Problem 1

Let $X = \{0,1\}^n$. Show that the Hamming distance:

$$\begin{aligned} d_H &: X \times X \to [0,\infty) \\ d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) &= \left|\{j \mid x_j \neq y_j\}\right| \end{aligned}$$

defines a metric on X.

Proof:

- Symmetry:

$$\begin{aligned} d_H\left((x_j)_{j=1}^n, (y_j)_{j=1}^n\right) &= \left|\{j \mid x_j \neq y_j\}\right| \\ &= \left|\{j \mid y_j \neq x_j\}\right| \\ &= d_H\left((y_j)_{j=1}^n, (x_j)_{j=1}^n\right) \end{aligned}$$

- Definiteness: it is only the case that $d_H(x_j, y_j) = 0$ if $x_j = y_j$ for all j, by the definition of the distance.
- Similarly, since $x_j = x_j$ for all j, $d_H(x_j, x_j) = 0$.
- Let $(x_j)_j$, $(y_j)_j$, and $(z_j)_j$ be sequences of bits. The set $\{j \mid x_j \neq z_j\}$ is formed by taking all the values $\{j \mid x_j \neq y_j\}$ along with $\{j \mid y_j \neq z_j\}$, net of particular indices where $x_j = z_j$, but $x_j \neq y_j$. Therefore,

$$d(x,z) \leq d(x,y) + d(y,z).$$

Problem 2

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on a vector space V, show that the induced metrics d and d' are equivalent.

Proof: Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms. Then, $\exists c_1, c_2 \in \mathbb{R}$ such that $\|v-w\|' \leq c_1 \|v-w\|$ and $\|v-w\| \leq c_2 \|v-w\|'$. However, this is the exact same statement as $d(v,w) \leq c_1 d'(v,w)$ and $d'(v,w) \leq c_2 d'(v,w)$. Thus, d and d' are equivalent metrics.

Problem 3

Let $\{X_k, d_k\}$ be a sequence of metric spaces with uniformly bounded metrics. Let

$$X := \prod_{k\geq 1} X_k$$

denote the product.

(a) Show that

$$\begin{aligned} D &: X \times X \to [0,\infty) \\ D(x,y) &:= \sum_{k\geq 1} 2^{-k} d_k(x_k, y_k) \end{aligned}$$

defines a metric on X.

(b) Consider the case where $\{X_k\} = \{0,2\}$ and $d_k(a,b) = |a-b|$ for every $k \geq 1$. We get the abstract Cantor set

$$\Delta := \prod_{k\geq 1}\{0,2\};$$

$$D(x,y) := \sum_{k=1}^{\infty} 3^{-k}|x_k - y_k|.$$

Prove that $D(x,z) = D(y,z)$ implies $x = y$.

Proof:

(i) Let D be defined as above. Then, $D((x_k)_k, (x_k)_k)$ is a sum of $d_k(x_k, x_k)$, all uniformly zero, meaning $D((x_k)_k, (x_k)_k) = 0$.

Similarly, $D((x_k)_k, (y_k)_k) = 0$ implies that $d_k(x_k, y_k) = 0$ for all x_k, y_k. Since d_k is a metric, this means $x_k = y_k$ for all k, implying that $(x_k)_k = (y_k)_k$.

Additionally, $d_k(x_k, y_k) = d_k(y_k, x_k)$, it is the case that $D((x_k)_k, (y_k)_k) = D((y_k)_k, (x_k)_k)$.

Finally, we must show the triangle inequality:

$$\begin{aligned} D((x_k)_k, (z_k)_k) &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, z_k) \\ &\leq \sum_{k=1}^{\infty} 2^{-k} (d_k(x_k, y_k) + d(y_k, z_k)) \\ &= \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, y_k) + \sum_{k=1}^{\infty} d(y_k, z_k) \\ &= D((x_k)_k, (y_k)_k) + D((y_k)_k, (z_k)_k). \end{aligned}$$

(ii) Suppose $x \neq y$. Let ℓ denote the smallest index where $x_\ell \neq y_\ell$. Suppose without loss of generality that $x_\ell = 2$ and $y_\ell = 0$. Then, $||x_\ell - z_\ell| - |y_\ell - z_\ell|| = 2 \cdot 3^{-\ell}$. Additionally,

$$\begin{aligned} 0 &\leq \sum_{k=\ell+1}^{\infty} 3^{-k} |x_k - z_k| \\ &\leq \sum_{k=\ell+1}^{\infty} 3^{-k}(2) \\ &= \frac{2}{3^{\ell+1}} \\ &< \frac{2}{3^{\ell}}. \end{aligned}$$

Thus, $D(x, z) \neq D(y, z)$.

