Introduction

Oh hey, it's another one of these independent studies. Me and a friend are going to be going through William Fulton's *Algebraic Curves*. It will be hard, it will be long, and it might not work out for me, but who cares.

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We will assume all rings are commutative with unity, where $\mathbb{Z}$ is the integers, $\mathbb{Q}$ is the rationals, $\mathbb{R}$ is the reals, and $\mathbb{C}$ is the complex numbers.

Any integral domain R has a quotient field K, which contains R as a subring, and any element in K may be written as a not necessarily unique ratio of two elements of R. Any one-to-one ring homomorphism from R to a field L extends uniquely to a ring homomorphism from K to L.

If R is a ring, then $R[x]$ is the ring of polynomials with coefficients in R. The degree of a nonzero polynomial $\sum a_i x^i$ is the largest integer d such that $a_d \neq 0$. The polynomial is monic if $a_d = 1$.

The ring of polynomials in n variables over R is $R[x_1, \dots, x_n]$. We write $R[x, y]$ and $R[x, y, z]$ if $n = 2$ and 3 respectively. Monomials in $R[x_1, \dots, x_n]$ are of the form $x^{(i)} := x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where i_j are nonnegative integers, and the degree of the monomial is $i_1 + \cdots i_n$. Every $F \in R[x_1, \dots, x_n]$ has a unique expression $F = \sum a_{(i)} x^{(i)}$, where $x^{(i)}$ are monomials, and $a_{(i)} \in R$. We say F is homogeneous of degree d if all $a_{(i)}$ are zero except for monomials of degree d. The polynomial F is written as $F = F_0 + F_1 + \cdots F_d$, where F_i is a form of degree i, and $d = \deg(F)$ for $F_d \neq 0$.

The ring R is a subring of $R[x_1, \dots, x_n]$, and the ring $R[x_1, \dots, x_n]$ is characterized by the following: if $\varphi \colon R \to S$ is a ring homomorphism, and $s_1, \dots, x_n$ are elements in S, then there is a unique extension of φ to a ring homomorphism $\overline{\varphi} \colon R[x_1, \dots, x_n] \to S$ such that $\overline{\varphi}(x_i) = s_i$. The image of F under $\overline{\varphi}$ is written $F(s_1, \dots, s_n)$. The ring $R[x_1, \dots, x_n]$ is canonically isomorphic to $R[x_1, \dots, x_{n-1}][x_n]$.

An element $a \in R$ is called irreducible if it is not a unit or zero, and any factorization $a = bc$ with $b, c \in R$ is such that either b or c is a unit. A domain R is a unique factorization domain (UFD) if every nonzero element in R can be factored uniquely up to units and ordering.

If R is a UFD with quotient field K, then any irreducible element $F \in R[x]$ remains irreducible when considered in $K[x]$.

Theorem (Gauss's Lemma for $\mathbb{Z}$)**:** If $F \in \mathbb{Z}[x]$ is a monic polynomial that is irreducible, then F is irreducible in $\mathbb{Q}[x]$.

If F and G are polynomials in $R[x]$ with no common factors in $R[x]$, then they have no common factors in $K[x]$.

If R is a UFD, then $R[x]$ is also a UFD, and consequently $k[x_1, \dots, x_n]$ is a UFD for any field k. The quotient field of $k[x_1, \dots, x_n]$ is written $k(x_1, \dots, x_n)$ is called the field of rational functions in n variables over k.

If $\varphi\colon R \to S$ is a ring homomorphism, $\ker(\varphi) := \varphi^{-1}(0)$. The kernel is an ideal in R. An ideal in R is proper if $I \neq R$, and a proper ideal is known as maximal if it is not contained in any larger proper ideal.[I] An ideal $\mathfrak{p}$ is prime if, whenever $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.[II]

Let k be a field and I a proper ideal in $k[x_1, \dots, x_n]$. The canonical homomorphism π from $k[x_1, \dots, x_n]$ to $k[x_1, \dots, x_n]/I$ restricts to a ring homomorphism from k to $k[x_1, \dots, x_n]/I$. We regard k as a subring of $k[x_1, \dots, x_n]/I$, which is a vector space over k.

