# Excess Area Identities and Operator Symbols in Bergman Spaces

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# Summary

# Contents

## Definition and Notations

- $\Omega$ : a region in $\mathbb{C}$ e.g. $\mathbb{D}, D(0, r), \mathbb{A}(0, r, 1), \mathbb{C}$
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$: weight function

### Definition ($\lambda$-weighted Square-Integrable Functions)

$$L^2(\Omega, \lambda) = \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(z)|^2 \lambda(z) \, dA(z) < \infty \right\}$$

- $L^2(\Omega, \lambda)$ forms a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega, \lambda)} = \int_\Omega f(z) \overline{g(z)} \lambda(z) \, dA(z)$$

inducing the norm

$$\|f\|_{L^2(\Omega, \lambda)}^2 = \int_\Omega |f(z)|^2 \lambda(z) \, dA(z)$$

# Definitions and Notations

## Definition (Holomorphic Function on $\Omega$)

$$h \in \mathcal{O}(\Omega) \iff \frac{\partial h(z)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial h(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}\right) = 0, \; \forall z \in \Omega$$

## Definition ($\lambda$-weighted Bergman Space)

$$A^2(\Omega, \lambda) := \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

# Definitions and Notations

## Definition ($A^{1,2}(\Omega, \lambda)$)

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \;\middle|\; \frac{\partial h}{\partial z} \in A^2(\Omega, \lambda) \right\}$$

## Definition (Weighted Image-Area)

Let $h \in A^{1,2}(\Omega, \lambda)$.

$$A_{\Omega, \lambda}(h) = \int_{\Omega} \left| \frac{\partial h}{\partial z} \right|^2 \lambda(z) \; dA(z)$$

# Definitions and Notations

- $A^2(\Omega, \lambda)$ has a reproducing kernel i.e $\exists! \, K_\Omega^\lambda(\cdot, z) \in A^2(\Omega, \lambda)$ :

$$h(z) = \left\langle h(\cdot), K_\Omega^\lambda(\cdot, z) \right\rangle_{L^2(\Omega, \lambda)}$$

- $A^2(\Omega, \lambda)$ is a closed subspace of $L^2(\Omega, \lambda)$.

## Definition (Orthogonal Projection)

Let $P^{\Omega, \lambda} : L^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$

$$\left( P^{\Omega, \lambda} h \right)(z) := \left\langle h(\cdot), K_\Omega^\lambda(\cdot, z) \right\rangle_{L^2(\Omega, \lambda)}$$

$$= \int_\Omega h(w) \overline{K_\Omega^\lambda(w, z)} \lambda(w) \, dA(w)$$

# Definitions and Notations

### Definition (Multiplication Operator)

Let $M_\varphi : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda)$ where $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

### Definition (Toeplitz Operator)

$T_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$, where $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega,\lambda} := P^{\Omega,\lambda} M_\varphi$$

# Definitions and Notations

## Definition (Commutator)

Let $\left[ P^{\Omega,\lambda}, M_\varphi \right] : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda)$

$$\left[ P^{\Omega,\lambda}, M_\varphi \right] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

## Definition (Hankel Operator)

Let $H_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \to (A^2(\Omega, \lambda))^\perp$

$$\begin{aligned}
H_\varphi^{\Omega,\lambda} &:= - \left[ P^{\Omega,\lambda}, M_\varphi \right] \Big|_{A^2(\Omega,\lambda)} \\
&= \left( I - P^{\Omega,\lambda} \right) M_\varphi \\
&= M_\varphi - P^{\Omega,\lambda} M_\varphi \\
&= M_\varphi - T_\varphi^{\Omega,\lambda}
\end{aligned}$$

# Contents

## Motivations

- $\{z^n\}_{n=0}^{\infty}$ form a *complete orthogonal basis* for $A^2(\mathbb{D})$
- If $h \in O(\mathbb{D})$, then $h$ is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

and

$$S_N := \sum_{n=0}^{N} h_n z^n$$

converges uniformly on compact subsets.

