Amenability: A (Somewhat) Brief Introduction

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March 20, 2025

Outline

Contents

Groups

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then we call the pair $(A, \star)$ a *group*.

We (usually) abbreviate $a \star b$ as ab. If $ab = ba$, then we say the group is *abelian*.

Subgroups, Quotient Groups

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* G/N.
- The *index* of a subgroup $H \leq G$ is the number of cosets, $gH := \{gh \mid h \in H\}$, written $[G : H]$.

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- The group $\mathrm{SO}(n)$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under matrix multiplication.
- The group $\mathrm{E}(3)$ consists of all translations, rotations, and flips in $\mathbb{R}^3$, and is also known as the *isometry group* of $\mathbb{R}^3$.

Group Actions

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Every group is equipped with a family of canonical actions, $\sigma_a\colon G \to G$ for each $a \in G$, given by $x \mapsto ax$, known as *left-multiplication*.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathcal{A}$;
2. for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
3. for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say $\mathcal{A}$ is a *σ-algebra* of subsets.

The most important σ-algebra, and the one we will be dealing with throughout this talk, is $P(G)$, where G is a group.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathcal{A}$ is a σ-algebra, then a map $\mu\colon \mathcal{A} \to [0,\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

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If $\{A_n\}_{n\geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

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we say μ is a measure. If $\mu(X) = 1$, then we say μ is a probability measure.

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- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We define $F(a,b)$ to be the set of all "words" in the alphabet $\left\{a,b,a^{-1},b^{-1}\right\}$, subject to the condition that, for $w,w' \in F(a,b)$,

$$waa^{-1}w' \sim wa^{-1}aw' \sim ww'$$
$$wbb^{-1}w' \sim wb^{-1}bw' \sim ww'.$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a,b)$.

A Curiosity

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- all words that start with a;
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- all words that start with b — think words that start with b^2 before you multiply b^{-1}.

Thus, all we need to do is add back $W\left(b^{-1}\right)$ to get $F(a,b)$ back.

$$F(a,b) = W\left(b^{-1}\right) \cup b^{-1}W(b).$$

A Curiosity, Cont'd

Similarly, we can do this for a, giving a decomposition of $F(a,b)$ in two separate ways:

$$\begin{aligned} F(a,b) &= b^{-1}W(b) \cup W\left(b^{-1}\right) \\ &= a^{-1}W(a) \cup W\left(a^{-1}\right). \end{aligned}$$

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Furthermore, note that $W(a), W(b), W\left(a^{-1}\right), W\left(b^{-1}\right)$ are disjoint.

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We're able to take part of the group $F(a,b)$, take some translations, and, miraculously, obtain the entire group back.

Paradoxical Decompositions of Groups

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$;

such that

$$G = \bigcup_{i=1}^{n} g_i A_i$$
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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is G-*paradoxical* if there exist

- pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq A$; and
- elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$

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Paradoxical Decompositions of Sets

If G acts on a set X, then a subset $A \subseteq X$ is *G-paradoxical* if there exist

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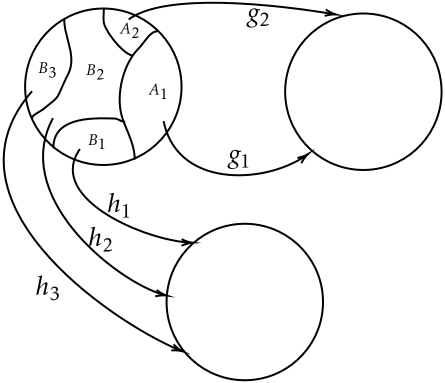
$$\begin{aligned} A &= \bigcup_{i=1}^{n} g_i \cdot A_i \\ &= \bigcup_{j=1}^{m} h_j \cdot B_j. \end{aligned}$$

A paradoxical group is a paradoxical set under the action of left-multiplication.

Depiction



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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of SO(3)

The following two matrices (and their inverses) generate a subgroup of SO(3) that is isomorphic to $F(a,b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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Thus, SO(3) is paradoxical — can we use it to find a paradoxical decomposition?

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B.

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Theorem (The Banach–Tarski Paradox)

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- In other words, not all subsets of $\mathbb{R}^3$ have a definite “volume” invariant under isometry.

Equidecomposability

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- finite partitions, $A_1, \ldots, A_n \subseteq A$, $B_1, \ldots, B_n \subseteq B$
- group elements $g_1, \ldots, g_n \in G$

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If A is G-paradoxical, then so too is B.

