

Amenability: A (Somewhat) Brief Introduction

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Outline

- 1 Definitions
- 2 Paradoxical Decompositions
- 3 From Paradoxical Decompositions to Amenability
- 4 Equivalent Definitions and Other Criteria
 - A Taste of Functional Analysis
 - Introducing Approximations
 - Approximations with Representations and Operators
 - Review
- 5 Remarks and Acknowledgments

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We (usually) abbreviate $a \star b$ as ab . If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups, and All That

Let G be a group.

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_{\sim}$ known as the *quotient group* G/N .

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- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

Some Groups

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- The group $\text{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group $E(3)$ consists of all translations, rotations, and flips composed with each other in \mathbb{R}^3 , and is also known as the *isometry group* of \mathbb{R}^3 .

Group Actions

Let G be a group, and X a set. Let $\rho: G \times X \rightarrow X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

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Then, we say ρ is an *action* of G on X . We write $\rho(g, x) = g \cdot x$. The above lines become $e_G \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$. If $g \cdot x = x$ only when $g = e_G$, then we say the action is *free*.

Every group is equipped with a family of canonical actions, $\sigma_a: G \rightarrow G$ for each $a \in G$, given by $x \mapsto ax$, known as *left-multiplication*.

σ -Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

- ① $\emptyset, X \in \mathcal{A}$;
- ② for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- ③ for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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The most important σ -algebra, and the one we will be dealing with throughout this talk, is $P(G)$, where G is a group.

σ -Algebras and Measures, Cont'd

If X is a set and \mathcal{A} is a σ -algebra, then a map $\mu: \mathcal{A} \rightarrow [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

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If $\{A_n\}_{n \geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $$\mu\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n),$$

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σ -Algebras and Measures, Cont'd

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- $$\mu\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n),$$

we say μ is a measure. If $\mu(X) = 1$, then we say μ is a probability measure.

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Motivating Questions

- If G is a group, is it possible to reconstruct G by using some subset of G ?

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- When may we find a finitely additive probability measure $\mu: P(G) \rightarrow [0, 1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We define $F(a, b)$ to be the set of all “words” in the alphabet $\{a, b, a^{-1}, b^{-1}\}$, subject to the condition that, for $w, w' \in F(a, b)$,

$$\begin{aligned} waa^{-1}w' &\sim wa^{-1}aw' \sim ww' \\ wbb^{-1}w' &\sim wb^{-1}bw' \sim ww'. \end{aligned}$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a, b)$.

A Curiosity

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Thus, all we need to do is add back $W(b^{-1})$ to get $F(a, b)$ back.

$$F(a, b) = W(b^{-1}) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a , giving a decomposition of $F(a, b)$ in two separate ways:

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Furthermore, note that $W(a), W(b), W(a^{-1}), W(b^{-1})$ are disjoint.

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Furthermore, note that $W(a), W(b), W(a^{-1}), W(b^{-1})$ are disjoint.

We're able to take part of the group $F(a, b)$, take some translations, and, miraculously, obtain the entire group back.

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$;

such that

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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Decompositions of Sets

If G acts on a set X , then a subset $A \subseteq X$ is *G-paradoxical* if there exist

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq A$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$

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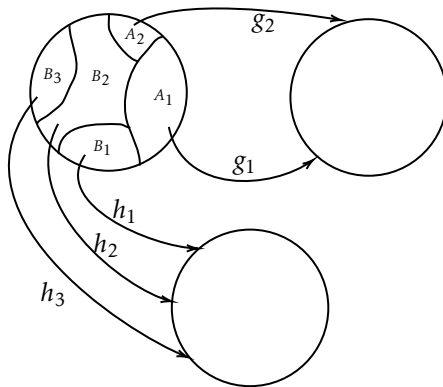
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A paradoxical group is a paradoxical set under the action of left-multiplication. If G is paradoxical and acts freely on X , then X is G -paradoxical.

Depiction



Some Paradoxical Groups

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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of $SO(3)$

The following two matrices (and their inverses) generate a subgroup of $SO(3)$ that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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Thus, $SO(3)$ is paradoxical — can we use it to find a paradoxical decomposition?

