Amenability: A (Somewhat) Brief Introduction

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Outline

Contents

Groups

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then we call the pair $(A, \star)$ a *group*.

We (usually) abbreviate $a \star b$ as ab. If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups

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- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

Some Groups

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- The group $\mathrm{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group $\mathrm{E}(3)$ consists of all translations, rotations, and flips in $\mathbb{R}^3$, and is also known as the *isometry group* of $\mathbb{R}^3$.

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Every group is equipped with a family of canonical actions, $\sigma_a\colon G \to G$ for each $a \in G$, given by $x \mapsto ax$, known as *left-multiplication*.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a σ*-algebra* of subsets.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

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If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$,

we say μ is a measure.

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- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We define $F(a,b)$ to be the set of all "words" in the alphabet $\{a,b,a^{-1},b^{-1}\}$, subject to the condition that, for $w,w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

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- all words that start with b — think words that start with b^2 before you multiply b^{-1}.

Thus, all we need to do is add back $W\left(b^{-1}\right)$ to get $F(a,b)$ back.

$$F(a,b) = W\left(b^{-1}\right) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

We're able to take part of the group $F(a,b)$, take some translations, and, miraculously, obtain the entire group back.

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$;

such that

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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is G-*paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
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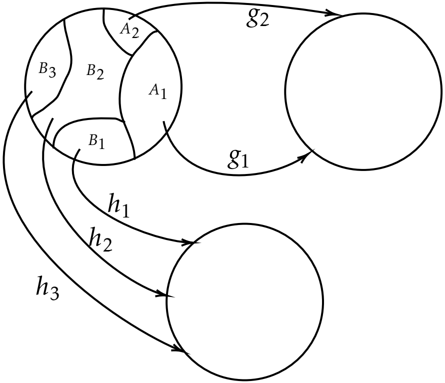
A paradoxical group is a paradoxical set under the action of left-multiplication.

Depiction



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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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Thus, SO(3) is paradoxical — can we use it to find a paradoxical decomposition?

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

- In other words, not all subsets of $\mathbb{R}^3$ have a definite “volume” invariant under isometry.

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Effectively, A and B are "equal" to each other up to the group action.

If A is G-paradoxical, then so too is B.

The Banach–Tarski Paradox: Proof Outline I

1. We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a,b)$.

The Banach–Tarski Paradox: Proof Outline II

2. We use the decomposition

$$F(a,b) = a^{-1}W(a) \cup W\left(a^{-1}\right)$$
$$= b^{-1}W(b) \cup W\left(b^{-1}\right)$$

   to duplicate the unit sphere in $\mathbb{R}^3$, S^2, except for a countable subset D. (The *Hausdorff Paradox*.)
3. We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable — there is thus a paradoxical decomposition of S^2.
4. We show that the unit ball, $B(0,1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group E(3).

The Banach–Tarski Paradox: Proof Outline III

5. Define a relation $A \preceq B$ if A is G-equidecomposable with a subset of B, and show that if $A \preceq B$ and $B \preceq A$, then A and B are G-equidecomposable.
6. Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

Contents

Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped "create" the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu : F(a,b) \to [0,1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$1 = \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right)$$

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$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\ &= \nu(F(a,b)) + \nu(F(a,b)) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned}
1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\
&= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\
&= \nu(F(a,b)) + \nu(F(a,b)) \\
&= 2.
\end{aligned}$$

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu\colon G \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

If G admits a mean, we say G is *amenable*.

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If G admits a mean, we say G is *amenable*.

- In other words, G is sufficiently "well-behaved."

Inheritance Properties of Amenability

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- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

  Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

  is a mean.
- Abelian groups are amenable.
- The free group, $F(a,b)$, is *not* amenable.

Paradoxical Groups and Amenability

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Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

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Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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On first glance, it may seem like we're finished, but we're really not.

Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

Normed Vector Spaces

Functional analysis is, of course, the study of normed vector spaces.

