, $D \subseteq \mathbb{R}$ is dense if given any open interval I , $I \cap D \neq \emptyset$.

A metric space is **separable** if it admits a *countable* dense subset.

Density of the Rationals

Statement: $\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Proof: Let $I = (a, b)$ be an open interval. We may assume that $a, b \in \mathbb{R}$ are finite.

Then, $b - a > 0$. By the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, meaning $1 < nb - na$.

There exists also an integer m such that $m - 1 \leq na < m$, implying that $a \frac{m}{n}$. Also, $m \leq na + 1 < nb$. Therefore, $\frac{m}{n} < b$.

So, $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$.

Density of the Irrationals

Statement: $\mathbb{R} \setminus \mathbb{Q}$ is dense.

Proof: Assume $\sqrt{2}$ exists. Let $I = (a, b)$ be any open interval. Then, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

Find $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$.

Then, $a < q\sqrt{2} < b$, where $q\sqrt{2} \in \mathbb{R}$ and $q\sqrt{2} \notin \mathbb{Q}$.

Uniqueness of $\sqrt{2}$

Statement:

$$\exists! x > 0 \text{ such that } x^2 = 2$$

Proof:

Existence: Let $S = \{t \in \mathbb{R} \mid 0 < t, t^2 < 2\}$. S is nonempty because $1 \in S$, and S is bounded above because $y > 2 \Rightarrow y^2 > 4$.

So, by the completeness axiom, $x := \sup(S)$ exists, and $x \geq 1$.

Claim: $x^2 = 2$

Contradiction 1: Assume $x^2 < 2$. We want to show that $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$. By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2 \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x + 1) \end{aligned}$$

We want to find $n \in \mathbb{N}$ with:

$$x^2 + \frac{1}{n}(2x + 1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

Therefore, by the Archimedean Property corollary, we know that n must exist.

Contradiction 2: We know that $x^2 \geq 2$. Since $x = \sup(S)$, $\forall m \in \mathbb{N}$, $\exists t_m \in S$ with $x - \frac{1}{m} < t_m$, so $(x - \frac{1}{m})^2 < t_m^2 < 2$.

Therefore, $x^2 - \frac{2x}{m} + \frac{1}{m^2} < 2$, so $x^2 - \frac{2x}{m} < 2$, so $0 \leq x^2 - 2 < \frac{2x}{m}$.

So, $0 \leq \frac{x^2 - 2}{2x} < \frac{1}{m}$, so $x^2 - 2 = 0$, so $x^2 = 2$.

Remark: If we had set $S' = \{t' \in \mathbb{Q} \mid t'^2 < 2, t' > 0\}$, we would have still obtained $\sup(S') = \sqrt{2}$. This means that \mathbb{Q} is *not* complete.

Intervals in \mathbb{R}

(*) Given any interval I , if $x_1, x_2 \in I$, with $x_1 < x_2$, then $[x_1, x_2] \in I$.

This seems like an obvious property, but this is the *characteristic property* of intervals.

Characterization of Intervals

Statement: Let $S \in \mathbb{R}$ be any nonempty subset of cardinality at least 2. Suppose S satisfies (*). Then, S is an interval.

Proof:

Case 1: Suppose S is bounded.

Let $a = \inf(S)$ and $b = \sup(S)$. Then, $S \subseteq [a, b]$. We will show that $(a, b) \subseteq S$. Once this is shown, $S = \{(a, b), [a, b], [a, b), (a, b]\}$.

Let $t \in (a, b)$. Since $a = \inf(S)$, $\exists x_1 \in S$, $x_1 \in (a, t)$. Similarly, since $b = \sup(S)$, $\exists x_2 \in S$, $x_2 \in (t, b)$.

By the hypothesis, $[x_1, x_2] \subseteq S$. Since $t \in [x_1, x_2]$, $t \in S$.

Case 2: Suppose S is bounded above, but not below.

Let $b = \sup(S)$. Clearly, $S \subseteq (-\infty, b]$. We will show that $(-\infty, b) \subseteq S$. Once this is shown, $S = \{(-\infty, b), (-\infty, b]\}$.

Let $t \in (-\infty, b)$. Since $b = \sup(S)$, $\exists x_2 \in S$, $x_2 \in (t, b)$.

Since S is not bounded below, $\exists x_1 \in S$ such that $x_1 < t$ (or else t would be a lower bound).

By the hypothesis, $[x_1, x_2] \in S$, and $t \in [x_1, x_2]$, so $t \in S$.

Case 3, 4: Left as an exercise for the reader.

Nested Intervals

A sequence of intervals $(I_n)_{n \geq 1}$ is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in $\bigcap I_n$.

(a) $\bigcap_{n=1} [0, 1/n] = \{0\}$.

(b) $\bigcap_{n=1} (0, 1/n) = \emptyset$

(c) $\bigcap_{n=1} [n, \infty) = \emptyset$

Measure

The **measure** of an interval is basically its "size."

$$(a) \quad m([a, b]) = b - a$$

$$(b) \quad m((a, b]) = b - a$$

$$(c) \quad m((a, b)) = b - a$$

$$(d) \quad m([a, b)) = b - a$$

Nested Intervals Theorem

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested sequence of intervals.

$$(1) \quad \bigcap_{n \geq 1} I_n \neq \emptyset$$

$$(2) \quad \text{If } \inf \{m(I_n) \mid n \geq 1\} = 0, \text{ then } \bigcap_{n \geq 1} I_n = \{\xi\}$$

Proof of (1): Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, we have that $a_1 \leq a_2 \leq a_3, \dots$, and $b_1 \geq b_2 \geq b_3 \geq \dots$.

We know that $\{a_n\}$ is bounded above and nonempty. Let $\xi = \sup(\{a_n\}_{n=1}^\infty)$.

We know that $\{b_n\}$ is bounded below. Let $\eta = \inf(\{b_n\}_{n=1}^\infty)$.

We claim that $\xi \leq b_n \forall n \geq 1$. Suppose toward contradiction that $\exists m$ such that $\xi > b_m$. Then, by the supremum property, $\exists a_i$ such that $\xi > a_i > b_m$. If $k \leq m$, $a_k \leq a_m \leq b_m < a_k$. If $m \leq k$, then $b_m \geq b_k \geq a_k > b_m$. \perp

Similarly, using the same argument, $a_n \leq \eta \forall n$.

Thus, $\xi \leq \eta$.

We claim that $\bigcap_{n \geq 1} I_n = [\xi, \eta]$. If $t \in [\xi, \eta]$, then $a_n \leq \xi \leq t \leq \eta \leq b_n$. So $t \in [a_n, b_n] \forall n$, so $t \in \bigcap_{n \geq 1} [a_n, b_n]$.

If $t \in \bigcap_{n \geq 1} I_n$, then $t \in [a_n, b_n] \forall n$, so $a_n \leq t \leq b_n \forall n$. So, t is an upper bound on a_n , and a lower bound on b_n . So, $\xi \leq t \leq \eta$ by definition of ξ and η .

Proof of (2): We have $\forall n \geq 1$

$$\begin{aligned} [\xi, \eta] &\subseteq [a_n, b_n] \\ \Rightarrow 0 &\leq \eta - \xi \leq b_n - a_n \\ &= m(I_n) \end{aligned}$$

So, if $\inf(\{m(I_n) \mid n \geq 1\}) = 0$, then $0 \leq \eta - \xi \leq 0$, so $\eta = \xi$.

Corollary to the Nested Intervals Theorem

Statement: $[0, 1]$ is uncountable.

Proof: Suppose it is possible to denumerate the interval $[0, 1] = \{t_1, t_2, \dots\}$.

We can find $[a_1, b_1] \subseteq [0, 1]$ with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$.

Then, we find $[a_2, b_2] \subseteq [a_1, b_1]$ with $a_2 < b_2$, $t_2 \notin [a_2, b_2]$.

Recursively, we find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $a_n < b_n$, $t_n \notin [a_n, b_n]$.

So, $I_n = ([a_n, b_n])_0^\infty$ is a sequence of nested intervals.

Therefore, $\exists \xi \in \bigcap I_n \subseteq [0, 1]$. However, $\xi \notin \{t_1, t_2, \dots\}$. \perp

Sequences and Convergence

Sequences in Metric Spaces

A sequence in a metric space M is a map

$$x : \mathbb{N} \rightarrow M$$

$M = \mathbb{R}$, usually

$$x = (x_n)_{n=1}^\infty$$

I. Sequences defined by a formula:

- (i) $x_n = t$ (the constant sequence)
- (ii) $x_n = 2n + 1$
- (iii) $x_n = \frac{1}{n-1}$ (here, $n \geq 2$)
- (iv) $c_n = n \sin\left(\frac{1}{n}\right)$
- (v) $d_n = \left(1 + \frac{1}{n}\right)^n$
- (vi) Geometric Sequence: for $b \neq 0$, $(b^n)_{n \geq 0} = (1, b, b^2, \dots)$
- (vii) $x_n = \frac{n!}{n^n}$
- (viii) Given any function

$$f : [0, \infty) \rightarrow \mathbb{R}$$

we can set $x_n = f(n)$.

II. Sequences defined recursively:

- (i) $a_1 = 1$, $a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
- (ii) Fibonacci: $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$. The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - (1 - \varphi)^n)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$

(iii) Let $f : M \rightarrow M$ be a self-map on a metric space. Fix $x_0 \in M$.

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

III. New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences, $t \in \mathbb{R}$. Then, we can do the following:

- $(a_n)_n + (b_n)_n = (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n(b_n)_n = (a_nb_n)_n$
- If $b_n \neq 0 \forall n$, $\left(\frac{a_n}{b_n}\right)$

(ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

(iii) We can look at ratios for $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

(iv) We can look at partial sums, given $(a_k)_{k=1}^\infty$.

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^n a_k \end{aligned}$$

The sequence $(s_n)_n$ is called the sequence of partial sums. For example, the sequence of partial sums for $(b^k)_{k=0}^\infty$ is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for $b \neq 1$, $(1-b)(1+b+b^2+\cdots+b^n) = 1-b^{n+1}$

Finding a Sequence

Statement: Let $a_k = \frac{1}{k(k+1)}$. Find $(s_n)_n$.

Solution: Via partial fraction decomposition, we get that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore, $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^\infty$

Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R}, a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k| \quad \text{Infinity Norm}$$

Statement: This norm has the traditional properties of the norm:

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty && \text{Triangle Inequality} \\ \|t(a_k)_k\|_\infty &= |t| \|(a_k)_k\|_\infty && \text{Scalar Multiplication} \\ \|(a_k)_k\|_\infty = 0 &\Leftrightarrow a_k = 0 \quad \forall k && \text{Zero Property} \\ \|(a_k)_k (b_k)_k\|_\infty &\leq \|(a_k)_k\|_\infty \|(b_k)_k\|_\infty && \text{Multiplication} \end{aligned}$$

Proof: Let $u = \|(a_k)_k\|_\infty$ and $v = \|(b_k)_k\|_\infty$.

Given any k ,

$$\begin{aligned} |a_k + b_k| &\leq |a_k| + |b_k| && \text{Triangle Inequality on } |\cdot| \\ &\leq u + v && \text{definition of supremum} \\ \Rightarrow \sup_{k \geq 1} |a_k + b_k| &\leq u + v \end{aligned}$$

Thus,

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &= \|((a_k + b_k)_k)_k\|_\infty \\ &= \sup_{k \geq 1} |a_k + b_k| \\ &\leq u + v \end{aligned}$$

Monotonicity

A sequence $(x_n)_n$ is **increasing** if

$$x_1 \leq x_2 \leq \dots \quad \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \dots \quad \forall n$$

The sequence is *eventually* increasing if $\exists m \in \mathbb{N}$ such that $x_n \leq x_{n+1}$ for $n > m$.

Similarly, the sequence is eventually decreasing if $\exists m \in \mathbb{N}$ such that $x_n \geq x_{n+1}$ for $n > m$.

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

Monotonicity Example

Statement: The sequence

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \frac{1}{2}a_n + 2 \end{aligned}$$

is increasing and bounded above.

Proof: We will prove the first statement via induction:

Base: $a_1 = 1$, $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

Inductive Hypothesis $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+2}$

Proof:

$$\begin{aligned}
 a_n &\leq a_{n+1} \\
 \frac{1}{2}a_n &\leq \frac{1}{2}a_{n+1} \\
 \frac{1}{2}a_n + 2 &\leq \frac{1}{2}a_{n+1} + 2 \\
 a_{n+1} &\leq a_{n+2}
 \end{aligned}$$

To prove the sequence is bounded above, we do the following:

$$\begin{aligned}
 a_1 &= 1 \leq 4 \\
 \frac{1}{2}a_1 &\leq 2 \\
 \frac{1}{2}a_1 + 2 &\leq 2 \\
 a_2 &\leq 4
 \end{aligned}$$

We claim that $\forall n, a_n \leq 4 \Rightarrow a_{n+1} \leq 4$, as we have shown the base case.

$$\begin{aligned}
 a_n &\leq 4 \\
 \frac{1}{2}a_n &\leq 2 \\
 \frac{1}{2}a_n + 2 &\leq 4 \\
 a_{n+1} &\leq 4
 \end{aligned}$$

Convergence of Sequences

Let $L \in \mathbb{R}$, $\varepsilon > 0$. Then, the ε -neighborhood of L is $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$.

$$\begin{aligned}
 x &\in V_\varepsilon(L) \\
 &\Leftrightarrow \\
 |x - L| &< \varepsilon \\
 L - \varepsilon &< x < L + \varepsilon
 \end{aligned}$$

With this in mind, we know the following:

Definition of Convergence

A real sequence $(x_n)_n$ converges to a number x if

$$(\forall \varepsilon > 0) (\exists N_\varepsilon \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such L exists, then $(x_n)_n$ is said to **diverge**.

A sequence $(x_n)_n$ in a metric space (X, d) converges to a point x if

$$(\forall \varepsilon > 0) (\exists N_\varepsilon \in \mathbb{N}) \text{ such that } d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an N such that the N th tail (i.e., x_N, x_{N+1}, \dots) are within ε of L for any ε .

Note: N usually depends on ε (the smaller the ε , the larger the N).

Convergence Proof 1**Statement:**

$$\left(\frac{1}{n}\right)_n \xrightarrow{n \rightarrow \infty} 0$$

Proof: We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given $\varepsilon > 0$, we want $\frac{1}{n} < \varepsilon$, meaning $n > \frac{1}{\varepsilon}$.**Proof:** Let $\varepsilon > 0$. By the Archimedean property corollary, find $N \in \mathbb{N}$ large such that $\frac{1}{N} < \varepsilon$.

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

so, if $n \geq N$, then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

Convergence Proof 2**Statement:** Show that

$$\left(\frac{5n-1}{3-n}\right)_{n \geq 4} \xrightarrow{n \rightarrow \infty} -5$$

Proof:

$$\begin{aligned} |x_n - L| &= \left|\frac{5n-1}{3-n} + 5\right| \\ &= \frac{14}{|3-n|} \\ &= \frac{14}{n-3} \\ &< \varepsilon \\ \frac{14}{n-3} &< \varepsilon \\ n &> \frac{14}{\varepsilon} + 3 \end{aligned}$$

Proof: Let $\varepsilon > 0$. Find $N' \in \mathbb{N}$ so large that $\frac{1}{N'} < \frac{\varepsilon}{14}$ (which exists by the Archimedean property corollary). Let $N = N' + 3$. If $n \geq N$, then

$$\begin{aligned} n-3 &\geq \frac{1}{N'} \\ \frac{1}{n-3} &\leq \frac{1}{N'} \\ &< \frac{\varepsilon}{14} \end{aligned}$$

whence

$$\begin{aligned} |x_n - L| &= \frac{14}{n-3} \\ &< \frac{14\epsilon}{14} \\ &= \epsilon \end{aligned}$$

Convergence and Distance

Statement: Let (X, d) be a metric space, and let $(x_n)_n$ be a sequence in the metric space. The following are equivalent:

- (i) $(x_n)_n \rightarrow x$
- (ii) $(d(x_n, x))_n \rightarrow 0$

Proof:

(i) \Rightarrow (b) Let $\epsilon > 0$. Find $N_\epsilon \in \mathbb{N}$ so large such that $d(x_n, x) < \epsilon$ whenever $n \geq N_\epsilon$.

So, $|d(x_n, x) - 0| = d(x_n, x) < \epsilon$ for all $\epsilon > 0$. Whence, $(d(x_n, x))_n \rightarrow 0$.

(ii) \Rightarrow (i) This direction is similar.

In \mathbb{R} , this implies that

$$\begin{aligned} (x_n)_n &\rightarrow x \\ &\Leftrightarrow \\ (|x_n - x|)_n &\rightarrow 0 \end{aligned}$$

Comparison Proposition

Statement: Let (X, d) be a metric space and let $x \in X$, and suppose $(x_n)_n$ is a sequence in X .

If $\exists c \geq 0$, $m \in \mathbb{N}$, and a sequence $(a_n)_n \in \mathbb{R}^+$ with $(a_n)_n \rightarrow 0$ and $d(x_n, x) \leq ca_n \forall n > m$. Then, $(x_n)_n \rightarrow x$.

Proof: Let $\epsilon > 0$. Note that $\frac{\epsilon}{c} > 0$.

Find $N_1 \in \mathbb{N}$ large such that $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\epsilon}{c}$, which is always possible since $(a_n)_n \rightarrow 0$.

Let $N = \max(N_1, m)$. Then, $n \geq N \Rightarrow n \geq N_1$ and $n \geq m$. So,

$$\begin{aligned} d(x_n, x) &\leq ca_n \\ &< c \frac{\epsilon}{c} \\ &= \epsilon \end{aligned}$$

so, $n \geq N \Rightarrow d(x_n, x) < \epsilon$, whence $(x_n)_n \rightarrow x$

Comparison Example 1**Statement:**

$$\left(\frac{\sin(n^2 - 1)}{n^2 + 3} \right)_n \rightarrow 0$$

Proof:

$$\begin{aligned} \left| \frac{\sin(n^2 - 1)}{n^2 + 3} - 0 \right| &= \frac{|\sin(n^2 - 1)|}{n^2 + 3} \\ &\leq \frac{1}{n^2 + 3} \\ &\leq \frac{1}{n^2} \\ &\leq \frac{1}{n} \end{aligned}$$

We know that $a_n = \frac{1}{n}$ converges to 0. So, by our comparison proposition, we are done.

Comparison Example 2**Prove:**

$$\left(\frac{1}{2^n} \right)_n \rightarrow 0$$

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &\geq 1 + n \\ &> n \end{aligned}$$

Bernoulli's Inequality

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since $a_n = \frac{1}{n}$ converges, we know that $\frac{1}{2^n}$ must converge.

Sequence Divergence

A sequence $(x_n)_n$ is **divergent** if it does not converge. $(x_n)_n \rightarrow 0$ if and only if

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbb{N}) \text{ such that } (\forall n \geq N_\epsilon) d(x_n, x) < \epsilon$$

So, $(x_n)_n$ diverges if and only if

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \rightarrow d(x_n, x) \geq \epsilon_0$$

Sequence Divergence 1**Statement:** Show that the following sequence diverges:

$$a_n = (-1)^n$$

Proof:

Step 1:

$$((-1)^n)_n \not\rightarrow 1$$

Take $\varepsilon_0 = 1/2$, given any $N \in \mathbb{N}$, we will find $n \geq N$ odd:

$$\begin{aligned} n &= 2N + 1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

Step 2:

$$((-1)^n)_n \not\rightarrow -1$$

by letting $\varepsilon_0 = 1/2$ and $n = 2N$.

Sequence Divergence 2

Statement: Does

$$a_n = (\sin(n))_n$$

converge?

Proof: It is not the case that $(\sin(n))_n \rightarrow L$ for any $L \in \mathbb{R}$.

Case 1 If $L > 1$, set $\varepsilon_0 = \frac{L-1}{2}$. Then, given any $N \in \mathbb{N}$, pick $n = N$.

$$\begin{aligned} |\sin(n) - L| &= L - \sin(n) \\ &\geq L - 1 \\ &> \frac{L-1}{2} \\ &= \varepsilon_0 \end{aligned}$$

Case 2 Similarly for $L < -1$

Case 3 Suppose $-1 < L < 1$.

Case 3.1 Suppose $L > 0$. Set $\varepsilon_0 = \frac{L}{2}$. Given any N , find $n \geq N$ with $\sin(n) < 0$.

We find k large such that $N < (2k+1)\pi$, which we can always do because we are finding $k > \frac{1}{2}(\frac{N}{\pi} - 1)$, which is always possible by the Archimedean property.

Note that $N < (2k+1)\pi < (2k+2)\pi$. Note that $\sin(x) < 0$ on the interval $((2k+1)\pi, (2k+2)\pi)$. Note that $|(2k+1)\pi - (2k+2)\pi| = \pi$. Let $n = \lceil (2k+1)\pi \rceil$. Then, $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

Case 3.2 Suppose $L < 0$, set $\varepsilon_0 = \frac{-L}{2}$. Given N , find $n \geq N$ with $\sin(n) > 0$. Using the same strategy as above, we find n such that $|L - \sin(n)| > \varepsilon_0$

Case 3.3 Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given $N \in \mathbb{N}$, find $n \geq N$ with $\sin(n) \geq \frac{1}{2}$. Then, $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$.

Showing that a sequence diverges is not easy — later, we will show divergence with non-uniqueness of convergent subsequences.

Alternating Sequence

Consider again

$$((-1)^n)_{n \geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to -1 :

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating sequence diverges.

Uniqueness of Limits

Statement: A sequence $(x_n)_n$ can converge to at most one limit.

Proof: Suppose toward contradiction that $(x_n)_n$ converges to L_1 and L_2 with $L_1 \neq L_2$.

WLOG, let $L_2 > L_1$. Take $\varepsilon = \frac{L_2 - L_1}{3}$.

Since $(x_n)_n$ converges to L_1 , $\exists N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \varepsilon$. Similarly, since $(x_n)_n$ converges to L_2 , $\exists N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \varepsilon$.

Let $N = \max N_1, N_2$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

So, $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$. So, $x_n \in V_\varepsilon(L_1)$, and $x_n \in V_\varepsilon(L_2)$, meaning $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$, but $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. \perp

Useful Lemmas for Convergence

Absolutely Convergent Sequences

Statement: Let $(x_n)_n$ be a real sequence. If x_n converges to x , then $|(x_n)_n| \rightarrow |x|$. However, the converse is not the case.

