Abstract

We discuss compactness in topological spaces, normed spaces, and weak compactness, covering results such as Tychonoff's Theorem, relations between norm-compactness and dimension, sequential compactness, the Banach–Alaoglu Theorem, and the Eberlein–Šmulian theorem.

Compactness in Topological Spaces

Traditionally, one is introduced to compactness in their first class on topology. There, the definition of compactness appears a bit strange — but we'll see soon enough that there are a variety of simpler, equivalent ways to use compactness that are just as powerful as the original definition. However, as is customary, we start with the standard definition.

Definition. Let X be a topological space. An *open cover* of X is a family of open sets $\{U_i\}_{i\in I}$ such that

$$X \subseteq \bigcup_{i\in I} U_i.$$

Definition. Let X be a topological space. We say X is *compact* if, for any open cover of X, $\{U_i\}_{i\in I}$, there is a finite $F \subseteq I$ such that

$$X \subseteq \bigcup_{i\in F} U_i.$$

In other words, X is compact if every open cover admits a finite subcover.

One of the early equivalent characterizations we encounter with compactness relates to what happens when we take complements of the definition.

Theorem: Let X be a topological space. The following are equivalent:

- X is compact;
- for every family of closed sets $\{C_i\}_{i\in I}$ in X with the finite intersection property — that is, the intersection of finitely many such C_i is nonempty — then

$$\bigcap_{i\in I} C_i \neq \varnothing.$$

Proof. Let X be a compact topological space, and let $\{C_i\}_{i\in I}$ be a family with the finite intersection property. Defining $U_i = C_i^c$, it is then the case that any finite union of U_i does not fully cover X. Since any open cover of X must admit a finite subcover, and the union of no finite subcollection of $\{U_i\}_{i\in I}$ covers X, it is then the case that

$$\begin{aligned} \varnothing &\neq \left(\bigcup_{i\in I} U_i\right)^c \\ &= \bigcap_{i\in I} C_i. \end{aligned}$$

Now, if X is not compact, there is an open cover $\{U_i\}_{i\in I}$ of X with no finite subcover, or that for any finite subcollection $\{U_i\}_{i\in F}$, their union "misses" at least one element of X. Defining $C_i = U_i^c$, this means that the family $\{C_i\}_{i\in I}$ has the finite intersection property, but since $\bigcup_{i\in I} U_i = X$, the intersection $\bigcap_{i\in I} C_i = \varnothing$. $\square$

The finite intersection property characterization is immensely useful in an analytical context — for instance, it allows us to establish the existence of choice functions.

Compactness through Nets

Here, we give a crash course in nets before discussing the first

Definition. Let X be a topological space. An *open neighborhood* of x is a subset $U \in \tau$ such that $x \in U$. The family of open neighborhoods of x is denoted $\mathcal{O}_x$.

A *neighborhood* of $x \in X$ is a subset $N \subseteq X$ such such that there exists $U \in \mathcal{O}_x$ such that $U \subseteq N$. The family of open neighborhoods of x is denoted $\mathcal{N}_x$, and is known as the *neighborhood system* at x.

Definition. Let A be a set, and let $\preceq$ be a relation on A. We say $(A, \preceq)$ is *directed* if, for any $a, b \in A$, there exists $c \in A$ such that $a \preceq c$ and $b \preceq c$.

Definition. A *net* is a function from a directed set to a topological space. We write $(x_\alpha)_\alpha \subseteq X$ to denote a net.

Definition. A net $(x_\alpha)_\alpha$ *converges to* $x \in X$ if, for every $U \in \mathcal{N}_x$, there exists $\beta \in A$ such that for all $\alpha \geq \beta$, $x_\alpha \in U$. We write $(x_\alpha)_\alpha \to x$.

An element $x \in X$ is known as a *cluster point* (or *accumulation point*) of $(x_\alpha)_\alpha$ if, for all $U \in \mathcal{O}_x$ and all $\beta \in A$, there exists $\alpha \in A$ such that $x_\alpha \in U$.

Definition. If A and B are directed sets, a map $\phi \colon B \to A$ is called *cofinal* if, for every $\alpha \in A$, there is $\beta_\alpha \in B$ such that $\alpha \preceq \phi(\beta)$ for all $\beta \in B$ where $\beta_\alpha \preceq \beta$.