If V is a vector space, then a *norm* on V is a map $\|\cdot\|\colon V \to [0,\infty)$ satisfying:

- definiteness: $\|v\| \geq 0$, with equality if and only if $v = 0$;
- homogeneity: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$;
- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sup_{t\in\Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t\in\Gamma} |f(t)| < \infty \right\};$$

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They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_1} := \sum_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_2} := \left(\sum_{t\in\Gamma} |f(t)|^2\right)^{1/2}$.

Linear Maps and Linear Functionals

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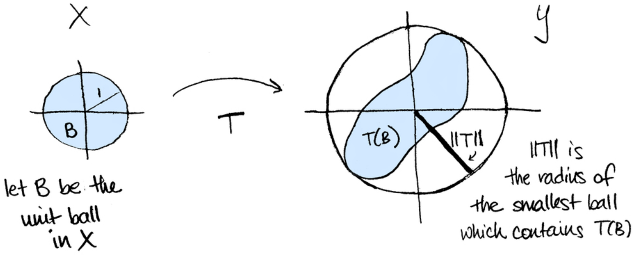
We call the quantity on the left the *operator norm*, denoted $\|T\|_{\text{op}}$.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Operator Norm Pictorial Depiction

Courtesy of Tai-Danae Bradley.



Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a bounded linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}}$.

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- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}}$.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

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If $\varphi : \ell_\infty(\Gamma) \to \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

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If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

Approximations and Amenability

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Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was "significantly" "bigger" than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

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But what does "bigger" actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n\to\infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

Approximate Means

The Følner condition allows us to find an “approximate” version of a mean.

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Keeping $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a "sufficient" finitely supported function g,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

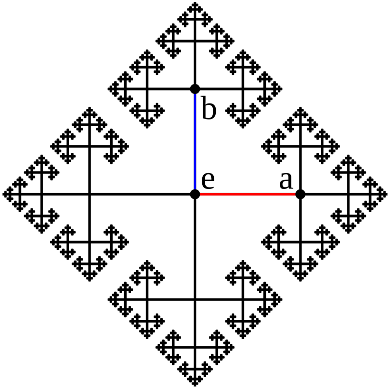
then use a "layer cake" decomposition to find our Følner sets:

$$g = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S, we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by "walking" along the generators.



Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

$$\inf\left\{ \frac{|N(S)|}{|S|} \,\middle|\, S \subseteq V(G), |S| \text{ finite} \right\} = 0.$$

- Essentially, the Cayley graph doesn't "get too big" "too fast."
- This is proven with the Følner condition.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies

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The inner product induces a norm $\|x\|^2 = \langle x,x\rangle$.

If $\mathcal{H}$ is complete with respect to this norm, we call $\mathcal{H}$ a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that "plays nicely" with the inner product:

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

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then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$
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All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma\colon \Gamma \to \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

As it turns out, the map $\lambda_s(f)(t) = f\left(s^{-1}t\right)$ is a unitary operator on $\ell_2(\Gamma)$, where $\lambda_s^* = \lambda_{s^{-1}}$.

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The map $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$ is a very special representation, known as the *left-regular representation*.

This is because it "encodes" the group's left-multiplication action, in the sense that $\lambda_s(\delta_t) = \delta_{st}$, where δ_t is the point mass at $t \in \Gamma$.

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* for $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ if

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_2} = 0.$$

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If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

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These make $\mathbb{B}(\mathcal{H})$ a C^*-*algebra*. However, there are other C^*-algebras.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

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This becomes a $*$-algebra when endowed with multiplication (by convolution) and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g\left(s^{-1} t\right)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^*-Algebra, cont'd

If we represent $\pi_\lambda : \mathbb{C}[\Gamma] \to \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\mathrm{op}},$$

we get the *reduced group C^*-algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

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There is a special kind of finite-dimensional approximation for C^*-algebras — which we can use yet again to establish amenability.

