Normed Vector Spaces

Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **vector space** over $\mathbb{F}$ is a nonempty set V equipped with two operations: vector addition and scalar multiplication.

$$\begin{aligned} V \times V &\xrightarrow{+} V \\ (v, w) &\mapsto v + w && \text{Vector Addition} \\ F \times V &\to V \\ (\alpha, v) &\mapsto \alpha v && \text{Scalar Multiplication} \end{aligned}$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

(i) $u + (v + w) = (u + v) + w$

(ii) $\exists 0_v \in V$ with $\forall v \in V,\ 0_v + v = v + 0_v = v$

(iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$

(iv) $\forall v, w \in V,\ v + w = w + v$

(v) $\alpha(v + w) = \alpha v + \alpha w,\ (\alpha + \beta)v = \alpha v + \beta v$

(vi) $\alpha(\beta w) = (\alpha\beta)w$

(vii) $1 \cdot v = v$

Remarks:

(a) 0_v is unique and known as the zero vector.

(b) The vector w in (iii) is unique, and denoted $-v$.

(c) $0 \cdot v = 0_v$

(d) $(-1) \cdot v = -v$

(e) Property (iv) follows from all the other axioms.

(f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

Subspaces

Let V be a vector space over $\mathbb{F}$. A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

(i) $w \in W, \alpha \in \mathbb{F} \to \alpha w \in W$.

(ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

Remark: 0_v is always a member of any subspace; a subspace is also a vector space.

Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of V, then, $\bigcap W_i$ is a subspace of V.

Proposition: Union of Subspaces

It is not the case that the union of subspaces of V also a subspace. For example, consider $\mathbb{R}^2$ with the traditional vector space operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$$
$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

If $W_1, W_2 \in V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Generated Subspaces

Let $S \subseteq V$ be any subset of a vector space V. Then,

$$\text{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

Remarks:

- $\text{span}(S) \subseteq V$ is a subspace.
- $\text{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\text{span}(S)$ is the "smallest" subspace containing S, or the subspace generated by S.

Proposition: Quotient Group on Vector Space

Let V be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

(1) $\sim_W$ is an equivalence relation.

(2) If $[v]_W$ denotes the equivalence class of v, then $[v]_W = v + W = \{v + w | w \in W\}$.

(3) $V/W := \{[v]_W | v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

Proof of (1):

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

Proof of (2):

$$\begin{aligned}
[v]_W &= \{u \in V \mid u \sim_W v\} \\
&= \{u \in V \mid u - v \in W\} \\
&= \{u \in V \mid u = v + w \text{ some } w \in W\} \\
&= \{v + w \mid w \in W\} \\
&= v + W
\end{aligned}$$

Proof of (3): Prove that the operation is well-defined.

Bases

Let V be a vector space and $S \subseteq V$ be a subset.

(1) S is said to be spanning for V if $\text{span}(S) = V$.

(2) S is linearly independent if, for $\sum_{j=1}^{n} \alpha_j v_j = 0_v$ with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

(3) S is a basis for V if S is linearly independent and spanning for V.

Proposition: Existence of Basis

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that B is a basis and $B \supseteq B_0$.

Background: A relation on a set X is a subset $R \subseteq X \times X$. If R is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \to x \sim z$), and antisymmetric ($x \sim y, y \sim x \to x = y$), then R is an ordering, and we write $x \leq y$.

If $\leq$ is an ordering of X such that $\forall x, y \in X,\ x \leq y$ or $y \leq x$, then $\leq$ is a total (or linear) ordering.

Let $\leq$ be an ordering of X, let $Y \subseteq X$. An upper bound for Y is an element $u \in X$ such that $y \leq u$ $\forall y \in Y$. A maximal element in X is an element $m \in X$ such that $x \in X,\ x \geq m \to x = m$.

Example: $\mathbb{N}$ under the division ordering defines $a \leq b \Leftrightarrow a|b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of A can be divided by 9 and 12). Meanwhile, $\mathbb{N}$ itself has no maximal elements.

This leads us to ask: given an ordered set, $(X, \leq)$, does X admit maximal elements.

Zorn's Lemma (or Axiom): Let $(X, \leq)$ be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in X. Then, X admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

Proof: Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with D linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that X has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i \in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \dots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \dots, v_n \in D_j$ because Y is totally ordered. However, by definition, D_j is a linearly independent set — therefore, $\alpha_k = 0$. Thus, D is linearly independent.

Since D is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. D is also an upper bound for Y. So, by Zorn's Lemma, X has a maximal element, B.

So, $B_0 \subseteq B \subseteq V$, B is independent, and B is maximal in X. We claim that B is a basis for V. Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and B' is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \dots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.

- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \operatorname{span}(B)$. $\perp$

Thus, we have a linearly independent set, B', with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of B. Therefore, $\operatorname{span}(B) = V$, and B is a basis for V.

Examples: Vector Spaces

(1) n-Dimensional Vectors:

$$\begin{aligned} \mathbb{F}^n &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\} \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} &= \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \\ \alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} &= \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix} \\ B &= \{e_1, \dots, e_n\} \end{aligned}$$

where e_i denotes the unit vector at position i.

(2) $m \times n$ Matrices:

$$\begin{aligned} \mathbb{M}_{m,n}(\mathbb{F}) &= \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\} \\ (a_{ij}) + (b_{ij}) &= (a_{ij} + b_{ij}) \\ \alpha(a_{ij}) &= (\alpha a_{ij}) \\ B &= \{e_{ij}\} \end{aligned}$$

where e_{ij} denotes a matrix of 0 everywhere except column i and row j.

(3) Functions with domain Ω:

$$\begin{aligned} \mathcal{F}(\Omega, \mathbb{F}) &= \{f \mid f : \Omega \to \mathbb{F}\} \\ (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}$$

(4) Bounded functions with domain Ω:

$$\begin{aligned} \ell_\infty(\Omega, \mathbb{F}) &= \{f \in \mathcal{F}(\Omega, \mathbb{F}) \mid \|f\|_u \leq \infty\} \\ \|f\|_u &= \sup_{x \in \Omega} |f(x)| \end{aligned}$$

Exercises:

- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha| \|f\|_u$

- Positive Definite: $\|f\|_u = 0 \Rightarrow f = \mathbb{0}$

Proof of Triangle Inequality: Given $x \in \Omega$,

$$\begin{aligned} |(f+g)(x)| &= |f(x) + g(x)| \\ &\leq |f(x)| + |g(x)| \\ &\leq \|f\|_u + \|g\|_u \end{aligned}$$

Therefore,

$$\begin{aligned} \sup |(f+g)(x)| &\leq \|f\|_u + \|g\|_u \\ \|f+g\|_u &\leq \|f\|_u + \|g\|_u \end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a,b],\mathbb{F}) = \{f : [a,b] \to \mathbb{F} \mid f \text{ continuous}\}$$

Check that $C([a,b],\mathbb{F}) \subseteq \ell_\infty([a,b],\mathbb{F})$ is a subspace.

(6) Let $f : [a,b] \to \mathbb{R}$ be any function. Let $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

$$\begin{aligned} \text{var}(f;\mathcal{P}) &:= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ \text{var}(f) &= \sup_{\mathcal{P}} \text{var}(f;\mathcal{P}) \\ \text{BV}([a,b]) &= \{f : [a,b] \to \mathbb{R} \mid \text{var}(f) < \infty\} \\ \|f\|_{\text{BV}} &= |f(a)| + \text{var}(f) \end{aligned}$$

$\text{BV}([a,b])$ is a vector space.

Question: Is $\mathbb{1}_{\mathbb{Q}} \in \text{BV}([0,1])$?

(7) Suppose $K \subseteq V$ is a *convex* subset of a vector space: $v, w \in K, t \in [0,1] \Rightarrow (1-t)v + tw \in K$. Let $\text{Aff}(K) = \{f : K \to \mathbb{R} \mid f \text{ is affine}\}$, where f is affine if $\forall v, w \in K, t \in [0,1], f((1-t)v + tw) = (1-t)f(v) + tf(w)$.

Exercise: Show that $\text{Aff}(K) \subseteq \mathcal{F}(K,\mathbb{R})$ is a subspace.

(8) Let S be defined as

$$S = \{(a_k)_{k=1}^{\infty} \mid a_k \in \mathbb{F}\}.$$

Under pointwise operations, S is a vector space.

$$\begin{aligned} (a_k)_k + (b_k)_k &= (a_k + b_k)_k \\ \alpha(a_k)_k &= (\alpha a_k)_k \end{aligned}$$

Note 1: $S = \mathcal{F}(\mathbb{N},\mathbb{F})$.

Note 2: $c_{00} \subseteq \ell_1 \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq S$.

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k \neq 0\}$
- $c_0 = \{(a_k)k \mid (a_k)_k \to 0\}$
- $c = \{(a_k)_k \mid (a_k)_k \to a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_u < \infty\}$
- $\ell_1 = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k| = a < \infty\right\}$

(9) $C_C(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \to \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ continuous}, \lim_{x\to\pm\infty} f(x) = 0\}$

(10) Let S be any non-empty set.

$$\mathbb{F}(S) := \{f : S \to \mathbb{F} \mid f \text{ finitely supported}\}$$
$$\operatorname{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \to \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t\in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\operatorname{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^{n} f(t_k)e_{t_k} \in \operatorname{span}(\xi)$. Therefore, ξ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^{n} \alpha_{t_k} e_{t_k} = \mathbb{0}$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\left(\sum_{k=1}^{\alpha_{t_k}} e_{t_k}\right) = \mathbb{0}(t_1)$$
$$\alpha_{t_1} = 0.$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \dots, n$. Therefore, ξ is linearly independent. Since ξ is linearly independent and spanning, ξ forms a basis for $\mathbb{F}(S)$.

Note: The free vector space, $\mathbb{F}(S)$, displays the universal property.

> There are functions $\iota : S \to \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \to V$ for V a vector space over $\mathbb{F}$, $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \to V$ such that $\iota \circ T_\varphi = \varphi$.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbb{F}(S) \arrow[d, "T_\varphi"] \\ & V \end{tikzcd}$$

Proof: Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^{n} f(t_k)e_{t_k}$, where $\operatorname{supp}(f) = \{t_1, \dots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^{n} f(t_k)\varphi(t_k)$$

Exercise: Show T_φ is linear and unique.

Exercise 2: Suppose V is a vector space over $\mathbb{F}$ with basis B. Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists\ T : V \to W$ such that T is bijective and linear.

Normed Spaces

To every vector $v \in V$, we want to assign a length to v, $\|v\|$.

A **norm** on a vector space V is a map

$$\|\cdot\| : V \to \mathbb{R}^+$$
$$v \mapsto \|v\| \geq 0$$

such that

(i) Homogeneity: $\|\alpha v\| = |\alpha|\|v\|$

(ii) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

(iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = \mathbb{0}_V$.

If $p : V \to \mathbb{R}^+$ satisfies (i) and (ii), then p is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\|v\| \leq c_1\|v\|'$$
$$\|v\|' \leq c_2\|v\|$$

Note: On $\mathbb{R}^n$, all norms are equivalent.

Exercise: If p is any seminorm on V, then $|p(v) - p(w)| \leq p(v - w)$.

Notation: If V is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.

Examples of Normed Spaces

(1) Given $V = \mathbb{F}^n$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have different norms:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$
$$\|x\|_\infty = \max_{1\leq j\leq n} |x_j|$$
$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}.$$

In general, for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}.$$

Exercise: Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

We want to show that $\|\cdot\|_p$ defines a norm for $1 \leq p < \infty$. If $1 \leq p < \infty$, its conjugate index $q \in [1, \infty]$ whereby $\frac{1}{p} + \frac{1}{q} = 1$. For example, if $p = 1$, then $q = \infty$, and if $p = \infty$, then $q = 1$.

Lemma 1: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $f : [0, \infty) \to \mathbb{R}$, $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then, $f(t) \geq 0$ for all $t \geq 0$.

Proof 1: We can see that $f'(t) = t^{p-1} - 1$. Then, $f'(t) = 0$ at $t = 1$; $f'(t) > 0$ for $t > 1$ and $f'(t) < 0$ for $t \in [0, 1)$.

So, since $f(t) \geq f(1)$ for all $t \geq 0$, and $f(1) = 0$, $f(t) \geq 0$ for all $t \geq 0$.

Lemma 2: For $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $z, y \geq 0$, $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

Proof 2: We know from Lemma 1, $t \leq \frac{1}{p}t^p + \frac{1}{q}$. Multiply by y^q to get

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Set $t = xy^{1-q}$. Then,

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq}y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $p - pq = -q$, so

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

With these two lemmas in mind, we get two important inequalities.

Hölder's Inequality: For $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$. Then, for $x, y \in \mathbb{F}^n$,

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \|x\|_p \|y\|_q.$$

Proof of Hölder's Inequality: For $p = 1$, the solution is as follows:

$$\begin{aligned}
\left|\sum_{j=1}^{n} x_j y_j\right| &\leq \sum_{j=1}^{n} |x_j||y_j| \\
&\leq \sum_{j=1}^{n} |x_j| \|y\|_\infty \\
&= \|x\|_q \|y\|_\infty,
\end{aligned}$$

and similarly for $p = \infty, q = 1$.

For $1 < p < \infty$, assume $\|x\|_p = \|y\|_q = 1$.

$$\begin{aligned}
\left|\sum_{j=1}^{n} x_j y_j\right| &\leq \sum_{j=1}^{\infty} |x_j||y_j| \\
&\leq \sum_{j=1}^{n} \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q\right) \\
&= \frac{1}{p}\left(\sum_{j=1}^{n} |x_j|^p\right) + \frac{1}{q}\left(\sum_{j=1}^{n} |y_j|^q\right) \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = \mathbb{0}_\mathbb{F}$ or $y = \mathbb{0}_\mathbb{F}$, the inequality still holds.

Assume $\|x\|_p \neq 0$, $\|y\|_p \neq 0$. Set

$$\begin{aligned}
x' &= \frac{x}{\|x\|_p} \\
y' &= \frac{y}{\|y\|_p}.
\end{aligned}$$

It can be verified that $\|x'\|_p = 1 = \|y'\|_q$. Therefore,

$$\left|\sum_{j=1}^{n} x_j' y_j'\right| \leq 1$$
$$\left|\sum_{j=1}^{n} \frac{x_j}{\|x\|_p}\frac{y_j}{\|y\|_q}\right| \leq 1$$
$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \|x\|_p \|y\|_q$$

Minkowski's Inequality: Given $x, y \in \mathbb{F}^n$, $1 \leq p \leq \infty$, $\frac{1}{p} = \frac{1}{q} = 1$,

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Proof of Minkowski's Inequality: We can verify for $p = 1, q = \infty$, and vice versa.

Assume $1 < p < \infty$. Then,

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{j=1}^{n} |x_j + y_j|^p \\
&= \sum_{j=1}^{\infty} |x_j + y_j||x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^{\infty} |x_j||x_j + y_j|^{p-1} + \sum_{j=1}^{n} |y_j||x_j + y_j|^{p-1} \\
&\leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}\left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p}\left(\sum_{j=1}^{n} |x_j + y_j|^{pq-q}\right)^{1/q} && \text{Hölder's Inequality} \\
&= \|x\|_p\|x+y\|_p^{p/q} + \|y\|_p\|x+y\|_p^{p/q} \\
&= \left(\|x\|_p + \|y\|_p\right)\|x+y\|_p^{p-1}
\end{aligned}$$

Divide by $\|x+y\|_p^{p-1}$ to get desired inequality.

(2) $\ell_\infty(\Omega, \mathbb{F})$ with $\|\cdot\|_u$. This includes subspaces that inherit the norm, such as

$$C([a,b]) \subseteq \ell_\infty(\Omega)$$
$$\ell_\infty(\mathbb{R}) \supseteq C_0(\mathbb{R}) \supseteq C_C(\mathbb{R})$$

Exercise: Show that $C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ is a subspace.

(3) $\Omega = \mathbb{N}$, $\ell_\infty = \ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$. Subspaces that inherit the norm are

$$c_{00} \subseteq c_0 \leq \ell_\infty.$$

(4) ℓ_1 with $\|\cdot\|_1$,

$$\|(a_k)_k\|_1 = \sum_{k=1}^{n} |a_k|.$$

(5) $C([a,b])$ with

$$\|f\|_1 = \int_a^b |f(x)|dx.$$

(6) Let $1 \leq p < \infty$.

$$\ell_p = \left\{ (a_k)_{k=1}^{\infty} \mid \sum_{k=1}^{\infty} |a_k|^p < \infty \right\}$$

is a normed space with

$$\|(a_k)_k\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}$$

We will show that the triangle inequality holds for this norm.

$$\begin{aligned}
\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} &= \left\| \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \right\|_{\ell_p^n} \\
&= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^n} \\
&\leq \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\| + \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|_{\ell_p^\infty} \\
&\leq \|(a_k)_k\|_p + \|(b_k)_k\|_p.
\end{aligned}$$

Taking the limit as $n \to \infty$ (by the definition of an infinite series), we find that $\|(a_k)_k + (b_k)_k\|_p \leq \|(a_k)_k\|_p + \|(b_k)_k\|_p$.

(7) $\text{BV}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid \text{Var}(f) < \infty\}$ with the norm $\|f\|_{\text{BV}} = |f(a)| + \text{Var}(f)$ is a normed space:

$$\begin{aligned}
\|f\|_{\text{BV}} &= 0 \\
|f(a)| &= 0 \\
\text{Var}(f) &= 0
\end{aligned}$$

given $t \in (a,b]$, look at the partition $a < t \leq b$. Then,

$$\begin{aligned}
\text{Var}(f) &\geq |f(t) - f(a)| + |f(b) - f(t)| \\
f(t) &= 0 \\
f &= \mathbb{0}_f.
\end{aligned}$$

(8) $\mathbb{M}_{m,n}(\mathbb{F})$ with

$$\|a\|_{\text{op}} = \sup_{\|\xi\|_{\ell_2^n} \leq 1} \|a\xi\|_{\ell_2^m}$$

is a normed vector space. If $\|a\|_{\text{op}} = 0$, then

$$ae_j = 0 \qquad \forall j \in \{1, \ldots, n\}.$$

take the dot product with $i \neq j$

$$\begin{aligned} ae_j \cdot e_i &= a_{ij} \\ &= 0 \end{aligned}$$

so $a_{ij} = 0$ for all a_{ij}, so a is the $\mathbb{0}$ matrix.

(9) Let V, W be vector spaces over $\mathbb{F}$. Then, $\mathcal{L}(V, W) = \{T \mid T : V \to W \text{ linear}\}$, where $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

$\mathcal{L}(V, W)$ is a vector space with operations

$$\begin{aligned} (T + S)(v) &= T(v) + S(v) \\ (\alpha T)(v) &= \alpha T(v). \end{aligned}$$

Notation: $\mathcal{L}(V) := \mathcal{L}(V, V)$ is all linear operators on V. $\mathcal{L}(V, \mathbb{F}) = V'$ is all linear functionals.

Suppose V and W are normed vector spaces. If $T : V \to W$, set

$$\begin{aligned} \|T\|_{\text{op}} &:= \sup_{\|v\|_v \leq 1} \|T(v)\|_W, \\ \mathbb{B}(V, W) &= \{T \in \mathcal{L}(V, W) \mid \|T\|_{op} \leq \infty\}, \end{aligned}$$

where $\mathbb{B}(V, W)$ is referred to as the set of all bounded linear maps from V to W. $\mathbb{B}(V, W)$ with $\|\cdot\|_{\text{op}}$ is a normed space.

- Homogeneity:

$$\begin{aligned} \|\alpha T\|_{[op]} &= \sup_{\|v\|_V \leq 1} \|\alpha T(v)\|_W \\ &= \sup_{\|v\|_V \leq 1} |\alpha| \|T(v)\|_W \\ &= |\alpha| \sup_{\|v\|_V \leq 1} \|T(v)\|_W \\ &= |\alpha| \|T\|_{\text{op}}. \end{aligned}$$

- Triangle Inequality: for $\|v\|_V \leq 1$,

$$\begin{aligned} \|\left(T + S\right)(v)\|_W &= \|T(v) + S(v)\|_W \\ &\leq \|T(v)\|_W + \|S(v)\|_W \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

  so

$$\begin{aligned} \|T + S\|_{\text{op}} &= \sup_{\|v\| \leq 1} \|T + S(v)\| \\ &\leq \|T\|_{\text{op}} + \|S\|_{\text{op}} \end{aligned}$$

- Positive Definite: If $\|T\|_{\text{op}} = 0$, then $T(v) = 0$ for all $v \in V$, $\|v\| \leq 1$.

  Let $v \in V$, $v \neq 0$. Then, $\frac{v}{\|v\|} \in B_V$.

$$\begin{aligned} T\left(\frac{v}{\|v\|}\right) &= 0 \\ \frac{1}{\|v\|} T(v) &= 0 \\ T(v) &= 0 \end{aligned}$$

Special Cases: $\mathbb{B}(V) = \mathbb{B}(V, V)$, $V^* = \mathbb{B}(V, \mathbb{F})$.

Exercise: $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \mathbb{B}(\ell_2^n, \ell_2^m)$.

(10) Inner Product Spaces (expanded upon below).

Inner Product Spaces

An inner product on a vector space V is a pairing

$$V \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}$$

that satisfies

(i) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$, $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$.

(ii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

(iii) $\langle v, v\rangle \geq 0$.

(iv) If $\langle v, v\rangle = 0$, then $v = 0$.

The pair $(V, \langle \cdot, \cdot\rangle)$ is known as an inner product space.

Remarks: $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$, $\langle v, \alpha w\rangle = \overline{\alpha}\langle v, w\rangle$.

If $\langle \cdot, \cdot\rangle$ is an inner product on a linear space V, then set

$$\|v\|_2 := \langle v, v\rangle^{1/2}.$$

Exercise: $\|\alpha v\|_2 = |\alpha| \|v_2\|$, $\|v\|_2 = 0 \Rightarrow v = 0$.

$v, w \in (V, \langle, \cdot, \cdot\rangle)$ are *orthogonal* if $\langle v, w\rangle = 0$.

The Pythagoran theorem states that for $v_1, \dots, v_n \in V$ mutually orthogonal, then

$$\left\|\sum_{i=1}^{n} v_i\right\|^2 = \sum_{j=1}^{n} \|v_j\|^2.$$

For two vectors $v, w \in V$, $P_w(v) = \frac{\langle v, w\rangle}{\langle w, w\rangle} w$.

Exercise: Check that $\langle P_w(v), v - P_w(v)\rangle$, meaning

$$\|v\|^2 = \|P_w(v)\|^2 + \|v - P_w(v)\|^2$$

Cauchy-Schwarz Inequality: In any inner product space,

$$|\langle v, w\rangle| \leq \|v\| \cdot \|w\|.$$

Proof of Cauchy-Schwarz: From the exercise,

$$\begin{aligned} \|v\| &\geq \|P_w(v)\| \\ \|v\| &\geq \left\|\frac{\langle v, w\rangle}{\langle w, w\rangle} w\right\| \\ &= \frac{|\langle v, w\rangle|}{\|w\|^2}\|w\| \end{aligned}$$

therefore,

$$\|v\|\|w\| \geq |\langle v, w\rangle|$$

The triangle inequality follows from the Cauchy-Schwarz inequality.

Proof of Triangle Inequality:

$$\begin{aligned}
\|v+w\|_2^2 &= \langle v+w, v+w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \|v\|^2 + \|w\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} \\
&= \|v\|^2 + \|w\|^2 + 2\text{Re}\,\langle v, w\rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w\rangle| \\
&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| && \text{Cauchy-Schwarz Inequality} \\
&= (\|v\| + \|w\|)^2.
\end{aligned}$$

Take square roots on both sides.

(1) $\ell_2^n = \mathbb{F}^n$ with

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Cauchy-Schwarz is found as

$$\left| \sum_{j=1}^{n} x_j \overline{y_j} \right| \leq \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} \left(\sum_{j=1}^{n} |y_j|^2 \right)^{1/2}.$$

(2) ℓ_2 with

$$\langle (a_j)_j, (b_j)_j \rangle = \sum_{j=1}^{\infty} a_j \overline{b}_j.$$

We can see that for any finite n, the Cauchy-Schwarz inequality in ℓ_2^n states

$$\begin{aligned}
\left| \sum_{j=1}^{n} a_j \overline{b_j} \right| &\leq \left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \\
&\leq \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |b_j|^2 \right)^{1/2}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we see that $\langle (a_j)_j, (b_j)_j \rangle$ is convergent.

(3) $C([a, b])$ with

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

(4) Let $V = \mathbb{M}_n(\mathbb{C})$.

Recall that if

$$a = (a_{ij})_{i,j},$$

then

$$a^* = (\overline{a_{ji}})_{i,j}.$$

Let $\text{Tr} : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$, $\text{Tr}((a_{ij})) = \sum_{i=1}^{n} a_{ii}$.

- $\text{Tr}(I_n) = n$
- $\text{Tr}(a + \alpha b) = \text{Tr}(a) + \alpha \text{Tr}(b)$
- $\text{Tr}(ab) = \text{Tr}(ba)$

Then, if $\text{Tr}(a^*a) = 0$, then $a = \mathbb{0}_{\mathbb{M}_n}$.

$$\begin{aligned}
a^*a &= (\overline{a_{ji}})_{i,j}(a_{ij})_{i,j}\\
&= \left(\sum_{k=1}^{n} \overline{ki}a_{kj}\right)_{i,j}\\
\text{Tr}(a^*a) &= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{a_{ki}}a_{ki}\\
&= \sum_{i,k=1}^{n} |a_{ki}|^2\\
&= \sum_{i,j=1}^{n} |a_{ij}|^2.
\end{aligned}$$

If $\text{Tr}(a^*a) = 0$, then $a_{ij} = 0$ for all i, j.

We define

$$\langle a, b\rangle_{\text{HS}} = \text{Tr}(b^*a).$$

(i) $(b_1 + b_2)^* = b_1^* + b_2^*$
(ii) $(\alpha b)^* = \overline{\alpha} b^*$
(iii) $(b_1 b_2)^* = b_2^* b_1^*$
(iv) $b^{**} = b$

The norm is defined as

$$\begin{aligned}
\|a\|_{\text{HS}} &= \langle a, a\rangle^{1/2}\\
&= \text{Tr}(a^*a)^{1/2}\\
&= \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}
\end{aligned}$$

Metric Spaces

We looked at normed spaces, where we attach a length $\|v\|$ to very vector v. We can also speak of the distance between two vectors, defined as $d(v, w) = \|v - w\|$.

Notice that the following hold:

- $d(v, w) \geq 0$
- $$\begin{aligned}
d(v, w) &= \|v - w\|\\
&= \|(-1)(w - v)\|\\
&= |-1|\,\|w - v\|\\
&= \|w - v\|
\end{aligned}$$

-
$$\begin{aligned} d(u,w) &= \|u-w\| \\ &= \|u-v+v-w\| \\ &\leq \|u-v\| + \|v-w\| \\ &= d(u,v) + d(v,w). \end{aligned}$$

- $d(v,v) = \|v-v\| = 0$. If $d(v,w) = 0$, then $\|v-w\| = 0$, so $v - w = \mathbb{0}$, so $v = w$.

In Real Analysis I, we studied the properties (such as convergence, limits, and continuity) of a particular normed vector space, namely $(\mathbb{R}, |\cdot|)$. We will expand these concepts to all metric spaces.

Definition of a Metric Space

Let X be a non-empty set. A **metric** on X is a map

$$\begin{aligned} d : X \times X &\to \mathbb{R}^+ \\ (x,y) &\mapsto d(x,y) \geq 0 \end{aligned}$$

such that

(i) Symmetry: $d(x,y) = d(y,x)$ for all $x,y \in X$.

(ii) Triangle Inequality: $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

(iii) Zero Distance: $d(x,x) = 0$

(iv) Definite: $d(x,y) = 0 \Rightarrow x = y$

If d satisfies (i), (ii), and (iii), then d is called a semi-metric. If d satisfies (iv) as well, then d is a metric.

If d is a (semi-)metric on X, the pair (X,d) is called a (semi-)metric space.

Two metrics, d and ρ, on X, are equivalent if $\exists c_1, c_2 \geq 0$ such that $d(x,y) \leq c_1\rho(x,y)$ and $\rho(x,y) \leq c_2 d(x,y)$ for all x,y.

Examples of Metric Spaces

(1) Discrete Metric:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for X any set.

(2) Hamming distance: between two bit strings of equal length. Let

$$\begin{aligned} X &= \{0,1\}^n \\ &= \{0,1\}\underbrace{\times\cdots\times}_{n \text{ times}}\{0,1\} \\ d_H((x_j)_1^n, (y_j)_1^n) &= \left|\{j \mid x_j \neq y_j\}\right|. \end{aligned}$$

(3) Any normed space $(V, \|\cdot\|)$ is a metric space.

$$d(v,w) = \|v-w\|.$$

Exercise: Show that if two norms are equivalent, their induced metrics are equivalent.

(4) Subset of Metric Space: If (X, d) is a metric space, and $Y \subseteq X$ is non-empty. Then, (Y, d) is a metric space.

(5) Paris metric: let (X, ρ) be a metric space. Let $p \in X$ be a fixed point.

$$\rho(x, y) := \begin{cases} 0 & x = y \\ \rho(x, p) + \rho(p, y) & x \neq y \end{cases}$$

(6) Bounded metric: Let ρ be a (semi-)metric on X. Set

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We claim that d is a (semi-)metric. Notice that $0 \leq d(x, y) \leq 1$.

Proof: Clearly, $d(x, y) = d(y, x)$. Additionally, $d(x, x) = 0$. If $d(x, y) = 0$ and ρ is a metric, then $\rho(x, y) = 0$, so $x = y$.

