Problem (Problem 1)**:** Describe the topology of the Grassmanian $\mathrm{Gr}(k, n)$ in a uniform way, so that $\mathbb{RP}^n$ becomes the special case of $\mathrm{Gr}(1, n)$.

Solution: We let elements of $\mathrm{Gr}(k, n)$ be defined as equivalence classes of linearly independent k-tuples of vectors in $\mathbb{R}^n$, where $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if $\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{w_1, \dots, w_k\}$.

By extending $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ to ordered bases $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$, we see that these k-tuples are equivalent if and only if there is an invertible linear transformation Q with matrix representation

$$Q = \begin{pmatrix} A & H \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n-k) \times (n-k)$ invertible matrix, so that

$$Q[v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n] = \left[w_1, \dots, w_k, w_{k+1}^*, \dots, w_n^*\right],$$

where $\left\{w_{k+1}^*, \dots, w_n^*\right\}$ is a basis for an $(n-k)$-dimensional complementary subspace. The subgroup of all such $Q \subseteq \mathrm{GL}_n(\mathbb{R})$, which we call P, is the stabilizer of $\mathrm{Gr}(k, n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $\mathrm{GL}_n(\mathbb{R})$ acts transitively on all ordered bases of $\mathbb{R}^n$), we obtain $\mathrm{Gr}(k, n) \cong \mathrm{GL}_n(\mathbb{R})/P$, where the latter coset space is given the quotient topology.

Given some element of $\mathbb{R}^{k(n-k)}$ viewed as a $(n-k) \times k$ matrix (with the standard basis), we consider the map

$$\mathbb{R}^{k(n-k)} \times P \to \mathrm{GL}_n(\mathbb{R})$$

given by taking $K \in \mathbb{R}^{k(n-k)} \cong \mathrm{Mat}_{(n-k)\times k}(\mathbb{R})$, extending to $\mathbb{R}^{n^2} \cong \mathrm{Mat}_n(\mathbb{R})$ by adding 0 above and to the right of all the entries in A, and adding to elements of P. This gives matrices of the form

$$M = \begin{pmatrix} A & H \\ K & B \end{pmatrix},$$

where $A \in \mathrm{GL}_k(\mathbb{R})$, $B \in \mathrm{GL}_{n-k}(\mathbb{R})$, H is any matrix in $\mathrm{Mat}_{k\times(n-k)}(\mathbb{R})$, and $K \in \mathrm{Mat}_{(n-k)\times k}(\mathbb{R})$. Notice that

$$\begin{aligned} \det(M) &= \det\begin{pmatrix} A & H \\ 0 & B \end{pmatrix} + \det\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \\ &= \det\begin{pmatrix} A & H \\ 0 & B \end{pmatrix} \\ &= \det(A)\det(B), \end{aligned}$$

meaning that, given some element in $\mathrm{GL}_n(\mathbb{R})$, we may sufficiently bound this element away from a singular matrix, and elements in $\mathbb{R}^{k(n-k)}$ viewed as a subspace of $\mathrm{GL}_n(\mathbb{R})$, under addition, do not affect the determinant when added to elements of P. This gives a diffeomorphism from $\mathbb{R}^{k(n-k)} \times P \to \mathrm{GL}_n(\mathbb{R})$, as matrix addition is differentiable.

In the case of $\mathbb{RP}^{(n-1)}$, where the matrix A in the definition of P is a 1×1 (or nonzero scalar), we may find a diffeomorphism from $(\mathbb{R}^n \setminus \{0\})/\mathbb{R}^\times$ and our expression $\mathbb{R}^{n-1} \times P$ by taking

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \times \begin{pmatrix} \alpha & H \\ 0 & B \end{pmatrix} \mapsto [\alpha : x_1 : \cdots : x_{n-1}].$$

This allows our definition of the topology to comport with the case of $\mathrm{Gr}(1, n)$.

Problem (Problem 2)**:** Fix an inner product on $\mathbb{R}^n$. Show that the map $V \mapsto V^\perp$ induces a C^∞ diffeomorphism $\mathrm{Gr}(k,n) \to \mathrm{Gr}(n-k,n)$.