- Relationship between $L^2$ norm of $h$ to the $\ell^2$ norm of $\{h_k\}_{k=0}^{\infty}$:

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 \, dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $\left[ T_{\bar{z}}^{\mathbb{D}} M_z, D M_z \right] (z^m) = 0$

# Problems

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

# Literature Review on Previous Results I

- D'Angelo's excess area identity [D'A19]

  Let $h \in A^{1,2}(\mathbb{D})$. Then,

  $$
  \begin{aligned}
  A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial(zh)}{\partial z} \right\|^2_{L^2(\mathbb{D})} - \left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\mathbb{D})} \\
  &= \frac{1}{2} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta \\
  &= \pi \left\| Sh \right\|^2_{L^2(b\mathbb{D})}
  \end{aligned}
  $$

  where $Sh$ is the restriction of $h$ to the unit circle.

# Literature Review on Previous Results II

- Excess area identity with Blaschke product multiplier
- 'Excess area' identity for harmonic functions [BÇGH22]

- Generating symbols for Toeplitz operators for a given initial p and target polynomial q on unit disc and polydisc, $T_\varphi^{\mathbb{D}}(p) = q$ and $T_\varphi^{\mathbb{D}^n}(p) = q$ [ÇDTR+24]

- Substituted derivatives for Toeplitz operators in excess area identity [ÇDTR+24]

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# Summary of Results

1. Results and Observations influenced by the Area Difference of the image of $\mathbb{D}$ between $zh$ and $h$:
   i. On $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2})$, $A^2(\mathbb{D}, \lambda)$, $A^2(D(0, r))$
   ii. On convergence of identities on certain weighted discs.
2. Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
   i. On unweighted and weighted Toeplitz operators relation
   ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on $A^2(\mathbb{D})$

## Methods Used

- Relation between $L^2$ norms of functions and $\ell^2$ norms of Taylor series:

$$\|h\|_{L^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- Integration by parts via Stokes's theorem on forms:

$$\oint_{b\Omega} f \, dz = \int_{\Omega} \overline{\frac{\partial f}{\partial z}} \, d\bar{z} \wedge dz$$

$$\oint_{b\Omega} f \, d\bar{z} = \int_{\Omega} \frac{\partial f}{\partial z} dz \wedge d\bar{z}.$$

- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Special functions e.g. beta, gamma, hypergeometric

# Excess Area on Fock Spaces

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta = \pi \left\| Sh \right\|_{L^2(b\mathbb{D})}^2$$

## Excess Area on Fock Space

Given $0 < \rho < 1$, $h \in \mathcal{F}^2$, let $h_\rho(z) := h(\rho z)$

$$A_{\mathcal{F}^2}\left(zh_\rho\right) - A_{\mathcal{F}^2}\left(h_\rho\right)$$
$$= \pi \left\| z T_{\bar{z}}^{\mathcal{F}^2}\left(h_\rho\right) \right\|_{\mathcal{F}^2}^2 + \pi \left\| T_{\bar{z}}^{\mathcal{F}^2}\left(h_\rho\right) \right\|_{\mathcal{F}^2}^2 + \pi \left\| H_{\bar{z}}^{\mathcal{F}^2}\left(h_\rho\right) \right\|_{\mathcal{F}^2}^2$$
$$= \pi \left\| z^2 h_\rho \right\|_{\mathcal{F}^2}^2 - 2\pi \left\| z h_\rho \right\|_{\mathcal{F}^2}^2 + \pi \left\| h_\rho \right\|_{\mathcal{F}^2}^2$$

Here, the restriction of $h$ to the unit circle in D'Angelo's identity is replaced with the Bergman projection on $\mathbb{C}$.

## Using Integration by Parts to find Excess Area
## Identity: Wedge Product I

We want to use Integration by Parts and Stokes's Theorem to prove this excess area identity.

