

This is a collection of old real analysis qualifier exam solutions.

## August 2019

### Problem 1

**Problem:** Let  $\mathcal{C}$  be the Cantor set on  $[0, 1]$ .

- (a) Show that  $\mathcal{C} + \mathcal{C} = [0, 2]$ .
- (b) Find two sets  $A, B \subseteq \mathbb{R}$  that are closed and have Lebesgue measure zero such that  $A + B = \mathbb{R}$ .
- (a) Recall that the Cantor set  $\mathcal{C}$  is defined to consist of all  $x \in [0, 1]$  such that  $x$  only contains 0 and 2 in the ternary expansion of  $x$ . Writing  $a \in [0, 2]$  as

$$a = \sum_{k=0}^{\infty} \frac{a_k}{3^k},$$

where  $a_k \in \{0, 1, 2\}$ , we may then find  $a_k$  at each ternary expansion slot for  $k$  as follows:

- if  $a_k = 0$ , we may find  $b_k, c_k \in \mathcal{C}$  such that  $b_k = c_k = 0$
- if  $a_k = 2$ , we may find  $b_k, c_k \in \mathcal{C}$  such that  $b_k = 2$  and  $c_k = 0$  or vice versa.
- if  $a_k = 1$ , we may find  $b_k, c_k \in \mathcal{C}$  such that  $b_{k+1} = c_{k+1} = 2$ .

Therefore, since every digit of every ternary expansion in  $[0, 2]$  can be obtained from  $\mathcal{C}$ , we see that  $\mathcal{C} + \mathcal{C} = [0, 2]$ .

- (b) We may set  $B$  to be the union of all integer translates of  $\mathcal{C}$ , and set  $A = \mathcal{C}$ . This yields closed subsets of  $\mathbb{R}$  with Lebesgue measure zero that sum to  $\mathbb{R}$ .

### Problem 2

**Problem:** Does there exist a finite measure space  $(X, \mathcal{F}, \mu)$  and a sequence  $(f_n)_n$  of  $\mu$ -measurable functions such that

- $f_n(x) \geq 0$ ;
- $f_n(x) \rightarrow 0$  for all  $x$ ;
- $\int_X f_n(x) d\mu(x) \rightarrow 0$  as  $n \rightarrow \infty$ ;
- $\Phi(x) = \sup_n f_n(x)$  has infinite integral?

Consider the sequence of functions

$$f_n(x) = n \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]},$$

defined on  $[0, 1]$ . This sequence is pointwise convergent everywhere to zero, as  $f_n(0) = 0$  and the Archimedean property give that for any  $x \in (0, 1]$ , there is some  $n$  large enough that gives  $\frac{1}{n} < x$ . Furthermore, we see that

$$\begin{aligned} \int f_n d\mu &= n \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{n+1} \\ &\rightarrow 0. \end{aligned}$$

Finally, we see that by taking suprema, we have the integral

$$\begin{aligned} \int \Phi d\mu &= \sum_{n=1}^{\infty} \frac{1}{n+1} \\ &\rightarrow \infty. \end{aligned}$$

### Problem 4

**Problem:** Let  $L_1(\mathbb{R})$  be the space of Lebesgue integrable functions on  $\mathbb{R}$ . Suppose  $f \in L_1(\mathbb{R})$  is positive. Show that  $\frac{1}{f(x)} \notin L_1(\mathbb{R})$ .

Suppose toward contradiction that both  $f$  and  $1/f$  are in  $L_1(\mathbb{R})$ . Then, from Hölder's Inequality, we have

$$\begin{aligned} \infty &= \int 1 \, d\mu \\ &\leq \left( \int f \, d\mu \right)^{1/2} \left( \int \frac{1}{f} \, d\mu \right)^{1/2} \\ &< \infty, \end{aligned}$$

which is a contradiction.

### Problem 5

**Problem:** Applying the Gram-Schmidt orthogonalization to  $\{1, x, x^2, \dots\}$  in the Hilbert space  $L_2([-1, 1])$  with Lebesgue measure, one gets the Legendre polynomials  $L_n(x)$ .

- (a) Show that the Legendre polynomials form a basis (complete orthogonal system) in the Hilbert space  $L_2([-1, 1])$ .
- (b) Show that the Legendre polynomials are given by the formula  $L_n(x) = c_n \frac{d^n}{dx^n} (x^2 - 1)^n$ .
- (a) Let  $f \in L_2([-1, 1])$ . We may find  $g \in C([-1, 1])$  such that  $\|f - g\|_{L_2} < \varepsilon/2$ . Similarly, we may find a polynomial  $p$  such that  $\|g - p\|_{L_2} < \varepsilon/4$ , meaning that  $|p(x) - g(x)| < \varepsilon/4$  for all  $x \in [-1, 1]$ . This yields

$$\begin{aligned} \|p - g\|_{L_2} &= \left( \int_{-1}^1 |p(x) - g(x)|^2 \, dx \right)^{1/2} \\ &< \left( \int_{-1}^1 \left( \frac{\varepsilon}{4} \right)^2 \, dx \right)^{1/2} \\ &= \left( \frac{\varepsilon^2}{8} \right)^{1/2} \\ &< \frac{\varepsilon}{2}, \end{aligned}$$

so  $\|f - p\|_{L_2} < \varepsilon$ , meaning that the closed linear span of the monomials is dense in  $L_2$ , and the Legendre polynomials form an orthonormal system.

- (b) We see that at every step in evaluating the expression

$$L_n(x) = c_n \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (*)$$

the degree of the polynomial increases by 1, so each  $L_n(x)$  has degree  $n$ . To verify that the polynomials generated from  $(*)$  are orthogonal to each other, we let  $n > m$  without loss of generality, and use integration by parts to obtain

$$\begin{aligned} \langle L_n, L_m \rangle &= \int_{-1}^1 \left( \frac{d^n}{dx^n} (x^2 - 1)^n \right) \left( \frac{d^m}{dx^m} (x^2 - 1)^m \right) dx \\ &= \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^m}{dx^m} (x^2 - 1)^m \Big|_{-1}^1 - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^{m+1}}{dx^{m+1}} (x^2 - 1)^m dx \\ &\vdots \end{aligned}$$

$$\begin{aligned}
&= (-1)^n \int_{-1}^1 \frac{d^{m+n}}{dx^{m+n}} (x^2 - 1)^m dx \\
&= (-1)^n \int_{-1}^1 \frac{d^n}{dx^n} \left( \frac{d^m}{dx^m} (x^2 - 1)^m \right) dx \\
&= (-1)^n \int \frac{d^n}{dx^n} L_m(x) dx \\
&= 0,
\end{aligned}$$

seeing as we are taking  $n$  derivatives of a degree  $m < n$  polynomial.

## January 2020

### Problem 1

**Problem:** Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$ , and let  $A \subseteq [0, 1]$  be Lebesgue-measurable.

- (a) Prove or show a counterexample to the assertion that

$$\mu(A) = \sup_{\substack{U \subseteq A \\ U \text{ open}}} \mu(U).$$

- (b) Prove or show a counterexample to the assertion that

$$\mu(A) = \inf_{\substack{A \subseteq U \\ U \text{ open}}} \mu(U).$$

- (a) This is false. If  $A \subseteq [0, 1]$  is the “fat Cantor set” constructed similar to the traditional Cantor, but obtained by deleting the middle fourth of each subinterval rather than the middle third, then  $\mu(A) = \frac{1}{2}$ , but  $A$  is nowhere dense, meaning that if  $U \subseteq A$  is open, then  $U = \emptyset$ .