Proof: Note that since $(x_n)_n \rightarrow x$, $d(x_n, x) \rightarrow 0$.

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

Convergence to Zero

Statement: Let a_n be a sequence.

$$\begin{aligned} (a_n)_n &\rightarrow 0 \\ &\Leftrightarrow \\ |(a_n)| &\rightarrow 0 \end{aligned}$$

Proof:

(\Rightarrow) If $(a_n)_n \rightarrow 0$, then we showed previously that $|(a_n)_n| \rightarrow |0| = 0$

(\Leftarrow) Suppose $|(a_n)_n| \rightarrow 0$. Given $\varepsilon > 0$, then $\exists N$ such that $n \geq N$ implies

$$||a_n| - 0| < \varepsilon$$

$$||a_n|| < \varepsilon$$

$$|a_n| < \varepsilon$$

$$|a_n - 0| < \varepsilon$$

So, $(a_n)_n \rightarrow 0$

Geometric Sequence

Statement: Let $b \in \mathbb{R}$. Consider

$$(b^n)_{n \geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided $-1 < b \leq 1$. Otherwise, the sequence is divergent.

Proof: If $b = 0$, then the sequence $(b^n)_n = (0, 0, 0, \dots)$ is convergent.

If $b = 1$, then the sequence $(b^n)_n = (1, 1, 1, \dots)$ is convergent.

If $b = -1$, then the sequence $(b^n)_n = (1, -1, 1, \dots)$ is divergent.

Case 1 Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, so $\frac{1}{b} = 1 + a$.

So, by Bernoulli's Inequality, $(\frac{1}{b})^n = (1 + a)^n \geq 1 + na > na$, so $b^n < \frac{1}{na}$.

$$\begin{aligned} |b^n - 0| &= b^n \\ &< \frac{1}{na} \\ &= \frac{1}{a} \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

So, $(b^n)_n \rightarrow 0$.

Case 2 Suppose $-1 < b < 0$. If we look at $|b^n| = |b|^n$, we know that $(|b|^n)_n \rightarrow 0$ by our work above. By the previous lemma, we know that since $|b^n| \rightarrow 0$, $b^n \rightarrow 0$.

Case 3 Suppose $b > 1$. Then, $b = 1 + a$ where $a > 0$.

$$\begin{aligned} b^n &= (1 + a)^n \\ &\geq 1 + na && \text{Bernoulli's Inequality} \\ &> na \end{aligned}$$

Suppose toward contradiction that $(b^n)_n \rightarrow L$. Let $\varepsilon_0 = 1$. Find $N \in \mathbb{N}$ such that $N > \frac{L+1}{a}$. N must exist by the Archimedean property.

Therefore, $(N)(a) > L + 1$. If $n \geq N$, then $(n)(a) > (N)(a) > L + 1$, so $|b^n - L| \geq na - L \geq \varepsilon_0$. \perp

Case 4 Suppose $b < -1$, and suppose toward contradiction that $(b^n)_n \rightarrow L$. By the previous lemma, we know that $|b^n| \rightarrow |L|$. So, $|b|^n \rightarrow |L|$. But, $|b| > 1$, which means our assumption contradicts the result from above. \perp

n th Root Convergence

Statement: If $c > 0$, then $(c^{1/n})_n \rightarrow 1$.

Proof:

Case 1: If $c = 1$, then we get $(c^{1/n})_n = (1, 1, 1, \dots)$, which clearly converges to one.

Case 2: Assume that $c > 1$. Then, $c^{1/n} > 1$, because if $d = c^{1/n} \leq 1$, then $d^n \leq 1$, so $c \leq 1$. We can write $c^{1/n} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c &= c^n \\ &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned} \quad \text{Bernoulli's Inequality}$$

So, $d_n \leq \frac{c}{n}$. Remember, $c^{1/n} = 1 + d_n$.

$$\begin{aligned} |c^{1/n} - 1| &= c^{1/n} - 1 \\ &= d_n \\ &\leq c \cdot \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore, $c^{1/n} \rightarrow 1$.

Case 3: Assume $0 < c < 1$. Then, $c^{1/n} < 1$, so $\frac{1}{c^{1/n}} > 1$.

So, we can write $\frac{1}{c^{1/n}} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c^{1/n} &= \frac{1}{1 + d_n} \\ 1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\ &= \frac{d_n}{1 + d_n} \\ &\leq d_n \end{aligned}$$

Remember, $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned} \frac{1}{c} &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned}$$

So, $d_n \leq \frac{1}{cn}$

$$\begin{aligned} |1 - c^{1/n}| &= 1 - c^{1/n} \\ &\leq d_n \\ &\leq \frac{1}{c} \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore, $(c^{1/n})_n \rightarrow 1$.

Positive Sequence Convergence

Statement: Let $(x_n)_n$ be a sequence with $x_n \in \mathbb{R}^+ \forall n \in \mathbb{N}$, with $(x_n)_n \rightarrow x$. Then, x is also positive, and $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.

Proof: Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0-x|}{2}$. Since $(x_n)_n$ converges to x , we know that $x_n \in V_\varepsilon(x)$ for large n . However, every member of $V_\varepsilon(x) < 0$, and $x_n > 0$. \perp

Case 1: If $x = 0$, we will show that $(\sqrt{x_n})_n \rightarrow 0$.

Let $\varepsilon > 0$, find $N \in \mathbb{N}$ large such that if $n \geq N$, we have

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ x_n &< \varepsilon^2 \\ \sqrt{x_n} &< \varepsilon \\ |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

Case 2: If $x > 0$, we will show that $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{1}{\sqrt{x}} |x_n - x| \\ &\rightarrow 0 \end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \rightarrow 0$, so $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.

n th Root of n Convergence

Show:

$$(n^{1/n})_n \rightarrow 1$$

Proof: We will make use of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Note that $n^{1/n} > 1$ for n past 1. So, we write

$$\begin{aligned} n^{1/n} &= 1 + d_n & d_n > 0 \\ n &= (1 + d_n)^n \\ &= \sum_{k=0}^n \binom{n}{k} d_n^k \\ &= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n \\ &\geq \binom{n}{0} + \binom{n}{2} d_n^2 & \text{as all terms are positive} \\ &= 1 + \frac{n(n-1)}{2} d_n^2 \end{aligned}$$

so

$$\begin{aligned} n-1 &\geq \frac{n(n-1)}{2} d_n^2 \\ \frac{2}{n} &\geq d_n^2 \\ \frac{\sqrt{2}}{\sqrt{n}} &\geq d_n \end{aligned}$$

So, we have

$$\begin{aligned} |n^{1/n} - 1| &= n^{1/n} - 1 \\ &= d_n \\ &\leq \sqrt{2} \frac{1}{\sqrt{n}} \\ &\rightarrow 0 \end{aligned}$$

by previous corollary

So, $(n^{1/n})_n \rightarrow 1$.

Multiplication by Geometric Sequence

Statement: Let $0 \leq b < 1$. Show that

$$(nb^n)_n \rightarrow 0$$

Proof: If $0 < b < 1$ (the 0 case is trivial). So, $\frac{1}{b} > 1$, meaning $\frac{1}{b} = 1 + d$ for some $d > 0$.

$$\begin{aligned} \frac{1}{b^n} &= (1 + d)^n \\ &\geq \frac{n(n-1)}{2} d^2 \\ \frac{2}{d^2(n-1)} &\geq b^n \\ nb^n &\leq \frac{2}{d^2(n-1)} \\ &\rightarrow 0 \end{aligned}$$

by previous corollary

Therefore, $(nb^n)_n \rightarrow 0$.

Boundedness and Convergence

Statement: If $(x_n)_n$ is a convergent sequence, x_n is bounded. The converse is false in general.

Proof: Suppose $(x_n)_n \rightarrow x$. Let $\varepsilon = 1$.

Then, $\exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq N$.

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$. If $n \geq N$, then $|x_n| \leq c$, because $x_n \in V_\varepsilon(x)$. If $n < N$, then $|x_n| \leq c$.

Together, we have $|x_n| \leq c$ for all n .

To show the converse is not true, consider $((-1)^n)_n$. This sequence is bounded but not convergent.

Algebraic Operations on Sequences

Statement: Let $(x_n)_n \rightarrow x$, $(y_n)_n \rightarrow y$, and $(z_n)_n \rightarrow z$ be convergent sequences. Let $t \in \mathbb{R}$. Then, the following are all true:

$$(1) (x_n \pm y_n)_n \rightarrow x \pm y$$

$$(2) (tx_n)_n \rightarrow tx$$

$$(3) (x_n y_n)_n \rightarrow xy$$

$$(4) \text{ Assume } z_n \neq 0 \forall n, \text{ and } z \neq 0. \text{ Then, } \left(\frac{1}{z_n}\right)_n \rightarrow \frac{1}{z}, \text{ and } \left(\frac{x_n}{z_n}\right)_n \rightarrow \frac{x}{z}.$$

Proof of (1): Let $\varepsilon > 0$. Since $x_n \rightarrow x$, $y_n \rightarrow y$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \rightarrow |x_n - x| < \frac{\varepsilon}{2}$, and $n \geq N_2 \rightarrow |y_n - y| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

$$\begin{aligned} |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Proof of (3): We have $(x_n)_n \rightarrow x$ and $(y_n)_n \rightarrow y$.

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |x_n - x||y| \end{aligned}$$

Since convergent sequences are bounded, $\exists c \in \mathbb{R}$ such that $|x_n| < c$, $\forall n$

$$\begin{aligned} &\leq c|y_n - y| + |x_n - x||y| \\ &\rightarrow 0 \end{aligned}$$

Therefore, $|x_n y_n - xy| \rightarrow 0$, so $x_n y_n \rightarrow xy$.

Proof of (4): We have $z_n \neq 0$ and $z \neq 0$. Let $\varepsilon > 0$.

$$\begin{aligned} \left| \frac{1}{z_n} - \frac{1}{z} \right| &= \frac{|z - z_n|}{|z_n z|} \\ &= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|} \end{aligned}$$

Let $\varepsilon = \frac{|z|}{2}$. Since $(z_n)_n \rightarrow z$, we know that $z_n \in V_\varepsilon(z)$ for $n \geq N \in \mathbb{N}$. For $n \geq N$, $|z_n| > \frac{|z|}{2}$, so $\frac{1}{|z_n|} < \frac{2}{|z|}$.

$$\begin{aligned} &\leq |z_n - z| \frac{2}{|z|^2} \\ &\rightarrow 0 \end{aligned}$$

So, $\left| \frac{1}{z_n} - \frac{1}{z} \right| \rightarrow 0$, so $\frac{1}{z_n} \rightarrow \frac{1}{z}$

Ordering of Limits

Statement: Let $(x_n)_n \rightarrow x$ and $(y_n)_n \rightarrow y$. If $x_n \leq y_n$ for all n , then $x \leq y$.

Proof: Suppose toward contradiction that $x > y$.

Let $\varepsilon = \frac{x-y}{2}$.

So, $\exists N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$.

Let $N = \max\{N_1, N_2\}$. Then, $x_N \in V_\varepsilon(x)$ and $y_N \in V_\varepsilon(y)$. But that means $x_N > y_N$. \perp

In particular, if $(x_n)_n \rightarrow x$, and $a \leq x_n \leq b$, then $a \leq x \leq b$.

Squeeze Theorem

Statement: Let $(x_n)_n \rightarrow x$, $(y_n)_n \rightarrow y$, and $(z_n)_n \rightarrow z$, where $x_n \leq y_n \leq z_n$ for all n .

If $L = x = z$, then $y = L$.

Proof: Let $\varepsilon > 0$. Find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow V_\varepsilon(L)$, and $n \geq N_2 \Rightarrow V_\varepsilon(L)$.

Let $N = \max\{N_1, N_2\}$. Then, $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$. Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so $y_n \in V_\varepsilon(L)$, so $(y_n)_n \rightarrow L$.

Squeeze Theorem Examples

For example, let $a_n = \frac{\sin(n)}{n}$. Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero, $a_n \rightarrow 0$

As another example, consider $a_n = (2^n + 3^n)^{1/n}$. Then,

$$\begin{aligned} 3^n &\leq 2^n + 3^n \leq 2 \cdot 3^n \\ 3 &\leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3 \end{aligned}$$

Since $2^{1/n} \rightarrow 1$, we have $a_n \rightarrow 3$.

Ratio Test

Statement: Let (x_n) be a sequence of strictly positive numbers, with $\left(\frac{x_{n+1}}{x_n}\right)_n \rightarrow r < 1$. Then, $(x_n)_n \rightarrow 0$.

Proof: Since $r < 1$, then $1 - r > 0$. Let $\rho = r + \frac{1-r}{2}$, and $\varepsilon = \rho - r = \frac{1-r}{2}$.

Since the sequence converges, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} - r \right| &< \varepsilon \\ \frac{x_{n+1}}{x_n} &< \rho \\ x_{n+1} &< \rho x_n \end{aligned}$$

In particular, $x_{N+1} < \rho x_N$, and $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$. Inductively, one can show that $\forall k \geq 1$, $x_{N+k} < \rho^k x_N$.

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as $k \rightarrow \infty$, both sides of the inequality go to 0, implying that $x_n \rightarrow 0$.

Monotone Convergence Theorem

Proof: Let $(x_n)_n$ be a monotone sequence. Then, $(x_n)_n$ is convergent if and only if it is bounded.

(a) If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n \rightarrow \sup(\{x_n \mid n \in \mathbb{N}\})$.

(b) If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n \rightarrow \inf(\{x_n \mid n \in \mathbb{N}\})$.

Proof: We have already shown that all convergent sequences are bounded.

Assume that $(x_n)_n$ is monotonic and bounded.

Case 1: Suppose $(x_n)_n$ is increasing. Let $\sup\{x_n \mid n \in \mathbb{N}\} := u$. We claim that $(x_n)_n \rightarrow u$.

Let $\varepsilon > 0$. By the definition of supremum, $\exists N \in \mathbb{N}$ such that $u - \varepsilon < x_N$. Note that $\forall n \geq N$, $u - \varepsilon < x_N \leq x_n \leq u$.

Therefore, if $n \geq N$, then $|x_n - u| < \varepsilon$.

Case 2: Suppose $(x_n)_n$ is decreasing. Let $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$. We claim that $(x_n)_n \rightarrow \ell$.

Let $\varepsilon > 0$. By the definition of infimum, $\exists N \in \mathbb{N}$ such that $\ell + \varepsilon > x_N$. Additionally, $\forall n \geq N$, $\ell \leq x_n \leq x_N < \ell + \varepsilon$.

Therefore, if $n \geq N$, $|x_n - \ell| < \varepsilon$.

Applications of the Monotone Convergence Theorem

Statement: If $(x_n)_n$ is a convergent sequence, and $m \in \mathbb{N}$, the m -th tail, $x_{(m)} = (x_{m+k})_{k=1}^\infty$ is also convergent. If $(x_n)_n \rightarrow L$ then $x_{(m)} \rightarrow L$.

Proof: Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. If $k \geq N$, then $m+k \geq N$, so $|x_{m+k} - L| < \varepsilon$.

Thus, $(x_{m+k})_k \rightarrow L$

Monotone Convergence Example 1

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that $x_n \geq 4 \forall n$.

$$x_1 = 8 \geq 4$$

Suppose $x_k \geq 4$. We will show that $x_{k+1} \geq 4$.

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore, $(x_n)_n$ is bounded below by 4.

We claim that $(x_n)_n$ is decreasing. $\forall n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} \leq x_n &\Leftrightarrow \\ \frac{1}{2}x_n + 2 &\leq x_n \\ &\Leftrightarrow 4 \leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that $(x_n)_n \rightarrow L$.

To find L , we use the recursive relationship and the lemma.

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2}x_n + 2 \right)_{n=1}^{\infty} \\ L &= \frac{1}{2}L + 2 \\ L &= 4 \end{aligned}$$

Monotone Convergence Example 2

Consider the following sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{4} \\ x_3 &= 1 + \frac{1}{4} + \frac{1}{9} \\ x_k &= \sum_{k=1}^n \frac{1}{k^2} \end{aligned}$$

We will show that $(x_n)_n$, the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$\begin{aligned} k^2 &\geq k(k-1) & k \geq 2 \\ \frac{1}{k^2} &\leq \frac{1}{k(k-1)} \\ &= \frac{1}{k-1} - \frac{1}{k} \\ \sum_{k=2}^n \frac{1}{k^2} &\leq \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ \sum_{k=1}^n \frac{1}{k^2} &\leq 1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) \end{aligned}$$

But

$$1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{n}$$

so, we have

$$\sum_{k=1}^n \frac{1}{k^2} \leq 2 - \frac{1}{n} < 2$$

So, $(x_n)_n$ is bounded above.

Alternative Proof of the Nested Intervals Theorem

Statement: Let $I_n = [a_n, b_n]$ be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

Proof: Since the intervals are nested, it must be the case that $a_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq b_1$.

Similarly, $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$.

So, $(a_n)_n$ is an increasing sequence bounded above by b_1 and $(b_n)_n$ is a decreasing sequence bounded below by a_1 . So, $(b_n)_n \rightarrow r$ and $(a_n)_n \rightarrow \ell$.

Note that $\ell = \sup(a_n)$ and $r = \inf(b_n)$.

Fix $n \in \mathbb{N}$, then for any $m \geq n$, $a_n \leq a_m \leq b_m \leq b_n$. So, $\sup(a_m) = \ell \leq b_n$. Unlocking n , we get that $\ell \leq \inf(b_n) = r$.

Calculating Square Roots

Let $a \in \mathbb{R}^+$. We will construct a sequence $(x_n)_n \rightarrow \sqrt{a}$.

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

We will prove that $x_n^2 \geq a$.

$$\begin{aligned} 2x_{n+1} &= x_n + \frac{a}{x_n} \\ 2x_{n+1}x_n &= x_n^2 + a \\ 0 &= x_n^2 - 2x_{n+1}x_n + a \end{aligned}$$

So, x_n is a real root, meaning

$$\begin{aligned} \Delta &= 4x_{n+1}^2 - 4a \\ x_{n+1}^2 &\geq a \end{aligned} \quad \forall n$$

So, $\forall n \geq 2$

$$x_n^2 \geq a$$

We will show that x_n is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \\ &= \frac{1}{2} \underbrace{\left(\frac{x_n^2 - a}{x_n} \right)}_{\geq 0 \quad \forall n \geq 2} \end{aligned}$$

So, we have that $(x_n)_n$ is decreasing and bounded below, meaning $(x_n)_n \rightarrow x$ for some $x \in \mathbb{R}$.

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \\ x &= \frac{1}{2} \left(x + \frac{a}{x} \right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} \end{aligned}$$

remember that $x > 0$

Euler's Number

Consider

$$\begin{aligned} (e_n)_n &= \left(1 + \frac{1}{n} \right)^n \\ &= \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} \end{aligned}$$

Similarly,

$$\begin{aligned} e_{n+1} &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n+1} \right) \right) \\ e_{n+1} &\geq e_n \end{aligned}$$

$\forall n$

We claim that $(e_n)_n$ is bounded above.

$$\begin{aligned}
 e_1 &= \left(1 + \frac{1}{1}\right)^1 \\
 2 &\leq e_n \\
 e_n &= \sum_{k=0}^n \left(\frac{1}{k!} \underbrace{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}_{\leq 1} \right) \\
 2^{k-1} &\leq k! & k \geq 2 \\
 \frac{1}{k!} &\leq \frac{1}{2^{k-1}} \\
 e_n &= \sum_{k=0}^n \frac{1}{k!} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right) \\
 &\leq \sum_{k=0}^n \frac{1}{k!} \\
 &\leq 2 + \sum_{\ell=1}^{n-1} \frac{1}{2^\ell} \\
 &< 3
 \end{aligned}$$

so, we have

$$2 \leq e_n \leq 3$$

so, by the monotone convergence theorem, $(e_n)_n$ converges

$$e := \sup_n \left(1 + \frac{1}{n}\right)^n$$

Monotone Divergence

A sequence that is increasing and *unbounded* diverges to infinity.

Let $M > 0$. Since $(x_n)_n$ is unbounded, $\exists N \in \mathbb{N}$ such that $x_N > M$

Thus, if $n \geq N$, then $x_n \geq x_N > M$.

Monotone Divergence Example

Consider

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We can see that $h_n < h_{n+1}$. The primary question is as to whether $(h_n)_n$ is bounded.

$$\begin{aligned}
 h_1 &= 1 \\
 &\geq 1 \\
 h_2 &= 1 + \frac{1}{2} \\
 &\geq 1 + \frac{1}{2} \\
 h_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
 &\geq 1 + \frac{1}{2} + \frac{1}{2} \\
 h_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
 &\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}
 \end{aligned}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^k \frac{1}{2}$$

Let M be large. Find n such that $n > 2(M-1)$. In this case, $n/2 + 1 > M$. Let $N = 2^n$. Then, for $m \geq N$, $h_m > M$.

Thus, $(h_n)_n$ diverges to infinity.

Subsequences and Bolzano-Weierstrass

Natural Sequences

A **natural sequence** is a strictly increasing sequence of natural numbers, $(n_k)_{k=1}^{\infty}$

$$n_1 < n_2 < n_3 < \dots$$

where $\forall k \in \mathbb{N}$, $n_k \in \mathbb{N}$.

Statement: Given $(n_k)_k$ natural sequence, show that $(n_k) \geq k$.

Proof:

Base Case: We know that $n_1 \leq 1$, as $n_1 \in \mathbb{N}$.

Inductive Step: To be continued...

Subsequences

Let $(x_n)_n$ be a sequence. A subsequence $(x_{n_k})_{k=1}^{\infty}$, where $(n_k)_k$ is a natural sequence.

For example, if $(x_n)_n = (-1)^n$. If $(n_k) = 2k$, then, $(x_{n_k}) = ((-1)^{2k})_k = (1, 1, 1, \dots)$. But, if $(n_k) = 2k + 1$, then $(x_{n_k}) = (-1, -1, -1, \dots)$.

If $(x_n) = (1/n)_n$, and $(n_k)_k = k^2$, then $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \dots)$.

If $(x_n)_n$ is a sequence, its m -th **tail** is $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \dots)$, where $n_k = m + k$.

