Problem 1

Let $(x_k)_k$ be a sequence of strictly positive numbers such that

$$(kx_k)_k \to L > 0.$$

Show that $\sum_k x_k$ diverges.

Since $(kx_k)_k \to L$, every subsequence of $(kx_k)_k$ converges to L. Let $n_k = 2^k$. Then,

$$(2^k x_{2^k})_k \to L > 0,$$

implying that

$$\sum_k 2^k x_{2^k} = \infty.$$

By the Cauchy Condensation test, this implies that $\sum_k x_k$ diverges.

Problem 2

Let $(x_k)_k$ be a sequence of strictly positive numbers. Show the following:

(i) If $\limsup_{k\to\infty} \frac{x_{k+1}}{x_k} < 1$, then $\sum_k x_k$ converges.

(ii) If $\liminf_{k\to\infty} \frac{x_{k+1}}{x_k} > 1$, then $\sum_k x_k$ diverges.

(a)

Let $\varepsilon > 0$.

$$\begin{aligned}\limsup_{k\to\infty} \frac{x_{k+1}}{x_k} &:= u < 1\\ &= \inf_{n\geq 1}\left(\sup_{k\geq n} \frac{x_{k+1}}{x_k}\right)\end{aligned}$$

By the definition of inf, we have that $\exists N \in N$ large such that

$$\sup_{k\geq N} \frac{x_{k+1}}{x_k} < u + \varepsilon.$$

By the definition of sup, we have that $\forall k \geq N$,

$$\begin{aligned}\frac{x_{k+1}}{x_k} &< u + \varepsilon\\ x_{k+1} &< (u+\varepsilon)x_k.\end{aligned}$$

Inductively on x_k, we have that

$$x_{k+m} < (L+\varepsilon)^m x_k,$$

and series-wise, we have

$$\sum_{k=N}^{\infty} x_k < x_N \sum_{m=1}^{\infty} (u+\varepsilon)^m.$$

For sufficiently small ε, the sum on the right-hand side converges, implying that the sum on the left-hand side must converge. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{N-1} x_k + \sum_{k=N}^{\infty} x_k\\ &< \sum_{k=1}^{N-1} x_k + x_n \sum_{m=1}^{\infty} (u+\varepsilon)^m,\end{aligned}$$

meaning that $\sum_k x_k$ is bounded above by a convergent series, so it is convergent.

(b)

Let $\varepsilon > 0$.

$$\begin{aligned}\liminf_{k\to\infty} \frac{x_{k+1}}{x_k} &:= \ell > 1\\ &= \sup_{n\geq 1}\left(\inf_{k\geq n} \frac{x_{k+1}}{x_k}\right)\end{aligned}$$

By the definition of sup, we have that for large $N \in \mathbb{N}$, and for $k \geq N$,

$$\inf_{k\geq n} \frac{x_{k+1}}{x_k} > \ell - \varepsilon.$$

By the definition of inf, we also have that

$$\begin{aligned}\frac{x_{k+1}}{x_k} &> \ell - \varepsilon\\ x_{k+1} &> (\ell-\varepsilon)x_k\end{aligned}$$

Inductively, we have that

$$x_{k+m} > (\ell-\varepsilon)^m x_k,$$

and via series, we have

$$\sum_{k=N}^{\infty} x_k > x_N \sum_{m=1}^{\infty}(\ell-\varepsilon)^m.$$

For sufficiently small ε, the sum on the right-hand side diverges. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{N-1} x_k + \sum_{k=N}^{\infty} x_k\\ &> x_N \sum_{k=1}^{\infty}(\ell-\varepsilon)^m + \sum_{k=1}^{N-1} x_k,\end{aligned}$$

and since $\sum_k x_k$ is bounded below by a divergent series, the sum diverges.

Problem 3

Consider the sequence of functions

$$\begin{aligned}f_n &: \mathbb{R} \to \mathbb{R};\\ f_n(x) &= \arctan(nx)\end{aligned}$$

(i) Show that $(f_n)_n \to \frac{\pi}{2}\text{sgn}$ point-wise.

(ii) Show that the convergence in (i) is nonuniform on $(0,\infty)$.

(iii) Show that the convergence in (i) is uniform on $[a,\infty)$ for a fixed $a > 0$.

