

Problem 1

Let X be a metric space. Show that X is second countable if and only if X is separable. Conclude that if X is a separable metric space, then every open set is the union of countably many open balls.

Proof:

(\Rightarrow): Let X be second countable. Then, X contains base $U_1, U_2, \dots \in \mathcal{B}$ such that each U_i is nonempty. Let $x_1 \in U_1, x_2 \in U_2, \dots$

The set $\{x_i\}_{i \geq 1}$ is countable, as each $x_i \in U_i$. For any $U \in \tau_X$ where $U \neq \emptyset$, $U = \bigcup_{i \in I} U_i$, meaning that $U \cap \{x_i\}_{i \geq 1} \neq \emptyset$. Thus, $\{x_i\}_{i \geq 1}$ is dense in X , meaning X is separable.

(\Leftarrow): Let X be separable, with countable dense subset $\{x_i\}_{i \geq 1}$. Let

$$\mathcal{B} = \{U(x_i, 1/n) \mid x_i \in \{x_i\}_{i \geq 1}, n \in \mathbb{N}\}.$$

Then, for every $U \in \tau_X$, since $U \cap \{x_i\}_{i \geq 1} \neq \emptyset$, and $\exists n$ such that $U(x_k, 1/n) \subseteq U$, it must be the case that \mathcal{B} is a base for τ_X . Thus, X is second countable.

If X is a separable metric space, then it admits a countable base, and any element of τ_X is a union of the elements of the base, implying that any element of τ_X is a union of countably many open balls.

Problem 2

Let (X, d) be a metric space, $(x_n)_n$ a sequence in x , and $x \in X$. The following are equivalent:

- (i) $(x_n)_n \rightarrow x$ in X ;
- (ii) $(d(x_n, x))_n \rightarrow 0$ in \mathbb{R} ;
- (iii) For every neighborhood $V \in \mathcal{N}_x$, there is an $N \in \mathbb{N}$ with $n \geq N \Rightarrow x_n \in V$.

Proof: Let $(x_n)_n \rightarrow x$ in X . Then, for any $\varepsilon > 0$, $\exists N$ large such that $n \geq N \Rightarrow d(x_n, x) < \varepsilon$. However, this is precisely the same as $|d(x_n, x) - 0| < \varepsilon$, which is true if and only if $(d(x_n, x))_n \rightarrow 0$.

Problem 4

Let $\{(X_k, d_k)\}$ be a family of metric spaces. Assume that for every $k \geq 1$, we have $d_k(x, y) \leq 1$ for all $x, y \in X_k$. Let

$$X := \prod_{k \geq 1} X_k$$

denote the product with metric

$$d(f, g) := \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k)).$$

Show that a sequence $(f_n)_n$ converges to f in X if and only if $(f_n(k))_n \rightarrow f(k)$ for every $k \geq 1$.

Proof: Let $(f_n)_n \rightarrow f$. Then, $(d(f_n, f))_n \rightarrow 0$. Therefore, for $\varepsilon > 0$, there exists an N large such that

$$\sum_{k=1}^{\infty} 2^{-k} d_k(f_n(k), f(k)) < \varepsilon$$

for $n \geq N$.

Problem 5

Let V be a normed space. Show that the operations

$$a : V \times V \rightarrow V;$$

$$a(v, w) = v + w$$

and

$$\mu : \mathbb{F} \times V \rightarrow V;$$

$$\mu(\alpha, v) = \alpha v$$

are continuous.

Proof:

- $a : V \times V \rightarrow V$, $a(v, w) = v + w$:

$$\begin{aligned} \|a(v, w) - a(v', w')\| &= \|v + w - (v' + w')\| \\ &= \|v - v' + w - w'\| \\ &\leq \|v - v'\| + \|w - w'\| \\ &= d(v, v') + d(w, w') \\ &= d_1((v, w), (v', w')), \end{aligned}$$

meaning a is Lipschitz.

- $\mu : \mathbb{F} \times V \rightarrow V$, $\mu(\alpha, v) = \alpha v$:

$$\begin{aligned} \|\mu(\alpha, v) - \mu(\beta, w)\| &= \|\alpha v - \beta w\| \\ &= \|\alpha v - \alpha w + \alpha w - \beta w\| \\ &\leq |\alpha| \|v - w\| + |\alpha - \beta| \|w\| \end{aligned}$$

If $(\alpha_n)_n \rightarrow \beta$ and $(v_n)_n \rightarrow w$, then

$$\begin{aligned} \|\alpha_n v_n - \beta w\| &\leq |\alpha_n| \|v_n - w\| + |\alpha_n - \beta| \|w\| \\ &\rightarrow 0. \end{aligned}$$

Problem 6

Let (X, d) be a metric space, $f, g : X \rightarrow \mathbb{F}$ continuous maps, and $\alpha \in \mathbb{F}$. Show that $f + g$, fg , and αf are continuous.

Proof: Let $(x_n)_n \rightarrow x \in X$. Then, we know that $|f(x_n) - f(x)| \rightarrow 0$ and $|g(x_n) - g(x)| \rightarrow 0$ (where $|\cdot|$ denotes absolute value in \mathbb{F}). Let $\varepsilon > 0$. Therefore, for N large, we know that

$$\begin{aligned} |f(x_n) + g(x_n) - (f(x) + g(x))| &\leq |f(x_n) - f(x)| + |g(x_n) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

meaning $|f(x_n) + g(x_n) - (f(x) + g(x))| \rightarrow 0$, so $(f(x_n) + g(x_n))_n \rightarrow f(x) + g(x)$. Thus, $f + g$ is continuous.

Similarly,

$$\begin{aligned} |f(x_n)g(x_n) - f(x)g(x)| &= |f(x_n)g(x_n) - f(x_n)g(x) + f(x_n)g(x) - f(x)g(x)| \\ &= |f(x_n)(g(x_n) - g(x)) + g(x)(f(x_n) - f(x))| \\ &\leq |f(x_n)||g(x_n) - g(x)| + |g(x)||f(x_n) - f(x)| \\ &\leq c|g(x_n) - g(x)| + |g(x)||f(x_n) - f(x)| \quad \text{convergent sequences are bounded} \\ &< \varepsilon \end{aligned}$$

so $(f(x_n)g(x_n))_n \rightarrow f(x)g(x)$.

Problem 9

Suppose $T : V \rightarrow W$ is a bijective linear map between normed spaces with $\|T\|_{\text{op}} \leq 1$ and $\|T^{-1}\|_{\text{op}} \leq 1$. Show that T is an isometry.

Proof: Since the operator norm for T is less than or equal to 1, we know that for $v, w \in V$,

$$\|T(v) - T(w)\|_W \leq \|v - w\|_V$$

and

$$\|T^{-1}(T(v)) - T^{-1}(T(w))\|_V \leq \|T(v) - T(w)\|_W$$

so, since T is bijective,

$$\|v - w\|_V \leq \|T(v) - T(w)\|_W$$

meaning

$$\|T(v) - T(w)\|_W = \|v - w\|_V$$

so T is an isometry.