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of \mathbb{R}^3 with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B .

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We need to introduce a few more concepts before we can show the proof of the paradox.

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Effectively, A and B are “equal” to each other up to the group action.

If A is G -paradoxical, then so too is B .

The Banach–Tarski Paradox: Proof Outline

- Use the two matrices

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- Copy $F(a, b)$ over by using the decomposition

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Removing fixed points yields a decomposition of $S^2 \setminus D$. This is known as the *Hausdorff paradox*.

The Banach–Tarski Paradox: Proof Outline, cont'd

- We show that S^2 and $S^2 \setminus D$ are $\text{SO}(3)$ -equidecomposable.

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This shows the *weak* Banach–Tarski paradox (i.e., that the unit ball in \mathbb{R}^3 can be duplicated).

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For the full paradox, we need one more thing.

The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \leq B$ if A is G -equidecomposable with *a subset of* B .

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This relation is reflexive (since all sets are subsets of themselves, take the identity action),

The Banach–Tarski Paradox: Proof Outline, cont'd

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The Banach–Tarski Paradox: Proof Outline, cont'd

Consider the relation $A \leq B$ if A is G -equidecomposable with *a subset of* B .

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This relation is reflexive (since all sets are subsets of themselves, take the identity action), transitive (compose the group action), and antisymmetric (this one takes a bit more work).

Thus, to show that A and B are G -equidecomposable, it suffices to show that $A \leq B$.

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Ill-Behaved Groups

- The way that our copy of $F(a, b)$ helped “create” the Banach–Tarski paradox suggests that $F(a, b)$ is a particularly ill-behaved group.

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- Let $\nu: F(a, b) \rightarrow [0, 1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a, b)$ and $E \subseteq F(a, b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a, b) = W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}),$$

we have

$$1 = \nu(W(a)) + \nu(W(a^{-1})) + \nu(W(b)) + \nu(W(b^{-1}))$$

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Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu: P(G) \rightarrow [0, 1]$ such that

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for all $t \in G$ and $E \subseteq G$.

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- In other words, G is sufficiently “well-behaved.”

Inheritance Properties of Amenability

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- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If G and H are amenable, then so is $G \times H$.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

is a mean.

- Abelian groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

Paradoxical Groups and Amenability

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Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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- and operator algebras.

Normed Vector Spaces

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- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of \mathbb{R}^n or \mathbb{C}^n with the Euclidean norm,

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However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \rightarrow \mathbb{C} \mid \sup_{t \in \Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \rightarrow \mathbb{C} \mid \sum_{t \in \Gamma} |f(t)| < \infty \right\};$$

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They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t \in \Gamma} |f(t)|;$
- $\|f\|_{\ell_1} := \sum_{t \in \Gamma} |f(t)|;$
- $\|f\|_{\ell_2} := \left(\sum_{t \in \Gamma} |f(t)|^2 \right)^{1/2}.$

Linear Maps and Linear Functionals

A linear transformation $T: V \rightarrow W$ is called *bounded* if

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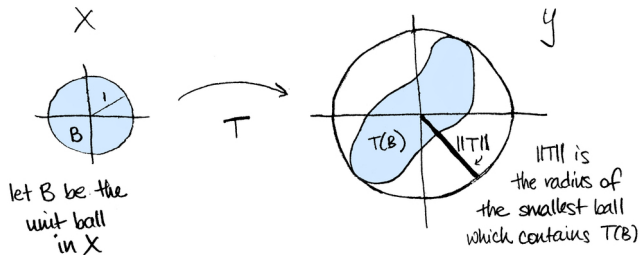
$$\sup_{\|v\|=1} \|T(v)\| < \infty.$$

We call the quantity on the left the *operator norm*, denoted $\|T\|_{\text{op}}$. A linear map is bounded if and only if it is continuous.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Operator Norm Pictorial Depiction

Courtesy of Tai-Danae Bradley.



Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi: \ell_\infty(\Gamma) \rightarrow \mathbb{C}$ is a linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbf{1}_\Gamma) = \|\varphi\|_{\text{op}}$.

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- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}}$. All positive linear functionals are automatically continuous.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s: \ell_\infty(\Gamma) \rightarrow \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f(s^{-1}t)$$

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If $\varphi: \ell_\infty(\Gamma) \rightarrow \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

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If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

Approximations and Amenability

There is actually one way that working with sets makes life easier.

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But what does “bigger” actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n \rightarrow \infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

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If $\Gamma = \langle S \rangle$ for some finite generating set S , we only need to check for all $s \in S$.

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We can find a Følner sequence by defining

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Then, since \mathbb{Z} is generated by 1, we verify

$$\frac{|(F_n + 1) \cap F_n|}{|F_n|} = \frac{2n-1}{2n+1} \rightarrow 1.$$

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Thus, \mathbb{Z} is amenable.

Approximate Means

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Keeping $\lambda_s(f)(t) = f(s^{-1}t)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k \rightarrow \infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|} \mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a “sufficient” finitely supported function g ,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

then use a “layer cake” decomposition to find our Følner sets:

$$g = \sum_{i=1}^n c_i \mathbb{1}_{F_i},$$

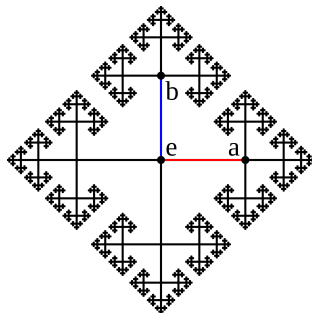
where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S , we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by “walking” along the generators.

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If G is the Cayley graph of Γ , then Γ is amenable if and only if

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- This is proven with the Følner condition.

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Hilbert Spaces

If \mathcal{H} is a vector space, an *inner product* on \mathcal{H} is a map $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ that satisfies

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The inner product induces a norm $\|x\|^2 = \langle x, x \rangle$.

If \mathcal{H} is complete with respect to this norm, we call \mathcal{H} a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T: \mathcal{H} \rightarrow \mathcal{H}$, include a special structure called an adjoint that “plays nicely” with the inner product:

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then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda: \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$

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All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma: \Gamma \rightarrow \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

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The map $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$ is a very special representation, known as the *left-regular representation*.

This is because it “encodes” the group’s left-multiplication action, in the sense that $\lambda_s(\delta_t) = \delta_{st}$, where δ_t is the point mass at $t \in \Gamma$.

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* for $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ if

$$\lim_{k \rightarrow \infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

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If $\lambda: \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

Introduction to C^* -Algebras

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These make $\mathbb{B}(\mathcal{H})$ a C^* -algebra. However, there are other C^* -algebras.

A Group C^* -Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t) \delta_t,$$

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where δ_t is the point mass at $t \in \Gamma$.

This becomes a $*$ -algebra when endowed with multiplication (by convolution) and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g(s^{-1}t)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^* -Algebra, cont'd

If we represent $\pi_\lambda: \mathbb{C}[\Gamma] \rightarrow \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\text{op}},$$

we get the *reduced group C^* -algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

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We can use these sufficient approximations to establish amenability.

Nuclearity

A C^* -algebra, A , is called *nuclear* if there exist two sequences of maps, $\varphi_n: A \rightarrow \text{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n: \text{Mat}_{k(n)}(\mathbb{C}) \rightarrow A$, such that

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- Essentially, any $a \in A$ is “close enough” to a certain family of finite-dimensional analogues.

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Specifically, by showing that the approximation of $\frac{|sF_n \cap F_n|}{|F_n|} \rightarrow 1$ corresponds to the existence of maps $\varphi_n: C_\lambda^*(\Gamma) \rightarrow \text{Mat}_{|F_n|}(\mathbb{C})$ and $\psi_n: \text{Mat}_{|F_n|}(\mathbb{C}) \rightarrow C_\lambda^*(\Gamma)$ that satisfy

$$\|x - \psi_n \circ \varphi_n(x)\| \xrightarrow{n \rightarrow \infty} 0.$$

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- there is a sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ such that $\|f_k - \lambda_s(f_k)\|_{\ell_2} \rightarrow 0$ (almost-invariant vectors);
- the reduced group C^* -algebra, $C_\lambda^*(\Gamma)$, is nuclear (nuclearity).

