Math 395: Homework 5
Due: October 22, 2024
Name: Avinash Iyer
Collaborators: Carly Venenciano, Gianluca Crescenzo, Noah Smith, Ben Langer Weida

Problem 6

Problem: Let $A \in \mathrm{Mat}_n(\mathbb{F})$ be a triangular matrix. Prove that the eigenvalues of A are the entries on the main diagonal.

Solution: Let

$$A = \begin{pmatrix} a_1 & \cdots & \cdots \\ 0 & \ddots & \vdots \\ 0 & 0 & a_n \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{F}).$$

Then, we have

$$\begin{aligned} c_A(x) &= \det(xI_n - A) \\ &= \det\left(\begin{pmatrix} x - a_1 & \cdots & \cdots \\ 0 & \ddots & \vdots \\ 0 & 0 & x - a_n \end{pmatrix}\right) \\ &= (x - a_1)(x - a_2)\cdots(x - a_n). \end{aligned}$$

Thus, $c_A(x)$ has roots at $a_1, \dots, a_n$, which are the entries on the main diagonal of A.

Problem 16

Problem: Let

$$A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

(a) Find the characteristic polynomial of A.

(b) Compute E_λ^j for all eigenvalues λ of A and all j.

(c) Give the Jordan canonical form of A and the Jordan basis $\mathcal{B}$ of $\mathbb{F}^4$.

Solution:

(a) Using computational assistance, we find

$$\begin{aligned} c_A(x) &= \det(A - xI_4) \\ &= x^4 - 12x^3 + 52x^2 - 96x + 64 \\ &= (x-4)^2(x-2)^2. \end{aligned}$$

(b) The eigenvalues of A are 2 and 4. Thus, we calculate

$$A - 2I_4 = \begin{pmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{pmatrix}.$$

In reduced row echelon form (with computational assistance), we get

$$\simeq \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $A - 2I_4$ is of rank 2, so $\dim(\ker(A - 2I_4)) = 2$. The vectors $v_1 = e_1 + \frac{1}{2}e_2 + e_4$ and $v_2 = \frac{1}{2}e_2 + e_3$ form a basis for E_2^1. Since the degree on the factor $(x - 2)$ is 2, this means $E_2^\infty = E_2^1$.

Now, we turn our attention to $A - 4I_4$. We have

$$A - 4I_4 = \begin{pmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{pmatrix},$$

which row-reduces to

$$\simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $A - 4I_4$ is of rank 3, so $\dim(\ker(A - 4I_4)) = 1$. The vector $v_3 = e_2 + e_3 + e_4$ forms a basis for E_4^1. However, since the degree on the factor $(x - 4)$ is 2, we must turn our attention to E_4^2.

We now examine $(A - I_4)^2$. We have

$$(A - 4I_4)^2 = \begin{pmatrix} 4 & 8 & -4 & -4 \\ 4 & 4 & 0 & -4 \\ 4 & 0 & 4 & -4 \\ 4 & 8 & -4 & -4 \end{pmatrix},$$

which row-reduces to

$$\simeq \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $(A - 4I_4)^2$ is of rank 2, meaning $\dim\left(\ker\left((A - 4I_4)^2\right)\right) = 2$, The vector $v_4 = e_1 + e_4$, along with v_3, forms a basis for E_4^2.

(c) Thus, via finding the generalized eigenspaces E_2^1 and E_4^2, we get the Jordan basis of

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

with the Jordan canonical form of

$$[T_A]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Problem 21

Problem: Prove that the matrices

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -4 & -1 & -4 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & 4 & 9 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 5 & 0 & -4 & -7 \\ 3 & -8 & 15 & -13 \\ 2 & -4 & 7 & -7 \\ 1 & 2 & -5 & 1 \end{pmatrix}$$

are similar.

Solution: We calculate the Jordan canonical form of A by calculating the characteristic polynomial.

$$c_A(x) = (x-1)^3 (x-2).$$

Evaluating the generalized eigenspaces, we find (with computational assistance) that

$$\begin{aligned} E_1^\infty &= E_1^3 \\ E_2^\infty &= E_2^1, \end{aligned}$$

meaning the Jordan canonical form of A is

$$\simeq \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Similarly, we find

$$c_B(x) = (x-1)^3 (x-2).$$

With computational assistance, we find that

$$\begin{aligned} E_1^\infty &= E_1^3 \\ E_2^\infty &= E_2^1, \end{aligned}$$

meaning the Jordan canonical form of B is

$$\simeq \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since A and B have the same Jordan canonical form, A and B are similar.

Problem 23

Problem: Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3 (x-3)^2$.

Solution: The following dimensions are possible for each of the generalized eigenspaces

$$\dim\left(E_2^\infty\right) = 1, 2, \text{ or } 3,$$

and

$$\dim\left(E_3^{\infty}\right) = 1 \text{ or } 2.$$

Thus, the potential Jordan canonical forms consist of the following matrices and the permutations of the Jordan blocks.

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}$$