This is a collection of old real analysis qualifier exam solutions, as well as some notes on useful results and proofs.

# Useful Results and Proofs

## Borel–Cantelli Lemma

**Theorem:** If $\{E_j\}_j \subseteq \mathcal{M}$ is a family of measurable sets, with

$$\sum_{j=1}^{\infty} \mu(E_j) < \infty,$$

then

$$\mu\left(\limsup_{j\to\infty} E_j\right) = 0.$$

*Proof.* Recall the definition of the limit superior of a collection of sets $\{E_j\}_j$ is defined by

$$\limsup(E_j) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j.$$

Defining

$$F_k = \bigcup_{j=k}^{\infty} E_j,$$

we see that $F_k \supseteq F_{k+1} \supseteq \cdots$. Furthermore, note that

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j),$$

and

$$\begin{aligned} \mu(F_1) &\leq \sum_{j=k}^{\infty} \mu(E_j) \\ &< \infty. \end{aligned}$$

Thus, by continuity from above, we see that

$$\begin{aligned} \mu\left(\limsup_{j\to\infty} E_j\right) &= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) \\ &= \lim_{k\to\infty} \mu(F_k) \\ &\leq \lim_{k\to\infty} \sum_{j=k}^{\infty} \mu(E_j) \\ &= 0, \end{aligned}$$

as the series converges. □

## Functions Defined as Integrals

**Theorem:** Let $f\colon X \times [a,b] \to \mathbb{C}$, where $-\infty < a < b < \infty$, suppose and $f(\cdot,t)\colon X \to \mathbb{C}$ is integrable for each $t \in [a,b]$. Let $F(t) = \int_X f(x,t)\, d\mu(x)$.

(a) If there exists $g \in L_1(\mu)$ such that $|f(x,t)| < g(x)$ for all $x, t$, and $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for every $x$, then $\lim_{t\to t_0} F(t) = F(t_0)$. In particular, if $f(x,\cdot)$ is continuous for every $x$, then $F$ is continuous.

(b) If $\frac{\partial f}{\partial t}$ exists, and there is $g \in L_1(\mu)$ such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ for all $x, t$. Then, $F$ is differentiable, and $F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x)$.

*Proof.*

(a) If $(t_n)_n \to t_0$, we may define

$$f_n(x) = f(x,t_n),$$

and use the Dominated Convergence Theorem to find that.

$$\begin{aligned}\lim_{n\to\infty} F(t_n) &= \lim_{n\to\infty} \int_X f(x,t_n)\, d\mu(x)\\ &= \int_X \lim_{n\to\infty} f(x,t_n)\, d\mu(x)\\ &= \int_X f(x,t)\, d\mu(x)\\ &= F(t),\end{aligned}$$

so $F(t)$ is continuous.

(b) Observe that

$$\frac{\partial f}{\partial t}(x,t_0) = \lim_{n\to\infty} \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0},$$

where $(t_n)_n \to t_0$. It follows that $\frac{\partial f}{\partial t}$ is measurable, and by mean value theorem,

$$\left|\frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}\right| \leq \sup_{t\in[a,b]} \left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x),$$

so by Dominated Convergence,

$$\begin{aligned}F'(t_0) &= \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0}\\ &= \lim_{n\to\infty} \int_X \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}\, d\mu(x)\\ &= \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x).\end{aligned}$$

□

## Completeness of $L_p$

**Theorem:** If $(X, \mathcal{M}, \mu)$ is a measure space, then $L_p(\mu)$ is complete.

*Proof.* Let $(f_n)_n \subseteq L_p(X, \mathcal{M}, \mu)$ be $L_p$-Cauchy. Then, we may extract a subsequence $(f_{n_k})_k$ such that

$$\left\| f_{n_{k+1}} - f_{n_k} \right\| < 2^{-k},$$

by recursively selecting $n_{k+1} > n_k$ and using the fact that $(f_n)_n$ is Cauchy.

Consider now the functions

$$s_n = \sum_{k=1}^{n} \left| f_{n_{k+1}} - f_{n_k} \right|$$
$$s = \sum_{k=1}^{\infty} \left| f_{n_{k+1}} - f_{n_k} \right|.$$

We see from Minkowski's Inequality that

$$\|s_n\| \leq \sum_{k=1}^{n} \left\| f_{n_{k+1}} - f_{n_k} \right\|,$$

and so by using Fatou's Lemma, that

$$\begin{aligned} \|s\| &\leq \liminf_{n\to\infty} \|s_n\| \\ &\leq \liminf_{n\to\infty} \sum_{k=1}^{n} \left\| f_{n_{k+1}} - f_{n_k} \right\| \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \\ &= 1, \end{aligned}$$

meaning that in particular, $s(x) < \infty$ for a.e. $x \in X$. Thus, the series

$$s = \sum_{k=1}^{\infty} \left| f_{n_{k+1}} - f_{n_k} \right|$$

converges absolutely for almost every $x$. Define now the function

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left( f_{n_{k+1}} - f_{n_k} \right)(x)$$

where $s$ converges, and 0 otherwise. Notice that

$$f_{n_k}(x) = f_{n_1}(x) + \sum_{k=1}^{n-1} f_{n_{k+1}}(x) - f_{n_k}(x),$$

so by telescoping, we see that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for almost every $x$.

> **Remark:** This is part of how we show that if a sequence $(f_n)_n \to f$ in $L_p$, then $(f_n)_n$ admits a sequence converging pointwise a.e. to $f$.

Now, we show that $f$ is the $L_p$ limit of $(f_n)_n$. To see this, let $\varepsilon > 0$, and let $N$ be such that for all $m, n \geq N$, $\|f_m - f_n\| < \varepsilon$. Then, we see that for all $m \geq N$,

$$\int_X |f - f_m|^p \, d\mu = \int_X \liminf_{k\to\infty} |f_{n_k} - f_m| \, d\mu$$

$$\begin{aligned} &\leq \liminf_{k\to\infty} \int_X |f_{n_k} - f_m| \, d\mu \\ &\leq \varepsilon^p, \end{aligned}$$

so that $\|f - f_m\| \leq \varepsilon$. Since Cauchy sequences are bounded, and for all $m \geq N$,

$$\begin{aligned} \|f\| &= \|f - f_m + f_m\| \\ &\leq \|f - f_m\| + \|f_m\| \\ &\leq \varepsilon + \|f_m\| \\ &< \infty, \end{aligned}$$

we see that $f \in L_p$, and we are done. □

## Modes of Convergence

**Definition:** Let $(f_n)_n$ be a sequence of measurable, complex-valued functions on $(X, \mathcal{M}, \mu)$.

(i) We say that $(f_n)_n$ is Cauchy in measure if, for all $\varepsilon > 0$, we have

$$\mu(\{x \mid |f_m(x) - f_n(x)| \geq \varepsilon\}) \to 0$$

as $m, n \to \infty$.

(ii) We say that $(f_n)_n \to f$ in measure if, for all $\varepsilon > 0$, the sequence

$$\mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) \to 0.$$

(iii) We say $(f_n)_n \to f$ in $L_1$ if the sequence

$$\int_X |f_n - f| \, d\mu \to 0.$$

**Proposition:** If $(f_n)_n \to f$ in $L_1$, then $(f_n)_n \to f$ in measure.

*Proof.* Let

$$E_{n,\varepsilon} = \{x \mid |f_n(x) - f(x)| \geq \varepsilon\}.$$

Then,

$$\begin{aligned} \int_X |f_n - f| \, d\mu &\geq \int_{E_{n,\varepsilon}} |f_n - f| \, d\mu \\ &\geq \varepsilon \mu(E_{n,\varepsilon}), \end{aligned}$$

so

$$\mu(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon} \int_X |f_n - f| \, d\mu.$$

□

**Remark:** We may have proven this by using Chebyshev's inequality, which is proved in effectively the same way. We will prove Chebyshev's inequality below.

**Theorem:** If $(f_n)_n$ is Cauchy in measure, then there is a measurable function $f$ such that $(f_n)_n \to f$ in measure, and $\left(f_{n_j}\right)_j \to f$ almost everywhere. Furthermore, these limits are essentially unique (i.e., any two such limits are equal almost everywhere).

*Proof.* We may choose a subsequence $\left(f_{n_j}\right)_j$ such that if we define

$$E_j = \left\{x \mid \left|f_{n_{j+1}}(x) - f_{n_j}(x)\right| \geq 2^{-j}\right\},$$

then

$$\mu(E_j) \leq 2^{-j}.$$

Note that this means

$$\begin{aligned}\sum_{j=1}^{\infty}\mu(E_j) &\leq 1\\ &< \infty,\end{aligned}$$

so $\mu\left(\limsup_{j\to\infty} E_j\right) = 0$ by the Borel–Cantelli Lemma. In particular, for all $x \notin \limsup_{j\to\infty} E_j$, $\left(f_{n_j}\right)_j$ is pointwise Cauchy, so we may set $f(x) = \lim_{j\to\infty} f_{n_j}(x)$, and we set $f(x) = 0$ for all $x \in \limsup_{j\to\infty} E_j$.

This means that $\left(f_{n_j}\right)_j \to f$ almost everywhere, and $\left|f_{n_j}(x) - f(x)\right| \leq 2^{1-j}$ for all $x \notin \bigcup_{j=k}^{\infty} E_j$ and $j \geq k$. It follows that $\left(f_{n_j}\right)_j \to f$ in measure.

Furthermore, since

$$\{x \mid |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \left\{x \;\middle|\; \left|f_n(x) - f_{n_j}(x)\right| \geq \frac{1}{2}\varepsilon\right\} \cup \left\{x \;\middle|\; \left|f_{n_j}(x) - f(x)\right| \geq \frac{1}{2}\varepsilon\right\},$$

and the latter two sets have small measure with sufficiently large $n$ and $j$, it follows that $(f_n)_n \to f$ in measure. □

**Theorem** (Egorov's Theorem)**:** If $\mu(X) < \infty$, and $(f_n)_n \to f$ pointwise a.e., then for any $\varepsilon > 0$, there is $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $E^c$.

*Proof.* We may assume that $(f_n)_n \to f$ everywhere on $X$. For $k, n \in \mathbb{N}$, define

$$E_{n,k} = \bigcup_{m=n}^{\infty}\left\{x \;\middle|\; |f_m(x) - f(x)| \geq \frac{1}{k}\right\}.$$

For fixed $k$, $E_{n,k}$ reduces as $n \to \infty$, and $\bigcap_{n=1}^{\infty} E_{n,k} = \emptyset$. Since $\mu(X) < \infty$, we may assume that $\mu(E_{n,k}) \to 0$ for fixed $k$ as $n \to \infty$. Thus, we may choose $n_k$ such that $\mu(E_{n_k,k}) < \varepsilon 2^{-k}$, and define $E = \bigcup_{k=1}^{\infty} E_{n_k,k}$. Then, $\mu(E) < \varepsilon$, and $|f_n(x) - f(x)| < \frac{1}{k}$ for all $n \geq n_k$ and $x \notin E$. Thus, $(f_n)_n \to f$ uniformly on $E^c$. □

## Signed Measures and Radon–Nikodym

**Definition:** A *signed measure* on a measurable space $(X, \mathcal{M})$ is a function $\nu \colon \mathcal{M} \to [-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$;
- $\nu$ does *not* attain at least one of $+\infty$ or $-\infty$;
- if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then

$$\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty}\nu(E_j),$$

where the latter sum converges absolutely when the former value is finite.

**Definition:** If $\nu$ is a signed measure, then we say $E \in \mathcal{M}$ is

- *positive* for $\nu$ if for all $F \in \mathcal{M}$ with $F \subseteq E$, $\nu(F) \geq 0$;
- *negative* for $\nu$ if for all $F \in \mathcal{M}$ with $F \subseteq E$, $\nu(F) \leq 0$;
- *null* for $\nu$ if for all $F \in \mathcal{M}$ with $F \subseteq E$, $\nu(F) = 0$.

**Theorem** (Hahn and Jordan Decomposition)**:** If $\nu$ is a signed measure on $(X, \mathcal{M})$, there exist measurable subsets $P$ and $N$ such that $P \cap N = \emptyset$, $P \cup N = X$, $P$ is positive for $\nu$, and $N$ is negative for $\nu$.

*Proof.* Without loss of generality, we may assume that for all $E \in \mathcal{M}$, $-\infty < \mu(E) \leq \infty$.

We note that the difference of two negative sets is negative, and the disjoint, countable union of negative sets is negative, so every countable union of negative sets is negative. We let $\beta = \inf(\mu(B))$ for all negative $B \in \mathcal{M}$. We let $(B_j)_j \subseteq \mathcal{M}$ be a sequence of measurable negative sets such that $\lim_{j\to\infty} \mu(B_j) = \beta$, and set $B = \bigcup_{j=1}^{\infty} B_j$. We see then that $B$ is a negative set for which $\mu(B)$ is minimal.

We now prove that $A = X \setminus B$ is a positive set. Suppose toward contradiction that there is $E_0 \subseteq A$ such that $\mu(E_0) < 0$. The set $E_0$ cannot be a negative set, or else $B \cup E_0$ would be a negative set with a smaller measure than $\mu(B)$, which is not possible.

Let $k_1$ be the smallest natural number such that there is $E_1 \subseteq E_0$ with $\mu(E_1) \geq \frac{1}{k_1}$. Since

$$\begin{aligned}
\mu(E_0 \setminus E_1) &= \mu(E_0) - \mu(E_1)\\
&\leq \mu(E_0) - \frac{1}{k_1}\\
&< 0.
\end{aligned}$$

The argument applied to $E_0$ is now applicable to $E_0 \setminus E_1$. Letting $k_2$ be the smallest natural number such that $E_0 \setminus E_1$ contains $E_2 \subseteq E_0 \setminus E_1$ with $\mu(E_2) \geq \frac{1}{k_2}$, and proceeding ad infinitum, we see that since $\mu$ is finitely valued for measurable subsets of $E_0$, we have $\lim_{n\to\infty} \frac{1}{k_n} = 0$.

It follows that for every measurable subset $F$ of

$$F_0 = E_0 \setminus \left(\bigcup_{j=1}^{\infty} E_j\right),$$

we have $\mu(F) \leq 0$ — i.e., $F_0$ is a measurable negative set. Since $F_0$ is disjoint from $B$, and

$$\begin{aligned}
\mu(F_0) &= \mu(E_0) - \sum_{j=1}^{\infty} \mu(E_j)\\
&\leq \mu(E_0)\\
&< 0,
\end{aligned}$$

this contradicts the minimality of $B$. Therefore, the hypothesis $\mu(E_0) < 0$ is untenable.

If $A_1 \sqcup B_1$ and $A_2 \sqcup B_2$ are two Hahn decompositions for $X$, then $A_1 \setminus A_2 \subseteq B_2$ and $A_1 \setminus A_2 \subseteq A_1$, meaning that $A_1 \setminus A_2$ is both positive and negative, hence null; similarly for $A_2 \setminus A_1$, so that $A_1 \triangle A_2$ is $\mu$-null, and similarly for $B_1 \triangle B_2$. □

**Definition:** We say two signed measures $\mu$ and $\nu$ are *mutually singular* on $X$ if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$, $E$ is null for $\mu$, and $F$ is null for $\nu$. We write $\nu \perp \mu$

**Theorem** (Jordan Decomposition)**:** If $\nu$ is a signed measure, there are positive measures $\nu^+$ and $\nu^-$ such that $\nu = \nu^+ - \nu^-$.