To show the triangle inequality, we examine the function

$$\begin{aligned} f(t) &= \frac{t}{1 + t} \\ f'(t) &= \frac{1}{(1 + t)^2} > 0. \end{aligned}$$

Since ρ satisfies the triangle inequality, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Apply f on both sides. Then,

$$\begin{aligned} \underbrace{\frac{\rho(x, z)}{1 + \rho(x, z)}}_{d(x,z)} &\leq \frac{\rho(x, y) + \rho(y, z)}{1 + (\rho(x, y) + \rho(y, z))} \\ &= \frac{\rho(x, y)}{1 + \rho(x, y) + \rho(y, z)} + \frac{\rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\ &\leq \underbrace{\frac{\rho(x, y)}{1 + \rho(x, y)}}_{d(x,y)} + \underbrace{\frac{\rho(y, z)}{1 + \rho(y, z)}}_{d(y,z)}. \end{aligned}$$

(7) If $d_1, \dots, d_n$ are metrics on X, $c_1, \dots, c_n \geq 0$. Then,

$$d(x, y) = \sum_{k=1}^{n} c_k d_k(x, y)$$

is a metric.

(8) Let $\{\rho_k\}_{k=1}^{\infty}$ be a family of semi-metrics. Assume the family is separating — for all $x \neq y$, there exists k such that $\rho_k(x, y) \neq 0$.

Let d_k be defined as

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Note that $\{d_k\}_{k=1}^{\infty}$ is also separating.

Then,

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric.

We will now define the Frechet Metric using this method. Let $X = C(\mathbb{R})$. For each $k = 1, 2, 3, \dots$, set $p_k(f) = \sup_{x \in [-k,k]} |f(x)|$.

We can verify that p_k defines a seminorm. We can then check $\rho_k(f, g) = p_k(f - g)$ is a semi-metric.

We claim that $\{\rho_k\}$ is separating: if $f \neq g$, then there exists $x_0 \in \mathbb{R}$ with $f(x_0) \neq g(x_0)$. Since f and g are continuous, there is a neighborhood $[x_0 - \delta, x_0 + \delta]$ such that $f(x) \neq g(x)$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Find k such that $[x_0 - \delta, x_0 + \delta] \subseteq [-k, k]$. Then, $\rho_k(f - g) > 0$.

Construct d_k as above, and then d as follows:

$$d_{\text{F}} = \sum \frac{2^{-k} p_k(f - g)}{1 + p_k(f - g)}$$

(9) Product of metric spaces: let $(X_k, \rho_k)_{k=1}^{\infty}$ be a countable family of metric spaces. For each k, let

$$d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}.$$

Remark: If the ρ_k are already uniformly bounded, let $d_k = \rho_k$.

Let

$$\begin{aligned} X &= \prod_{k=1}^{\infty} X_k \\ &= \{(x_k)_k \mid x_k \in X_k\} \\ &= \left\{ f : \mathbb{N} \to \bigsqcup_{k=1}^{\infty} X_k \mid f(k) \in X_k \right\}. \end{aligned}$$

Define $D : X \times X \to [0, \infty)$ as

$$\begin{aligned} D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} \rho_k(x_k, y_k), \\ D(f, g) &= \sum_{k=1}^{\infty} 2^{-k} \rho(f(k), g(k)). \end{aligned}$$

For example, for each k, let $X_k = \{0, 1\}$ with the discrete metric. Let

$$\begin{aligned} \Delta &= \prod_{k \in \mathbb{N}} \{0, 1\} \\ &= \{(x_k)_k \mid x_k \in \{0, 1\}\} \\ D(x, y) &= \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k| \qquad (x_k)_k, (y_k)_k \in \Delta. \end{aligned}$$

Δ is known as the abstract Cantor set; every compact metric space is a surjective image of the abstract Cantor set.

(10) Geodesic Distance: let $\langle \cdot, \cdot \rangle$ be the standard dot product on $\mathbb{R}^3(\mathbb{R}^n)$, then

$$S^2 = \left\{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\right\}$$
$$S^{n-1} = \left\{x \in \mathbb{R}^n \mid \|x\|_2 = 1\right\}.$$

To find the geodesic distance, we take $d(x, y) = \arccos(\langle x, y\rangle)$. We claim d is a metric.

- Symmetry: self-evident.
- $d(x, x) = \arccos(1) = 0$. Suppose $d(x, y) = 0$. Then, $\langle x, y\rangle = 1$, meaning $\|x - y\|^2 = 0$, so $x = y$.
- Let $\theta = \arccos(\langle x, y\rangle)$, $\varphi = \arccos(\langle y, z\rangle)$, where $\theta, \varphi \in [0, \pi]$.

  $$\begin{aligned} p_x &= \frac{\langle x, y\rangle}{\langle y, y\rangle} y \\ &= \cos(\theta) y \\ x &= \cos(\theta) y + \sin(\theta) u \end{aligned}$$

  where

  $$u = \frac{x - p_x}{\|x - p_x\|}.$$

  Similarly, we can take

  $$z = \cos(\varphi) y + \sin(\varphi) v$$

  where

  $$v = \frac{z - p_z}{\|z - p_z\|}.$$

  So,

  $$\begin{aligned} \langle x, z\rangle &= \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)\langle u, v\rangle \\ &\geq \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) && \langle u, v\rangle \geq -1 \\ &= \cos(\theta + \varphi). \end{aligned}$$

  Since arccos is decreasing,

  $$\begin{aligned} \arccos(\langle x, z\rangle) &\leq \arccos(\cos(\theta + \varphi)) \\ &= \theta + \varphi \\ &= \arccos(\langle x, y\rangle) + \arccos(\langle y, z\rangle). \end{aligned}$$

  Therefore, $d(x, y) \leq d(x, y) + d(y, z)$.
- Let $\Gamma = (V, E)$ be a simple connected graph. We define $d : V \times V \to [0, \infty)$ to be the length of the shortest path between vertices u and v.

  Exercise: Show this is a metric.

(11) Let (X, d) be any metric space. If $E \subseteq X$, define $\operatorname{diam}(E) = \sup_{x,y\in E} d(x, y)$. E is bounded if $\operatorname{diam}(E) < \infty$.

Exercise: If $(V, \|\cdot\|)$ is a normed space, $E \subseteq V$ is a subset, show the following are equivalent:

(i) E is bounded (in the metric sense)
(ii) $\sup_{v\in E} \|v\| < \infty$
(iii) $\exists r > 0$ such that $E \subseteq rB_V$.

Let Ω be any set. The function $f : \Omega \to X$ is bounded if $f(\Omega) \subseteq X$ is bounded. We let.

$$\text{Bd}(\Omega, X) = \{f : \Omega \to X \mid f \text{ is bounded}\}.$$

Remark: $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty(\Omega, \mathbb{F})$.

(12) $\text{Bd}(\Omega, X)$ with

$$D_u(f, g) = \sup_{x\in\Omega} d(f(x), g(x)).$$

Exercise: Show that D_u defines a metric.

Consider $\text{Bd}(\Omega, \mathbb{F}) = \ell_\infty$. Look at the subset

$$E = \{f \in \text{Bd}(\Omega, \mathbb{F}) \mid f(x) \in \{0, 1\}\}.$$

Then,

$$\begin{aligned} D_u(f, g) &= \sup_{x\in\Omega} |f(x) - g(x)|. \\ &= \begin{cases} 1 & f \neq g \\ 0 & f = g \end{cases}. \end{aligned}$$

When we take a particular subset of $D_u(f, g)$, we find that we get the discrete metric.

Taking an overview of the concepts we have learned so far, we see

$$\text{Inner Product Spaces} \subseteq \text{Normed Vector Spaces} \subseteq \text{Metric Spaces}$$

Topology of Metric Spaces

Throughout this section, let (X, d) be a metric space.

(1) Let $x_0 \in X$, $\delta > 0$.

(i) We say

$$U(x_0, \delta) = \{x \in X \mid d(x, x_0) < \delta\}$$

is the open ball centered at x_0 with radius δ.

(ii) We say

$$B(x_0, \delta) = \{x \in X \mid d(x, x_0) \leq \delta\}$$

is the closed ball.

(iii) We say

$$S(x_0, \delta) = \{x \in X \mid d(x, x_0) = \delta\}$$

is the sphere.

(2) $U \subseteq X$ is open if

$$(\forall x \in U)(\exists \delta > 0) \text{ such that } U(x, \delta) \subseteq U.$$

Let

$$\begin{aligned} \tau_X &= \{U \subseteq X \mid U \text{ open}\} \\ &\subseteq \mathcal{P}(X). \end{aligned}$$

(3) $D \subseteq X$ is closed if D^c is open.

(4) If $x \in U \in \tau_X$, then U is called an open neighborhood of x. If $x \in U \subseteq N$, where $U \in \tau_X$, then N is a neighborhood of x.

$$\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$$

(5) Let $A \subseteq X$. The interior of A is

$$A^\circ = \bigcup \{V \mid V \subseteq A, V \text{ open}\}.$$

The closure of A is

$$\overline{A} = \bigcap \{D \mid A \subseteq D, D \text{ closed}\}.$$

The boundary of A is

$$\partial A = \overline{A} \setminus A^0.$$

Exercise: $\overline{A^c} = (A^\circ)^c$, $(\overline{A})^c = (A^c)^\circ$.

Remarks: A° is the largest open set contained in A. So, if V is open and $V \subseteq A$, then $V \subseteq A^\circ$. Similarly, $\overline{D}$ is the smallest closed set containing D. If C is closed and $D \subseteq C$, then $\overline{D} \subseteq C$.

- For example, $(a,b]^\circ = (a,b)$. This is because (a,b) is open and contained in $(a,b]$, so $(a,b) \subseteq (a,b]^\circ$.
- We will show that $\overline{A^c} \subseteq (A^\circ)^c$.

$$A^\circ \subseteq A$$
$$(A^\circ)^c \supseteq A^c$$

The union of open sets is open, so A° is open, so $(A^\circ)^c$ is closed by definition. Therefore,

$$(A^\circ)^c \supseteq \overline{A^c}.$$

Topology of Open Sets in a Metric Space

The open sets τ_X form a topology:

(i) $\emptyset, X \in \tau_X$.

(ii) If $\{V_i\}_{i \in I} \subseteq \tau_x$, then

$$\bigcup_{i \in I} V_i \in \tau_X.$$

(iii) If $V_i, \dots, V_n \in \tau_X$, then

$$\bigcap_{i=1}^{n} V_i \in \tau_X.$$

Remark: This is only true of finite intersections. For a counterexample, if $V_n = (-1/n, 1/n) \subseteq \mathbb{R}$ with the Euclidean metric, then the infinite intersection yields $\{0\}$, which is closed in $\mathbb{R}$ with the Euclidean metric.

Proof:

(1) Clearly, $\emptyset$ (by vacuous truth) and X are open.
(2) Let $x \in \bigcup_{i\in I} V_i$. Then, $\exists i_0 \in I$ with $x \in V_{i_0}$. Since V_{i_0} is open, $\exists \varepsilon > 0$ such that $U(x,\varepsilon) \subseteq V_{i_0} \subseteq \bigcup V_i$.
(3) Let $x \in \bigcap_{i=1}^n V_i$. Then, $x \in V_i$ for all $i \in 1,\dots,n$. Since each V_i is open, $\exists \varepsilon_1,\dots,\varepsilon_n$ with $U(x,\varepsilon_i) \subseteq V_i$ for each $i = 1,\dots,n$. Set $\varepsilon = \min\{\varepsilon_i\}_{i=1}^n$. Then, $U(x,\varepsilon) \subseteq U(x,\varepsilon_i) \subseteq V_i$ for all i. Therefore, $U(x,\varepsilon) \subseteq \bigcap_{i=1}^n V_i$.

Exercise: Show all open balls are open. In particular, show all open intervals are open.

Exercise: Show the following:

(1) $X, \emptyset$ are closed.
(2) If $\{C_i\}_{i\in I}$ is a family of closed sets, then $\bigcap_{i\in I} C_i$ is closed.
(3) For $C_1,\dots,C_n$ closed, then $\bigcup_{i=1}^n C_i$ is closed.
(4) Closed balls are closed. Spheres are closed.

Let $x \in X$. Recall that $\mathcal{N}_x$ is the set of all neighborhoods of x.

(i) $N \in \mathcal{N}_x \Leftrightarrow \exists \delta > 0 : U(x,\delta) \in N$

(ii) $N \in \mathcal{N}_x, N \subseteq M \Rightarrow M \in \mathcal{N}_x$

(iii) $N_1, N_2 \in \mathcal{N}_x \Rightarrow N_1 \cap N_2 \in \mathcal{N}_x$

In this sense, $\mathcal{N}_x$ is a directed set with reverse inclusion.

Pointwise Characterization of Subsets

Let $A \subseteq X$.

(i) $x \in A^\circ \Leftrightarrow \exists \delta > 0 : U(x,\delta) \subseteq A$.

(ii) $x \in \overline{A} \Leftrightarrow \forall \delta > 0 : U(x,\delta) \cap A \neq \emptyset$.

(iii) $x \in \partial A \Leftrightarrow \forall \delta > 0 : U(x,\delta) \cap A \neq \emptyset$ and $U(x,\delta) \cap A^c \neq \emptyset$.

Proof: Let $A \subseteq X$

(i)

$$\begin{aligned} x \in A^\circ &\Leftrightarrow x \in \bigcup_{\substack{V\in\tau_X \\ V\subseteq A}} V \\ &\Leftrightarrow \exists V \in \tau_X, V \subseteq A, x \in V \\ &\Leftrightarrow \exists \delta > 0 : U(x,\delta) \subseteq A. \end{aligned}$$

(ii)

$$\begin{aligned} x \notin \overline{A} &\Leftrightarrow x \in (\overline{A})^c \\ &\Leftrightarrow x \in (A^c)^\circ \\ &\Leftrightarrow \exists \delta > 0 : U(x,\delta) \subseteq A^c \\ &\Leftrightarrow \exists \delta > 0 : U(x,\delta) \cap A = \emptyset. \end{aligned}$$

We negate both sides.

(iii)

$$\begin{aligned} x \in \partial A &\Leftrightarrow x \in \overline{A} \setminus A^{\circ} \\ &\Leftrightarrow x \in \overline{A} \cap (A^{0})^{c} \\ &\Leftrightarrow x \in \overline{A} \cap \overline{A}^{c} \\ &\Leftrightarrow x \in \overline{A} \text{ and } x \in \overline{A}^{c} \\ &\Leftrightarrow \forall \delta > 0 : U(x,\delta) \cap A \neq \emptyset, U(x,\delta) \cap A^{c} \neq \emptyset \end{aligned}$$

Remark: $\overline{U(v,\delta)} = B(v,\delta)$ in a normed space. $\partial U(v,\delta) = \partial B(v,\delta) = S(v,\delta)$ in a normed space. Also, $B(v,\delta)^{\circ} = U(v,\delta)$.

Proof: We show that $\overline{U}(v,\delta) = B(v,\delta)$. Since $B(v,\delta)$ is closed, and $U(v,\delta) \subseteq B(v,\delta)$, we know $\overline{U(v,\delta)} \subseteq B(v,\delta)$.

Let $w \in B(v,\delta)$. If $\|w - v\| < \delta$, then $w \in U(v,\delta)$. Assume $\|w - v\| = \delta$. Let $u_t = (1-t)v + tw$, where $t \in [0,1]$.

$$\begin{aligned} \|w - u_t\| &= \|w - (1-t)v - tw\| \\ &= \|(1-t)(w-v)\| \\ &= (1-t)\|w - v\| \\ &= (1-t)\delta. \end{aligned}$$

Let $\varepsilon > 0$. Let $t \in (0,1)$ such that $(1-t)\delta < \varepsilon$. Then, $u_t \in U(w,\varepsilon) \cap U(v,\delta)$. Therefore, $w \in \overline{U(v,\delta)}$.

Unions and Intersections of Closure/Interior

Let (X,d) be a metric space.

(i)

$$\left(\bigcup_{i\in I} A_i\right)^{\circ} \supseteq \bigcup_{i\in I} A_i^{\circ} \qquad \text{may be strict}$$

(ii)

$$\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} \overline{A_i}$$

(iii)

$$\bigcap_{k=1}^{n} A_k^{\circ} = \left(\bigcap_{k=1}^{n} A_k\right)^{0}$$

(iv)

$$\overline{\bigcup_{k=1}^{n} D_k} = \bigcup_{k=1}^{n} \overline{D_k}$$

Proof:

(i)

$$\begin{aligned}
A_i^\circ &\subseteq A_i \\
\bigcup_{i\in I} A_i^\circ &\subseteq \bigcup_{i\in I} A_i \\
\bigcup_{i\in I} A_i^\circ &\subseteq \left(\bigcup_{i\in I} A_i\right)^\circ
\end{aligned}$$

Remark: We claim $\overline{\mathbb{Q}} = \mathbb{R}$ under the absolute value metric. We know that $\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{R}$ is closed, meaning $\overline{\mathbb{Q}} \subseteq \mathbb{R}$. Let $t \in \mathbb{R}$, $\delta > 0$. We know that $(t-\delta, t+\delta) \cap \mathbb{Q} \neq \emptyset$. Therefore, $t \in \overline{\mathbb{Q}}$. Thus, $\overline{\mathbb{Q}} = \mathbb{R}$.

Properties of Boundary

Let $A \subseteq X$.

(1) ∂A is closed.

(2) $\partial A = \partial A^c$

(3) $\overline{A} = A \cup \partial A$

(4) $A \setminus \partial A = A^\circ$

Proof:

(1)

$$\begin{aligned}
\partial A &= \overline{A} \setminus A^\circ \\
&= \overline{A} \cap (A^\circ)^c.
\end{aligned}$$

(2) Follows from pointwise characterization.

(3) Clearly, $A \cup \partial A \subseteq \overline{A}$. Let $x \in \overline{A}$. If $x \in A$, we're done. Otherwise, $x \in \overline{A} \setminus A \subseteq \overline{A} \setminus A^\circ = \partial A$.

(4)

$$\begin{aligned}
A \setminus \partial A &= A \cap (\partial A)^c \\
&= A \cap (\overline{A} \setminus A^\circ)^c \\
&= A \cap \left(\overline{A} \cap (A^\circ)^c\right)^c \\
&= A \cap \left(\overline{A}^c \cup A^\circ\right) \\
&= \left(A \cap \overline{A}^c\right) \cup (A \cap A^\circ) \\
&= A^\circ
\end{aligned}$$

Density and Separability

Let (X, d) be a metric space.

(1) $A \subseteq X$ is d-dense if $\overline{A} = X$.

(2) $N \subseteq X$ is nowhere dense if $(\overline{N})^\circ = \emptyset$.

(3) (X, d) is separable if there is a countable dense subset.

Exercise: If $N \subseteq X$ is closed, then N is nowhere dense if and only if N^c is dense.

Exercise: The following are equivalent.

(1) $A \subseteq X$ is dense.
(2) $\forall \emptyset \neq U \in \tau_X$, $U \cap A \neq \emptyset$.
(3) $\forall x \in X, \forall \varepsilon > 0$, $U(x, \varepsilon) \cap A \neq \emptyset$.
(4) $\forall x \in X, \forall \varepsilon > 0, \exists a \in A$ such that $d(x, a) < \varepsilon$.

Let X be a metric space.

(1) A base for τ_X is a family of open subsets $\mathcal{B}$ such that:

$$(\forall U \in \tau_X)\,(\forall x \in U)\exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U.$$

Equivalently,

$$\forall U \in \tau_X, U = \bigcup_{i \in I} B_i. \qquad B_i \in \mathcal{B}$$

(2) We say that (X, d) is second countable if τ_X admits a countable base.

- For any (X, d) a metric space, $\mathcal{B} = \{U(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a base. Indeed, given any $x \in U \subseteq \tau_X$, by definition, $\exists \varepsilon > 0$ such that $U(x, \varepsilon) \subseteq U$. Alternatively, $\mathcal{B}' = \{U(x, 1/n) \mid x \in X, n \geq 1\}$ is a topological base.
- Let $X = \mathbb{R}^d$ with the Euclidean metric. Then, for $\mathcal{B} = \{U(q, 1/n) \mid n \geq 1, q \in \mathbb{Q}^d\}$, we claim this is a base.

  Let $V \subseteq \mathbb{R}^d$ be open, $r \in V$. Since V is open, $\exists \delta > 0$ with $U(r, \delta) \subseteq V$. Find n large such that $1/n < \delta$. Find $q \in \mathbb{Q}^d$ with $\|r - q\| < 1/2n$. This is always possible as $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$.

  Consider $U(q, 1/2n)$. Then, $r \subseteq U(q, 1/2n) \subseteq U(r, \delta) \subseteq V$ because $\|r - q\| < 1/2n$, and if $t \in U(q, 1/2n)$, then

$$\begin{aligned}\|t - r\| &\leq \|t - q\| + \|q - r\| \\ &< 1/2n + 1/2n \\ &= 1/n \\ &< \delta.\end{aligned}$$

Separable, Non-Separable, Dense, and Non-Dense Sets

(1) $\left(R^d, \|\cdot\|_p\right)$ is separable for any $p \in [1, \infty]$. Indeed, $\mathbb{Q}^d \subseteq \mathbb{R}^d$ is the countable dense subset of $\mathbb{R}^d$.

Let $r = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \in \mathbb{R}^d$. Find $q = \begin{bmatrix} q_1 \\ \vdots \\ q_d \end{bmatrix} \in \mathbb{Q}^d$ with $|r_j - q_j| < \varepsilon/d$. Then,

$$\begin{aligned}\|r - q\|_1 &= \sum_{j=1}^{d} |r_j - q_j| \\ &< d.\end{aligned}$$

We know that for any vector $r \in \mathbb{R}^d$, we can find a vector q such that

$$\|q - r\|_p \leq c\,\|q - r\|_1,$$

so for arbitrary p, find q such that $\|q - r\|_1 < \varepsilon/c$.

(2) Similarly, $\mathbb{C}_{\mathbb{Q}} = \{a + bi \mid a, b \in \mathbb{Q}\}$ is also countable, meaning $\mathbb{C}_{\mathbb{Q}}^d \subseteq \mathbb{C}^d$ is dense and $\mathbb{C}^d$ is dense.

Proposition: Separable Subsets

If (X, d) is separable, and $Y \subseteq X$, then (Y, d) is also separable.

Let $\{a_k\}$ be a countable dense subset in X. Let $N = \{(m, n) \mid U(a_m, 1/n) \cap Y \neq \emptyset\}$. Clearly, N is nonempty. For each $(m, n) \in N$, choose $b_{(m,n)} \in Y \cap U(a_m, 1/n)$. We claim $\{b_{(m,n)} \mid m, n \geq 1\}$ is dense in Y.

Let $y \in Y$, $\varepsilon > 0$. Find N large so that $\frac{1}{n} < \varepsilon/2$. Since $A \subseteq X$ is dense, find $U(y, 1/n) \cap A \neq \emptyset$. Suppose $d(a_m, y) < 1/n$. Then,

$$\begin{aligned} d(b_{(m,n)}, y) &\leq d(b_{(m,n)}, a_m) + d(a_m, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n} \\ &< \varepsilon. \end{aligned}$$

(1) ℓ_p^n is separable.

(2) $c_{00} = \{(a_k)_{k=1}^n \mid \text{finitely many } a_k \neq 0\}$ with $\|\cdot\|_u$ is separable.

Recall that $e_k = (0, 0, \dots, 1, 0, 0, \dots)$ where 1 is at position k. Consider $E = \mathbb{Q}\text{-span}\{e_k \mid k \geq 1\}$,

$$E = \left\{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{Q}, n \geq 1 \right\}.$$

The set E is countable. If we fix $n \geq 1$, we have

$$E_n = \left\{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\}.$$

Then, $E = \bigcup E_n$. Note

$$\underbrace{\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}}_{n} \to E_n$$

$$(\alpha_1, \dots, \alpha_n) \mapsto \sum_{k=1}^{n} \alpha_k e_k.$$

Thus, E_n is countable, and E is a countable union of countable sets.

We claim that E is dense. Given $z \in c_{00}$, $\varepsilon > 0$, we know that $z = \sum_{k=1}^{n} a_k e_k$ for some n and $a_k \in \mathbb{R}$. Find $\alpha_k \in \mathbb{Q}$ such that $|\alpha_k - a_k| < \varepsilon$. Set $w = \sum_{k=1}^{n} \alpha_k e_k$. Then, $\|z - w\|_u = \sup |\alpha_k - a_k| < \varepsilon$.

(3) c_0 with $\|\cdot\|_u$ is separable.

(4) ℓ_∞ is not separable.

Suppose ℓ_∞ were separable. Consider $E = \{(a_k)_k \in \ell_\infty \mid a_k \in \{0, 1\}\}$. Then, E is separable. Recall that $(E, \|\cdot\|_u)$ has the discrete metric.

In the discrete metric, every subset is open, meaning every subset is closed. Therefore, if X is separable and discrete, then X is countable.

However, E is not countable by Cantor's theorem. $\text{card}(E) = 2^{\aleph_0}$.

Alternatively, we can show that

$$(a_k)_k \mapsto \sum_{k=1}^{\infty} 2^{-k} a_k$$

is onto.

Exercise: ℓ_p is separable for $1 \leq p < \infty$.

(5) We will show that

$$\mathbb{P}[0,1] \left\{ \sum_{k=1}^{n} a_k x^k \mid a_k \in \mathbb{R}, n \geq 1 \right\}$$

is $\|\cdot\|_u$-dense in $C([0,1])$ (see: Stone-Weierstrass Theorem). Using this, we can show that $(C([0,1]), \|\cdot\|_u)$ is separable.

The Cantor Set

$$\begin{aligned}
C_0 &= [0,1] \\
C_1 &= [0,1/3] \cup [2/3,1] \\
C_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \\
C_3 &= [0,1/27] \cup [2/27,1/9] \cup \cdots \cup [26/27,1] \\
&\vdots
\end{aligned}$$

In each step, we delete the middle third of each interval. This process repeated ad infinitum yields the Cantor set.

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right).$$

(i) $\mathcal{C}$ is closed as it is the intersection of closed sets.

(ii) $\text{length}(\mathcal{C}) = 0$. Look at the total length of the removed intervals,

$$\begin{aligned}
l &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots \\
&= \sum_{k=1}^{\infty} \left(\frac{2^{k-1}}{3^k} \right) \\
&= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{2}{3} \right)^k \\
&= 1.
\end{aligned}$$

Thus, $\text{length}(\mathcal{C}) = 0$.

(iii) $\mathcal{C}$ is nowhere dense — $(\overline{\mathcal{C}})^\circ = \emptyset$. Since $\mathcal{C}$ is closed, $\mathcal{C}^\circ = \emptyset$.

Suppose $\mathcal{C}^\circ \neq \emptyset$. Then, $\exists x \in \mathcal{C}, \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{C}$. So, $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{C}_n$ for all n.

Note $\mathcal{C}_n$ is the disjoint union of 2^n subintervals, each with length $1/3^n$. Find m so large such that $3^{-m} < \varepsilon$. We know that $(x-\varepsilon, x+\varepsilon) \subseteq C_m$.

However, $(x-\varepsilon, x+\varepsilon)$ has length $2\varepsilon > \frac{2}{3^m}$. Each subinterval in C_m has length $1/3^m$. This implies C_m contains an interval of length greater than $\frac{2}{3^m}$. $\perp$

(iv) $\text{card}(\mathcal{C}) = \text{card}(\mathbb{R})$

Claim 1: Given $n \geq 1$,

$$E_n = \left\{\sum_{k=1}^{n} \frac{w_k}{3^k} \mid w_k \in \{0,2\}\right\}$$

is precisely the set of *left* endpoints of the subintervals of C_n.

For $n = 1$, if $w_1 = 0$, then we get 0, and $w_1 = 2$ yields 2/3. Meanwhile, if $n = 2$, then we have

$$\begin{aligned} w_1 = 0, w_2 = 0 &\mapsto 0 \\ w_1 = 0, w_2 = 2 &\mapsto 2/9 \\ w_1 = 2, w_2 = 0 &\mapsto 2/3 \\ w_1 = 2, w_2 = 2 &\mapsto 8/9. \end{aligned}$$

By induction, we have shown for $n = 1, 2$. Assume this is true for n.

$$\sum_{k=1}^{n+1} w_k 3^{-k} = \underbrace{\sum_{k=1}^{n} w_k 3^{-k}}_{(1)} + \underbrace{w_{n+1}3^{-(n+1)}}_{(2)}$$

Part (1) denotes one of the left endpoints of C_n, called $C_{n,k}$ for some $1 \leq k \leq 2^n$. Then, if $w_{n+1} = 0$, we get the left endpoint of $C_{n+1,2k-1}$, and if $w_n = 2$, we get the left endpoint of $C_{n+1,2k}$.

Claim 2:

$$C = \left\{\sum_{k=1}^{\infty} w_k 3^{-k} \mid w_k \in \{0,2\}\right\}$$

is precisely the Cantor set.

Let $x = \sum_{k=1}^{\infty} w_k 3^{-k}$. We will show that $x \in C_n$ for all n. Fix $n \geq 1$. Then,

$$x = \underbrace{\sum_{k=1}^{n} w_k 3^{-k}}_{y} + \underbrace{\sum_{k>n} w_k 3^{-k}}_{z}.$$

From our previous claim, y is the left endpoint of some subinterval of C_n. Additionally,

$$\begin{aligned} z &= \sum_{k>n} w_k 3^{-k} \\ &\leq 2\sum_{k>n} 3^{-k} \\ &= \frac{2}{3^{n+1}}\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right) \\ &= \frac{1}{3^n}. \end{aligned}$$

Since the length of a subinterval in C_n is exactly 3^{-n}, it is the case that $x = y+z$ remains an element of $C_{n,k}$.

Let $x \in \mathcal{C}$. Then, $x \in C_n$ for all n. Then, $x \in C_1$, so let x_1 be the left endpoint of the interval $C_{1,j}$ that contains x. Then, $|x - x_1| < \frac{1}{3}$, and $x_1 = w_1 3^{-1}$ for some $w_1 \in \{0, 2\}$.

Let x_2 be the left endpoint of the subinterval $C_{2,j}$ that contains x. Then, $|x - x_2| < \frac{1}{3^2}$. Therefore,

$$\begin{aligned} x_2 &= x_1 + w_2 3^{-2} \\ &= w_1 3^{-1} + w_2 3^{-2}. \end{aligned}$$

Iterating, we have x_n, the left endpoint of the subinterval $C_{n,j}$ that contains x.

$$x_n = \sum_{k=1}^{n} w_k 3^{-k}.$$

We have that $|x - x_n| < 3^{-n}$.