Nuclearity

A C^*-algebra, A, is called *nuclear* if there exist two sequences of maps, $\varphi_n \colon A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n \colon \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$, such that

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- Essentially, any $a \in A$ is "close enough" to a certain family of finite-dimensional analogues.

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Specifically, by showing that the approximation of $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ corresponds to the existence of maps $\varphi_n \colon C^*_\lambda(\Gamma) \to \mathrm{Mat}_{|F_n|}(\mathbb{C})$ and $\psi_n \colon \mathrm{Mat}_{|F_n|}(\mathbb{C}) \to C^*_\lambda(\Gamma)$ that satisfy

$$\left\| x - \psi_n \circ \varphi_n(x) \right\| \xrightarrow{n \to \infty} 0.$$

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Nuclear C^*-algebras are classified, so active research areas primarily concern whether or not certain classes of C^*-algebras are nuclear (hence classifiable).

There are also a lot of other directions that amenability can take the eager student, but I think this was a pretty nice overview of some of the ways that amenability touches all sorts of other fields of math.

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- friends, family, and acquaintances both in the math major and outside;
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References I

[AB06] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis*. Third. A Hitchhiker's Guide. Springer, Berlin, 2006, pp. xxii+703. ISBN: 978-3-540-32696-0.

[Alu09] Paolo Aluffi. *Algebra: Chapter 0*. Vol. 104. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xx+713. ISBN: 978-0-8218-4781-7. DOI: 10.1090/gsm/104. URL: https://doi.org/10.1090/gsm/104.

[BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*. Vol. 11. New Mathematical Monographs. Cambridge University Press, Cambridge, 2008, pp. xiv+472. ISBN: 978-0-521-88720-5. DOI: 10.1017/CBO9780511542749. URL: https://doi.org/10.1017/CBO9780511542749.

References II

[Bla06] B. Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4. DOI: 10.1007/3-540-28517-2. URL: https://doi.org/10.1007/3-540-28517-2.

[BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004, pp. xiv+716. ISBN: 0-521-83378-7. DOI: 10.1017/CBO9780511804441. URL: https://doi.org/10.1017/CBO9780511804441.

References III

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+509. ISBN: 978-0-8218-4381-9. DOI: 10.1090/gsm/088. URL: https://doi.org/10.1090/gsm/088.

[CE78] Man-Duen Choi and Edward G. Effros. "Nuclear C^*-Algebras and the Approximation Property". In: *American Journal of Mathematics* 100.1 (1978), pp. 61–79. ISSN: 00029327. URL: http://www.jstor.org/stable/2373876 (visited on 02/07/2025).

[DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. Third. John Wiley & Sons, Inc., Hoboken, NJ, 2004, pp. xii+932. ISBN: 0-471-43334-9.

References IV

[Enc25] The Editors of Encyclopaedia Britannica. *Ship of Theseus*. Accessed: 2025-02-06. 2025. URL: https://www.britannica.com/topic/Ship-of-Theseus.

[Fol84] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984, pp. xiv+350. ISBN: 0-471-80958-6.

[Hal06] Paul R. Halmos. "How to write mathematics". In: *Butl. Soc. Catalana Mat.* 21.1 (2006). Translation of Enseignement Math. (2) **16** (1970), 123–152 [MR0277319], pp. 53–79, 158. ISSN: 0214-316X,2013-9829.

References V

[Hal66] James D. Halpern. “Bases in vector spaces and the axiom of choice”. In: *Proc. Amer. Math. Soc.* 17 (1966), pp. 670–673. ISSN: 0002-9939,1088-6826. DOI: 10.2307/2035388. URL: https://doi.org/10.2307/2035388.

[Har00] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000, pp. vi+310. ISBN: 0-226-31719-6.

[Jec03] Thomas Jech. *Set theory*. millennium. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

References VI

[Jus22] Kate Juschenko. *Amenability of discrete groups by examples*. Vol. 266. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2022, pp. xi+165. ISBN: 978-1-4704-7032-6. DOI: 10.1090/surv/266. URL: https://doi.org/10.1090/surv/266.