The area is integrated with respect to $dA = dx \wedge dy$. The wedge product has the following properties:

$$(a + b) \wedge c = a \wedge c + b \wedge c$$
$$a \wedge b = -b \wedge a$$
$$a \wedge a = 0.$$

## Using Integration by Parts to find Excess Area
## Identity: Wedge Product II

With $z = x + iy$, $\overline{z} = x - iy$, the substitution $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$ yields

$$\begin{aligned}
dx \wedge dy &= \frac{1}{2i} \left( d\overline{z} \wedge dz \right) \\
&= -\frac{1}{2i} \left( dz \wedge d\overline{z} \right).
\end{aligned}$$

The area integral is now rewritten as:

$$\begin{aligned}
\left\langle \frac{\partial h}{\partial z}, \frac{\partial h}{\partial z} \right\rangle_{L^2(\Omega,\lambda)} &= \int_\Omega \left( \overline{\frac{\partial h}{\partial z}} \right) \left( \frac{\partial h}{\partial z} \right) \lambda\left(|z|\right) \; dx \wedge dy \\
&= \frac{1}{2i} \int_\Omega \lambda\left(|z|\right) \left( \left( \overline{\frac{\partial h}{\partial z}} \right) d\overline{z} \right) \wedge \left( \left( \frac{\partial h}{\partial z} \right) dz \right)
\end{aligned}$$

# Using Integration by Parts to find Excess Area Identity: Stokes's Theorem I

In particular,

$$\overline{\frac{\partial}{\partial z}} \left( \left( \lambda \left( |z| \right) \right) \overline{h} \frac{\partial h}{\partial z} \right) d\overline{z} \wedge dz = \underbrace{\left( \lambda \left( |z| \right) \right) \overline{\frac{\partial h}{\partial z}} d\overline{z} \wedge \frac{\partial h}{\partial z} dz}_{\text{area integrand}} + \left( \overline{\frac{\partial}{\partial z}} \lambda \left( |z| \right) \right) \overline{h} \wedge \frac{\partial h}{\partial z} dz$$

meaning

$$\frac{1}{2i} \int_\Omega \frac{\partial h}{\partial z} \overline{\frac{\partial h}{\partial z}} \lambda \left( |z| \right) \ d\overline{z} \wedge dz = \underbrace{\frac{1}{2i} \int_\Omega \overline{\frac{\partial}{\partial z}} \left( \lambda \left( |z| \right) \overline{h} \frac{\partial h}{\partial z} \right) \ d\overline{z} \wedge dz}_{\text{Integral } A}$$

$$- \frac{1}{2i} \int_\Omega \overline{h} \frac{\partial h}{\partial z} \left( \overline{\frac{\partial}{\partial z}} \lambda \left( |z| \right) \right) \ d\overline{z} \wedge dz.$$

# Using Integration by Parts to find Excess Area
## Identity: Stokes's Theorem II

Turning our attention to Integral $A$,

$$\frac{1}{2i} = \int_{\Omega} d\left(\lambda\left(|z|\right)\overline{h}\frac{\partial h}{\partial z}\right) \, d\overline{z} \wedge dz$$

$$= \underbrace{\int_{b\Omega} \lambda\left(|z|\right)\overline{h}\frac{\partial h}{\partial z} \, dz}_{=0}.$$

With this, the area integral is now

$$\frac{1}{2i}\int \frac{\partial h}{\partial z}\overline{\frac{\partial h}{\partial z}}\lambda\left(|z|\right) \, d\overline{z} \wedge dz = -\frac{1}{2i}\int \overline{h}\frac{\partial h}{\partial z}\left(\overline{\frac{\partial}{\partial z}}\lambda\left(|z|\right)\right) \, d\overline{z} \wedge dz$$

# Excess Area on $A^2(\mathbb{D}, \lambda)$

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta = \pi \|Sh\|_{L^2(b\mathbb{D})}^2$$

### Excess Area on $A^2(\mathbb{D}, \lambda)$

Let $h \in A^{1,2}(\mathbb{D}, \lambda)$, $\lambda(z) = 1 - |z|^2$. Then,

$$A_{\mathbb{D}, \lambda}\left(z^{m+1}h\right) - A_{\mathbb{D}, \lambda}\left(z^m h\right) = \pi \|z^m h\|_{L^2(\mathbb{D}, \lambda)}^2.$$

Here, the restriction of $h$ to the unit circle is replaced with the function itself.