Therefore, $(x_n)_n \to x$. Also,

$$\begin{aligned} x_n &= \sum_{k=1}^{n} w_k 3^{-k} \\ &\to \sum_{k=1}^{n} w_k 3^{-k}. \end{aligned}$$

Thus,

$$x = \sum_{k=1}^{\infty} w_k 3^{-k}.$$

To prove $\text{card}(\mathcal{C}) = \text{card}(\mathbb{R})$, we will show that $\text{card}\left(\{0,1\}^{\mathbb{N}}\right) = \text{card}(\mathcal{C})$.

$$(a_k)_k \mapsto \sum_{k=1}^{\infty} 2a_k 3^{-k}.$$

Relative (or Subspace) Topology

We know that if (X, d) is a metric space, and $Y \subseteq X$ is any subset, then (Y, d) is a metric space. The question now is: what are the open sets of Y?

For example, let $X = \mathbb{R}$, $Y = [0, 1]$. Consider $U = [0, 1/2)$. U is not open in $\mathbb{R}$, as if $x = 0$, then there is no open ball completely contained in U. However, in Y, U is open.

Let (X, d) be a metric space, $Y \subseteq X$ any subset. $V \subseteq Y$ is open if and only if $\exists U \subseteq X$ open such that $V = U \cap Y$. That is, $\tau_Y = \{U \cap Y \mid U \in \tau_X\}$.

Let V be open in Y. Then, $\forall x \in V$, $\exists \delta_x > 0$ such that $U_Y(x, \delta_x) \subseteq V$. We have $U_Y(x, \delta_x) = \{y \in Y \mid d(y, x) <$

$\delta_x\}$. Let

$$\begin{aligned} U &= \bigcup_{x\in V} U_X(x,\delta_x) \\ U\cap Y &= \left(\bigcup_{x\in V} U_X(x,\delta_x)\right)\cap Y \\ &= \bigcup_{x\in V} U_X(x,\delta_x)\cap Y \\ &= \bigcup_{x\in V} U_Y(x,\delta_x). \end{aligned}$$

Let U be open in X. Then, for $x \in U\cap Y$, $\exists \delta_x$ such that $U(x,\delta_x)\subseteq U$.

(1) ℓ_∞ is not a discrete metric space. However, $E = \{(a_k)_k \mid a_k \in \{0,1\}\}$ with the induced metric. Then, E is a discrete metric space.

Convergent Sequences

Fix a metric space (X,d). A sequence in X is a map $x : \mathbb{N} \to X$, $n \mapsto x(n) = x_n$.

A natural sequence $(n_k)_k$ is a sequence in $\mathbb{N}$ with $n_k \geq k$ for all k. A subsequence of $(x_n)_n$ is a sequence $(x_{n_k})_k$, where $(n_k)_k$ is a natural sequence.

A sequence $(x_n)_n$ converges to $x \in X$ if $\forall\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $n \geq N_\varepsilon$ implies $d(x_n,x) < \varepsilon$. We write $(x_n)_n \xrightarrow{d} x$.

Exercise: A sequence can have at most one limit, as metric spaces are Hausdorff.

Proposition: Equivalent Definitions of Convergence

Given $(x_n)_n \in X$, $x \in X$, the following are equivalent.

(i) $(x_n)_n \to x$ in X

(ii) $(d(x_n,x))_n \to 0$ in $\mathbb{R}$

(iii) $\forall V \in \mathcal{N}_x$, $\exists N \in \mathbb{N}$ with $n \geq N \Rightarrow x_n \in V$.

Exercise: Let (X,ρ) be a metric space, let $d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$. A sequence $(x_n)_n \xrightarrow{d} x$ if and only if $(x_n)_n \xrightarrow{\rho} x$.

Proposition: Convergent Sequences are Bounded

Let $(x_n)_n \to x$ in (X,d). Let $\varepsilon = 1$. Then, $\exists N\in\mathbb{N}$ large such that for $n \geq N$, $d(x_n,x) < 1$.

If $m,n \geq N$, then $d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) < 2$. Let $c = \max_{1\leq n,m\leq N} d(x_n,x_m)$. Then,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_N) + d(x_n,x_m) \\ &\leq 1 + c. \end{aligned}$$

Let $k = \max\{1+c, 2\}$. Then, $\operatorname{diam}(\{x_n\}) \leq k$.

Convergence in Different Metric Spaces

Convergence for Bounded Functions: Recall that for (Y, d) a metric space is

$$\begin{aligned} \mathrm{Bd}(\Omega, Y) &= \{f : \Omega \to Y \mid f \text{ bounded}\} \\ D_u(f, g) &= \sup_{x\in\Omega} d(f(x), g(x)). \end{aligned}$$

Then, $(f_n)_n \to f$ in $\mathrm{Bd}(\Omega, Y)$ if and only if $D_u(f_n, f) \to 0$ in $\mathbb{R}$.

$$\begin{aligned} &(\forall\varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow D_u(f_n, f) < \varepsilon \\ &\Leftrightarrow \\ &(\forall\varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow \sup_{x\in\Omega} d(f_n(x), f(x)) < \varepsilon \\ &\Leftrightarrow \\ &(\forall\varepsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } n \geq N \Rightarrow \forall x,\ d(f_n(x), f(x)) < \varepsilon. \end{aligned}$$

This is exactly the definition of uniform convergence.

Since $\ell_\infty(\Omega) = \mathrm{Bd}(\Omega, \mathbb{F})$, convergence in $\ell_\infty(\Omega)$ is uniform convergence. This is also the case for subspaces, such as c, c_0, and c_{00}.

Convergence in the Frechet Metric: Consider a separating family of semimetrics ρ_k on a set X. Set $d_k = \frac{\rho_k}{1+\rho_k}$. We saw that

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric on X.

We claim that $(x_n)_n \to x$ in (X, d) if and only if for all $k \geq 1$, $\rho_k(x_n, x) \to 0$.

In the forward direction, we know that $(x_n)_n \to x$ with respect to d if and only if $d(x_n, x) \to 0$ in $\mathbb{R}$. Since $0 \leq 2^{-k} d_k(x_n, x) \leq d(x_n)$ for fixed k, we have that

$$0 \leq d_k(x_n, x) \leq 2^k d(x_n, x),$$

and as $n \to \infty$, $d(x_n, x) \to 0$, meaning $d_k(x_n, x) \to 0$. Therefore, $\rho_k(x_n, x) \to 0$.

In the reverse direction, suppose $\rho_k(x_n, x) \to 0$ in $\mathbb{R}$ as $n \to \infty$ for all $k \geq 1$. Thus, $d_k(x_n, x) \to 0$ as $n \to \infty$ for all $k \geq 1$.

Let $\varepsilon > 0$. Let K be so large such that

$$\sum_{k\geq K} 2^{-k} < \varepsilon/2.$$

Therefore, $d_k(x_n, x) \to 0$ for all $k = 1, \ldots, K$. Therefore, $\exists N_1, \ldots, N_K$ such that for $n \geq N_k$,

$$d_k(x_n, x) < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, \ldots, N_K\}$. Therefore, for $n \geq N$,

$$d_k(x_n, x) < \frac{\varepsilon}{2}$$

for all $k = 1, \dots, K$.

Thus, for all $n \geq N$,

$$\begin{aligned} d(x_n, x) &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_n, x) \\ &= \sum_{k=1}^{K} 2^{-k} d_k(x_n, x) + \sum_{k=K+1}^{\infty} 2^{-k} d_k(x_n, x) \\ &\leq \frac{\varepsilon}{2} \sum_{k=1}^{K} 2^{-k} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Therefore, $(x_n)_n \to x$.

Recall that, for the Frechet metric, our set was $X = C(\mathbb{R})$. For $k = 1, 2, 3, \dots$, we had

$$p_k(f) = \sup_{[-k,k]} |f(x)|$$

as our seminorm, and our semimetric was

$$\rho_k(f, g) = p_k(f - g).$$

We also showed that the ρ_k family is separating. We make $d_k(f, g) = \frac{\rho_k(f,g)}{1+\rho_k(f,g)}$ as the bounded family of separating metrics, and

$$d_F(f, g) = \sum_{k=1}^{\infty} \frac{2^{-k} \rho_k(f - g)}{1 + \rho_k(f - g)}.$$

In $(C(\mathbb{R}), d_F)$, $(f_n)_n \to f$ if and only if $\rho_k(f_n, f) \to 0$ for all k, meaning $(f_n)_n \to f$ uniformly on $[-k, k]$ for all k.

This is known as convergence on compact subsets.

Convergence in a Product Space: Let (X, d) and (Y, ρ) be metric spaces. Then,

$$\begin{aligned} X \times Y &= \{(x, y) \mid x \in X, y \in Y\}, \\ D_1((x, y), (x', y')) &= d(x, x') + \rho(y, y') \\ D_\infty((x, y), (x', y')) &= \max\{d(x, x'), \rho(y, y')\}. \end{aligned}$$

Both D_1 and D_∞ are equivalent metrics.

Exercise: $((x_n, y_n))_n \to (x, y)$ if and only if $(x_n)_n \xrightarrow{d} x$ and $(y_n)_n \xrightarrow{\rho} y$.

Series in a Normed Vector Space

Let $(V, \|\cdot\|)$ be a normed vector space. Consider a sequence $(v_k)_k$ of vectors.

$$\begin{aligned} s_1 &= v_1 \\ s_2 &= v_1 + v_2 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} v_k. \end{aligned}$$

If $s_n \to s$ in $(V, \|\cdot\|)$, meaning $\|s_n - s\| \to 0$, then we say the series $\sum_{k=1}^{\infty} v_k$ converges to s. We write

$$\sum_{k=1}^{\infty} v_k = s.$$

The series converges absolutely if

$$\sum_{k=1}^{\infty} \|v_k\|$$

converges in $\mathbb{R}$.

Proposition: Sequential Characterization of Closure

Let (X, d) be a metric space with $A \subseteq X$. $x \in \overline{A}$ if and only if $\exists (a_n)_n$ in A with $(a_n)_n \to X$.

In the forward direction, recall that $x \in \overline{A}$ if and only if $\forall \delta > 0$, $U(x, \delta) \cap A \neq \emptyset$. If $x \in \overline{A}$, then set $\varepsilon_n = 1/n$, and since $U(x, 1/n) \cap A \neq \emptyset$. Let $a_n \in U(x, 1/n) \cap A$. Then, $d(a_n, x) < 1/n \to 0$, meaning $a_n \to x$ and $a_n \in A$.

In the reverse direction, if $(a_n)_n \to x$ and $\varepsilon > 0$, $\exists N$ with $n \geq N \Rightarrow a_n \in U(x, \varepsilon) \cap A$. Thus, $x \in \overline{A}$.

Proposition: Sequential Characterization of Closed Sets

If (X, d) is a metric space, $A \subseteq X$, then the following are equivalent:

(i) A is closed.

(ii) Whenever $(a_n)_n$ in A with $(a_n)_n \xrightarrow{d} x$ in X, then $x \in A$.

Continuous Bounded Functions: $C([a, b]) \subseteq \ell_\infty([a, b])$ is closed under $\|\cdot\|_u$, since if $(f_n)_n \to f$ uniformly, and f_n is continuous, then f is continuous.

Sequence Closure: $c_0 \subseteq \ell_\infty$ is closed under $\|\cdot\|_u$. Let $(f_n)_n$ be a sequence

$$\begin{aligned} f_1 &= (f_1(1), f_1(2), \dots) \\ f_2 &= (f_2(1), f_2(2), \dots) \\ \lim_{k \to \infty} f_n(k) &= 0 && \forall n \end{aligned}$$

Suppose $(f_n)_n \xrightarrow{\|\cdot\|_\infty} f \in \ell_\infty$.

Let $\varepsilon > 0$. Then, $\exists n \in \mathbb{N}$ such that for $n \geq N$, $\|f - f_n\|_\infty < \varepsilon/2$. Also, $\lim_{k \to \infty} f_N(k) = 0$. Then, $\exists K \in \mathbb{N}$ such that for $k \geq K$, $|f_N(k)| < \varepsilon/2$. Thus, for $k \geq K$,

$$\begin{aligned} |f(k)| &= |f(k) - f_N(k) + f_N(k)| \\ &\leq |f(k) - f_N(k)| + |f_N(k)| \\ &\leq \|f - f_N\|_\infty + |f_N(k)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, $f \in c_0$.

Distance to a Set

Let (X, d) be a metric space, $A \subseteq X$. Then, $\text{dist}_A : X \to [0, \infty)$ is defined as

$$\text{dist}_A(x) = \inf_{a \in A} d(x, a).$$

(1) $\overline{A} = \{x \mid \text{dist}_A = 0\}$

(2) $\text{dist}_A(\cdot) = \text{dist}_{\overline{A}}(\cdot)$

(3) $|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y)$

Proof of (1): Let $x \in \overline{A}$. Then, $\exists (a_n)_n$ such that $(a_n)_n \to x$. Then, $d(a_n, x) \to 0$. Since $0 \leq \text{dist}_A(x) \leq d(x, a_n)$, $\text{dist}_A(x) = 0$.

Let x be such that $\text{dist}_A(x) = 0$. By the definition of inf, we construct a_n by finding $a_n \in U(x, 1/n) \cap A$. Thus, $d(a_n, x) \to 0$, meaning $(a_n)_n \to x$, so $x \in \overline{A}$.

Proof of (2): Exercise; use (1).

Proof of (3): For all $a \in A$,

$$\begin{aligned} \text{dist}_A(x) &\leq d(x, a) \\ &\leq d(x, y) + d(y, a). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{dist}_A(x) - d(x, y) &\leq d(y, a) \\ \text{dist}_A(x) - d(x, y) &\leq \inf_{a \in A} d(y, a) \\ &= \text{dist}_A(y) \\ \text{dist}_A(x) - \text{dist}_A(y) &\leq d(x, y). \end{aligned}$$

Similarly,

$$\text{dist}_A(y) - \text{dist}_A(x) \leq d(y, x) = d(x, y)$$

meaning

$$|\text{dist}_A(y) - \text{dist}_A(x)| \leq d(x, y).$$

Continuity

Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$

(1) is continuous at $x_0 \in X$ if

$$\begin{aligned} (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon \\ (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } x \in U_X(x_0, \delta) \Rightarrow f(x) \in U_Y(f(x_0), \varepsilon) \\ (\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } f(U_X(x_0, \delta)) \subseteq U_Y(f(x_0), \varepsilon). \end{aligned}$$

(2) is continuous if f is continuous at every $x_0 \in X$.

Proposition: Equivalent Continuity Criteria

Let $f : (X, d) \to (Y, \rho)$, $x_0 \in X$. The following are equivalent:

(1) f is continuous at x_0;

(2) $(\forall V \in \mathcal{N}_{f(x_0)})(U \in \mathcal{N}_{x_0})$ such that $f(U) \subseteq V$.

(3) $\forall (x_n)_n \to x_0$, $(f(x_n))_n \to f(x_0)$.

(1) $\Leftrightarrow$ (2): Clearly follows from definitions.

(1) $\Rightarrow$ (3): Let $(x_n)_n \to x_0$. Let $\varepsilon > 0$. Then, $\exists \delta > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \varepsilon$.

Thus, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x_0) < \delta$. So, if $n \geq N$, $d(x_n, x_0) < \delta$, implying $\rho(f(x_n), f(x_0)) < \varepsilon$. So, $(f(x_n))_n \to f(x_0)$.

(3) $\Rightarrow$ (1): Suppose toward contradiction that $\exists \varepsilon_0 > 0$ such that for $\delta = 1/n$ where $n \in \mathbb{N}$, $\exists (x_n)_n : d(x_n, x_0) < \delta$ and $\rho(f(x_n), f(x_0)) \geq \varepsilon_0$. Then, $(x_n)_n \to x_0$, but $f(x_n)_n \nrightarrow f(x_0)$. $\perp$

Proposition: Topological Criterion for Continuity

Let $f : (X, d) \to (Y, \rho)$. The following are equivalent:

(1) f is continuous.

(2) $\forall V \in \tau_Y$, $f^{-1}(V) \in \tau_X$.

(3) $\forall x \in X, \forall (x_n)_n \to x$, we have $(f(x_n))_n \to f(x)$.

Proof: Exercise.

Proposition: Composition of Functions

Let $(X, d) \xrightarrow{f} (Y, \rho) \xrightarrow{g} (Z, p)$. If f and g are continuous, then $g \circ f$ is continuous.

Proof: Exercise.

Uniform Continuity

Let $f : (X, d) \to (Y, \rho)$.

(1) f is uniformly continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } \forall x, x' \in X, d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$$

(2) f is Lipschitz if $\exists c > 0$ with

$$\rho(f(x), f(x')) \leq c d(x, x')$$

for all $x, x' \in X$.

(3) If $\rho(f(x), f(x')) = d(x, x')$, then f is an isometry. Isometries are always injective.

Exercise:

$$\text{Isometry} \Rightarrow \text{Lipschitz} \Rightarrow \text{Uniformly Continuous} \Rightarrow \text{Continuous}.$$

For example, $f(x) = x^2$ on $[0, \infty)$ is continuous but not uniformly continuous, and $\sqrt{x}$ on $[0, 1]$ is uniformly continuous but not Lipschitz.

If $(V, \|\cdot\|)$ is a normed space, we might want to care that the following operations are continuous:

- $a : V \times V \to V$, $a(v, w) = v + w$:

$$\begin{aligned}\|a(v,w) - a(v',w')\| &= \|v + w - (v' + w')\| \\ &= \|v - v' + w - w'\| \\ &\leq \|v - v'\| + \|w - w'\| \\ &= d(v, v') + d(w, w') \\ &= d_1((v,w),(v',w')),\end{aligned}$$

  meaning a is Lipschitz.

- $m : \mathbb{F} \times V \to V$, $m(\alpha, v) = \alpha v$;

$$\begin{aligned}\|m(\alpha, v) - m(\beta, w)\| &= \|\alpha v - \beta w\| \\ &= \|\alpha v - \alpha w + \alpha w - \beta w\| \\ &\leq |\alpha|\,\|v - w\| + |\alpha - \beta|\,\|w\|\end{aligned}$$

  If $(\alpha_n)_n \to \beta$ and $(v_n)_n \to w$, then

$$\begin{aligned}\|\alpha_n v_n - \beta w\| &\leq |\alpha_n|\,\|v_n - w\| + |\alpha_n - \beta|\,\|w\| \\ &\to 0.\end{aligned}$$

- $\|:\| V \to \mathbb{F}$:

$$|\|v\| - \|w\|| \leq \|v - w\|,$$

  meaning $\|\cdot\|$ is Lipschitz.

Let (X, d) be a metric space. Then, $\text{dist}_A : X \to [0, \infty)$, $\text{dist}_A(x) = \inf_{a \in A} d(x, a)$ is continuous. We had shown

$$|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y),$$

meaning dist_A is Lipschitz.

Proposition: Normal Property of Metric Spaces

Given $A, B \subseteq X$ with $A \cap B = \emptyset$, then $\exists U, V \in \tau_X$ with $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Proof: Set

$$f(x) = \frac{\text{dist}_A(x)}{\text{dist}_A(x) + \text{dist}_B(x)}.$$

Note that $\text{dist}_A(x) + \text{dist}_B(x) = 0$ if and only if $x \in \overline{A} = A$ and $x \in \overline{B} = B$. Therefore, the denominator in $f(x)$ is always positive.

Additionally, $f : X \to [0, 1]$ is continuous. Note that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Let $U = f^{-1}((-1/2, 1/2)) = f^{-1}([0, 1/2))$, and $V = f^{-1}((1/2, 3/2)) = f((1/2, 1])$. Obviously, $U \subseteq A$ and $V \subseteq B$, and $U \cap V = \emptyset$.

Proposition: Quotient Space

Let $(V, \|\cdot\|)$ be a normed space, and let $W \subseteq V$ be a closed subspace. Then, V/W is a normed space with

$$\begin{aligned} \|v + W\| &= \mathrm{dist}_W(v) \\ &= \inf_{w \in W} \|v - w\|. \end{aligned}$$

Proposition: Uniform Continuity of Linear Transformations

Let $T : V \to W$ be a linear transformation between two normed spaces. The following are equivalent:

(1) T is continuous at 0_V.

(2) T is continuous.

(3) T is uniformly continuous.

(4) T is Lipschitz.

(5) $\exists c \geq 0$ such that $\|T(v)\| \leq c\|v\|$ for all $v \in V$.

(6) $\|T\|_{\text{op}} = \sup_{\|v\| \leq 1} \|T(v)\| < \infty$. In other words, T is bounded linear.

Proof:

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1): Obvious.

(6) $\Rightarrow$ (5) Let $v \in V$. If $v = 0_V$, then $T(v) = 0_W$. Suppose $v \neq 0_V$. We know

$$\begin{aligned} \left\| T\left(\frac{v}{\|v\|}\right) \right\| &\leq \|T\|_{\text{op}} \\ \frac{1}{\|v\|} \|T(v)\| &\leq \|T\|_{\text{op}} \\ \|T(v)\| &\leq \|T\|_{\text{op}} \|v\|. \end{aligned}$$

Therefore, $c = \|T\|_{\text{op}}$.

(5) $\Rightarrow$ (6): We will have $\|T(v)\| \leq c$ for all $v \in B_V$. Thus, $\|T\|_{\text{op}} \leq c$ for such c.

(5) $\Rightarrow$ (4): Let $v, w \in V$. Then,

$$\begin{aligned} \|T(v) - T(w)\| &= \|T(v - w)\| \\ &\leq c\|v - w\|, \end{aligned}$$

meaning T is Lipschitz.

(1) $\Rightarrow$ (5): Let $\varepsilon = 1$. Then, $\exists \delta$ such that

$$T(U_V(0, \delta)) \subseteq U(T(0), 1).$$

Since T is linear,

$$T(U_V(0, \delta)) \subseteq U_W(0, 1).$$

Let $v \in V \neq 0_V$. We know $\frac{\delta v}{2\|v\|} \in U_V(0, \delta)$. Then,

$$\begin{aligned} \left\| T\left(\frac{\delta v}{2\|v\|}\right) \right\| &\leq 1, \\ \frac{\delta}{2\|v\|} \|T(v)\| &\leq 1 \\ \|T(v)\| &\leq \frac{2}{\delta} \|v\|. \end{aligned}$$

Set $c = \frac{2}{\delta}$. Clearly, $\|T(0)\| \leq \frac{2}{\delta}\|0\|$.

A corollary to this is that any linear map $T : \ell_p^n \to W$ for W a normed space is uniformly continuous.

Proposition: Continuous Functions on Dense Sets

Let (X, d), (Y, ρ) be metric spaces, and $A \subseteq X$ dense. If $f, g : X \to Y$ and $f(A) = g(A)$, then $f(X) = g(X)$.

Proof: Given $x \in X$, there exists $(a_n)_n \to x$. We know that $(g(a_n))_n \to g(x)$ and $(f(a_n))_n \to f(x)$. Since $f(a_n) = g(a_n)$ for all a_n, it is the case that $f(x) = g(x)$.

Morphisms in the Category of Metric Spaces

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$ a map.

(1) f is a homeomorphism if f is bijective, continuous, and has a continuous inverse. We write $X \cong Y$ are homeomorphic.

(2) f is a uniformism if f is bijective, uniformly continuous, and has a uniformly continuous inverse. We write $X \cong Y$ are uniformly isomorphic.

(3) f is a metric isomorphism if f is bijective, Lipschitz, and has a Lipschitz inverse. We write $X \cong Y$ are metrically isomorphic.

(4) f is an isometric isomorphism if f is bijective and isometric. We write $X \cong Y$ are isometrically isomorphic.

For example, $R \cong (-\pi/2, \pi/2)$ are homeomorphic (using $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$). However, $\mathbb{R}$ is not uniformly isomorphic to $(-\pi/2, \pi/2)$.

Suppose $f : (-\pi/2, \pi/2) \to \mathbb{R}$ is a uniformism. Let $(x_n)_n = \pi/2 - 1/n$. Then, $(x_n)_n$ is Cauchy. Therefore, $(f(x_n))_n$ is Cauchy. Since $\mathbb{R}$ is complete, $(f(x_n))_n \to y$ for some $y \in \mathbb{R}$. Then, $f^{-1}(f(x_n))_n \to f^{-1}(y)$, meaning $(x_n)_n \to f^{-1}(y) \in (-\pi/2, \pi/2)$. However, $(x_n)_n \to \pi/2 \notin (-\pi/2, \pi/2)$.

Completeness

Proposition: Weierstrass M-Test

Let V be a Banach space (complete normed vector space). Suppose $(v_k)_k$ is such that $\sum \|v_k\|$ is convergent. Then, $(s_n)_n = \sum_{k=1}^{n} v_k$ converges in V. Additionally,

$$\left\| \sum_{k=1}^{\infty} v_k \right\| \leq \sum_{k=1}^{\infty} \|v_k\| .$$

Proof: Let $s_n = \sum_{k=1}^{n} v_k$, and $t_n = \sum_{k=1}^{n} \|v_k\|$. Let $n > m$. Then,

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^{n} v_k \right\| \\ &\leq \sum_{k=m+1}^{n} \|v_k\| \\ &= |t_n - t_m|. \end{aligned}$$

Since $(t_n)_n$ converges, it is Cauchy, and thus s_n is Cauchy. Since V is complete, $(s_n)_n$ converges.

$$\begin{aligned}\|s_n\| &= \left\|\sum_{k=1}^{n} v_k\right\| \\ &\leq \sum_{k=1}^{n} \|v_k\| \\ &\leq \sum_{k=1}^{\infty} \|v_k\|.\end{aligned}$$

Let $n \to \infty$. Using the continuity of the norm, we get

$$\left\|\sum_{k=1}^{\infty} v_k\right\| \leq \sum_{k=1}^{\infty} \|v_k\|.$$

Proposition: Convergence in Hilbert Space

Let H be a Hilbert space (inner product space with a complete norm). Let $(e_n)_n$ be an orthonormal sequence in H. Let $(t_k)_k$ be a sequence in ℓ_2. Then, $\sum_{k=1}^{\infty} t_k e_k$ converges in H, but not absolutely.

Proof: Let $s_n = \sum_{k=1}^{n} t_k e_k$. For $n > m$,

$$\begin{aligned}\|s_n - s_m\|^2 &= \left\|\sum_{k=m+1}^{n} t_k e_k\right\|^2 \\ &= \sum_{k=m+1}^{n} \|t_k e_k\|^2 && \text{Pythagorean Theorem} \\ &= \sum_{k=m+1}^{n} |t_k|^2\end{aligned}$$

Since $(t_k)_k \in \ell_2$, we know that $(t_k)_k$ is convergent and thus Cauchy. Thus, $(s_n)_n$ is Cauchy.

Note that for $t_k = \frac{1}{k}$, $(t_k)_k$ is square-summable, but not summable in absolute value.

Exercise: Show that

$$\left\|\sum_{k=1}^{\infty} t_k e_k\right\|^2 = \sum_{k=1}^{\infty} |t_k|^2.$$

This result is known as Parseval's Theorem.

Extensions of Continuous Functions

Lemma: Cauchy Sequences in Uniformly Continuous Functions

Let $f : (X, d) \to (Y, \rho)$ be uniformly continuous. If $(x_n)_n$ is Cauchy, then $(f(x_n))_n$ is Cauchy.

Proof: Let $\varepsilon > 0$. Then, $\exists \delta > 0$ such that

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon.$$

Similarly, there exists $N \in \mathbb{N}$ such that for $p, q \geq N$, $d(x_p, x_q) < \delta$. So, for $p, q \geq N$, $d(f(x_p), f(x_q)) < \varepsilon$.

Remark: This is not true for continuous functions. For example, if $f(t) = 1/t$ on $(0, 1)$, $x_n = 1/n$ is Cauchy but not convergent.

Theorem: Extension on a Dense Subset

Let (X, d) be a metric space with $A \subseteq X$ dense. Suppose $f : A \to Y$ is uniformly continuous with (Y, ρ) complete. Then, $\exists!$ uniformly continuous extension, $\tilde{f} : X \to Y$ that agrees with f on A.

Proof: Let $x \in X$. Then, $\exists (a_n)_n \in A$ with $(a_n)_n \to x$. Therefore, $(a_n)_n$ is Cauchy, and since f is uniformly continuous, we know that $(f(a_n))_n$ is Cauchy. Thus, $\lim_{n\to\infty}(f(a_n))_n = \tilde{f}(x)$ exists.

To show $\tilde{f}$ is well-defined, suppose $(b_n)_n$ is another sequence in A with $(b_n)_n \to x$. Consider $(c_n)_n = (a_1, b_1, a_2, b_2, \dots)$. It must be the case that $(c_n)_n \to x$. Thus, $(f(c_n))_n$ converges to $y \in Y$. The subsequence of $(f(a_n))_n \to y$ and $(f(b_n))_n \to y$. So, we must have $\lim f(a_n) = \lim f(b_n)$.

Note that $\tilde{f}(a) = f(a)$ for all $a \in A$, by choosing the sequence $(a, a, a, \dots)$.

We claim that $\tilde{f}$ is uniformly continuous. Let $\varepsilon > 0$. We know $\exists \delta > 0$ such that for any $a, b \in A$, with $d(a, b) < \delta$, then $\rho(f(a), f(b)) < \varepsilon/2$. Now, let $x, x' \in X$ with $d(x, x') < \delta/4$. Find sequences $(a_n)_n \to x$ and $(b_n)_n \to x'$ with $(a_n)_n, (b_n)_n \in A$. Find N large such that $n \geq N$ implies $d(a_n, x) < \delta/4$ and $d(b_n, x') < \delta/4$. For $n \geq N$, we have

$$\begin{aligned} d(a_n, b_n) &\leq d(a_n, x) + d(x, x') + d(x', b_n) \\ &< \frac{3\delta}{4} \\ &< \delta \end{aligned}$$

Thus, for $n \geq N$, $\rho(f(a_n), f(b_n)) < \varepsilon/2$. By continuity of ρ, taking $n \to \infty$, we get $\rho(\tilde{f}(x), \tilde{f}(x')) < \varepsilon/2$. Therefore, we have $d(x, x') < \delta/4 \Rightarrow d(\tilde{f}(x), \tilde{f}(x')) < \varepsilon$. Therefore, $\tilde{f}$ is uniformly continuous.

