Contents

Set Theory

Naive Set Theory

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \dots,\} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \dots,\} \\
\mathbb{Z}_+ &= \{0, 1, 2, \dots,\} \\
\mathbb{Q} &= \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\} \\
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} \\
\mathbb{C}_q &= \{a + bi \mid a, b \in \mathbb{Q}\}
\end{aligned}$$

Recall: given sets X and Y, a relation from X to Y is a subset of $X \times Y$, where $\times$ denotes the cartesian product of X and Y.

A relation $f \subseteq X \times Y$ is a function from X to Y such that $\forall x \in X$, $\exists! y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$, and denote f as $f : X \to Y$.

X is the **domain** of f and Y is the **codomain**. The range $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$.

The graph of a function $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Function Examples

Identity Function:

$$\text{id}_x : X \to X, \text{id}_X(x) = x$$

The Characteristic Function: If $A \subseteq X$

$$\mathbb{1}_A : X \to \mathbb{R},\ \mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Function Operations

Let X be any set, and $(X; \mathbb{R}) = \{f : X \to \mathbb{R}\}$ represent the function space of X with codomain $\mathbb{R}$.

Addition: Let $f, g \in \mathcal{F}(X; \mathbb{R})$. Then, $(f + g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$.

Scalar Multiplication: If $t \in \mathbb{R}$, then $(tf)(x) = tf(x)$.

Function Multiplication: If $g(x) \neq 0 \forall x \in X$, then $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$.

Composition: If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f(x) = g(f(x))$.

Injective, Surjective, Bijective

A function $f : X \to Y$ is a **injective** map, then, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.For example, the shift map $S : \mathbb{N} \to \mathbb{N},\ S(n) = n + 1$ is injective.

Any strictly increasing function $f : I \to \mathbb{R}$, where I is any interval, is injective.

A function f is **surjective** if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 2x + 1$. We can show that this function is surjective because $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$. Due to the intermediate value theorem, we get that $\text{ran}(f) = \mathbb{R}$.

f is **bijective** if it is injective and surjective.

Invertibility

Let $f : X \to Y$ be a function. f is **left-invertible** if $\exists g : Y \to X$ such that $g \circ f = \text{id}_X$. f is **right-invertible** if $\exists h : Y \to X$ such that $f \circ h = \text{id}_Y$.

f is **invertible** if $\exists k : Y \to X$ such that $f \circ k = \text{id}_Y$ and $k \circ f = \text{id}_X$.

For example, the function $\text{Sin}(x)$ defined as $\sin(x)$ restricted to $[-\pi/2, \pi/2]$ has an inverse, $\arcsin(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

Definition of Invertibility

Statement: f is invertible if and only if f is left and right invertible.

Proof:

($\Rightarrow$) This is via the definition of invertibility.

($\Leftarrow$) Suppose g is a left-inverse of f, and h is a right-inverse of f. Therefore, $g \circ f = \text{id}_X$, and $f \circ h = \text{id}_Y$. Observe that $g = g \circ \text{id}_Y$. Therefore, $g = g \circ (f \circ h)$. Via associativity, $g = (g \circ f) \circ h = \text{id}_X \circ h = h$.

Injection and Surjection Invertibility

Statement: If $f : X \to Y$ is a function:

(1) f is injective $\Leftrightarrow$ f is left-invertible.

(2) f is surjective $\Leftrightarrow$ f is right-invertible.

(3) f is bijective $\Leftrightarrow$ f is invertible.

Proof: $(1), (\Rightarrow)$ — suppose f is injective. Given $y \in \text{ran}(f)$, we know that $\exists! x_y \in X$ such that $f(x_y) = Y$, by the definition of injective.

Let $g : Y \to X$. We will define g as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where x_0 is an arbitrary point in X. We can see that $g \circ f = \text{id}_X$.

Cardinality and Countability

Introduction to Cardinality

Which set is "larger," $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$? $\mathbb{N}$ or $\mathbb{N}_0$? $\mathbb{Z}$ or $\mathbb{Q}$?

In order to prove that one set is "the same size" as the other, we can create pairs. For two sets A and B, we can show that A is the same size as B by creating a function. For example, to show that $\mathbb{N}$ and $\mathbb{N}_0$ have the same size, we create $s : \mathbb{N} \to \mathbb{N}_0$, $s(n) = n + 1$.

Equivalent Cardinality

Sets A and B have the same **cardinality** if $\exists$ bijection $f : A \to B$. We write $\text{card}(A) = \text{card}(B)$.

Equivalent Cardinalities of Intervals

Statement: Given $a < b$ and $c < d$, we know that $\text{card}\left([a, b]\right) = \text{card}\left([c, d]\right)$.

Proof: We can create a linear function from $[a, b]$ to $[c, d]$, and since linear functions are bijections, we know that $\text{card}\left([a, b]\right) = \text{card}\left([c, d]\right)$.

Intervals and Real Numbers

$$\text{card}\left((0, 1)\right) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection:
  - tan is strictly increasing (and thus injective)
  - $\lim_{x\to\infty} \tan(x) = \infty$ and $\lim_{x\to-\infty} \tan(x) = -\infty$, and by intermediate value theorem, tan is surjective
- $\ell : (0, 1) \to (-\pi/2, \pi/2)$ is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is $\tan \circ \ell : (0, 1) \to \mathbb{R}$.

Finitude and Infinitude

A set F is **finite** if F is empty or $\exists n \in \mathbb{N}$ such that $\text{card}(F) = \text{card}\left(\{1, 2, \dots, n\}\right)$. A non-finite set is called infinite.

We can *enumerate* F by creating a function $\sigma : \{1, 2, \dots, n\} \to F$, such that $x_j = \sigma(j)$ for $F = \{x_1, x_2, \dots, x_n\}$.

Inequality of Finite Sets

Statement: If $m \neq n$, then $\text{card}\{1, 2, \dots, m\} = \text{card}\{1, 2, \dots, n\}$.

Proof: WLOG, suppose $m > n$.

Suppose toward contradiction that $f : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is our bijection. This means there are m "pigeons" and n "holes."

One hole, j, must contain at least two pigeons (i.e., $f(i) = f(k) = j$ for some $i \neq k \in \{1, 2, \dots, m\}$). Since f is assumed to be injective, this is a contradiction.

Infinitude of the Naturals

Statement: $\mathbb{N}$ is infinite.

Proof: Suppose toward contradiction that $\mathbb{N}$ is finite. Thus, $\exists m \in \mathbb{N}$ such that $f : \mathbb{N} \to \{1, 2, \dots, m\}$ is a bijection.

Consider the inclusion $i : \{1, 2, \dots, m + 1\} \to \mathbb{N}$. i is injective.

Then, $f \circ i : \{1, 2, \dots, m+1\} \to \{1, 2, \dots, m\}$ is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

Infinitude of a Set

Statement: If A is infinite, $\exists i : \mathbb{N} \hookrightarrow A$.

Proof:

$$\begin{aligned} &\exists a_1 \in A && A \neq \emptyset \\ &\exists a_2 \in A \setminus \{a_1\} && A \setminus \{a_1\} \neq \emptyset \\ &\exists a_3 \in A \setminus \{a_1, a_2\} && A \setminus \{a_1, a_2\} \neq \emptyset \\ &\quad\vdots \end{aligned}$$

We thus get a sequence $\{a_1, a_2, \dots\}$ of distinct elements of A.

Consider $f : \mathbb{N} \to A$, $f(n) = a_n$. f is injective as a_n are distinct.

Integers and Power Sets

Cardinality of Integers and Natural Numbers

Statement:

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

Proof:

$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

f is a bijection as $g : \mathbb{N} \to \mathbb{Z},\ g(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor$ is the inverse of f.

Power Set and 2^X

Given any set X, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is the **power set** of X.

$$2^X := \{f \mid f : X \to \{0, 1\}\}.$$

Statement:

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Proof: Let $\varphi : \mathcal{P}(X) \to 2^X$.

For $A \subseteq X$, put $\varphi(A) := \mathbb{1}_A$.

Consider $\psi : 2^X \to \mathcal{P}(X)$. $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$.

Then, $\psi \circ \varphi(A) = \psi(\mathbb{1}_A) = \mathbb{1}^{-1}(\{1\}) = A$,

and, we claim $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbb{1}_{f^{-1}(\{1\})} = f$.

Cantor's Theorem

Statement:

$$\nexists \text{ surjection } \mathbb{N} \to (0,1)$$

Proof: From calculus we know $\forall \sigma \in (0,1)$, σ can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where $\sigma_k \in \{0, 1, \dots, 9\}$ and not terminating in 9s.

Suppose toward contradiction that $\exists r : \mathbb{N} \to (0,1)$ that is a surjection. Write $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, and $\sigma_j(n) \in \{0, 1, \dots, 9\}$, and not terminating in 9s.

Consider $\tau : \mathbb{N} \to \{0, 1, \dots, 9\}$:

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$. Since r is surjective, $\exists m \in \mathbb{N}$ such that $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$.

This implies that $\sigma_m(m) = \tau(m)$, which is definitionally not true, which is our contradiction.

Comparing Cardinality

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example, $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$ because $i : X \hookrightarrow Y, i(x) = x$ is an injection.

Since the composition of two injective functions is injective, if $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.
Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

Cardinality of the Power Set

Statement: For any set A, $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Proof: Let us construct a function: $f : A \to \mathcal{P}(A)$, where $a \mapsto \{a\}$.

f is injective, as if $\{a\} = \{a'\}$, $a = a'$. So, $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$.

Claim: $\nexists g : A \to \mathcal{P}(A)$, g is surjective.

> Suppose toward contradiction that such a g exists. Consider $S : \{a \in A \mid a \notin g(a)\}$.
>
> Since g is onto, $\exists a_0 \in A$ with $g(a_0) = S$. $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$. $\perp$

Equivalent Cardinality Comparisons

(i) $\text{card}(A) \leq \text{card}(B)$

(ii) $\exists f : A \hookrightarrow B$

(iii) $\exists g : B \to A$, g surjection.

Proof:

(ii) $\Rightarrow$ (iii) If $f : A \hookrightarrow B$, f is left-invertible, and thus $\exists g : B \to A$ with $g \circ f = id_A$. So, g is right-invertible, so g is surjective.

(iii) $\Rightarrow$ (ii) If $g : B \to A$ is surjective, then g is right-invertible, so $\exists f : A \to B$ such that $g \circ f = id_B$. So, f is left-invertible, so f is injective.

From the above, we can see that, if $f : A \to B$ is any map, $\text{card}(f(A)) \leq \text{card}(A)$, by considering $g : A \to f(A)$ defined as $g(a) = f(a)$, which is onto, meaning $\exists$ an injection $f(A) \hookrightarrow A$.

Cardinality Rules

(i) $\text{card}(A) \leq \text{card}(A)$ (Reflexivity)

(ii) $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$ (Transitivity)

(iii) $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$ (Cantor-Schröder-Bernstein)

(iv) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

Proof of (iii): We have injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.

Let $A_0 \setminus \text{ran}(g)$. Let $A_1 = g \circ f(A_0)$. Note that $A_0 \cap A_1 = \emptyset$. Let $A_2 = g \circ f(A_1)$. Note that $A_0 \cap A_2 = \emptyset$.

Claim: We claim $A_1 \cap A_2 = \emptyset$. If $\exists z \in A_1 \cap A_2$, then $z = g(f(x_0))$ for some $x_0 \in A_0$, and $z = g(f(x_1))$ where $x_1 \in A_1$. However, g and f are injective, so $g \circ f$ is injective, so $x_0 = x_1$, but $A_0 \cap A_1$. $\perp$

We let $A_n = g \circ f(A_{n-1})$ for arbitrary n, and $A_\infty = \bigcup_{n \geq 0} A_n$. If $a \notin A_\infty$, then $a \notin A_0$, so $a \in \text{ran}(g)$. Define $h : A \to B$.

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where y_x is the unique element in B with $g(y_x) = x$.

Claim: We claim h is the desired bijection.

Proof of Injection: Suppose $h(x_1) = h(x_2)$.

If $x_1, x_2 \in A_\infty$, then by the definition of H, $f(x_1) = f(x_2)$, f is injective, so $x_1 = x_2$.

Suppose $x_1, x_2 \notin A_\infty$. Then, by definition, $h(x_1) = y_{x_1}$ and $h(x_2) = y_{x_2}$, then $g(y_{x_1}) = g(y_{x_2})$, so $x_1 = x_2$.

WLOG, suppose $x_1 \in A_\infty$, and $x_2 \notin A_\infty$. $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$. Then, $g(f(x_1)) \in A_\infty = g(y(x_2)) = x_2 \notin A_\infty$. This case is not possible.

Thus, h is injective.

Proof of Surjection: Let $y \in B$. Set $x := g(y)$.

Suppose $x \notin A_\infty$. Then, $h(x) = y_x$, where y_x is the unique element in B with $g(y_x) = x = g(y)$, so $y = y_x$, so $h(x) = y$.

If $x \in A_\infty$. We know that $x \notin A_0$, as $x \in \text{ran}(g)$. So, $x = g(f(z))$ for some $z \in A_{m-1}$. Since g is injective, $y = f(z),\ z \in A_\infty$. Thus, $h(z) = f(z) = y$.

Cardinality of Canonical Sets

Consider the map $q : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, q(m,n) = \frac{m}{n}$. This map is *not* injective, as $2/4 = 1/2$. However, it is surjective, meaning $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$.

Earlier, we showed that $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$. Consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $H(m,n) = (h(m), n)$.

Claim: H is a bijection.

Proof of Injection: If $H(m_1, n_1) = H(m_2, n_2)$, then $h(m_1) = h(m_2)$, and $n_1 = n_2$, and since h is bijective, $m_1 = m_2$, and $n_1 = n_2$, so $(m_1, n_1) = (m_2, n_2)$.

Proof of Surjection: Let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. We want to find $(m,n) \in \mathbb{Z} \times \mathbb{N}$ such that $H(m,n) = (k, \ell)$. Set $n = \ell$, and since h is surjective, set $m \in \mathbb{Z}$ such that $h(m) = k$.

Therefore $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$.

We claim that $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. First, we need to find $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$. Consider $\varphi(m,n) = 2^m \cdot 3^n$. By the Fundamental Theorem of Arithmetic, φ is injective.

Bringing together our inequalities, we have:

$$\begin{aligned}\text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\ &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\ &= \text{card}(\mathbb{N} \times \mathbb{N}) \\ &\leq \text{card}(\mathbb{N})\end{aligned}$$

Countability and the Continuum Hypothesis

A set X is *countable* if $\exists f : x \hookrightarrow \mathbb{N}$ ($\text{card}(X) \leq \text{card}(\mathbb{N})$). $\text{card}(\mathbb{N}) = \aleph_0$. If X is countable and infinite, X is *denumerable*.

Corollary to Cantor-Schröder-Bernstein

Statement: If X is denumerable, then $\text{card}(X) = \aleph_0$.

Proof: Since X is infinite, $\exists f : \mathbb{N} \hookrightarrow X$. Since X is countable, $\exists g : X \hookrightarrow \mathbb{N}$. By Cantor-Schröder-Bernstein, $\text{card}(X) = \text{card}(\mathbb{N})$, so $\text{card}(X) = \aleph_0$.

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

Countability under Union

Statement: The countable union of countable sets is countable. If I is a countable indexing set and for each $i \in I$, A_i is countable, then $\bigcup\limits_{i\in I} A_i$ is countable.

Proof: Since each A_i is countable, $\exists \pi_i : \mathbb{N} \twoheadrightarrow A_i$. Consider the function

$$\pi : I \times \mathbb{N} \to \bigcup_{i\in I} A_i$$

defined as $\pi(i, j) = \pi_i(j)$.

Claim 1: π is a surjection.

Proof 1: Let $x \in \bigcup_{i\in I} A_i$. $\exists i_0$ such that $x \in A_{i_0}$. Since π_{i_0} is surjective, $\exists k \in \mathbb{N}$ with $\pi_{i_0}(k) = x$. $\pi_{i_0}(k) = \pi(i_0, k)$. Therefore, π is surjective.

Claim 2: $I \times \mathbb{N}$ is countable.

Proof 2: We know $\exists f : I \hookrightarrow \mathbb{N}$ since I is countable. Thus, $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$, $(i, n) \mapsto (f(i), n)$. Recall, $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$, $(m, n) \mapsto 2^m \cdot 3^n$ is an injection. By composing these maps, $I \times \mathbb{N} \hookrightarrow \mathbb{N}$. Since π is onto, and $I \times \mathbb{N}$ is countable, $\bigcup_{i\in I} A_i$ is countable.

Continuum Hypothesis

Statement:

$$\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}}),$$

where I is any non-degenerate interval.

Proof:

Lemma 1: $\text{card}([0, 1]) \leq \text{card}(2^{\mathbb{N}})$.

Proof 1: Every $t \in [0, 1]$ has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where $\sigma_k \in \{0, 1\}$.

Consider $2^{\mathbb{N}} \xrightarrow{\varphi} [0, 1]$, defined as $\phi(f) = \sum\limits_{k=1}^{\infty} \dfrac{f(k)}{2^k}$. Set $f : \mathbb{N} \to \{0, 1\}$, $f(k) = \sigma_k$.

Therefore, φ is surjective, so $\exists\{0, 1\} \hookrightarrow 2^{\mathbb{N}}$, so $\text{card}([0, 1]) \leq 2^{\mathbb{N}}$

Lemma 2: $\text{card}([0, 1]) = \text{card}(\mathbb{R})$.

Proof 2: We have $[0, 1] \overset{i}{\hookrightarrow} \mathbb{R}$ via inclusion, so $\text{card}([0, 1]) \leq \text{card}(\mathbb{R})$.

Also, $\text{card}(\mathbb{R}) = \text{card}((0, 1)) \leq \text{card}([0, 1])$, so by Cantor-Schröder-Bernstein, $\text{card}(\mathbb{R}) = \text{card}([0, 1])$.

Lemma 3: Any two non-degenerate intervals I and J have the same cardinality.

Proof 3: We can create injections $I \hookrightarrow J$ and vice-versa.

Lemma 4: $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0, 1])$.

Proof 4: $\psi : 2^{\mathbb{N}} \to [0,1]$. Where $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$.

ψ is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$

We claim ψ is injective. Suppose $f \neq g$ in $2^{\mathbb{N}}$. Let $k_0 = \min\{k \mid f(k) \neq g(k)\}$. WLOG, $f(k_0) = 0, g(k_0) = 1$. Let $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$, $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$.

Therefore, $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$, and $\psi(g) = \sum_{k=1}^{k_0-1} + \frac{1}{3^{k_0}} + t_g$.

Suppose toward contradiction $\psi(f) = \psi(g)$. Then, $t_f = \frac{1}{3^{k_0}} + t_g$, or $t_f - t_g = \frac{1}{3^{k_0}}$.

$$\begin{aligned}
|t_f - t_g| &= |\sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k}| \\
&\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\
&\leq \sum_{k>k_0} \frac{1}{3^k} \\
&= \frac{(1/3)^{k_0+1}}{1-(1/3)} \\
&= \frac{1}{2} \cdot \frac{1}{3^{k_0}}
\end{aligned}$$

$\perp$

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0,1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than $\aleph_0$ and strictly less than $2^{\aleph_0}$ is an axiom (i.e., it can be an assumption or not).

Field Ordering

Ordering Relations

Let X be a non-empty set. A relation on X is a subset of $X \times X$.

- R is *reflexive* if $\forall x \in X,\ (x,x) \in R$.
- R is *transitive* if $(x,y),(y,z) \in R \to (x,z) \in R$.
- If R is *antisymmetric* $(x,y),(y,x) \in R \to x = y$.

If R is reflexive, transitive, and antisymmetric, then R is an *ordering* of X.

If R is an ordering of X, then we write:

$$(x, y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \ \forall x \in X$
- $x \leq_R y,\ y \leq_R z \to x \leq_R z$
- $x \leq_R y,\ y \leq_R x \to x = y$

Additionally, $x <_R y$ means $x \leq_R y$ and $x \neq y$.

Examples of Orderings

Algebraic Ordering of $\mathbb{N}_0$: $n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0$ such that $n + k = m$

$\mathbb{N}$ ordered via division: $n \leq_D m \Leftrightarrow n|m$; under this definition, it is false that $2 \leq_D 5$, but it is true that $4 \leq_D 20$.

Inclusion: Let S be any set, and let $X = \mathcal{P}(S)$. For $A, B \in \mathcal{P}(S)$, we define $A \leq_i B \Leftrightarrow A \subseteq B$.

Containment: With X defined as above, $A \leq_c B \Leftrightarrow A \supseteq B$.

Functions: For $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \to \mathbb{R}\}$, we can define $f \leq g \Leftrightarrow f(x) \leq g(x) \ \forall x \in X$.

Total and Directed Orderings

- An ordering $\leq$ of X is *total* or *linear* if $\forall x, y \in X, x \leq y$ or $y \leq x$.
- An ordering is *directed* if $\forall x, y \in X \ \exists z \in X$ such that $x \leq z$ and $y \leq z$.

If X is a totally ordered set, X is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D),\ (\mathcal{P}(S), \leq_i),\ (\mathcal{P}(S), \leq_c)$$

Upper and Lower Bounds

(i) Let $(X, \leq)$ be an ordered set, $A \subseteq X$. A is bounded above if $\exists v \in X$ with $a \leq v \ \forall a \in A$. Such a v is an upper bound.

(ii) A is bounded below if $\exists \ell \in X$ such that $a \geq \ell \ \forall a \in A$. Such a w is a lower bound.

(iii) If v is an upper bound of A and $v \in A$, then v is the greatest element of A, or $\max(A) = v$.

(iv) If ℓ is a lower bound for A and $\ell \in A$, then ℓ is the least element of A, or $\min(A) = \ell$.

(v) If u is an upper bound for A, and $u \leq v$ for all other upper bounds v of A, then u is the *least upper bound* of A, or $\sup(A) = u$ (for *supremum*).

(vi) If ℓ is a lower bound for A, and $\ell \leq g$ for all other lower bounds g of A, then ℓ is the *greatest lower bound* of A, or $\inf(A) = \ell$ (for *infimum*).

(vii) If A is bounded above and below, then A is bounded.

An ordered set $(X, \leq)$ is *complete* if for all $A \subseteq X$ bounded, $\inf(A)$ and $\sup(A)$ exist.

For example, $\mathbb{Q}$ is *not* complete, as there is not a largest rational number less than $\sqrt{2}$, for example.

Well-Ordering Principle: With $(\mathbb{N}, \leq_a)$, every nonempty $A \subseteq \mathbb{N}$ has a least element.

Examples

For $A \subseteq (\mathbb{N}, \leq_a)$, $A = \{2, 3, \dots, 12\}$, we have the following:

Bounded Above? Yes.

Upper Bounds $12, 13, 14, \dots$

Greatest Element 12

For $A \subseteq (\mathbb{N}, \leq_D)$, $A = \{2, 3, \dots, 10\}$

Bounded Above? Yes.

Upper Bounds $10!$

Greatest Element? No.

Supremum $2^3 \cdot 3^2 \cdot 5 \cdot 7$

Bounded Below? Yes.

Lower Bound 1

Least Element? No.

Infimum 1

For $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$, $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$.

Supremum $\bigcup_{i \in I} A_i$

Infimum $\bigcap_{i \in I} A_i$

Ordering of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0,\ n + k = m$$

This defines a total and complete ordering.

Define $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

Properties of $\mathbb{Z}^+$

(i) $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+,\ m \cdot n \in \mathbb{Z}^+$

(ii) $m \in \mathbb{Z}$, then $m \in \mathbb{Z}^+$ or $-m \in \mathbb{Z}^+$

(iii) $m, -m \in \mathbb{Z}^+$, then $m = 0$

(iv) $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

Statement:

(1) $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$

(2) $m \leq_a n$ and $p \leq_a q \Rightarrow m + p \leq_a n + q$

(3) $m \leq_a n$ and $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$

(4) $m \leq_a n \Rightarrow -m\ {}_a{\geq} n$

(5) $\leq_a$ is total.

(6) If $a\ {}_a{>}0$, and $ab\ {}_a{\geq}0$, then $b\ {}_a{\geq}0$

(7) If $a > 0$ and $ab\ {}_a{\geq}ac$, then $b \geq c$.

Proof of (3):

$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0$ with $m + k = n$.
$\Rightarrow pm + pk = pn$
$pk \in \mathbb{N}_0$ by the properties of $\mathbb{Z}^+$. So, $pm \leq_a pn$

Proof of (5):

Let $m, n \in \mathbb{Z}$. Consider $m - n$.
By (ii), $m - n \in \mathbb{Z}^+$ or $-(m - n) \in \mathbb{Z}^+$. Thus, $m - n = k$ for some $k \in \mathbb{Z}^+$, or $-(m - n) = \ell$ for some $\ell \in \mathbb{Z}^+$.
Thus, $n \leq_a m$ in the first case, or $m \leq_a n$ in the second case.

Creating the Rationals

Recall that $Q = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z},\ b \in \mathbb{Z}^+,\ b \neq 0\}$. Consider the equivalence relation:

$$(a, b) \sim (c, d) \overset{\text{def}}{\iff} ad = bc$$

We will let $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in Q\}$ be the set of all equivalence classes in Q. We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined: $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{a'd' + c'b'}{b'd'} = \frac{ad+bc}{bd}$.

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make $\mathbb{Q}$ a **field**.

Fields

A ring is a nonempty set set R equipped with two binary operations:

- $+ : R \times R \to R$, $(a, b) \mapsto a + b$ ("addition")
- $\cdot : R \times R \to R$, $(a, b) \mapsto a \cdot b$ ("multiplication")

such that the following hold:

(1) $(a + b) + c = a + (b + c)$

(2) $\exists z \in R$ such that $a + z = a = z + a\ \forall a \in R$; there is at most one such z. Set $z = 0_R$.

(3) $\forall a \in R, \exists b \in R$ such that $a + b = 0_R = b + a$; there is at most one such b. Set $b = -a$.

(4) $\forall a, b \in R,\ a + b = b + a$.

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If $(R, +, \cdot)$ satisfies $ab = ba$, R is a commutative ring.

If there exists $u \in R$ such that $ua = au = a\ \forall a \in R$, R is a unital ring; there is at most one unit. Set $u = 1_R$

An integral domain is a unital, commutative ring such that $ab = 0 \Rightarrow a = 0 \vee b = 0$. For example, $\mathbb{Z}$ is an integral domain. However, $c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ is a unital, commutative ring, but there exist two functions such that $f, g \neq \mathbf{o}$, but $f \cdot g = \mathbf{o}$.

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R, \text{ with } ab = 1_R$$

There is only one such b. Set $b = a^{-1}$.

Ordering of $\mathbb{Q}$

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

$\leq$ is a well-defined total ordering of $\mathbb{Q}$, and $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$, $j(n) = \frac{n}{1}$ is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

Properties of $\mathbb{Q}^+$

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

(i) $q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+,\ q_1 q_2 \in \mathbb{Q}^+$

(ii) $q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$

(iii) $\pm q \in \mathbb{Q}^+, q = 0$

(iv) $x \leq y, u \leq v \Rightarrow x + u \leq y + v$

(v) $x \leq y,\ 0 \leq z \Rightarrow zx \leq zy$

Ordered Fields and the Ordering of $\mathbb{R}$

An **ordered field** is a field F equipped with a total ordering $\leq_F$ such that:

(i) if $s \leq_F t$, and $x \leq_F y$, then $s + x \leq_F t + y$

(ii) if $s \leq_F t$ and $0 \leq_F z$, then $zs \leq_F zt$

For example, $\mathbb{Q}$ with its ordering is an ordered field.

Statement: If $(F, \leq_F)$ is an ordered field, we define $F^+ = \{x \in F \mid x\ {}_F{\geq} 0\}$ with the following properties:

(1) $x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$

(2) $x \in F \Rightarrow x \in F^+, -x \in F^+$

(3) $\pm x \in F^+ \Rightarrow x = 0_F$

Proofs:

(1) Let $x, y \in F^+$. Then, $x \geq 0$ and $y \geq 0$, so by property (i) of an ordered field, $x + y \geq 0 + 0 = 0$, so $x + y \in F^+$. Additionally, we have $x \cdot y \geq x \cdot 0 = 0$, so $xy \in F^+$.

(2) Let $x \in F$. Since the ordering on F is total, $x \geq 0$ or $0 \geq x$. In the first case, $x \in F^+$. In the second case, we add $-x$ to both sides, so by (i), $-x \geq 0$, so $-x \in F^+$.

(3) We have $x \geq 0$ and $-x \geq 0$. So $x \geq 0$ and $x + (-x) \geq x + 0$, so $x \geq 0$ and $0 \geq x$. So, $x = 0$ by antisymmetry.

Note: $x \leq_F y \Leftrightarrow y - x \in F^+$.

Statement: Let F be an ordered field. Then, the following is true:

(1) $\forall a \in F$, $a^2 \in F^+$

(2) $0, 1 \in F^+$

(3) If $n \in \mathbb{N}$, $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$

(4) If $x \in F^+$, and $x \neq 0$, then $x^{-1} \in F^+$

(5) If $xy > 0$, then $x, y \in F^+$, or $-x, -y \in F^+$

(6) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$

(7) If $x \leq y$, then $-y \leq -x$

(8) $x \geq 1 \Rightarrow x^2 \geq x \geq 1$, and $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$.

Proof:

(1) Let $a \in F$. Then, $a \in F^+$ or $-a \in F^+$.

Case 1 If $a \in F^+$, then by the previous proposition, $a^2 \in F^+$.

Case 2 If $-a \in F^+$, then by the previous proposition, $(-a)(-a) = a^2 \in F^+$.

(2) $0 \geq 0$, so $0 \in F+$. $1 = 1 \cdot 1 = 1^2 \in F^+$ by the previous result.

(3) $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots 1_F}_{n \text{ times}} \in F^+$ by the previous proposition.