To see that  $A$  is nowhere dense, we see that  $A$  is closed, so if  $x \in A \subseteq [0, 1]$ , and  $\varepsilon > 0$ , we may show that the interval  $(x - \varepsilon, x + \varepsilon)$  is not contained in  $A$ . In the recursive construction of  $A$ , we may see that there is some step  $n_1$  such that  $\frac{1}{4^{n_1}} < 2\varepsilon$ , implying that  $(x - \varepsilon, x + \varepsilon)$  is not contained in the recursive construction at  $n_1$ . Therefore  $A^\circ = \emptyset$ .

- (b) This is true. By the definition of the Lebesgue outer measure, for any  $\varepsilon > 0$ , there are  $\{(a_k, b_k)\}_{k=1}^\infty$  such that

$$\mu(A) + \varepsilon < \mu\left(\bigcup_{k=1}^\infty (a_k, b_k)\right),$$

so by setting

$$U = \bigcup_{k=1}^\infty (a_k, b_k),$$

we have that  $U$  is open, meaning that by the definition of infimum, we get

$$\mu(A) = \inf\{\mu(U) \mid A \subseteq U, U \text{ open}\}.$$

**Remark:** Part (a) can be solved by selecting  $A = \mathbb{R} \setminus \mathbb{Q} \cap [0, 1]$ .

### Problem 3

**Problem:** Let  $X$  be a compact metric space,  $C(X)$  the space of real-valued continuous functions on  $X$  with the supremum norm. Assume that  $\mathcal{A} \subseteq C(X)$  satisfies

- (algebra) for all  $f, g \in \mathcal{A}$ ,  $\alpha, \beta \in \mathbb{R}$ , we have  $\alpha f + \beta g \in \mathcal{A}$  and  $fg \in \mathcal{A}$ ;
- (separates points) for any  $x \neq y$  in  $X$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .

- Show by example that  $\mathcal{A}$  need not be dense in  $C(X)$ .
  - In order to conclude that  $\mathcal{A}$  is dense by the Stone–Weierstrass Theorem, what additional condition(s) should be added.
- Consider the algebra of polynomials on  $[0, 1]$  without a constant term. Then, since linear combinations and multiplications still yield polynomials without constant term, and  $f(x) = x$  separates points in  $[0, 1]$ , this algebra satisfies the requirements of the question. Yet, since all elements of this algebra are equal to zero at  $x = 0$ , the uniform closure of the algebra yields all the continuous functions on  $[0, 1]$  with  $f(0) = 0$ .
  - In order to satisfy the requirements of the Stone–Weierstrass theorem, we need the algebra  $\mathcal{A}$  to include the constant functions.

### Problem 4

**Problem:** Let  $\mu$  be a measure on  $(\mathbb{R}, \mathcal{B})$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra. Let  $\mu(\mathbb{R}) = 1$ . Next, let  $\mathcal{F} \subseteq \mathcal{B}$  be the sub- $\sigma$ -algebra generated by symmetric intervals.

Let  $f \in L_1(\mathbb{R}, \mathcal{B}, \mu)$ . Find a function  $g$  such that:

- $g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$  (in particular,  $g$  is  $\mathcal{F}$ -measurable);
- for all  $E \in \mathcal{F}$ ,  $\int_E g \, d\mu = \int_E f \, d\mu$ .

We consider the signed measure on  $\mathcal{F}$  defined by

$$\nu(E) = \int_E f \, d\mu,$$

meaning that  $\nu \ll \mu$ , so the function  $g := \frac{d\nu}{d\mu}$ , where  $\frac{d\nu}{d\mu}$  denotes the Radon–Nikodym derivative of  $\nu$  with respect to  $\mu$  (where we restrict  $\mu$  to  $\mathcal{F}$ ), is  $\mathcal{F}$ -measurable (by Radon–Nikodym) and in  $L_1(\mathbb{R}, \mathcal{F}, \mu)$ . This gives, for all  $E \in \mathcal{F}$ ,

$$\begin{aligned} \int_E g \, d\mu &= \int_E \frac{d\nu}{d\mu} \, d\mu \\ &= \int_E d\nu \\ &= \nu(E) \\ &= \int_E f \, d\mu. \end{aligned}$$

### Problem 5

**Problem:** Let  $\mu$  be a finite measure on  $(X, \mathcal{F})$ . Show that a sequence of  $\mathcal{F}$ -measurable functions  $(f_n)_n$  converges to  $f$  in measure if and only if

$$\int_X \min\{1, |f_n - f|\} \, d\mu(x) \rightarrow 0.$$

Let  $M = \mu(X)$ .

Let  $(f_n)_n \rightarrow f$  in measure, and let  $\varepsilon > 0$ . If we let

$$\begin{aligned} A &= \{x \mid |f_n(x) - f(x)| > \varepsilon/2M\} \\ B &= \{x \mid |f_n(x) - f(x)| \leq \varepsilon/2M\}, \end{aligned}$$

we have

$$\begin{aligned} \int_X \min(1, |f_n - f|) \, d\mu &= \int_A \min(1, |f_n - f|) \, d\mu + \int_B \min(1, |f_n - f|) \, d\mu \\ &\leq \mu(A) + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Meanwhile, if

$$\int_X \min(1, |f_n - f|) \, d\mu \rightarrow 0,$$

then by Chebyshev's Inequality, we have, for a fixed  $0 < \varepsilon \leq 1$ ,

$$\begin{aligned} \mu(\{x \mid |f_n - f| \geq \varepsilon\}) &= \mu(\{x \mid \min(1, |f_n - f|) \geq \varepsilon\}) \\ &\leq \frac{1}{\varepsilon} \int_X \min(1, |f_n - f|) \, d\mu \\ &\rightarrow 0, \end{aligned}$$

so  $(f_n)_n \rightarrow f$  in measure.

## August 2020

### Problem 1

**Problem:** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and almost everywhere differentiable such that  $f'(x) = 1$  almost everywhere. Does this imply that  $f(2) - f(1) = 1$ ?

This is false. To see this, let  $\mathcal{C}(x)$  denote the Cantor–Lebesgue function, and let

$$h(x) = \sum_{n=-\infty}^{\infty} \mathcal{C}(x - n) + n.$$

Then, since  $\mathcal{C}(x)$  has derivative zero almost everywhere, the sum of a number of translates of  $\mathcal{C}(x)$  still has derivative zero almost everywhere. Then, setting

$$f(x) = h(x) + x,$$

we get that  $f(x)$  has derivative equal to 1 almost everywhere. However, at the same time,  $f(2) - f(1) = 2$ .

### Problem 2

**Problem:** Prove or provide a counterexample to the assertion that every open set in  $\mathbb{R}^2$  is a countable union of closed sets.

We show the inverse problem, which is that every closed set in  $\mathbb{R}^2$  is  $G_\delta$ . To do this, we let  $A \subseteq \mathbb{R}^2$  be closed, nonempty, and proper (if  $A = \emptyset$  or  $A = \mathbb{R}^2$  the answer is trivial).

Then, there is some  $x \in A^c$ , and specifically there is  $x \in A^c$  with rational coordinates (else, select  $y \in \mathbb{Q}^2$  within the ball of radius  $\varepsilon$  that allows  $A^c$  to be open). Furthermore, since  $\mathbb{R}^2$  is a metric space,  $\mathbb{R}^2$  is regular, so there are open  $U_x$  and  $V_x$  such that  $A \subseteq U_x$ ,  $x \in V_x$ , and  $U_x \cap V_x = \emptyset$ .

Therefore, we get

$$A = \bigcap \{U_x \mid x \in \mathbb{Q}^2 \setminus A\},$$

meaning that  $A$  is  $G_\delta$ . Taking complements, we thus get that every open set is  $F_\sigma$ .

### Problem 3

**Problem:** Let  $\mathcal{H}$  be a separable complex Hilbert space with basis  $(f_n)_n$ . Define  $P(f_n) = f_{n+1}$ .

(a) Find  $P^*$ , the adjoint to  $P$ .

(b) Find  $PP^*$  and  $P^*P$ .

