Introduction

Consider the equations

$$\frac{d^2y}{dx^2} + y(x) = e^x \tag{1}$$

$$\frac{d^{17}y}{dx^{17}}(x) + \sin(y(x)) = (x^x)^x \tag{2}$$

Before we want to solve these equations, we need to understand what these equations *are*.

(1) This is a second order, inhomogeneous, linear ordinary differential equation.

(2) This is a 17th order, inhomogeneous, nonlinear ordinary differential equation.

Generally, when we have a nonlinear equation, we convert it (using the Jacobian) to the "nearest" corresponding linear equation using Taylor approximations. In this case, converting equation (2), we have

$$\frac{d^{17}y}{dx^{1y}}(x) + y(x) = (x^x)^x. \tag{2'}$$

Now, equation (2′) is linear, so it is able to be solved. It may not be pretty,[1] but it can be solved, using Laplace Transforms or other methods.

Ordinary Differential Equations

Returning to our equation (1),

$$\frac{d^2y}{dx^2} + y(x) = e^x, \tag{1}$$

there is one more fact that we can see — this is an equation with constant coefficients. The most general form of a nth order linear ordinary differential equation is of the form

$$a_n(x)\frac{d^ny}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}(x) + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = g(x). \tag{†}$$

Specifically, we also require $a_k(x) \in C(I)$, where I is some interval (specifics will be detailed later).

Theorem (Existence and Uniqueness Theorem)**:** Any ordinary differential equation of the form (†) has unique solutions in the interval I.

There are n linearly independent solutions for $g(x) = 0$.

The corresponding homogeneous equation for (1) is

$$\frac{d^2y}{dx^2} + y(x) = 0. \tag{1'}$$

The equations (1) and (1′) are related by the linearity principle. In particular, if $y_0(x)$ is a solution to (1′), then we can add $\alpha y_0(x)$ to any solution $y_p(x)$ of (1), then we have all the solutions for (1). In particular, the solutions to (1′) are

$$y_1(x) = \sin(x)$$
$$y_2(x) = \cos(x).$$

[1] Citation needed.

To evaluate that these solutions are linearly independent, we consider the differential operator L from (†) defined by

$$L[y] = \sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k}.$$

We rewrite (†) as

$$L[y] = g(x).$$

The operator L is linear, so L has the following properties:

- $L[y_1 + y_2]$;
- $L[cy] = cL[y]$.

Now, in (1) and (1′), if we set $L[y] = \frac{d^2y}{dx^2} + y(x)$, then evaluating our solutions y_1 and y_2 to (1′), we get

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= c_1L[y_1] + c_2L[y_2] \\ &= 0. \end{aligned}$$

Now, we get

$$y_0(x) = c_1\sin(x) + c_2\sin(x)$$

as our general solution to (1′). By the linearity principle, all we need is one solution to $L[y] = e^x$ to find all solutions to (1).

Evaluating (†) in the most general form, we have the general solution

$$y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{\text{homogeneous solution}} + y_p(x),$$

where $y_p(x)$ is the particular solution. In other words, our general solution is

$$y(x) = \operatorname{span}(y_1(x), y_2(x), \dots, y_n(x)) + y_p(x).$$

For this to work, we need the set $\{y_1, \dots, y_n\}$ to be linearly independent. To do this, we evaluate the Wronskian:

$$W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{pmatrix}.$$

Specifically, the set $\{y_1, \dots, y_n\}$ is linearly independent if $W(x) \neq 0$ for all $x \in I$.

Example. Consider the equation

$$\frac{d^2y}{dx^2} - y(x) = e^x \tag{1}$$

We want to find the general solution to this constant coefficient equation.

We start by finding two linearly independent homogeneous solutions to the equation, take their span, then add a particular solution.

The characteristic equation of the homogeneous equation for (1) is

$$r^2 - 1 = 0$$

We get $r = \pm 1$, which by the definition of the characteristic equation yields $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. To verify that this this solution set is linearly independent

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \\ &= -2 \\ &\neq 0. \end{aligned}$$

Thus, our solutions are linearly independent. We get the general form of

$$y(x) = c_1 e^x + c_2 e^{-x} + y_p(x).$$

Now, we only have to find a particular solution. This is, unfortunately, the hard part.

We begin by guessing. But, in a way that doesn't suck. Specifically, we let $y_p(x) = Axe^x$. Evaluating, we get

$$\begin{aligned} \frac{dy_p}{dx} &= A(x+1)e^x \\ \frac{d^2 y_p}{dx^2} &= A(x+2)e^x \\ \frac{d^2 y_p}{dx^2} - y_p(x) &= A(x+2)e^x - Axe^x \\ &= 2Ae^x, \end{aligned}$$

so $2A = 1$, and $A = \frac{1}{2}$. Thus, we have the end result of

$$y(x) = c_1 e^x + c_2 e^x + \frac{1}{2} x e^x.$$

Evaluating in Mathematica, we take

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DSolve[y''[x] - y[x] == Exp[x], y[x], x]
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and we get

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{4}(2x - 1)e^x,$$

corroborating our solution.[II]

Example. Consider the equation

$$\frac{d^3 y}{dx^3} - y(x) = 0.$$

The particular solution to this equation is $y(x) = 0$. The characteristic equation for this equation is

$$r^3 - 1 = 0.$$

[II]Only slightly different, but they're the same solution.

Factoring, we get

$$(r-1)\left(r^2+r+1\right)=0$$
$$(r-1)(r-\zeta_3)\left(r-\zeta_3^2\right)=0.$$

Thus, we get

$$r=\left\{1,e^{\frac{2\pi i}{3}},e^{\frac{4\pi i}{3}}\right\}.$$

Thus, our solutions are of the form

$$y(x)=c_1e^x+c_2e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right)+c_3e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Recall that the most general second order constant-coefficient linear differential equation is

$$y''+ay'+by=0,$$

with characteristic equation

$$r^2+ar+b=0.$$

The solutions to the characteristic equation are

$$r=-\frac{a}{2}\pm\frac{\sqrt{a^2-4b}}{2}.$$

There are a few cases:

(1) $r_1\neq r_2$ with $r_1,r_2\in\mathbb{R}$;

(2) $r_1=r_2$ with $r_1,r_2\in\mathbb{R}$;

(3) $r_1=c+id$, $r_2=c-id$, where $c,d\in\mathbb{R}$.

The solutions are $y_1=c_1e^{r_1x}$ and $y_2=c_2e^{r_2x}$.

Example (Solving Second-Order Equations)**.**

(1) Let

$$y''-3y'+2y=0.$$

The characteristic equation is $r^2-3r+2=0$, whose solutions are $r=1,r=2$. The general solution is, thus,

$$y(x)=c_1e^x+c_2e^{2x} \tag{†}$$

The Wronskian is

$$\begin{aligned}W(x)&=\det\begin{pmatrix}e^x & e^{2x}\\ e^x & 2e^{2x}\end{pmatrix}\\&=2e^{3x}-e^{3x}\\&=e^{3x}\\&\neq 0.\end{aligned}$$

Thus, the solution is indeed (†).

(2) Let

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with solution $r = -3, -3$. Currently, we only have the solution $y_1(x) = c_1 e^{-3x}$.

Note that in an nth order linear ordinary differential equation, we always have n linearly independent solutions. Let's guess. Consider the equation $y_2(x) = c_2 x e^{-3x}$.

