

3

The graph  $G$  in Figure 50 is connected and contains no bridges. Find a strong orientation of  $G$ .

4

Suppose that  $D$  is an orientation of a connected graph  $G$  such that for each vertex  $v$  of  $G$ , some edge is directed toward  $v$  and some edge is directed away from  $v$ . Is  $D$  a strong orientation on  $G$ .

Since  $G$  is a connected graph, there must be a path  $P$  between any two vertices  $v_1$  and  $v_2$ . Call this path  $P$ , travelling along the orientation  $D$ . Then, the path “exits”  $v_1$  and “enters”  $v_2$ . Suppose that there is no path from  $v_2$  back to  $v_1$ .

Then, within  $G - P$  it must be the case that either  $v_1$  or  $v_2$  are of degree 0, or there is a point in  $G - P$  wherein the interior vertices have no edges directed “out” both of which would contradict the assumptions. Additionally, since there is at least one edge directed “out” from  $v_2$  and directed “in”  $v_1$ .

## Extra Problem 1

Determine whether the following statements are true, and prove if so.

- (a) A graph  $G$  has a strong orientation if and only if  $G$  is connected and has an orientation such that every pair of distinct vertices in  $G$  is in a directed cycle.
- (b) A graph  $G$  has a strong orientation if and only if  $G$  is connected and has an orientation such that every pair of distinct vertices in  $G$  is in a directed circuit.
- (c) A graph  $G$  has a strong orientation if and only if  $G$  is connected and has an orientation such that every pair of distinct vertices in  $G$  is in a directed closed walk.

(a)

( $\Rightarrow$ ) If  $G$  has a strong orientation, then for  $u, v \in V(G)$  distinct,  $\exists P = u, \dots, v$ , and  $P' = v, \dots, u \in G - P$  paths. Therefore, by appending  $P$  and  $P'$  together, we find that  $u$  and  $v$  are in a directed cycle.

( $\Leftarrow$ ) Let  $u, v \in V(G)$  distinct such that  $u, v \in C$ , where  $C$  is a directed cycle. Thus, there must be no bridge between  $u$  and  $v$ , as deletion of any edge must allow a path in  $C - e$  — so, by the condition of Robbin's Theorem, it must be the case that there is a strong orientation on  $G$ .

(b)

( $\Rightarrow$ ) Suppose  $G$  has a strong orientation. Then, every pair of distinct vertices  $u, v \in V(G)$  must have a path  $P = u, \dots, v$  and a path  $P' = v, \dots, u \in G - P$ . By appending these paths together, we get a directed cycle, which is also a directed circuit.

$\Leftarrow$  Suppose  $G$  is connected and has an orientation such that for every  $u, v \in V(G)$  distinct,  $u, v$  are in a directed circuit  $C'$ . This means there is a directed  $u, v$  trail in  $G$  — and thus, a directed  $u, v$  path  $P$  in  $G$ . Similarly, in  $G - P$ , there must be a directed  $v, u$  trail, and thus a directed  $v, u$  path. So, by the conditions of Robbin's Theorem, it must be the case that  $G$  has a strong orientation.

(c)

Since a closed walk is able to repeat edges, it is not necessarily the case that  $G$  is a bridgeless graph, and thus has a strong orientation.

## Extra Problem 2

Let  $K_n$  be a strong tournament with  $n \geq 3$ .

- (a) Prove that for every  $j$  in  $\{2, \dots, n-2\}$ ,  $K_n$  has a directed cycle of length  $1+j$  or  $1+n-j$ .
- (b) Prove that for every  $j$  in  $\{2, \dots, n-2\}$ ,  $K_n$  has  $n$  distinct directed cycles  $C_1, \dots, C_n$  such that each  $C_i$  has length  $1+j$  or  $1+n-j$ .