Suppose $g : X \to Y$ is another continuous extension of f. Therefore, $g(a) = \tilde{f}(a)$ for all $a \in A$. However, A is dense. Therefore, $g = \tilde{f}$.

Completion of a Metric Space

Let (X, d) be a fixed metric space. A completion of X is a pair $((Z, \rho), i)$.

(i) (Z, ρ) is a complete metric space.

(ii) $i : X \to Z$ is an isometry.

(iii) $\overline{i(X)}^{\rho} = Z$.

For example, the completion of $(0, 1)$ is $(([0, 1], |\cdot|), i(t) = t)$.

Isometric Isomorphism of Completions

Given $((Z, \rho), i)$ and $((Z', \rho'), j)$ completions of X, then there exists a unique isometric isomorphism $\varphi : Z \to Z'$ such that the following diagram commutes.

$$\begin{tikzcd} & Z \arrow[dd, "\varphi"] \\ X \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & Z' \end{tikzcd}$$

Corollary: Isometric Map and Completion of Metric Space

If (X, d) is a metric space, and $i : (X, d) \to (Y, \rho)$ is an isometry into a complete metric space, then $\left((\overline{i(X)}, \rho), i\right)$ is the completion of X.

Theorem: Every Metric Space has a Completion

Consider the Banach space $(C_b(X), \|\cdot\|_u)$. We embed $X \hookrightarrow C_b(X)$ as follows. Fix $x_0 \in X$. Given $x \in X$, $i(x) = X \to \mathbb{F}$ where $i(x)(t) = d(t, x) - d(t, x_0)$.

Clearly, $i(x)$ is continuous for all x as the distance function is continuous. Also,

$$\begin{aligned} |i(x)(t)| &= |d(t, x) - d(t, x_0)| \\ &\leq d(x, x_0) \\ \|i(x)\|_u &\leq d(x, x_0). \end{aligned}$$

We need only show that $i(x)$ is an isometry.

$$\begin{aligned} \|i(x) - i(y)\|_u &= \sup_{t \in X} |i(x)(t) - i(y)(t)| \\ &= \sup_{t \in X} |d(t, x) - d(t, y)| \\ &= d(x, y). \end{aligned}$$

Nowhere Dense Sets

Let (X, d) be a metric space. Recall that a subset A if $(\overline{A})^\circ = \emptyset$. For example, $G = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ is nowhere dense.

Proposition: Equivalent Conditions for Nowhere Dense Sets

For a $A \subseteq X$, the following are equivalent:

(i) A is nowhere dense.

(ii) $\exists F \subseteq X$ closed with $F^\circ = \emptyset$, $A \subseteq F$.

(iii) $\exists U \subseteq X$ open and dense with $U \subseteq A^c$.

Proof:

(i) $\Rightarrow$ (ii): Take $F = \overline{A}$.

(ii) $\Rightarrow$ (i): $\overline{A} \subseteq \overline{F}$, so $\overline{A}^\circ \subseteq \overline{F}^\circ = \emptyset$

(ii) $\Rightarrow$ (iii): Take $U = F^c$. Note that $U = F^c \subseteq A^c$. Then, $\overline{U} = \overline{F^c} = (F^\circ)^c = X$. Therefore, U is dense and open, and U is contained in A^c.

(iii) $\Rightarrow$ (ii): Take $F = U^c$.

A point $x \in X$ is isolated if $\exists \varepsilon > 0$ such that $U(x, \varepsilon) = \{x\}$.

Proposition: Extension of Nowhere Dense Sets

Let (X, d) be a metric space.

(i) If $A \subseteq X$ is nowhere dense and $B \subseteq A$, then B is nowhere dense.

(ii) If $A \subseteq X$ is nowhere dense, then $\overline{A}$ is nowhere dense.

(iii) Let $A_1, \dots, A_n$ be nowhere dense. Then, $\bigcup A_i$ is nowhere dense.

(iv) If X has no isolated points, then every finite set is nowhere dense.

Proof:

(i) $B \subseteq A$ implies $\overline{B} \subseteq \overline{A}$, so $\overline{B}^{\circ} = \emptyset$, so B is nowhere dense.

(ii) If A is nowhere dense, then $\overline{\overline{A}}^{\circ} = \overline{A}^{\circ} = \emptyset$.

(iii) Let A_1 and A_2 be nowhere dense. By the alternate characterization, $U_1 \subseteq A_1^c$, where U_1 is open and dense. Similarly, $U_2 \subseteq A_2^c$, where U_2 is open and dense.

$$\begin{aligned}(A_1 \cup A_2)^c &= A_1^c \cap A_2^c \\ &\supseteq U_1 \cap U_2\end{aligned}$$

We know $U_1 \cap U_2$ is open. We claim that $U_1 \cap U_2$ is dense.

Let $x \in X$, $\varepsilon > 0$. We want to show that $U(x, \varepsilon) \cap (U_1 \cap U_2) \neq \emptyset$. Since U_1 is dense, we know $U_1 \cap U(x, \varepsilon) \neq \emptyset$. Let $z \in U_1 \cap U(x, \varepsilon)$. Therefore, $\exists \delta > 0$ such that $U(z, \delta) \subseteq U_1 \cap U(x, \varepsilon)$. Since U_2 is dense, $U(z, \delta) \cap U_2 \neq \emptyset$. Therefore, $\emptyset \neq U(z, \delta) \cap U_2 \subseteq U(x, \varepsilon) \cap (U_1 \cap U_2)$.

By induction, assuming $A_1 \cup \dots \cup A_{n-1}$ are nowhere dense, then $(A_1 \cup \dots \cup A_{n-1}) \cup A_n$ is nowhere dense.

(iv) Since X has no isolated points, $\{x\}$ is closed but not open. Therefore, $(\overline{\{x\}})^{\circ} = \emptyset$. Use (iii).

Remark: Note that $\mathbb{Q}$ is not nowhere dense, but $\mathbb{Q}$ is the countable union of nowhere dense sets.

Meager Sets

Let (X, d) be a metric space.

(i) $A \subseteq X$ is meager if A is the countable union of nowhere dense sets. Or, A is of the first category.

(ii) $B \subseteq X$ is called residual if B^c is meager.

Examples: $\mathbb{Q} \subseteq \mathbb{R}$ is meager, so $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$ is residual. $\mathbb{Z} \subseteq \mathbb{R}$ is meager, but $\mathbb{Z} \subseteq \mathbb{Z}$ is not meager.

Proposition: Extension of Meager Sets

(i) If A is meager, and $B \subseteq A$, then B is meager.

(ii) If A_k is meager for $k = 1, \dots,$ then A_k is meager.

(iii) If X has no isolated points, then every countable set is meager.

Proof:

(i) $A = \bigcup A_k$, with A_k nowhere dense. Then, $B = B \cap A = \bigcup B \cap A_k$.

(ii) Each A_k is meager, meaning $A_k = \bigcup A_{k_j}$ with A_{k_j} nowhere dense. Thus, $A = \bigcup A_k$ is the countable union of A_{k_j}. Thus, A is meager.

(iii) Since singleton sets are nowhere dense, we write the countable set as the union of singleton sets.

Proposition: Cantor's Intersection Theorem

Let (X, d) be a complete metric space, and $F_1 \supseteq F_2 \supseteq \cdots$ be a sequence of closed, nonempty sets with $(\text{diam}(F_n))_n \to 0$. Then, $\bigcap F_n = \{x\}$ for some $x \in X$.

Proof: Let $x_n \in F_n$ for $n \geq 1$. Note that $(x_n)_n$ is Cauchy. For $\varepsilon > 0$, let N be large such that $n \geq N \Rightarrow \text{diam}(F_n) < \varepsilon$. For $m, n \geq N$, $d(x_n, x_m) < \varepsilon$ because $x_n, x_m \in F_N$. Therefore, $(x_n)_n \to x$ for $x \in X$.

We claim that $\{x\} = \bigcap F_n$. To see this, fix $m \in \mathbb{N}$, and consider $(x_{m+k})_k \in F_m$. The tail sequence $(x_{m+k})_k \to x$. Since F_m is closed, we know $x \in F_m$. Therefore, since m is arbitrary, $x \in \bigcap F_n$.

Now, suppose $\exists x, x' \in \bigcap F_n$ distinct. Then, $d(x, x') > 0$. However, $\exists N \in \mathbb{N}$ large with $\text{diam}(F_N) < d(x, x')$. However, $x, x' \in F_N$, which is a contradiction. Therefore, $\bigcap F_n = \{x\}$.

Baire's Theorem

Let (X, d) be a complete metric space.

(i) If $\{V_k\}_{k\geq 1}$ is a countable family of open and dense subsets, then $\bigcap V_k$ is dense.

(ii) X is not meager.

Proof:

(i) Let U_0 be any open ball. Since V_1 is open and dense, $U_0 \cap V_1$ is open and nonempty. So, $\exists U_1$ with $B_1 = \overline{U_1} \subseteq U_0 \cap V_1$. We can assure that $\text{diam}(B_1) < 1$.

Consider $U_1 \cap V_2$. Since V_2 is dense and open, $U_1 \cap V_2$ is open and nomempty. Therefore, there must be $B_2 = \overline{U_2} \subseteq U_1 \cap V_2$. We can insure that $\text{diam}(B_2) < 1/2$.

Now, with $U_2 \cap V_3$, we have $B_3 = \overline{U_3} \subseteq U_2 \cap V_3$, with $\text{diam}(B_3) < 1/3$.

Inductively, we have $U_1, \dots, U_{n-1}$ and $B_1, \dots, B_{n-1}$, we see that $U_{n-1} \cap V_n$ is open and nonempty, so we have U_n with $B_n = \overline{U_n} \subseteq U_{n-1} \cap V_n$, with $\text{diam}(B_n) < 1/n$.

Observe that we have $B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \cdots$. In particular, $\{B_n\}_{n\geq 1}$ is a nested sequence of closed sets with $\text{diam}(B_n) \to 0$. Therefore, $\bigcap B_n = \{x\}$.

We claim that $x \in U_0 \cap (\bigcap V_k)$. Note that $B_n \subseteq U_{n-1} \cap V_n \subseteq V_n$, Therefore, $x \in \bigcap B_n$ implies $x \in \bigcap V_n$. Also, $x \in B_1 = \overline{U_1} \subseteq U_0 \cap V_n \subseteq U_0$. Therefore, $\bigcap V_k$ is dense.

(ii) Suppose $X = \bigcup A_k$ for A_k nowhere dense. Therefore, $\exists V_k$ open and dense with $V_k \subseteq A_k^c$. Then,

$$\begin{aligned}
\emptyset &= X^c \\
&= \left(\bigcup A_k\right)^c \\
&= \bigcap A_k^c \\
&\supseteq \bigcap V_k.
\end{aligned}$$

Therefore, by the previous result, $\bigcap V_k$ is open and dense, which is a contradiction. Therefore, X is not meager.

Question: Is $\mathbb{Q} \subseteq \mathbb{R}$ meager? Yes, $\mathbb{Q}$ is the countable union of singleton sets. Is $\mathbb{R} \setminus \mathbb{Q}$ meager? The answer is no — otherwise, we would write $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be a union of meager sets, but $\mathbb{R}$ is complete.

Applying Baire's Theorem

Let (X, d) be a metric space.

(i) $G \subseteq X$ is a G_δ-set if

$$G = \bigcap_{k \geq 1} V_k$$

with V_k open.

(ii) $F \subseteq X$ is a F_σ-set if

$$F = \bigcup_{k \geq 1} C_k$$

with C_k closed.

For example, $\mathbb{Q} \subseteq \mathbb{R}$ is F_σ, since $\mathbb{Q}$ is the countable union of singleton sets (which are closed in $\mathbb{R}$). It can be shown that A is F_σ if and only if A^c is G_δ.

We claim that $\mathbb{Q}$ is not G_δ.

Proof: If $\mathbb{Q}$ is G_δ, then $\mathbb{R} \setminus \mathbb{Q}$ is F_σ, so

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup F_k$$

for F_k closed. Thus,

$$\begin{aligned}\mathbb{R} &= \mathbb{Q} \setminus \mathbb{R} \setminus \mathbb{Q} \\ &= \bigcup \{q_k\} \cup \bigcup F_k.\end{aligned}$$

Therefore, $\mathbb{R}$ is the countable union of closed sets. Since $\mathbb{R}$ is complete, by Baire's Theorem, we must have $\{q_k\}^\circ \neq \emptyset$, or that $F_k^\circ \neq \emptyset$ for some k. However, $\{q_k\}^\circ = \emptyset$, and $F_k^\circ = \emptyset$ since $F_k \subseteq \mathbb{R} \setminus \mathbb{Q}$, and $\mathbb{R} \setminus \mathbb{Q}$ cannot contain an interval. Therefore, $\mathbb{Q}$ is not G_δ

Let (X, d) be a metric space. If A is closed, then A is G_δ.

Proof: Recall $\text{dist}_A : X \to \mathbb{R}$ is continuous. Therefore, $\text{dist}_A^{-1}((-1/n, 1/n)) = \{x \mid \text{dist}_A(x) < 1/n\}$ is open. Recall that $x \in A$ if and only if $\text{dist}_A(x) = 0$.

Therefore, we can write

$$A = \bigcap_{n \geq 1} \{x \mid \text{dist}_A(x) < 1/n\}.$$

Therefore, A is G_δ.

It follows that if A is open, then A is F_σ.

Theorem: Set of Continuities

Let $f : (X, d) \to (Y, \rho)$ be a map. Then, $C_f := \{x \in X \mid f \text{ is continuous at } x\}$ is a G_δ set.

Oscillation of a Function

Let $f : (X, d) \to (Y, \rho)$. Fix $x_0 \in X$. The oscillation $\omega_f(x_0) = \inf_{\delta>0}\text{diam}(f(U(x, \delta)))$, or

$$\omega_f(x_0) = \inf_{\delta>0}\left(\sup_{x,x'\in U(x,\delta)} \rho(f(x), f(x'))\right).$$

Note that $\omega_f(x_0) \in [0, \infty]$.

(i) f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

(ii) Given $c > 0$, $\{x \mid \omega_f(x_0) < c\} \subseteq X$ is open.

Proof:

(i) Suppose f is continuous at x_0. Let $\varepsilon > 0$. Then, $\exists\delta > 0$ such that $d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon/2$. Therefore,

$$\text{diam}(f(U(x_0, \delta))) \leq \varepsilon,$$

since for $x, x' \in U(x_0, \delta)$, we have

$$\begin{aligned}\rho(f(x), f(x')) &\leq \rho(f(x), f(x_0)) + \rho(f(x_0), f(x')) \\ &< \varepsilon.\end{aligned}$$

In particular, $\omega(f(x_0)) \leq \varepsilon$. Since ε was arbitrary, we have $\omega_f(x_0) = 0$.

Suppose $\omega_f(x_0) = 0$. Let $\varepsilon > 0$. By the property of infimum, then $\exists\delta > 0$ such that

$$\text{diam}(f(U(x_0, \delta))) < \varepsilon.$$

In particular, if $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \varepsilon$. Thus, f is continuous at x_0.

(ii) Let $V = \{x \mid \omega_f(x_0) < c\}$. Let $x_0 \in V$. Since $x_0 \in V$, $\omega_f(x_0) < c$. By the property of infimum, $\exists\delta > 0$ such that $\text{diam}(f(U(x_0, \delta))) < c$. Let $\varepsilon = \delta/2$. We claim that $U(x_0, \varepsilon) \subseteq V$.

Let $z \in U(x_0, \varepsilon)$. Note that $U(z, \delta/2) \subseteq U(x_0, \delta)$. Therefore, $f(U(z, \delta/2)) \subseteq f(U(x_0, \delta))$. Thus, $\text{diam}(f(U(z, \delta))) \leq \text{diam}(f(U(x_0, \delta))) < c$.

By property of oscillation, $\omega_f(z) < c$. So, $U(x_0, \varepsilon) \subseteq V$.

Proof of Theorem:

$$\begin{aligned}C_f &= \{x \mid f \text{ is continuous at } x\} \\ &= \bigcap_{n\geq 1} \underbrace{\{x \mid \omega_f(x) < 1/n\}}_{\text{open sets}}\end{aligned}$$

meaning $x \in C_f \leftrightarrow \omega_f(x) = 0 \leftrightarrow \omega_f(x) < 1/n$ for all n.

Applying Set of Continuities

There does exist a function continuous at every irrational point and discontinuous at every rational point. Recall from Real Analysis that such f is

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

However, there does not exist $f : \mathbb{R} \to \mathbb{R}$ with $C_f = \mathbb{Q}$, since the set of continuities is always a G_δ set.

Nowhere Differentiable Functions

Does there exist a function $f : [0,1] \to \mathbb{R}$ such that f is continuous on $[0,1]$ but differentiable nowhere? The answer is yes.

$$f(x) = \sum_{n\geq 1} a^n \cos(b^n x),$$

where $0 < a < 1$ and $ab > 1$ is such a function. This is known as the Weierstrass function.

Such functions are not rare at all.

In the complete normed vector space $X = (C[0,1], \|\cdot\|_u)$, $\{f \in X \mid f \text{ differentiable nowhere}\}$ is the complement of a meager set (meaning it is topologically "big").

Compactness

Compactness can best be analogized to finite dimensionality in a metric space.

Let (X, d) be a metric space, and let $K \subseteq X$.

(1) A cover for K is a family of subsets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathcal{P}(X)$ with $K \subseteq \bigcup U_i$.

The cover $\mathcal{U}$ is called an open cover if each $U_i \subseteq X$ is open. The cover $\mathcal{U}$ is called finite if I is finite. If $\mathcal{U}$ is a cover for K, a subcover of $\mathcal{U}$ is a subfamily $\mathcal{V} = \{U_i\}_{i\in J}$, with $J \subseteq I$, and $K \subseteq \bigcup_{j\in J} U_j$.

(2) K is called compact if every open cover of K admits a finite subcover. If $\{U_i\}_{i\in I}$ is any family that covers K, then there exists a finite $F \subseteq I$ such that $\{U_i\}_{i\in F}$ covers K.

For example, the set $(0,1] \subseteq \mathbb{R}$ is not compact, because

$$(0,1] \subseteq \bigcup_{n\in\mathbb{N}} (1/n, 3/2)$$

does not admit a finite subcover.

Any finite set is compact.

A discrete metric space is X is compact if and only if X is finite.

Let (X, d) be a metric space, and $Y \subseteq X$. Let $K \subseteq Y$; K is compact in X if and only if K is compact in Y. This can be shown by taking the relative topology of Y on every open cover of K in X.

Proposition: Properties of Compactness

Let (X, d) be a metric space.

(1) If $K \subseteq X$ is compact, then K is closed and bounded.

(2) If X is a compact metric space, and $K \subseteq X$ is closed, then K is compact.

Proof of (2): Let $K \subseteq \bigcup U_i$, with $U_i \subseteq X$ open. Then, $X = (X \setminus K) \cup \left(\bigcup_{i\in I} U_i\right)$. This is an open cover for X, meaning it admits a finite subcover $F \subseteq I$ such that $X = (X \setminus K) \cup \bigcup_{i\in F} U_i$. Clearly, $K \subseteq \bigcup_{i\in F} U_i$. Thus, K is compact.

Proof of (1): Let $K \subseteq X$ be compact. Then,

$$K \subseteq \bigcup_{x \in K} \bigcup U(x, 1).$$

Since K is compact, there exist $\{x_1, \dots, x_n\}$ with $K \subseteq \bigcup_{j=1}^{n} U(x_j, 1)$. Let $c = \max d(x_i, x_j)$. If $x, y \in K$, then $x \in U(x_i, 1)$ and $y \in U(x_j, 1)$ for some x_i, x_j. Then,

$$\begin{aligned} d(x, y) &\leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \\ &< 1 + c + 1 = 2 + c. \end{aligned}$$

Thus, $\operatorname{diam}(K) < \infty$.

We will show that K^c is open. Let $x_0 \notin K$. For each $x \in K$, there exist $\delta_x > 0$ with $U(x, \delta_x) \cap U(x_0, \delta_x) = \emptyset$. Then,

$$K \subseteq \bigcup_{x \in K} U(x, \delta_x).$$

Since K is compact, there exist $\{x_1, \dots, x_n\}$ with $K \subseteq \bigcup U(x_j, \delta_{x_j})$. Let $\delta = \min\{\delta_{x_j}\} > 0$. Then, $U(x_0, \delta) \subseteq K^c$.

Proposition: Compactness and Intersections of Closed Sets

Let (X, d) be a metric space. The following are equivalent.

(1) X is compact;

(2) If $\{C_i\}_{i \in I}$ is a family of closed sets with the finite intersection property (i.e., the intersection of finitely many elements of $\{C_i\}$ is non-empty), then $\bigcap_{i \in I} C_i \neq \emptyset$.

Proposition: Separability of Compact Metric Spaces

Let (X, d) be a compact metric space. Then, (X, d) is separable.

Proof: For fixed $n \geq 1$, consider the cover

$$X = \bigcup U(x, 1/n).$$

By compactness, there exist $\{x_{n,1}, \dots, x_{n,m_n}\}$ with

$$X = \bigcup_{j=1}^{m_n} U(x_{n,j}, 1/n).$$

Let $S = \{x_{n,j} \mid n \in \mathbb{N}, 1 \leq j \leq m_n\}$. Then, S is countable.

Let $x \in X$, $\varepsilon > 0$. Let N be large such that $N^{-1} < \varepsilon$. So,

$$x \in \bigcup_{j=1}^{m_N} U(x_{N,j}, 1/N),$$

so $x \in U(x_{N,j}, 1/N)$ for some j, whence $d(x, x_{N,j}) < 1/N < \varepsilon$, so $x_{N,j} \in U(x, \varepsilon)$. So, $\overline{S} = X$.

Proposition: Sequential Compactness

Let (X, d) be a metric space, $K \subseteq X$. We say K is sequentially compact if every sequence in K admits a convergent subsequence in K.

From Bolzano-Weierstrass, we know that $[a, b] \subseteq \mathbb{R}$ is sequentially compact.

If K is compact, then K is sequentially compact.

Proof: Let $(x_k)_k \in K$. Let $C_0 = \overline{\{x_1, x_2, \dots\}}$, $C_1 = \overline{\{x_2, x_3, \dots\}}$, etc. such that $C_n = \overline{\{x_{n+1}, x_{n+2}, \dots\}}$.

Observe that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$. Additionally, $\{C_n\}$ has the finite intersection property. Since K is compact, the previous proposition states that $\bigcap C_n \neq \emptyset$. Let $x \in \bigcap C_n$.

$x \in C_1$, meaning $\exists k_1 > 1$ with $d(x, x_{k_1}) < 1$. $x \in C_{k_1}$, meaning $\exists k_2 > k_1$ with $d(x, x_{k_2}) < 1/2$. $x \in C_{k_2}$, meaning $\exists k_3 > k_2$ with $d(x, x_{k_3}) < 1/3$. Continuing, we have $(x_{k_j})_j \in K$ with $d(x, x_{k_j}) < 1/j$. Thus, $(x_{k_j})_j \to x$.

If (X, d) is sequentially compact, then X is complete.

Lemma: If $(x_n)_n$ is Cauchy, and $(x_n)_n$ admits a convergent subsequence, then $(x_n)_n$ is convergent.

Proof of Lemma: Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for $p, q \geq N$, $d(x_p, x_q) < \varepsilon/2$.

Also, suppose $(x_{n_k})_k \to x$. Then, $\exists K \in \mathbb{N}$ large such that for $k \geq K$, $d(x_{n_k}, x) < \varepsilon/2$.

Therefore, for $n \geq N$, find $k \geq \max\{N, K\}$, we have

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Proof: If (X, d) is sequentially compact, for $(x_n)_n$ a Cauchy sequence in (X, d), we have that $(x_n)_n$ admits a convergent subsequence. Then, we use the lemma.

Total Boundedness

Let (X, d) be a metric space. $K \subseteq X$ is totally bounded if $\forall \delta > 0$, $\exists x_1, \dots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} U(x_i, \delta)$.

Exercise: If K is totally bounded, then K is bounded. If $L \subseteq K$, and K is totally bounded, then L is totally bounded.

Sequential Compactness and Total Boundedness

Let (X, d) be a metric space. Let $K \subseteq X$ be sequentially compact. Then, K is totally bounded.

Proof: Suppose K is not totally bounded. Then, $\exists \delta_0 > 0$ such that $K \nsubseteq \bigcup_{x \in F} U(x, \delta_0)$ for any finite F.

Let $x_1 \in K$. Since $K \nsubseteq U(x_1, \delta_0)$, so let $x_2 \in K \setminus U(x_1, \delta_0)$. Since $K \nsubseteq U(x_1, \delta_0) \cup U(x_2, \delta_0)$, let $x_3 \in K \setminus (U(x_1, \delta_0) \cup U(x_2, \delta_0))$. Continuing, we find $x_n \in K \setminus \bigcup_{j=1}^{n-1} U(x_j, \delta_0)$.

Thus, we have a sequence $(x_n)_n$. By sequential compactness, $(x_n)_n$ admits $(x_{n_k})_k \to x \in K$. Since $(x_{n_k})_k$ is convergent, $(x_{n_k})_k$ is Cauchy. But, $d(x_p, x_q) \geq \delta_0$, since, without loss of generality, for $p > q$, $x_p \notin U(x_q, \delta_0)$. $\perp$

Corollary: Compact Subsets of Real Numbers

If $K \subseteq \mathbb{R}$ is compact, $\sup K \in K$ and $\inf K \in K$.

Proof: We can always construct sequences $(x_n)_n \to \sup K$ and $(y_n)_n \to \inf K$ in K. Since $\sup K < \infty$ and $\inf K < \infty$, since K is compact, and thus bounded.

Since K is also closed, $\sup K \in K$ and $\inf K \in K$.

Theorem: Equivalence of Compactness Definitions

Let (X, d) be a metric space. The following are equivalent.

(1) X is compact.

(2) X is sequentially compact.

(3) X is complete and totally bounded.

Proof: We proved that $(1) \Rightarrow (2) \Rightarrow (3)$. We will now prove $(3) \Rightarrow (1)$.

Suppose $\mathcal{V}$ is an open cover of X that fails to admit a finite subcover. Let $\varepsilon = 1$. Since X is totally bounded $X = \bigcup_{j=1}^{m_1} U_{1,j}$, where $U_{1,j}$ are open balls of radius 1.

There must be some open ball among the $U_{1,j}$ not covered by finitely many members of $\mathcal{V}$. Call this ball $U(x_1, 1)$. Let $\varepsilon = 1/2$. By total boundedness, $X = \bigcup_{j=1}^{m_2} U_{2,j}$, where $U_{2,j}$ are open balls of radius $1/2$. Then, $U(x_1, 1) = \bigcup (U(x_1, 1) \cap U_{2,j})$. So, there must be an open ball of radius $1/2$, $U(x_2, 1/2)$, such that $U(x_1, 1) \cap U(x_2, 1/2)$ cannot be covered by finitely many members of $\mathcal{V}$.

Continuing, we have a sequence $(x_n)_n$, where $F_n = U(x_1, 1) \cap U(x_2, 1/2) \cap \cdots \cap U(x_n, 1/n)$ cannot be covered by finitely many members of $\mathcal{V}$.

Let $C_n = \overline{F_n}$. Notice that $F_1 \supseteq F_2 \supseteq \dots$, meaning $C_1 \supseteq C_2 \supseteq \dots$. We see that $\operatorname{diam}(C_n) = \operatorname{diam}(F_n) \leq 2/n$. Applying Cantor's intersection theorem, we have $\bigcap C_n = \{x\}$.

Since $\mathcal{V}$ is an open cover, locate $V \in \mathcal{V}$ such that $x \in V$. Since V is open, there exists $\varepsilon > 0$ such that $U(x, \varepsilon) \subseteq V$. Choose N large such that $2/N < \varepsilon$. Since $x \in C_N$, $d(z, x) \leq 2/N < \varepsilon$ for all $z \in C_N$, meaning $F_N \subseteq C_N \subseteq U(x, \varepsilon) \subseteq V$.

Therefore, $\{V\}$ is a cover for F_N. $\perp$

Proposition: Multi-dimensional Bolzano-Weierstrass Theorem

Let $\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] = \prod_{j=1}^{d} [a_j, b_j] \subseteq \ell_p^d$. Then, $\mathcal{R}$ is sequentially compact, so $\mathcal{R}$ is compact.

Proof: The proof in $\mathbb{R}^d$ works similarly to the proof in $\mathbb{R}^2$. Consider $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_y : \mathbb{R}^2 \to \mathbb{R}$. We saw that $(v_n)_n \to v$ in ℓ_p^2 if and only if $(\pi_x(v_n))_n \to \pi_x(v)$ and $(\pi_y(v_n))_n \to \pi_y(v)$.

If $(v_n)_n \in \mathcal{R}$, then $(\pi_x(v_n))_n \in [a_1, b_1]$. By Bolzano-Weierstrass, there is a convergent subsequence $(\pi_x(v_{n_k}))_k \to x \in [a_1, b_1]$.

Now, consider $(\pi_y(v_{n_k}))_k \in [a_2, b_2]$. By Bolzano-Weierstrass, there is a convergent subsequence $\left(\pi_y(v_{n_{k_j}})\right)_j \to y \in [a_2, b_2]$. Thus, $(v_{n_{k_j}})_j \to (x, y)$ in $\mathcal{R}$.

Heine-Borel Theorem

Let $K \subseteq \mathbb{R}^d$. The following are equivalent:

(i) K is compact;

(ii) K is sequentially compact;

(iii) K is closed and bounded.

Proof: We have (i) $\Leftrightarrow$ (ii), and (i) $\Rightarrow$ (iii). We will show (iii) $\Rightarrow$ (ii).

> If K is bounded, then $K \subseteq \mathcal{R} = \prod_{j=1}^{d} [a_j, b_j]$. Let $(v_n)_n$ be a sequence in K. By the previous proposition, there exists a subsequence $(v_{n_k})_k \to v \in \mathcal{R}$. Since K is closed, $v \in K$. Therefore, K is sequentially compact.