Solution: Due to the inner product, we make the identification $v \mapsto v^*$ such that $v^*(w) = \langle v, w\rangle$. In particular, we have isomorphisms $V \cong V^*$ and $V^\perp \cong (V^\perp)^*$. Therefore, given an element $T \in \mathrm{Hom}(V, V^\perp)$, dualization gives the transpose map $T^* \in \mathrm{Hom}\big((V^\perp)^*, V^*\big)$.

Now, given any chart (U_V, φ_V) in $\mathrm{Gr}(k,n)$, we identify $T \in \mathrm{Hom}(V, V^\perp) \cong U_V$ to $T^* \in \mathrm{Hom}\big((V^\perp)^*, V^*\big) \cong U_{V^\perp}$, and identify subspaces $W \in U_V$ with their annihilators

$$W^0 = \{w^* \in (\mathbb{R}^n)^* \mid w^*(v) = 0 \text{ for all } v \in W\},$$

so that $W^0 \cap V^* = 0$. Finally, we define $\varphi_{V^\perp}$ by

$$\varphi_{V^\perp} = P_{V^*} \circ P_{(V^\perp)^*}|_{W^0}^{-1}.$$

Since every $W \in \mathrm{Gr}(k,n)$ has a unique annihilator subspace $W^0 \in \mathrm{Gr}(n-k,n)$, we have the series of bijective correspondences

$$\begin{aligned}\mathrm{Hom}\big(V, V^\perp\big) &\xleftrightarrow{\varphi_V} U_V \\ &\xleftrightarrow{W \leftrightarrow W^0} U_{V^\perp} \\ &\xleftrightarrow{\varphi_{V^\perp}} \mathrm{Hom}\big(\big(V^\perp\big)^*, V^*\big) \\ &\xleftrightarrow{\langle\cdot,\cdot\rangle} \mathrm{Hom}\big(V^\perp, V\big),\end{aligned}$$

meaning that this identification is a C^∞ diffeomorphism.

Problem (Problem 3)**:** Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Solution: Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^k map that is a C^1 diffeomorphism. In order to show that f is a C^k diffeomorphism, we need to show that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ exists and is of class C^k.

First, by the inverse function theorem, since f is a C^1 diffeomorphism, we see that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ exists, is continuous, and is such that $D\big(f^{-1}\big)$ is continuous.

We observe that for any $y \in \mathbb{R}^n$, $D_y\big(f^{-1}\big)$ exists and is continuous, where $D_y\big(f^{-1}\big) = \big(D_y f\big(f^{-1}(y)\big)\big)^{-1}$ by the inverse function theorem. Since $f^{-1}(y)$ is continuously differentiable, $D_y f$ is C^{k-1}, and inversion is C^∞, we see that $D_y\big(f^{-1}\big)$ is C^1, meaning that f^{-1} is C^2. Inductively, this gives that f^{-1} is C^k, meaning f is a C^k diffeomorphism.

Problem (Problem 4)**:** Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either $\mathbb{R}$ or S^1, depending on whether it is compact or not.

Solution: Let M be a connected, paracompact manifold of dimension 1, and let $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ be a locally finite atlas, where without loss of generality, each of the U_i are connected, and $\varphi_i(U_i) = (0,1)$. We will show that this atlas allows us to define a homeomorphism between M and either S^1 or $\mathbb{R}$.

Consider two open sets, U_1 and U_2 with respective charts φ_1 and φ_2. Suppose $U_1 \cap U_2$ admits one connected component, and assume that $U_j \setminus U_i \neq \emptyset$. We will show that this allows us to, in a sense, "amalgamate" their respective coordinate maps φ_1 and φ_2, so that we may reduce to the case of two subsets if our atlas is finite. Since $U_1 \cap U_2$ is an open subset of U_1, the coordinate map $\varphi_1\colon U_1 \to (0,1)$ restricts to an embedding $\varphi_1\colon U_1 \cap U_2 \to (0,1)$. Note that since φ_1 is continuous, there is at most one cluster point for $\varphi_1(U_1 \cap U_2)$ within $(0,1)$, seeing as φ_1 is not defined on $U_2 \setminus U_1$. Thus, we may assume that $\varphi_1(U_1 \cap U_2) = (b, 1)$ for some $b \in (0,1)$. Similarly, we may assume that $\varphi_2(U_1 \cap U_2) = (0, a)$, so on

$U_1 \cup U_2$, we may define $\varphi_{1,2}\colon U_1 \cup U_2 \to (0,1)$ by $\varphi_1(U_1 \setminus U_2) = (0, a/(a+1)]$ and $\varphi_2(U_2) = (a/(a+1), 1)$, which is our desired amalgamation.