Contents

- ① Definitions
- ② Paradoxical Decompositions
- ③ From Paradoxical Decompositions to Amenability
- ④ Equivalent Definitions and Other Criteria
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Additionally, other questions related to finite-dimensional approximations of C^* -algebras, such as quasidiagonality, were only resolved recently. A paper in 2015 proved that if Γ is amenable, then $C_\lambda^*(\Gamma)$ is quasidiagonal (the reverse direction was shown in 1987).

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- the professors of the math department;
- friends, family, and acquaintances both in the math major and outside;
- everyone in attendance.

References I

- [AB06] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis*. Third. A Hitchhiker's Guide. Springer, Berlin, 2006, pp. xxii+703. ISBN: 978-3-540-32696-0.
- [Alu09] Paolo Aluffi. *Algebra: Chapter 0*. Vol. 104. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xx+713. ISBN: 978-0-8218-4781-7. DOI: 10.1090/gsm/104. URL: <https://doi.org/10.1090/gsm/104>.
- [BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*. Vol. 11. New Mathematical Monographs. Cambridge University Press, Cambridge, 2008, pp. xiv+472. ISBN: 978-0-521-88720-5. DOI: 10.1017/CB09780511542749. URL: <https://doi.org/10.1017/CB09780511542749>.

References II

- [Bla06] B. Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of C^* -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4. DOI: 10.1007/3-540-28517-2. URL: <https://doi.org/10.1007/3-540-28517-2>.
- [BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004, pp. xiv+716. ISBN: 0-521-83378-7. DOI: 10.1017/CB09780511804441. URL: <https://doi.org/10.1017/CB09780511804441>.

References III

- [BO08] Nathanial P. Brown and Narutaka Ozawa. *C^* -algebras and finite-dimensional approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+509. ISBN: 978-0-8218-4381-9. DOI: 10.1090/gsm/088. URL: <https://doi.org/10.1090/gsm/088>.
- [CE78] Man-Duen Choi and Edward G. Effros. “Nuclear C^* -Algebras and the Approximation Property”. In: *American Journal of Mathematics* 100.1 (1978), pp. 61–79. ISSN: 00029327. URL: <http://www.jstor.org/stable/2373876> (visited on 02/07/2025).
- [DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. Third. John Wiley & Sons, Inc., Hoboken, NJ, 2004, pp. xii+932. ISBN: 0-471-43334-9.

References IV

- [Enc25] The Editors of Encyclopaedia Britannica. *Ship of Theseus*. Accessed: 2025-02-06. 2025. URL: <https://www.britannica.com/topic/Ship-of-Theseus>.
- [Fol84] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984, pp. xiv+350. ISBN: 0-471-80958-6.
- [Hal06] Paul R. Halmos. “How to write mathematics”. In: *Butl. Soc. Catalana Mat.* 21.1 (2006). Translation of Enseignement Math. (2) **16** (1970), 123–152 [MR0277319], pp. 53–79, 158. ISSN: 0214-316X, 2013-9829.

References V

- [Hal66] James D. Halpern. “Bases in vector spaces and the axiom of choice”. In: *Proc. Amer. Math. Soc.* 17 (1966), pp. 670–673. ISSN: 0002-9939, 1088-6826. DOI: 10.2307/2035388. URL: <https://doi.org/10.2307/2035388>.
- [Har00] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000, pp. vi+310. ISBN: 0-226-31719-6.
- [Jec03] Thomas Jech. *Set theory*. millennium. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

