Problem 1

Let $v_1, \ldots, v_n$ be mutually orthogonal vectors in an inner product space V. Show that

$$\left\|\sum_{k=1}^{n} v_k\right\|^2 = \sum_{k=1}^{n} \|v_k\|^2 .$$

Proof:

$$\begin{aligned}
\left\|\sum_{k=1}^{n} v_k\right\|^2 &= \left\langle \sum_{k=1}^{n} v_k, \sum_{k=1}^{n} v_k \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \sum_{k=1}^{n} v_k, v_i \right\rangle \\
&= \sum_{i=1}^{n} \langle v_i, v_i\rangle && \text{since for } i \neq j,\ \langle v_i, v_j\rangle = 0 \\
&= \sum_{i=1}^{n} \|v_i\|^2
\end{aligned}$$

Problem 2

Let V be an inner product space and fix $w \neq 0$ in V. We define the one-dimensional projection

$$P_w : V \to V; P_w(v) := \frac{\langle v, w\rangle}{\langle w, w\rangle} w.$$

(i) Prove that $v - P_w(v) \perp P_w(v)$.

(ii) Show that $P_w : V \to V$ is a linear operator with $\|P_w\|_{\text{op}} = 1$.

(iii) Show that $P_w \circ P_w = P_w$.

Proof of (i):

$$\begin{aligned}
\langle v - P_w(v), P_w(v)\rangle &= \langle v, P_w(v)\rangle - \langle P_w(v), P_w(v)\rangle \\
&= \langle v, P_w(v)\rangle - \|P_w(v)\|^2 \\
&= \left\langle v, \frac{\langle v, w\rangle}{\langle w, w\rangle} w \right\rangle - \|P_w(v)\|^2 \\
&= \frac{\overline{\langle v, w\rangle}}{\langle w, w\rangle} \langle v, w\rangle - \|P_w(v)\|^2 \\
&= \frac{|\langle v, w\rangle|^2}{\|w\|^2} - \frac{|\langle v, w\rangle|}{\|w\|^2} \\
&= 0
\end{aligned}$$

Proof of (ii):

$$\begin{aligned}
\|P_w\|_{\text{op}} &= \sup_{v \leq 1} \left\| \frac{\langle v, w\rangle}{\langle w, w\rangle} w \right\| \\
&= \sup_{v \leq 1} \frac{|\langle v, w\rangle|}{\|w\|} \\
&\leq \sup_{v \leq 1} \frac{\|v\|\,\|w\|}{\|w\|} \\
&= 1
\end{aligned}$$

Proof of (iii):

$$\begin{aligned} P_w(P_w(v)) &= P_w\left(\frac{\langle v, w\rangle}{\langle w, w\rangle} w\right) \\ &= \frac{\left\langle \frac{\langle v,w\rangle}{\langle w,w\rangle} w, w\right\rangle}{\langle w, w\rangle} w \\ &= \frac{\langle v, w\rangle}{\langle w, w\rangle} w \\ &= P_w(v). \end{aligned}$$

Problem 3

Let V be an inner product space. Prove the reverse Cauchy-Schwarz Inequality which states

$$v, w \in V, \text{ and } |\langle v, w\rangle| = \|v\|\,\|w\| \Rightarrow v = \alpha w.$$

Proof: If $\|w\| = 0$, then $w = 0$, so $\langle v, w\rangle = 0$ and $\alpha = 0$. Suppose $\|w\| \neq 0$. Then,

$$\begin{aligned} |\langle v, w\rangle| &= \|v\|\,\|w\| \\ \|w\| \left|\frac{\langle v, w\rangle}{\langle w, w\rangle}\right| &= \|v\|, \end{aligned}$$

so $P_w(v) = v$, meaning $w = \alpha v$.

Problem 4

Let V be an inner product space. Then, for any $v, w \in V$, show that

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

Proof:

$$\begin{aligned} \langle v + w, v + w\rangle + \langle v - w, v - w\rangle &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle + \langle v, v\rangle - \langle w, v\rangle - \langle v, w\rangle + \langle -w, -w\rangle \\ &= \langle v, v\rangle + \langle v, v\rangle + \langle w, w\rangle + \langle w, w\rangle \\ &= 2\|v\|^2 + 2\|w\|^2 \end{aligned}$$

Problem 5

Let $\lambda = (\lambda_k)_k$ belong to ℓ_∞. Show that the map

$$D_\lambda : \ell_2 \to \ell_2; D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$

Proof:

Well-Defined: Let $(\zeta_k)_k = 0$ for all $k \in \mathbb{N}$. Then,

$$\begin{aligned} D_\lambda((\zeta_k)_k) &= (\lambda_k \zeta_k)_k \\ &= ((\lambda_k)(0))_k \\ &= 0 \end{aligned}$$