(a) We see that

$$\begin{aligned} \langle Pf_i, f_j \rangle &= \delta_{i+1,j} \\ &= \delta_{i,j-1} \\ &= \langle f_i, f_{j-1} \rangle \\ &= \langle f_i, P^*f_j \rangle, \end{aligned}$$

so that  $Pf_n = f_{n-1}$  if  $n > 1$ . Else, if  $n = 1$ , then  $P^*f_n = 0$ .

(b) We see that, acting on the orthonormal basis  $(f_n)_n$ ,  $P^*P(f_n) = f_n$ , and

$$PP^*(f_n) = \begin{cases} 0 & n = 1 \\ 1 & \text{else,} \end{cases}$$

so that  $P^*P = I$  and  $PP^*$  is as above.

### Problem 4

**Problem:** Let  $(X, \mathcal{F}, \mu)$  be a measure space with  $\mu(X) = 1$ . Let  $f_n : X \rightarrow \mathbb{R}$  be measurable functions such that

$$\lim_{n \rightarrow \infty} \mu(\{x \mid f_n(x) \leq t\}) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

Show that  $f_n \rightarrow 0$  in measure.

We see that

$$\mu(\{x \mid f_n(x) > t\}) = \mu(X) - \mu(\{x \mid f_n(x) \leq t\}),$$

so by taking limits, we find that

$$\lim_{n \rightarrow \infty} \mu(\{x \mid f_n(x) > t\}) = \begin{cases} 1 & t < 0 \\ 0 & t \geq 0 \end{cases}.$$

So, if  $\varepsilon > 0$ , then

$$\begin{aligned} \mu(\{x \mid |f_n(x)| > \varepsilon\}) &= \mu(\{x \mid f_n(x) < -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\leq \mu(\{x \mid f_n(x) \leq -\varepsilon\}) + \mu(\{x \mid f_n(x) > \varepsilon\}) \\ &\rightarrow 0. \end{aligned}$$

## January 2021

### Problem 1

**Problem:** Let  $(f_n)_n, f$  be measurable functions on  $(\Omega, \mathcal{F}, \mu)$  such that  $f_n \rightarrow f$  in measure. Does this imply that there exists a measurable set  $A \subseteq \Omega$  with  $\mu(\Omega \setminus A) = 0$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in A$ .

This is not true. To see this, consider the family of functions defined by

$$\begin{aligned} f_1 &= \mathbb{1}_{[0,1]} \\ f_2 &= \mathbb{1}_{[0,1/2]} \\ f_3 &= \mathbb{1}_{[1/2,1]} \\ &\vdots \end{aligned}$$

where  $f_n$  is of width  $\frac{1}{2^k}$  when  $2^k \leq n < 2^{k+1}$ , moving along  $[0, 1]$ . Then, since  $\mu(\{x \mid |f_n(x)| > 0\}) = \frac{1}{2^k}$ , we have that  $(f_n)_n \rightarrow 0$  in measure. Yet, since for any  $x \in [0, 1]$  there are infinitely many such  $n$  such that  $f_n(x) = 1$ , the family  $(f_n)_n$  does not converge to 0 pointwise anywhere on  $[0, 1]$ .

### Problem 2

**Problem:** Let  $B$  be a measurable subset of the two-dimensional plane such that the intersection of  $B$  with every vertical line is either finite or countable. Find  $\mu(B)$ , where  $\mu$  is the two-dimensional Lebesgue measure.

Note that the two-dimensional Lebesgue measure is the completion of  $m \times m$ , where  $m \times m$  is the product measure on the product  $\sigma$ -algebra  $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ . If  $B \in \mathcal{L}(\mathbb{R}^2)$ , then  $B = C \cup N$ , where  $N$  is a  $\mu$ -null set and  $C \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ . Therefore, if we show that  $(m \times m)(C) = 0$ , we then show that  $\mu(B) = 0$ .

To see that  $(m \times m)(C) = 0$ , note that by our assumption,  $B^x = \{y \in \mathbb{R} \mid (x, y) \in B\}$  is either finite or countable, so since  $C^x \subseteq B^x$ , we must have that  $C^x$  is either finite or countable. By Tonelli's Theorem, since  $\mathbb{1}_C$  is positive, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \mathbb{1}_C \, d(m \times m) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{C^x} \, dy \, dx \\ &= \int_{\mathbb{R}} m(C^x) \, dx \\ &= 0, \end{aligned}$$

so  $(m \times m)(C^x) = 0$ , meaning

$$\begin{aligned} \mu(B) &= \mu(C) + \mu(N) \\ &= (m \times m)(C) + \mu(N) \\ &= 0. \end{aligned}$$

### Problem 3

**Problem:** Let  $(\Omega, \mathcal{F})$  be a measurable space,  $\mu, \nu, \rho$  finite positive measures with  $\mu \ll \nu$ . Show that there exists a measurable function  $f$  on  $\Omega$  such that for all  $E \in \mathcal{F}$ ,

$$\mu(E) = \int_E f \, d\nu + \int_E (f - 1) \, d\rho.$$

Since  $\mu \ll \nu$ , and  $\rho \ll \nu$ , we have  $\mu + \rho \ll \nu + \rho$ , so by Radon–Nikodym, there is some measurable  $f$  such that

$$\mu(E) + \rho(E) = \int_E f \, d(\nu + \rho),$$

so

$$\mu(E) = \int_E f \, d\nu + \int_E (f - 1) \, d\rho.$$

#### Problem 4

**Problem:** Let  $f, g$  be nonnegative measurable functions on  $[0, 1]$ , and let  $a, b, c, d \geq 0$  be arbitrary nonnegative numbers. Show that

$$\left( ac + bd + \int_0^1 f(x)g(x) \, dx \right)^3 \leq \left( a^3 + b^3 + \int_0^1 (f(x))^3 \, dx \right) \left( c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx \right)^2.$$

Since all of  $f, g, a, b, c, d$  are positive, we may show

$$ac + bd + \int_0^1 f(x)g(x) \, dx \leq \left( a^3 + b^3 + \int_0^1 (f(x))^3 \, dx \right)^{1/3} \left( c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx \right)^{2/3}.$$

To do this, we use Hölder's Inequality three times:

$$\begin{aligned} ac + bd + \int_0^1 f(x)g(x) \, dx &\leq \left( a^3 + b^3 \right)^{1/3} \left( c^{3/2} + d^{3/2} \right)^{2/3} + \int_0^1 f(x)g(x) \, dx \\ &\leq \left( a^3 + b^3 \right)^{1/3} \left( c^{3/2} + d^{3/2} \right)^{2/3} + \left( \int_0^1 (f(x))^3 \, dx \right)^{1/3} \left( \int_0^1 (g(x))^{3/2} \, dx \right)^{2/3} \\ &\leq \left( a^3 + b^3 + \int_0^1 (f(x))^3 \, dx \right)^{1/3} \left( c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx \right)^{2/3}. \end{aligned}$$

#### Problem 5

**Problem:** Let  $f(x)$  be a continuous function on  $[0, 1]$ . Show that for every  $\varepsilon > 0$  there exists  $n \in \mathbb{Z}_{\geq 0}$  and  $a_0, a_1, \dots, a_n \in \mathbb{R}$  such that for

$$D := \sum_{k=0}^n a_k \left( \frac{d}{dx} \right)^k,$$

we have

$$\left| f(x) - e^{x^2} \left( D e^{-x^2} \right) \right| < \varepsilon$$

for all  $x \in [0, 1]$ .

