This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

Useful Results and Proofs

Analytic Functions

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A function $f\colon U \to \mathbb{C}$ is called *analytic* if, for any $z_0 \in U$, there is $r > 0$ and $(a_k)_k \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in U(z_0, r)$.

Analytic functions form a $\mathbb{C}$-algebra.

Theorem (Identity Theorem)**:** Let $f, g\colon U \to \mathbb{C}$ be analytic functions defined a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in U, then $f = g$ on U.

Proof. To begin, we show that if $f\colon U \to \mathbb{C}$ is an analytic function that is not uniformly zero, then for any $z_0 \in U$, there is $\rho > 0$ such that f is nonzero on $\dot{U}(z_0, \rho) \subseteq U$. Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all $z \in U(z_0, r)$, some $r > 0$, and since f is not uniformly zero, there is some minimal ℓ such that $a_\ell \neq 0$. This yields

$$f(z) = (z - z_0)^\ell \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function $h\colon U(z_0, r) \to \mathbb{C}$ given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as f and is not zero at z_0, so that g is not zero on some $U(z_0, \rho)$ as g is continuous.

Now, we let V_1 be the set of accumulation points of A in U, and let $V_2 = U \setminus V_1$.

If $z \in V_2$, then there is some $r_1 > 0$ such that $\dot{U}(z_0, r_1) \cap A = \emptyset$, or that $\dot{U}(z_0, r_1) \subseteq A^c$. Meanwhile, since U is open, there is some $r_2 > 0$ such that $U(z_0, r_2) \subseteq U$, meaning that if $r = \min\{r_1, r_2\}$, then $U(z_0, r) \subseteq U \setminus A$. Thus, V_2 is open.

Meanwhile, if $z \in V_1$, then since $V_1 \subseteq U$, it follows that there is $r > 0$ such that $U(z, r)$ and $(a_k)_k$ such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all $w \in U(z,r)$. We claim that $f(w) - g(w)$ is uniformly zero on $U(z,r)$. Else, if there were $w_0 \in U(z,r)$ such that $f(w_0) \neq g(w_0)$, then it would follow that there is $0 < s \leqslant r$ such that $f(w) \neq g(w)$ for all $w \in \dot{U}(w_0, s)$. Yet, this would contradict the assumption that z is an accumulation point, meaning that V_1 is open.

Since V_1 and V_2 are disjoint open sets whose union is equal to U, it follows that either $V_1 = U$ or $V_2 = U$. If $A \neq \emptyset$, then the identity theorem follows. □

Differentiability

Definition: If $U \subseteq \mathbb{C}$ is an open set, then we say f is differentiable at $z_0 \in U$ if

$$\lim_{w \to z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of f at z_0, and usually write $f'(z_0)$.

If f is differentiable at every $z_0 \in U$, we say f is differentiable on U.

If f is continuous and admits a continuous derivative, then we say f is *holomorphic*.

Note that the limit must be independent of direction. That is, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(w) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Now, given $U \subseteq \mathbb{C}$, write $z = x + iy$ and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x,y) + iv(x,y), \end{aligned}$$

where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$. Observe then that if f is differentiable at $x_0 + iy_0 \in U$, then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x+h,y) + iv(x+h,y)) - (u(x,y) + iv(x,y))}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{(u(x,y+h) + iv(x,y+h)) - (u(x,y) + iv(x,y))}{ih} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

Definition: The system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if f is differentiable, then the u and v in the definition of f satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly f is differentiable or holomorphic.

Cauchy's Integral Formula and its Consequences

Old Exams

Notation

- $U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leqslant r\}$
- $S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$
- $\dot{U}(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$
- $A(z_0, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$