Convergence of Subsequences

Statement: If $(x_n)_n \rightarrow x$, then for any natural sequence $(n_k)_k$,

$$(x_{n_k})_k \rightarrow x$$

Proof: Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ large such that $n \geq N$, $|x_n - x| < \varepsilon$.

Take $K = N$. Then,

$$\begin{aligned} n_k &\geq k \\ &\geq K \\ &= N \\ \Rightarrow |x_{n_k} - x| &< \varepsilon \end{aligned}$$

Corollary to Convergence of Subsequences

Given a sequence $(x_n)_n$, if there are two subsequences $(x_{n_k})_k \rightarrow x$, $(x_{n_\ell})_\ell \rightarrow x'$, where $x \neq x'$, then $(x_n)_n$ is divergent.

Convergence of Subsequences Example

Recall the geometric sequence

$$(b^n)_{n=1}^\infty \rightarrow 0$$

if $0 < b < 1$.

The sequence $(1, b, b^2, \dots)$ is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem, $(b^n)_n \rightarrow \ell$.

Given $n = 2k$, we know that $(b^{2k})_k \rightarrow \ell$.

$$\begin{aligned} b^{2k} &= (b^k)^2 \\ (b^k)^2 &\rightarrow \ell^2 \\ \ell^2 &= \ell \\ \ell &= \{0, 1\} \end{aligned}$$

since $b < 1$

$$\ell = 0$$

Divergence and Subsequences

If $(x_n)_n \not\rightarrow x$, then

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) (\exists n \geq N) \text{ such that } |x_n - x| \geq \varepsilon_0$$

We can use this to construct a sequence to show divergence.

Statement: Let $(x_n)_n$ be a sequence, and $x \in \mathbb{R}$.

$$\begin{aligned} (x_n)_n &\not\rightarrow x \\ \Leftrightarrow \\ (\exists \varepsilon_0 > 0) &(\exists (x_{n_k})_k) \end{aligned}$$

with

$$|x_{n_k} - x| \geq \varepsilon_0$$

Proof:

(\Rightarrow) We know $\exists \varepsilon_0 > 0$ as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \varepsilon_0$$

$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \varepsilon_0$$

$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \varepsilon_0$$

Assume we have $n_1 < n_2 < \dots, n_k$ with

$$|x_{n_j} - x| \geq \varepsilon_0$$

$$j = 1, 2, \dots, k$$

$$N = n_k + 1 \Rightarrow n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \varepsilon_0$$

Iteratively, we have our desired subsequence $(x_{n_k})_k$.

(\Leftarrow) If $(x_n)_n \rightarrow x$, any subsequence converges to x .

By assumption, $(\exists \varepsilon_0 > 0) (\exists (n_k)_k)$ with $|x_{n_k} - x| \geq \varepsilon_0$. Thus, $(x_{n_k})_k \not\rightarrow x$.

Bolzano-Weierstrass Theorem

Statement: If $(x_n)_n$ is a bounded sequence, then $(x_n)_n$ admits a convergent subsequence.

Proof:

Lemma: Let $(x_n)_n$ be any real sequence. Then, $\exists n_k$ such that $(x_{n_k})_k$ is monotone.

Proof of Lemma: A **peak** of a sequence $(x_n)_n$ is an x_m such that $x_m \geq x_n \forall n \geq m$.

Case 1: There are infinitely many peaks, $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$, where $n_1 < n_2 < \dots$.

Then, $(x_{n_k})_k$ is decreasing.

Case 2: There are finitely many peaks. Let these peaks be $x_{m_1}, x_{m_2}, \dots, x_{m_r}$.

Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, $\exists n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since x_{n_2} is not a peak, $\exists n_3 > n_2$ such that $x_{n_3} > x_{n_2}$.

Iteratively, we have an increasing sequence of non-peaks $(x_{n_k})_k$.

Since $(x_n)_n$ admits a monotone subsequence, and $(x_{n_k})_k$ is bounded as $(x_n)_n$ is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

Limit Superior and Limit Inferior

Limit Points

Let $X = (x_n)_n$ be a bounded real sequence. By Bolzano-Weierstrass, $(x_n)_n$ admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R}, t = \lim_{k \rightarrow \infty} x_{n_k} \right\} \quad \text{for any subsequence } (x_{n_k})_k$$

Then, $t \in \overline{X}$ is called a **limit point** of X .

Finding the Limit Points

Let $u_1 = \sup_{n \geq 1} (x_n)$, $\ell_1 = \inf_{n \geq 1} (x_n)$. Clearly, $\ell_1 \leq u_1$, and $\overline{X} \subseteq [\ell_1, u_1]$.

Let $u_2 = \sup_{n \geq 2} (x_n)$ and $\ell_2 = \inf_{n \geq 2} (x_n)$.

Since u_1 is an upper bound for $(x_n)_n$, it is an upper bound for $(x_n)_{n \geq 2}$, so $u_2 \leq u_1$. Similarly, since ℓ_1 is a lower bound for $(x_n)_n$, it is a lower bound for $(x_n)_{n \geq 2}$, so $\ell_2 \geq \ell_1$.

As a result, we can see that $\overline{X} \subseteq [\ell_2, u_2]$.

We continue, letting $u_m = \sup_{n \geq m} (x_n)$, and $\ell_m = \inf_{n \geq m} (x_n)$. We get $\ell_1 \leq \ell_2 \leq \dots$, and $u_1 \geq u_2 \geq \dots$, and $\overline{X} \subseteq [\ell_m, u_m]$, $\forall m$.

We get a nested sequence of intervals $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \dots$. By the Nested Intervals Theorem, we know that

$$\begin{aligned} \overline{X} &\subseteq \bigcap_{m \geq 1} [\ell_m, u_m] \\ &= [\ell, u] \end{aligned}$$

where $\ell = \sup(\ell_m)$ and $u = \inf(u_m)$.

Defining Limit Superior and Limit Inferior

Given a bounded sequence $(x_n)_n = X$,

$$\begin{aligned} u &= \inf_{m \geq 1} (u_m) \\ &= \inf_{m \geq 1} \left(\sup_{n \geq m} x_n \right) \end{aligned}$$

called the **limit superior** of $(x_n)_n$

$$u = \limsup_{n \rightarrow \infty} x_n$$

and

$$\begin{aligned} \ell &= \sup_{m \geq 1} (\ell_m) \\ &= \sup_{m \geq 1} \left(\inf_{n \geq m} (x_n) \right) \end{aligned}$$

called the **limit inferior** of $(x_n)_n$

$$\ell = \liminf_{n \rightarrow \infty} x_n$$

Fundamental Results in Limit Superior and Limit Inferior

Statement: Let $(x_n)_n$ be bounded. Then,

- (1) $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$
- (2) $(x_n)_n \rightarrow x \Leftrightarrow \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$

Proof of (1): This was proven with the Nested Intervals Theorem

Proof of (2): Let $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$.

We know that $u_m = \sup_{n \geq m} x_n$. If $m \geq N$, then $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$. Therefore, $|u_m - x| \leq \varepsilon/2 < \varepsilon$, so $(u_m)_m \rightarrow x = \limsup_{n \rightarrow \infty} x_n$.

Similarly, we know that $\ell_m = \inf_{n \geq m} x_n$. If $m \geq N$, then $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$. So, $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$, so $(\ell_m)_m \rightarrow x = \liminf_{n \rightarrow \infty} x_n$.

Applying Limit Superior and Limit Inferior

Consider the sequence

$$x_n = \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases} \\ = (-1, 5/2, -1/3, 9/4, -1/5, \dots)$$

We begin by constructing the u_m sequence: $(5/2, 5/2, 9/4, 9/4, \dots)$. We can see that $u_m \rightarrow 2$.

Then, we construct the ℓ_m sequence: $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$. We can see that $\ell_m \rightarrow 0$.

Exercise: If $(x_n)_n$ and $(y_n)_n$ are sequences with $x_n \leq y_n \forall n$, then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

Ratio Test and Root Test: Equivalent Convergence

Statement: If $(a_n)_n$ is a sequence of strictly positive terms such that

$$\left(\frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$$

then,

$$\left(a_n^{1/n} \right)_{n=1}^{\infty} \rightarrow \rho$$

Proof: Let $\varepsilon > 0$. Then, $\exists N$ large such that $\forall n \geq N$,

$$\begin{aligned}
 \left| \frac{a_{n+1}}{a_n} - \rho \right| &< \varepsilon & \forall n \geq N \\
 \Rightarrow \frac{a_{n+1}}{a_n} &< \rho + \varepsilon & \forall n \geq N \\
 a_{n+1} &< a_n(\rho + \varepsilon) & \forall n \geq N \\
 a_n &< a_N(\rho + \varepsilon)^{n-N} & \forall n \geq N \\
 a_n &< (\rho + \varepsilon)^n \cdot \frac{a_N}{(\rho + \varepsilon)^N} \\
 a_n^{1/n} &< (\rho + \varepsilon) \left(\frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\
 \limsup a_n^{1/n} &\leq \limsup (\rho + \varepsilon) \left(\frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\
 \limsup_{n \rightarrow \infty} a_n^{1/n} &\leq \rho + \varepsilon
 \end{aligned}$$

Case 1: If $\rho = 0$, the case is trivial.

Case 2: Suppose $\rho > 0$. Find $\varepsilon > 0$ small such that $0 < \varepsilon < \rho$.

Since $\left(\frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$, find N large such that $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$. So, $\forall n \geq N$,

$$\begin{aligned}
 a_{n+1} &\geq a_n(\rho - \varepsilon) \\
 a_n &\geq a_N(\rho - \varepsilon)^{n-N} \\
 a_n^{1/n} &\geq (\rho - \varepsilon) \left(\frac{a_N}{(\rho - \varepsilon)^N} \right)^{1/n} \\
 \liminf a_n^{1/n} &\geq \rho - \varepsilon
 \end{aligned}$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together, $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$, so $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$, whence $\left(a_n^{1/n} \right) \rightarrow \rho$

Properties of \overline{X}

Statement: We found earlier that $\overline{X} \subseteq [\ell, u]$. We claim that

$$\begin{aligned}
 \sup \overline{X} &= u \\
 \sup \overline{X} &= \ell
 \end{aligned}$$

Proof: We have shown that u is an upper bound for \overline{X} . The goal is to show that u is the least upper bound.

Let $\varepsilon > 0$. We need to find a $t \in \overline{X}$ with $u - \varepsilon < t$. Note that $u - \varepsilon < u_m \forall m$.

We know that $u - \varepsilon < u_1$. Since $u_1 = \sup_{n \geq 1} x_n$, we know $\exists n_1 \in \mathbb{N}$ with $u - \varepsilon < x_{n_1} < u_1$.

Consider $u_{n_1+1} = \sup_{n > n_1} x_n$. We know that $u - \varepsilon < u_{n_1+1}$. Therefore, $\exists x_{n_2}$ with $n_2 > n_1$ and $u - \varepsilon < x_{n_2} < u_{n_1+1}$.

Then, we use u_{n_2+1} . Then, $\exists n_3 > n_2$ with $u - \varepsilon < x_{n_3} < u_{n_2+1}$.

We get a subsequence from the natural sequence n_1, n_2, \dots , where $u - \varepsilon < x_{n_k} \forall k$.

Also, $x_{n_k} < u_1 \forall k$. Therefore, $(x_{n_k})_k$ is a bounded sequence. By Bolzano-Weierstrass, \exists a convergent subsequence

$$(x_{n_{k_j}})_j \rightarrow t$$

We know that $u - \varepsilon \leq t$. Note that $t \in \overline{X}$.

Exercise: Show that $\inf \overline{X} = \ell$.

Cauchy and Contractive Sequences

Cauchy Sequences

A sequence $(x_n)_n$ in a metric space (X, d) is Cauchy if

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \text{ such that } p, q \geq N \Rightarrow d(x_p, x_q) < \varepsilon$$

if $(X, d) = (\mathbb{R}, |\cdot|)$:

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \text{ such that } p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon$$

Consider the sequence $(x_n)_n = \frac{1}{n}$. Then,

$$\begin{aligned} |x_p - x_q| &= \left| \frac{1}{p} - \frac{1}{q} \right| \\ &= \frac{1}{q} - \frac{1}{p} \\ &\leq \frac{1}{q} \end{aligned}$$

Given $\varepsilon > 0$, find N large such that $\frac{1}{N} < \varepsilon$. Then, $p, q \geq N$ implies

$$\begin{aligned} \left| \frac{1}{p} - \frac{1}{q} \right| &< \frac{1}{q} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

To show that any sequence is not Cauchy, we use the following negation of the definition:

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) \text{ such that } p, q \geq N \Rightarrow |x_p - x_q| \geq \varepsilon_0$$

Boundedness of Cauchy Sequences

Statement: Cauchy sequences are bounded.

Proof: Let $\varepsilon = 1$. Then, by the Cauchy criterion, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < 1$.

In particular, $\forall n \geq N$,

$$\begin{aligned} |x_n| &= |x_n - x_N + x_N| \\ &\leq |x_n - x_N| + |x_N| \\ &< 1 + |x_N| \end{aligned}$$

Triangle Inequality

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$. Then, $x_n \leq c \forall n \geq 1$. Thus, x_n is bounded.

Convergent Subsequences and Cauchy Sequences

Statement: If $(x_n)_n$ is Cauchy and $(x_n)_n$ admits a convergent subsequence, then $(x_n)_n$ is convergent.

Proof: Say $(x_{n_k})_k \rightarrow x$ for some natural sequence $(n_k)_k$. We claim that $(x_n)_n \rightarrow x$.

Let $\varepsilon > 0$. Since $(x_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon/2$.

Also, since $(x_{n_k})_k \rightarrow x$, then $\exists K \in \mathbb{N}$ and $K \geq N$ with $k \geq K \Rightarrow |x_{n_k} - x| < \varepsilon/2$.

For all $k \geq K$,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \end{aligned}$$

Let $N_1 = \max\{N, K\}$. Then,

$$\begin{aligned} n \geq N_1 &\Rightarrow n \geq N && \text{by max} \\ &\Rightarrow n_k \geq k \geq K \geq N && \text{def. of natural sequence} \\ |x_n - x| &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Cauchy Sequence Convergence in the Reals

Statement: Let $(x_n)_n$ be any sequence in \mathbb{R} . The following are equivalent:

- (1) $(x_n)_n$ converges.
- (2) $(x_n)_n$ is Cauchy.

Proof:

(1) \Rightarrow (2) (Holds in any metric space). Suppose $(x_n)_n \rightarrow x$. Find N large such that $n \geq N \rightarrow d(x_n, x) < \varepsilon/2$.

Then, $p, q \geq N \Rightarrow$

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

(2) \Rightarrow (1) If $(x_n)_n$ is Cauchy, then $(x_n)_n$ converges.

By Bolzano-Weierstrass, $(x_n)_n$ admits a convergent subsequence, so by our previous lemma, $(x_n)_n$ must converge.

Note: To show (2) \Rightarrow (1), we used Bolzano-Weierstrass, which requires the monotone convergence theorem, which itself requires the completeness axiom. This is why we cannot show (2) \Rightarrow (1) converges.

Complete Metric Spaces

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Remark: All convergent sequences are Cauchy, and all Cauchy sequences are bounded. We showed that \mathbb{R} under the absolute value metric is complete.

\mathbb{Q} under $d(s, t) = |s - t|$ is not complete; similarly, $A = (0, 1)$ under the metric inherited from \mathbb{R} is not complete; $x_n = \frac{1}{n}$ is Cauchy but not convergent in A .

Finding Cauchy Sequences and Convergence in \mathbb{R}

Cauchy Sequences and Convergence 1

Consider the harmonic sequence

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We claim that h_n is not convergent.

Let $p > q$. Then,

$$\begin{aligned} |h_p - h_q| &= \left| \sum_{k=1}^p \frac{1}{k} - \sum_{k=1}^q \frac{1}{k} \right| \\ &= \frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p} \\ &\geq \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} \\ &= \frac{p-q}{p} \\ &= 1 - \frac{q}{p} \end{aligned}$$

set $p = 2q$:

$$\begin{aligned} |h_{2q} - h_q| &\geq 1 - \frac{q}{2q} \\ &= 1/2 \end{aligned}$$

Therefore, h_n is not Cauchy, and thus not convergent.

Cauchy Sequences and Convergence 2

Consider a sequence of partial sums

$$x_n = \sum_{k=0}^n \frac{(-1)^k}{k!}$$

We claim that $(x_n)_n$ is Cauchy, and thus convergent. Let $p > q$. Then, we have

$$\begin{aligned} |x_p - x_q| &= \left| \sum_{k=q+1}^p \frac{(-1)^k}{k!} \right| \\ &\leq \sum_{k=q+1}^p \frac{1}{k!} \\ &\leq \sum_{k=q+1}^p \frac{1}{2^{k-1}} \\ &= \frac{1}{2^q} + \frac{1}{2^{q+1}} + \cdots + \frac{1}{2^{p-1}} \\ &= \frac{1}{2^q} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-q-1}} \right) \\ &\leq \frac{1}{2^{q-1}} \end{aligned}$$

Given $\varepsilon > 0$, choose N large such that $\frac{1}{2^{N-1}} < \varepsilon$. When $p > q > N$, then $|x_p - x_q| \leq \frac{1}{2^{q-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$.

Thus, the sequence is convergent.

Contractive Sequences

A sequence $(x_n)_n$ in a metric space (X, d) is **contractive** if

$$\exists 0 < \rho < 1 \text{ such that } d(x_{n+1}, x_n) \leq \rho d(x_n, x_{n-1}) \quad \forall n \geq 1$$

In \mathbb{R} , the definition is

$$|x_{n+1} - x_n| \leq \rho |x_n - x_{n-1}|$$

Contractive and Cauchy

We claim that every contractive sequence is Cauchy.

From examination, we arrive at the following:

$$|x_n - x_{n-1}| \leq \rho^{n-2} |x_2 - x_1| \quad (*)$$

If $p > q$, then

$$\begin{aligned} |x_p - x_q| &= |x_p - x_{p-1} + x_{p-1} - x_{p-2} + \cdots + x_{q+1} - x_q| \\ &\leq |x_p - x_{p-1}| + \cdots + |x_{q+1} - x_q| && \text{Triangle Inequality} \\ &\leq |x_2 - x_1| (\rho^{p-2} + \rho^{p-3} + \cdots + \rho^{q-1}) \\ &= |x_2 - x_1| \rho^{q-1} (1 + \rho + \rho^2 + \cdots + \rho^{p-q-1}) \\ &= |x_2 - x_1| \rho^{q-1} \frac{1 - \rho^{p-q}}{1 - \rho} && \text{Finite Geometric Sequence} \\ &\leq |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} \end{aligned}$$

Given $\varepsilon > 0$, we can find N large such that

$$q \geq N \Rightarrow |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} < \varepsilon$$

Thus, $p > q \geq N \Rightarrow |x_p - x_q| < \varepsilon$.

Applying Contractive Sequences 1

Consider $(f_n)_n$ defined as follows:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_{n+1} &= f_n + f_{n-1} \end{aligned}$$

Consider x_n defined as follows:

$$x_n = \frac{f_{n+1}}{f_n}$$

We can rewrite x_n as:

$$\begin{aligned} x_n &= \frac{f_n + f_{n-1}}{f_n} \\ &= 1 + \frac{f_{n-1}}{f_n} \\ &= 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \\ &= 1 + \frac{1}{x_{n-1}} \end{aligned}$$

We claim that $3/2 \leq x_n \leq 2 \forall n \geq 2$.

$$x_2 = 2$$

Inductive Hypothesis: suppose $3/2 \leq x_n \leq 2$

$$\begin{aligned} &: \frac{3}{2} \leq x_n \leq 2 \\ &\frac{2}{3} \geq \frac{1}{x_n} \geq \frac{3}{2} \\ 2 \geq \frac{5}{3} &\geq 1 + \frac{1}{x_n} \geq \frac{3}{2} \end{aligned}$$

We now claim that $(x_n)_n$ is contractive.

$$\begin{aligned} |x_{n+1} - x_n| &= \left| \left(1 + \frac{1}{x_n}\right) - \left(1 + \frac{1}{x_{n-1}}\right) \right| \\ &= \left| \frac{1}{x_n} - \frac{1}{x_{n-1}} \right| \\ &= \left| \frac{x_{n-1} - x_n}{x_{n-1}x_n} \right| \\ &\leq \frac{4}{9} |x_n - x_{n-1}| \end{aligned}$$

Therefore, $\rho = \frac{4}{9}$ is our constant of contraction. Thus, $(x_n)_n$ is Cauchy, so it converges in \mathbb{R} .

$$\begin{aligned} x_{n+1} &= 1 + \frac{1}{x_n} & (n \rightarrow \infty, x_n \rightarrow \varphi) \\ \varphi &= 1 + \frac{1}{\varphi} \\ \varphi^2 - \varphi - 1 &= 0 \\ \varphi &= \frac{1 + \sqrt{5}}{2} \end{aligned}$$

Applying Contractive Sequences 2

Let $x_1 = 0$, $x_2 = 1$, and

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_n + x_{n-1}) \\ (x_n)_n &= (0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, \dots) \end{aligned}$$

While the sequence is not monotone, we can show that the sequence is contractive.

$$\begin{aligned} |x_{n+1} - x_n| &= \left| \frac{1}{2} (x_n + x_{n-1}) - x_n \right| \\ &= \left| \frac{1}{2} (x_{n-1} - x_n) \right| \\ &= \frac{1}{2} |x_n - x_{n-1}| \end{aligned}$$

Since the constant of contraction is equal to $1/2$, this sequence is Cauchy, and thus converges in the real numbers.

Since $(x_n)_n \rightarrow x$, every subsequence converges to x . Therefore, $(x_{2k+1})_k \rightarrow x$.

$$\begin{aligned} x_{2k+1} &= \sum_{j=1}^k \frac{1}{2^{2j-1}} \\ &= 2 \sum_{j=1}^k \frac{1}{4^j} \\ &= 2 \frac{1 - \frac{1}{4^{k+1}}}{1 - \frac{1}{4}} \\ &= \frac{2}{3} \end{aligned} \quad k \rightarrow \infty$$

Sequence Divergence

Properly Divergent Sequences

Let $(x_n)_n$ be a real sequence. $(x_n)_n$ properly diverges to $+\infty$ if

$$(\forall \alpha > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow x_n \geq \alpha$$

We write that $(x_n)_n \rightarrow +\infty$. Similarly, $(x_n)_n$ properly diverges to $-\infty$ if

$$(\forall \beta < 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow x_n \leq \beta$$

and $(x_n)_n \rightarrow -\infty$. We say that $(x_n)_n$ is properly divergent if $(x_n)_n \rightarrow \pm\infty$.

