Problem (Problem 1)**:** Let I, J, K be ideals of R.

(a) Show that $(IJ)K = I(JK)$.

(b) Show that $(I + J)K = IK + JK$.

Problem (Problem 4)**:** Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and let $\iota_{S_i} : R \to S_i^{-1}R$ be the corresponding localization homomorphisms. Use the universal property of localization to show that there exists a unique ring homomorphism $\iota' : S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. Provide an explicit description of this ring homomorphism. Use this to show that if R is an integral domain and S an arbitrary multiplicative subset of R, then $S^{-1}R$ injects into the fraction field $K = \operatorname{frac}(R)$.

Solution: We observe that $\iota_{S_2} : R \to S_2^{-1}R$ maps elements of S_1 to units in $S_2^{-1}R$, as the units in $S_2^{-1}R$ are elements of the form $\frac{s}{s'}$ with $s, s' \in S_2$, so by the universal property, there is a unique ring homomorphism $\iota' : S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. In particular, this is the map $\left[\frac{r}{1}\right]_{S_1^{-1}R} \mapsto \left[\frac{r}{1}\right]_{S_2^{-1}R}$.

Since any arbitrary multiplicative subset $S \subseteq R$ of an integral domain is contained in $R \setminus \{0\}$, it follows that $S^{-1}R$ injects into $(R \setminus \{0\})^{-1}R =: \operatorname{frac}(R)$.

Problem (Problem 5)**:** Let $R = \mathbb{Q} \times \mathbb{Q}$ and $S = \{(1,1)\} \cup (\mathbb{Q}^\times \times \{0\})$. The goal of this problem is to identify the localization $S^{-1}R$.

(a) Describe explicitly when $\frac{(a_1,a_2)}{(s_1,s_2)}$ is equal to $\frac{(b_1,b_2)}{(t_1,t_2)}$ in $S^{-1}R$.

(b) Use your result from part (a) to show that the localization $S^{-1}R$ is isomorphic to the localization $T^{-1}\mathbb{Q}$, where $T = \mathbb{Q} \setminus \{0\}$, hence is isomorphic to $\mathbb{R}$.

(c) Find the kernel of the localization homomorphism $\iota_S : R \to S^{-1}R$.

Solution:

(a) By the definition of the equivalence relation, we must have an element $(r_1, r_2) \in S$ such that

$$(r_1(a_1t_1 - b_1s_1), r_2(a_2t_2 - b_2s_2)) = (0, 0).$$

In particular, since $r_1 \in \mathbb{Q}^\times$, and we may always select $r_2 = 0$, it follows that

$$r_1(a_1t_1 - b_1s_1) = 0,$$

so that $a_1t_1 - b_1s_1 = 0$ (as $\mathbb{Q}$ is an integral domain).

(b) We consider the map $\pi_1 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, which maps $(a_1, a_2) \mapsto a_1$. Observe then that $S^{-1}R$ satisfies the universal property for localization, as we may write $S = (\mathbb{Q}^\times \times \{0\}) \cup (\mathbb{Q}^\times \cup \{1\})$, which clearly maps to $\mathbb{Q}^\times \subseteq \mathbb{Q}$ under this projection map. Additionally, we see that $T^{-1}\mathbb{Q}$ satisfies the universal property for localization when restricted to the first coordinate; yet, this restriction to the first coordinate is exactly our original homomorphism, so both $T^{-1}\mathbb{Q}$ and $S^{-1}R$ satisfy the universal property for localization. Thus, they must be isomorphic.

(c)

Problem (Problem 7)**:** Let $S \subseteq R$ be a multiplicative subset, and let $\iota_S : R \to S^{-1}R$ be the corresponding localization homomorphism. Consider the map

$$\begin{aligned} \alpha : \left\{P' \;\middle|\; P' \text{ is a prime ideal of } S^{-1}R\right\} &\to \left\{P \;\middle|\; P \text{ is a prime ideal of } R \text{ such that } S \cap P = \emptyset\right\} \\ P' &\mapsto \iota_S^{-1}(P'). \end{aligned}$$

(a) Verify that α is well-defined.

(b) Define an inverse map β by $\beta(P) = P \cdot S^{-1}R$. Show that β is well-defined. That is, $\beta(P)$ is a prime ideal of $S^{-1}R$.

(c) Show that α and β are mutual inverses.