Part 1

2.1, Problem 1

(i) This system is the one with small prey, since, in the expression

$$\frac{dy}{dt} = y\left(-5 + \frac{x}{20}\right)$$

we see that one of the equilibrium points occurs when $x = 100$ and, at that equilibrium rate,

$$\frac{dx}{dt} = (10)(100)\left(1 - \frac{100}{10}\right) - 20\,(100)\,y$$

has equilibrium solution at $y = 4.5$, so there are a large amount of prey and a small quantity of predators.

(ii) Similarly, this system has an equilibrium solution with $y = 30$ and $x = 1.2$, meaning there are a large quantity of predators and a small quantity of prey.

2.1, Problem 2

(i) Starting with

$$\frac{dx}{dt} = 10x\left(1 - \frac{x}{10}\right) - 20xy,$$

we take $y = 0$ (see below), and find

$$\frac{dx}{dt}\Big|_{y=0} = 10x\left(1 - \frac{x}{10}\right),$$

which has equilibrium solutions at $x = 0$ and $x = 10$.

Now, turning our attention to

$$\frac{dy}{dt} = y\left(-5 + \frac{x}{20}\right),$$

we have an equilibrium solution at $y = 0$ as well as $x = 100$, which, substituting back into $\frac{dx}{dt}$, we get

$$\frac{dx}{dt}\Big|_{x=100} = 10\,(100)\left(1 - \frac{100}{10}\right) - 20\,(100)\,y,$$

and have $y = -4.5$, which is not allowed. Thus our equilibrium solutions are at $(0, 0)$ and $(0, 10)$.

(ii) Starting with

$$\frac{dx}{dt} = x\left(0.3 - \frac{y}{100}\right),$$

we have equilibrium solutions at $x = 0$ and $y = 30$. Substituting $x = 0$ into $\frac{dy}{dt}$, we get

$$\frac{dy}{dt}\Big|_{x=0} = 15y\left(1 - \frac{y}{15}\right),$$

which has equilibrium solutions at $y = 0$ and $y = 15$. Substituting $y = 30$ into $\frac{dy}{dt}$, we get

$$\frac{dy}{dt}\Big|_{y=30} = 15\,(30)\left(1 - \frac{30}{15}\right) + 25\,(15)\,x,$$

which has an equilibrium solution at $x = 1.2$. Thus, our equilibrium solutions are at $(0, 0), (0, 15), (1.2, 30)$.

2.1, Problem 3

1. If $y(t_0) = 0$, then

$$\begin{aligned}\frac{dy}{dt}\Big|_{y(t_0)=0} &= y(t_0)\left(-5 + \frac{x}{20}\right)\\ &= 0.\end{aligned}$$

   Thus, the predator population stays at 0.

2. If $y(t_0) = 0$, then

$$\begin{aligned}\frac{dy}{dt}\Big|_{y(t_0)=0} &= 15y(t_0)\left(1 - \frac{y(t_0)}{15}\right) + 25xy(t_0)\\ &= 0.\end{aligned}$$

   Thus, the predator population stays at 0.

2.1, Problem 5

(i) If $x(t_0) = 0$, then

$$\begin{aligned}\frac{dx}{dt}\Big|_{x(t_0)=0} &= 10x(t_0)\left(1 - \frac{x(t_0)}{10}\right) - 20x(t_0)y\\ &= 0.\end{aligned}$$

Thus, the prey population stays at 0.

(ii) If $x(t_0) = 0$, then

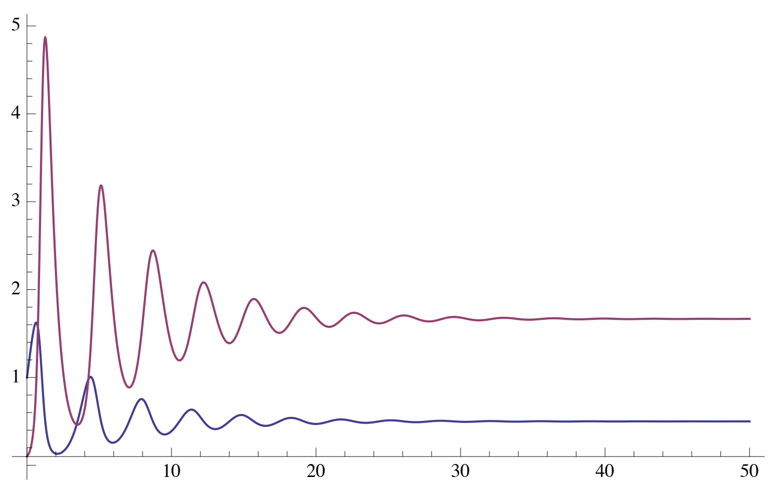
$$\begin{aligned}\frac{dx}{dt}\Big|_{x(t_0)=0} &= 0.3x(t_0) - \frac{x(t_0)y}{100}\\ &= 0.\end{aligned}$$

Thus, the prey population stays at 0.