(4) Let $x \neq 0, x \in F^+$. Suppose toward contradiction that $x^{-1} \notin F^+$, then $-x^{-1} \in F^+$. Thus, $x \cdot (-x^{-1}) \in F^+$, so $-1 \in F^+$, but $1 \in F^+$, so $1 = 0$. $\perp$

(5) Let $xy > 0$, meaning $xy \in F^+$. Suppose toward contradiction that $x > 0$ and $y < 0$. So, $x > 0$ and $-y > 0$, so $(x)(-y) > 0$, so $-(xy) \in F^+0$, so $xy = 0$. $\perp$

(6) Let $0 < x \leq y$. We know $x^{-1} \in F^+$, so $x^{-1}x \leq x^{-1}y$. So $1 \leq x^{-1}y$. We also know $y \in F^+$, so $y^{-1} \in F^+$. So, $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$.

(7) Let $x \leq y$. Then, $0 \leq y - x$, so $-y \leq -x$.

(8) Let $x \geq 1$. Then, $x \cdot x \geq 1 \cdot x \geq 1$.

Order Axiom: $\mathbb{R}$ is an ordered field. The injection $\mathbb{Q} \hookrightarrow \mathbb{R}$, $i(q) = q$ is an order embedding.

Rational Orderings

Statement: If $a \leq b$, then $a \leq \frac{1}{2}(a+b) \leq b$.

Proof: $2a = a + a \leq a + b \leq b + b$, all by property (i) of an ordered field.

Therefore, $2a \leq a + b \leq 2b$. Since $2 = 1 + 1 \in \mathbb{R}^+$, $2^{-1} \in \mathbb{R}^+$, so $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$, so $a \leq \frac{1}{2}(a+b) \leq b$.

Statement: If $a \geq 0$ and $(\forall \varepsilon > 0), a \leq \varepsilon$, then $a = 0$.

Proof: Suppose toward contradiction that $a \geq 0$ and $a \neq 0$, so $a > 0$. So, we have that $\frac{1}{2}a < a$. Let $\varepsilon = \frac{1}{2}a$. We also have that $a \leq \varepsilon = \frac{1}{2}a < a$, so $a < a$. $\perp$

Important Inequalities

Arithmetic and Geometric Means

Given $a_1, a_2, \dots, a_n \in \mathbb{R}^+$:

Arithmetic Mean

$$= \frac{\sum_{i=1}^{n} a_i}{m}$$

Geometric Mean

$$= \sqrt[m]{a_1 a_2 \cdots a_m}$$

Arithmetic Mean-Geometric Mean Inequality

Statement: Let $a, b \geq 0$.

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

Proof: If $x, y \geq 0$, $x \leq y \Leftrightarrow x^2 \leq y^2$.

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y \qquad \text{by property (ii) of ordered fields}$$

Therefore,

$$\begin{aligned}
(ab)^{1/2} &\leq \frac{1}{2}(a+b) \\
ab &\leq \frac{1}{4}(a^2 + 2ab + b^2) \\
4ab &\leq a^2 + 2ab + b^2 \\
0 &\leq a^2 - 2ab + b^2 \\
0 &\leq (a-b)^2 && \text{by definition}
\end{aligned}$$

Challenge: Prove for m.

Remark: The harmonic mean is defined as:

$$\frac{n}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i}}$$

Bernoulli's Inequality

Statement: If $x \geq -1$, then $(1+x)^n \geq 1 + nx$, for any $n \in \mathbb{N}_0$.

Proof: By induction, we know that for $n = 0$ and $n = 1$, this holds.

Assume the inequality holds for some $m \geq 1$.

$$\begin{aligned}
(1+x)^{m+1} &= (1+x)^m(1+x) \\
&\geq (1+mx)(1+x) && \text{by the inductive hypothesis} \\
&= 1 + x + mx + mx^2 \\
&= 1 + (m+1)x + mx^2 \\
&\geq 1 + (m+1)x
\end{aligned}$$

Cauchy's Inequality

Statement: Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. Then

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

In linear algebra language, this is equivalent to $\|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\| \cdot \|\vec{w}\|$.

Proof: Consider $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^{n} (a_j - b_j x)^2$$

We know that $f(x) \geq 0$ for all $x \in \mathbb{R}$

$$\begin{aligned}
&= \sum_{i=1}^{n} (a_j^2 - 2a_j b_j x + b_j^2 x^2) \\
&= \left(\sum_{j=1}^{n} b_j^2\right) x^2 + \left(\sum_{j=1}^{n} -2a_j b_j\right) x + \sum_{j=1}^{n} a_j^2 \\
&= Ax^2 + Bx + C
\end{aligned}$$

Therefore, $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned}
\left(-2\sum_{j=1}^{n} a_j b_j\right)^2 &\leq 4\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right) \\
\left|\sum_{j=1}^{n} a_j b_j\right| &= \left(\sum_{j=1}^{n} a_j\right)^{1/2}\left(\sum_{j=1}^{n} b_j\right)^{1/2}
\end{aligned}$$

As we know from linear algebra, the way we get equality is when $\vec{v} = c\vec{w}$, or that $a_j = cb_j \ \forall j$ for some $c \in \mathbb{R}$.

Triangle Inequality

Statement: Given $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left(\sum_{j=1}^{n}(a_j + b_j)^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

In linear algebra, this is equivalent to $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

Proof:

$$\begin{aligned}
\sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\
&\leq \sum a_j^2 + 2\left(\sum a_j^2\right)^{1/2}\left(\sum b_j^2\right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\
&= \left(\left(\sum a_j^2\right)^{1/2} + \left(\sum b_j^2\right)^{1/2}\right)^2
\end{aligned}$$

we take square roots to get our end result

Metrics, Norms, and Bounds

Metrics and Norms on $\mathbb{R}^n$

Consider $|\cdot| : \mathbb{R} \to \mathbb{R}$, defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

Theorems about Absolute Value:

(i) $|ab| = |a||b|$
(ii) $|a^2| = |a|^2$
(iii) $|-a| = |a|$
(iv) $|a| \in \mathbb{R}^+$
(v) $-|a| \leq a \leq |a|$
(vi) $|a| \leq \delta \Rightarrow -\delta \leq a \leq \delta$ for $\delta > 0$
(vii) $|a + b| \leq |a| + |b|$, $|a - b| \leq |a| + |b|$, $||a| - |b|| \leq |a - b|$

Proof of (i):

Case 1: If $a, b \in \mathbb{R}^+$, then $|a| = a$, and $|b| = b$, and $ab \in \mathbb{R}^+$, so $|ab| = ab$

Case 2: If $a, b \notin \mathbb{R}^+$, then $|a| = -a$, and $|b| = -b$. Additionally, $(-a)(-b) = ab \in \mathbb{R}^+$, so $|ab| = ab$. The LHS $= ab$, and the RHS $= ab$.

Case 3: $a \in \mathbb{R}^+$, $-b \in \mathbb{R}^+$. Then, $|a||b| = (a)(-b) = -ab$. Then, since $a(-b) \in \mathbb{R}^+$, $-ab \in \mathbb{R}^+$, so $|ab| = -ab$. Therefore, the LHS and RHS are equal.

Proof of (vii): Having established that $|a + b| \leq |a| + |b|$, we will show that $||a| - |b|| \leq |a - b|$.

$$\begin{aligned}
|a| &= |a - b + b| \\
&\leq |a - b| + |b| \\
|a| - |b| &\leq |a - b|
\end{aligned}$$

Similarly, by exchanging a for b

$$\begin{aligned} |b| - |a| &\leq |b - a| \\ |b| - |a| &\leq |a - b| \end{aligned}$$

Let $t = |a| - |b|$. We have shown that

$$\begin{aligned} \pm t &\leq |a - b| \\ -|a - b| \leq t &\leq |a - b| \\ |t| &\leq |a - b| \end{aligned}$$

Bounded Sets

A subset $A \subseteq \mathbb{R}$ is **bounded** $\Leftrightarrow \exists c \geq 0$ such that $\forall x \in A,\ |x| \leq c$.

($\Rightarrow$) Suppose $A \subseteq \mathbb{R}$ is bounded. Then, $\exists \ell, u \in \mathbb{R}$ such that $\ell \leq x \leq u\ \forall x \in A$. Let $c := \max\{|\ell|, |u|\}$.

Since $|u| \leq c$, we have that $x \leq c$.

Since $|\ell| \leq c$, and $-|\ell| \leq x$, we get that $-x \leq |\ell| \leq c$.

Since $x \leq c$ and $-x \leq c$, $|x| \leq c$.

($\Leftarrow$) If such a c exists, then $|x| \leq c$ if and only if $-c \leq x \leq c$. Thus, $-c$ is a lower bound and c is a upper bound.

Bounded Functions

Let D be any set. A function $f : D \to \mathbb{R}$ is bounded if $\text{Ran}(D) \subseteq \mathbb{R}$ is bounded.
For example, let $f : [3, 7] \to \mathbb{R}$, $f(x) = \frac{x^2+2x+1}{x-1}$. We will show that f is bounded.

$3 \leq x \leq 7 \Rightarrow 2 \leq x - 1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}$.

Also, $4 \leq x + 1 \leq 8 \Rightarrow 16 \leq x^2 + 2x + 1 \leq 64 \Rightarrow |x^2 + 2x + 1| \leq 64$.

So, $|f(x)| \leq 32$.

Distance Metrics

For $s, t \in \mathbb{R}$, we will define $d(s, t) = |s - t|$ to be the **distance** between s and t.

Properties:

(i)

$$\begin{aligned} d :& \mathbb{R} \times \mathbb{R} \to [0, \infty) \\ & (s, t) \mapsto d(s, t) \geq 0 \end{aligned}$$

(ii) $d(s, t) = d(t, s)$

(iii) $d(s, r) \leq d(s, t) + d(t, r)$

(iv) $d(s, s) = 0$

(v) If $d(s, t) = 0$, then $s = t$.

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^{n} |x_j|$$

- ∞-norm:

$$\|v\|_\infty = \max_{j=1}^{n} |x_j|$$

- 2-norm:

$$\|v\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

Properties of Norms

Statement: With v, w above, let $p = 1, 2, \infty$. The following are true:

(1) $\|v\|_p \geq 0$

(2) $\|v + w\|_p \leq \|v\|_p + \|w\|_p$

(3) $\|\vec{0}\|_p = 0$

(4) $\|v\|_p = 0 \Rightarrow v = \vec{0}$

(5) $\forall t \in \mathbb{R}$, $\|tv\|_p = |t|\|v\|_p$

Proofs: Let $p = \infty$. We will prove (2).

Say $\|v\|_{infty} = |x_i|$ and $\|w\|_\infty = |y_k|$. We want to show that $\|v + w\|_\infty = \max_{j=1}^{n} |x_j + y_j| \leq |x_i| + |y_k|$.

Note that $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| && \text{Triangle Inequality} \\ &\leq |x_i| + |y_k| \\ &= \|v\|_\infty + \|w\|_\infty \end{aligned}$$

Therefore, $\|v + w\|_\infty \leq \|v\|_\infty + \|w\|_\infty$.

Relating Distance Metrics and Norms

A **norm** on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$, $v \mapsto \|v\|$, satisfying the following properties for $v \in \mathbb{R}^n$:

(1) $\|v\| \geq 0$

(2) $\|v + w\| \leq \|v\| + \|w\|$

(3) $\|\vec{0}\| = 0$

(4) $\|v\| = 0 \Rightarrow v = \vec{0}$

(5) $\forall t \in \mathbb{R},\ \|tv\| = |t|\|v\|$

If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$ is a norm, we define $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$, defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for $v, w \in \mathbb{R}^n$.

The properties of distance in $\mathbb{R}$ still hold for distance in $\mathbb{R}^n$:

(1) $d(v, w) = d(w, v)$

(2) $d(u, w) \leq d(u, v) + d(v, w)$

(3) $d(v, v) = 0$

(4) $d(v, w) = 0 \Rightarrow v = w$

Metric Spaces

A **metric space** is a nonempty set X equipped with a function $d : X \times X \to \mathbb{R}^+$, $(x, y) \mapsto d(x, y) \geq 0$. The metric has the following properties:

(1) $d(x, y) = d(y, x)\ \forall x, y \in X$

(2) $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$

(3) $d(x, x) = 0$

(4) $d(x, y) = 0 \Leftrightarrow x = y$

The map d is called a *metric* on X.

Examples of Metric Spaces

- $\mathbb{R}$ with $d(x, y) = |x - y|$.
- $\mathbb{R}^n$ with the *Euclidean metric*:

$$\begin{aligned} d_2(v, w) &= \|v - w\|_2 \\ &= \left(\sum_{j=1}^{n}(x_j - y_j)^2\right)^{1/2} \end{aligned}$$

- $\mathbb{R}^n$ with the 1-norm:

$$\begin{aligned} d_1(v, w) &= \|v - w\|_1 \\ &= \sum_{j=1}^{n} |x_j - y_j| \end{aligned}$$

- $\mathbb{R}^n$ with the ∞-norm:

$$\begin{aligned} d_\infty(v, w) &= \|v - w\|_\infty \\ &= \max_{j=1}^{n} |x_j - y_j| \end{aligned}$$

Open and Closed Sets in Metric Spaces

Let (X, d) be a metric space.

(1) The **open ball** centered at $x_0 \in X$ with radius δ is:

$$V(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

(2) The **closed ball** centered at $x_0 \in X$ with radius δ is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

(3) A set $U \subseteq X$ is **open** if $\forall x \in U$, $\exists \delta > 0$ such that $V(x, \delta) \subseteq U$.

(4) A set $C \subseteq X$ is **closed** if $\overline{C} = X - C \subseteq X$ is open.

For example,

In $\mathbb{R}$ with $d(s, t) = |s - t|$:

$$\begin{aligned} V(x_0, \delta) &= \{y \in \mathbb{R} \mid d(y, x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0, \delta) &= [x_0 - \delta, x_0 + \delta] \end{aligned}$$

The interval $A = [1, \infty)$ is not open, as $\forall \delta > 0$, $U(1, \delta) \not\subseteq [1, \infty)$.

However, A is closed, as $\overline{A} = (-\infty, 1)$ is open: given $t \in \overline{A}$, choose $\delta = 1 - t$. Let $s \in V_\delta(t)$. Then, $s \in (t - \delta, t + \delta)$, so $s \in (t - (1 - t), t + (1 - t))$, or $s \in (2t - 1, 1)$, so $s < 1$.

In $(\mathbb{R}^2, d_2)$, $B(0_{\mathbb{R}^2}, 1)$ is the **unit disc** centered at $(0, 0)$.

However, in $(\mathbb{R}^2, d_\infty)$:

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

Supremum, Infimum, and Completeness

Finding a Supremum

Statement: Let $0 \neq A \subseteq \mathbb{R}$. Let $u \in \mathbb{R}$ be an upper bound for A. The following are equivalent:

(i) $u = \sup(A)$

(ii) If $t < u$, then $\exists a_t \in A$ such that $a_t > t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $u - \varepsilon < a_\varepsilon$

Proof:

(i) $\Rightarrow$ (ii): Given $t < u$, if no such $a \in A$ with $t < a$ exists, then $a \leq t$ $\forall a \in A$. Thus t would be an upper bound. However, $t < u$ and u is the supremum of A. $\perp$

(ii) $\Rightarrow$ (iii): Given $\varepsilon > 0$, set $t = u - \varepsilon < u$. So, by (ii), $\exists a_t$ with $t < a_t$. Thus, $u - \varepsilon \leq a_t$. Set $a_\varepsilon = a_t$.

(iii) $\Rightarrow$ (i): Let v be an upper bound for A. Suppose $v < u$. Then, set $\varepsilon = u - v > 0$. By (iii), $\exists a_\varepsilon \in A$ with $u - \varepsilon < a_\varepsilon$. So $u - (u - v) < a_\varepsilon$, so $v < a_\varepsilon$, meaning v cannot be an upper bound. $\perp$

Supremum Example

$\sup[0,1) = 1$: Certainly, 1 is an upper bound for $[0,1)$. Let $\varepsilon > 0$.

> If $\varepsilon \geq 1$, pick $t = \frac{1}{2}$. Then, $1 - \varepsilon < 0 < \frac{1}{2}$
>
> If $0 < \varepsilon < 1$, let $t = (1-\varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$. Then, $t \in [0,1)$, and $1 - \varepsilon < 1 - \varepsilon/2 = t$

Finding an Infimum

Let $\emptyset \neq A \subseteq \mathbb{R}$. Let $\ell \in \mathbb{R}$ be a lower bound for A. The following are equivalent:

(i) $\ell = \inf(A)$

(ii) If $t > \ell$, $\exists a_t$ such that $t > a_t$

(iii) $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$ with $\ell + \varepsilon > a_\varepsilon$

Infimum Example

$\inf \left\{ \frac{1}{n} \mid n \geq 1 \right\}$: Clearly, $0 < \frac{1}{n}\ \forall n \geq 1$. Let $\varepsilon > 0$.

> We need to find $a \in \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ with $\varepsilon > a$. By the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Let $a_\varepsilon = \frac{1}{m}$.

Properties of Supremum and Infimum

- If $A \subseteq \mathbb{R}$ and $\max(A) = u$, then $u = \sup(A)$: u is an upper bound of A by the definition of max, and if $v \neq u$ is any upper bound of A, then $u < v$ since $u \in A$.
- If $\min(A) = \ell$, then $\ell = \inf(A)$ (by the same logic).
- If A is not bounded above, $\sup(A) = +\infty$, and if A is not bounded below, then $\inf(A) = -\infty$.
- If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.
- If $A \subseteq B$, then $\inf(A) \geq \inf(B)$: Let $\ell_A = \inf(A)$ and $\ell_B = \inf(B)$. By definition, $\ell_B \leq b\ \forall b \in B$. Since $A \subseteq B$, $\ell_B \leq a\ \forall a \in A$. Thus, ℓ_B is a lower bound for A. By definition of ℓ_A, $\ell_B \leq \ell_A$.

Let $A, B \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Then, the following are also sets:

(1) $A + B = \{a + b \mid a \in A, b \in B\}$

(2) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$

(3) $t \cdot A = \{ta \mid a \in A\}$

(4) $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

Completeness Axiom

If $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above, then $\sup(A)$ exists.

Well-Ordering Property: if $\emptyset \neq S \subseteq \mathbb{N}$, then $\min(S)$ exists.

Therefore, we can prove that if $F \subseteq \mathbb{Z}$ is bounded, then F has a least and greatest element.

Archimedean Property

Statement: If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ such that $x \leq n_x$.
Proof: Suppose there exists no natural number greater than x, then $\mathbb{N}$ is bounded above by X. Let $u = \sup(\mathbb{N})$. By the Completeness Axiom, $u \in \mathbb{R}$ exists. Let $\varepsilon = 1$. Then, $\exists n \in \mathbb{N}$ with $u - 1 < n$. So, $u < n + 1$, but $n + 1 \in \mathbb{N}$, so u cannot be an upper bound.

Corollary to the Archimedean Property

$$\forall t > 0 \ \exists n \in \mathbb{N} \text{ such that } \frac{1}{n} < t$$

Corollary to the Corollary: Powers of 2

Statement:

$$\forall t > 0 \ \exists m \in \mathbb{N} \text{ such that } \frac{1}{2^m} < t$$

Proof: By the corollary to the Archimedean Property, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < t$. By Bernoulli's inequality with $x = 1$, we have $2^n \geq n$, so $2^{-n} < n^{-1} < t$.

Corollary to the Corollary: In Between Integers

Statement:

$$\forall x \in \mathbb{R} \ \exists n_x \in \mathbb{Z} \text{ such that } n_x - 1 \leq x < n_x$$

Proof: Assume $x \geq 0$. Let $S_x = \{n \mid n \in \mathbb{N} \ x < n\}$.

$S_x \neq \emptyset$ by the Archimedean Property. By the well-ordering property, let $n_x = \min(S_x)$.

Therefore, $x < n_x$. Also, $n_x - 1 \notin S_x$. Therefore $n_x - 1 \leq x$.

Density

Let (X, d) be any metric space. A subset $D \subseteq X$ is **dense** if $\forall x \in X, \ \varepsilon > 0, \ U(x, \varepsilon) \cap D \neq \emptyset$.

In the case of $X = \mathbb{R}$ and $d(s, t) = |s - t|$, $D \subseteq \mathbb{R}$ is dense if given any open interval I, $I \cap D \neq \emptyset$.

A metric space is **separable** if it admits a *countable* dense subset.

Density of the Rationals

Statement: $\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Proof: Let $I = (a, b)$ be an open interval. We may assume that $a, b \in \mathbb{R}$ are finite.

Then, $b - a > 0$. By the Archimedean property corollary, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, meaning $1 < nb - na$.

There exists also an integer m such that $m - 1 \leq na < m$, implying that $a\frac{m}{n}$. Also, $m \leq na + 1 < nb$. Therefore, $\frac{m}{n} < b$.

So, $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$.

Density of the Irrationals

Statement: $\mathbb{R} \setminus \mathbb{Q}$ is dense.
Proof: Assume $\sqrt{2}$ exists. Let $I = (a, b)$ be any open interval. Then, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

Find $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$.

Then, $a < q\sqrt{2} < b$, where $q\sqrt{2} \in \mathbb{R}$ and $q\sqrt{2} \notin \mathbb{Q}$.

Uniqueness of $\sqrt{2}$

Statement:

$$\exists! x > 0 \text{ such that } x^2 = 2$$

Proof:

Existence: Let $S = \{t \in \mathbb{R} \mid 0 < t,\ t^2 < 2\}$. S is nonempty because $1 \in S$, and S is bounded above because $y > 2 \Rightarrow y^2 > 4$.

So, by the completeness axiom, $x := \sup(S)$ exists, and $x \geq 1$.

Claim: $x^2 = 2$

Contradiction 1: Assume $x^2 < 2$. We want to show that $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$. By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x + 1) \end{aligned}$$

We want to find $n \in \mathbb{N}$ with:

$$x^2 + \frac{1}{n}(2x + 1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

Therefore, by the Archimedean Property corollary, we know that n must exist.

Contradiction 2: We know that $x^2 \geq 2$. Since $x = \sup(S)$, $\forall m \in \mathbb{N}$, $\exists t_m \in S$ with $x - \frac{1}{m} < t_m$, so $\left(x - \frac{1}{m}\right)^2 < t_m^2 < 2$.

Therefore, $x^2 - \frac{2x}{m} + \frac{1}{m^2}$, so $x^2 - \frac{2x}{m} < 2$, so $0 \leq x^2 - 2 < \frac{2x}{m}$.

So, $0 \leq \frac{x^2 - 2}{2x} < \frac{1}{m}$, so $x^2 - 2 = 0$, so $x^2 = 2$.

Remark: If we had set $S' = \{t' \in \mathbb{Q} \mid t^2 < 2,\ t > 0\}$, we would have still obtained $\sup(S') = \sqrt{2}$. This means that $\mathbb{Q}$ is *not* complete.

Intervals in $\mathbb{R}$

$(*)$ Given any interval I, if $x_1, x_2 \in I$, with $x_1 < x_2$, then $[x_1, x_2] \in I$.

This seems like an obvious property, but this is the *characteristic property* of intervals.

Characterization of Intervals

Statement: Let $S \in \mathbb{R}$ be any nonempty subset of cardinality at least 2. Suppose S satisfies $(*)$. Then, S is an interval.

Proof:

Case 1: Suppose S is bounded.

Let $a = \inf(S)$ and $b = \sup(S)$. Then, $S \subseteq [a, b]$. We will show that $(a, b) \subseteq S$. Once this is shown, $S = \{(a, b), [a, b], [a, b), (a, b]\}$.

Let $t \in (a, b)$. Since $a = \inf(S)$, $\exists x_1 \in S,\ x_1 \in (a, t)$. Similarly, since $b = \sup(S)$, $\exists x_2 \in S,\ x_1 \in (t, b)$.

By the hypothesis, $[x_1, x_2] \subseteq S$. Since $t \in [x_1, x_2]$, $t \in S$.

Case 2: Suppose S is bounded above, but not below.

Let $b = \sup(S)$. Clearly, $S \subseteq (-\infty, b]$. We will show that $(-\infty, b) \subseteq S$. Once this is shown, $S = \{(-\infty, b), (-\infty, b]\}$.

Let $t \in (-\infty, b)$. Since $b = \sup(S)$, $\exists x_2 \in S,\ x_2 \in (t, b)$.

Since S is not bounded below, $\exists x_1 \in S$ such that $x_1 < t$ (or else t would be a lower bound).

By the hypothesis, $[x_1, x_2] \in S$, and $t \in [x_1, x_2]$, so $t \in S$.

Case 3, 4: Left as an exercise for the reader.

Nested Intervals

A sequence of intervals $(I_n)_{n\geq 1}$ is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in $\bigcap I_n$.

(a) $\bigcap_{n=1} [0, 1/n) = \{0\}$.

(b) $\bigcap_{n=1} (0, 1/n) = \emptyset$

(c) $\bigcap_{n=1} [n, \infty) = \emptyset$

Measure

The **measure** of an interval is basically its "size."

(a) $m([a, b]) = b - a$

(b) $m((a, b]) = b - a$

(c) $m((a, b)) = b - a$

(d) $m([a, b)) = b - a$

Nested Intervals Theorem

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ be a nested sequence of intervals.

(1) $\bigcap_{n\geq 1} I_n \neq \emptyset$

(2) If $\inf \{m(I_n) \mid n \geq 1\} = 0$, then $\bigcap_{n\geq 1} I_n = \{\xi\}$

Proof of (1): Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, we have that $a_1 \leq a_2 \leq a_3, \dots$, and $b_1 \geq b_2 \geq b_3 \geq \cdots$.

We know that $\{a_n\}$ is bounded above and nonempty. Let $\xi = \sup(\{a_n\}_{n=1}^{\infty})$.

We know that $\{b_n\}$ is bounded below. Let $\eta = \inf(\{b_n\}_{n=1}^{\infty})$.

We claim that $\xi \leq b_n \ \forall n \geq 1$. Suppose toward contradiction that $\exists m$ such that $\xi > b_m$. Then, by the supremum property, $\exists a_i$ such that $\xi > a_i > b_m$. If $k \leq m$, $a_k \leq a_m \leq b_m < a_k$. If $m \leq k$, then $b_m \geq b_k \geq a_k > b_m$. $\perp$

Similarly, using the same argument, $a_n \leq \eta \ \forall n$.

Thus, $\xi \leq \eta$.

We claim that $\bigcap_{n\geq 1} I_n = [\xi, \eta]$. If $t \in [\xi, \eta]$, then $a_n \leq \xi \leq t \leq \eta \leq b_n$. So $t \in [a_n, b_n] \ \forall n$, so $t \in \bigcap_{n\geq 1}[a_n, b_n]$.

If $t \in \bigcap_{n\geq 1} I_n$, then $t \in [a_n, b_n] \ \forall n$, so $a_n \leq t \leq b_n \ \forall n$. So, t is an upper bound on a_n, and a lower bound on b_n. So, $\xi \leq t \leq \eta$ by definition of ξ and η.

Proof of (2): We have $\forall n \geq 1$

$$\begin{aligned} & [\xi, \eta] \subseteq [a_n, b_n] \\ \Rightarrow\ & 0 \leq \eta - \xi \leq b_n - a_n \\ & = m(I_n) \end{aligned}$$

So, if $\inf(\{m(I_n) \mid n \geq 1) = 0$, then $0 \leq \eta - \xi \leq 0$, so $\eta = \xi$.

Corollary to the Nested Intervals Theorem

Statement: $[0, 1]$ is uncountable.

Proof: Suppose it is possible to denumerate the interval $[0, 1] = \{t_1, t_2, \dots, \}$.

We can find $[a_1, b_1] \subseteq [0, 1]$ with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$.

Then, we find $[a_2, b_2] \in [a_1, b_1]$ with $a_2 < b_2,\ t_2 \notin [a_2, b_2]$.

Recursively, we find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $a_n < b_n,\ t_n \notin [a_n, b_n]$.

So, $I_n = ([a_n, b_n])_0^\infty$ is a sequence of nested intervals.

Therefore, $\exists \xi \in \bigcap I_n \subseteq [0, 1]$. However, $\xi \notin \{t_1, t_2, \dots\}$. $\perp$

Sequences and Convergence

Sequences in Metric Spaces

A sequence in a metric space M is a map

$$x : \mathbb{N} \to M \qquad M = \mathbb{R}, \text{ usually}$$
$$x = (x_n)_{n=1}^\infty$$

I. Sequences defined by a formula:

(i) $x_n = t$ (the constant sequence)
(ii) $x_n = 2n + 1$
(iii) $x_n = \frac{1}{n-1}$ (here, $n \geq 2$)
(iv) $c_n = n \sin\left(\frac{1}{n}\right)$
(v) $d_n = \left(1 + \frac{1}{n}\right)^n$
(vi) Geometric *Sequence*: for $b \neq 0$, $(b^n)_{n\geq 0} = (1, b, b^2, \dots)$
(vii) $x_n = \frac{n!}{n^n}$
(viii) Given any function

$$f : [0, \infty) \to \mathbb{R}$$

we can set $x_n = f(n)$.

II. Sequences defined recursively:

(i) $a_1 = 1,\ a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
(ii) Fibonacci: $f_1 = 1,\ f_2 = 1,\ f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$. The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (1 - \varphi)^n\right)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$

(iii) Let $f : M \to M$ be a self-map on a metric space. Fix $x_0 \in M$.

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

III. New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences, $t \in \mathbb{R}$. Then, we can do the following:

- $(a_n)_n + (b_n)_n + (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n(b_n)_n = (a_n b_n)_n$
- If $b_n \neq 0\ \forall n$, $\left(\frac{a_n}{b_n}\right)$

(ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

(iii) We can look at ratios for $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

(iv) We can look at partial sums, given $(a_k)_{k=1}^{\infty}$.