We note that for each  $n$ ,

$$\left( \frac{d}{dx} \right)^n \left( e^{-x^2} \right) = P_n(x) e^{-x^2}$$

where  $P_n(x)$  is a degree  $n$  polynomial. To see this,

$$\begin{aligned} \frac{d}{dx} \left( e^{-x^2} \right) &= -2xe^{-x^2} \\ \frac{d}{dx} \left( P_n(x) e^{-x^2} \right) &= P'_n(x) e^{-x^2} - 2xP_n(x) e^{-x^2} \\ &:= P_{n+1}(x) e^{-x^2}. \end{aligned}$$

Therefore,

$$e^{x^2} \left( \frac{d}{dx} \right)^n \left( e^{-x^2} \right) = P_n(x).$$



Since each  $P_n(x)$  is linearly independent (as they have different degrees of polynomials), the linear combinations of  $P_n$  form polynomials. By Stone–Weierstrass, there is some linear combination  $\sum_{k=0}^n a_k P_k(x)$  such that  $|f(x) - \sum_{k=0}^n a_k P_k(x)| < \varepsilon$  for all  $x \in [0, 1]$ . Thus,

$$\left| f(x) - e^{x^2} \left( D e^{-x^2} \right) \right| < \varepsilon$$

for the particular expression

$$D = \sum_{k=0}^n a_k \left( \frac{d}{dx} \right)^k.$$

## August 2022

### Problem 1

**Problem:** Compute

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx.$$

We note that

$$\begin{aligned} \left| \frac{n \sin(x/n)}{x(1+x^2)} \right| &\leq \left| \frac{n(x/n)}{x(1+x^2)} \right| \\ &= \frac{1}{1+x^2}, \end{aligned}$$

and since  $\frac{1}{1+x^2}$  is integrable, we may use Dominated Convergence to switch limit and integral, giving

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx &= \int_0^\infty \lim_{n \rightarrow \infty} \frac{n \sin(x/n)}{x(1+x^2)} dx \\ &= \int_0^\infty \lim_{h \rightarrow 0} \frac{\frac{1}{h} \sin(hx)}{x(1+x^2)} dx \\ &= \int_0^\infty \frac{x}{x(1+x^2)} dx \\ &= \frac{\pi}{2}. \end{aligned}$$

### Problem 2

**Problem:** Fix  $a < b$  in  $\mathbb{R}$ . For a Lipschitz function  $g: [a, b] \rightarrow \mathbb{C}$ , set

$$\|g\|_{\text{Lip}} = \sup_{x \neq y \in [a, b]} \frac{|g(x) - g(y)|}{|x - y|}.$$

- (a) Show that  $f: [a, b] \rightarrow \mathbb{C}$  is Lipschitz if and only if  $f$  is absolutely continuous and  $f' \in L_\infty([a, b])$ .
- (b) If  $f: [a, b] \rightarrow \mathbb{C}$  is Lipschitz, show that  $\|f\|_{\text{Lip}} = \|f'\|_{L_\infty}$ .

- (a) Let  $f$  be Lipschitz, and let  $M$  denote the Lipschitz constant — i.e.,  $|f(x) - f(y)| \leq |x - y|$  for all  $x, y \in [a, b]$ . Set  $\delta = \frac{\varepsilon}{M}$ . Then, if  $\{(a_j, b_j)\}_{j=1}^k$  is a partition such that  $\sum_{j=1}^k |b_j - a_j| < \delta$ , we have

$$\sum_{j=1}^k |f(b_j) - f(a_j)| \leq M \sum_{j=1}^k |b_j - a_j|$$

$$< \varepsilon.$$

Thus,  $f$  is absolutely continuous. Now, if  $x, x + h \in [a, b]$ , we have that

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq M,$$

meaning that

$$\begin{aligned} |f'(x)| &= \lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x)}{h} \right| \\ &\leq M, \end{aligned}$$

and since  $f'(x)$  exists for a.e.  $x \in [a, b]$ , we have that  $\text{ess sup}_{x \in [a, b]} |f'(x)| \leq M$ , so  $f' \in L_\infty([a, b])$ .

Let  $f$  be absolutely continuous with bounded derivative. Then, if  $M$  is the essential supremum of the  $f'$ , the fundamental theorem of calculus gives

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_x^y f'(t) dt \right| \\ &\leq \int_x^y |f'(t)| dt \\ &\leq \int_x^y M dx \\ &= M|y - x|, \end{aligned}$$

so  $f$  is Lipschitz.

(b) If  $f$  is such that  $f'(x)$  exists, then for  $x, x + h \in [a, b]$ , we have

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq \|f'\|_{Lip},$$

so by taking limits, we have

$$|f'(x)| \leq \|f'\|_{Lip}.$$

Thus, this ordering must respect essential suprema, meaning

$$\|f'\|_{L_\infty} \leq \|f'\|_{Lip}.$$

Furthermore, if  $\varepsilon > 0$ , there are  $x, y \in [a, b]$  with  $x < y$  such that

$$\begin{aligned} \|f'\|_{Lip} - \varepsilon &< \left| \frac{f(y) - f(x)}{y - x} \right| \\ &= \frac{1}{|y - x|} \left| \int_x^y f'(t) dt \right| \\ &\leq \frac{1}{|y - x|} \int_x^y |f'(t)| dt \\ &\leq \frac{1}{|y - x|} \int_x^y \|f'\|_{L_\infty} dt \\ &= \|f'\|_{L_\infty}, \end{aligned}$$

and since  $\varepsilon$  is arbitrary, we have  $\|f'\|_{Lip} \leq \|f'\|_{L_\infty}$ .

### Problem 3

**Problem:** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. Show that if  $f, g \in L_1(X, \mu)$  with  $0 \leq f, g$  almost everywhere, then

$$\|f - g\|_{L_1} = \int_0^\infty \mu(\{x \mid f(x) > t\} \Delta \{x \mid g(x) > t\}) dt.$$

We start by showing that

$$|a - b| = \int_0^\infty |\mathbb{1}_{(t, \infty)}(a) - \mathbb{1}_{(t, \infty)}(b)| dt$$

for all  $a, b \in [0, \infty)$ . Without loss of generality,  $a \leq b$ . To see this, note that there are three cases:

$$|\mathbb{1}_{(t, \infty)}(a) - \mathbb{1}_{(t, \infty)}(b)| = \begin{cases} 0 & t < a, b \\ 1 & a \leq t < b, \\ 0 & a, b \leq t \end{cases}$$

giving

$$\begin{aligned} \int_0^\infty \mathbb{1}_{[a, b)} dt &= \mu([a, b)) \\ &= b - a \\ &= |a - b|. \end{aligned}$$

Now, we have

$$\begin{aligned} \|f - g\|_{L_1} &= \int_X |f(x) - g(x)| d\mu(x) \\ &= \int_X \int_0^\infty |\mathbb{1}_{(t, \infty)}(f(x)) - \mathbb{1}_{(t, \infty)}(g(x))| dt d\mu(x), \end{aligned}$$

and by Tonelli's Theorem, we have

$$\begin{aligned} &= \int_0^\infty \int_X |\mathbb{1}_{f^{-1}((t, \infty))} - \mathbb{1}_{g^{-1}((t, \infty))}| d\mu(x) dt \\ &= \int_0^\infty \int_X \mathbb{1}_{f^{-1}((t, \infty)) \Delta g^{-1}((t, \infty))} d\mu(x) dt \\ &= \int_0^\infty \mu(f^{-1}((t, \infty)) \Delta g^{-1}((t, \infty))) dt. \end{aligned}$$

### Problem 4

**Problem:** Let  $(X, \Sigma)$  be a measurable space. Suppose that  $\mu, \nu$  are signed measures on  $\Sigma$  such that  $\|\mu\|_{TV}, \|\nu\|_{TV} < \infty$ , and  $|\mu| \perp |\nu|$ .