*Proof.* If $P \cup N$ is a Hahn Decomposition for $\nu$, we may define

$$\begin{aligned}\nu^{+}(E) &= \nu(E \cap P)\\ \nu^{-}(E) &= -\nu(E \cap N).\end{aligned}$$

This yields the Jordan decomposition. □

**Definition:** If $\nu = \nu^{+} - \nu^{-}$ is a Jordan decomposition for a signed measure, then the *total variation* of $\nu$ is defined to be

$$|\nu| = \nu^{+} + \nu^{-}.$$

**Definition:** If $\nu$ is a signed measure on $(X, \mathcal{M})$, and $\mu$ is a positive measure, then we say that $\nu$ is *absolutely continuous* with respect to $\mu$, written $\nu \ll \mu$, if $\mu(E) = 0$ implies that $\nu(E) = 0$.

**Proposition:** If $\nu$ is a signed measure on $(X, \mathcal{M})$, $\mu$ is a positive measure, then $\nu \ll \mu$ if and only if for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $\mu(E) < \delta$, $|\nu(E)| < \varepsilon$.

*Proof.* Absolute continuity clearly implies the $\varepsilon$-$\delta$ condition. To see the reverse direction, suppose there were some $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, we may find $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon_0$. Then, since

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty,$$

the Borel–Cantelli Lemma provides that $\mu(\limsup_{n\to\infty} E_n) = 0$. Yet, we must also have that

$$\begin{aligned}\nu\left(\limsup_{n\to\infty} E_n\right) &= \lim_{k\to\infty} \nu\left(\bigcup_{j=k}^{\infty} E_j\right)\\ &\geq \varepsilon_0,\end{aligned}$$

so $\nu \not\ll \mu$. □

**Theorem** (Lebesgue–Radon–Nikodym Theorem)**:** If $\nu$ is a signed measure, and $\mu$ is a positive measure, then there exists a measure $\lambda$ and a measurable function $f\colon X \to \mathbb{R}$ such that $\lambda \perp \rho$, and

$$\nu(E) = \lambda(E) + \int_E f\, d\mu.$$

If $\nu \ll \mu$, we say the particular function $f$ is the *Radon–Nikodym derivative* of $\nu$ with respect to $\mu$, and write $\frac{d\nu}{d\mu} = f$.

## Differentiation

**Definition:** A measurable function $f\colon \mathbb{R}^n \to \mathbb{C}$ is called locally integrable if $f\mathbb{1}_B$ is integrable for every closed ball $B \subseteq \mathbb{R}^n$.

**Theorem** (Vitali Covering Lemma)**:** If $\{B_1, \dots, B_N\}$ are a finite collection of open balls in $\mathbb{R}^n$, then there exists a disjoint subset $\{B_{n_1}, \dots, B_{n_k}\}$ such that

$$m\left(\bigcup_{j=1}^{N} B_j\right) \leq 3^n \sum_{j=1}^{k} m\left(B_{n_j}\right).$$

**Definition** (Hardy–Littlewood Maximal Function)**:** Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function. The Hardy–Littlewood Maximal Function $f^{*}\colon \mathbb{R}^n \to \mathbb{C}$ is defined by

$$f^{*}(x) = \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls $B$ that contain $x \in \mathbb{R}^n$.

**Theorem** (Maximal Theorem)**:** If $f\colon \mathbb{R}^n \to \mathbb{C}$ is locally integrable, then

(i) $f^*$ is measurable;

(ii) $f^*$ is finite a.e.;

(iii) $f^*(x) \geq |f(x)|$ a.e.;

(iv) if $f$ is integrable, then $f^*$ satisfies

$$m(\{x \in \mathbb{R}^n \mid f^*(x) > \alpha\}) \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

**Definition:** A family of subsets $\{E_r\}_{r>0}$ is said to *shrink nicely* to $x$ if

- $E_r \subseteq U(x,r)$ for each $r$;
- there is $\alpha > 0$ independent of $r$ such that $m(E_r) > \alpha m(U(x,r))$.

**Theorem** (Lebesgue Differentiation Theorem)**:** If $f\colon \mathbb{R}^n \to \mathbb{C}$ is locally integrable, then for a.e. $x \in \mathbb{R}^n$, we have

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_B f(y)\, dy = f(x)$$

for every family $\{E_r\}_{r>0}$ that shrinks nicely to $x$.

**Corollary:** If $\nu$ is a regular complex Borel measure on $\mathbb{R}^n$, with Lebesgue–Radon–Nikodym decomposition $d\nu = d\lambda + f\, dm$, then for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every family $\{E_r\}_{r>0}$ that shrinks nicely to $x$.

## Cavalieri's Principle

**Theorem:** Let $(X, \mathcal{M}, \mu)$ be a measure space, $f\colon X \to [0,\infty]$ a nonnegative measurable function. Then, the function $g\colon [0,\infty) \to [0,\infty]$ given by $\alpha \mapsto \mu(\{x \in X \mid f(x) > \alpha\})$ is a measurable function, and

$$\int_X f(x)\, d\mu(x) = \int_0^\infty \mu(\{x \mid f(x) > \alpha\})\, d\alpha.$$

*Proof.* Since $g$ is a decreasing function, we may set $b = \sup g^{-1}([0,a))$, so $g^{-1}([0,a))$ is either $[0,b]$ or $[0,b)$, both of which are Borel sets.

To show the integral identity, let $f = \sum_{k=1}^K a_k \mathbb{1}_{A_k}$, where the $a_k$ are distinct and $A_k$ are disjoint, so $f(x) = 0$ if $x \in X \setminus \bigcup_{k=1}^K A_k$. From linearity, we have

$$\int_X \sum_{k=1}^K a_k \mathbb{1}_{A_k}\, d\mu(x) = \sum_{k=1}^K a_k \mu(A_k).$$

Meanwhile,

$$\mu\left(\left\{x \;\middle|\; \sum_{k=1}^K a_k \mathbb{1}_{A_k} > \alpha\right\}\right) = \sum_{k=1}^K \mu(A_k) \mathbb{1}_{[0,a_k)}(x),$$

so

$$\int_0^\infty \mu\left(\left\{x \;\middle|\; \sum_{k=1}^K a_k \mathbb{1}_{A_k}(x)\right\}\right) d\alpha = \sum_{k=1}^K a_k \mu(A_k)$$

$$= \int_X f(x)\,d\mu(x).$$

Finally, if $f$ is a nonnegative $\mathcal{M}$-measurable function, there exists a pointwise increasing sequence of simple functions $(\varphi_m)_m$ that converge pointwise to $f$, meaning that by monotone convergence,

$$\begin{aligned}\int_X f(x)\,d\mu(x) &= \lim_{m\to\infty}\int_X \varphi_m(x)\,d\mu(x)\\ &= \lim_{m\to\infty}\int_0^\infty \mu(\{x \mid \varphi_m(x) > \alpha\})\,d\alpha\\ &= \int_0^\infty \mu(\{x \mid f(x) > \alpha\})\,d\alpha.\end{aligned}$$

□

### Chebyshev's Inequality

**Theorem:** If $f \in L_p(X, \mathcal{M}, \mu)$, then for any $\alpha > 0$, we have

$$\mu(\{x \mid |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha^p}\int_X |f|^p\,d\mu.$$

*Proof.* Let $E_\alpha = \{x \mid |f(x)| \geq \alpha\}$. Then,

$$\begin{aligned}\int_X |f|^p\,d\mu &\geq \int_{E_\alpha} |f|^p\,d\mu\\ &\geq \alpha^p \int_X \mathbb{1}_{E_\alpha}\,d\mu\\ &= \alpha^p\mu(\{x \mid |f(x)| > \alpha\}).\end{aligned}$$

□

## August 2019

**Problem** (Problem 1)**:** Let $\mathcal{C}$ be the Cantor set on $[0,1]$.

(a) Show that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) Find two sets $A, B \subseteq \mathbb{R}$ that are closed and have Lebesgue measure zero such that $A + B = \mathbb{R}$.

**Solution:**

(a) Recall that the Cantor set $\mathcal{C}$ is defined to consist of all $x \in [0,1]$ such that $x$ only contains 0 and 2 in the ternary expansion of $x$. Writing $a \in [0,2]$ as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where $a_k \in \{0,1,2\}$, we may then find $a_k$ at each ternary expansion slot for $k$ as follows:

- if $a_k = 0$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = c_k = 0$
- if $a_k = 2$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_k = 2$ and $c_k = 0$ or vice versa.
- if $a_k = 1$, we may find $b_k, c_k \in \mathcal{C}$ such that $b_{k+1} = c_{k+1} = 2$.

Therefore, since every digit of every ternary expansion in $[0,2]$ can be obtained from $\mathcal{C}$, we see that $\mathcal{C} + \mathcal{C} = [0,2]$.

(b) We may set $B$ to be the union of all integer translates of $\mathcal{C}$, and set $A = \mathcal{C}$. This yields closed subsets of $\mathbb{R}$ with Lebesgue measure zero that sum to $\mathbb{R}$.

**Problem** (Problem 2)**:** Does there exist a finite measure space $(X, \mathcal{F}, \mu)$ and a sequence $(f_n)_n$ of $\mu$-measurable functions such that

- $f_n(x) \geq 0$;
- $f_n(x) \to 0$ for all $x$;
- $\int_X f_n(x)\, d\mu(x) \to 0$ as $n \to \infty$;
- $\Phi(x) = \sup_n f_n(x)$ has infinite integral?

**Solution:** Consider the sequence of functions

$$f_n(x) = n\mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on $[0,1]$. This sequence is pointwise convergent everywhere to zero, as $f_n(0) = 0$ and the Archimedean property give that for any $x \in (0,1]$, there is some $n$ large enough that gives $\frac{1}{n} < x$. Furthermore, we see that

$$\begin{aligned}\int f_n\, d\mu &= n\left(\frac{1}{n} - \frac{1}{n+1}\right)\\ &= \frac{1}{n+1}\\ &\to 0.\end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned}\int \Phi\, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1}\\ &\to \infty.\end{aligned}$$

**Problem** (Problem 3)**:** Let $\mu$ be a signed measure in $\mathbb{R}^n$ that is bounded on bounded sets. Suppose that

$$\int_{\mathbb{R}^n} f\, d\mu = 0$$

for all continuous functions $f$ with bounded support. Show that $\mu = 0$.

**Solution:** Fix $r > 0$, and consider the family of continuous functions $f_n$ defined by

$$f_n = \begin{cases} 1 & x \in B(0,r) \\ 0 \leq f_n(x) \leq 1 & x \in B(0, r+1/n) \setminus B(0,r) \\ 0 & x \in B(0, r+1/n)^c \end{cases}$$

Since each $f_n$ is continuous with bounded support, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} f_n\, d\mu &= 0\\ &= \int_{B(0,r)} f_n\, d\mu + \int_{B(0,r+1/n)\setminus B(0,r)} f_n\, d\mu.\end{aligned}$$

We may define $K_n = B(0, r+1/n) \setminus B(0,r)$. Therefore,

$$|\mu(B(0,r))| = \left|\int_{B(0,r)} f_n\, d\mu\right|$$

$$
\begin{aligned}
&= \left|\int_{K_n} f_n \, d\mu\right| \\
&\leq \int_{K_n} f_n \, d|\mu| \\
&\leq \int_{K_n} d|\mu| \\
&= \mu^+(K_n) + \mu^-(K_n).
\end{aligned}
$$

Now, since $K_n$ is bounded, we see that $\mu^+$ and $\mu^-$ are both finite. Thus, since $\bigcap_{n\geq 1} K_n = \emptyset$, we see that $|\mu(B(0,r))| \leq \lim_{n\to\infty}(\mu^+(K_n) + \mu^-(K_n)) = 0$, so $\mu(B(0,r))$. Since the Borel $\sigma$-algebra is generated by the closed balls, $\mu = 0$ for all Borel sets.

**Problem** (Problem 4)**:** Let $L_1(\mathbb{R})$ be the space of Lebesgue integrable functions on $\mathbb{R}$. Suppose $f \in L_1(\mathbb{R})$ is positive. Show that $\frac{1}{f(x)} \notin L_1(\mathbb{R})$.

**Solution:** Suppose toward contradiction that both $f$ and $1/f$ are in $L_1(\mathbb{R})$. Then, from Hölder's Inequality, we have

$$
\begin{aligned}
\infty &= \int 1 \, d\mu \\
&\leq \left(\int f \, d\mu\right)^{1/2} \left(\int \frac{1}{f} \, d\mu\right)^{1/2} \\
&< \infty,
\end{aligned}
$$

which is a contradiction.

**Problem** (Problem 5)**:** Applying the Gram–Schmidt orthogonalization to $\{1, x, x^2, \dots\}$ in the Hilbert space $L_2([-1,1])$ with Lebesgue measure, one gets the Legendre polynomials $L_n(x)$.

(a) Show that the Legendre polynomials form a basis (complete orthogonal system) in the Hilbert space $L_2([-1,1])$.

(b) Show that the Legendre polynomials are given by the formula $L_n(x) = c_n \frac{d^n}{dx^n}(x^2-1)^n$.

**Solution:**

(a) Let $f \in L_2([-1,1])$. We may find $g \in C([-1,1])$ such that $\|f-g\|_{L_2} < \varepsilon/2$. Similarly, we may find a polynomial $p$ such that $\|g-p\|_u < \varepsilon/4$, meaning that $|p(x) - g(x)| < \varepsilon/4$ for all $x \in [-1,1]$. This yields

$$
\begin{aligned}
\|p-g\|_{L_2} &= \left(\int_{-1}^{1} |p(x) - g(x)|^2 \, dx\right)^{1/2} \\
&< \left(\int_{-1}^{1} \left(\frac{\varepsilon}{4}\right)^2 dx\right)^{1/2} \\
&= \left(\frac{\varepsilon^2}{8}\right)^{1/2} \\
&< \frac{\varepsilon}{2},
\end{aligned}
$$

so $\|f-p\|_{L_2} < \varepsilon$, meaning that the closed linear span of the monomials is dense in $L_2$, and the Legendre polynomials form an orthonormal system.

(b) We see that at every step in evaluating the expression

$$
L_n(x) = c_n \frac{d^n}{dx^n}(x^2-1)^n, \tag{$*$}
$$

the degree of the polynomial increases by 1, so each $L_n(x)$ has degree $n$. To verify that the polynomials generated from $(*)$ are orthogonal to each other, we let $n > m$ without loss of generality, and use integration by parts to obtain

$$\begin{aligned}
\langle L_n, L_m\rangle &= \int_{-1}^{1}\left(\frac{d^n}{dx^n}\left(x^2-1\right)^n\right)\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx\\
&= \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^m}{dx^m}\left(x^2-1\right)^m\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^{m+1}}{dx^{m+1}}\left(x^2-1\right)^m dx\\
&\vdots\\
&= (-1)^n\int_{-1}^{1}\frac{d^{m+n}}{dx^{m+n}}\left(x^2-1\right)^m dx\\
&= (-1)^n\int_{-1}^{1}\frac{d^n}{dx^n}\left(\frac{d^m}{dx^m}\left(x^2-1\right)^m\right)dx\\
&= (-1)^n\int\frac{d^n}{dx^n}L_m(x)\,dx\\
&= 0,
\end{aligned}$$

seeing as we are taking $n$ derivatives of a degree $m < n$ polynomial.

# January 2020

**Problem** (Problem 1)**:** Let $\mu$ be the Lebesgue measure on $\mathbb{R}$, and let $A \subseteq [0,1]$ be Lebesgue-measurable.

(a) Prove or show a counterexample to the assertion that

$$\mu(A) = \sup_{\substack{U\subseteq A\\ U \text{ open}}}\mu(U).$$

(b) Prove or show a counterexample to the assertion that

$$\mu(A) = \inf_{\substack{A\subseteq U\\ U \text{ open}}}\mu(U).$$

**Solution:**

(a) This is false. If $A \subseteq [0,1]$ is the "fat Cantor set" constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then $\mu(A) = \frac{1}{2}$, but $A$ is nowhere dense, meaning that if $U \subseteq A$ is open, then $U = \emptyset$.

