

## Introduction: naive set theory

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \dots\} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \dots\} \\
\mathbb{Z}_+ &= \{0, 1, 2, \dots\} \\
\mathbb{Q} &= \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\} \\
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} \\
\mathbb{C}_q &= \{a + bi \mid a, b \in \mathbb{Q}\}
\end{aligned}$$

Recall: given sets  $X$  and  $Y$ , a relation from  $X$  to  $Y$  is a subset of  $X \times Y$ , where  $\times$  denotes the cartesian product of  $X$  and  $Y$ .

A relation  $f \subseteq X \times Y$  is a function from  $X$  to  $Y$  such that  $\forall x \in X, \exists! y \in Y$  such that  $(x, y) \in f$ . We write  $f(x) = y$ , and denote  $f$  as  $f : X \rightarrow Y$ .

$X$  is the **domain** of  $f$  and  $Y$  is the **codomain**. The range  $\text{Ran}(f) = \{f(x) \mid x \in X\} \subseteq Y$ .

The graph of a function  $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ .

## Examples

$$\text{id}_X : X \rightarrow X, \text{id}_X(x) = x$$

This is the identity function.

The Characteristic Function: If  $A \subseteq X$

$$\mathbf{1}_A : X \rightarrow \mathbb{R}, \mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

## Algebra of Functions

Let  $X$  be any set, and  $(X; \mathbb{R}) = \{f : X \rightarrow \mathbb{R}\}$  represent the function space of  $X$  with codomain  $\mathbb{R}$ .

Let  $f, g \in \mathcal{F}(X; \mathbb{R})$ . Then,  $(f + g)(x) = f(x) + g(x)$ , and  $(f \cdot g)(x) = f(x) \cdot g(x)$ .

If  $t \in \mathbb{R}$ , then  $(tf)(x) = tf(x)$  (scalar multiplication). If  $g(x) \neq 0 \forall x \in X$ , then  $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$ .

Finally, we have composition. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are functions, then  $g \circ f(x) = g(f(x))$ .

## Injective, Subjective, and Bijective

A function  $f : X \rightarrow Y$  is a **injective** map, then, if  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$ . For example, the shift map  $S : \mathbb{N} \rightarrow \mathbb{N}$ ,  $S(n) = n + 1$  is injective.

Any strictly increasing function  $f : I \rightarrow \mathbb{R}$ , where  $I$  is any interval, is injective.

A function  $f$  is **surjective** if  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$ .

Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^3 - 2x + 1$ . We can show that this function is surjective because  $\lim_{x \rightarrow \infty} f(x) = \infty$ ,  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ . Due to the intermediate value theorem, we get that  $\text{ran}(f) = \mathbb{R}$ .

$f$  is **bijective** if it is injective and surjective.

## Invertibility

Let  $f : X \rightarrow Y$  be a function.  $f$  is **left-invertible** if  $\exists g : Y \rightarrow X$  such that  $g \circ f = \text{id}_X$ .  $f$  is **right-invertible** if  $\exists h : Y \rightarrow X$  such that  $f \circ h = \text{id}_Y$ .

$f$  is **invertible** if  $\exists k : Y \rightarrow X$  such that  $f \circ k = \text{id}_Y$  and  $k \circ f = \text{id}_X$ .

## Invertibility Definition

$f$  is invertible if and only if  $f$  is left and right invertible.

Forward direction: This is via the definition of invertibility.

Reverse direction: Suppose  $g$  is a left-inverse of  $f$ , and  $h$  is a right-inverse of  $f$ . Therefore,  $g \circ f = \text{id}_X$ , and  $f \circ h = \text{id}_Y$ . Observe that  $g = g \circ \text{id}_Y$ . Therefore,  $g = g \circ (f \circ h)$ . Via associativity,  $g = (g \circ f) \circ h = \text{id}_X \circ h = h$ .

## Injection and Surjection Invertibility

If  $f : X \rightarrow Y$  is a function:

1.  $f$  is injective  $\Leftrightarrow f$  is left-invertible.
2.  $f$  is surjective  $\Leftrightarrow f$  is right-invertible.
3.  $f$  is bijective  $\Leftrightarrow f$  is invertible.

We will prove the first proposition in the forward direction. Suppose  $f$  is injective. Given  $y \in \text{ran}(f)$ , we know that  $\exists! x_y \in X$  such that  $f(x_y) = y$ , by the definition of injective.

Let  $g : Y \rightarrow X$ . We will define  $g$  as follows:

$$g(y) = \begin{cases} x_y & y \in \text{ran}(f) \\ x_0 & y \notin \text{ran}(f) \end{cases}$$

Where  $x_0$  is an arbitrary point in  $X$ . We can see that  $g \circ f = \text{id}_X$ .

For example, the function  $\text{Sin}(x)$  defined as  $\sin(x)$  restricted to  $[-\pi/2, \pi/2]$  has an inverse,  $\arcsin(x) : [-1, 1] \rightarrow [-\pi/2, \pi/2]$ .

## Cardinality and Finitude

Which set is “larger,”  $\{1, 2, 3\}$  or  $\{1, 2, 3, 4\}$ ?  $\mathbb{N}$  or  $\mathbb{N}_0$ ?  $\mathbb{Z}$  or  $\mathbb{Q}$ ?

In order to prove that one set is “the same size” as the other, we can create pairs. For two sets  $A$  and  $B$ , we can show that  $A$  is the same size as  $B$  by creating a function. For example, to show that  $\mathbb{N}$  and  $\mathbb{N}_0$  have the same size, we create  $s : \mathbb{N} \rightarrow \mathbb{N}_0$ ,  $s(n) = n + 1$ .

## Cardinality

Sets  $A$  and  $B$  have the same **cardinality** if  $\exists$  bijection  $f : A \rightarrow B$ . We write  $\text{card}(A) = \text{card}(B)$ .

## Equivalent Cardinalities of Intervals

Given  $a < b$  and  $c < d$ , we know that  $\text{card}([a, b]) = \text{card}([c, d])$ .

We can create a linear function from  $[a, b]$  to  $[c, d]$ , and since linear functions are bijections, we know that  $\text{card}([a, b]) = \text{card}([c, d])$ .

## Intervals and Real Numbers

$$\text{card}((0, 1)) = \text{card}(\mathbb{R})$$

- $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$  is a bijection:
  - $\tan$  is strictly increasing (and thus injective)
  - $\lim_{x \rightarrow \infty} \tan(x) = \infty$  and  $\lim_{x \rightarrow -\infty} \tan(x) = -\infty$ , and by intermediate value theorem,  $\tan$  is surjective
- $\ell : (0, 1) \rightarrow (-\pi/2, \pi/2)$  is a bijection as it is a linear function between two intervals.
- Therefore, our bijection is  $\tan \circ \ell : (0, 1) \rightarrow \mathbb{R}$ .

## Finitude

A set  $F$  is **finite** if  $F$  is empty or  $\exists n \in \mathbb{N}$  such that  $\text{card}(F) = \text{card}(\{1, 2, \dots, n\})$ . A non-finite set is called infinite.

We can *enumerate*  $F$  by creating a function  $\sigma : \{1, 2, \dots, n\} \rightarrow F$ , such that  $x_j = \sigma(j)$  for  $F = \{x_1, x_2, \dots, x_n\}$ .

## Inequality of Finite Sets

If  $m \neq n$ , then  $\text{card}\{1, 2, \dots, m\} \neq \text{card}\{1, 2, \dots, n\}$ .

WLOG, suppose  $m > n$ .

Suppose toward contradiction that  $f : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$  is our bijection. This means there are  $m$  “pigeons” and  $n$  “holes.”

One hole,  $j$ , must contain at least two pigeons (i.e.,  $f(i) = f(k) = j$  for some  $i \neq k \in \{1, 2, \dots, m\}$ ). Since  $f$  is assumed to be injective, this is a contradiction.

## Infinitude of the Naturals

$\mathbb{N}$  is infinite.

Suppose toward contradiction that  $\mathbb{N}$  is finite. Thus,  $\exists m \in \mathbb{N}$  such that  $f : \mathbb{N} \rightarrow \{1, 2, \dots, m\}$  is a bijection.

Consider the inclusion  $i : \{1, 2, \dots, m+1\} \rightarrow \mathbb{N}$ .  $i$  is injective.

Then,  $f \circ i : \{1, 2, \dots, m+1\} \rightarrow \{1, 2, \dots, m\}$  is an injection, but by the pigeonhole principle, this cannot be. Therefore, we have reached a contradiction.

## Proposition

If  $A$  is infinite,  $\exists i : \mathbb{N} \hookrightarrow A$ .

$\exists a_1 \in A$ , as  $A \neq \emptyset$ .

$A \setminus \{a_1\} \neq \emptyset$ , so  $\exists a_2 \in A \setminus \{a_1\}$ .

$A \setminus \{a_1, a_2\} \neq \emptyset$ , so  $\exists a_3 \in A \setminus \{a_1, a_2\}$ .

$\vdots$

We thus get a sequence  $\{a_1, a_2, \dots\}$  of distinct elements of  $A$ .

Consider  $f : \mathbb{N} \rightarrow A$ ,  $f(n) = a_n$ .  $f$  is injective as  $a_n$  are distinct.

## Cardinality of Integers and Natural Numbers

$$\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$$

$$f : \mathbb{Z} \rightarrow \mathbb{N}$$

$$f(m) = \begin{cases} 2m+1 & m \geq 0 \\ -2m & m < 0 \end{cases}$$

$f$  is a bijection as  $g : \mathbb{N} \rightarrow \mathbb{Z}$ ,  $g(n) = (-1)^{n+1} \lfloor \frac{n}{2} \rfloor$  is the inverse of  $f$ .

## Power Set

Given any set  $X$ ,  $\mathcal{P}(X) = \{A \mid A \subseteq X\}$  is the **power set** of  $X$ .

$$2^X := \{f \mid f : X \rightarrow \{0, 1\}\}.$$

Power Set and  $2^X$ 

$$\text{card}(\mathcal{P}(X)) = \text{card}(2^X)$$

Let  $\varphi : \mathcal{P}(X) \rightarrow 2^X$ .

For  $A \subseteq X$ , put  $\varphi(A) := \mathbf{1}_A$ .

Consider  $\psi : 2^X \rightarrow \mathcal{P}(X)$ .  $\psi(f) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$ .