- (a) If  $\mu = \mu_1 - \mu_2$  and  $\nu = \nu_1 - \nu_2$  with  $\mu_1 \perp \mu_2$  and  $\nu_1 \perp \nu_2$ , show that  $\mu_i \perp \nu_j$  for all  $i, j \in \{1, 2\}$ .
- (b) Show that

$$\|\mu + \nu\|_{TV} = \|\mu\|_{TV} + \|\nu\|_{TV}.$$

- (a) Since  $|\mu| \perp |\nu|$ , there are  $U, V \subseteq X$  such that  $|\mu|$  is concentrated on  $U$  and  $|\nu|$  is concentrated on  $V$ , with  $U \cap V = \emptyset$ .

Note that by the Jordan decompositions, we have  $|\mu| = \mu_1 + \mu_2 \geq \mu_{1,2}$  so  $\mu_{1,2}$  are concentrated on  $U$ , and similarly  $\nu_{1,2}$  are concentrated on  $V$ , so  $\mu_i \perp \nu_j$ .

(b) We show that the measures  $\mu_1 + \nu_1$  and  $\mu_2 + \nu_2$  are mutually singular. To see this, note the following:

- $\mu_1 = 0$  on  $N_\mu \cup V$ ;
- $\nu_1 = 0$  on  $N_\nu \cup U$ ;
- $\mu_2 = 0$  on  $P_\mu \cup V$ ;
- $\nu_2 = 0$  on  $P_\nu \cup U$ ,

so  $\mu_1 + \nu_1 = 0$  on  $A = (N_\mu \cup V) \cap (N_\nu \cup U)$ , and  $\mu_2 + \nu_2 = 0$  on  $B = (P_\mu \cup V) \cap (P_\nu \cup U)$ . Therefore, since

$$\begin{aligned} A \cup B &= (N_\mu \cap N_\nu) \cup (N_\mu \cap U) \cup (N_\nu \cap V) \\ &\quad \cup (P_\mu \cap P_\mu) \cup (P_\mu \cap U) \cup (P_\nu \cap V) \\ &= X \end{aligned}$$

$$\begin{aligned} A \cap B &= (N_\mu \cup V) \cap (N_\nu \cup U) \\ &\quad \cap (P_\mu \cup V) \cap (P_\nu \cup U) \\ &= \emptyset, \end{aligned}$$

the measures  $\mu_1 + \nu_1$  and  $\mu_2 + \nu_2$  are mutually singular, so  $A \sqcup B$  forms a Hahn decomposition for  $\mu + \nu$  with corresponding Jordan decomposition of  $(\mu_1 + \nu_1) - (\mu_2 + \nu_2)$ . Thus,

$$\begin{aligned} \|\mu + \nu\|_{TV} &= |\mu + \nu|(X) \\ &= (\mu_1 + \nu_1)(X) + (\mu_2 + \nu_2)(X) \\ &= (\mu_1 + \mu_2)(X) + (\nu_1 + \nu_2)(X) \\ &= |\mu|(X) + |\nu|(X) \\ &= \|\mu\|_{TV} + \|\nu\|_{TV}. \end{aligned}$$

## Problem 5

**Problem:**

(a) For  $f \in L_1([0, 1])$ , let  $L_f$  be the set of all  $x \in [0, 1]$  such that

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)| dy = 0.$$

State the conclusion of the Lebesgue differentiation theorem regarding  $L_f$ .

(b) For  $n \in \mathbb{N}$ ,  $0 \leq j \leq 2^n - 1$ , set  $I_{n,j} = [j2^{-n}, (j+1)2^{-n})$ . For  $f \in L_1([0, 1])$ , define

$$E_n f = \sum_{j=0}^{2^n-1} \left( \frac{1}{m(I_{n,j})} \int_{I_{n,j}} f(t) dt \right) \mathbb{1}_{I_{n,j}}.$$

Show that  $\lim_{n \rightarrow \infty} (E_n f)(x) = f(x)$  for a.e.  $x \in [0, 1]$ .

(a) The conclusion of the Lebesgue differentiation theorem states that  $\mu([0, 1] \setminus L_f) = 0$ .

(b) Let  $x \in [0, 1]$ . We note that  $x$  must be in exactly one such interval  $[j2^{-n}, (j+1)2^{-n}]$  since these intervals are disjoint. If we select  $r > 0$  such that  $\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$ , then we note the following:

- $I_{n,j} \subseteq U(x, r)$  for exactly one such  $j$ ;
- $m(U(x, r)) \leq 4m(I_{n,j})$ .

If  $x \in L_f$ , then for any  $\varepsilon > 0$ , there is some  $\delta > 0$  such that when  $r < \delta$ , then

$$\frac{1}{\mu(U(x, r))} \int_{U(x, r)} |f(t) - f(x)| dt < \varepsilon,$$

by the Lebesgue Differentiation Theorem. If  $n$  is such that  $\frac{1}{2^{n-1}} < \delta$ , then when  $\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$ , then for any  $x \in L_f$ , we have

$$\begin{aligned} |E_n f(x) - f(x)| &= \left| \frac{1}{m(I_{n,j})} \int_{I_{n,j}} f(t) dt - f(x) \right| \\ &\leq \frac{1}{m(I_{n,j})} \int_{I_{n,j}} |f(t) - f(x)| dt \\ &\leq \frac{1}{m(I_{n,j})} \int_{U(x, r)} |f(t) - f(x)| dt \\ &\leq \frac{4}{\mu(U(x, r))} \int_{U(x, r)} |f(t) - f(x)| dt \\ &< 4\varepsilon, \end{aligned}$$

so  $\lim_{n \rightarrow \infty} E_n f(x) = f(x)$  for all  $x \in L_f$ , meaning that it holds for a.e.  $x \in [0, 1]$ .

## January 2023

### Problem 1

**Problem:** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space,  $p \in [1, \infty)$ . Let  $(f_n)_n$  be a sequence in  $L_p(X, \mu)$ , and suppose  $\|f_n\|_{L_p} \leq 1$ ,  $(f_n)_n \rightarrow f$  almost everywhere. Show that  $\|f\|_p \leq 1$ .

By using Fatou's Lemma, and assuming WLOG that  $(f_n)_n \rightarrow f$  pointwise everywhere, we get

$$\begin{aligned} \int_X |f|^p d\mu &= \int_X \liminf_{n \rightarrow \infty} |f_n|^p d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p d\mu \\ &\leq 1, \end{aligned}$$

so  $\|f\|_{L_p} \leq 1$ .

### Problem 2

**Problem:** Let  $\mu$  be an atomless Borel probability measure on  $\mathbb{R}$ . Suppose  $E \subseteq \mathbb{R}$  is a Borel set with  $\mu(E) > 0$ . Show that there is  $t \in \mathbb{R}$  with  $\mu(E \cap (-\infty, t)) = \frac{1}{2}\mu(E)$ .

Let

$$f(t) = \mu(E \cap (-\infty, t)),$$

and for any sequence  $(t_n)_n$ , define

$$E_n = E \cap (-\infty, t_n).$$

We will show that  $f$  is left- and right-continuous, hence continuous. To start, if  $(t_n)_n \searrow t$ , then

$$\bigcap_{n \in \mathbb{N}} E_n = E \cap (-\infty, t],$$

so

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n \setminus \{t\}\right) \\ &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) - \mu(\{t\}). \end{aligned}$$

Since  $\mu$  is atomless, we see that  $\mu(\{t\}) = 0$ , so since  $\mu(E) < \infty$ ,

$$\begin{aligned} f(t) &= \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \lim_{n \rightarrow \infty} f(t_n). \end{aligned}$$

Thus,  $f$  is right-continuous. Similarly, if  $f$  is left-continuous, and  $(t_n)_n \nearrow t$ , then

$$\bigcup_{n \in \mathbb{N}} E_n = E \cap (-\infty, t),$$

so by continuity from below,

$$\begin{aligned} f(t) &= \mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \lim_{n \rightarrow \infty} f(t_n). \end{aligned}$$

Therefore,  $f$  is continuous. Since

$$\begin{aligned} \lim_{t \rightarrow -\infty} f(t) &= 0 \\ \lim_{t \rightarrow \infty} f(t) &= \mu(E) \\ &> 0, \end{aligned}$$

the intermediate value theorem gives some  $t_0 \in \mathbb{R}$  such that

$$\begin{aligned} f(t_0) &= \mu(E \cap (-\infty, t_0)) \\ &= \frac{1}{2} \mu(E). \end{aligned}$$

## Problem 4

**Problem:** Fix  $p \in [1, \infty)$ . Let  $W_p([0, 1])$  be the space of absolutely continuous functions on  $[0, 1]$  such that  $f' \in L_p([0, 1])$ . For all  $f \in W_p([0, 1])$ , define

$$\|f\|_{W_p} = |f(0)| + \|f'\|_{L_p}.$$

Show that  $\|\cdot\|_{W_p}$  is a norm that makes  $W_p([0, 1])$  into a Banach space. You are allowed to use the fact that  $L_p([0, 1])$  is a Banach space.