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^{n} a_k \end{aligned}$$

The sequence $(s_n)_n$ is called the sequence of partial sums. For example, the sequence of partial sums for $(b^k)_{k=0}^{\infty}$ is:

$$1 + b + b^2 + \cdots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for $b \neq 1$, $(1-b)(1+b+b^2+\cdots+b^n) = 1 - b^{n+1}$

Finding a Sequence

Statement: Let $a_k = \frac{1}{k(k+1)}$. Find $(s_n)_n$.

Solution: Via partial fraction decomposition, we get that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore, $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^{\infty}$

Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R},\ a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k| \qquad \text{Infinity Norm}$$

Statement: This norm has the traditional properties of the norm:

$$\|(a_k)_k + (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty \qquad \text{Triangle Inequality}$$
$$\|t(a_k)_k\|_\infty = |t|\|(a_k)_k\|_\infty \qquad \text{Scalar Multiplication}$$
$$\|(a_k)_k\|_\infty = 0 \Leftrightarrow a_k = 0\ \forall k \qquad \text{Zero Property}$$
$$\|(a_k)_k(b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty\|(b_k)_k\|_\infty \qquad \text{Multiplication}$$

Proof: Let $u = \|(a_k)_k\|_\infty$ and $v = \|(b_k)_k\|_\infty$.

Given any k,

$$\begin{aligned}
|a_k + b_k| &\leq |a_k| + |b_k| && \text{Triangle Inequality on } |\cdot| \\
&\leq u + v && \text{definition of supremum} \\
\Rightarrow \sup_{k\geq 1}|a_k + b_k| &\leq u + v
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(a_k)_k + (b_k)_k\|_\infty &= \|\left((a_k + b_k)_k\right)_k\|_\infty \\
&= \sup_{k\geq 1}|a_k + b_k| \\
&\leq u + v
\end{aligned}$$

Monotonicity

A sequence $(x_n)_n$ is **increasing** if

$$x_1 \leq x_2 \leq \cdots\ \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \cdots\ \forall n$$

The sequence is *eventually* increasing if $\exists m \in \mathbb{N}$ such that $x_n \leq x_{n+1}$ for $n > m$.

Similarly, the sequence is eventually decreasing if $\exists m \in \mathbb{N}$ such that $x_n \geq x_{n+1}$ for $n > m$.

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

Monotonicity Example

Statement: The sequence

$$\begin{aligned}
a_1 &= 1 \\
a_{n+1} &= \frac{1}{2}a_n + 2
\end{aligned}$$

is increasing and bounded above.

Proof: We will prove the first statement via induction:

Base: $a_1 = 1$, $a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

Inductive Hypothesis $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+1}$

Proof:

$$\begin{aligned}
a_n &\leq a_{n+1} \\
\frac{1}{2}a_n &\leq \frac{1}{2}a_{n+1} \\
\frac{1}{2}a_n + 2 &\leq \frac{1}{2}a_{n+1} + 2 \\
a_{n+1} &\leq a_{n+2}
\end{aligned}$$

To prove the sequence is bounded above, we do the following:

$$\begin{aligned}
a_1 = 1 &\leq 4 \\
\frac{1}{2}a_1 &\leq 2 \\
\frac{1}{2}a_1 + 2 &\leq 2 \\
a_2 &\leq 4
\end{aligned}$$

We claim that $\forall n,\ a_n \leq 4 \Rightarrow a_{n+1} \leq 4$, as we have shown the base case.

$$\begin{aligned}
a_n &\leq 4 \\
\frac{1}{2}a_n &\leq 2 \\
\frac{1}{2}a_n + 2 &\leq 4 \\
a_{n+1} &\leq 4
\end{aligned}$$

Convergence of Sequences

Let $L \in \mathbb{R}$, $\varepsilon > 0$. Then, the ε-**neighborhood** of L is $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$.

$$\begin{aligned}
x &\in V_\varepsilon(L) \\
&\Leftrightarrow \\
|x - L| &< \varepsilon \\
L - \varepsilon < x &< L + \varepsilon
\end{aligned}$$

With this in mind, we know the following:

Definition of Convergence

A real sequence $(x_n)_n$ converges to a number x if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such L exists, then $(x_n)_n$ is said to **diverge**.

A sequence $(x_n)_n$ in a metric space (X, d) converges to a point x if

$$(\forall \varepsilon > 0)\,(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an N such that the Nth tail (i.e., $x_N, x_{N+1}, \cdots$) are within ε of L for any ε.

Note: N usually depends on ε (the smaller the ε, the larger the N).

Convergence Proof 1

Statement:

$$\left(\frac{1}{n}\right)_n \xrightarrow{n\to\infty} 0$$

Proof: We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given $\varepsilon > 0$, we want $\frac{1}{n} < \varepsilon$, meaning $n > \frac{1}{\varepsilon}$.

Proof: Let $\varepsilon > 0$. By the Archimedean property corollary, find $N \in \mathbb{N}$ large such that $\frac{1}{N} < \varepsilon$.

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

so, if $n \geq N$, then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

Convergence Proof 2

Statement: Show that

$$\left(\frac{5n-1}{3-n}\right)_{n\geq 4} \xrightarrow{n\to\infty} -5$$

Proof:

$$\begin{aligned} |x_n - L| &= \left|\frac{5n-1}{3-n} + 5\right| \\ &= \frac{14}{|3-n|} \\ &= \frac{14}{n-3} \\ &< \varepsilon \\ \frac{14}{n-3} &< \varepsilon \\ n &> \frac{14}{\varepsilon} + 3 \end{aligned}$$

Proof: Let $\varepsilon > 0$. Find $N' \in \mathbb{N}$ so large that $\frac{1}{N'} < \frac{\varepsilon}{14}$ (which exists by the Archimedean property corollary). Let $N = N' + 3$. If $n \geq N$, then

$$\begin{aligned} n - 3 &\geq \frac{1}{N'} \\ \frac{1}{n-3} &\leq \frac{1}{N'} \\ &< \frac{\varepsilon}{14} \end{aligned}$$

whence

$$\begin{aligned}|x_n - L| &= \frac{14}{n-3}\\ &< \frac{14\varepsilon}{14}\\ &= \varepsilon\end{aligned}$$

Convergence and Distance

Statement: Let (X, d) be a metric space, and let $(x_n)_n$ be a sequence in the metric space. The following are equivalent:

(i) $(x_n)_n \to x$

(ii) $(d(x_n, x))_n \to 0$

Proof:

(i) $\Rightarrow$ (b) Let $\varepsilon > 0$. Find $N_\varepsilon \in \mathbb{N}$ so large such that $d(x_n, x) < \varepsilon$ whenever $n \geq N_\varepsilon$.

So, $|d(x_n, x) - 0| = d(x_n, x) < \varepsilon$ for all $\varepsilon > 0$. Whence, $(d(x_n, x))_n \to 0$.

(ii) $\Rightarrow$ (i) This direction is similar.

In $\mathbb{R}$, this implies that

$$\begin{aligned}(x_n)_n &\to x\\ &\Leftrightarrow\\ (|x_n - x|)_n &\to 0\end{aligned}$$

Comparison Proposition

Statement: Let (X, d) be a metric space and let $x \in X$, and suppose $(x_n)_n$ is a sequence in X.

If $\exists c \geq 0$, $m \in \mathbb{N}$, and a sequence $(a_n)_n \in \mathbb{R}^+$ with $(a_n)_n \to 0$ and $d(x_n, x) \leq c a_n$ $\forall n > m$. Then, $(x_n)_n \to x$.

Proof: Let $\varepsilon > 0$. Note that $\frac{\varepsilon}{c} > 0$.

Find $N_1 \in \mathbb{N}$ large such that $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\varepsilon}{c}$, which is always possible since $(a_n)_n \to 0$.

Let $N = \max(N_1, m)$. Then, $n \geq N \Rightarrow n \geq N_1$ and $n \geq m$. So,

$$\begin{aligned}d(x_n, x) &\leq c a_n\\ &< c\frac{\varepsilon}{c}\\ &= \varepsilon\end{aligned}$$

so, $n \geq N \Rightarrow d(x_n, x) < \varepsilon$, whence $(x_n)_n \to x$

Comparison Example 1

Statement:

$$\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \to 0$$

Proof:

$$\begin{aligned}
\left|\frac{\sin(n^2-1)}{n^2+3} - 0\right| &= \frac{|\sin(n^2-1)|}{n^2+3} \\
&\leq \frac{1}{n^2+3} \\
&\leq \frac{1}{n^2} \\
&\leq \frac{1}{n}
\end{aligned}$$

We know that $a_n = \frac{1}{n}$ converges to 0. So, by our comparison proposition, we are done.

Comparison Example 2

Prove:

$$\left(\frac{1}{2^n}\right)_n \to 0$$

$$\begin{aligned}
2^n &= (1+1)^n \\
&\geq 1+n && \text{Bernoulli's Inequality} \\
&> n
\end{aligned}$$

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since $a_n = \frac{1}{n}$ converges, we know that $\frac{1}{2^n}$ must converge.

Sequence Divergence

A sequence $(x_n)_n$ is **divergent** if it does not converge. $(x_n)_n \to 0$ if and only if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } (\forall n \geq N_\varepsilon) d(x_n, x) < \varepsilon$$

So, $(x_n)_n$ diverges if and only if

$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \to d(x_n, x) \geq \varepsilon_0$$

Sequence Divergence 1

Statement: Show that the following sequence diverges:

$$a_n = (-1)^n$$

Proof:

Step 1:

$$((-1)^n)_n \not\to 1$$

Take $\varepsilon_0 = 1/2$, given any $N \in \mathbb{N}$, we will find $n \geq N$ odd:

$$\begin{aligned} n &= 2N + 1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

Step 2:

$$((-1)^n)_n \not\to -1$$

by letting $\varepsilon_0 = 1/2$ and $n = 2N$.

Sequence Divergence 2

Statement: Does

$$a_n = (\sin(n))_n$$

converge?

Proof: It is not the case that $(\sin(n))_n \to L$ for any $L \in \mathbb{R}$.

Case 1 If $L > 1$, set $\varepsilon_0 = \frac{L-1}{2}$. Then, given any $N \in \mathbb{N}$, pick $n = N$.

$$\begin{aligned} |\sin(n) - L| &= L - \sin(n) \\ &\geq L - 1 \\ &> \frac{L-1}{2} \\ &= \varepsilon_0 \end{aligned}$$

Case 2 Similarly for $L < -1$

Case 3 Suppose $-1 < L < 1$.

Case 3.1 Suppose $L > 0$. Set $\varepsilon_0 = \frac{L}{2}$. Given any N, find $n \geq N$ with $\sin(n) < 0$.

We find k large such that $N < (2k+1)\pi$, which we can always do because we are finding $k > \frac{1}{2}\left(\frac{N}{\pi} - 1\right)$, which is always possible by the Archimedean property.

Note that $N < (2k+1)\pi < (2k+2)\pi$. Note that $\sin(x) < 0$ on the interval $((2k+1)\pi, (2k+2)\pi)$. Note that $|(2k+1)\pi - (2k+2)\pi| = \pi$. Let $n = \lceil (2k+1)\pi \rceil$. Then, $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

Case 3.2 Suppose $L < 0$, set $\varepsilon_0 = \frac{-L}{2}$. Given N, find $n \geq N$ with $\sin(n) > 0$. Using the same strategy as above, we find n such that $|L - \sin(n)| > \varepsilon_0$

Case 3.3 Suppose $L = 0$. Set $\varepsilon_0 = 1/2$. Given $N \in \mathbb{N}$, find $n \geq N$ with $\sin(n) \geq \frac{1}{2}$. Then, $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$.

Showing that a sequence diverges is not easy — later, we will show divergence with non-uniqueness of convergent subsequences.

Alternating Sequence

Consider again

$$((-1)^n)_{n\geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to -1:

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating sequence diverges.

Uniqueness of Limits

Statement: A sequence $(x_n)_n$ can converge to at most one limit.

Proof: Suppose toward contradiction that $(x_n)_n$ converges to L_1 and L_2 with $L_1 \neq L_2$.

WLOG, let $L_2 > L_1$. Take $\varepsilon = \frac{L_2 - L_1}{3}$.

Since $(x_n)_n$ converges to L_1, $\exists N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \varepsilon$. Similarly, since $(x_n)_n$ converges to L_2, $\exists N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \varepsilon$.

Let $N = \max N_1, N_2$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

So, $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$. So, $x_n \in V_\varepsilon(L_1)$, and $x_n \in V_\varepsilon(L_2)$, meaning $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$, but $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. $\perp$

Useful Lemmas for Convergence

Absolutely Convergent Sequences

Statement: Let $(x_n)_n$ be a real sequence. If x_n converges to x, then $|(x_n)_n| \to |x|$. However, the converse is not the case.

Proof: Note that since $(x_n)_n \to x$, $d(x_n, x) \to 0$.

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

Convergence to Zero

Statement: Let a_n be a sequence.

$$\begin{aligned} (a_n)_n &\to 0 \\ &\Leftrightarrow \\ |(a_n)| &\to 0 \end{aligned}$$

Proof:

($\Rightarrow$) If $(a_n)_n \to 0$, then we showed previously that $|(a_n)_n| \to |0| = 0$

($\Leftarrow$) Suppose $|(a_n)_n| \to 0$. Given $\varepsilon > 0$, then $\exists N$ such that $n \geq N$ implies

$$\begin{aligned} ||a_n| - 0| &< \varepsilon \\ ||a_n|| &< \varepsilon \\ |a_n| &< \varepsilon \\ |a_n - 0| &< \varepsilon \end{aligned}$$

So, $(a_n)_n \to 0$

Geometric Sequence

Statement: Let $b \in \mathbb{R}$. Consider

$$(b^n)_{n\geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided $-1 < b \leq 1$. Otherwise, the sequence is divergent.

Proof: If $b = 0$, then the sequence $(b^n)_n = (0, 0, 0, \dots)$ is convergent.

If $b = 1$, then the sequence $(b^n)_n = (1, 1, 1, \dots)$ is convergent.

If $b = -1$, then the sequence $(b^n)_n = (1, -1, 1, \dots)$ is divergent.

Case 1 Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, so $\frac{1}{b} = 1 + a$.

So, by Bernoulli's Inequality, $\left(\frac{1}{b}\right)^n = (1 + a)^n \geq 1 + na > na$, so $b^n < \frac{1}{na}$.

$$\begin{aligned} |b^n - 0| &= b^n \\ &< \frac{1}{na} \\ &= \frac{1}{a}\frac{1}{n} \\ &\to 0 \end{aligned}$$

So, $(b^n)_n \to 0$.

Case 2 Suppose $-1 < b < 0$. If we look at $|b^n| = |b|^n$, we know that $(|b|^n)_n \to 0$ by our work above. By the previous lemma, we know that since $|b^n| \to 0$, $b^n \to 0$.

Case 3 Suppose $b > 1$. Then, $b = 1 + a$ where $a > 0$.

$$\begin{aligned} b^n &= (1 + a)^n \\ &\geq 1 + na && \text{Bernoulli's Inequality} \\ &> na \end{aligned}$$

Suppose toward contradiction that $(b^n)_n \to L$. Let $\varepsilon_0 = 1$. Find $N \in \mathbb{N}$ such that $N > \frac{L+1}{a}$. N must exist by the Archimedean property.

Therefore, $(N)(a) > L + 1$. If $n \geq N$, then $(n)(a) > (N)(a) > L + 1$, so $|b^n - L| \geq na - L \geq \varepsilon_0$. $\perp$

Case 4 Suppose $b < -1$, and suppose toward contradiction that $(b^n)_n \to L$. By the previous lemma, we know that $|b^n| \to |L|$. So, $|b|^n \to |L|$. But, $|b| > 1$, which means our assumption contradicts the result from above. $\perp$

nth Root Convergence

Statement: If $c > 0$, then $\left(c^{1/n}\right)_n \to 1$.
Proof:

Case 1: If $c = 1$, then we get $\left(c^{1/n}\right)_n = (1, 1, 1, \dots)$, which clearly converges to one.

Case 2: Assume that $c > 1$. Then, $c^{1/n} > 1$, because if $d = c^{1/n} \leq 1$, then $d^n \leq 1$, so $c \leq 1$. We can write $c^{1/n} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c &= c^n \\ &= (1 + d_n)^n \\ &\geq 1 + nd_n && \text{Bernoulli's Inequality} \\ &> nd_n \end{aligned}$$

So, $d_n \leq \frac{c}{n}$. Remember, $c^{1/n} = 1 + d_n$.

$$\begin{aligned} |c^{1/n} - 1| &= c^{1/n} - 1 \\ &= d_n \\ &\leq c \cdot \frac{1}{n} \\ &\to 0 \end{aligned}$$

Therefore, $c^{1/n} \to 1$.

Case 3: Assume $0 < c < 1$. Then, $c^{1/n} < 1$, so $\frac{1}{c^{1/n}} > 1$.

So, we can write $\frac{1}{c^{1/n}} = (1 + d_n)$, where $d_n > 0$.

$$\begin{aligned} c^{1/n} &= \frac{1}{1 + d_n} \\ 1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\ &= \frac{d_n}{1 + d_n} \\ &\leq d_n \end{aligned}$$

Remember, $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned} \frac{1}{c} &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned}$$

So, $d_n \leq \frac{1}{cn}$

$$\begin{aligned} |1 - c^{1/n}| &= 1 - c^{1/n} \\ &\leq d_n \\ &\leq \frac{1}{c}\frac{1}{n} \\ &\to 0 \end{aligned}$$

Therefore, $\left(c^{1/n}\right)_n \to 1$.

Positive Sequence Convergence

Statement: Let $(x_n)_n$ be a sequence with $x_n \in \mathbb{R}^+$ $\forall n \in \mathbb{N}$, with $(x_n)_n \to x$. Then, x is also positive, and $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.
Proof: Suppose toward contradiction that $x < 0$. Let $\varepsilon = \frac{|0-x|}{2}$. Since $(x_n)_n$ converges to x, we know that $x_n \in V_\varepsilon(x)$ for large n. However, every member of $V_\varepsilon(x) < 0$, and $x_n > 0$. $\perp$

Case 1: If $x = 0$, we will show that $\left(\sqrt{x_n}\right)_n \to 0$.

Let $\varepsilon > 0$, find $N \in \mathbb{N}$ large such that if $n \geq N$, we have

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ x_n &< \varepsilon^2 \\ \sqrt{x_n} &< \varepsilon \\ |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

Case 2: If $x > 0$, we will show that $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

$$\begin{aligned} \left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{\left(\sqrt{x_n} - \sqrt{x}\right)\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x_n}}\right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{1}{\sqrt{x}}|x_n - x| \\ &\to 0 \end{aligned}$$

Therefore, $|\sqrt{x_n} - \sqrt{x}| \to 0$, so $\left(\sqrt{x_n}\right)_n \to \sqrt{x}$.

nth Root of n Convergence

Show:

$$\left(n^{1/n}\right)_n \to 1$$

Proof: We will make use of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Note that $n^{1/n} > 1$ for n past 1. So, we write

$$\begin{aligned} n^{1/n} &= 1 + d_n && d_n > 0 \\ n &= (1 + d_n)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} d_n^k \\ &= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n \\ &\geq \binom{n}{0} + \binom{n}{2} d_n^2 && \text{as all terms are positive} \\ &= 1 + \frac{n(n-1)}{2} d_n^2 \end{aligned}$$

so

$$\begin{aligned} n-1 &\geq \frac{n(n-1)}{2}d_n^2 \\ \frac{2}{n} &\geq d_n^2 \\ \frac{\sqrt{2}}{\sqrt{n}} &\geq d_n \end{aligned}$$

So, we have

$$\begin{aligned} |n^{1/n}-1| &= n^{1/n}-1 \\ &= d_n \\ &\leq \sqrt{2}\frac{1}{\sqrt{n}} \\ &\to 0 && \text{by previous corollary} \end{aligned}$$

So, $\left(n^{1/n}\right)_n \to 0$.

Multiplication by Geometric Sequence

Statement: Let $0 \leq b < 1$. Show that

$$\left(nb^n\right)_n \to 0$$

Proof: If $0 < b < 1$ (the 0 case is trivial). So, $\frac{1}{b} > 1$, meaning $\frac{1}{b} = 1 + d$ for some $d > 0$.

$$\begin{aligned} \frac{1}{b^n} &= (1+d)^n \\ &\geq \frac{n(n-1)}{2}d^2 \\ \frac{2}{d^2(n)(n-1)} &\geq b^n \\ nb^n &\leq \frac{2}{d^2(n-1)} \\ &\to 0 && \text{by previous corollary} \end{aligned}$$

Therefore, $(nb^n)_n \to 0$.

Boundedness and Convergence

Statement: If $(x_n)_n$ is a convergent sequence, x_n is bounded. The converse is false in general.

Proof: Suppose $(x_n)_n \to x$. Let $\varepsilon = 1$.

Then, $\exists N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq N$.

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$. If $n \geq N$, then $|x_n| \leq c$, because $x_n \in V_\varepsilon(x)$. If $n < N$, then $|x_n| \leq c$.

Together, we have $|x_n| \leq c$ for all n.

To show the converse is not true, consider $((-1)^n)_n$. This sequence is bounded but not convergent.

Algebraic Operations on Sequences

Statement: Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$ be convergent sequences. Let $t \in \mathbb{R}$. Then, the following are all true:

(1) $(x_n \pm y_n)_n \to x \pm y$

(2) $(tx_n)_n \to tx$

(3) $(x_n y_n)_n \to xy$

(4) Assume $z_n \neq 0\ \forall n$, and $z \neq 0$. Then, $\left(\frac{1}{z_n}\right)_n \to \frac{1}{z}$, and $\left(\frac{x_n}{z_n}\right)_n \to \frac{x}{z}$.

Proof of (1): Let $\varepsilon > 0$. Since $x_n \to x$, $y_n \to y$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \to |x_n - x| < \frac{\varepsilon}{2}$, and $n \geq N_2 \to |x_n - x| \leq \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$.

$$\begin{aligned}
|(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Proof of (3): We have $(x_n)_n \to x$ and $(y_n)_n \to y$.

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) + y(x_n - x)| \\
&\leq |x_n(y_n - y)| + |y(x_n - x)| \\
&= |x_n||y_n - y| + |x_n - x||y|
\end{aligned}$$

Since convergent sequences are bounded, $\exists c \in \mathbb{R}$ such that $|x_n| < c,\ \forall n$

$$\begin{aligned}
&\leq c|y_n - y| + |x_n - x||y| \\
&\to 0
\end{aligned}$$

Therefore, $|x_n y_n - xy| \to 0$, so $x_n y_n \to xy$.

Proof of (4): We have $z_n \neq 0$ and $z \neq 0$. Let $\varepsilon > 0$.

$$\begin{aligned}
\left|\frac{1}{z_n} - \frac{1}{z}\right| &= \frac{|z - z_n|}{|z_n z|} \\
&= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|}
\end{aligned}$$

Let $\varepsilon = \frac{|z|}{2}$. Since $(z_n)_n \to z$, we know that $z_n \in V_\varepsilon(z)$ for $n \geq N \in \mathbb{N}$. For $n \geq N$, $|z_n| > \frac{|z|}{2}$, so $\frac{1}{|z_n|} < \frac{2}{|z|}$.

$$\begin{aligned}
&\leq |z_n - z| \frac{2}{|z|^2} \\
&\to 0
\end{aligned}$$

So, $\left|\frac{1}{z_n} - \frac{1}{z}\right| \to 0$, so $\frac{1}{z_n} \to \frac{1}{z}$

Ordering of Limits

Statement: Let $(x_n)_n \to x$ and $(y_n)_n \to y$. If $x_n \leq y_n$ for all n, then $x \leq y$.
Proof: Suppose toward contradiction that $x > y$.

Let $\varepsilon = \frac{x-y}{2}$.

So, $\exists N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$.

Let $N = \max\{N_1, N_2\}$. Then, $x_N \in V_\varepsilon(x)$ and $y_N \in V_\varepsilon(y)$. But that means $x_N > y_N$. $\perp$

In particular, if $(x_n)_n \to x$, and $a \leq x_n \leq b$, then $a \leq x \leq b$.

Squeeze Theorem

Statement: Let $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$, where $x_n \leq y_n \leq z_n$ for all n.

If $L = x = z$, then $y = L$.
Proof: Let $\varepsilon > 0$. Find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow V_\varepsilon(L)$, and $n \geq N_2 \Rightarrow V_\varepsilon(L)$.

Let $N = \max\{N_1, N_2\}$. Then, $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$. Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so $y_n \in V_\varepsilon(L)$, so $(y_n)_n \to L$.

Squeeze Theorem Examples

For example, let $a_n = \frac{\sin(n)}{n}$. Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero, $a_n \to 0$

As another example, consider $a_n = (2^n + 3^n)^{1/n}$. Then,

$$3^n \leq 2^n + 3^n \leq 2 \cdot 3^n$$
$$3 \leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3$$

Since $2^{1/n} \to 1$, we have $a_n \to 3$.

Ratio Test

Statement: Let (x_n) be a sequence of strictly positive numbers, with $\left(\frac{x_{n+1}}{x_n}\right)_n \to r < 1$. Then, $(x_n)_n \to 0$.
Proof: Since $r < 1$, then $1 - r > 0$. Let $\rho = r + \frac{1-r}{2}$, and $\varepsilon = \rho - r = \frac{1-r}{2}$.

Since the sequence converges, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\left|\frac{x_{n+1}}{x_n} - r\right| < \varepsilon$$
$$\frac{x_{n+1}}{x_n} < \rho$$
$$x_{n+1} < \rho x_n$$

In particular, $x_{N+1} < \rho x_N$, and $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$. Inductively, one can show that $\forall k \geq 1,\ x_{N+k} < \rho^k x_N$.

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as $k \to \infty$, both sides of the inequality go to 0, implying that $x_n \to 0$.

Monotone Convergence Theorem

Proof: Let $(x_n)_n$ be a monotone sequence. Then, $(x_n)_n$ is convergent if and only if it is bounded.

(a) If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n \to \sup(\{x_n \mid n \in \mathbb{N}\})$.

(b) If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n \to \inf(\{x_n \mid n \in \mathbb{N}\})$.

Proof: We have already shown that all convergent sequences are bounded.

Assume that $(x_n)_n$ is monotonic and bounded.

Case 1: Suppose $(x_n)_n$ is increasing. Let $\sup\{x_n \mid n \in \mathbb{N}\} := u$. We claim that $(x_n)_n \to u$.

> Let $\varepsilon > 0$. By the definition of supremum, $\exists N \in \mathbb{N}$ such that $u - \varepsilon < x_N$. Note that $\forall n \geq N$, $u - \varepsilon < x_N \leq x_n \leq u$.
>
> Therefore, if $n \geq N$, then $|x_n - u| < \varepsilon$.

Case 2: Suppose $(x_n)_n$ is decreasing. Let $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$. We claim that $(x_n)_n \to \ell$.

> Let $\varepsilon > 0$. By the definition of infimum, $\exists N \in \mathbb{N}$ such that $\ell + \varepsilon > x_N$. Additionally, $\forall n \geq N$, $\ell \leq x_n \leq x_N < \ell + \varepsilon$.
>
> Therefore, if $n \geq N$, $|x_n - \ell| < \varepsilon$.

Applications of the Monotone Convergence Theorem

Statement: If $(x_n)_n$ is a convergent sequence, and $m \in \mathbb{N}$, the m-th tail, $x_{(m)} = (x_{m+k})_{k=1}^{\infty}$ is also convergent. If $(x_n)_n \to L$ then $x_{(m)} \to L$.
Proof: Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. If $k \geq N$, then $m + k \geq N$, so $|x_{m+k} - L| < \varepsilon$.

Thus, $(x_{m+k})_k \to L$

Monotone Convergence Example 1

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that $x_n \geq 4\ \forall n$.

$$x_1 = 8 \geq 4$$

Suppose $x_k \geq 4$. We will show that $x_{k+1} \geq 4$.

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore, $(x_n)_n$ is bounded below by 4.

We claim that $(x_n)_n$ is decreasing. $\forall n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} \leq x_n &\Leftrightarrow \\ \frac{1}{2}x_n + 2 \leq x_n \\ &\Leftrightarrow 4 \leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that $(x_n)_n \to L$.

To find L, we use the recursive relationship and the lemma.

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2}x_n + 2\right)_{n=1}^{\infty} \\ L &= \frac{1}{2}L + 2 \\ L &= 4 \end{aligned}$$

Monotone Convergence Example 2

Consider the following sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{4} \\ x_3 &= 1 + \frac{1}{4} + \frac{1}{9} \\ x_k &= \sum_{k=1}^{n} \frac{1}{k^2} \end{aligned}$$

We will show that $(x_n)_n$, the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$\begin{aligned} k^2 &\geq k(k-1) && k \geq 2 \\ \frac{1}{k^2} &\leq \frac{1}{k(k-1)} \\ &= \frac{1}{k-1} - \frac{1}{k} \\ \sum_{k=2}^{n} \frac{1}{k^2} &\leq \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ \sum_{k=1}^{n} \frac{1}{k^2} &\leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \end{aligned}$$

But

$$1+\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)=2-\frac{1}{n}$$

so, we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k^2}&\leq 2-\frac{1}{n}\\&<2\end{aligned}$$

So, $(x_n)_n$ is bounded above.

Alternative Proof of the Nested Intervals Theorem

Statement: Let $I_n=[a_n,b_n]$ be a countable family of nested intervals. Then,

$$\bigcap I_n\neq\emptyset$$

Proof: Since the intervals are nested, it must be the case that $a_1\leq a_2\leq\cdots\leq a_n\leq b_n\leq b_1$.

Similarly, $a_1\leq a_n\leq b_n\leq b_{n-1}\leq\cdots\leq b_2\leq b_1$.

So, $(a_n)_n$ is an increasing sequence bounded above by b_1 and $(b_n)\,n$ is a decreasing sequence bounded below by a_1. So, $(b_n)_n\to r$ and $(a_n)\to\ell$
Note that $\ell=\sup(a_n)$ and $r=\inf(b_n)$.