Then,  $\psi \circ \varphi(A) = \psi(\mathbf{1}_A) = \mathbf{1}^{-1}(\{1\}) = A$ ,

and, we claim  $\varphi(\psi(f)) = \varphi(f^{-1}(\{1\})) = \mathbf{1}_{f^{-1}(\{1\})} = f$ .

## Cantor's Theorem

⚡ surjection  $\mathbb{N} \rightarrow (0, 1)$

Fact from calculus:  $\forall \sigma \in (0, 1)$ ,  $\sigma$  can be written uniquely as a decimal expansion.

$$\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$$

Where  $\sigma_k \in \{0, 1, \dots, 9\}$  and not terminating in 9s.

Suppose toward contradiction that  $\exists r : \mathbb{N} \rightarrow (0, 1)$  that is a surjection. Write  $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$ , and  $\sigma_j(n) \in \{0, 1, \dots, 9\}$ , and not terminating in 9s.

Consider  $\tau : \mathbb{N} \rightarrow \{0, 1, \dots, 9\}$ :

$$\tau(n) = \begin{cases} 3 & \sigma_n(n) = 2 \\ 2 & \sigma_n(n) \neq 2 \end{cases}$$

Let  $\tau = 0.\tau(1)\tau(2)\tau(3)\dots$ . Since  $r$  is surjective,  $\exists m \in \mathbb{N}$  such that  $r(m) = 0.\sigma_1(m)\sigma_2(m)\dots\sigma_m(m)\dots = \tau = 0.\tau(1)\tau(2)\dots\tau(m)\dots$ .

This implies that  $\sigma_m(m) = \tau(m)$ , which is definitionally not true, which is our contradiction.

## Comparing Cardinalities

- $\text{card}(A) \leq \text{card}(B) \Rightarrow \exists f : A \hookrightarrow B$
- $\text{card}(A) < \text{card}(B) \Rightarrow \text{card}(A) \leq \text{card}(B), \text{card}(A) \neq \text{card}(B)$

For example,  $X \subseteq Y \Rightarrow \text{card}(X) \leq \text{card}(Y)$  because  $i : X \hookrightarrow Y, i(x) = x$  is an injection.

## Transitive Property

If  $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$ , then  $\text{card}(A) \leq \text{card}(C)$ .

The composition of two injective functions is injective.

## Canonical Set Comparisons

Via the inclusion map, we know the following:

$$\text{card}(\mathbb{N}) \leq \text{card}(\mathbb{Z}) \leq \text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{R})$$

## Cardinality of the Power Set

For any set  $A$ ,  $\text{card}(A) < \text{card}(\mathcal{P}(A))$ .

Let us construct a function:  $f : A \rightarrow \mathcal{P}(A)$ , where  $a \mapsto \{a\}$ .

$f$  is injective, as if  $\{a\} = \{a'\}$ ,  $a = a'$ . So,  $\text{card}(A) \leq \text{card}(\mathcal{P}(A))$ .

**Claim**  $\nexists g : A \rightarrow \mathcal{P}(A)$ ,  $g$  is surjective.

Suppose toward contradiction that such a  $g$  exists. Consider  $S : \{a \in A \mid a \notin g(a)\}$ .

Since  $g$  is onto,  $\exists a_0 \in A$  with  $g(a_0) = S$ .  $a_0 \in g(a_0) \Leftrightarrow a_0 \in S \Leftrightarrow a_0 \notin g(a_0)$ .  $\perp$

## Equivalent Propositions

- (i)  $\text{card}(A) \leq \text{card}(B)$
- (ii)  $\exists f : A \hookrightarrow B$
- (iii)  $\exists g : B \rightarrow A$ ,  $g$  surjection.

By definition, (i)  $\Leftrightarrow$  (ii).

- (ii)  $\Rightarrow$  (iii) If  $f : A \hookrightarrow B$ ,  $f$  is left-invertible, and thus  $\exists g : B \rightarrow A$  with  $g \circ f = \text{id}_A$ . So,  $g$  is right-invertible, so  $g$  is surjective.
- (iii)  $\Rightarrow$  (ii) If  $g : B \rightarrow A$  is surjective, then  $g$  is right-invertible, so  $\exists f : A \rightarrow B$  such that  $g \circ f = \text{id}_B$ . So,  $f$  is left-invertible, so  $f$  is injective.

## Corollary

If  $f : A \rightarrow B$  is any map,  $\text{card}(f(A)) \leq \text{card}(A)$ .

Consider  $g : A \rightarrow f(A)$ , where  $g(a) = f(a)$ . So,  $g$  is onto, so  $\exists$  an injection  $f(A) \hookrightarrow A$ .

## More Cardinality of Canonical Sets

Consider the map  $q : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ ,  $q(m, n) = \frac{m}{n}$ . This map is *not* injective, as  $2/4 = 1/2$ . However, it is surjective, meaning  $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{Z} \times \mathbb{N})$ .

Earlier, we showed that  $\exists h : \mathbb{Z} \leftrightarrow \mathbb{N}$ . Consider  $H : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ , defined as  $H(m, n) = (h(m), n)$ .

**Claim**  $H$  is a bijection.

**Proof of Injection** If  $H(m_1, n_1) = H(m_2, n_2)$ , then  $h(m_1) = h(m_2)$ , and  $n_1 = n_2$ , and since  $h$  is bijective,  $m_1 = m_2$ , and  $n_1 = n_2$ , so  $(m_1, n_1) = (m_2, n_2)$ .

**Proof of Surjection** Let  $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ . We want to find  $(m, n) \in \mathbb{Z} \times \mathbb{N}$  such that  $H(m, n) = (k, \ell)$ . Set  $n = \ell$ , and since  $h$  is surjective, set  $m \in \mathbb{Z}$  such that  $h(m) = k$ .

Therefore  $\text{card}(\mathbb{Z} \times \mathbb{N}) = \text{card}(\mathbb{N} \times \mathbb{N})$ .

We claim that  $\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$ . First, we need to find  $\varphi : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ . Consider  $\varphi(m, n) = 2^m \cdot 3^n$ . By the Fundamental Theorem of Arithmetic,  $\varphi$  is injective.

Bringing together our inequalities, we have:

$$\begin{aligned}
 \text{card}(\mathbb{N}) &\leq \text{card}(\mathbb{Q}) \\
 &\leq \text{card}(\mathbb{Z} \times \mathbb{N}) \\
 &= \text{card}(\mathbb{N} \times \mathbb{N}) \\
 &\leq \text{card}(\mathbb{N})
 \end{aligned}$$

## Cardinality Rules

- (i)  $\text{card}(A) \leq \text{card}(A)$  (Reflexivity)
- (ii)  $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \Rightarrow \text{card}(A) \leq \text{card}(C)$  (Transitivity)
- (iii)  $\text{card}(A) \leq \text{card}(B)$  and  $\text{card}(B) \leq \text{card}(A) \Rightarrow \text{card}(A) = \text{card}(B)$  (Cantor-Schröder-Bernstein)
- (iv) Either  $\text{card}(A) \leq \text{card}(B)$  or  $\text{card}(B) \leq \text{card}(A)$ .

**Proof of (iii)** We have injections  $f : A \hookrightarrow B$  and  $g : B \hookrightarrow A$ .

Let  $A_0 \setminus \text{ran}(g)$ . Let  $A_1 = g \circ f(A_0)$ . Note that  $A_0 \cap A_1 = \emptyset$ . Let  $A_2 = g \circ f(A_1)$ . Note that  $A_0 \cap A_2 = \emptyset$ .

**Claim** We claim  $A_1 \cap A_2 = \emptyset$ . If  $\exists z \in A_1 \cap A_2$ , then  $z = g(f(x_0))$  for some  $x_0 \in A_0$ , and  $z = g(f(x_1))$  where  $x_1 \in A_1$ . However,  $g$  and  $f$  are injective, so  $g \circ f$  is injective, so  $x_0 = x_1$ , but  $A_0 \cap A_1 = \emptyset$ .  $\perp$

We let  $A_n = g \circ f(A_{n-1})$  for arbitrary  $n$ , and  $A_\infty = \bigcup_{n \geq 0} A_n$ . If  $a \notin A_\infty$ , then  $a \notin A_0$ , so  $a \in \text{ran}(g)$ . Define  $h : A \rightarrow B$ .

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ y_x & x \notin A_\infty \end{cases}$$

Where  $y_x$  is the unique element in  $B$  with  $g(y_x) = x$ .

**Claim** We claim  $h$  is the desired bijection.

**Proof of Injection** Suppose  $h(x_1) = h(x_2)$ .

If  $x_1, x_2 \in A_\infty$ , then by the definition of  $H$ ,  $f(x_1) = f(x_2)$ ,  $f$  is injective, so  $x_1 = x_2$ .

Suppose  $x_1, x_2 \notin A_\infty$ . Then, by definition,  $h(x_1) = y_{x_1}$  and  $h(x_2) = y_{x_2}$ , then  $g(y_{x_1}) = g(y_{x_2})$ , so  $x_1 = x_2$ .

WLOG, suppose  $x_1 \in A_\infty$ , and  $x_2 \notin A_\infty$ .  $h(x_1) = f(x_1) = h(x_2) = y_{x_2}$ . Then,  $g(f(x_1)) \in A_\infty = g(y_{x_2}) = x_2 \notin A_\infty$ . This case is not possible.

Thus,  $h$  is injective.

**Proof of Surjection** Let  $y \in B$ . Set  $x := g(y)$ .

Suppose  $x \notin A_\infty$ . Then,  $h(x) = y_x$ , where  $y_x$  is the unique element in  $B$  with  $g(y_x) = x = g(y)$ , so  $y = y_x$ , so  $h(x) = y$ .

If  $x \in A_\infty$ . We know that  $x \notin A_0$ , as  $x \in \text{ran}(g)$ . So,  $x = g(f(z))$  for some  $z \in A_{m-1}$ . Since  $g$  is injective,  $y = f(z)$ ,  $z \in A_\infty$ . Thus,  $h(z) = f(z) = y$ .

Therefore, we have  $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$ .

## Countability

A set  $X$  is *countable* if  $\exists f : x \hookrightarrow \mathbb{N}$  ( $\text{card}(X) \leq \text{card}(\mathbb{N})$ ).  $\text{card}(\mathbb{N}) = \aleph_0$ . If  $X$  is countable and infinite,  $X$  is *denumerable*.

## Corollary to Cantor-Schröder-Bernstein

If  $X$  is denumerable, then  $\text{card}(X) = \aleph_0$ .