We start by showing that  $\|\cdot\|_{W_p}$  is indeed a norm. To see that  $\|\cdot\|_{W_p}$  is positive definite, if

$$\|f\|_{W_p} = 0,$$

then  $|f(0)| = 0$  and  $\|f'\|_{L_p} = 0$ . Since  $\|f'\|_{L_p} = 0$ ,  $f' = 0$  a.e. as  $L_p$  is a Banach space. Note that, by the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t) dt,$$

so  $f(x) = 0$  almost everywhere, hence  $f(x) = 0$  in  $L_p$ .

Next, to see homogeneity, we have for all  $\alpha \in \mathbb{C}$ ,

$$\begin{aligned} \|\alpha f\|_{W_p} &= |\alpha f(0)| + \|(\alpha f)'\|_{L_p} \\ &= |\alpha| \left( |f(0)| + \|f'\|_{L_p} \right) \\ &= |\alpha| \|f\|_{W_p}, \end{aligned}$$

as  $\|\cdot\|_{L_p}$  is a norm. Finally, we have

$$\begin{aligned} \|f + g\|_{W_p} &= |(f + g)(0)| + \|(f + g)'\|_{L_p} \\ &\leq |f(0)| + |g(0)| + \|f'\|_{L_p} + \|g'\|_{L_p} \\ &= \|f\|_{W_p} + \|g\|_{W_p}, \end{aligned}$$

as  $\|\cdot\|_{L_p}$  is a norm, so the triangle inequality holds. Thus,  $\|\cdot\|_{W_p}$  is a norm.

Let  $(f_n)_n$  be Cauchy in  $W_p([0, 1])$ . Then, for all  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,

$$\begin{aligned} \|f_n - f_m\|_{W_p} &= |f_n(0) - f_m(0)| + \|f'_n - f'_m\|_{L_p} \\ &< \varepsilon, \end{aligned}$$

meaning that both

$$\begin{aligned} |f_n(0) - f_m(0)| &< \varepsilon \\ \|f'_n - f'_m\|_{L_p} &< \varepsilon. \end{aligned}$$

Since  $\mathbb{C}$  and  $L_p([0, 1])$  are complete, there is  $c \in \mathbb{C}$  and  $g \in L_p([0, 1])$  such that

$$\begin{aligned} f_n(0) &\rightarrow c \\ f'_n &\rightarrow g. \end{aligned}$$

Define

$$f(x) = c + \int_0^x g(t) dt.$$

Then, we note that by the Fundamental Theorem of Calculus,

$$\begin{aligned} f'(x) &= g(x) \\ &\in L_p([0, 1]), \end{aligned}$$

so  $f \in W_p([0, 1])$ . Finally, we see that

$$\begin{aligned} \|f_n - f\|_{W_p([0,1])} &= |f_n(0) - f(0)| + \|f'_n - f'\|_{L_p} \\ &= |f_n(0) - c| + \|f'_n - g\|_{L_p} \\ &\rightarrow 0, \end{aligned}$$

so  $(f_n)_n \rightarrow f$  in  $W_p$ , meaning  $W_p$  is complete.

## Problem 5

**Problem:** Let  $m$  be Lebesgue measure on  $\mathbb{R}$ ,  $\Omega = \{\mathbb{1}_E \mid E \subseteq \mathbb{R} \text{ Borel}, m(E) < \infty\}$  be regarded as a subset of  $L_1(\mathbb{R})$ . We regard  $\Omega$  as a metric space with the  $L_1$  distance.

- (i) If  $a < b$  are real numbers, show that the function  $\Omega \rightarrow \mathbb{R}$  given by

$$\mathbb{1}_E \mapsto m(E \cap [a, b])$$

is a continuous function.

- (ii) If  $a < b$  are real numbers, let  $U_{a,b}$  be the subset of  $\Omega$  consisting of all  $\mathbb{1}_E$  where  $E \subseteq \mathbb{R}$  is Borel, and

$$0 < m(E \cap [a, b]) < b - a.$$

Show that  $U_{a,b}$  is open and dense in  $\Omega$ .

- (iii) Let  $D$  be the set of all  $\mathbb{1}_E$  where  $E \subseteq \mathbb{R}$  is Borel, and for every interval  $I$  of positive measure, we have

$$0 < m(E \cap I) < m(I).$$

Show that there is a countable collection  $\{U_j\}_{j \in J}$  of open and dense subsets of  $\Omega$  with  $\bigcap_{j \in J} U_j \subseteq D$ .

- (i) Letting  $f: \Omega \rightarrow \mathbb{R}$  be defined by  $f(\mathbb{1}_E) = m(E \cap [a, b])$ , we have

$$\begin{aligned} |m(E \cap [a, b]) - m(F \cap [a, b])| &= \left| \int_a^b \mathbb{1}_E - \mathbb{1}_F \, dm \right| \\ &\leq \int_a^b |\mathbb{1}_E - \mathbb{1}_F| \, dm \\ &\leq \int_{\mathbb{R}} |\mathbb{1}_E - \mathbb{1}_F| \, dm \\ &= \|\mathbb{1}_E - \mathbb{1}_F\|_{L_1}, \end{aligned}$$

meaning that  $f$  is Lipschitz, hence continuous.

- (ii) Let  $\mathbb{1}_F \in \Omega$ . Then,  $0 \leq \mu(F \cap [a, b]) \leq b - a$ . If these inequalities are strict, then  $F \in U_{a,b}$ . Else, we let  $\varepsilon > 0$ , and see two cases:

- if  $\mu(F \cap [a, b]) = b - a$ , then we may set  $E = F \setminus ([a, a + \varepsilon) \cup (b - \varepsilon/2, b])$ , so that  $0 < \mu(E \cap [a, b]) < b - a$ , and  $\|\mathbb{1}_E - \mathbb{1}_F\|_{L_1} = \mu(E \Delta F) \leq \varepsilon$ ;
- if  $\mu(F \cap [a, b]) = 0$ , then we may set  $E = F \cup ([a, a + \varepsilon/2) \cup [b - \varepsilon/2, b])$ , meaning that  $0 < \mu(E \cap [a, b]) < b - a$ , and  $\mu(E \Delta F) \leq \varepsilon$ .

Therefore,  $U_{a,b}$  is dense in  $\Omega$ . To see that  $U_{a,b}$  is open, notice that for any  $\mathbb{1}_E \in U_{a,b}$ , we may find  $\varepsilon > 0$  such that  $0 < \mu(E \cap [a, b]) - \varepsilon < \mu(E \cap [a, b]) < \mu(E \cap [a, b]) + \varepsilon < b - a$ , and for all  $F$  with  $\|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} < \varepsilon$ , we have

$$\begin{aligned} |\mu(F \cap [a, b]) - \mu(E \cap [a, b])| &\leq \|\mathbb{1}_F - \mathbb{1}_E\|_{L_1} \\ &< \varepsilon, \end{aligned}$$

so  $0 < \mu(F \cap [a, b]) < b - a$ . Thus,  $U_{a,b}$  is also open.