Fix $n\in\mathbb{N}$, then for any $m\geq n$, $a_n\leq a_m\leq b_m\leq b_n$. So, $\sup(a_m)=\ell\leq b_n$. Unlocking n, we get that $\ell\leq\inf(b_n)=r$.

Calculating Square Roots

Let $a\in\mathbb{R}^+$. We will construct a sequence $(x_n)_n\to\sqrt{a}$.

Let

$$x_1=1$$

Define

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right).$$

We will prove that $x_n^2\geq a$.

$$\begin{aligned}2x_{n+1}&=x_n+\frac{a}{x_n}\\2x_{n+1}x_n&=x_n^2+a\\0&=x_n^2-2x_{n+1}x_n+a\end{aligned}$$

So, x_n is a real root, meaning

$$\begin{aligned}\Delta&=4x_{n+1}^2-4a\\x_{n+1}^2&\geq a && \forall n\end{aligned}$$

So, $\forall n\geq 2$

$$x_n^2\geq a$$

We will show that x_n is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &= \underbrace{\frac{1}{2}\left(\frac{x_n^2 - a}{x_n}\right)}_{\geq 0\ \forall n \geq 2} \end{aligned}$$

So, we have that $(x_n)_n$ is decreasing and bounded below, meaning $(x_n)_n \to x$ for some $x \in \mathbb{R}$.

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ x &= \frac{1}{2}\left(x + \frac{a}{x}\right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} && \text{remember that } x > 0 \end{aligned}$$

Euler's Number

Consider

$$\begin{aligned} (e_n)_n &= \left(1 + \frac{1}{n}\right)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \end{aligned}$$

Similarly,

$$\begin{aligned} e_{n+1} &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n+1}\right)\right) \\ e_{n+1} &\geq e_n && \forall n \end{aligned}$$

We claim that $(e_n)_n$ is bounded above.

$$\begin{aligned}
e_1 &= \left(1+\frac{1}{1}\right)^1\\
2 &\leq e_n\\
e_n &= \sum_{k=0}^{n}\left(\frac{1}{k!}\underbrace{\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)}_{\leq 1}\right)\\
2^{k-1} &\leq k! && k\geq 2\\
\frac{1}{k!} &\leq \frac{1}{2^{k-1}}\\
e_n &= \sum_{k=0}^{n}\frac{1}{k!}\cdot\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)\\
&\leq \sum_{k=0}^{n}\frac{1}{k!}\\
&\leq 2+\sum_{\ell=1}^{n-1}\frac{1}{2^{\ell}}\\
&< 3
\end{aligned}$$

so, we have

$$2 \leq e_n \leq 3$$

so, by the monotone convergence theorem, $(e_n)_n$ converges

$$e := \sup_n \left(1+\frac{1}{n}\right)^n$$

Monotone Divergence

A sequence that is increasing and *unbounded* diverges to infinity.
Let $M > 0$. Since $(x_n)_n$ is unbounded, $\exists N \in \mathbb{N}$ such that $x_N > M$

Thus, if $n \geq N$, then $x_n \geq x_N > M$.

Monotone Divergence Example

Consider

$$h_n = \sum_{k=1}^{n}\frac{1}{k}$$

We can see that $h_n < h_{n+1}$. The primary question is as to whether $(h_n)_n$ is bounded.

$$\begin{aligned}
h_1 &= 1 \\
&\geq 1 \\
h_2 &= 1 + \frac{1}{2} \\
&\geq 1 + \frac{1}{2} \\
h_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&\geq 1 + \frac{1}{2} + \frac{1}{2} \\
h_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}
\end{aligned}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^{k} \frac{1}{2}$$

Let M be large. Find n such that $n > 2(M-1)$. In this case, $n/2 + 1 > M$. Let $N = 2^n$. Then, for $m \geq N$, $h_m > M$.

Thus, $(h_n)_n$ diverges to infinity.

Subsequences and Bolzano-Weierstrass

Natural Sequences

A **natural sequence** is a strictly increasing sequence of natural numbers, $(n_k)_{k=1}^{\infty}$

$$n_1 < n_2 < n_3 < \dots$$

where $\forall k \in \mathbb{N},\ n_k \in \mathbb{N}$.

Statement: Given $(n_k)_k$ natural sequence, show that $(n_k) \geq k$.

Proof:

Base Case: We know that $n_1 \leq 1$, as $n_1 \in \mathbb{N}$.

Inductive Step: To be continued...

Subsequences

Let $(x_n)_n$ be a sequence. A subsequence $(x_{n_k})_{k=1}^{\infty}$, where $(n_k)_k$ is a natural sequence.

For example, if $(x_n)_n = (-1)^n$. If $(n_k) = 2k$, then, $(x_{n_k}) = \left((-1)^{2k}\right)_k = (1, 1, 1, \dots)$. But, if $(n_k) = 2k+1$, then $(x_{n_k}) = (-1, -1, -1, \dots)$.

If $(x_n) = (1/n)_n$, and $(n_k)_k = k^2$, then $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \dots)$.

If $(x_n)_n$ is a sequence, its m-th **tail** is $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \dots)$, where $n_k = m + k$.

Convergence of Subsequences

Statement: If $(x_n)_n \to x$, then for any natural sequence $(n_k)_k$,

$$(x_{n_k})_k \to x$$

Proof: Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ large such that $n \geq N$, $|x_n - x| < \varepsilon$.

Take $K = N$. Then,

$$\begin{aligned} n_k &\geq k \\ &\geq K \\ &= N \\ \Rightarrow |x_{n_k} - x| &< \varepsilon \end{aligned}$$

Corollary to Convergence of Subsequences

Given a sequence $(x_n)_n$, if there are two subsequences $(x_{n_k})_k \to x$, $(x_{n_\ell})_\ell \to x'$, where $x \neq x'$, then $(x_n)_n$ is divergent.

Convergence of Subsequences Example

Recall the geometric sequence

$$(b^n)_{n=1}^{\infty} \to 0$$

if $0 < b < 1$.

The sequence $(1, b, b^2, \dots)$ is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem, $(b^n)_n \to \ell$.

Given $n = 2k$, we know that $(b^{2k})_k \to \ell$.

$$\begin{aligned} b^{2k} &= \left(b^k\right)^2 \\ \left(b^k\right)^2 &\to \ell^2 \\ \ell^2 &= \ell \\ \ell &= \{0, 1\} \end{aligned}$$

since $b < 1$

$$\ell = 0$$

Divergence and Subsequences

If $(x_n)_n \nrightarrow x$, then

$$(\exists \varepsilon_0 > 0)\,(\forall N \in \mathbb{N})\,(\exists n \geq N) \text{ such that } |x_n - x| \geq \varepsilon_0$$

We can use this to construct a sequence to show divergence.

Statement: Let $(x_n)_n$ be a sequence, and $x \in \mathbb{R}$.

$$\begin{gathered} (x_n)_n \nrightarrow x \\ \Leftrightarrow \\ (\exists \varepsilon_0 > 0)(\exists (x_{n_k})_k) \end{gathered}$$

with

$$|x_{n_k} - x| \geq \varepsilon_0$$

Proof:

($\Rightarrow$) We know $\exists \varepsilon_0 > 0$ as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \varepsilon_0$$
$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \varepsilon_0$$
$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \varepsilon_0$$

Assume we have $n_1 < n_2 < \dots, n_k$ with

$$|x_{n_j} - x| \geq \varepsilon_0 \qquad j = 1, 2, \dots, k$$
$$N = n_k + 1 \Rightarrow n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \varepsilon_0$$

Iteratively, we have our desired subsequence $(x_{n_k})_k$.

($\Leftarrow$) If $(x_n)_n \to x$, any subsequence converges to x.

By assumption, $(\exists \varepsilon_0 > 0)\,(\exists (n_k)_k)$ with $|x_{n_k} - x| \geq \varepsilon_0$. Thus, $(x_{n_k})_k \not\to x$.

Bolzano-Weierstrass Theorem

Statement: If $(x_n)_n$ is a bounded sequence, then $(x_n)_n$ admits a convergent subsequence.

Proof:

Lemma: Let $(x_n)_n$ be any real sequence. Then, $\exists n_k$ such that $(x_{n_k})_k$ is monotone.

Proof of Lemma: A **peak** of a sequence $(x_n)_n$ is an x_m such that $x_m \geq x_n \ \forall n \geq m$.

Case 1: There are infinitely many peaks, $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$, where $n_1 < n_2 < \dots$.

Then, $(x_{n_k})_k$ is decreasing.

Case 2: There are finitely many peaks. Let these peaks be $x_{m_1}, x_{m_2}, \dots, x_{m_r}$.

Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, $\exists n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since x_{n_2} is not a peak, $\exists n_3 > n_2$ such that $x_{n_3} > x_{n_2}$.

Iteratively, we have an increasing sequence of non-peaks $(x_{n_k})_k$.

Since $(x_n)_n$ admits a monotone subsequence, and $(x_{n_k})_k$ is bounded as $(x_n)_n$ is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

Limit Superior and Limit Inferior

Limit Points

Let $X = (x_n)_n$ be a bounded real sequence. By Bolzano-Weierstrass, $(x_n)_n$ admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R},\ t = \lim_{k\to\infty} x_{n_k} \right\} \qquad \text{for any subsequence } (x_{n_k})_k$$

Then, $t \in \overline{X}$ is called a **limit point** of X.

Finding the Limit Points

Let $u_1 = \sup_{n\geq 1}(x_n)$, $\ell_1 = \inf_{n\geq 1}(x_n)$. Clearly, $\ell_1 \leq u_1$, and $\overline{X} \subseteq [\ell_1, u_1]$.

Let $u_2 = \sup_{n\geq 2}(x_n)$ and $\ell_2 = \inf_{n\geq 2}(x_n)$.

Since u_1 is an upper bound for $(x_n)_n$, it is an upper bound for $(x_n)_{n\geq 2}$, so $u_2 \leq u_1$. Similarly, since ℓ_1 is a lower bound for $(x_n)_n$, it is a lower bound for $(x_n)_{n\geq 2}$, so $\ell_2 \geq \ell_1$.

As a result, we can see that $\overline{X} \subseteq [\ell_2, u_2]$.

We continue, letting $u_m = \sup_{n\geq m}(x_n)$, and $\ell_m = \inf_{n\geq m}(x_n)$. We get $\ell_1 \leq \ell_2 \leq \cdots$, and $u_1 \geq u_2 \geq \cdots$, and $\overline{X} \in [\ell_m, u_m]$, $\forall m$.

We get a nested sequence of intervals $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \cdots$. By the Nested Intervals Theorem, we know that

$$\begin{aligned} \overline{X} &\subseteq \bigcap_{m\geq 1} [\ell_m, u_m] \\ &= [\ell, u] \end{aligned}$$

where $\ell = \sup(\ell_m)$ and $u = \inf(u_m)$.

Defining Limit Superior and Limit Inferior

Given a bounded sequence $(x_n)_x = X$,

$$\begin{aligned} u &= \inf_{m\geq 1}(u_m) \\ &= \inf_{m\geq 1}\left(\sup_{n\geq m} x_n\right) \end{aligned}$$

called the **limit superior** of $(x_n)_n$

$$u = \limsup_{n\to\infty} x_n$$

and

$$\begin{aligned} \ell &= \sup_{m\geq 1}(\ell_m) \\ &= \sup_{m\geq 1}\left(\inf_{n\geq m}(x_n)\right) \end{aligned}$$

called the **limit inferior** of $(x_n)_n$

$$\ell = \liminf_{n\to\infty} x_n$$

Fundamental Results in Limit Superior and Limit Inferior

Statement: Let $(x_n)_n$ be bounded. Then,

(1) $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$

(2) $(x_n)_n \to x \Leftrightarrow \liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n = x$

Proof of (1): This was proven with the Nested Intervals Theorem

Proof of (2): Let $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$.

We know that $u_m = \sup_{n\geq m} x_n$. If $m \geq N$, then $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$. Therefore, $|u_m - x| \leq \varepsilon/2 < \varepsilon$, so $(u_m)_m \to \varepsilon x \limsup_{n\to\infty} x_n$.

Similarly, we know that $\ell_m = \inf_{n\geq m} x_n$. If $m \geq N$, then $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$. So, $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$, so $(\ell_m)_m \to x = \liminf_{n\to\infty} x_n$.

Applying Limit Superior and Limit Inferior

Consider the sequence

$$\begin{aligned} x_n &= \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases} \\ &= (-1, 5/2, -1/3, 9/4, -1/5, \dots) \end{aligned}$$

We begin by constructing the u_m sequence: $(5/2, 5/2, 9/4, 9/4, \dots)$. We can see that $u_m \to 2$.

Then, we construct the ℓ_m sequence: $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$. We can see that $\ell_m \to 0$.

Exercise: If $(x_n)_n$ and $(y_n)_n$ are sequences with $x_n \leq y_n\ \forall n$, then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

Ratio Test and Root Test: Equivalent Convergence

Statement: If $(a_n)_n$ is a sequence of strictly positive terms such that

$$\left(\frac{a_{n+1}}{a_n}\right)_n \to \rho$$

then,

$$\left(a_n^{1/n}\right)_{n=1}^{\infty} \to \rho$$

Proof: Let $\varepsilon > 0$. Then, $\exists N$ large such that $\forall n \geq N$,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n} - \rho\right| &< \varepsilon && \forall n \geq N\\
\Rightarrow \frac{a_{n+1}}{a_n} &< \rho + \varepsilon && \forall n \geq N\\
a_{n+1} n a_n &(\rho + \varepsilon) && \forall n \geq N\\
a_n &< a_N (\rho+\varepsilon)^{n-N} && \forall n \geq N\\
a_n &< (\rho+\varepsilon)^n \cdot \frac{a_N}{(\rho+\varepsilon)^N}\\
a_n^{1/n} &< (\rho+\varepsilon)\left(\frac{a_N}{(\rho+\varepsilon)^N}\right)^{1/n}\\
\limsup a_n^{1/n} &\leq \limsup (\rho+\varepsilon)\left(\frac{a_N}{(\rho+\varepsilon)^N}\right)^{1/n}\\
\limsup_{n\to\infty} a_n^{1/n} &\leq \rho + \varepsilon
\end{aligned}$$

Case 1: If $\rho = 0$, the case his trivial.

Case 2: Suppose $\rho > 0$. Find $\varepsilon > 0$ small such that $0 < \varepsilon < \rho$.

Since $\left(\frac{a_{n+1}}{a_n}\right)_n \to \rho$, find N large such that $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$. So, $\forall n \geq N$,

$$\begin{aligned}
a_{n+1} &\geq a_n (\rho - \varepsilon)\\
a_n &\geq a_N (\rho-\varepsilon)^{n-N}\\
a_n^{1/n} &\geq (\rho-\varepsilon)\left(\frac{a_N}{(\rho-\varepsilon)^N}\right)^{1/n}\\
\liminf a_n^{1/n} &\geq \rho - \varepsilon
\end{aligned}$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together, $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$, so $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$, whence $\left(a_n^{1/n}\right) \to \rho$

Properties of $\overline{X}$

Statement: We found earlier that $\overline{X} \subseteq [\ell, u]$. We claim that

$$\begin{aligned}
\sup \overline{X} &= u\\
\sup \overline{X} &= \ell
\end{aligned}$$

Proof: We have shown that u is an upper bound for $\overline{X}$. The goal is to show that u is the least upper bound.

Let $\varepsilon > 0$. We need to find a $t \in \overline{X}$ with $u - \varepsilon < t$. Note that $u - \varepsilon < u_m\ \forall m$.

We know that $u - \varepsilon < u_1$. Since $u_1 = \sup_{n\geq 1} x_n$, we know $\exists n_1 \in \mathbb{N}$ with $u - \varepsilon < x_{n_1} < u_1$.

Consider $u_{n_1+1} = \sup_{n>n_1} x_n$. We know that $u-\varepsilon < u_{n_1+1}$. Therefore, $\exists x_{n_2}$ with $n_2 > n_1$ and $u-\varepsilon < x_{n_2} < u_{n_1+1}$.

Then, we use u_{n_2+1}. Then, $\exists n_3 > n_2$ with $u - \varepsilon < x_{n_3} < u_{n_2+1}$.

We get a subsequence from the natural sequence $n_1, n_2, \dots,$ where $u-\varepsilon < x_{n_k}\ \forall k$.

Also, $x_{n_k} < u_1\ \forall k$. Therefore, $(x_{n_k})_k$ is a bounded sequence. By Bolzano-Weierstrass, $\exists$ a convergent subsequence

$$\left(x_{n_{k_j}}\right)_j \to t$$

We know that $u - \varepsilon \leq t$. Note that $t \in \overline{X}$.

Exercise: Show that $\inf \overline{X} = \ell$.

Cauchy and Contractive Sequences

Cauchy Sequences

A sequence $(x_n)_n$ in a metric space (X,d) is Cauchy if

$$(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N}) \text{ such that } p,q \geq N \Rightarrow d(x_p,x_q) < \varepsilon$$

if $(X,d) = (\mathbb{R}, |\cdot|)$:

$$(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N}) \text{ such that } p,q \geq N \Rightarrow |x_p - x_q| < \varepsilon$$

Consider the sequence $(x_n)_n = \frac{1}{n}$. Then,

$$\begin{aligned} |x_p - x_q| &= \left|\frac{1}{p} - \frac{1}{q}\right| \\ &= \frac{1}{q} - \frac{1}{p} \\ &\leq \frac{1}{q} \end{aligned}$$

Given $\varepsilon > 0$, find N large such that $\frac{1}{N} < \varepsilon$. Then, $p,q \geq N$ implies

$$\begin{aligned} \left|\frac{1}{p} - \frac{1}{q}\right| &< \frac{1}{q} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

To show that any sequence is not Cauchy, we use the following negation of the definition:

$$(\exists \varepsilon_0 > 0)\,(\forall N \in \mathbb{N}) \text{ such that } p,q \geq N \Rightarrow |x_p - x_q| \geq \varepsilon_0$$

Boundedness of Cauchy Sequences

Statement: Cauchy sequences are bounded.

Proof: Let $\varepsilon = 1$. Then, by the Cauchy criterion, $\exists N \in \mathbb{N}$ such that $p,q \geq N \Rightarrow |x_p - x_q| < 1$.

In particular, $\forall n \geq N$,

$$\begin{aligned} |x_n| &= |x_n - x_N + x_N| \\ &\leq |x_n + x_N| + |x_N| && \text{Triangle Inequality} \\ &< 1 + |x_N| \end{aligned}$$

Let $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$. Then, $x_n \leq c\ \forall n \geq 1$. Thus, x_n is bounded.

Convergent Subsequences and Cauchy Sequences

Statement: If $(x_n)_n$ is Cauchy and $(x_n)_n$ admits a convergent subsequence, then $(x_n)_n$ is convergent.

Proof: Say $(x_{n_k}) \to x$ for some natural sequence $(n_k)_k$. We claim that $(x_n)_n \to x$.

Let $\varepsilon > 0$. Since $(x_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ such that $p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon/2$.

Also, since $(x_{n_k})_k \to x$, then $\exists K \in \mathbb{N}$ and $K \geq N$ with $k \geq K \Rightarrow |x_{n_k} - x| < \varepsilon/2$.

For all $k \geq K$,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \end{aligned}$$

Let $N_1 = \max\{N, K\}$. Then,

$$\begin{aligned} n \geq N_1 &\Rightarrow n \geq N && \text{by max} \\ &\Rightarrow n_k \geq k \geq K \geq N && \text{def. of natural sequence} \\ |x_n - x| &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Cauchy Sequence Convergence in the Reals

Statement: Let $(x_n)_n$ be any sequence in $\mathbb{R}$. The following are equivalent:

(1) $(x_n)_n$ converges.

(2) $(x_n)_n$ is Cauchy.

Proof:

$(1) \Rightarrow (2)$ (Holds in any metric space). Suppose $(x_n)_n \to x$. Find N large such that $n \geq N \to d(x_n, x) < \varepsilon/2$.

Then, $p, q \geq N \Rightarrow$

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

$(2) \Rightarrow (1)$ If $(x_n)_n$ is Cauchy, then $(x_n)_n$ converges.

By Bolzano-Weierstrass, $(x_n)_n$ admits a convergent subsequence, so by our previous lemma, $(x_n)_n$ must converge.

Note: To show $(2) \Rightarrow (1)$, we used Bolzano-Weierstrass, which requires the monotone convergence theorem, which itself requires the completeness axiom. This is why we cannot show $(2) \Rightarrow (1)$ converges.

Complete Metric Spaces

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Remark: All convergent sequences are Cauchy, and all Cauchy sequences are bounded. We showed that $\mathbb{R}$ under the absolute value metric is complete.

$\mathbb{Q}$ under $d(s, t) = |s - t|$ is not complete; similarly, $A = (0, 1)$ under the metric inherited from $\mathbb{R}$ is not complete; $x_n = \frac{1}{n}$ is Cauchy but not convergent in A.

Finding Cauchy Sequences and Convergence in $\mathbb{R}$

Cauchy Sequences and Convergence 1

Consider the harmonic sequence

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

We claim that h_n is not convergent.

Let $p > q$. Then,

$$\begin{aligned}
|h_p - h_q| &= \left| \sum_{1}^{p} \frac{1}{k} - \sum_{1}^{q} \frac{1}{k} \right| \\
&= \frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p} \\
&\geq \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} \\
&= \frac{p-q}{p} \\
&= 1 - \frac{q}{p}
\end{aligned}$$

set $p = 2q$:

$$\begin{aligned}
|h_{2q} - h_q| &\geq 1 \frac{q}{2q} \\
&= 1/2
\end{aligned}$$

Therefore, h_n is not Cauchy, and thus not convergent.

Cauchy Sequences and Convergence 2

Consider a sequence of partial sums

$$x_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

We claim that $(x_n)_n$ is Cauchy, and thus convergent. Let $p > q$. Then, we have

$$\begin{aligned}
|x_p - x_q| &= \left| \sum_{k=q+1}^{p} \frac{(-1)^k}{k!} \right| \\
&\leq \sum_{k=q+1}^{p} \frac{1}{k!} \\
&\leq \sum k = q + 1^p \frac{1}{2^{k-1}} \\
&= \frac{1}{2^q} + \frac{1}{2^{q+1}} + \cdots + \frac{1}{2^{p-1}} \\
&= \frac{1}{2^q} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-q-1}} \right) \\
&\leq \frac{1}{2^{q-1}}
\end{aligned}$$

Given $\varepsilon > 0$, choose N large such that $\dfrac{1}{2^{N-1}} < \varepsilon$. When $p > q > N$, then $|x_p - x_q| \leq \dfrac{1}{2^{q-1}} \leq \dfrac{1}{2^{N-1}} < \varepsilon$.

Thus, the sequence is convergent.

Contractive Sequences

A sequence $(x_n)_n$ in a metric space (X, d) is **contractive** if

$$\exists 0 < \rho < 1 \text{ such that } d(x_{n+1}, x_n) \leq \rho d(x_n, x_{n-1}) \qquad \forall n \geq 1$$

In $\mathbb{R}$, the definition is

$$|x_{n+1} - x_n| \leq \rho |x_n - x_{n-1}|$$

Contractive and Cauchy

We claim that every contractive sequence is Cauchy.

From examination, we arrive at the following:

$$|x_n - x_{n-1}| \leq \rho^{n-2}|x_2 - x_1| \qquad (*)$$

If $p > q$, then

$$\begin{aligned}
|x_p - x_q| &= |x_p - x_{p-1} + x_{p-1} - x_{p-1} + \cdots + x_{q+1} - x_q| \\
&\leq |x_p - x_{p-1}| + \cdots + |x_{q+1} - x_q| && \text{Triangle Inequality} \\
&\leq |x_2 - x_1| \left(\rho^{p-2} + \rho^{p-3} + \cdots + \rho^{q-1}\right) \\
&= |x_2 - x_1| \rho^{q-1} \left(1 + \rho + \rho^2 + \cdots + \rho^{p-q-1}\right) \\
&= |x_2 - x_1| \rho^{q-1} \frac{1 - \rho^{p-q}}{1 - x} && \text{Finite Geometric Sequence} \\
&\leq |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho}
\end{aligned}$$

Given $\varepsilon > 0$, we can find N large such that

$$q \geq N \Rightarrow |x_2 - x_1| \frac{\rho^{q-1}}{1 - \rho} < \varepsilon$$

Thus, $p > q \geq N \Rightarrow |x_p - x_q| < \varepsilon$.

Applying Contractive Sequences 1

Consider $(f_n)_n$ defined as follows:

$$\begin{aligned}
f_0 &= 1 \\
f_1 &= 1 \\
f_{n+1} &= f_n + f_{n-1}
\end{aligned}$$

Consider x_n defined as follows:

$$x_n = \frac{f_{n+1}}{f_n}$$

We can rewrite x_n as:

$$\begin{aligned} x_n &= \frac{f_n + f_{n-1}}{f_n} \\ &= 1 + \frac{f_{n-1}}{f_n} \\ &= 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \\ &= 1 + \frac{1}{x_{n-1}} \end{aligned}$$

We claim that $3/2 \leq x_n \leq 2 \ \forall n \geq 2$.

$$x_2 = 2$$

Inductive Hypothesis: suppose $3/2 \leq x_n \leq 2$

$$\begin{aligned} :\frac{3}{2} \leq x_n \leq 2 \\ \frac{2}{3} \geq \frac{1}{x_n} \geq \frac{3}{2} \\ 2 \geq \frac{5}{3} \geq 1 + \frac{1}{x_n} \geq \frac{3}{2} \end{aligned}$$

We now claim that $(x_n)_n$ is contractive.

$$\begin{aligned} |x - n + 1 - x_n| &= \left|\left(1 + \frac{1}{x_n}\right) - \left(1 + \frac{1}{x_{n-1}}\right)\right| \\ &= \left|\frac{1}{x_n} - \frac{1}{x_{n-1}}\right| \\ &= \left|\frac{x_{n-1} - x_n}{x_{n-1}x_n}\right| \\ &\leq \frac{4}{9}|x_n - x_{n-1}| \end{aligned}$$

Therefore, $\rho = \frac{4}{9}$ is our constant of contraction. Thus, $(x_n)_n$ is Cauchy, so it converges in $\mathbb{R}$.

$$\begin{aligned} x_{n+1} &= 1 + \frac{1}{x_n} && (n \to \infty,\ x_n \to \varphi) \\ \varphi &= 1 + \frac{1}{\varphi} \\ \varphi^2 - \varphi - 1 &= 0 \\ \varphi &= \frac{1 + \sqrt{5}}{2} \end{aligned}$$

Applying Contractive Sequences 2

Let $x_1 = 0$, $x_2 = 1$, and

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_n + x_{n-1}) \\ (x_n)_n &= (0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, \dots) \end{aligned}$$

While the sequence is not monotone, we can show that the sequence is contractive.

$$\begin{aligned}
|x_{n+1} - x_n| &= \left|\frac{1}{2}\left(x_n + x_{n-1}\right) - x_n\right| \\
&= \left|\frac{1}{2}\left(x_{n-1} - x_n\right)\right| \\
&= \frac{1}{2}|x_n - x_{n-1}|
\end{aligned}$$

Since the constant of contraction is equal to 1/2, this sequence is Cauchy, and thus converges in the real numbers.

Since $(x_n)_n \to x$, every subsequence converges to x. Therefore, $(x_{2k+1})_k \to x$.

$$\begin{aligned}
x_{2k+1} &= \sum_{j=1}^{k} \frac{1}{2^{2j-1}} \\
&= 2\sum_{j=1}^{k} \frac{1}{4^j} \\
&= 2\frac{1 - \frac{1}{4^{k+1}}}{1 - \frac{1}{4}} \\
&= \frac{2}{3} && k \to \infty
\end{aligned}$$

Sequence Divergence

Properly Divergent Sequences

Let $(x_n)_n$ be a real sequence. $(x_n)_n$ *properly* diverges to $+\infty$ if

$$(\forall \alpha > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow x_n \geq \alpha$$

We write that $(x_n)_n \to +\infty$. Similarly, $(x_n)_n$ properly diverges to $-\infty$ if

$$(\forall \beta < 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow x_n \leq \beta$$

and $(x_n)_n \to -\infty$. We say that $(x_n)_n$ is properly divergent if $(x_n)_n \to \pm\infty$.

If $(x_n)_n$ and $(y_n)_n$ are sequences such that $x_n \geq y_n \; \forall n$, and $(y_n)_n \to +\infty$, then $(x_n)_n \to +\infty$.

Divergence of the Geometric Sequence

In the geometric sequence, if $b > 1$, we can show that $(b^n) \to +\infty$.

Write $b = 1 + a$ for some $a > 0$. Then, by Bernoulli's inequality, we have

$$\begin{aligned}
b^n &= (1 + a)^n \\
&\geq 1 + na \\
&\geq na
\end{aligned}$$

Given any $\alpha > 0$, find N large such that $N > \frac{\alpha}{a}$, which is always possible by the Archimedean property. Then, for $Na \geq \alpha$. If $n \geq N$, then $na \geq Na > \alpha$.

Since $b^n > na$, we have that $(b^n)_n \to +\infty$.

Monotone Divergence

By the Monotone Convergence Theorem, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \to x \Leftrightarrow (x_n)_n \text{ bounded}$$

Negating, we have that if $(x_n)_n$ is monotone, then

$$(x_n)_n \text{ divergent} \Leftrightarrow (x_n)_n \text{ unbounded}$$

However, we can make this statement stronger.

Statement: Let $(x_n)_n$ be monotone. $(x_n)_n$ is unbounded if and only if $(x_n)_n$ is properly divergent.

Proof:

($\Leftarrow$) If $(x_n)_n$ is properly divergent, then $(x_n)_n$ is divergent, and thus unbounded.