If $(x_\alpha)_\alpha$ and $(y_\beta)_\beta$ are nets in X, then we say $(y_\beta)_\beta$ is a subnet of $(x_\alpha)_\alpha$ if $y_\beta = x_{\phi(\beta)}$ for some cofinal map $\phi \colon B \to A$. We write $\left(x_{\alpha_\beta}\right)_\beta$.

There are a variety of characterizations of topological qualities that can be rephrased in terms of nets. We state them here and leave their proofs as exercises.

Theorem: Let X and Y be topological spaces, let $f \colon X \to Y$ be a map, and let $A \subseteq X$ be some subset.

- We have $x \in \overline{A}$ if and only if there is a net $(x_\alpha)_\alpha \subseteq A$ such that $(x_\alpha)_\alpha \to x$.
- A point $x \in X$ is a cluster point for $(x_\alpha)_\alpha$ if and only if $(x_\alpha)_\alpha$ admits a subnet, $\left(x_{\alpha_\beta}\right)_\beta \to x$.
- The map $f \colon X \to Y$ is continuous if and only if, for all nets $(x_\alpha)_\alpha \to x$ in X, we have $(f(x_\alpha))_\alpha \to f(x)$.
- A topological space X is Hausdorff if and only if any net converges to at most one point.

The one we will focus on today is, of course, compactness.

Theorem: Let X be a topological space. Then, X is compact if and only if every net $(x_\alpha)_\alpha$ admits a cluster point.

Proof. Let X be compact, and set $V_\alpha = \{x_\beta \mid \beta \geq \alpha\}$. Then, the family $\left\{\overline{V_\alpha}\right\}_{\alpha \in A}$ satisfies the finite intersection property — for any $\alpha_1, \dots, \alpha_n \in A$, there is some $\alpha \in A$ such that $\alpha_i \preceq \alpha$ for all i, meaning that $x_\alpha \in \bigcap_{i=1}^{n} \overline{V_{\alpha_i}}$. Therefore, there is some $x \in X$ such that $x \in \bigcap_{\alpha \in A} \overline{V_\alpha}$. Let $U \in \mathcal{O}_x$, and let $\alpha \in A$. Then, since $x \in \overline{V_\alpha}$, there is some $\beta \geq \alpha$ such that $x_\beta \in U$. Therefore, x is a cluster point of $(x_\alpha)_\alpha$.

Now, let X be noncompact, and let $\{U_i\}_{i \in I}$ be an open cover of X that does not admit any finite subcover. Let $\mathcal{F}$ be a family of finite subsets of I directed by inclusion. For each $F \in \mathcal{F}$, there is some $x_F \in X \setminus \bigcup_{i \in F} U_i$ by assumption. The net $(x_F)_{F \in \mathcal{F}}$ does not admit any accumulation point, since for any $i_0 \in I$, and for all S such that $\{i_0\} \subseteq S$, we have defined x_S such that $x_S \notin U_{i_0}$. $\square$

Compactness through Filters and Ultrafilters

Nets are nice and all, but there is another way to discuss convergence (that is actually in one-to-one correspondence with nets): filters.

Definition. Let X be a set, and let $\mathcal{F}$ be a family of subsets. We say $\mathcal{F}$ is a *filter* if

(a) $\varnothing \notin \mathcal{F}$ and $X \in \mathcal{F}$;

(b) if $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$;

(c) if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

We say the filter $\mathcal{F}$ is *free* if $\bigcap_{A \in \mathcal{F}} A = \varnothing$. If $\mathcal{F}$ is not free, we say $\mathcal{F}$ is *principal*.

If $\mathcal{F}$ and $\mathcal{G}$ are two filters, we say $\mathcal{G}$ is a *subfilter* of $\mathcal{F}$ if $\mathcal{G} \supseteq \mathcal{F}$ — i.e., that $\mathcal{G}$ is a "finer" filter than $\mathcal{F}$.

Example.

- If x is a set, then the neighborhood system at x, $\mathcal{N}_x$, is a filter.
- If $S \subseteq X$ is some nonempty set, the family $\mathcal{F} = \{A \subseteq X \mid S \subseteq A\}$ is a filter.
- If X is an infinite set, the family of cofinite sets — i.e., $A \in \mathcal{F}$ if A^c is finite — is a free filter.

