Chapter 23 Problems

23.1

1. Since
$$\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and all matrices are representations of linear transformations over a given basis, it is the case that $\hat{M}$ is a linear operator.

2. $\hat{T}$ is not a linear operator.

3. $\hat{S}$ is not a linear operator.

4. $\hat{N}$ is not a linear operator.

5. Since
$$\begin{aligned} \hat{C}\left(\alpha\mathbf{A} + \mathbf{B}\right) &= \alpha\nabla\times\mathbf{A} + \nabla\times\mathbf{B} \\ &= \alpha\hat{C}\left(\mathbf{A}\right) + \hat{C}\left(\mathbf{B}\right), \end{aligned}$$
$\hat{C}$ is a linear operator.

6. Since
$$\begin{aligned} \hat{D}\left(\alpha f + g\right) &= \frac{d\left(\alpha f + g\right)}{dx} \\ &= \alpha\frac{df}{dx} + \frac{dg}{dx} \\ &= \alpha\hat{D}\left(f\right) + \hat{D}\left(g\right), \end{aligned}$$
$\hat{D}$ is a linear operator.

7. Since the derivative is linear, multiple applications of the derivative are still linear.

8. Since the derivative and multiplication by x^2 are linear operators, their sum is linear.

9. $\hat{D}^2 + \sin^2$ is not a linear operator.

10. Since
$$\begin{aligned} \hat{K}\left(\alpha f + g\right) &= \int_{P_1}^{P_2} K\left(x,t\right)\left(\alpha f(t) + g(t)\right)\,dt \\ &= \alpha\int_{P_1}^{P_2} K(x,t)f(t)\,dt + \int_{P_1}^{P_2} K\left(x,t\right)g(t)\,dt, \end{aligned}$$
it is the case that $\hat{K}$ is a linear operator.

23.2

Linear combinations of solutions to the Schrödinger equation are also solutions to the Schrödinger equation since the equation consists entirely of linear operators.

23.3

Let L be a linear operator, $v \in V$. Then,

$$\begin{aligned} L(0) &= L(v - v) \\ &= L(v) - L(v) \\ &= 0. \end{aligned}$$

Chapter 24 Problems

24.1

(a) The set of scalars is a vector space.

(b) The set of two-dimensional column vectors whose first element is smaller than the second element is a vector space.

(c) The set of n-dimensional column vectors consisting of integer-valued elements is a $\mathbb{Z}$-module, but since $\mathbb{Z}$ is not a field, it is not a vector space.

(d) The set of n-dimensional column vectors whose elements sum to zero is a vector space.

(e) The set of $n \times n$ antisymmetric matrices combining under addition is a vector space.

(f) The set of $n \times n$ matrices combining under multiplication does not form a vector space.

(g) The set of polynomials with real coefficients is an $\mathbb{R}$-vector space but is not a $\mathbb{C}$-vector space.

(h) The set of periodic functions with $f(0) = 1$ is not a vector space.

(i) The set of periodic functions with $f(0) = 0$ is a vector space.

(j) The set of functions with $cf(a) + df'(a) = 0$ is a vector space.

24.2

The product space $U = V \times W$ under the operation

$$a(v_1, w_1) + b(v_2, w_2) = (av_1 + bv_2, aw_1 + bw_2)$$

is a vector space.

24.3

(a) This is a linearly independent set.

(b) This is not a linearly independent set.

(c) This is a linearly independent set.

(d) This is a linearly independent set.

(e) This is a linearly independent set.

24.4

Consider $z = bi$. Then, $\overline{z} = -bi$, and $z + \overline{z} = 0$, so z is not necessarily linearly independent of $\overline{z}$.

24.5

Calculating the determinant, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 6,$$

so the vectors form a basis.

24.6

Let $\{v_i\}_{i\in I}$ be a basis for a given $\mathbb{C}$-vector space V, and let $w \in V$ be expressed by

$$\begin{aligned} w &= \sum_{i\in I} a_i v_i \\ &= \sum_{i\in I} b_i v_i, \end{aligned}$$

where the sums are finite linear combinations. Then,

$$\begin{aligned} 0 &= (w - w) \\ &= \sum_{i\in I} (a_i - b_i) v_i, \end{aligned}$$

and since the $\{v_i\}_{i\in I}$ form a basis, this can only be the case if $a_i = b_i$ for each i.

24.7

(a)

$$\begin{aligned} v &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ w &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \end{aligned}$$

(b)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$

(c)

$$\begin{aligned} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} &= k \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}. \end{aligned}$$

(d)

$$\begin{aligned} \begin{pmatrix} a_1 - a_1 \\ a_2 - a_2 \\ a_3 - a_3 \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

24.8

We can let

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the matrix representation for $\{|R\rangle, |G\rangle, |B\rangle\}$. We have

$$\begin{aligned} |\text{white}\rangle &= 255\,|R\rangle + 255\,|G\rangle + 255\,|B\rangle \\ |\text{black}\rangle &= 0\,|R\rangle + 0\,|G\rangle + 0\,|B\rangle \\ |\text{orange}\rangle &= 255\,|R\rangle + 165\,|G\rangle + 0\,|B\rangle. \end{aligned}$$