Problem (Problem 1)**:** Let F be a field, $a(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in F[x]$ a nonconstant monic polynomial, and let $A = C_{a(x)}$ be its companion matrix.

Prove by direct computation that $\mathrm{SNF}(xI - A) = \mathrm{diag}(1, \dots, 1, a(x))$.

Solution: We observe that

$$xI - A = \begin{pmatrix} x & 0 & \cdots & 0 & a_0 \\ -1 & x & \cdots & 0 & a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x + a_{n-1} \end{pmatrix}.$$

Focusing on the bottom 2 rows, we use the following reduction method

$$\begin{pmatrix} x & a_{n-2} \\ -1 & x + a_{n-1} \end{pmatrix} \xrightarrow{R_{n-1} \leftarrow xR_n + R_{n-1}} \begin{pmatrix} 0 & x^2 + a_{n-1}x + a_{n-2} \\ -1 & x + a_{n-1} \end{pmatrix}$$
$$\xrightarrow{C_n \leftarrow (x + a_{n-1})C_{n-1} + C_n} \begin{pmatrix} 0 & x^2 + a_{n-1}x + a_{n-2} \\ -1 & 0 \end{pmatrix}.$$

Inductively repeating this reduction method, we say at step i that we perform the following two operations consecutively

- $R_{n-i} \leftarrow xR_{n-i+1} + R_{n-i}$;
- $C_{n-i+1} \leftarrow \left(x^i + a_{n-1}x^{i-1} + \cdots + a_{n-i}\right)C_{n-i} + C_{n-i+1}$

Upon completion of this process at step n, we obtain a matrix consisting entirely of -1 along the subdiagonal and $a(x)$ in position $(1, n)$. Next, we perform the following procedure as i ranges from 1 to $n - 1$.

- $R_i \leftarrow (-1)R_{i+1} + R_i$;
- $R_{i+1} \leftarrow R_i + R_{i+1}$.

This gives a matrix with 1 along the diagonal and $a(x)$ along column n. Then, upon performing the operation

- $R_i \leftarrow (-1)R_n + R_i$

for each $1 \leq i \leq n - 1$, we obtain our desired diagonal matrix in Smith normal form, where we have $\mathrm{diag}(1, \dots, 1, a(x))$.

Problem (Problem 2)**:** Prove that the constant term in the characteristic polynomial of the $n \times n$ matrix A is $(-1)^n \det(A)$, and that the coefficient of x^{n-1} is the negative of the sum of the diagonal entries of A. Prove that $\det(A)$ is the product of the eigenvalues of A and that the trace of A is the sum of the eigenvalues of A.

Solution: We start by showing that this holds for a companion matrix, $A = C_{a(x)}$. Note that in our computation showing that $\mathrm{SNF}(xI - A) = \mathrm{diag}(1, 1, \dots, a(x))$, we exclusively used row and column operations (and employed no flips); as a result, it follows that the characteristic polynomial of a companion matrix for $a(x)$ is exactly $a(x)$. Then, we observe that

$$\begin{aligned} a_0 &= \chi_A(0) \\ &= \det(-A) \\ &= \det((-I)A) \\ &= \det(-I)\det(A) \\ &= (-1)^n \det(A), \end{aligned}$$

and that the coefficient on the x^{n-1} is equal to a_{n-1}, or $-(-a_{n-1})$, which is the trace of the companion matrix.

In the general case, we observe that A is similar to a matrix in rational canonical form,

$$A \sim \operatorname{diag}(A_1, \dots, A_r),$$

and has

$$\chi_A(x) = \chi_{A_1}(x) \cdots \chi_{A_r}(x),$$

where we use the fact that characteristic polynomials are invariant under similarity transformation, so that

$$\begin{aligned}
\chi_A(0) &= \chi_{A_1}(0) \cdots \chi_{A_r}(0) \\
&= a_{0,1} \cdots a_{0,r} \\
&= (-1)^{n_1} \det(A_1) \cdots (-1)^{n_r} \det(A_r) \\
&= (-1)^n \det(A_1) \cdots \det(A_r) \\
&= (-1)^n \det(A),
\end{aligned}$$

where we let n_i denote the dimension of the specific companion matrix A_i. Additionally, we observe that the coefficient on the $n-1$ degree term on $\chi_A(x)$ is given summing the coefficient of an $n_i - 1$ degree term with the n_j degree terms for all $j \neq i$. In particular, this means that we get

$$\begin{aligned}
a_{n-1} &= \sum_{i=1}^{r} a_{n_i - 1} \\
&= \sum_{i=1}^{r} -\operatorname{Tr}(A_i) \\
&= -\operatorname{Tr}(A).
\end{aligned}$$

From basic properties of polynomials, we know that the constant term of a polynomial of degree n is equal to $(-1)^n$ multiplied by the product of the roots, while the coefficient on the degree $n-1$ term is equal to -1 multiplied by the sum of the roots. In particular, applying this to the characteristic polynomial, we get that the trace is the sum of the eigenvalues of A and the determinant is the product of the eigenvalues.