Linear:

$$\begin{aligned} D_\lambda((\alpha\xi_k)_k + (\beta\zeta_k)_k) &= D_\lambda((\alpha\xi_k + \beta\zeta_k)_k) \\ &= (\lambda_k(\alpha\xi_k + \beta\zeta_k))_k \\ &= (\alpha\lambda_k\xi_k + \alpha\lambda_k\zeta_k)_k \\ &= (\alpha\lambda_k\xi_k)_k + (\beta\lambda_k\zeta_k) \\ &= \alpha(\lambda_k\xi_k)_k + \beta(\lambda_k\zeta_k)_k \\ &= \alpha D_\lambda((\xi_k)_k) + \beta D_\lambda((\zeta_k)_k) \end{aligned}$$

Bounded:

$$
\begin{aligned}
\|D_\lambda\|_{\text{op}} &= \sup_{\|\xi_k\|_k \leq 1} \|D_\lambda((\xi_k)_k)\| \\
\|D_\lambda((\xi_k)_k)\| &= \left(\sum_{k=1}^{\infty} |\lambda_k \xi_k|^2\right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} \left|\sup_{k\in\mathbb{N}} |\lambda_k| \xi_k\right|^2\right)^{1/2} \\
&= \|\lambda\|_\infty \left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} \\
&= \|\lambda\|_\infty \|\xi_k\|
\end{aligned}
$$

Therefore,

$$\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty .$$

Problem 6

Consider the vector space $C([0, 2\pi])$ equipped with

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt.$$

(i) Show that this pairing defines an inner product on $C([0, 2\pi])$.

Proof: We will show that $\langle f, g\rangle$ satisfies the axioms of the inner product.

Addition:

$$
\begin{aligned}
\langle f_1 + f_2, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (f_1(t) + f_2(t))\overline{g(t)}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(f_1(t)\overline{g(t)} + f_2(t)\overline{g(t)}\right) dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f_1(t)\overline{g(t)}dt + \frac{1}{2\pi}\int_0^{2\pi} f_2(t)\overline{g(t)}dt \\
&= \langle f_1, g\rangle + \langle f_2, g\rangle .
\end{aligned}
$$

Scalar Multiplication:

$$
\begin{aligned}
\langle \alpha f, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (\alpha f(t))\overline{g(t)}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \alpha\left(f(t)\overline{g(t)}\right) dt \\
&= \alpha\left(\frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt\right) \\
&= \alpha\langle f, g\rangle .
\end{aligned}
$$

Conjugation:

$$
\begin{aligned}
\overline{\langle g, f\rangle} &= \frac{1}{2\pi}\int_0^{2\pi} \overline{g(t)\overline{f(t)}}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt \\
&= \langle f, g\rangle .
\end{aligned}
$$

Positive Definition:

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt\\ &\geq 0.\end{aligned}$$

For $\langle f, f\rangle = 0$, we have that the integral equals zero — since f is continuous, it means that if $|f(t)|^2 > 0$ for some $t_0 \in [0, 2\pi]$, then $|f(t)|^2 \neq 0$ on some interval $[t_0 - \delta, t_0 + \delta]$, meaning the integral can only equal zero if f is $\mathbb{0}_f$ on $[0, 2\pi]$.

(ii) For $n \in \mathbb{Z}$, set $e_n(t) = \cos(nt) + i\sin(nt)$. Show that the family $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal.

Proof: We will show that $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal by showing that $\langle e_n, e_n\rangle = 1$ and $\langle e_n, e_m\rangle = 0$ for $m \neq n$.

$$\begin{aligned}\langle e_n, e_n\rangle &= \frac{1}{2\pi}\int_0^{2\pi}(\cos(nt) + i\sin(nt))(\cos(nt) - i\sin(nt))dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\cos^2(nt) + \sin^2(nt)\right)dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} dt\\ &= 1\\ \langle e_n, e_m\rangle &= \frac{1}{2\pi}\int_0^{2\pi}(\cos(nt) + i\sin(nt))(\cos(mt) - i\sin(mt))dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}(\cos(mt)\cos(nt) + i\sin(nt)\cos(mt) - i\sin(mt)\cos(nt) + \sin(nt)\sin(mt))\,dt\\ &= \frac{1}{2\pi}\left(\int_0^{2\pi}(\cos(mt)\cos(nt) + \sin(nt)\sin(mt))dt + i\int_0^{2\pi}(\sin(nt)\cos(mt) - \sin(mt)\cos(nt))dt\right)\\ &= 0.\end{aligned}$$

Problem 7

Let V be any normed space, $p \in [1, \infty]$, and suppose $T : \ell_p^n \to V$ is linear. Show that T is bounded.

