Solution (21.1)**:**

(a) Doing a partial fraction decomposition, we find

$$\frac{1}{(z-1)(z+2)} = \frac{1}{3}\frac{1}{z-1} - \frac{1}{3}\frac{1}{z+2},$$

giving giving a residue of $\frac{1}{3}$ at $z = 1$ and a residue of $-\frac{1}{3}$ at $z = -2$.

(b) Evaluating the residue at $z = 1$, we may use the cover-up method to find

$$\text{Res}[f(z), 1] = \frac{e^{2i}}{27}.$$

To evaluate the residue at $z = -2$, we use the formula to calculate residues, giving

$$\begin{aligned}\text{Res}[f(z), -2] &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{e^{2iz}}{z-1}\right)\Big|_{z=-2} \\ &= \frac{38}{27}e^{-4i}\end{aligned}$$

(c) Note that $\sin(z)$ is a simple zero at $z = n\pi$. Therefore, we evaluate

$$\text{Res}[f(z), n\pi] = (-1)^n e^{n\pi}.$$

(d) Using the Laurent series for $e^{1/z}$, we find that

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots,$$

so that

$$\text{Res}[f(z), 0] = 1.$$

(e) Note that $e^{2z} + 1 = 0$ whenever $z = i(2n+1)\pi/2$. These are all simple zeros, so we may evaluate

$$\begin{aligned}\text{Res}[f(z), i(2n+1)\pi/2] &= \frac{-(2n+1)^2\pi^2(-1)}{4(-2)} \\ &= -\frac{(2n+1)^2\pi^2}{8}.\end{aligned}$$

Solution (21.2)**:** Since $\lim_{|z|\to\infty} f(z) = 0$, we may evaluate the residue at the pole at infinity by evaluating (and accounting for the sign flip)

$$\lim_{|z|\to\infty} zf(z) = -1.$$

Pairing with the (negative of the) residue at $z = 3$, we have

$$\begin{aligned}\oint_{|z|=2} f(z)\, dz &= 2\pi i\left(-1 + \frac{1}{6}\right) \\ &= -\frac{5\pi}{3}i.\end{aligned}$$

Solution (21.6)**:** We start by factoring and using the cover-up method to obtain

$$\begin{aligned}f(z) &= \frac{4-2z}{(z-i)(z+i)(z-1)^2} \\ &= \frac{1}{(z-1)^2} + \left(1 - \frac{1}{2}i\right)\frac{1}{z-i} + \left(1 + \frac{1}{2}i\right)\frac{1}{z+i} + \frac{B}{z-1},\end{aligned}$$

where

$$B = \text{Res}[f(z), 1]$$

$$
\begin{aligned}
&= \frac{d}{dz}\left(\frac{4-2z}{z^2+1}\right)\Big|_{z=1}\\
&= -\frac{3}{2}.
\end{aligned}
$$

Thus, we obtain the partial fraction decomposition of

$$f(z) = \frac{1}{(z-1)^2} - \frac{3}{2}\left(\frac{1}{z-1}\right) + \left(1-\frac{1}{2}i\right)\frac{1}{z-i} + \left(1+\frac{1}{2}i\right)\frac{1}{z+i}.$$

Solution (21.8)**:** Closing the contour in the upper half plane, we evaluate the residues of the roots at $e^{i\pi/6}$ and $e^{i5\pi/6}$, giving

$$
\begin{aligned}
\operatorname{Res}\left[f(z), e^{i\pi/6}\right] &= \frac{1}{3\left(e^{i\pi/6}\right)^2}\\
&= \frac{1}{3e^{i\pi/3}}\\
\operatorname{Res}\left[f(z), e^{i5\pi/6}\right] &= \frac{1}{3\left(e^{i5\pi/6}\right)}\\
&= -\frac{1}{3e^{i2\pi/3}}.
\end{aligned}
$$

Thus, calculating the integral, we have

$$\oint_C \frac{1}{z^3-i}\,dz = \frac{2i\pi}{3}.$$

Solution (21.10)**:** We start by doing a partial fraction decomposition of f, giving

$$f(z) = \frac{1}{8}\left(\frac{1}{z-1}\right) + \frac{1}{8}\left(\frac{1}{z-i}\right) - \frac{1}{8}\left(\frac{1}{z+i}\right) - \frac{1}{8}\left(\frac{1}{z+1}\right) - \frac{i}{8}\left(\frac{1}{(z+i)^2}\right) + \frac{i}{8}\left(\frac{1}{(z-i)^2}\right).$$

(a) If we evaluate the integral inside the circle $|z| < 1/2$, then there are no poles inside the contour, so the integral evaluates to 0.

(b) If we evaluate the integral inside $|z| < 2$, then all the poles are inside the contour, so the integral once again evaluates to 0.

(c) If we evaluate the integral inside the contour $|z-i| < 1$, only the pole at $z = i$ is inside the contour, meaning we have the integral of $\frac{\pi i}{4}$.

(d) If we evaluate the integral inside the elliptical contour, only the poles at $z = \pm 1$ are inside the contour, yet again meaning the integral evaluates to 0.

