Problem (Problem 1)**:**

(a) Show that $\mathbb{R}$ is not a free $\mathbb{Z}$-module.

(b) Compute $\hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ and $\hom_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z})$.

Solution:

(a) Suppose toward contradiction that $\mathbb{R}$ were a free $\mathbb{Z}$-module. Then, there would be some unique $\mathbb{Z}$-linear combination

$$1 = z_1 b_1 + \cdots + z_n b_n,$$

with $b_1, \dots, b_n \in B$, where B is the basis for $\mathbb{R}$. We observe now that for any $k \in \mathbb{Z}_{>0}$,

$$\frac{1}{k} = z_1' b_1' + \cdots + z_m' b_m'$$

for some other basis elements $b_1', \dots, b_m' \in B$ and integers $z_1', \dots, z_m'$. Suppose toward contradiction that there were some b_i' such that $b_i' \notin \{b_1, \dots, b_n\}$. Then, we would have

$$\begin{aligned} 1 &= k\left(z_1' b_1' + \cdots + z_m' b_m'\right) \\ &= k z_1' b_1' + \cdots + k z_m' b_m', \end{aligned}$$

implying that 1 has a non-unique expression of integral linear combinations of basis elements, contradicting the assumption that $\mathbb{R}$ is free over $\mathbb{Z}$.

There is some submodule $Y \supseteq \mathbb{Q}$ of $\mathbb{R}$ defined by $\mathbb{Z}\langle b_1, \dots, b_n\rangle$. The map

$$\begin{aligned} \nu \colon \mathbb{Z}^n &\to Y \\ (z_1, \dots, z_n) &\mapsto z_1 b_1 + \cdots + z_n b_n \end{aligned}$$

is thus an isomorphism, as it is injective by the assumption that B is a basis and surjective by definition. Now, since $\mathbb{Q} \subseteq Y$ is a submodule, we observe that $\nu^{-1}(\mathbb{Q}) \subseteq \mathbb{Z}^n$ is a submodule, as for any $w_1, w_2 \in \nu^{-1}(\mathbb{Q})$, we have $\nu(w_1), \nu(w_2) \in \mathbb{Q}$, whence $\nu(w_1 + w_2) \in \mathbb{Q}$, so that $w_1 + w_2 \in \nu^{-1}(\mathbb{Q})$, and $\nu(z w_1) = z\nu(w_1) \in \mathbb{Q}$ for any $z \in \mathbb{Z}$, whence $z w_1 \in \nu^{-1}(\mathbb{Q})$.

Now, since each $\mathbb{Z}$ is a PID (hence Noetherian), it follows that every $\mathbb{Z}$-submodule(/ideal) of $\mathbb{Z}^n$ is also finitely generated, as it is of the form $I_1 \times \cdots \times I_n$ for ideals $I_1, \dots, I_n \in \mathbb{Z}$. Thus, it follows that $\mathbb{Q} \cong \nu^{-1}(\mathbb{Q})$, whence $\mathbb{Q}$ is then isomorphic to a finitely generated $\mathbb{Z}$-module, which is a contradiction as it has been well-established that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module.

(b) We claim that both $\hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ and $\hom_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z})$ are zero. Toward this end, observe that

$$\varphi\left(\frac{a}{b}\right) = k\varphi\left(\frac{a}{kb}\right)$$

for all $\frac{a}{b} \in \mathbb{Q}$ with $\frac{a}{b} \neq 0$ and all $k \in \mathbb{Z}_{>0}$. Yet, this can only be the case if $\varphi\left(\frac{a}{b}\right) = 0$, whence $\hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \{0\}$. Similarly, if $r \in \mathbb{R}$ is real with $r \neq 0$, then

$$\varphi(r) = k\varphi\left(\frac{r}{k}\right),$$

for all $k \in \mathbb{Z}_{>0}$, so that $\varphi(r) = 0$, and thus $\hom_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z}) \cong \{0\}$.

Problem (Problem 2)**:** Let R be a commutative ring with 1. Suppose there are integers m_1 and m_2 such that $R^{m_1} \cong R^{m_2}$. Prove that $m_1 = m_2$.