2.1, Problem 7

(a) Based on this image, the prey and predator populations approach the equilibrium solution of approximately 1.67 for the predator and 0.5 for the prey.

(b) Confirmed with Mathematica below.



2.1, Problem 10

We would add an extra $-\alpha F$ term to $\frac{dF}{dt}$ to account for the effect of hunting predators.

2.1, Problem 14

If the prey move out at a rate proportional to the predators, we add an extra $-\beta F$ term to $\frac{dR}{dt}$ to account for the effect.

2.2, Problem 7

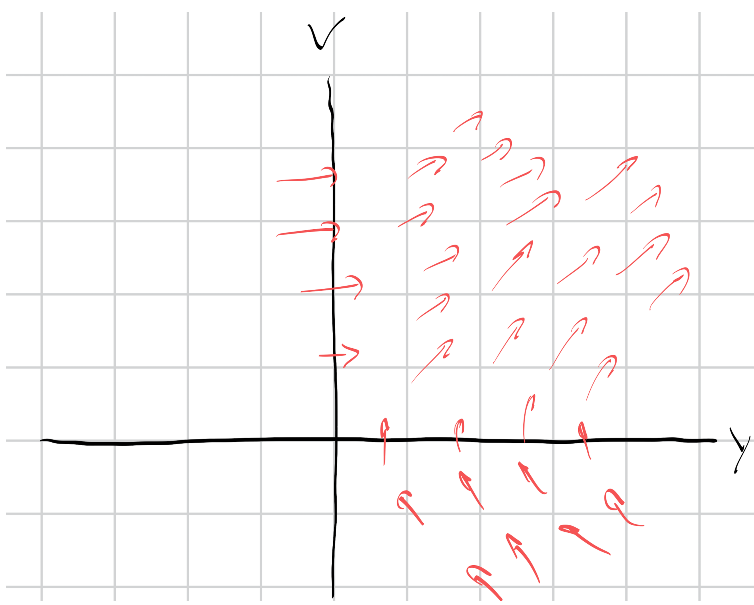
(a) We have

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = y,$$

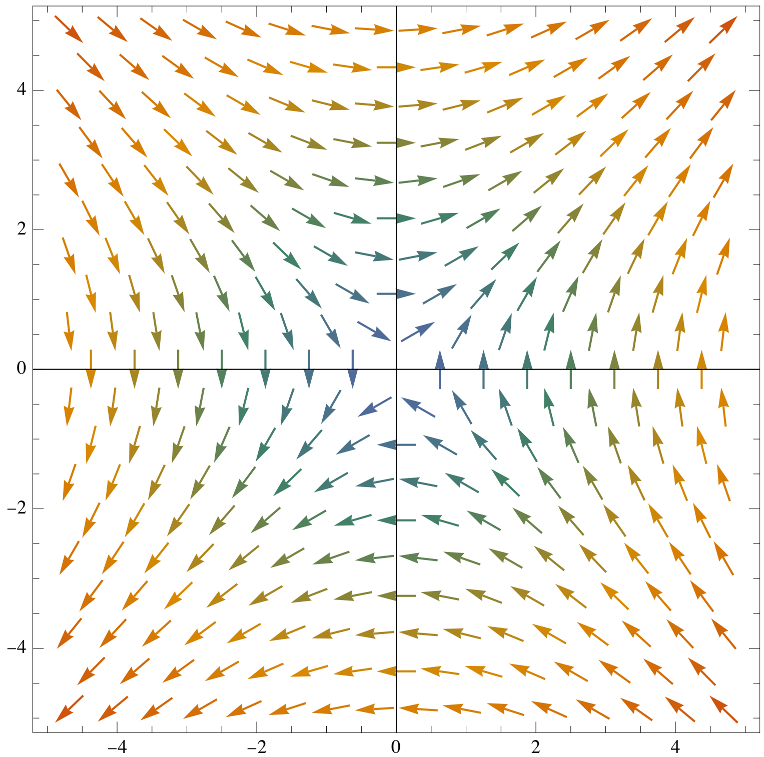
so the vector field associated with this first order system is

$$\begin{pmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} v(t) \\ y(t) \end{pmatrix}.$$

