Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: Let $(f_n)_n$ be a Cauchy sequence in $C_0(\mathbb{R})$. Since each $f_k \in C_0(\mathbb{R})$, it must be the case that each f_k is uniformly continuous. For each $x \in \mathbb{R}$, it is thus the case that $(f_n(x))_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(f_n(x))_n \to f(x)$ for each $x \in \mathbb{R}$, and since each f_k is uniformly continuous, it must be the case that $f(x)$ is continuous.

For $\varepsilon > 0$, there must be N large such that for $m, n \geq N$ and $m \geq n$, it must be the case that $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Letting $m \to \infty$, we have $|f_n(x) - f(x)| < \varepsilon$, meaning $(f_n)_n \to f$. Thus, $f \in C_0(\mathbb{R})$.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $\|x\|_2 = \langle x, x \rangle^{1/2}$ for $x \in \ell_2$. Let $\varepsilon > 0$. Let $(x_n)_n$ be a Cauchy sequence in ℓ_2. Then, for N large and $m, n \geq N$,

$$\|x_m - x_n\|^2 < \varepsilon$$
$$\langle x_m - x_n, x_m - x_n \rangle = \langle x_m, x_m \rangle + \langle x_n, x_n \rangle - 2 \langle x_m, x_n \rangle$$