Solution: Let I be a maximal ideal of R, and let $K = R/I$. We claim that if $M_1 \cong M_2$ are isomorphic R-modules, then $M_1/IM_1 \cong M_2/IM_2$ are isomorphic as R/I-vector spaces. Toward this end, we let

$$\psi\colon M_1 \to M_2/IM_2$$

be a surjective homomorphism of R-modules defined by $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\pi} M_2/IM_2$, whence $\ker(\psi) = IM_1$, as

$$\psi(v_1) = 0 + IM_2$$

if and only if $\varphi(v_1) \in IM_2$, whence $\varphi(v_1) = i\varphi(w_1)$ with $i \in I$, or that $\varphi(iw_1) \in IM_2$, so $iw_1 \in IM_1$. The reverse inclusion follows from the first isomorphism theorem, as $IM_1 \subseteq \ker(\psi)$ by observation. Thus, we have an isomorphism $\overline{\psi}\colon M_1/IM_1 \to M_2/IM_2$.

We claim that the action

$$(r + I) \cdot (m + IM_1) = r \cdot m + IM_1$$

is a well-defined action of R/I on M_1/IM_1. Toward this end, we let $r_1 + I = r_2 + I$, whence $r_1 - r_2 \in I$. For any $m + IM_1 \in M_1/IM_1$, we have (as the quotient module M_1/IM_1 is well-defined)

$$\begin{aligned}(r_1 + I) \cdot (m + IM_1) &= r_1 \cdot m + IM_1\\ &= (r_1 - r_2 + r_2) \cdot m + IM_1\\ &= ((r_1 - r_2) \cdot m + IM_1) + (r_2 \cdot m + IM_1)\\ &= (0 + IM_1) + (r_2 \cdot m + IM_1)\\ &= r_2 \cdot m + IM_1.\end{aligned}$$

The rest of the axioms for the action of R/I on M_1/IM_1 follow from the axioms of R-modules.

Thus, it follows that if $R^{m_1} \cong R^{m_2}$, then we have

$$\begin{aligned}R^{m_1}/IR^{m_1} &\cong R^{m_2}/IR^{m_2}\\ K^{m_1} &\cong K^{m_2},\end{aligned}$$

whence $m_1 = m_2$ by the invariance of dimension for vector spaces.

Problem (Problem 4)**:** Let R be a local ring with maximal ideal I.

(a) Show that if M is a finitely generated module with $I \cdot M = M$, then $M = \{0\}$.

(b) If M is a finitely generated R-module, and $y_1, \dots, y_m \in M$ are such that $\overline{y_1}, \dots, \overline{y_m} \in M/IM$ generate M/IM, then $y_1, \dots, y_m$ generate M.

Solution:

(a) Let $M = \langle x_1, \dots, x_n \rangle$, and suppose $IM = M$. Then, it follows that there are $v_1, \dots, v_n \in I$ such that

$$x_n = v_1 \cdot x_1 + \cdots + v_n \cdot x_n,$$

whence

$$(1 - v_n) \cdot x_n = v_1 \cdot x_1 + \cdots + v_{n-1} \cdot x_{n-1},$$

whence, since I is a local ring,

$$x_n = (1 - v_n)^{-1}(v_1 \cdot x_1 + \cdots + v_{n-1} \cdot x_{n-1}),$$

meaning that $M = \langle x_1, \dots, x_{n-1} \rangle$. Inductively, any generating subset of M can be reduced in this fashion until $M = \{0\}$.

(b) Let $N = \langle y_1, \dots, y_m \rangle$. We wish to show that

$$M = N + IM.$$

Toward this end, let $v \in M$. If $v \in IM$, then we are done. Else, if $v \notin IM$, it follows that $v + IM \neq 0 + IM$, so there are $\alpha_1, \dots, \alpha_m$ such that

$$\begin{aligned} v + IM &= \alpha_1 \cdot (y_1 + IM) + \dots + \alpha_m \cdot (y_1 + IM) \\ &= (\alpha_1 \cdot y_1 + \dots + \alpha_m \cdot y_m) + IM. \end{aligned}$$

In particular, this means there is some $q \in IM$ such that

$$v = (\alpha_1 \cdot y_1 + \dots + \alpha_m \cdot y_m) + q,$$

whence $M = N + IM$.

