Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: We know that $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$, meaning we need show $C_0(\mathbb{R})$ is closed under the uniform norm.

Let $(f_n)_n \to f$, with $(f_n)_n \in C_0(\mathbb{R})$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then,

$$|f(x)| = |f(x) - f_n(x) + f_n(x)|$$

$$\leq |f_n(x) - f(x)| + |f_n(x)|$$

$$\leq ||f_n - f||_{\mathcal{U}} + |f_n(x)|$$

By the definition of uniform convergence, for all $n \ge N_{\varepsilon}$, $||f_n - f|| < \varepsilon/2$ and by the definition of vanishing at $\pm \infty$, for all |x| > M, $|f_n(x)| < \varepsilon/2$. Thus,

$$< \varepsilon$$
.

meaning $f(x) \in C_0(\mathbb{R})$, so $C_0(\mathbb{R})$ is closed, so it is complete.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $(x_n)_n$ be a Cauchy sequence in ℓ_2 . Let $x_n(k)$ denote the index k of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$|x_n(k) - x_m(k)| \le c \|x_m(k) - x_n(k)\|_2$$

 $\to 0$ since $(x_n)_n$ is Cauchy in ℓ_2 .

Since \mathbb{R} (or \mathbb{C}) is complete, $x_n(k) \to x(k)$ as $k \to \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \to x(k)$ for each n.

We must show that $||x_n - x||_2 \to 0$ as $n \to \infty$. This is equivalent to

$$\lim_{N \to \infty} \sum_{k=1}^{N} \lim_{m \to \infty} |x_n(k) - x_m(k)|^2 \le \lim_{m \to \infty} \sup_{m \ge M} ||x_n - x_m||^2$$

$$\le \varepsilon^2 \qquad \text{since } (x_n)_n \text{ is Cauchy.}$$

Thus, $||x_n - x_m|| \to 0$ as $m \to \infty$ and $n \to \infty$, so $||x_n - x|| \to 0$ as $n \to \infty$.

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \ge n$. Notice that $d(x_n, x_{n-1}) \le \theta^{n-2} d(x_2, x_1)$. Thus,

$$d(x_{m}, x_{n}) \leq d(x_{m}, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_{n})$$

$$\leq d(x_{2}, x_{1}) \left(\theta^{m-2} + \theta^{m-3} + \dots + \theta^{n-1}\right)$$

$$= d(x_{2}, x_{1})\theta^{n-1} \left(1 + \theta + \theta^{2} + \dots + \theta^{p-q-1}\right)$$

$$\leq d(x_{2}, x_{1}) \frac{\theta^{n-1}}{1 - \theta}.$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in \mathbb{R} , meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.

Problem 4

Let (X, d) be a complete metric space, and suppose $f: X \to X$ is a contractive map — i.e., there is a $\theta \in (0, 1)$ with

$$d(f(x), f(y)) \le \theta d(x, y).$$

Prove that f has a unique fixed point.

Proof: Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all n, we have

$$d(x_n, x_{n-1}) \le \theta^n d(x_1, x_0).$$

Therefore, for x_m , x_n arbitrary in X with m > n, we have

$$d(x_{m}, x_{n}) \leq d(x_{m}, x_{m-1}) + \dots + d(x_{n+1}, x_{n})$$

$$\leq \theta^{m} d(x_{1}, x_{0}) + \dots + \theta^{n+1} d(x_{1}, x_{0})$$

$$= d(x_{1}, x_{0}) \theta^{n+1} \left(1 + \theta + \dots + \theta^{m-n-1} \right)$$

$$\leq d(x_{1}, x_{0}) \frac{\theta^{n+1}}{1 - \theta}.$$

Since $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \to 0$ in $\mathbb R$ as $n \to \infty$, it must be the case that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Since X is complete, this means $(x_n)_n \to x$ for some $x \in X$, meaning f(x) = x.

Suppose it were the case that there existed s, t distinct with f(s) = s and f(t) = t. Then, $d(f(s), f(t)) = d(s, t) \le \theta d(s, t)$, but $\theta < 1$, which is a contradiction. Thus, x is unique.

Problem 5

If $(f_k)_k$ is an orthonormal sequence in a Hilbert space \mathcal{H} , show that the map

$$\varphi: \ell_2 \to \mathcal{H}$$

$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_k$$

is a well-defined isometry.

Proof: Let ξ , $\eta \in \ell_2$. Then,

$$d(\xi, \eta) = \|\xi - \eta\|_{2}$$

$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_{k}$$

$$\varphi(\eta) = \sum_{k=1}^{\infty} \eta(k) f_{k}$$

$$d(\varphi(\xi), \varphi(\eta)) = \left(\sum_{k=1}^{\infty} \langle \xi(k) - \eta(k), f_{k} \rangle\right)^{1/2}$$

$$= \|\xi - \eta\|_{2}$$

Parseval's Identity.