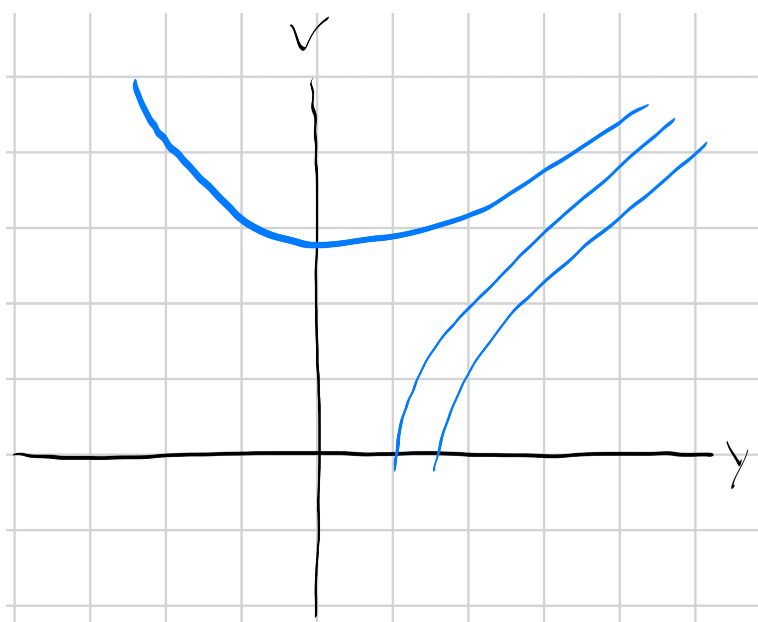
(b)



(c) Using Mathematica:



(d)



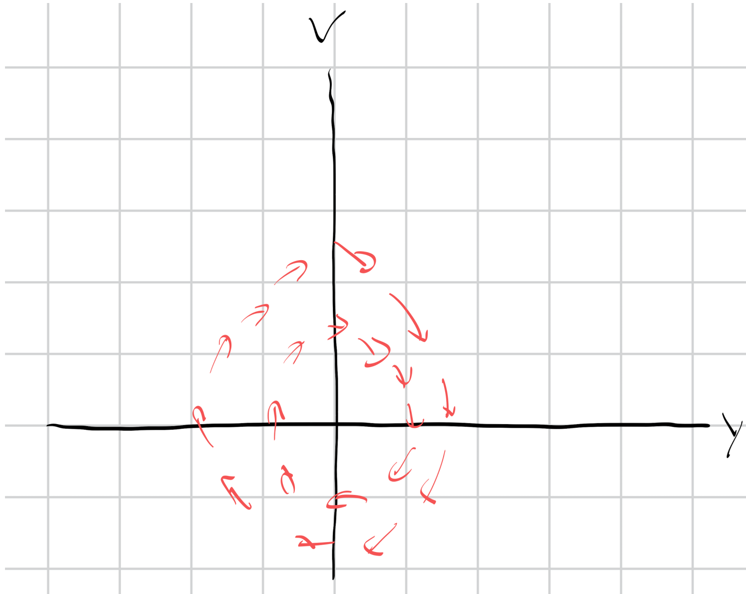
(e) As t goes to infinity, the solutions approach a "blow up" case.

