Complex Analysis

Analyticity and Path-Independence in the Complex Plane

Baby's First Complex Function Theory

We are interested in functions of the form $f(z)$, where $z = x + iy$ is some complex number. Note that this is specifically different from a function $g\colon \mathbb{R}^2 \to \Omega$ for some domain Ω; in the latter case, we have independent variables x and y, while in the former case, we must express $z = x + iy$.

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z)\, dz &= \oint_C w(z)\,(dx + i dy)\\ &= \oint_C w(z)\, dx + i \oint_C w(z)\, dy.\end{aligned}$$

Taking $A_x = w(z)$ and $A_y = iw(z)$, we have

$$= \oint_C \mathbf{A} \cdot\, d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla \times \mathbf{A})\, d\mathbf{a},$$

and when this integral is zero. Note that $(\nabla \times \mathbf{A}) \cdot \hat{n} = 0$, so $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$.

Note that we can take

$$w(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$.

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

Theorem (Cauchy–Riemann Equations)**:**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that w is analytic,[1] which leads to Cauchy's theorem.

Theorem (Cauchy's Theorem)**:** If C is a simple closed curve in a simply connected region, then w is analytic if and only if

$$\oint_C w(z)\, dz = 0. \tag{$\dagger$}$$

Fact. The function $w(z)$ is analytic inside the simply connected region R if any of these hold:

- w satisfies the Cauchy–Riemann equations;

[1] Equal to its Taylor series, also holomorphic.

- $w'(z)$ is unique and exists;
- $\frac{\partial w}{\partial \overline{z}} = 0$.
- w can be expanded in a Taylor series convergent on some open neighborhood of z: $w(z) = \sum_{n\geqslant 0} c_n(z-a)^n$;[II]
- $w(z)$ is path-independent everywhere in R: $\oint_C w(z)\,dz = 0$.

Example. Considering $w(z) = z$, we have $u = x$ and $v = y$, so it satisfies the Cauchy–Riemann equations. However, neither $\operatorname{Re}(z)$ nor $\operatorname{Im}(z)$ are analytic, and neither is $\overline{z} = x - iy$.

Remark: Whenever we say "analytic at p," we mean "analytic in a neighborhood of p."

Note that since $\mathbb{C}$ is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of $\mathbb{C}$, by adjoining a point $\{\infty\}$, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. This compactified $\mathbb{C}^*$ is often represented as a unit sphere with the north pole, determined by $(0,0,1)$, is the point at infinity. The correspondence between $\mathbb{C}^* \setminus \{\infty\}$ and $\mathbb{C}$ is evaluated via stereographic projection.

We define $\frac{z}{\infty} = 0$ and $\frac{z}{0} = \infty$ for any $z \neq 0, \infty$. The correspondence between $z = x + iy$ in the plane to Z on the Riemann sphere with $\mathbb{R}^3$ coordinates (ξ_1, ξ_2, ξ_3) is

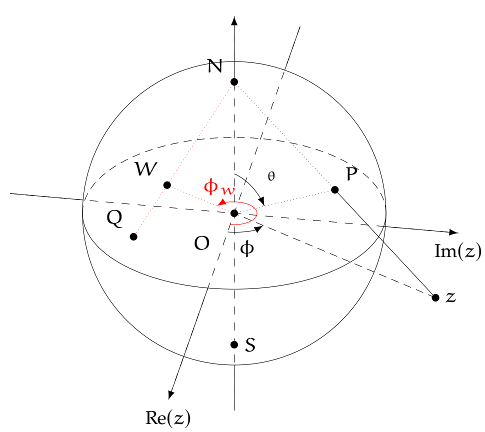
$$\begin{aligned} \xi_1 &= \frac{2\operatorname{Re}(z)}{|z|^2+1} \\ \xi_2 &= \frac{2\operatorname{Im}(z)}{|z|^2+1} \\ \xi_3 &= \frac{|z|^2-1}{|z|^2+1}. \end{aligned}$$

Inverting, we may find

$$\begin{aligned} x &= \frac{\xi_1}{1-\xi_3} \\ y &= \frac{\xi_2}{1-\xi_3}, \end{aligned}$$

and with polar coordinates,

$$z = \cot(\theta/2)e^{i\phi}.$$



To determine analyticity at ∞, we set $\zeta = \frac{1}{z}$, and analyze the analyticity of $\tilde{w}(\zeta) = w(1/z)$ at 0.

[II]This is the real definition of analytic.

Cauchy's Integral Formula

Consider the function $w(z) = c/z$, integrated around a circle of radius R. Then, writing $z = Re^{i\varphi}$, we get

$$\begin{aligned}\oint_{\Gamma} w(z)\,dz &= C\int_{0}^{2\pi}\frac{e^{-i\varphi}}{R}\underbrace{iRe^{i\varphi}\,d\varphi}_{dz}\\ &= ic\int_{0}^{2\pi}\,d\varphi\\ &= 2\pi ic.\end{aligned}$$

If our contour C runs around our singularity at $z = 0$ a total of n times, then we pick up a factor of n.