To see that $A$ is nowhere dense, we see that $A$ is closed, so if $x \in A \subseteq [0,1]$, and $\varepsilon > 0$, we may show that the interval $(x-\varepsilon, x+\varepsilon)$ is not contained in $A$. In the recursive construction of $A$, we may see that there is some step $n_1$ such that $\frac{1}{4^{n_1}} < 2\varepsilon$, implying that $(x-\varepsilon, x+\varepsilon)$ is not contained in the recursive construction at $n_1$. Therefore $A^\circ = \emptyset$.

(b) This is true. By the definition of the Lebesgue outer measure, for any $\varepsilon > 0$, there are $\{(a_k,b_k)\}_{k=1}^{\infty}$ such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^{\infty}(a_k,b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^{\infty} (a_k, b_k),$$

we have that $U$ is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\{U \mid A \subseteq U, U \text{ open}\}.$$

**Remark:** Part (a) can be solved by selecting $A = \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$.

**Problem** (Problem 3)**:** Let $X$ be a compact metric space, $C(X)$ the space of real-valued continuous functions on $X$ with the supremum norm. Assume that $\mathcal{A} \subseteq C(X)$ satisfies

- (algebra) for all $f, g \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$, we have $\alpha f + \beta g \in \mathcal{A}$ and $fg \in \mathcal{A}$;
- (separates points) for any $x \neq y$ in $X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

(a) Show by example that $\mathcal{A}$ need not be dense in $C(X)$.

(b) In order to conclude that $\mathcal{A}$ is dense by the Stone–Weierstrass Theorem, what additional condition(s) should be added.

**Solution:**

(a) Consider the algebra of polynomials on $[0, 1]$ without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and $f(x) = x$ separates points in $[0, 1]$, this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at $x = 0$, the uniform closure of the algebra yields all the continuous functions on $[0, 1]$ with $f(0) = 0$.

(b) In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra $\mathcal{A}$ to include the constant functions.

**Problem** (Problem 4)**:** Let $\mu$ be a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra. Let $\mu(\mathbb{R}) = 1$. Next, let $\mathcal{F} \subseteq \mathcal{B}$ be the sub-$\sigma$-algebra generated by symmetric intervals.

Let $f \in L_1(\mathbb{R}, \mathcal{B}, \mu)$. Find a function $g$ such that:

- $g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$ (in particular, $g$ is $\mathcal{F}$-measurable);
- for all $E \in \mathcal{F}$, $\int_E g\, d\mu = \int_E f\, d\mu$.

**Solution:** We consider the signed measure on $\mathcal{F}$ defined by

$$\nu(E) = \int_E f\, d\mu,$$

meaning that $\nu \ll \mu$, so the function $g := \frac{d\nu}{d\mu}$, where $\frac{d\nu}{d\mu}$ denotes the Radon–Nikodym derivative of $\nu$ with respect to $\mu$ (where we restrict $\mu$ to $\mathcal{F}$), is $\mathcal{F}$-measurable (by Radon–Nikodym) and in $L_1(\mathbb{R}, \mathcal{F}, \mu)$. This gives, for all $E \in \mathcal{F}$,

$$\begin{aligned}\int_E g\, d\mu &= \int_E \frac{d\nu}{d\mu}\, d\mu \\ &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f\, d\mu.\end{aligned}$$

**Problem** (Problem 5)**:** Let $\mu$ be a finite measure on $(X,\mathcal{F})$. Show that a sequence of $\mathcal{F}$-measurable functions $(f_n)_n$ converges to $f$ in measure if and only if

$$\int_X \min\{1,|f_n-f|\}\,d\mu(x)\to 0.$$

**Solution:** Let $M=\mu(X)$.

Let $(f_n)_n\to f$ in measure, and let $\varepsilon>0$. If we let

$$\begin{aligned} A &= \{x \mid |f_n(x)-f(x)|>\varepsilon/2M\}\\ B &= \{x \mid |f_n(x)-f(x)|\leq\varepsilon/2M\}, \end{aligned}$$

we have

$$\begin{aligned} \int_X \min(1,|f_n-f|)\,d\mu &= \int_A \min(1,|f_n-f|)\,d\mu + \int_B \min(1,|f_n-f|)\,d\mu\\ &\leq \mu(A)+\varepsilon/2\\ &< \varepsilon/2+\varepsilon/2\\ &= \varepsilon. \end{aligned}$$

Meanwhile, if

$$\int_X \min(1,|f_n-f|)\,d\mu\to 0,$$

then by Chebyshev's Inequality, we have, for a fixed $0<\varepsilon\leq 1$,

$$\begin{aligned} \mu(\{x \mid |f_n-f|\geq\varepsilon\}) &= \mu(\{x \mid \min(1,|f_n-f|)\geq\varepsilon\})\\ &\leq \frac{1}{\varepsilon}\int_X \min(1,|f_n-f|)\,d\mu\\ &\to 0, \end{aligned}$$

so $(f_n)_n\to f$ in measure.

# August 2020

**Problem** (Problem 1)**:** Let $f\colon\mathbb{R}\to\mathbb{R}$ be continuous and almost everywhere differentiable such that $f'(x)=1$ almost everywhere. Does this imply that $f(2)-f(1)=1$?

**Solution:** This is false. To see this, let $\mathfrak{C}(x)$ denote the Cantor–Lebesgue function, and let

$$h(x)=\sum_{n=-\infty}^{\infty}\mathfrak{C}(x-n)+n.$$

Then, since $\mathfrak{C}(x)$ has derivative zero almost everywhere, the sum of a number of translates of $\mathfrak{C}(x)$ still has derivative zero almost everywhere. Then, setting

$$f(x)=h(x)+x,$$

we get that $f(x)$ has derivative equal to 1 almost everywhere. However, at the same time, $f(2)-f(1)=2$.

**Problem** (Problem 2)**:** Prove or provide a counterexample to the assertion that every open set in $\mathbb{R}^2$ is a countable union of closed sets.

**Solution:** We show the inverse problem, which is that every closed set in $\mathbb{R}^2$ is $G_\delta$. To do this, we let $A\subseteq\mathbb{R}^2$ be closed, nonempty, and proper (if $A=\emptyset$ or $A=\mathbb{R}^2$ the answer is trivial).

Then, there is some $x \in A^c$, and specifically there is $x \in A^c$ with rational coordinates (else, select $y \in \mathbb{Q}^2$ within the ball of radius $\varepsilon$ that allows $A^c$ to be open). Furthermore, since $\mathbb{R}^2$ is a metric space, $\mathbb{R}^2$ is regular, so there are open $U_x$ and $V_x$ such that $A \subseteq U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$.

Therefore, we get

$$A = \bigcap \{U_x \mid x \in \mathbb{Q}^2 \setminus A\},$$

meaning that $A$ is $G_\delta$. Taking complements, we thus get that every open set is $F_\sigma$.

**Problem** (Problem 3)**:** Let $\mathcal{H}$ be a separable complex Hilbert space with basis $(f_n)_n$. Define $P(f_n) = f_{n+1}$.

(a) Find $P^*$, the adjoint to $P$.

(b) Find $PP^*$ and $P^*P$.

**Solution:**

(a) We see that

$$\begin{aligned}\langle Pf_i, f_j\rangle &= \delta_{i+1,j}\\ &= \delta_{i,j-1}\\ &= \langle f_i, f_{j-1}\rangle\\ &= \langle f_i, P^* f_j\rangle,\end{aligned}$$

so that $Pf_n = f_{n-1}$ if $n > 1$. Else, if $n = 1$, then $P^* f_n = 0$.

(b) We see that, acting on the orthonormal basis $(f_n)_n$, $P^*P(f_n) = f_n$, and

$$PP^*(f_n) = \begin{cases} 0 & n = 1\\ 1 & \text{else,}\end{cases}$$

so that $P^*P = I$ and $PP^*$ is as above.

**Problem** (Problem 4)**:** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $f_n \colon X \to \mathbb{R}$ be measurable functions such that

$$\lim_{n\to\infty} \mu(\{x \mid f_n(x) \leq t\}) = \begin{cases} 0 & t < 0\\ 1 & t \geq 0\end{cases}.$$

Show that $f_n \to 0$ in measure.

**Solution:** We see that

$$\mu(\{x \mid f_n(x) > t\}) = \mu(X) - \mu(\{x \mid f_n(x) \leq t\}),$$

so by taking limits, we find that

$$\lim_{n\to\infty} \mu(\{x \mid f_n(x) > t\}) = \begin{cases} 1 & t < 0\\ 0 & t \geq 0\end{cases}.$$

So, if $\varepsilon > 0$, then

$$\begin{aligned}\mu(\{x \mid |f_n(x)| > \varepsilon\}) &= \mu(\{x \mid f_n(x) < -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\})\\ &\leq \mu(\{x \mid f_n(x) \leq -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\})\\ &\to 0.\end{aligned}$$

# January 2021

**Problem** (Problem 1)**:** Let $(f_n)_n$, $f$ be measurable functions on $(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ in measure. Does this imply that there exists a measurable set $A \subseteq \Omega$ with $\mu(\Omega \setminus A) = 0$ such that $f_n(x) \to f(x)$ for all $x \in A$.

**Solution:** This is not true. To see this, consider the family of functions defined by

$$\begin{aligned} f_1 &= \mathbb{1}_{[0,1]} \\ f_2 &= \mathbb{1}_{[0,1/2]} \\ f_3 &= \mathbb{1}_{[1/2,1]} \\ &\vdots \end{aligned}$$

where $f_n$ is of width $\frac{1}{2^k}$ when $2^k \leq n < 2^{k+1}$, moving along $[0,1]$. Then, since $\mu(\{x \mid |f_n(x)| > 0\}) = \frac{1}{2^k}$, we have that for any $\varepsilon > 0$, $(\mu(\{x \mid |f_n(x)| > \varepsilon\}))_n \leq (\mu(A_n))_n$, where we have defined $A_n$ to be the particular set with width $\frac{1}{2^k}$ when $2^k \leq n \leq 2^{k+1}$. Yet, since for any $x \in [0,1]$ there are infinitely many such $n$ such that $f_n(x) = 1$, the family $(f_n)_n$ does not converge to 0 pointwise anywhere on $[0,1]$.

**Problem** (Problem 2)**:** Let $B$ be a measurable subset of the two-dimensional plane such that the intersection of $B$ with every vertical line is either finite or countable. Find $\mu(B)$, where $\mu$ is the two-dimensional Lebesgue measure.

**Solution:** Note that the two-dimensional Lebesgue measure is the completion of $m \times m$, where $m \times m$ is the product measure on the product $\sigma$-algebra $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. If $B \in \mathcal{L}(\mathbb{R}^2)$, then $B = C \cup N$, where $N$ is a $\mu$-null set and $C \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. Therefore, if we show that $(m \times m)(C) = 0$, we then show that $\mu(B) = 0$.

To see that $(m \times m)(\mathbb{C}) = 0$, note that by our assumption, $B^x = \{y \in \mathbb{R} \mid (x,y) \in B\}$ is either finite or countable, so since $C^x \subseteq B^x$, we must have that $C^x$ is either finite or countable. By Tonelli's Theorem, since $\mathbb{1}_C$ is positive, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \mathbb{1}_C \, d(m \times m) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{C^x} \, dy \, dx \\ &= \int_{\mathbb{R}} m(C^x) \, dx \\ &= 0, \end{aligned}$$

so $(m \times m)(C^x) = 0$, meaning

$$\begin{aligned} \mu(B) &= \mu(C) + \mu(N) \\ &= (m \times m)(C) + \mu(N) \\ &= 0. \end{aligned}$$

**Problem** (Problem 3)**:** Let $(\Omega, \mathcal{F})$ be a measurable space, $\mu, \nu, \rho$ finite positive measures with $\mu \ll \nu$. Show that there exists a measurable function $f$ on $\Omega$ such that for all $E \in \mathcal{F}$,

$$\mu(E) = \int_E f \, d\nu + \int_E (f-1) \, d\rho.$$

**Solution:** Since $\mu \ll \nu$, and $\rho \ll \rho$, we have $\mu + \rho \ll \nu + \rho$, as $(\nu + \rho)(E) = 0$ if and only if $\nu(E) = 0$ and $\rho(E) = 0$, meaning that $\mu(E) = 0$ and $\rho(E) = 0$, so by Radon–Nikodym, there is some measurable $f$ such that

$$\mu(E) + \rho(E) = \int_E f \, d(\nu + \rho),$$

so by rearranging, we get

$$\mu(E) = \int_E f\, d\nu + \int_E (f-1)\, d\rho.$$

**Problem** (Problem 4)**:** Let $f, g$ be nonnegative measurable functions on $[0,1]$, and let $a, b, c, d \geq 0$ be arbitrary nonnegative numbers. Show that

$$\left(ac + bd + \int_0^1 f(x)g(x)\, dx\right)^3 \leq \left(a^3 + b^3 + \int_0^1 (f(x))^3\, dx\right)\left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2}\, dx\right)^2.$$

**Solution:** Since all of $f, g, a, b, c, d$ are positive, we may show

$$ac + bd + \int_0^1 f(x)g(x)\, dx \leq \left(a^3 + b^3 + \int_0^1 (f(x))^3\, dx\right)^{1/3}\left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2}\, dx\right)^{2/3}.$$

To do this, we use Hölder's Inequality three times:

$$\begin{aligned}
ac + bd + \int_0^1 f(x)g(x)\, dx &\leq \left(a^3 + b^3\right)^{1/3}\left(c^{3/2} + d^{3/2}\right)^{2/3} + \int_0^1 f(x)g(x)\, dx\\
&\leq \left(a^3 + b^3\right)^{1/3}\left(c^{3/2} + d^{3/2}\right)^{2/3} + \left(\int_0^1 (f(x))^3\, dx\right)^{1/3}\left(\int_0^1 (g(x))^{3/2}\, dx\right)^{2/3}\\
&\leq \left(a^3 + b^3 + \int_0^1 (f(x))^3\, dx\right)^{1/3}\left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2}\, dx\right)^{2/3}.
\end{aligned}$$

**Problem** (Problem 5)**:** Let $f(x)$ be a continuous function on $[0,1]$. Show that for every $\varepsilon > 0$ there exists $n \in \mathbb{Z}_{\geq 0}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$ such that for

$$D := \sum_{k=0}^{n} a_k \left(\frac{d}{dx}\right)^k,$$

we have

$$\left|f(x) - e^{x^2}\left(De^{-x^2}\right)\right| < \varepsilon$$

for all $x \in [0,1]$.

**Solution:** We note that for each $n$,

$$\left(\frac{d}{dx}\right)^n\left(e^{-x^2}\right) = P_n(x)e^{-x^2}$$

where $P_n(x)$ is a degree $n$ polynomial. To see this, using induction on $n$, we get

$$\begin{aligned}
\left(\frac{d}{dx}\right)^0\left(e^{-x^2}\right) &= (1)e^{-x^2}\\
&=: P_0(x)e^{-x^2}\\
\frac{d}{dx}\left(P_n(x)e^{-x^2}\right) &= P_n'(x)e^{-x^2} - 2xP_n(x)e^{-x^2}\\
&=: P_{n+1}(x)e^{-x^2}.
\end{aligned}$$

Therefore,

$$e^{x^2}\left(\frac{d}{dx}\right)^n\left(e^{-x^2}\right) = P_n(x).$$

Since each $P_n(x)$ is linearly independent (as they have different degrees of polynomials), and consist of polynomials of each degree for all $n \geq 0$, they span $\mathbb{C}[x]$. Then, for any $\varepsilon > 0$, by Stone–Weierstrass, there is some polynomial $p(x)$ such that

$$\sup_{x\in[0,1]} |f(x) - p(x)| < \varepsilon.$$

Since $\{P_n(x)\}_{n\geq 0}$ forms a basis for $\mathbb{C}[x]$, there are $a_0, \dots, a_n$ such that $p(x) = \sum_{k=0}^{n} a_k P_k(x)$. Setting

$$D = \sum_{k=0}^{n} a_k \left(\frac{d}{dx}\right)^k,$$

we obtain that

$$\left|f(x) - e^{x^2}\left(De^{-x^2}\right)\right| < \varepsilon.$$

## January 2022

**Problem** (Problem 1)**:** Let $(f_n)_n, f \subseteq L_1(X, \mu)$ be nonnegative functions, and let $(f_n)_n \to f$ pointwise, as well as

$$\left(\int_X f_n \, d\mu\right)_n \to \int_X f \, d\mu.$$

Show that $(f_n)_n \to f$ in $L_1$.

