Problem (Problem 1)**:** Describe the topology of the Grassmanian $\mathrm{Gr}(k, n)$ in a uniform way, so that $\mathbb{RP}^n$ becomes the special case of $\mathrm{Gr}(1, n)$.

Solution: We let elements of $\mathrm{Gr}(k, n)$ be defined as equivalence classes of linearly independent k-tuples of vectors in $\mathbb{R}^n$, where $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if $\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{w_1, \dots, w_k\}$.

By extending $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ to ordered bases $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$, we see that these k-tuples are equivalent if and only if there is a change of basis transformation Q with matrix representation

$$Q = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is a $k \times k$ invertible matrix, and B is a $(n-k) \times (n-k)$ matrix. The subgroup of all such $Q \subseteq \mathrm{GL}_n(\mathbb{R})$, which we call P, is the stabilizer of $\mathrm{Gr}(k, n)$ as we have defined it, so by the orbit-stabilizer theorem (seeing as $\mathrm{GL}_n(\mathbb{R})$ acts transitively on all ordered bases of $\mathbb{R}^n$), we obtain $\mathrm{Gr}(k, n) \cong \mathrm{GL}_n(\mathbb{R})/P$, where the latter coset space is given the quotient topology.

Note that this definition comports with the definition of $\mathbb{RP}^n$ as the space of one-dimensional subspaces, as the invertible 1×1 matrices are precisely the nonzero scalars.

Problem (Problem 2)**:** Fix an inner product on $\mathbb{R}^n$. Show that the map $V \mapsto V^\perp$ induces a C^∞ diffeomorphism $\mathrm{Gr}(k, n) \to \mathrm{Gr}(n-k, n)$.

Solution: We know that, since there is an inner product, we may express the smooth atlas of $\mathrm{Gr}(n, k)$ by $\{(U_V, \varphi_V)\}$, where

$$U_V = \left\{ W \in \mathrm{Gr}(k, n) \;\middle|\; W \cap V^\perp = 0 \right\},$$

and $\varphi = P_{V^\perp} P_V|_W^{-1}$ is the sequence of projections. By pre-composing with the map $V \mapsto V^\perp$, we get the atlas $\{(U_{V^\perp}, \varphi_{V^\perp})\}$ for $\mathrm{Gr}(n-k, n)$ consisting of charts of the form

$$\begin{aligned} U_{V^\perp} &= \{W \in \mathrm{Gr}(n-k, n) \mid W \cap V = 0\} \\ \varphi_{V^\perp} &= P_V P_{V^\perp}|_W^{-1}, \end{aligned}$$

Since the maps $\varphi_V \circ (V \mapsto V^\perp) \circ \varphi_{V^\perp}^{-1}$ are a composition of smooth bijections with smooth inverses, we see that this is a C^∞ diffeomorphism between $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k, n)$.

Problem (Problem 3)**:** Prove that a C^k map which is a C^1 diffeomorphism is necessarily a C^k diffeomorphism.

Solution: Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^k map that is a C^1 diffeomorphism. In order to show that f is a C^k diffeomorphism, we need to show that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ exists and is of class C^k.

First, by the inverse function theorem, since f is a C^1 diffeomorphism, we see that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ exists, is continuous, and is such that $D\left(f^{-1}\right)$ is continuous.

Now, we observe that the association $y \mapsto D_y\left(f^{-1}\right)$ can be written as

$$y \mapsto f^{-1}(y) \mapsto D_y f\left(f^{-1}(y)\right) \mapsto \left(D_y f\left(f^{-1}(y)\right)\right)^{-1} = D_y\left(f^{-1}\right),$$

where we observe that f^{-1} is of class C^1, the derivative D_f is of class C^{k-1}, and matrix inversion is C^∞; since $D\left(f^{-1}\right)$ is a composition of C^1 functions, $D\left(f^{-1}\right)$ is C^1, so f^{-1} is C^2. Inductively, we see that f^{-1} is also of class C^k, so f is a C^k diffeomorphism.

Problem (Problem 4)**:** Recall that a topological space is paracompact if every open cover admits a locally finite refinement. Prove that a connected, paracompact manifold of dimension one is either $\mathbb{R}$ or S^1, depending on whether it is compact or not.

Solution: Let M be a connected, paracompact manifold with dimension 1, and let $\{(U_i, \varphi_i)\}_{i\in I}$ be an atlas for M, where φ_i are homeomorphisms between U_i and $\mathbb{R}$.

