Basic Properties

Definition: A topological space M is called a *manifold* if it satisfies the following:

- M is Hausdorff (points can be separated by open sets);
- M is second countable (the basis for the topology of M is countable);
- M is locally Euclidean (every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$ for some n).

In particular, the third condition says that for every $p \in M$, there is $U \in \mathcal{O}_p$ and a homeomorphism $\varphi\colon U \to \mathbb{R}^n$. The value of n is called the *dimension* of the manifold M.

Definition: Let M be an n-manifold. A *chart* on M is a pair (U, φ) such that $U \subseteq M$ is open, $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism.

A family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ is known as an *atlas* if

$$M = \bigcup_{i\in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point $p \in M$ and two charts (U, φ_U) and (V, φ_V) such that $p \in U$ and $p \in V$. The functions $\varphi_U\colon U \to \mathbb{R}^n$ and $\varphi_V\colon V \to \mathbb{R}^n$ are homeomorphism, meaning that $\varphi_V \circ \varphi_U^{-1}\colon \varphi_U(U \cap V)^n \to \mathbb{R}^n$ defined on the (nonempty) $U \cap V$ is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs $\varphi_V \circ \varphi_U^{-1}$ are *diffeomorphisms*. To do this, we need to first develop the derivative in $\mathbb{R}^n$.

Definition: Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say f is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear map $L \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\frac{\|f(p+h) - f(p) - Lh\|}{\|h\|} \to 0$$

as $h \to 0$.

The *derivative* of f is the association $f \mapsto L$ for each $p \in \mathbb{R}^n$. We write $D_p f$ to denote this map. Note that we consider elements of $\operatorname{Mat}_n(\mathbb{R})$ as points in $\mathbb{R}^{n^2}$ with the standard topology on $\mathbb{R}^{n^2}$.

A function f is called a *diffeomorphism* if it is continuously differentiable and has a continuously differentiable inverse.

Definition: If (U, φ_U) and (V, φ_V) are charts such that $U \cap V \neq \emptyset$, the function $\varphi_V \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is known as the *transition map* between φ_U and φ_V.

A *smooth structure* for M is an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that for all $i, j \in I$, the transition maps $\varphi_j \circ \varphi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$ are diffeomorphisms where defined (if not defined, then it is vacuously so). If M admits a smooth structure, then we call M a smooth manifold.

Note: From now on, we use "manifold" to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

Definition: A map $f\colon M \to N$ between manifolds is called *smooth* if for any chart (U, φ_U) in M and corresponding chart (V, φ_V) in N, the map $\varphi_V \circ f \circ \varphi_U^{-1}\colon \mathbb{R}^n \to \mathbb{R}^k$ is continuously differentiable.

The function f is a *diffeomorphism* if f is a smooth bijection with smooth inverse, and we say the manifolds M and N are diffeomorphic if they admit a diffeomorphism.

In order to replace manifolds with linear maps, we need to understand smooth maps on $\mathbb{R}^n$. The most important theorems in this regard are the inverse function theorem and the implicit function theorem.

Theorem (Inverse Function Theorem)**:** Let $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. If $D_p f$ is invertible as a linear map, then f has a local, continuously differentiable inverse $f^{-1}\colon V \to W$, where $p \in W \subseteq U$ and $f(p) \in V \subseteq \mathbb{R}^n$.

The proof uses the contraction mapping theorem. Recall that if X is a complete metric space, and $f\colon X \to X$ is a strict uniform contraction — that is, there exists $0 \leqslant \lambda < 1$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for all $x, y \in X$ — then f has a unique fixed point.

We begin with a technical lemma.

Lemma: If $U(0, r) \subseteq V$ for some $r > 0$ where V is a normed vector space, $g\colon V \to V$ is a uniform contraction, and $f = \mathrm{id} + g$, then the following hold:

- $(1-\lambda)\|x-y\| \leqslant \|f(x) - f(y)\|$ (in particular, f is injective);
- if $g(0) = 0$, then
$$U(0, (1-\lambda)r) \subseteq f(U(0, r)) \subseteq U(0, (1+\lambda)r).$$