Definition. An *ultrafilter* is a maximal proper filter. Equivalently, a filter $\mathcal{U}$ is an ultrafilter if $\mathcal{U}$ is a filter and, for all $A \subseteq X$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

If there is a finite union $A_1 \cup \cdots \cup A_n \in \mathcal{U}$ for an ultrafilter $\mathcal{U}$, then at least one of the A_i is an element of $\mathcal{U}$. This can be proven by induction and using the maximality of the ultrafilter.

Theorem (Ultrafilter Lemma)**:** If $\mathcal{F}$ is a filter, then there is an ultrafilter $\mathcal{U}$ that contains $\mathcal{F}$.

Proof. Consider the partially ordered set of all (proper) filters on X that contain $\mathcal{F}$. This family is partially ordered by inclusion, and for any chain $\mathcal{C}$ of this family, the family $\{A \mid A \in \mathcal{G} \text{ for some } \mathcal{G} \in \mathcal{C}\}$ is a filter that serves as an upper bound for $\mathcal{C}$. By Zorn's Lemma, this family admits a maximal element. □

Definition. If $\mathcal{B}$ is a nonempty collection of subsets of X, we say $\mathcal{B}$ is a *filter base* if

(a) $\varnothing \notin \mathcal{B}$;

(b) if $A, B \in \mathcal{B}$, there is some $C \in \mathcal{B}$ such that $C \subseteq A \cap B$ — i.e., $\mathcal{B}$ is directed by containment.

The collection of sets

$$\mathcal{F}_{\mathcal{B}} = \{A \subseteq X \mid B \subseteq A \text{ for some } B \in \mathcal{B}\}$$

is a filter known as the filter *generated by* $\mathcal{B}$.

Example. If X is a topological space, and $x \in X$, the family $\mathcal{O}_x$ is a filter base that generates the neighborhood system, $\mathcal{N}_x$.

Proposition: Any collection $\{S_i\}_{i \in I}$ with the finite intersection property is a filter base.

Similar to the case of nets, there are definitions of convergence and cluster points.

Definition. Let $\mathcal{F}$ be a filter on a topological space X, and let $x \in X$.

- We say $\mathcal{F}$ *converges to* x if $\mathcal{N}_x \subseteq \mathcal{F}$. In other words, if $\mathcal{F}$ is a subfilter of the neighborhood system at x, then we say the filter converges to x.
- We say x is a *cluster point* of $\mathcal{F}$ if, for all $F \in \mathcal{F}$, $x \in \overline{F}$. Equivalently, x is a cluster point for $\mathcal{F}$ if, for all $U \in \mathcal{O}_x$ and for all $F \in \mathcal{F}$, $U \cap F \neq \varnothing$.

Theorem: If $\mathcal{F}$ is a filter in a topological space X, an element $x \in X$ is a cluster point of $\mathcal{F}$ if and only if $\mathcal{F}$ admits a subfilter, $\mathcal{G}$, such that $\mathcal{G}$ converges to x.

One of the most important facts about filters is that every filter has a corresponding net, and vice versa.

Definition. Let $(x_\alpha)_\alpha$ be a net in the topological space X.

- For each α, the *tail* of $(x_\alpha)_\alpha$ is $F_\alpha := \{x_\beta \mid \alpha \preceq \beta\}$.
- The family $\mathcal{B} = \{F_\alpha \mid \alpha \in A\}$ serves as a filter base for the *section filter* of $(x_\alpha)_\alpha$

Theorem: The net $(x_\alpha)_\alpha$ and its corresponding section filter $\mathcal{F}$ have the same cluster points.

Proof. If x is a cluster point of $(x_\alpha)_\alpha$, then there is a subnet $\left(x_{\alpha_\beta}\right)_\beta \to x$. By cofinality, the section filter $\mathcal{G}$ generated by $\left(x_{\alpha_\beta}\right)_\beta$ is a subfilter of $\mathcal{F}$ with $\mathcal{G}$ converging to x.

