Problem (Problem 1)**:** Show that if $1 < \lambda < \infty$, then the equation

$$ze^{\lambda - z} = 1$$

has precisely one solution in $\mathbb{D}$.

Solution: Write $f(z) = ze^{\lambda - z} - 1$. Our task is to show that $f(z)$ has exactly one solution in $\mathbb{D}$. Consider the function

$$g(z) = ze^{\lambda - z}.$$

We observe that $g(0) = 0$, and for any $z \neq 0$, $g(z) \neq 0$. Furthermore, since $e^{\lambda - z} \neq 0$ for all $z \in \mathbb{D}$, we observe that g has exactly one zero at $z = 0 \in \mathbb{D}$.

Let $\Gamma = S^1 = \partial\mathbb{D}$. We then observe that g and f are never zero on S^1, and that

$$\begin{aligned} |f(z) - g(z)| &= 1 \\ &< e^{\lambda - 1} \\ &< e^{\lambda - \operatorname{Re}(z)} \\ &= \left|ze^{\lambda - z}\right| \\ &= |g(z)|, \end{aligned}$$

whence f and g have the same number of zeros in $\mathbb{D}$.

Problem (Problem 2)**:**

(a) Prove that for any constants $a_0, a_1, a_2 \in \mathbb{C}$, the following inequality holds:

$$\max_{|z|=1}\left|z^7 + a_2 z^2 + a_1 z + a_0\right| \geqslant 1.$$

(b) Let $U \subseteq \mathbb{C}$ be open with $B(0,1) \subseteq U$, and let $f\colon U \to \mathbb{C}$ be holomorphic. Suppose that

$$\max_{|z|=1}\left|f(z) - \frac{1}{z^2}\right| < 1.$$

Show that f is not a polynomial.

Problem (Problem 3)**:** Let $U \subseteq \mathbb{C}$ be open containing $B(0,1)$, and let $f, g\colon U \to \mathbb{C}$ be holomorphic such that $\operatorname{ord}_0(f) = 1$ and $\operatorname{ord}_z(f) = 0$ for all $z \in B(0,1) \setminus \{0\}$. For $w \in \mathbb{C}$, define $f_w(z) = f(z) + wg(z)$.

(a) Show that there exists some $r > 0$ dependent on g such that if $w \in U(0,r)$, then f_w has a unique zero in $B(0,1)$, which we call z_w.

(b) Show that $\lim_{w\to 0} z_w = 0$.

(c) Show that

$$z_w = \frac{1}{2\pi i}\oint_{S(0,1)} \frac{f_w'(\xi)}{f_w(\xi)}\xi \, d\xi.$$

Problem (Problem 4)**:** For all $n \in \mathbb{N}$ find the residue at $z = 0$ for each of the following functions.

(a) $\dfrac{e^{z^2}}{z^{2n} + 1}$;

(b) $z^{-n}e^{\alpha z}$ for $\alpha \in \mathbb{Z}$;

(c) $\dfrac{z^{n-1}}{\sin^n(z)}$.

Solution:

(a) We observe that $e^{z^2} \neq 0$ for all z, whence f has a pole of order $2n+1$. Using the Taylor expansion for e^{z^2}, we find that

$$\begin{aligned}\frac{1}{z^{2n+1}}e^{z^2} &= \frac{1}{z^{2n+1}}\sum_{k=0}^{\infty}\frac{z^{2k}}{k!}\\ &= \sum_{k=0}^{\infty}\frac{z^{2k-2n-1}}{k!},\end{aligned}$$

meaning that the coefficient at a_{-1} is $\frac{1}{n!}$.

(b) We have a pole of order n at $z = 0$, as $e^{\alpha z} \neq 0$ for all z. Thus, computing the residue directly, we find

$$\begin{aligned}\operatorname{Res}(f;0) &= \frac{1}{(n-1)!}\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}(e^{\alpha z})\\ &= \frac{\alpha^{n-1}}{(n-1)!}.\end{aligned}$$

(c) We observe that the order of the numerator at $z = 0$ is $n-1$, while the order in the denominator at $z = 0$ is n, meaning that we have a simple pole at $z = 0$. Therefore, we compute

$$\begin{aligned}\operatorname{Res}(f;0) &= \lim_{z\to 0}\frac{z^n}{\sin^n(z)}\\ &= \left(\lim_{z\to 0}\frac{z}{\sin(z)}\right)^n\\ &= 1.\end{aligned}$$

Problem (Problem 5)**:** For each positive $n \in \mathbb{N}$, let γ_N be the loop consisting of the square with vertices at $\left(N+\frac{1}{2}\right)(-1-i)$, $\left(N+\frac{1}{2}\right)(1-i)$, $\left(N+\frac{1}{2}\right)(1+i)$, and $\left(N+\frac{1}{2}\right)(-1+i)$.

Let $f(z) = \frac{\pi\cot(\pi z)}{z^4}$. By evaluating $\oint_{\gamma_N} f(z)\,dz$, determine

$$\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Solution: We observe that the poles of $f(z)$ are at $-N, -N+1, \dots, 0, \dots, N-1, N$. To compute the residue at each of these poles, we separate into the case of $z = 0$ and of $z \neq 0$. For the case with $z \neq 0$, we find that f has a simple pole at $z = k$ for each such k, whence

$$\begin{aligned}\operatorname{Res}(f;k) &= \lim_{z\to k}\frac{\pi\cos(\pi z)}{z^4\frac{d}{dz}|_{z=n}\pi\sin(\pi z)}\\ &= \frac{1}{k^4}.\end{aligned}$$

Since $z^4\sin(\pi z)$ has a zero of order 5 at 0, and $\cos(\pi z)$ does not have a zero at $z = 0$, it follows that

$$f(z) = \frac{\pi\cos(\pi z)}{z^4\sin(\pi z)}$$

has a pole of order 5 at 0. We compute

$$\begin{aligned}\operatorname{Res}(f;0) &= \frac{1}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\left(z^5 f(z)\right)\\ &= \frac{1}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}(\pi z\cot(\pi z)).\end{aligned}$$

Upon tedious computation, we find that

$$\operatorname{Res}(f;0) = -\frac{\pi^4}{45}.$$

Therefore, we find that

$$\frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz = 2\sum_{k=1}^{N}\frac{1}{k^4} - \frac{\pi^4}{45}.$$

Now, we want to evaluate

$$\lim_{N\to\infty}\frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz.$$

Toward this end, we observe that on the square γ_N, that by the definition of the hyperbolic cotangent,

$$\begin{aligned}|\cot(\pi z)| &\leqslant \left|\cot\left(\pi\left(N+\frac{1}{2}\right)i\right)\right|\\ &= \coth\left(\left(N+\frac{1}{2}\right)\right),\end{aligned}$$

whence

$$\begin{aligned}\left|\frac{\pi\cot(\pi z)}{z^4}\right| &\leqslant \frac{\pi\coth\left(\pi\left(N+\frac{1}{2}\right)\right)}{\left(2\left(N+\frac{1}{2}\right)^2\right)^2}\\ &= \frac{\pi\coth\left(\pi\left(N+\frac{1}{2}\right)\right)}{4\left(N+\frac{1}{2}\right)^4}\\ &=: M_N.\end{aligned}$$

Therefore, we observe that

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz\right| &\leqslant \ell(\gamma_N)\frac{\pi\coth\left(\pi\left(N+\frac{1}{2}\right)\right)}{4\left(N+\frac{1}{2}\right)^4}\\ &\to 0,\end{aligned}$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$