By taking a countable basis for the topology of M (as all connected, paracompact topological spaces are second-countable), using the fact that $\{(U_i)\}_{i\in I}$ is locally finite, and amalgamating the charts via this process for the finitely many elements of $\{U_i\}_{i\in I}$ that intersect elements of this topological basis, we may assume that the atlas $\mathcal{A}' = \{(V_k, \psi_k)\}_{k\geqslant 1}$ is countable. There are then two cases.

If M is compact, then M is covered by finitely many of these charts, $\left\{(V_j, \psi_j)\right\}_{j=1}^{n}$, so by using the amalgamation process once again, we are left with two charts. Without loss of generality, we call them (V_1, ψ_1) and (V_2, ψ_2). Observe that $V_1 \cap V_2$ *must* have two connected components; if there is one connected component, we may use this amalgamation process one more time, yielding a homeomorphism between the compact manifold M and the non-compact interval $(0,1)$, a contradiction, and if there are no connected components, then M is disconnected. Thus, if we are able to develop a continuous bijection between M and S^1, since S^1 is Hausdorff and M is compact, we automatically find $M \cong S^1$.

From earlier, we know that if W_1 and W_2 are the connected components of $V_1 \cap V_2$, then we may take $\psi_1(W_1) = (0,a)$ and $\psi_1(W_2) = (b,1)$. Similarly, we may take $\psi_2(W_2) = (0,c)$ and $\psi_2(W_2) = (d,1)$. We define the continuous bijection $r\colon M \to S^1$ piecewise, by taking

$$r(x) = \begin{cases} (\cos(\pi\psi_1(x)), \sin(\pi\psi_1(x))) & x \in V_1 \\ \left(\cos\left(\frac{\pi}{d-c}\psi_2(x) + \pi\right), \sin\left(\frac{\pi}{d-c}\psi_2(x) + \pi\right)\right) & x \in V_2 \setminus V_1. \end{cases}$$

If M is not compact, then via some rearrangement, cutting, and compositions, we may assume that $V_k \cap V_{k+1}$ has one connected component, and $V_k \cap V_{k+n}$ for $n \geqslant 2$ has no connected components, and that $\psi_k(V_k) = (k-1, k+1)$ for each k. Then, we define $r^*\colon M \to (0,\infty)$ by

$$r(x) = \begin{cases} \psi_1(x) & x \in V_1 \\ \psi_k(x) & x \in V_k \setminus V_{k-1}. \end{cases}$$

This is a homeomorphism, so by composing with a homeomorphism between $(0,\infty)$ and $\mathbb{R}$, we find that M is homeomorphic to $\mathbb{R}$.

Problem (Problem 5)**:** In this problem, we prove a weak version of the Whitney Embedding Theorem.

(a) Find a C^∞ function λ on $\mathbb{R}^n$ with values in $[0,1]$ such that λ takes the value 1 on the closed ball $B(0,1)$, and vanishes outside the closed ball $B(0,2)$.

(b) Suppose M is a compact C^k manifold of dimension n. Find a C^k atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that $\varphi_i(U_i)$ contains $B(0,2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0,1))^\circ$.