(i)

Let $\varepsilon > 0$. We know that, $\exists N \in N$ such that $\forall n \geq N$, $|\arctan(n) - \pi/2| < \varepsilon$.

Case 1: Let $x = 0$. Then,

$$\arctan(nx) = 0 \qquad \forall n \geq 1$$

Case 2: Let $x > 0$. Then, set $N' = \lceil N/x \rceil$. So, for $n' \geq N'$, we have

$$\begin{aligned} |\arctan(nx) - \pi/2| &= |\arctan(n') - \pi/2| \\ &< \varepsilon \end{aligned}$$

implying that $\arctan(nx) \to \pi/2$ when $x > 0$.

Case 3: Let $x < 0$. Then, set $x^* = -x$, and we have the same result as in Case 2, where $\arctan(nx^*) \to \pi/2$.

Since $\arctan(nx^*) = \arctan(n(-x)) = -\arctan(nx)$, we have that $\arctan(nx) \to -\pi/2$.

(ii)

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Set $\varepsilon_0 = \frac{\pi}{4}$. Then, we have that

$$\begin{aligned} |\arctan(n_k x_k) - \pi/2| &= \left|\arctan\left(k\frac{1}{k}\right) - \frac{\pi}{2}\right| \\ &= \left|\arctan(1) - \frac{\pi}{2}\right| \\ &= \left|\frac{\pi}{4} - \frac{\pi}{2}\right| \\ &= \frac{\pi}{4} \\ &\geq \varepsilon_0. \end{aligned}$$

(iii)

Let $x \in \Omega = [a, \infty)$, where $a > 0$, and let $\varepsilon > 0$. Then, since $\arctan(n) \to \frac{\pi}{2}$,

$$\begin{aligned} \left\|\arctan(nx) - \frac{\pi}{2}\right\|_u &= \frac{\pi}{2} - \arctan(na) \\ &< \varepsilon. \qquad \text{for sufficiently large } n \end{aligned}$$

Therefore, $\arctan(nx)$ is uniformly convergent to $\frac{\pi}{2}$ on $[a, \infty)$.

Problem 4

Consider the sequence of functions

$$\begin{aligned} f_n &: [0, \infty) \to \mathbb{R}; \\ f_n(x) &= \frac{\sin(nx)}{1 + nx}. \end{aligned}$$

(i) Show that $(f_n)_n \to 0$ pointwise.

(ii) Show that the convergence in (i) is nonuniform on $[0, \infty)$.

(iii) Show that the convergence in (i) is uniform on $[a, \infty)$ for a fixed $a > 0$.

(i)

We know that $f_n(0) = 0\ \forall n \in \mathbb{N}$. For all $x > 0$, we have:

$$\begin{aligned}\left|\frac{\sin(nx)}{1+nx} - \mathbf{o}(x)\right| &\leq \frac{1}{1+nx}\\ &< \frac{1}{nx}\\ &\to 0.\end{aligned}$$

So,

$$f_n \xrightarrow{\text{p.w.}} \mathbf{o}.$$

(ii)

Let $n_k = k$ and $x_k = \frac{\pi}{2k}$. Set $\varepsilon_0 = 1/4$. Then,

$$\begin{aligned}\left|f_{n_k}(x_k) - \mathbf{o}(x_k)\right| &= \frac{\sin\left(k\frac{\pi}{2k}\right)}{1+k\frac{\pi}{2k}}\\ &= \frac{1}{1+\frac{\pi}{2}}\\ &\geq \varepsilon_0\end{aligned}$$

(iii)

On $[a,\infty)$, we have

$$\begin{aligned}\left|\frac{\sin(nx)}{1+nx} - \mathbf{o}(x)\right| &\leq \frac{1}{1+nx}\\ &\leq \frac{1}{1+na}\\ &\leq \frac{1}{na}\\ \sup\left|\frac{\sin(nx)}{1+nx} - \mathbf{o}(x)\right| &\leq \frac{1}{na}\\ &\to 0\end{aligned}$$

So, $\frac{\sin(nx)}{1+nx} \to \mathbf{o}$ on $[a,\infty)$ uniformly.

Problem 5

Show that the sequence of functions

$$\begin{aligned}f_n &: [0,\infty) \to \mathbb{R};\\ f_n(x) &= x^2e^{-nx}\end{aligned}$$

converges uniformly to 0.

