Problem (Problem 1)**:** Let $a_1, \dots, a_n \in \mathbb{R}$. Suppose that for each $i \in \{1, \dots, n\}$, we are given $m_i \geqslant 0$ and $m+1$ numbers $b_{i0}, \dots, b_{im_i} \in \mathbb{R}$. Use the Chinese Remainder Theorem to show that there exists a polynomial $f(x) \in \mathbb{R}[x]$ such that

$$\begin{aligned} f(a_i) &= b_{i0} \\ f'(a_i) &= b_{i1} \\ &\vdots \\ f^{(m_i)} &= b_{im_i}. \end{aligned}$$

Solution: We observe that if we take

$$f(x) = q_{01}(x)(x - a_1) + b_{10},$$

then

$$f'(x) = q_{01}(x) + q'_{01}(x)(x - a_1),$$

so that

$$f'(a_1) = q_{01}(a_1)$$

and

$$f'(x) = q_{11}(x)(x - a_1) + b_{11},$$

meaning

$$f(x) = (q_{11}(x)(x - a_1) + b_{11})(x - a_1) + b_{10}.$$

Inductively, we thus get the system of congruences

$$\begin{aligned} f(x) &\equiv b_{10} + b_{11}(x - a_1) + \cdots + b_{1m_1}(x - a_1)^{m_1 - 1} \bmod (x - a_1)^{m_1} \\ &\equiv b_{20} + b_{21}(x - a_2) + \cdots + b_{2m_2}(x - a_2)^{m_2 - 1} \bmod (x - a_2)^{m_2} \\ &\vdots \\ &\equiv b_{n0} + b_{n1}(x - a_n) + \cdots + b_{nm_n}(x - a_n)^{m_n - 1} \bmod (x - a_n)^{m_n}. \end{aligned}$$

Since the family of ideals $\left\{\left((x - a_1)^{m_1}\right), \dots, \left((x - a_n)^{m_n}\right)\right\}$ are pairwise coprime, the Chinese Remainder Theorem implies that some $f(x) \in \mathbb{R}[x]$ satisfies this system of congruences.

Problem (Problem 4)**:**

(a) Let R, S be commutative rings with 1, and let $f\colon R \to S$ be a ring homomorphism such that $f(1_R) = 1_S$. Show that for any prime ideal $P \subseteq S$, the preimage $f^{-1}(P)$ is a prime ideal of R.

(b) Give an example of a ring homomorphism $f\colon R \to S$ with $f(1_R) = 1_S$ and a maximal ideal $M \subseteq S$ such that $f^{-1}(M)$ is not a maximal ideal of R.

Solution:

(a)

(b) Let $R = \mathbb{Z}$ and $S = \mathbb{Q}$, with $f\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$ being the natural inclusion. Since $\mathbb{Q}$ is a field, the only maximal ideal of $\mathbb{Q}$ is $\{0\}$, but $\{0\} = f^{-1}(\{0\})$ is not maximal in $\mathbb{Z}$ since there are other proper ideals in $\mathbb{Z}$.