There are many examples of closed and bounded sets that are not compact (in infinite-dimensional vector spaces).

For example, in $\ell_1 = \left\{ a = (a_k)_k \mid \sum_{k=1}^{\infty} |a_k| < \infty \right\}$, we have $e_n = (0, 0, \dots, 0, 1, 0, \dots)$, with 1 at the nth coordinate. For the sequence $(e_n)_n$, $\|e_k\|_1 = 1$ for all e_k, so $(e_n)_n \in B_{\ell_1}$, which is closed and bounded. Observe that $\|e_n - e_m\| = 2$ for all $m \neq n$, so there does not exist a convergent subsequence. Thus, ℓ_1 is not sequentially compact.

Remark: We will show that for a normed space, $(V, \|\cdot\|)$, B_V is compact if and only if $\dim(V) < \infty$.

Proposition: Continuous Image of Compact Sets

If $f : (X, d) \to (Y, \rho)$ is continuous, and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

Proof: Let $\bigcup_{i \in I} V_i$ be an open cover for $f(K)$, where $V_i \subseteq Y$ open. Taking the preimage, we have

$$\begin{aligned} K &\subseteq f^{-1}(f(K)) \\ &\subseteq f^{-1}\left(\bigcup_{i \in I} V_i\right) \\ &= \bigcup_{i \in I} f^{-1}(V_i) \end{aligned}$$

> since f is continuous, $f^{-1}(V_i) \subseteq X$ are open. By compactness, there exists $F \subseteq I$ finite such that
>
> $$K \subseteq \bigcup_{i \in F} f^{-1}(V_i).$$
>
> Taking the image, we have
>
> $$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i \in F} f^{-1}(V_i)\right) \\ &= \bigcup_{i \in F} f(f^{-1}(V_i)) \\ &= \bigcup_{i \in F} V_i. \end{aligned}$$
>
> Thus, $f(K)$ has a finite subcover.

Corollary: Compactness under Topologically Equivalent Metrics

Let d_1 and d_2 be topologically equivalent ($\text{id}_X : (X, d_1) \to (X, d_2)$ is a homeomorphism). Then, $K \subseteq X$ is d_1-compact if and only if K is d_2-compact.

Corollary: Heine-Borel Theorem Extension

For $K \subseteq \ell_p^n$, K is compact if and only if K is closed and bounded.

Extreme Value Theorem

Let (X, d) be a metric space, $K \subseteq X$ compact, and $f : X \to \mathbb{R}$ continuous. Then, $\sup_{x \in X} f(x) = f(x_M)$ and $\inf_{x \in X} f(x) = f(x_m)$ for some $x_M, x_m \in K$.

Proof: We know that $f(K) \subseteq \mathbb{R}$ is compact. Then, $\inf f(K)$ and $\sup f(K)$ are elements of $f(K)$.

Proposition: Compactness of Closed Unit Ball

Let V be a finite-dimensional vector space over $\mathbb{F}$.

(1) All norms on V are equivalent.

(2) For any norm, $\|\cdot\|$ on V, $B_{(V, \|\cdot\|)} = \{v \in V \mid \|v\| \leq 1\}$ is compact.

Proof of (1): Let $\{v_1, \dots, v_n\}$ be a linear basis for V. Define

$$\left\| \sum_{j=1}^{n} t_j v_j \right\|_1 = \sum_{j=1}^{n} |t_j|.$$

This is a norm on V.

Then, $\varphi : \ell_1^n \to V$

$$\varphi \left(\sum_{j=1}^{n} t_j e_j \right) = \sum_{j=1}^{n} t_j v_j$$

is a linear isometric isomorphism. Since $B_{\ell_1^n}$ is compact, so too is $\varphi(B_{\ell_1^n})$, so $B_{(V, \|\cdot\|)}$ is compact.

Then, $S_1 := \{v \in V \mid \|v\|_1 = 1\}$ is compact since $S_1 \subseteq B_{(V, \|\cdot\|)}$ is closed.

Let $\|\cdot\|$ be any norm on V. We will show that $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. Note that

$$\begin{aligned}
\left\| \sum_{j=1}^{n} t_j v_j \right\| &\leq \sum_{j=1}^{n} |t_j| \, \|v_j\| \\
&\leq c \sum_{j=1}^{n} |t_j| \\
&= c \left\| \sum_{j=1}^{n} t_j v_j \right\|_1
\end{aligned}$$

where $c = \max \|v_j\|$. Consider $g : (V, \|\cdot\|_1) \to \mathbb{R}$, with $g(v) = \|v\|$.

$$\begin{aligned} |g(v) - g(w)| &= |\, \|v\| - \|w\|\, | \\ &\leq \|v - w\| \\ &\leq c \|v - w\|_1 \end{aligned}$$

so g is Lipschitz, and thus continuous. S_1 is compact in $(V, \|\cdot\|)$, so by the extreme value theorem, $\inf_{v \in S_1} g(v) = g(v_0) = \|v_0\|$ for some $v_0 \in S_1$. Note that $D := \|v_0\| > 0$, else $v_0 = 0$. Thus, $g(v) \geq D$ for all $v \in S_1$

$$\|v\| \geq D \qquad \forall v \in S_1$$

Let $0 \neq v$. Then,

$$\begin{aligned} \frac{v}{\|v\|_1} &\in S_1 \\ \left\| \frac{v}{\|v\|_1} \right\| &\geq D \end{aligned}$$

so

$$\|v\| \geq D \|v\|_1 .$$

Therefore, we have $\|v\|_1 \leq \frac{1}{D} \|v\|$. Thus, any two norms on V are equivalent.

Proof of (2): Exercise.

Corollary: Finite-Dimensional Subspaces

Let V be a normed space, and $W \subseteq V$ finite-dimensional. Then, $W \subseteq V$ is closed.

Proof: We know there is a linear uniformism $\varphi : W \to \ell_1^n$, for $\dim(W) = n$. If $(w_n)_n \to v \in V$, where $(w_n)_n \in W$, then $(w_n)_n$ is Cauchy. Therefore, $(\varphi(w_n))_n$ is Cauchy in ℓ_1^n. Since ℓ_1^n is complete, $(\varphi(w_n))_n \to z \in \ell_1^n$. Since φ^{-1} is uniformly continuous, $(w_n)_n = \left(\varphi^{-1}(\varphi(w_n))\right)_n \to \varphi^{-1}(z) \in W$. Thus, $\varphi^{-1}(z) = v$, so $v \in W$.

Proposition: Uncountable Basis of Banach Space

If V is an infinite-dimensional Banach space, then $\dim(V)$ is uncountable.

Proof: Let $\{e_n\}$ be a linearly independent set. Let $W_n = \operatorname{span}\{e_1, \dots, e_n\}$. So, W_n is closed, and $W_n \neq V$. We can see that $W_1 \subseteq W_2 \subseteq \cdots$.

We claim that $W_n^\circ = \emptyset$. Suppose $\exists U(x, \varepsilon) \subseteq W_n$ for some $\varepsilon > 0$. Given any $v \in V$ with $v \neq 0$, we take $\frac{\varepsilon}{2} \frac{v}{\|v\|} + x \in W_n$. Thus, we have $\frac{\varepsilon}{2} \frac{v}{\|v\|} \in W_n$, so $v \in W_n$, meaning $V \subseteq W_n$.

By Baire's Theorem, $\bigcup W_n \neq V$.

Proposition: Compact Unit Ball and Finite Dimensions

Let V be a normed space, and $B_V := \{v \mid \|v\| \leq 1\}$. The following are equivalent:

(i) B_V is compact;

(ii) $\dim(V) < \infty$.

Riesz's Lemma: Let V be a normed space, and W a proper closed subspace. For every $t \in (0,1)$, there exists $v_t \in V$ with $\|v_t\| = 1$ and $\text{dist}_W(v_t) \geq t$.

Proof of Riesz's Lemma: Find $v_0 \in V \setminus W$. We know $\text{dist}_W(v_0) := \delta > 0$. Recall that $\text{dist}_W(v_0) = \inf_{w\in W} \|v_0 - w\|$. Note that $t\delta < \delta$. So, $\delta < \frac{\delta}{t}$. Find $w_0 \in W$ with $\delta \leq \|v_0 - w_0\| < \frac{\delta}{t}$. Let $v_t = \frac{v_0 - w_0}{\|v_0 - w_0\|}$. Then, $\|v_t\| = 1$. We claim that v_t satisfies the lemma.

If $w \in W$ arbitrary, then

$$\begin{aligned}
\|v_t - w\| &= \left\| \frac{v_0 - w_0}{\|v_0 - w_0\|} - w \right\| \\
&= \frac{1}{\|v_0 - w_0\|} \left\| v_0 - \underbrace{(w_0 + w\|v_0 - w_0\|)}_{\in W} \right\| \\
&> \frac{t}{\delta} \cdot \delta \\
&= t.
\end{aligned}$$

Thus, $\text{dist}_W(v_t) \geq t$.

Proof: To show (i) $\Rightarrow$ (ii), we need Riesz's Lemma. Let B_V be compact. Suppose toward contradiction that $\dim(V) = \infty$.

Choose $v_1 \in V$ with $\|v_1\| = 1$. Let $W_1 = \text{span}\{v_1\} \subset V$. Then, W is closed and proper, meaning $\exists v_2 \in V$ with $\|v_2\| = 1$ with $\text{dist}_{W_1}(v_2) \geq 1/2$. Let $W_2 = \text{span}\{v_1, v_2\}$. Then, W_2 is a proper, closed subspace, meaning $\exists v_3 \in V$ with $\|v_3\| = 1$ and $\text{dist}_{W_2}(v_3) \geq 1/2$.

Continuing, we find $\exists v_n \in V$ with $\|v_n\| = 1$ and $\text{dist}_{W_{n-1}}(v_n) \geq 1/2$, where $W_{n-1} = \text{span}\{v_1, \dots, v_{n_1}\}$. We have a sequence $(v_n)_n \in B_V$. Since B_V is compact, $\exists (v_{n_k})_k \to v \in B_V$, meaning B_V is Cauchy. However, since $\|v_n - v_m\| \geq 1/2$ for all n and m. $\perp$

Proposition: Compact Domain and Uniform Continuity

If $f : (X,d) \to (Y,\rho)$ is continuous, and X is compact, then f is uniformly continuous.

Proof: Let $\varepsilon > 0$. For each $x \in X$, we have $\exists \delta_x > 0$ such that for $d(z,x) < \delta_x \Rightarrow \rho(f(z), f(x)) < \varepsilon/2$.

Since $X = \bigcup_{x\in X} U(x, \delta_x/2)$, by compactness, we have $x_1, \dots, x_n$ with $X = \bigcup_{j=1}^{n} U(x_j, \delta_{x_j}/2)$. Take $\delta = \min\{\delta_{x_j}/2\}$.

Let $x, x' \in X$ arbitrary with $d(x,x') < \delta$. Locate $x \in U(x_j, \delta_{x_j}/2)$ for some j. Then,

$$\begin{aligned}
d(x', x_j) &\leq d(x', x) + d(x, x_j) \\
&< \delta + \delta_{x_j}/2 \\
&\leq \delta_{x_j}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(f(x), f(x')) &\leq \rho(f(x), f(x_j)) + \rho(f(x_j), f(x')) \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Compactness and Uniform Convergence

(1) Let $f_n : (0,1) \to \mathbb{R}$ with $f_n(t) = t^n$. Pointwise, $(f_n)_n \to \mathbb{0}$, meaning for $(f_n(t))_n \to \mathbb{0}(t) = 0$ for all $t \in (0,1)$. However, the convergence is not uniform. We have $\|f_n - \mathbb{0}\|_u = \|f_n\|_u = 1$.

Note that $f_n(t)$ decreases pointwise to 0 for all $t \in (0,1)$, meaning $f_1(t) \geq f_2(t) \geq f_3(t) \geq \cdots$.

(2) Consider the sequence of functions defined by

$$f_n(x) = \begin{cases} 0 & x \in (-\infty, n) \\ x - n & x \in [n, n+1] \\ 1 & x \in (n_1, \infty) \end{cases}.$$

Notice that $f_n(t)$ is decreasing in n for all t and $(f_n)_n \to \mathbb{0}$ pointwise, but convergence is not uniform, as $\|f_n\|_u = 1$ for all n.

Dini's Theorem

If (X, d) is a compact metric space, and $(f_n : X \to \mathbb{R})_n$ is a sequence of continuous real-valued functions with $\forall x \in X$, $(f_n(x))_n \to 0$ is decreasing. Then, $(f_n)_n \to \mathbb{0}$ uniformly.

Proof: Let $\varepsilon > 0$. For each $n \geq 1$, take $U_n = \{x \mid f_n(x) < \varepsilon/2\}$. Then $U_n = f_n^{-1}((-\infty, \varepsilon/2))$. Since f_n is continuous, and $(-\infty, \varepsilon/2)$, so too is U_n in X.

Notice that $U_1 \subseteq U_2 \subseteq \cdots$, as if $x \in U_n$, then $f_{n+1}(x) \leq f_n(x) < \varepsilon/2$, meaning $x \in U_{n+1}$. Then, we have that $\bigcup U_n = X$, as for all x, $f_n(x) \to 0$. Since X is compact, we have $X = \bigcup U_{n_k} = U_{n_K}$. For any $x \in X$, $f_{n_K}(x) < \varepsilon/2$. Thus, $\|f_{n_K}\| \leq \varepsilon/2 < \varepsilon$, so we have uniform convergence.

Compactness in $C(X)$

If X is a compact metric space, then, by the Extreme Value Theorem, $C(X) = C_b(X)$. We can see that $C_b(X)$ is complete under $\|\cdot\|_u$. We may ask when $\mathcal{F} \subseteq C(X)$ is compact.

A family $\mathcal{F} \subseteq C(X)$ is equicontinuous if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in X$ with $d(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$.

Exercise: For $\mathcal{F} \subseteq C(X)$ with $\mathcal{F}$ finite, then $\mathcal{F}$ is always equicontinuous.

Since every $f \in \mathcal{F}$ is uniformly continuous, take the minimum value of δ.

Arzelà-Ascoli Theorem

Let (X, d) be a compact metric space. The family $\mathcal{F} \subseteq C(X)$ is compact if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.

Proof: Let $\mathcal{F}$ be compact. Then, $\mathcal{F}$ is complete, and thus closed and totally bounded, meaning $\mathcal{F}$ is bounded. Thus, we need to show $\mathcal{F}$ is equicontinuous.

Let $\varepsilon > 0$. By total boundedness, $\exists f_1, \dots, f_n \in \mathcal{F}$ with $\mathcal{F} \subseteq \bigcup_{j=1}^{n} U(f_j, \varepsilon/3)$. Each f_j is uniformly continuous since X is compact. Thus, $\exists \delta_j$ with $x, y \in X$ and $d(x, y) \leq \delta_j$, then $|f_j(x) - f_j(y)| < \varepsilon/3$.

Let $\delta = \min\{\delta_j\}$. Given any $f \in \mathcal{F}$, we have $\mathcal{F} \in U(f_j, \varepsilon/3)$ for some j. For any $x, y \in X$ with $d(x,y) < \delta$, we have

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &\leq \|f - f_j\|_u + |f_j(x) - f_j(y)| + \|f - f_j\|_u \\ &< 2\varepsilon/3 + \varepsilon/3 \\ &= \varepsilon\end{aligned}$$

Let $\mathcal{F}$ be closed, bounded, and equicontinuous. Since $\mathcal{F} \subseteq C(X)$ is closed, $\mathcal{F}$ is complete. We need only show $\mathcal{F}$ is totally bounded.

Let $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta$, then $|f(x) - f(y)| < \varepsilon/4$ for any $f \in \mathcal{F}$.

Since X is compact, X is totally bounded, so $\exists x_1, \dots, x_n \in X$ with $X \subseteq \bigcup_{j=1}^{n} U(x_j, \delta)$. Consider the set $C_{\mathcal{F}} := \{(f(x_1), \dots, f(x_n)) \,|\, f \in \mathcal{F}\} \subseteq \mathbb{R}^n$.

Since $\mathcal{F}$ is bounded, we have that $\|f\|_u \leq M$ for all $f \in \mathcal{F}$ for some $M > 0$. Thus, $|f(x_j)| \leq \|f\|_u \leq M$ for $j = 1, \dots, n$. Thus, $C_{\mathcal{F}}$ is bounded in $\mathbb{R}^n$.

Exercise: $S \subseteq \mathbb{R}^n$ is bounded if and only if S is totally bounded.

Thus, $C_{\mathcal{F}}$ is totally bounded. Therefore, $\exists f_1, \dots, f_m \in \mathcal{F}$ with $C_{\mathcal{F}} \subseteq \bigcup_{i=1}^{m} U\left((f_i(x_1), \dots, f_i(x_n)), \varepsilon/4\right)$.

If $f \in \mathcal{F}$, then $\exists i = 1, \dots, m$ $(*)$ such that $\|(f(x_1), \dots, f(x_n)) - (f_i(x_1), \dots, f_i(x_n))\|_1 < \varepsilon/4$. Thus,

$$\sum_{j=1}^{n} |f(x_j) - f_i(x_j)| < \varepsilon/4.$$

We claim that $F \subseteq \bigcup_{i=1}^{m} U(f_i, \varepsilon)$. Let $f \in \mathcal{F}$ and $x \in X$. Pick i as in $(*)$, and j with $x \in U(x_j, \delta)$. Then,

$$\begin{aligned}|f(x) - f_i(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| \\ &< 3\varepsilon/4\end{aligned}$$

so

$$\begin{aligned}\|f - f_i\| &\leq 3\varepsilon/4 \\ &< \varepsilon.\end{aligned}$$

Stone-Weierstrass Theorem

Let (X, d) be a compact metric space. Suppose $A \subseteq C(X; \mathbb{R})$ with

- $f, g \in A \Rightarrow f + g \in A$;
- $f \in A, \alpha \in \mathbb{F} \Rightarrow \alpha f \in A$;
- $f, g \in A \Rightarrow fg \in A$;
- $\mathbb{1}_X \in A$;
- A is separating — if $x \neq y$ in X, then $\exists f \in A$ with $f(x) \neq f(y)$.

We say A is a unital separating subalgebra of $C(X)$.

Then, $\overline{A}^{\|\cdot\|_u} = C(X; \mathbb{R})$ (A is uniformly dense).

Uniform Approximation by Polynomials

For example, considering $\mathcal{P} = \left\{x \mapsto \sum_{k=0}^{n} a_k x^k \mid a_k \in \mathbb{R}\right\} \subseteq C([0,1])$. We can see that $\mathcal{P}$ is a separating unital subalgebra. Thus, $\mathcal{P}$ is dense.

Let $f(x) = |x|$ on $[-1,1]$. Consider the sequence $P_n(x)$ given by

$$\begin{aligned} P_0(x) &= 0 \\ P_{n+1}(x) &= P_n(x) + \frac{x^2 - (P_n(x))^2}{2}. \end{aligned}$$

For example, $P_1(x) = x^2/2$, $P_2(x) = \frac{x^2}{2} + \frac{x^2 - x^4/4}{2}$. Then, $(P_n)_n \xrightarrow{\|\cdot\|_u} f$.

Proof: We claim that $0 \leq P_n(x) \leq f(x)$ for all $x \in [-1,1]$. Clearly, $0 \leq P_0(x) \leq |x|$, and $0 \leq P_1(x) \leq |x|$. Assume it is the case that $0 \leq P_n(x) \leq |x|$. Then,

$$\begin{aligned} 0 \leq P_n(x) &\leq |x| \\ 0 \leq P_n^2(x) &\leq x^2 \\ x^2 - P_n^2(x) &\geq 0 \\ P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2} &\geq 0 \end{aligned}$$

and

$$\begin{aligned} |x| - P_{n+1}(x) &= |x| - P_n(x) - \frac{|x|^2 - P_n(x)}{2} \\ &= |x| - P_n(x) - \frac{(|x| - P_n(x))(|x| + P_n(x))}{2} \\ &= (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right) \\ &\geq 0 \end{aligned}$$

Observe that $P_n(x) \leq P_{n+1}(x)$. For every x, $(P_n(x))_n$ is increasing and bounded above by $|x|$. So, $P_n(x) \to L_x$.

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + \frac{x^2 - P_n^2(x)}{2} \\ L_x &= L_x + \frac{x^2 - L_x^2}{2} \\ L_x &= \sqrt{x^2} = |x|. \end{aligned}$$

Thus, $(P_n)_n$ converges pointwise on $[-1,1]$. So, $(f - P_n) \to 0$ is decreasing pointwise. Whence, by Dini's Theorem, $\|f - P_n\|_u \to 0$.

Connectedness

Let (X,d) be a metric space.

(1) Let $Y \subseteq X$. A splitting for Y in X is an inclusion $Y \subseteq U \cup V$, where $U, V \in \tau_X$ with $Y \cap U \cap V = \emptyset$.

Remark: If we set $U_1 = U \cap Y$ and $V_1 = V \cap Y$, then U_1 and V_1 are open in Y with the relative topology. We have $Y = U_1 \sqcup V_1$. Also note that U_1 and V_1 are clopen in Y.

(2) A splitting for Y is called trivial if either $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$.

(3) Y is connected in X if every splitting for Y in X is trivial. Otherwise, we say Y is disconnected.

Exercise: Suppose $C \subseteq Y \subseteq X$. C is connected in Y if and only if C is connected in X.

Connectedness of Subsets in $\mathbb{R}$

We have $[a, b] \subseteq \mathbb{R}$ is connected.

Proof: Suppose $[a, b] \subseteq U \cup V$ is a splitting.

- If $a = b$ or $a > b$, clearly the splitting is trivial.
- Assume $a < b$. Without loss of generality, $a \in U$. Suppose toward contradiction that $[a, b] \cap V \neq \emptyset$. Set $c = \inf [a, b] \cap V$.

  We claim that $a < c$; since U is open, $\exists \varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq U$. So, $V \cap [a, b] \subseteq [a + \varepsilon, b]$. Therefore, $c \geq a + \varepsilon$. Thus, $[a, c) \subseteq U$.

  We claim $c \in V$. Since U is open, we cannot have $c < b$ and $c \in U$. Also, if $c \in U$ and $c = b$, then $[a, b] \cap V = \emptyset$.

  Since V is open, $\exists \delta > 0$ with $(c - \delta, c + \delta) \subseteq V$. However, this means $c \neq \inf V \cap [a, b]$.

  Thus, $V \cap [a, b] = \emptyset$.

We have that $\mathbb{Q} \subseteq \mathbb{R}$ is disconnected.

Proof: We have $\mathbb{Q} \subseteq (-\infty, \pi) \cup (\pi, \infty)$ is a non-trivial splitting.

Proposition: Intervals in $\mathbb{R}$

Every interval $I \subseteq \mathbb{R}$ is connected.

Proof: Let $I \subseteq U \cup V$ be a non-trivial splitting. Therefore, $U \cap I \neq \emptyset$, and $V \cap I \neq \emptyset$. Let $a \in I \cap U$ and $b \in I \cap V$. Without loss of generality, $a < b$.Then, by the definition of an interval, $[a, b] \subseteq I \subseteq U \cup V$.

However, at the same time, $[a, b] \cap U \cap V \subseteq I \cap U \cap V = \emptyset$. So, we have a splitting for $[a, b]$. This splitting for $[a, b]$ is non-trivial, since $[a, b] \cap U \neq \emptyset$ and $[a, b] \cap V \neq \emptyset$. However, we had shown that $[a, b]$ is connected.

If $I \subseteq \mathbb{R}$ is connected, then I is an interval.

Proof: Let $a = \inf I$ and $b = \sup I$. It is possible for a to equal $-\infty$ and b to equal $+\infty$. We claim that $(a, b) \subseteq I$.

If $\exists c \in I$ with $c \notin (a, b)$, then we have a non-trivial splitting $I \subseteq (-\infty, c) \cup (c, \infty)$, which would contradict the assumption that I is connected. Thus, $(a, b) \subseteq I$.

If $s, t \in I$ with $s \leq t$, then $s \geq a$ or $s > a$, or $t \leq b$ or $t < b$. By cases, we find$[s, t] \subseteq I$, meaning I is an interval.

Exercise: If $Y \subseteq X$ is connected, then $\overline{Y}$ is connected.

Connected Components and Clopen Sets

Let (X, d) be a metric space. We define $\sim_X$ on X as $x \sim_X y$ if there is a connected $C \subseteq X$ with $x, y \in C$. This is an equivalence relation.

We have that $x \sim_X x$ by taking $C = \{x\}$, so the relation is reflexive. Clearly, the relation is symmetric. To show transitivity, we need the following lemma:

Lemma: If $Y_1, Y_2 \subseteq X$ are connected with $Y_1 \cap Y_2 \neq \emptyset$, then $Y_1 \cup Y_2$ is connected.

Proof of Lemma: Let $Y_1 \cup Y_2 \subseteq U \cup V$ be a splitting. Note that $Y_i \subseteq U \cup V$, and $Y_i \cap U \cap V =\subseteq (Y_1 \cup Y_2) \cap U \cap V = \emptyset$. For $i = 1, 2$, since Y_i are connected, so we have splittings for Y_i. Since the Y_i are connected, these splittings are trivial.

Since the splitting for Y_1 is trivial, $Y_1 \subseteq U$, or $Y_1 \subseteq V$. Similarly, since the splitting for Y_2 is trivial, $Y_2 \subseteq U$ or $Y_2 \subseteq V$.

Suppose $Y_1 \subseteq U$ and $Y_2 \subseteq U$. Then, $Y_1 \cup Y_2 \subseteq U$, and our original splitting is trivial.

Suppose $Y_1 \subseteq U$ and $Y_1 \subseteq V$. Then, $\emptyset \neq Y_1 \cap Y_2 = (Y_1 \cap U) \cap (Y_2 \cap V) = (Y_1 \cap Y_2) \cap (U \cap V) \subseteq (Y_1 \cup Y_2) \cap U \cap V = \emptyset$.

Other cases follow similarly.

If $x \sim_X y \sim_X z$, then there exist connected subsets $C, D \subseteq X$ with $x, y \in C$ and $y, z \in D$. Since $y \in C \cap D$, we have that $C \cup D$ is connected, so $x, z \in C \cup D$, which is connected.

The equivalence classes of X under $\sim_X$ are called components.

Remark: $[x]_\sim = \{y \in X \mid y \sim_X x\} = \bigcup_{x \in C} C$ with C connected. This is the largest connected subset of X containing x. We have that $X = \bigsqcup_{i \in I} [x_i]_\sim$.

If (X, d) is a metric space, and $C \subseteq X$ is clopen and connected, then C is a component in X.

Proof: Let $x \in C$. We claim that $C = [x]_\sim$.

Clearly, $C \subseteq [x]_\sim$. Suppose $y \in [x]_\sim$ and $y \notin C$.

Since $y \in [x]_\sim$, there is a connected $D \subseteq X$ with $x, y \in D$. We have that $D \subseteq C \cup (X \setminus C)$. This is a non-trivial splitting for D, meaning D is disconnected. $\perp$

Totally Disconnected Metric Spaces

Consider the set $X = \{0\} \cup \{1/n \mid n \geq 1\}$ with the topology inherited from $\mathbb{R}$. We want to find the connected components.

Solution: The set $\{1/n\}$ for each n is connected in $\mathbb{R}$, meaning it is connected in X. Since $\{1/n\}$ is closed in $\mathbb{R}$, it is also closed in X. We also have that $\{1/n\} = X \cap (1/n - \delta_n, 1/n + \delta_n)$, with $\delta_n = \frac{1}{n(n+1)}$.

Since each $\{1/n\}$ is clopen and connected, each $\{1/n\}$ is a component. Additionally, $\{0\}$ is necessarily a component of X since it is left over after we take $X \setminus \{1/n \mid n \geq 1\}$. We see that every connected component of X is a singleton.

For $X = \mathbb{Z}$, we see that the components are singletons.

For $X = \mathbb{Q}$, we need a little bit more machinery to find the components.

Solution: Suppose $q, r \in \mathbb{Q}$ with $r \sim_\mathbb{Q} q$. Then, $\exists D \subseteq \mathbb{Q}$ connected with $r, q \in D$. If $r \neq q$, then let $x \in \mathbb{R} \setminus \mathbb{Q}$ with x strictly between r and q. Without loss of generality, $r < q$. Then, $D \subseteq ((-\infty, x) \cap \mathbb{Q}) \cup ((x, \infty) \cap \mathbb{Q})$ is a non-trivial splitting, meaning D is not connected.

Therefore, $r = q$, meaning the components of $\mathbb{Q}$ are singletons.

If (X, d) is a metric space where every connected component is a singleton, then X is totally disconnected.

Exercise: The Cantor set is totally disconnected.

Proposition: Open Sets in $\mathbb{R}$

If $U \subseteq \mathbb{R}$ is open, then $U = \bigsqcup_{i \in I} V_i$, where each $V_i \subseteq \mathbb{R}$ is an open interval and I is countable.

Proof: Let U be the metric space with the topology inherited from $\mathbb{R}$. Then, $U = \bigsqcup_{i \in I} V_i$, with $V_i \subseteq U$ are the connected components in U.

Since V_i is connected in U, V_i is connected in $\mathbb{R}$. Thus, V_i is an interval. We will show that each V_i is open in $\mathbb{R}$.

Let $x \in V_i$. Since U is open, $\exists \varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \subseteq U$. Since $x \in (x - \varepsilon, x + \varepsilon)$, and $(x - \varepsilon, x + \varepsilon)$, it is the case that $(x - \varepsilon, x + \varepsilon) \subseteq [x]_{\sim_U} = V_i$. Thus, V_i is open.

Now, we need to show that I is countable. Consider $N : I \to \mathbb{Q}$; $N(i) = q_i \in V_i$, with $q_i \in \mathbb{Q}$. If $i \neq j$, then $N(i) = N(j)$ since $V_i \cap V_j \neq \emptyset$. Hence, N is injective, so I is countable.

Proposition: Connectedness and Continuity

If $f : X_1 \to X_2$ is continuous and $Y \subseteq X_1$ is connected, then $f(Y) \subseteq X_2$ is connected.