If $(x_n)_n$ and $(y_n)_n$ are sequences such that $x_n \geq y_n \forall n$, and $(y_n)_n \rightarrow +\infty$, then $(x_n)_n \rightarrow +\infty$.

Divergence of the Geometric Sequence

In the geometric sequence, if $b > 1$, we can show that $(b^n)_n \rightarrow +\infty$.

Write $b = 1 + a$ for some $a > 0$. Then, by Bernoulli's inequality, we have

$$\begin{aligned} b^n &= (1 + a)^n \\ &\geq 1 + na \\ &\geq na \end{aligned}$$

Given any $\alpha > 0$, find N large such that $N > \frac{\alpha}{a}$, which is always possible by the Archimedean property. Then, for $Na \geq \alpha$. If $n \geq N$, then $na \geq Na > \alpha$.

Since $b^n > na$, we have that $(b^n)_n \rightarrow +\infty$.

Monotone Divergence

By the Monotone Convergence Theorem, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \rightarrow x \Leftrightarrow (x_n)_n \text{ bounded}$$

Negating, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \text{ divergent} \Leftrightarrow (x_n)_n \text{ unbounded}$$

However, we can make this statement stronger.

Statement: Let $(x_n)_n$ be monotone. $(x_n)_n$ is unbounded if and only if $(x_n)_n$ is properly divergent.

Proof:

(\Leftarrow) If $(x_n)_n$ is properly divergent, then $(x_n)_n$ is divergent, and thus unbounded.

(\Rightarrow) Let $(x_n)_n$ be unbounded and increasing. Then, given $\alpha > 0$, $\exists n_\alpha$ with $x_{n_\alpha} > \alpha$. If $n \geq n_\alpha$, then $x_n \geq x_{n_\alpha} > \alpha$, so $(x_n)_n$ is properly divergent to $+\infty$.

Sequence Comparison Test

Let $(x_n)_n$ and $(y_n)_n$ be sequences with $x_n > 0$ and $y_n > 0$. Suppose that

$$\left(\frac{x_n}{y_n}\right)_n \rightarrow L > 0$$

Then, $(x_n)_n \rightarrow +\infty \Leftrightarrow (y_n)_n \rightarrow \infty$.

Let $\varepsilon = L/2$. Since

$$\left(\frac{x_n}{y_n}\right)_n \rightarrow L,$$

$\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} \frac{L}{2} &\leq \frac{x_n}{y_n} \leq \frac{3L}{2} \\ \frac{L}{2}y_n &\leq x_n \\ \frac{2}{3L}x_n &\leq y_n \end{aligned}$$

If $(y_n)_n \rightarrow \infty$, then so too does $(L/2)(y_n)$, so $(x_n)_n \rightarrow \infty$. Similarly, if $(x_n)_n \rightarrow \infty$, then so too does $(2/3L)x_n$, so $(y_n)_n \rightarrow \infty$.

Applying the Sequence Comparison Test

Problem: Show that

$$\left(\sqrt{4n^2 - 3n + 1}\right)_n \rightarrow +\infty$$

Solution: We will compare to $y_n = n$. Then

$$\begin{aligned} \frac{x_n}{y_n} &= \frac{\sqrt{4n^2 - 3n + 1}}{n} \\ &= \sqrt{4 - \frac{3}{n} + \frac{1}{n^2}} \\ &\rightarrow 2 \geq 0 \end{aligned}$$

Since y_n is properly divergent to $+\infty$, so too is x_n .

Series Convergence and Divergence

Introduction to Infinite Series

An **infinite series** is a sequence of partial sums s_n , where s_n is formed from x_k as follows:

$$s_n = \sum_{k=1}^n x_k$$

alternatively,

$$\begin{aligned} s_1 &= x_1 \\ s_n &= s_{n-1} + x_n \end{aligned}$$

The limit of the sequence $(s_n)_n$ is the value of

$$\sum_{n=1}^{\infty} x_n$$

The infinite series converges to s if $(s_n)_n \rightarrow s$.

If $(s_n)_n$ diverges, then so too does the series. If $(s_n)_n$ is properly divergent to $\pm\infty$, then we write that the series is equal to $\pm\infty$.

Convergence of a Series of Positive Terms

Statement: Let $(x_k)_k$ be a sequence of positive terms. The following are equivalent:

- (a) $\sum x_k$ converges.
- (b) The sequence of partial sums $(s_n)_n$ is bounded above.
- (c) A subsequence of the sequence of partial sums $(s_{n_j})_j$ is bounded above.

Proof:

(1) \Rightarrow (2): $\sum x_k$ is convergent $\Rightarrow (s_n)_n$ is convergent $\Rightarrow (s_n)_n$ is bounded.

(2) \Rightarrow (3): If $(s_n)_n$ is bounded, so is any subsequence $(s_{n_j})_j$.

(3) \Rightarrow (2): Suppose $s_{n_j} \leq c$. If m is arbitrary, $\exists j$ such that $n_j \geq m$. Take $j = m$. Then, $s_m \leq s_{n_j} \leq c$. Therefore, $(s_n)_n$ is bounded above.

(2) \Rightarrow (1) Let $(s_n)_n$ be bounded above. We know that $(s_n)_n$ is increasing as $x_k \geq 0$. By the Monotone Convergence theorem, $(s_n)_n$ converges, meaning $\sum x_k$ converges.

Corollary to Convergence of a Series of Positive Terms

Let $(x_k)_k$ be a sequence with $x_k \geq 0$. Then,

$$\sum x_k \text{ properly diverges} \Leftrightarrow (s_n)_n \text{ is unbounded}$$

Applying Convergence of a Series of Positive Terms 1

Recall that for $x_k = 1/k$, we proved that $(s_n)_n$ is unbounded, and also that $(s_n)_n$ is not Cauchy, meaning $\sum_{k=1}^{\infty} 1/k$ is properly divergent.

Applying Convergence of a Series of Positive Terms 2

Additionally, we saw that for $x_k = 1/k^2$, $(s_n)_n$ is increasing and bounded above.

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^n \frac{1}{k} - \frac{1}{k-1} \\ &= 2 - \frac{1}{n} \end{aligned}$$

Applying Convergence of a Series of Positive Terms 3

Let $b \in \mathbb{R}$. Let $x_k = b^k$. Then, we have

$$\begin{aligned} s_n &= \sum_{k=0}^n b^k \\ &= \frac{1 - b^{n+1}}{1 - b} \end{aligned} \quad b \neq 1$$

Therefore, we know the end behavior of the series:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{1 - b^{n+1}}{1 - b} \\ &= \frac{1}{1 - b} \left(1 - b \lim_{n \rightarrow \infty} b^n \right) \\ &= \begin{cases} \frac{1}{1-b} & |b| < 1 \\ \text{diverges} & |b| > 1 \end{cases} \end{aligned}$$

Series Comparison Test

Statement: Let $0 \leq x_k \leq y_k$.

- If $\sum y_k$ converges, then so too does $\sum x_k$
- If $\sum x_k$ diverges, then so too does $\sum y_k$.

Proof:

(\Rightarrow) If $\sum y_k$ converges, then $t_n = \sum_{k=1}^n y_k$ is bounded.

Setting $s_n = \sum_{k=1}^n x_k$, we see that $0 \leq s_n \leq t_n$. Seeing as t_n is bounded, so too is s_n . Therefore, $\sum x_k$ is convergent.

For example, consider the series

$$\sum \frac{1}{k^2 + k}$$

Since $\frac{1}{k^2} \geq \frac{1}{k^2 + k}$, we know that, seeing as $\frac{1}{k^2}$ converges, so does $\frac{1}{k^2 + k}$.

Limit Comparison Test

Statement: Let x_k and y_k be strictly positive sequences. Suppose that

$$\lim_{k \rightarrow \infty} \frac{x_k}{y_k} = L$$

(a) If $L > 0$, then $\sum x_k$ converges if and only if $\sum y_k$ converges.

(b) If $L = 0$, then $\sum y_k$ converges $\Rightarrow \sum x_k$ converges.

Proof:

(a) Since

$$\frac{x_k}{y_k} \rightarrow L$$

Set $\varepsilon = L$. We know $\exists K$ such that $k \geq K \Rightarrow y_k \leq \frac{2}{L}x_k$. Let $s_n = \sum_{k=1}^n x_k$ and $t_n = \sum_{k=1}^n y_k$. Then,

$$\begin{aligned} t_n &= \sum_{k=1}^{K-1} y_k + \sum_{k=K}^n y_k \\ &\leq t_{K-1} + \frac{2}{L} \sum_{k=K}^n x_k \\ &\leq t_{K-1} + \frac{2}{L} s_n \\ &\leq t_{K-1} + c, \end{aligned}$$

implying that t_n is bounded, so $\sum y_k$ converges.

(b) Since

$$\frac{x_k}{y_k} \rightarrow 0,$$

$\exists K$ such that $\frac{x_k}{y_k} \leq 1 \forall k \geq K$, meaning $x_k < y_k \forall k \geq K$.

Letting $s_n = \sum_{k=1}^n x_k$ and $t_n = \sum_{k=1}^n y_k$. Thus,

$$\begin{aligned} s_n &= \sum_{k=1}^{K-1} x_k + \sum_{k=K}^n x_k \\ &= s_{K-1} + \sum_{k=K}^n y_k \\ &\leq s_{K-1} + t_n \\ &\leq s_{K-1} + c \end{aligned}$$

Thus, s_n is bounded, meaning $\sum x_k$ is convergent.

Applying the Limit Comparison Test

Consider

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}}$$

Letting $x_n = \frac{1}{\sqrt{n^2-1}}$, and $y_n = \frac{1}{n}$, we have

$$\begin{aligned}\frac{x_n}{y_n} &= \frac{n}{\sqrt{n^2-1}} \\ &\rightarrow 1 > 0\end{aligned}$$

Since $\sum y_n$ diverges, so too does $\sum x_n$.

n th Term Divergence Test

If $\sum x_k$ is convergent, then $(x_k)_k \rightarrow 0$. Conversely, if $(x_k)_k \not\rightarrow 0$, then $\sum x_k$ diverges. Recall that $s_n = s_{n-1} + x_n$. If $\sum x_k$ converges, then $(x_n)_n \rightarrow 0$. So,

$$\begin{aligned}x_n &= s_n - s_{n-1} \\ (s_n)_n &\rightarrow s \\ x_n &\rightarrow s - s \\ &= 0\end{aligned}$$

For example, we can find that

$$\sum_{k=1}^{\infty} \frac{1}{\arctan k}$$

diverges, as $\lim_{k \rightarrow \infty} \frac{1}{\arctan k} = \frac{2}{\pi} \neq 0$

Cauchy Condensation Test

Statement: Let $(x_k)_k$ be a decreasing sequence of positive numbers. Then,

$$\sum_k x^k \text{ converges} \Leftrightarrow \sum_k 2^k x_{2^k} \text{ converges}$$

Proof: Look at the partial sum s_{2^n} ,

$$\begin{aligned}s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\leq x_1 + 2x_2 + 4x_4 + \cdots + 2^{n-1}x_{2^{n-1}} + x_{2^n} \\ &= \sum_{k=1}^{n-1} 2^k x_{2^k} + x_{2^n}\end{aligned}$$

If $\sum_k 2^k x_{2^k}$ converges, then its partial sums are bounded, and we have that $x_{2^n} \rightarrow 0$. Then, s_{2^n} is bounded, and thus $\sum x_k$ converges.

$$\begin{aligned}2s_{2^n} &= \sum_{k=1}^{2^n} x_k \\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &\quad + x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_{15}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n} \\ &= (x_1 + x_1) + (x_2 + x_2) + (x_3 + x_3 + x_4 + x_4) + \cdots + (x_{2^{n-1}} + x_{2^{n-1}} + \cdots + x_{2^n} + x_{2^n}) \\ &\geq x_1 + 2x_2 + 4x_4 + \cdots + 2^n x_{2^n} \\ &= \sum_{k=0}^n 2^k x_{2^k}\end{aligned}$$

therefore, we get that

$$\frac{1}{2} \sum_{k=0}^n 2^k a_{2^k} \leq s_{2^n}$$

If $\sum x_k$ converges, then s_n is bounded, so s_{2^n} is bounded, so $\sum_{k=0}^n 2^k x_{2^k}$ is bounded, so the series $\sum_{k=0}^n 2^k x_{2^k}$ is convergent.

p -Series

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad p \in \mathbb{R}$$

By the Cauchy Condensation Test, we see that the series converges if and only if the following series converges:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{2^{np}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n(p-1)}} \right)^n \\ &\Leftrightarrow \frac{1}{2^{p-1}} < 1 \\ &\Leftrightarrow 2^{p-1} > 1 \\ &\Leftrightarrow p > 1 \end{aligned}$$

Sequences and Series of Functions

Pointwise Convergence

Fix a nonempty set Ω . Then,

$$\mathcal{F}(\Omega, \mathbb{R}) = \{f \mid f : \Omega \rightarrow \mathbb{R}\}$$

A sequence of functions $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$ converges pointwise to $f \in \mathcal{F}(\Omega, \mathbb{R})$ if

$$\forall x \in \Omega, (f_n(x))_n \xrightarrow{n \rightarrow \infty} f(x)$$

Alternatively, using ε , we have:

$$\begin{aligned} (f_n)_n \rightarrow f \text{ pointwise } &\in \mathcal{F}(\Omega, \mathbb{R}) \\ \Leftrightarrow & \\ (\forall x \in \Omega)(\forall \varepsilon > 0)(\exists N_{x,\varepsilon} \in \mathbb{N}) \text{ such that } &n \geq N_{x,\varepsilon} \Rightarrow |f_n(x) - f(x)| < \varepsilon \end{aligned}$$

Applying Pointwise Convergence

Example 1: Let $f_n : [0, 1] \rightarrow \mathbb{R}$, and $f_n(x) = x^n$. Note that $(f_n)_n \rightarrow \delta_1$, where

$$\delta_1(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Example 2: Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$, where

$$f_n(x) = \frac{nx}{1 + n^2 x^2}$$

Claim: $f_n \rightarrow \mathbf{o}$.

If $x = 0$, then $f_n(0) = \mathbf{o} \forall n \geq 1$.

Otherwise, we have

$$\begin{aligned} |f_n(x) - \mathbf{o}(x)| &= \frac{n|x|}{1+n^2x^2} \\ &\leq \frac{n|x|}{n^2x^2} \\ &= \frac{1}{n|x|} \\ &\rightarrow 0 \end{aligned}$$

Example 3: Let $h_n : [0, \infty) \rightarrow \mathbb{R}$, where $h_n(x) = x^{1/n}$. We claim that

$$\begin{aligned} h_n &\rightarrow h \\ h(x) &= \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases} \\ &= \mathbb{1}_{(0, \infty)} \end{aligned}$$

Since, for any $b > 0$, $(b^{1/n}) \rightarrow 1$

Example 4: Let $g_n : [0, \infty) \rightarrow \mathbb{R}$, where $g_n(x) = \frac{x^n}{1+x^n}$. We claim that $g_n \rightarrow g$, where $g : [0, \infty) \rightarrow \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}$$

When $x > 1$, we have

$$\begin{aligned} |g_n(x) - 1| &= \left| \frac{x^n}{1+x^n} - 1 \right| \\ &= \left| \frac{-1}{1+x^n} \right| \\ &= \frac{1}{1+x^n} \\ &\rightarrow 0 \end{aligned}$$

Uniform Convergence

A sequence of functions $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$ converges uniformly to $f \in \mathcal{F}(\Omega, \mathbb{R})$ if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } (n \geq N_\varepsilon)(\forall x \in \Omega) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Equivalently,

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } n \geq N_\varepsilon \Rightarrow \sup_{x \in \Omega} |f_n(x) - f(x)| < \varepsilon.$$

Applying Uniform Convergence

Example 1: Let $f_n : [0, 4] \rightarrow \mathbb{R}$.

$$f_n(x) = \frac{x}{x+n}$$

We claim that

$$f_n \rightarrow \mathbf{0} \text{ uniformly.}$$

We start by examining the maximum size of $f_n(x)$:

$$\begin{aligned} |f_n(x) - \mathbf{0}(x)| &= \frac{x}{x+n} \\ &\leq \frac{x}{n} \\ &\leq \frac{4}{n} \end{aligned}$$

so,

$$\sup_{x \in [0,4]} |f_n(x) - \mathbf{0}(x)| \leq \frac{4}{n}.$$

Given $\varepsilon > 0$, find N so large such that $\frac{1}{N} < \frac{\varepsilon}{4}$. Then, for $n \geq N$,

$$\begin{aligned} \sup_{x \in \Omega} |f_n(x) - f(x)| &\leq \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \varepsilon \end{aligned}$$

Negating Uniform Convergence

Statement:

$$\begin{aligned} (f_n)_n &\nrightarrow f \text{ uniformly} \\ \Leftrightarrow & \\ (\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N}) \text{ such that } &(\exists n_0 \geq N)(\exists x_0 \in \Omega) |f_{n_0}(x_0) - f(x_0)| \geq \varepsilon_0 \\ \Leftrightarrow & \\ (\exists \varepsilon_0 > 0)(\forall N)(\exists (x_k)_k \in \Omega)(\exists (f_{n_k})_k) \text{ such that } &|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \end{aligned}$$

Proof:

(\Rightarrow) We know $\exists \varepsilon_0$ satisfying condition (1). Let $N = 1$. We know $\exists n_1 \geq 1$ such that $\exists x_1 \in \Omega$ with $|f_{n_1}(x_1) - f(x_1)| \geq \varepsilon_0$.

Now, set $N = n_1 + 1$. Then, $\exists n_2 \geq N$ and $x_2 \in \Omega$ satisfying condition (1).

Defining n_k and x_k recursively, we have a natural sequence $(n_k)_k$, and thus a subsequence of f_n , thereby satisfying condition (2).

Negating Uniform Convergence 1

Statement: Does $(f_n)_n \rightarrow f$ uniformly converge on $[0, 1]$, where $f_n(x) = x^n$, $f = \delta_1$?

Proof: Let $x_k = (\frac{1}{2})^k$, $n_k = k$.

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= |f_{n_k}(x_k)| \\ &= \left(\frac{1}{2^{1/k}}\right)^k \\ &= \frac{1}{2} \end{aligned}$$

Setting $\varepsilon_0 = 1/2$, we have that it does *not* converge uniformly.

Changing Domain and Uniform Convergence

Recall $g_n : [0, \infty) \rightarrow \mathbb{R}$, where

$$g_n(x) = \frac{nx}{1 + n^2 x^2}$$

We saw that $(g_n)_n \rightarrow \mathbf{0}$ pointwise. However, it is *not* uniformly convergent. Take $x_k = \frac{1}{k}$, and $n_k = k$. Then,

$$\begin{aligned} |g_{n_k}(x_k) - \mathbf{0}(x_k)| &= \frac{k \cdot \frac{1}{k}}{1 + k^2 \cdot \frac{1}{k^2}} \\ &= 1/2 \\ &= \varepsilon_0. \end{aligned}$$

However, $g_n \rightarrow g$ on $[a, \infty)$ where $a > 0$. Let $x \in [a, \infty)$

$$\begin{aligned} |g_n(x) - \mathbf{0}(x)| &= \frac{nx}{1 + n^2 x^2} \\ &\leq \frac{nx}{n^2 x^2} \\ &= \frac{1}{nx} \\ &\leq \frac{1}{na} \end{aligned}$$

therefore,

$$\sup_{x \in [a, \infty)} |g_n(x) - \mathbf{0}(x)| \leq \frac{1}{na}$$

Negating Uniform Convergence 2

Consider the family of functions

$$\begin{aligned} f_n : [0, \infty) &\rightarrow \mathbb{R} \\ f_n(x) &= e^{-nx} \end{aligned}$$

Upon examination, we can see that:

$$f_n \xrightarrow{\text{p.w.}} \delta_0 = \mathbb{1}_{\{0\}}.$$

However, the convergence is *not* uniform.

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Then, setting $\varepsilon_0 = e^{-1}$

$$\begin{aligned} |f_{n_k}(x_k) - \delta_0(x_k)| &= \left| f_k\left(\frac{1}{k}\right) \right| \\ &= e^{-1} \\ &\geq \varepsilon_0 \end{aligned}$$

Uniform Norm

For $f \in \mathcal{F}(\Omega, \mathbb{R})$, the **uniform norm** or **infinity norm** is defined as:

$$\|f\|_u = \sup_{x \in \Omega} |f(x)|.$$

Importantly, the value of the uniform norm is dependent on Ω .

The uniform norm satisfies the rules we desire of any other norm:

- Scalar multiplication: $\forall t \in \mathbb{R}, \|tf\|_u = |t|\|f\|_u$
- Triangle Inequality: $\|f + g\|_u \leq \|f\|_u + \|g\|_u$
- Zero Property: $\|f\|_u = 0 \Leftrightarrow f = \mathbf{0}_{\mathbb{R}}$
- Algebraic Property: $\|fg\|_u \leq \|f\|_u \cdot \|g\|_u$.

$$\ell_\infty(\Omega) = \{f \in \mathcal{F}(\Omega, \mathbb{R}) \mid \|f\|_u < \infty\}$$

is a normed vector space.