Conversely, if x is a cluster point of $\mathcal{F}$, then for each α and for each $V \in \mathcal{N}_x$, we have $V \cap F_\alpha \neq \varnothing$. We may choose $y_{\alpha,V} \in V \cap F_\alpha$; the net $(y_{\alpha,V})_{\alpha \in A, V \in \mathcal{N}_x}$ is a subnet of $(x_\alpha)_\alpha$ that converges to x, meaning x is a cluster point of $(x_\alpha)_\alpha$ $\square$

Definition. If $\mathcal{F}$ is a filter in a topological space X, then we define the set $A = \{(\alpha_S, S) \mid S \in \mathcal{F}, \alpha_S \in S\}$, with direction $(\alpha_S, S) \succeq (\alpha_T, T)$ if $S \subseteq T$. The family $\left(x_{(\alpha_S,S)}\right)_{(\alpha_S,S)\in A}$, where $x_{(\alpha_S,S)} = \alpha_S$, is a net in X known as the *net generated by the filter* $\mathcal{F}$.

Note that the net generated by the section filter of a net $(x_\alpha)_\alpha$ is $(x_\alpha)_\alpha$.

Now, we turn towards characterizations of compactness using filters and ultrafilters.

Theorem: Let X be a topological space. Then, X is compact if and only if every filter in X has a cluster point.

Proof. Let X be compact, and let $\mathcal{F}$ be a filter on X. Define $\mathcal{S} = \left\{\overline{A} \mid A \in \mathcal{F}\right\}$. Since all filters are filter bases, the family $\mathcal{S}$ has the finite intersection property. Since X is compact, $\bigcap_{\overline{A} \in \mathcal{S}} \overline{A} = \bigcap_{A \in \mathcal{F}} \overline{A} \neq \varnothing$, meaning $\mathcal{F}$ admits a cluster point.

Suppose every filter on X admits a cluster point. If $\mathcal{C}$ is a family of closed subsets of X with the finite intersection property, then $\mathcal{C}$ is a filter base for some filter $\mathcal{F}$, which has a cluster point. Thus, $\bigcap_{C \in \mathcal{C}} C = \bigcap_{A \in \mathcal{F}} \overline{A} \neq \varnothing$, so X is compact. $\square$

Theorem: Let X be a topological space. Then, X is compact if and only if every ultrafilter in X converges to at least one point.

Proof. We know from the previous theorem that X is compact if and only if every filter on X admits a cluster point, meaning that every filter $\mathcal{F}$ on X has a convergent subfilter $\mathcal{G} \to x$. Using the Ultrafilter Lemma, we may expand $\mathcal{G}$ to an ultrafilter $\mathcal{U} \to x$.

Now, suppose toward contradiction that X is compact and there is an ultrafilter $\mathcal{U}$ on X that does not converge to any point. Then, by definition, for each x, there is some $U_x \in \mathcal{O}_x$ such that $U_x \notin \mathcal{U}$. The family $\{U_x\}_{x \in X}$ is an open cover of X, which admits the finite subcover $X = \bigcup_{i=1}^n U_{x_i}$. However, since $\mathcal{U}$ is an ultrafilter, and $X \in \mathcal{U}$, it must be the case that one of the U_{x_i} is an element of $\mathcal{U}$, which is a contradiction. $\square$

Tychonoff's Theorem

In the realm of compactness, there is probably no more powerful a theorem than Tychonoff's theorem. There are many proofs of Tychonoff's theorem (including one using the Stone–Čech compactification, apparently), but we will provide the proof using filters.

First, we need to review some notions from general topology.

Definition. If $\{Y_i\}_{i\in I}$, X is a topological space, and $f_i\colon X \to Y_i$ is a family of maps, the *initial topology* on X is the coarsest topology on X such that all the f_i are continuous.

A topological base for the initial topology on X induced by $\{f_i\}_{i\in I}$ is given by

$$\mathcal{B} = \left\{ f_{i_1}^{-1}(U_1) \cap \cdots \cap f_{i_n}^{-1}(U_n) \mid U_j \subseteq Y_j \text{ is open} \right\}.$$

Proposition: If τ is the initial topology on X_i induced by $\{f_i\}_{i\in I}$, then a net $(x_\alpha)_\alpha$ converges to x if and only if $(f_i(x_\alpha))_\alpha \to f_i(x)$ for all i.