**Solution:** Consider the function $g_n(x) = \min(f_n, f)$, also written as

$$g_n = \frac{1}{2}(f_n + f - |f_n - f|).$$

Note that $|g_n| \leq f$, and $(g_n)_n \to f$ pointwise, so by dominated convergence, we have

$$\begin{aligned}
\int_X f \, d\mu &= \lim_{n\to\infty} \int_X g_n \, d\mu \\
&= \frac{1}{2} \lim_{n\to\infty} \left(\int_X f_n \, d\mu + \int_X f \, d\mu - \int_X |f_n - f| \, d\mu\right) \\
&= \int_X f \, d\mu - \frac{1}{2} \lim_{n\to\infty} \int_X |f_n - f| \, d\mu,
\end{aligned}$$

so

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0,$$

and $(f_n)_n \to f$ in $L_1$.

**Problem** (Problem 2)**:** Let $p \in [1, \infty)$.

(a) Show that if $(f_n)_n \to f$ in $L_p$, then there is $(f_{n_k})_k$ such that for $\mu$-a.e. $x \in X$, $(f_{n_k})_k \to f$ pointwise.

(b) Let $h$ be a measurable function, and let $D$ be defined such that

$$D = \{f \in L_p(X, \mu) \mid hf \in L_p(X, \mu)\}.$$

Suppose $(f_n)_n \to f$ in $L_p$, and $(hf_n)_n \to g$ in $L_p$. Show that $f \in D$ and $g = hf$.

**Solution:**

(a) Since $(f_n)_n \to f$ in $L_p$, the sequence $(f_n)_n$ is $L_p$-Cauchy, so we may find a subsequence $(f_{n_k})_k$ such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\| < \frac{1}{2^k}.$$

Defining

$$\begin{aligned} s_n &= \sum_{k=1}^{n}\left|f_{n_{k+1}} - f_{n_k}\right| \\ s &= \sum_{k=1}^{n}\left|f_{n_{k+1}} - f_{n_k}\right|, \end{aligned}$$

we see that by Minkowski's Inequality,

$$\begin{aligned} \|s_n\| &\leq \sum_{k=1}^{n}\left\|f_{n_{k+1}} - f_{n_k}\right\| \\ &\leq 1. \end{aligned}$$

So, by applying Fatou's Lemma to $s_n^p$, we see that

$$\|s\| \leq 1,$$

meaning that in particular, $s(x) < \infty$ almost everywhere, and $(s_n)_n$ converges absolutely almost everywhere. Defining

$$g(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}\left(f_{n_{k+1}} - f_{n_k}\right)(x)$$

for all $x$ where $s(x)$ is defined, and 0 otherwise, we see that by telescoping, $g(x) = \lim_{k\to\infty} f_{n_k}(x)$. Now, we show that $\|g - f\| = 0$, meaning that $g = f$ under the $\mu$-a.e. equivalence relation. Computing, we have

$$\begin{aligned} \int_X |g - f|^p \, d\mu &= \int_X \liminf_{k\to\infty}|f_{n_k} - f|^p \, d\mu \\ &\leq \liminf_{k\to\infty} \int_X |f_{n_k} - f|^p \, d\mu \\ &= \liminf_{k\to\infty}\|f_{n_k} - f\|^p \\ &= 0, \end{aligned}$$

as for any subsequence $(f_{n_k})_k$, $(f_{n_k})_k \to f$ in $L_p$. Thus, $(f_{n_k})_k \to f$ for $\mu$-almost every $x$.

(b) Since $(f_n)_n \to f$ in $L_p$, there is a subsequence $(f_{n_k})_k \to f$ pointwise almost everywhere. Thus, by multiplying $h(x)$, we see that $(hf_{n_k})_k \to hf$ pointwise almost everywhere.

Now, since $(hf_n)_n \to g$ in $L_p$, this applies for every subsequence of $(hf_n)_n$; in particular, it applies to $(hf_{n_k})_k$, meaning that $(hf_{n_k})_k \to g$ in $L_p$, and admits a subsequence $\left(hf_{n_{k_j}}\right)_j \to g$ pointwise almost everywhere.

Returning to the convergence $(hf_{n_k})_k \to hf$ pointwise almost everywhere, this applies for every subsequence, so in particular, it applies to $\left(hf_{n_{k_j}}\right)_j$.

Set

$$E_1 = \left\{x \;\middle|\; \left(\left(hf_{n_{k_j}}\right)(x)\right)_j \not\to g(x)\right\}$$
$$E_2 = \left\{x \;\middle|\; \left(\left(hf_{n_{k_j}}\right)(x)\right)_j \not\to (hf)(x)\right\}.$$

Then, $\mu(E_1) = \mu(E_2) = 0$, so $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2) = 0$, and so $g(x) = (hf)(x)$ for almost every $x$ (as $\mathbb{C}$ is Hausdorff). In particular, this means that $[g] = [hf]$ under the almost everywhere equivalence relation. Since $L_p$ is complete, and $(hf_n)_n \to g$ in $L_p$, we have $g \in L_p$, so $hf \in L_p$, and $f \in D$.

**Problem** (Problem 3)**:** Let $\mu$ be a Borel probability measure on $\mathbb{R}$, and define

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx}\, d\mu(x).$$

(a) Show that $\hat{\mu}(t)$ is bounded and continuous.

(b) If $\delta > 0$, show that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\, dt = \int_{\mathbb{R}} 1 - \operatorname{sinc}(\delta x)\, d\mu(x).$$

(c) Show that

$$1 - \operatorname{sinc}(u) \geq \frac{1}{2}\mathbb{1}_{(-\infty,2)\cup(2,\infty)}(u),$$

and deduce that

$$\mu(\{x \mid |x| > 2/\delta\}) \leq \frac{1}{\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\, dt.$$

(d) Let $(\mu_n)_n$ be a sequence of Borel probability measures on $\mathbb{R}$. Suppose that for all $t$, $\Phi(t) = \lim_{n\to\infty} \widehat{\mu_n}(t)$ exists, and $\Phi(t)$ is continuous at $t = 0$. Show that for all $\varepsilon > 0$, there is a compact $K$ such that for all $n$, $\mu_n(K) \geq 1 - \varepsilon$.

**Solution:**

(a) We see that $\hat{\mu}$ is bounded, as

$$\begin{aligned}|\hat{\mu}(t)| &= \left|\int_{\mathbb{R}} e^{itx}\, d\mu(x)\right| \\ &\leq \int_{\mathbb{R}} \left|e^{itx}\right| d\mu(x) \\ &\leq 1,\end{aligned}$$

since $\mu$ is a probability measure. Furthermore, using dominated convergence with $g(t) = 1$, we see that if $(t_n)_n \to t$, then $e^{it_nx} \to e^{itx}$ as the exponential function is continuous, so $\hat{\mu}(t_n) \to \hat{\mu}(t)$, and $\hat{\mu}$ is continuous.

(b) We note that $\operatorname{Re}(\hat{\mu}(t)) = \int_{\mathbb{R}} \cos(tx)\, d\mu(x) \leq 1$ for all $t$, meaning that $1 - \operatorname{Re}(\hat{\mu}(t)) \geq 0$ for all $t$. Writing our integral, we then get

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\, dt = \frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}\int_{\mathbb{R}} e^{itx}\, d\mu(x)\, dt$$

and using the fact that $\mu(\mathbb{R}) = 1$,

$$= \frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\mathbb{R}} 1 - \cos(tx)\,d\mu(x)\,dt.$$

By Tonelli's Theorem, we may switch the order of integration, so

$$\begin{aligned} &= \frac{1}{2\delta}\int_{\mathbb{R}}\int_{-\delta}^{\delta} 1 - \cos(tx)\,dt\,d\mu(x)\\ &= \int_{\mathbb{R}} 1 - \int_{-\delta}^{\delta}\frac{1}{2\delta}\cos(tx)\,dt\,d\mu(x). \end{aligned}$$

Now, evaluating the inner integral, we see that

$$\int_{-\delta}^{\delta}\frac{1}{2\delta}\cos(tx)\,dt = \begin{cases} 1 & x = 0\\ \frac{\sin(\delta x)}{\delta x} & x \neq 0 \end{cases},$$

so

$$= \int_{\mathbb{R}} 1 - \operatorname{sinc}(\delta x)\,d\mu(x).$$

(c) We see that $1 - \operatorname{sinc}(u) \geq 0$ for all $u$, so when $|u| \leq 2$, the inequality is satisfied. Similarly, if $|u| > 2$, then

$$\begin{aligned} 1 - \operatorname{sinc}(u) &= 1 - \frac{\sin(u)}{u}\\ &\geq 1 - \frac{1}{2}\\ &= \frac{1}{2}, \end{aligned}$$

so the inequality is satisfied when $|u| > 2$. Thus, we see that

$$\begin{aligned} \frac{1}{\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\hat{\mu}(t))\,dt &= 2\int_{\mathbb{R}} 1 - \operatorname{sinc}(\delta x)\,d\mu(x)\\ &\geq \int_{\mathbb{R}} \mathbb{1}_{(-\infty,2)\cup(2,\infty)}(\delta x)\,d\mu(x)\\ &= \int_{\mathbb{R}} \mathbb{1}_{(\infty,2/\delta)\cup(2/\delta,\infty)}(x)\,d\mu(x)\\ &= \mu(\{x \mid |x| > 2/\delta\}). \end{aligned}$$

(d) Let $\varepsilon > 0$. Since $\Phi(t)$ is continuous at 0, and $\Phi(0) = \lim_{n\to\infty}\widehat{\mu_n}(0) = 1$, there is $\delta$ such that whenever $|t| < \delta$, $|1 - \Phi(t)| < \varepsilon/2$. Note that this implies that $1 - \operatorname{Re}(\Phi(t)) < \varepsilon/2$ for all $t$ with $|t| < \delta$.

Next, we see that $1 - \operatorname{Re}(\widehat{\mu}_n(t)) \to 1 - \operatorname{Re}(\Phi(t))$, so by using the dominating function $g(t) = 2$, the dominated convergence theorem implies that

$$\begin{aligned} \frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\widehat{\mu_n}(t))\,dt &\to \frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\Phi(t))\,dt\\ &< \varepsilon/2, \end{aligned}$$

meaning that there is $N$ such that for all $n \geq N$,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} 1 - \operatorname{Re}(\widehat{\mu_n}(t))\,dt < \varepsilon/2.$$

Thus, by using part (c), we see that

$$\mu_n(\{x \mid |x| > 2/\delta\}) < \varepsilon,$$

so for all $n \geq N$,

$$\mu_n([-2/\delta, 2/\delta]) \geq 1 - \varepsilon.$$

Next, for each $n \leq N$, we find $k_n \in \mathbb{N}$ such that $\mu([-k_n, k_n]) \geq 1 - \varepsilon$; the existence of such a $k_n$ follows from continuity from below, as for each $n$,

$$\begin{aligned} 1 &= \mu_n(\mathbb{R}) \\ &= \mu_n\left(\bigcup_{k \geq 1} [-k, k]\right) \\ &= \sup_{k \geq 1} \mu_n([-k, k]). \end{aligned}$$

Set $K_N = \max\left(\{k_n\}_{n=1}^N\right)$, and let $K = [-K_N, K_N] \cup [-2/\delta, 2/\delta]$. Then, for all $n$, we find that

$$\mu_n(K) \geq 1 - \varepsilon.$$

## August 2022

**Problem** (Problem 1)**:** Compute

$$\lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1 + x^2)} \, dx.$$

**Solution:** We note that

$$\begin{aligned} \left|\frac{n \sin(x/n)}{x(1 + x^2)}\right| &\leq \left|\frac{n(x/n)}{x(1 + x^2)}\right| \\ &= \frac{1}{1 + x^2}, \end{aligned}$$

and since $\frac{1}{1+x^2}$ is integrable, we may use Dominated Convergence to switch limit and integral, giving

$$\begin{aligned} \lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1 + x^2)} \, dx &= \int_0^\infty \lim_{n \to \infty} \frac{n \sin(x/n)}{x(1 + x^2)} \, dx \\ &= \int_0^\infty \lim_{h \to 0} \frac{\frac{1}{h} \sin(hx)}{x(1 + x^2)} \, dx \\ &= \int_0^\infty \frac{x}{x(1 + x^2)} \, dx \\ &= \frac{\pi}{2}. \end{aligned}$$

### Problem 2

**Problem** (Problem 2)**:** Fix $a < b$ in $\mathbb{R}$. For a Lipschitz function $g\colon [a, b] \to \mathbb{C}$, set

$$\|g\|_{\text{Lip}} = \sup_{x \neq y \in [a,b]} \frac{|g(x) - g(y)|}{|x - y|}.$$

(a) Show that $f\colon [a, b] \to \mathbb{C}$ is Lipschitz if and only if $f$ is absolutely continuous and $f' \in L_\infty([a, b])$.

(b) If $f\colon [a,b] \to \mathbb{C}$ is Lipschitz, show that $\|f\|_{\text{Lip}} = \|f'\|_{L_\infty}$.

**Solution:**

(a) Let $f$ be Lipschitz, and let $M$ denote the Lipschitz constant — i.e., $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [a,b]$. Set $\delta = \frac{\varepsilon}{M}$. Then, if $\{(a_j, b_j)\}_{j=1}^{k}$ is a partition such that $\sum_{j=1}^{k} |b_j - a_j| < \delta$, we have

$$\begin{aligned}\sum_{j=1}^{k} |f(b_j) - f(a_j)| &\leq M \sum_{j=1}^{k} |b_j - a_j| \\ &< \varepsilon.\end{aligned}$$

Thus, $f$ is absolutely continuous. Now, if $x, x+h \in [a,b]$, we have that

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq M,$$

meaning that

$$\begin{aligned}|f'(x)| &= \lim_{h\to 0}\left|\frac{f(x+h) - f(x)}{h}\right| \\ &\leq M,\end{aligned}$$

and since $f'(x)$ exists for a.e. $x \in [a,b]$, we have that $\operatorname{ess\,sup}_{x\in[a,b]} |f'(x)| \leq M$, so $f' \in L_\infty([a,b])$.

Let $f$ be absolutely continuous with bounded derivative. Then, if $M$ is the essential supremum of the $f'$, the fundamental theorem of calculus gives

$$\begin{aligned}|f(y) - f(x)| &= \left|\int_x^y f'(t)\, dt\right| \\ &\leq \int_x^y |f'(t)|\, dt \\ &\leq \int_x^y M\, dx \\ &= M|y - x|,\end{aligned}$$

so $f$ is Lipschitz.