The Banach–Tarski Paradox: Proof Outline I

1. We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a, b)$.

The Banach–Tarski Paradox: Proof Outline II

2. We use the decomposition

$$\begin{aligned} F(a,b) &= a^{-1}W(a) \cup W\left(a^{-1}\right) \\ &= b^{-1}W(b) \cup W\left(b^{-1}\right) \end{aligned}$$

   to duplicate the unit sphere in $\mathbb{R}^3$, S^2, except for a countable subset D. (The *Hausdorff Paradox*.)
3. We show that S^2 and $S^2 \setminus D$ are SO(3)-equidecomposable — there is thus a paradoxical decomposition of S^2.
4. We show that the unit ball, $B(0,1) \subseteq \mathbb{R}^3$, is paradoxical under the isometry group E(3).

The Banach–Tarski Paradox: Proof Outline III

5. Define a relation $A \preceq B$ if A is G-equidecomposable with a subset of B, and show that if $A \preceq B$ and $B \preceq A$, then A and B are G-equidecomposable.
6. Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

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Ill-Behaved Groups

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Ill-Behaved Groups

- The way that our copy of $F(a,b)$ helped "create" the Banach–Tarski paradox suggests that $F(a,b)$ is a particularly ill-behaved group.
- Let $\nu\colon F(a,b) \to [0,1]$ be a probability measure — we will show that ν *cannot* be translation-invariant (i.e., $\nu(tE) = \nu(E)$ for all $t \in F(a,b), E \subseteq F(a,b)$).

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$1 = \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right)$$

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$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\ &= \nu(F(a,b)) + \nu(F(a,b)) \end{aligned}$$

Ill-Behaved Groups, Cont'd

Suppose such a translation-invariant ν exists. Taking

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right),$$

we have

$$\begin{aligned} 1 &= \nu(W(a)) + \nu\left(W\left(a^{-1}\right)\right) + \nu(W(b)) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a)\right) + \nu\left(W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b)\right) + \nu\left(W\left(b^{-1}\right)\right) \\ &= \nu\left(a^{-1}W(a) \sqcup W\left(a^{-1}\right)\right) + \nu\left(b^{-1}W(b) \sqcup W\left(b^{-1}\right)\right) \\ &= \nu(F(a,b)) + \nu(F(a,b)) \\ &= 2. \end{aligned}$$

Amenability

Let G be a group. A *mean* is a finitely additive probability measure $\nu : P(G) \to [0,1]$ such that

$$\nu(tE) = \nu(E)$$

for all $t \in G$ and $E \subseteq G$.

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If G admits a mean, we say G is *amenable*.

- In other words, G is sufficiently "well-behaved."

Inheritance Properties of Amenability

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- If $H \leq G$ is an amenable subgroup such that $[G : H] < \infty$, then G is amenable.
- If $N \trianglelefteq G$ and G/N are amenable, then G is amenable.
- If $(G_i, \varphi_i)_{i \in I}$ is a directed system of amenable groups, then the union $G = \bigcup_{i \in I} G_i$ is amenable.

Examples

- Finite groups are amenable: let δ_t be the point mass at $t \in G$,

$$\delta_t(s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}.$$

Then,

$$\nu = \frac{1}{|G|} \sum_{t \in G} \delta_t$$

is a mean.

- Abelian groups are amenable.
- The free group, $F(a, b)$, is *not* amenable.

Paradoxical Groups and Amenability

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Theorem (Tarski's Theorem)

Let G be a group. Then, G is non-paradoxical if and only if G is amenable.

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Unfortunately, the proof that every non-paradoxical group is amenable is significantly harder.

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On first glance, it may seem like we're finished, but we're really not.

Our methods so far — the existence of a mean, or showing non-paradoxicality — are quite difficult to establish.

As it turns out, amenability touches a variety of fields:

- functional analysis;
- geometric group theory;
- representation theory;
- operator algebras.

Normed Vector Spaces

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- homogeneity: $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in \mathbb{C}$;
- triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

A Normed Vector Space

The best example is that of $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

However, we need a few more dimensions in order to get to where we're going.

Function Spaces

There are three main function spaces that we're concerned with for our studies:

$$\ell_\infty(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sup_{t\in\Gamma} |f(t)| < \infty \right\};$$

$$\ell_1(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \;\middle|\; \sum_{t\in\Gamma} |f(t)| < \infty \right\};$$

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They are equipped with the respective norms of

- $\|f\|_{\ell_\infty} := \sup_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_1} := \sum_{t\in\Gamma} |f(t)|$;
- $\|f\|_{\ell_2} := \left(\sum_{t\in\Gamma} |f(t)|^2\right)^{1/2}$.