Proof of Lemma. To see the first item, we notice that by the triangle inequality,

$$\begin{aligned}\|x-y\| - \|f(x) - f(y)\| &\leqslant \|x-y\| - \|x-y\| + \|g(x) - g(y)\| \\ &\leqslant \lambda\|x-y\|,\end{aligned}$$

so $(1-\lambda)\|x-y\| \leqslant \|f(x) - f(y)\|$, and f is injective. Furthermore, we see that if $g(0) = 0$, then

$$\begin{aligned}f(U(0, r)) &= U(0, r) + g(U(0, r)) \\ &\subseteq U(0, r) + \lambda U(0, r) \\ &= U(0, (1+\lambda)r).\end{aligned}$$

Finally, if $y \in U(0, (1-\lambda)r)$, then we want to find x such that $y = f(x) = x + g(x)$; equivalently, we see that we want x such that $x = y - g(x)$. Since the function $F(x) = y - g(x)$ is a translation of a uniform contraction, $F(x)$ is a contraction, so there is a fixed point, meaning $y \in f(U(0, r))$. □

Note: We will use $|\cdot|$ to denote the norm on $\mathbb{R}^n$.

Proof of the Inverse Function Theorem. By using a series of affine maps — first by translating p to 0, then translating $f(p)$ to 0, then inverting $D_0 f$ as per our assumption, we may safely assume that $p = f(p) = 0$ and $D_0 f = \mathrm{Id}$.

Set $g = f - \mathrm{Id}$. We will show that g is a contraction in a sufficiently small ball. Fixing $x, y \in \mathbb{R}^n$, consider the map $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto g(x + t(y-x))$. Notice that by the Fundamental Theorem of Calculus,

$$|g(y) - g(x)| \leqslant |y - x| \sup_{0 \leqslant t \leqslant 1} |g'(x + t(y-x))|.$$

Furthermore, since $g'(0) = \mathbf{0}$ by the fact that $D_0 f = \mathrm{Id}$ and $(\mathrm{Id})' = \mathrm{Id}$, and since f is continuously differentiable, there is $r > 0$ such that

$$|g(y) - g(x)| \leqslant \frac{1}{2}|y - x|$$

for all $x, y \in U(0, r)$. Thus, g is a strict contraction on $U(0, r)$. By the previous lemma, we see that

$$U(0, r/2) \subseteq f(U(0, r));$$

by setting $U = U(0, r) \cap f^{-1}(U(0, r))$, we see that the map $f|_U\colon U \to V := U(0, r/2)$ is a bijection. The inverse function $f^{-1}\colon V \to U$ thus exists.

Now, we let $h = f^{-1}$, $x \in U$, $y \in V$ such that $h(x) = y$, and $A = D_x f$. We will show that $A^{-1} = D_y h$, which is enough to show that h is continuously differentiable, as we assume the map $x \mapsto D_x f$ is continuous, and inversion is continuous in $GL_n(\mathbb{R})$.

For sufficiently small vectors s and k, since f and h are bijections, we have

$$h(y+k) = x + s,$$

so

$$f(x+s) = y + k.$$

Furthermore, by unraveling the definitions of $f = g + \mathrm{Id}$, s, and k, and the fact that g is a uniform contraction on U, we get

$$\begin{aligned}|s-k| &= |(f(x+s) - f(x)) - s| \\ &= |(x+s+g(x+s)) - (x+g(x)) - s| \\ &= |g(x+s) - g(x)| \\ &\leqslant \frac{|s|}{2}.\end{aligned}$$

In particular, since

$$\begin{aligned}|s| &\leqslant |s-k| + |k| \\ &\leqslant |k| + \frac{|s|}{2},\end{aligned}$$

we see that $|s|/2 \leqslant |k|$. We calculate

$$\begin{aligned}\left|h(y+k) - h(y) - A^{-1}k\right| &= \left|x + s - x - A^{-1}(f(x+s) - f(x))\right| \\ &= \left|s - A^{-1}(f(x+s) - f(x))\right| \\ &\leqslant \left\|A^{-1}\right\|_{op} |As - f(x+s) - f(x)|.\end{aligned}$$

Thus, since $|s|/2 \leqslant |k|$,

$$\begin{aligned}\frac{\left|h(y+k) - h(y) - A^{-1}k\right|}{|k|} &\leqslant \frac{2\left\|A^{-1}\right\|_{op}|As - f(x+s) - f(x)|}{|s|} \\ &\to 0,\end{aligned}$$

so $D_y h = A^{-1}$. $\square$