We know that $\forall n \in \mathbb{N}$, $f_n(0) = 0$. Otherwise, we have that

$$\begin{aligned}\sup\left(x^2e^{-nx}\right) \Rightarrow \frac{df_n}{dx} &= 0\\ 2xe^{-nx} - nx^2e^{-nx} &= 0\\ xe^{-nx}\left(2-nx\right) &= 0\\ x &= \frac{2}{n}\\ f(x) &= \frac{4}{n^2e^2}.\end{aligned}$$

Additionally, we have

$$\begin{aligned} n^2 &\geq n \\ \frac{e^2n^2}{4} &\geq \frac{e^2n}{4} \\ \frac{4}{e^2n^2} &\leq \frac{4}{e^2n}, \end{aligned}$$

so,

$$\sup(x^2e^{-nx}) \to 0.$$

Therefore, $f_n(x)$ converges to 0 uniformly.

Problem 6

Let $f_n = \mathbf{1}_{n,n+1}$. Show that $(f_n)_n \to \mathbf{0}$ pointwise on $\mathbb{R}$. Is the convergence uniform?

$\forall x \in \mathbb{R}$, find $N \in \mathbb{N}$ so large such that $x < N$, which is always true by the Archimedean property. Then, $|f_n(x) - \mathbf{0}(x)| = 0 < \varepsilon$.

However, since $\sup(f_n) = 1\ \forall n$, it must be the case that $(f_n)_n$ does not converge to $\mathbf{0}$ uniformly.

Problem 7

Let $(f_n)_n$ and $(g_n)_n$ be sequences in $\ell_\infty(\Omega)$ with $(f_n)_n \to f$ and $(g_n)_n \to g$ uniformly on Ω. Prove that $(f_ng_n)_n \to fg$ uniformly on Ω.

$$\begin{aligned} \|f_n(x)g_n(x) - f(x)g(x)\|_u &= \|f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)\|_u \\ &= \|f_n(x)\left(g_n(x) - g(x)\right) + g(x)\left(f_n(x) - f(x)\right)\|_u \\ &\leq \|f_n(x)\|_u \cdot \|g_n(x) - g(x)\|_u + \|g(x)\|_u \|f_n(x) - f(x)\|_u && \text{Triangle Inequality} \\ &\leq c\|f_n(x) - f(x)\|_u + d\|g_n(x) - g(x)\| && \text{Definition of Supremum} \\ &\to 0 \end{aligned}$$

Problem 8

Find a sequence of functions with $(f_n)_n$ defined on $[0,\infty)$ such that $|f_n|_u \geq n$, but $(f_n)_n \to 0$ pointwise.

Let f_n be defined as δ_n, where δ_n is defined as follows:

$$\delta_n(x) = \begin{cases} n & x = n \\ 0 & \text{otherwise} \end{cases}.$$

Then, $(f_n)_n \xrightarrow{\text{p.w.}} \mathbf{0}$, but $\sup(f_n) = n \geq n$.

Problem 9

Show that the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely and uniformly on any closed and bounded interval $[a,b]$.

Let $s_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, and $x \in [a,b]$. Then,

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left|\sum_{k=n+1}^{m} \frac{x^k}{k!}\right| \\ &\leq \sum_{k=n+1}^{m} \frac{|x|^k}{k!}. && \text{Triangle Inequality} \end{aligned}$$

Let $c = \max\{|a|,|b|\}$. Then,

$$|s_m(x) - s_n(x)| \leq \sum_{k=n+1}^{m} \frac{c^k}{k!},$$

We will show that $\displaystyle\sum_{k=0}^{\infty} \frac{c^k}{k!}$ converges:

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right| &= \lim_{k\to\infty} \frac{c}{k+1} \\
&\leq \lim_{k\to\infty} \frac{c}{k} \\
&= 0 < 1.
\end{aligned}$$

Therefore, for $\varepsilon > 0$ and N large,

$$\begin{aligned}
|s_m(x) - s_n(x)| &\leq \sum_{k=n+1}^{m} \frac{c^k}{k!} \\
&< \varepsilon. && m > n \geq N
\end{aligned}$$

Taking $m \to \infty$, we have

$$|s(x) - s_n(x)| < \varepsilon,$$

so,

$$\sup_{x\in[a,b]} |s(x) - s_n(x)| < \varepsilon.$$

So by the Weierstrass M-test, we can see that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges uniformly and absolutely on $[a, b]$.