Since  $X$  is infinite,  $\exists f : \mathbb{N} \hookrightarrow X$ . Since  $X$  is countable,  $\exists g : X \hookrightarrow \mathbb{N}$ . By Cantor-Schröder-Bernstein,  $\text{card}(X) = \text{card}(\mathbb{N})$ , so  $\text{card}(X) = \aleph_0$ .

Thus, we have:

$$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$$

(as shown earlier)

### Countability under Union

The countable union of countable sets is countable. If  $I$  is a countable indexing set and for each  $i \in I$ ,  $A_i$  is countable, then  $\bigcup_{i \in I} A_i$  is countable.

Since each  $A_i$  is countable,  $\exists \pi_i : \mathbb{N} \rightarrow A_i$ . Consider the function

$$\pi : I \times \mathbb{N} \rightarrow \bigcup_{i \in I} A_i$$

defined as  $\pi(i, j) = \pi_i(j)$ .

**Claim 1**  $\pi$  is a surjection.

**Proof 1** Let  $x \in \bigcup_{i \in I} A_i$ .  $\exists i_0$  such that  $x \in A_{i_0}$ . Since  $\pi_{i_0}$  is surjective,  $\exists k \in \mathbb{N}$  with  $\pi_{i_0}(k) = x$ .  $\pi_{i_0}(k) = \pi(i_0, k)$ . Therefore,  $\pi$  is surjective.

**Claim 2**  $I \times \mathbb{N}$  is countable.

**Proof 2** We know  $\exists f : I \hookrightarrow \mathbb{N}$  since  $I$  is countable. Thus,  $g : I \times \mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$ ,  $(i, n) \mapsto (f(i), n)$ . Recall,  $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ ,  $(m, n) \mapsto 2^m \cdot 3^n$  is an injection. By composing these maps,  $I \times \mathbb{N} \hookrightarrow \mathbb{N}$ . Since  $\pi$  is onto, and  $I \times \mathbb{N}$  is countable,  $\bigcup_{i \in I} A_i$  is countable.

### Continuum Hypothesis

We saw that  $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(2^{\mathbb{N}})$ , where  $2^{\mathbb{N}} = \{f \mid f : \mathbb{N} \rightarrow \{0, 1\}\}$ .

**Theorem**  $\text{card}(\mathbb{R}) = \text{card}(I) = \text{card}(2^{\mathbb{N}})$ , where  $I$  is any non-degenerate interval.

**Lemma 1**  $\text{card}([0, 1]) \leq \text{card}(2^{\mathbb{N}})$ .

**Proof 1** Every  $t \in [0, 1]$  has a binary expansion.

$$t = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$$

where  $\sigma_k \in \{0, 1\}$ .

Consider  $2^{\mathbb{N}} \xrightarrow{\varphi} [0, 1]$ , defined as  $\phi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$ . Set  $f : \mathbb{N} \rightarrow \{0, 1\}$ ,  $f(k) = \sigma_k$ .

Therefore,  $\varphi$  is surjective, so  $\exists \{0, 1\} \hookrightarrow 2^{\mathbb{N}}$ , so  $\text{card}([0, 1]) \leq 2^{\mathbb{N}}$

**Lemma 2**  $\text{card}([0, 1]) = \text{card}(\mathbb{R})$ .

**Proof 2** We have  $[0, 1] \xhookrightarrow{i} \mathbb{R}$  via inclusion, so  $\text{card}([0, 1]) \leq \text{card}(\mathbb{R})$ .

Also,  $\text{card}(\mathbb{R}) = \text{card}((0, 1)) \leq \text{card}([0, 1])$ , so by Cantor-Schröder-Bernstein,  $\text{card}(\mathbb{R}) = \text{card}([0, 1])$ .

**Lemma 3** Any two non-degenerate intervals  $I$  and  $J$  have the same cardinality.

**Proof 3** We can create injections  $I \hookrightarrow J$  and vice-versa.

**Lemma 4**  $\text{card}(2^{\mathbb{N}}) \leq \text{card}([0, 1])$ .

**Proof 4**  $\psi : 2^{\mathbb{N}} \rightarrow [0, 1]$ . Where  $\psi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$ .

$\psi$  is well-defined:

$$0 \leq \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2} \leq 1$$



We claim  $\psi$  is injective. Suppose  $f \neq g$  in  $2^{\mathbb{N}}$ . Let  $k_0 = \min\{k \mid f(k) \neq g(k)\}$ . WLOG,  $f(k_0) = 0, g(k_0) = 1$ . Let  $t_f = \sum_{k>k_0}^{\infty} \frac{f(k)}{3^k}$ ,  $t_g = \sum_{k>k_0}^{\infty} \frac{g(k)}{3^k}$ .

Therefore,  $\psi(f) = \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} + 0 + t_f$ , and  $\psi(g) = \sum_{k=1}^{k_0-1} \frac{g(k)}{3^k} + \frac{1}{3^{k_0}} + t_g$ .

Suppose toward contradiction  $\psi(f) = \psi(g)$ . Then,  $t_f = \frac{1}{3^{k_0}} + t_g$ , or  $t_f - t_g = \frac{1}{3^{k_0}}$ .

$$\begin{aligned} |t_f - t_g| &= \left| \sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{k>k_0} \frac{g(k)}{3^k} \right| \\ &\leq \sum_{k>k_0} \frac{|f(k) - g(k)|}{3^k} \\ &\leq \sum_{k>k_0} \frac{1}{3^k} \\ &= \frac{(1/3)^{k_0+1}}{1 - (1/3)} \\ &= \frac{1}{2} \cdot \frac{1}{3^{k_0}} \end{aligned}$$

⊥

We have thus shown:

$$\text{card}(\mathbb{R}) = \text{card}([0, 1]) = \text{card}(2^{\mathbb{N}})$$

We know that

$$\aleph_0 = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{Z}) < 2^{\aleph_0} = \text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R}) = \text{card}(I)$$

However, the existence of an infinity with cardinality strictly greater than  $\aleph_0$  and strictly less than  $2^{\aleph_0}$  is an axiom (i.e., it can be an assumption or not).

## Ordering

Let  $X$  be a non-empty set. A relation on  $X$  is a subset of  $X \times X$ .

- $R$  is *reflexive* if  $\forall x \in X, (x, x) \in R$ .
- $R$  is *transitive* if  $(x, y), (y, z) \in R \rightarrow (x, z) \in R$ .
- If  $R$  is *antisymmetric*  $(x, y), (y, x) \in R \rightarrow x = y$ .

If  $R$  is reflexive, transitive, and antisymmetric, then  $R$  is an *ordering* of  $X$ .

If  $R$  is an ordering of  $X$ , then we write:

$$(x, y) \in R \Leftrightarrow xRy \Leftrightarrow x \leq_R y$$

- $x \leq_R x \quad \forall x \in X$
- $x \leq_R y, y \leq_R z \rightarrow x \leq_R z$
- $x \leq_R y, y \leq_R x \rightarrow x = y$

Additionally,  $x <_R y$  means  $x \leq_R y$  and  $x \neq y$ .

Algebraic ordering of  $\mathbb{N}_0$

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0 \text{ such that } n + k = m$$

$\mathbb{N}$  ordered via division

$$n \leq_D m \Leftrightarrow n|m$$

Under this definition, it is false that  $2 \leq_D 5$ , but it is true that  $4 \leq_D 20$ .

**Inclusion** Let  $S$  be any set, and let  $X = \mathcal{P}(S)$ . For  $A, B \in \mathcal{P}(S)$ , we define  $A \leq_i B \Leftrightarrow A \subseteq B$ .

**Containment** With  $X$  defined as above,  $A \leq_c B \Leftrightarrow A \supseteq B$ .

For  $\mathcal{F}(X, \mathbb{R}) = \{f \mid f : X \rightarrow \mathbb{R}\}$ , we can define  $f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in X$ .

## Types of Orderings

- An ordering  $\leq$  of  $X$  is *total* or *linear* if  $\forall x, y \in X, x \leq y$  or  $y \leq x$ .
- An ordering is *directed* if  $\forall x, y \in X \exists z \in X$  such that  $x \leq z$  and  $y \leq z$ .

If  $X$  is a totally ordered set,  $X$  is directed.

For example, all the following orderings are directed but not total:

$$(\mathbb{N}_0, \leq_D), (\mathcal{P}(S), \leq_i), (\mathcal{P}(S), \leq_c)$$

## Upper/Lower Bounds

- Let  $(X, \leq)$  be an ordered set,  $A \subseteq X$ .  $A$  is bounded above if  $\exists v \in X$  with  $a \leq v \forall a \in A$ . Such a  $v$  is an upper bound.
- $A$  is bounded below if  $\exists \ell \in X$  such that  $a \geq \ell \forall a \in A$ . Such a  $w$  is a lower bound.
- If  $v$  is an upper bound of  $A$  and  $v \in A$ , then  $v$  is the greatest element of  $A$ , or  $\max(A) = v$ .
- If  $\ell$  is a lower bound for  $A$  and  $\ell \in A$ , then  $\ell$  is the least element of  $A$ , or  $\min(A) = \ell$ .
- If  $u$  is an upper bound for  $A$ , and  $u \leq v$  for all other upper bounds  $v$  of  $A$ , then  $u$  is the *least upper bound* of  $A$ , or  $\sup(A) = u$  (for *supremum*).
- If  $\ell$  is a lower bound for  $A$ , and  $\ell \leq g$  for all other lower bounds  $g$  of  $A$ , then  $\ell$  is the *greatest lower bound* of  $A$ , or  $\inf(A) = \ell$  (for *infimum*).
- If  $A$  is bounded above and below, then  $A$  is bounded.

## Well-Ordering Principle

With  $(\mathbb{N}, \leq_a)$ , every nonempty  $A \subseteq \mathbb{N}$  has a least element.

## Examples

## Example 1

For  $A \subseteq (\mathbb{N}, \leq_a)$ ,  $A = \{2, 3, \dots, 12\}$ , we have the following:

**Bounded Above?** Yes.

**Upper Bounds** 12, 13, 14, ...

**Greatest Element** 12

## Example 2

For  $A \subseteq (\mathbb{N}, \leq_D)$ ,  $A = \{2, 3, \dots, 10\}$

**Bounded Above?** Yes.

**Upper Bounds** 10!

**Greatest Element?** No.

**Supremum**  $2^3 \cdot 3^2 \cdot 5 \cdot 7$

**Bounded Below?** Yes.

**Lower Bound** 1

**Least Element?** No.

**Infimum** 1

## Example 3

For  $\mathcal{A} \subseteq (\mathcal{P}(S), \leq_i)$ ,  $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$ .