Proof: Let $f(Y) \subseteq U \cup V$ is a splitting of $f(Y) \subseteq X_2$.

Taking the preimage, we have $Y \subseteq f^{-1}(f(Y)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. We have that $f^{-1}(U)$ and $f^{-1}(V)$ are open in X_1. Additionally,

$$\begin{aligned} Y \cap f^{-1}(U) \cap f^{-1}(V) &= Y \cap f^{-1}(U \cap V) \\ &\subseteq f^{-1}\left(f(Y)\right) \cap f^{-1}(U \cap V) \\ &\subseteq f^{-1}\left(f(Y) \cap f^{-1}(U \cap V)\right) \\ &= \emptyset. \end{aligned}$$

Thus, $Y \subseteq f^{-1}(U) \cup f^{-1}(V)$ is a splitting. Since Y is connected, the splitting is trivial, meaning without loss of generality, $Y \subseteq f^{-1}(U)$. So, $f(Y) \subseteq U$.

Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ is continuous. If $f(a) \leq \lambda \leq f(b)$, then $\lambda \in f([a, b])$.

Proof: Since $[a, b]$ is compact and connected, and f is continuous, $f([a, b]) \subseteq \mathbb{R}$ is also connected. So, $f([a, b])$ is a compact and connected interval.

Since $f(a), f(b) \in f([a, b])$, and $f([a, b])$ is an interval, $\lambda \in f([a, b])$.

Proposition: Continuous Map to Totally Disconnected Set

Let X be connected, Y totally disconnected, and $f : X \to Y$ continuous. Then, f is a constant map.

Proof: The continuous image of a connected set is connected, and the only connected sets in Y are singletons, meaning the image of X is a singleton.

Path-Connectedness

Let (X, d) be a metric space.

(i) A path in X is a continuous map $\gamma : [0,1] \to X$. If $\gamma(0) = x_0$ and $\gamma(1) = x_1$, we say the path connects x_0 to x_1.

(ii) X is said to be path-connected if for any two points x_0 and x_1, there exists a path. $Y \subseteq X$ is path connected if Y is connected.

(1) Let V be any normed space, and $C \subseteq V$ convex. By definition, C is path-connected. Indeed, $\gamma(t) = (1-t)x_0 + x_1$.

(2) The metric space $\mathbb{R}^2 \setminus \{0\}$ is path-connected.

Proposition: Composition of Paths

Let $\gamma : [0,1] \to X$ is a path from x_0 to x_1, and $\sigma : [0,1] \to X$ is a path from x_1 to x_2. Then, the following are all true.

(1) $\gamma^{-1} : [0,1] \to X$, with $\gamma^{-1}(t) = \gamma(1-t)$, is a path from x_1 to x_0.

(2) $\sigma \cdot \gamma : [0,1] \to X$ is a path from x_0 to x_2, with $\sigma \cdot \gamma(t)$ defined as follows:

$$\sigma \cdot \gamma(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \sigma(2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

Lemma: Base Point and Path-Connectedness

Let (X, d) be a metric space, and $x_0 \in X$ fixed. Suppose $\forall x$, $\exists$ a path from x_0 to x. Then, X is path-connected.

(1) The unitary group is path-connected.

$$\begin{aligned} U_n(\mathbb{C}) &= \{U \in \mathbb{M}_n(\mathbb{C}) \mid U^*U = I_n = UU^*\} \\ d(U,V) &= \|U - V\|_{\text{op}} \end{aligned}$$

Let $U \in U_n(\mathbb{C})$. By the spectral theorem via a unitary; there exists $V \in U_n(\mathbb{C})$ with $V^*UV = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, with $|\lambda_j| = 1$. Write $\lambda_j = e^{i\theta_j}$, with $\theta_j \in [0, 2\pi)$.

Consider $U_t = V\operatorname{diag}\left(e^{it\theta_1}, \dots, e^{it\theta_n}\right)V^*$. Clearly, $U_t \in \mathbb{M}_n(\mathbb{C})$. Additionally, $U_0 = I_n$, and $U_1 = U$. We have

$$\begin{aligned} \|U_s - U_t\| &= \|V^*\Lambda_s V - V\Lambda_t V^*\| \\ &= \|V(\Lambda_s - \Lambda_t)V^*\| \\ &\leq \|V\| \, \|\Lambda_s - \Lambda_t\| \, \|V^*\| \\ &= \|\Lambda_s - \Lambda_t\| \\ &\to 0. \end{aligned}$$

Thus, U_t is continuous, meaning we have a path from I_n to U. Thus, $U_n(\mathbb{C})$ is path-connected.

Proposition: Path-Connectedness implies Connectedness

If (X, d) is a path-connected metric space, then X is connected.

Proof: Let $X = U \sqcup V$ be a splitting. Suppose $\exists x_0 \in U$ and $x_1 \in V$. We know $\exists \gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Since $[0,1]$ is connected and γ is continuous, $\gamma([0,1]) \subseteq X$ is connected. However, $\gamma([0,1]) \subseteq U \cup V$ is a non-trivial splitting. $\perp$

Exercise: If $f : X_1 \to X_2$ is continuous, and $Y \subseteq X_1$ is path-connected, then $f(Y) \subseteq X_2$ is path-connected.

Proof of Exercise: Let $f(y_1), f(y_2) \in f(Y)$. We have that $\gamma : [0,1] \to Y$ is a path. Thus, $f \circ \gamma : [0,1] \to f(Y)$ is a path.

A Connected Space that is not Path-Connected

Set $Y_0 = \{0\} \times [-1,1] \subseteq \mathbb{R}^2$, and $Y_1 = \{(x, \sin(1/x)) \mid x \in (0,1]\}$. Let $Y = Y_0 \cup Y_1$. This space is known as the topologist's sine curve, and it is connected but not path-connected.

Proof: We can see that Y_1 is the continuous image of a connected set, so Y_1 is connected.

We also see that Y is connected, as $Y = \overline{Y_1}$.

We claim that Y is not path-connected. There does not exist a path $\gamma : [0,1] \to Y$ with $\gamma(0) \in Y_0$ and $\gamma(1) \in Y_1$. Suppose toward contradiction that such a path existed. Let $\gamma^{-1}(Y_0) := F$, with γ^{-1} being the inverse image (not inverse path). Since Y_0 is closed, we have $F \subseteq [0,1]$ is closed, so $u = \sup F \in F$, and $u < 1$.

By replacing $[0,1]$ by $[u,1]$, we may assume a new path $\gamma' : [0,1] \to Y$ is a path with $\gamma_1(t) \in (0,1]$, for $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$.

Let $r > 0$ be small such that $[-1,1] \supset [\gamma_2'(0) - r, \gamma_2'(0) + r]$. Since γ_2' is continuous at $t = 0$, we know $\exists \varepsilon > 0$ with $\gamma_2'([0,\varepsilon]) \subseteq (\gamma_2'(0) - r, \gamma_2'(0) + r)$.

Since $\gamma_1'([0,\varepsilon])$ is connected, and hence an interval, and $\gamma_1'(t) > 0$ for all $t \in (0,1]$, we can find δ small such that $[0,\delta] \subseteq \gamma_1'([0,\varepsilon))$.

We have that $\gamma_2'(t) = \sin\left(\frac{1}{\gamma_1'(t)}\right)$ for $t > 0$. Therefore,

$$\begin{aligned}
[-1,1] &= \left\{\sin\left(\frac{1}{x}\right) \mid 0 < x < \delta\right\} \\
&\subseteq \left\{\sin\left(\frac{1}{\gamma_1'(t)}\right) \mid 0 < t < \varepsilon\right\} \\
&= \gamma_2'((0,\varepsilon)) \\
&\subseteq (\gamma_2'(0) - r, \gamma_2'(0) + r) \\
&\subset [-1,1].
\end{aligned}$$

Proposition: Connectedness in a Normed Space

Let V be a normed space, and $Y \subseteq V$ is open and connected, then Y is path-connected.

Proof: Fix $y_0 \in Y$. Consider the set $W = \{y \in Y \mid \exists \gamma \text{ from } y_0 \text{ to } y\}$. We claim that W is open in Y.

> Let $y \in W$. Since Y is open, $\exists \delta > 0$ with $U(y, \delta) \subseteq Y$. If $w \in U(y, \delta)$, $\exists \gamma$ from y to w. Concatenating, we get a path from y_0 to w. Thus, $U(y, \delta) \subseteq W$.
>
> We also claim W is closed in Y.

Measurable Spaces

The theory of integration is tied to notions of length, area, volume, etc. The Riemann integral

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right),$$

is defined through the length of a subinterval. We took the interval $[0, 1]$, calculated base multiplied by height, and found the area of the rectangle.

It's easy to compute the length of an interval. However, Lebesgue integration does the opposite; it subdivides the range of f into subintervals I_k, and calculates the "length" of $f^{-1}(I_k)$.

We need a more rigorous treatment of length (or area, or volume) to deal with Lebesgue integration.

Given $E \subseteq \mathbb{R}^n$, with E "sufficiently nice," we want to assign an extended positive real number $\lambda(E) \in [0, \infty]$, such that certain natural properties are satisfied.

- $\lambda(\emptyset) = 0$
- $\lambda\left(\prod_{j=1}^{n} [a_j, b_j]\right) = \prod_{j=1}^{n} (b_j - a_j)$
- $\lambda(x + E) = \lambda(E)$
- $\lambda\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{n} E_k$
- if $E \subseteq F$, then $\lambda(E) \leq \lambda(F)$

Proposition: Non-existence of λ

There is no $\lambda : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ that satisfies the properties above.

Proof: Consider the equivalence relation on $[0, 1]$, with $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.

> So, $[0, 1] = \bigsqcup_{i\in I} [x_i]$, with $x_i \in [0, 1]$. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [-1, 1]$. Let $N = \{x_i\}_{i\in I}$ (possible with the axiom of choice).
>
> Consider the set $E_k = r_k + N$.
>
> - E_k are pairwise disjoint; if $r_k + x_i = r_\ell + x_j$, then $x_j - x_i = r_k - r_\ell \in \mathbb{Q}$, meaning $x_i \sim x_j$.
> - $E_k \subseteq [-1, 2]$.

If $t \in [0,1]$, then $t \sim x_i$ for some $i \in I$. So, $t - x_i \in \mathbb{Q}$, and $t - x_i \in [-1,1]$, so $t - x_i = r_k$ for some k. Thus, $t \in E_k$. Thus, we have shown that $[0,1] \subseteq \bigsqcup E_k \subseteq [-1,2]$.

If λ were such a mapping, we have

$$\begin{aligned} 1 &= \lambda([0,1]) \\ &\leq \lambda(\bigsqcup E_k) \\ &= \sum \lambda(E_k) \\ &= \sum \lambda(r_k + N) \\ &= \sum \lambda(N). \end{aligned}$$

If $E = \bigsqcup E_k$, then $\lambda(E) \leq 3$ and $\lambda(E) = \sum \lambda(N)$. $\perp$.

Thus, we conclude that some sets are not measurable. We might then ask what sets are able to be measured.

- Intervals;
- open sets;
- closed sets.

We will eventually define a class of measurable sets, $\mathcal{L}$, and we will also construct a measure $\lambda : \mathcal{L} \to [0,\infty]$ satisfying the above properties.

Measurable Spaces and σ-Algebras

Let $\Omega \neq \emptyset$.

(1) An algebra of subsets of Ω is a nonempty family $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ such that

- If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$;
- If $E, F \in \mathcal{M}$, then $E \cup F \in \mathcal{M}$

(2) A nonempty collection $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra of subsets of Ω if

(i) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$;

(ii) If $\{E_k\}_{k=1}^{\infty} \in \mathcal{M}$, then $\bigcup E_k \in \mathcal{M}$.

(3) A measurable space is a pair $(\Omega, \mathcal{M})$ with $\Omega \neq \emptyset$ a set and $\mathcal{M}$ is a σ-algebra.

Let $\mathcal{M}$ be an algebra of subsets of Ω. Then, the following are true.

(i) $\emptyset, \Omega \in \mathcal{M}$;

(ii) If $E_1, \dots, E_n \in \mathcal{M}$, then $\bigcup E_k \in \mathcal{M}$;

(iii) If $E_1, \dots, E_n \in \mathcal{M}$, then $\bigcap E_k \in \mathcal{M}$;

(iv) If $E, F \in \mathcal{M}$, then $E \setminus F \in \mathcal{M}$.

Proof:

(i) Since $\mathcal{M}$ is not empty, there is an $E \in \mathcal{M}$, so $E^c \in \mathcal{M}$, so $E \cup E^c = \Omega \in \mathcal{M}$,and $(E \cup E^c)^c = \emptyset \in \mathcal{M}$.

(ii) Induction.

(iii) We have $\bigcap E_k = \left(\bigcup_{i=1}^{\infty} E_k^c\right)^c \in \mathcal{M}$.

(iv) We have $E \setminus F = E \cap F^c \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra, then (1) through (4) hold for countable families as well.

(1) $(\Omega, \mathcal{P}(\Omega))$ is a measurable space.

(2) $(\Omega, \{\emptyset, \Omega\})$ is a measurable space.

(3) For Ω uncountable, let $\mathcal{M} = \{E \subseteq \Omega \mid E \text{ countable or } E^c \text{ countable}\}$. Then, $(\Omega, \mathcal{M})$ is a measurable space.

(4) If $\{\mathcal{M}_i\}_{i \in I}$ is a family of σ-algebras on Ω, then $\bigcap \mathcal{M}_i$ is a σ-algebra on Ω.

If $0 \neq \mathcal{E} \subseteq \mathcal{P}(\Omega)$, the σ-algebra generated by $\mathcal{E}$ is

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{M}_i\ \sigma\text{-algebra} \\ \mathcal{E} \subseteq \mathcal{M}_i}} \mathcal{M}_i.$$

Borel σ-Algebra

Let (X, d) be a metric space. Let $\tau_d = \{U \mid U \subseteq X \text{ open}\}$. The Borel σ-algebra on X is

$$\mathcal{B}_X = \sigma(\tau_d).$$

Remark: $\mathcal{B}_X$ contains all open sets, closed sets, F_σ sets, G_δ sets, etc.

Proposition: Borel σ-Algebra on $\mathbb{R}$

Consider the families of $\mathcal{P}(\mathbb{R})$,

$$\begin{aligned}
\mathcal{E}_1 &= \{(a, b) \mid a < b\} \\
\mathcal{E}_2 &= \{[a, b] \mid a < b\} \\
\mathcal{E}_3 &= \{(a, b] \mid a < b\} \\
\mathcal{E}_4 &= \{[a, b) \mid a < b\} \\
\mathcal{E}_5 &= \{(-\infty, b) \mid b \in \mathbb{R}\} \\
\mathcal{E}_6 &= \{(-\infty, b] \mid b \in \mathbb{R}\} \\
\mathcal{E}_7 &= \{(a, \infty) \mid a \in \mathbb{R}\} \\
\mathcal{E}_8 &= \{[a, \infty) \mid a \in \mathbb{R}\}.
\end{aligned}$$

For $i = 1, \dots, 8$, we have $\sigma(\mathcal{E}_i) = \mathcal{B}_{\mathbb{R}}$.

Proof: Note that $\mathcal{E}_1 \subseteq \tau_d \subseteq \sigma(\tau_d) \subseteq \mathcal{B}_{\mathbb{R}}$. Thus, $\sigma(\mathcal{E}_1) \subseteq \mathcal{B}_{\mathbb{R}}$. Let $U \in \mathbb{R}$ be open. Then, $U = \bigsqcup I_j$, with I_j open. Consider any open interval I. If I is bounded, then $I \in \mathcal{E}_1$. If I is not bounded, then $I = \bigcup_{k=1}^{\infty} J_k$ with J_k bounded open intervals. Since each $J_k \in \mathcal{E}_1$, then $I \in \sigma(\mathcal{E}_1)$. Therefore, each $I_j \in \sigma(\mathcal{E}_1)$, so $U \in \sigma(\mathcal{E})_1$. Thus, $\tau_d \subseteq \sigma(\mathcal{E}_1)$, so $\mathcal{B}_{\mathbb{R}} \subseteq \sigma(\mathcal{E}_1)$.

Thus, $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{E}_1)$.

We have that $[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right) \in \sigma(\mathcal{E}_1)$. Therefore, $\mathcal{E}_4 \in \sigma(\mathcal{E}_1)$, thus $\sigma(\mathcal{E}_4) \subseteq \sigma(\mathcal{E}_1)$. Additionally, $(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right] \in \sigma(\mathcal{E}_4)$. So, $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_4) = \mathcal{B}_{\mathbb{R}}$.

Measure and Measure Spaces

Let $(\Omega, \mathcal{M})$ be a measurable space.

(1) A function $\mu : \mathcal{M} \to [0, \infty]$ is a measure on $(\Omega, \mathcal{M})$ if

  (i) $\mu(\emptyset) = 0$;

  (ii) if $\{E_k\}_{k\geq 1} \in \mathcal{M}$ are pairwise disjoint, then $\mu\left(\bigsqcup E_k\right) = \sum \mu(E_k)$. Notice that $\mu(E_k) \geq 0$ for all E_k, so the order of the sum does not matter.

(2) If $\mathcal{M}$ is an algebra (or σ-algebra), and μ satisfies $\mu(E \sqcup F) = \mu(E) + \mu(F)$ for $E, F \in \mathcal{M}$, then μ is called a finitely additive measure.

(3) A measure space is a triple $(\Omega, \mathcal{M}, \mu)$, where $(\Omega, \mathcal{M})$ is a measurable space and μ is a measure.

(4) A measure space $(\Omega, \mathcal{M}, \mu)$ is called finite if $\mu(\Omega) < \infty$. If $\mu(\Omega) = 1$, then $(\Omega, \mathcal{M}, \mu)$ is called a probability space, with Ω the sample space and $\mathcal{M}$ the collection of events.

(5) A measure μ is σ-finite if there exists $\{E_k\}_{k\geq 1} \subseteq \mathcal{M}$ with $\Omega = \bigcup E_k$ and $\mu(E_k) < \infty$ for each k.

(6) A measure μ on $(\Omega, \mathcal{M})$ is semi-finite if $\forall E \in \mathcal{M}$ with $\mu(E) = \infty$, $\exists F \subseteq E$ with $0 < \mu(F) < \infty$.

Exercise: Show that σ-finite implies semi-finite.

Examples of Measure Spaces

(i) Consider $(\Omega, \mathcal{P}(\Omega))$. Fix $x \in \Omega$, with $\delta_x : \mathcal{P}(\Omega) \to [0, \infty]$, with

$$\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

We can see that δ_x is a probability measure, known as the Dirac measure.

(ii) If μ is a measure on $(\Omega, \mathcal{M})$, and $t \in [0, \infty)$, then $(t\mu)(E) = t(\mu(E))$ is a measure.

(iii) If $\mu_1, \dots, \mu_n$ are measures on $(\Omega, \mathcal{M})$, then $\mu(E) = \sum \mu_j(E)$ is a measure.

(iv) If $0 \leq t_1, \dots, t_n \leq 1$ with $\sum t_j = 1$, and $x_1, \dots, x_n \in X$, we have

$$\mu(E) = \sum t_j \delta_{x_j}$$

is a probability measure on $(\Omega, \mathcal{P}(\Omega))$.

(v) Suppose $f : \Omega \to [0, \infty]$ is any function. We get a measure on $(\Omega, \mathcal{P}(\Omega))$. We get that

$$\mu(E) = \sum_{x\in E} f(x) := \sup\left\{\sum_{x\in F} f(x) \mid F \subseteq \Omega \text{ finite}\right\}.$$

If $f(x) = 1$ for all elements of Ω, then μ is called the counting measure, with $\mu(E) = \text{card}(E)$.

Proposition: Properties of Measures

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

(i) Monotonality: let $E, F \subseteq \mathcal{M}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$

(ii) Subadditivity: let $\{E_k\}_k \geq 1 \subseteq M$. Then, $\mu\left(\bigcup E_k\right) \leq \sum \mu(E_k)$.

(iii) Continuity (from below): say $\{E_k\}_{k\geq 1} \subseteq \mathcal{M}$ with $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$. Then,

$$\begin{aligned}\mu\left(\bigcup E_k\right) &= \lim_{k\to\infty} \mu(E_k)\\ &= \sup \mu(E_k).\end{aligned}$$

(iv) Set subtraction: if $E, F \subseteq \mathcal{M}$ with $E \subseteq F$ and $\mu(F) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.

(v) Continuity (from above): let $\{E_k\}_{k\geq 1} \subseteq \mathcal{M}$ with $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ and $\mu(E_1) < \infty$. Then,

$$\begin{aligned}\mu\left(\bigcap E_k\right) &= \lim_{k\to\infty} \mu(E_k)\\ &= \inf \mu(E_k).\end{aligned}$$

Proof:

(i) We have that $\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$.

(ii) Let $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $F_3 = E_3 \setminus (E_1 \cup E_2)$. Continuing, we have $F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k$. Notice

$$\begin{aligned}\bigsqcup F_k &= \bigcup E_k\\ \mu\left(\bigcup E_k\right) &= \mu\left(\bigsqcup F_k\right)\\ &= \sum \mu(F_k)\\ &\leq \sum \mu(E_k).\end{aligned}$$

(iii) Let $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, etc. with $F_k = E_k \setminus E_{k-1}$. Notice that

$$\begin{aligned}\mu\left(\bigsqcup F_k\right) &= \mu\left(\bigcup E_k\right)\\ \mu\left(\bigcup E_k\right) &= \sum \mu(F_k)\\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k)\\ &= \lim_{n\to\infty} \mu\left(\bigsqcup_{k=1}^{n} F_k\right)\\ &= \lim_{n\to\infty} \mu(E_n).\end{aligned}$$

(iv) For $E \subseteq F$, we have $\mu(F) = \mu(E) + \mu(F \setminus E)$. Subtracting, we have $\mu(F) \setminus \mu(E) = \mu(F \setminus E)$, provided $\mu(F)$ is finite.

(v) Exercise.

Complete Measure Spaces

If $(\Omega, \mathcal{M}, \mu)$ is a measures space, a subset $N \subseteq \Omega$ is μ-null if $N \in \mathcal{M}$ and $\mu(N) = 0$.

Remark: If N is μ-null, and $M \subseteq N$, then M is not necessarily μ-null, because we do not know if $M \in \mathcal{M}$.

A measure space $(\Omega, \mathcal{M}, \mu)$ is said to be complete if for any N μ-null and $M \subseteq N$, then M is μ-null.

If $(\Omega, \mathcal{M}, \mu)$, and $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$, we set

$$\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M}, F \subseteq N \in \mathcal{N} \text{ for some } N \in \mathcal{N}\}.$$

We have that $\overline{\mathcal{M}}$ is a σ-algebra with $\mathcal{M} \subseteq \overline{\mathcal{M}}$ and $\exists! \overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$, with $\overline{\mu}(E) = \mu(E)$ for all $E \in \mathcal{M}$, such that $(\Omega, \overline{\mathcal{M}}, \overline{\mu})$ is a complete measure space.

Outer Measures

An outer measure on a set Ω is a map $\theta : \mathcal{P}(\Omega) \to [0, \infty]$ such that

(i) $\theta(\emptyset) = 0$

(ii) $E \subseteq F \Rightarrow \theta(E) \leq \theta(F)$

(iii) $\theta\left(\bigcup E_k\right) \leq \sum \theta(E_k)$

Remark: Any measure is an outer measure.

We will construct outer measures from covering families equipped with a notion of measure.

Proposition: Constructing an Outer Measure

Let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ be a "covering family" — $\forall A \subseteq \Omega,\ A \subseteq \bigcup_{k \geq 1} E_k$, where $E_k \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$.

Set $\theta_\rho : \mathcal{P}(\Omega) \to [0, \infty]$; set

$$\theta_\rho(A) = \inf\left\{\sum \rho(E_k) \mid A \subseteq \bigcup E_k, E_k \in \mathcal{E}\right\}.$$

Then, θ_ρ is an outer measure.

Proof: Clearly, $\theta_\rho(\emptyset) = 0$.

Suppose $A \subseteq B$. If $B \subseteq \bigcup E_k$, then $A \subseteq \bigcup E_k$. Therefore, $\theta_\rho(A) \leq \sum \rho(E_k)$. By definition, it is then the case that $\theta_\rho(A) \leq \theta_\rho(B)$.

Let $\{A_k\}_{k \geq 1} \subseteq \mathcal{P}(\Omega)$. Let $\varepsilon > 0$. For each k, we can find a cover $A_k \subseteq \bigcup_{j=1}^{\infty} E_{k,j}$ such that

$$\theta_\rho(A_k) + \frac{\varepsilon}{2^k} > \sum_{j=1}^{\infty} \rho(E_{k_j})$$

$$\sum_{k=1}^{\infty} \theta_\rho(A_k) + \varepsilon > \sum_{j,k=1}^{\infty} \rho(E_{k,j}).$$

Since $\bigcup A_k \subseteq \bigcup_{k,j=1}^{\infty} E_{k,j}$, it must be the case that

$$\theta_\rho\left(\bigcup A_k\right) \leq \sum_{k,j=1}^{\infty} \rho(E_{k,j}).$$

Therefore, we have

$$\theta_\rho\left(\bigcup A_k\right) \leq \sum \theta_\rho(A_k) + \varepsilon.$$

Since ε was arbitrary, it must be the case that we get countable subadditivity.

Measurable Sets in Outer Measures

Let θ be an outer measure on Ω.

A subset $M \subseteq \Omega$ is said to be θ-measurable if $\forall E \subseteq \Omega$, $\theta(E \cap M) + \theta(E \cap M^c) = \theta(E)$. Essentially, M is a good "cookie-cutter" for any subset of Ω.

Remark: We always have $\theta(E) = \theta\left((E \cap M) \cup (E \cap M^c)\right) \leq \theta(E \cap M) + \theta(E \cap M^c)$. So, in order to show M is θ-measurable, all we need show is that $\theta(E \cap M) + \theta(E \cap M^c) \leq \theta(E)$.

This inequality always holds if $\theta(E) = \infty$.

Carathéodory's Theorem

Let $\theta : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be an outer measure on Ω.

(i) $\mathcal{M}_\theta = \{M \subseteq \Omega \mid M \text{ is } \theta\text{-measurable}\}$ is a σ-algebra.

(ii) $\theta|_{\mathcal{M}_\theta} : \mathcal{M}_\theta \rightarrow [0, \infty]$ is a complete measure.

Proof: We will show systematically via a series of claims.

Claim 1: $\mathcal{M}_\theta$ is an algebra of subsets.

- We have that $\emptyset \in \mathcal{M}_\theta$.

$$\begin{aligned}\theta(E) &\geq \theta(E \cap \emptyset) + \theta(E \cap \emptyset^c)\\ &= 0 + \theta(E).\end{aligned}$$

- Let $M \in \mathcal{M}_\theta$. Clearly, M^c is measurable, since the definition of measurable is symmetric.
- Suppose M_1, M_2 are measurable. We will show that $M_1 \cap M_2$ is measurable.

$$\begin{aligned}\theta(E) &\geq \theta(E \cap M_1) + \theta(E \cap M_1^c)\\ &\geq \theta(E \cap M_1 \cap M_2) + \theta(E \cap M_1 \cap M_2^c) + \theta(E \cap M_1^c \cap M_2) + \theta(E \cap M_1^c \cap M_2^c).\\ &\geq \theta(E \cap M_1 \cap M_2) + \theta\left(E \cap ((M_1 \cap M_2^c) \cup (M_1^c \cap M_2) \cup (M_1^c \cap M_2^c))\right)\\ &= \theta(E \cap M_1 \cap M_2) + \theta(E \cap (M_1 \cap M_2)^c).\end{aligned}$$

  Thus, $M_1 \cap M_2$ is measurable.

Claim 2: $\theta|_{\mathcal{M}_\theta}$ is a finitely additive measure. Let $M_1, M_2 \in \mathcal{M}_\theta$ with $M_1 \cap M_2 = \emptyset$.

$$\begin{aligned}\theta(M_1 \sqcup M_2) &= \theta(M_1 \sqcup M_2 \cap M_1) + \theta(M_1 \sqcup M_2 \cap M_1^c)\\ &= \theta(M_1) + \theta(M_2).\end{aligned}$$

Claim 3: If $\{M_k\}_{k\geq 1} \subseteq \mathcal{M}_\theta$ are pairwise disjoint, then $\forall E \subseteq \Omega$, $\theta\left(E \cap \bigsqcup M_k\right) = \sum \theta(E \cap M_k)$.

$$\theta\left(E \cap \bigsqcup_{k=1}^{n} M_k\right) = \theta\left(\bigsqcup_{k=1}^{n} E \cap M_k\right)$$

cutting with M_n, we have

$$\begin{aligned}&= \theta\left(\bigsqcup_{k=1}^{n} E \cap M_k \cap M_n\right) + \theta(\bigsqcup_{k=1}^{n} E \cap M_k \cap M_n^c)\\ &= \theta\left(E \cap M_n\right) + \theta\left(\bigsqcup_{k=1}^{n-1} E \cap M_k\right)\end{aligned}$$

cutting with M_{n-1}, we get

$$= \theta(E \cap M_n) + \theta(E \cap M_{n-1}) + E\left(\bigsqcup_{k=1}^{n-2} E \cap M_k\right).$$

Continuing inductively, we have

$$\theta\left(E \cap \bigsqcup_{k=1}^{n} M_k\right) = \sum_{k=1}^{n} \theta(E \cap M_k).$$

In the infinite case,

$$\begin{aligned}\sum \theta(E \cap M_k) &\geq \theta\left(\bigsqcup E \cap M_k\right)\\ &= \bigsqcup \theta\left(E \cap \bigsqcup M_k\right)\\ &\geq \theta\left(E \cap \bigsqcup_{k=1}^{n} M_k\right)\\ &= \bigsqcup_{k=1}^{n} \theta(E \cap M_k).\end{aligned}$$

Letting $n \to \infty$, we are done.

Claim 4: $\mathcal{M}_\theta$ is a σ-algebra. Additionally, $\theta|_{\mathcal{M}_\theta}$ is a measure.