(c) Let λ_i be defined by $\lambda \circ \varphi_i$ on U_i, and 0 outside U_i. Let $f_i\colon M \to \mathbb{R}^n$ be defined by $\lambda_i \cdot \varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Solution:

(a) Consider the function $H\colon \mathbb{R} \to \mathbb{R}$ given by

$$H(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leqslant 0, \end{cases}$$

which is a C^∞ function on $\mathbb{R}$, as $e^{-1/t}$ is C^∞ for all $t > 0$, and the derivative is well-defined at $t = 0$. Next, we see that the function

$$G(t) = \frac{H\left(4 - t^2\right)}{H(4-t^2) + H(t^2 - 1)}$$

takes on the value 1 whenever $-1 \leqslant t \leqslant 1$ and is supported on $[-2,2]$. Furthermore, it is a C^∞ function, as it is a rational function of C^∞ functions where the denominator never takes the value 0. Therefore, if we replace t with $|x|$, when $x \in \mathbb{R}^n$, since the norm is a C^∞ function, we obtain a C^∞ function that is supported on $B(0,2)$ and is equal to 1 on $B(0,1)$, given by

$$\lambda(x) = \frac{H\left(4-|x|^2\right)}{H\left(4-|x|^2\right) + H\left(|x|^2-1\right)}.$$

(b) Let M be a compact C^k manifold, and let $\{(V_i, \psi_i)\}_{i \in I}$ be a C^k atlas for M, where $\{V_i\}_{i\in I}$ is an open cover, the $\psi_i \colon V_i \to \mathbb{R}^n$ are homeomorphisms, and the $\psi_j \circ \psi_i^{-1}$ are C^k diffeomorphisms.

Since M is compact, we have a finite subcover $\left\{V_j\right\}_{j=1}^n$ and an exhaustion by compact subsets via

$$U_j = \bigcup_{k=1}^{j} V_k$$
$$M = \bigcup_{j=1}^{n} U_j,$$

where, without loss of generality, $\overline{U_j} \subsetneq U_{j+1}$.

Now, for each $p \in \overline{U_j} \setminus U_{j-1}$ (define $U_0 = U_1 = \emptyset$), we may find i_p with a corresponding C^k chart $\left(V_{i_p}, \psi_p\right)$ mapping $\psi_p\left(V_{i_p}\right) = \mathbb{R}^n$. Without loss of generality, $\psi_p(p) = 0$ (compose with a translation if not), and let $W_p = \psi_p^{-1}(U(0,1))$.

Clearly, $B(0,2) \subseteq \psi_{i_p}\left(V_{i_p}\right)$, and by finitely enumerating the elements p_{j_k} in $\overline{U_j} \setminus U_{j-1}$, we have an open cover $\left\{W_{p_{j_k}}\right\}_{k=1}^m = \left\{\psi_{p_{j_k}}^{-1}(U(0,1))\right\}_{k=1}^m$ of M, and $\left\{\left(V_{i_{p_{j_k}}}, \psi_{p_{j_k}}\right)\right\}_{k=1}^m$ are C^k charts such that $B(0,2) \subseteq \psi_{p_{j_k}}\left(V_{i_{p_{j_k}}}\right)$.

(c) We rename the finite atlas from part (b), $\left\{\left(V_{i_{p_{j_k}}}, \psi_{p_{j_k}}\right)\right\}_{k=1}^m$, to $\{(V_k, \psi_k)\}_{k=1}^m$. Note that the $W_k = \psi_k^{-1}(U(0,1))$ is the open cover we use to define m. We may redefine each W_k to be equal to its closure.

Now, if $f_k = \lambda_k \cdot \psi_k$, then by setting $g_k = (f_k, \lambda_k)$, we find that for any $x \in W_k$, $g_k(x) = (\psi_k(x), 1)$, so $g_k(W_k) = (\psi_k(W_k), 1)$, and if $x \notin W_k$, then $g_k(x) = (\psi_k(x), 0)$. It is clear that $g \colon M \to \mathbb{R}^{m(n+1)}$ given by $g \equiv (g_1, \dots, g_m)$ is continuous. It remains to show that g is injective. To see this, if $x \neq y$, there are two cases:

- if $x, y \in W_k$, then since $\psi_k \colon V_k \to \mathbb{R}^n$ is a bijection, we must have $g_k(x) \neq g_k(y)$;
- if $x \in W_k$ and $y \notin W_k$, then since $\lambda_k(x) = 1$ and $\lambda_k(y) = 0$, we must have $g_k(x) \neq g_k(y)$.

Since the W_k cover M, we must have that g is injective. Thus, $M \hookrightarrow \mathbb{R}^{m(n+1)}$ given by $x \mapsto g(x)$ is our desired embedding.