Let $\{V_j\}_{j\in J}$ be a locally finite refinement of $\{U_i\}_{i\in I}$, where the restrictions $\psi_j := \varphi_i|_{V_j}$ are homeomorphisms to $O_j \subseteq \mathbb{R}$. We see that for any $p \in M$, since the family of V_j with $p \in V_j$, which we call $\mathcal{V}_p = \{V_j \mid p \in V_j\}$, is finite, the intersection $\bigcap \mathcal{V}_p$ is open; similarly, the intersection $\bigcap \mathcal{O}_p \subseteq \mathbb{R}$ is open, where $\mathcal{O}_p = \left\{\varphi|_{V_j}(V_j) \subseteq \mathbb{R} \,\middle|\, V_j \in \mathcal{V}_p\right\}$.

We see that $M = \bigcup_{p\in M} \bigcap \mathcal{V}_p$. Note that for any distinguished point p_1, the corresponding sets $\bigcap \mathcal{V}_{p_1}$ and $\bigcup_{p\neq p_1} \bigcap \mathcal{V}_p$ must have nonempty (open) intersection, by the assumption that M is connected. Thus, the corresponding union $\bigcup_{p\in M} \bigcap \mathcal{O}_p$ is an open and connected subset of $\mathbb{R}$. We may similarly map $\bigcup_{p\in M} \bigcap \mathcal{O}_p$ into $\S^1$ by composing with the quotient map.

Now, if M is compact, then $\bigcup_{p\in M} \bigcap \mathcal{V}_p$ covers M, so there is a finite subcover $M = \bigcup_{i=1}^{n} \bigcap \mathcal{V}_{p_i}$, so that $\bigcup_{i=1}^{n} \bigcap \mathcal{O}_{p_i}$ fully covers the corresponding range, meaning that, composing with the quotient map $\bigcup_{i=1}^{n} \bigcap \mathcal{O}_{p_i}$, we have that $M \cong S^1$. Similarly, if M is non-compact, then $\bigcup_{p\in M} \bigcap \mathcal{O}_p$ is an open and connected subset of $\mathbb{R}$ that does not admit any finite subcover, hence it is homeomorphic to $\mathbb{R}$.

Problem (Problem 5)**:** In this problem, we prove a weak version of the Whitney Embedding Theorem.

(a) Find a C^∞ function λ on $\mathbb{R}^n$ with values in $[0, 1]$ such that λ takes the value 1 on the closed ball $B(0, 1)$, and vanishes outside the closed ball $B(0, 2)$.

(b) Suppose M is a compact C^k manifold of dimension n. Find a C^k atlas $\{U_i, \varphi_i\}_{i\in I}$ such that $\varphi_i(U_i)$ contains $B(0, 2)$, and such that M is covered by the union of $\varphi_i^{-1}(B(0, 1))^\circ$.

(c) Let λ_i be defined by $\lambda \circ \varphi_i$ on U_i, and 0 outside U_i. Let $f_i : M \to \mathbb{R}^n$ be defined by $\lambda_i \circ \varphi_i$ on U_i and zero otherwise. Use these functions to embed M as a submanifold of some Euclidean space.

Problem (Problem 6)**:** Use the ideas of the previous exercise to prove that a C^k manifold admits a C^k partition of unity subordinate to any locally finite cover.

Problem (Problem 7)**:** Let X and Y be topological spaces, and let $C(X, Y)$ be the set of continuous maps from X to Y. Equip $C(X, Y)$ with the compact-open topology, where the basic open sets are

$$U_{K,V} = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

If Y is a metric space, and if X is compact, prove that this topology is the same as the topology of uniform convergence.

Solution: Let Y be a metric space and let X be compact. We note that a neighborhood basis in the topology of uniform convergence on $C(X, Y)$ consists of sets of the form

$$U_{f,\varepsilon} = \left\{g \,\middle|\, \sup_{x\in X} d(f(x), g(x)) < \varepsilon\right\}.$$

Similarly, a subbase for the compact open topology consists of sets of the form

$$U_{f,K,\varepsilon} = \left\{g \,\middle|\, \sup_{x\in K} d(f(x), g(x)) < \varepsilon\right\};$$

the fact that Y is a metric space allows us to take this refinement of the compact-open topology.

Thus, to prove that the compact-open topology and the topology of uniform convergence are equivalent, we show that any basis element of the topology of uniform convergence is contained in a basis element of the compact-open topology, and vice versa.

First, we see that almost by definition, if $K \subseteq X$ is any compact subset, then

$$U_{f,\varepsilon} \subseteq U_{f,K,\varepsilon},$$

as any function whose supremum distance is less than ε over X must have that supremum distance hold over $K \subseteq X$.