Problem (Problem 3)**:** Determine the number of possible RCFs of 8×8 matrices over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$.

Solution: Factoring over $\mathbb{Q}$, we have that

$$\chi_A(x) = x^4 \left(x^2 + 1\right)(x - 1)(x + 1).$$

In order to determine the possible rational canonical forms, we need to determine the possible invariant factors, $a_1(x) | a_2(x) | \cdots | a_d(x)$, subject to the constraint that $a_d(x) = \mu_A(x)$ has the same roots as $\chi_A(x)$. In particular, we must have that $\mu_A(x)$ can only be one of the following, where we observe that we cannot have any of $x^2 + 1$, $x + 1$, or $x - 1$ in the invariant factor decomposition outside of the minimal polynomial since they each have multiplicity 1:

- $p_1(x) = x\left(x^2 + 1\right)(x - 1)(x + 1)$;
- $p_2(x) = x^2\left(x^2 + 1\right)(x - 1)(x + 1)$;
- $p_3(x) = x^3\left(x^2 + 1\right)(x - 1)(x + 1)$;
- $p_4(x) = x^4\left(x^2 + 1\right)(x - 1)(x + 1)$.

We find that the possible decompositions are thus

$$\begin{aligned}
A_1 &= [x, x, x, p_1(x)] \\
A_2 &= \left[x, x^2, p_1(x)\right] \\
A_3 &= \left[x^3, p_1(x)\right]
\end{aligned}$$

$$\begin{aligned} B_1 &= [x, x, p_2(x)] \\ B_2 &= \left[x^2, p_2(x)\right] \\ C &= [x, p_3(x)] \\ D &= [p_4(x)]. \end{aligned}$$

Problem (Problem 4)**:** Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing this does not hold for 4×4 matrices.

Solution: Suppose A and B are 3×3 matrices with characteristic polynomial $\chi(x)$ and minimal polynomial $\mu(x)$. The characteristic polynomial has degree 3, so we may consider the degree(s) of the minimal polynomial.

If $\mu(x)$ has degree 1, then it is of the form $\mu(x) = x - a$; this is a prime in $F[x]$, and since the degree of the characteristic polynomial is 3 and all the invariant factors must divide $\mu(x)$, it follows that A and B have invariant factors given by

$$a_i(x) = [(x-a), (x-a), (x-a)],$$

so since they have the same invariant factors, they have the same rational canonical form and are thus similar.

If $\mu(x)$ has degree 2, then the lower 2×2 submatrix of both A and B are equal, and both of them admit invariant factors given by

$$a_i(x) = \left[\frac{\chi(x)}{\mu(x)}, \mu(x)\right].$$

Finally, if $\mu(x)$ has degree 3, then both A and B admit the same rational canonical form as both of them have the invariant factor $\mu(x)$.

As a counter-example in the 4×4 case, consider the matrices with minimal polynomial $\mu(x) = (x-1)^2$ and characteristic polynomial $\chi(x) = (x-1)^4$. These matrices have invariant factor decompositions

$$\begin{aligned} a_i(x) &= \left[(x-1), (x-1), (x-1)^2\right] \\ b_i(x) &= \left[(x-1)^2, (x-1)^2\right], \end{aligned}$$

admitting rational canonical forms

$$\begin{aligned} A &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & -1 & 2 \end{pmatrix} \\ B &= \begin{pmatrix} & 1 & & \\ -1 & 2 & & \\ & & & 1 \\ & & -1 & 2 \end{pmatrix}. \end{aligned}$$

Since these rational canonical forms differ, these matrices are necessarily not similar.

Problem (Problem 5)**:** Find the number of distinct conjugacy classes in the group $\mathrm{GL}_3(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the field with two elements, and specify one element in each conjugacy class.