(b) If $f$ is such that $f'(x)$ exists, then for $x, x+h \in [a,b]$, we have

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq \|f\|_{\text{Lip}},$$

so by taking limits, we have

$$|f'(x)| \leq \|f\|_{\text{Lip}}.$$

Thus, this ordering must respect essential suprema, meaning

$$\|f'\|_{L_\infty} \leq \|f\|_{\text{Lip}}.$$

Furthermore, if $\varepsilon > 0$, there are $x, y \in [a,b]$ with $x < y$ such that

$$\begin{aligned}\|f\|_{\text{Lip}} - \varepsilon &< \left|\frac{f(y) - f(x)}{y - x}\right| \\ &= \frac{1}{|y - x|}\left|\int_x^y f'(t)\, dt\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{|y-x|}\int_{x}^{y}|f'(t)|\,dt\\
&\leq \frac{1}{|y-x|}\int_{x}^{y}\|f'\|_{L_\infty}\,dt\\
&= \|f'\|_{L_\infty},
\end{aligned}
$$

and since $\varepsilon$ is arbitrary, we have $\|f\|_{\text{Lip}} \leq \|f'\|_{L_\infty}$.

**Problem** (Problem 3)**:** Let $(X,\mu)$ be a $\sigma$-finite measure space. Show that if $f,g \in L_1(X,\mu)$ with $0 \leq f,g$ almost everywhere, then

$$\|f-g\|_{L_1} = \int_{0}^{\infty}\mu(\{x \mid f(x) > t\}\triangle\{x \mid g(x) > t\})\,dt.$$

**Solution:** We start by showing that

$$|a-b| = \int_{0}^{\infty}\left|\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)\right|dt$$

for all $a,b \in [0,\infty)$. Without loss of generality, $a \leq b$. To see this, note that there are three cases:

$$\left|\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)\right| = \begin{cases}0 & t < a,b\\ 1 & a \leq t < b,\\ 0 & a,b \leq t\end{cases}$$

giving

$$
\begin{aligned}
\int_{0}^{\infty}\mathbb{1}_{[a,b)}\,dt &= \mu([a,b))\\
&= b-a\\
&= |a-b|.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\|f-g\|_{L_1} &= \int_{X}|f(x)-g(x)|\,d\mu(x)\\
&= \int_{X}\int_{0}^{\infty}\left|\mathbb{1}_{(t,\infty)}(f(x)) - \mathbb{1}_{(t,\infty)}(g(x))\right|dt\,d\mu(x),
\end{aligned}
$$

and by Tonelli's Theorem, we have

$$
\begin{aligned}
&= \int_{0}^{\infty}\int_{X}\left|\mathbb{1}_{f^{-1}((t,\infty))} - \mathbb{1}_{g^{-1}((t,\infty))}\right|d\mu(x)\,dt\\
&= \int_{0}^{\infty}\int_{X}\mathbb{1}_{f^{-1}((t,\infty))\triangle g^{-1}((t,\infty))}\,d\mu(x)\,dt\\
&= \int_{0}^{\infty}\mu\left(f^{-1}((t,\infty))\triangle g^{-1}((t,\infty))\right)dt.
\end{aligned}
$$

**Problem** (Problem 4)**:** Let $(X,\Sigma)$ be a measurable space. Suppose that $\mu,\nu$ are signed measures on $\Sigma$ such that $\|\mu\|_{TV}, \|\nu\|_{TV} < \infty$, and $|\mu| \perp |\mu|$.

(a) If $\mu = \mu_1 - \mu_2$ and $\nu = \nu_1 - \nu_2$ with $\mu_1 \perp \mu_2$ and $\nu_1 \perp \nu_2$, show that $\mu_i \perp \nu_j$ for all $i,j \in \{1,2\}$.

(b) Show that

$$\|\mu+\nu\|_{\text{TV}} = \|\mu\|_{\text{TV}} + \|\nu\|_{\text{TV}}.$$

**Solution:**

(a) Since $|\mu| \perp |\nu|$, there are $U, V \subseteq X$ such that $|\mu|$ is concentrated on $U$ and $|\nu|$ is concentrated on $V$, with $U \cap V = \emptyset$.

Note that by the Jordan decompositions, we have $|\mu| = \mu_1 + \mu_2 \geq \mu_{1,2}$ so $\mu_{1,2}$ are concentrated on $U$, and similarly $\nu_{1,2}$ are concentrated on $V$, so $\mu_i \perp \nu_j$.

(b) We show that the measures $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ are mutually singular. To see this, note the following:

- $\mu_1 = 0$ on $N_\mu \cup V$;
- $\nu_1 = 0$ on $N_\nu \cup U$;
- $\mu_2 = 0$ on $P_\mu \cup V$;
- $\nu_2 = 0$ on $P_\nu \cup U$,

so $\mu_1 + \nu_1 = 0$ on $A = (N_\mu \cup V) \cap (N_\nu \cup U)$, and $\mu_2 + \nu_2 = 0$ on $B = (P_\mu \cup V) \cap (P_\nu \cup U)$. Therefore, since

$$\begin{aligned} A \cup B &= (N_\mu \cap N_\nu) \cup (N_\mu \cap U) \cup (N_\nu \cap V) \\ &\quad \cup (P_\mu \cap P_\mu) \cup (P_\mu \cap U) \cup (P_\nu \cap V) \\ &= X \end{aligned}$$

$$\begin{aligned} A \cap B &= (N_\mu \cup V) \cap (N_\nu \cup U) \\ &\quad \cap (P_\mu \cup V) \cap (P_\nu \cup U) \\ &= \emptyset, \end{aligned}$$

the measures $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ are mutually singular, so $A \sqcup B$ forms a Hahn decomposition for $\mu + \nu$ with corresponding Jordan decomposition of $(\mu_1 + \nu_1) - (\mu_2 + \nu_2)$. Thus,

$$\begin{aligned} \|\mu + \nu\|_{\text{TV}} &= |\mu + \nu|(X) \\ &= (\mu_1 + \nu_1)(X) + (\mu_2 + \nu_2)(X) \\ &= (\mu_1 + \mu_2)(X) + (\nu_1 + \nu_2)(X) \\ &= |\mu|(X) + |\nu|(X) \\ &= \|\mu\|_{\text{TV}} + \|\nu\|_{\text{TV}}. \end{aligned}$$

**Problem** (Problem 5)**:**

(a) For $f \in L_1([0,1])$, let $L_f$ be the set of all $x \in [0,1]$ such that

$$\lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)|\, dy = 0.$$

State the conclusion of the Lebesgue differentiation theorem regarding $L_f$.

(b) For $n \in \mathbb{N}$, $0 \leq j \leq 2^n - 1$, set $I_{n,j} = [j2^{-n}, (j+1)2^{-n})$. For $f \in L_1([0,1])$, define

$$E_n f = \sum_{j=0}^{2^n - 1} \left( \frac{1}{m(I_{n,j})} \int_{I_{n_j}} f(t)\, dt \right) \mathbb{1}_{I_{n_j}}.$$

Show that $\lim_{n \to \infty} (E_n f)(x) = f(x)$ for a.e. $x \in [0,1]$.

**Solution:**

(a) The conclusion of the Lebesgue differentiation theorem states that $\mu([0,1] \setminus L_f) = 0$.

(b) Let $x \in [0,1]$. We note that $x$ must be in exactly one such interval $(j2^{-n}, (j+1)2^{-n}]$ since these intervals are disjoint. If we select $r > 0$ such that $\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$, then we note the following:

- $I_{n,j} \subseteq U(x,r)$ for exactly one such $j$;
- $m(U(x,r)) \leq 4\mu(I_{n,j})$.

If $x \in L_f$, then for any $\varepsilon > 0$, there is some $\delta > 0$ such that when $r < \delta$, then

$$\frac{1}{\mu(U(x,r))} \int_{U(x,r)} |f(t) - f(x)|\, dt < \varepsilon,$$

by the Lebesgue Differentiation Theorem. If $n$ is such that $\frac{1}{2^{n-1}} < \delta$, then when $\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$, then for any $x \in L_f$, we have

$$\begin{aligned} |E_n f(x) - f(x)| &= \left| \frac{1}{m(I_{n,j})} \int_{I_{n,j}} f(t)\, dt - f(x) \right| \\ &\leq \frac{1}{m(I_{n,j})} \int_{I_{n,j}} |f(t) - f(x)|\, dt \\ &\leq \frac{1}{m(I_{n,j})} \int_{U(x,r)} |f(t) - f(x)|\, dt \\ &\leq \frac{4}{U(x,r)} \int_{U(x,r)} |f(t) - f(x)|\, dt \\ &< 4\varepsilon, \end{aligned}$$

so $\lim_{n\to\infty} E_n f(x) = f(x)$ for all $x \in L_f$, meaning that it holds for a.e. $x \in [0,1]$.

## January 2023

**Problem** (Problem 1)**:** Let $(X, \mu)$ be a $\sigma$-finite measure space, $p \in [1, \infty)$. Let $(f_n)_n$ be a sequence in $L_p(X, \mu)$, and suppose $\|f_n\|_{L_p} \leq 1$, $(f_n)_n \to f$ almost everywhere. Show that $\|f\|_p \leq 1$.

**Solution:** By using Fatou's Lemma, and assuming without loss of generality that $(f_n)_n \to f$ pointwise everywhere, we get

$$\begin{aligned} \int_X |f|^p\, d\mu &= \int_X \liminf_{n\to\infty} |f_n|^p\, d\mu \\ &\leq \liminf_{n\to\infty} \int_X |f_n|^p\, d\mu \\ &\leq 1, \end{aligned}$$

so $\|f\|_{L_p} \leq 1$.

**Problem** (Problem 2)**:** Let $\mu$ be an atomless Borel probability measure on $\mathbb{R}$. Suppose $E \subseteq \mathbb{R}$ is a Borel set with $\mu(E) > 0$. Show that there is $t \in \mathbb{R}$ with $\mu(E \cap (-\infty, t)) = \frac{1}{2}\mu(E)$.

**Solution:** Let

$$f(t) = \mu(E \cap (-\infty, t)),$$

and for any sequence $(t_n)_n$, define

$$E_n = E \cap (-\infty, t_n).$$

We will show that $f$ is left- and right-continuous, hence continuous. To start, if $(t_n)_n \searrow t$, then

$$\bigcap_{n\in\mathbb{N}} E_n = E \cap (-\infty, t],$$

so

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n \setminus \{t\}\right) \\ &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) - \mu(\{t\}). \end{aligned}$$

Since $\mu$ is atomless, we see that $\mu(\{t\}) = 0$, so since $\mu(E) < \infty$,

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) \\ &= \lim_{n\to\infty} \mu(E_n) \\ &= \lim_{n\to\infty} f(t_n). \end{aligned}$$

Thus, $f$ is right-continuous. Similarly, if $f$ is left-continuous, and $(t_n)_n \nearrow t$, then

$$\bigcup_{n\in\mathbb{N}} E_n = E \cap (-\infty, t),$$

so by continuity from below,

$$\begin{aligned} f(t) &= \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \\ &= \lim_{n\to\infty} \mu(E_n) \\ &= \lim_{n\to\infty} f(t_n). \end{aligned}$$

Therefore, $f$ is continuous. Since

$$\begin{aligned} \lim_{t\to-\infty} f(t) &= 0 \\ \lim_{t\to\infty} f(t) &= \mu(E) \\ &> 0, \end{aligned}$$

the intermediate value theorem gives some $t_0 \in \mathbb{R}$ such that

$$\begin{aligned} f(t_0) &= \mu(E \cap (-\infty, t_0)) \\ &= \frac{1}{2}\mu(E). \end{aligned}$$

**Problem** (Problem 3)**:** Let $X$ be a set equipped with a $\sigma$-algebra $\Sigma$. Suppose $\mu, \nu \colon \Sigma \to [0, \infty)$ are finite measures with $\lambda = \mu + \nu$. Define $f$ such that

$$\nu(E) = \int_E f \, d\lambda.$$

(i) Show that $0 \leq f \leq 1$ $\lambda$-a.e.

(ii) If $F = \{x \mid f(x) = 1\}$, show that $\mu(F) = 0$.

(iii) If $A \subseteq \{x \mid 0 \leq f(x) < 1\}$ is such that $\mu(A) = 0$, show that $\nu(A) = 0$.

**Solution:**

(i) Consider the sets $E_n$, for each $n \in \mathbb{N}$, defined by

$$E_n = \left\{x \,\middle|\, f(x) < -\frac{1}{n}\right\},$$

so that $E_n \subseteq E_{n+1}$, and

$$\begin{aligned} E &= \bigcup_{n=1}^{\infty} E_n \\ &= \{x \mid f(x) < 0\}. \end{aligned}$$

Then, we see that

$$\begin{aligned} 0 &\geq -\frac{1}{n}\lambda(E_n) \\ &= -\frac{1}{n}\int_{E_n} d\lambda \\ &> \int_{E_n} f\, d\lambda \\ &= \nu(E_n) \\ &\geq 0, \end{aligned}$$

meaning that $\lambda(E_n) = 0$ for each $n$, so by continuity from below, $\lambda(E) = \lim_{n\to\infty} \lambda(E_n) = 0$.

Now, the set

$$F = \{x \mid f(x) > 1\}$$

has

$$\begin{aligned} \lambda(F) &= \int_F d\lambda \\ &< \int_F f\, d\lambda \\ &= \nu(F) \\ &\leq \nu(F) + \mu(F) \\ &= \lambda(F), \end{aligned}$$

meaning that $\lambda(F) = 0$, and $0 \leq f \leq 1$ $\lambda$-a.e.

(ii) If $F = \{x \mid f(x) = 1\}$, then

$$\begin{aligned} \lambda(F) &= \int_F d\lambda \\ &= \int_F f\, d\lambda \\ &= \nu(F), \end{aligned}$$

so $\mu(F) = 0$.

(iii) Let $A \subseteq \{x \mid 0 \leq f(x) < 1\}$ be such that $\mu(A) = 0$. Then, we have

$$\begin{aligned}\nu(A) &= \int_A f\,d\lambda\\ &= \int_A f\,d\nu + \int_A f\,d\mu\\ &< \int_A f\,d\nu + \int_A\,d\mu\\ &= \int_A f\,d\nu + \mu(A)\\ &= \int_A f\,d\nu\\ &\leq \int_A f\,d\lambda\\ &= \nu(A),\end{aligned}$$

so $\nu(A) = 0$, else we reach a contradiction.

**Problem** (Problem 4)**:** Fix $p \in [1,\infty)$. Let $W_p([0,1])$ be the space of absolutely continuous functions on $[0,1]$ such that $f' \in L_p([0,1])$. For all $f \in W_p([0,1])$, define

$$\|f\|_{W_p} = |f(0)| + \|f'\|_{L_p}.$$

Show that $\|\cdot\|_{W_p}$ is a norm that makes $W_p([0,1])$ into a Banach space. You are allowed to use the fact that $L_p([0,1])$ is a Banach space.

**Solution:** We start by showing that $\|\cdot\|_{W_p}$ is indeed a norm. To see that $\|\cdot\|_{W_p}$ is positive definite, if

$$\|f\|_{W_p} = 0,$$

then $|f(0)| = 0$ and $\|f'\|_{L_p} = 0$. Since $\|f'\|_{L_p} = 0$, $f' = 0$ a.e. as $L_p$ is a Banach space. Note that, by the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t)\,dt,$$

so $f(x) = 0$ almost everywhere, hence $f(x) = 0$ in $L_p$.

Next, to see homogeneity, we have for all $\alpha \in \mathbb{C}$,

$$\begin{aligned}\|\alpha f\|_{W_p} &= |\alpha f(0)| + \left\|(\alpha f)'\right\|_{L_p}\\ &= |\alpha|\left(|\alpha| + \|f'\|_{L_p}\right)\\ &= |\alpha|\|f\|_{W_p},\end{aligned}$$

as $\|\cdot\|_{L_p}$ is a norm. Finally, we have

$$\begin{aligned}\|f+g\|_{W_p} &= |(f+g)(0)| + \left\|(f+g)'\right\|_{L_p}\\ &\leq |f(0)| + |g(0)| + \|f'\|_{L_p} + \|g'\|_{L_p}\\ &= \|f\|_{W_p} + \|g\|_{W_p},\end{aligned}$$

as $\|\cdot\|_{L_p}$ is a norm, so the triangle inequality holds. Thus, $\|\cdot\|_{W_p}$ is a norm.