Solution (21.12)**:**

(a) Since the integrand goes to zero at infinity, we may close the contour in the upper half-plane, giving two poles inside the contour at $e^{i\pi/4}$ and $e^{i3\pi/4}$. Evaluating the residues, we have

$$
\begin{aligned}
\operatorname{Res}\left[f(z), e^{i\pi/4}\right] &= \frac{i}{4e^{i3\pi/4}}\\
\operatorname{Res}\left[f(z), e^{i3\pi/4}\right] &= \frac{-i}{4e^{i\pi/4}}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx &= \oint_C \frac{z^2}{z^4+1}\,dz\\
&= 2\pi i\left(\frac{-\sqrt{2}i}{4}\right)\\
&= \frac{\sqrt{2}}{2}\pi.
\end{aligned}
$$

(b) The integrand goes to zero at infinity, so we may close the contour in the upper half-plane, giving the residues of i and $2i$. Evaluating these residues, we have

$$\begin{aligned}
\operatorname{Res}[f(z), i] &= \frac{d}{dz}\left(\frac{1}{z^2+4}\right)\Big|_{z=i}\\
&= -\frac{2z}{\left(z^2+4\right)^2}\Big|_{z=i}\\
&= -\frac{2i}{9}\\
\operatorname{Res}[f(z), 2i] &= \frac{1}{\left(z^2+1\right)^2(z+2i)}\Big|_{z=2i}\\
&= -\frac{i}{36}.
\end{aligned}$$

Thus, we have the integral of

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\left(x^2+1\right)^2\left(x^2+4\right)}\,dx &= \oint_C \frac{1}{\left(z^2+1\right)^2\left(z^2+4\right)}\,dz\\
&= 2\pi i\left(-\frac{2i}{9} - \frac{i}{36}\right)\\
&= \frac{\pi}{2}.
\end{aligned}$$

(c) Using the cubic factorization, we have

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{x^2\left(x^2-1\right)}{x^6-1}\,dx,$$

with removable discontinuity at $x = \pm 1$. Furthermore, since the integrand goes to zero at infinity, we may close the contour in the upper half-plane, giving residues at $z = e^{i\pi/3}$ and $z = e^{i2\pi/3}$. Evaluating at these residues using the cover-up method, we find

$$\begin{aligned}
\operatorname{Res}\left[f(z), e^{i\pi/3}\right] &= \frac{e^{i2\pi/3}}{\left(e^{i\pi/3}-e^{i2\pi/3}\right)\left(e^{i\pi/3}-e^{i4\pi/3}\right)\left(e^{i\pi/3}-e^{i5\pi/3}\right)}\\
&= \frac{1}{2\sqrt{3}}e^{-i\pi/6}\\
\operatorname{Res}\left[f(z), e^{i2\pi/3}\right] &= \frac{e^{i4\pi/3}}{\left(e^{i2\pi/3}-e^{i\pi/3}\right)\left(e^{i2\pi/3}-e^{i4\pi/3}\right)\left(e^{i2\pi/3}-e^{i5\pi/3}\right)}\\
&= -\frac{1}{2\sqrt{3}}e^{i\pi/6}.
\end{aligned}$$

Therefore, our integral gives

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{x^4+x^2+1}\,dx &= \oint_C \frac{z^2}{z^4+z^2+1}\,dz\\
&= 2\pi i\left(\frac{1}{2\sqrt{3}}(-2i\sin(\pi/6))\right)\\
&= \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

(d) We have that

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2+1}\,dx = \frac{1}{2}\operatorname{Re}\left(\oint_C \frac{e^{i\pi z}}{(z-i)(z+i)}\,dz\right).$$

Since $\pi > 0$, we close our contour in the upper half-plane, giving

$$\oint_C \frac{e^{i\pi z}}{(z-i)(z+i)}\,dz = 2\pi i\operatorname{Res}[f(z), i]$$

$$
\begin{aligned}
&= 2\pi i\left(\frac{e^{-\pi}}{2i}\right)\\
&= \pi e^{-\pi}.
\end{aligned}
$$

Solution (21.16)**:**

Solution (21.17)**:** We evaluate

$$I = \int_{0}^{\infty} \frac{x}{1+x^4}\,dx$$

by considering

$$I' = \oint_{C} \frac{z}{1+z^4}\,dz,$$

where C is the contour that goes to a large radius r and returns along the imaginary axis. The integral along this component is equal to $-e^{i\pi/2}I$, giving $I' = \left(1 - e^{i\pi/2}\right)I$.

$$
\begin{aligned}
\int_{0}^{\infty} \frac{x}{1+x^4}\,dx &= \frac{1}{1-i}\oint_{C} \frac{z}{1+z^4}\,dz\\
&= \left(\frac{1}{1-i}\right)\left(2\pi i\,\mathrm{Res}\left[f(z), e^{i\pi/4}\right]\right)\\
&= 2\pi i\left(\frac{e^{i\pi/4}}{\left(e^{i\pi/4} - e^{i3\pi/4}\right)\left(e^{i\pi/4} - e^{i5\pi/4}\right)\left(e^{i\pi/4} - e^{i7\pi/4}\right)(1-i)}\right)\\
&= \frac{\pi}{4}.
\end{aligned}
$$

Solution (21.22)**:**