**Supremum**  $\bigcup_{i \in I} A_i$

**Infimum**  $\bigcap_{i \in I} A_i$

## Complete Sets

An ordered set  $(X, \leq)$  is *complete* if for all  $A \subseteq X$  bounded,  $\inf(A)$  and  $\sup(A)$  exist.

For example,  $\mathbb{Q}$  is *not* complete, as there is not a largest rational number less than  $\sqrt{2}$ , for example.

Ordering of  $\mathbb{Z}$ 

$$n \leq_a m \Leftrightarrow \exists k \in \mathbb{N}_0, n + k = m$$

This defines a total and complete ordering.

Define  $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$

Properties of  $\mathbb{Z}^+$ 

- (i)  $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, m \cdot n \in \mathbb{Z}^+$
- (ii)  $m \in \mathbb{Z}$ , then  $m \in \mathbb{Z}^+$  or  $-m \in \mathbb{Z}^+$
- (iii)  $m, -m \in \mathbb{Z}^+$ , then  $m = 0$
- (iv)  $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

Ordering of  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ 

Recall the ordering of  $\mathbb{Z}$ :

$$n \leq_a m \stackrel{\text{def}}{\Leftrightarrow} \exists k \in \mathbb{N}_0 \text{ with } n + k = m$$

**Claim**  $\leq_a$  is an ordering of  $\mathbb{Z}$

We claim that  $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid 0 \leq_a m\}$ . Thus,  $\mathbb{Z}^+ = \mathbb{N}_0$ .

Properties of  $\mathbb{Z}^+$ 

- (i)  $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}^+, m \cdot n \in \mathbb{Z}^+$
- (ii)  $m \in \mathbb{Z}$ , then  $m \in \mathbb{Z}^+$  or  $-m \in \mathbb{Z}^+$
- (iii)  $m, -m \in \mathbb{Z}^+$ , then  $m = 0$
- (iv)  $m \leq_a n \Leftrightarrow n - m \in \mathbb{Z}^+$

Other Properties of  $\mathbb{Z}$ 

- (1)  $n \leq_a m \Leftrightarrow m - n \in \mathbb{Z}^+$
- (2)  $m \leq_a n$  and  $p \leq_a q \Rightarrow m + p \leq_a n + q$
- (3)  $m \leq_a n$  and  $p \in \mathbb{Z}^+ \Rightarrow pm \leq_a pn$
- (4)  $m \leq_a n \Rightarrow -m \geq_a n$
- (5)  $\leq_a$  is total.
- (6) If  $a_a > -$ , and  $ab_a \geq 0$ , then  $b_a > 0$
- (7) If  $a > 0$  and  $ab_a \geq ac$ , then  $b \geq c$ .

**Proof of (3):**

$m \leq_a n \Rightarrow \exists k \in \mathbb{N}_0$  with  $m + k = n$ .  
 $\Rightarrow pm + pk = pn$   
 $pk \in \mathbb{N}_0$  by the properties of  $\mathbb{Z}^+$ . So,  $pm \leq_a pn$

**Proof of (5):**

Let  $m, n \in \mathbb{Z}$ . Consider  $m - n$ .  
 By (ii),  $m - n \in \mathbb{Z}^+$  or  $-(m - n) \in \mathbb{Z}^+$ . Thus,  $m - n = k$  for some  $k \in \mathbb{Z}^+$ , or  $-(m - n) = \ell$  for some  $\ell \in \mathbb{Z}^+$ .  
 Thus,  $n \leq_a m$  in the first case, or  $m \leq_a n$  in the second case.

We now want an ordering on  $\mathbb{Q}$ .

### Creating the Rationals

Recall that  $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^* = \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$ . Consider the equivalence relation:

$$(a, b) \sim (c, d) \stackrel{\text{def}}{\iff} ad = bc$$

We will let  $\mathbb{Q} = \{[(a, b)] \mid (a, b) \in \mathbb{Q}\}$  be the set of all equivalence classes in  $\mathbb{Q}$ . We write:

$$[(a, b)] = \frac{a}{b}$$

We define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We must check that addition is well-defined:  $\frac{a'}{b'} = \frac{a}{b}$  and  $\frac{c'}{d'} = \frac{c}{d}$ , then  $\frac{a'd' + c'b'}{b'd'} = \frac{ad + bc}{bd}$ .

We define multiplication as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

These operations make  $\mathbb{Q}$  a **field**:

### Fields

A ring is a nonempty set  $R$  equipped with two binary operations:

- $+: R \times R \rightarrow R, (a, b) \mapsto a + b$  ("addition")
- $\cdot: R \times R \rightarrow R, (a, b) \mapsto a \cdot b$  ("multiplication")

such that the following hold:

- (1)  $(a + b) + c = a + (b + c)$
- (2)  $\exists z \in R$  such that  $a + z = a = z + a \forall a \in R$ ; there is at most one such  $z$ . Set  $z = 0_R$ .
- (3)  $\forall a \in R, \exists b \in R$  such that  $a + b = 0_R = b + a$ ; there is at most one such  $b$ . Set  $b = -a$ .
- (4)  $\forall a, b \in R, a + b = b + a$ .
- (5)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (6)  $a \cdot (b + c) = a \cdot b + a \cdot c, (a + b) \cdot c = a \cdot c + b \cdot c$

The above six rules define a ring. If  $(R, +, \cdot)$  satisfies  $ab = ba$ ,  $R$  is a commutative ring.

If there exists  $u \in R$  such that  $ua = au = a \forall a \in R$ ,  $R$  is a unital ring; there is at most one unit. Set  $u = 1_R$ .

An integral domain is a unital, commutative ring such that  $ab = 0 \Rightarrow a = 0 \vee b = 0$ . For example,  $\mathbb{Z}$  is an integral domain. However,  $c(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  is a unital, commutative ring, but there exist two functions such that  $f, g \neq \mathbf{0}$ , but  $f \cdot g = \mathbf{0}$ .

A field is a unital, commutative ring such that every element has a multiplicative inverse.

$$\forall a \in R, a \neq 0_R, \exists b \in R, \text{ with } ab = 1_R$$

There is only one such  $b$ . Set  $b = a^{-1}$ .

### Proof that $\mathbb{Q}$ is a Field:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Provided that  $\frac{a}{b} \neq 0_{\mathbb{Q}}$ .

Additionally,  $\mathbb{Z} \xrightarrow{j} \mathbb{Q}, j(n) = \frac{n}{1}$  is injective.

Ordering of  $\mathbb{Q}$ 

$$\frac{a}{b} \leq_a \frac{c}{d} \Leftrightarrow ad \leq_a bc \in \mathbb{Z}$$

Prove that this ordering is well-defined.

## Order Embedding

$\leq$  is a well-defined total ordering of  $\mathbb{Q}$ , and  $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$ ,  $j(n) = \frac{n}{1}$  is an order embedding.

$$j(n) \leq j(m) \Leftrightarrow n \leq_a m \in \mathbb{Z}$$

Properties of  $\mathbb{Q}^+$ 

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0_{\mathbb{Q}}\}$$

- (i)  $q_1, q_2 \in \mathbb{Q}^+ \Rightarrow q_1 + q_2 \in \mathbb{Q}^+, q_1 q_2 \in \mathbb{Q}^+$
- (ii)  $q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}^+ \vee -q \in \mathbb{Q}^+$
- (iii)  $\pm q \in \mathbb{Q}^+, q = 0$
- (iv)  $x \leq y, u \leq v \Rightarrow x + u \leq y + v$
- (v)  $x \leq y, 0 \leq z \Rightarrow zx \leq zy$

Ordering of  $\mathbb{R}$ , cont'd

An **ordered field** is a field  $F$  equipped with a total ordering  $\leq_F$  such that:

- (i) if  $s \leq_F t$ , and  $x \leq_F y$ , then  $s + x \leq_F t + y$
- (ii) if  $s \leq_F t$  and  $0 \leq_F z$ , then  $zs \leq_F zt$

For example,  $\mathbb{Q}$  with its ordering is an ordered field.

**Proposition 1:** If  $(F, \leq_F)$  is an ordered field, we define  $F^+ = \{x \in F, x_F \geq 0\}$  with the following properties:

- (1)  $x, y \in F^+ \Rightarrow x + y \in F^+, xy \in F^+$
- (2)  $x \in F \Rightarrow x \in F^+, -x \in F^+$
- (3)  $\pm x \in F^+ \Rightarrow x = 0_F$

## Proofs

- (1) Let  $x, y \in F^+$ . Then,  $x \geq 0$  and  $y \geq 0$ , so by property (i) of an ordered field,  $x + y \geq 0 + 0 = 0$ , so  $x + y \in F^+$ . Additionally, we have  $x \cdot y \geq x \cdot 0 = 0$ , so  $xy \in F^+$ .
- (2) Let  $x \in F$ . Since the ordering on  $F$  is total,  $x \geq 0$  or  $0 \geq x$ . In the first case,  $x \in F^+$ . In the second case, we add  $-x$  to both sides, so by (i),  $-x \geq 0$ , so  $-x \in F^+$ .
- (3) We have  $x \geq 0$  and  $-x \geq 0$ . So  $x \geq 0$  and  $x + (-x) \geq x + 0$ , so  $x \geq 0$  and  $0 \geq x$ . So,  $x = 0$  by antisymmetry.

**Note:**  $x \leq_F y \Leftrightarrow y - x \in F^+$ .

**Proposition 2:** Let  $F$  be an ordered field. Then, the following is true:

- (1)  $\forall a \in F, a^2 \in F^+$
- (2)  $0, 1 \in F^+$
- (3) If  $n \in \mathbb{N}$ ,  $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}}$

- (4) If  $x \in F^+$ , and  $x \neq 0$ , then  $x^{-1} \in F^+$
- (5) If  $xy > 0$ , then  $x, y \in F^+$ , or  $-x, -y \in F^+$
- (6) If  $0 < x \leq y$ , then  $0 < y^{-1} \leq x^{-1}$
- (7) If  $x \leq y$ , then  $-y \leq -x$
- (8)  $x \geq 1 \Rightarrow x^2 \geq x \geq 1$ , and  $0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1$ .