Let $(f_n)_n$ be Cauchy in $W_p([0,1])$. Then, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m,n \geq N$,

$$\|f_n - f_m\|_{W_p} = |f_n(0) - f_m(0)| + \|f_n' - f_m'\|_{L_p}$$

$$< \varepsilon,$$

meaning that both

$$\begin{aligned} |f_n(0) - f_m(0)| &< \varepsilon \\ \|f_n' - f_m'\|_{L_p} &< \varepsilon. \end{aligned}$$

Since $\mathbb{C}$ and $L_p([0,1])$ are complete, there is $c \in \mathbb{C}$ and $g \in L_p([0,1])$ such that

$$\begin{aligned} f_n(0) &\to c \\ f_n' &\to g. \end{aligned}$$

Define

$$f(x) = c + \int_0^x g(t)\,dt.$$

Then, we note that by the Fundamental Theorem of Calculus,

$$\begin{aligned} f'(x) &= g(x) \\ &\in L_p([0,1]), \end{aligned}$$

so $f \in W_p([0,1])$. Finally, we see that

$$\begin{aligned} \|f_n - f\|_{W_p([0,1])} &= |f_n(0) - f(0)| + \|f_n' - f'\|_{L_p} \\ &= |f_n(0) - c| + \|f_n' - g\|_{L_p} \\ &\to 0, \end{aligned}$$

so $(f_n)_n \to f$ in $W_p$, meaning $W_p$ is complete.

**Problem** (Problem 5)**:** Let $m$ be Lebesgue measure on $\mathbb{R}$, $\Omega = \{\mathbb{1}_E \mid E \subseteq \mathbb{R} \text{ Borel}, m(E) < \infty\}$ be regarded as a subset of $L_1(\mathbb{R})$. We regard $\Omega$ as a metric space with the $L_1$ distance.

(i) If $a < b$ are real numbers, show that the function $\Omega \to \mathbb{R}$ given by

$$\mathbb{1}_E \mapsto m(E \cap [a,b])$$

is a continuous function.

(ii) If $a < b$ are real numbers, let $U_{a,b}$ be the subset of $\Omega$ consisting of all $\mathbb{1}_E$ where $E \subseteq \mathbb{R}$ is Borel, and

$$0 < m(E \cap [a,b]) < b - a.$$

Show that $U_{a,b}$ is open and dense in $\Omega$.

(iii) Let $D$ be the set of all $\mathbb{1}_E$ where $E \subseteq \mathbb{R}$ is Borel, and for every interval $I$ of positive measure, we have

$$0 < m(E \cap I) < m(I).$$

Show that there is a countable collection $\{U_j\}_{j \in J}$ of open and dense subsets of $\Omega$ with $\bigcap_{j \in J} U_j \subseteq D$.

**Solution:**

(i) Letting $f\colon \Omega \to \mathbb{R}$ be defined by $f(\mathbb{1}_E) = m(E \cap [a,b])$, we have

$$\begin{aligned}|m(E\cap[a,b]) - m(F\cap[a,b])| &= \left|\int_a^b \mathbb{1}_E - \mathbb{1}_F\,dm\right| \\ &\leq \int_a^b |\mathbb{1}_E - \mathbb{1}_F|\,dm \\ &\leq \int_{\mathbb{R}} |\mathbb{1}_E - \mathbb{1}_F|\,dm \\ &= \|\mathbb{1}_E - \mathbb{1}_F\|_{L_1},\end{aligned}$$

meaning that $f$ is Lipschitz, hence continuous.

(ii) Let $\mathbb{1}_F \in \Omega$. Then, $0 \leq \mu(F\cap[a,b]) \leq b-a$. If these inequalities are strict, then $F \in U_{a,b}$. Else, we let $\varepsilon > 0$, and see two cases:

- if $\mu(F\cap[a,b]) = b-a$, then we may set $E = F \setminus ([a, a+\varepsilon/) \cup (b-\varepsilon/2, b])$, so that $0 < \mu(E\cap[a,b]) < b-a$, and $\|\mathbb{1}_E - \mathbb{1}_F\|_{L_1} = \mu(E \triangle F) \leq \varepsilon$;
- if $\mu(F\cap[a,b]) = 0$, then we may set $E = F \cup ([a, a+\varepsilon/2) \cup [b-\varepsilon/2, b))$, meaning that $0 < \mu(E\cap[a,b]) < b-a$, and $\mu(E\triangle F) \leq \varepsilon$.

Therefore, $U_{a,b}$ is dense in $\Omega$. To see that $U_{a,b}$ is open, notice that for any $\mathbb{1}_E \in U_{a,b}$, we may find $\varepsilon > 0$ such that $0 < \mu(E\cap[a,b]) - \varepsilon < \mu(E\cap[a,b]) < \mu(E\cap[a,b]) + \varepsilon < b-a$, and for all $F$ with $\|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} < \varepsilon$, we have

$$\begin{aligned}|\mu(F\cap[a,b]) - \mu(E\cap[a,b])| &\leq \|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} \\ &< \varepsilon,\end{aligned}$$

so $0 < \mu(F\cap[a,b]) < b-a$. Thus, $U_{a,b}$ is also open.

(iii) If $\{[a_k, b_k]\}$ is an enumeration of rational-endpoint intervals in $\mathbb{R}$, then for any interval $I$, there is some rational-endpoint interval $[a_k, b_k] \subseteq I$ by density and the characterization of an interval. For any $\mathbb{1}_E \in U_{a_k,b_k}$, we have that for an interval $[a,b] \subseteq I$ with $a_k \geq a$ and $b_k \leq b$,

$$\begin{aligned}m(E\cap[a,b]) &= m(E\cap[a,a_k]) + m(E\cap[a_k,b_k]) + m(E\cap[b_k,b]) \\ &< a_k - a + b_k - a_k + b - b_k \\ &= b-a,\end{aligned}$$

so $U_{a_k,b_k} \subseteq D$. Thus, since this holds for all intervals of positive measure for each $a_k, b_k$, we get

$$\bigcap_{k=1}^{\infty} U_{a_k,b_k} \subseteq D.$$

# August 2023

**Problem** (Problem 1)**:** Let $(X,\mu)$ be a $\sigma$-finite Borel measure space. Let $(f_n)_n$ be a sequence in $L_2(X,\mu)$, and $f \in L_2(X,\mu)$ such that for every $g \in L_2(X,\mu)$, we have

$$\lim_{n\to\infty} \int_X f_n(x)g(x)\,d\mu(x) = \int_X f(x)g(x)\,d\mu(x).$$

Furthermore, suppose that

$$\lim_{n\to\infty} \|f_n\|_{L_2} = \|f\|_{L_2}.$$

Prove that there is a subsequence $\left(f_{n_j}\right)_j$ and a subset $E \subseteq X$ with $\mu(E) = 0$ such that for all $x \in X \setminus E$,

$$\lim_{j\to\infty} \left|f_{n_j}(x) - f(x)\right| = 0.$$

**Solution:** In order to show that $\left(f_{n_j}\right)_j \to f$ pointwise a.e., we show that $(f_n)_n \to f$ in measure; it has been well-established that if $(f_n)_n \to f$ in measure, then $(f_n)_n$ admits a subsequence that converges to $f$ pointwise almost everywhere.

By Chebyshev's Inequality, we have that

$$\begin{aligned}\mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) &\leq \frac{1}{\varepsilon^2}\|f_n - f\|_{L_2}^2 \\ &= \frac{1}{\varepsilon^2}\int_X |f_n - f|^2\,d\mu.\end{aligned}$$

Focusing on the integral,

$$\begin{aligned}\int_X |f_n - f|^2\,d\mu &= \int_X (f_n - f)\overline{(f_n - f)}\,d\mu \\ &= \int_X |f_n|^2 - f_n\overline{f} - \overline{f_n}f + |f|^2\,d\mu \\ &= \int_X |f_n|^2\,d\mu - \int_X f_n\overline{f}\,d\mu + \int_X |f|^2\,d\mu - \overline{\int_X f_n\overline{f}\,d\mu}.\end{aligned}$$

Now, we note the following:

- $\lim_{n\to\infty}\int_X |f_n|^2\,d\mu = \int_X |f|^2\,d\mu$; and
- if $f \in L_2(X,\mu)$, then so too is $\overline{f}$.

Thus, by taking limits, we have

$$\begin{aligned}\lim_{n\to\infty}\int_X |f_n - f|^2\,dx &= \lim_{n\to\infty}\left(\int_X |f_n|^2\,d\mu - \int_X f_n\overline{f}\,d\mu + \int_X |f|^2\,d\mu - \overline{\int_X f_n\overline{f}\,d\mu}\right) \\ &= \int_X |f|^2\,d\mu - \int_X |f|^2\,d\mu + \int_X |f|^2\,d\mu - \overline{\int_X |f|^2\,d\mu} \\ &= 0,\end{aligned}$$

so $\|f_n - f\|_{L_2}^2 \to 0$. Thus, $(f_n)_n \to f$ in measure, and thus there is a subsequence $\left(f_{n_j}\right)_j \to f$ pointwise almost everywhere.

**Problem** (Problem 3)**:** Let $X$ be a LCH space. Recall that $g\colon X \to \mathbb{C}$ vanishes at infinity if for every $\varepsilon > 0$, there is a compact $K_\varepsilon \subseteq X$ such that for all $x \in X \setminus K_\varepsilon$, $|g(x)| < \varepsilon$. Show that $C_0(X)$ is complete with respect to the sup norm.

**Solution:** Let $(f_n)_n$ be Cauchy in the sup norm. Then, for all $\varepsilon > 0$, there is $N$ such that for all $m, n \geq N$, $\|f_m - f_n\| < \varepsilon$. Therefore, for all $x \in X$, we have $|f_n(x) - f_m(x)| < \varepsilon$, meaning that the sequence $(f_n(x))_n$ is Cauchy in $\mathbb{C}$. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x$.

We must now show that

- $(f_n)_n \to f$ in the supremum norm;
- $f \in C_0(X)$.

For the first point, we see that for $\varepsilon > 0$, there is $N$ such that for all $n, m \geq N$ and all $x \in X$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Taking the limit as $m \to \infty$, we have

$$|f_n(x) - f(x)| \leq \varepsilon.$$

Thus, by taking suprema, we get that

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon,$$

so $\|f_n - f\| \leq \varepsilon$, meaning that $(f_n)_n \to f$ in the sup norm, implying that $f$ is continuous as it is the uniform limit of continuous functions.

Finally, we let $N_1$ be such that for all $n \geq N_1$, $\|f_n - f\| < \varepsilon/2$. Note that since $f_{N_1} \in C_0(X)$, we have a $K_{\varepsilon/2}$ such that for all $x \in X \setminus K_{\varepsilon/2}$, $|f_N(x)| < \varepsilon/2$. Therefore, for all $x \in X \setminus K_{\varepsilon/2}$, we have

$$\begin{aligned} |f(x)| &\leq |f_{N_1}(x) - f(x)| + |f_{N_1}(x)| \\ &\leq \|f_{N_1} - f\| + |f_{N_1}(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

so $f \in C_0(X)$. Thus, $C_0(X)$ is complete.

**Problem** (Problem 4)**:** Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that for any $n \geq 1$, and any $A_1, \dots, A_n, B_1, \dots, B_n$ $\mathcal{A}$,

$$\mu((A_1 \cup \cdots \cup A_n) \triangle (B_1 \cup \cdots \cup B_n)) \leq \sum_{j=1}^{n} \mu(A_j \triangle B_j).$$

**Solution:** We start off by noting that the symmetric difference $A \triangle B$ can be written as

$$A \triangle B = A \cup B \setminus (A \cap B).$$

This is evident from unwinding the definition $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Now, writing the left-hand side of our desired inequality, we get

$$\mu((A_1 \cup \cdots \cup A_n) \triangle (B_1 \cup \cdots \cup B_n)) = \mu(A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n) - \mu((A_1 \cup \cdots \cup A_n) \cap (B_1 \cup \cdots \cup B_n)).$$

Distributing the second term on the right-hand side and rearranging the first term, we get

$$= \mu\left(\bigcup_{j=1}^{n} (A_j \cup B_j)\right) - \mu\left(\bigcup_{j=1}^{n} (A_1 \cup \cdots \cup A_n) \cap B_j\right).$$

Using subadditivity on the first term, we get

$$\leq \sum_{j=1}^{n} \mu(A_j \cup B_j) - \mu\left(\bigcup_{j=1}^{n} (A_1 \cup \cdots \cup A_n) \cap B_j\right).$$

Finally, using monotonicity and subadditivity on the second term, and exercising the fact that

$$A_j \cap B_j \subseteq \bigcap_{j=1}^{n} (A_1 \cup \cdots \cup A_n) \cap B_j,$$

we get

$$\leq \sum_{j=1}^{n} \mu(A_j \cup B_j) - \sum_{j=1}^{n} \mu(A_j \cap B_j)$$

$$= \sum_{j=1}^{n} \mu(A_j \triangle B_j).$$

**Problem** (Problem 5)**:** Let $(X, \mu)$ be a nonnegative measure space and $f$ a measurable function on $(X, \mu)$ such that

$$\sup_{\lambda > 0} \mu(\{x \mid |f(x)| > \lambda\}) < \infty.$$

Prove that there is a finite constant $C$ such that for every finite measure subset, we have

$$\int_E |f(x)|\, d\mu(x) \leq C\mu(E)^{1/2}.$$

**Theorem** (Cavalieri's Principle)**:**

$$\int_X |f|\, d\mu = \int_0^\infty \mu(\{x \in E \mid |f| > \lambda\})\, d\lambda.$$

**Solution:** Using Cavalieri's Principle, we get

$$\begin{aligned}
\int_E |f|\, d\mu &\leq \int_0^\alpha \mu(\{x \in E \mid |f| > \lambda\})\, d\lambda + \int_\alpha^\infty \mu(\{x \in E \mid |f| > \lambda\})\, d\lambda \\
&\leq \alpha\mu(E) + \int_\alpha^\infty \frac{M}{\lambda^2}\, d\lambda \\
&= \alpha\mu(E) + \frac{M}{\alpha} \\
&\leq (M+1)\mu(E)^{1/2},
\end{aligned}$$

where we selected $\alpha = \frac{1}{\mu(E)^{1/2}}$, and $M$ denotes the given supremum.

## January 2024

**Problem** (Problem 1)**:** Let $(X, \mu)$ be a $\sigma$-finite measure space, and suppose $(f_n)_n$ is a sequence in $L_2(X, \mu)$ such that $\sup_{n \geq 1} \|f_n\|_{L_2} < \infty$ and $(f_n)_n \to f$ $\mu$-almost everywhere. Prove that $f \in L_2(X, \mu)$.

**Solution:** Applying Fatou's Lemma, we find that

$$\begin{aligned}
\int_X |f|^2\, d\mu &= \int_X \liminf_{n\to\infty} |f_n|^2\, d\mu \\
&\leq \liminf_{n\to\infty} \int_X |f_n|^2\, d\mu \\
&\leq \limsup_{n\to\infty} \int_X |f_n|^2\, d\mu \\
&\leq \sup_{n\geq 1} \int_X |f_n|^2\, d\mu \\
&< \infty.
\end{aligned}$$

**Problem** (Problem 2)**:** Let $(X, \mu)$ be a measure space, and let $p \in [1, \infty)$. Let $(f_n)_n \to f$ in $L_p$.