References VI

- [Jus22] Kate Juschenko. *Amenability of discrete groups by examples*. Vol. 266. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2022, pp. xi+165. ISBN: 978-1-4704-7032-6. DOI: 10.1090/surv/266. URL: <https://doi.org/10.1090/surv/266>.
- [Kes59a] Harry Kesten. “Full Banach Mean Values on Countable Groups”. In: *Mathematica Scandinavica* 7.1 (1959), pp. 146–156. ISSN: 00255521. URL: <http://www.jstor.org/stable/24489015> (visited on 02/05/2025).
- [Kes59b] Harry Kesten. “Symmetric Random Walks on Groups”. In: *Transactions of the American Mathematical Society* 92.2 (1959), pp. 336–354. ISSN: 00029947. URL: <http://www.jstor.org/stable/1993160> (visited on 02/05/2025).
- [Knu09] Søren Knudby. “The Banach-Tarski Paradox”. 2009.

References VII

- [Löh17] Clara Löh. *Geometric group theory*. Universitext. An introduction. Springer, Cham, 2017, pp. xi+389. ISBN: 978-3-319-72253-5. DOI: 10.1007/978-3-319-72254-2. URL: <https://doi.org/10.1007/978-3-319-72254-2>.
- [Mon17] Mehdi Sangani Monfared. “Følner’s condition and expansion of Cayley graphs for group actions”. In: *New York Journal of Mathematics* 23 (Sept. 2017), pp. 1295–1306.
- [Mon13] Nicolas Monod. “Groups of piecewise projective homeomorphisms”. In: *Proc. Natl. Acad. Sci. USA* 110.12 (2013), pp. 4524–4527. ISSN: 0027-8424,1091-6490. DOI: 10.1073/pnas.1218426110. URL: <https://doi.org/10.1073/pnas.1218426110>.

References VIII

- [Pau02] Vern Paulsen. *Completely bounded maps and operator algebras*. Vol. 78. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+300. ISBN: 0-521-81669-6.
- [Rai23] Timothy Rainone. “Functional Analysis-En Route to Operator Algebras”. 2023.
- [Rud73] Walter Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, pp. xiii+397.
- [Run02] Volker Runde. *Lectures on amenability*. Vol. 1774. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. xiv+296. ISBN: 3-540-42852-6. DOI: 10.1007/b82937. URL: <https://doi.org/10.1007/b82937>.

References IX

- [Run05] Volker Runde. *A taste of topology*. Universitext. Springer, New York, 2005, pp. x+176. ISBN: 978-0387-25790-7.
- [Run20] Volker Runde. *Amenable Banach algebras*. Springer Monographs in Mathematics. A panorama. Springer-Verlag, New York, 2020, pp. xvii+462. ISBN: 978-1-0716-0351-2. DOI: 10.1007/978-1-0716-0351-2. URL: <https://doi.org/10.1007/978-1-0716-0351-2>.
- [Tak64] Masamichi Takesaki. “On the cross-norm of the direct product of C^* -algebras”. In: *Tohoku Math. J. (2)* 16 (1964), pp. 111–122. ISSN: 0040-8735,2186-585X. DOI: 10.2748/tmj/1178243737. URL: <https://doi.org/10.2748/tmj/1178243737>.

References X

- [Tao09] Terence Tao. *245B, notes 2: Amenability, the ping-pong lemma, and the Banach-Tarski paradox (optional)*.
<https://terrytao.wordpress.com/2009/01/08/245b-notes-2-amenability-the-ping-pong-lemma-and-the-banach-tarski-paradox-optional/>. 2009.
- [Tit72] J Tits. “Free subgroups in linear groups”. In: *Journal of Algebra* 20.2 (1972), pp. 250–270. ISSN: 0021-8693. DOI: [https://doi.org/10.1016/0021-8693\(72\)90058-0](https://doi.org/10.1016/0021-8693(72)90058-0). URL: <https://www.sciencedirect.com/science/article/pii/0021869372900580>.

References XI

- [Wei80] Joachim Weidmann. *Linear operators in Hilbert spaces*. Vol. 68. Graduate Texts in Mathematics. Translated from the German by Joseph Szücs. Springer-Verlag, New York-Berlin, 1980, pp. xiii+402. ISBN: 0-387-90427-1.