## Problem 1

Let $v_1, \dots, v_n$ be mutually orthogonal vectors in an inner product space $V$. Show that

$$\left\| \sum_{k=1}^{n} v_k \right\|^2 = \sum_{k=1}^{n} \|v_k\|^2 .$$

**Proof:**

$$\begin{aligned} \left\| \sum_{k=1}^{n} v_k \right\|^2 &= \left\langle \sum_{k=1}^{n} v_k, \sum_{k=1}^{n} v_k \right\rangle \\ &= \sum_{i=1}^{n} \left\langle \sum_{k=1}^{n} v_k, v_i \right\rangle \\ &= \sum_{i=1}^{n} \langle v_i, v_i \rangle && \text{since for } i \neq j,\ \langle v_i, v_j \rangle = 0 \\ &= \sum_{i=1}^{n} \|v_i\|^2 \end{aligned}$$

## Problem 2

Let $V$ be an inner product space and fix $w \neq 0$ in $V$. We define the one-dimensional projection

$$P_w : V \to V; P_w(v) := \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

(i) Prove that $v - P_w(v) \perp P_w(v)$.

(ii) Show that $P_w : V \to V$ is a linear operator with $\|P_w\|_{\text{op}} = 1$.

(iii) Show that $P_w \circ P_w = P_w$.

## Problem 3

Let $V$ be an inner product space. Prove the reverse Cauchy-Schwarz Inequality which states

$$v, w \in V, \text{ and } |\langle v, w \rangle| = \|v\| \|w\| \Rightarrow v = \alpha w.$$

## Problem 4

Let $V$ be an inner product space. Then, for any $v, w \in V$, show that

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

## Problem 5

Let $\lambda = (\lambda_k)_k$ belong to $\ell_\infty$. Show that the map

$$D_\lambda : \ell_2 \to \ell_2; D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$

## Problem 6

Consider the vector space $C([0, 2\pi])$ equipped with

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt.$$

(i) Show that this pairing defines an inner product on $C([0, 2\pi])$.

**Proof:** We will show that $\langle f, g\rangle$ satisfies the axioms of the inner product.

Addition:

$$\begin{aligned}\langle f_1 + f_2, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (f_1(t) + f_2(t))\overline{g(t)}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(f_1(t)\overline{g(t)} + f_2(t)\overline{g(t)}\right) dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} f_1(t)\overline{g(t)}dt + \frac{1}{2\pi}\int_0^{2\pi} f_2(t)\overline{g(t)}dt\\ &= \langle f_1, g\rangle + \langle f_2, g\rangle .\end{aligned}$$

Scalar Multiplication:

$$\begin{aligned}\langle \alpha f, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (\alpha f(t))\overline{g(t)}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} \alpha \left(f(t)\overline{g(t)}\right) dt\\ &= \alpha\left(\frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt\right)\\ &= \alpha\langle f, g\rangle .\end{aligned}$$

Conjugation:

$$\begin{aligned}\overline{\langle g, f\rangle} &= \frac{1}{2\pi}\int_0^{2\pi} \overline{g(t)\overline{f(t)}}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt\\ &= \langle f, g\rangle .\end{aligned}$$

Positive Definition:

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt\\ &\geq 0.\end{aligned}$$

For $\langle f, f\rangle = 0$, we have that the integral equals zero — since $f$ is continuous, it means that if $|f(t)|^2 > 0$ for some $t_0 \in [0, 2\pi]$, then $|f(t)|^2 \neq 0$ on some interval $[t_0 - \delta, t_0 + \delta]$, meaning the integral can only equal zero if $f$ is $\mathbb{0}_f$ on $[0, 2\pi]$.

(ii) For $n \in \mathbb{Z}$, set $e_n(t) = \cos(nt) + i\sin(nt)$. Show that the family $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal.

**Proof:** We will show that $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal by showing that $\langle e_n, e_n\rangle = 1$ and $\langle e_n, e_m\rangle = 0$ for $m \neq n$.

$$\begin{aligned}\langle e_n, e_n\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (\cos(nt) + i\sin(nt))(\cos(nt) - i\sin(nt))dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\cos^2(nt) + \sin^2(nt)\right) dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} dt\\ &= 1\\ \langle e_n, e_m\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (\cos(nt) + i\sin(nt))(\cos(mt) - i\sin(mt))dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} (\cos(mt)\cos(nt) + i\sin(nt)\cos(mt) - i\sin(mt)\cos(nt) + \sin(nt)\sin(mt))\, dt\\ &= \frac{1}{2\pi}\left(\int_0^{2\pi} (\cos(mt)\cos(nt) + \sin(nt)\sin(mt))dt + i\int_0^{2\pi} (\sin(nt)\cos(mt) - \sin(mt)\cos(nt))dt\right)\\ &= 0.\end{aligned}$$

## Problem 7

Let $V$ be any normed space, $p \in [1, \infty]$, and suppose $T : \ell_p^n \to V$ is linear. Show that $T$ is bounded.

## Problem 8

Let $\mathbb{P}[0,1] = \left\{\sum_0^n a_k x^k \mid a_k \in \mathbb{C}\right\} \subseteq C([0,1])$ denote the linear subspace of all polynomial functions equipped with the uniform norm $\|\cdot\|_u$ inherited from $C([0,1])$. We define the map

$$\begin{aligned} D &: \mathbb{P}[0,1] \to \mathbb{P}[0,1] \\ D(p(x)) &= p'(x). \end{aligned}$$

Show that $D$ is unbounded.

## Problem 9

Let $V$ be an infinite-dimensional normed space. Show that there is a linear functional $\varphi : V \to \mathbb{F}$ that is unbounded.

## Problem 10

Let $a, b \in \mathbb{M}_n$. Show the following properties of the operator norm.

(i) $\|a\|_{\text{op}} = \sup\left\{ |\langle a\xi, \eta\rangle| \mid \xi, \eta \in B_{\ell_2^n}\right\}$

(ii) $\|a^*\|_{\text{op}} = \|a\|_{\text{op}}$

(iii) $\|ab\|_{\text{op}} \leq \|a\|_{\text{op}} \|b\|_{\text{op}}$

(iv) $\|a^*a\|_{\text{op}} = \|a\|_{\text{op}}^2$