

Problem 1

(a)

$$\begin{aligned}\int_{C_1} (x^2 + y^2) \, d\ell &= \int_0^1 x^2 \, dx + \int_0^1 y^2 + 1 \, dy \\ &= \frac{5}{3}.\end{aligned}$$

(b)

$$\begin{aligned}\int_{C_2} (x^2 + y^2) \, d\ell &= \int_0^1 2x^2 \, dx \\ &= \frac{2}{3}.\end{aligned}$$

(c)

$$\begin{aligned}\int_{C_3} (x^2 + y^2) \, d\ell &= \int_0^1 x^2 + x^4 \, dx \\ &= \frac{8}{15}.\end{aligned}$$

Problem 2(a) Since $\oint_C d\ell$ “adds up” the infinitesimal lengths along C , this integral gives the length of C .(b) Since $\oint_C d\vec{\ell}$ is a vector-valued integral along C , and since C is closed, this integral gives 0.**Problem 3**(a) We have $d\ell = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, and $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$, so

$$\begin{aligned}\int_C d\ell &= \int \sqrt{1 + \frac{x^2}{a^2 - x^2}} \, dx \\ &= \int \frac{1}{\sqrt{a^2 - x^2}} \, dx \\ &= a \arcsin\left(\frac{x}{a}\right).\end{aligned}$$

Evaluated from $x = -a$ to $x = a$, we get that $\int_C d\ell = \pi a$.(b) We have $d\ell = \sqrt{dr^2 + r^2 d\theta^2}$, so

$$\begin{aligned}\int d\ell &= \int_0^\pi a \, d\theta \\ &= \pi a\end{aligned}$$

Problem 7(a) $\oint_S dA$ gives the area of the sphere, as we do not have to integrate with respect to a direction.(b) $\oint_S d\mathbf{A}$ yields zero, as \hat{n} is symmetrical with respect to S .

Problem 11

$$\begin{aligned}
\int_S \mathbf{r} \cdot d\mathbf{A} &= \int (R\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} (R^2 d\Omega) \\
&= R^3 \int_0^{2\pi} \int_0^{\pi/2} \sin \theta \, d\phi \, d\theta \\
&= 2\pi R^3
\end{aligned}$$

Problem 18

- (a) Since $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ are both zero, this field is both surface independent and path independent.
- (b) Since both the curl and divergence of $\hat{\mathbf{r}}$ is zero, this field is both surface independent and path independent.
- (c)

$$\begin{aligned}
\nabla \cdot (-\sin x \cosh y \hat{\mathbf{i}} + \cos x \sinh y \hat{\mathbf{j}}) &= -\cos x \cosh y + \cos x \cosh y \\
&= 0 \\
\nabla \times (-\sin x \cosh y \hat{\mathbf{i}} + \cos x \sinh y \hat{\mathbf{j}}) &= (-\sin x \sinh y + \sin x \sinh y) \hat{\mathbf{k}} \\
&= 0.
\end{aligned}$$

Thus, the field is both surface independent and path independent.

(d)

$$\begin{aligned}
\nabla \cdot (xy^2 \hat{\mathbf{i}} - x^2 y \hat{\mathbf{j}}) &= 0 \\
\nabla \times (xy^2 \hat{\mathbf{i}} - x^2 y \hat{\mathbf{j}}) &= -4xy \hat{\mathbf{k}}.
\end{aligned}$$

Thus, the field is surface independent but not path independent.

(e)

$$\begin{aligned}
\nabla \cdot (\rho z \hat{\phi}) &= 0 \\
\nabla \times (\rho z \hat{\phi}) &\neq 0.
\end{aligned}$$

Thus, the field is surface independent but not path independent.

Problem 19

The integral along the path dba is -5 and the integral along the path ecdb is 3 .