($\Rightarrow$) Let $(x_n)_n$ be unbounded and increasing. Then, given $\alpha > 0$, $\exists n_\alpha$ with $x_{n_\alpha} > \alpha$. If $n \geq n_\alpha$, then $x_n \geq x_{n_\alpha} > \alpha$, so $(x_n)_n$ is properly divergent to $+\infty$.

Sequence Comparison Test

Let $(x_n)_n$ and $(y_n)_n$ be sequences with $x_n > 0$ and $y_n > 0$. Suppose that

$$\left(\frac{x_n}{y_n}\right)_n \to L > 0$$

Then, $(x_n)_n \to +\infty \Leftrightarrow (y_n)_n \to \infty$.

Let $\varepsilon = L/2$. Since

$$\left(\frac{x_n}{y_n}\right)_n \to L,$$

$\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} \frac{L}{2} \leq \frac{x_n}{y_n} &\leq \frac{3L}{2} \\ \frac{L}{2} y_n &\leq x_n \\ \frac{2}{3L} x_n &\leq y_n \end{aligned}$$

If $(y_n)_n \to \infty$, then so too does $(L/2)(y_n)$, so $(x_n)_n \to \infty$. Similarly, if $(x_n)_n \to \infty$, then so too does $(2/3L)x_n$, so $(y_n)_n \to \infty$.

Applying the Sequence Comparison Test

Problem: Show that

$$\left(\sqrt{4n^2 - 3n + 1}\right)_n \to +\infty$$

Solution: We will compare to $y_n = n$. Then

$$\begin{aligned} \frac{x_n}{y_n} &= \frac{\sqrt{4n^2 - 3n + 1}}{n} \\ &= \sqrt{4 - \frac{3}{n} + \frac{1}{n^2}} \\ &\to 2 \geq 0 \end{aligned}$$

Since y_n is properly divergent to $+\infty$, so too is x_n.

Series Convergence and Divergence

Introduction to Infinite Series

An **infinite series** is a sequence of partial sums s_n, where s_n is formed from x_k as follows:

$$s_n = \sum_{k=1}^{n} x_k$$

alternatively,

$$s_1 = x_1$$
$$s_n = s_{n-1} + x_n$$

The limit of the sequence $(s_n)_n$ is the value of

$$\sum_{n=1}^{\infty} x_n$$

The infinite series converges to s if $(s_n)_n \to s$.

If $(s_n)_n$ diverges, then so too does the series. If $(s_n)_n$ is properly divergent to $\pm\infty$, then we write that the series is equal to $\pm\infty$.

Convergence of a Series of Positive Terms

Statement: Let $(x_k)_k$ be a sequence of positive terms. The following are equivalent:

(a) $\sum x_k$ converges.

(b) The sequence of partial sums $(s_n)_n$ is bounded above.

(c) A subsequence of the sequence of partial sums $(s_{n_j})_j$ is bounded above.

Proof:

(1) $\Rightarrow$ (2)**:** $\sum x_k$ is convergent $\Rightarrow$ $(s_n)_n$ is convergent $\Rightarrow$ $(s_n)_n$ is bounded.

(2) $\Rightarrow$ (3)**:** If $(s_n)_n$ is bounded, so is any subsequence $(s_{n_j})_j$.

(3) $\Rightarrow$ (2)**:** Suppose $s_{n_j} \leq c$. If m is arbitrary, $\exists j$ such that $n_j \geq m$. Take $j = m$. Then, $s_m \leq s_{n_j} \leq c$. Therefore, $(s_n)_n$ is bounded above.

(2) $\Rightarrow$ (1) Let $(s_n)_n$ be bounded above. We know that $(s_n)_n$ is increasing as $x_k \geq 0$. By the Monotone Convergence theorem, $(s_n)_n$ converges, meaning $\sum x_k$ converges.

Corollary to Convergence of a Series of Positive Terms

Let $(x_k)_k$ be a sequence with $x_k \geq 0$. Then,

$$\sum x_k \text{ properly diverges} \Leftrightarrow (s_n)_n \text{ is unbounded}$$

Applying Convergence of a Series of Positive Terms 1

Recall that for $x_k = 1/k$, we proved that $(s_n)_n$ is unbounded, and also that $(s_n)_n$ is not Cauchy, meaning $\sum_{k=1}^{\infty} 1/k$ is properly divergent.

Applying Convergence of a Series of Positive Terms 2

Additionally, we saw that for $x_k = 1/k^2$, $(s_n)_n$ is increasing and bounded above.

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{k-1} \\ &= 2 - \frac{1}{n} \end{aligned}$$

Applying Convergence of a Series of Positive Terms 3

Let $b \in \mathbb{R}$. Let $x_k = b^k$. Then, we have

$$\begin{aligned} s_n &= \sum_{k=0}^{n} b^k \\ &= \frac{1-b^{n+1}}{1-b} && b \neq 1 \end{aligned}$$

Therefore, we know the end behavior of the series:

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{1-b^{n+1}}{1-b} \\ &= \frac{1}{1-b}\left(1 - b \lim_{b\to\infty} b^n\right) \\ &= \begin{cases} \frac{1}{1-b} & |b| < 1 \\ \text{diverges} & |b| > 1 \end{cases} \end{aligned}$$

Series Comparison Test

Statement: Let $0 \leq x_k \leq y_k$.

- If $\sum y_k$ converges, then so too does $\sum x_k$
- If $\sum x_k$ diverges, then so too does $\sum y_k$.

Proof:

$(\Rightarrow)$ If $\sum y_k$ converges, then $t_n = \sum_{k=1}^{n} y_k$ is bounded.

Setting $s_n = \sum_{k=1}^{n} x_k$, we see that $0 \leq s_n \leq t_n$. Seeing as t_n is bounded, so too is s_n. Therefore, $\sum x_k$ is convergent.

For example, consider the series

$$\sum \frac{1}{k^2+k}$$

Since $\frac{1}{k^2} \geq \frac{1}{k^2+k}$, we know that, seeing as $\frac{1}{k^2}$ converges, so does $\frac{1}{k^2+k}$.

Limit Comparison Test

Statement: Let x_k and y_k be strictly positive sequences. Suppose that

$$\lim_{k\to\infty}\frac{x_k}{y_k} = L$$

(a) If $L > 0$, then $\sum x_k$ converges if and only if $\sum y_k$ converges.

(b) If $L = 0$, then $\sum y_k$ converges $\Rightarrow \sum x_k$ converges.

Proof:

(a) Since

$$\frac{x_k}{y_k} \to L$$

Set $\varepsilon = L$. We know $\exists K$ such that $k \geq K \Rightarrow y_k \leq \frac{2}{L}x_k$. Let $s_n = \sum_{k=1}^{n} x_k$ and $t_n = \sum_{k=1}^{n} y_k$. Then,

$$\begin{aligned} t_n &= \sum_{k=1}^{K-1} y_k + \sum_{k=K}^{n} \\ &\leq t_{K-1} + \frac{2}{L}\sum_{k=K}^{n} \\ &\leq t_{K-1} + \frac{2}{L}s_n \\ &\leq t_{K-1} + c, \end{aligned}$$

implying that t_n is bounded, so $\sum y_k$ converges.

(b) Since

$$\frac{x_k}{y_k} \to 0,$$

$\exists K$ such that $\frac{x_k}{y_k} \leq 1\ \forall k \geq K$, meaning $x_k < y_k\ \forall k \geq K$.

Letting $s_n = \sum_{k=1}^{n} x_k$ and $t_n = \sum_{k=1}^{n} y_k$. Thus,

$$\begin{aligned} s_n &= \sum_{k=1}^{K-1} x_k + \sum_{k=K}^{n} x_k \\ &= s_{K-1} + \sum_{k=K}^{n} y_k \\ &\leq s_{K-1} + t_n \\ &\leq s_{K-1} + c \end{aligned}$$

Thus, s_n is bounded, meaning $\sum x_k$ is convergent.

Applying the Limit Comparison Test

Consider

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}}$$

Letting $x_n = \frac{1}{\sqrt{n^2-1}}$, and $y_n = \frac{1}{n}$, we have

$$\begin{aligned}\frac{x_n}{y_n} &= \frac{n}{\sqrt{n^2-1}}\\ &\to 1 > 0\end{aligned}$$

Since $\sum y_n$ diverges, so too does $\sum x_n$.

nth Term Divergence Test

If $\sum x_k$ is convergent, then $(x_k)_k \to 0$. Conversely, if $(x_k)_k \not\to 0$, then $\sum x_k$ diverges. Recall that $s_n = s_{n-1} + x_n$. If $\sum x_k$ converges, then $(x_n)_n \to 0$. So,

$$\begin{aligned}x_n &= s_n - s_{n-1}\\ (s_n)_n &\to s\\ x_n &\to s - s\\ &= 0\end{aligned}$$

For example, we can find that

$$\sum_{k=1}^{\infty} \frac{1}{\arctan k}$$

diverges, as $\lim_{k\to\infty} \frac{1}{\arctan k} = \frac{2}{\pi} \neq 0$

Cauchy Condensation Test

Statement: Let $(x_k)_k$ be a decreasing sequence of positive numbers. Then,

$$\sum_k x^k \text{ converges } \Leftrightarrow \sum_k 2^k x_{2^k} \text{ converges}$$

Proof: Look at the partial sum s_{2^n},

$$\begin{aligned}s_{2^n} &= \sum_{k=1}^{2^n} x_k\\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_15) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n}\\ &\leq x_1 + 2x_2 + 4x_4 + \cdots + 2^{n-1}x_{2^{n-1}} + x_{2^n}\\ &= \sum_{k=1}^{n-1} 2^k x_{2^k} + x_{2^n}\end{aligned}$$

If $\sum_k 2^k x_{2^k}$ converges, then its partial sums are bounded, and we have that $x_{2^n} \to 0$. Then, s_{2^n} is bounded, and thus $\sum x_k$ converges.

$$\begin{aligned}2s_{2^n} &= \sum_{k=1}^{2^n} x_k\\ &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_15) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n}\\ &+ x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + (x_8 + \cdots + x_15) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1}) + x_{2^n}\\ &= (x_1 + x_1) + (x_2 + x_2) + (x_3 + x_3 + x_4 + x_4) + \cdots (x_{2^{n-1}} + x_{2^{n-1}} + \cdots + x_{2^n} + x_{2^n})\\ &\geq x_1 + 2x_2 + 4x_4 + \cdots + 2^n x_{2^n}\\ &= \sum_{k=0}^{n} 2^k x_{2^k}\end{aligned}$$

therefore, we get that

$$\frac{1}{2}\sum_{k=0}^{n} 2^k a_{2^k} \leq s_{2^n}$$

If $\sum x_k$ converges, then s_n is bounded, so s_{2^n} is bounded, so $\sum_{k=0}^{n} 2^k x_{2^k}$ is bounded, so the series $\sum_{k=0}^{n} 2^k x_{2^k}$ is convergent.

p-Series

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \in \mathbb{R}$$

By the Cauchy Condensation Test, we see that the series converges if and only if the following series converges:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{np-1}}\right)^n \\
&\Leftrightarrow \frac{1}{2^{p-1}} < 1 \\
&\Leftrightarrow 2^{p-1} > 1 \\
&\Leftrightarrow p > 1
\end{aligned}$$

Sequences and Series of Functions

Pointwise Convergence

Fix a nonempty set Ω. Then,

$$\mathcal{F}(\Omega, \mathbb{R}) = \{f \mid f : \Omega \to \mathbb{R}\}$$

A sequence of functions $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$ converges pointwise to $f \in \mathcal{F}(\Omega, \mathbb{R})$ if

$$\forall x \in \Omega,\ (f_n(x))_n \xrightarrow{n\to\infty} f(x)$$

Alternatively, using ε, we have:

$$\begin{gathered}
(f_n)_n \to f \text{ pointwise } \in \mathcal{F}(\Omega, \mathbb{R}) \\
\Leftrightarrow \\
(\forall x \in \Omega)(\forall \varepsilon > 0)(\exists N_{x,\varepsilon} \in \mathbb{N}) \text{ such that } n \geq N_{x,\varepsilon} \Rightarrow |f_n(x) - f(x)| < \varepsilon
\end{gathered}$$

Applying Pointwise Convergence

Example 1: Let $f_n : [0, 1] \to \mathbb{R}$, and $f_n(x) = x^n$. Note that $(f_n)_n \to \delta_1$, where

$$\delta_1(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Example 2: Let $f_n : \mathbb{R} \to \mathbb{R}$, where

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

Claim: $f_n \to \mathbf{o}$.

If $x = 0$, then $f_n(0) = \mathbf{o}\ \forall n \geq 1$.

Otherwise, we have

$$\begin{aligned}|f_n(x) - \mathbf{o}(x)| &= \frac{n|x|}{1 + n^2x^2}\\ &\leq \frac{n|x|}{n^2x^2}\\ &= \frac{1}{n|x|}\\ &\to 0\end{aligned}$$

Example 3: Let $h_n : [0, \infty) \to \mathbb{R}$, where $h_n(x) = x^{1/n}$. We claim that

$$\begin{aligned}h_n &\to h\\ h(x) &= \begin{cases}1 & x > 0\\ 0 & x = 0\end{cases}\\ &= \mathbb{1}_{(0,\infty)}\end{aligned}$$

Since, for any $b > 0$, $\left(b^{1/n}\right) \to 1$

Example 4: Let $g_n : [0, \infty) \to \mathbb{R}$, where $g_n(x) = \frac{x^n}{1+x^n}$. We claim that $g_n \to g$, where $g : [0.\infty) \to \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases}0 & 0 \leq x < 1\\ \frac{1}{2} & x = 1\\ 1 & x > 1\end{cases}$$

When $x > 1$, we have

$$\begin{aligned}|g_n(x) - 1| &= \left|\frac{x^n}{1 + x^n} - 1\right|\\ &= \left|\frac{-1}{1 + x^n}\right|\\ &= \frac{1}{1 + x^n}\\ &\to 0\end{aligned}$$

Uniform Convergence

A sequence of functions $(f_n)_n \in \mathcal{F}(\Omega, \mathbb{R})$ converges uniformly to $f \in \mathcal{F}(\Omega, \mathbb{R})$ if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } (n \geq N_\varepsilon)\,(\forall x \in \Omega) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Equivalently,

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \text{ such that } n \geq N_\varepsilon \Rightarrow \sup_{x\in\Omega} |f_n(x) - f(x)| < \varepsilon.$$

Applying Uniform Convergence

Example 1: Let $f_n : [0,4] \to \mathbb{R}$.

$$f_n(x) = \frac{x}{x+n}$$

We claim that

$$f_n \to \mathbf{o} \text{ uniformly.}$$

We start by examining the maximum size of $f_n(x)$:

$$\begin{aligned} |f_n(x) - \mathbf{o}(x)| &= \frac{x}{x+n} \\ &\leq \frac{x}{n} \\ &\leq \frac{4}{n} \end{aligned}$$

so,

$$\sup_{x\in[0,4]} |f_n(x) - \mathbf{o}(x)| \leq \frac{4}{n}.$$

Given $\varepsilon > 0$, find N so large such that $\frac{1}{N} < \frac{\varepsilon}{4}$. Then, for $n \geq N$,

$$\begin{aligned} \sup_{x\in\Omega} |f_n(x) - f(x)| &\leq \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \varepsilon \end{aligned}$$

Negating Uniform Convergence

Statement:

$$\begin{gathered} (f_n)_n \nrightarrow f \text{ uniformly} \\ \Leftrightarrow \\ (\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N}) \text{ such that } (\exists n_0 \geq N)(\exists x_0 \in \Omega)\, |f_{n_0}(x_0) - f(x_0)| \geq \varepsilon_0 \\ \Leftrightarrow \\ (\exists \varepsilon_0 > 0)(\forall N)(\exists (x_k)_k \in \Omega)(\exists\, (f_{n_k})\, k) \text{ such that } |f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \end{gathered}$$

Proof:

($\Rightarrow$) We know $\exists \varepsilon_0$ satisfying condition (1). Let $N = 1$. We know $\exists n_1 \geq 1$ such that $\exists x_1 \in \Omega$ with $|f_{n_1}(x_1) - f(x_1)| \geq \varepsilon_0$.

Now, set $N = n_1 + 1$. Then, $\exists n_2 \geq N$ and $x_2 \in \Omega$ satisfying condition (1).

Defining n_k and x_k recursively, we have a natural sequence $(n_k)_k$, and thus a subsequence of f_n, thereby satisfying condition (2).

Negating Uniform Convergence 1

Statement: Does $(f_n)_n \to f$ uniformly converge on $[0,1]$, where $f_n(x) = x^n,\ f = \delta_1$?

Proof: Let $x_k = \left(\frac{1}{2}\right)^k$, $n_k = k$.

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= |f_{n_k}(x_k)| \\ &= \left(\frac{1}{2^{1/k}}\right)^k \\ &= \frac{1}{2}\end{aligned}$$

Setting $\varepsilon_0 = 1/2$, we have that it does *not* converge uniformly.

Changing Domain and Uniform Convergence

Recall $g_n : [0,\infty) \to \mathbb{R}$, where

$$g_n(x) = \frac{nx}{1+n^2x^2}$$

We saw that $(g_n)_n \to \mathbf{o}$ pointwise. However, it is *not* uniformly convergent. Take $x_k = \frac{1}{k}$, and $n_k = k$. Then,

$$\begin{aligned}|g_{n_k}(x_k) - \mathbf{o}(x_k)| &= \frac{k\cdot\frac{1}{k}}{1+k^2\cdot\frac{1}{k^2}} \\ &= 1/2 \\ &= \varepsilon_0.\end{aligned}$$

However, $g_n \to g$ on $[a,\infty)$ where $a > 0$. Let $x \in [a,\infty)$

$$\begin{aligned}|g_n(x) - \mathbf{o}(x)| &= \frac{nx}{1+n^2x^2} \\ &\leq \frac{nx}{n^2x^2} \\ &= \frac{1}{nx} \\ &\leq \frac{1}{na}\end{aligned}$$

therefore,

$$\sup_{x\in[a,\infty)} |g_n(x) - \mathbf{o}(x)| \leq \frac{1}{na}$$

Negating Uniform Convergence 2

Consider the family of functions

$$\begin{aligned}f_n &: [0,\infty) \to \mathbb{R} \\ f_n(x) &= e^{-nx}\end{aligned}$$

Upon examination, we can see that:

$$f_n \xrightarrow{\text{p.w.}} \delta_0 = \mathbb{1}_{\{0\}}.$$

However, the convergence is *not* uniform.

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Then, setting $\varepsilon_0 = e^{-1}$

$$\begin{aligned} |f_{n_k}(x_k) - \delta_0(x_k)| &= \left|f_k\left(\frac{1}{k}\right)\right| \\ &= e^{-1} \\ &\geq \varepsilon_0 \end{aligned}$$

Uniform Norm

For $f \in \mathcal{F}(\Omega, \mathbb{R})$, the **uniform norm** or **infinity norm** is defined as:

$$\|f\|_u = \sup_{x\in\Omega} |f(x)|.$$

Importantly, the value of the uniform norm is dependent on Ω.

The uniform norm satisfies the rules we desire of any other norm:

- Scalar multiplication: $\forall t \in \mathbb{R}$, $\|tf\|_u = |t|\|f\|_u$
- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Zero Property: $\|f\|_u = 0 \Leftrightarrow f = \mathbf{0}_{\mathbb{R}}$
- Algebraic Property: $\|fg\|_u \leq \|f\|_u \cdot \|g\|_u$.

$$\ell_\infty(\Omega) = \{f \in \mathcal{F}(\Omega, \mathbb{R}) \mid \|f\|_u < \infty\}$$

is a normed vector space.

Given $(f_k)_k,\ f \in \ell_\infty(\Omega)$, we have

$$(f_k)_k \xrightarrow{\text{uniformly}} f \Leftrightarrow (\|f_k - f\|_u)_k \to 0$$

Applying Uniform Norm 1

Let

$$\begin{aligned} g_n &: [0,1] \to \mathbb{R} \\ g_n(x) &= x^n(1-x) \end{aligned}$$

Clearly, $(g_n)_n$ belongs to $\ell_\infty([0,1])$. We can see that

$$(g_n)_n \xrightarrow{\text{p.w.}} \mathbf{0}$$

To show that the convergence is uniform, we must find

$$\|g_n - \mathbf{0}\|_u \xrightarrow{n\to\infty} \mathbf{0},$$

or

$$\begin{aligned}
\sup_{x\in[0,1]} x^n(1-x) &\to 0\\
\frac{d}{dx}(x^n(1-x)) &= nx^{n-1}-(n+1)x^n\\
nx^{n-1} &= (n+1)x^n\\
x &= \frac{n}{n+1}\\
\sup_{x\in[0,1]} x^n(1-x) &= \left(\frac{n}{n+1}\right)^n\left(1-\frac{n}{n+1}\right)\\
&= \frac{1}{(1+1/n)^n}\left(\frac{1}{n+1}\right)\\
&\to 0
\end{aligned}$$

Root Test and Series Convergence

Statement: Let

$$\limsup_{k\to\infty} |x_k|^{1/k} = \rho.$$

If $\rho < 1$, then $\sum_k x_k$ converges absolutely. If $\rho > 1$, then $\sum_k x_k$ diverges.

Proof: Suppose $\rho < 1$. Let $\rho < r < 1$. By property of inf, $\exists N \in \mathbb{N}$ large such that $r \geq \sup_{k\geq N} |x_k|^{1/k}$.

Therefore, $\forall k \geq N$, we have

$$\begin{aligned}
x_k^{1/k} &\leq r\\
x_k &\leq r^k && \forall k \geq N
\end{aligned}$$

Therefore,

$$\sum_k x^k \leq \underbrace{\sum_{k=1}^{N-1} x_k + \sum_{k\geq N} r^k}_{\text{converges: } r<1}$$

If $\limsup |x_k|^{1/k} = \rho > 1$, we can find a subsequence $(x_{k_\ell})^{1/k_\ell} \xrightarrow{\ell\to\infty} \rho$. We cannot have $((x_k)_k)^{1/k} \to 0$. Thus, the series diverges.

Absolute Convergence

Statement: A series $\sum_k x_k$ converges absolutely if $\sum_k |x_k|$ converges. If a series converges absolutely, then it always converges.

Proof: Let $s_n = \sum_{k=1}^n x_k$, $t_n = \sum_{k=1}^n |x_k|$. Let $m > n$. Then,

$$\begin{aligned}
|s_m - s_n| &= \left|\sum_{k=n+1}^m x_k\right|\\
&\leq \sum_{k=n+1}^m |x_k| && \text{Triangle Inequality}\\
&= |t_m - t_n|
\end{aligned}$$

By assumption, $(t_n)_n$ converges, and thus is Cauchy. By the above inequality, $(s_n)_n$ is Cauchy, and thus convergent.

Series of Functions

Given a sequence of functions $(f_k)_k \in \mathcal{F}(\Omega, \mathbb{R})$, we say that the series

$$\sum_k f_k$$

converges pointwise to f in $\mathcal{F}(\Omega, \mathbb{R})$ if

$$s_n = \left(\sum_{k=1}^{n} f_k\right)_n$$

converges to f pointwise.

$$\sum_{k=1}^{\infty} f_k(x) = f(x) \qquad \forall x \in \Omega$$

$\sum f_k$ converges to f **uniformly** if

$$s_n = \left(\sum_{k=1}^{n} f_k\right)_n$$

converges to f uniformly.

Applying Pointwise Convergence of Series of Functions

Let $f_k : (-1, 1) \to \mathbb{R}$, where $f_k = x^k$. Then,

$$\sum_{k=0}^{\infty} f_k \to f(x) = \frac{1}{1-x}$$

Applying Uniform Convergence of Series of Functions

Statement: We know that $\sum_{k=0}^{\infty} x_k$ converges pointwise to $s(x) = \frac{1}{1-x}$ on $(-1, 1)$. Does it converge *uniformly* on the same interval?

Proof:
We claim the convergence is not uniform on $(-1, 1)$, but convergence is uniform on $[a, b]$, where $-1 < a \leq b < 1$.

Let $s_n(x) = \sum_{k=0}^{n} x^k$.

$$\begin{aligned} |s_n(x) - s(x)| &= \left| \frac{1 - x^{n+1}}{1-x} - \frac{1}{1-x} \right| \\ &= \frac{|x|^{n+1}}{1-x} \end{aligned}$$

Let $c = \max\{|a|, |b|\} < 1$

$$\leq \frac{c^{n+1}}{1-b} \qquad \forall a \leq x \leq b$$

$$\begin{aligned} \sup_{x \in [a,b]} |s_n(x) - s(x)| &\leq \frac{c^{n+1}}{1-b} \\ &\to 0 \end{aligned}$$

To show non-uniform convergence on $(-1,1)$, let $x_\ell = 1 - \frac{1}{\ell}$, and let $n_\ell = \ell$.

$$\begin{aligned}
|s_{n_\ell}(x_\ell) - s(x_\ell)| &= \frac{\left(1-\frac{1}{\ell}\right)^{\ell+1}}{\frac{1}{\ell}}\\
&= \ell\left(1-\frac{1}{\ell}\right)^{\ell}\left(1-\frac{1}{\ell}\right)\\
&= (\ell-1)\left(1-\frac{1}{\ell}\right)^{\ell}\\
&\to \infty
\end{aligned}$$

since $\left(1-\frac{1}{\ell}\right)^{\ell} \to \frac{1}{e}$.

Weierstrass M-test

Statement: Consider a sequence of functions $(f_k)_k$ in $\ell_\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}$.

If $\sum_{k=1}^{\infty} \|f\|_u$ converges, then $\sum_k f_k$ converges uniformly and absolutely on Ω.

Proof: Set $M_k = \|f_k\|_u$. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\sum_{n+1}^{m} M_k < \varepsilon \qquad \forall m > n \geq N$$

since $\sum_{k=1}^{\infty} M_k$ is convergent, and thus Cauchy.

Let $s_n(x) = \sum_{k=1}^{n} f_k(x)$. So,

$$\begin{aligned}
|s_n(x) - s_m(x)| &= \left|\sum_{k=n+1}^{m} f_k(x)\right|\\
&\leq \sum_{k=n+1}^{m} |f_k(x)|\\
&\leq \sum_{k=n+1}^{m} M_k\\
&< \varepsilon && \text{whenever } m > n \geq N
\end{aligned}$$

For every $x \in \Omega$, $s_n(x)$ is Cauchy. So, $\forall x \in \Omega$, $s(x) := \lim s_n(x)$ exists.

Additionally, $\forall x \in \Omega$,

$$|s_m(x) - s_n(x)| < \varepsilon.$$

Let $m \to \infty$. Then,

$$\begin{aligned}
|s(x) - s_n(x)| &< \varepsilon && \forall x \in \Omega,\ \forall n \geq N\\
\sup_{x\in\Omega} |s(x) - s_n(x)| &< \varepsilon. && \forall n \geq N
\end{aligned}$$

Applying the Weierstrass M-test

Consider

$$\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2},$$

where $f_k : \mathbb{R} \to \mathbb{R}$. Then, $\|f_k\|_u \leq \frac{1}{k^2}$. So,

$$\begin{aligned}\sum \|f_k\|_u &\leq \sum \frac{1}{k^2}\\ &< \infty.\end{aligned}$$

Thus, $\sum \frac{1}{x^2+k^2}$ converges absolutely and uniformly.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges $\forall x \in \mathbb{R}$, and converges *uniformly* on any closed and bounded interval $[a, b]$.

Power Series

A **power series** *centered* at c in $\mathbb{R}$ is a formal series of functions

$$\sum_{k=0}^{\infty} a_k (x-c)^k.$$

We want to examine the convergence and the uniformity of such convergence of these power series.

Given $\sum a_k (x-c)^k$, set $\rho = \limsup |a_k|^{1/k}$ and $r = 1/\rho$.

Cauchy-Hadamard Theorem

Statement: A power series

$$\sum_{k=1}^{\infty} a_k (x-c)^k$$

converges absolutely on $(c-r, c+r)$, diverges on $\overline{[c-r, c+r]}$, and uniformly convergent on $[a, b]$, $c - r < a \leq b < c + r$.

Proof: Let $\sum_{k=1}^{\infty} a_k (x-c)^k$, where $x_k = a_k (x-c)^k$.

$$|x_k|^{1/k} = |a_k|^{1/k} |x - c|$$

Root test:

$$\begin{aligned}\limsup_{k\to\infty} |x_k|^{1/k} &= |x-c| \limsup_{k\to\infty} |a_k|^{1/k}\\ &= |x-c|\rho\end{aligned}$$

Absolute Convergence:

$$\begin{aligned}|x-c|\rho &< 1\\ |x-c| &< \frac{1}{\rho}\end{aligned}$$

Divergence:

$$\begin{aligned}|x-c|\rho &> 1\\ |x-c| &> \frac{1}{\rho}\end{aligned}$$

Let $[a, b] \subset (c - r, c + r)$. Set $d = \max\{|a - c|, |b - c|\}$. So,

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{k=n+1}^{m} a_k (x - c)^k \right| \\ &\leq \sum_{k=n+1}^{m} |a_k||x - c|^k \\ &\leq \sum_{k=n+1}^{m} |a_k||d|^k \end{aligned}$$

we know that $d < r \Rightarrow d/r < 1 \Rightarrow d\rho < 1 \Rightarrow \rho < 1/d$. Pick $\rho < p < 1/d$. So, $\exists N \in \mathbb{N}$ with

$$\begin{aligned} \sup_{k \geq N} |a_k|^{1/k} &< p \\ |a_k| &< p^k \end{aligned}$$

So, if $m > n \geq N$, we have

$$\begin{aligned} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^{m} (rd)^k \\ \sup_{x \in [a,b]} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^{m} (rd)^k \end{aligned}$$

Given $\varepsilon > 0$, find $N_1 \in \mathbb{N}$ with $m > n \geq N_1$ meaning

$$\begin{aligned} \sup_{x \in [a,b]} |s_m(x) - s_n(x)| &\leq \sum_{n+1}^{m} (rd)^k \\ &< \varepsilon \end{aligned}$$

Let $K = \max\{N, N_1\}$. With $m > n \geq K$, we have

$$\sup_{x \in [a,b]} |s_m(x) - s_n(x)| < \varepsilon$$

Letting $m \to \infty$, we have

$$\sup_{x \in [a,b]} |s(x) - s_n(x)| < \varepsilon.$$

So, $(s_n(x))_n \to s(x)$ uniformly on $[a, b]$.

