Section 4.1

Solution (Problem 4)**:** Evaluating with the initial conditions, we get

$$\begin{aligned} c_1 - c_2 &= 0 \\ -c_3 &= 2 \\ c_2 &= -1. \end{aligned}$$

We see that $c_1 = -1$, $c_2 = -1$, and $c_3 = -2$. This yields the particular solution of

$$y = -1 - \cos x - 2\sin x.$$

Solution (Problem 10)**:** The interval $(-\pi, \pi)$ contains a unique solution to the initial value problem.

Solution (Problem 14)**:**

(a) We have

$$\begin{aligned} c_1 + c_2 + 3 &= 0 \\ c_1 + c_2 + 3 &= 4, \end{aligned}$$

which is not possible.

(b) We have

$$\begin{aligned} 3 &= 0 \\ c_1 + c_2 + 3 &= 2, \end{aligned}$$

which is yet again not possible.

(c) We have

$$\begin{aligned} 3 &= 3 \\ c_1 + c_2 + 3 &= 0, \end{aligned}$$

meaning that the solution set is all pairs (c_1, c_2) such that $c_1 + c_2 = -3$.

(d) We have

$$\begin{aligned} c_1 + c_2 + 3 &= 3 \\ 4c_1 + 16c_2 + 3 &= 15, \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 &= 0 \\ 4c_1 + 16c_2 &= 12 \end{aligned}$$

meaning

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1. \end{aligned}$$

Solution (Problem 22)**:** Since

$$\sinh(x) = \frac{1}{2}(e^x + e^{-x}),$$

the functions are not linearly independent anywhere on $(-\infty, \infty)$.

Solution (Problem 28)**:** First, we verify that both solutions work.

$$\begin{aligned} x^2\frac{d^2}{dx^2}(\cos(\ln(x))) + x\frac{d}{dx}(\cos(\ln(x))) + \cos(\ln(x)) &= x^2\left(-\frac{\cos(\ln(x))}{x^2} + \frac{\sin(\ln(x))}{x^2}\right) + x\left(-\frac{\sin(\ln(x))}{x}\right) + \cos(\ln(x)) \\ &= 0 \end{aligned}$$

$$\begin{aligned}x^2\frac{d^2}{dx^2}(\sin(\ln(x))) + x\frac{d}{dx}(\sin(\ln(x))) + \sin(\ln(x)) &= x^2\left(-\frac{\cos(\ln(x))}{x^2} - \frac{\sin(\ln(x))}{x^2}\right) + x\left(\frac{\cos(\ln(x))}{x}\right) + \sin(\ln(x))\\ &= 0.\end{aligned}$$

Additionally, we find that

$$\begin{aligned}\det\begin{pmatrix}\cos(\ln(x)) & \sin(\ln(x))\\ \frac{-\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x}\end{pmatrix} &= \frac{1}{x}\\ &\neq 0,\end{aligned}$$

so the solutions are linearly independent. Since the differential equation $x^2y'' + xy' + y' = 0$ is a second order equation, there are no other linearly independent solutions. Thus, we have the general solution of

$$y = \alpha\cos(\ln(x)) + \beta\sin(\ln(x)).$$

Solution (Problem 30)**:** I'm not checking the Wronskian on this one, they're clearly linearly independent. However, I will be doing the derivatives.

$$\begin{aligned}\frac{d^4}{dx^4}(1) + \frac{d^2}{dx^2}(1) &= 0\\ \frac{d^4}{dx^4}(x) + \frac{d^2}{dx^2}(x) &= 0\\ \frac{d^4}{dx^4}(\cos(x)) + \frac{d^2}{dx^2}(\cos(x)) &= \cos(x) - \cos(x)\\ &= 0\\ \frac{d^4}{dx^4}(\sin(x)) + \frac{d^2}{dx^2}(\sin(x)) &= \sin(x) - \sin(x)\\ &= 0.\end{aligned}$$

Thus, since the solutions are linearly independent and have dimension 4, they form a basis for the general solution of $y^{(4)} + y'' = 0$. The general solution is

$$y(x) = c_1 + c_2x + c_3\cos(x) + c_4\sin(x).$$

Solution (Problem 36)**:**

(a) We have $y = 5$ is a particular solution to $y'' + 2y = 10$.

(b) We have $y = -2x$ is a particular solution to $y'' + 2y = 10$.

(c) Using linearity, we get that $y = -2x + 5$ is a particular solution to $y'' + 2y = -4x + 10$.

(d) Using a similar process, we have a particular solution of $y = 4x + \frac{5}{2}$.

(e) Neither of these linear combinations are general solutions of the differential equation, as the linearity principle only applies to solutions of the corresponding homogeneous equation.

Section 4.2

Solution (Problem 2)**:**

Solution (Problem 8)**:**

Solution (Problem 16)**:**

Solution (Problem 20)**:**

Solution (Problem 22)**:**

Section 4.3

Solution (Problem 4)**:**

Solution (Problem 6)**:**

Solution (Problem 12)**:**

Solution (Problem 16)**:**

Solution (Problem 22)**:**

Solution (Problem 36)**:**

Solution (Problem 38)**:**

Solution (Problem 50)**:**