We can see that $y_2(x)$ is also a solution to this equation,[III] but we need to verify linear independence. Taking the Wronskian, we get

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{pmatrix} \\ &= e^{-6x}\begin{pmatrix} 1 & x \\ -3 & -3x+1 \end{pmatrix} \\ &= e^{-6x}(-3x + 1 + 3x) \\ &= e^{-6x} \\ &\neq 0. \end{aligned}$$

Thus, we have two linearly independent solutions, with the general solution of

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

(3) Let

$$y'' + 4y' + 5 = 0.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with solutions of $r = -2 \pm i$. We then have the solutions

$$\begin{aligned} y_1(x) &= e^{(-2+i)x} \\ y_2(x) &= e^{(-2-i)x}. \end{aligned}$$

Unfortunately, we cannot just let these equations stand on their own, because we want *real* solutions. Let's use Euler's theorem, $e^{ix} = \cos x + i \sin x$. Then, we get

$$\begin{aligned} y(x) &= c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x} \\ &= e^{-2x}\left(c_1 e^{ix} + c_2 e^{-ix}\right). \end{aligned}$$

Let $f(x) = c_1 e^{ix} + c_2 e^{-ix}$. Using the even/odd decomposition, we get

$$\begin{aligned} f(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\ &= (c_1 + c_2)\cos(x) + i(c_1 - c_2)\sin(x). \end{aligned}$$

We "real"-ize our solution by just dropping the value of i in $f(x)$. Thus, we get the full general solution

$$y(x) = e^{-2x}(d_1 \cos(x) + d_2 \sin(x)).$$

[III] Exercise left for the reader.

(4) If we have the equation

$$y^{(4)} - 25y'',$$

then using a similar process, we get the solution

$$y(x) = c_1 + c_2x + c_3e^{5x} + c_4e^{-5x}.$$

(5) Considering the equation

$$y^{(5)} + 4y''' + 4y' = 0,$$

we take the characteristic equation $r^5 + 4r^3 + 4r = 0$. Factoring, we get solutions of $r = 0, r = \pm i\sqrt{2}$. Thus, we get the solution of

$$y(x) = c_1 + c_2\cos\left(\sqrt{2}x\right) + c_3\sin\left(\sqrt{2}x\right) + c_4x\cos\left(\sqrt{2}x\right) + c_5x\sin\left(\sqrt{2}x\right).$$

Reducing our Orders

Let

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0.$$

Suppose we know $y_1(x)$. Can we find $y_2(x)$? The answer is yes. We presume

$$y_2(x) = v(x)y_1(x).$$

Now, we have

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'', \end{aligned}$$

and inserting into the equation, we get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{=0} \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Now, we have

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p. \tag{$*$}$$

Integrating, we get

$$\ln(v') = -2\ln(y_1) - \int p(x)\,dx.$$

Taking powers, we get

$$\begin{aligned} v' &= e^{-2\ln(y_1) - \int p(x)\,dx} \\ &= y_1^{-2}e^{-\int p(x)\,dx} \\ &= \frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \\ v &= \int \frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\,dx \end{aligned}$$

Example. Consider the equation

$$\cos^2(x)\frac{d^2y}{dx^2} - \sin(x)\cos(x)y' - y(x) = 0.$$

Putting our equation into standard form, we may be able to find another solution.

$$y'' - \tan(x)y' - \sec^2(x)y = 0.$$

Guessing $y(x) = \tan(x)$, we get $y' = \sec^2(x)$ and $y'' = 2\sec^2(x)\tan(x)$. This is also another solution, $y_2(x) = \tan(x)$.

We don't want to guess anymore. Let $y_2(x) = v(x)y_1(x)$. We get

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx.$$

We have $-p(x) = \tan(x)$, so $-\int p(x)\,dx = \ln(\sec(x))$. Thus, $e^{-\int p(x)\,dx} = \sec(x)$. Thus, we get

$$\begin{aligned}
v(x) &= \int \frac{\sec(x)}{\tan^2(x)}\,dx \\
&= \int \frac{\cos(x)}{\sin^2(x)}\,dx \\
&= \int \frac{1}{u^2}\,du && u = \sin(x) \\
&= -\frac{1}{u} \\
&= -\csc(x).
\end{aligned}$$

Thus, we have $y_2(x) = -\csc(x)\tan(x) = -\sec(x)$.

Example. Consider the equation

$$x^2(\ln(x) - 1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{dy}{dx} = 0.$$

We can use the power of inspection to find one solution, $y_1(x) = x$. Dividing out, we have

$$y'' - \frac{1}{x(\ln(x) - 1)}y' + \frac{1}{x^2(\ln(x) - 1)}y = 0.$$

Using the reduction of order, we guess $y_2(x) = v(x)y_1(x)$, and have

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

Noting that $-p(x) = \frac{1}{x(\ln(x)-1)}$, we have $\int \frac{1}{x(\ln(x)-1)}\,dx = \ln(\ln(x) - 1)$.

Now, we have

$$\begin{aligned}
v(x) &= \int \frac{\ln(x) - 1}{x^2}\,dx \\
&= \frac{1 - \ln(x)}{x} - \int -\frac{1}{x^2}\,dx && u = \ln(x) - 1, dv = x^{-2}
\end{aligned}$$

$$\begin{aligned} &= \frac{-\ln(x)}{x} - \frac{1}{x} \\ &= -\frac{\ln(x)}{x}. \end{aligned}$$

Thus, we get $y_2(x) = -\ln(x)$, and the general solution of $y(x) = c_1 x + c_2 \ln(x)$.

Example (Cauchy–Euler Equation)**.** A second-order Cauchy–Euler equation is of the form

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy(x) = 0.$$

More generally,

$$\sum_{k=0}^{n} c_k x^k y^{(k)}(x) = 0.$$

We guess $y(x) = x^r$. Then, $\frac{dy}{dx} = rx^{r-1}$ and $\frac{d^2y}{dx^2} = r(r-1)x^{r-2}$. This yields

$$\begin{aligned} a(r)(r-1)x^r + brx^r + cx^r &= x^r\left(a\left(r^2 - r\right) + br + c\right) \\ &= 0. \end{aligned}$$

Example (Solving a Cauchy–Euler Equation)**.** Consider the equation

$$x^2y'' + xy' - y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 1 = 0,$$

so our general solution is $y(x) = c_1 x + c_2/x$.

Example (Solving another Cauchy–Euler Equation)**.** Consider the equation

$$x^2y'' - 3xy' + 4y = 0.$$

Substituting the characteristic equation, we get

$$r^2 - 4r + 4 = 0,$$

so our solutions are x^2 and x^2. This is not good enough, we need another solution.

Now, we place our equation into standard form.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y' = 0.$$

Thus, we get $p(x) = -\frac{3}{x}$. Using reduction of order, we get $y_2(x) = \nu(x)y_1(x)$,

$$\begin{aligned} \nu(x) &= \int \frac{e^{-\int -3/x \, dx}}{x^4} \, dx \\ &= \int \frac{e^{3\ln(x)}}{x^4} \, dx \\ &= \int \frac{x^3}{x^4} \, dx \\ &= \ln(x). \end{aligned}$$

Thus, we have the solution $y_2(x) = \ln(x)x^2$, and the general solution of $y(x) = c_1 x^2 + c_2 \ln(x) x^2$.

Example. Consider the equation

$$x^2y'' + 3xy' + 5y = 0.$$

We get the characteristic equation of

$$\begin{aligned} 0 &= r^2 - 4r + 5 \\ r &= 2 \pm i. \end{aligned}$$

Now, we need to figure out what $x^{2\pm i}$ means.

