Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: We know that $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$, meaning we need show $C_0(\mathbb{R})$ is closed under the uniform norm.

Let $(f_n)_n \to f$, with $(f_n)_n \in C_0(\mathbb{R})$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then,

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f_n(x) - f(x)| + |f_n(x)|. \end{aligned}$$

By the definition of uniform convergence, for all $n \geq N_\varepsilon$ it is the case that $|f_n(x) - f(x)| < \varepsilon/2$, and by the definition of vanishing at $\pm\infty$, for all $|x| > M$, $|f_n(x)| < \varepsilon/2$. Thus,

$$< \varepsilon,$$

meaning $f(x) \in C_0(\mathbb{R})$, so $C_0(\mathbb{R})$ is closed, so it is complete.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $(x_n)_n$ be a Cauchy sequence in ℓ_2. Let $x_n(k)$ denote the index k of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$\begin{aligned} |x_n(k) - x_m(k)| &\leq c \left\| x_m(k) - x_n(k) \right\|_2 \\ &\to 0 && \text{since } (x_n)_n \text{ is Cauchy in } \ell_2. \end{aligned}$$

Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, $x_n(k) \to x(k)$ as $k \to \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \to x(k)$ for each n.

We must show that $\left\| x_n - x \right\|_2 \to 0$ as $n \to \infty$. This is equivalent to

$$\begin{aligned} \lim_{N\to\infty} \sum_{k=1}^{N} \lim_{m\to\infty} |x_n(k) - x_m(k)|^2 &\leq \lim_{m\to\infty} \sup_{m \geq M} \left\| x_n - x_m \right\|^2 \\ &\leq \varepsilon^2 && \text{since } (x_n)_n \text{ is Cauchy.} \end{aligned}$$

Thus, $\|x_n - x_m\| \to 0$ as $m \to \infty$ and $n \to \infty$, so $\|x_n - x\| \to 0$ as $n \to \infty$.

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \geq n$. Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq d(x_2, x_1) \left(\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}\right) \\ &= d(x_2, x_1) \theta^{n-1} \left(1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}\right) \\ &\leq d(x_2, x_1) \frac{\theta^{n-1}}{1-\theta}. \end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$, meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.

Problem 4

Let (X, d) be a complete metric space, and suppose $f : X \to X$ is a contractive map — i.e., there is a $\theta \in (0, 1)$ with

$$d(f(x), f(y)) \leq \theta d(x, y).$$

Prove that f has a unique fixed point.

Proof: Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all n, we have

$$d(x_n, x_{n-1}) \leq \theta^n d(x_1, x_0).$$

Therefore, for x_m, x_n arbitrary in X with $m > n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0) \\ &= d(x_1, x_0)\theta^{n+1}\left(1 + \theta + \cdots + \theta^{m-n-1}\right) \\ &\leq d(x_1, x_0)\frac{\theta^{n+1}}{1-\theta}. \end{aligned}$$

Since $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \to 0$ in $\mathbb{R}$ as $n \to \infty$, it must be the case that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Since X is complete, this means $(x_n)_n \to x$ for some $x \in X$, meaning $f(x) = x$.

Suppose it were the case that there existed s, t distinct with $f(s) = s$ and $f(t) = t$. Then, $d(f(s), f(t)) = d(s, t) \leq \theta d(s, t)$, but $\theta < 1$, which is a contradiction. Thus, x is unique.

Problem 5

If $(f_k)_k$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, show that the map

$$\varphi : \ell_2 \to \mathcal{H}$$
$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi(k) f_k$$

is a well-defined isometry.

Proof: Let $\xi, \eta \in \ell_2$. Then,

$$\begin{aligned} d(\xi, \eta) &= \|\xi - \eta\|_2 \\ \varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k \\ \varphi(\eta) &= \sum_{k=1}^{\infty} \eta(k) f_k \\ d(\varphi(\xi), \varphi(\eta)) &= \left(\sum_{k=1}^{\infty} \langle \xi(k) - \eta(k), f_k \rangle\right)^{1/2} \\ &= \|\xi - \eta\|_2 && \text{Parseval's Identity.} \end{aligned}$$

Problem 6

Let $T : V \to W$ be a continuous linear map between normed spaces which is bounded below; that is, there is a $C > 0$ such that $\|T(v)\| \geq C\|v\|$ for all $v \in V$. If V is complete, show that $\operatorname{Ran}(T) \subseteq W$ is a closed subspace, and that $V \cong \operatorname{Ran}(T)$ are uniformly isomorphic.

Proof: Let $(v_n)_n$ be a Cauchy sequence in V.