Now, when we consider

$$I = \oint_{C}\frac{dz}{z^n},$$

this integral actually yields 0 for any $n \neq 1$, despite the fact that 0 is a singularity for $f(z) = \frac{1}{z^n}$. This 0 is not a reflection of (†), but of the fact that

$$z^{-n} = \frac{d}{dz}\left(\frac{z^{-n+1}}{n+1}\right),$$

meaning that z^{-n} is an exact differential, so integrating along a closed curve yields zero change. However, $\frac{1}{z} = \frac{d}{dz}(\ln z)$ may be an exact differential, but for complex z, $\ln z = \ln|z| + i\arg(z) = \ln r + i\varphi$. This yields

$$\begin{aligned}\oint_{C}\frac{c}{z}dz &= c\oint_{C}d(\ln z)\\ &= c(i(\varphi + 2\pi) - \varphi)\\ &= 2\pi ic.\end{aligned}$$

Ultimately, what this shows is that when we integrate any analytic function $f(\zeta)$ along a closed contour with a singularity at z, only the coefficient on $\frac{1}{\zeta - z}$ will remain. This coefficient is known as the residue at 0.

Theorem (Cauchy's Integral Formula)**:** If w is analytic in a simply connected region and C is a closed contour winding once around a point z in the region, then

$$w(z) = \frac{1}{2\pi i}\oint_{C}\frac{w(\zeta)}{\zeta - z}\,d\zeta.$$

Furthermore, this shows that any once-differentiable function is infinitely differentiable, as by differentiating under the integral sign, we get

$$\frac{d^n w}{dz^n} = \frac{n!}{2\pi i}\oint_{C}\frac{w(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Example (Deriving Liouville's Theorem)**.** Consider a circle C centered at radius r centered at at z, $\zeta - z = Re^{i\varphi}$. We take $d\zeta = iRe^{i\varphi}\,d\varphi$, and taking derivatives, we have

$$w'(z) = \frac{1}{2\pi R}\int_{0}^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi.$$

If w is bounded — i.e., $|w(z)| \leqslant M$ for all z in a given region — then

$$|w'(z)| = \left|\frac{1}{2\pi R}\int_{0}^{2\pi} w\left(z + Re^{i\varphi}\right)e^{-i\varphi}\,d\varphi\right|$$

$$\begin{aligned}
&\leqslant \frac{1}{2\pi R}\int_0^{2\pi}\left|w\left(z + Re^{i\varphi}\right)\right|\,d\varphi\\
&\leqslant \frac{M}{R}
\end{aligned}$$

for all R within the analytic region.

In the case where w is entire (i.e., analytic on $\mathbb{C}$), then this inequality holds for all $R \to \infty$. Thus, $|w'(z)| = 0$ for all z, meaning that w is constant.

This is known as Liouville's theorem — every bounded entire function is constant. This can be used to prove the fundamental theorem of algebra.

What Liouville's theorem tells us is that any nontrivial behavior will emerge from a function's singularities.

Singularities and Branches

To understand nontrivial behavior on the complex plane, we need to understand singularities. This will require us to develop understanding of Laurent series.

Taylor Series

We want to integrate $w(z)$ around some point a in an analytic region of $w(z)$. This yields the form

$$\begin{aligned}
w(z) &= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{\zeta - z}\,d\zeta\\
&= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - a) - (z - a)}\,d\zeta\\
&= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - a)\left(1 - \frac{z-a}{\zeta-a}\right)}\,d\zeta. \qquad (\ddagger)
\end{aligned}$$

Since ζ is on the contour and z is in the contour, $\left|\frac{z-a}{\zeta-a}\right| < 1$, we may expand as a geometric series. Thus, we get

$$= \frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - a)}\left(\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n\right)d\zeta.$$

Since the series is uniformly convergent, we are allowed to exchange sum and integral, yielding

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_C \frac{w(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta\right)}_{=c_n}(z-a)^n\\
&= \sum_{n=0}^{\infty} c_n(z-a)^n,
\end{aligned}$$

where

$$c_n = \frac{1}{n!}\left.\frac{d^n w}{dz^n}\right|_{z=a}.$$

If our Taylor series reduces to a known series on the real axis, we find this very desirable. We say this is a type of analytic continuation from the real axis to the complex plane. For example,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is an analytic continuation of e^x.

However, more interestingly,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for all $s > 1$. However, we have also shown that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

converges for complex s for all real part greater than 1. Since values of this integral agree with the series representation of $\zeta(s)$ on real axis, we have that this is an analytic continuation of $\zeta(s)$ to the subset of $\mathbb{C}$ defined by $\mathrm{Re}(s) > 1$.

Laurent Series

Now, what happens if, at (‡), we have $\left|\frac{z-a}{\zeta-a}\right| > 1$. The series as constructed would not converge, but what if we have a series that converges everywhere *outside* C? This would entail an expansion in reciprocal integer powers of $z - a$. This yields

$$\begin{aligned}
w(z) &= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{(z-a)\left(1 - \frac{\zeta-a}{z-a}\right)}\, d\zeta \\
&= -\frac{1}{2\pi i} \oint_C \frac{w(\zeta)}{z-a} \left(\sum_{n=0}^{\infty} \left(\frac{\zeta - a}{z-a}\right)^n\right) d\zeta \\
&= -\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C w(\zeta - a)^n\, d\zeta\right) \frac{1}{(z-a)^{n+1}} \\
&= \sum_{n=1}^{\infty} \underbrace{\left(-\frac{1}{2\pi i} \oint_C w(\zeta - a)^{n-1}\, d\zeta\right)}_{=c_{-n}} \frac{1}{(z-a)^n} \\
&= \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}
\end{aligned}$$

Note that this series has a singularity at $z = a$, but since our series is only defined outside a particular region, that doesn't matter. We call a series in reciprocal powers a Laurent series.