Limits

Cluster Points

Recall: If $c \in \mathbb{R}$, and $\delta > 0$, then $V_\delta(x) = (c - \delta, c + \delta)$.

The *deleted neighborhood* $\dot{V}_\delta = (c - \delta, c) \cup (c, c + \delta) = V_\delta \setminus \{c\}$.

(i) $x \in V_\delta(c) \Leftrightarrow |x - c| < \delta$

(ii) $x \in \dot{V}_\delta(c) \Leftrightarrow 0 < |x - c| < \delta$

Let $D \subseteq \mathbb{R}$. A number $c \in \mathbb{R}$ is a *cluster point* or *limit point* of D if

$$(\forall \delta > 0)(\exists x \in D \cap \dot{V}(c)) \Leftrightarrow \forall \delta > 0,\ \dot{V}(c) \cap D \neq \emptyset$$

Remarks If c is a cluster point of D, c may or may not belong to D. If $c \in D$, then c is not necessarily a cluster point.

Examples:

- Let $D = (0, 1)$. Is $c = 0$ a cluster point of D?

  Yes — given any $\delta > 0$, $\dot{V}_\delta(0) \cap (0, 1) = (0, \min(1.\delta))$. We have that $[0, 1]$ is the set of all limit points of D.
- Let $D = \mathbb{N}$. Then, D admits no cluster points.
- Additionally, all finite sets have no cluster points.
- If $D = \mathbb{Q}$, then the set of cluster points of $\mathbb{Q}$ is $\mathbb{R}$.

  Given any $t \in \mathbb{R}$, $\delta > 0$,

  $$\dot{V}_\delta \cap \mathbb{Q} \neq \emptyset$$

  because $\mathbb{Q}$ is dense.
- If $D = \left\{\frac{1}{n} \mid n \geq 1\right\}$, then $\{0\}$ is the set of cluster points of D.

Sequential Criterion of Cluster Points

Statement: Let $D \subseteq \mathbb{R}$, $c \in \mathbb{R}$. The following are equivalent:

(1) c is a limit point of D.

(2) $\exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$

Proof:

(2) $\Rightarrow$ (1) Let $\delta > 0$. Then, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $0 < |x_n - c| < \delta$. Thus $x_N \in \dot{V}_\delta(c) \cap D$.

(1) $\Rightarrow$ (2) Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in D \cap \dot{V}_{1/n}(c)$. So, $x_n \neq c$, $x_n \in D$, and $|x_n - c| < 1/n$. So, $(x_n)_n \to c$.

Definition of a Limit

Let $f : D \to \mathbb{R}$, and c a limit point of D. Let $L \in \mathbb{R}$.

$$\lim_{x\to c} f(x) = L \overset{\text{defn.}}{\Longleftrightarrow} (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } \forall x \in \dot{V}_\delta(c) \cap D,\ f(x) \in V_\varepsilon(L)$$

Applying the Limit Definition: Linear Function

$$\lim_{x\to c} ax + b = ac + b \qquad a \neq 0$$

Preliminary Work:

$$\begin{aligned} |f(x) - L| &= |ax + b - (ac + b)| \\ &= |ax - ac| \\ &= |a||x - c| \end{aligned}$$

Proof: Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{|a|}$.

$$\begin{aligned} 0 < |x - c| &< \delta \\ 0 < |x - c| &< \frac{\varepsilon}{|a|} \\ |f(x) - L| &= |a||x - c| \\ &< |a|\frac{\varepsilon}{|a|} \\ &= \varepsilon \end{aligned}$$

Applying the Limit Definition: Quadratic Function

$$\lim_{x\to c} x^2 = c^2$$

Preliminary Work:

$$\begin{aligned} |f(x) - L| &= |x^2 - c^2| \\ &= |x - c||x + c| \end{aligned}$$

If $0 < \delta < 1$, and $|x - c| < \delta$, then $|x + c| \leq |x| + |c| \leq 2|c| + 1$. In this case,

$$|f(x) - L| \leq (2|c| + 1)\,|x - c|.$$

Proof: Given $\varepsilon > 0$, let $\delta = \frac{1}{2}\min\left(1, \frac{\varepsilon}{2|c|+1}\right)$. This guarantees $\delta < 1$. So, if $|x - c| < \delta$,

$$\begin{aligned} |f(x) - L| &\leq (2|c| + 1)\,|x - c| \\ &< (2|c| + 1)\,|x - c| \\ &< (2|c| + 1)\,\frac{\varepsilon}{2|c| + 1} \\ &= \varepsilon \end{aligned}$$

Applying the Limit Definition: Rational Function

$$\lim_{x\to c}\frac{1}{x} = \frac{1}{c} \qquad c \neq 0$$

Preliminary Work:

$$\begin{aligned} |f(x) - L| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\ &= \frac{1}{|x|}\frac{1}{|c|}|x - c| \end{aligned}$$

If $x \in \left(c - \frac{|c|}{2}, c + \frac{|c|}{2}\right)$, then $|x| \geq |c|/2$, so $\frac{1}{|x|} \leq \frac{2}{|c|}$. So,

$$\frac{1}{|x|}\frac{1}{|c|}|x - c| \leq \frac{2}{|c|^2}|x - c|$$

Proof: Given $\varepsilon > 0$, let $\delta = \frac{1}{2}\min\left(\frac{|c|}{2}, \frac{|c|^2}{2}\varepsilon\right)$. If

$$\begin{aligned} 0 < |x - c| &< \delta \\ |f(x) - L| &\leq \frac{2}{|c|^2}|x - c| \\ &< \frac{2}{|c|^2}\frac{|c|^2}{2}\varepsilon \\ &= \varepsilon \end{aligned}$$

Uniqueness of Limits

Statement: Let $f : D \to \mathbb{R}$ with c a limit point of D. Then, f can have at most one limit.

Proof: Suppose toward contradiction that $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, where $L_1 \neq L_2$.

Let ε be small such that $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$. So, $\exists \delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow f(x) \in V_\varepsilon(L_1),$$

and $\exists \delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow f(x) \in V_\varepsilon(L_2).$$

Set $\delta = \min(\delta_1, \delta_2)$. Then,

$$0 < |x - c| < \delta \Rightarrow f(x) \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$$

Sequential Criterion for Limits

Statement: Let $f : D \to \mathbb{R}$, c a cluster point of D. The following are equivalent:

(i) $\lim_{x\to c} f = L$

(ii) $\forall (x_n)_n \in D \setminus \{c\}$ where $(x_n)_n \to c$, we have $(f(x_n))_n \to L$

Proof:

(⇐) Assume $\lim_{x\to c} f(x) \neq L$. Then, $(\exists \varepsilon_0)\,(\forall \delta > 0)(\exists x \in \dot{V}(c) \cap D)$ with $|f(x) - L| \geq \varepsilon_0$.

Let $\delta_n = \frac{1}{n}$. Then, $\exists x_n \in \dot{V}_{1/n}(c) \cap D$, with $|f(x_n) - L| \geq \varepsilon_0$.

Note that $0 < |x - c| < 1/n$. So, $(x_n)_n \in D \setminus \{c\}$, and $(x_n)_n \to c$. By (ii), it must be the case that $(f(x_n))_n \to L$.

However, $|f(x_n) - L| \geq \varepsilon_0$. $\perp$

Limit Divergence and Non-Existence

Statement: Let $f : D \to \mathbb{R}$, and c a cluster point of D. Let $L \in \mathbb{R}$. The following are true:

(1) $\lim_{x\to c} f(x) \neq L \Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$ but $f(x_n) \nrightarrow L$

(2) $\lim_{x\to c} f(x)$ DNE $\Leftrightarrow \exists (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$ and $(f(x_n))_n$ divergent.

Proof:

(1) This is a direct negation of the Sequential Definition.

(2)

(⇒) Suppose toward contradiction, $\forall (x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$, we have $(f(x_n))_n$ is convergent.

Pick any two such sequences, $(x_n)_n$ and $(y_n)_n$. We know $(f(x_n))_n \to L_1$, and $(f(y_n))_n \to L_2$.

Consider $(z_n)_n = (x_1, y_1, x_2, y_2, \dots)$. We know that $(z_n)_n \to c$, meaning $(f(z_n))_n \to M$.

The sequence $(f(z_n))_n$ admits two subsequences $(f(x_n))_n \to M$ and $(f(x_n))_n \to M$. Thus, $L_1 = L_2$.

We showed that, for any sequence $(x_n)_n \to c$, $(f(x_n))_n \to L$. Thus, $\lim_{x\to c} f(x)$ exists. $\perp$

Applying Limit Divergence using Sequences

We want to find $\lim_{x\to c} \mathbb{1}_{\mathbb{Q}}$. Consider two sequences $(r_n)_n \to c$, where $r_n \in \mathbb{Q}$ — this is always possible since the rationals are dense — and $(t_n)_n \to c$, where $t_n \notin \mathbb{Q}$.

Let $(x_n)_n = (r_1, t_1, r_2, t_2, \dots)$. Then, $(x_n) \to c$, but $\left(\mathbb{1}_{\mathbb{Q}}(x_n)\right)_n = (1, 0, 1, 0, \dots)$. Thus, $\lim_{x\to c} \mathbb{1}_{\mathbb{Q}}$ DNE.

Bounded Functions and Cluster Points

Statement: Recall that $f : D \to \mathbb{R}$ is bounded on $E \subseteq D$ if $\sup_{x\in E} |f(x)| < \infty$.

If $f : D \to \mathbb{R}$ and c is a cluster point of D, if $\lim_{x\to c} f(x) = L$, then $\exists\delta > 0$ such that f is bounded on $\dot{V}_\delta(c) \cap D$.

Proof: Let $\varepsilon = 1$. Then, $\exists\delta > 0$ such that $x \in \dot{V}_\delta(c) \cap D \Rightarrow |f(x) - L| < 1$. Then,

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L|, \end{aligned}$$

so,

$$\sup_{x\in\dot{V}_\delta(c)} |f(x)| \leq 1 + |L|$$

Operations with Limits

Statement: Let $f, g : D \to \mathbb{R}$, and c is a cluster point of D. Let $\alpha \in \mathbb{R}$.

(a) If $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then

  (i) $\lim_{x\to c}(f \pm g) = L \pm M$
  (ii) $\lim_{x\to c}(\alpha f) = \alpha L$
  (iii) $\lim_{x\to c}(fg) = LM$
  (iv) $\lim_{x\to c} \left(\frac{f}{g}\right) = \frac{L}{M}$ if $M \neq 0$

(b) $\lim_{x\to c} |f(x)| = |L|$

(c) $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$, provided $f(x) \geq 0$

(d) If $f(x)$ is a polynomial, then $\lim_{x\to c} f(x) = f(c)$.

(e) If $f(x)$ is rational, then $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, provided $q(c) \neq 0$.

Proof of (a)(iii): Let $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$. Then, $(f(x_n))_n \to L$, $(g(x_n))_n \to M$. Then,

$$\begin{aligned} (f \cdot g(x_n)) &= (f(x_n)g(x_n))_n \\ &\to LM && \text{by sequence properties} \end{aligned}$$

Proof of (a)(iv): Let $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$. Then, by the properties of sequences,

$$\begin{aligned} \left(\frac{f}{g}(x_n)\right) &= \left(\frac{f(x_n)}{g(x_n)}\right)_n \\ &\to \frac{L}{M} && \text{provided } M \neq 0 \end{aligned}$$

Proof of (d): Let $p(x) = \sum_{k=0}^{n} a_k x^k$. Then,

$$
\begin{aligned}
\lim_{x\to c} p(x) &= \lim_{x\to c}\left(\sum_{k=0}^{n} a_k x^k\right) \\
&= \sum_{k=0}^{n} \lim_{x\to c} a_k x^k && \text{(a)(i)} \\
&= \sum_{k=0}^{n} a_k \lim_{x\to c} x^k && \text{(a)(ii)} \\
&= \sum_{k=0}^{n} a_k \left(\lim_{x\to c} x\right)^k && \text{(a)(i)} \\
&= p(c)
\end{aligned}
$$

Proof of (b) Using the properties of sequence, we can show that $(|f(x_n)|)_n \to |L|$ for $(x_n)_n \in D \setminus \{c\}$ with $(x_n)_n \to c$

Squeeze Theorem

Statement: If $f : D \to \mathbb{R}$, c is a cluster point of D.

(i) If $f(x) \leq b$ for x in a deleted neighborhood of c, and if $\lim_{x\to c} f(x) = L$, then $L \leq b$.

(ii) If $f(x) \geq a$ for all x in a deleted neighborhood of c, and if $\lim_{x\to c} f(x) = L$, then $L \geq a$.

(iii) If $f, g, h : D \to \mathbb{R}$, and c is a cluster point of D. Suppose

$$g(x) \leq f(x) \leq h(x)$$

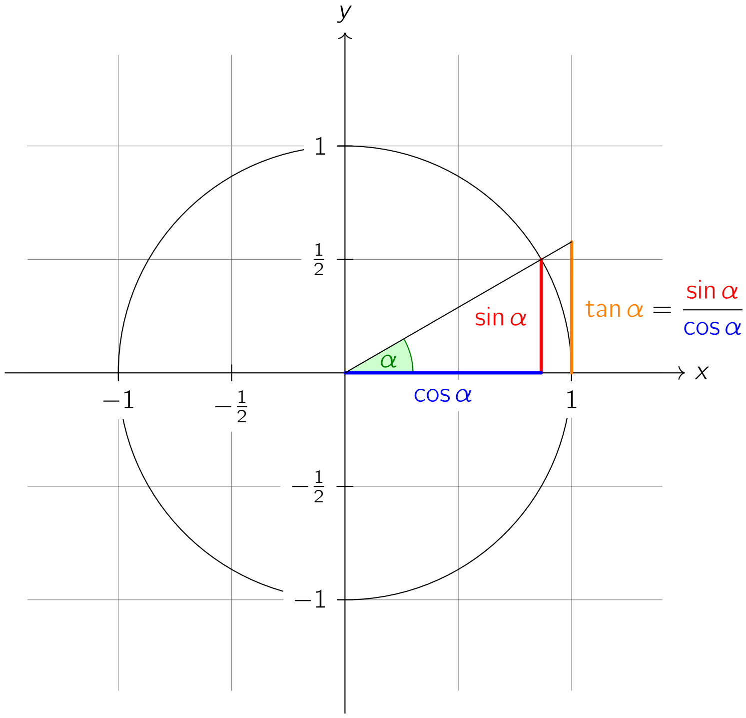
for all x in some deleted neighborhood of c. Suppose $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then, $\lim_{x\to c} f(x) = L$.

Proof of (iii) Let $(x_n)_n \in D \setminus \{c\}$, with $(x_n)_n \to c$. Then, as $n \to \infty$,

$$
\begin{aligned}
g(x_n) \leq f(x_n) \leq h(x_n) \\
L \leq f(x_n) \leq L,
\end{aligned}
$$

so $f(x_n)_n \to L$.

Trigonometric Limits



We know that

$$0 \leq \sin(x) \leq x$$

so as $x \to 0^+$, $\sin(x) \to 0$. Similarly, if $x \to 0^-$, then

$$\begin{aligned}\lim_{x\to 0^-} \sin(x) &= \lim_{y\to 0^+} \sin(-y)\\ &= -\lim_{y\to 0^+} \sin(y)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 0^+} \cos(x) &= \lim_{x\to 0^+} \sqrt{1-\sin^2(x)}\\ &= 1\\ \lim_{x\to 0^-} \cos(x) &= \lim_{y\to 0^+} \cos(-y)\\ &= \lim_{y\to 0^+} \cos(y)\\ &= 1\end{aligned}$$

Claim:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Proof: Let $x \to 0$

$$\frac{\sin(x)}{2} \leq \frac{x}{2} \leq \frac{\tan(x)}{2}$$
$$0 \leq \frac{\sin(x)}{x} \leq 1$$
$$\cos(x) \leq \frac{\sin(x)}{x}$$
$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$
$$1 \leq \frac{\sin(x)}{x} \leq 1$$

Strictly Positive Limits

Statement: Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$. Let c be a cluster point of D. If $\lim_{x\to c} f(x) = L > 0$, then $\exists \delta > 0$ and $\exists t > 0$ such that $f(x) > t$ for $x \in \dot{V}_\delta(c) \cap D$.

Proof: Let $\varepsilon = \frac{L}{2}$. Then, $V_\varepsilon = (L/2, 3L/2)$. So, $\exists \delta > 0$ such that $x \in \dot{V}_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$. Set $t = L/2$.

One-Sided Limits

Let $f : D \to \mathbb{R}$.

Cluster Points:

(i) A number $c \in D$ is a right cluster point if $\forall \delta > 0$, $\exists x \in (c, c+\delta) \cap D$

(ii) A number $c \in D$ is a left cluster point if $\forall \delta > 0$, $\exists x \in (c-\delta, c) \cap D$.

Limits:

(i) $\lim\limits_{x\to c^+} f(x) = L \overset{\text{def}}{\Longleftrightarrow}$

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in (c, c+\delta) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

(ii) $\lim\limits_{x\to c^-} f(x) = L \overset{\text{def}}{\Longleftrightarrow}$

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in (c-\delta, c) \cap D \Rightarrow f(x) \in V_\varepsilon(L)$$

Sequential Criterion:

(i) Let c be a right cluster point of D. $\lim_{x\to c^+} f(x) = L$ if and only if $\forall (x_n)_n \in D \cap (c, \infty)$ with $(x_n)_n \to c$, we have $(f(x_n))_n \to L$

(ii) Let c be a left cluster point of D. $\lim_{x\to c^-} f(x) = L$ if and only if $\forall (x_n)_n \in (-\infty, c) \cap D$ with $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

Limit Equality

Let $f : D \to \mathbb{R}$. Let c be a cluster point of D.

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Infinite Limits

Let $f : D \to \mathbb{R}$, and c be a limit point of D. Then,

$$\lim_{x\to c} f(x) = \infty \overset{\text{def}}{\iff} (\forall M \geq 0)(\exists \delta > 0) \text{ such that } x \in \dot{V}_\delta(c) \cap D \Rightarrow f(x) \geq M$$

We can also define

$$\lim_{x\to c} f(x) = -\infty$$
$$\lim_{x\to c^\pm} f(x) = \pm\infty$$

Applying Infinite Limits

Statement:

$$\lim_{x\to 1^-} \frac{1}{1-x} = -\infty$$

Proof: Let $M \geq 0$ be large. We want $f(x) \geq M$.

$$\frac{1}{1-x} \geq M$$
$$1 - x \leq \frac{1}{M}$$
$$x \geq 1 - \frac{1}{M}$$

Set $\delta = \frac{1}{M}$. If $x \in (1-\delta, 1)$, then $x \geq 1 - \frac{1}{M}$. So, by our work above, $f(x) \geq M$.

Limits at Infinity

Let $f : [a, \infty) \to \mathbb{R}$, $L \in \mathbb{R}$. Then,

$$\lim_{x\to\infty} f(x) = L \overset{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \geq a) \text{ such that } x \geq K \Rightarrow f(x) \in V_\varepsilon(L)$$

Similarly, we can define for $f : (-\infty, b] \to \mathbb{R}$, $L \in \mathbb{R}$

$$\lim_{x\to-\infty} f(x) = L \overset{\text{def}}{\iff} (\forall \varepsilon > 0)(\exists K \leq b) \text{ such that } x \leq K \Rightarrow f(x) \in V_\varepsilon(L)$$

and for $f : [a, \infty)$ where

$$\lim_{x\to\infty} f(x) = \infty \overset{\text{def}}{\iff} (\forall M \geq 0)(\exists K \geq a) \text{ such that } x \geq K \Rightarrow f(x) \geq M$$

and the respective sequential definitions.

Applying Limits at Infinity 1

Statement: Let $n \in \mathbb{N}$.

$$\lim_{x\to\infty} x^n = \infty$$

Proof: Let M be large. We want $x^n \geq M$. Then, $x \geq M^{1/n}$. Set $K = M^{1/n}$.

Applying limits at Infinity: Polynomials

$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty, & n = 2k \\ -\infty, & n = 2k+1 \end{cases}$$

$$p(x) = \sum_{k=1}^{n} a_k x^k$$

$$\lim_{x\to\infty} p(x) = \begin{cases} +\infty, & a_n > 0 \\ -\infty & a_n < 0 \end{cases}$$

Let $g(x) = x^n$.

$$\frac{p(x)}{g(x)} = a_n + a_{n-1}\frac{1}{x} + \cdots + a_0\frac{1}{x^n}$$

$$\lim_{x\to\infty} \frac{p(x)}{g(x)} = a_n$$

Lemma: If $f, g : [a, \infty) \to \mathbb{R}$, and $g(x) > 0$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$$

(1) If $L > 0$, then $\lim_{x\to\infty} f(x) = \infty \Leftrightarrow \lim_{x\to\infty} g(x) = \infty$

(2) If $L < 0$, then $\lim_{x\to\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to\infty} g(x) = +\infty$

Apply the lemma to $p(x)$, x^n.

Continuity and Uniform Continuity

Continuity

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$. Let $c \in D$. The function f is continuous at c if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in V_\delta(c) \cap D \Rightarrow f(x) \in V_\varepsilon(f(c))$$

Remark: Here, c may not be a cluster point of D.

For example, let

$$f(x) = \begin{cases} x & x = -1 \\ x^2 & x \geq 0 \end{cases}$$

$$D = \{-1\} \cup [0, \infty)$$

Here, f is continuous at $c = -1$. Given any $\varepsilon > 0$, let $\delta = 1/2$. Then, if $x \in V_{1/2}(-1) \cap D$, $x = -1$, meaning $|f(x) - f(-1)| = 0 < \varepsilon$

Continuity and Limits

If $f : D \to \mathbb{R}$, $c \in D$ and c a cluster point of D, the following are equivalent:

(i) f is continuous at c

(ii) $\lim_{x\to c} f(x) = f(c)$

Remark: We are deign to use the second definition as *the* definition of continuity due to the fact that it removes the possibility of those mentioned above.

Sequential Criterion of Continuity

Let $f : D \to \mathbb{R}$, $c \in D$. The following are equivalent:

(i) f is continuous at $x = c$

(ii) $\forall (x_n)_n$ in D with $(x_n)_n \to c$, we have $(f(x_n))_n \to f(c)$

Left and Right Continuity

Let $f : D \to \mathbb{R}$, $c \in D$.

- f is left-continuous if

$$\begin{aligned}(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } 0 \leq c - x < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \\ \forall (x_n)_n \in D,\ x_n \leq c,\ (x_n)_n \to c \text{ we have } (f(x_n))_n \to f(c)\end{aligned}$$

- f is right-continuous if

$$\begin{aligned}(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } 0 \leq x - c < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \\ \forall (x_n)_n \in D,\ x_n \geq c,\ (x_n)_n \to c \text{ we have } (f(x_n))_n \to f(c)\end{aligned}$$

Continuity on Sets

Let $f : D \to \mathbb{R}$.

(1) f is continuous on $E \subseteq D$ if f is continuous at each $c \in E$.

(2) f is continuous on $[a, b]$ if

  (i) f is continuous on (a, b)
  (ii) f is left-continuous at b
  (iii) f is right-continuous at a

Applying Continuity on Sets

(1) Polynomials are continuous on $\mathbb{R}$ because $\lim_{x\to c} p(x) = p(c)$.

(2) Rational functions are continuous on their domain.

(3) $f : \mathbb{1}_{\mathbb{Q}}$ is continuous nowhere:

Case 1: Suppose $c \in \mathbb{Q}$. Let $(t_n)_n \to c$ with $t_n \in \mathbb{R} \setminus \mathbb{Q}$. Then, $(f(t_n))_n = 0 \to 0 \neq f(c) = 1$
Case 2: Let $c \in \mathbb{R} \setminus \mathbb{Q}$. Let $(r_n)_n \to c$ with $r_n \in \mathbb{Q}$. Then, $(f(r_n))_n = 1 \to 1 \neq f(c) = 0$

Discontinuity

$f : D \to \mathbb{R}$ is not continuous at $x = c$ if $\exists (x_n)_n$ in D with $(x_n)_n \to c$ and $(f(x_n))_n \nrightarrow f(c)$

Discontinuity of the Sign Function

$$\operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at $x = 0$, since $(x_n)_n = \frac{1}{n} \to 0$ but $(f(x_n))_n = 1 \neq 0$.

Discontinuity of Thomae's Function

Statement: Let

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \in \mathbb{Q},\ b \in \mathbb{N},\ \gcd(a, b) = 1 \\ 1 & x = 0 \end{cases}$$

Proof:

Claim 1: f is not continuous at $x \in \mathbb{Q}$: find a sequence $(t_n)_n$ of irrationals with $(t_n)_n \to x$. Then, $(f(t_n))_n = 0 \neq f(x) = \frac{1}{b}$

Claim 2: f is continuous at $t \in \mathbb{R} \setminus \mathbb{Q}$: let $t \in \mathbb{R} \setminus \mathbb{Q}$, $t > 0$. Let $n \in \mathbb{N}$. Consider

$$A_n = \left\{ \frac{a}{b} \mid 1 \leq b \leq n \right\} \cap (t - 1, t + 1).$$

We claim that A_n is finite.

$$\begin{aligned} t - 1 < \frac{a}{b} < t + 1 \\ b(t - 1) < a < b(t + 1) \\ t - 1 < a < n(t + 1), \end{aligned}$$

so there are finitely many values of a and finitely many values of b — therefore, A_n is finite. One can find $\delta > 0$ such that $(t - \delta, t + \delta) \cap A_n = \emptyset$

Given $\varepsilon > 0$, find $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \varepsilon$. Let δ be such that $(t - \delta, t + \delta) \cap A_{n_0} = \emptyset$. If $x \in (t - \delta, t + \delta)$,

$$\begin{aligned} |f(x) - f(t)| &= |f(x)| \\ &= \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{b} & x = \frac{a}{b} \text{ lowest terms} \end{cases} \end{aligned}$$

but $\frac{1}{b} < \varepsilon$ because $x \notin A_{n_0}$, meaning $b > n_0$.

Extension of a Function

Consider

$$g(x) = \sin\left(\frac{1}{x}\right) \qquad x \neq 0$$

Assuming that g is continuous on its domain, can we find a $\tilde{g} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{g}(x) = g(x) \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

If such a $\tilde{g}$ existed, we would expect that $\lim_{x \to 0} \tilde{g}(x) = \tilde{g}(0)$. But, $\lim_{x \to 0} \tilde{g}(x) = \lim_{x \to 0} g(x)$. However, since $\lim_{x \to 0} g(x)$ DNE, so such an extension does not exist.

Therefore, $x = 0$ is known as a non-removable discontinuity (i.e., we cannot create an extension of the function that "fills in" the function).

However, not all discontinuities involving $\sin(1/x)$ are non-extendible:

$$\begin{aligned} f(x) &= x \sin\left(\frac{1}{x}\right) \\ \tilde{f}(x) &= \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned}$$

Jump Discontinuities

Suppose $\lim_{x\to c^-} f(x) = L$, $\lim_{x\to c^+} f(x) = R$. If $L \neq R$, then $x = c$ is a jump discontinuity.

Lipschitz Functions

A function $f : D \to \mathbb{R}$ is called Lipschitz if $\exists c \geq 0$ with

$$|f(x) - f(y)| \leq c|x - y| \qquad \forall x, y \in D$$

The linear function $f(x) = ax + b$ is a Lipschitz function. Additionally, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $\|T(\vec{v}) - T(\vec{w})\| \leq c\|\vec{v} - \vec{w}\|$ for any norm on $\mathbb{R}^n$ and $\mathbb{R}^m$.

- If $c < 1$, then f is a contraction.
- If $c = 1$ and $|f(x) - f(y)| = |x - y|$, f is called an isometry.

Lipschitz functions are continuous on their domain:

Proof: Let $c \in D$, let $\varepsilon > 0$. Set $\delta = \varepsilon/c$.

$$\begin{aligned} |x - c| &< \delta \\ |f(x) - f(c)| &\leq c|x - c| \\ |f(x) - f(c)| &< c\delta \\ &= \varepsilon \end{aligned}$$

If $f(x) = \sin(x)$, then

$$\begin{aligned} |\sin(x) - \sin(y)| &= \left|2\sin\left(\frac{1}{2}(x-y)\right)\cos\left(\frac{1}{2}(x+y)\right)\right| \\ &\leq 2\frac{1}{2}|x - y| \\ &= |x - y| \end{aligned}$$

Properties of Continuous Functions

Equality over Dense Subsets

Statement: Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. Let $E \subseteq \mathbb{R}$. If $f(x) = g(x)\ \forall x \in E$, then $f = g$.

Proof: Let $t \in \mathbb{R}$. Since E is dense, $\exists (x_n)_n \in E$ such that $(x_n)_n \to t$. So, $(f(x_n))_n \to t$ because f is continuous, and $(g(x_n))_n \to g(t)$ because g is continuous.

However, since $f(x_n) = g(x_n)\ \forall x_n$, it must be the case that $f(t) = g(t)$.

Boundedness over a Dense Subset

Statement: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f|_E$ is bounded. That is, $\exists c$ such that

$$|f(x)| \leq c. \qquad \forall x \in E$$

Then, f is bounded.

Proof: Let $t \in \mathbb{R}$. Since E is dense, $\exists (x_n)_n \in E$ such that $(f(x_n))_n \to t$. Then,

$$|f(x_n)_n| \leq c,$$

meaning that $f(t) \leq c$.