[Kes59a] Harry Kesten. "Full Banach Mean Values on Countable Groups". In: *Mathematica Scandinavica* 7.1 (1959), pp. 146–156. ISSN: 00255521. URL: http://www.jstor.org/stable/24489015 (visited on 02/05/2025).

[Kes59b] Harry Kesten. "Symmetric Random Walks on Groups". In: *Transactions of the American Mathematical Society* 92.2 (1959), pp. 336–354. ISSN: 00029947. URL: http://www.jstor.org/stable/1993160 (visited on 02/05/2025).

[Knu09] Søren Knudby. "The Banach-Tarski Paradox". 2009.

References VII

[Löh17] Clara Löh. *Geometric group theory*. Universitext. An introduction. Springer, Cham, 2017, pp. xi+389. ISBN: 978-3-319-72253-5. DOI: 10.1007/978-3-319-72254-2. URL: https://doi.org/10.1007/978-3-319-72254-2.

[Mon17] Mehdi Sangani Monfared. "Følner's condition and expansion of Cayley graphs for group actions". In: *New York Journal of Mathematics* 23 (Sept. 2017), pp. 1295–1306.

[Mon13] Nicolas Monod. "Groups of piecewise projective homeomorphisms". In: *Proc. Natl. Acad. Sci. USA* 110.12 (2013), pp. 4524–4527. ISSN: 0027-8424,1091-6490. DOI: 10.1073/pnas.1218426110. URL: https://doi.org/10.1073/pnas.1218426110.

References VIII

[Pau02] Vern Paulsen. *Completely bounded maps and operator algebras*. Vol. 78. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+300. ISBN: 0-521-81669-6.

[Rai23] Timothy Rainone. "Functional Analysis-En Route to Operator Algebras". 2023.

[Rud73] Walter Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, pp. xiii+397.

[Run02] Volker Runde. *Lectures on amenability*. Vol. 1774. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. xiv+296. ISBN: 3-540-42852-6. DOI: 10.1007/b82937. URL: https://doi.org/10.1007/b82937.

References IX

[Run05] Volker Runde. *A taste of topology*. Universitext. Springer, New York, 2005, pp. x+176. ISBN: 978-0387-25790-7.

[Run20] Volker Runde. *Amenable Banach algebras*. Springer Monographs in Mathematics. A panorama. Springer-Verlag, New York, 2020, pp. xvii+462. ISBN: 978-1-0716-0351-2. DOI: 10.1007/978-1-0716-0351-2. URL: https://doi.org/10.1007/978-1-0716-0351-2.

[Tak64] Masamichi Takesaki. “On the cross-norm of the direct product of C^*-algebras”. In: *Tohoku Math. J. (2)* 16 (1964), pp. 111–122. ISSN: 0040-8735,2186-585X. DOI: 10.2748/tmj/1178243737. URL: https://doi.org/10.2748/tmj/1178243737.

References X

[Tao09] Terence Tao. *245B, notes 2: Amenability, the ping-pong lemma, and the Banach-Tarski paradox (optional).* https://terrytao.wordpress.com/2009/01/08/245b-notes-2-amenability-the-ping-pong-lemma-and-the-banach-tarski-paradox-optional/. 2009.

[Tit72] J Tits. "Free subgroups in linear groups". In: *Journal of Algebra* 20.2 (1972), pp. 250–270. ISSN: 0021-8693. DOI: https://doi.org/10.1016/0021-8693(72)90058-0. URL: https://www.sciencedirect.com/science/article/pii/0021869372900580.

References XI

[Wei80] Joachim Weidmann. *Linear operators in Hilbert spaces*. Vol. 68. Graduate Texts in Mathematics. Translated from the German by Joseph Szücs. Springer-Verlag, New York-Berlin, 1980, pp. xiii+402. ISBN: 0-387-90427-1.