Consider the subspace $I(M/N)$ of M/N. We seek to show that $I(M/N) = M/N$. Let $v + N \in M/N$. Since $v \in M$, it follows that there are $r_1, \dots, r_n \in I$ and $q \in IM$ such that

$$v = \sum_{i=1}^{n} r_i \cdot y_i + q.$$

In particular, this means that $v + N = q + N$. Since $q + N = ip + N$ for some $p \in M$, we have $i(p + N) = v + N$, whence $I(M/N) = M/N$, meaning that by part (a), we have $M/N \cong \{0\}$, or that $M = N$. Thus, $y_1, \dots, y_n$ generate N.

Problem (Problem 6)**:** Let R be a ring, M an R-module, and let $S \subseteq R$ be multiplicative.

(a) Mimic the construction of the localization $S^{-1}R$ to define the localization $S^{-1}M$ making it into an R-module.

(b) Show that $S^{-1}M$ gains the structure of an $S^{-1}R$-module.

Solution:

(a) Let $\overline{M} = M \times S$, and define a relation $(m_1, s_1) \sim (m_2, s_2)$ if there exists $s \in S$ such that $s(s_2 m_1 - s_1 m_2) = 0$.

We claim that this is an equivalence relation.

- Reflexivity is clear from the fact that we may choose $s = 1$, whence $(m_1, s_1) \sim (m_1, s_1)$ if and only if $s_1 m_1 - s_1 m_1 = 0$.
- Symmetry follows from the fact that, if $(m_1, s_1) \sim (m_2, s_2)$, then

$$\begin{aligned} s(s_2 m_1 - s_1 m_2) &= 0 \\ &= (-1) \cdot 0 \\ &= (-1)(s(s_2 m_1 - s_1 m_2)) \\ &= s(s_1 m_2 - s_2 m_1), \end{aligned}$$

  meaning that $(m_2, s_2) \sim (m_1, s_1)$.
- Finally, for transitivity, we let $(m_1, s_1) \sim (m_2, s_2)$ and $(m_2, s_2) \sim (m_3, s_3)$. Then,

$$\begin{aligned} s(s_2 m_1 - s_1 m_2) &= 0 \\ t(s_3 m_2 - s_2 m_3) &= 0. \end{aligned}$$

  We seek to find $r \in S$ such that $r(s_3 m_1 - s_1 m_3) = 0$. Toward this end, we multiply the first equation by ts_3 and the second equation by ss_1. This gives

$$sts_3(s_2 m_1 - s_1 m_2) = 0$$

$$sts_1(s_3m_2 - s_2m_3) = 0.$$

Distributing these sums out, we get

$$\begin{aligned} sts_2s_3m_1 - sts_1s_3m_2 &= 0 \\ sts_1s_3m_2 - sts_1s_2m_3 &= 0. \end{aligned}$$

Adding, we get

$$sts_2s_3m_1 - sts_2s_1m_3 = 0,$$

whence

$$sts_2(s_3m_1 - s_1m_3) = 0,$$

and since $s, t, s_2 \in S$, so too is sts_2, whence $(m_1, s_1) \sim (m_3, s_3)$.

We write $\frac{m}{s} \equiv [(m, s)]$. We may define R-operations by taking

$$\begin{aligned} r \cdot \left(\frac{m}{s}\right) &= \frac{rm}{s} \\ \frac{m_1}{s_1} + \frac{m_2}{s_2} &= \frac{s_2m_1 + s_1m_2}{s_1s_2}. \end{aligned}$$

We claim that both of these operations are well-defined. To start, if $(m_1, s_1) \sim (m_2, s_2)$, then

$$\begin{aligned} s(s_2m_1 - s_1m_2) &= 0 \\ rs(s_2m_1 - s_1m_2) &= 0 \\ s(s_2rm_1 - s_1rm_2) &= 0, \end{aligned}$$

whence $\frac{rm_1}{s_1} = \frac{rm_2}{s_2}$.