Bounding Away From 0

Statement: If f is continuous at $x = c$ and $f(c) > 0$, then $\exists \delta > 0$ and $\exists m > 0$ with $f(x) \leq m\ \forall x \in V_\delta(c)$. Similarly for the negative case.

Proof: Let $\varepsilon = f(c)/2 > 0$. Then, $\exists \delta > 0$ such that $\forall x \in V_\delta(c)$, $f(x) \in V_\varepsilon(f(c)) = (f(c)/2, 3f(c)/2)$. Set $m = f(c)/2$.

Continuity over Operations

Let $f, g : D \to \mathbb{R}$, $c \in D$.

(1) If f, g are continuous at $x = c$, then $f \pm g$ are continuous at $x = c$. Similarly, if f, g are continuous on D, then $f \pm g$ is continuous on D.

(2) Let $\alpha \in \mathbb{R}$. If f is continuous at $x = c$ or on D, then αf is continuous at $x = c$ or D respectively.

(3) If f, g are continuous at $x = c$ or on D, then $f \cdot g$ is continuous on $x = c$ or D respectively.

(4) If f, g are continuous at $x = c$, and $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c. Likewise, if f, g are continuous on D and $g(x) \neq 0\ \forall x \in D$, then $\frac{f}{g}$ is continuous.

(5) If g is continuous at $x = c$ and f is continuous at $d = g(c)$, then $f \circ g$ is continuous at $x = c$. If $\text{ran}(g) \subseteq \text{dom}(f)$, with f, g continuous on their domain, then $f \circ g$ is continuous.

(6) If $f : D \to \mathbb{R}$ is continuous, and $f(x) \geq 0$ on D, then $\sqrt{f(x)}$ is continuous on D.

(7) If $f : D \to \mathbb{R}$ is continuous on D, then $|f(x)|$ is continuous.

(8) Polynomials and Rational functions are continuous on their domain.

(9) If $f(x), g(x)$ are continuous, then $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$.

Remark on (4): If $g(c) \neq 0$, then $g \neq 0$ on a δ-neighborhood of c.

Proof of (5): Let $(x_n)_n \to c$. Then, $g(x_n)_n \to g(c)$. So, $(f(g(x_n)))_n \to f(g(c))$.

Fundamental Theorem of Continuous Functions on $[a, b]$

Boundedness Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then $\|f\|_u < \infty$.

> **Proof:** Suppose it is not the case. Given any $n \geq 1$, $\exists x_n \in [a, b]$ with $|f(x_n)| \leq n$. We thus have a sequence $(x_n)_n \in [a, b]$.
>
> By Bolzano-Weierstrass, $\exists (x_{n_k})_k \to x \in [a, b]$. So, $f(x_{n_k}) \to f(x)$. In particular, $(f(x_{n_k}))_k$ is bounded; however, $f(x_{n_k}) \geq k$. $\perp$
>
> **Note:** It is possible for f to be bounded on an infinite interval where it does not attain the supremum or infimum.

Let $f : D \to \mathbb{R}$.

(1) f has an absolute maximum on D if $\exists x_M \in D$ with $f(x) \leq f(x_M)\ \forall x \in D$. Notably, this means $\sup_{x \in D} f(x) = f(x_M)$.

(2) f has an absolute minimum on D if $\exists x_m \in D$ with $f(x_m) \leq f(x)\ \forall x \in D$. Notably, this means $\inf_{x \in D} f(x) = f(x_m)$.

Extreme Value Theorem (EVT): If $f : [a, b] \to \mathbb{R}$ is continuous, then f admits an absolute minimum and absolute maximum.

Proof: We know that $\sup_{x\in[a,b]} f(x) = u < \infty$ by the boundedness theorem. For each $n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that

$$u - \frac{1}{n} < f(x_n) \leq u.$$

Thus, there is a sequence $(x_n)_n \in [a, b]$ — by Bolzano-Weierstrass, $\exists (x_{n_k})_k \to x^*$ for some $x^* \in [a, b]$. So, for each k,

$$\begin{aligned} u - \frac{1}{n_k} &< f(x_{n_k}) \leq u \\ u &< f(x^*) \leq u. && \text{since } f \text{ is continuous} \end{aligned}$$

So, by the squeeze theorem, $f(x^*) = u$ is our absolute max.

Corollary to the Extreme Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous with $f(x) > 0\ \forall x \in [a, b]$, then $\exists \alpha > 0$ such that $f(x) \geq \alpha\ \forall x \in [a, b]$.

Proof: By the previous theorem, we know $\exists x_m \in [a, b]$ such that $f(x) \geq f(x_m)\ \forall x \in [a, b]$. But $\alpha := f(x_m) > 0$ by definition.

Location of Roots: We will use this to prove the Intermediate Value Theorem. Let $f : [a, b] \to \mathbb{R}$ be continuous, Suppose $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. Then, $\exists c \in (a, b)$ such that $f(c) = 0$.

Proof: Assume $f(a) < 0$ and $f(b) > 0$. Let $N = \{x \in [a, b] \mid f(x) \geq 0\}$. Since $b \in N$, $N \neq \emptyset$. Let $z = \inf N$. We claim that $f(z) = 0$.

We know that $\exists (x_n)_n \in N$ with $x_n \to z$. Since $(x_n)_n \in N$, $f(x_n) \geq 0\ \forall n \geq 1$. However, $f(x_n) \to f(z)$ since f is continuous. So, $f(z) \geq 0$.

Suppose toward contradiction that $f(z) > 0$. So, $\exists \delta > 0$ such that $f(x) \geq \frac{f(z)}{2}$ on $(z - \delta, z + \delta)$. Then, $z - \frac{\delta}{2} \in N$. $\perp$

Intermediate Value Theorem (IVT): Let $f : I \to \mathbb{R}$, where I is any interval. Suppose $\exists x_1, x_2 \in I$ and $k \in \mathbb{R}$, with $f(x_1) < k < f(x_2)$. Then, $\exists \xi$ strictly between x_1 and x_2, with $f(\xi) = k$.

Proof: Clearly, $x_1 \neq x_2$. Suppose $x_1 < x_2$. Consider $g : [x_1, x_2] \to \mathbb{R}$, $g(x) = f(x) - k$. So, g is continuous (as f is continuous), and $g(x_1) = f(x_1) - k < 0$, and $g(x_2) = f(x_2) - k > 0$. Thus, $\exists \xi \in [x_1, x_2]$ with $g(\xi) = 0$, whence $f(\xi) = k$.

Corollary to IVT and EVT: Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\inf_{[a,b]} f \leq k \leq \sup_{[a,b]} f$, then $\exists c \in [a, b]$ with $f(c) = k$.

Proof: We know that by EVT, $\exists x_m, x_M$ with $\inf_{[a,b]} f = f(x_m)$ and $\sup_{[a,b]} f = f(x_M)$. So, $f(x_m) \leq k \leq f(x_M)$. Apply IVT.

Preservation of Intervals 1: If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b]) = [c, d]$.

Proof: Set $c = \inf_{[a,b]} f$ and $d = \inf_{[c,d]} f$. By definition, $c \leq f(x) \leq d$, meaning $f([a, b]) \subseteq [c, d]$. By the previous corollary, if $k \in [c, d]$, then $\exists \xi \in [a, b]$ with $f(\xi) = k$. Thus, $[c, d] \subseteq f([a, b])$.

Preservation of Intervals 2: Let I be any interval, and $f : I \to \mathbb{R}$ continuous. Then, $f(I)$ is an interval.

> **Proof:** Let $\alpha, \beta \in f(I)$. WLOG, $\alpha < \beta$. We will show that $[\alpha, \beta] \in f(I)$. Say $f(a) = \alpha$ and $f(b) = \beta$ for some $a, b \in I$. Note that $a \neq b$. Let $\alpha < k < \beta$. By IVT, $\exists \xi$ strictly between a and b with $f(\xi) = k$. If $a < b$, then $[a, b] \subseteq I$, and if $b < a$, then $[b, a] \subseteq I$. Thus, $\xi \in I$.

Uniform Continuity

A function $f : D \to \mathbb{R}$ is **uniformly continuous** on D if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } u, v \in D,\ |u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$$

Uniform continuity is different from continuity in that f is continuous at a point $x = c$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

In (non-uniform) continuity, $\delta = \delta(\varepsilon, c)$.

Illustrating Non-Uniform Continuity

For example, if $f(x) = \frac{1}{x}$ and $D = (0, \infty)$, we will show that f is continuous at $x = c > 0$.

$$\begin{aligned} |f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\ &= \frac{1}{c}\frac{1}{x}|x - c| \end{aligned}$$

if $0 < \delta < c/2$ and $|x - c| < \delta$, then $x \geq c/2$. Thus,

$$\begin{aligned} |f(x) - f(c)| &= \frac{1}{c}\frac{2}{c}|x - c| \\ &= \frac{2}{c^2}|x - c|. \end{aligned}$$

Given $\varepsilon > 0$, pick $\delta = \frac{1}{2}\min\left(\frac{c}{2}, \frac{2}{c^2}\varepsilon\right)$. Thus, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

Specifically, we can see that on this domain, we require that δ be a function of ε and c.

Proving Uniform Continuity 1

However, if we look at $f(x) = \frac{1}{x}$ on $[1, \infty)$, we can see that for $u, v \geq 1$,

$$\begin{aligned} |f(u) - f(v)| &= \left|\frac{1}{u} - \frac{1}{v}\right| \\ &= \frac{1}{uv}|v - u| \\ &\leq |v - u| \end{aligned}$$

Given $\varepsilon > 0$, set $\delta = \varepsilon$. If $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon$.

Here, we see that $\delta = \delta(\varepsilon)$.

Proving Uniform Continuity 2

We will show that $f(x) = x^2$ is uniformly continuous on $[1, 4]$.

$$\begin{aligned} |f(u) - f(v)| &= |u^2 - v^2| \\ &= |u - v||u + v| \\ &\leq |u - v|\left(|u| + |v|\right) && \text{Triangle Inequality} \\ &\leq 8|u - v| \end{aligned}$$

Given $\varepsilon > 0$, set $\delta = \varepsilon/8$. Whenever $u, v \in [1, 4]$, with $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon$

Lipschitz and Uniform Continuity

Statement: If $f : D \to \mathbb{R}$ is Lipschitz, then f is uniformly continuous.
Proof: If $f : D \to \mathbb{R}$ is Lipschitz, then $\exists c > 0$ such that $\forall u, v \in D$,

$$|f(u) - f(v)| \leq c|x - y|.$$

Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{c}$. Whenever $|u - v| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Uniform Continuity and Continuity

Statement: If $f : D \to \mathbb{R}$ is uniformly continuous, then f is continuous on D.

Proof: Let $c \in D$. Given $\varepsilon > 0$, by uniform continuity, $\exists \delta > 0$ such that

$$\begin{aligned} |u - v| < \delta &\Rightarrow |f(u) - f(v)| < \varepsilon \\ |x - c| < \delta &\Rightarrow |f(x) - f(c)| < \varepsilon \end{aligned}$$

Negating Uniform Continuity

Statement: The following are equivalent for $f : D \to \mathbb{R}$

(i) f is *not* uniformly continuous

(ii) $\exists \varepsilon_0$ such that $\forall \delta > 0$, $\exists u_\delta, v_\delta$ such that $|u_\delta - v_\delta| < \delta$ and $|f(u_\delta) - f(v_\delta)| \geq \varepsilon_0$

(iii) $\exists \varepsilon_0$ such that $\exists (u_n)_n, (v_n)_n \in D$ with $(u_n - v_n)_n \to 0$ and $|f(u_n) - f(v_n)| \geq \varepsilon_0$

Proof:

(i) $\Leftrightarrow$ (ii): Negating definition.

(ii) $\Rightarrow$ (iii): Set $\delta_n = 1/n$ in (ii). Given δ_n, it must be the case that

$$|u_n - v_n| < \frac{1}{n}$$

so $(u_n - v_n)_n \to 0$, and

$$|f(u_n) - f(v_n)| \geq \varepsilon_0.$$

(iii) $\Rightarrow$ (ii): Let $\delta > 0$. Then, $\exists N \in \mathbb{N}$ large such that $|u_N - v_N| < \delta$, by the definition of sequence convergence. Set $u_\delta = u_N$ and $v_\delta = v_N$.

Applying Non-Uniform Continuity 1

We will show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

Set $u_n = 1/n$, and $v_n = \frac{1}{n+1}$. Then,

$$\begin{aligned}|f(u) - f(v)| &= |n - (n+1)| \\ &= 1 \\ &= \varepsilon_0 \\ |u_n - v_n| &= \left|\frac{1}{n} - \frac{1}{n+1}\right| \\ &= \frac{1}{n(n+1)} \\ &\to 0\end{aligned}$$

Applying Non-Uniform Continuity 2

Consider $f(x) = x^2$ on $[0,\infty)$. We will show that f is not uniformly continuous.

Let $u_n = n$ and $v_n = n + \frac{1}{n}$. Clearly, $(u_n - v_n)_n \to 0$.

$$\begin{aligned}|f(u_n) - f(v_n)| &= \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| \\ &= \left|n^2 - n^2 - 2 - \frac{1}{n^2}\right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2\end{aligned}$$

Uniform Continuity Theorem

Statement: If $f : [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Proof: Suppose toward contradiction that f is not uniformly continuous. Then, $\exists (u_n)_n, (v_n)_n \in [a,b]$ and $\varepsilon_0 > 0$ such that $(u_n - v_n)_n \to 0$ and $|f(u_n) - f(v_n)| \geq \varepsilon_0 > 0$.

Since $(u_n)_n$ is bounded, $\exists n_k$ such that $(u_{n_k})_k \to z$ by the Bolzano-Weierstrass. We claim that $(v_{n_k})_k \to z$:

$$\begin{aligned}|v_{n_k} - z| &= |v_{n_k} - u_{n_k} + u_{n_k} - z| \\ &\leq |v_{n_k} - u_{n_k}| + |u_{n_k} - z| \\ &\to 0.\end{aligned}$$

So,

$$\begin{aligned}0 < \varepsilon_0 &\leq |f(u_{n_k}) - f(v_{n_k})| \\ &\to 0\end{aligned}$$

since $(f(u_k))_k \to f(z)$ and $(f(v_k))_k \to f(z)$.

Uniform Continuity and Lipschitz

The function $f(x) = \sqrt{x}$ on $[0,1]$ is uniformly continuous. However, $f(x) = \sqrt{x}$ is not Lipschitz.

Suppose toward contradiction that f is Lipschitz.

$$|f(x) - f(y)| \leq c|x - y| \qquad \forall x, y \in [0,1]$$

Take $y = 0$.

$$\begin{aligned} \sqrt{x} &\leq cx \\ 0 < \frac{1}{c} &\leq \sqrt{x} \end{aligned}$$

Uniform Limit Theorem

Statement: If $f_n : D \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to $f : D \to \mathbb{R}$. Then, f is continuous.

Proof: Let $x, c \in D$ with, for $\varepsilon > 0$ and N large, $|x - c| < \delta$. Then,

$$\begin{aligned} |f_N(x) - f(x)| &< \varepsilon/3 && \text{by uniform convergence} \\ |f_N(c) - f(c)| &< \varepsilon/3 && \text{by uniform convergence} \\ |f_N(x) - f_N(c)| &< \varepsilon/3 && \text{by continuity} \end{aligned}$$

so, for $|x - c| < \delta$,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f_N(x) - f(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| && \text{Triangle Inequality} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

Lemma: Uniform Continuity and Cauchy Sequences

Statement: Let $f : D \to \mathbb{R}$ be uniformly continuous. If $(x_n)_n \in D$ is Cauchy, then $(f(x_n))_n$ is Cauchy.

This is not true for mere continuity. For example, for $f(x) = \frac{1}{x}$ in $(0, \infty)$, $(x_n)_n = \frac{1}{n}$ is Cauchy in $(0.\infty)$, but $f(x_n) = n$ is not Cauchy.

Proof: Let $(x_n)_n$ be Cauchy. Let $\varepsilon > 0$. Then, $\exists \delta > 0$ such that $\forall u, v \in D$ with $|u - v| < \delta$, we have $|f(u) - f(v)| < \varepsilon$.

Since $(x_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ such that for $p, q \geq N$, $|x_p - x_q| < \delta$. So, $|f(x_p) - f(x_q)| < \varepsilon$. So, $(f(x_n))_n$ is Cauchy.

Continuous Extension Theorem

Statement: Let $f : (a,b) \to \mathbb{R}$ be a map. The following are equivalent:

(1) f is uniformly continuous.

(2) $\exists \tilde{f} : [a,b] \to \mathbb{R}$ such that

- $\tilde{f}$ is continuous
- $\tilde{f}(x) = f(x)\ \forall x \in (a,b)$

Proof:

(2) $\Rightarrow$ (1)**:** Since $\tilde{f}$ is continuous on $[a,b]$, $\tilde{f}$ is uniformly continuous on $[a,b]$. So, $\tilde{f}$ is uniformly continuous on (a,b). But, $\tilde{f}=f$ on (a,b). So, f is uniformly continuous.

(1) $\Rightarrow$ (2)**:** Let $f:(a,b)\to\mathbb{R}$ be uniformly continuous.

Claim: $\lim_{x\to a^+} f(x)$ exists. Let $(x_n)_n$ be any sequence where $x_n > a$ and $(x_n)_n \to a$. Then, $(x_n)_n$ is Cauchy. So, by the lemma, $(f(x_n))_n$ is Cauchy. Since $\mathbb{R}$ is complete, $\exists L\in\mathbb{R}$ such that $(f(x_n))_n \to L$.

We claim that the limit is L. Let $(y_n)_n$ be any sequence with $y_n > a$, $(y_n)_n \to a$. By our work above, $(f(y_n))_n \to L'$ for some $L'\in\mathbb{R}$. Consider $z_n = (x_1, y_1, x_2, y_2, \dots)$. Then, $z_n > a$ with $(z_n)_n \to a$. By our work above, $(f(z_n)) \to L''$, for some $L''\in\mathbb{R}$. Since $(f(x_n))_n$ is a subsequence of $(f(z_n))_n$, $(f(x_n))_n \to L''$, so $L = L''$, and similarly, $L' = L''$.

Therefore, $L = L'$. So, we have $\lim_{x\to a^+} f(x) = L$.

Similarly, $\lim_{x\to b^-} f(x) = R$ exists. Set $\tilde{f}:[a,b]\to\mathbb{R}$ such that

$$\tilde{f}(x) = \begin{cases} f(x) & x\in(a,b) \\ L & x = a \\ R & x = b \end{cases}$$

Then, $\tilde{f}$ is the desired continuous extension.

Applying the Continuous Extension Theorem

If $f(x)=\sin(1/x)$, then $f(x)$ is not uniformly continuous on $(0,1)$. This is because $\lim_{x\to 0^+} f(x)$ does not exist.

Meanwhile, $g(x) = x\sin(1/x)$ is uniformly continuous on $(0,1)$, since we can define $\tilde{g}(x)$ as follows:

$$\tilde{g}(x) = \begin{cases} 0 & x = 0 \\ g(x) & 0 < x < 1 \end{cases}$$

Approximation by Step Function

A map $s:[a,b]\to\mathbb{R}$ is called a step function if

(1) $[a,b] = \bigsqcup_{j=1}^{n} I_j$ where I_j are intervals.

(2) $\exists c_1,\dots,c_n\in\mathbb{R}$ such that $s(x) = c_j\ \forall x\in I_j$.

Alternatively, this is equivalent to:

$$s = \sum_{j=1}^{n} c_j \mathbb{1}_{I_j}$$

Statement: If $f:[a,b]\to\mathbb{R}$ is uniformly continuous and $\varepsilon > 0$, then $\exists s:[a,b]\to\mathbb{R}$ with $\|f-s\|_u < \varepsilon$.

Proof: We know that f is uniformly continuous. Given $\varepsilon > 0$, $\exists \delta > 0$ with $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$. Choose N large such that

$$\begin{aligned}\Delta_n &= \frac{b-a}{N}\\ &< \delta.\end{aligned}$$

Set $x_j = j\Delta_N$. Set $I_j = [x_j, x_{j+1})$ with $0 \leq j \leq N-1$.

Set $c_j = f(x_j)$,

$$s = \sum_{j=0}^{N-1} c_j \mathbb{1}_{I_j}.$$

If $x \in [a, b]$, $x \in I_k$ for some $k = 0, \dots, N-1$. Then,

$$\begin{aligned}|f(x) - s(x)| &= |f(x) - c_k|\\ &\leq |f(x) - f(x_k)|\\ &< \varepsilon\end{aligned}$$

since

$$\begin{aligned}|x - x_k| &< \Delta_N\\ &< \delta\end{aligned}$$

so,

$$\|f - s\|_u < \varepsilon$$

Approximation by Piecewise Linear Function

A function g is piecewise linear if

(a) $[a, b] = \bigsqcup_{j=1}^{n} I_j$, where I_j are intervals.

(b) $g|_{I_j}$ is linear; $\exists a_1, b_1, \dots, a_n, b_n$ with $g(x) = a_j + b_j x \ \forall x \in I_j$.

Statement: If $f : [a, b] \to \mathbb{R}$ is uniformly continuous and $\varepsilon > 0$, then there is a continuous piecewise linear $g : [a, b] \to \mathbb{R}$ with $\|f - g\|_u < \varepsilon$.

Proof: We know that f is uniformly continuous. Given $\varepsilon > 0$, $\exists \delta > 0$ with $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon/2$. Choose N large such that

$$\begin{aligned}\Delta_n &= \frac{b-a}{N}\\ &< \delta.\end{aligned}$$

Set $x_j = j\Delta_N$. Set $I_j = [x_j, x_{j+1})$ with $0 \leq j \leq N-1$.

Set $g(x) = \sum_{k=0}^{N-1} g_k(x) \mathbb{1}_{I_k}$, where

$$g_k(x) = f(x_k) + \left(\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}\right)(x - x_k)$$

We observe that if $x \in I_k$, then

$$\begin{aligned} |f(x) - g(x)| &= \left| f(x) - f(x_k) - \left(\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right) (x - x_k) \right| \\ &\leq |f(x) - f(x_k)| + |f(x_{k+1}) - f(x_k)| \frac{|x - x_k|}{|x_{k+1} - x_k|} \\ &\leq |f(x) - f(x_k)| + |f(x_{k+1}) - f(x_k)| \\ &< \varepsilon \end{aligned}$$

so,

$$\|f - g\| < \varepsilon$$

Monotone Functions

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$.

(1) f is increasing if $x_1, x_2 \in D$ with $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$.

(2) f is strictly increasing if $x_1, x_2 \in D$ with $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

(3) f is monotone if f is increasing or decreasing.

(4) f is strictly monotone if f is strictly increasing or strictly decreasing.

If $f : D \to \mathbb{R}$ is increasing or strictly increasing, then $-f : D \to \mathbb{R}$ is decreasing or strictly decreasing (respectively).

Additionally, monotone functions are not always continuous. However, one-sided limits always exist.

Statement: Let I be an interval, $f : I \to \mathbb{R}$ increasing. Let $c \in I$, where c is not an endpoint. Then,

(1) $\lim_{x \to c^-} f(x) = \sup_{x \in I, x < c} f(x)$

(2) $\lim_{x \to c^+} f(x) = \inf_{x \in I, x < c} f(x)$

are both existent and finite:

Proof of (1): Since c is not an endpoint, $\{x \mid x \in I, x < c\} \neq \emptyset$ and is bounded above by c. Therefore, $\{f(x) \mid x \in I, x < c\}$ is nonempty and bounded above by $f(c)$ (since f is increasing). So, $u = \sup_{x \in I, x < c} f(x)$ exists.

Let $\varepsilon > 0$. $\exists x_\varepsilon \in I$ with $x_\varepsilon < c$ such that $u - \varepsilon < f(x_\varepsilon)$. Set $\delta = c - x_\varepsilon > 0$. If $x \in I, c - x < \delta$, then $x_\varepsilon < x < c$, so $f(x_\varepsilon) \leq f(x) \leq f(c)$. So, $u - f(x) \leq u - f(x_\varepsilon) < \varepsilon$. But, $u \geq f(x)$, so $u - f(x) = |u - f(x)|$. Thus, $0 < c - x < \delta \Rightarrow |u - f(x)| < \varepsilon$. Thus, $u = \lim_{x \to c^-} f(x)$.

Limits and Continuity with Monotone Functions

Let I be an interval, $f : I \to \mathbb{R}$ increasing. Suppose $c \in I$ is not an endpoint. The following are equivalent:

(1) f is continuous at $x = c$.

(2) $\lim_{x \to c} f(x) = f(c)$.

(3) $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+}$.

(4) $\sup_{x \in I, x < c} f(x) = f(c) = \inf x \in I, x > c f(x)$.

Suppose c is a right endpoint of I. The following are equivalent:

(1) f is continuous at $x = c$.

(2) $\lim_{x\to c^-} f(x) = f(c)$.

(3) $\sup_{x\in I, x<c} f(x) = f(c)$.

Suppose c is a left endpoint of I. The following are equivalent:

(1) f is continuous at $x = c$.

(2) $\lim_{x\to c^+} f(x) = f(c)$.

(3) $\inf_{x\in I, x>c} f(x) = f(c)$.

We can make a similar set of corollaries with decreasing functions.

Jump of a Function

Let I be an interval, $f : I \to \mathbb{R}$ increasing.

(1) If c is not an endpoint of I, we define the jump of f at $x = c$ as:

$$j_f(c) = \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x)$$

(2) If c is a left endpoint of I, we define the jump of f at $x = c$ as:

$$j_f(c) = \lim_{x\to c^+} f(x) - f(c)$$

(3) If c is a right endpoint of I, we define the jump of f at $x = c$ as:

$$j_f(c) = f(c) - \lim_{x\to c^-} f(x)$$

Statement: We claim that f is continuous at $c \in I$ if and only if $j_f(c) = 0$.

Proof: If c is not an endpoint, then f is continuous at $x = c$ if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

If c is a left endpoint, then f is continuous at $x = c$ if and only if $f(c) = \lim_{x\to c^+ f(x)}$, if and only if $j_f(c) = 0$.

Countability of Monotone Function Discontinuities

Statement: Let $I \subseteq \mathbb{R}$ be any interval. Let $f : I \to \mathbb{R}$ be monotone. Then, $D = \{x \in I \mid f \text{ not continuous at } x = c\}$ is countable.

Proof: For the sake of simplicity, we will assume that f is monotone increasing.

Lemma: Let $\{x_1, x_2, \dots, x_n\}$ be a partition of $I = [a, b]$, where $a \leq x_1 < x_2 < \cdots < x_n \leq b$. Then, $f(a) + \sum_{i=1}^{n} j_f(x_i) \leq f(b)$.

Proof of Lemma: By induction on n, if $x_1 = a$, then

$$\begin{aligned}
f(a) + j_f(x_1) &= f(a) + j_f(a) \\
&= f(a) + \lim_{x\to a^+} f(x) - f(a) \\
&= \lim_{x\to a^+} \\
&\leq f(b).
\end{aligned}$$

If $x_1 = b$, then

$$\begin{aligned}
f(a) + j_f(x_1) &= f(a) + j_f(b)\\
&= f(a) + f(b) - \lim_{x\to b^-} f(b)\\
&= f(b) - (\lim_{x\to b^-} f(x) - a)\\
&\leq f(b).
\end{aligned}$$

If $a < x_1 < b$, then

$$\begin{aligned}
f(a) + j_f(x_1) &= f(a) + \lim_{x\to x_1^+} f(x) - \lim_{x\to x_1^-} f(x)\\
&\leq f(a) - \lim_{x\to x_1^-} f(x) + f(b)\\
&\leq f(b)
\end{aligned}$$

Assume the formula holds for n. Then, for the $n+1$ case:

$$\begin{aligned}
f(a) + \sum_{i=1}^{n+1} j_f(x_i) &= f(a) + \sum_{i=1}^{n} f(x_i) + j_f(x_{n+1})\\
&\leq f(x_n) + j_f(x_{n+1})\\
&\leq f(b)
\end{aligned}$$

Case 1: Suppose $I = [a, b]$. Consequently,

$$\sum_{i=1}^{n} j_f(x_i) \leq f(b) - f(a)$$

Let $G_k = \left\{x \in [a, b] \mid j_f(x) \geq \frac{f(b)-f(a)}{k}\right\}$. By the lemma, $|G_k| \leq k$. This is because, if $x_1, \dots, x_n \in G_k$ with $n > k$, then

$$\begin{aligned}
\sum_{i=1}^{n} j_f(x_i) &\geq \frac{n(f(b) - f(a))}{k}\\
&> f(b) - f(a)
\end{aligned}$$

contradicting the lemma.

Recall that f is discontinuous at $x = c$ if and only if $j_f(c) > 0$. Therefore, we have that

$$D = \bigcup_{k=1}^{\infty} G_k,$$

So for k large enough, $j_f(x) \geq \frac{f(b)-f(a)}{k}$. Since each G_k is a finite set, D is a countable union of countable sets, and is thus countable.

Case 2: $I = (a, b]$. Write I as

$$I = \bigcup_{n=1}^{\infty} [a + 1/n, b].$$

Let $D_n = \{x \in [a + 1/n, b] \mid f \text{ discontinuous at } x\}$. By case 1, D_n is countable. Let $D = \{x \in (a, b] \mid f \text{ discontinuous at } x\}$. Note that $D = \cup D_n$. Therefore, D is countable.

Case 3: $I = [a, b)$. Write I as

$$I = \cup_{n\geq 1}[a, b - 1/n].$$

Proceed as with case 2.

Case 4: $I = (a, b)$. Write I as

$$I = (a, b - \delta] \cup [b - \delta, b).$$

Apply case 2 and case 3.

Case 5: $I = (-\infty, b)$ or $I = (-\infty, b]$. Write I as

$$I = \bigcup_{n\geq 1}(b - n, b)$$

or

$$I = \bigcup_{n\geq 1}(b - n, b].$$

Proceed via the countable union of countable sets.