Problem 20

- (a) Since $\nabla \times \mathbf{E} = 0$, we find the integral by taking

$$\int_C \mathbf{E} \cdot d\vec{\ell} = \int_{C_1} \mathbf{E} \cdot d\vec{\ell}$$

$$\begin{aligned}
&= \int_{C_2} \mathbf{E} \cdot d\vec{\ell} \\
&= \frac{1}{2} x^2 y^2 \Big|_{(0,a)}^{(a,0)} \\
&= 0.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\int_{C_1} \mathbf{B} \cdot d\vec{\ell} &= \int_{C_1} B_x dx + B_y dy \\
&= \sqrt{2} \int_0^a x^2 (-x + a) + \sqrt{2} \int_0^a (-y + a) y^2 dy \\
&= 2\sqrt{2} \int_0^a t^2 (-t + a) dt \\
&= 2\sqrt{2} \int_0^a at^2 - t^3 dt \\
&= 2\sqrt{2} \left(\frac{a^4}{3} - \frac{a^4}{4} \right) \\
&= \frac{a^4}{6} \sqrt{2}.
\end{aligned}$$

$$\begin{aligned}
\int_{C_2} \mathbf{B} \cdot d\vec{\ell} &= \int_0^{\pi/2} \begin{pmatrix} a^3 \sin^2 t \cos t \\ -a^3 \sin t \cos^2 t \end{pmatrix} \cdot \begin{pmatrix} a \cos t \\ -a \sin t \end{pmatrix} dt \\
&= a^4 \int_0^{\pi/2} 2 \sin^2 t \cos^2 t dt \\
&= \frac{a^4}{4} \int_0^{\pi/2} \sin^2(2t) dt \\
&= \frac{\pi}{4} a^4.
\end{aligned}$$

Problem 21

(a)

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\vec{\ell} &= \int_{C_1+C_2+C_3+C_4} x dx - \int_{C_1+C_2+C_3+C_4} y dy \\
&= 0.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\vec{\ell} &= \int_0^{2\pi} \left(-a^2 \cos t \sin t \right) + \left(-a^2 \sin^2 t \right) + \left(3a^3 \cos^2 t \sin t \right) dt \\
&= -2\pi a^2.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\vec{\ell} &= \int_0^{2\pi} R^2 d\phi \\
&= 2\pi R^2.
\end{aligned}$$

(d)

$$\begin{aligned}\int_C \mathbf{F} \cdot d\vec{\ell} &= \int_0^{2\pi} r^2 \sin^2 \theta \sin^2 \phi \Big|_{r=R, \theta=\alpha} d\phi \\ &= \pi R^2 \sin^2 \alpha.\end{aligned}$$

As α approaches π , this value approaches 0, which would be expected as we are integrating about a point at $\alpha = \pi$.

Problem 22

(a)

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{a} &= \int_{S_1 + \dots + S_6} \mathbf{F} \cdot d\mathbf{a} \\ &= 6a^5.\end{aligned}$$

(b)

$$\begin{aligned}\int_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot d\mathbf{a} &= \int r\hat{r} \cdot d\mathbf{a} \\ &= a^3 \int d\Omega \\ &= 4\pi a^3.\end{aligned}$$

(c)

$$\begin{aligned}\int_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot d\mathbf{a} &= \int (r\hat{r} + z\hat{k}) \cdot \hat{r} dA \\ &= 4\pi a.\end{aligned}$$

(d)

$$\int_S r^2 \hat{r} \cdot d\mathbf{a} = R^3.$$

(e)

$$\int_S (\hat{i} + \hat{j} + z(x^2 + y^2)\hat{k}) \cdot d\mathbf{a} = 0.$$

Problem 26

Since $\nabla \cdot \mathbf{E} \neq 0$, \mathbf{E} is not surface-independent.

Problem 28

(a) Inside the sphere,

$$Q_0 = \int \rho dV$$

$$\begin{aligned} &= \int_0^\pi \int_0^{2\pi} \int_0^s \alpha r^3 \sin \theta \, dr d\phi d\theta \\ &= \pi \alpha a^4, \end{aligned}$$

meaning that

$$q_{\text{encl}}(r < a) = \pi \alpha r^4,$$

so we have

$$\mathbf{E}(r > a) = \frac{Q_0}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{E}(r < a) = \frac{\alpha r^2}{4\epsilon_0} \hat{\mathbf{r}}$$

(b) I don't know how to do this problem.