#### Proofs

- (1) Let  $a \in F$ . Then,  $a \in F^+$  or  $-a \in F^+$ .  
 CASE 1 If  $a \in F^+$ , then by the previous proposition,  $a^2 \in F^+$ .  
 CASE 2 If  $-a \in F^+$ , then by the previous proposition,  $(-a)(-a) = a^2 \in F^+$ .
- (2)  $0 \geq 0$ , so  $0 \in F^+$ .  $1 = 1 \cdot 1 = 1^2 \in F^+$  by the previous result.
- (3)  $n \cdot 1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ times}} \in F^+$  by the previous proposition.
- (4) Let  $x \neq 0, x \in F^+$ . Suppose toward contradiction that  $x^{-1} \notin F^+$ , then  $-x^{-1} \in F^+$ . Thus,  $x \cdot (-x^{-1}) \in F^+$ , so  $-1 \in F^+$ , but  $1 \in F^+$ , so  $1 = 0$ .  $\perp$
- (5) Let  $xy > 0$ , meaning  $xy \in F^+$ . Suppose toward contradiction that  $x > 0$  and  $y < 0$ . So,  $x > 0$  and  $-y > 0$ , so  $(x)(-y) > 0$ , so  $-(xy) \in F^+$ , so  $xy = 0$ .  $\perp$
- (6) Let  $0 < x \leq y$ . We know  $x^{-1} \in F^+$ , so  $x^{-1}x \leq x^{-1}y$ . So  $1 \leq x^{-1}y$ . We also know  $y \in F^+$ , so  $y^{-1} \in F^+$ . So,  $1 \cdot y^{-1} \leq x^{-1} \cdot y \cdot y^{-1}$ .
- (7) Let  $x \leq y$ . Then,  $0 \leq y - x$ , so  $-y \leq -x$ .
- (8) Let  $x \geq 1$ . Then,  $x \cdot x \geq 1 \cdot x \geq 1$ .

#### Order Axiom

$\mathbb{R}$  is an ordered field. The injection  $\mathbb{Q} \hookrightarrow \mathbb{R}$ ,  $i(q) = q$  is an order embedding.

#### Rational Orderings

**Proposition 1:** If  $a \leq b$ , then  $a \leq \frac{1}{2}(a+b) \leq b$

##### Proof

$2a = a + a \leq a + b \leq b + b$ , all by property (i) of an ordered field.

Therefore,  $2a \leq a+b \leq 2b$ . Since  $2 = 1+1 \in \mathbb{R}^+$ ,  $2^{-1} \in \mathbb{R}^+$ , so  $(2a)/2 \leq \frac{1}{2}(a+b) \leq (2b)/2$ , so  $a \leq \frac{1}{2}(a+b) \leq b$ .

**Proposition 2:** If  $a \geq 0$  and  $(\forall \varepsilon > 0), a \leq \varepsilon$ .

##### Proof

If  $a \geq 0$  and  $a \neq 0$ , then  $a > 0$ . So, we have that  $\frac{1}{2}a < a$ . Let  $\varepsilon = \frac{1}{2}a$ . We also have that  $a \leq \varepsilon = \frac{1}{2}a < a$ , so  $a < a$ .  $\perp$

#### Arithmetic and Geometric Means

Given  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ :

##### Arithmetic Mean

$$= \frac{\sum_{i=1}^n a_i}{m}$$

**Geometric Mean**

$$= \sqrt[n]{a_1 a_2 \cdots a_m}$$

**Arithmetic Mean-Geometric Mean Inequality**

Let  $a, b \geq 0$ .

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

If  $x, y \geq 0$ ,  $x \leq y \Leftrightarrow x^2 \leq y^2$ .

$$0 \leq x \cdot x \leq x \cdot y \leq y \cdot y$$

by property (ii) of ordered fields

Therefore,

$$(ab)^{1/2} \leq \frac{1}{2}(a+b)$$

$$ab \leq \frac{1}{4}(a^2 + 2ab + b^2)$$

$$4ab \leq a^2 + 2ab + b^2$$

$$0 \leq a^2 - 2ab + b^2$$

$$0 \leq (a-b)^2$$

by definition

**Challenge:** Prove for  $m$ .

**Remark:** The harmonic mean is defined as:

$$\frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$$

**Bernoulli's Inequality**

If  $x \geq -1$ , then  $(1+x)^n \geq 1+nx$ , for any  $n \in \mathbb{N}_0$

By induction, we know that for  $n=0$  and  $n=1$ , this holds.

Assume the inequality holds for some  $m \geq 1$ .

$$\begin{aligned} (1+x)^{m+1} &= (1+x)^m(1+x) \\ &\geq (1+mx)(1+x) \\ &= 1+x+mx+mx^2 \\ &= 1+(m+1)x+mx^2 \\ &\geq 1+(m+1)x \end{aligned}$$

by the inductive hypothesis

**Cauchy's Inequality**

Let  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ . Then

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left( \sum_{j=1}^n a_j^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra language, this is equivalent to  $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$ .



Consider  $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^n (a_i - b_i x)^2$$

We know that  $f(x) \geq 0$  for all  $x \in \mathbb{R}$

$$\begin{aligned} &= \sum_{i=1}^n (a_i^2 - 2a_i b_i x + b_i^2 x^2) \\ &= \left( \sum_{j=1}^n b_j^2 \right) x^2 + \left( \sum_{j=1}^n -2a_j b_j \right) x + \sum_{j=1}^n a_j^2 \\ &= Ax^2 + Bx + C \end{aligned}$$

Therefore,  $\Delta = B^2 - 4AC \leq 0 \Rightarrow B^2 \leq 4AC$

$$\begin{aligned} \left( -2 \sum_{j=1}^n a_j b_j \right)^2 &\leq 4 \left( \sum_{j=1}^n a_j^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\ \left| \sum_{j=1}^n a_j b_j \right| &= \left( \sum_{j=1}^n a_j^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2} \end{aligned}$$

As we know from linear algebra, the way we get equality is when  $\vec{v} = c\vec{w}$ , or that  $a_j = cb_j \forall j$  for some  $c \in \mathbb{R}$ .

### Triangle Inequality

Given  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

$$\left( \sum_{j=1}^n (a_j + b_j)^2 \right)^{1/2} \leq \left( \sum_{j=1}^n a_j^2 \right)^{1/2} + \left( \sum_{j=1}^n b_j^2 \right)^{1/2}$$

In linear algebra, this is equivalent to  $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ .

$$\begin{aligned} \sum (a_j + b_j)^2 &= \sum a_j^2 + \sum 2a_j b_j + \sum b_j^2 \\ &\leq \sum a_j^2 + 2 \left( \sum a_j^2 \right)^{1/2} \left( \sum b_j^2 \right)^{1/2} + \sum b_j^2 && \text{by Cauchy} \\ &= \left( \left( \sum a_j^2 \right)^{1/2} + \left( \sum b_j^2 \right)^{1/2} \right)^2 \end{aligned}$$

we take square roots to get our end result

### Metrics and Norms on $\mathbb{R}^n$

Consider  $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$ , defined as follows:

$$|x| := \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

#### Theorems about Absolute Value:

- (i)  $|ab| = |a||b|$
- (ii)  $|a^2| = |a|^2$
- (iii)  $|-a| = |a|$
- (iv)  $|a| \in \mathbb{R}^+$
- (v)  $-|a| \leq a \leq |a|$

$$(vi) \quad |a| \leq \delta \Rightarrow -\delta \leq a \leq \delta \text{ for } \delta > 0$$

$$(vii) \quad |a + b| \leq |a| + |b|, |a - b| \leq |a| + |b|, ||a| - |b|| \leq |a - b|$$

### Proofs

Proof of (i)

**Case 1:** If  $a, b \in \mathbb{R}^+$ , then  $|a| = a$ , and  $|b| = b$ , and  $ab \in \mathbb{R}^+$ , so  $|ab| = ab$

**Case 2:** If  $a, b \notin \mathbb{R}^+$ , then  $|a| = -a$ , and  $|b| = -b$ . Additionally,  $(-a)(-b) = ab \in \mathbb{R}^+$ , so  $|ab| = ab$ . The LHS =  $ab$ , and the RHS =  $ab$ .

**Case 3:**  $a \in \mathbb{R}^+$ ,  $-b \in \mathbb{R}^+$ . Then,  $|a||b| = (a)(-b) = -ab$ . Then, since  $a(-b) \in \mathbb{R}^+$ ,  $-ab \in \mathbb{R}^+$ , so  $|ab| = -ab$ . Therefore, the LHS and RHS are equal.

Proof of (vii) Having established that  $|a + b| \leq |a| + |b|$ , we will show that  $||a| - |b|| \leq |a - b|$ .

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a - b| + |b| \\ |a| - |b| &\leq |a - b| \end{aligned}$$

Similarly, by exchanging  $a$  for  $b$

$$\begin{aligned} |b| - |a| &\leq |b - a| \\ |b| - |a| &\leq |a - b| \end{aligned}$$

Let  $t = |a| - |b|$ . We have shown that

$$\begin{aligned} \pm t &\leq |a - b| \\ -|a - b| &\leq t \leq |a - b| \\ |t| &\leq |a - b| \end{aligned}$$

### Absolute Values, cont'd

Recall:

$$|x| = \begin{cases} x, & x \in \mathbb{R}^+ \\ -x, & x \notin \mathbb{R}^+ \end{cases}$$

If we want to find all  $x \in \mathbb{R}$  such that  $|x - 1| \leq |x|$ , we would split up into cases:

$x \leq 0$   $x - 1 \leq -1$ , so  $|x| = -x$  and  $|x - 1| = 1 - x$ , so  $1 - x \leq -x$ , so  $0 \geq 1$ .  $\perp$

$0 < x \leq 1$   $|x| = x$  and  $|x - 1| = 1 - x$ , so  $1 - x \leq x$ , so  $x \geq \frac{1}{2}$ , so  $\frac{1}{2} \leq x \leq 1$ .

$1 < x$   $|x| = x$  and  $|x - 1| = x - 1$ , so  $x - 1 \leq x$ , so  $-1 \leq 0$ , which is true  $\forall \mathbb{R}$  in the interval, so  $x > 1$ .

Therefore, we have  $x \in (\frac{1}{2}, \infty)$  as that which satisfies this inequality.

### Bounded Sets

A subset  $A \subseteq \mathbb{R}$  is **bounded**  $\Leftrightarrow \exists c \geq 0$  such that  $\forall x \in A, |x| \leq c$ .