To solve this part, we keep the positive exponent, so we only need to try to understand $y = x^{2+i}$. Now, we get $y = x^2x^i$. To evaluate x^i, we take $x = \left(e^{\ln x}\right)^i = e^{i\ln x}$. Using Euler's identity, we get

$$y = x^2(\cos(\ln x) + i\sin(\ln x)).$$

Since our solutions are real, get

$$y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x).$$

Example. Consider the equation

$$x^4y^{(4)} - 2x^2y'' + y = 2.$$

We have the particular solution $y_p(x) = 2$. Substituting into our method for the Cauchy–Euler equation, we have

$$r(r-1)(r-2)(r-3) - 2r(r-1) + 1 = 0.$$

Factoring, we have

$$r(r-1)^2(r-4) + 1 = 0.$$

Unfortunately, to go forward from here we need Mathematica.

This has the solution set of of

$$\begin{aligned} r_1 &= \frac{3}{2} - \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\ r_2 &= \frac{3}{2} - \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad + \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} - \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \\ r_3 &= \frac{3}{2} + \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &\quad - \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \end{aligned}$$

$$\begin{aligned} r_4 &= \frac{3}{2} + \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}}} \\ &+ \frac{1}{2}\sqrt{6 - \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} - \frac{\sqrt[3]{45 + 2\sqrt{249}}}{3^{2/3}} + \frac{8}{\sqrt{3 + \frac{1}{3}\sqrt[3]{135 - 6\sqrt{249}} + \frac{\sqrt[3]{45+2\sqrt{249}}}{3^{2/3}}}}} \end{aligned}$$

Varying our Parameters

Given a set of n linearly independent homogeneous solutions, we want to find a particular solution.

To find this, we start with the general second-order inhomogeneous equation in standard form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = g(x).$$

Given y_1, y_2, we find $y_p(x)$ by taking

$$y_p = v_1y_1 + v_2y_2.$$

Finding the derivatives, we have

$$\begin{aligned} y_p' &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2 \\ y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2. \end{aligned}$$

Substituting, we have

$$\begin{aligned} y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2 \\ py_p' &= pv_1y_1' + pv_1'y_1 + pv_2y_2' + pv_2'y_2 \\ qy_p &= qv_1y_1 + qv_2y_2 \end{aligned}$$

$$g(x) = v_1\overbrace{\left(y_1'' + py_1' + qy_1\right)}^{=0} + v_2\overbrace{\left(y_2'' + py_2' + qy_2\right)}^{=0} + v_1'\left(2y_1' + py_1\right) + v_1''y_1 + v_2\left(2y_2' + py_2\right) + v_2''y_2$$

$$g(x) = v_1'\left(2y_1' + py_1\right) + v_1''y_1 + v_2\left(2y_2' + py_2\right) + v_2''y_2.$$

We suppose that $v_1'y_1 + v_2'y_2 = 0$. Then,

$$\begin{aligned} \frac{d}{dx}\left(v_1'y_1 + v_2'y_2\right) &= 0 \\ v_1''y_1 + v_1'y_1' + v_2''y_2 + v_2'y_2 &= 0. \end{aligned}$$

Plugging into our earlier expression, we get the expression of

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_2'y2' + v_2'y_2' &= g(x). \end{aligned}$$

Plugging into matrix form, we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

Since the Wronskian is nonzero, we have

$$\begin{aligned}\begin{pmatrix}\frac{dv_1}{dx}\\ \frac{dv_2}{dx}\end{pmatrix} &= \begin{pmatrix}y_1 & y_2\\ y_1' & y_2'\end{pmatrix}^{-1}\begin{pmatrix}0\\ g(x)\end{pmatrix}\\ &= \frac{1}{y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx}}\begin{pmatrix}-y_2(x)g(x)\\ y_1(x)g(x)\end{pmatrix} \qquad (\ddagger)\end{aligned}$$

Example. Let

$$y'' - 2y' + y = e^x.$$

Solving the homogeneous solution, we have the characteristic equation of $r^2 - 2r + 1 = 0$. Thus, $y_1(x) = e^x$ and $y_2(x) = xe^x$.

To find $y_p(x)$, we guess $y_p(x) = x^2e^x$. Using the power of computation in Sage, we get the answer of

Avoiding Variation of Parameters

```
de = diff(y,x,2) - 2*diff(y,x) + y - e^(x)
g = desolve(de,y)
latex(expand(g))
```

$$y_p(x) = K_2xe^x + K_1e^x + \frac{1}{2}x^2e^x.$$

However, this is a very unsatisfying method.

Using (‡), we can find a different solution. We find

$$\begin{aligned}\frac{dv_1}{dx} &= \frac{1}{e^{2x}}((-1)(xe^x)(e^x))\\ &= -x,\end{aligned}$$

yielding

$$v_1(x) = -\frac{x^2}{2} + c_2.$$

Similarly, we get

$$\begin{aligned}\frac{dv_2}{dx} &= \frac{1}{e^{2x}}(e^x)(e^x)\\ v_2(x) &= x + c_2.\end{aligned}$$

This gives

$$y_p(x) = \frac{1}{2}x^2e^x.$$

Example. Let

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = x + e^x.$$

Using the characteristic equation, we have $y_1(x) = 1$, $y_2(x) = e^x$, and $y_3(x) = e^{-x}$.

Now, using the Wronskian, we get

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ x + e^x \end{pmatrix}.$$

This would suck, but we would be able to find a solution nonetheless.

In the general form, with linearly independent homogeneous solutions $y_1, \dots, y_n$, we have the solution of

$$\begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ g(x) \end{pmatrix}$$

$$y(x) = \sum_{i=1}^{n} c_i y_i(x) + \sum_{i=1}^{n} v_i(x) y_i(x).$$

Systems of Homogeneous Equations

We will consider solving systems of equations.

Example (Solving a Coupled System)**.** Before we can start using variation of parameters for systems, we need to recall how to solve constant-coefficient systems.

$$\begin{aligned} x'(t) &= 3x(t) + y(t) \\ y'(t) &= x(t) + 3y(t). \end{aligned}$$

Here, setting

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

we get system of linear equations

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Remark: In the matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

the eigenvalues are

$$\begin{aligned} \lambda_1 &= a + b \\ \lambda_2 &= a - b \end{aligned}$$

with eigenvectors of

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example (General n-dimensional System of Differential Equations)**.** Consider the system of equations defined by

$$\begin{aligned} x_1'(t) &= g_1(t, x_1(t), \dots, x_n(t)) \\ &\vdots \\ x_n'(t) &= g_n(t, x_1(t), \dots, x_n(t)). \end{aligned}$$

We will refine this slightly so as to be a system of *linear* equations. Let

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \\ \frac{d\mathbf{x}}{dt} &= \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \\ \mathbf{F} &= \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \\ \mathbf{x}_{t_0} &= \begin{pmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}. \end{aligned}$$

Now, we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where $\mathbf{x}(t_0) = \mathbf{x}_{t_0}$ and A is some matrix that represents some linear transformation.

Furthermore, we may make an inhomogeneous equation by

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{F}.$$

Example. Going back to our example of

$$\frac{d\mathbf{x}}{dt} = \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}_{A}\mathbf{x}.$$

We find eigenvalues of $\lambda_1 = 4, \lambda_2 = 2$ and eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This gives

$$\mathbf{x}_1 = e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_2 = e^{2t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

In general, if we have two distinct eigenvalues, then our solutions are

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

Define

$$\begin{aligned}\Phi_A(t) &= \begin{pmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{pmatrix}\\ &= \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}.\end{aligned}$$

We call Φ_A a fundamental matrix for A.