Solution: We start by finding all the polynomials of degree 3 (representing all the possible characteristic polynomials) over $\mathbb{F}_2$ as follows:

(i) x^3;

(ii) $x^3+1=(x+1)\left(x^2+x+1\right)$;

(iii) $x^3+x=x(x+1)^2$;

(iv) $x^3+x^2=x^2(x+1)$;

(v) x^3+x+1;

(vi) x^3+x^2+1;

(vii) $x^3+x^2+x=x\left(x^2+x+1\right)$;

(viii) $x^3+x^2+x+1=(x+1)^3$.

Before we start the process of listing the conjugacy classes, we start by observing that if $x|\chi(x)$, then the matrix admits an eigenvalue of 0, so $x|\mu(x)$. In this scenario, we observe that such matrices cannot be invertible, so we may disregard these cases.

We start with the cases of the irreducible polynomials in this list:

(C1) $\left[x^3+x+1\right]=\begin{pmatrix}0&0&1\\1&0&1\\0&1&0\end{pmatrix}$;

(C2) $\left[x^3+x^2+1\right]=\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$.

Next, we observe that the invariant factors for (ii) must divide either $(x+1)$ or $\left(x^2+x+1\right)$, but since both of these are irreducible in $\mathbb{F}_2[x]$, and their product is of degree 3, it follows that the minimal polynomial is equal to $(x+1)\left(x^2+x+1\right)$, meaning that we get the following rational canonical form:

(C3) $\left[x^3+1\right]=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$.

The remaining case is that of (viii). This admits three different invariant factor decompositions, admitting three different minimal polynomials:

(C4) $[x+1,x+1,x+1]=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$;

(C4) $\left[x+1,(x+1)^2\right]=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$;

(C5) $\left[(x+1)^3\right]=\begin{pmatrix}0&0&1\\1&0&1\\0&1&1\end{pmatrix}$.

Therefore, these are representatives of the distinct conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_2)$.

Problem (Problem 6)**:** Prove that there is no matrix $A\in\mathrm{Mat}_{10}(\mathbb{Q})$ satisfying $A^4=-I$.

Solution: We observe that equivalently, we have that $A^4+I=0$, so that $\mu_A(x)|x^4+1$. Since x^4+1 is irreducible, it follows that $\mu_A(x)=x^4+1$, and that the invariant factors of A must divide x^4+1. Yet, this means that the invariant factors of A must be equal to x^4+1. This yields a contradiction since the product of the invariant factors of A is equal to the characteristic polynomial of A, which has degree 10, but 4 does not divide 10.

Problem (Problem 7)**:** Prove that the matrices

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 5 & 2 & -8 & -8 \\ -6 & -3 & 8 & 8 \\ -3 & -1 & 3 & 4 \\ 3 & 1 & -4 & -5 \end{pmatrix}$$

both have characteristic polynomial $(x-3)(x+1)^3$. Determine whether they are similar and determine the Jordan canonical form for each matrix.

Solution: We observe that

$$xI - A = \begin{pmatrix} x & -1 & -1 & -1 \\ -1 & x & -1 & -1 \\ -1 & -1 & x & -1 \\ -1 & -1 & -1 & x \end{pmatrix}$$
$$xI - B = \begin{pmatrix} x-5 & -2 & 8 & 8 \\ 6 & x+3 & -8 & -8 \\ 3 & 1 & x-3 & -4 \\ -3 & -1 & 4 & x+5 \end{pmatrix}$$

To resolve these determinants, we use the elementary row and column operations. First, we start with the case of $xI - A$, giving