( $\Rightarrow$ ) Suppose  $A \subseteq \mathbb{R}$  is bounded. Then,  $\exists \ell, u \in \mathbb{R}$  such that  $\ell \leq x \leq u \quad \forall x \in A$ . Let  $c := \max\{|\ell|, |u|\}$ .

Since  $|u| \leq c$ , we have that  $x \leq c$ .

Since  $|\ell| \leq c$ , and  $-\ell \leq x$ , we get that  $-x \leq |\ell| \leq c$ .

Since  $x \leq c$  and  $-x \leq c$ ,  $|x| \leq c$ .

( $\Leftarrow$ ) If such a  $c$  exists, then  $|x| \leq c$  if and only if  $-c \leq x \leq c$ . Thus,  $-c$  is the lower bound and  $c$  is the upper bound.

## Bounded Functions

Let  $D$  be any set. A function  $f : D \rightarrow \mathbb{R}$  is bounded if  $\text{Ran}(D) \subseteq \mathbb{R}$  is bounded.

## Example

Let  $f : [3, 7] \rightarrow \mathbb{R}$ ,  $f(x) = \frac{x^2+2x+1}{x-1}$ . Show that  $f$  is bounded.

$$3 \leq x \leq 7 \Rightarrow 2 \leq x-1 \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{x-1} \leq \frac{1}{2} \Rightarrow \frac{1}{|x-1|} \leq \frac{1}{2}.$$

$$\text{Also, } 4 \leq x+1 \leq 8 \Rightarrow 16 \leq x^2+2x+1 \leq 64 \Rightarrow |x^2+2x+1| \leq 64.$$

$$\text{So, } |f(x)| \leq 32.$$

## Distance Metrics

For  $s, t \in \mathbb{R}$ , we will define  $d(s, t) = |s - t|$  to be the **distance** between  $s$  and  $t$ .

## Properties:

(i)

$$\begin{aligned} d : \mathbb{R} \times \mathbb{R} &\rightarrow [0, \infty) \\ (s, t) &\mapsto d(s, t) \geq 0 \end{aligned}$$

(ii)  $d(s, t) = d(t, s)$

(iii)  $d(s, r) \leq d(s, t) + d(t, r)$

(iv)  $d(s, s) = 0$

(v) If  $d(s, t) = 0$ , then  $s = t$ .

Let  $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ,  $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ .

- 1-norm:

$$\|v\|_1 = \sum_{j=1}^n |x_j|$$

- $\infty$ -norm:

$$\|v\|_\infty = \max_{j=1}^n |x_j|$$

- 2-norm:

$$\|v\|_2 = \left( \sum_{j=1}^n x_j^2 \right)^{1/2}$$

## Properties of the Norms

**Properties:** With  $v, w$  above, let  $p = 1, 2, \infty$ . The following are true:

- (1)  $\|v\|_p \geq 0$
- (2)  $\|v + w\|_p \leq \|v\|_p + \|w\|_p$
- (3)  $\|\vec{0}\|_p = 0$
- (4)  $\|v\|_p = 0 \Rightarrow v = \vec{0}$
- (5)  $\forall t \in \mathbb{R}, \|tv\|_p = |t|\|v\|_p$

## Proofs

Let  $p = \infty$ . We will prove (2).

Say  $\|v\|_{\infty} = |x_i|$  and  $\|w\|_{\infty} = |y_k|$ . We want to show that  $\|v + w\|_{\infty} = \max_{j=1}^n |x_j + y_j| \leq |x_i| + |y_k|$ .

Note that  $\forall j$

$$\begin{aligned} |x_j + y_j| &\leq |x_j| + |y_j| && \text{Triangle Inequality} \\ &\leq |x_i| + |y_k| \\ &= \|v\|_{\infty} + \|w\|_{\infty} \end{aligned}$$

Therefore,  $\|v + w\|_{\infty} \leq \|v\|_{\infty} + \|w\|_{\infty}$ .

## Distances and Norms

A **norm** on  $\mathbb{R}^n$  is a function  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+, v \mapsto \|v\|$ , satisfying the following properties for  $v \in \mathbb{R}^n$ :

- (1)  $\|v\| \geq 0$
- (2)  $\|v + w\| \leq \|v\| + \|w\|$
- (3)  $\|\vec{0}\| = 0$
- (4)  $\|v\| = 0 \Rightarrow v = \vec{0}$
- (5)  $\forall t \in \mathbb{R}, \|tv\| = |t|\|v\|$

If  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^+$  is a norm, we define  $d_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ , defined as follows:

$$d_{\|\cdot\|}(v, w) = \|v - w\|$$

for  $v, w \in \mathbb{R}^n$ .

The properties of distance in  $\mathbb{R}$  still hold for distance in  $\mathbb{R}^n$ :

- (1)  $d(v, w) = d(w, v)$
- (2)  $d(u, w) \leq d(u, v) + d(v, w)$
- (3)  $d(v, v) = 0$
- (4)  $d(v, w) = 0 \Rightarrow v = w$

## Metric Spaces

A **metric space** is a nonempty set  $X$  equipped with a function  $d : X \times X \rightarrow \mathbb{R}^+, (x, y) \mapsto d(x, y) \geq 0$ . The metric has the following properties:

- (1)  $d(x, y) = d(y, x) \forall x, y \in X$
- (2)  $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X$

$$(3) \ d(x, x) = 0$$

$$(4) \ d(x, y) = 0 \Leftrightarrow x = y$$

The map  $d$  is called a *metric* on  $X$ .

### Metric Spaces, Open Sets, and Closed Sets

Examples of Metric Spaces:

- $\mathbb{R}$  with  $d(x, y) = |x - y|$ .
- $\mathbb{R}^n$  with the *Euclidean metric*:

$$\begin{aligned} d_2(v, w) &= \|v - w\|_2 \\ &= \left( \sum_{j=1}^n (x_j - y_j)^2 \right)^{1/2} \end{aligned}$$

- $\mathbb{R}^n$  with the 1-norm:

$$\begin{aligned} d_1(v, w) &= \|v - w\|_1 \\ &= \sum_{j=1}^n |x_j - y_j| \end{aligned}$$

- $\mathbb{R}^n$  with the  $\infty$ -norm:

$$\begin{aligned} d_\infty(v, w) &= \|v - w\|_\infty \\ &= \max_{j=1}^n |x_j - y_j| \end{aligned}$$

Let  $(X, d)$  be a metric space.

- (1) The **open ball** centered at  $x_0 \in X$  with radius  $\delta$  is:

$$U(x_0, \delta) := \{x \in X \mid d(x, x_0) < \delta\}$$

- (2) The **closed ball** centered at  $x_0 \in X$  with radius  $\delta$  is:

$$B(x_0, \delta) := \{x \in X \mid d(x, x_0) \leq \delta\}$$

- (3) A set  $U \subseteq X$  is **open** if  $\forall x \in U, \exists \delta > 0$  such that  $U(x, \delta) \subseteq U$ .

- (4) A set  $C \subseteq X$  is **closed** if  $\overline{C} = X - C \subseteq X$  is open.

#### Examples

In  $\mathbb{R}$  with  $d(s, t) = |s - t|$ :

$$\begin{aligned} U(x_0, \delta) &= \{y \in \mathbb{R} \mid d(y, x_0) < \delta\} \\ &= \{y \in \mathbb{R} \mid |y - x_0| < \delta\} \\ &= (x_0 - \delta, x_0 + \delta) \\ B(x_0, \delta) &= [x_0 - \delta, x_0 + \delta] \end{aligned}$$

The interval  $A = [1, \infty)$  is not open, as  $\forall \delta > 0, U(1, \delta) \not\subseteq [1, \infty)$ .

However,  $A$  is closed, as  $\overline{A} = (-\infty, 1)$  is open: given  $t \in \overline{A}$ , choose  $\delta = 1 - t$ . Let  $s \in V_\delta(t)$ . Then,  $s \in (t - \delta, t + \delta)$ , so  $s \in (t - (1 - t), t + (1 - t))$ , or  $s \in (2t - 1, 1)$ , so  $s < 1$ .

## Exercises

Show that the following are open:

- $(a, b)$
- $(a, \infty)$
- $(-\infty, b)$

and that the following are closed:

- $[a, b]$
- $[a, \infty)$
- $(-\infty, b]$

In  $(\mathbb{R}^2, d_2)$ ,  $B(0_{\mathbb{R}^2}, 1)$  is the **unit disc** centered at  $(0, 0)$ .

However, in  $(\mathbb{R}^2, d_\infty)$ :

$$\begin{aligned} B(0_{\mathbb{R}^2}, 1) &= \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max\{|x|, |y|\} \leq 1 \right\} \end{aligned}$$

is the **unit square**.

## Finding a Supremum

Let  $0 \neq A \subseteq \mathbb{R}$ . Let  $u \in \mathbb{R}$  be an upper bound for  $A$ . The following are equivalent:

- (i)  $u = \sup(A)$
- (ii) If  $t < u$ , then  $\exists a_t \in A$  such that  $a_t > t$
- (iii)  $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$  with  $u - \varepsilon < a_\varepsilon$

## Proofs

- (i)  $\Rightarrow$  (ii): Given  $t < u$ , if no such  $a \in A$  with  $t < a$  exists, then  $a \leq t \forall a \in A$ . Thus  $t$  would be an upper bound. However,  $t < u$  and  $u$  is the supremum of  $A$ .  $\perp$
- (ii)  $\Rightarrow$  (iii): Given  $\varepsilon > 0$ , set  $t = u - \varepsilon < u$ . So, by (ii),  $\exists a_t$  with  $t < a_t$ . Thus,  $u - \varepsilon \leq a_t$ . Set  $a_\varepsilon = a_t$ .
- (iii)  $\Rightarrow$  (i): Let  $v$  be an upper bound for  $A$ . Suppose  $v < u$ . Then, set  $\varepsilon = u - v > 0$ . By (iii),  $\exists a_\varepsilon \in A$  with  $u - \varepsilon < a_\varepsilon$ . So  $u - (u - v) < a_\varepsilon$ , so  $v < a_\varepsilon$ , meaning  $v$  cannot be an upper bound.  $\perp$

## Supremum Example

$\sup[0, 1) = 1$ : Certainly, 1 is an upper bound for  $[0, 1)$ . Let  $\varepsilon > 0$ .