Given $(f_k)_k$, $f \in \ell_\infty(\Omega)$, we have

$$(f_k)_k \xrightarrow{\text{uniformly}} f \Leftrightarrow (\|f_k - f\|_u)_k \rightarrow 0$$

Applying Uniform Norm 1

Let

$$\begin{aligned} g_n &: [0, 1] \rightarrow \mathbb{R} \\ g_n(x) &= x^n(1-x) \end{aligned}$$

Clearly, $(g_n)_n$ belongs to $\ell_\infty([0, 1])$. We can see that

$$(g_n)_n \xrightarrow{\text{p.w.}} \mathbf{0}$$

To show that the convergence is uniform, we must find

$$\|g_n - \mathbf{0}\|_u \xrightarrow{n \rightarrow \infty} \mathbf{0},$$

or

$$\begin{aligned}
 \sup_{x \in [0,1]} x^n(1-x) &\rightarrow 0 \\
 \frac{d}{dx}(x^n(1-x)) &= nx^{n-1} - (n+1)x^n \\
 nx^{n-1} &= (n+1)x^n \\
 x &= \frac{n}{n+1} \\
 \sup_{x \in [0,1]} x^n(1-x) &= \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \\
 &= \frac{1}{(1+1/n)^n} \left(\frac{1}{n+1}\right) \\
 &\rightarrow 0
 \end{aligned}$$

Root Test and Series Convergence

Statement: Let

$$\limsup_{k \rightarrow \infty} |x_k|^{1/k} = \rho.$$

If $\rho < 1$, then $\sum_k x_k$ converges absolutely. If $\rho > 1$, then $\sum_k x_k$ diverges.

Proof: Suppose $\rho < 1$. Let $\rho < r < 1$. By property of \inf , $\exists N \in \mathbb{N}$ large such that $r \geq \sup_{k \geq N} |x_k|^{1/k}$.

Therefore, $\forall k \geq N$, we have

$$\begin{aligned}
 |x_k|^{1/k} &\leq r \\
 |x_k| &\leq r^k \quad \forall k \geq N
 \end{aligned}$$

Therefore,

$$\sum_k x_k \leq \underbrace{\sum_{k=1}^{N-1} x_k + \sum_{k \geq N} r^k}_{\text{converges: } r < 1}$$

If $\limsup |x_k|^{1/k} = \rho > 1$, we can find a subsequence $(x_{k_\ell})^{1/k_\ell} \xrightarrow{\ell \rightarrow \infty} \rho$. We cannot have $((x_k)_k)^{1/k} \rightarrow 0$. Thus, the series diverges.

Absolute Convergence

Statement: A series $\sum_k x_k$ converges absolutely if $\sum_k |x_k|$ converges. If a series converges absolutely, then it always converges.

Proof: Let $s_n = \sum_{k=1}^n x_k$, $t_n = \sum_{k=1}^n |x_k|$. Let $m > n$. Then,

$$\begin{aligned}
 |s_m - s_n| &= \left| \sum_{k=n+1}^m x_k \right| \\
 &\leq \sum_{k=n+1}^m |x_k| \quad \text{Triangle Inequality} \\
 &= |t_m - t_n|
 \end{aligned}$$

By assumption, $(t_n)_n$ converges, and thus is Cauchy. By the above inequality, $(s_n)_n$ is Cauchy, and thus convergent.

Series of Functions

Given a sequence of functions $(f_k)_k \in \mathcal{F}(\Omega, \mathbb{R})$, we say that the series

$$\sum_k f_k$$

converges pointwise to f in $\mathcal{F}(\Omega, \mathbb{R})$ if

$$s_n = \left(\sum_{k=1}^n f_k \right)_n$$

converges to f pointwise.

$$\sum_{k=1}^{\infty} f_k(x) = f(x) \quad \forall x \in \Omega$$

$\sum f_k$ converges to f **uniformly** if

$$s_n = \left(\sum_{k=1}^n f_k \right)_n$$

converges to f uniformly.

Applying Pointwise Convergence of Series of Functions

Let $f_k : (-1, 1) \rightarrow \mathbb{R}$, where $f_k = x^k$. Then,

$$\sum_{k=0}^{\infty} f_k \rightarrow f(x) = \frac{1}{1-x}$$

Applying Uniform Convergence of Series of Functions

Statement: We know that $\sum_{k=0}^{\infty} x^k$ converges pointwise to $s(x) = \frac{1}{1-x}$ on $(-1, 1)$. Does it converge *uniformly* on the same interval?

Proof:

We claim the convergence is not uniform on $(-1, 1)$, but convergence is uniform on $[a, b]$, where $-1 < a \leq b < 1$.

Let $s_n(x) = \sum_{k=0}^n x^k$.

$$\begin{aligned} |s_n(x) - s(x)| &= \left| \frac{1 - x^{n+1}}{1 - x} - \frac{1}{1 - x} \right| \\ &= \frac{|x|^{n+1}}{1 - x} \end{aligned}$$

Let $c = \max\{|a|, |b|\} < 1$

$$\begin{aligned} &\leq \frac{c^{n+1}}{1 - b} \quad \forall a \leq x \leq b \\ \sup_{x \in [a, b]} |s_n(x) - s(x)| &\leq \frac{c^{n+1}}{1 - b} \\ &\rightarrow 0 \end{aligned}$$

To show non-uniform convergence on $(-1, 1)$, let $x_\ell = 1 - \frac{1}{\ell}$, and let $n_\ell = \ell$.

$$\begin{aligned} |s_{n_\ell}(x_\ell) - s(x_\ell)| &= \frac{\left(1 - \frac{1}{\ell}\right)^{\ell+1}}{\frac{1}{\ell}} \\ &= \ell \left(1 - \frac{1}{\ell}\right)^\ell \left(1 - \frac{1}{\ell}\right) \\ &= (\ell - 1) \left(1 - \frac{1}{\ell}\right)^\ell \\ &\rightarrow \infty \end{aligned}$$

since $\left(1 - \frac{1}{\ell}\right)^\ell \rightarrow \frac{1}{e}$.

Weierstrass M -test

Statement: Consider a sequence of functions $(f_k)_k$ in $\ell_\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}$.

If $\sum_{k=1}^{\infty} \|f_k\|_u$ converges, then $\sum_k f_k$ converges uniformly and absolutely on Ω .

Proof: Set $M_k = \|f_k\|_u$. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\sum_{n+1}^m M_k < \varepsilon \quad \forall m > n \geq N$$

since $\sum_{k=1}^{\infty} M_k$ is convergent, and thus Cauchy.

Let $s_n(x) = \sum_{k=1}^n f_k(x)$. So,

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left| \sum_{k=n+1}^m f_k(x) \right| \\ &\leq \sum_{k=n+1}^m |f_k(x)| \\ &\leq \sum_{k=n+1}^m M_k \\ &< \varepsilon \end{aligned} \quad \text{whenever } m > n \geq N$$

For every $x \in \Omega$, $s_n(x)$ is Cauchy. So, $\forall x \in \Omega$, $s(x) := \lim s_n(x)$ exists.

Additionally, $\forall x \in \Omega$,

$$|s_m(x) - s_n(x)| < \varepsilon.$$

Let $m \rightarrow \infty$. Then,

$$\begin{aligned} |s(x) - s_n(x)| &< \varepsilon \\ \sup_{x \in \Omega} |s(x) - s_n(x)| &< \varepsilon. \end{aligned} \quad \forall x \in \Omega, \forall n \geq N$$

Applying the Weierstrass M -test

Consider

$$\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2},$$

where $f_k : \mathbb{R} \rightarrow \mathbb{R}$. Then, $\|f_k\|_u \leq \frac{1}{k^2}$. So,

$$\sum \|f_k\|_u \leq \sum \frac{1}{k^2} < \infty.$$

Thus, $\sum \frac{1}{x^2+k^2}$ converges absolutely and uniformly.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges $\forall x \in \mathbb{R}$, and converges *uniformly* on any closed and bounded interval $[a, b]$.

Power Series

A **power series** centered at c in \mathbb{R} is a formal series of functions

$$\sum_{k=0}^{\infty} a_k(x - c)^k.$$

We want to examine the convergence and the uniformity of such convergence of these power series.

Given $\sum a_k(x - c)^k$, set $\rho = \limsup |a_k|^{1/k}$ and $r = 1/\rho$.

Cauchy-Hadamard Theorem

Statement: A power series

$$\sum_{k=1}^{\infty} a_k(x - c)^k$$

converges absolutely on $(c - r, c + r)$, diverges on $\overline{[c - r, c + r]}$, and uniformly convergent on $[a, b]$, $c - r < a \leq b < c + r$.

Proof: Let $\sum_{k=1}^{\infty} a_k(x - c)^k$, where $x_k = a_k(x - c)^k$.

$$|x_k|^{1/k} = |a_k|^{1/k} |x - c|$$

Root test:

$$\begin{aligned} \limsup_{k \rightarrow \infty} |x_k|^{1/k} &= |x - c| \limsup_{k \rightarrow \infty} |a_k|^{1/k} \\ &= |x - c| \rho \end{aligned}$$

Absolute Convergence:

$$\begin{aligned} |x - c| \rho &< 1 \\ |x - c| &< \frac{1}{\rho} \end{aligned}$$

Divergence:

$$\begin{aligned} |x - c| \rho &> 1 \\ |x - c| &> \frac{1}{\rho} \end{aligned}$$

Let $[a, b] \subset (c - r, c + r)$. Set $d = \max\{|a - c|, |b - c|\}$. So,

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{k=n+1}^m a_k (x - c)^k \right| \\ &\leq \sum_{k=n+1}^m |a_k| |x - c|^k \\ &\leq \sum_{k=n+1}^m |a_k| d^k \end{aligned}$$

we know that $d < r \Rightarrow d/r < 1 \Rightarrow dp < 1 \Rightarrow p < 1/d$. Pick $p < p < 1/d$. So, $\exists N \in \mathbb{N}$ with

$$\begin{aligned} \sup_{k \geq N} |a_k|^{1/k} &< p \\ |a_k| &< p^k \end{aligned}$$

So, if $m > n \geq N$, we have

$$\begin{aligned} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^m (rd)^k \\ \sup_{x \in [a, b]} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^m (rd)^k \end{aligned}$$

Given $\varepsilon > 0$, find $N_1 \in \mathbb{N}$ with $m > n \geq N_1$ meaning

$$\begin{aligned} \sup_{x \in [a, b]} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^m (rd)^k \\ &< \varepsilon \end{aligned}$$

Let $K = \max\{N, N_1\}$. With $m > n \geq K$, we have

$$\sup_{x \in [a, b]} |s_m(x) - s_n(x)| < \varepsilon$$

Letting $m \rightarrow \infty$, we have

$$\sup_{x \in [a, b]} |s(x) - s_n(x)| < \varepsilon.$$

So, $(s_n(x))_n \rightarrow s(x)$ uniformly on $[a, b]$.

Limits

Cluster Points

Recall: If $c \in \mathbb{R}$, and $\delta > 0$, then $V_\delta(x) = (c - \delta, c + \delta)$.

The *deleted neighborhood* $\dot{V}_\delta = (c - \delta, c) \cup (c, c + \delta) = V_\delta \setminus \{c\}$.

$$(i) \quad x \in V_\delta(c) \Leftrightarrow |x - c| < \delta$$

$$(ii) \quad x \in \dot{V}_\delta(c) \Leftrightarrow 0 < |x - c| < \delta$$

Let $D \subseteq \mathbb{R}$. A number $c \in \mathbb{R}$ is a *cluster point* or *limit point* of D if

$$(\forall \delta > 0)(\exists x \in D \cap \dot{V}_\delta(c)) \Leftrightarrow \forall \delta > 0, \dot{V}_\delta(c) \cap D \neq \emptyset$$

Remarks If c is a cluster point of D , c may or may not belong to D . If $c \in D$, then c is not necessarily a cluster point.

Examples:

- Let $D = (0, 1)$. Is $c = 0$ a cluster point of D ?

Yes — given any $\delta > 0$, $\dot{V}_\delta(0) \cap (0, 1) = (0, \min(1, \delta))$. We have that $[0, 1]$ is the set of all limit points of D .

- Let $D = \mathbb{N}$. Then, D admits no cluster points.
- Additionally, all finite sets have no cluster points.
- If $D = \mathbb{Q}$, then the set of cluster points of \mathbb{Q} is \mathbb{R} .

Given any $t \in \mathbb{R}$, $\delta > 0$,

$$\dot{V}_\delta \cap \mathbb{Q} \neq \emptyset$$

because \mathbb{Q} is dense.

- If $D = \{\frac{1}{n} \mid n \geq 1\}$, then $\{0\}$ is the set of cluster points of D .

Sequential Criterion of Cluster Points

Statement: Let $D \subseteq \mathbb{R}$, $c \in \mathbb{R}$. The following are equivalent:

- (1) c is a limit point of D .
- (2) $\exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$

Proof:

(2) \Rightarrow (1) Let $\delta > 0$. Then, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $0 < |x_n - c| < \delta$. Thus $x_N \in \dot{V}_\delta(c) \cap D$.

(1) \Rightarrow (2) Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in D \cap \dot{V}_{1/n}(c)$. So, $x_n \neq c$, $x_n \in D$, and $|x_n - c| < 1/n$. So, $(x_n)_n \rightarrow c$.

Definition of a Limit

Let $f : D \rightarrow \mathbb{R}$, and c a limit point of D . Let $L \in \mathbb{R}$.

$$\lim_{x \rightarrow c} f(x) = L \xLeftrightarrow{\text{defn.}} (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } \forall x \in \dot{V}_\delta(c) \cap D, f(x) \in V_\varepsilon(L)$$

Applying the Limit Definition: Linear Function

$$\lim_{x \rightarrow c} ax + b = ac + b \qquad a \neq 0$$

Preliminary Work:

$$\begin{aligned} |f(x) - L| &= |ax + b - (ac + b)| \\ &= |ax - ac| \\ &= |a||x - c| \end{aligned}$$

Proof: Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{|a|}$.

$$\begin{aligned} 0 &< |x - c| < \delta \\ 0 &< |x - c| < \frac{\varepsilon}{|a|} \\ |f(x) - L| &= |a||x - c| \\ &< |a| \frac{\varepsilon}{|a|} \\ &= \varepsilon \end{aligned}$$

Applying the Limit Definition: Quadratic Function

$$\lim_{x \rightarrow c} x^2 = c^2$$

Preliminary Work:

$$\begin{aligned} |f(x) - L| &= |x^2 - c^2| \\ &= |x - c||x + c| \end{aligned}$$

If $0 < \delta < 1$, and $|x - c| < \delta$, then $|x + c| \leq |x| + |c| \leq 2|c| + 1$. In this case,

$$|f(x) - L| \leq (2|c| + 1)|x - c|.$$

Proof: Given $\varepsilon > 0$, let $\delta = \frac{1}{2} \min \left(1, \frac{\varepsilon}{2|c|+1} \right)$. This guarantees $\delta < 1$. So, if $|x - c| < \delta$,

$$\begin{aligned} |f(x) - L| &\leq (2|c| + 1)|x - c| \\ &< (2|c| + 1)|x - c| \\ &< (2|c| + 1) \frac{\varepsilon}{2|c| + 1} \\ &= \varepsilon \end{aligned}$$

Applying the Limit Definition: Rational Function

$$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c} \quad c \neq 0$$

Preliminary Work:

$$\begin{aligned} |f(x) - L| &= \left| \frac{1}{x} - \frac{1}{c} \right| \\ &= \frac{1}{|x|} \frac{1}{|c|} |x - c| \end{aligned}$$

If $x \in \left(c - \frac{|c|}{2}, c + \frac{|c|}{2} \right)$, then $|x| \geq |c|/2$, so $\frac{1}{|x|} \leq \frac{2}{|c|}$. So,

$$\frac{1}{|x|} \frac{1}{|c|} |x - c| \leq \frac{2}{|c|^2} |x - c|$$

Proof: Given $\varepsilon > 0$, let $\delta = \frac{1}{2} \min \left(\frac{|c|}{2}, \frac{|c|^2}{2} \varepsilon \right)$. If

$$\begin{aligned} 0 &< |x - c| < \delta \\ |f(x) - L| &\leq \frac{2}{|c|^2} |x - c| \\ &< \frac{2}{|c|^2} \frac{|c|^2}{2} \varepsilon \\ &= \varepsilon \end{aligned}$$

Uniqueness of Limits

Statement: Let $f : D \rightarrow \mathbb{R}$ with c a limit point of D . Then, f can have at most one limit.

Proof: Suppose toward contradiction that $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, where $L_1 \neq L_2$.

Let ε be small such that $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. So, $\exists \delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow f(x) \in V_\varepsilon(L_1),$$

and $\exists \delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow f(x) \in V_\varepsilon(L_2).$$

Set $\delta = \min(\delta_1, \delta_2)$. Then,

$$0 < |x - c| < \delta \Rightarrow f(x) \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$$

Sequential Criterion for Limits

Statement: Let $f : D \rightarrow \mathbb{R}$, c a cluster point of D . The following are equivalent:

- (i) $\lim_{x \rightarrow c} f = L$
- (ii) $\forall (x_n)_n \in D \setminus \{c\}$ where $(x_n)_n \rightarrow c$, we have $(f(x_n))_n \rightarrow L$

Proof:

(\Leftarrow) Assume $\lim_{x \rightarrow c} f(x) \neq L$. Then, $(\exists \varepsilon_0) (\forall \delta > 0) (\exists x \in \dot{V}(c) \cap D)$ with $|f(x) - L| \geq \varepsilon_0$.

Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in \dot{V}_{1/n}(c) \cap D$, with $|f(x_n) - L| \geq \varepsilon_0$.

Note that $0 < |x - c| < 1/n$. So, $(x_n)_n \in D \setminus \{c\}$, and $(x_n)_n \rightarrow c$. By (ii), it must be the case that $(f(x_n))_n \rightarrow L$.

However, $|f(x_n) - L| \geq \varepsilon_0$. \perp

Limit Divergence and Non-Existence

Statement: Let $f : D \rightarrow \mathbb{R}$, and c a cluster point of D . Let $L \in \mathbb{R}$. The following are true:

- (1) $\lim_{x \rightarrow c} f(x) \neq L \Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$ but $f(x_n) \not\rightarrow L$
- (2) $\lim_{x \rightarrow c} f(x)$ DNE $\Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$ and $(f(x_n))_n$ divergent.

Proof:

(1) This is a direct negation of the Sequential Definition.

(2)

(\Rightarrow) Suppose toward contradiction, $\forall (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$, we have $(f(x_n))_n$ is convergent.

Pick any two such sequences, $(x_n)_n$ and $(y_n)_n$. We know $(f(x_n))_n \rightarrow L_1$, and $(f(y_n))_n \rightarrow L_2$.

Consider $(z_n)_n = (x_1, y_1, x_2, y_2, \dots)$. We know that $(z_n)_n \rightarrow c$, meaning $(f(z_n))_n \rightarrow M$.

The sequence $(f(z_n))_n$ admits two subsequences $(f(x_n))_n \rightarrow M$ and $(f(y_n))_n \rightarrow M$. Thus, $L_1 = L_2$.

We showed that, for any sequence $(x_n)_n \rightarrow c$, $(f(x_n))_n \rightarrow L$. Thus, $\lim_{x \rightarrow c} f(x)$ exists. \perp

Applying Limit Divergence using Sequences

We want to find $\lim_{x \rightarrow c} \mathbb{1}_{\mathbb{Q}}$. Consider two sequences $(r_n)_n \rightarrow c$, where $r_n \in \mathbb{Q}$ — this is always possible since the rationals are dense — and $(t_n)_n \rightarrow c$, where $t_n \notin \mathbb{Q}$.

Let $(x_n)_n = (r_1, t_1, r_2, t_2, \dots)$. Then, $(x_n) \rightarrow c$, but $(\mathbb{1}_{\mathbb{Q}}(x_n))_n = (1, 0, 1, 0, \dots)$. Thus, $\lim_{x \rightarrow c} \mathbb{1}_{\mathbb{Q}}$ DNE.

Bounded Functions and Cluster Points

Statement: Recall that $f : D \rightarrow \mathbb{R}$ is bounded on $E \subseteq D$ if $\sup_{x \in E} |f(x)| < \infty$.

If $f : D \rightarrow \mathbb{R}$ and c is a cluster point of D , if $\lim_{x \rightarrow c} f(x) = L$, then $\exists \delta > 0$ such that f is bounded on $\dot{V}_\delta(c) \cap D$.

Proof: Let $\varepsilon = 1$. Then, $\exists \delta > 0$ such that $x \in \dot{V}_\delta(c) \cap D \Rightarrow |f(x) - L| < 1$. Then,

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L|, \end{aligned}$$

so,

$$\sup_{x \in \dot{V}_\delta(c)} |f(x)| \leq 1 + |L|$$

Operations with Limits

Statement: Let $f, g : D \rightarrow \mathbb{R}$, and c is a cluster point of D . Let $\alpha \in \mathbb{R}$.

(a) If $\lim_{x \rightarrow c} f(x) = L$, and $\lim_{x \rightarrow c} g(x) = M$, then

- (i) $\lim_{x \rightarrow c} (f \pm g) = L \pm M$
- (ii) $\lim_{x \rightarrow c} (\alpha f) = \alpha L$
- (iii) $\lim_{x \rightarrow c} (fg) = LM$
- (iv) $\lim_{x \rightarrow c} \left(\frac{f}{g} \right) = \frac{L}{M}$ if $M \neq 0$

(b) $\lim_{x \rightarrow c} |f(x)| = |L|$

(c) $\lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{L}$, provided $f(x) \geq 0$

(d) If $f(x)$ is a polynomial, then $\lim_{x \rightarrow c} f(x) = f(c)$.

(e) If $f(x)$ is rational, then $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, provided $q(c) \neq 0$.

Proof of (a)(iii): Let $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$. Then, $(f(x_n))_n \rightarrow L$, $(g(x_n))_n \rightarrow M$. Then,

$$\begin{aligned} (f \cdot g(x_n)) &= (f(x_n)g(x_n))_n \\ &\rightarrow LM \end{aligned} \quad \text{by sequence properties}$$

Proof of (a)(iv): Let $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$. Then, by the properties of sequences,

$$\begin{aligned} \left(\frac{f}{g}(x_n) \right) &= \left(\frac{f(x_n)}{g(x_n)} \right)_n \\ &\rightarrow \frac{L}{M} \end{aligned} \quad \text{provided } M \neq 0$$

Proof of (d): Let $p(x) = \sum_{k=0}^n a_k x^k$. Then,

$$\begin{aligned}
 \lim_{x \rightarrow c} p(x) &= \lim_{x \rightarrow c} \left(\sum_{k=0}^n a_k x^k \right) \\
 &= \sum_{k=0}^n \lim_{x \rightarrow c} a_k x^k && \text{(a)(i)} \\
 &= \sum_{k=0}^n a_k \lim_{x \rightarrow c} x^k && \text{(a)(ii)} \\
 &= \sum_{k=0}^n a_k \left(\lim_{x \rightarrow c} x \right)^k && \text{(a)(i)} \\
 &= p(c)
 \end{aligned}$$

Proof of (b) Using the properties of sequence, we can show that $(|f(x_n)|)_n \rightarrow |L|$ for $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \rightarrow c$

Squeeze Theorem

Statement: If $f : D \rightarrow \mathbb{R}$, c is a cluster point of D .

