Math 395

Homework 3

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Problem 1

Let $\varphi : R \to S$ be a ring homomorphism. Let $\mathfrak{p} \in \operatorname{Spec}(S)$. We will prove that $\varphi^{-1}(\mathfrak{p}) \subset R$ is an element of $\operatorname{Spec}(R)$.

Let $\mathfrak{p} \in \operatorname{Spec}(S)$. Let $ab \in \varphi^{-1}(\mathfrak{p})$. Then, $\varphi(ab) \in \mathfrak{p}$. So, $\varphi(a)\varphi(b) \in \mathfrak{p}$, meaning either $\varphi(a) \in \mathfrak{p}$ or $\varphi(b) \in \mathfrak{p}$. Therefore, $a \in \varphi^{-1}(\mathfrak{p})$ or $b \in \varphi^{-1}(\mathfrak{p})$. Therefore, $\varphi^{-1}(\mathfrak{p})$.

Problem 5

Define $\varphi : \mathbb{F}_p \to \mathbb{F}_p$, where $\varphi(x) = x^p$ for $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We will show that φ is an isomorphism.

We will start by showing that φ is a well-defined homomorphism. Let $[a]_p = [b]_p$. Then, $a = b + kp$ for some $k \in \mathbb{Z}$. By Fermat's Little Theorem, $\varphi(a) = a^p \equiv [a]_p$, and $\varphi(b + kp) = b^p + p(\ell)$ for some ℓ, so $\varphi(b) \equiv [b]_p$ as well. Thus, $\varphi([a]_p) = \varphi([b]_p)$.

Since φ is well-defined, we find that, for $a, b \in \mathbb{F}_p$,

$$\begin{aligned}
\varphi(a+b) &= ([a+b]_p)^p \\
&\equiv [a+b]_p \\
&= [a]_p + [b]_p \\
&\equiv ([a]_p)^p + ([b]_p)^p \\
&= \varphi(a) + \varphi(b),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(ab) &= ([ab]_p)^p \\
&\equiv [ab]_p \\
&= [a]_p[b]_p \\
&\equiv ([a]_p)^p\,([b]_p)^p \\
&= \varphi(a)\varphi(b),
\end{aligned}$$

meaning φ is a homomorphism.

Since, for all $x \in \mathbb{F}_p$, $x \equiv x^p$, it is the case that φ is surjective. Finally, since $\varphi(0) = 0$, and for $x \neq 0$, $\varphi(x) \neq 0$, it is the case that $\ker(\varphi) = \{0\}$, meaning φ is injective.

Since φ is a bijective homomorphism, φ is an isomorphism.