$$\begin{aligned}
\begin{pmatrix} x & -1 & -1 & -1 \\ -1 & x & -1 & -1 \\ -1 & -1 & x & -1 \\ -1 & -1 & -1 & x \end{pmatrix} &\xmapsto{R_2 \leftarrow -R_1 + R_2} \begin{pmatrix} x & -1 & -1 & -1 \\ -x-1 & x+1 & 0 & 0 \\ -1 & -1 & x & -1 \\ -1 & -1 & -1 & x \end{pmatrix} \\
&\xmapsto{C_1 \leftarrow -C_2 + C_1} \begin{pmatrix} x+1 & -1 & -1 & -1 \\ -2x-2 & x+1 & 0 & 0 \\ 0 & -1 & x & -1 \\ 0 & -1 & -1 & x \end{pmatrix} \\
&\xmapsto{R_2 \leftarrow 2R_1 + R_2} \begin{pmatrix} x+1 & -1 & -1 & -1 \\ 0 & x-1 & -2 & -2 \\ 0 & -1 & x & -1 \\ 0 & -1 & -1 & x \end{pmatrix} \\
&\xmapsto{R_2 \leftarrow -R_3 + R_2} \begin{pmatrix} x+1 & -1 & -1 & -1 \\ 0 & x & -x-2 & -1 \\ 0 & -1 & x & -1 \\ 0 & -1 & -1 & x \end{pmatrix} \\
&\xmapsto{R_2 \leftarrow -R_4 + R_2} \begin{pmatrix} x+1 & -1 & -1 & -1 \\ 0 & x+1 & -x-1 & -x-1 \\ 0 & -1 & x & -1 \\ 0 & -1 & -1 & x \end{pmatrix} \\
&\xmapsto{C_3 \leftarrow C_2 + C_3} \begin{pmatrix} x+1 & -1 & -2 & -1 \\ 0 & x+1 & 0 & -x-1 \\ 0 & -1 & x-1 & -1 \\ 0 & -1 & -2 & x \end{pmatrix}
\end{aligned}$$

$$\xmapsto{C_4 \leftarrow C_2 + C_4} \begin{pmatrix} x+1 & -1 & -2 & -2 \\ 0 & x+1 & 0 & 0 \\ 0 & -1 & x-1 & -2 \\ 0 & -1 & -2 & x-1 \end{pmatrix},$$

from which we see that we get the characteristic polynomial $(x-3)(x+1)^3$.

Similarly, reducing $xI - B$ gives

$$\begin{pmatrix} x-5 & -2 & 8 & 8 \\ 6 & x+3 & -8 & -8 \\ 3 & 1 & x-3 & -4 \\ -3 & -1 & 4 & x+5 \end{pmatrix} \xmapsto{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} x-5 & -2 & 8 & 8 \\ x+1 & x+1 & 0 & 0 \\ 3 & 1 & x-3 & -4 \\ -3 & -1 & 4 & x+5 \end{pmatrix}$$

$$\xmapsto{C_2 \leftarrow -3C_2 + C_1} \begin{pmatrix} x+1 & -2 & 8 & 8 \\ -2x-2 & x+1 & 0 & 0 \\ 0 & 1 & x-3 & -4 \\ 0 & -1 & 4 & x+5 \end{pmatrix}$$

$$\xmapsto{R_4 \leftarrow R_3 + R_4} \begin{pmatrix} x+1 & -2 & 8 & 8 \\ -2x-2 & x+1 & 0 & 0 \\ 0 & 1 & x-3 & -4 \\ 0 & 0 & x+1 & x+1 \end{pmatrix}$$

$$\xmapsto{C_3 \leftarrow -C_4 + C_3} \begin{pmatrix} x+1 & -2 & 0 & 8 \\ -2x-2 & x+1 & 0 & 0 \\ 0 & 1 & x+1 & -4 \\ 0 & 0 & 0 & x+1 \end{pmatrix}.$$

Therefore, by using the cofactor expansion along the bottom row, we find that the characteristic polynomial is equal to

$$\begin{aligned} \det(xI - B) &= (x+1)\det\begin{pmatrix} x+1 & -2 & 0 \\ -2x-2 & x+1 & 0 \\ 0 & 1 & x+1 \end{pmatrix} \\ &= (x+1)^2 \det\begin{pmatrix} x+1 & -2 \\ -2x-2 & x+1 \end{pmatrix} \\ &= (x+1)^2\left((x+1)^2 - 4x - 4\right) \\ &= (x+1)^2(x^2 - 2x - 3) \\ &= (x+1)^3(x-3). \end{aligned}$$

Now, computing multiplicities, we observe that

$$\begin{aligned} (-1)I - A &= \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \\ (-1)I - B &= \begin{pmatrix} -6 & -2 & 8 & 8 \\ 6 & 2 & -8 & -8 \\ 3 & 1 & -4 & -4 \\ -3 & -1 & 4 & 4 \end{pmatrix}, \end{aligned}$$

meaning that the dimensions of the kernels of both $(-1)I - A$ and $(-1)I - B$ are three. In particular, this means that the geometric multiplicity and algebraic multiplicity of both A and B are identical, meaning they are diagonalizable and thus admit identical Jordan canonical forms

$$J = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Problem (Problem 8)**:** Show that the following matrices are similar in $\mathrm{Mat}_p(\mathbb{F}_p)$

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Solution: We observe that the matrix A is in rational canonical form, and in particular, it is the companion matrix for the polynomial

$$a(x) = x^p - 1.$$

Note then that this means the minimal polynomial of A is also $\mu(x) = x^p - 1$ since the minimal polynomial is the largest invariant factor of A, which is equal to $a(x)$ since A is a companion matrix. Note that by the Frobenius endomorphism, we have that $\mu(x) = (x-1)^p$, meaning that the multiplicity in $\mu(x)$ of the eigenvalue 1 is equal to p. This means there is one Jordan block in the Jordan canonical form of A, giving that A and B are similar.