The general solution to the system is given by

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\\ &= c_1\begin{pmatrix}e^{4t}\\ e^{4t}\end{pmatrix} + c_2\begin{pmatrix}e^{2t}\\ -e^{2t}\end{pmatrix}\\ &= \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}.\end{aligned}$$

Example. Consider the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$A = \begin{pmatrix}4 & 2 & 1\\ 0 & 4 & 2\\ 0 & 0 & 4\end{pmatrix} \tag{A}$$

Notice that we have a triple-repeated eigenvalue,

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= 4\\ \lambda_3 &= 4.\end{aligned}$$

Unfortunately, to find the eigenvectors, this will be a bit harder.

$$\begin{aligned}(A - 4I)\mathbf{v} &= 0\\ \begin{pmatrix}0 & 2 & 1\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} &= \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.\end{aligned}$$

This gives

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so $b = c = 0$, and our eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We may need some more eigenvectors. Currently, our solution is

$$\mathbf{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We need to go into the realm of generalized eigenvectors. If λ is repeated, we need to do the following.

(1) Find all the eigenvectors for which $(A - \lambda I)\mathbf{v} = 0$. If we come up short, then we have a defective system.

(2) For the remaining eigenvectors, we solve the system

$$(A - \lambda I)\mathbf{v}_j = \mathbf{v}_k,$$

where $\mathbf{v}_k$ is known, and we desire $\mathbf{v}_j$. The $\mathbf{v}_j$ are known as generalized eigenvectors.

(3) Continue this process until we are done.

Now, in this case, we get

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and a generalized eigenvector of

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}.$$

Going at it again, we have

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix},$$

giving the equation

$$\begin{pmatrix} 2b + c \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix},$$

giving

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1/8 \\ 1/4 \end{pmatrix}.$$

Note that when we take generalized eigenvectors, we "integrate" with respect to t before adding. For instance

$$\begin{aligned} \mathbf{x}_1 &= e^{\lambda t}\mathbf{v}_1 \\ \mathbf{x}_2 &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) \\ \mathbf{x}_3 &= e^{\lambda t}\left(\frac{t^2}{t}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right). \end{aligned}$$

Now, our linearly independent solutions to the system in (A) is of the form

$$\begin{aligned} \mathbf{x}_1(t) &= e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \mathbf{x}_2(t) &= e^{4t}\left(t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}\right) \\ \mathbf{x}_3(t) &= e^{4t}\left(\frac{t^2}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/8 \\ 1/4 \end{pmatrix}\right). \end{aligned}$$

This gives the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{4t} & te^{4t} & \frac{t^2}{2}e^{4t} \\ 0 & \frac{1}{2}e^{4t} & e^{4t}\left(\frac{t}{2} - \frac{1}{8}\right) \\ 0 & 0 & \frac{1}{4}e^{4t} \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}.$$

The general solution is, then,

$$\mathbf{x}(t) = e^{At}\mathbf{c},$$

where $\mathbf{c}$ is a constant vector, and e^{At} is the matrix exponential of A.

Example. Consider A as the matrix with eigenvalue λ and eigenvector $\mathbf{v}_1$ and generalized eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$. Then, the solution set

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v}_1\\
\mathbf{x}_2(t) &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)\\
\mathbf{x}_3(t) &= e^{\lambda t}\left(\frac{t^2}{2}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \lambda e^{\lambda t}\mathbf{v}_1\\
A\mathbf{x}_1(t) &= Ae^{\lambda t}\mathbf{v}_1\\
&= e^{\lambda t}A\mathbf{v}_1\\
&= \lambda e^{\lambda t}\mathbf{v}_1.
\end{aligned}$$

Now, recalling that $A\mathbf{v}_1 = \lambda\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$, we have

$$\begin{aligned}
\frac{d\mathbf{x}_2}{dt} &= \lambda e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) + e^{\lambda t}\mathbf{v}_1\\
A\mathbf{x}_2(t) &= Ae^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)\\
&= e^{\lambda t}(tA\mathbf{v}_1 + A\mathbf{v}_2)\\
&= e^{\lambda t}(t\lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_1)\\
&= \lambda e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) + e^{\lambda t}\mathbf{v}_1.
\end{aligned}$$

Finally, we have $A\mathbf{v}_3 = \lambda\mathbf{v}_3 + \mathbf{v}_2$.

Example. We assume A is a $n \times n$ real matrix. Then, all complex eigenvalues of A come in conjugate pairs, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$.

Then, our eigenvectors are of the form $\mathbf{v}_1 = \mathbf{u} + i\mathbf{w}$ and $\mathbf{v}_2 = \mathbf{u} - i\mathbf{w}$.

Note that if we find the solution for λ_1 and $\mathbf{v}_2$. This gives

$$\begin{aligned}
e^{\lambda t}\mathbf{v} &= e^{(a+ib)}(\mathbf{u} + i\mathbf{w})\\
&= e^{at}(\cos(bt) + i\sin(bt))(\mathbf{u} + i\mathbf{w})\\
&= e^{at}((\cos(bt)\mathbf{u} - \sin(bt)\mathbf{w}) + i(\cos(bt)\mathbf{w} + \sin(bt)\mathbf{u})).
\end{aligned}$$

Example. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -4\\ 0 & 3 & 0\\ 2 & 0 & 5 \end{pmatrix}$$

for the system of equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Using the power of computation, we have

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= 3+2i\\ \lambda_3 &= 3-2i,\end{aligned}$$

and eigenvectors of

$$\begin{aligned}\mathbf{v}_1 &= \begin{pmatrix}0\\1\\0\end{pmatrix}\\ \mathbf{v}_2 &= \begin{pmatrix}-4\\0\\2+2i\end{pmatrix}\\ \mathbf{v}_3 &= \begin{pmatrix}-4\\0\\2-2i\end{pmatrix}.\end{aligned}$$

Now, we see that

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\lambda_1 t}\mathbf{v}_1\\ &= \begin{pmatrix}0\\e^{3t}\\0\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_2(t) &= e^{3t}\left(\cos(2t)\begin{pmatrix}-4\\0\\2\end{pmatrix} - \sin(2t)\begin{pmatrix}0\\0\\2\end{pmatrix}\right)\\ \mathbf{x}_3(t) &= e^{3t}\left(\cos(2t)\begin{pmatrix}0\\0\\2\end{pmatrix} + \sin(2t)\begin{pmatrix}-4\\0\\2\end{pmatrix}\right).\end{aligned}$$

This gives the matrix

$$\begin{aligned}\Phi(t) &= \begin{pmatrix}0 & -4e^{3t}\cos(2t) & -4e^{3t}\sin(2t)\\ e^{3t} & 0 & 0\\ 0 & 2e^{3t}(\cos(2t)-\sin(2t)) & 2e^{3t}(\sin(2t)+\cos(2t))\end{pmatrix}\\ W(t) &= \det(\Phi(t))\\ &= -e^{3t}\left(-8e^{6t}\left(\cos(2t)\sin(2t)+\cos^2(2t)\right) + 8e^{6t}\left(\sin(2t)\cos(2t)-\sin^2(2t)\right)\right)\\ &= 8e^{9t}\\ &\neq 0.\end{aligned}$$

Example. We wish to solve $\frac{dx}{dt} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

To find our eigenvalues and eigenvectors, we begin by finding

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)^2 \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 5 \\ 0 & -2 & -1-\lambda \end{pmatrix} \\ &= (2-\lambda)^3 \det\begin{pmatrix} 1-\lambda & 5 \\ -2 & -1-\lambda \end{pmatrix} \\ &= (2-\lambda)^3((1-\lambda)(-1-\lambda)+10). \end{aligned}$$