Linear Maps and Linear Functionals

A linear transformation $T\colon V \to W$ is called *bounded* if

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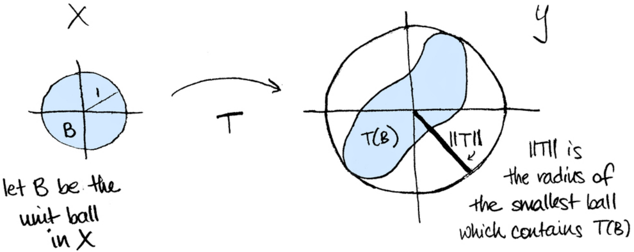
We call the quantity on the left the *operator norm*, denoted $\|T\|_{\mathrm{op}}$.

If $W = \mathbb{C}$, then we call T a *linear functional*.

Operator Norm Pictorial Depiction

Courtesy of Tai-Danae Bradley.



Positive Linear Functionals on $\ell_\infty(\Gamma)$

If $\varphi\colon \ell_\infty(\Gamma) \to \mathbb{C}$ is a linear functional, we say φ is *positive* if, for any $f \in \ell_\infty(\Gamma)$ with $f \geq 0$, $\varphi(f) \geq 0$.

- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\mathrm{op}}$.

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- It can be shown that φ is positive if and only if $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}}$. All positive linear functionals are automatically continuous.
- If $\varphi(\mathbb{1}_\Gamma) = \|\varphi\|_{\text{op}} = 1$, then we say φ is a *state*.

Translations of $\ell_\infty(\Gamma)$

If $f \in \ell_\infty(\Gamma)$, we define the translation $\lambda_s : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\lambda_s(f)(t) = f\left(s^{-1}t\right)$$

for all $t \in \Gamma$ and fixed $s \in \Gamma$.

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If $\varphi : \ell_\infty(\Gamma) \to \mathbb{C}$ is a state such that $\varphi(\lambda_s(f)) = \varphi(f)$ for all $f \in \ell_\infty(\Gamma)$, then we say φ is an *invariant state*.

Invariant States and Means

Invariant states and means are interchangeable.

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If φ is an invariant state on $\ell_\infty(\Gamma)$, define

$$\mu(E) = \varphi(\mathbb{1}_E)$$

for all $E \subseteq \Gamma$.

Approximations and Amenability

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Remember when we decomposed

$$F(a,b) = W(a) \sqcup W\left(a^{-1}\right) \sqcup W(b) \sqcup W\left(b^{-1}\right).$$

Translating $W(a) \mapsto a^{-1}W(a)$ gave us a set that was "significantly" "bigger" than $W\left(a^{-1}\right)$; specifically, it gave us $F(a,b) \setminus W\left(a^{-1}\right)$.

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But what does “bigger” actually mean?

Følner's Condition

Theorem (Følner's Theorem)

Let Γ be a countable, discrete group. Then, Γ is amenable if and only if there exists a sequence of finite subsets $(F_n)_n$ such that

$$\lim_{n\to\infty} \frac{|sF_n \cap F_n|}{|F_n|} = 1$$

for all $s \in \Gamma$.

Approximate Means

The Følner condition allows us to find an “approximate” version of a mean.

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Keeping $\lambda_s(f)(t) = f\left(s^{-1}t\right)$, if $(f_k)_k \subseteq \ell_1(\Gamma)$ is such that

$$\lim_{k\to\infty} \|f_k - \lambda_s(f_k)\|_{\ell_1} = 0,$$

then we say $(f_k)_k$ is an *approximate mean*.

Approximate Means, Cont'd

This is equal to Følner's condition.

In one direction, we take

$$f_k = \frac{1}{|F_k|}\mathbb{1}_{F_k},$$

Approximate Means, Cont'd

In the other direction, we arbitrarily approximate $f \in \ell_1(\Gamma)$ with a "sufficient" finitely supported function g,

$$\|g - f\|_{\ell_1} < \varepsilon/2,$$

then use a "layer cake" decomposition to find our Følner sets:

$$g = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i},$$

where $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$.

Graphs and Amenability

Given a group Γ with generating set S, we may define a graph — known as the Cayley graph — with vertices consisting of group elements and edges defined by "walking" along the generators.

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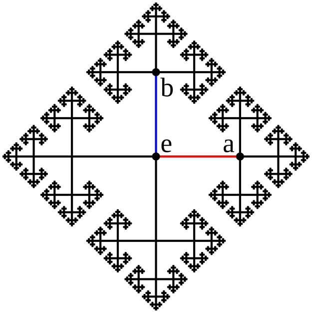

Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

Graphs and Amenability, cont'd

If $S \subseteq V(G)$ is a subset of vertices of a graph G, the *neighbor vertex set*, $N(S)$, is the set of vertices in G that are adjacent to S (not including elements of S).

If G is the Cayley graph of Γ, then Γ is amenable if and only if

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- Essentially, the Cayley graph doesn't "get too big" "too fast."
- This is proven with the Følner condition.