Now, we observe that addition as defined is commutative, so we only need to check well-definedness in the case of one summand. Therefore, if $(m_1, s_1) \sim (n_1, t_1)$, we have some $v \in S$ such that

$$v(t_1m_1 - s_1n_1) = 0.$$

We claim now that

$$\frac{s_2m_1 + s_1m_2}{s_1s_2} = \frac{s_2n_1 + t_1m_2}{t_1s_2}.$$

Indeed, we observe that

$$\begin{aligned} v\left(t_1s_2^2m_1 + t_1s_1s_2m_2 - s_1s_2^2n_1 - t_1s_1s_2m_2\right) &= vs_2^2(t_1m_1 - s_1n_1) \\ &= s_2^2(v(t_1m_1 - s_1n_1)) \\ &= 0. \end{aligned}$$

Now, addition is associative, since

$$\begin{aligned} \frac{m_1}{s_1} + \left(\frac{m_2}{s_2} + \frac{m_3}{s_3}\right) &= \frac{m_1}{s_1} + \left(\frac{s_3m_2 + s_2m_3}{s_2s_3}\right) \\ &= \frac{s_2s_3m_1 + s_1s_3m_2 + s_1s_2m_3}{s_1s_2s_3} \\ &= \frac{s_2m_1 + s_1m_2}{s_1s_2} + \frac{m_3}{s_3} \end{aligned}$$

$$= \left(\frac{m_1}{s_1} + \frac{m_2}{s_2}\right) + \frac{m_3}{s_3}.$$

Furthermore, we observe that scalar multiplication comports with addition in both R and $S^{-1}M$, as

$$\begin{aligned}
(r_1 + r_2)\frac{m}{s} &= \frac{(r_1 + r_2)m}{s}\\
&= \frac{r_1 m + r_2 m}{s}\\
&= \frac{r_1 m}{s} + \frac{r_2 m}{s}\\
&= r_1\frac{m}{s} + r_2\frac{m}{s}\\
r\left(\frac{m_1}{s_1} + \frac{m_2}{s_2}\right) &= r\left(\frac{s_2 m_1 + s_1 m_2}{s_1 s_2}\right)\\
&= \frac{r s_2 m_1 + r s_1 m_2}{s_1 s_2}\\
&= \frac{s_2 r m_1 + s_1 r m_2}{s_1 s_2}\\
&= \frac{r m_1}{s_1} + \frac{r m_2}{s_2}\\
&= r\frac{m_1}{s_1} + r\frac{m_2}{s_2}.
\end{aligned}$$

Finally, we observe that

$$\frac{m_1}{s_1} + \frac{0}{1} = \frac{m_1}{s_1},$$

whence $\frac{0}{1}$ is the additive identity in $S^{-1}M$, and

$$1\frac{m}{s} = \frac{m}{s},$$

whence $1 \cdot v = v$ in $S^{-1}M$. Thus, we find that $S^{-1}M$ takes on a structure as an R-module.

(b) To extend the structure of $S^{-1}M$ to yield an $S^{-1}M$-module, we take the scalar multiplication

$$\frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}.$$

Now, we observe that

$$\begin{aligned}
\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}\right)\frac{m}{t} &= \left(\frac{r_1 s_2 + r_2 s_1}{s_1 s_2}\right)\frac{m}{t}\\
&= \frac{(r_1 s_2 + r_2 s_1)m}{s_1 s_2 t}\\
&= \frac{r_1 s_2 m + r_2 s_1 m}{s_1 s_2 t}\\
&= \frac{r_1 m}{s_1 t} + \frac{r_2 m}{s_2 t}\\
&= \frac{r_1}{s_1}\frac{m}{t} + \frac{r_2}{s_2}\frac{m}{t}\\
\frac{r}{s}\left(\frac{m_1}{t_1} + \frac{m_2}{t_2}\right) &= \frac{r}{s}\left(\frac{t_2 m_1 + t_1 m_2}{t_1 t_2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{rt_2m_1 + rt_1m_2}{st_1t_2} \\
&= \frac{rm_1}{st_1} + \frac{rm_2}{st_2} \\
&= \frac{r}{s}\frac{m_1}{t_1} + \frac{r}{s}\frac{m_2}{t_2},
\end{aligned}$$

meaning that scalar multiplication by elements of $S^{-1}R$ comports with addition in $S^{-1}M$ and vice versa. Finally, we also observe that

$$\begin{aligned}
\frac{0}{1}\frac{m}{s} &= \frac{0}{s} \\
&= \frac{0}{1} \\
\frac{1}{1}\frac{m}{s} &= \frac{m}{s}.
\end{aligned}$$

Thus, $S^{-1}M$ takes on the structure of an $S^{-1}R$-module.