If R is an integral domain, then $\operatorname{char}(R)$, the characteristic of R, is the smallest integer p such that

$$\underbrace{1 + 1 \cdots + 1}_{p \text{ times}} = 0.$$

If p exists, we say $\operatorname{char}(R) = p$, else 0.

Note that if $\varphi\colon \mathbb{Z} \to R$ is the unique ring homomorphism from $\mathbb{Z}$ to R,[III] then $\ker(\varphi) = \langle p \rangle$, so $\operatorname{char}(R)$ is prime or 0.

If R is a ring, and $F \in R[x]$, and a is a root of F, then $F = (x - a)G$ for some unique polynomial $G \in R[x]$. A field k is algebraically closed if any nonconstant $F \in k[x]$ has a root.

Exercise (Exercise 1.1)**:** Let R be an integral domain.

(a) If F and G are forms of degree r and s respectively in $R[x_1, \dots, x_n]$, show that FG is a form of degree $r + s$.

(b) Show that any factor of a form in $R[x_1, \dots, x_n]$ is also a form.

Exercise (Exercise 1.2)**:** Let R be a UFD and K the quotient field of R. Show that every element $z \in K$ may be written as $z = a/b$, where $a, b \in R$ have no common factors. This representative is unique up to units of R.

Solution: Since $K = \operatorname{Frac}(R)$, we know that every $z \in K$ is of the form $z = \frac{a}{b}$. Since R a unique factorization domain, $\gcd(a, b)$ is unique and well-defined. Set $c \cdot \gcd(a, b) = a$ and $d \cdot \gcd(a, b) = b$. Then,

$$\begin{aligned} z &= \frac{a}{b} \\ &= \frac{c \cdot \gcd(a, b)}{d \cdot \gcd(a, b)} \\ &= \frac{c}{d}. \end{aligned}$$

We show that this is unique up to units. Suppose

$$\begin{aligned} z &= \frac{c}{d} \\ &= \frac{c'}{d'}. \end{aligned}$$

Then, by the properties of the field of fractions, we know that

$$c'd = cd',$$

and since R is a UFD, we know that $\gcd(c, d) = \gcd(c', d') = 1$, so $c = u_1 c'$ and $d = u_2 d'$.

Exercise (Exercise 1.3)**:** Let R be a principal ideal domain, and let P be a nonzero proper prime ideal in R.

(a) Show that P is generated by an irreducible element.

[I] Alternatively, an ideal I is maximal if the quotient ring R/M is a field.

[II] Alternatively, an ideal $\mathfrak{p}$ is prime if $R/\mathfrak{p}$ is an integral domain.

[III] This is because $\mathbb{Z}$ is initial in the category of rings. See Aluffi.

(b) Show that P is maximal.

Solution:

(a) Since P is principal, we know that $P = \langle a \rangle$ for some $a \in R$. We know that a cannot be a unit, as otherwise $P = R$, contradicting the assumption that P is proper, and that $a \neq 0$ as P is not zero.

Suppose toward contradiction that $\langle a \rangle \subsetneq \langle b \rangle$ for some $b \in R$. Then, $a = bc$ for some $c \in R$. If $c \notin \langle a \rangle$, then since $\langle a \rangle$ is prime, we must have $b \in \langle a \rangle$, contradicting strict inclusion. Thus, $c \in \langle a \rangle$, so $c = at$ for some $t \in R$. Therefore, we have $a = abt$, so $bt = 1_R$, and $\langle b \rangle = R$.

(b) Since R is a PID, and P is prime, we know that $P = \langle a \rangle$ is generated by an irreducible element. Thus, if $\langle a \rangle \subseteq \langle b \rangle$, then $a = bc$ for some $c \in R$. Since we have unique factorization (as all PIDs are UFDs), and a is irreducible, this means either b or c is a unit. If b is a unit, then $\langle b \rangle = R$, and if c is a unit, then $\langle b \rangle = \langle a \rangle$. Thus, $\langle a \rangle$ is maximal.