- (i) If $f(x) \leq b$ for x in a deleted neighborhood of c , and if $\lim_{x \rightarrow c} f(x) = L$, then $L \leq b$.
- (ii) If $f(x) \geq a$ for all x in a deleted neighborhood of c , and if $\lim_{x \rightarrow c} f(x) = L$, then $L \geq a$.
- (iii) If $f, g, h : D \rightarrow \mathbb{R}$, and c is a cluster point of D . Suppose

$$g(x) \leq f(x) \leq h(x)$$

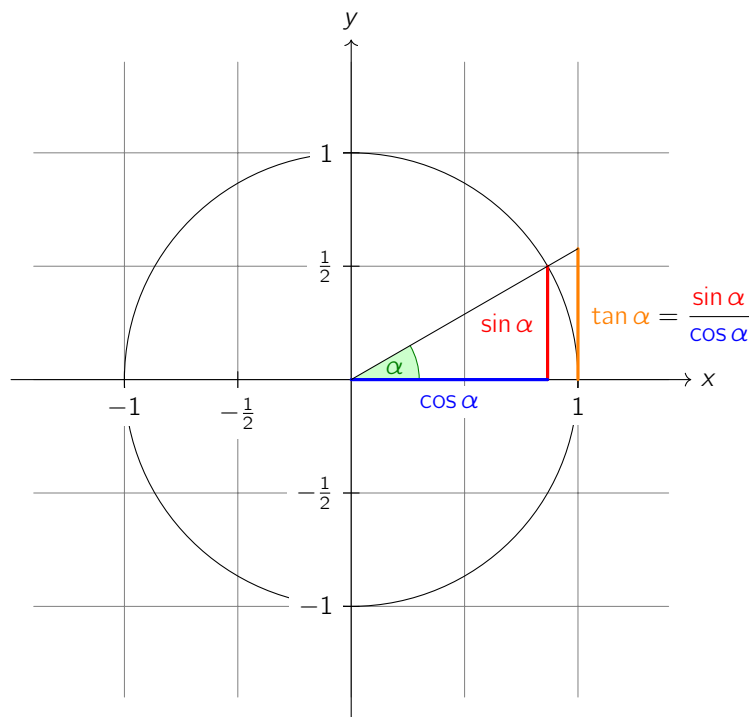
for all x in some deleted neighborhood of c . Suppose $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$. Then, $\lim_{x \rightarrow c} f(x) = L$.

Proof of (iii) Let $(x_n)_n \in D \setminus \{c\}$, with $(x_n)_n \rightarrow c$. Then, as $n \rightarrow \infty$,

$$\begin{aligned}
 g(x_n) &\leq f(x_n) \leq h(x_n) \\
 L &\leq f(x_n) \leq L,
 \end{aligned}$$

so $f(x_n)_n \rightarrow L$.

Trigonometric Limits



We know that

$$0 \leq \sin(x) \leq x$$

so as $x \rightarrow 0^+$, $\sin(x) \rightarrow 0$. Similarly, if $x \rightarrow 0^-$, then

$$\begin{aligned} \lim_{x \rightarrow 0^-} \sin(x) &= \lim_{y \rightarrow 0^+} \sin(-y) \\ &= - \lim_{y \rightarrow 0^+} \sin(y) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow 0^+} \cos(x) &= \lim_{x \rightarrow 0^+} \sqrt{1 - \sin^2(x)} \\ &= 1 \\ \lim_{x \rightarrow 0^-} \cos(x) &= \lim_{y \rightarrow 0^+} \cos(-y) \\ &= \lim_{y \rightarrow 0^+} \cos(y) \\ &= 1 \end{aligned}$$

Claim:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

Proof: Let $x \rightarrow 0$

$$\begin{aligned}\frac{\sin(x)}{2} &\leq \frac{x}{2} \leq \frac{\tan(x)}{2} \\ 0 &\leq \frac{\sin(x)}{x} \leq 1 \\ \cos(x) &\leq \frac{\sin(x)}{x} \\ \cos(x) &\leq \frac{\sin(x)}{x} \leq 1 \\ 1 &\leq \frac{\sin(x)}{x} \leq 1\end{aligned}$$

Strictly Positive Limits

Statement: Let $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$. Let c be a cluster point of D . If $\lim_{x \rightarrow c} f(x) = L > 0$, then $\exists \delta > 0$ and $\exists t > 0$ such that $f(x) > t$ for $x \in \dot{V}_\delta(c) \cap D$.

Proof: Let $\varepsilon = \frac{L}{2}$. Then, $V_\varepsilon = (L/2, 3L/2)$. So, $\exists \delta > 0$ such that $x \in \dot{V}_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$. Set $t = L/2$.

One-Sided Limits

Let $f : D \rightarrow \mathbb{R}$.

Cluster Points:

- (i) A number $c \in D$ is a right cluster point if $\forall \delta > 0$, $\exists x \in (c, c + \delta) \cap D$
- (ii) A number $c \in D$ is a left cluster point if $\forall \delta > 0$, $\exists x \in (c - \delta, c) \cap D$.

Limits:

$$(i) \lim_{x \rightarrow c^+} f(x) = L \stackrel{\text{def}}{\iff}$$

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in (c, c + \delta) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

$$(ii) \lim_{x \rightarrow c^-} f(x) = L \stackrel{\text{def}}{\iff}$$

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in (c - \delta, c) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

Sequential Criterion:

- (i) Let c be a right cluster point of D . $\lim_{x \rightarrow c^+} f(x) = L$ if and only if $\forall (x_n)_n \in D \cap (c, \infty)$ with $(x_n)_n \rightarrow c$, we have $(f(x_n))_n \rightarrow L$
- (ii) Let c be a left cluster point of D . $\lim_{x \rightarrow c^-} f(x) = L$ if and only if $\forall (x_n)_n \in (-\infty, c) \cap D$ with $(x_n)_n \rightarrow c$, we have $(f(x_n))_n \rightarrow L$.

Limit Equality

Let $f : D \rightarrow \mathbb{R}$. Let c be a cluster point of D .

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

Infinite Limits

Let $f : D \rightarrow \mathbb{R}$, and c be a limit point of D . Then,

$$\lim_{x \rightarrow c} f(x) = \infty \stackrel{\text{def}}{\iff} (\forall M \geq 0)(\exists \delta > 0) \text{ such that } x \in \dot{V}_\delta(c) \cap D \Rightarrow f(x) \geq M$$

We can also define

$$\begin{aligned} \lim_{x \rightarrow c} f(x) &= -\infty \\ \lim_{x \rightarrow c^\pm} f(x) &= \pm\infty \end{aligned}$$

Applying Infinite Limits

Statement:

$$\lim_{x \rightarrow 1^-} \frac{1}{1-x} = -\infty$$

Proof: Let $M \geq 0$ be large. We want $f(x) \geq M$.

$$\begin{aligned} \frac{1}{1-x} &\geq M \\ 1-x &\leq \frac{1}{M} \\ x &\geq 1 - \frac{1}{M} \end{aligned}$$

Set $\delta = \frac{1}{M}$. If $x \in (1 - \delta, 1)$, then $x \geq 1 - \frac{1}{M}$. So, by our work above, $f(x) \geq M$.

Limits at Infinity

Let $f : [a, \infty) \rightarrow \mathbb{R}$, $L \in \mathbb{R}$. Then,

$$\lim_{x \rightarrow \infty} f(x) = L \stackrel{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \geq a) \text{ such that } x \geq K \Rightarrow f(x) \in V_\varepsilon(L)$$

Similarly, we can define for $f : (-\infty, b] \rightarrow \mathbb{R}$, $L \in \mathbb{R}$

$$\lim_{x \rightarrow -\infty} f(x) = L \stackrel{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \leq b) \text{ such that } x \leq K \Rightarrow f(x) \in V_\varepsilon(L)$$

and for $f : [a, \infty)$ where

$$\lim_{x \rightarrow \infty} f(x) = \infty \stackrel{\text{def}}{\iff} (\forall M \geq 0)(\exists K \geq a) \text{ such that } x \geq K \Rightarrow f(x) \geq M$$

and the respective sequential definitions.

Applying Limits at Infinity 1

Statement: Let $n \in \mathbb{N}$.

$$\lim_{x \rightarrow \infty} x^n = \infty$$

Proof: Let M be large. We want $x^n \geq M$. Then, $x \geq M^{1/n}$. Set $K = M^{1/n}$.

Applying limits at Infinity: Polynomials

$$\lim_{x \rightarrow -\infty} x^n = \begin{cases} +\infty, & n = 2k \\ -\infty, & n = 2k + 1 \end{cases}$$

$$p(x) = \sum_{k=1}^n a_k x^k$$

$$\lim_{x \rightarrow \infty} p(x) = \begin{cases} +\infty, & a_n > 0 \\ -\infty, & a_n < 0 \end{cases}$$

Let $g(x) = x^n$.

$$\frac{p(x)}{g(x)} = a_n + a_{n-1} \frac{1}{x} + \cdots + a_0 \frac{1}{x^n}$$

$$\lim_{x \rightarrow \infty} \frac{p(x)}{g(x)} = a_n$$

Lemma: If $f, g : [a, \infty) \rightarrow \mathbb{R}$, and $g(x) > 0$. If

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L \neq 0$$

- (1) If $L > 0$, then $\lim_{x \rightarrow \infty} f(x) = \infty \Leftrightarrow \lim_{x \rightarrow \infty} g(x) = \infty$
- (2) If $L < 0$, then $\lim_{x \rightarrow \infty} f(x) = -\infty \Leftrightarrow \lim_{x \rightarrow \infty} g(x) = +\infty$

Apply the lemma to $p(x)$, x^n .

Continuity and Uniform Continuity

Continuity

Let $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$. Let $c \in D$. The function f is continuous at c if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in V_\delta(c) \cap D \Rightarrow f(x) \in V_\varepsilon(f(c))$$

Remark: Here, c may not be a cluster point of D .

For example, let

$$f(x) = \begin{cases} x & x = -1 \\ x^2 & x \geq 0 \end{cases}$$

$$D = \{-1\} \cup [0, \infty)$$

Here, f is continuous at $c = -1$. Given any $\varepsilon > 0$, let $\delta = 1/2$. Then, if $x \in V_{1/2}(-1) \cap D$, $x = -1$, meaning $|f(x) - f(-1)| = 0 < \varepsilon$

Continuity and Limits

If $f : D \rightarrow \mathbb{R}$, $c \in D$ and c a cluster point of D , the following are equivalent:

- (i) f is continuous at c
- (ii) $\lim_{x \rightarrow c} f(x) = f(c)$

Remark: We are deign to use the second definition as *the* definition of continuity due to the fact that it removes the possibility of those mentioned above.

Sequential Criterion of Continuity

Let $f : D \rightarrow \mathbb{R}$, $c \in D$. The following are equivalent:

- (i) f is continuous at $x = c$
- (ii) $\forall (x_n)_n$ in D with $(x_n)_n \rightarrow c$, we have $(f(x_n))_n \rightarrow f(c)$

Left and Right Continuity

Let $f : D \rightarrow \mathbb{R}$, $c \in D$.

- f is left-continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } 0 \leq c - x < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\forall (x_n)_n \in D, x_n \leq c, (x_n)_n \rightarrow c \text{ we have } (f(x_n))_n \rightarrow f(c)$$

- f is right-continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } 0 \leq x - c < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\forall (x_n)_n \in D, x_n \geq c, (x_n)_n \rightarrow c \text{ we have } (f(x_n))_n \rightarrow f(c)$$

Continuity on Sets

Let $f : D \rightarrow \mathbb{R}$.

- (1) f is continuous on $E \subseteq D$ if f is continuous at each $c \in E$.
- (2) f is continuous on $[a, b]$ if
 - (i) f is continuous on (a, b)
 - (ii) f is left-continuous at b
 - (iii) f is right-continuous at a

Applying Continuity on Sets

- (1) Polynomials are continuous on \mathbb{R} because $\lim_{x \rightarrow c} p(x) = p(c)$.
- (2) Rational functions are continuous on their domain.
- (3) $f : \mathbb{1}_{\mathbb{Q}}$ is continuous nowhere:

Case 1: Suppose $c \in \mathbb{Q}$. Let $(t_n)_n \rightarrow c$ with $t_n \in \mathbb{R} \setminus \mathbb{Q}$. Then, $(f(t_n))_n = 0 \rightarrow 0 \neq f(c) = 1$

Case 2: Let $c \in \mathbb{R} \setminus \mathbb{Q}$. Let $(r_n)_n \rightarrow c$ with $r_n \in \mathbb{Q}$. Then, $(f(r_n))_n = 1 \rightarrow 1 \neq f(c) = 0$

Discontinuity

$f : D \rightarrow \mathbb{R}$ is not continuous at $x = c$ if $\exists (x_n)_n$ in D with $(x_n)_n \rightarrow c$ and $(f(x_n))_n \nrightarrow f(c)$

Discontinuity of the Sign Function

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at $x = 0$, since $(x_n)_n = \frac{1}{n} \rightarrow 0$ but $(f(x_n))_n = 1 \neq 0$.

Discontinuity of Thomae's Function

Statement: Let

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \in \mathbb{Q}, \ b \in \mathbb{N}, \ \gcd(a, b) = 1 \\ 1 & x = 0 \end{cases}$$

Proof:

Claim 1: f is not continuous at $x \in \mathbb{Q}$: find a sequence $(t_n)_n$ of irrationals with $(t_n)_n \rightarrow x$. Then, $(f(t_n))_n = 0 \neq f(x) = \frac{1}{b}$

Claim 2: f is continuous at $t \in \mathbb{R} \setminus \mathbb{Q}$: let $t \in \mathbb{R} \setminus \mathbb{Q}$, $t > 0$. Let $n \in \mathbb{N}$. Consider

$$A_n = \left\{ \frac{a}{b} \mid 1 \leq b \leq n \right\} \cap (t-1, t+1).$$

We claim that A_n is finite.

$$\begin{aligned} t-1 &< \frac{a}{b} < t+1 \\ b(t-1) &< a < b(t+1) \\ t-1 &< a < n(t+1), \end{aligned}$$

so there are finitely many values of a and finitely many values of b — therefore, A_n is finite. One can find $\delta > 0$ such that $(t-\delta, t+\delta) \cap A_n = \emptyset$

Given $\varepsilon > 0$, find $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \varepsilon$. Let δ be such that $(t-\delta, t+\delta) \cap A_{n_0} = \emptyset$. If $x \in (t-\delta, t+\delta)$,

$$\begin{aligned} |f(x) - f(t)| &= |f(x)| \\ &= \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \text{ lowest terms} \end{cases} \end{aligned}$$

but $\frac{1}{b} < \varepsilon$ because $x \notin A_{n_0}$, meaning $b > n_0$.

Extension of a Function

Consider

$$g(x) = \sin\left(\frac{1}{x}\right) \quad x \neq 0$$

Assuming that g is continuous on its domain, can we find a $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\tilde{g}(x) = g(x) \quad \forall x \in \mathbb{R} \setminus \{0\}$$

If such a \tilde{g} existed, we would expect that $\lim_{x \rightarrow 0} \tilde{g}(x) = \tilde{g}(0)$. But, $\lim_{x \rightarrow 0} \tilde{g}(x) = \lim_{x \rightarrow 0} g(x)$. However, since $\lim_{x \rightarrow 0} g(x)$ DNE, so such an extension does not exist.

Therefore, $x = 0$ is known as a non-removable discontinuity (i.e., we cannot create an extension of the function that “fills in” the function).

However, not all discontinuities involving $\sin(1/x)$ are non-extendible:

$$\begin{aligned} f(x) &= x \sin\left(\frac{1}{x}\right) \\ \tilde{f}(x) &= \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned}$$

Jump Discontinuities

Suppose $\lim_{x \rightarrow c^-} f(x) = L$, $\lim_{x \rightarrow c^+} f(x) = R$. If $L \neq R$, then $x = c$ is a jump discontinuity.

Lipschitz Functions

A function $f : D \rightarrow \mathbb{R}$ is called Lipschitz if $\exists c \geq 0$ with

$$|f(x) - f(y)| \leq c|x - y| \quad \forall x, y \in D$$

The linear function $f(x) = ax + b$ is a Lipschitz function. Additionally, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then $\|T(\vec{v}) - T(\vec{w})\| \leq c\|\vec{v} - \vec{w}\|$ for any norm on \mathbb{R}^n and \mathbb{R}^m .

- If $c < 1$, then f is a contraction.
- If $c = 1$ and $|f(x) - f(y)| = |x - y|$, f is called an isometry.

Lipschitz functions are continuous on their domain:

Proof: Let $c \in D$, let $\varepsilon > 0$. Set $\delta = \varepsilon/c$.

$$\begin{aligned} |x - c| &< \delta \\ |f(x) - f(c)| &\leq c|x - c| \\ |f(x) - f(c)| &< c\delta \\ &= \varepsilon \end{aligned}$$

If $f(x) = \sin(x)$, then

$$\begin{aligned} |\sin(x) - \sin(y)| &= \left| 2 \sin\left(\frac{1}{2}(x - y)\right) \cos\left(\frac{1}{2}(x + y)\right) \right| \\ &\leq 2 \frac{1}{2} |x - y| \\ &= |x - y| \end{aligned}$$

Properties of Continuous Functions

Equality over Dense Subsets

Statement: Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Let $E \subseteq \mathbb{R}$. If $f(x) = g(x) \forall x \in E$, then $f = g$.

Proof: Let $t \in \mathbb{R}$. Since E is dense, $\exists (x_n)_n \in E$ such that $(x_n)_n \rightarrow t$. So, $(f(x_n))_n \rightarrow t$ because f is continuous, and $(g(x_n))_n \rightarrow g(t)$ because g is continuous.

However, since $f(x_n) = g(x_n) \forall x_n$, it must be the case that $f(t) = g(t)$.

Boundedness over a Dense Subset

Statement: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose $f|_E$ is bounded. That is, $\exists c$ such that

$$|f(x)| \leq c. \quad \forall x \in E$$

Then, f is bounded.

Proof: Let $t \in \mathbb{R}$. Since E is dense, $\exists (x_n)_n \in E$ such that $(f(x_n))_n \rightarrow t$. Then,

$$|f(x_n)_n| \leq c,$$

meaning that $f(t) \leq c$.

Bounding Away From 0

Statement: If f is continuous at $x = c$ and $f(c) > 0$, then $\exists \delta > 0$ and $\exists m > 0$ with $f(x) \geq m \forall x \in V_\delta(c)$. Similarly for the negative case.

Proof: Let $\varepsilon = f(c)/2 > 0$. Then, $\exists \delta > 0$ such that $\forall x \in V_\delta(c)$, $f(x) \in V_\varepsilon(f(c)) = (f(c)/2, 3f(c)/2)$. Set $m = f(c)/2$.

Continuity over Operations

Let $f, g : D \rightarrow \mathbb{R}$, $c \in D$.

- (1) If f, g are continuous at $x = c$, then $f \pm g$ are continuous at $x = c$. Similarly, if f, g are continuous on D , then $f \pm g$ is continuous on D .
- (2) Let $\alpha \in \mathbb{R}$. If f is continuous at $x = c$ or on D , then αf is continuous at $x = c$ or D respectively.
- (3) If f, g are continuous at $x = c$ or on D , then $f \cdot g$ is continuous on $x = c$ or D respectively.
- (4) If f, g are continuous at $x = c$, and $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c . Likewise, if f, g are continuous on D and $g(x) \neq 0 \forall x \in D$, then $\frac{f}{g}$ is continuous.
- (5) If g is continuous at $x = c$ and f is continuous at $d = g(c)$, then $f \circ g$ is continuous at $x = c$. If $\text{ran}(g) \subseteq \text{dom}(f)$, with f, g continuous on their domain, then $f \circ g$ is continuous.
- (6) If $f : D \rightarrow \mathbb{R}$ is continuous, and $f(x) \geq 0$ on D , then $\sqrt{f(x)}$ is continuous on D .
- (7) If $f : D \rightarrow \mathbb{R}$ is continuous on D , then $|f(x)|$ is continuous.
- (8) Polynomials and Rational functions are continuous on their domain.
- (9) If $f(x), g(x)$ are continuous, then $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$.

Remark on (4): If $g(c) \neq 0$, then $g \neq 0$ on a δ -neighborhood of c .

Proof of (5): Let $(x_n)_n \rightarrow c$. Then, $g(x_n)_n \rightarrow g(c)$. So, $(f(g(x_n)))_n \rightarrow f(g(c))$.

Fundamental Theorem of Continuous Functions on $[a, b]$

Boundedness Theorem: If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $\|f\|_u < \infty$.

Proof: Suppose it is not the case. Given any $n \geq 1$, $\exists x_n \in [a, b]$ with $|f(x_n)| \leq n$. We thus have a sequence $(x_n)_n \in [a, b]$.

By Bolzano-Weierstrass, $\exists (x_{n_k})_k \rightarrow x \in [a, b]$. So, $f(x_{n_k}) \rightarrow f(x)$. In particular, $(f(x_{n_k}))_k$ is bounded; however, $f(x_{n_k}) \geq k$. \perp

Note: It is possible for f to be bounded on an infinite interval where it does not attain the supremum or infimum.

Let $f : D \rightarrow \mathbb{R}$.

- (1) f has an absolute maximum on D if $\exists x_M \in D$ with $f(x) \leq f(x_M) \forall x \in D$. Notably, this means $\sup_{x \in D} f(x) = f(x_M)$.
- (2) f has an absolute minimum on D if $\exists x_m \in D$ with $f(x_m) \leq f(x) \forall x \in D$. Notably, this means $\inf_{x \in D} f(x) = f(x_m)$.

Extreme Value Theorem (EVT): If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f admits an absolute minimum and absolute maximum.