Proof. Let $(x_\alpha)_\alpha \to x$. Since each $f_i\colon X \to Y$ is continuous, $(f_i(x_\alpha))_\alpha \to f(x)$ for each i.

Let $(f_i(x_\alpha))_\alpha \to x$ for all i. Then, if $U \in \mathcal{O}_x$, the base of the initial topology gives open $U_j \subseteq Y_j$ such that

$$\begin{aligned} x &\in f_{i_1}^{-1}(U_1) \cap \cdots \cap f_{i_n}^{-1}(U_n) \\ &\subseteq U. \end{aligned}$$

Since $\left(f_{i_j}(x_\alpha)\right)_\alpha \to f_{i_j}(x)$ for each $j = 1, \dots, n$, there are $\alpha_1, \dots, \alpha_n$ such that, for each j and for all $\alpha \succeq \alpha_j$, $f_{i_j}(x_\alpha) \in U_j$.
Since A is directed, there is some $\alpha_N \succeq \alpha_j$ for each j, so for all $\alpha \succeq \alpha_N$,

$$\begin{aligned} x_\alpha &\in f_{i_1}^{-1}(U_1) \cap \cdots \cap f_{i_n}^{-1}(U_n) \\ &\subseteq U, \end{aligned}$$

so $(x_\alpha)_\alpha \to x$. □

Definition. If $\{X_i\}_{i\in I}$ is a family of topological spaces, the *product topology* is the initial topology on $\prod_{i\in I} X_i$ such that the family of projection maps, $\pi_j\colon \prod_{i\in I} X_i \to X_j$, defined by $(x_i)_{i\in I} \mapsto x_j$, is continuous.

A base for this topology consists of all sets of the form

$$\mathcal{B} = \left\{ \prod_{i\in I} U_i \;\middle|\; U_i \subseteq X_i \text{ is open}, U_i = X_i \text{ for all but finitely many } U_i \right\}$$

This definition of the product topology, along with the idea of the pushforward of a filter/ultrafilter, will allow us to prove Tychonoff's Theorem.

Definition. Let X, Y be sets, and let $f\colon X \to Y$ be a map. Let $\mathcal{F}$ be a filter on X. Then, the collection

$$\mathcal{G} := \left\{ V \subseteq Y \mid f^{-1}(V) \in \mathcal{F} \right\}$$

defines a filter on Y known as the *pushforward* of $\mathcal{F}$ with respect to f. We write $f_*\mathcal{F}$.

Proposition: If $\mathcal{F}$ is an ultrafilter on X, $f\colon X \to Y$ is a map, and $f_*\mathcal{F}$ is the pushforward of $\mathcal{F}$, then $f_*\mathcal{F}$ is an ultrafilter on Y.

Proof. Let $B \subseteq Y$. Since $\mathcal{F}$ is an ultrafilter, we have either $f^{-1}(B) \in \mathcal{F}$ or $f^{-1}(B)^c \in \mathcal{F}$, so that either $f^{-1}(B) \in \mathcal{F}$ or $f^{-1}(B^c) \in \mathcal{F}$. By definition, this means either $B \in f_*\mathcal{F}$ or $B^c \in f_*\mathcal{F}$, so $f_*\mathcal{F}$ is an ultrafilter on Y. □

Theorem (Tychonoff's Theorem)**:** Let $\{X_i\}_{i\in I}$ be a family of nonempty compact topological spaces. Then, $\prod_{i\in I} X_i$, endowed with the product topology, is compact.

Proof. Let $\mathcal{U}$ be an ultrafilter on $\prod_{i\in I} X_i$. Pushing forward this ultrafilter to each X_i, we have that $(\pi_i)_*\mathcal{U}$ converges to x_i for some $x_i \in X$.

By the definition of the product topology, $\mathcal{U} \to (x_i)_i$, so $\mathcal{U}$ converges, meaning $\prod_{i\in I} X_i$ is compact. $\square$

Compactness in Normed Spaces and Metric Spaces

Compactness and Dimension

Compactness and Sequential Compactness

Compactness in Continuous Function Spaces

Weak Compactness

The Banach–Alaoglu Theorem

Goldstine's Theorem

The Eberlein–Šmulian Theorem