# Excess Area on $A^2(D(0, r))$

# Excess Area on $A^2(D(0,r))$, cont'd

## Excess Area Identity for Harmonic Functions on $D(0,r)$, $0 < r < 1$

For a harmonic function $u \in L^2(D(0,r))$, $\exists v \in L^2(D(0,r))$ harmonic conjugate [BÇGH22]. Let $h = u + iv$ be the corresponding holomorphic function. Then,

$$\left\| \frac{\partial(zu)}{\partial z} \right\|_{L^2(D(0,r))}^2 - r^2 \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D(0,r))}^2$$

$$= \frac{1}{4} \left( \underbrace{r^2\pi \|Sh\|_{L^2(bD(0,r))}^2}_{A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h)} + 2r^2\pi\mathfrak{R}(h_0^2) + \|h\|_{L^2(D(0,r))}^2 \right).$$

# Dilation and Contraction from $A^2(D(0,r))$ to $A^2(\mathbb{D})$

Contracting $h \in A^{1,2}(\mathbb{D})$ by taking $h_r = h(rz)$ for some $0 < r < 1$,

$$A_{\mathbb{D}}(zh_r) - A_{\mathbb{D}}(h_r) = \pi \|Sh_r\|_{L^2(b\mathbb{D})}^2 \tag{1}$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \|Sh\|_{L^2(bD(0,r))}^2. \tag{2}$$

Dilating $h \in A^{1,2}(D(0,r))$ by taking $h_{\frac{1}{r}} = h(\frac{z}{r})$ for some $0 < r < 1$

$$A_{D(0,r)}(zh_{1/r}) - r^2 A_{D(0,r)}(h_{1/r}) = \pi r^2 \|Sh_{1/r}\|_{L^2(bD(0,r))}^2 \tag{3}$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \|Sh\|_{L^2(b\mathbb{D})}^2 \tag{4}$$

# Dilation from $A^2\left(D(0,r),\lambda_r\right)$ to $\mathcal{F}^2$

## Weighted Area on $D(0,r)$

Let $\lambda_r(z) = \chi_{D(0,r)}\left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$ where $r > 0$. Then,

$$A_{D(0,r),\lambda_r}(h) = \int_{D(0,r)} |h'(z)|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} \, dA(z)$$

We find that, as $r \to \infty$, $A_{D(0,r),\lambda_r}(h) \to A_{\mathcal{F}^2}(h)$.

Additionally, we know that

$$A_{\mathcal{F}^2}\left(h_\rho\right) = \left\| T_{\bar{z}}^{\mathcal{F}^2} h_\rho \right\|_{\mathcal{F}^2}^2$$

# Berezin Transform Convergence

## Reproducing Kernel on $A^2\left(D(0,r), \lambda_r\right)$

$$K^{\lambda_r}_{D(0,r)}(w,z) = \frac{1}{\left(1 - \frac{\overline{z}w}{r^2}\right)^{r^2+2}}$$

$K^{\lambda_r}_{D(0,r)}(w,z)$ uniformly converges on compact subsets of $D(0,r)$.

## Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w,z) = e^{\overline{z}w}$$

# Berezin Transform Convergence, Cont'd

## Definition (Berezin Transform ([Zhu07])

Let
$$k_z^{\Omega,\lambda}(w) := \frac{K_\Omega^\lambda(w,z)}{\sqrt{K_\Omega^\lambda(z,z)}}$$

Then, for some bounded operator T on $L^2(\Omega, \lambda)$, define
$\mathcal{B}^{\Omega,\lambda} : B(L^2(\Omega, \lambda)) \to L^2(\Omega, \lambda)$

$$(\mathcal{B}^{\Omega,\lambda}T)(z) := \left\langle Tk_z^{\Omega,\lambda}, k_z^{\Omega,\lambda} \right\rangle_{L^2(\Omega,\lambda)}$$

# Berezin Transform Convergence, Cont'd

- For $\varphi \in L^\infty(\Omega, \lambda)$, $\mathcal{B}^{\Omega,\lambda} T_\varphi = \mathcal{B}^{\Omega,\lambda} M_\varphi$. (see Axler and Zheng, [AZ98a]).