We have five eigenvalues,

$$\lambda = \pm 3i, 2, 2, 2.$$

For $\lambda_{1,2} = \pm 3i$, then

$$\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now for $\lambda_3 = 2$, we have

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now, we have

$$(A - 2I)\mathbf{v}_3 = 0$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ 3 \end{pmatrix} = 0.$$

From this equation, we have

$$\begin{aligned} c &= 0 \\ -b + 5d &= 0 \\ -2b - 3d &= 0. \end{aligned}$$

Now, we have independent a and e. This gives

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that both $\mathbf{v}_3$ and $\mathbf{v}_4$ are regular eigenvectors. Now, we wish to find one generalized eigenvector. We find this generalized eigenvector, $\mathbf{w}$, by observing that the 1 in entry $A_{1,3}$ effectively ties our vector $\mathbf{v}_4$ to vector $\mathbf{v}_{1,2}$. Thus, we get

$$(A - 2I)\mathbf{w} = \mathbf{v}_4$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, solving this, we get $c = 1$, giving the generalized eigenvector of

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we have

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{v}_4 \to \mathbf{w}$ is a chain of length 2. This gives the JCF of

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1+3i & 1-3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3i & 0 & 0 & 0 & 0 \\ 0 & -3i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1+3i & 1-3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^{-1}.$$

We get the solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ -\cos(3t) - 3\sin(3t) \\ 0 \\ 2\cos(3t) \\ 0 \end{pmatrix}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} 0 \\ 3\cos(3t) - \sin(3t) \\ 0 \\ 2\sin(3t) \\ 0 \end{pmatrix}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{2t} \end{pmatrix}$$

$$\mathbf{x}_4(t) = \begin{pmatrix} e^{2t} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_5(t) = \begin{pmatrix} te^{2t} \\ 0 \\ e^{2t} \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{x}_5(t) = e^{2t}(t\mathbf{v}_4 + \mathbf{v}_5)$.

The fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} 0 & 0 & 0 & e^{2t} & te^{2t} \\ -\cos(3t) + 3\sin(3t) & 3\cos(3t) - \sin(3t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2t} \\ 2\cos(3t) & 2\sin(3t) & 0 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \end{pmatrix}.$$

Now, we want to find $\Phi(0)$, or $\Phi(t_0)$. Furthermore, we need to find $\Phi^{-1}(0)$, or $\Phi^{-1}(t_0)$.

Example (Implementing Initial Conditions)**.** Looking back at our equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

we may apply the initial condition of

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

We use the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

with the initial condition

$$\mathbf{x}_0 = \begin{pmatrix} 4 \\ 15 \end{pmatrix}.$$

Generally our approach to solving this kind of problem, we take the eigenvectors and eigenvalues, giving

$$\begin{aligned}\lambda_1 &= 4\\ \mathbf{v}_1 &= \begin{pmatrix}1\\1\end{pmatrix}\\ \lambda_2 &= 2\\ \mathbf{v}_2 &= \begin{pmatrix}1\\-1\end{pmatrix},\end{aligned}$$

and associated solutions of

$$\begin{aligned}\mathbf{x}_1 &= \begin{pmatrix}e^{4t}\\e^{4t}\end{pmatrix}\\ \mathbf{x}_2 &= \begin{pmatrix}e^{2t}\\-e^{2t}\end{pmatrix}.\end{aligned}$$

Then, we form a fundamental matrix of solutions:

$$\Phi(t) = \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}.$$

Note that, for any vector of constants $\mathbf{c}$, we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}$$

is a solution of $\frac{dx}{dt} = A\mathbf{x}$.

To find $\mathbf{c}$, we see that

$$\mathbf{x}_0 = \Phi(0)\mathbf{c},$$

so that

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0$$

is the solution to our initial value problem.

Calculating

$$\Phi(0) = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},$$

we find

$$\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Thus, we get the solutions of

$$\mathbf{x}(t) = \begin{pmatrix}e^{4t} & e^{2t}\\ e^{4t} & -e^{2t}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}4\\15\end{pmatrix}$$

$$= \begin{pmatrix} \frac{19}{2}e^{4t} - \frac{11}{2}e^{2t} \\ \frac{19}{2}e^{4t} + \frac{11}{2}e^{2t} \end{pmatrix}.$$

Note that we may define

$$\Psi(t) = \Phi(t)\Phi^{-1}(0),$$

giving

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0.$$

We may calculate

$$\begin{aligned}\Psi(t) &= \Phi(t)\Phi^{-1}(0)\\ &= \frac{1}{2}\begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix} e^{4t} + e^{2t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{4t} + e^{2t} \end{pmatrix}.\end{aligned}$$

The Matrix Exponential

Example (The Matrix Exponential)**.** When we have a single first-order equation, such as

$$\frac{dy}{dt} = 3y,$$

with initial condition $y(0)$, we solve it by taking $y(t) = \pi e^{3t}$.

Similarly, if we're given

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

we may want to know if there is an analogous e^{At}.

In fact, there is. Using the Taylor expansion, we have

$$\begin{aligned}e^{At} &= I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots\\ &= \sum_{k=0}^{\infty} A^k\frac{t^k}{k!}.\end{aligned}$$

Note that we may take P to be the matrix of unit eigenvectors of A, and D to be the matrix of eigenvalues corresponding to column eigenvectors

$$A = PDP^{-1}.$$

This is assuming A can be diagonalized. This gives $D = P^{-1}AP$.

Now, if A can be diagonalized, we can take

$$e^{At} = I + \left(PDP^{-1}\right)t + \left(PDP^{-1}\right)^2\frac{t^2}{2} + \cdots$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\left(PDP^{-1}\right)^{k}\frac{t^k}{k!}\\
&= \sum_{k=0}^{\infty} PD^kP^{-1}\frac{t^k}{k!}\\
&= P\left(\sum_{k=0}^{\infty} D^k\frac{t^k}{k!}\right)P^{-1}.
\end{aligned}
$$

We can find the power on any diagonal matrix much more easily than we can on a general matrix. In particular, this gives

$$
e^{At} = P\begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t}\end{pmatrix}P^{-1}
$$

Given A in the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, we wish to find e^{At}, and show that $e^{At} = \Psi(t)$, where $\Psi(t) = \Phi(t)\Phi^{-1}(0)$.

Example. Let

$$
A = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}.
$$

We see that A has repeated eigenvalues of 4 and 4.

Our first eigenvector is

$$
\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Now, evaluating

$$
\begin{aligned}
(A-4I)\mathbf{v} &= 0\\
\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
$$

giving $2b = 0$.

Thus, we're going to need a generalized eigenvector. We have

$$
\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

giving $(A-\lambda I)\mathbf{w} = \mathbf{v}_1$. Thus, we have $2b = 1$. Thus, we have

$$
\mathbf{w} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.
$$

Now, we have

$$
\mathbf{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
&= \begin{pmatrix} e^{4t} \\ 0 \end{pmatrix} \\
\mathbf{x}_2(t) &= te^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{4t}\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \\
&= \begin{pmatrix} te^{4t} \\ \frac{1}{2}e^{4t} \end{pmatrix}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\
&= \begin{pmatrix} e^{4t} & te^{4t} \\ 0 & \frac{1}{2}e^{4t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.
\end{aligned}
$$

Now, we see that

$$
\begin{aligned}
\Phi(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\
\Phi^{-1}(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.
\end{aligned}
$$

Therefore, our matrix exponential is

$$
\begin{aligned}
\Psi(t) &= \begin{pmatrix} e^{4t} & te^{4t} \\ 0 & \frac{1}{2}e^{4t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} e^{4t} & 2te^{4t} \\ 0 & e^{4t} \end{pmatrix}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{x}(t) &= \Psi(t)\mathbf{x}_0 \\
&= \begin{pmatrix} e^{4t} & 2te^{4t} \\ 0 & e^{4t} \end{pmatrix}\mathbf{x}_0.
\end{aligned}
$$

We often refer to $\Psi(t)$ as the flow matrix.