Proof: We know that $\sup_{x \in [a, b]} f(x) = u < \infty$ by the boundedness theorem. For each $n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that

$$u - \frac{1}{n} < f(x_n) \leq u.$$

Thus, there is a sequence $(x_n)_n \in [a, b]$ — by Bolzano-Weierstrass, $\exists (x_{n_k})_k \rightarrow x^*$ for some $x^* \in [a, b]$. So, for each k ,

$$\begin{aligned} u - \frac{1}{n_k} &< f(x_{n_k}) \leq u \\ u &< f(x^*) \leq u. \end{aligned} \quad \text{since } f \text{ is continuous}$$

So, by the squeeze theorem, $f(x^*) = u$ is our absolute max.

Corollary to the Extreme Value Theorem: If $f : [a, b] \rightarrow \mathbb{R}$ is continuous with $f(x) > 0 \forall x \in [a, b]$, then $\exists \alpha > 0$ such that $f(x) \geq \alpha \forall x \in [a, b]$.

Proof: By the previous theorem, we know $\exists x_m \in [a, b]$ such that $f(x) \geq f(x_m) \forall x \in [a, b]$. But $\alpha := f(x_m) > 0$ by definition.

Location of Roots: We will use this to prove the Intermediate Value Theorem. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Suppose $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. Then, $\exists c \in (a, b)$ such that $f(c) = 0$.

Proof: Assume $f(a) < 0$ and $f(b) > 0$. Let $N = \{x \in [a, b] \mid f(x) \geq 0\}$. Since $b \in N$, $N \neq \emptyset$. Let $z = \inf N$. We claim that $f(z) = 0$.

We know that $\exists (x_n)_n \in N$ with $x_n \rightarrow z$. Since $(x_n)_n \in N$, $f(x_n) \geq 0 \forall n \geq 1$. However, $f(x_n) \rightarrow f(z)$ since f is continuous. So, $f(z) \geq 0$.

Suppose toward contradiction that $f(z) > 0$. So, $\exists \delta > 0$ such that $f(x) \geq \frac{f(z)}{2}$ on $(z - \delta, z + \delta)$. Then, $z - \frac{\delta}{2} \in N$. \perp

Intermediate Value Theorem (IVT): Let $f : I \rightarrow \mathbb{R}$, where I is any interval. Suppose $\exists x_1, x_2 \in I$ and $k \in \mathbb{R}$, with $f(x_1) < k < f(x_2)$. Then, $\exists \xi$ strictly between x_1 and x_2 , with $f(\xi) = k$.

Proof: Clearly, $x_1 \neq x_2$. Suppose $x_1 < x_2$. Consider $g : [x_1, x_2] \rightarrow \mathbb{R}$, $g(x) = f(x) - k$. So, g is continuous (as f is continuous), and $g(x_1) = f(x_1) - k < 0$, and $g(x_2) = f(x_2) - k > 0$. Thus, $\exists \xi \in [x_1, x_2]$ with $g(\xi) = 0$, whence $f(\xi) = k$.

Corollary to IVT and EVT: Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. If $\inf_{[a, b]} f \leq k \leq \sup_{[a, b]} f$, then $\exists c \in [a, b]$ with $f(c) = k$.

Proof: We know that by EVT, $\exists x_m, x_M$ with $\inf_{[a, b]} f = f(x_m)$ and $\sup_{[a, b]} f = f(x_M)$. So, $f(x_m) \leq k \leq f(x_M)$. Apply IVT.

Preservation of Intervals 1: If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $f([a, b]) = [c, d]$.

Proof: Set $c = \inf_{[a, b]} f$ and $d = \sup_{[a, b]} f$. By definition, $c \leq f(x) \leq d$, meaning $f([a, b]) \subseteq [c, d]$. By the previous corollary, if $k \in [c, d]$, then $\exists \xi \in [a, b]$ with $f(\xi) = k$. Thus, $[c, d] \subseteq f([a, b])$.

Preservation of Intervals 2: Let I be any interval, and $f : I \rightarrow \mathbb{R}$ continuous. Then, $f(I)$ is an interval.

Proof: Let $\alpha, \beta \in f(I)$. WLOG, $\alpha < \beta$. We will show that $[\alpha, \beta] \subseteq f(I)$. Say $f(a) = \alpha$ and $f(b) = \beta$ for some $a, b \in I$. Note that $a \neq b$. Let $\alpha < k < \beta$. By IVT, $\exists \xi$ strictly between a and b with $f(\xi) = k$. If $a < b$, then $[a, b] \subseteq I$, and if $b < a$, then $[b, a] \subseteq I$. Thus, $\xi \in I$.

Uniform Continuity

A function $f : D \rightarrow \mathbb{R}$ is **uniformly continuous** on D if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } u, v \in D, |u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$$

Uniform continuity is different from continuity in that f is continuous at a point $x = c$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

In (non-uniform) continuity, $\delta = \delta(\varepsilon, c)$.

Illustrating Non-Uniform Continuity

For example, if $f(x) = \frac{1}{x}$ and $D = (0, \infty)$, we will show that f is continuous at $x = c > 0$.

$$\begin{aligned} |f(x) - f(c)| &= \left| \frac{1}{x} - \frac{1}{c} \right| \\ &= \frac{1}{c} \frac{1}{x} |x - c| \end{aligned}$$

if $0 < \delta < c/2$ and $|x - c| < \delta$, then $x \geq c/2$. Thus,

$$\begin{aligned} |f(x) - f(c)| &= \frac{1}{c} \frac{2}{c} |x - c| \\ &= \frac{2}{c^2} |x - c|. \end{aligned}$$

Given $\varepsilon > 0$, pick $\delta = \frac{1}{2} \min\left(\frac{\varepsilon}{2}, \frac{2}{c^2} \varepsilon\right)$. Thus, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Specifically, we can see that on this domain, we require that δ be a function of ε and c .

Proving Uniform Continuity 1

However, if we look at $f(x) = \frac{1}{x}$ on $[1, \infty)$, we can see that for $u, v \geq 1$,

$$\begin{aligned} |f(u) - f(v)| &= \left| \frac{1}{u} - \frac{1}{v} \right| \\ &= \frac{1}{uv} |v - u| \\ &\leq |v - u| \end{aligned}$$

Given $\varepsilon > 0$, set $\delta = \varepsilon$. If $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon$.

Here, we see that $\delta = \delta(\varepsilon)$.

Proving Uniform Continuity 2

We will show that $f(x) = x^2$ is uniformly continuous on $[1, 4]$.

$$\begin{aligned}
 |f(u) - f(v)| &= |u^2 - v^2| \\
 &= |u - v||u + v| \\
 &\leq |u - v|(|u| + |v|) \\
 &\leq 8|u - v|
 \end{aligned}$$

Triangle Inequality

Given $\varepsilon > 0$, set $\delta = \varepsilon/8$. Whenever $u, v \in [1, 4]$, with $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon$

Lipschitz and Uniform Continuity

Statement: If $f : D \rightarrow \mathbb{R}$ is Lipschitz, then f is uniformly continuous.

Proof: If $f : D \rightarrow \mathbb{R}$ is Lipschitz, then $\exists c > 0$ such that $\forall u, v \in D$,

$$|f(u) - f(v)| \leq c|x - y|.$$

Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{c}$. Whenever $|u - v| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Uniform Continuity and Continuity

Statement: If $f : D \rightarrow \mathbb{R}$ is uniformly continuous, then f is continuous on D .

Proof: Let $c \in D$. Given $\varepsilon > 0$, by uniform continuity, $\exists \delta > 0$ such that

$$\begin{aligned}
 |u - v| < \delta &\Rightarrow |f(u) - f(v)| < \varepsilon \\
 |x - c| < \delta &\Rightarrow |f(x) - f(c)| < \varepsilon
 \end{aligned}$$

Negating Uniform Continuity

Statement: The following are equivalent for $f : D \rightarrow \mathbb{R}$

- (i) f is *not* uniformly continuous
- (ii) $\exists \varepsilon_0$ such that $\forall \delta > 0$, $\exists u_\delta, v_\delta$ such that $|u_\delta - v_\delta| < \delta$ and $|f(u_\delta) - f(v_\delta)| > \varepsilon$
- (iii) $\exists \varepsilon_0$ such that $\exists (u_n)_n, (v_n)_n \in D$ with $(u_n - v_n)_n \rightarrow 0$ and $|f(u_n) - f(v_n)| \geq \varepsilon_0$

Proof:

(i) \Leftrightarrow (ii): Negating definition.

(ii) \Rightarrow (iii): Set $\delta_n = 1/n$ in (ii). Given δ_n , it must be the case that

$$|u_n - v_n| < \frac{1}{n}$$

so $(u_n - v_n)_n \rightarrow 0$, and

$$|f(u_n) - f(v_n)| \geq \varepsilon_0.$$

(iii) \Rightarrow (ii): Let $\delta > 0$. Then, $\exists N \in \mathbb{N}$ large such that $|u_N - v_N| < \delta$, by the definition of sequence convergence.

Set $u_\delta = u_N$ and $v_\delta = v_N$.

Applying Non-Uniform Continuity 1

We will show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Set $u_n = 1/n$, and $v_n = \frac{1}{n+1}$. Then,

$$\begin{aligned} |f(u) - f(v)| &= |n - (n+1)| \\ &= 1 \\ &= \varepsilon_0 \\ |u_n - v_n| &= \left| \frac{1}{n} - \frac{1}{n+1} \right| \\ &= \frac{1}{n(n+1)} \\ &\rightarrow 0 \end{aligned}$$

Applying Non-Uniform Continuity 2

Consider $f(x) = x^2$ on $[0, \infty)$. We will show that f is not uniformly continuous.

Let $u_n = n$ and $v_n = n + \frac{1}{n}$. Clearly, $(u_n - v_n)_n \rightarrow 0$.

$$\begin{aligned} |f(u_n) - f(v_n)| &= \left| n^2 - \left(n + \frac{1}{n} \right)^2 \right| \\ &= \left| n^2 - n^2 - 2 - \frac{1}{n^2} \right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2 \end{aligned}$$

Uniform Continuity Theorem

Statement: If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f is uniformly continuous.

Proof: Suppose toward contradiction that f is not uniformly continuous. Then, $\exists (u_n)_n, (v_n)_n \in [a, b]$ and $\varepsilon_0 > 0$ such that $(u_n - v_n)_n \rightarrow 0$ and $|f(u_n) - f(v_n)| \geq \varepsilon_0$.

Since $(u_n)_n$ is bounded, $\exists n_k$ such that $(u_{n_k})_k \rightarrow z$ by the Bolzano-Weierstrass. We claim that $(v_{n_k})_k \rightarrow z$:

$$\begin{aligned} |v_{n_k} - z| &= |v_{n_k} - u_{n_k} + u_{n_k} - z| \\ &\leq |v_{n_k} - u_{n_k}| + |u_{n_k} - z| \\ &\rightarrow 0. \end{aligned}$$

So,

$$\begin{aligned} 0 < \varepsilon_0 &\leq |f(u_{n_k}) - f(v_{n_k})| \\ &\rightarrow 0 \end{aligned}$$

since $(f(u_k))_k \rightarrow f(z)$ and $(f(v_k))_k \rightarrow f(z)$.

Uniform Continuity and Lipschitz

The function $f(x) = \sqrt{x}$ on $[0, 1]$ is uniformly continuous. However, $f(x) = \sqrt{x}$ is not Lipschitz.

Suppose toward contradiction that f is Lipschitz.

$$|f(x) - f(y)| \leq c|x - y| \quad \forall x, y \in [0, 1]$$

Take $y = 0$.

$$\begin{aligned} \sqrt{x} &\leq cx \\ 0 &< \frac{1}{c} \leq \sqrt{x} \end{aligned}$$

Lemma: Uniform Continuity and Cauchy Sequences

Statement: Let $f : D \rightarrow \mathbb{R}$ be uniformly continuous. If $(x_n)_n \in D$ is Cauchy, then $(f(x_n))_n$ is Cauchy.

This is not true for mere continuity. For example, for $f(x) = \frac{1}{x}$ in $(0, \infty)$, $(x_n)_n = \frac{1}{n}$ is Cauchy in $(0, \infty)$, but $f(x_n) = n$ is not Cauchy.

Proof: Let $(x_n)_n$ be Cauchy. Let $\varepsilon > 0$. Then, $\exists \delta > 0$ such that $\forall u, v \in D$ with $|u - v| < \delta$, we have $|f(u) - f(v)| < \varepsilon$.

Since $(x_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ such that for $p, q \geq N$, $|x_p - x_q| < \delta$. So, $|f(x_p) - f(x_q)| < \varepsilon$. So, $(f(x_n))_n$ is Cauchy.

Continuous Extension Theorem

Statement: Let $f : (a, b) \rightarrow \mathbb{R}$ be a map. The following are equivalent:

- (1) f is uniformly continuous.
- (2) $\exists \tilde{f} : [a, b] \rightarrow \mathbb{R}$ such that
 - \tilde{f} is continuous
 - $\tilde{f}(x) = f(x) \quad \forall x \in (a, b)$

Proof:

(2) \Rightarrow (1): Since \tilde{f} is continuous on $[a, b]$, \tilde{f} is uniformly continuous on $[a, b]$. So, \tilde{f} is uniformly continuous on (a, b) . But, $\tilde{f} = f$ on (a, b) . So, f is uniformly continuous.

(1) \Rightarrow (2): Let $f : (a, b) \rightarrow \mathbb{R}$ be uniformly continuous.

Claim: $\lim_{x \rightarrow a^+} f(x)$ exists. Let $(x_n)_n$ be any sequence where $x_n > a$ and $(x_n)_n \rightarrow a$. Then, $(x_n)_n$ is Cauchy. So, by the lemma, $(f(x_n))_n$ is Cauchy. Since \mathbb{R} is complete, $\exists L \in \mathbb{R}$ such that $(f(x_n))_n \rightarrow L$.

We claim that the limit is L . Let $(y_n)_n$ be any sequence with $y_n > a$, $(y_n)_n \rightarrow a$. By our work above, $(f(y_n))_n \rightarrow L'$ for some $L' \in \mathbb{R}$. Consider $z_n = (x_1, y_1, x_2, y_2, \dots)$. Then, $z_n > a$ with $(z_n)_n \rightarrow a$. By our work above, $(f(z_n))_n \rightarrow L''$, for some $L'' \in \mathbb{R}$. Since $(f(x_n))_n$ is a subsequence of $(f(z_n))_n$, $(f(x_n))_n \rightarrow L''$, so $L = L''$, and similarly, $L' = L''$.

Therefore, $L = L'$. So, we have $\lim_{x \rightarrow a^+} f(x) = L$.

Similarly, $\lim_{x \rightarrow b^-} f(x) = R$ exists. Set $\tilde{f} : [a, b] \rightarrow \mathbb{R}$ such that

$$\tilde{f}(x) = \begin{cases} f(x) & x \in (a, b) \\ L & x = a \\ R & x = b \end{cases}$$

Then, \tilde{f} is the desired continuous extension.

Applying the Continuous Extension Theorem

If $f(x) = \sin(1/x)$, then $f(x)$ is not uniformly continuous on $(0, 1)$. This is because $\lim_{x \rightarrow 0^+} f(x)$ does not exist.

Meanwhile, $g(x) = x \sin(1/x)$ is uniformly continuous on $(0, 1)$, since we can define $\tilde{g}(x)$ as follows:

$$\tilde{g}(x) = \begin{cases} 0 & x = 0 \\ g(x) & 0 < x < 1 \end{cases}$$

Approximation by Step Function

A map $s : [a, b] \rightarrow \mathbb{R}$ is called a step function if

$$(1) [a, b] = \bigcup_{j=1}^n I_j \text{ where } I_j \text{ are intervals.}$$

$$(2) \exists c_1, \dots, c_n \in \mathbb{R} \text{ such that } s(x) = c_j \forall x \in I_j.$$

Alternatively, this is equivalent to:

$$s = \sum_{j=1}^n c_j \mathbb{1}_{I_j}$$

Statement: If $f : [a, b] \rightarrow \mathbb{R}$ is uniformly continuous and $\varepsilon > 0$, then $\exists s : [a, b] \rightarrow \mathbb{R}$ with $\|f - s\|_u < \varepsilon$.

Proof: We know that f is uniformly continuous. Given $\varepsilon > 0$, $\exists \delta > 0$ with $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$. Choose N large such that

$$\Delta_n = \frac{b-a}{N} < \delta.$$

Set $x_j = j\Delta_n$. Set $I_j = [x_j, x_{j+1})$ with $0 \leq j \leq N-1$.

Set $c_j = f(x_j)$,

$$s = \sum_{j=0}^{N-1} c_j \mathbb{1}_{I_j}.$$

If $x \in [a, b]$, $x \in I_k$ for some $k = 0, \dots, N-1$. Then,

$$\begin{aligned} |f(x) - s(x)| &= |f(x) - c_k| \\ &\leq |f(x) - f(x_k)| \\ &< \varepsilon \end{aligned}$$

since

$$\begin{aligned} |x - x_k| &< \Delta_N \\ &< \delta \end{aligned}$$

so,

$$\|f - s\|_u < \varepsilon$$

Approximation by Piecewise Linear Function

A function g is piecewise linear if

$$(a) [a, b] = \bigcup_{j=1}^n I_j, \text{ where } I_j \text{ are intervals.}$$

$$(b) g|_{I_j} \text{ is linear; } \exists a_1, b_1, \dots, a_n, b_n \text{ with } g(x) = a_j + b_j x \forall x \in I_j.$$

Statement: If $f : [a, b] \rightarrow \mathbb{R}$ is uniformly continuous and $\varepsilon > 0$, then there is a continuous piecewise linear $g : [a, b] \rightarrow \mathbb{R}$ with $\|f - g\|_u < \varepsilon$.

Proof: We know that f is uniformly continuous. Given $\varepsilon > 0$, $\exists \delta > 0$ with $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon/2$. Choose N large such that

$$\begin{aligned} \Delta_n &= \frac{b - a}{N} \\ &< \delta. \end{aligned}$$

Set $x_j = j\Delta_n$. Set $I_j = [x_j, x_{j+1})$ with $0 \leq j \leq N - 1$.

Set $g(x) = \sum_{k=0}^{N-1} g_k(x) \mathbb{1}_{I_k}$, where

$$g_k(x) = f(x_k) + \left(\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right) (x - x_k)$$

We observe that if $x \in I_k$, then

$$\begin{aligned} |f(x) - g(x)| &= \left| f(x) - f(x_k) - \left(\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right) (x - x_k) \right| \\ &\leq |f(x) - f(x_k)| + |f(x_{k+1}) - f(x_k)| \frac{|x - x_k|}{|x_{k+1} - x_k|} \\ &\leq |f(x) - f(x_k)| + |f(x_{k+1}) - f(x_k)| \\ &< \varepsilon \end{aligned}$$

so,

$$\|f - g\| < \varepsilon$$

Monotone Functions

Let $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$.

- (1) f is increasing if $x_1, x_2 \in D$ with $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$.
- (2) f is strictly increasing if $x_1, x_2 \in D$ with $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- (3) f is monotone if f is increasing or decreasing.
- (4) f is strictly monotone if f is strictly increasing or strictly decreasing.

If $f : D \rightarrow \mathbb{R}$ is increasing or strictly increasing, then $-f : D \rightarrow \mathbb{R}$ is decreasing or strictly decreasing (respectively).

Additionally, monotone functions are not always continuous. However, one-sided limits always exist.

Statement: Let I be an interval, $f : I \rightarrow \mathbb{R}$ increasing. Let $c \in I$, where c is not an endpoint. Then,

$$(1) \lim_{x \rightarrow c^-} f(x) = \sup_{x \in I, x < c} f(x)$$

$$(2) \lim_{x \rightarrow c^+} f(x) = \inf_{x \in I, x > c} f(x)$$

are both existent and finite:

Proof of (1): Since c is not an endpoint, $\{x \mid x \in I, x < c\} \neq \emptyset$ and is bounded above by c . Therefore, $\{f(x) \mid x \in I, x < c\}$ is nonempty and bounded above by $f(c)$ (since f is increasing). So, $u = \sup_{x \in I, x < c} f(x)$ exists.

Let $\varepsilon > 0$. $\exists x_\varepsilon \in I$ with $x_\varepsilon < c$ such that $u - \varepsilon < f(x_\varepsilon)$. Set $\delta = c - x_\varepsilon > 0$. If $x \in I, c - x < \delta$, then $x_\varepsilon < x < c$, so $f(x_\varepsilon) \leq f(x) \leq f(c)$. So, $u - f(x) \leq u - f(x_\varepsilon) < \varepsilon$. But, $u \geq f(x)$, so $u - f(x) = |u - f(x)|$. Thus, $0 < c - x < \delta \Rightarrow |u - f(x)| < \varepsilon$. Thus, $u = \lim_{x \rightarrow c^-} f(x)$.

Limits and Continuity with Monotone Functions

Let I be an interval, $f : I \rightarrow \mathbb{R}$ increasing. Suppose $c \in I$ is not an endpoint. The following are equivalent:

- (1) f is continuous at $x = c$.
- (2) $\lim_{x \rightarrow c} f(x) = f(c)$.
- (3) $\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$.
- (4) $\sup_{x \in I, x < c} f(x) = f(c) = \inf_{x \in I, x > c} f(x)$.

Suppose c is a right endpoint of I . The following are equivalent:

- (1) f is continuous at $x = c$.
- (2) $\lim_{x \rightarrow c^-} f(x) = f(c)$.
- (3) $\sup_{x \in I, x < c} f(x) = f(c)$.

Suppose c is a left endpoint of I . The following are equivalent:

- (1) f is continuous at $x = c$.
- (2) $\lim_{x \rightarrow c^+} f(x) = f(c)$.
- (3) $\inf_{x \in I, x > c} f(x) = f(c)$.

We can make a similar set of corollaries with decreasing functions.

Jump of a Function

Let I be an interval, $f : I \rightarrow \mathbb{R}$ increasing.

- (1) If c is not an endpoint of I , we define the jump of f at $x = c$ as:

$$j_f(c) = \lim_{x \rightarrow c^+} f(x) - \lim_{x \rightarrow c^-} f(x)$$

(2) If c is a left endpoint of I , we define the jump of f at $x = c$ as:

$$j_f(c) = \lim_{x \rightarrow c^+} f(x) - f(c)$$

(3) If c is a right endpoint of I , we define the jump of f at $x = c$ as:

$$j_f(c) = f(c) - \lim_{x \rightarrow c^-} f(x)$$

Statement: We claim that f is continuous at $c \in I$ if and only if $j_f(c) = 0$.

