

The primary text for Algebra II is Dummit and Foote's *Abstract Algebra*, and will cover the following topics:

- modules and advanced linear algebra;
- representation theory of finite groups;
- field theory and Galois theory.

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Modules and Advanced Linear Algebra

Tensor Products of Modules

To motivate tensor products, we recall a basic fact from linear algebra. If we assume that R is a field, and M, N are finite-dimensional R -vector spaces, then the following equation necessarily holds:

$$\dim(M \oplus N) = \dim(M) + \dim(N).$$

We want to construct a similar operation on vector spaces, $M \otimes N$, that satisfies

$$\dim(M \otimes N) = \dim(M) \dim(N).$$

For now, we will label this by $M \bar{\otimes} N$, where we use the $\bar{\otimes}$ to refer to the fact that this is a temporary definition. Naively, we might seek to define $M \bar{\otimes} N$ as follows. If we let $\{x_1, \dots, x_k\}$ be a basis for M and $\{y_1, \dots, y_\ell\}$ a basis for N , then we will define $M \bar{\otimes} N$ to be all the formal R -linear combinations over the basis

$$B = \{x_i \otimes y_j \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}.$$

While this is technically correct — as in, this does yield a vector space with

$$\dim(M \bar{\otimes} N) = \dim(M) \dim(N),$$

the issue is that this definition is not canonical, in that it depends on chosen bases for M and N . Furthermore, it is not clear how one may generalize from this definition to modules over arbitrary rings, which do not necessarily have bases. To resolve this issue, we will go about defining a construction that “extends,” in a sense, this definition of $M \bar{\otimes} N$.

To start, we define the simple tensor $m \otimes n$ for any $m \in M$ and $n \in N$. If we let

$$m = \sum_{i=1}^k \lambda_i x_i$$

$$n = \sum_{j=1}^{\ell} \mu_j y_j,$$

then we will define

$$m \otimes n = \sum_{i=1}^k \sum_{j=1}^{\ell} \lambda_i \mu_j (x_i \otimes y_j).$$

We observe that every element of $M \bar{\otimes} N$ is a sum (i.e., an *integral* linear combination) of simple tensors, as by regrouping we may take

$$\sum_{i=1}^k \sum_{j=1}^{\ell} \lambda_{ij} (x_i \otimes y_j) = \sum_{i=1}^k \sum_{j=1}^{\ell} (\lambda_{ij} x_i) \otimes y_j.$$

The simple tensors satisfy the following relations:

$$(R1) \quad (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n;$$

$$(R2) \quad m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2;$$

$$(R3) \quad (\alpha m) \otimes n = m \otimes (\alpha n)$$

for $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $\alpha \in R$.

Proposition: These are the defining relations for $M \bar{\otimes} N$ in the category of abelian groups.

We will simply take this proposition as fact.

Now, let

$$\begin{aligned} Q &= M \times N \\ &= \{(m, n) \mid m \in M, n \in N\} \end{aligned}$$

be the Cartesian product of M and N as sets. We will then take $\mathbb{Z}[Q]$ to be the standard free \mathbb{Z} -module (i.e., free abelian group) on Q . That is, $\mathbb{Z}[Q]$ is the set of formal linear combinations

$$v = \sum_{q \in Q} \lambda_q q,$$

where $\lambda_q \in \mathbb{Z}$ and only finitely many coefficients are nonzero. By the universal property of free abelian groups, the map $(m, n) \mapsto m \otimes n$ descends to a unique homomorphism $\varphi: \mathbb{Z}[Q] \rightarrow M \bar{\otimes} N$. Such a homomorphism is necessarily surjective as every element of $M \bar{\otimes} N$ is an integral linear combination of simple tensors, meaning that we have

$$M \bar{\otimes} N \cong \mathbb{Z}[Q] / \ker(\varphi)$$

as abelian groups.

Now, consider the subgroup of $\mathbb{Z}[Q]$, which we denote K , that is generated by the following elements:

$$(I) \quad (m_1 + m_2, n) - (m_1, n) - (m_2, n);$$

$$(II) \quad (m, n_1 + n_2) - (m, n_1) - (m, n_2);$$

$$(III) \quad (\alpha m, n) - (m, \alpha n)$$

for $m_1, m_2, m \in M$, $n_1, n_2, n \in N$, and $\alpha \in R$. Then, from proposition that the relations (R1) through (R3) define $M \bar{\otimes} N$, it follows that $K = \ker(\varphi)$. Thus, we may define the tensor product canonically as follows.

Definition: Letting M, N, Q, K be as above, we define

$$M \otimes N := \mathbb{Z}[Q] / K,$$

and define $m \otimes n = (m, n) + K$.