Now, because $\Psi(0) = I$, we have

$$\Psi(t)\Psi(-t) = I,$$

meaning that $\Psi(-t) = \Psi(t)^{-1}$.

Example. Consider

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

as the set of eigenvector equations, giving

$$\begin{aligned}\begin{pmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_1\lambda_1 & \mathbf{v}_2\lambda_2 & \cdots & \mathbf{v}_n\lambda_n \end{pmatrix}\\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},\end{aligned}$$

giving the expression $AP = PD$, where P is the set of eigenvectors.

Now, if we have generalized eigenvectors, we have a different case.

Consider the case of a chain. We know that

$$(A - \lambda I)\mathbf{v}_1 = 0$$

is the expression of an eigenvector. Now, if we have repeated eigenvalues, we get the second equation of

$$\begin{aligned}(A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1\\ (A - \lambda I)\mathbf{v}_3 &= \mathbf{v}_1\\ &\vdots\\ (A - \lambda I)\mathbf{v}_n &= \mathbf{v}_{n-1}.\end{aligned}$$

We start by changing these equations to give

$$\begin{aligned}A\mathbf{v}_1 &= \lambda\mathbf{v}_1\\ A\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1\\ A\mathbf{v}_3 &= \lambda\mathbf{v}_3 + \mathbf{v}_2\\ &\vdots\\ A\mathbf{v}_n &= \lambda\mathbf{v}_n + \mathbf{v}_{n-1}.\end{aligned}$$

We see that these are effectively the eigenvalue equations with a small perturbation. Constructing the matrix, we have

$$\begin{aligned}\begin{pmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 & \cdots & A\mathbf{v}_n \end{pmatrix} &= \begin{pmatrix} \lambda\mathbf{v}_1 & \lambda\mathbf{v}_2 + \mathbf{v}_1 & \lambda\mathbf{v}_3 + \mathbf{v}_2 & \cdots\lambda\mathbf{v}_n + \mathbf{v}_{n-1} \end{pmatrix}\\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n \end{pmatrix}\underbrace{\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}}_{J}.\end{aligned}$$

We call the matrix J the Jordan canonical form of A, and we get the expression $AP = PJ$, where P is the matrix of generalized eigenvectors as columns.

Now, if we have multiple chains, we get multiple blocks. For instance, if we have the chains $\mathbf{v}_1 \to \mathbf{v}_2 \to \mathbf{v}_3$, $\mathbf{v}_4 \to \mathbf{v}_5 \to \mathbf{v}_6 \to \mathbf{v}_7$, and $\mathbf{v}_8$ being standalone, all for the same eigenvalue λ. This gives the Jordan canonical form of

$$J = \begin{pmatrix} \lambda & 1 & 0 & & & & & \\ 0 & \lambda & 1 & 0 & & & & \\ 0 & 0 & \lambda & 0 & & & & \\ & & & \lambda & 1 & 0 & 0 & \\ & & & 0 & \lambda & 1 & 0 & \\ & & & 0 & 0 & \lambda & 1 & \\ & & & 0 & 0 & 0 & \lambda & \\ & & & & & & & \lambda \end{pmatrix}$$

The reason block matrices are useful is that they simplify calculations massively. We may consider the block matrices as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

For instance, if we have 4×4 matrices, we convert this multiplication into 8 2×2 matrix multiplications. On first glance, this doesn't seem more efficient, but if there are a lot of zeros, it does actually become more efficient.

Example. In the general case, our flow matrix is of the form

$$\begin{aligned} \Psi(t) &= e^{At} \\ &= Pe^{Jt}P^{-1}. \end{aligned}$$

If

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

we now want to find e^{Jt}.

Now, if we had

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$

then

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{pmatrix}.$$

Note that the Jordan–Chevalley decomposition allows us to take

$$\begin{aligned} e^{Jt} &= e^{(D+N)t} \\ &= e^{Dt}e^{Nt}. \end{aligned}$$

where

$$\begin{aligned} N &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ N^2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ N^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We may calculate

$$\begin{aligned} e^{Nt} &= I + Nt + \frac{N^2}{2}t^2 + \frac{N^3}{6}t^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$e^{Jt} = \begin{pmatrix} e^{\lambda t} & & \\ & e^{\lambda t} & \\ & & e^{\lambda t} \end{pmatrix}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Example (Constructing Differential Equations)**.** Consider

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

with solutions of $y_2(x) = \cos^2(x)$ and $y_2(x) = \sin^2(x)$.

We start by computing some derivatives.

$$\begin{aligned} \frac{dy_1}{dx} &= -2\cos(x)\sin(x) \\ &= -\sin(2x) \\ \frac{d^2y_1}{dx^2} &= -2\cos(2x) \\ &= -2\sin^2(x) + 2\cos^2(x) \end{aligned}$$

$$\begin{aligned}\frac{dy_2}{dx} &= 2\sin(x)\cos(x)\\ &= \sin(2x)\\ \frac{d^2y_2}{dx^2} &= 2\cos(2x)\\ &= 2\cos^2(x) - 2\sin^2(x).\end{aligned}$$

Substituting, we get

$$\begin{aligned}-2\sin^2(x) + 2\cos^2(x) - 2p(x)\cos(x)\sin(x) + q(x)\cos^2(x) &= 0\\ 2\sin^2(x) + 2\cos^2(x) + 2p(x)\cos(x)\sin(x) + q(x)\sin^2(x) &= 0.\end{aligned}$$

If we add these two equations together, we get $q(x) = 0$, meaning that a constant is a solution — note that $\cos^2(x) + \sin^2(x) = 1$, so we know this is fair.

If we subtract the second equation from the first equation, we get

$$\begin{aligned}-4\sin^2(x) + 4\cos^2(x) - 4p(x)\cos(x)\sin(x) &= 0\\ 4\cos(2x) - 2p(x)\sin(2x) &= 0.\end{aligned}$$

Thus, we want

$$2p(x)\sin(2x) = 4\cos(2x),$$

giving

$$p(x) = 2\cot(2x).$$

Therefore, our equation is

$$\frac{d^2y}{dx^2} + 2\cot(2x)\frac{dy}{dx} = 0.$$

Example. Consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix}1 & -2\\ 5 & 3\end{pmatrix}\mathbf{x}.$$

We see that

$$\begin{aligned}\lambda &= 2 \pm 3i\\ \mathbf{v} &= \begin{pmatrix}-1\\ 5\end{pmatrix} \pm i\begin{pmatrix}3\\ 0\end{pmatrix}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}e^{\lambda t} &= e^{2t}(\cos(3t) + i\sin(3t))\\ e^{\lambda t}\mathbf{v} &= e^{2t}(\cos(3t) + i\sin(3t))\left(\begin{pmatrix}-1\\ 5\end{pmatrix} + i\begin{pmatrix}3\\ 0\end{pmatrix}\right)\\ \mathbf{x}_1(t) &= e^{2t}\begin{pmatrix}-\cos(3t) - 3\sin(3t)\\ 5\cos(3t)\end{pmatrix}\end{aligned}$$

$$\mathbf{x}_2(t) = e^{2t}\begin{pmatrix} 3\cos(3t) - \sin(3t) \\ 5\sin(3t) \end{pmatrix}.$$

Now, we want to find the fundamental matrix, $\Psi_A(t)$, and the matrix exponential e^{At}.