If  $\varepsilon \geq 1$ , pick  $t = \frac{1}{2}$ . Then,  $1 - \varepsilon < 0 < \frac{1}{2}$

If  $0 < \varepsilon < 1$ , let  $t = (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \varepsilon/2$ . Then,  $t \in [0, 1)$ , and  $1 - \varepsilon < 1 - \varepsilon/2 = t$

## Finding an Infimum

Let  $\emptyset \neq A \subseteq \mathbb{R}$ . Let  $\ell \in \mathbb{R}$  be a lower bound for  $A$ . The following are equivalent:

- (i)  $\ell = \inf(A)$
- (ii) If  $t > \ell$ ,  $\exists a_t$  such that  $t > a_t$

(iii)  $(\forall \varepsilon > 0)(\exists a_\varepsilon \in A)$  with  $\ell + \varepsilon > a_\varepsilon$

#### Infimum Example

$\inf \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ : Clearly,  $0 < \frac{1}{n} \forall n \geq 1$ . Let  $\varepsilon > 0$ .

We need to find  $a \in \left\{ \frac{1}{n} \mid n \geq 1 \right\}$  with  $\varepsilon > a$ . By the Archimedean Property,  $\exists m \in \mathbb{N}$  such that  $\frac{1}{m} < \varepsilon$ . Let  $a_\varepsilon = \frac{1}{m}$ .

#### More on Supremum/Infimum

- If  $A \subseteq \mathbb{R}$  and  $\max(A) = u$ , then  $u = \sup(A)$ :  $u$  is an upper bound of  $A$  by the definition of  $\max$ , and if  $v \neq u$  is any upper bound of  $A$ , then  $u < v$  since  $u \in A$ .
- If  $\min(A) = \ell$ , then  $\ell = \inf(A)$  (by the same logic).
- If  $A$  is not bounded above,  $\sup(A) = +\infty$ , and if  $A$  is not bounded below, then  $\inf(A) = -\infty$ .
- If  $A \subseteq B$ , then  $\sup(A) \leq \sup(B)$ .
- If  $A \subseteq B$ , then  $\inf(A) \geq \inf(B)$ : Let  $\ell_A = \inf(A)$  and  $\ell_B = \inf(B)$ . By definition,  $\ell_B \leq b \forall b \in B$ . Since  $A \subseteq B$ ,  $\ell_B \leq a \forall a \in A$ . Thus,  $\ell_B$  is a lower bound for  $A$ . By definition of  $\ell_A$ ,  $\ell_B \leq \ell_A$ .

Let  $A, B \subseteq \mathbb{R}$  and  $t \in \mathbb{R}$ . Then, the following are also sets:

- (1)  $A + B = \{a + b \mid a \in A, b \in B\}$
- (2)  $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$
- (3)  $t \cdot A = \{ta \mid a \in A\}$
- (4)  $A + t = \{a + t \mid a \in A\}$

For example, we have the following results:

- $\sup(A + B) = \sup(A) + \sup(B)$
- $\sup(A + t) = \sup(A) + t$
- $\inf(-A) = -\sup(A)$

#### Completeness Axiom

If  $\emptyset \neq A \subseteq \mathbb{R}$  is bounded above, then  $\sup(A)$  exists.

Well-Ordering Property: if  $\emptyset \neq S \subseteq \mathbb{N}$ , then  $\min(S)$  exists.

Therefore, we can prove that if  $F \subseteq \mathbb{Z}$  is bounded, then  $F$  has a least and greatest element.

##### Archimedean Property: Proof

If  $x \in \mathbb{R}$ , then  $\exists n_x \in \mathbb{N}$  such that  $x \leq n_x$ .

Suppose there exists no natural number greater than  $x$ , then  $\mathbb{N}$  is bounded above by  $X$ . Let  $u = \sup(\mathbb{N})$ . By the Completeness Axiom,  $u \in \mathbb{R}$  exists. Let  $\varepsilon = 1$ . Then,  $\exists n \in \mathbb{N}$  with  $u - 1 < n$ . So,  $u < n + 1$ , but  $n + 1 \in \mathbb{N}$ , so  $u$  cannot be an upper bound.

##### Corollary to the Archimedean Property

$$\forall t > 0 \exists n \in \mathbb{N} \ni \frac{1}{n} < t$$

## Corollary: Powers of 2

$$\forall t > 0 \exists m \in \mathbb{N} \ni \frac{1}{2^m} < t$$

By the corollary to the Archimedean Property, we know that  $\exists n \in \mathbb{N}$  such that  $\frac{1}{n} < t$ . By Bernoulli's inequality with  $x = 1$ , we have  $2^n \geq n$ , so  $2^{-n} < n^{-1} < t$ .

## Corollary: In Between Integers

$$\forall x \in \mathbb{R} \exists n_x \in \mathbb{Z} \ni n_x - 1 \leq x < n_x$$

Assume  $x \geq 0$ . Let  $S_x = \{n \mid n \in \mathbb{N} \ x < n\}$ .

$S_x \neq \emptyset$  by the Archimedean Property. By the well-ordering property, let  $n_x = \min(S_x)$ .

Therefore,  $x < n_x$ . Also,  $n_x - 1 \notin S_x$ . Therefore  $n_x - 1 \leq x$ .

## Density Theorems

Let  $(X, d)$  be any metric space. A subset  $D \subseteq X$  is **dense** if  $\forall x \in X, \varepsilon > 0, U(x, \varepsilon) \cap D \neq \emptyset$ .

In the case of  $X = \mathbb{R}$  and  $d(s, t) = |s - t|$ ,  $D \subseteq \mathbb{R}$  is dense if given any open interval  $I$ ,  $I \cap D \neq \emptyset$ .

A metric space is **separable** if it admits a *countable* dense subset.

## Density of the Rationals

$\mathbb{Q} \subseteq \mathbb{R}$  is dense.

Let  $I = (a, b)$  be an open interval. We may assume that  $a, b \in \mathbb{R}$  are finite.

Then,  $b - a > 0$ . By the Archimedean property corollary,  $\exists n \in \mathbb{N}$  such that  $\frac{1}{n} < b - a$ , meaning  $1 < nb - na$ .

There exists also an integer  $m$  such that  $m - 1 \leq na < m$ , implying that  $a \frac{m}{n}$ . Also,  $m \leq na + 1 < nb$ . Therefore,  $\frac{m}{n} < b$ .

So,  $\frac{m}{n} \in \mathbb{Q} \cap (a, b)$ .

## Density of the Irrationals

$\mathbb{R} \setminus \mathbb{Q}$  is dense.

Assume  $\sqrt{2}$  exists. Let  $I = (a, b)$  be any open interval. Then,  $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ .

Find  $q \in \mathbb{Q}$  such that  $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$ .

Then,  $a < q\sqrt{2} < b$ , where  $q\sqrt{2} \in \mathbb{R}$  and  $q\sqrt{2} \notin \mathbb{Q}$ .



Uniqueness of  $\sqrt{2}$ 

$$\exists! x > 0 \ x^2 = 2$$

Existence: Let  $S = \{t \in \mathbb{R} \mid 0 < t, t^2 < 2\}$ .  $S$  is nonempty because  $1 \in S$ , and  $S$  is bounded above because  $y > 2 \Rightarrow y^2 > 4$ .

So, by the completeness axiom,  $x := \sup(S)$  exists, and  $x \geq 1$ .

Claim:  $x^2 = 2$

Contradiction 1: Assume  $x^2 < 2$ . We want to show that  $\exists n \in \mathbb{N}$  such that  $x + \frac{1}{n} \in S$ . By this assumption, we find that

$$\begin{aligned} 0 < x + \frac{1}{n} \in S &\Leftrightarrow \left(x + \frac{1}{n}\right)^2 < 2 \\ &\Leftrightarrow x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2 \end{aligned}$$

Observe:

$$\begin{aligned} x^2 + \frac{2x}{n} + \frac{1}{n^2} &\leq x^2 + \frac{2x}{n} + \frac{1}{n} \\ &= x^2 + \frac{1}{n}(2x + 1) \end{aligned}$$

We want to find  $n \in \mathbb{N}$  with:

$$x^2 + \frac{1}{n}(2x + 1) < 2 \Leftrightarrow \frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

Therefore, by the Archimedean Property corollary, we know that  $n$  must exist.

Contradiction 2: We know that  $x^2 \geq 2$ . Since  $x = \sup(S)$ ,  $\forall m \in \mathbb{N}$ ,  $\exists t_m \in S$  with  $x - \frac{1}{m} < t_m$ , so  $\left(x - \frac{1}{m}\right)^2 < t_m^2 < 2$ .

Therefore,  $x^2 - \frac{2x}{m} + \frac{1}{m^2} < 2$ , so  $x^2 - \frac{2x}{m} < 2$ , so  $0 \leq x^2 - 2 < \frac{2x}{m}$ .

So,  $0 \leq \frac{x^2 - 2}{2x} < \frac{1}{m}$ , so  $x^2 - 2 = 0$ , so  $x^2 = 2$ .

**Remark:** If we had set  $S' = \{t' \in \mathbb{Q} \mid t'^2 < 2, t' > 0\}$ , we would have still obtained  $\sup(S') = \sqrt{2}$ . This means that  $\mathbb{Q}$  is *not* complete.

## Intervals and Nested Intervals

(\*) Given any interval  $I$ , if  $x_1, x_2 \in I$ , with  $x_1 < x_2$ , then  $[x_1, x_2] \in I$ .

This seems like an obvious property, but this is the *characteristic property* of intervals.

### Characterization of Intervals

Let  $S \subseteq \mathbb{R}$  be any nonempty subset of cardinality at least 2. Suppose  $S$  satisfies (\*). Then,  $S$  is an interval.

**Case 1:** Suppose  $S$  is bounded.

Let  $a = \inf(S)$  and  $b = \sup(S)$ . Then,  $S \subseteq [a, b]$ . We will show that  $(a, b) \subseteq S$ . Once this is shown,  $S = \{(a, b), [a, b], [a, b), (a, b]\}$ .

Let  $t \in (a, b)$ . Since  $a = \inf(S)$ ,  $\exists x_1 \in S$ ,  $x_1 \in (a, t)$ . Similarly, since  $b = \sup(S)$ ,  $\exists x_2 \in S$ ,  $x_2 \in (t, b)$ .

By the hypothesis,  $[x_1, x_2] \subseteq S$ . Since  $t \in [x_1, x_2]$ ,  $t \in S$ .

