Problem (Problem 2)**:**

(a) Determine the Laurent series expansion of the function

$$f(z) = \frac{z}{(z-3)^2(z-4)}$$

that converges on $A(0,3,4)$.

(b) Show that there does not exist a holomorphic function $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ satisfying $|f(z)| \geqslant |z|^{-2/3}$.

Solution:

(a) We start by taking a partial fraction decomposition of f to yield

$$\begin{aligned} f(z) &= \frac{4}{z-4} - \frac{4}{z-3} - \frac{3}{(z-3)^2} \\ &= \frac{4}{z-4} - \frac{4}{z-3} + 3\frac{d}{dz}\left(\frac{1}{z-3}\right) \end{aligned}$$

We seek to expand about $z = 0$ within the ball $U(0,4)$ and outside the closed ball $B(0,3)$. This means that the first term in our partial fraction expansion becomes

$$a_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{4^n}.$$

The expansion in the second and third terms will require a little bit more work. Dividing out by z, we find that the second term becomes

$$\begin{aligned} a_2(z) &= -\frac{4}{z\left(1 - \frac{3}{z}\right)} \\ &= -\frac{4}{z}\sum_{n=0}^{\infty} \frac{3^n}{z^n} \\ &= -\sum_{n=1}^{\infty} \frac{4 \cdot 3^{n-1}}{z^n} \\ &= -\sum_{n=-\infty}^{-1} 12(3^{-n})z^n. \end{aligned}$$

Finally, for the third term, we observe that, using term-by-term differentiation (allowable as the series is uniformly convergent), we have

$$\begin{aligned} 3\frac{d}{dz}\left(\frac{1}{z-3}\right) &= 3\frac{d}{dz}\left(\sum_{n=1}^{\infty} 3^{n-1}z^{-n}\right) \\ &= \sum_{n=1}^{\infty} -n3^n z^{-(n+1)} \\ &= \sum_{n=-\infty}^{-1} n3^{-n}z^{n-1}. \end{aligned}$$

This yields a Laurent series expansion of

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{4^n} + \sum_{n=-\infty}^{-1} \left(12(3^{-n})z^n + n3^{-n}z^{n-1}\right).$$

(b) Suppose toward contradiction that there were such an $f(z)$. Since $|z|^{-2/3}$ is strictly greater than zero along its domain, it would follow that $|f(z)|$ would not have any zero along its domain. This means that $g(z) = \frac{1}{f(z)} \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ would be defined on its entire domain. Furthermore, we would have

$$|g(z)| \leqslant |z|^{2/3},$$

and on $U(0, \varepsilon)$, we know that $|z|^{2/3}$ is bounded above by $\varepsilon^{2/3}$ as $|z|^{2/3} \colon \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ is an increasing function. Thus, since g would be locally bounded around 0, it would follow that g has a removable singularity at 0. This means that there is a holomorphic extension $h \colon \mathbb{C} \to \mathbb{C}$ that agrees with g on $\mathbb{C} \setminus \{0\}$. In particular, we would have $|h(z)| \leqslant |z|^{2/3}$ for all $z \in \mathbb{C} \setminus \{0\}$.

Now, let $R > 0$. Using the Cauchy estimate on $S(0, R)$, we have, for any fixed $n > 0$,

$$\begin{aligned} \left|h^{(n)}(z)\right| &\leqslant \frac{n!}{R^n} \sup_{|z|=R} |h(z)| \\ &\leqslant \frac{n!}{R^n} \sup_{|z|=R} |z|^{2/3} \\ &= \frac{n!}{R^{n-2/3}}. \end{aligned}$$

Yet, since R is arbitrary, it follows that $\left|h^{(n)}(z)\right| = 0$ for all $n > 0$, whence h is constant. Yet, since $|h(z)| \leqslant |z|^{2/3}$ for all $z \in \mathbb{C} \setminus \{0\}$, it follows that $|h(z)| \leqslant \varepsilon^{2/3}$ for any $\varepsilon > 0$, whence $|h(z)| = 0$ for all $z \in \mathbb{C}$. At the same time, we explicitly defined $g(z)$ in a manner such that it could never equal zero, meaning that such an f cannot exist.

Problem (Problem 3)**:** Let $0 < r < R$. Show that there does not exist a holomorphic bijection $f \colon \mathbb{D} \setminus \{0\} \to A(0, r, R)$.

Solution: Suppose there were such a holomorphic bijection. Notice that for all $z \in \mathbb{D} \setminus \{0\}$, we would then have $|f(z)| < R$, meaning that f is necessarily locally bounded close to 0. Thus, the singularity at 0 is removable, so there is a unique holomorphic function $g \colon \mathbb{D} \to \mathbb{C}$ with $g|_{\mathbb{D}\setminus\{0\}} = f$.

We notice that g is an injection, as $g|_{\mathbb{D}\setminus\{0\}}$ is a bijection and $g(0)$ is uniquely defined. As a result, it follows that the restriction $g \colon \mathbb{D} \to \operatorname{im}(g)$ is a holomorphic bijection. Furthermore, we also notice that

$$\begin{aligned} \lim_{z\to 0} |g(z)| &= \lim_{z\to 0} |f(z)| \\ &\geqslant r \\ &> 0, \end{aligned}$$

meaning that g is nonvanishing on $\mathbb{D}$. In particular, there is a logarithm $h(z) \colon \mathbb{D} \to \mathbb{C}$ such that

$$g(z) = e^{h(z)},$$

and $f(z) = e^{h(z)}$ when restricted to $\mathbb{D} \setminus \{0\}$. Now, since the identity map $\mathrm{id} \colon A(0, r, R) \to A(0, r, R)$ is a bijective holomorphic map with holomorphic inverse, it follows that

$$e^{h(z)} = \mathrm{id}(f(z)).$$

Yet, this means that

$$\mathrm{id}(z) = e^{h\left(f^{-1}(z)\right)},$$

meaning that id admits a holomorphic logarithm. Yet, $A(0, r, R)$ is not simply connected, while id is non-constant, which is a contradiction. Thus, no such f exists.

Problem (Problem 4)**:** Show that if f is entire and satisfies $\lim_{z\to\infty} f(z) = \infty$, then f is a polynomial.

Solution: Consider the function $g\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ given by $g(z) = f\left(\frac{1}{z}\right)$. Since f is entire and $\frac{1}{z}$ is holomorphic on $\mathbb{C}\setminus\{0\}$, it follows that, given the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have the Laurent series expansion

$$g(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

Observe that the limit $\lim_{z\to\infty} f(z)$ is equivalent to $\lim_{z\to 0} f\left(\frac{1}{z}\right)$, whence $\lim_{z\to 0} g(z) = \infty$. Therefore, g has a pole of order k at 0, whence

$$g(z) = \sum_{n=0}^{k} a_n z^{-n}.$$

Since $g\left(\frac{1}{z}\right) = f(z)$, it then follows that

$$f(z) = \sum_{n=0}^{k} a_n z^n.$$