2.2, Problem 8

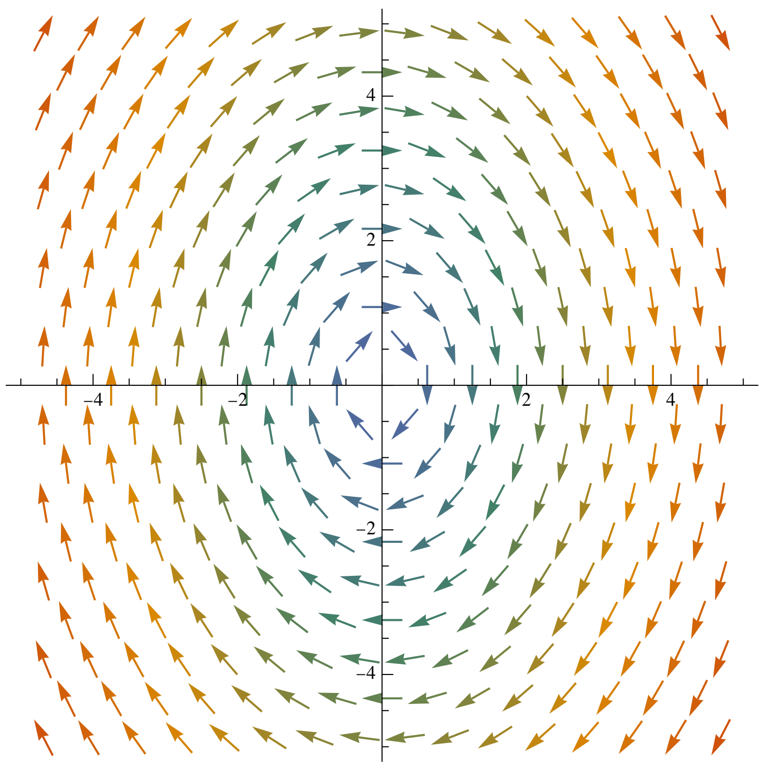
(a) We have

$$\begin{pmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} v(t) \\ -2y(t) \end{pmatrix}$$

(b)



(c)



(d)

2.2, Problem 11

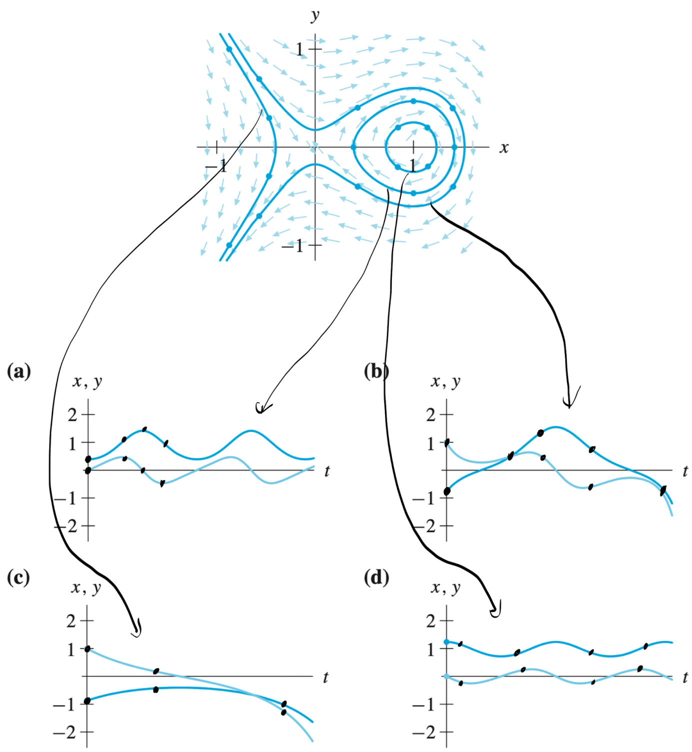
(a) Since the x components are 0 at $x = \pm 1$, the two options are either (ii) or (vii). Since the slopes are negative for $x > 1$ and $y > 0$, it is the case that this slope field is (vii).

(b) This slope field corresponds to (viii), as the slope fields point to the origin for $y = x$.

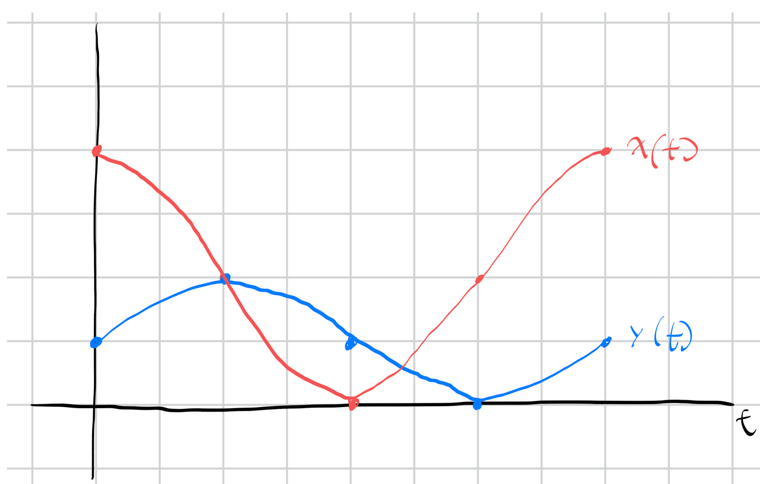
(c) This slope field corresponds to system (iv), as it necessarily shoots away from the origin.