Proof: Let T be a linear transformation from ℓ_p^n to V. Let $\xi = \sum_{k=1}^n \alpha_k e_k$ where $\|\xi\|_p = 1$. Then,

$$\begin{aligned}\|T(\xi)\| &= \left\|T\left(\sum_{k=1}^n \alpha_k e_k\right)\right\|\\ &= \left\|\sum_{k=1}^n \alpha_k T(e_k)\right\|\\ &\leq \sum_{k=1}^n |\alpha_k|\,\|T(e_k)\|\\ &\leq \sum_{k=1}^n \sup|\alpha_k|\,\|T(e_k)\|\\ &\leq \sum_{k=1}^n \|T(e_k)\|\\ &\leq \sum_{k=1}^n \max_k \|T(e_k)\|\\ &= n\,\|T(e_M)\|\\ &< \infty.\end{aligned}$$

Problem 8

Let $\mathbb{P}[0,1] = \left\{\sum_0^n a_k x^k \mid a_k \in \mathbb{C}\right\} \subseteq C([0,1])$ denote the linear subspace of all polynomial functions equipped with the uniform norm $\|\cdot\|_u$ inherited from $C([0,1])$. We define the map

$$\begin{aligned} D &: \mathbb{P}[0,1] \to \mathbb{P}[0,1] \\ D(p(x)) &= p'(x). \end{aligned}$$

Show that D is unbounded.

Proof: Let $p(x) = x^n$. Then, in $\mathbb{P}[0,1]$,

$$\begin{aligned} \|p\|_u &= 1 \\ \|D(p)\|_u &= n. \end{aligned}$$

For any $L \in \mathbb{R}$, we can find a $n \in \mathbb{N}$ sufficiently large such that $\|D(p)\|_u = n > L$, by the Archimedean property. Therefore, D is unbounded.

Problem 9

Let V be an infinite-dimensional normed space. Show that there is a linear functional $\varphi : V \to \mathbb{F}$ that is unbounded.

Proof: Let $B = \{x_n\}$ be the basis for V. We define $\varphi : V \to \mathbb{F}$ as $\varphi(x) = \sum_n n\alpha_n$ for the $\alpha_n x_n$ component in x. Then, φ is linear and unbounded, as the values n takes are not bounded, seeing as V is infinite-dimensional.

Problem 10

Let $a, b \in \mathbb{M}_n$. Show the following properties of the operator norm.

(i) $\|a\|_{\text{op}} = \sup\left\{|\langle a\xi, \eta\rangle| \mid \xi, \eta \in B_{\ell_2^n}\right\}$

(ii) $\|a^*\|_{\text{op}} = \|a\|_{\text{op}}$

(iii) $\|ab\|_{\text{op}} \leq \|a\|_{\text{op}} \|b\|_{\text{op}}$

(iv) $\|a^*a\|_{\text{op}} = \|a\|_{\text{op}}^2$

Proof:

(i)

$$\begin{aligned} \langle a\xi, \eta\rangle &\leq \|a\xi\| \|\eta\| \\ &= \|a\xi\| \\ &\leq \sup_{\xi \in B_{\ell_2^n}} \|a\xi\| \\ &= \|a\|_{\text{op}}. \\ \|a\| \text{ op} &= \sup_{\xi \in B_{\ell_2^n}} \|a\xi\| \end{aligned}$$

Set $\eta = \frac{a\xi}{\|a\xi\|}$. Then,

$$\begin{aligned} &= \sup_{\xi \in B_{\ell_2^n}} \frac{1}{\|a\xi\|} \langle a\xi, \eta\rangle \\ &= \sup\left\{\langle a\xi, \eta\rangle \mid \xi, \eta \in B_{\ell_2^n}\right\}. \end{aligned}$$

(ii)

$$\begin{aligned}
\|a^*\|_{\text{op}} &= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle a^*\xi,\eta\rangle| \\
&= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle \xi, a^{**}\eta\rangle| && \text{definition of conjugate transpose}\\
&= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle a\xi,\eta\rangle| && \text{by absolute value}\\
&= \|a\|_{\text{op}}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\|ab\|_{\text{op}} &= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle (ab)\xi,\eta\rangle| \\
&= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle a(b\xi),\eta\rangle| \\
&= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle b\xi, a^*\eta\rangle| \\
&\leq \sup_{\xi\in B_{\ell_2^n}} \|b\xi\| \sup_{\eta\in B_{\ell_2^n}} \|a^*\eta\| \\
&= \|b\|_{\text{op}} \|a^*\|_{\text{op}} \\
&= \|a\| \|b\|.
\end{aligned}$$

(iv)

$$\begin{aligned}
\|a^*a\|_{\text{op}} &= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle (a^*a)\xi,\eta\rangle| \\
&= \sup_{\xi,\eta\in B_{\ell_2^n}} |\langle a\xi, a^{**}\eta\rangle| \\
&= \sup_{\xi\in B_{\ell_2^n}} \|a\xi\|^2 \\
&= \|a\|_{\text{op}}^2
\end{aligned}$$