Problem (Problem 6)**:** Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Solution: Let $\{U_i\}_{i\in I}$ be a locally finite open cover of M, and let $\{(U_i, \varphi_i)\}_{i\in I}$ be the corresponding C^k

atlas for M where $B(0,2) \subseteq \varphi_i(U_i)$, and M is covered by $\varphi_i^{-1}(U(0,1))$. Then, we may define

$$f_i = \begin{cases} G \circ \varphi_i & \text{on } U_i \\ 0 & \text{on } U_i^c, \end{cases}$$

where

$$G(x) = \frac{e^{\frac{1}{4-|x|^2}}}{e^{\frac{1}{4-|x|^2}} + e^{\frac{1}{|x|^2-1}}}$$

is a C^∞ function supported on $B(0,2)$ and equal to 1 on $U(0,1)$. Defining

$$f = \sum_{i \in I} f_i,$$

we see that $f \neq 0$ everywhere, as M is covered by the family $\varphi_i^{-1}(U(0,1))$, where G is nonzero on $U(0,1)$, and since $\{U_i\}_{i\in I}$ is locally finite, f is also C^k as each f_i is the composition of a C^k diffeomorphism and a C^∞ function. The functions

$$g_i = \frac{f_i}{f}$$

are thus C^k functions, $0 \leqslant g_i \leqslant 1$, and $\sum_{i\in I} g_i = 1$.

Problem (Problem 7)**:** Let X and Y be topological spaces, and let $C(X,Y)$ be the set of continuous maps from X to Y. Equip $C(X,Y)$ with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space and let X be compact. We note that a neighborhood basis in the topology of uniform convergence on $C(X,Y)$ consists of sets of the form

$$U_{f,\varepsilon} = \left\{ g \;\middle|\; \sup_{x\in X} d(f(x), g(x)) < \varepsilon \right\}.$$

Similarly, a neighborhood basis for the compact open topology consists of sets of the form

$$U_{f,K,\varepsilon} = \left\{ g \;\middle|\; \sup_{x\in K} d(f(x), g(x)) < \varepsilon \right\};$$

the fact that Y is a metric space allows us to take this refinement of the compact-open topology.

Thus, to prove that the compact-open topology and the topology of uniform convergence are equivalent, we show that any basis element of the topology of uniform convergence is contained in a basis element of the compact-open topology, and vice versa.

First, we see that almost by definition, if $K \subseteq X$ is any compact subset, then

$$U_{f,\varepsilon} \subseteq U_{f,K,\varepsilon},$$

as any function whose supremum distance is less than ε over X must have that supremum distance hold over $K \subseteq X$.

Now, in the reverse direction, we fix f and ε. We wish to show that there is a finite family of subsets U_{K_i,V_i} with $f \in U_{K_i,V_i}$ for each i, and their intersection lies in $U_{f,\varepsilon}$. We see that every point $x \in X$ has a pre-compact open neighborhood U_x such that $f\left(\overline{U_x}\right) \subseteq U(f(x), \varepsilon/3)$, which follows from the fact that compact subsets of Y are bounded. The family $\{U_x \mid x \in X\}$ is an open cover for X, so admits a finite subcover $\{U_{x_i}\}_{i=1}^{n}$. Since each $\left\{\overline{U_{x_i}}\right\}_{i=1}^{n}$ is compact, and for each i, $f \in U_{\overline{U_{x_i}},U(f(x_i),\varepsilon/3)}$, we see that

$$V = \bigcap_{i=1}^{n} U_{\overline{U_{x_i}},U(f(x_i),\varepsilon/3)}$$

is a nonempty open subset in the compact-open topology on $C(X, Y)$ that contains f. Now, for any $g \in V$ and for any $x \in X$, we see that there is some U_{x_j} such that $x \in U_{x_j}$, and since $g \in U_{\overline{U_{x_j}},U\left(f\left(x_j\right),\varepsilon/3\right)}$, we have that

$$\begin{aligned} d(g(x), f(x)) &\leqslant d\left(g(x), f\left(x_j\right)\right) + d\left(f\left(x_j\right), f(x)\right) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon, \end{aligned}$$

so $V \subseteq U_{f,\varepsilon}$, meaning the topologies are equal.