- (iii) If  $\{[a_k, b_k]\}$  is an enumeration of rational-endpoint intervals in  $\mathbb{R}$ , then for any interval  $I$ , there is some rational-endpoint interval  $[a_k, b_k] \subseteq I$  by density and the characterization of an interval, meaning that  $U_{a_k, b_k}$  is such that for the given interval  $I$ ,  $0 < \mu(E \cap I) < \mu(I)$  for all  $E \in U_{a_k, b_k}$ . Thus, taking the intersection of all such  $U_{a_k, b_k}$ , we have

$$\bigcap_{k=1}^{\infty} U_{a_k, b_k} \subseteq D.$$



## August 2023

### Problem 1

**Problem:** Let  $(X, \mu)$  be a  $\sigma$ -finite Borel measure space. Let  $(f_n)_n$  be a sequence in  $L_2(X, \mu)$ , and  $f \in L_2(X, \mu)$  such that for every  $g \in L_2(X, \mu)$ , we have

$$\lim_{n \rightarrow \infty} \int_X f_n(x)g(x) \, d\mu(x) = \int_X f(x)g(x) \, d\mu(x).$$

Furthermore, suppose that

$$\lim_{n \rightarrow \infty} \|f_n\|_{L_2} = \|f\|_{L_2}.$$

Prove that there is a subsequence  $(f_{n_j})_j$  and a subset  $E \subseteq X$  with  $\mu(E) = 0$  such that for all  $x \in X \setminus E$ ,

$$\lim_{j \rightarrow \infty} |f_{n_j}(x) - f(x)| = 0.$$

In order to show that  $(f_{n_j})_j \rightarrow f$  pointwise a.e., we show that  $(f_n)_n \rightarrow f$  in measure; it has been well-established that if  $(f_n)_n \rightarrow f$  in measure, then  $(f_n)_n$  admits a subsequence that converges to  $f$  pointwise almost everywhere.

By Chebyshev's Inequality, we have that

$$\begin{aligned} \mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) &\leq \frac{1}{\varepsilon^2} \|f_n - f\|_{L_2}^2 \\ &= \frac{1}{\varepsilon^2} \int_X |f_n - f|^2 \, d\mu. \end{aligned}$$

Focusing on the integral,

$$\begin{aligned} \int_X |f_n - f|^2 \, d\mu &= \int_X (f_n - f)(\overline{f_n - f}) \, d\mu \\ &= \int_X |f_n|^2 - f_n \bar{f} - \overline{f_n} f + |f|^2 \, d\mu \\ &= \int_X |f_n|^2 \, d\mu - \int_X f_n \bar{f} \, d\mu + \int_X |f|^2 \, d\mu - \overline{\int_X f_n \bar{f} \, d\mu}. \end{aligned}$$

Now, we note the following:

- $\lim_{n \rightarrow \infty} \int_X |f_n|^2 \, d\mu = \int_X |f|^2 \, d\mu$ ; and
- if  $f \in L_2(X, \mu)$ , then so too is  $\bar{f}$ .

Thus, by taking limits, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_X |f_n - f|^2 \, dx &= \lim_{n \rightarrow \infty} \left( \int_X |f_n|^2 \, d\mu - \int_X f_n \bar{f} \, d\mu + \int_X |f|^2 \, d\mu - \overline{\int_X f_n \bar{f} \, d\mu} \right) \\ &= \int_X |f|^2 \, d\mu - \int_X |f|^2 \, d\mu + \int_X |f|^2 \, d\mu - \overline{\int_X |f|^2 \, d\mu} \\ &= 0, \end{aligned}$$

so  $\|f_n - f\|_{L_2}^2 \rightarrow 0$ . Thus,  $(f_n)_n \rightarrow f$  in measure, and thus there is a subsequence  $(f_{n_j})_j \rightarrow f$  pointwise almost everywhere.

### Problem 3

**Problem:** Let  $X$  be a LCH space. Recall that  $g: X \rightarrow \mathbb{C}$  vanishes at infinity if for every  $\varepsilon > 0$ , there is a compact  $K_\varepsilon \subseteq X$  such that for all  $x \in X \setminus K_\varepsilon$ ,  $|g(x)| < \varepsilon$ . Show that  $C_0(X)$  is complete with respect to the sup norm.

Let  $(f_n)_n$  be Cauchy in the sup norm. Then, for all  $\varepsilon > 0$ , there is  $N$  such that for all  $m, n \geq N$ ,  $\|f_m - f_n\| < \varepsilon$ . Therefore, for all  $x \in X$ , we have  $|f_n(x) - f_m(x)| < \varepsilon$ , meaning that the sequence  $(f_n(x))_n$  is Cauchy in  $\mathbb{C}$ . Define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for each  $x$ .

We must now show that

- $(f_n)_n \rightarrow f$  in the supremum norm;
- $f \in C_0(X)$ .

For the first point, if  $x \in X$ , we see that for  $\varepsilon > 0$ , there is  $N$  such that for all  $n \geq N$ ,

$$|f_n(x) - f(x)| < \varepsilon.$$

Taking suprema, this gives

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon,$$

so  $\|f_n - f\| \leq \varepsilon$ , meaning that  $(f_n)_n \rightarrow f$  in the sup norm.

Finally, we let  $N_1$  be such that for all  $n \geq N_1$ ,  $\|f_n - f\| < \varepsilon/2$ . Note that since  $f_{N_1} \in C_0(X)$ , we have a  $K_{\varepsilon/2}$  such that for all  $x \in X \setminus K_{\varepsilon/2}$ ,  $|f_{N_1}(x)| < \varepsilon/2$ . Therefore, for all  $x \in X \setminus K_{\varepsilon/2}$ , we have

$$\begin{aligned} |f(x)| &\leq |f_{N_1}(x) - f(x)| + |f_{N_1}(x)| \\ &\leq \|f_{N_1} - f\| + |f_{N_1}(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

so  $f \in C_0(X)$ . Thus,  $C_0(X)$  is complete.

## January 2024

### Problem 1

**Problem:** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space, and suppose  $(f_n)_n$  is a sequence in  $L_2(X, \mu)$  such that  $\sup_{n \geq 1} \|f_n\|_{L_2} < \infty$  and  $(f_n)_n \rightarrow f$   $\mu$ -almost everywhere. Prove that  $f \in L_2(X, \mu)$ .

Applying Fatou's Lemma, we find that

$$\begin{aligned} \int_X |f|^2 d\mu &= \int_X \liminf_{n \rightarrow \infty} |f_n|^2 d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X |f_n|^2 d\mu \\ &\leq \limsup_{n \rightarrow \infty} \int_X |f_n|^2 d\mu \\ &\leq \sup_{n \geq 1} \int_X |f_n|^2 d\mu \\ &< \infty. \end{aligned}$$

## Problem 2

**Problem:** Let  $(X, \mu)$  be a measure space, and let  $p \in [1, \infty)$ . Let  $(f_n)_n \rightarrow f$  in  $L_p$ .