We start by finding $\Phi_A(t)$ to give

$$\Phi_A(t) = \begin{pmatrix} -e^{2t}\cos(3t) - 3e^{2t}\sin(3t) & 3e^{2t}\cos(3t) - e^{2t}\sin(3t) \\ 5e^{2t}\cos(3t) & 5e^{2t}\sin(3t) \end{pmatrix}$$

We have already found the solution

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}.$$

However, if we want to apply initial conditions, we need $\Psi_A(t)$ to obtain $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$.

$$\begin{aligned}\Psi_A(t) &= \Phi(t)\Phi_A^{-1}(0) \\ &= \begin{pmatrix} -e^{2t}\cos(3t) - 3e^{2t}\sin(3t) & 3e^{2t}\cos(3t) - e^{2t}\sin(3t) \\ 5e^{2t}\cos(3t) & 5e^{2t}\sin(3t) \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}^{-1}.\end{aligned}$$

Notice that $\Phi_A(0)$ has columns equal to the real and imaginary eigenvectors.

$$= -\frac{1}{15}\begin{pmatrix} -e^{2t}\cos(3t) - 3e^{2t}\sin(3t) & 3e^{2t}\cos(3t) - e^{2t}\sin(3t) \\ 5e^{2t}\cos(3t) & 5e^{2t}\sin(3t) \end{pmatrix}\begin{pmatrix} 0 & -3 \\ -5 & -1 \end{pmatrix}.$$

After many error-prone computations, we obtain

$$\Psi_A(t) = \begin{pmatrix} e^{2t}\cos(3t) - \frac{1}{3}e^{2t}\sin(3t) & -\frac{2}{3}e^{2t}\sin(3t) \\ \frac{5}{3}e^{2t}\sin(3t) & e^{2t}\cos(3t) + \frac{1}{3}e^{2t}\sin(3t) \end{pmatrix}.$$

To ensure that this is a (plausible) $\Psi(t)$, we check $\Psi_A(0)$, and see that $\Psi_A(0) = I$. Also, it can be verified[IV] that

$$\frac{d\Psi_A}{dt} = A\Psi_A(t).$$

Now, finding Ψ_A was quite difficult. What if there's a better way?

Recall from the eigenvector with conjugate eigenvalues $\lambda = a \pm ib$ equation that

$$A(\mathbf{u} + i\mathbf{w}) = \left(e^{at}\cos(bt) + ie^{at}\sin(bt)\right)(\mathbf{u} + i\mathbf{w}).$$

Now, examining the real part, we have

$$\begin{aligned}A\mathbf{u} &= e^{at}\cos(bt)\mathbf{u} - e^{at}\sin(bt)\mathbf{w} \\ A\mathbf{w} &= e^{at}\cos(bt)\mathbf{w} + e^{at}\sin(bt)\mathbf{u}.\end{aligned}$$

We will construct a similarity transform

$$AP = PD.$$

[IV] I don't think this is a fun thing to do, but it can indeed be verified.

Now, we may take

$$\begin{aligned} P &= \begin{pmatrix} \mathrm{Re}(\mathbf{v}) & \mathrm{Im}(\mathbf{v}) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u} & \mathbf{w} \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We want to find $e^{At} = Pe^{Dt}P^{-1}$. Note that, after much tedious calculation (or noticing that our vector D is a matrix expression for the complex number $a + bi$), we obtain

$$e^{Dt} = \begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}.$$

Inhomogeneous Systems of Equations

Now, we discuss inhomogeneous systems. Consider the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{F}(t).$$

We look at the homogeneous system,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

which has the solutions of

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{c} \\ &= e^{At}\mathbf{x}_0. \end{aligned}$$

When we deal with inhomogeneous solutions in the one-dimensional case, we use methods like variation of parameters or undetermined coefficients.

Here, we will assume a perturbation — i.e., our solution is close enough to our fundamental solution.

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{u}(t) \\ \frac{d\mathbf{x}}{dt} &= \frac{d\Phi}{dt}\mathbf{u}(t) + \Phi(t)\frac{d\mathbf{u}}{dt}. \end{aligned}$$

Plugging in, we have

$$\frac{d\Phi}{dt}\mathbf{u}(t) + \Phi(t)\frac{d\mathbf{u}}{dt} = A\Phi(t)\mathbf{u}(t) + \mathbf{F}(t).$$

We want to cancel some things out. Recall that $\frac{d\Phi}{dt} = A\Phi$. Therefore, we obtain

$$\Phi(t)\frac{d\mathbf{u}}{dt} = \mathbf{F}(t).$$

Multiplying on both sides by $\Phi^{-1}(t)$, we get

$$\frac{d\mathbf{u}}{dt} = \Phi^{-1}(t)\mathbf{F}(t)$$

$$\mathbf{u}(t) = \int \Phi^{-1}(t)\mathbf{F}(t)\,dt,$$

where the integral is taken coordinatewise. Thus,

$$\mathbf{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{F}(t)\,dt.$$

Therefore, presumably, we have the general solution

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} + \Phi(t)\int \Phi^{-1}(t)\mathbf{F}(t)\,dt.$$

Instead, we may consider $\Psi_A(t)$ rather than $\Phi(t)$, as $\Psi_A^{-1}(t) = \Psi(-t)$. This gives

$$\mathbf{x}(t) = \Psi_A(t)\mathbf{c} + \Psi_A(t)\int \Psi_A(-t)F(t)\,dt.$$

Now, if we want to implement the condition $\mathbf{x}_0 = \mathbf{x}(0)$, we want to modify the integral slightly.

$$\begin{aligned}\mathbf{x}(t) &= \Psi_A(t)\mathbf{c} + \Psi_A(t)\int_0^t \Psi_A(-s)\mathbf{F}(s)\,ds\\ \mathbf{x}(0) &= \mathbf{c}\end{aligned}$$

meaning

$$\mathbf{x}(t) = \Psi_A(t)\mathbf{x}_0 + \Psi_a(t)\int_0^t \Psi_A(-s)\mathbf{F}(s)\,dt.$$

Example (Solving an Inhomogeneous System)**.** Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 3x + y + 2\\ \frac{dy}{dt} &= x + 3y + e^{7t}.\end{aligned}$$

Writing in matrix form, we have

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\ e^{7t}\end{pmatrix}.$$

Note that we don't have to use an initial condition, so we only need the formula

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} + \Psi(t)\int \Psi(-t)\mathbf{F}(t)\,dt$$

Now, note that

$$\Psi(t) = \begin{pmatrix}\frac{1}{2}\left(e^{4t}+e^{2t}\right) & \frac{1}{2}\left(e^{4t}-e^{2t}\right)\\ \frac{1}{2}\left(e^{4t}-e^{2t}\right) & \frac{1}{2}\left(e^{4t}+e^{2t}\right)\end{pmatrix}.$$

Calculating the inverse, we have

$$\begin{aligned}\Psi^{-1}(t) &= \Psi(-t)\\ &= \begin{pmatrix}\frac{1}{2}\left(e^{-4t}+e^{-2t}\right) & \frac{1}{2}\left(e^{-4t}-e^{-2t}\right)\\ \frac{1}{2}\left(e^{-4t}-e^{-2t}\right) & \frac{1}{2}\left(e^{-4t}+e^{-2t}\right)\end{pmatrix}.\end{aligned}$$

Computing, we have

$$\Psi^{-1}\mathbf{F}(t) = \begin{pmatrix} e^{-4t} + e^{-2t} + \frac{1}{2}\left(e^{3t} - e^{5t}\right) \\ e^{-4t} - e^{-2t} + \frac{1}{2}\left(e^{3t} + e^{5t}\right) \end{pmatrix}.$$

Integrating coordinatewise, we get

$$\int \Psi^{-1}\mathbf{F}(t)\, dt = \begin{pmatrix} -\frac{1}{4}e^{-4t} - \frac{1}{2}e^{-2t} + \frac{1}{6}e^{3t} - \frac{1}{10}e^{5t} \\ -\frac{1}{4}e^{-4t} + \frac{1}{2}e^{-2t} + \frac{1}{6}e^{3t} + \frac{1}{10}e^{5t} \end{pmatrix}$$

We find the general solution by multiplying by $\Psi(t)$ and adding the homogeneous solution.

