Excess Area Identities and Operator Symbols in Bergman Spaces

Avinash Iyer

Occidental College

Summary

Contents

Weighted Square-Integrable Functions

- Domain: $\Omega = \mathbb{D}, D(0,r), \mathbb{C}$.

Definition (L^2 Functions)

$$L^2(\Omega) = \left\{ f \,\middle|\, \int_\Omega |f(z)|^2 \, dA < \infty \right\}$$

- $\lambda(z) = \lambda(|z|)$

Definition (Weighted L^2 Functions)

$$L^2(\Omega, \lambda) = \left\{ f \,\middle|\, \int_\Omega |f(z)|^2 \lambda(z) \, dA < \infty \right\}$$

Weighted Square-Integrable Functions, Cont'd

- $L^2(\Omega, \lambda)$ forms a Hilbert space.

Definition (Weighted L^2 Inner Product)

For $f, g \in L^2(\Omega, \lambda)$,

$$\langle f, g \rangle_{L^2(\Omega,\lambda)} = \int_\Omega f(z)\overline{g(z)}\lambda(z)\, dA.$$

Definition (Weighted L^2 Norm)

For $f \in L^2(\Omega, \lambda)$,

$$\|f\|^2_{L^2(\Omega,\lambda)} = \int_\Omega |f(z)|^2\, \lambda(z)\, dA.$$

Bergman Spaces

Definition ($\mathcal{O}(\Omega)$)

$$
\begin{aligned}
f \in \mathcal{O}(\Omega) &\iff \frac{\partial f}{\partial \overline{z}} = 0 \\
&\iff \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0
\end{aligned}
$$

- If $\Omega \underset{\text{open}}{\subseteq} \mathbb{C}$, then f is holomorphic iff f is analytic.

Bergman Spaces, Cont'd

Definition (Bergman Space)

$$A^2(\Omega, \lambda) = L^2(\Omega, \lambda) \cap \mathcal{O}(\Omega)$$

Definition ($A^{1,2}(\Omega, \lambda)$)

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \middle| \frac{\partial h}{\partial z} \in A^2(\Omega, \lambda) \right\}$$

Definition (Image Area)

Let $h \in A^{1,2}(\Omega, \lambda)$. Then,

$$A(h) = \int_\Omega |h'(z)|^2 \lambda(z)\, dA$$

Projection Operator

- $A^2(\Omega) \subseteq L^2(\Omega)$ is closed and has a reproducing kernel $K_z(\cdot)$
- $f(z) = \langle f(\cdot), K_z(\cdot)\rangle_{L^2(\Omega,\lambda)}$

Definition (Projection Operator)

Let $h \in L^2(\Omega, \lambda)$. Then,

$$P^{\Omega,\lambda}(h) = \int_\Omega (h(w)) \left(\overline{K_z(w)}\right) (\lambda(w)) \, dA$$

Definition (Toeplitz Operator)

Let $\varphi \in L^\infty(\Omega, \lambda)$, $h \in L^2(\Omega, \lambda)$. Then,

$$T_\varphi^{\Omega,\lambda}(h) = P^{\Omega,\lambda}(\varphi h) = \int_\Omega (\varphi(w)h(w)) \left(\overline{K_z(w)}\right) (\lambda(w)) \, dA$$

Commutator and Hankel Operators

Definition (Commutator)

Let $M_\varphi(h) = \varphi h$ for $\varphi \in L^\infty(\Omega, \lambda)$.

Then, $\left[P^{\Omega,\lambda}, M_\varphi\right] : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda)$ is defined by

$$\left[P^{\Omega,\lambda}, M_\varphi\right](h) = P^{\Omega,\lambda}(\varphi h) - \varphi P^{\Omega,\lambda}(h).$$

Definition (Hankel Operator)

Let $\varphi \in L^\infty$. Then, $H_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \to \left(A^2(\Omega, \lambda)\right)^\perp$ is defined by

$$\begin{aligned} H_\varphi^{\Omega,\lambda}(h) &= -\left[P^{\Omega,\lambda}, M_\varphi\right]\Big|_{A^2(\Omega,\lambda)} \\ &= \left(I - P^{\Omega,\lambda}\right)(\varphi h) \\ &= \left(M_\varphi - P^{\Omega,\lambda}\right)(h) \end{aligned}$$

Contents

Abstract Motivations

- Relationship between L^2 norms of functions and ℓ^2 norms of Taylor coefficients. For $h \in A^2(\mathbb{D})$, $h = \sum_{k=0}^{\infty} h_k z^k$

$$\|h\|_{L^2}^2 = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}.$$

- $\{z^k\}_{k=0}^{\infty}$ forms an orthogonal basis for A^2.
- On bounded domains,

$$\int_{\Omega} \frac{\partial f}{\partial \overline{z}}\, d\overline{z} \wedge dz = \int_{\partial\Omega} f\, dz$$
$$\int_{\Omega} \frac{\partial f}{\partial z}\, dz \wedge d\overline{z} = \int_{\partial\Omega} f\, d\overline{z}.$$

- $[T_{\overline{z}} M_z, DM_z]\left(z^k\right) = 0$, where $D = \frac{\partial}{\partial z}$.

Literature Review

- In [D'A19], John D'Angelo proved the following identity regarding the excess area of the image of a function $h \in A^{1,2}(\mathbb{D})$.

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \frac{1}{2}\int_0^{2\pi} \left|h\left(e^{i\theta}\right)\right|^2 \, d\theta \\ &= \pi \left\|Sh\right\|^2_{L^2(\partial\mathbb{D})} \end{aligned}$$

Literature Review, Cont'd

- In [BÇGH22], the excess area identity was extended to include Blaschke product multipliers.
- Additionally, [BÇGH22] formulated an excess area identity of the form

$$\left\|\frac{\partial}{\partial z}(zu)\right\|_{L^2(\mathbb{D})}^2 - \left\|\frac{\partial}{\partial z}(u)\right\|_{L^2(\mathbb{D})}^2,$$

  where u is a harmonic function.
- In [ÇDTR+24], an algorithm to formulate a harmonic symbol φ such that $T_\varphi^{\mathbb{D}}(p) = q$ for holomorphic polynomials p and q, $p \neq 0$.
- Additionally, [ÇDTR+24] substituted derivatives for Toeplitz operators in the excess area identity.

Contents

Contents

Contents

References I

Haley K. Bambico, Mehmet Çelik, Sarah T. Gross, and Francis Hall, *Generalization of the excess area and its geometric interpretation*, Zbl **28** (2022). MR 4474183

Mehmet Çelik, Luke Duane-Tessier, Ashley M. Rodriguez, Daniel Rodriguez, and Aden Shaw, *Area differences under analytic maps and operators*, Czechoslovak Math. J. (2024).

John P. D'Angelo, *Hermitian analysis*, second ed., Cornerstones, Birkhäuser/Springer, Cham, 2019, From Fourier series to Cauchy-Riemann geometry. MR 3931729