(d) This slope field corresponds to equation (vi), as for $x > 1$ and $y > 0$, the slopes are negative.

2.2, Problem 21



2.2, Problem 24



Part 2

2.4, Problem 2

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left(3e^{2t} + e^{t}\right)\\ &= 6e^{2t} + e^{t}\\ &\neq 2\left(3e^{2t} + e^{t}\right) + 2\left(-e^{t} + e^{4t}\right).\end{aligned}$$

2.4, Problem 5

The solution for $\frac{dy}{dt} = -y$ is of the form y_0e^{-t}, which this proposed solution does not have.

2.4, Problem 7

Solving for y, we get $y(t) = k_2e^{-t}$. Substituting back in to the first equation, we have

$$\begin{aligned}\frac{dx}{dt} &= 2x + k_2e^{-t}\\ \frac{dx}{dt} - 2x &= k_2e^{-t}\\ e^{-2t}\frac{dx}{dt} - 2xe^{-2t} &= k_2e^{-3t}\\ \frac{d}{dt}\left(xe^{-2t}\right) &= k_2e^{-3t}\\ xe^{-2t} &= -3k_2e^{-3t} + C\\ x &= -3k_2e^{-3t} + k_1e^{2t}.\end{aligned}$$

Thus, the solution is

$$\vec{Y}(t) = \begin{pmatrix}-3k_2e^{-t} + k_1e^{2t}\\ k_2e^{-t}\end{pmatrix}.$$

2.4, Problem 8

(a) If we select $k_2 = 3$, then $-3k_2e^{-t} = -9e^{-t}$, which cannot equal e^{-t}.

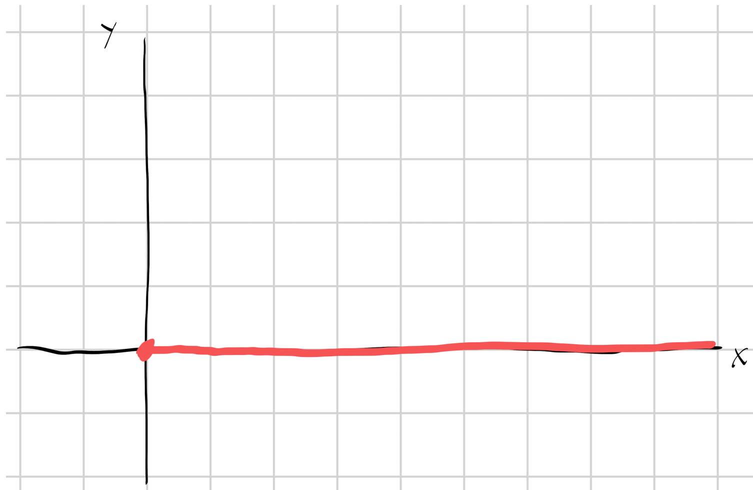
(b) There is no x dependence in the expression $\frac{dy}{dt}$.

2.4, Problem 9

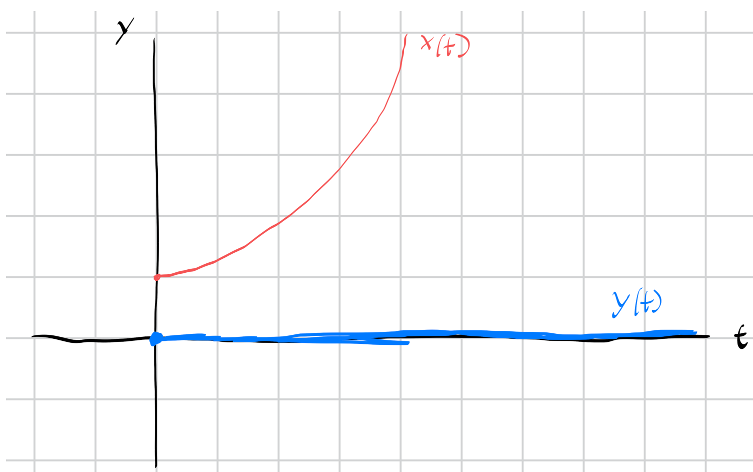
(a) If $y(0) = 0$, then $k_2 = 0$, meaning $k_1 = 1$. Thus, we get

$$\vec{Y}(t) = \begin{pmatrix}e^{2t}\\ 0\end{pmatrix}.$$

(b)