Letting V be a p-dimensional vector space over $\mathbb{F}_p$, and T the underlying transformation corresponding to A and/or B, we may equip V with the structure of an $\mathbb{F}_p[x]$-module by setting $x \cdot v = Tv$; by our assumption work above, this has the invariant factor decomposition consisting exclusively of the form

$$V \cong \mathbb{F}_p[x]/(x-1)^p,$$

which is also its elementary divisor decomposition. There are two canonical bases in this case; we have

$$\beta = \left\{\overline{1}, \overline{x}, \dots, \overline{x}^{p-1}\right\}$$
$$\gamma = \left\{\overline{(x-1)}^{p-1}, \dots, \overline{(x-1)}, \overline{1}\right\},$$

where β corresponds to the rational canonical form (that A is in) and γ corresponds to the Jordan canonical form (that B is in). Therefore, we may let P be the change of basis matrix between β and γ, which will give

$$PAP^{-1} = [\mathrm{id}]_\beta^\gamma [T]_\beta [\mathrm{id}]_\gamma^\beta.$$

We have that the entries of each column of P is given by

$$P\left(\overline{(x-1)}^\ell\right) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \overline{x}^k,$$

where the scalars are modulo p

Problem (Problem 9)**:** Let V be an n-dimensional vector space over F, and let $T: V \to V$ be a nilpotent F-linear map. Prove that $T^n = 0$ in two different ways:

(a) using JCF;

(b) without using JCF or RCF, but instead looking at the sequence of kernels $\ker\left(T^k\right)$.

Solution:

(a) We let m be the smallest index such that $T^m = 0$. If $m \leq n$, then we are done. We will show now that if $m \geq n$, then $m = n$. Letting $\mu_T(x)$ denote the minimal polynomial for T, we observe that $\mu_T(x) | x^m$ by definition. Since x is prime, it follows that $\mu_T(x)$ must be of the form x^k for some $k \geq 0$ as $F[x]$ is a PID.

Yet, since the minimal polynomial is the smallest degree polynomial satisfying $\mu_T(T) = 0$, and we assume that m is the smallest such index satisfying this criterion, it follows that $\mu_T(x) = x^m$. Finally, since we know that, since the degree of x in the expression for $\mu_T(x)$ denotes the size of the largest Jordan block with eigenvalue 0, we cannot have that $\mu_T(x)$ has a Jordan block with eigenvalue 0 of size larger than n, and since $m \geq n$, it follows that we must have $m = n$ and $T^n = 0$.

(b) The sequence of kernels $\ker\left(T^k\right)$ defines an increasing sequence of subspaces, where we have $\ker(T) \subseteq \ker\left(T^2\right) \subseteq \cdots$ etc. First, observe that since V is a finite-dimensional vector space, this sequence of subspaces must admit some $1 \leq m \leq n$ such that $\ker(T^m) = \ker\left(T^{m+1}\right)$, as else we would have the sequence of increasing dimensions would exceed n. We will show that for any j, we have $\ker(T^m) = \ker\left(T^{m+j}\right)$.

The base case has already been assumed, so we have the inductive hypothesis that for all $j \leq N$, we have $\ker\left(T^{m+j}\right) = \ker(T^m)$. Suppose $v \in \ker\left(T^{m+N+1}\right)$. Then, we have

$$\begin{aligned} 0 &= T^{m+N+1} v \\ &= T^{m+1}\left(T^N v\right), \end{aligned}$$

meaning that $T^N v \in \ker\left(T^{m+1}\right) = \ker(T^m)$. Yet, since by assumption, we have $\ker\left(T^{m+N}\right) = \ker(T^m)$, we have $v \in \ker(T^m)$, so the sequence $\ker\left(T^k\right)$ stabilizes at T^m; since T is nilpotent, it follows that this stabilization value must be $T^m = 0$. Since $m \leq n$, it follows that $T^n = 0$.