Case 6: $I = [a, \infty)$ or $I = (a, \infty)$. Write I as

$$I = \bigcup_{n\geq 1}(a, a + n)$$

or

$$I = \bigcup_{n\geq 1}[a, a + n].$$

Proceed via the countable union of countable sets.

Case 7: $I = \mathbb{R}$. Write I as

$$I = \bigcup_{n\geq 1}[-n, n].$$

Proceed via the countable union of countable sets.

Continuous Inverse Theorem

Statement: Let $I \in \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be continuous and strictly monotone. Then,

(1) $J = f(I)$ is an interval. (Proved earlier.)

(2) $f : I \to J$ is bijective and thus invertible.

(3) $f^{-1} : J \to I$ is continuous and strictly monotone.

Assume f is continuous and strictly increasing.

Proof of (3): First, we prove $g : J \to I$ is also strictly increasing. To see this, let $y_1, y_2 \in J$, with $y_1 < y_2$. If

$$g(y_1) \geq g(y_2)$$

then,

$$\begin{aligned} f(g(y_1)) &\geq f(g(y_2)) \\ y_1 &\geq y_2, \end{aligned} \qquad \perp$$

So $g(y_1) < g(y_2)$.

Now, we will show that g is continuous. Note that since $f(I) = J$, it must be the case that $g(J) = I$. Suppose toward contradiction that g is discontinuous at $x = c \in J$. Then, $j_g(c) = \lim_{x\to c^+} g(x) - \lim_{x\to c^-} g(x) > 0$.

So, we find $x \in I$ with $\lim_{x\to c^-} g(x) < x < \lim_{x\to c^+} g(x)$. However, since g is strictly increasing, it follows that $x \notin \operatorname{Ran} g$. If $y < c$, then $g(y) \leq \lim_{x\to c^-} g(x)$, and if $z > c$, then $g(z) \geq \lim_{x\to c^+} g(x)$. However, we know that $g(J) = I$. $\perp$

The nth Root Function

Let n be even, $f : [0,\infty) \to \mathbb{R}$ where $f(x) = x^n$. Clearly, f is continuous, and f is also strictly increasing.

- $\operatorname{Ran}(f) = [0,\infty)$. To see this, we see that $f(0) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$. By the Intermediate Value Theorem, f must obtain every value in $[0,\infty)$.

Thus, $f : [0,\infty) \to [0,\infty)$ is invertible, and we write $g : [0,\infty) \to [0,\infty)$, where $g(x) = x^{1/n}$.

If $x, y > 0$, then $(xy)^{1/n} = x^{1/n}y^{1/n}$. Note that $f(uv) = f(u)f(v)$.

If $x = f(u)$ and $y = f(v)$, then $f\left((xy)^{1/n}\right) = f(g(xy)) = xy = f(g(x))f(g(y)) = f(x^{1/n})f(y^{1/n}) = f(x^{1/n}y^{1/n})$.

If $x > 0$, then $(x^n)^{1/n} = x = \left(x^{1/n}\right)^n$, following from the fact that $g \circ f(x) = x = f \circ g(x)$. If $x < 0$, then $(x^n)^{1/n} = |x|$.

Since $x < 0$, we can write

$$\begin{aligned}(x^n)^{1/n} &= ((-|x|)^n)^{1/n}\\ &= ((-1)^n\,|x|^n)^{1/n}\\ &= |x|\end{aligned}$$

Note that if $x < 0$, $\left(x^{1/n}\right)^n$ is not defined.

If n is odd, then $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ is continuous and strictly increasing with range $\mathbb{R}$. By the continuous inverses theorem, $f^{-1} = g$ is continuous and strictly increasing. We write $g(x) = x^{1/n}$.

Similarly as to the even case, we can show that

- $(xy)^{1/n} = x^{1/n}y^{1/n}$
- $\forall x \in \mathbb{R}$, $\left(x^{1/n}\right)^n = x = (x^n)^{1/n}$

Recall that if $x \neq 0$ in $\mathbb{R}$, then x^{-1} is defined as the unique value such that $xx^{-1} = 1$.

If $x \neq 0$ and $n \in \mathbb{N}$, then $(x^n)^{-1} = (x^{-1})^n$. We write x^{-n} as the common value.

(1) If n is even and $x > 0$, then $\left(x^{1/n}\right)^{-1} = \left(x^{-1}\right)^{1/n}$

(2) If n is odd, and $x \neq 0$, then $\left(x^{1/n}\right)^{-1} = \left(x^{-1}\right)^{1/n}$.

Proof: If $x > 0$, then $x^{1/n} > 0$. So,

$$\begin{aligned}x^{1/n}\left(x^{-1}\right)^{1/n} &= (x \cdot x^{-1})^{1/n}\\ &= 1\end{aligned}$$

So by the uniqueness of inverses, the theorem follows.

Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

(1) If n is even, $x > 0$, then $(x^m)^{1/n} = \left(x^{1/n}\right)^m$

(2) If n is odd, $x \neq 0$, then $(x^m)^{1/n} = \left(x^{1/n}\right)^m$

We define the unique values as $x^{m/n}$.

Derivatives

In this context, I always refers to an interval, and $c \in I$.

Definition of Differentiation

A function f is differentiable at c if

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite.

In that case, we denote the limit as $f'(c)$. The value $f'(c)$ is called the derivative of f at c.

Like with continuity, f is differentiable on I if $f'(c)$ exists $\forall c \in I$.

Applying Differentiation 1

Let $f(x) = ax + b$, $c \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{x\to c} \frac{f(x) - f(c)}{x - c} &= \frac{(ax + b) - (ac + b)}{x - c}\\ &= \frac{a(x - c)}{x - c}\\ &= a\end{aligned}$$

Applying Differentiation 2

Let $f(x) = x^2$, $c \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{x\to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x\to c} \frac{x^2 - c^2}{x - c}\\ &= \lim_{x\to c} x + c\\ &= 2c\end{aligned}$$

Applying Differentiation 3

Let $f(x) = \sqrt{x}$, $c \geq 0$. Then,

$$\begin{aligned}\lim_{x\to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x\to c} \frac{\sqrt{x} - \sqrt{c}}{x - c}\\ &= \lim_{x\to c} \frac{1}{\sqrt{x} + \sqrt{c}}\\ &= \begin{cases} \frac{1}{2\sqrt{c}} & c \neq 0 \\ +\infty & c = 0 \end{cases}\end{aligned}$$

Therefore, $f'(c)$ exists only when $c \geq 0$.

Applying Differentiation 4

For example, $f(x) = |x|$ is *not* differentiable at $c = 0$.

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{|x|}{x}$$

Let $(x_n)_n = \frac{(-1)^n}{n}$. Then, $(x_n)_n \to 0$. However, $\frac{|x_n|}{x_n} = (-1)^n$, which diverges. Therefore, the limit does not exist.

Applying Differentiation 5

Let

$$g(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then,

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \sin(1/x).$$

Let $(x_n)_n = \frac{2}{\pi n}$. Then, $(x_n)_n \to 0$, but $\sin(1/x_n)$ is divergent.

Applying Differentiation 6

Let $f(x) = \sin(x)$, $c \in \mathbb{R}$. Then,

$$f'(c) = \lim_{x\to c} \frac{\sin(x) - \sin(c)}{x - c}$$

Let $h = x - c$. Then, $x \to c \Leftrightarrow h \to 0$. Then,

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{\sin(h + c) - \sin(c)}{h} \\ &= \lim_{h\to 0} \frac{\sin(h)\cos(c) + \cos(h)\sin(c) - \sin(c)}{h} \\ &= \lim_{h\to 0} \cos(c)\frac{\sin(h)}{h} + \sin(c)\frac{\cos(h) - 1}{h} \\ &= \cos(c) \end{aligned}$$

Differentiability and Continuity

Statement: If $f : I \to \mathbb{R}$ is differentiable at $x = c$, then f is continuous at $x = c$.

Proof:

$$\begin{aligned} \lim_{x\to c} (f(x) - f(c)) &= \lim_{x\to c} \left((x - c)\, \frac{f(x) - f(c)}{x - c} \right) \\ &= \lim_{x\to c} (x - c) f'(c) \\ &= 0 \end{aligned}$$

Thus, $\lim_{x\to c} f(x) = f(c)$, and f is continuous.

Operations with Differentiation

Statement: Let $I \in \mathbb{R}$ be an interval, $c \in I$. Let $f, g : I \to \mathbb{R}$ be differentiable at $x = c$. Let $\alpha \in \mathbb{R}$. Then,

(1) $(\alpha f)'(c) = \alpha f'(c)$

(2) $(f + g)'(c) = f'(c) + g'(c)$

(3) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(4) $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$, provided $g(c) \neq 0$.

Proof of (4):

$$\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x - c} \\
&= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(x)}{(x-c)g(x)g(c)} \\
&= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{(x-c)g(x)g(c)} \\
&= \lim_{x\to c} \frac{g(c)(f(x) - f(c))}{(x-c)g(x)g(c)} - \lim_{x\to c} \frac{f(c)(g(x) - g(c))}{(x-c)g(x)g(c)} \\
&= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} && \text{since } \lim_{x\to c} g(x) = g(c)
\end{aligned}$$

Power Rule

Statement: Let $f_n(x) = x^n$, where $n \in \mathbb{Z}$. Then, $f_n'(x) = nx^{n-1}$.

Proof: Let $n \geq 1$. We have already proved the linear case ($n = 1$). Inductively assume true for n.

Then,

$$\begin{aligned}
f_{n+1}'(x) &= (x \cdot f_n(x))' \\
&= f_n(x) + xf_n'(x) \\
&= x^n + x \cdot nx^{n-1} \\
&= (n+1)x^n
\end{aligned}$$

Similarly, the proof is clear for $n = 0$. Using the quotient rule, we can show the similar case for $n < 0$.

$$f_{-n}(x) = \frac{1}{f_n(x)} \qquad n = 1, 2, 3, \dots$$

Carathéodory's Theorem

Statement: If $f : I \to \mathbb{R}$, $c \in I$. f is differentiable at $x = c$ if and only if $\exists \varphi : I \to \mathbb{R}$ continuous at c such that $\forall x \in I$, $f(x) - f(c) = \phi(x) \cdot (x - c)$. In this case, $f'(c) = \phi(c)$.

For example, if $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Fix $c \in \mathbb{R}$. Then, $f(x) - f(c) = (x-c)(x^2 + cx + c^2)$. Let $\varphi(x) = x^2 + cx + c$. Then, $\varphi(c) = 3c^2$.

Proof:

($\Rightarrow$): Suppose $\exists \varphi : I \to \mathbb{R}$ such that $f(x) - f(c) = \varphi(x)(x-c)$ $\forall x \in I$. Then,

$$\begin{aligned}\lim_{x\to c} \frac{f(x)-f(c)}{x-c} &= \lim_{x\to c} \varphi(x)\\ &= \varphi(c)\end{aligned}$$

So, f is differentiable and $f'(c) = \varphi(c)$.

($\Leftarrow$) Assume f is differentiable at $x = c$. Let $\varphi : I \to \mathbb{R}$

$$\varphi(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c\end{cases}$$

It is the case that φ is continuous at $x = c$ since $\lim_{x\to c} \varphi(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) = \varphi(c)$.

Clearly, $f(x) - f(c) = \varphi(x)(x-c)$.

Chain Rule

Statement: Let $J \xrightarrow{f} I \xrightarrow{g} \mathbb{R}$, where I and J are intervals. Let $c \in J$ and $d = f(c) \in I$. Assume f is differentiable at $x = c$, and g is differentiable at $d = f(c)$. Then, $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof: We know that $\exists \varphi : J \to \mathbb{R}$ with $\forall x \in J,\ f(x) - f(c) = \varphi(x)(x-c)$, with φ continuous at $x = c$. Similarly, $\exists \psi : I \to \mathbb{R}$ with $\forall y \in I,\ g(y) - g(d) = \psi(y)(y-d)$.

In particular, $\forall x \in J$,

$$\begin{aligned} g(f(x)) - g(f(c)) &= \psi(f(x))(f(x) - f(c))\\ g(f(x)) - g(f(c)) &= \psi(f(x))\varphi(x)(x-c),\end{aligned}$$

so

$$g \circ f(x) - g \circ f(c) = \lambda(x)(x-c) \qquad \text{where } \lambda(x) = \psi(f(x))\varphi(x)$$

Note that $\lambda : J \to \mathbb{R}$ is continuous at $x = c$ because

- φ is continuous at $x = c$
- f is differentiable at $x = c$, and thus continuous at $x = c$
- ψ is continuous at $d = f(c)$

Therefore, by Carathéodory's theorem, $g \circ f$ is differentiable at $x = c$.

Additionally,

$$\begin{aligned}(g \circ f)'(c) &= \lambda(c)\\ &= \psi(f(c))\varphi(c)\\ &= \psi(d)\varphi(c)\\ &= g'(d)f'(c).\end{aligned}$$

Inverse Functions

Let I be an interval, $f : I \to \mathbb{R}$ strictly monotone and continuous, $f(I) = J$. Let $g : J \to I$ be the inverse map.

- J is an interval
- g is continuous and strictly monotone
- If f is differentiable at $c \in I$, and $f'(c) \neq 0$, then g is differentiable at $y = d = f(c)$, and

$$g'(d) = \frac{1}{f'(c)}$$

Applying Inverse Functions 1

Let $T : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $T(x) = \tan(x)$. Since T is strictly monotone, continuous, and $\lim_{x\to\pi/2^-} T(x) = +\infty$, and $\lim_{x\to-\pi/2^+} T(x) = -\infty$, T is bijective.

Let $A : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\begin{aligned}
A'(d) &= \frac{1}{T'(c)} \\
T(c) &= d \\
A'(d) &= \frac{1}{\sec^2(c)} \\
&= \frac{1}{1+\tan^2(c)} \\
&= \frac{1}{1+d^2}
\end{aligned}$$

Applying Inverse Functions 2

Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = x^n$, where n is odd. Since f is strictly monotone, continuous, and surjective, f is bijective. Let $g_n : \mathbb{R} \to \mathbb{R}$ be the inverse. Then, $g_n(y) = y^{1/n}$. Let $f_n(c) = d$.

$$\begin{aligned}
g_n'(d) &= \frac{1}{f_n'(c)} \\
&= \frac{1}{nc^{n-1}} \\
&= \frac{1}{nd^{1-\frac{1}{n}}} \\
&= \frac{1}{n}d^{\frac{1}{n}-1}
\end{aligned}$$

The same idea works when n is even on $(0, \infty)$.

Exercise: Let $\frac{m}{n} \in \mathbb{Q}$. Show that $\frac{d}{dx}x^{m/n} = \frac{m}{n}x^{m/n-1}$.

We can write this as a composition and use the chain rule.

Fermat's Theorem

Statement: Let $f : I \to \mathbb{R}$, c an interior point of I. Suppose f has a local maximum or minimum at $x = c$ Then,

(1) $f'(c)$ does not exist.

(2) $f'(c) = 0$.

Proof: If $f'(c)$ does not exist, there is nothing to prove. Assume $f'(c)$ does exist.

Suppose toward contradiction that $f'(c) \neq 0$.

Case 1: $f'(c) > 0$. So,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0,$$

Meaning $\exists \delta$ such that $x \in \dot{V}_\delta(c)$ implies

$$\frac{f(x) - f(c)}{x - c} > 0.$$

So, if $x \in (c - \delta, c)$,

$$\begin{aligned} f(x) - f(c) &= \frac{f(x) - f(c)}{x - c}(x - c) \\ &< 0 \\ f(x) &< f(c), \end{aligned} \tag{$*$}$$

and if $x \in (c, c + \delta)$,

$$\begin{aligned} f(x) - f(c) &= \frac{f(x) - f(c)}{x - c}(x - c) \\ &> 0 \\ f(x) &> f(c). \end{aligned} \tag{$**$}$$

If c is a local minimum, $(*)$ violates the assumption, and if c is a local maximum, $(**)$ violates the assumption. $\perp$

Warning: Fermat's theorem does not run in converse: $f(x) = x^3$, $f'(0) = 0$ but $x = 0$ is not a local minimum or maximum. Similarly, $f(x) = x^{1/3}$, $f'(0) = 0$ but $x = 0$ is not a local minimum or maximum.

Rolle's Theorem

Statement: Let $f : [a, b] \to \mathbb{R}$ with f continuous on $[a, b]$ and f differentiable on (a, b). If $f(a) = f(b)$, $\exists c \in (a, b)$ with $f'(c) = 0$.

Proof: If f is a constant function, we are done.

Suppose f is not a constant function.

Case 1: $\exists x \in (a, b)$ with $f(x) > f(a)$. By the extreme value theorem and the hypothesis, $\exists x_M \in (a, b)$ with $f(x_M) = \sup_{x \in [a,b]} f(x)$. By Fermat's Theorem, $f'(x_M) = 0$.

Case 2: $\exists x \in (a, b)$ with $f(x) < f(a)$. By the extreme value theorem, $\exists x_m \in (a, b)$ with $f(x_m) = \inf_{x \in [a,b]} f(x)$. By Fermat's Theorem, $f'(x_m) = 0$.

Applying Rolle's Theorem

Problem: Suppose $f : [a, b] \in \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a)f(b) < 0$, and $f'(x) \neq 0$. Show f has a unique real root in $[a, b]$.

Solution: Without loss of generality, $f(a) < 0$ and $f(b) > 0$. By the intermediate value theorem, $\exists z \in (a, b)$ with $f(z) = 0$.

Suppose toward contradiction $\exists z' \in (a, b)$ with $z' \neq z$. Use Rolle's theorem on $[z, z']$ or $[z', z]$.

Mean Value Theorem

Statement: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then, $\exists c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Consider the function $g : [a, b] \to \mathbb{R}$ given by

$$\begin{aligned} g(x) &= f(x) - \ell(x) \\ \ell(x) &= f(a) + \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned}$$

Since g is continuous on $[a, b]$ and differentiable on (a, b), and

$$\begin{aligned} g(a) &= 0 \\ g(b) &= 0, \end{aligned}$$

by Rolle's Theorem there must be a point $c \in (a, b)$ with

$$g'(c) = 0,$$

so,

$$\begin{aligned} g'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

Corollary to the Mean Value Theorem: Constant Functions

Statement: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and $f'(x) = 0$, $\forall x \in (a, b)$, then f is constant.

Proof: Let $x_1, x_2 \in [a, b]$, with $x_1 < x_2$.

Then, applying the Mean Value Theorem on $[x_1, x_2]$, we get that $\exists c \in (x_1, x_2)$ with $0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, implying $f(x_2) = f(x_1)$.

Corollary to the Mean Value Theorem: Identical Derivatives

Statement: Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), with $f'(x) = g'(x)$ on (a, b). Then, $f = g + k$ for some $k \in \mathbb{R}$.

Proof: Apply the constant functions corollary to $h = f - g$.

Corollary to the Mean Value Theorem: Increasing Functions

Statement: Let I be any interval with $f : I \to \mathbb{R}$ differentiable on the interval.

(i) f is increasing on $I \Leftrightarrow f'(x) \geq 0\ \forall x \in I$

(ii) f is decreasing on $I \Leftrightarrow f'(x) \leq 0\ \forall x \in I$

(iii) $f'(x) > 0$ on $I \Rightarrow f$ is strictly increasing on I

(iv) $f'(x) < 0$ on $I \Rightarrow f$ is strictly decreasing on I

Proof of (i):

($\Rightarrow$) Let $c \in I$. If $x < c$, then

$$\frac{f(x) - f(c)}{x - c} \geq 0,$$

and if $x > c$, then

$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

Therefore,

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &\geq 0 \end{aligned}$$

($\Leftarrow$) Let $x_1, x_2 \in I$, $x_1 < x_2$. Apply the Mean Value Theorem on $[x_1, x_2]$. Then,

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad c \in (x_1, x_2)$$

Assuming $f'(c) \geq 0$,

$$\begin{aligned} 0 &\leq f(x_2) - f(x_1) \\ f(x_1) &\leq f(x_2) \end{aligned}$$

Using Mean Value Theorem for Inequalities: Lipschitz

Problem:

$$|\cos(x) - \cos(y)| \leq |x - y| \qquad \forall x, y \in \mathbb{R}$$

Solution: Let $x < y$. Apply the Mean Value Theorem to $[x, y]$. Then, $\exists c \in (x, y)$ with

$$\begin{aligned} \sin(c) &= \frac{\cos(y) - \cos(x)}{y - x} \\ \left|\frac{\cos(y) - \cos(x)}{y - x}\right| &= |\sin(c)| \\ &\leq 1 \\ |\cos(y) - \cos(x)| &\leq |y - x| \end{aligned}$$

Using Mean Value Theorem for Inequalities: Logarithms

Assume the existence of $L : (0, \infty) \to \mathbb{R}$, with

- $L(1) = 0$
- $L'(x) = \frac{1}{x}$

$$L(x) = \int_1^x \frac{1}{t} dt$$

Problem: Show

$$\frac{x - 1}{x} \leq L(x) \leq x - 1 \qquad \text{for } x \geq 1$$

Solution: For $x = 1$, $\frac{x-1}{x} = L(x) = x - 1 = 0$.

For $x > 1$, apply the Mean Value Theorem to $[1, x]$. Then, for some $c \in (x, 1)$

$$\begin{aligned}
\frac{L(x) - L(1)}{x - 1} &= L'(c) \\
\frac{L(x)}{x - 1} &= \frac{1}{c} \\
&< x - 1 \\
L(x) &< x - 1
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{L(x)}{x - 1} &> \frac{1}{x} && c < x \\
L(x) &> \frac{x - 1}{x}
\end{aligned}$$

Using Mean Value Theorem for Inequalities: Extension of Bernoulli's Inequality

Statement: Let $r \in \mathbb{Q}$, $r \geq 1$, $x > -1$. Then,

$$(1 + x)^r \geq 1 + rx$$

Proof: Consider $h(x) = (1 + x)^r$ defined on $[-1, \infty)$.
If $x = 0$, we are done. Otherwise, let $x > 0$. Apply the Mean Value Theorem on $[0, x]$. So, for some $c \in (0, x)$,

$$\begin{aligned}
\frac{h(x) - h(0)}{x - 0} &= h'(c) \\
\frac{(1 + x)^r - 1}{x} &= r(1 + c)^{r-1} \\
&\geq r \\
(1 + x)^r &\geq rx + 1
\end{aligned}$$

Let $x \in (-1, 0)$. Apply the Mean Value Theorem to $[x, 0]$. So, for some $c \in (x, 0)$,

$$\begin{aligned}
\frac{h(0) - h(x)}{0 - x} &= h'(c) \\
\frac{1 - (1 + x)^r}{-x} &= r(1 + c)^{r-1} \\
&\leq r \\
1 - (1 + x)^r &\leq -rx \\
1 + rx &\leq (1 + x)^r
\end{aligned}$$

Remark: Bernoulli's Inequality works for $\alpha \geq 1$ where $\alpha \in \mathbb{R}$, and $x > -1$.

First Derivative Test

Statement: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, $c \in (a, b)$. Assume f is differentiable on $(a, b) \setminus c$.

(1) If $\exists \delta > 0$ with $f'(x) \geq 0$ on $(c - \delta, c)$ and $f'(x) \leq 0$ on $(c, c + \delta)$, then $f(c)$ is a local maximum.

(2) If $\exists \delta > 0$ with $f'(x) \leq 0$ on $(c - \delta, c)$ and $f'(x) \geq 0$ on $(c, c + \delta)$, then $f(c)$ is a local minimum.

Proof of (1): Let $x \in (c - \delta, c)$. Apply the Mean Value Theorem to $[x, c]$. So, $\exists \xi \in (x, c)$ with $f'(\xi) = \frac{f(c) - f(x)}{c - x}$. Since $\xi \in (c - \delta, c)$, $f'(\xi) \geq 0$.

Since $c - x > 0$, we have $f(c) - f(x) \geq 0$, so $f(c) \geq f(x)$.

Let $x \in (c, c+\delta)$. Apply the Mean Value Theorem to $[c, x]$, ...

Thus, $f(c)$ is a local maximum on $V_\delta(c)$.

Darboux's Theorem

Lemma: Let $I \in \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $c \in I$, and f differentiable at c.

(i) If $f'(c) > 0$, $\exists\delta$ such that $x \in (c, c+\delta)$, $f(x) > f(c)$.
(ii) If $f'(c) < 0$, $\exists\delta$ such that $x \in (c-\delta, c)$, $f(x) > f(c)$.

Proof of Lemma:

(i)

$$0 < f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

so, $\exists\delta > 0$ such that for $x \in \dot{V}_\delta(c)$,

$$0 < \frac{f(x) - f(c)}{x - c}.$$

In particular, if $x \in (c, c+\delta)$,

$$\begin{aligned} 0 &< \frac{f(x) - f(c)}{x - c} \\ 0 &< f(x) - f(c) \\ f(c) &< f(x) \end{aligned}$$

(ii) Similar.

Statement: If $f : [a, b] \to \mathbb{R}$ differentiable, and k is between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b)$ with $f'(c) = k$.

Proof: Consider the function $h(x) = kx - f(x)$ on $[a, b]$. It is the case that h is continuous on $[a, b]$, meaning that by the Extreme Value Theorem, h attains its supremum: $\exists c \in [a, b]$ with $h(c) \geq h(x)$ $\forall x \in [a, b]$.

$$\begin{aligned} h'(a) &= k - f'(a) \\ h'(b) &= k - f'(b). \end{aligned}$$

WLOG, $f'(a) < f'(b)$. So, $k \in (f'(a), f'(b))$. Therefore, $h'(a) > 0$ and $h'(b) < 0$.

By the lemma, $\exists\delta > 0$ such that $x \in (a, a+\delta) \Rightarrow h(x) > h(a)$ — therefore $a \neq c$.

Similarly, $\exists\delta > 0$ such that $x \in (b-\delta, b) \Rightarrow h(x) > h(b)$ — therefore, $b \neq c$.

So, $c \in (a, b)$. Therefore, by Fermat's theorem, $h'(c) = 0$.

Applying Darboux's Theorem 1

Problem: Consider $g : [-1, 1] \to \mathbb{R}$, $g(x) = \text{sgn}(x)$. Does there exist a function $f : [-1, 1] \to \mathbb{R}$ with $f' = g$?

Solution: By Darboux's Theorem, this is not the case, since g does not satisfy the intermediate value property.

Corollary to Darboux's Theorem

Statement: Let $f : I \to \mathbb{R}$, differentiable, and $f' \neq 0$ on I. Show that f is either strictly increasing on I or strictly decreasing on I.

Proof: If $f'(x) > 0 \; \forall x \in I$, then f is strictly increasing on I, and if $f'(x) < 0 \; \forall x \in I$, then f is strictly decreasing on I.

If not, then $f'(x_1) > 0$, $f'(x_2) < 0$ for some $x_1, x_2 \in I$. Applying Darboux's theorem, $\exists c$ between x_1 and x_2 with $f'(c) = 0$.

Taylor's Theorem

Suppose $f : I \to \mathbb{R}$ is differentiable on I.

(1) If $f' : I \to \mathbb{R}$ is differentiable at $x = c$, then we write $f''(c) = (f')'(c)$ is the second derivative of f at $x = c$. We say f is *twice differentiable* at $x = c$ if $f''(c)$ exists.

(2) Similarly, $f^{(n)}(c)$ is defined as $(f^{(n-1)})'(c)$, where $f^{(n-1)}(x)$ is differentiable on I.

(3) $C^n(I) = \{f : I \to \mathbb{R} \mid f^{(n)}$ exists and is continuous on $I\}$

(4) $C^\infty(I) = \{f : I \to \mathbb{R} \mid f$ infinitely differentiable on $I\}$

Let $f : I \to \mathbb{R}$ with $f^{(n)}(c)$ existing for some $c \in I$. The nth Taylor polynomial

$$\begin{aligned} T_n(f, c) &: I \to \mathbb{R} \\ T_n(f, c)(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k \end{aligned}$$

Lemma: $T_n(f, c)(c) = f(c),\; T_n(f, c)'(c) = f'(c), \dots, T_n(f, c)^{(k)}(c) = f^{(k)}(c)$.

Statement: Let $f \in C^{n+1}(I)$. Let $c \in I$. Given $x \in I$, $\exists \xi_x$ between x and c with

$$f(x) = T_n(f, c)(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - c)^{n+1}.$$

Remark: The term $R_n(f, c)(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - c)^{n+1}$ is known as the Lagrange remainder.

Applying Taylor's Theorem: $\sin(x)$

Let $f(x) = \sin(x),\; c = 0$. Then,

$$T_8(f, c)(x) = x - \frac{x^3}{3!} + \frac{x^3}{5!} - \frac{x^7}{7!}.$$

So,

$$\begin{aligned} |R_n(f, 0)(x)| &= \left| \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - c)^{n+1} \right| \\ &\leq \frac{|x|^{n+1}}{(n+1)!} \\ &\to 0 \end{aligned}$$

We say that $\sin(x)$ is *analytic* if its Lagrange remainder tends to zero as $n \to 0$.

Applying Taylor's Theorem: Approximating e

We want to approximate e to an error under 10^{-5}.

Let $f(x) = e^x$, $c = 0$. Then,

$$\begin{aligned}
T_n(f,0)(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\
e &= f(1) \\
&= T_n(f,0)(1) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{e^{\xi}}{(n+1)!} \\
\frac{e^{\xi}}{(n+1)} &< 10^{-5}
\end{aligned}$$

Since $e < 3$, and $0 < \xi < 1$,

$$\begin{aligned}
e^{\xi} &< 3 \\
\frac{e^{\xi}}{(n+1)!} &< \frac{3}{(n+1)!} \\
&< 10^{-5}
\end{aligned}$$

which works for $n = 8$. Therefore,

$$\begin{aligned}
e &\approx 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{8!} \\
&= 2.71828
\end{aligned}$$