Problem 6

Let $T:V\to W$ be a continuous linear map between normed spaces which is bounded below; that is, there is a C>0 such that $\|T(v)\|\geq C\|v\|$ for all $v\in V$. If V is complete, show that $\operatorname{ran}(T)\subseteq W$ is a closed subspace, and that $V\cong\operatorname{ran}(T)$ are uniformly isomorphic.

Proof: Since T is bounded below, we know that $||T||_{op} > 0$, meaning T is not the zero transformation.

Let $(v_n)_n$ be a Cauchy sequence in V. Since V is complete, $(v_n)_n \to v \in V$. Since T is continuous, we have that $(T(v_n))_n \to T(v)$. Thus, $(T(v_n))_n$ is Cauchy in W, and thus T is uniformly continuous.

It is also apparent that for any sequence $(v_n)_n \in V$, then since $(T(v_n))_n \to T(v)$ (as T is continuous), any sequence in T(V) is contained in T(V), so $T(V) \subseteq W$ is closed.

Since $T': V \to ran(T)$ is surjective, it is bijective, so it must be the case that $V \cong ran(T)$ are uniformly isomorphic.

Problem 7

Let X and Y be metric spaces with completions (\tilde{X}, ι_X) and (\tilde{Y}, ι_Y) respectively. If $f: X \to Y$ is an isometry, show that there is a unique isometry $\tilde{f}: \tilde{X} \to \tilde{Y}$ that extends f. That is, the following diagram commutes:

$$\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}$$

$$\iota_X \qquad \qquad \downarrow^{\iota_Y}$$

$$X \xrightarrow{f} Y$$

Proof: We have that ι_X and ι_Y are isometries for X and Y into \tilde{X} and \tilde{Y} respectively. Consider the map $\ell: \iota_X(X) \to \tilde{Y}$, with $\iota_X(X) \mapsto \iota_Y(f(X))$. We have that since ι_Y and f are isometries, $\iota_Y \circ f$ is an isometry, hence uniformly continuous.

Thus, we can extend ℓ to $\overline{\iota(X)} = \tilde{X}$, yielding $\tilde{f} : \tilde{X} \to \tilde{Y}$. Since f is an isometry on X, and $X \subseteq \tilde{X}$ is dense, \tilde{f} must also be an isometry.

Problem 8

Let V be a normed space, W a Banach space, and $U \subseteq V$ a dense linear subspace. Let $T_0: U \to W$ be a bounded linear map. Show that there is a unique bounded linear map $T: V \to W$ that extends T_0 .

Proof: Since T_0 is bounded linear, T_0 is uniformly continuous. Thus, we have that T_0 is uniquely extendible to V, yielding $T:V\to W$ is continuous, and thus bounded linear.

Problem 9

Let X be a metric space. Show that the following are equivalent:

- (i) Every meager set has empty interior.
- (ii) The complement of a meager set is dense.

Moreover, show that these equivalent statements hold true if the metric space is complete.

Proof: Let $A = \bigcup_{i \ge 1} A_i$ be a meager subset of X. Suppose $A^\circ = \emptyset$. Then, $\overline{A^c} = (A^\circ)^c = X$, so A^c is dense in X.

Suppose
$$\overline{A^c} = X$$
. Then, $(A^\circ)^c = X$, so $A^\circ = \emptyset$.

Let $A \subseteq X$ be meager in X a complete metric space. Since X is complete, it cannot be the case that $X \subseteq \bigcup A_i = A$ by Baire's theorem. Thus, for any $Y \subseteq X$ such that $Y^{\circ} \neq \emptyset$, it cannot be the case that $Y \subseteq \bigcup A_i$, so $A^{\circ} = \emptyset$.

Problem 10

Let V be an infinite-dimensional normed space with linear basis B.

- (i) If $W \subset V$ is a proper subspace, show that $W^{\circ} = \emptyset$.
- (ii) If V is a Banach space, show that B is uncountable. You may used the fact that finite-dimensional subspaces are always closed.
- **Proof of (i):** Let $W \subset V$ be proper. Suppose $U(x, \varepsilon) \subseteq W$ for some $x \in V$ and $\varepsilon > 0$. Then, for $v \in V$, we have that $\frac{\varepsilon}{2} \frac{v}{\|v\|} + x \in U(x, \varepsilon)$, meaning $v \in W$, so $V \subseteq W$, which is a contradiction. Thus, $W^{\circ} = \emptyset$.
- **Proof of (ii):** Let $\{e_n\}_{n\geq 1}$ be a countable, linearly independent set. Let $W_1=\operatorname{span}\{e_1\}$, $W_2=\operatorname{span}\{e_1,e_2\}$, and so on. We have that each $W_n\subseteq V$ is closed (by assumption), and $W_1\subseteq W_2\subseteq\cdots$. Since each W_n has empty interior, it cannot be the case that $V=\bigcup W_n$ by Baire's Theorem.