(i) Prove that there exists a subsequence $(f_{n_k})$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L_p} < 2^{-k}$.

(ii) Show that for $\mu$-almost every $x$, we have $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

**Solution:**

(i) Since $(f_n)_n \to f$ in $L_p$, we see that $(f_n)_n$ is $L_p$-Cauchy, so we may extract a subsequence as follows.

Let $f_{n_1} = f_1$, and find $f_{n_2}$ with $n_2 > 1$ such that

$$\|f_{n_2} - f_{n_1}\| < \frac{1}{2}.$$

Inductively, we may use the fact that $(f_n)_n$ is Cauchy to find $n_{k+1} > n_k$ such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\| < \frac{1}{2^k}.$$

(ii) Consider the sequence $(s_n)_n$ given by

$$s_n = \sum_{k=1}^{n} \left|f_{n_{k+1}} - f_{n_k}\right|.$$

Then, by Minkowski's Inequality, we find that

$$\|s_n\|_{L_p} \leq \sum_{k=1}^{n} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{L_p}.$$

In particular, $\|s_n\|_{L_p} \leq 1$ for all $n$, meaning that by dominated convergence, $s = \lim_{n\to\infty} s_n$ is in $L_p$, and in particular, $s(x) < \infty$ for almost every $x$. Notice that this means that

$$h(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

converges for almost every $x$. Defining $h(x) = 0$ for all $x$ where this sum does not converge absolutely, we notice that

$$f_{n_1}(x) + \sum_{k=1}^{m} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right) = f_{n_{m+1}}(x),$$

meaning that $h$ is the pointwise (almost everywhere) limit of the sequence $(f_{n_k})_k$; by Minkowski's Inequality, and applying Fatou's Lemma, as earlier, we also find that

$$\begin{aligned}
\|h\|_{L_p} &\leq \|f_{n_1}\|_{L_p} + \sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{L_p} \\
&\leq \|f_{n_1}\|_{L_p} + 1 \\
&< \infty,
\end{aligned}$$

meaning $h \in L_p(X, \mu)$. All we need to do now is show that $\|f - h\|_{L_p} = 0$, meaning that $[f] = [h]$ under the pointwise almost everywhere equivalence relation. To see this,

$$\begin{aligned}
\int_X |h - f|^p \, d\mu &= \int_X \liminf_{k\to\infty} |f_{n_k} - f|^p \, d\mu \\
&\leq \liminf_{k\to\infty} \int_X |f_{n_k} - f|^p \, d\mu \\
&= \liminf_{k\to\infty} \|f_{n_k} - f\|_{L_p}^p \\
&= 0,
\end{aligned}$$

where the last equality is derived from the fact that $(f_n)_n \to f$ in $L_p$, so every subsequence of $(f_n)_n$ converges to $f$ in $L_p$.

**Problem** (Problem 3)**:** Let $f$ be Lebesgue-integrable on $\mathbb{R}$, and let $g$ be a bounded continuous function on $\mathbb{R}$. Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy$$

is a continuous function on $\mathbb{R}$.

**Solution:** Let $M = \sup_{x\in\mathbb{R}}|g(x)|$. Now, since $f \in L_1$, there is a compactly supported continuous function $h \in C_c(\mathbb{R})$ such that $\|h - f\|_{L_1} < \frac{\varepsilon}{3M}$. If we let $K = \operatorname{supp}(h)$, then since $h$ is compactly supported, $h$ is uniformly continuous, so there is $\delta > 0$ such that whenever $|x - y| < \delta$, we have

$$|h(x) - h(y)| < \frac{\varepsilon}{3Mm(K)},$$

where $m(K)$ is the Lebesgue measure of $K$ in $\mathbb{R}$. Therefore, if $|x - y| < \delta$, we have

$$\begin{aligned}
|(f*g)(x) - (f*g)(y)| &= \left|\int_{\mathbb{R}} (f(x-t) - f(y-t))g(t)\,dt\right| \\
&\leq \int_{\mathbb{R}} |f(x-t) - f(y-t)||g(t)|\,dt \\
&\leq \int_{\mathbb{R}} |f(x-t) - h(x-t)||g(t)|\,dt \\
&\quad + \int_{\mathbb{R}} |h(x-t) - h(y-t)||g(t)|\,dt \\
&\quad + \int_{\mathbb{R}} |h(y-t) - f(y-t)||g(t)|\,dt.
\end{aligned}$$

Using Hölder's Inequality on the first and third integrals, we get

$$\leq M\left(\frac{\varepsilon}{3M}\right) + \int_{\mathbb{R}} |h(x-t) - h(y-t)||g(t)|\,dt + M\left(\frac{\varepsilon}{3M}\right),$$

and using the uniform continuity of $h$, we get

$$\begin{aligned}
&\leq \frac{2\varepsilon}{3} + M(m(K))\frac{\varepsilon}{3M(m(K))} \\
&= \varepsilon.
\end{aligned}$$

**Solution** (Alternative Solution)**:** We know that $f$ is integrable on $\mathbb{R}$, and $g$ is bounded and continuous. We will show that if $(x_n)_n \to x_0$, then $((f*g)(x_n))_n \to (f*g)(x_0)$.

Now, if $(x_n)_n \to x_0$, then $g(x_n) \to g(x_0)$, since $g$ is continuous. Since $f$ is integrable, $f$ is finite almost everywhere, meaning that $f(y)g(x_n - y) \to f(y)g(x_0 - y)$ almost everywhere.

Furthermore, since $g$ is bounded, we have $|g| \leq M$ for some $M > 0$. Writing our convolution integrand, we have

$$|f(y)g(x_n - y)| \leq M|f(y)|.$$

Since $f$ is integrable, we may use the dominated convergence theorem to find that

$$\lim_{n\to\infty} \int f(y)g(x_n - y)\,dy = \int f(y)g(x_0 - y)\,dy.$$

## August 2024

**Problem** (Problem 1)**:** Let $A \subseteq \mathbb{R}$ be a Lebesgue-measurable subset of finite measure. For $r \in \mathbb{R} \setminus \{0\}$, let $rA = \left\{x \in \mathbb{R} \mid r^{-1}x \in A\right\}$, and let $A \triangle rA = (A \setminus rA) \cup (rA \setminus A)$. Show that

$$\lim_{r \to 1} m(A \triangle rA) = 0.$$

**Solution:** We want to evaluate

$$\begin{aligned}\lim_{r \to 1} m(A \triangle rA) &= \int_{\mathbb{R}} |\mathbb{1}_A - \mathbb{1}_{rA}| \, dm \\ &= \int_{\mathbb{R}} \left|\mathbb{1}_A(x) - \mathbb{1}_A\left(r^{-1}x\right)\right| \, dx.\end{aligned}$$

Let $g$ be a continuous function with compact support such that

$$\|g - \mathbb{1}_A\|_{L_1} < \varepsilon.$$

Then, setting $\delta_r g(x) = g\left(r^{-1}x\right)$, we have for any $1/2 \leq r \leq 2$,

$$\begin{aligned}\|\mathbb{1}_A - \mathbb{1}_{rA}\|_{L_1} &\leq \|\mathbb{1}_A - g\|_{L_1} + \|g - \delta_r g\|_{L_1} + \|\delta_r g - \mathbb{1}_{rA}\|_{L_1} \\ &= (|r| + 1)\|\mathbb{1}_A - g\| + \|g - \delta_r g\|_{L_1} \\ &\leq 3\varepsilon + \|g - \delta_r g\|_{L_1}.\end{aligned}$$

Now, since $g$ is compactly supported, there is some $R > 0$ such that $\operatorname{supp}(g) \subseteq [-R, R]$, meaning that

$$\|\delta_r g - g\|_{L_1} \leq 2R \sup_{x \in [-R,R]} \left|g\left(r^{-1}x\right) - g(x)\right|.$$

Since $g$ is continuous, $g$ is uniformly continuous, so for all $1/2 \leq r \leq 2$, since $\left|r^{-1}x - x\right| \leq |x|\left|r^{-1} - 1\right| \leq R\left|r^{-1} - 1\right|$, there is some $\delta > 0$ such that

$$\sup_{x \in [-R,R]} \left|g\left(r^{-1}x\right) - g(x)\right| \leq \frac{\varepsilon}{R},$$

whenever $\left|r^{-1} - 1\right| \leq \delta$, meaning that whenever $\left|r^{-1} - 1\right| \leq \delta$, we have that

$$\|\mathbb{1}_A - \mathbb{1}_{rA}\|_{L_1} \leq 5\varepsilon,$$

so by a scaling argument, we see that $\mathbb{1}_{rA} \to \mathbb{1}_A$ in the $L_1$ norm as $r \to 1$.

**Problem** (Problem 2)**:** Let $\mu$ be a finite Borel measure on $\mathbb{R}$. For $\xi \in \mathbb{R}$, define

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} \, d\mu(x).$$

Suppose

$$\lim_{\xi \to 0} \frac{\widehat{\mu}(\xi) - \widehat{\mu}(0)}{\xi^2} = 0.$$

(a) Show that

$$\int_{\mathbb{R}} x^2 \, d\mu(x) = 0.$$

(b) Deduce that for any open interval $(a, b) \subseteq \mathbb{R}$,

$$\mu((a,b)) = \begin{cases} \mu(\mathbb{R}) & 0 \in (a,b) \\ 0 & 0 \notin (a,b) \end{cases}.$$

**Solution:**

(a) Since the given limit is equal to zero, it holds for both the real and imaginary components; thus, we see that

$$\begin{aligned} 0 &= \lim_{\xi \to 0} \frac{\hat{\mu}(\xi) - \hat{\mu}(0)}{\xi^2} \\ &= \lim_{\xi \to 0} \frac{\int_{\mathbb{R}} (\cos(2\pi x\xi) - 1)\, d\mu(x)}{\xi^2} \\ &= \lim_{\xi \to 0} \frac{\int_{\mathbb{R}} -2\sin^2(\pi x \xi)\, d\mu(x)}{\xi^2} \\ &= -2 \lim_{\xi \to 0} \int \left(\frac{\sin(\pi x\xi)}{\xi}\right)^2 d\mu(x). \end{aligned}$$

Define

$$f_\xi(x) = \left(\frac{\sin(\pi x\xi)}{\xi}\right)^2.$$

Now, by Fatou's Lemma, we have

$$\begin{aligned} \int_{\mathbb{R}} (\pi x)^2\, d\mu(x) &= \int_{\mathbb{R}} \lim_{\xi \to 0} \left(\frac{\sin(\pi x\xi)}{\xi}\right)^2 d\mu \\ &= \int_{\mathbb{R}} \liminf_{\xi \to 0} \left(\frac{\sin(\pi x\xi)}{\xi}\right)^2 d\mu(x) \\ &\leq \liminf_{\xi \to 0} \int_{\mathbb{R}} \left(\frac{\sin(\pi x\xi)}{\xi}\right)^2 d\mu(x) \\ &= \lim_{\xi \to 0} \int_{\mathbb{R}} \left(\frac{\sin(\pi x\xi)}{\xi}\right)^2 d\mu(x) \\ &= 0, \end{aligned}$$

so that

$$\int_{\mathbb{R}} x^2\, d\mu(x) = 0.$$

(b) We want to evaluate

$$\mu((a,b)) = \int_{\mathbb{R}} \mathbb{1}_{(a,b)}\, d\mu(x).$$

We fix $(a,b) \subseteq \mathbb{R}$, where $0 \notin (a,b)$. There are four cases.

- If $0 < a < b$, then we see that

  $$a^2 \int_{\mathbb{R}} \mathbb{1}_{(a,b)}\, d\mu \leq \int_{\mathbb{R}} x^2\, d\mu(x),$$

  meaning that $\mu((a,b)) = 0$.

- If $a < b < 0$, then we see that
$$|b|^2 \int_{\mathbb{R}} \mathbb{1}_{(a,b)}\, d\mu \leq \int_{\mathbb{R}} x^2\, d\mu(x),$$
so $\mu((a,b)) = 0$.
- If $0 = a < b$, then by continuity from below, we see that
$$\begin{aligned}\mu((a,b)) &= \lim_{n\to\infty} \mu((1/n, b))\\ &= 0.\end{aligned}$$
- Similarly, if $a < b = 0$, then by continuity from below, we see that
$$\begin{aligned}\mu((a,b)) &= \lim_{n\to\infty} \mu((a, -1/n))\\ &= 0.\end{aligned}$$

Meanwhile, if $0 \in (a,b)$, then we may split
$$\int_{\mathbb{R}} d\mu(x) = \int_{\mathbb{R}} \mathbb{1}_{(a,b)}(x)\, d\mu(x) + \int_{\mathbb{R}} \mathbb{1}_{(-\infty,a]}(x)\, d\mu(x) + \int \mathbb{1}_{[b,\infty)}(x)\, d\mu(x)$$
and, by continuity from above/below,
$$\begin{aligned}&= \int_{\mathbb{R}} \mathbb{1}_{(a,b)}(x)\, d\mu(x) + \lim_{n\to\infty} \mu((-\infty, a+1/n)) + \lim_{n\to\infty} \mu((b-1/n,\infty))\\ &= \int_{\mathbb{R}} \mathbb{1}_{(a,b)}(x)\, d\mu(x).\end{aligned}$$

**Problem** (Problem 3)**:** Fix $\alpha \in (0,1]$. The space $C^{0,\alpha}$ of Hölder-continuous functions consists of functions $f\colon [0,1] \to \mathbb{C}$ such that
$$\|f\| \coloneqq |f(0)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$$
is finite.

(a) Show that $\|\cdot\|$ is a norm on $C^{0,\alpha}([0,1])$.

(b) Show that $\left(C^{0,\alpha}([0,1]), \|\cdot\|\right)$ is a Banach space.

**Solution:**

(a) Clearly, the value $\|f\| \geq 0$ for all $f \in C^{0,\alpha}([0,1])$, as $|f(0)| \geq 0$ and, for all $x,y \in [0,1]$, $\frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \geq 0$. The value $\|f\| = 0$ if and only if $|f(0)| = 1$, and for all $x,y \in [0,1]$, we have
$$\sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} = 0,$$
or that $|f(x)-f(y)| = 0$ for all $x \neq y$, implying that $f$ is a constant function. Combined, this means that $f = 0$.

To see homogeneity, we let $\alpha \in \mathbb{C}$, and see that
$$\|\alpha f\| = |(\alpha f)(0)| + \sup_{x\neq y\in[0,1]} \frac{|(\alpha f)(x)-(\alpha f)(y)|}{|x-y|^{\alpha}}$$

$$
\begin{aligned}
&= |\alpha||f(0)| + \sup_{x\neq y\in[0,1]} \frac{|\alpha(f(x)-f(y))|}{|x-y|^{\alpha}}\\
&= |\alpha|\left(|f(0)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}\right)\\
&= |\alpha|\|f\|.
\end{aligned}
$$

Finally, if $f$ and $g$ are elements of $C^{0,\alpha}([0,1])$, then

$$
\begin{aligned}
\|f+g\| &= |(f+g)(0)| + \sup_{x\neq y\in[0,1]} \frac{|(f+g)(x)-(f+g)(y)|}{|x-y|^{\alpha}}\\
&\leq |f(0)| + |g(0)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|+|g(x)-g(y)|}{|x-y|^{\alpha}}\\
&\leq |f(0)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} + |g(0)| + \sup_{x\neq y\in[0,1]} \frac{|g(x)-g(y)|}{|x-y|^{\alpha}}\\
&= \|f\| + \|g\|.
\end{aligned}
$$

Thus, $\|\cdot\|$ is a bona fide norm on $C^{0,\alpha}([0,1])$.