Problem 4

Let $(V, \|\cdot\|)$ be a normed space, and suppose $E \subseteq V$. Show that the following are equivalent:

(1) E is bounded — $\operatorname{diam}(E) < \infty$;

(2) $\sup_{v \in E} \|v\| < \infty$;

(3) there is an $r > 0$ such that $E \subseteq B(0, r)$.

Proof: (i) $\Rightarrow$ (ii): Let E be bounded. Then,

$$\begin{aligned} |\|v\| - \|w\|| &\leq \|v - w\| \\ \sup_{v,w \in E} |\|v\| - \|w\|| &\leq \sup_{v,w \in E} \|v - w\| \\ \sup_{v \in E} \|v\| - \inf_{w \in E} \|w\| &\leq c \\ \sup_{v \in E} \|v\| &\leq c + \inf_{w \in E} \|w\|. \end{aligned}$$

(ii) $\Rightarrow$ (iii): Since, for $v \in E$, $\sup \|v\| < \infty$, if we set $r = \sup \|v\| + 1$, then $v \in B(0, r)$, meaning $E \subseteq B(0, r)$.

(iii) $\Rightarrow$ (i): Let E be such that $E \subseteq B(0, r)$ for some r. Then, $\forall v, w \in B(0, r)$, $\|v - w\| \leq 2r$, meaning that $\forall v, w \in E, \|v - w\| \leq 2r$, meaning $\operatorname{diam}(E) < \infty$.

Problem 5

Let (X, d) be a metric space and suppose $A \subseteq X$. Show:

(i) $\overline{A^c} = (A^\circ)^c$

(ii) $(\overline{A})^c = (A^c)^\circ$

Proof:

(i) We have previously established that $\overline{A^c} \subseteq (A^\circ)^c$. Let $x \in (A^\circ)^c$. Then, $x \notin A^\circ$, meaning $\forall \delta > 0$, $U(x,\delta) \cap A^c \neq \emptyset$. Thus, $x \in \overline{A^c}$.

(ii) Let $x \in \overline{A}^c$. Then, $x \notin \overline{A}$, meaning $\exists \delta > 0$ such that $U(x,\delta) \cap A = \emptyset$. Thus, $U(x,\delta) \subseteq A^c$, meaning $x \in (A^c)^\circ$.

Let $x \in (A^c)^\circ$. Then, $\exists \delta > 0$ such that $U(x,\delta) \subseteq A^c$. Therefore, $U(x,\delta) \cap A = \emptyset$, meaning $x \notin \overline{A}$, so $x \in \overline{A}^c$.

Problem 6

In any metric space, show that open balls are open, closed balls are closed, and spheres are closed. Moreover, in a normed space, show that $\partial U(v,r) = \partial B(v,r) = S(v,r)$.

Proof:

(i) Let $\delta > 0$, and $A = U(x,\delta)$ for some $x \in X$. Then, for any $y \in A$, set $\varepsilon = \min\{d(x,y), \delta - d(x,y)\}$. Then, $U(y,\varepsilon) \subseteq A$.

(ii) Let $M = B(x,\delta)$. Let $y \in M^c$. Set $\varepsilon = d(x,y) - \delta$. Then, $U(y,\varepsilon) \subseteq M^c$, meaning M^c is open, and M is thus closed.

(iii) Let $A = S(x,\delta)$ for some $\delta > 0$. Then, $A^c = U(x,\delta) \cup (B(x,\delta))^c$, meaning A^c is a union of open sets, which is open. Thus, A is closed.

(iv) We have previously established that, in a normed space, $\overline{U(v,r)} = B(v,r)$. Therefore,

$$\begin{aligned}
\partial U(v,r) &= \overline{U(v,r)} \setminus U(v,r)^\circ \\
&= \{x \mid d(x,v) \leq r\} \setminus \{x \mid d(x,v) < r\} \\
&= \{x \mid d(x,v) = r\} \\
&= S(v,r).
\end{aligned}$$

Similarly, in a normed vector space, $B(v,r)^\circ = U(v,r)$. Therefore,

$$\begin{aligned}
\partial B(v,r) &= \overline{B(v,r)} \setminus B(v,r)^\circ \\
&= \{x \mid d(x,v) \leq r\} \setminus \{x \mid d(x,v) < r\} \\
&= \{x \mid d(x,v) = r\} \\
&= S(v,r).
\end{aligned}$$

Problem 7

Let (X,d) be a metric space, and suppose $A \subseteq X$. Show that the following are equivalent:

(i) A is dense in X;

(ii) For all $U \in \tau_X$, $U \cap A \neq \emptyset$;

(iii) For all $x \in X$ and for all $\varepsilon > 0$, $U(x,\varepsilon) \cap A \neq \emptyset$;

(iv) For all $x \in X$ and for all $\varepsilon > 0$, there is an $a \in A$ with $d(x,a) < \varepsilon$.