Now, in the reverse direction, we fix f and ε. We wish to show that there is a finite family of subsets U_{K_i,V_i} with $f \in U_{K_i,V_i}$ for each i, and their intersection lies in $U_{f,\varepsilon}$. We see that every point $x \in X$ has a pre-compact open neighborhood U_x such that $f\left(\overline{U_x}\right) \subseteq U(f(x), \varepsilon/3)$. The family $\{x \in X\}U_x$ is an open cover for X, so admits a finite subcover $\{U_{x_i}\}_{i=1}^{n}$. Since each $\overline{U_{x_i}}_{i=1}^{n}$ is compact, and $f \in U_{\overline{U_{x_i}},U(f(x_i),\varepsilon/3)}$ for each i, we see that

$$V = \bigcap_{i=1}^{n} U_{\overline{U_{x_i}},U(f(x_i),\varepsilon/3)}$$

is an open subset in the compact-open topology on $C(X, Y)$ that contains f and is contained in $U_{f,\varepsilon}$, so the topologies are thus equal.

Problem (Problem 8)**:** Let $C^k(M, N)$ be the set of C^k maps from M to N. The compact-open topology on $C^k(M, N)$ is defined similarly. Let $f \in C^k(M, N)$, (U, φ) and (V, ψ) charts on M and N, let $K \subseteq U$ be compact such that $f(K) \subseteq V$, and let $\varepsilon > 0$. We obtain a basic neighborhood $N(f, U, \varphi, V, \psi, K, \varepsilon)$ by looking at all the maps $g \in C^k(M, N)$ such that $g(K) \subseteq V$, and

$$\left\|D^r\left(\psi f \varphi^{-1}\right)(x) - D^r\left(\psi g \varphi^{-1}\right)(x)\right\|_{op} \leqslant \varepsilon \tag{$*$}$$

for all integers $0 \leqslant r \leqslant k$.

The Whitney topology is slightly different. Let $\Phi = \{(U_i, \varphi_i)\}_{i\in I}$ be a locally finite atlas on M, let $K_i \subseteq U_i$ be compact for all i, let Ψ be an atlas on N, and let $\{\varepsilon_i\}_{i\in I}$ be a family of positive numbers. A basic neighborhood of $f \in C^k(M, N)$ in this topology is given by all g such that $g(K_i) \subseteq V_i$ for all i, and

$$\left\|D^r\left(\psi_i f \varphi_i^{-1}\right)(x) - D^r\left(\psi_i g \varphi_i^{-1}\right)(x)\right\|_{op} \leqslant \varepsilon_i \tag{$**$}$$

for all $x \in \varphi_i(K_i)$ and all integers $0 \leqslant r \leqslant k$.

For infinite values of k, we take the compact-open and Whitney topologies on $C^\infty(M, N)$ to be the union of these topologies via the inclusion $C^\infty(M, N) \subseteq C^k(M, N)$. Show the following:

(a) these basic neighborhoods actually give a basis for a topology in both cases;

(b) if M is compact, these two topologies coincide;

(c) if M is compact and has no boundary, then the C^k diffeomorphisms from M to N are open in $C^k(M, N)$ in the Whitney topology.

Solution:

(a) Clearly, in both the compact open topology and the Whitney topology, the respective neighborhoods cover $C^k(M, N)$, so we only need to verify the condition that if $X_1, X_2 \subseteq C^k(M, N)$ are open subsets such that $f \in X_1 \cap X_2$, then there is $X_3 \subseteq C^k(M, N)$ open such that $X_3 \subseteq X_1 \cap X_2$.

We start with the case of the compact-open topology. Let $f \in X_1 \cap X_2$, where X_1 and X_2 are open in the compact-open topology. Since $f \in X_1$, there is a chart (U_1, φ_1) of M, a chart (V_1, ψ_1) of N, $K_1 \subseteq U_1$ compact such that $f(K_1) \subseteq V_1$, and $\varepsilon_1 > 0$ such that $(*)$ holds and $N(f, U_1, \varphi_1, V_1, \psi_1, \varepsilon_1) \subseteq X_1$. Similarly, since $f \in X_2$, there are charts (U_2, φ_2) and (V_2, ψ_2) of M and N respectively, $K_2 \subseteq U_2$

compact with $f(K_2) \subseteq V_2$, and $\varepsilon_2 > 0$ such that $(*)$ holds, and $N(f, U_2, \varphi_2, V_2, \psi_2, \varepsilon_2) \subseteq X_2$. Note that by the characterization, $(*)$ holds for the supremum over all $x \in \varphi_j\left(K_j\right)$ for $j = 1, 2$.