(b) We will use the absolute convergence criterion for a Banach space. That is, we will show that if

$$\sum_{n=1}^{\infty}\|f_n\| < \infty,$$

then the sum (without norm) converges to some $f \in C^{0,\alpha}([0,1])$. To start, we see that for any $x \in [0,1]$ and any $f \in C^{0,\alpha}([0,1])$,

$$
\begin{aligned}
|f(x)| &\leq |f(0)| + |f(x)-f(0)|\\
&= |f(0)| + \frac{|f(x)-f(0)|}{|x-0|^{\alpha}}|x|^{\alpha}\\
&\leq |f(0)| + \frac{|f(x)-f(0)|}{|x-0|^{\alpha}}\\
&\leq |f(0)| + \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}},
\end{aligned}
$$

meaning that $\|f\|_u \leq \|f\|$.

Now, let

$$\sum_{n=1}^{\infty}\|f_n\| < \infty,$$

so that

$$\sum_{n=1}^{\infty}\|f_n\|_u < \infty.$$

Since $C([0,1])$ is complete with respect to the uniform norm, there is some $f \in C([0,1])$ such that $f = \sum_{n=1}^{\infty} f_n$, where the convergence is with respect to the uniform norm. Notice, however, that

$$\frac{|f(x)-f(y)|}{|x-y|^{\alpha}} = \frac{|\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(y)|}{|x-y|^{\alpha}}$$

$$\begin{aligned}&\leq \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{|x-y|^{\alpha}}\\ &\leq \sum_{n=1}^{\infty} \|f_n\|\\ &< \infty,\end{aligned}$$

meaning that $f \in C^{0,\alpha}([0,1])$.

Finally, to see that the convergence is with respect to the $\alpha$-norm, we see that

$$\begin{aligned}\left\| \sum_{n=1}^{N} f_n - f \right\| &= \left\| \sum_{n=N+1}^{\infty} f_n \right\|\\ &\leq \sum_{n=N+1}^{\infty} \|f_n\|\\ &\to 0,\end{aligned}$$

by series convergence.

**Problem** (Problem 4)**:** Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, $p \in [1, \infty)$, and $f \in L_p(X, \mathcal{M}, \mu)$.

(a) Show that

$$\int_X |f(x)|^p \, d\mu(x) = \int_0^{\infty} \alpha^{p-1} \mu(\{x \mid |f(x)| > \alpha\}) \, d\alpha.$$

(b) Show that, for all $\alpha > 0$,

$$\mu(\{x \mid |f(x)| > \alpha\}) \leq \frac{\|f\|_{L_p}^p}{\alpha^p}.$$

**Solution:**

(a) We use Cavalieri's Principle using $\phi(t) = t^p$,

$$\begin{aligned}\int_X |f(x)|^p \, d\mu(x) &= \int_X (\phi \circ |f|)^p \, d\mu(x)\\ &= \int_0^{\infty} \mu(\{x \mid \phi \circ |f| > t\}) \, dt\\ &= \int_0^{\infty} \mu(\{x \mid |f| > \alpha\}) \phi'(\alpha) \, d\alpha\\ &= \int_0^{\infty} p\alpha^{p-1} \mu(\{x \mid |f| > \alpha\}) \, d\alpha.\end{aligned}$$

$$\int_X |f(x)|^p \, d\mu(x) = p \int_0^{\infty} \alpha^{p-1} \mu(\{x \mid |f(x)| > \alpha\}) \, d\alpha.$$

(b) Set $E_\alpha = \{x \mid |f(x)| > \alpha\}$. Then,

$$\begin{aligned}\int_X |f|^p \, d\mu &\geq \int_{E_\alpha} |f|^p \, d\mu\\ &\geq \alpha^p \int_X \mathbb{1}_{E_\alpha} \, d\mu\end{aligned}$$

$$
\begin{aligned}
&= \alpha^p \mu(E_\alpha) \\
&= \alpha^p \mu(\{x \mid |f(x)| > \alpha\}).
\end{aligned}
$$

By rearranging, we get that

$$\mu(\{x \mid |f(x)| > \alpha\}) \leq \frac{\|f\|_{L_p}^p}{\alpha^p}.$$

**Problem** (Problem 5)**:** Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed Schwartz function. The Fourier multiplier $M$ acts on functions $f \in L_2(\mathbb{R}^n)$ by

$$\widehat{Mf}(\xi) = \phi(\xi)\hat{f}(\xi).$$

That is, $M$ is equal to the operation of taking the Fourier transform of $f$, multiplying by $\phi(\xi)$, then taking the inverse Fourier transform. Show that $M$ is a bounded linear map from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$.

**Solution:** Rewriting $M$, we see that

$$Mf(x) = \mathcal{F}^{-1}(M_\phi \mathcal{F}(f(x))),$$

where $M_\phi$ is the multiplication operator that multiplies by $\phi$. We see thus that

$$\|M\|_{\text{op}} = \left\|\mathcal{F}^{-1} M_\phi \mathcal{F}\right\|_{\text{op}}.$$

The norm of a multiplication operator is bounded above by the essential supremum of its symbol, as

$$
\begin{aligned}
\|M_\phi(f)\|_{L_2} &= \left(\int_{\mathbb{R}^n} |\phi f|^2 \, dm\right)^{1/2} \\
&\leq \left(\|\phi\|_{L_\infty}^2 \int_{\mathbb{R}^n} |f|^2 \, dm\right)^{1/2} \\
&= \|\phi\|_{L_\infty} \|f\|_{L_2}.
\end{aligned}
$$

Since $\phi$ is a Schwartz function, all of its derivatives are bounded, and thus so is its zeroth derivative, so $\phi \in L_\infty$. Furthermore, by the Plancherel Theorem, the Fourier transform on $L_2$ is unitary, so by applying the properties of the operator norm, we see that

$$
\begin{aligned}
\|M\|_{\text{op}} &\leq \left\|\mathcal{F}^{-1}\right\|_{\text{op}} \|M_\phi\|_{\text{op}} \|\mathcal{F}\|_{\text{op}} \\
&\leq \|\phi\|_{L_\infty} \\
&< \infty,
\end{aligned}
$$

so $M$ is a bounded linear operator.

# January 2025

**Problem** (Problem 1)**:** Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_n$ and $(g_n)_n$ sequences of functions in $L_1(X, \mathcal{M}, \mu)$ in $L_1(X, \mathcal{M}, \mu)$ that converge pointwise almost everywhere to $f, g \in L_1(X, \mathcal{M}, \mu)$. Suppose $|f_n| \leq g_n$ almost everywhere, and

$$\lim_{n \to \infty} \int_X g_n \, d\mu = \int_X g \, d\mu.$$

Show that

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

**Solution:** We see that

$$\begin{aligned}\int_X g + f\, d\mu &= \int_X \liminf_{n\to\infty}(g_n + f_n)\, d\mu\\ &\leq \liminf_{n\to\infty}\left(\int_X g_n\, d\mu + \int_X f_n\, d\mu\right)\\ &= \int_X g\, d\mu + \liminf_{n\to\infty}\int_X f_n\, d\mu,\end{aligned}$$

so

$$\int_X f\, d\mu \leq \liminf_{n\to\infty}\int_X f_n\, d\mu.$$

Similarly, we also see that

$$\begin{aligned}\int_X g - f\, d\mu &= \int_X \liminf_{n\to\infty}(g_n - f_n)\, d\mu\\ &\leq \liminf_{n\to\infty}\left(\int_X g_n\, d\mu - \int_X f_n\, d\mu\right)\\ &= \int_X g\, d\mu - \limsup_{n\to\infty}\int_X f_n\, d\mu,\end{aligned}$$

meaning that

$$\limsup_{n\to\infty}\int_X f_n\, d\mu \leq \int_X f\, d\mu,$$

meaning that

$$\int_X f\, d\mu = \lim_{n\to\infty}\int_X f_n\, d\mu.$$

**Problem** (Problem 2)**:**

(a) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show that if $p, p' \in [1,\infty]$ with $p < p'$, then $L_p(X,\mathcal{M},\mu) \supseteq L_{p'}(X,\mathcal{M},\mu)$.

(b) Show that if $p, p' \in [1,\infty]$ are such that $p < p'$, then $L_p(\mathbb{R}) \setminus L_{p'}(\mathbb{R})$ and $L_{p'}(\mathbb{R}) \setminus L_p(\mathbb{R})$ are both nonempty.

**Solution:** (a) Let $f \in L_{p'}(X,\mathcal{M},\mu)$. Then, we use Hölder's inequality on with $\frac{p'}{p}$ and $1 - \frac{p'}{p}$ as our Hölder conjugates to obtain

$$\begin{aligned}\int_X |f|^p\, d\mu &= \int_X |f|^p(1)\, d\mu\\ &\leq \left(\int_X |f|^{p(p'/p)}\, d\mu\right)^{p/p'}\left(\int_X d\mu\right)^{1-p/p'}\\ &= \left(\int_X |f|^{p'}\, d\mu\right)^{p/p'}\mu(X)^{1-p/p'}\\ &< \infty,\end{aligned}$$

so $f \in L_p$.

(b) To see that $L_{p'} \setminus L_p$ is nonempty, we consider a function given by

$$f = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \mathbb{1}_{[n,n+1)},$$

where we see that

$$\begin{aligned} \int_{\mathbb{R}} |f|^p \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty \\ \int_X |f|^{p'} \, d\mu &= \sum_{n=1}^{\infty} \frac{1}{n^{p'/p}} \\ &< \infty. \end{aligned}$$

Now, to see that $L_p \setminus L_{p'}$ is nonempty, we consider the function

$$f = x^{-1/p'} \mathbb{1}_{(0,1]}.$$

Then,

$$\begin{aligned} \int_{\mathbb{R}} |f|^p \, d\mu &= \int_0^1 x^{-(p/p')} \, d\mu \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} |f|^{p'} \, d\mu &= \int_0^1 x^{-1} \, d\mu \\ &= \infty. \end{aligned}$$

Thus, $f \in L_p \setminus L_{p'}$

**Problem** (Problem 3)**:** Let $\mathcal{H}$ be a separable Hilbert space. A sequence $(v_m)_m$ is said to converge weakly to $v \in \mathcal{H}$ if

$$\lim_{m \to \infty} \langle v_m, w \rangle = \langle v, w \rangle$$

for every $w \in \mathcal{H}$. Show that for any sequence $(v_m)_m \subseteq \mathcal{H}$ for which $\sup_{m \in \mathbb{N}} \|v_m\|$ is finite, there exists a subsequence $(v_{m_k})_k \to v \in \mathcal{H}$ weakly.

**Solution:** We see that, since $\mathcal{H}$ is a Hilbert space, $\mathcal{H} \cong \mathcal{H}^{**}$, where $\mathcal{H}^{**}$ is the double dual of $\mathcal{H}$ (Hilbert spaces are reflexive). This means that, if $(v_m)_m \subseteq \mathcal{H}$, there is an isometric isomorphism to the sequence $(\hat{v}_m)_m \subseteq \mathcal{H}^{**}$, where $\hat{v}_m$ is the linear functional such that $\hat{v}_m(\varphi) = \varphi(v_m)$ for all $\varphi \in \mathcal{H}$.

Letting $M = \sup_{m \in \mathbb{N}} \|v_m\|$, we see that $\frac{1}{M}(\hat{v}_m)_m \subseteq B_{\mathcal{H}^{**}}$. By the Banach–Alaoglu theorem, there is a subsequence $\frac{1}{M}(\hat{v}_{m_k})_k \to \frac{1}{M}\hat{v}$, where the convergence is in the $w^{**}$ topology — i.e., for all $\varphi \in \mathcal{H}^*$, $\hat{v}_{m_k}(\varphi) \to \hat{v}(\varphi)$; rewriting, we then get that $\varphi(v_{m_k}) \to \varphi(v)$ for all $\varphi \in \mathcal{H}^*$.

By the Riesz Representation Theorem, any $\varphi \in \mathcal{H}^*$ corresponds to an inner product with a unique $w \in \mathcal{H}$, so we have

$$\langle v_{m_k}, w \rangle \to \langle v, w \rangle$$

for all $w \in \mathcal{H}$, so $(v_{m_k})_k \to v$ weakly.

**Problem** (Problem 4)**:** Let $\delta_0$ be the Dirac measure at 0, defined by

$$\delta_0(A) = \begin{cases} 1 & 0 \in A \\ 0 & \text{else} \end{cases}.$$

For each $r > 0$, define $\nu_r$ to be the measure defined by

$$\nu_r(A) = \frac{1}{2r} m(A \cap [-r, r]).$$

Show that for every continuous function $f\colon \mathbb{R} \to \mathbb{C}$, we have

$$\lim_{r \searrow 0} \int_{\mathbb{R}} f(x)\, d\nu_r(x) = \int_{\mathbb{R}} f(x)\, d\delta_0(x).$$

**Solution:** We consider the family $\{E_r\}_{r>0}$ defined by $E_r = [-r, r]$. We notice that $\frac{1}{2r} m(A \cap E_r) = \nu(E_r)$. Furthermore, we also see that $E_r \subseteq (-4/3r, 4/3r)$, and $E_r \supseteq \frac{3}{8}(-4/3r, 4/3r)$, so that by a scaling argument, $\{E_r\}_{r>0}$ is a family that shrinks nicely to $x = 0$.

Furthermore, we see that

$$\begin{aligned} \int_{\mathbb{R}} f(x)\, d\delta_0(x) &= \int_{\{0\}} f(x)\, d\delta_0(x) \\ &= f(0). \end{aligned}$$

Finally, since $f$ is continuous, for any compact $K \subseteq \mathbb{R}$, $f$ is bounded, so that

$$\begin{aligned} \int_K |f(x)|\, dx &\leq \int_K \sup_{x \in K} |f(x)|\, dx \\ &\leq m(K) \sup_{x \in K} |f(x)| \\ &< \infty, \end{aligned}$$

as $m$ is regular. Thus, $f$ is locally integrable, meaning that by the Lebesgue Differentiation Theorem,

$$\begin{aligned} \lim_{r \to 0} \int_{\mathbb{R}} f(x)\, d\nu_r(x) &= \lim_{r \to 0} \frac{1}{2r} \int_{E_r} f(x)\, dx \\ &= f(0), \end{aligned}$$

so

$$\int_{\mathbb{R}} f(x)\, d\nu_r(x) = \int_{\mathbb{R}} f(x)\, d\delta_0(x).$$

**Problem** (Problem 5)**:**

(a) State the Riemann–Lebesgue lemma for the Fourier transform on $\mathbb{R}^n$.

(b) Show that there does not exist a function $g \in L_1(\mathbb{R}^n)$ such that $f * g = f$ for all $f \in L_1(\mathbb{R}^n)$.

**Solution:**

(a) The Riemann–Lebesgue Lemma for the Fourier transform on $\mathbb{R}^n$ states that if $f \in L_1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

(b) Suppose toward contradiction that there were such a $g$. Then, it would also be the case that $g * g = g$, and since the Fourier transform on $L_1(\mathbb{R}^n)$ is injective, by the convolution property of the Fourier transform, we have $\hat{g}(k)\hat{g}(k) = \hat{g}(k)$ for all $k \in \mathbb{R}^n$, implying that $\hat{g}(k) = 0$ or $\hat{g}(k) = 1$ for all $k$, depending on if $\hat{g}(k)$ is zero or not.

However, by the Riemann–Lebesgue Lemma, $\hat{g}$ is continuous and vanishes at infinity, so since $\hat{g}$ takes on either 0 or 1 everywhere, we must have $\hat{g}(k) = 0$ for all $k$, implying that $g = 0$; yet, this is absurd, as $f * 0 = 0$, yet there are nonzero functions in $L_1(\mathbb{R}^n)$.