3.3.10

For every graph G, prove that $\beta(G) \leq \alpha'(G)$. For each $k \in \mathbb{N}$, construct a simple graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$.

Let M be a matching with cardinality $\alpha'(G)$. Let K be the set of vertices containing all the vertices in M — so, K is of size $2\alpha'(G)$. We posit that K is a vertex cover. Suppose toward contradiction that it were not. Then, there would exist $e = xy$ such that $e \in G$, $e \notin M$, and $x, y \notin K$. However, this would mean that M would not be a maximum matching, as we would be able to add e to it, which yields our desired contradiction. Since K is a vertex cover, we know that the minimum vertex cover must be of size less than or equal to K. Therefore, we have that $\beta(G) \leq 2\alpha'(G)$.

For every value of $k \in \mathbb{N}$, we can find a graph where $\alpha'(G) = k$ and $\beta(G) = 2k$ by using the disjoint union of k copies of C_3.

3.3.24

Let G be a simple graph of even order n with set S of size k such that $q(G - S) > k$. Prove that G has at most $\binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$ edges. Use this to determine the maximum size of a simple n-vertex graph with no 1-factor.