(c)



2.4, Problem 13

(a) We solve $\frac{dy}{dt} = -3y$, yielding $y(t) = k_2 e^{-3t}$. Substituting into $\frac{dx}{dt}$, we get

$$\begin{aligned}
\frac{dx}{dt} &= 2x - 8k_2 e^{-6t} \\
\frac{dx}{dt} - 2x &= -8k_2 e^{-6t} \\
e^{-2t}\frac{dx}{dt} - 2xe^{-2t} &= -8k2e^{-8t} \\
xe^{-2t} &= 64k_2 e^{-8t} + k_1 \\
x &= 64k_2 e^{-8t} + k_1 e^{2t}.
\end{aligned}$$

(b) We have $\frac{dy}{dt} = 0$ only if $k_2 = 0$, meaning $\frac{dx}{dt} = 0$ only if $\frac{dx}{dt} = 2x = 0$, so the only equilibrium point is at $(0, 0)$.

(c) Substituting into the expression for $y(t)$, we get $y(t) = e^{-3t}$. Substituting for x, we get

$$\begin{aligned}
0 &= 64 + k_1 e^{2t} \\
k_1 &= -64,
\end{aligned}$$

meaning the solution that satisfies this initial condition is

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 64e^{-8t} - 64e^{2t} \\ e^{-3t} \end{pmatrix}.$$

(d) I can't solve using Mathematica.

2.5, Problem 2

(a) Since the system is fully decoupled, and e^{2t} is a system for $\frac{dx}{dt} = 2x$, while $3e^{t}$ is a solution to $\frac{dy}{dt} = y$, this is a solution.

(b) At $t = 2$ the approximate solution is $(16, 15.19)$, while the exact solution is $(54.60, 22.17)$.

At $t = 4$, the approximate solution is $(256, 76.88)$, while the exact solution is $(2980, 163.8)$.

At $t = 6$, the approximate solution is $(4096, 389.24)$, while the exact solution is $(162755, 1210.3)$.

(c) At $t = 2$, the approximate solution is $(38.33, 20.18)$.

At $t = 4$, the approximate solution is $(1470, 136)$.

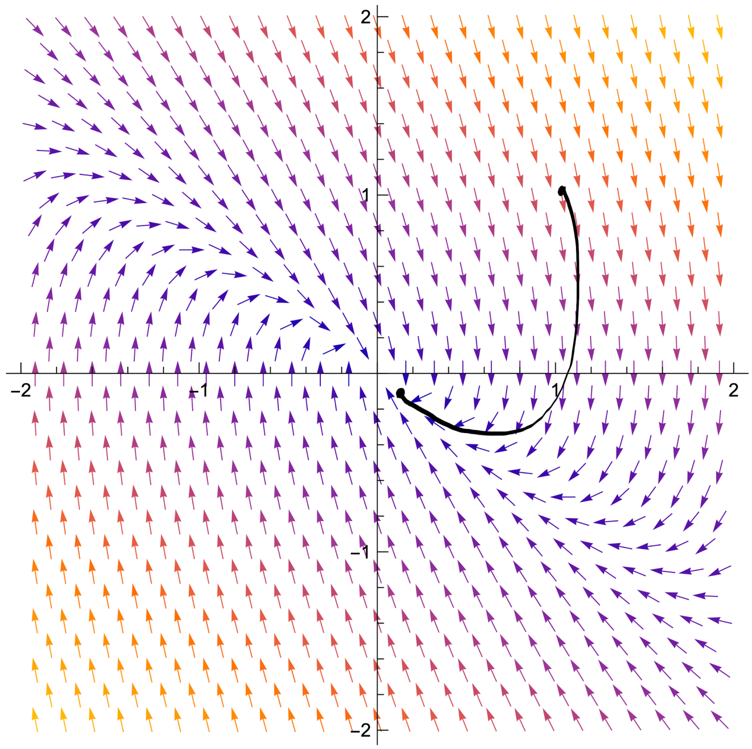
At $t = 6$ the approximate solution is $(56347, 913.44)$.

(d) The approximations are generally on the basis of slope, and so diverge as the exponential function grows much faster than any linear approximation.

2.5, Problem 3

(a) The estimated output from Euler's method has a final result of $(0.09375, -0.09375)$.

(b)



(c)