Hilbert Spaces

If $\mathcal{H}$ is a vector space, an *inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ that satisfies

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The inner product induces a norm $\|x\|^2 = \langle x, x \rangle$.

If $\mathcal{H}$ is complete with respect to this norm, we call $\mathcal{H}$ a Hilbert space.

Operators on Hilbert Spaces

Bounded linear maps on Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$, include a special structure called an adjoint that "plays nicely" with the inner product:

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$$\begin{aligned} U^*U &= I \\ UU^* &= I, \end{aligned}$$

then we call U a *unitary operator*. The space of unitary operators, $\mathcal{U}(\mathcal{H})$, is a group under composition.

Representations

A map $\lambda\colon \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies

$$\lambda(st) = \lambda(s)\lambda(t)$$
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All discrete groups are able to be unitarily represented by the trivial representation $1_\Gamma\colon \Gamma \to \mathbb{C}$, given by $1_\Gamma(s) = 1$.

The Left-Regular Representation

As it turns out, the map $\lambda_s(f)(t) = f\left(s^{-1}t\right)$ is a unitary operator on $\ell_2(\Gamma)$, where $\lambda_s^* = \lambda_{s^{-1}}$.

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The map $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$, given by $s \mapsto \lambda_s$ is a very special representation, known as the *left-regular representation*.

This is because it “encodes” the group’s left-multiplication action, in the sense that $\lambda_s(\delta_t) = \delta_{st}$, where δ_t is the point mass at $t \in \Gamma$.

The Left-Regular Representation and Amenability

A sequence $(f_k)_k \subseteq \ell_2(\Gamma)$ is known as an *almost-invariant vector* for $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ if

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If $\lambda\colon \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ admits an almost-invariant vector, then Γ is amenable.

Introduction to C^*-Algebras

The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\text{op}}$, is a very special vector space.

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The space of *all* bounded linear operators, $T\colon \mathcal{H} \to \mathcal{H}$, written $\mathbb{B}(\mathcal{H})$, along with the norm $\|\cdot\|_{\mathrm{op}}$, is a very special vector space. The adjoint map satisfies:

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These make $\mathbb{B}(\mathcal{H})$ a C^*-*algebra*. However, there are other C^*-algebras.

A Group C^*-Algebra

If Γ is a group, we may define a vector space, $\mathbb{C}[\Gamma]$, by finite sums

$$x = \sum_{t \in \Gamma} x(t)\delta_t,$$

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where δ_t is the point mass at $t \in \Gamma$.

This becomes a $*$-algebra when endowed with multiplication (by convolution) and involution:

$$f * g(s) = \sum_{t \in \Gamma} f(t) g\left(s^{-1} t\right)$$
$$f^*(t) = \overline{f(t^{-1})}.$$

A Group C^*-Algebra, cont'd

If we represent $\pi_\lambda : \mathbb{C}[\Gamma] \to \mathbb{B}(\ell_2(\Gamma))$ by mapping $\delta_t \mapsto \lambda_t \in \mathcal{U}(\ell_2(\Gamma))$, extending linearly, and taking

$$\|x\|_\lambda = \|\pi_\lambda(x)\|_{\mathrm{op}},$$

we get the *reduced group C^*-algebra* on Γ (upon norm completion).

Finite-Dimensional Approximations

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We can use these sufficient approximations to establish amenability.

Nuclearity

A C^*-algebra, A, is called *nuclear* if there exist two sequences of maps, $\varphi_n\colon A \to \mathrm{Mat}_{k(n)}(\mathbb{C})$ and $\psi_n\colon \mathrm{Mat}_{k(n)}(\mathbb{C}) \to A$, such that

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- Essentially, any $a \in A$ is "close enough" to a certain family of finite-dimensional analogues.

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Specifically, by showing that the approximation of $\frac{|sF_n \cap F_n|}{|F_n|} \to 1$ corresponds to the existence of maps $\varphi_n \colon C^*_\lambda(\Gamma) \to \mathrm{Mat}_{|F_n|}(\mathbb{C})$ and $\psi_n \colon \mathrm{Mat}_{|F_n|}(\mathbb{C}) \to C^*_\lambda(\Gamma)$ that satisfy

$$\left\| x - \psi_n \circ \varphi_n(x) \right\| \xrightarrow{n \to \infty} 0.$$

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Nuclear C^*-algebras are classified, so active research areas primarily concern whether or not certain classes of C^*-algebras are nuclear (hence classifiable).

There are also a lot of other directions that amenability can take the eager student, but I think this was a pretty nice overview of some of the ways that amenability touches all sorts of other fields of math.

Acknowledgments

A large thank you goes to

- the professors of the math department;
- friends, family, and acquaintances both in the math major and outside;
- everyone in attendance.

References I