Let $\{M_k\}_{k\geq 1} \subseteq \mathcal{M}_\theta$ be pairwise disjoint. Let $M = \bigsqcup M_k$. We will show M is measurable. Let $P_n = \bigsqcup_{k=1}^{n} M_k$. For $E \subseteq \Omega$,

$$\begin{aligned}\theta(E) &\geq \theta(E \cap P_n) + \theta(E \cap P_n^c) && \text{Claim 1}\\ &\geq \sum_{k=1}^{n} \theta(E \cap M_k) + \theta(E \cap M^c). && \text{Monotonicity}\end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned}\theta(E) &\geq \sum_{k=1}^{\infty} \theta(E \cap M_k) + \theta(E \cap M_c) && \text{Claim 3}\\ &= \theta(E \cap M) + \theta(E \cap M^c)\end{aligned}$$

Thus, M is measurable. Taking $E = \Omega$ in Claim 3, we show that $\theta|_{\mathcal{M}_\theta}$ is a measure.

Claim 5: $\theta|_{\mathcal{M}_\theta}$ is complete.

Let $N \subseteq \Omega$ with $\theta(N) = 0$. Then, for all $E \subseteq \Omega$,

$$\begin{aligned}\theta(E \cap N) + \theta(E \cap N^c) &\leq \theta(N) + \theta(E)\\ &= \theta(E).\end{aligned}$$

Thus, $N \in \mathcal{M}_\theta$. If $M \in \mathcal{M}_\theta$ and $\theta(M) = 0$, and $N \subseteq M$, then by monotonicity we have $\theta(N) = 0$, so $N \in \mathcal{M}_\theta$.

Remark: If $\theta(N) = 0$, then $N \in \mathcal{M}_\theta$, and $\theta(E \cup N) = \theta(E)$ and $\theta(E \setminus N) = \theta(E)$.

$$\begin{aligned}\theta(E) &\leq \theta(E \cup N)\\ &\leq \theta(E) + \theta(N)\\ &= \theta(E)\\ \theta(E) &= \theta(N \cup (E \setminus N))\\ &\leq \theta(N) + \theta(E \setminus N)\\ &= \theta(E \setminus N)\\ &\leq \theta(E)\end{aligned}$$

Lebesgue Measure over $\mathbb{R}$

Consider the family $\mathcal{E} = \{(a, b) \mid a \leq b\}$. Let $\lambda_0 : \mathcal{E} \to [0, \infty]$, with $\lambda_0((a, b)) = b - a$.

We see that $\mathcal{E}$ is a covering family with $\emptyset \in \mathcal{E}$. Notice that $\lambda_0(\emptyset) = 0$. As a result, we get the Lebesgue *outer* measure, $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, with

$$\lambda^*(E) := \inf\left\{\sum_{k=1}^{\infty} \lambda_0(I_k) \mid E \subseteq \bigcup_{k=1}^{\infty} I_k,\ I_k \in \mathcal{E}\right\}.$$

We thus define the Lebesgue σ-algebra as

$$\mathcal{L} = \{E \subseteq \mathbb{R} \mid E \text{ is } \lambda^*\text{-measurable}\}.$$

The Lebesgue measure is $\lambda := \lambda^*|_{\mathcal{L}}$. We know from Carathéodory's theorem that λ is complete.

Properties of the Lebesgue Measure

Proposition: Countable Subsets are Lebesgue Measurable

If $D \subseteq \mathbb{R}$ is countable, then $D \in \mathcal{L}$ and $\lambda(D) = 0$.

Proof: It suffices to show that for $t \in \mathbb{R}$, $\{t\}$ is Lebesgue measurable.

We have, for any $\varepsilon > 0$,

$$\{t\} \subseteq \left(t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}\right) \in \mathcal{E}.$$

Thus, $\lambda^*(\{t\}) \leq \lambda_0\left(\left(t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}\right)\right) = \varepsilon$. Since ε was arbitrary, we have that $\lambda^*(\{t\}) = 0$.

Thus, we have $\{t\} \in \mathcal{L}$. If $D = \{t_k\}_{k\geq 1}$ is countable, since each $\{t_k\} \in \mathcal{L}$, we have

$$\begin{aligned} D &= \bigcup_{k=1}^{\infty} \{t_k\} \in \mathcal{L}, \\ \lambda(D) &= \sum_{k=1}^{\infty} \lambda(\{t_k\}) \\ &= 0. \end{aligned}$$

The converse is not true: the Cantor set has measure 0.

Proposition: Borel Sets are Lebesgue Measurable

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}.$$

Proof: We show that $(-\infty, b) = I \in \mathcal{L}$ for any $b \in \mathbb{R}$. This is because $\sigma\left(\{(-\infty, b) \mid b \in \mathbb{R}\}\right) = \mathcal{B}_{\mathbb{R}}$, we will have that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$.

Let $E \subseteq \mathbb{R}$. Let $F = E \setminus \{b\}$. Let $F_1 = F \cap I = F \cap (-\infty, b)$, $F_2 = F \cap I^c = F \cap [b, \infty) = F \cap (b, \infty)$. Assume $F \subseteq \bigcup_{k=1}^{\infty} I_k$, with I_k open.

Let $L_k = (-\infty, b) \cap I_k$, $U_k = (b, \infty) \cap I_k$. Notice that L_k and U_k are open intervals, and $F_1 \subseteq \bigcup_{k=1}^{\infty} L_k$, $F_2 \subseteq \bigcup_{k=1}^{\infty} U_k$.

$$\begin{aligned} \lambda^*(F \cap I) + \lambda^*(F \cap I^c) &= \lambda^*(F_1) + \lambda^*(F_2) \\ &\leq \sum_{k=1}^{\infty} \lambda_0(L_k) + \sum_{k=1}^{\infty} \lambda_0(U_k) \\ &= \sum_{k=1}^{\infty} \left(\lambda_0(L_k) + \lambda_0(U_k)\right) \\ &= \sum_{k=1}^{\infty} \lambda_0(I_k) \end{aligned}$$

meaning

$$\lambda^*(F \cap I) + \lambda^*(F \cap I^c) \leq \lambda^*(F).$$

Therefore, F is λ^*-measurable. Notice that $E \cap I = F \cap I = F_1$, and $E \cap I^c = E \cap [b, \infty) \subseteq F_2 \cup \{b\}$. We have

$$\begin{aligned} \lambda^*(E \cap I) + \lambda^*(E \cap I^c) &\leq \lambda^*(F_1) + \lambda^*(F_2 \cup \{b\}) \\ &\leq \lambda^*(F_1) + \lambda^*(F_2) + \lambda^*(\{b\}) \\ &= \lambda^*(F_1) + \lambda^*(F_2) \\ &\leq \lambda^*(F) \\ &\leq \lambda^*(E). \end{aligned}$$

Thus, E is λ^*-measurable.

Remark: Every Borel set, including closed sets, open sets, compact sets, F_{σ}-sets, G_{δ}-sets, etc., is Lebesgue measurable.

Proposition: Measure of an Interval

If I is any interval, then $\lambda(I)$ is equal to the length of I.

Proof: Let $I = [a, b]$. For all $\varepsilon > 0$, we have

$$I \subseteq \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right),$$

meaning $\lambda^*(I) \leq (b-a) + \varepsilon$. Thus, we have $\lambda(I) = \lambda^*(I) \leq b - a$. To show the reverse direction, let

$$I \subseteq \bigcup_{k=1}^{\infty} I_k \qquad I_k \text{ open.}$$

It suffices to show that

$$\sum_{k=1}^{\infty} \lambda_0(I_k) \geq b - a.$$

Since I is compact, $\exists n$ with

$$I \subseteq \bigcup_{k=1}^{n} I_k.$$

Let $\ell = \sum_{k=1}^{n} I_k$ $(*)$.

Without loss of generality, let $a \in I_1 = (a_1, b_1)$. If $b_1 \geq b$, we are done. If not, we have $a_1 < a < b_1 < b$.

Now, $b_1 \in I \setminus I_1$. Without loss of generality, $b_1 \in I_2 = (a_2, b_2)$. If $b_2 \geq b$, we are done, as

$$\begin{aligned}
\ell &\geq (b_1 - a_1) + (b_2 - a_2) \\
&= b_2 - (a_2 - b_1) - a_1 \\
&\geq b - a_1 \\
&\geq b - a.
\end{aligned}$$

We continue this process; it must terminate, as there are finitely many such intervals, meaning $b_m \geq b$ for some m. We have a subcollection $\{(a_k, b_k)\}_{k=1}^{m}$, with $a_1 < a$, $a_2 < b_1 < b_2$, etc. all the way to $a_m < b_{m-1} < b_m$, and $b_m \geq b$.

$$\begin{aligned}
\ell &\geq \sum_{k=1}^{m} \lambda_0(a_k - b_k) \\
&= (b_m - a_m) + (b_{m-1} - a_{m-1}) + \cdots + (b_1 - a_1) \\
&= b_m - (a_m - b_{m-1}) - (a_{m-1} - b_{m-1}) - (a_2 - b_1) - a_1 \\
&= b_m + (b_{m-1} - a - m) + (b_{m-2} + a_{m-1}) + \cdots + (b_1 - a_2) - a_1 \\
&\geq b_m - a_1 \\
&\geq b - a_1 \\
&\geq b - a.
\end{aligned}$$

Thus, $\lambda^*([a, b]) = \lambda([a, b]) = b - a$.

Let $I = (a, b]$. Let $I_n = [a + 1/n, b]$. Then, $I = \bigcup_{n=1}^{\infty}[a + 1/n, b]$.

$$\begin{aligned}\lambda(I) &= \lambda\left(\bigcup_{n=1}^{\infty} I_n\right)\\ &= \lim_{n\to\infty}\lambda(I_n)\\ &= \lim_{n\to\infty}(b-a)-(1/n)\\ &= b-a.\end{aligned}$$

Similarly for $\lambda([a, b)) = b - a$, and $\lambda((a, b)) = b - a$.

If I is unbounded, for every n, we can find a closed and bounded $I_n \subseteq I$ with $\lambda(I_n) = n$. Therefore, $\lambda(I) \geq \lambda(I_n) = n$. Therefore, $\lambda(I) = \infty$.

Lemma: Translation-Invariance of the Outer Measure

For $E \subseteq \mathbb{R}$, $t \in \mathbb{R}$, $\lambda^*(E + t) = \lambda * *(E)$.

Proof: Given that $E \subseteq \bigcup_{k\geq 1} I_k$, then $E + t \subseteq \bigcup_{k\geq 1}(I_k + t)$. We have that $I_k + t$ are still open intervals. Note that $\lambda_0(I_k + t) = \lambda_0(I_k)$.

Therefore,

$$\begin{aligned}\lambda^*(E+t) &\leq \sum_{k=1}^{\infty}\lambda_0(I_k+t)\\ &= \sum_{k=1}^{\infty}\lambda_0(I_k).\end{aligned}$$

By definition, $\lambda^*(E + t) \leq \lambda^*(E)$.

Additionally,

$$\begin{aligned}\lambda^*(E) &= \lambda^*(E+t-t)\\ &\leq \lambda^*(E+t).\end{aligned}$$

Proposition: Translation-Invariance of the Lebesgue Measure

If $M \in \mathcal{L}$, and $t \in \mathbb{R}$, then $M + t \in \mathcal{L}$ and $\lambda(M + t) = \lambda(M)$.

Proof: Let $E \subseteq \mathbb{R}$.

$$\begin{aligned}\lambda^*(E) &= \lambda^*(E-t)\\ &= \lambda^*\left((E-t)\cap M\right) + \lambda^*\left((E-t)\cap M^c\right) && M\in\mathcal{L}\\ &= \lambda^*\left((E-t)\cap M + t\right) + \lambda^*\left((E-t)\cap M^c + t\right)\\ &= \lambda^*\left(E\cap(M+t)\right) + \lambda^*\left(E\cap(M+t)^c\right).\end{aligned}$$

Therefore, $M + t \in \mathcal{L}$, and

$$\begin{aligned}\lambda(M+t) &= \lambda^*(M+t)\\ &= \lambda^*(M)\\ &= \lambda(M).\end{aligned}$$

Thus, we have our measure space, $(\mathbb{R}, \mathcal{L}, \lambda)$, with

- λ complete;
- $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$;
- $\lambda(I) = \text{length}(I)$;
- $\lambda(E + t) = \lambda(E)$;
- λ is σ-finite.

Regularity of the Lebesgue Measure

Theorem: Approximating a Measurable Set

Let $M \in \mathcal{L}$.

(1) $\forall \varepsilon > 0$, $\exists U \subseteq \mathbb{R}$ open with $M \subseteq U$ and $\lambda(U \setminus M) < \varepsilon$.

(2) There is a G_δ set $V \subseteq \mathbb{R}$ with $M \subseteq V$ and $\lambda(V \setminus M) = 0$.

(3) $\forall \varepsilon > 0$, $\exists C \subseteq \mathbb{R}$ closed with $C \subseteq M$ and $\lambda(M \setminus C) < \varepsilon$.

(4) There is a F_σ set with $F \subseteq M$ and $\lambda(M \setminus F) = 0$.

Proof of (1): If $M \in \mathcal{L}$, then $\lambda(M) = \lambda^*(M)$. By definition, given $\varepsilon > 0$, $\exists M \subseteq \bigcup_{k\geq 1} I_k$, with I_k open, and

$$\begin{aligned}
\lambda(M) + \varepsilon &> \sum_{k=1}^{\infty} \lambda_0(I_k) \\
&= \sum_{k=1}^{\infty} \lambda(I_k) \\
&\geq \lambda\left(\bigcup_{k=1}^{\infty} I_k\right).
\end{aligned}$$

Set $U = \bigcup_{k=1}^{\infty} I_k$.

If $\lambda(M) < \infty$, then $\lambda(U \setminus M) = \lambda(U) - \lambda(M) < \varepsilon$. Otherwise, if $\lambda(M) = \infty$, then $M = \bigsqcup_{k=1}^{\infty} M_k$, where each $\lambda(M_k) < \infty$.

For each M_k, find U_k open with $U_K \supseteq M_k$ and $\lambda(U_k \setminus M_k) = \lambda(U_k) \setminus \lambda(M_k) < \varepsilon \cdot 2^{-k}$. Set $U = \bigcup U_k \supseteq M$. Then,

$$\begin{aligned}
\lambda(U \setminus M) &= \lambda\left(\bigcup_{k=1}^{\infty} \setminus \bigsqcup_{k=1}^{\infty} M_k\right) \\
&= \lambda\left(\bigcup_{k=1}^{\infty} (U_k \setminus M_k^c)\right) \\
&\leq \sum_{k=1}^{\infty} \lambda(U_k \setminus M_k) \\
&\leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} \\
&= \varepsilon.
\end{aligned}$$

Proof of (2): For every $n \geq 1$, find an open $U_n \subseteq \mathbb{R}$ with $U_n \supseteq M$ and $\lambda(U_n \setminus M) < 1/n$. Set $V = \bigcap_{n=1}^{\infty} U_n$.

$$\begin{aligned}\lambda(V \setminus M) &= \lambda(V \cap M^c)\\ &= \lambda\left(\bigcap_{n=1}^{\infty}(U_n \setminus M)\right)\\ &\leq \lambda(U_n \setminus M)\\ &< 1/n && \forall n\end{aligned}$$

meaning

$$\lambda(V \setminus M) = 0.$$

Proof of (3): $M^c \in \mathcal{L}$. Use (1) to prove.

Proof of (4): Use (3).

Corollary: Completion of the Borel Measure Space

$$(\mathbb{R}, \mathcal{L}, \lambda) = \left(\mathbb{R}, \overline{\mathcal{B}}_{\mathbb{R}}, \overline{\mu}\right),$$

where $\mu = \lambda|_{\mathcal{B}_{\mathbb{R}}}$.

Proof: We want to show that if $M \in \mathcal{L}$, then $M = B \cup E$, where $E \subseteq N \in \mathcal{B}_{\mathbb{R}}$, where $\mu(N) = 0$.

We know $\exists V$ G_δ and F F_σ, with $F \subseteq M \subseteq V$, $\lambda(M \setminus F) = \lambda(V \setminus M) = \emptyset$.

Set $M = F \cup (M \setminus F)$. We have F Borel, and $M \setminus F \subseteq V \setminus F$. We know that $\mu(V \setminus F) = 0$.

Corollary: Inner and Outer Regularity

Let $M \in \mathcal{L}$. Then,

- Outer regularity: $\lambda(M) = \inf\{\lambda(U) \mid U \supseteq M,\ U \text{ open}\}$
- Inner regularity: $\lambda(M) = \sup\{\lambda(K) \mid K \subseteq M,\ K \text{ compact}\}$

Proof of (1): If $\lambda(M) = \infty$, the proof is clear.

Assume $\lambda(M) < \infty$. The $\leq$ direction is clear.

Let $\varepsilon > 0$; we have $\exists U \subseteq \mathbb{R}$ with $M \subseteq U$ and $\lambda(U \setminus M) < \varepsilon$, so $\lambda(U) < \lambda(M) + \varepsilon$.

Proof of (2): Assume M is bounded. Given $\varepsilon > 0$, find C closed with $C \subseteq M$ and $\lambda(M \setminus C) < \varepsilon$. Since C is bounded, we have C is compact. Since M is bounded, $\lambda(M) < \infty$. Therefore, $\lambda(C) < \infty$, meaning $\lambda(M) - \lambda(C) < \varepsilon$, meaning $\lambda(M) - \varepsilon < \lambda(C)$. Since $\lambda(M)$ is an upper bound for the right hand side, we are done.

Suppose M is not bounded. Set $M_n = M \cap [-n, n]$. Notice that $M_1 \subseteq M_2 \subseteq \cdots$, with $\bigcup M_n = M$. Therefore,

$$\lambda(M) = \sup M_n.$$

Case 1: $\lambda(M) = +\infty$. For every n, find a compact $K_n \subseteq M_n$ (which is possible as the M_n are bounded) and $\lambda(M_n) - 1 < \lambda(K_n)$. Letting $n \to \infty$, we have $\lambda(K_n) \to \infty$. Therefore, $\sup \lambda(K_n) = \infty$.

Case 2: $\lambda(M) < \infty$. Given $\varepsilon > 0$, find n with $\lambda(M) - \varepsilon/2 < \lambda(M_n)$. There is a compact K with $K \subseteq M_n$ and $\lambda(M_n) - \varepsilon/2 < \lambda(K)$. Therefore, $K \subseteq M$ with $\lambda(M) - \varepsilon < \lambda(K)$.

Proposition: Symmetric Difference Approximation

Let $M \in \mathcal{L}$ with $\lambda(M) < \infty$. Given $\varepsilon > 0$, there is an open $V = \bigsqcup_{j=1}^{n}(a_j, b_j)$ such that $\lambda(M \triangle V) < \varepsilon$.

Proof: There is an open set U with $M \subseteq U$ and $\lambda(U \setminus M) = \lambda(U) - \lambda(M) < \varepsilon/2$. Since every open set is a disjoint union of open intervals, we have $U = \bigsqcup_{j=1}^{\infty}(a_j, b_j)$. Therefore, $\sum \lambda(a_j, b_j) \leq \lambda(M)$. Thus, $\exists n$ large such that $\sum_{j=n+1}^{\infty} \lambda(a_j, b_j) < \varepsilon/2$. Set $V = \bigsqcup_{j=1}^{n}(a_j, b_j)$.

Vitali's Theorem

Given $E \subseteq \mathbb{R}$ with $\lambda^*(E) > 0$, $\exists N \subseteq E$ with $N \notin \mathcal{L}$.

Proof: Assume E is bounded; $E \subseteq [-a, a]$. Put an equivalence relation on E: $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Therefore, $E = \bigsqcup_{i \in I}[x_i]$, with $x_i \in E$. Set $N = \{x_i\}_{i \in I}$. We claim that N is not measurable.

Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of the rationals inside $\mathbb{Q} \cap [-2a, 2a]$. Notice that $\{r_K + N\}_{k=1}^{\infty}$ are pairwise disjoint. Also, $E \subseteq \bigsqcup_{k=1}^{\infty} r_k + N$, since, given $x \in E$, we have $x \sim x_i$ for some $x_i \in N$, so $x - x_i \in \mathbb{Q} \cap [-2a, 2a]$, meaning $x - x_i = r_k$ for some $r_k \in \mathbb{Q} \cap [-2a, 2a]$, meaning $x \in r_k + N$.

If N were measurable, then

$$
\begin{aligned}
0 &< \lambda^*(E) \\
&\leq \lambda^*\left(\bigsqcup_{k=1}^{\infty} r_k + N\right) \\
&= \sum_{k=1}^{\infty} \lambda(r_k + N) \\
&= \sum_{k=1}^{\infty} \lambda(N).
\end{aligned}
$$

We also have $\lambda(N) = \lambda^*(N) \leq \lambda^*(E) \leq 2a$. Together, we arrive at a contradiction.

If E is not bounded, let $E_n = \cap[-n, n]$. Then,

$$
\begin{aligned}
0 &< \lambda^*(E) \\
&= \lambda^*(E_n) \\
&\leq \sum_{n=1}^{\infty} \lambda^*(E_n).
\end{aligned}
$$

Since $\lambda^*(E) > 0$, there must exist some E_n with $\lambda^*(E_n) > 0$, meaning E_n contains a non-measurable subset, so E has a non-measurable subset.

Cantor-Lebesgue Function

To find a non-Borel, Lebesgue-measurable set, we must construct and explore the properties of the Cantor-Lebesgue function.

Proof: Consider the Cantor set:

$$\begin{aligned} C_0 &= [0,1]\\ C_1 &= [0,1/3]\cup[2/3,1]\\ C_2 &= [0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]\\ C_n &= \frac{1}{3}\left(C_{n-1}\cup\left(2+C_{n-1}\right)\right)\\ \mathcal{C} &= \bigcap_{n=0}^{\infty} C_n. \end{aligned}$$

To find $\lambda(\mathcal{C})$, notice that $\mathcal{C}$ is closed (and thus Borel), meaning $\lambda(\mathcal{C}) = \lim_{n\to\infty}\lambda(C_n)$, meaning $\lambda(\mathcal{C}) = 0$.

We will build a function from the removed intervals of the Cantor set. Let

- $G_1 = C_0 \setminus C_1 = \underbrace{(1/3,2/3)}_{I_{1,1}}$
- $G_2 = C_1\setminus C_2 = \underbrace{(1/9,2/9)}_{I_{2,1}}\sqcup\underbrace{(7/9,8/9)}_{I_{2,2}}$
- $G_3 = C_2\setminus C_3 = \underbrace{(1/27,2/27)}_{I_{3,1}}\cup\underbrace{(7/27,8/27)}_{I_{3,2}}\cup\underbrace{(19/27,20/27)}_{I_{3,3}}\cup\underbrace{(25/27,26/27)}_{I_{3,4}}.$

At each step, we have $G_k = C_{k-1}\setminus C_k = \bigsqcup_{j=1}^{2^{k-1}} I_{k,j}$. If we let $L_k = \bigsqcup_{j=1}^{k} G_j$. Notice that $L_k \sqcup C_k = [0,1]$.

Let

$$\begin{aligned} g_k &= \sum_{j=1}^{2^{k-1}}\frac{2j-1}{2^k}\mathbb{1}_{I_{k,j}}\\ g_1 &= \frac{1}{2}\mathbb{1}_{(1/3,2/3)}\\ g_2 &= \frac{1}{4}\mathbb{1}_{(1/9,2/9)}+\frac{3}{4}\mathbb{1}_{(7/9,8/9)}\\ g_3 &= \frac{1}{8}\mathbb{1}_{(1/27,2/27)}+\frac{3}{8}\mathbb{1}_{(7/27,8/27)}+\frac{5}{8}\mathbb{1}_{(19/27,20/27)}+\frac{7}{8}\mathbb{1}_{(25/27,26/27)}. \end{aligned}$$

Now, let $f_n = \sum_{k=1}^{n} g_k$.

Let $\varphi_n : [0,1]\to[0,1]$ be the unique continuous extension of f_n, where $\varphi(0) = 0, \varphi(1) = 1$, and φ_n is linear on C_n.

We claim that $(\varphi_n)_n$ are uniformly Cauchy. Note that

$$|\varphi_{k+1}(x)-\varphi_k(x)| < \frac{1}{2^k}.$$

So, for $m > n$,

$$\begin{aligned} |\varphi_m(x)-\varphi_n(x)| &\leq |\varphi_m(x)-\varphi_{m-1}(x)| + \cdots |\varphi_{n+1}(x)-\varphi_n(x)|\\ &\leq 2^{1-m}+\cdots+2^{-n}\\ &\leq 2^{1-n}\\ \left\|\varphi_m(x)-\varphi_n(x)\right\|_u &\leq 2^{1-n}. \end{aligned}$$

Since $C([0,1])$ is complete, we must have that $(\varphi_n)_n \xrightarrow{\|\cdot\|_u} \varphi\in C([0,1])$. We call φ the Cantor-Lebesgue Function.

Properties of the Cantor-Lebesgue Function:

(1) φ is increasing;

(2) φ is constant on $[0,1] \setminus \mathcal{C}$;

(3) $\varphi([0,1]) = [0,1]$;

(4) $\varphi(\mathcal{C}) = [0,1]$.

Proof of Properties of Cantor-Lebesgue Function:

(1) If $x \leq y$, then $\varphi_n(x) \leq \varphi_n(y)$, meaning $\varphi(x) \leq \varphi(y)$ as $n \to \infty$.

(2) If $x \notin \mathcal{C}$, then $x \in \bigcup_{k=1}^{\infty} L_k$. Let $x \in L_\ell$. Thus, $\varphi(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} f_n(x) = f_\ell(x)$.

(3) Intermediate value theorem.

(4) We can see that $\mathcal{C} \sqcup \underbrace{[0,1] \setminus \mathcal{C}}_{L} = [0,1]$. Thus,

$$\varphi([0,1]) = \varphi(\mathcal{C}) \sqcup \varphi(L).$$

We can see that $\lambda(\varphi(L)) = 0$, since $\varphi(L)$ is a countable set. Thus, $\lambda(\varphi(\mathcal{C})) = 1$.

Since C is compact, $\varphi(C)$ is compact, and thus closed.

If $\exists t \in [0,1] \setminus \varphi(\mathcal{C})$, then $\exists \delta > 0$ such that $(t-\delta, t+\delta) \in [0,1] \setminus \varphi(\mathcal{C})$, implying $\lambda(\varphi(\mathcal{C})) < 1$.

Properties of a New Function: Let $\psi : [0,1] \to [0,2]$, $\psi(x) = x + \psi(x)$.

(1) ψ is strictly increasing

(2) $\psi : [0,1] \to [0,2]$ is bijective

(3) $\lambda(\psi(\mathcal{C})) = 1$

Proof of Properties of New Function:

(1) Trivial.

(2) Intermediate Value Theorem.

(3)

$$\begin{aligned}\psi(L) &= \psi\left(\bigsqcup L_k\right)\\ &= \bigsqcup \psi(L_k).\end{aligned}$$

Notice that

$$\begin{aligned}\psi(I_{k,j}) &= \{x + \varphi(x) \mid x \in I_{k,j}\}\\ &= \left\{x + \frac{2j-1}{2^k} \mid x \in I_{k,j}\right\}\\ \lambda(\psi(I_{k,j})) &= \lambda(I_{k,j}).\end{aligned}$$

Therefore, we see that

$$\begin{aligned}\lambda(\psi(G_k)) &= \lambda\left(\psi\left(\bigsqcup_{j=1}^{2k-1} I_{k,j}\right)\right)\\ &= \lambda\left(\bigsqcup \psi(I_{k,j})\right)\\ &= \sum_{j=1}^{2k-1} \lambda(\psi(I_{k,j}))\\ &= \sum_{j=1}^{2k-1} \lambda(I_{k,j})\\ &= \lambda(G_k).\end{aligned}$$

Thus,

$$\begin{aligned}\lambda(\psi(L)) &= \sum \lambda(\psi(G_k))\\ &= \sum \lambda(G_k)\\ &= \sum \lambda\left(\bigsqcup G_k\right)\\ &= 1,\end{aligned}$$

meaning

$$\begin{aligned}[0,2] &= \psi([0,1])\\ &= \psi(\mathcal{C} \sqcup L)\\ &= \psi(\mathcal{C}) \sqcup \psi(L)\\ &\Rightarrow \lambda(\psi(\mathcal{C})) = 1.\end{aligned}$$

Proposition: There is a set $E \in \mathcal{L} \setminus \mathcal{B}_{\mathbb{R}}$.

Proof of Proposition: We had $\psi : [0,1] \to [0,2]$ a continuous bijection with $\lambda(\psi(\mathcal{C})) = 1$. By Vitali's theorem, $\exists N \in \psi(\mathcal{C})$ with $N \notin \mathcal{L}$.

Set $E = \psi^{-1}(N) \subseteq \mathcal{C}$. Since $\lambda(\mathcal{C}) = 0$, and $E \subseteq \mathcal{C}$, $E \in \mathcal{L}$ (the Lebesgue measure is complete). Assume $E \in \mathcal{B}_{\mathbb{R}}$.

Since $\beta = \psi^{-1}$ is a continuous bijection, we have $N = \beta^{-1}(E)$ is Borel. $\perp$

Exercise: Let $f : X \to Y$ is a continuous map between two metric spaces. If $B \in \mathcal{B}_Y$, then $f^{-1}(B) \in \mathcal{B}_X$.

Measurable Functions

Measurable functions are morphisms in the category of measurable spaces.

Let $(\omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. $f : \Omega \to \Lambda$ is called $\mathcal{M}$-$\mathcal{N}$-measurable if $E \subseteq \mathcal{N}$ implies $f^{-1}(E) \in \mathcal{M}$.

When mapping into $\mathbb{R}$ or $\mathbb{C}$, we assume the codomain is equipped with the Borel σ-algebra.

(1) $f : \Omega \to \mathbb{F}$ is measurable if $f^{-1}(B) \in \mathcal{M}\ \forall B \in \mathcal{B}_{\mathbb{F}}$.

(2) For $S \subseteq \mathbb{R}$, $f : S \to \mathbb{F}$ is measurable if it is $\mathcal{L}_S - \mathcal{B}_{\mathbb{F}}$-measurable.