- $\varphi$ is harmonic if and only if $\mathcal{B}^{\Omega,\lambda} M_\varphi = \varphi$ (proof by Engliš, [Eng94]).

- We find that, for $T_\varphi^{D(0,r),\lambda_r} = P^{D(0,r),\lambda_r} M_\varphi$, the Berezin transform $\mathcal{B}^{D(0,r),\lambda_r} T_\varphi^{D(0,r),\lambda_r}$ converges pointwise to $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$ as $r \to \infty$ from Göğüş and Şahutoğlu ([Gc20])

- This convergence is uniform on compact subsets of $\mathbb{C}$ (proof inspired by Göğüş and Şahutoğlu in [Gc20]).

# Unweighted and Weighted Toeplitz Operators Relation I

Using an extension of [ÇDTR$^+$24, Lemma 2.1]

For weight $\lambda(z) = \left(1 - |z|^2\right)^{\alpha}$ ($\alpha \geqslant 0$) on the unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

$$\frac{T_{\bar{z}^m}^{\mathbb{D},\lambda_\alpha}(z^n)}{T_{\bar{z}^m}^{\mathbb{D}}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leqslant n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T_{\bar{z}^m}^{\mathbb{D},\lambda_\alpha}(z^n) = s_{n,m,\alpha} T_{\bar{z}^m}^{\mathbb{D}}(z^n), \text{ and } \lim_{n\to\infty} s_{n,m,\alpha} = 1$$

# Unweighted and Weighted Commutator on $A^2(\mathbb{D})$

### Existence of Commutator Symbols

Given $p$ and $q$ are harmonic polynomials and $\frac{\partial}{\partial z}(p) \neq 0$, there does not exist a polynomial symbol $\phi$, such that $\left[P^{\mathbb{D}}, M_\phi\right](p) = q$ or $\left[P^{\mathbb{D},\lambda}, M_\phi\right](p) = q$.

Compare to [ÇDTR⁺24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

# Unweighted and Weighted Hankel Operator on $A^2(\mathbb{D})$

## Existence of Hankel Operator Symbols

Given some holomorphic polynomials $p, q$ where $p$ is not constant, there does not exist a polynomial symbol $\phi$ such that $H_\phi^{\mathbb{D}}(p) = \overline{q}$ or $H_\phi^{\mathbb{D},\lambda}(p) = \overline{q}$

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# Remarks on the Annulus

## Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0, r, 1))$

For all integers $m$ and $n$,

$$T_{\overline{z}^m}^{\mathbb{A}(0,r,1)}(z^n) = \begin{cases} \frac{2mr^{2m}\ln(r)}{(r^{2m}-1)}z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m}-1}{2m\ln(r)}z^{-1} & \text{if } n = m-1 \\ \frac{(n-m+1)(1-r^{2n+2})}{(n+1)(1-r^{2n-2m+2})}z^{n-m} & \text{else} \end{cases}.$$

We attempted to find an algorithm to generate $\varphi \in L^\infty(\mathbb{A}(0, r, 1))$ such that $T_\varphi^{\mathbb{A}(0,r,1)}(p) = q$ for given holomorphic Laurent polynomials $p$ and $q$, but ran into trouble beyond the case where $p$ has roots outside $\mathbb{A}(0, r, 1)$.

# Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial p and target polynomial q on $\mathbb{A}(0, r, 1)$, $T_\varphi^{\mathbb{A}(0,r,1)}(p) = q$

- Extension of 'excess area' identity to harmonic functions in $L^2\left(\mathbb{C}, e^{-|z|^2}\right)$.

- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential, $(1 - |z|^2)^A e^{\frac{-B}{(1-|z|^2)^\alpha}} (A \geqslant 0, B > 0, \alpha > 0)$.

# Contents

# Experience at the REU

- Group consisted of Sakia Akamah (Rose–Hulman Institute of Technology), Jennifer Yuan (NYU Abu Dhabi), and myself.
- Spent 7 weeks doing various calculations, final week spent preparing the final report and presentation.
- Other projects at the REU included machine learning, coding theory, and mathematical biology,

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- Work was not all that we did, though.

# Pictures

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