Exercise (Exercise 1.4)**:** Let k be an infinite field, $f \in k[x_1, \dots, x_n]$. Suppose $F(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. Show that $f = 0$.

Exercise (Exercise 1.5)**:** Let k be any field. Show that there are an infinite number of irreducible monic polynomials in $k[x]$.

Solution: Suppose $F_1, \dots, F_n$ were all the irreducible monic polynomials in $k[x]$. Consider the polynomial $P = F_1F_2 \cdots F_n + 1$. We note that P is monic. We will show that P is irreducible.

Suppose toward contradiction that P were reducible. We know that $k[x]$ is a principal ideal domain, so $P \in \langle F_i \rangle$ for some irreducible monic F_i. However, we know that, for any F_i, $1 \leqslant i \leqslant n$, $P \nmid F_i$, as, applying the division algorithm to P, we get

$$P = (F_i) \prod_{j \neq i} F_j + 1,$$

where $r \neq 0$. Thus, P is not reducible and monic, so there are infinitely many irreducible monic polynomials in $k[x]$.

Exercise (Exercise 1.6)**:** Show that any algebraically closed field is infinite.

Solution: Note that if k is any field, then there are infinitely many irreducible monic polynoimals in $k[x]$. If k is algebraically closed, then $(x - a)$, for $a \in k$, is the only irreducible monic polynomial. Since there are infinitely many irreducible monic polynomials in $k[x]$, there are infinitely many $a \in k$ such that $(x - a)$ is irreducible in $k[x]$. Thus, k is infinite.

Exercise (Exercise 1.7)**:** Let k be any field, and $F \in k[x_1, \dots, x_n]$, with $a_1, \dots, a_n \in k$.

(a) Show that

$$F = \sum \lambda_{(i)}(x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n},$$

where $\lambda_{(i)} \in k$.

(b) If $F(a_1, \dots, a_n) = 0$, show that $F = \sum_{i=1}^{n}(x_i - a_i)G_i$ for some not necessarily unique $G_i \in k[x_1, \dots, x_n]$.

Solution:

(a) We let

$$G = F(x_1 + a_1, x_2 + a_2, \dots, x_n + a_n).$$

Then, since $G \in k[x_1, \dots, x_n]$, we have

$$G = \sum \lambda_{(i)} x_1^{i_1} \cdots x_n^{i_n}.$$

Then, we have

$$F = \sum \lambda_{(i)}(x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

(b) Note that if $F(a_1, \dots, a_n) = 0$, then $(x_i - a_i) \mid F(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$. Thus, we have

$$F(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n) = (x_i - a_i)\underbrace{g(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)}_{G_i}.$$

This yields

$$F(x_1, \dots, x_n) = \sum_{i=1}^{n} (x_i - a_i)G_i.$$

Affine Space and Algebraic Sets

Definition. If k is a field, then when we write $\mathbb{A}^n(k)$, or $\mathbb{A}^n$, to be the cartesian product of k with itself n times.

We call $\mathbb{A}^n(k)$ the affine n-space over k. Its elements are called points. We call $\mathbb{A}^1(k)$ the affine line and $\mathbb{A}^2(k)$ the affine plane.

Definition. If $F \in k[x_1, \dots, x_n]$, then $P = (a_1, \dots, a_n) \in \mathbb{A}^n(k)$ is called a zero of F if $F(P) = (a_1, \dots, a_n) = 0$.

If F is not constant, then the zeros of F are called the hypersurface defined by F, defined by $V(F)$. A hypersurface in $\mathbb{A}^2(k)$ is called an affine plane curve.

If F is a polynomial of degree 1, then $V(F)$ is called a hyperplane in $\mathbb{A}^n(k)$; if $n = 2$, then an affine hyperplane is a line.

Definition. If S is any set of polynomials in $k[x_1, \dots, x_n]$, then $V(S) = \{P \in \mathbb{A}^n \mid F(P) = 0 \text{ for all } F \in S\}$. In other words, $V(S) = \bigcap_{F \in S} V(F)$. If $S = \{F_1, \dots, F_r\}$, we write $V(F_1, \dots, F_r)$.