Proof:

(i) $\Leftrightarrow$ (iii): $\overline{A} = X \Leftrightarrow \forall x \in \overline{A}, \forall \varepsilon > 0, U(x,\varepsilon) \cap A \neq \emptyset \Leftrightarrow \forall x \in X, \forall \varepsilon > 0, U(x,\varepsilon) \cap A \neq \emptyset$.

(iii) $\Leftrightarrow$ (iv): $\forall x \in X, \forall \varepsilon > 0, U(x,\varepsilon) \cap A \neq \emptyset \Leftrightarrow \forall x \in X, \exists a \in A \ni a \in U(x,\varepsilon) \Leftrightarrow \forall x \in x, \exists a \in a \ni d(x,a) < \varepsilon$.

(iii) $\Rightarrow$ (ii): Suppose $\forall x \in x, U(x,\varepsilon) \cap A \neq \emptyset$. Then, since $U(x,\varepsilon) \subseteq U$ for some $U \in \tau_X$, it is the case that $U \cap A \neq \emptyset$.

(ii) $\Rightarrow$ (iii): Since $U(x,\varepsilon) \in \tau_X$, it is the case that for any $x \in X$ and any $\varepsilon > 0$, $U(x,\varepsilon) \cap A \neq \emptyset$.

Problem 8

Let $U \subseteq \mathbb{R}$ be an open set. For each $x \in U$, we put

$$I_x := \bigcup \{(a,b) \mid -\infty \leq a < b \leq \infty, x \in (a,b) \subseteq U\}.$$

(i) Show that I_x is an open interval contained in U with $x \in I_x$.

(ii) Given $x, y \in U$, show that $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

(iii) Prove that $U = \bigsqcup_{j \in J} I_j$, where J is a countable set and I_j are open intervals.

Proof of (i): By construction, $x \in I_x$, as $x \in (a, b)$ for all (a, b) that satisfy inclusion into I_x. Additionally, since $x \in U$, $\exists \delta_x$ such that $(x - \delta_x, x + \delta_x) \subseteq U$. Since I_x is the union of all such intervals, it must be the case that $(x - \delta_x, x + \delta_x) \subseteq I_x$, meaning I_x is an interval by the characterization of intervals.

Proof of (ii): Suppose that $\exists a_1, b_1$ such that $x \in (a_1, b_1)$ and $y \in (a_1, b_1)$. Then, it must be the case that $(a_1, b_1) \subseteq I_x$ and $(a_1, b_1) \subseteq I_y$. Thus, $I_x \subseteq I_y$ and $I_y \subseteq I_x$, meaning $I_x = I_y$.

If $\nexists (a, b)$ that satisfy the condition, then it must be the case that $\exists \delta_1$ and δ_2 such that $(x - \delta_1, x + \delta_1) \subseteq U$ and $(y - \delta_2, y + \delta_2) \subseteq U$, but $(x - \delta_1, x + \delta_1) \cap (y - \delta_2, y + \delta_2) = \emptyset$. Therefore, $I_y \cap I_x = \emptyset$.

Problem 9

Show that c_0 with $\|\cdot\|_u$ is separable.

Proof: Let $z \in c_0$. Set $\varepsilon_1 > 0$, then finding N_1 large such that for all $n > N_1$, $z_n < \varepsilon_1$. Set $z' \in c_{00}$ to be equal to z on $1, \dots, N_1$ and equal to 0 for all $n > N_1$.

Recall that for

$$E_n = \left\{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{Q} \right\},$$
$$E = \bigcup E_n,$$

E is dense in c_{00}, meaning that there exists some $w \in c_{00}$ such that $\|z' - w\| < \varepsilon$ for any $\varepsilon > 0$. However, since $z' = z$ for all n from $1, \dots, N_1$, and the index of $\|z\|_u$ is contained in $1, \dots, N_1$, this means $\|z - w\| < \varepsilon$, meaning E is dense in c_0.

Since E is countable, this means c_0 is countable.

Problem 10

Let $\mathcal{C}$ denote the Cantor set. Show that $\mathcal{C}$ is nowhere dense.

Proof: We know that $\mathcal{C}$ is closed, meaning all we need show is that $\mathcal{C}^0 = \emptyset$.

Suppose toward contradiction that $\mathcal{C}^0$ is not empty. Then, $\exists x \in \mathcal{C}$ and $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{C}$.

Find m so large such that $3^{-m} < \varepsilon$. Then, $(x - \varepsilon, x + \varepsilon)$ must be contained in a subinterval with length $\frac{1}{3^m}$. However, $2\varepsilon > \frac{1}{3^m}$, and every subinterval in the element $\mathcal{C}_m$ has length $\frac{1}{3^m}$.