- (i) Prove that there exists a subsequence  $(f_{n_k})$  such that  $\|f_{n_{k+1}} - f_{n_k}\|_{L_p} < 2^{-k}$ .
- (ii) Show that for  $\mu$ -almost every  $x$ , we have  $\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x)$ .
- (i) Since  $(f_n)_n \rightarrow f$  in  $L_p$ , we see that  $(f_n)_n$  is  $L_p$ -Cauchy, so we may extract a subsequence as follows. Let  $f_{n_1} = f_1$ , and find  $f_{n_2}$  with  $n_2 > 1$  such that

$$\|f_{n_2} - f_{n_1}\| < \frac{1}{2}.$$

Inductively, we may use the fact that  $(f_n)_n$  is Cauchy to find  $n_{k+1} > n_k$  such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}.$$

- (ii) Consider the sequence  $(s_n)_n$  given by

$$s_n = \sum_{k=1}^n |f_{n_{k+1}} - f_{n_k}|.$$

Then, by Minkowski's Inequality, we find that

$$\|s_n\|_{L_p} \leq \sum_{k=1}^n \|f_{n_{k+1}} - f_{n_k}\|_{L_p}.$$

In particular, since  $s := \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$  is the limit of  $(s_n)_n$ , we may apply Fatou's Lemma to find that

$$\begin{aligned} \|s\|_{L_p} &= \left\| \liminf_{n \rightarrow \infty} s_n \right\|_{L_p} \\ &\leq \liminf_{n \rightarrow \infty} \|s_n\|_{L_p} \\ &\leq \liminf_{n \rightarrow \infty} \sum_{k=1}^n \|f_{n_{k+1}} - f_{n_k}\|_{L_p} \\ &\leq \liminf_{n \rightarrow \infty} \sum_{k=1}^n 2^{-k} \\ &= 1, \end{aligned}$$

meaning in particular that  $s(x)$  is finite almost everywhere, so  $\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$  converges almost everywhere. Therefore, the sum

$$h(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges for almost every  $x$ . Defining  $h(x) = 0$  for all  $x$  where this sum does not converge absolutely, we notice that

$$f_{n_1}(x) + \sum_{k=1}^m (f_{n_{k+1}}(x) - f_{n_k}(x)) = f_{n_{m+1}}(x),$$

meaning that  $h$  is the pointwise (almost everywhere) limit of the sequence  $(f_{n_k})_k$ ; by Minkowski's Inequality, and applying Fatou's Lemma, as earlier, we also find that

$$\begin{aligned} \|h\|_{L_p} &\leq \|f_{n_1}\|_{L_p} + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_{L_p} \\ &\leq \|f_{n_1}\|_{L_p} + 1 \\ &< \infty, \end{aligned}$$

meaning  $h \in L_p(X, \mu)$ . All we need to do now is show that  $\|f - h\|_{L_p} = 0$ , meaning that  $[f] = [h]$  under the pointwise almost everywhere equivalence relation. To see this,

$$\begin{aligned} \int_X |h - f|^p d\mu &= \int_X \liminf_{k \rightarrow \infty} |f_{n_k} - f|^p d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X |f_{n_k} - f|^p d\mu \\ &= \liminf_{n \rightarrow \infty} \|f_{n_k} - f\|_{L_p}^p \\ &= 0, \end{aligned}$$

where the last equality is derived from the fact that  $(f_n)_n \rightarrow f$  in  $L_p$ , so every subsequence of  $(f_n)_n$  converges to  $f$  in  $L_p$ .

### Problem 3

**Problem:** Let  $f$  be Lebesgue-integrable on  $\mathbb{R}$ , and let  $g$  be a bounded continuous function on  $\mathbb{R}$ . Prove that the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x - y) dy$$

is a continuous function on  $\mathbb{R}$ .

Let  $M = \sup_{x \in \mathbb{R}} |g(x)|$ . We have

$$\begin{aligned} |(f * g)(x) - (f * g)(y)| &= \left| \int_{\mathbb{R}} f(t)g(x - t) - g(y - t) dt \right| \\ &\leq \int_{\mathbb{R}} |f(t)||g(x - t) - g(y - t)| dt. \end{aligned} \tag{*}$$

Now, since  $f \in L_1(\mathbb{R})$ , this means that the sequence

$$\left( \int_{B(0, n)} |f(t)| dt \right)_n \rightarrow \int_{\mathbb{R}} |f(t)| dt,$$

so that there is some value of  $N$  such that, for any  $\varepsilon > 0$  and for all  $n \geq N$ ,

$$\int_{\mathbb{R} \setminus B(0, n)} |f(t)| dt < \varepsilon/2M.$$

Splitting the right side of (\*), we have

$$\begin{aligned} \int_{\mathbb{R}} |f(t)||g(x - t) - g(y - t)| dt &= \int_{\mathbb{R} \setminus B(0, N)} |f(t)||g(x - t) - g(y - t)| dt + \int_{B(0, N)} |f(t)||g(x - t) - g(y - t)| dt \\ &\leq 2M \int_{\mathbb{R} \setminus B(0, N)} |f(t)| dt + \int_{B(0, N)} |f(t)||g(x - t) - g(y - t)| dt \end{aligned}$$

$$= \varepsilon + \int_{B(0,N)} |f(t)| |g(x-t) - g(y-t)| dt. \quad (**)$$

Now, for the integral in (\*\*), we use the fact that  $B(0, N)$  is compact, so there is some  $\delta$  such that whenever  $|x - y| = |(x - t) - (y - t)| < \delta$  with  $x, y \in B(0, N)$ , we have

$$|g(x - t) - g(y - t)| < \frac{\varepsilon}{\int_{B(0,N)} |f(t)| dt}.$$

Thus, rewriting (\*\*), we have, whenever  $x, y \in B(0, N)$  with  $|x - y| < \delta$ ,

$$\begin{aligned} \varepsilon + \int_{B(0,N)} |f(t)| |g(x-t) - g(y-t)| dt &< \varepsilon + \int_{B(0,N)} |f(t)| \frac{\varepsilon}{\int_{B(0,N)} |f(t)| dt} dt \\ &= 2\varepsilon. \end{aligned}$$

Now, we may prove the desired result. Let  $\varepsilon > 0$ . Let  $N \in \mathbb{N}$  be such that for all  $n \geq N$

$$\int_{\mathbb{R} \setminus B(0,n)} |f(t)| dt < \varepsilon/4M,$$

Now, for a given  $n \geq N$ , since  $B(0, n)$  is compact, we may find  $\delta$  such that whenever  $x, y \in B(0, n)$  with  $|x - y| < \delta$ , we have  $|g(x) - g(y)| < \frac{\varepsilon}{2 \int_{B(0,n)} |f(t)| dt}$ . Thus, whenever  $|x - y| < \delta$  in  $B(0, n)$ , it holds for all  $t$  where  $x - t, y - t \in B(0, n)$  that  $|x - t - (y - t)| < \delta$ . This gives

$$\begin{aligned} |(f * g)(x) - (f * g)(y)| &= \left| \int_{\mathbb{R}} f(t)(g(x-t) - g(y-t)) dt \right| \\ &\leq \int_{\mathbb{R} \setminus B(0,n)} |f(t)| |g(x-t) - g(y-t)| dt + \int_{B(0,n)} |f(t)| |g(x-t) - g(y-t)| dt \\ &\leq 2M \left( \frac{\varepsilon}{4M} \right) + \frac{\varepsilon}{2 \int_{B(0,n)} |f(t)| dt} \int_{B(0,n)} |f(t)| dt \\ &= \varepsilon, \end{aligned}$$

meaning that  $f * g$  is uniformly continuous on  $B(0, n)$  for all  $n \geq N$ . Since for any  $x \in \mathbb{R}$ , there is some  $n$  such that  $x \in B(0, n)$ , where  $n \geq N$ ,  $f * g$  is continuous at  $x$  for all  $x \in \mathbb{R}$ , and we are done.

## Problem 4

**Problem:** Let  $(a_n)_n$  be a sequence of complex numbers such that  $|a_n| < 1$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = 0$ .

- Show that if  $\sum_{n \geq 1} |a_n| < \infty$ , then the sequence  $(p_n)_n$  defined by  $p_n = \prod_{i=1}^n (1 + a_i)$  is convergent.
- Does the converse hold? In other words, is it true that if  $(p_n)_n$  is convergent, we must have  $\sum_{n \geq 1} |a_n| < \infty$ ? Recall the conditions that  $|a_n| < 1$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = 0$ .