Example (Nonconstant Coefficients)**.** Consider

$$\begin{aligned} \frac{dx}{dt} &= (2t+1)x + y \\ \frac{dy}{dt} &= x + (2t+1). \end{aligned}$$

Writing in matrix form, we have

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2t+1 & 1 \\ 1 & 2t+1 \end{pmatrix},$$

with eigenvalues

$$\begin{aligned} \lambda_1 &= 2t+2 \\ \lambda_2 &= 2t \\ \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Solving, we can take

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{Dt} \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right). \end{aligned}$$

Now, calculating e^{Dt}, we need to take integrals of our expressions $\lambda_1(t)$ and $\lambda_2(t)$ for our exponent expressions, giving

$$\begin{aligned} e^{At} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{\int \lambda_1(t)\, dt} & 0 \\ 0 & e^{\int \lambda_2(t)\, dt} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{t^2+2t} & 0 \\ 0 & e^{t^2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{x}(t) &= Pe^{Dt}P^{-1}\mathbf{x}_0 \\ &= \frac{1}{2}\begin{pmatrix} e^{t^2+2t} + e^{t^2} & e^{t^2+2t} - e^{t^2} \\ e^{t^2+2t} - e^{t^2} & e^{t^2+2t} + e^{t^2} \end{pmatrix}\mathbf{x}_0. \end{aligned}$$

Introducing Partial Differential Equations

Consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{$*$}$$

There are a lot of solutions to this equation.

- $u(x,t) = 5$;
- $u(x,t) = x - t$;
- $u(x,t) = e^{x}e^{-t}$.

We want to try to understand solutions to this equation given an initial value. Note that this means that the "initial values" are actually functions of x. Upon defining our domains in x and t, we have an initial condition of the form

$$u(x,0) = f(x).$$

Now, considering ($*$), we may consider the initial condition

$$u(x,0) = \sin(x).$$

Then, all our proposed solutions from earlier no longer apply. Using the power of inspection, we may consider the solution

$$\begin{aligned} u(x,t) &= \sin(x-t) \\ \frac{\partial u}{\partial t} &= -\cos(x-t) \\ \frac{\partial u}{\partial x} &= \cos(x-t), \end{aligned}$$

meaning that our solution for $u(x,t)$ works.

This gives an initial value problem of

$$\begin{aligned} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

We now need to know if there is an existence/uniqueness condition for partial differential equations. Consider the proposed solution

$$\begin{aligned} u(x,t) &= f(x-at) \\ \frac{\partial u}{\partial x} &= f(x-at) \\ \frac{\partial u}{\partial t} &= -af(x-at), \end{aligned}$$

which solves our equation. The main condition on f here is that f has to be differentiable.

We call ($*$) the *transport equation*.

Example. Consider

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F, G are functions of x and y.

This is the most general second-order two-variable inhomogeneous linear partial differential equation.

Example. In the case of $(*)$, we see that $A = B = C = F + G = 0$. For the transport equation, we defined the initial condition $u(x,0) := u_0(x)$, and a domain $-\infty < x < \infty$, $t \geqslant 0$.

The transport equation is a homogeneous first-order linear constant-coefficient partial differential equation, which is much easier to solve.

Equations of the type seen in the transport equation, which admit only initial conditions (and no boundary conditions) are known as Cauchy problems.

Example. The equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

is a *nonlinear* homogeneous first-order partial differential equation, known as Burgers' equation.

Definition. Consider the general partial differential equation(s) of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = 0.$$

- If $(B(x,y))^2 - 4(A(x,y)C(x,y)) > 0$, then we call this equation a hyperbolic partial differential equation.
- If $(B(x,y))^2 - 4(A(x,y)C(x,y)) = 0$, then we call this equation a parabolic partial differential equation.
- If $(B(x,y))^2 - 4(A(x,y)C(x,y)) < 0$, then we call this equation an elliptic partial differential equation.

Example (The Heat Equation)**.** Consider the equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

This is known as the heat equation (in one dimension). In multiple dimensions, the heat equation would have the form

$$\frac{\partial u}{\partial t} = \sum_i k_i \frac{\partial^2 u}{\partial x_i^2}.$$

We will focus on solving the equation in one dimension.

For example, the following equations are solutions to the heat equation:

$$\begin{aligned} u(x,t) &= 0 \\ u(x,t) &= 3x + 5 \\ u(x,t) &= kt - k\frac{x^2}{2}. \end{aligned}$$

Unfortunately, there are an infinite number of solutions to the heat equation, which is not helpful.

We implement some conditions in order to solve this equation. We start by implementing the domain $a \leqslant x \leqslant b$ and $t \geqslant 0$.

Now, we may implement the boundary conditions of $u(a,t) = c_1$ and $u(b,t) = c_2$, which allow us to express the fact that we don't want the very ends of our heated rod to change.

Example (Solving a Heat Equation)**.** Consider the heat equation of the form

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2},$$

with domain $0 \leqslant x \leqslant \pi$, $t \geqslant 0$, and initial condition $u(x, 0) = \sin(2x)$. The initial conditions yield the boundary condition of $u(0, t) = 0$ and $u(\pi, t) = 0$.

To solve this equation, we assume[v] a solution of the form

$$u(x, t) = X(x)T(t).$$

Solving this solution, we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= X(x)\frac{\partial T}{\partial t} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 X}{\partial x^2}T(t). \end{aligned}$$

Thus, we get the equation of the form

$$X(x)\frac{\partial T}{\partial t} = 3\frac{\partial^2 X}{\partial x^2}T(t).$$

Dividing by $X(x)T(t)$, we have

$$\begin{aligned} \left(\frac{\partial T}{\partial t}\right)\left(\frac{1}{T(t)}\right) &= 3\left(\frac{\partial^2 X}{\partial x^2}\right)\left(\frac{1}{X(x)}\right) \\ \frac{1}{3}\left(\frac{\partial T}{\partial t}\right)\left(\frac{1}{T(t)}\right) &= \left(\frac{\partial^2 X}{\partial x^2}\right)\left(\frac{1}{X(x)}\right) \\ &= C. \end{aligned}$$

Now, examining the equation

$$\left(\frac{\partial^2 X}{\partial x^2}\right)\left(\frac{1}{X(x)}\right) = C,$$

we see that X has solutions of the form

$$\begin{aligned} X(x) &= \sin\left(\sqrt{C}x\right) \\ &= \cos\left(\sqrt{C}x\right) \\ &= \cosh\left(\sqrt{C}x\right) \\ &= \sinh\left(\sqrt{C}x\right) \\ &= Ax + B \\ &= B \end{aligned}$$

Now, we have three classes of solutions. If $C < 0$, we have the trigonometric functions, if $C > 0$, then we have the hyperbolic trigonometric functions, and if $C = 0$, then we have the linear functions. Alternatively, we write $C = \lambda^2$.

[v] Also known as guessing.