Proof: If c is not an endpoint, then f is continuous at $x = c$ if and only if $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$.

If c is a left endpoint, then f is continuous at $x = c$ if and only if $f(c) = \lim_{x \rightarrow c^+} f(x)$, if and only if $j_f(c) = 0$.

Countability of Monotone Function Discontinuities

Statement: Let $I \subseteq \mathbb{R}$ be any interval. Let $f : I \rightarrow \mathbb{R}$ be monotone. Then, $D = \{x \in I \mid f \text{ not continuous at } x = c\}$ is countable.

Proof: For the sake of simplicity, we will assume that f is monotone increasing.

Lemma: Let $\{x_1, x_2, \dots, x_n\}$ be a partition of $I = [a, b]$, where $a \leq x_1 < x_2 < \dots < x_n \leq b$. Then, $f(a) + \sum_{i=1}^n j_f(x_i) \leq f(b)$.

Proof of Lemma: By induction on n , if $x_1 = a$, then

$$\begin{aligned} f(a) + j_f(x_1) &= f(a) + j_f(a) \\ &= f(a) + \lim_{x \rightarrow a^+} f(x) - f(a) \\ &= \lim_{x \rightarrow a^+} f(x) \\ &\leq f(b). \end{aligned}$$

If $x_1 = b$, then

$$\begin{aligned} f(a) + j_f(x_1) &= f(a) + j_f(b) \\ &= f(a) + f(b) - \lim_{x \rightarrow b^-} f(x) \\ &= f(b) - (\lim_{x \rightarrow b^-} f(x) - a) \\ &\leq f(b). \end{aligned}$$

If $a < x_1 < b$, then

$$\begin{aligned} f(a) + j_f(x_1) &= f(a) + \lim_{x \rightarrow x_1^+} f(x) - \lim_{x \rightarrow x_1^-} f(x) \\ &\leq f(a) - \lim_{x \rightarrow x_1^-} f(x) + f(b) \\ &\leq f(b) \end{aligned}$$

Assume the formula holds for n . Then, for the $n + 1$ case:

$$\begin{aligned} f(a) + \sum_{i=1}^{n+1} j_f(x_i) &= f(a) + \sum_{i=1}^n j_f(x_i) + j_f(x_{n+1}) \\ &\leq f(x_n) + j_f(x_{n+1}) \\ &\leq f(b) \end{aligned}$$

Case 1: Suppose $I = [a, b]$. Consequently,

$$\sum_{i=1}^n j_f(x_i) \leq f(b) - f(a)$$

Let $G_k = \left\{ x \in [a, b] \mid j_f(x) \geq \frac{f(b)-f(a)}{k} \right\}$. By the lemma, $|G_k| \leq k$. This is because, if $x_1, \dots, x_n \in G_k$ with $n > k$, then

$$\begin{aligned} \sum_{i=1}^n j_f(x_i) &\geq \frac{n(f(b) - f(a))}{k} \\ &> f(b) - f(a) \end{aligned}$$

contradicting the lemma.

Recall that f is discontinuous at $x = c$ if and only if $j_f(c) > 0$. Therefore, we have that

$$D = \bigcup_{k=1}^{\infty} G_k,$$

So for k large enough, $j_f(x) \geq \frac{f(b)-f(a)}{k}$. Since each G_k is a finite set, D is a countable union of countable sets, and is thus countable.

Case 2: $I = (a, b]$. Write I as

$$I = \bigcup_{n=1}^{\infty} [a + 1/n, b].$$

Let $D_n = \{x \in [a + 1/n, b] \mid f \text{ discontinuous at } x\}$. By case 1, D_n is countable. Let $D = \{x \in (a, b] \mid f \text{ discontinuous at } x\}$. Note that $D = \bigcup D_n$. Therefore, D is countable.

Case 3: $I = [a, b)$. Write I as

$$I = \bigcup_{n \geq 1} [a, b - 1/n].$$

Proceed as with case 2.

Case 4: $I = (a, b)$. Write I as

$$I = (a, b - \delta] \cup [b - \delta, b).$$

Apply case 2 and case 3.

Case 5: $I = (-\infty, b)$ or $I = (-\infty, b]$. Write I as

$$I = \bigcup_{n \geq 1} (b - n, b)$$

or

$$I = \bigcup_{n \geq 1} (b - n, b].$$

Proceed via the countable union of countable sets.

Case 6: $I = [a, \infty)$ or $I = (a, \infty)$. Write I as

$$I = \bigcup_{n \geq 1} (a, a + n)$$

or

$$I = \bigcup_{n \geq 1} [a, a + n].$$

Proceed via the countable union of countable sets.

Case 7: $I = \mathbb{R}$. Write I as

$$I = \bigcup_{n \geq 1} [-n, n].$$

Proceed via the countable union of countable sets.

Continuous Inverse Theorem

Statement: Let $I \in \mathbb{R}$ be an interval, and let $f : I \rightarrow \mathbb{R}$ be continuous and strictly monotone. Then,

- (1) $J = f(I)$ is an interval. (Proved earlier.)
- (2) $f : I \rightarrow J$ is bijective and thus invertible.
- (3) $f^{-1} : J \rightarrow I$ is continuous and strictly monotone.

Assume f is continuous and strictly increasing.

Proof of (3): First, we prove $g : J \rightarrow I$ is also strictly increasing. To see this, let $y_1, y_2 \in J$, with $y_1 < y_2$. If

$$g(y_1) \geq g(y_2)$$

then,

$$f(g(y_1)) \geq f(g(y_2))$$

$$y_1 \geq y_2,$$

⊥

So $g(y_1) < g(y_2)$.

Now, we will show that g is continuous. Note that since $f(I) = J$, it must be the case that $g(J) = I$. Suppose toward contradiction that g is discontinuous at $x = c \in J$. Then, $j_g(c) = \lim_{x \rightarrow c^+} g(x) - \lim_{x \rightarrow c^-} g(x) > 0$.

So, we find $x \in I$ with $\lim_{x \rightarrow c^-} g(x) < x < \lim_{x \rightarrow c^+} g(x)$. However, since g is strictly increasing, it follows that $x \notin \text{Rang}$. If $y < c$, then $g(y) \leq \lim_{x \rightarrow c^-} g(x)$, and if $z > c$, then $g(z) \geq \lim_{x \rightarrow c^+} g(x)$. However, we know that $g(J) = I$. ⊥

The n th Root Function

Let n be even, $f : [0, \infty) \rightarrow \mathbb{R}$ where $f(x) = x^n$. Clearly, f is continuous, and f is also strictly increasing.

- $\text{Ran}(f) = [0, \infty)$. To see this, we see that $f(0) = 0$ and $\lim_{x \rightarrow +\infty} f(x) = +\infty$. By the Intermediate Value Theorem, f must obtain every value in $[0, \infty)$.

Thus, $f : [0, \infty) \rightarrow [0, \infty)$ is invertible, and we write $g : [0, \infty) \rightarrow [0, \infty)$, where $g(x) = x^{1/n}$.

If $x, y > 0$, then $(xy)^{1/n} = x^{1/n}y^{1/n}$. Note that $f(uv) = f(u)f(v)$.

If $x = f(u)$ and $y = f(v)$, then $f((xy)^{1/n}) = f(g(xy)) = xy = f(g(x))f(g(y)) = f(x^{1/n})f(y^{1/n}) = f(x^{1/n}y^{1/n})$.

If $x > 0$, then $(x^n)^{1/n} = x = (x^{1/n})^n$, following from the fact that $g \circ f(x) = x = f \circ g(x)$. If $x < 0$, then $(x^n)^{1/n} = |x|$.

Since $x < 0$, we can write

$$\begin{aligned}(x^n)^{1/n} &= ((-|x|)^n)^{1/n} \\ &= ((-1)^n |x|^n)^{1/n} \\ &= |x|\end{aligned}$$

Note that if $x < 0$, $(x^{1/n})^n$ is not defined.

If n is odd, then $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^n$ is continuous and strictly increasing with range \mathbb{R} . By the continuous inverses theorem, $f^{-1} = g$ is continuous and strictly increasing. We write $g(x) = x^{1/n}$.

Similarly as to the even case, we can show that

- $(xy)^{1/n} = x^{1/n}y^{1/n}$
- $\forall x \in \mathbb{R}, (x^{1/n})^n = x = (x^n)^{1/n}$

Recall that if $x \neq 0$ in \mathbb{R} , then x^{-1} is defined as the unique value such that $xx^{-1} = 1$.

If $x \neq 0$ and $n \in \mathbb{N}$, then $(x^n)^{-1} = (x^{-1})^n$. We write x^{-n} as the common value.

(1) If n is even and $x > 0$, then $(x^{1/n})^{-1} = (x^{-1})^{1/n}$

(2) If n is odd, and $x \neq 0$, then $(x^{1/n})^{-1} = (x^{-1})^{1/n}$.

Proof: If $x > 0$, then $x^{1/n} > 0$. So,

$$\begin{aligned}x^{1/n} (x^{-1})^{1/n} &= (x \cdot x^{-1})^{1/n} \\ &= 1\end{aligned}$$

So by the uniqueness of inverses, the theorem follows.

Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

(1) If n is even, $x > 0$, then $(x^m)^{1/n} = (x^{1/n})^m$

(2) If n is odd, $x \neq 0$, then $(x^m)^{1/n} = (x^{1/n})^m$

We define the unique values as $x^{m/n}$.

Derivatives

In this context, I always refers to an interval, and $c \in I$.

Definition of Differentiation

A function f is differentiable at c if

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite.

In that case, we denote the limit as $f'(c)$. The value $f'(c)$ is called the derivative of f at c .

Like with continuity, f is differentiable on I if $f'(c)$ exists $\forall c \in I$.

Applying Differentiation 1

Let $f(x) = ax + b$, $c \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \frac{(ax + b) - (ac + b)}{x - c} \\ &= \frac{a(x - c)}{x - c} \\ &= a\end{aligned}$$

Applying Differentiation 2

Let $f(x) = x^2$, $c \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \rightarrow c} x + c \\ &= 2c\end{aligned}$$

Applying Differentiation 3

Let $f(x) = \sqrt{x}$, $c \geq 0$. Then,

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{\sqrt{x} - \sqrt{c}}{x - c} \\ &= \lim_{x \rightarrow c} \frac{1}{\sqrt{x} + \sqrt{c}} \\ &= \begin{cases} \frac{1}{2\sqrt{c}} & c \neq 0 \\ +\infty & c = 0 \end{cases}\end{aligned}$$

Therefore, $f'(c)$ exists only when $c \geq 0$.

Applying Differentiation 4

For example, $f(x) = |x|$ is *not* differentiable at $c = 0$.

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}$$

Let $(x_n)_n = \frac{(-1)^n}{n}$. Then, $(x_n)_n \rightarrow 0$. However, $\frac{|x_n|}{x_n} = (-1)^n$, which diverges. Therefore, the limit does not exist.

Applying Differentiation 5

Let

$$g(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then,

$$\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \sin(1/x).$$

Let $(x_n)_n = \frac{2}{\pi n}$. Then, $(x_n)_n \rightarrow 0$, but $\sin(1/x_n)$ is divergent.

Applying Differentiation 6

Let $f(x) = \sin(x)$, $c \in \mathbb{R}$. Then,

$$f'(c) = \lim_{x \rightarrow c} \frac{\sin(x) - \sin(c)}{x - c}$$

Let $h = x - c$. Then, $x \rightarrow c \Leftrightarrow h \rightarrow 0$. Then,

$$\begin{aligned} f'(c) &= \lim_{h \rightarrow 0} \frac{\sin(h+c) - \sin(c)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(h)\cos(c) + \cos(h)\sin(c) - \sin(c)}{h} \\ &= \lim_{h \rightarrow 0} \cos(c) \frac{\sin(h)}{h} + \sin(c) \frac{\cos(h) - 1}{h} \\ &= \cos(c) \end{aligned}$$

Differentiability and Continuity

Statement: If $f : I \rightarrow \mathbb{R}$ is differentiable at $x = c$, then f is continuous at $x = c$.

Proof:

$$\begin{aligned} \lim_{x \rightarrow c} (f(x) - f(c)) &= \lim_{x \rightarrow c} \left((x - c) \frac{f(x) - f(c)}{x - c} \right) \\ &= \lim_{x \rightarrow c} (x - c) f'(c) \\ &= 0 \end{aligned}$$

Thus, $\lim_{x \rightarrow c} f(x) = f(c)$, and f is continuous.

Operations with Differentiation

Statement: Let $I \subseteq \mathbb{R}$ be an interval, $c \in I$. Let $f, g : I \rightarrow \mathbb{R}$ be differentiable at $x = c$. Let $\alpha \in \mathbb{R}$. Then,

$$(1) (\alpha f)'(c) = \alpha f'(c)$$

$$(2) (f + g)'(c) = f'(c) + g'(c)$$

$$(3) (fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

$$(4) \left(\frac{f}{g} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}, \text{ provided } g(c) \neq 0.$$

Proof of (4):

$$\begin{aligned} \left(\frac{f}{g} \right)'(c) &= \lim_{x \rightarrow c} \frac{(f/g)(x) - (f/g)(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(c) - f(c)g(x)}{(x - c)g(x)g(c)} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{(x - c)g(x)g(c)} \\ &= \lim_{x \rightarrow c} \frac{g(c)(f(x) - f(c))}{(x - c)g(x)g(c)} - \lim_{x \rightarrow c} \frac{f(c)(g(x) - g(c))}{(x - c)g(x)g(c)} \\ &= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad \text{since } \lim_{x \rightarrow c} g(x) = g(c) \end{aligned}$$

Power Rule

Statement: Let $f_n(x) = x^n$, where $n \in \mathbb{Z}$. Then, $f'_n(x) = nx^{n-1}$.

Proof: Let $n \geq 1$. We have already proved the linear case ($n = 1$). Inductively assume true for n .

Then,

$$\begin{aligned} f'_{n+1}(x) &= (x \cdot f_n(x))' \\ &= f'_n(x) + x f'_n(x) \\ &= x^n + x \cdot n x^{n-1} \\ &= (n+1)x^n \end{aligned}$$

Similarly, the proof is clear for $n = 0$. Using the quotient rule, we can show the similar case for $n < 0$.

$$f_{-n}(x) = \frac{1}{f_n(x)} \quad n = 1, 2, 3, \dots$$

Carathéodory's Theorem

Statement: If $f : I \rightarrow \mathbb{R}$, $c \in I$. f is differentiable at $x = c$ if and only if $\exists \varphi : I \rightarrow \mathbb{R}$ continuous at c such that $\forall x \in I$, $f(x) - f(c) = \varphi(x) \cdot (x - c)$. In this case, $f'(c) = \varphi(c)$.

For example, if $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$. Fix $c \in \mathbb{R}$. Then, $f(x) - f(c) = (x - c)(x^2 + cx + c^2)$. Let $\varphi(x) = x^2 + cx + c$. Then, $\varphi(c) = 3c^2$.

Proof:

(\Rightarrow): Suppose $\exists \varphi : I \rightarrow \mathbb{R}$ such that $f(x) - f(c) = \varphi(x)(x - c) \forall x \in I$. Then,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \varphi(x) \\ &= \varphi(c) \end{aligned}$$

So, f is differentiable and $f'(c) = \varphi(c)$.

(\Leftarrow): Assume f is differentiable at $x = c$. Let $\varphi : I \rightarrow \mathbb{R}$

$$\varphi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & x \neq c \\ f'(c) & x = c \end{cases}$$

It is the case that φ is continuous at $x = c$ since $\lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) = \varphi(c)$.

Clearly, $f(x) - f(c) = \varphi(x)(x - c)$.

Chain Rule

Statement: Let $J \xrightarrow{f} I \xrightarrow{g} \mathbb{R}$, where I and J are intervals. Let $c \in J$ and $d = f(c) \in I$. Assume f is differentiable at $x = c$, and g is differentiable at $d = f(c)$. Then, $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof: We know that $\exists \varphi : J \rightarrow \mathbb{R}$ with $\forall x \in J$, $f(x) - f(c) = \varphi(x)(x - c)$, with φ continuous at $x = c$. Similarly, $\exists \psi : I \rightarrow \mathbb{R}$ with $\forall y \in I$, $g(y) - g(d) = \psi(y)(y - d)$.

In particular, $\forall x \in J$,

$$\begin{aligned} g(f(x)) - g(f(c)) &= \psi(f(x))(f(x) - f(c)) \\ g(f(x)) - g(f(c)) &= \psi(f(x))\varphi(x)(x - c), \end{aligned}$$

so

$$g \circ f(x) - g \circ f(c) = \lambda(x)(x - c) \quad \text{where } \lambda(x) = \psi(f(x))\varphi(x)$$

Note that $\lambda : J \rightarrow \mathbb{R}$ is continuous at $x = c$ because

- φ is continuous at $x = c$
- f is differentiable at $x = c$, and thus continuous at $x = c$
- ψ is continuous at $d = f(c)$

Therefore, by Carathéodory's theorem, $g \circ f$ is differentiable at $x = c$.

Additionally,

$$\begin{aligned} (g \circ f)'(c) &= \lambda(c) \\ &= \psi(f(c))\varphi(c) \\ &= \psi(d)\varphi(c) \\ &= g'(d)f'(c). \end{aligned}$$

Inverse Functions

Let I be an interval, $f : I \rightarrow \mathbb{R}$ strictly monotone and continuous, $f(I) = J$. Let $g : J \rightarrow I$ be the inverse map.

- J is an interval
- g is continuous and strictly monotone
- If f is differentiable at $c \in I$, and $f'(c) \neq 0$, then g is differentiable at $y = d = f(c)$, and

$$g'(d) = \frac{1}{f'(c)}$$

Applying Inverse Functions 1

Let $T : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$, $T(x) = \tan(x)$. Since T is strictly monotone, continuous, and $\lim_{x \rightarrow \pi/2^-} T(x) = +\infty$, and $\lim_{x \rightarrow -\pi/2^+} T(x) = -\infty$, T is bijective.

Let $A : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$\begin{aligned} A'(d) &= \frac{1}{T'(c)} \\ T(c) &= d \\ A'(d) &= \frac{1}{\sec^2(c)} \\ &= \frac{1}{1 + \tan^2(c)} \\ &= \frac{1}{1 + d^2} \end{aligned}$$

Applying Inverse Functions 2

Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $f_n(x) = x^n$, where n is odd. Since f is strictly monotone, continuous, and surjective, f is bijective. Let $g_n : \mathbb{R} \rightarrow \mathbb{R}$ be the inverse. Then, $g_n(y) = y^{1/n}$. Let $f_n(c) = d$.

$$\begin{aligned} g'_n(d) &= \frac{1}{f'_n(c)} \\ &= \frac{1}{nc^{n-1}} \\ &= \frac{1}{nd^{1-\frac{1}{n}}} \\ &= \frac{1}{n} d^{\frac{1}{n}-1} \end{aligned}$$

The same idea works when n is even on $(0, \infty)$.

Exercise: Let $\frac{m}{n} \in \mathbb{Q}$. Show that $\frac{d}{dx} x^{m/n} = \frac{m}{n} x^{m/n-1}$.

We can write this as a composition and use the chain rule.

Fermat's Theorem

Statement: Let $f : I \rightarrow \mathbb{R}$, c an interior point of f . Suppose f has a local maximum or minimum at $x = c$ ($\exists \delta \ni x \in V_\delta(c) \Rightarrow f(x) \leq f(c)$ or $\exists \delta \ni x \in V_\delta(c) \Rightarrow f(x) \geq f(c)$). Then,

(1) $f'(c)$ does not exist.

(2) $f'(c) = 0$.

Proof: If $f'(c)$ does not exist, there is nothing to prove. Assume $f'(c)$ does exist.

Suppose toward contradiction that $f'(c) \neq 0$.

Case 1: $f'(c) > 0$. So,

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} > 0,$$

Meaning $\exists \delta$ such that $x \in V_\delta(c)$ implies

$$\frac{f(x) - f(c)}{x - c} > 0.$$

So, if $x \in (c - \delta, c)$,

$$\begin{aligned} f(x) - f(c) &= \frac{f(x) - f(c)}{x - c} (x - c) \\ &< 0 \\ f(x) &< f(c), \end{aligned} \tag{*}$$

and if $x \in (c, c + \delta)$,

$$\begin{aligned} f(x) - f(c) &= \frac{f(x) - f(c)}{x - c} (x - c) \\ &> 0 \\ f(x) &> f(c). \end{aligned} \tag{**}$$

If c is a local minimum, (*) violates the assumption, and if c is a local maximum, (**) violates the assumption. \perp

Warning: Fermat's theorem does not run in converse: $f(x) = x^3$, $f'(0) = 0$ but $x = 0$ is not a local minimum or maximum. Similarly, $f(x) = x^{1/3}$, $f'(0) = 0$ but $x = 0$ is not a local minimum or maximum.

Rolle's Theorem

Statement: Let $f : [a, b] \rightarrow \mathbb{R}$ with f continuous on $[a, b]$ and f differentiable on (a, b) . If $f(a) = f(b)$, $\exists c \in (a, b)$ with $f'(c) = 0$.

Proof: If f is a constant function, we are done.

Suppose f is not a constant function.

Case 1: $\exists x \in (a, b)$ with $f(x) > f(a)$. By the extreme value theorem and the hypothesis, $\exists x_M \in (a, b)$ with $f(x_M) = \sup_{x \in [a, b]} f(x)$. By Fermat's Theorem, $f'(x_M) = 0$.

Case 2: $\exists x \in (a, b)$ with $f(x) < f(a)$. By the extreme value theorem, $\exists x_m \in (a, b)$ with $f(x_m) = \inf_{x \in [a, b]} f(x)$. By Fermat's Theorem, $f'(x_m) = 0$.

Applying Rolle's Theorem

Problem: Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) . Suppose $f(a)f(b) < 0$, and $f'(x) \neq 0$. Show f has a unique real root in $[a, b]$.

Solution: Without loss of generality, $f(a) < 0$ and $f(b) > 0$. By the intermediate value theorem, $\exists z \in (a, b)$ with $f(z) = 0$.

Suppose toward contradiction $\exists z' \in (a, b)$ with $z' \neq z$. Use Rolle's theorem on $[z, z']$ or $[z', z]$.