

Problem (Problem 1): If X is a connected space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy-equivalent to a wedge sum of 1-spheres and 2-spheres.

Solution: We consider X to have a CW complex structure consisting of 0-cells at each intersection point, 1-cells connecting each intersection point, and 2-cells “constructing” the sphere. Our primary method of construction will be via collapsing along select 1-cells, which since these 1-cells are contractible, will yield a homotopy-equivalent CW complex.

If we have a straight line of spheres connected one after another by their equator, we collapse along one side of this equator to yield a wedge sum of 2-spheres (as in Example 0.9 in Hatcher).

Else, we create a homotopy equivalent CW complex consisting of a straight line of spheres as in the previous case, introducing 1-cells along one side of the equator to denote identifications among spheres adjacent in X that are not adjacent in the straight line. By collapsing along the equator of this straight line as in the previous case, this yields a wedge sum of 1-spheres (the extra 1-cells that are not in the original straight line of spheres) and 2-spheres.