Solution (12.4, Problem 6)**:** Upon separation of variables, we get

$$\frac{1}{a^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$
$$\begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} Ae^{akt} & k^2 \\ At + B & 0 \\ A\cos(akt) + B\sin(akt) & -k^2 \end{cases}$$
$$X(x) = \begin{cases} Ce^{kx} & k^2 \\ Cx + D & 0 \\ C\cos(kx) + D\sin(kx) & -k^2 \end{cases}.$$

By plugging in the boundary conditions of $u(0,t) = u(1,t) = 0$, we quickly remove the former two cases, we are of the form

$$T(t) = A\cos(akt) + B\sin(akt)$$
$$X(x) = C\cos(kx) + D\sin(kx).$$

Since $X(0) = 0$, we must have $C = 0$, and since $X(1) = 0$, we have $k = n\pi$, $n \in \mathbb{Z}$. Thus, we have functions of the form

$$u_n(x,t) = (A_n\cos(n\pi at) + B_n\sin(n\pi at))\sin(n\pi x),$$

and the general solution of

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(n\pi at) + B_n\sin(n\pi at))\sin(n\pi x).$$

Plugging in the initial condition, we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n\sin(n\pi x)$$
$$= \frac{1}{100}\sin(3\pi x),$$

so that $A_n = \frac{1}{100}$ at $x = 3$ and 0 elsewhere. Writing our amended solution, we have

$$u(x,0) = \left(\frac{1}{100}\cos(3\pi at) + B_3\sin(3\pi at)\right)\sin(3\pi ax).$$

Taking derivatives, we have

$$\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = B_3\sin(3\pi ax)$$
$$= 0,$$

so $B_3 = 0$, and we arrive at the solution

$$u(x,t) = \frac{1}{100}\cos(3\pi at)\sin(3\pi x).$$

Solution (12.4, Problem 8)**:** Upon separation of variables, we get

$$\frac{1}{a^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$
$$\begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} Ae^{akt} & k^2 \\ At + B & 0 \\ A\cos(akt) + B\sin(akt) & -k^2 \end{cases}$$
$$X(x) = \begin{cases} Ce^{kx} & k^2 \\ Cx + D & 0 \\ C\cos(kx) + D\sin(kx) & -k^2 \end{cases}.$$

We plug in the boundary conditions of $\frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=L} = 0$ to obtain

$$X_n(x) = \begin{cases} C_n\cos\left(\frac{n\pi}{L}x\right) & -k^2 \\ Cx + D & 0 \end{cases}$$
$$T_n(t) = \begin{cases} B_n\cos\left(\frac{n\pi a}{L}t\right) & -k^2 \\ At + B & 0 \end{cases}$$

We may evaluate the solution

$$u(x,t) = X_0(x)T_0(t) + \sum_{n=1}^{\infty} D_n\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right).$$

To do this, we start with the initial condition, giving $T_0(t) = 1$ and $X_0(x) = x$. Taking the partial derivative with respect to t, we get

$$\frac{\partial u}{\partial t} = X_0(x)\frac{dT_0}{dt} - \sum_{n=1}^{\infty} D_n\cos\left(\frac{n\pi}{L}x\right)\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi a}{L}t\right).$$

Therefore,

$$u(x,t) = x$$

Solution (12.5, Problem 2)**:** Separating variables, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$
$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

Thus, we have

$$X_n = A_n\cos(\lambda x) + B_n\sin(\lambda x).$$

Using the boundary conditions of $X_n(a) = X_n(0) = 0$, we simplify to

$$X_n = B_n\sin\left(\frac{n\pi}{a}x\right).$$

Similarly, we have

$$Y_n(y) = C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right).$$

Applying the boundary condition of $\left.\frac{\partial u}{\partial y}\right|_{(x,0)} = 0$, we have $D_n = 0$, and

$$u(x,y) = \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right).$$

We have

$$\begin{aligned} f(x) &= u(x,b) \\ &= \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi}{a}x\right). \end{aligned}$$

Using the expansion of Fourier coefficients, we have

$$K_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)\,dx.$$

Solution (12.5, Problem 4)**:**

Solution (12.5, Problem 6)**:**

Solution (12.5, Problem 8)**:**

Solution (12.6, Problem 2)**:**

Solution (12.6, Problem 4)**:**

Solution (12.6, Problem 10)**:**

Solution (Extra Problems)**:**