Proposition: Measurability On a Generated σ-Algebra

Suppose $\sigma(\mathcal{E}) = \mathcal{N}$. Then, $f : \Omega \Rightarrow *$ is measurable if and only if $f^{-1}(E) \in \mathcal{M}\ \forall E \in \mathcal{E}$.

Proof: The forward direction is clear.

In the reverse direction, let $\mathcal{F} = \left\{F \subseteq \Lambda \mid f^{-1}(F) \in \mathcal{M}\right\}$. We have that $\mathcal{E} \subseteq \mathcal{F}$. All we need do is show that $\mathcal{F}$ is a σ-algebra, so $\mathcal{N} \subseteq \mathcal{F}$.

Thus, if $f : X \to Y$ is continuous between metric spaces, then f is $\mathcal{B}_X$-$\mathcal{B}_Y$-measurable.

Additionally, $f : \Omega \to \mathbb{R}$ is measurable if and only if $[f < b] := \{x \in \Omega \mid f(x) < b\}$, or $[f < b] = f^{-1}\left((-\infty, b)\right)$ is in $\mathcal{M}$ for all $b \in \mathbb{R}$.

Extended Real Numbers

We sometimes work with the extended real numbers $\overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$. It isn't a field, but $-\infty \leq a \leq \infty$ for all $a \in \overline{\mathbb{R}}$.

Exercise: $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subseteq \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$ is a σ-algebra on $\overline{\mathbb{R}}$.

A member of $\mathcal{B}_{\overline{\mathbb{R}}}$ looks like $B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{\pm\infty\}$ where $B \in \mathcal{B}_{\mathbb{R}}$.

(1) $f : \Omega \to \overline{\mathbb{R}}$ is measurable if it is $\mathcal{M}$-$\mathcal{B}_{\overline{\mathbb{R}}}$-measurable.

(2) $S \subseteq \mathbb{R}$, $f : S \to \overline{\mathbb{R}}$ is measurable if f is $\mathcal{L}_S$-$\mathcal{B}_{\overline{\mathbb{R}}}$-measurable.

Proposition: Preservation of Measurability under Operations

If $f, g : \Omega \to \mathbb{R}$ are measurable, then

(1) $\alpha \in \mathbb{R}$, αf is measurable;

(2) $f \pm g$ is measurable;

(3) fg is measurable;

(4) $\frac{f}{g}$ is measurable provided $g \neq 0$ on Ω.

Proof of (2): Fix $b \in \mathbb{R}$. We want to show that $[f + g < b] \in \mathcal{M}$. Let $x \in \Omega$ such that $f(x) + g(x) < b$. Then, $f(x) < b - g(x)$.

So, $\exists r \in \mathbb{Q}$ with $f(x) < r < b - g(x)$. So, $g(x) < b - r$. Therefore, $[f + g < b] \subseteq \bigcup_{r \in \mathbb{Q}} \left([f < r] \cap [g < b - r]\right)$. Reverse inclusion is straightforward.

Proof of (3): First, we will show that f^2 is measurable.

If $b \leq 0$, then $[f^2 < b] = \emptyset$.

Let $b > 0$. Then, $[f^2 < b] = [-\sqrt{b} < f < \sqrt{b}] = f^{-1}\left((-\sqrt{b}, \sqrt{b})\right)$.

We have $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$, so from (1), (2), and above, we have fg is measurable.

Exercise: $\sigma\left(\{[-\infty, b) \mid b \in \mathbb{R}\}\right) = \sigma\left(\{[-\infty, b] \mid b \in \mathbb{R}\}\right) = \mathcal{B}_{\overline{\mathbb{R}}}$. When checking if $f : \Omega \to \overline{\mathbb{R}}$ is measurable, we need only check $f^{-1}([-\infty, b])$ is measurable.

Proposition: More Preservation of Measurability

Let $f, g : \Omega \to \overline{\mathbb{R}}$. Then,

(1) $\max(f, g)$ is measurable;

(2) $\min(f, g)$ is measurable;

(3) $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are measurable;

(4) $|f|$ is measurable.

Proof: Fix $b \in \mathbb{R}$.

(1)
$$[\max(f, g) < b] = [f < b] \cap [g < b].$$

(2)
$$[\min(f, g) < b] = [f < b] \cup [g < b].$$

(3)
$$\begin{gathered}[\max(f, 0) < b] = [f < b] \cap [0 < b] \\ [-\min(f, 0) < b] = [-f < b] \cup [0 < b].\end{gathered}$$

(4) $|f| = f^+ + f^-$.

Proposition: Sequence of Measurable Functions

Let $\left(f_n : \Omega \to \overline{\mathbb{R}}\right)_n$ be a sequence of measurable functions. Then,

(1) $\sup f_n$ is measurable;

(2) $\inf f_n$ is measurable;

(3) $\limsup f_n$ is measurable;

(4) $\liminf f_n$ is measurable.

Proof: Let $b \in \mathbb{R}$.

(1)
$$[\sup f_n \leq b] = \bigcap_{n=1}^{\infty} [f_n \leq b]$$

(2)
$$[\inf f_n < b] = \bigcup_{n=1}^{\infty} [f_n < b]$$

(3)
$$\limsup f_n = \inf_{m \geq 1} \left(\sup_{n \geq m} f_n\right)$$

(4)
$$\liminf f_n = \sup_{m \geq 1} \left(\sup_{n \geq m} f_n\right)$$

Proposition: Pointwise Convergence of Measurable Functions

Let $\left(f_n : \Omega \to \overline{\mathbb{R}}\right)$ be a sequence of measurable functions with $(f_n)_n \to f$ pointwise. Then, f is measurable.

Proof: If $(f_n)_n \to f$ pointwise, then $f = \limsup f_n = \liminf f_n$.

Simple Functions

(1) for $E \subseteq \Omega$, then $\mathbb{1}_E : \Omega \to \mathbb{R}$, the characteristic function of E, is defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

If $E = \{x_0\}$, then we write $\mathbb{1}_E = \delta_{x_0}$.

(2) A simple function $\phi : \Omega \to \mathbb{R}$ is a linear combination of characteristic functions.

$$\phi = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k}. \qquad c_k \in \mathbb{R},\ E_k \subseteq \Omega$$

Remark: ϕ can assume finitely many values, specifically at most 2^n.

If $\operatorname{ran}(\phi) = \{d_1, \dots, d_m\}$, where d_j are distinct. Write $D_j = \phi^{-1}(\{d_j\})$. Then,

$$\phi = \sum_{j=1}^{m} d_j \mathbb{1}_{D_j}$$

is known as the *standard form*, where d_j are distinct, and $\bigsqcup D_j = \Omega$.

Exercise 1: Given $\phi = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k}$, ϕ is measurable if and only if $E_k \in \mathcal{M}$.

Exercise 2: If X is a metric space, $\mathbb{1}_E$ is continuous if and only if E is clopen in X.

Proposition: Properties of Characteristic Functions

(1)

$$\mathbb{1}_{\bigsqcup D_j} = \sum_{j=1}^{m} \mathbb{1}_{D_j}$$

(2)

$$\mathbb{1}_E \cdot \mathbb{1}_F = \mathbb{1}_{E \cap F}$$

(3)

$$\Sigma(\Omega) := \{\phi \mid \phi : \Omega \to \mathbb{R} \text{ simple}\}$$
$$\Sigma(\Omega, \mathcal{M}) := \{\phi \mid \phi : \Omega \to \mathbb{R} \text{ simple and measurable}\}$$

is a unital separating subalgebra of $\mathcal{F}(\Omega, \mathbb{R})$.

(4) Let X be a compact, totally disconnected metric space. Then,

$$\mathfrak{C} := \operatorname{span}\{\mathbb{1}_E \mid E \subseteq X \text{ clopen}\}$$

is a unital separating subalgebra for $C(X)$.

Therefore, $\overline{\mathfrak{C}}^{\|\cdot\|_u} = C(X)$.

Theorem: Pointwise Convergence of Simple Measurable Functions

If $(\Omega, \mathcal{M})$ is a measurable space, and $f : \Omega \to \overline{\mathbb{R}}$ is measurable, there is a sequence $(\phi_n)_n$ of simple measurable functions such that $\phi_n(x) \to f(x)$ for all $x \in \Omega$.

If $f \geq 0$, we can take $(\phi_n)_n$ to be pointwise increasing.

If f is bounded, then $\left(\|f - \phi_n\|_u\right)_n \to 0$, and ϕ_n are uniformly bounded: $\sup \|\phi_n\|_u < \infty$.

Proof: Assume that $f \geq 0$. For each n, partition $[0, 2^n]$ into subintervals of length 2^{-n}. We will have 2^{2n} subintervals:

$$
\begin{aligned}
I_{n,0} &= \left[0, \frac{1}{2^n}\right] \\
I_{n,k} &= \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right],
\end{aligned}
$$

with $k = 1, 2, \dots, 2^{2n} - 1$. We define $J_n = (2^n, \infty]$.

Let $E_{n,k} = f^{-1}(I_{n,k})$, with $k = 1, 2, \dots, 2^{2n} - 1$. Let $F_n = f^{-1}(J_n)$.

Notice that $\left(\bigsqcup_{k=1}^{2^{2n}-1} E_{n,k}\right) \sqcup F_n = \Omega$, and $E_{n,k}$, F_n are measurable.

Let

$$
\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n}.
$$

We see that ϕ_n are simple, measurable, and positive.

Fix $x \in \Omega$. If $f(x)$ is finite, there is a large N with $f(x) \leq 2^N$. Fix $n \geq N$. Then, $\exists! k$ with $x \in E_{n,k}$, meaning that $\frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}$.

Thus, we have

$$
\begin{aligned}
|f(x) - \phi_n(x)| &= \left|f(x) - \frac{k}{2^n}\right| \\
&\leq \frac{1}{2^n}.
\end{aligned}
$$

Thus, as $n \to \infty$, we have $(\phi_n(x)) \to f(x)$.

If $f(x) = +\infty$, then $x \in F_n$ for all n. So, $\phi_n(x) = 2^n$ for all n, which converges to $f(x)$.

If f is bounded, then for large n, $F_n = \emptyset$. So, $\|f - \phi_n\|_u \leq 2^{-n}$, since our choice of N above works for all x. Thus, $(\varphi_n)_n \xrightarrow{\|\cdot\|_u} f$, and clearly $\sup \|\phi_n\|_u \leq \|f\|_u$.

If $f : \Omega \to \overline{\mathbb{R}}$ is measurable, then $f = f^+ - f^-$, where f^+ and f^- are positive and measurable. Perform the above procedure for f^+ and f^-, and subtract.

Proposition: Measure on set of Measurable Functions

Let $(\Omega, \mathcal{M})$ be a measurable space.

$$L_0(\Omega, \mathcal{M}) := \{f : \Omega \to \mathbb{R} \mid f \text{ measurable}\}$$

is a unital, commutative algebra. Let μ be a measure on $(\Omega, \mathcal{M})$. Define a relation on $L_0(\Omega, \mathcal{M})$:

$$f \sim_\mu g \Leftrightarrow \mu\left(\underbrace{\{x \mid f(x) \neq g(x)\}}_{((f-g)^{-1}(\{0\}))^c}\right) = 0.$$

Then, $\sim_\mu$ is an equivalence relation.

We define

$$L(\Omega, \mathcal{M}, \mu) := L_0(\Omega, \mathcal{M}) / \sim_\mu$$

is a unital, commutative algebra.

$$\begin{aligned} [f]_\mu + [g]_\mu &= [f+g]_\mu \\ \alpha[f]_\mu &= [\alpha f]_\mu \\ [f]_\mu \cdot [g]_\mu &= [fg]_\mu. \end{aligned}$$

Proof: Reflexivity and symmetry are clear.

Let $f \sim_\mu g \sim_\mu h$. Let $N := \{x \mid f(x) \neq g(x)\}$ and $M = \{x \mid g(x) \neq h(x)\}$. We know that $\mu(N) = 0 = \mu(M)$.

$$\begin{aligned} N^c \cap M^c &\subseteq \{x \mid f(x) = h(x)\}. \\ \{x \mid f(x) \neq h(x)\} &\subseteq N \cup M. \end{aligned}$$

Since $\mu(N \cup M) = 0$, so too is $\mu(\{x \mid f(x) \neq h(x)\})$.

Essentially Bounded Functions

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Suppose $f \in L_0(\Omega, \mathcal{M})$.

(1) $c \geq 0$ is an essential bound for f if $\mu(\{x \mid |f(x)| > c\}) = 0$. If f admits an essential bound, f is called essentially bounded.

(2) The essential supremum, $\operatorname{ess\,sup}(f) = \inf(\{c \mid c \text{ is an essential bound}\})$. We say $\operatorname{ess\,sup}(f) = \infty$ if f has no essential bound.

For example, if $f = \mathbb{1}_{\mathbb{Q}}$, then $\operatorname{ess\,sup}(f) = 0$. At the same time, $\|f\|_u = 1$.

Lemma: Essential Supremum Property

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. For $f \in L_0(\Omega, \mathcal{M})$, $|f(x)| \leq \operatorname{ess\,sup}(f)$ for almost every $x \in \Omega$ (μ-almost everywhere). We say μ-a.e. if $x \in \Omega$ means $\forall x \in \Omega \setminus N$, where $\mu(N) = 0$.

Proof: If $\operatorname{ess\,sup}(f) = \infty$, then we are done.

Suppose $c_f = \operatorname{ess\,sup}(f) < \infty$. For $n \geq 1$, $\exists$ essential bound c_n for f such that $c_f + 1/n > c_n$.

Let $N_n = \{x \mid |f(x)| > c_n\}$. Since c_n is an essential bound, $\mu(N_n) = 0$.

$$\begin{aligned}
\mu\left(\{x \mid |f(x)| \leq c_f\}^c\right) &= \mu\left(\{x \mid |f(x)| > c_f\}^c\right)\\
&= \mu\left(\bigcup_{n\geq 1}\{x \mid |f(x)| > c_f + 1/n\}\right)\\
&\subseteq \mu\left(\bigcup_{n\geq 1}\{x \mid |f(x)| > c_n\}\right)\\
&= \mu\left(\bigcup_{n\geq 1} N_n\right)\\
&= 0.
\end{aligned}$$

Proposition: Arithmetic Operations of Essential Supremum

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and $f, g \in L_0(\Omega, \mathcal{M})$. Then,

(1) $\operatorname{ess\,sup}(f+g) \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g)$

(2) $\operatorname{ess\,sup}(\alpha f) = |\alpha| \operatorname{ess\,sup}(f)$

(3) $\operatorname{ess\,sup}(fg) \leq (\operatorname{ess\,sup}(f))(\operatorname{ess\,sup}(g))$

(4) $\operatorname{ess\,sup}(f) = 0 \Rightarrow f = 0$ μ-a.e., so $[f]_0 = L(\Omega, \mathcal{M}, \mu)$

(5) $\operatorname{ess\,sup}(\mathbb{1}_\Omega) = 1$

(6) $\operatorname{ess\,sup}(f) \leq \|f\|_u$

(7) $f \sim_\mu g \Rightarrow \operatorname{ess\,sup}(f) = \operatorname{ess\,sup}(g)$.

Proof of (1): Assume $c_f = \operatorname{ess\,sup}(f)$, $c_g = \operatorname{ess\,sup}(g)$, with $c_f, c_g < \infty$.

Let $N = \{x \mid |f(x)| > c_f\}$ and $M = \{x \mid |g(x)| > c_g\}$. Both N and M are μ-null, by the lemma.

$$\underbrace{\{x \mid |(f+g)(x)| > c_f + c_g\}}_{\mu\text{-null } (f+g \text{ measurable})} \subseteq N \cup M.$$

Therefore, $c_f + c_g$ is an essential bound for $f+g$. Thus, $\operatorname{ess\,sup}(f+g) \leq c_f + c_g$.

Proof of (7): Let $N = \{x \mid f(x) \neq g(x)\}$. It is the case that $\mu(N) = 0$. Let $c_f = \operatorname{ess\,sup}(f)$ and $N_f = \{x \mid |f(x)| > c_f\}$, which is is μ-null by the lemma.

Then, $\{x \mid |g(x)| > c_f\} \subseteq N_f \cup N$ is μ-null.

Therefore, c_f is an essential bound for g. Thus, $\operatorname{ess\,sup}(g) \leq c_f$.

Similarly, $\operatorname{ess\,sup}(f) \leq c_g$.

Proposition: Properties of L_∞

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

$$\{[f] \in L(\Omega, \mathcal{M}, \mu) \mid \operatorname{ess\,sup}(f) < \infty\}$$

is a unital commutative Banach algebra with norm $\|[f]_\mu\|_\infty = \operatorname{ess\,sup}(f)$. It is denoted $L_\infty(\Omega, \mu)$.

Proof: All we need show is completeness.

Let $(f_n)_n$ be Cauchy in $L_\infty(\Omega, \mu)$. Then, $|f_n(x)| \leq \|f_n\|_\infty$ for all $x \in N_n^c$, where $\mu(N_n) = 0$. Let $N = \bigcup_{n\geq 1} N_n$. Then, $\mu(N) = 0$.

For all $x \in N^c$, we have $|f_n(x)| \leq \|f_n\|_\infty$ for all n. Set

$$g_n(x) = \begin{cases} f_n(x) & x \in N^c \\ 0 & x \in N \end{cases}.$$

Then, $g_n = f_n$ in $L_\infty(\Omega, \mu)$. Note that $(g_n : \Omega \to \mathbb{R})_{n\geq 1}$ are uniformly Cauchy in $\ell_\infty(\Omega)$ (in N^c, $|g_n - g_m| = |f_n - f_m| < \varepsilon$, and in N, $|g_n - g_m| = 0$).

Since $\ell_\infty(\Omega)$ is complete, we know $(g_n)_n \to g$ in $\ell_\infty(\Omega)$. Certainly, $g \in L_\infty(\Omega, \mu)$. Thus,

$$\begin{aligned} \|f_n - g\|_\infty &= \|g_n - g\|_\infty \\ &\leq \|g_n - g\|_u \\ &\to 0, \end{aligned}$$

so $L_\infty(\Omega, \mu)$ is complete.

Lebesgue Integration

Fix a measure space $(\Omega, \mathcal{M}, \mu)$.

Define $\phi : \Omega \to [0, \infty)$ be simple, positive, and measurable, given by

$$\phi = \sum_{k=1}^{n} d_k \mathbb{1}_{D_k}. \qquad \text{Standard Form}$$

Then,

$$\int_\Omega \phi \, d\mu := \sum_{k=1}^{n} d_k \mu(D_k),$$

with the convention that $0 \cdot \infty = 0$.

Fact: If $\phi = \sum_{j=1}^{m} c_j \mathbb{1}_{E_j}$, with $c_j \geq 0$ and $E_j \in \mathcal{M}$, not necessarily in standard form. Then,

$$\int_\Omega \phi \, d\mu = \sum_{j=1}^{m} c_j \mu(E_j).$$

Properties of Integral of Simple Functions

Let $\phi, \psi : \Omega \to [0, \infty)$ be simple, measurable, and positive. Then,

(i)

$$\int_{\Omega} (\phi + \psi)\ d\mu = \int_{\Omega} \phi\ d\mu + \int_{\Omega} \psi\ d\mu$$

(ii) For $\alpha \geq 0$

$$\int_{\Omega} \alpha\phi\ d\mu = \alpha \int_{\Omega} \phi\ d\mu.$$

(iii) If $0 \leq \phi \leq \psi$, then

$$\int_{\Omega} \phi\ d\mu \leq \int_{\Omega} \psi\ d\mu$$

Proof of (iii): Let

$$\begin{aligned} \phi &= \sum_{k=1}^{n} c_k \mathbb{1}_{E_k} \\ \psi &= \sum_{\ell=1}^{m} d_\ell \mathbb{1}_{F_\ell} \end{aligned}$$

be standard representations. Consider a common refinement $\{E_k \cap F_\ell\}_{k,\ell}$. Then,

$$\begin{aligned} \mathbb{1}_{E_k} &= \mathbb{1}_{\bigsqcup_\ell E_k \cap F_\ell} \\ &= \sum_{\ell=1}^{m} \mathbb{1}_{E_k \cap F_\ell}. \end{aligned}$$

Thus,

$$\begin{aligned} \phi &= \sum_{k=1}^{n} c_k \sum_{\ell=1}^{m} \mathbb{1}_{E_k \cap F_\ell} \\ &= \sum_{k,\ell} c_{k,\ell} \mathbb{1}_{E_k \cap F_\ell}, \end{aligned}$$

where

$$c_{k,\ell} = \begin{cases} 0 & E_k \cap F_\ell = \emptyset \\ c_k & E_k \cap F_\ell \neq \emptyset \end{cases}.$$

Similarly,

$$\begin{aligned} \psi &= \sum_{k,\ell} d_{k,\ell} \mathbb{1}_{E_k \cap F_\ell}, \\ d_{k,\ell} &= \begin{cases} d_\ell & E_k \cap F_\ell \neq \emptyset \\ 0 & E_k \cap F_\ell = \emptyset \end{cases}. \end{aligned}$$

Then, $c_{k,\ell} \leq d_{k,\ell}$.

$$\begin{aligned} \int_{\Omega} \phi\ d\mu &= \sum_{k,\ell} c_{k,\ell} \mu(E_k \cap F_\ell) \\ &\leq \sum_{k,\ell} d_{k,\ell} \mu(E_k \cap F_\ell) \\ &= \int_{\Omega} \psi\ d\mu. \end{aligned}$$

Definition of the Lebesgue Integral

Let $f : \Omega \to [0, \infty]$ be measurable. Then,

$$\int_{\Omega} f \, d\mu := \sup \left\{ \int_{\Omega} \phi \, d\mu \mid 0 \leq \phi \leq f, \text{ simple, measurable} \right\}$$

For $E \in \mathcal{M}$, we define

$$\int_{E} f \, d\mu = \int_{\Omega} (f) \left(\mathbb{1}_E\right) \, d\mu.$$

We say f is (Lebesgue) integrable if $\int_{\Omega} f \, d\mu < \infty$.

Exercise:

$$\int_{(0,1]} \frac{1}{x} \, d\lambda = +\infty.$$

Proposition: Properties of the Lebesgue Integral

The following follow from the results about simple functions. Let $f, g : \Omega \to [0, \infty]$ measurable.

(1) For $\alpha \geq 0$

$$\int_{\Omega} (\alpha f) \, d\mu = \alpha \int_{\Omega} f \, d\mu;$$

(2) For $0 \leq f \leq g$

$$\int_{\Omega} f \, d\mu \leq \int_{\Omega} g \, d\mu.$$

Theorem: Monotone Convergence of Lebesgue Integral

Suppose $(f_n : \Omega \to [0, \infty])_{n \geq 1}$ are positive, measurable, and pointwise increasing. Let $f : \Omega \to [0, \infty]$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$. Then,

$$\begin{aligned} \int_{\Omega} f \, d\mu &= \lim_{n\to\infty} \int_{\Omega} f_n \, d\mu \\ &= \sup \int_{\Omega} f_n \, d\mu. \end{aligned}$$

Proof: Note that $\lim_{n\to\infty} f_n(x) \in [0, \infty]$ always exists, since $(f_n(x))_n$ is an increasing sequence.

Also, f is measurable (pointwise limit of measurable functions). Moreover,

$$\int_{\Omega} f_n \, d\mu \leq \int_{\Omega} f_{n+1} \, d\mu.$$

Therefore,

$$\lim_{n\to\infty} \int_{\Omega} f_n \, d\mu = \sup \int_{\Omega} f_n \, d\mu$$

exists in $[0, \infty]$. Note that $\int_{\Omega} f_n \, d\mu \leq \int_{\Omega} f \, d\mu$, since $f_n \leq f$. Thus, $\sup \int_{\Omega} f_n \, d\mu \leq \int_{\Omega} f \, d\mu$.

Let $0 < t < 1$. Let $\phi : \Omega \to [0, \infty)$ be simple and measurable with $0 \leq \phi \leq f$.

Set $E_n = \{x \in \Omega \mid f_n(x) \geq t\phi(x)\}$. Note $E_1 \subseteq E_2 \subseteq \cdots$ (since f_n are increasing). Additionally,

$$\bigcup_{n\geq 1} E_n = \Omega.$$

Notice that E_n are also measurable.

If $A \subseteq \Omega$ is any measurable set. Then, $A \cap E_1 \subseteq A \cap E_2 \subseteq \cdots$, and $\bigcup_{n\geq 1}(A \cap E_n) = A$. Therefore, $(\mu(A \cap E_n))_n \to \mu(A)$ by continuity of μ.

Suppose $\phi = \sum_{k=1}^{m} a_k \mathbb{1}_{A_k}$. Then,

$$\begin{aligned}
\phi\mathbb{1}_{E_n} &= \sum_{k=1}^{m} a_k \mathbb{1}_{A_k\cap E_n}\\
\int_{\Omega} \phi\mathbb{1}_{E_n}\, d\mu &= \sum_{k=1}^{m} a_k \mu(A_k \cap E_n)\\
&\to \sum_{k=1}^{m} a_k \mu(A_k)\\
&= \int_{\Omega} \phi\, d\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\Omega} f_n\, d\mu &\geq \int_{\Omega} f_n \mathbb{1}_{E_n}\, d\mu\\
&\geq \int_{\Omega} t\phi\mathbb{1}_{E_n}\, d\mu\\
&= t\int_{\Omega} \phi\mathbb{1}_{E_n}\, d\mu\\
\lim_{n\to\infty} \int_{\Omega} f_n\, d\mu &\geq t\int_{\Omega} \phi\, d\mu.
\end{aligned}$$

Taking the supremum over all ϕ,

$$t\int_{\Omega} f\, d\mu \leq \lim_{n\to\infty} \int_{\Omega} f_n\, d\mu,$$

and taking the supremum over all t, we get

$$\int_{\Omega} f\, d\mu \leq \lim_{n\to\infty} \int_{\Omega} f_n\, d\mu.$$

Remark: Given $f : \Omega \to [0, \infty]$ measurable, we proved that there exists a sequence $(\phi_n)_n$ of positive, simple, measurable functions with $(\phi_n)_n \to \phi$ pointwise increasing. Thus, by the monotone convergence theorem, $\int_{\Omega} \phi\, d\mu \to \int_{\Omega} f\, d\mu$.

Linearity of the Lebesgue Integral over $[0, \infty]$

Let $f, g : \Omega \to [0, \infty]$ be measurable. Then,

$$\int_{\Omega} (f + g)\, d\mu = \int_{\Omega} f\, d\mu + \int_{\Omega} g\, d\mu.$$

Proof: Use the Monotone Convergence Theorem and the earlier remark.

Lebesgue Integral over $\overline{\mathbb{R}}$

Let $f : \Omega \to \overline{\mathbb{R}}$ be measurable.

(1) If either f^+ or f^- is measurable, then

$$\int_\Omega f\ d\mu := \int_\Omega f^+\ d\mu - \int_\Omega f^-\ d\mu.$$

(2) f is said to be integrable if both f^+ and f^- are integrable.

Lemma: Absolute Value of Integrable Function

f is integrable if and only if $|f|$ is integrable.

Proof: If f is integrable, then f^+ and f^- are integrable, meaning

$$|f| = f^+ + f^-$$
$$\int_\Omega |f|\ d\mu = \int_\Omega f^+\ d\mu + \int_\Omega f^-\ d\mu.$$

If $|f|$ is integrable, then $\int_\Omega f\ d\mu \leq \int_\Omega |f|\ d\mu < \infty$.

Proposition: Linearity of the Lebesgue Integral over $\mathbb{R}$

Let $f, g : \Omega \to \mathbb{R}$ be integrable.

(1)

$$\int_\Omega \alpha f\ d\mu = \alpha \int_\Omega f\ d\mu$$

(2)

$$\int_\Omega (f+g)\ d\mu = \int_\Omega f\ d\mu + \int_\Omega g\ d\mu.$$

(3) If $f \leq g$, then

$$\int_\Omega f\ d\mu \leq \int_\Omega g\ d\mu$$

(4)

$$\left|\int_\Omega f\ d\mu\right| \leq \int_\Omega |f| d\ \mu$$

Proof of (2): Write h as $f + g$. Note that $|h| \leq |f| + |g|$, so h is integrable. Then,

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$
$$h^+ + f^- + g^- = f^+ + g^+ + h^-.$$

Integrating and using linearity, we get

$$\int h^+\ d\mu + \int g^-\ d\mu + \int f^-\ d\mu = \int f^+\ d\mu + \int g^+\ d\mu + \int h^-\ d\mu$$
$$\int h^+ d\mu - \int h^-\ d\mu = \int f^+\ d\mu - \int f^-\ d\mu + \int g^+\ d\mu - \int g^-\ d\mu$$
$$\int h\ d\mu = \int f\ d\mu + \int g\ d\mu.$$

Proof of (3): If $f \leq g$, then $g - f \geq 0$, so $\int (g-f)\ d\mu \geq 0$, meaning $\int g\ d\mu - \int f\ d\mu \geq 0$.

Proof of (4): $-|f| \leq f \leq |f|$. Using (3) and (1),

$$
\begin{gathered}
-\int |f|\ d\mu \leq \int f\ d\mu \leq \int |f|\ d\mu \\
\left|\int f\ d\mu\right| \leq \int |f|\ d\mu.
\end{gathered}
$$

Proposition: Integrable Function over Extended Real Line

Let $f : \Omega \to \overline{\mathbb{R}}$ is integrable. Then, f is finite μ-almost everywhere.

Proof: Let $E = \{x \mid |f(x)| = \infty\}$. For any $n \in \mathbb{N}$, $|f| \geq n\mathbb{1}_E$.

Therefore, we have $\infty > \int_\Omega f\ d\mu \geq \int_\Omega n\mathbb{1}_E = n\mu(E)$. Since this is true for any n, it must be the case that $\mu(E) = 0$.

Chebyshev's Inequality

Let $f : \Omega \to [0, \infty]$ be integrable. Then, $\mu\left(\{x \mid f(x) \geq t\}\right) \leq \frac{1}{t}\int_\Omega f\ d\mu$.

Proof: Let $E = \{x \mid f(x) \geq t\}$. Thus, $f \geq t\mathbb{1}_E$.