A subset $X \subseteq \mathbb{A}^n(k)$ is an affine algebraic set (or algebraic set) if $X = V(S)$ for some S.

Proposition:

(1) If I is the ideal in $k[x_1, \dots, x_n]$ generated by S, then $V(S) = V(I)$; thus, every algebraic set is equal to $V(I)$ for some ideal I.

(2) If $\{I_\alpha\}$ is a collection of ideals, then $V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$.

(3) If $I \subseteq J$, then $V(I) \supseteq V(J)$.

(4) For any polynomialsF, G, $V(FG) = V(F) \cup V(G)$. Furthermore, $V(I) \cup V(J) = V(\{FG \mid F \in I, G \in J\})$.

(5) We have that $V(0) = \mathbb{A}^n(k)$, $V(1) = \emptyset$, $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\}$ for $a_i \in k$. Thus, any finite subset of $\mathbb{A}^n(k)$ is an algebraic set.

Exercise (Exercise 1.8)**:** Show that the algebraic subsets of $\mathbb{A}^1(k)$ are just the finite subsets together with $\mathbb{A}^1(k)$ itself.

Solution: Since $k[x]$ is a principal ideal domain, we know that the zero set $V(S)$ for any $S \subseteq k[x]$ is of the form $V(\langle f \rangle) = V(f)$, where $f \in k[x]$. Since f is a polynomial, f has finitely many roots, so there are finitely many elements in the algebraic subset.

Additionally, since $0 \in k[x]$, we know that k is also an algebraic subset.

Exercise (Exercise 1.14)**:** Let F be a nonconstant polynomial in $k[x_1, \dots, x_n]$, where k is algebraically closed. Show that $\mathbb{A}^n(k) \setminus V(F)$ is infinite if $n \geqslant 1$ and that $V(F)$ is infinite if $n \geqslant 2$. Conclude that the complement of any proper algebraic set is infinite.

Solution: We know that k is infinite as k is algebraically closed.

Let $F \in k[x_1, \dots, x_n] \cong k[x_1, \dots, x_{n-1}][x_n]$. In the base case with $n = 1$, we know that there are finitely many roots in $\mathbb{A}^1(k)$, so we have the base case. If $n \geqslant 2$, then we write $F = \sum G_i x_n^i$. We know that since F is nonzero, then there is at least one nonzero G_i. We showed in Exercise 1.4 that there is some $a_1, \dots, a_{n-1} \in k$ such that $G_i(a_1, \dots, a_{n-1}) \neq 0$. Thus, $F(a_1, \dots, a_{n-1}, x_n)$ is not the zero polynomial, meaning there are finitely many roots, and thus infinitely many non-roots. Thus, there are infinitely many $a_1, \dots, a_n \in k$ with $a_1, \dots, a_n \neq 0$.

We write $F = \sum G_i x_n^i$. We know that if all the G_i are constant, then we have a single-variable polynomial in x_n, and any choice of $a_1, \dots, a_{n-1} \in k$ provide other elements of $V(F)$. We assume that there is some G_i that is a nonconstant polynomial in $x_1, \dots, x_{n-1}$. Since G_i is nonzero, we may use the previous paragraph to state that G_i has infinitely many non-roots, and for each choice of those $a_1, \dots, a_{n-1}$, we have a polynomial in x_n. This polynomial has a root, meaning there are infinitely many roots.

Exercise (Exercise 1.15)**:** Let $V \subseteq \mathbb{A}^n(k)$ and $W \subseteq \mathbb{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \mid (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbb{A}^{n+m}(k)$. It is called the product of V and W.

Solution: Consider the set of polynomials in $k[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}]$ given by $P = F(x_1, \dots, x_n) + G(x_{n+1}, \dots, x_m)$, where F is a polynomial in the ideal whose algebraic set is V and G is an ideal in the algebraic set whose ideal is W. Then, the collection of zeros are those of the form $(a_1, \dots, a_n, b_1, \dots, b_m)$, where $(a_1, \dots, a_n) \in V$ and $(b_1, \dots, b_m) \in W$.