**Case 2:** Suppose  $S$  is bounded above, but not below.

Let  $b = \sup(S)$ . Clearly,  $S \subseteq (-\infty, b]$ . We will show that  $(-\infty, b) \subseteq S$ . Once this is shown,  $S = \{(-\infty, b), (-\infty, b]\}$ .

Let  $t \in (-\infty, b)$ . Since  $b = \sup(S)$ ,  $\exists x_2 \in S$ ,  $x_2 \in (t, b)$ .

Since  $S$  is not bounded below,  $\exists x_1 \in S$  such that  $x_1 < t$  (or else  $t$  would be a lower bound).

By the hypothesis,  $[x_1, x_2] \subseteq S$ , and  $t \in [x_1, x_2]$ , so  $t \in S$ .

**Case 3, 4:** Left as an exercise for the reader.

A sequence of intervals  $(I_n)_{n \geq 1}$  is called *nested* if

$$I_1 \supseteq I_2 \supseteq \dots I_n \supseteq I_{n+1} \supseteq \dots$$

We are primarily interested in  $\bigcap I_n$ .

- (a)  $\bigcap_{n=1}^{\infty} [0, 1/n] = \{0\}$ .
- (b)  $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$
- (c)  $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$

### Measure

The **measure** of an interval is basically its “size.”

- (a)  $m([a, b]) = b - a$
- (b)  $m((a, b]) = b - a$
- (c)  $m((a, b)) = b - a$
- (d)  $m([a, b)) = b - a$

### Nested Intervals Theorem

Let  $I_n = [a_n, b_n]$  for  $n \in \mathbb{N}$  be a nested sequence of intervals.

- (1)  $\bigcap_{n \geq 1} I_n \neq \emptyset$
- (2) If  $\inf \{m(I_n) \mid n \geq 1\} = 0$ , then  $\bigcap_{n \geq 1} I_n = \{\xi\}$

(a)

Since  $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$ , we have that  $a_1 \leq a_2 \leq a_3, \dots$ , and  $b_1 \geq b_2 \geq b_3 \geq \dots$ .

We know that  $\{a_n\}$  is bounded above and nonempty. Let  $\xi = \sup(\{a_n\}_{n=1}^\infty)$ .

We know that  $\{b_n\}$  is bounded below. Let  $\eta = \inf(\{b_n\}_{n=1}^\infty)$ .

We claim that  $\xi \leq b_n \forall n \geq 1$ . Suppose toward contradiction that  $\exists m$  such that  $\xi > b_m$ . Then, by the supremum property,  $\exists a_i$  such that  $\xi > a_i > b_m$ . If  $k \leq m$ ,  $a_k \leq a_m \leq b_m < a_k$ . If  $m \leq k$ , then  $b_m \geq b_k \geq a_k > b_m$ .  $\perp$

Similarly, using the same argument,  $a_n \leq \eta \forall n$ .

Thus,  $\xi \leq \eta$ .

We claim that  $\bigcap_{n \geq 1} I_n = [\xi, \eta]$ . If  $t \in [\xi, \eta]$ , then  $a_n \leq \xi \leq t \leq \eta \leq b_n$ . So  $t \in [a_n, b_n] \forall n$ , so  $t \in \bigcap_{n \geq 1} [a_n, b_n]$ .

If  $t \in \bigcap_{n \geq 1} I_n$ , then  $t \in [a_n, b_n] \forall n$ , so  $a_n \leq t \leq b_n \forall n$ . So,  $t$  is an upper bound on  $a_n$ , and a lower bound on  $b_n$ . So,  $\xi \leq t \leq \eta$  by definition of  $\xi$  and  $\eta$ .

(b)

We have  $\forall n \geq 1$

$$\begin{aligned} [\xi, \eta] &\subseteq [a_n, b_n] \\ \Rightarrow 0 &\leq \eta - \xi \leq b_n - a_n \\ &= m(I_n) \end{aligned}$$

So, if  $\inf(\{m(I_n) \mid n \geq 1\}) = 0$ , then  $0 \leq \eta - \xi \leq 0$ , so  $\eta = \xi$ .

#### Corollary to the Nested Intervals Theorem

$[0, 1]$  is uncountable.

Suppose it is possible to denumerate the interval  $[0, 1] = \{t_1, t_2, \dots\}$ .

We can find  $[a_1, b_1] \subseteq [0, 1]$  with:

- $a_1 < b_1$
- $t_1 \notin [a_1, b_1]$ .

Then, we find  $[a_2, b_2] \subseteq [a_1, b_1]$  with  $a_2 < b_2$ ,  $t_2 \notin [a_2, b_2]$ .

Recursively, we find  $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$  with  $a_n < b_n$ ,  $t_n \notin [a_n, b_n]$ .

So,  $I_n = ([a_n, b_n])_0^\infty$  is a sequence of nested intervals.

Therefore,  $\exists \xi \in \bigcap I_n \subseteq [0, 1]$ . However,  $\xi \notin \{t_1, t_2, \dots\}$ .  $\perp$

#### Sequences in Metric Spaces

A sequence in a metric space  $M$  is a map

$$\begin{aligned} x : \mathbb{N} &\rightarrow M & M = \mathbb{R}, \text{ usually} \\ x &= (x_n)_{n=1}^\infty \end{aligned}$$

I. Sequences defined by a formula:

- (i)  $x_n = t$  (the constant sequence)
- (ii)  $x_n = 2n + 1$

- (iii)  $x_n = \frac{1}{n-1}$  (here,  $n \geq 2$ )
- (iv)  $c_n = n \sin\left(\frac{1}{n}\right)$
- (v)  $d_n = \left(1 + \frac{1}{n}\right)^n$
- (vi) Geometric Sequence: for  $b \neq 0$ ,  $(b^n)_{n \geq 0} = (1, b, b^2, \dots)$
- (vii)  $x_n = \frac{n!}{n^n}$
- (viii) Given any function

$$f : [0, \infty) \rightarrow \mathbb{R}$$

we can set  $x_n = f(n)$ .

## II. Sequences defined recursively:

- (i)  $a_1 = 1$ ,  $a_{n+1} = 2a_n + 1 = (1, 3, 7, 15, \dots)$
- (ii) Fibonacci:  $f_1 = 1$ ,  $f_2 = 1$ ,  $f_{n+1} = f_n + f_{n-1} = (1, 1, 2, 3, 5, 8, \dots)$ . The closed form equation is:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - (1 - \varphi)^n)$$

where  $\varphi = \frac{1+\sqrt{5}}{2}$

- (iii) Let  $f : M \rightarrow M$  be a self-map on a metric space. Fix  $x_0 \in M$ .

$$x_n = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}(x_0)$$

## III. New sequences from old:

- (i) Let  $(a_n)_n$  and  $(b_n)_n$  be sequences,  $t \in \mathbb{R}$ . Then, we can do the following:

- $(a_n)_n + (b_n)_n = (a_n + b_n)_n$
- $t(a_n)_n = (ta_n)_n$
- $(a_n)_n (b_n)_n = (a_n b_n)_n$
- If  $b_n \neq 0 \forall n$ ,  $\left(\frac{a_n}{b_n}\right)$

- (ii) We can also shift a sequence:

$$x_{n+1} = (x_2, x_3, \dots)$$

- (iii) We can look at ratios for  $a_n \neq 0$

$$r_n = \frac{a_{n+1}}{a_n}$$

- (iv) We can look at partial sums, given  $(a_k)_{k=1}^\infty$ .

$$\begin{aligned} s_1 &= a_1 \\ s_n &= s_{n-1} + a_n \\ &= \sum_{k=1}^n a_k \end{aligned}$$

The sequence  $(s_n)_n$  is called the sequence of partial sums. For example, the sequence of partial sums for  $(b^k)_{k=0}^\infty$  is:

$$1 + b + b^2 + \dots + b^n = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1 \\ n+1 & b = 1 \end{cases}$$

because for  $b \neq 1$ ,  $(1-b)(1+b+b^2+\dots+b^n) = 1-b^{n+1}$

### Exercise

Let  $a_k = \frac{1}{k(k+1)}$ . Find  $(s_n)_n$ .

Via partial fraction decomposition, we get that  $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ . Therefore,  $(s_n)_n = \left(1 - \frac{1}{n+1}\right)_{n=1}^\infty$

## Bounded Sequences

$$\ell_\infty = \{(a_k)_k \mid a_k \in \mathbb{R}, a_k \text{ bounded}\}$$

We define

$$\|(a_k)_k\|_\infty = \sup_{k \geq 1} |a_k| \quad \text{Infinity Norm}$$

This norm has the traditional properties of the norm:

$$\|(a_k)_k + (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty + \|(b_k)_k\|_\infty \quad \text{Triangle Inequality}$$

$$\|t(a_k)_k\|_\infty = |t| \|(a_k)_k\|_\infty \quad \text{Scalar Multiplication}$$

$$\|(a_k)_k\|_\infty = 0 \Leftrightarrow a_k = 0 \quad \forall k \quad \text{Zero Property}$$

$$\|(a_k)_k (b_k)_k\|_\infty \leq \|(a_k)_k\|_\infty \|(b_k)_k\|_\infty \quad \text{Multiplication}$$

## Proof

Let  $u = \|(a_k)_k\|_\infty$  and  $v = \|(b_k)_k\|_\infty$ .

Given any  $k$ ,

$$\begin{aligned} |a_k + b_k| &\leq |a_k| + |b_k| && \text{Triangle Inequality on } |\cdot| \\ &\leq u + v && \text{definition of supremum} \\ \Rightarrow \sup_{k \geq 1} |a_k + b_k| &\leq u + v \end{aligned}$$

Thus,

$$\begin{aligned} \|(a_k)_k + (b_k)_k\|_\infty &= \|((a_k + b_k)_k)_k\|_\infty \\ &= \sup_{k \geq 1} |a_k + b_k| \\ &\leq u + v \end{aligned}$$

## Monotonicity

A sequence  $(x_n)_n$  is **increasing** if

$$x_1 \leq x_2 \leq \cdots \quad \forall n$$

and is **decreasing** if

$$x_1 \geq x_2 \geq \cdots \quad \forall n$$

The sequence is *eventually* increasing if  $\exists m \in \mathbb{N} \ni x_n \leq x_{n+1}$  for  $n > m$ .

Similarly, the sequence is eventually decreasing if  $\exists m \in \mathbb{N} \ni x_n \geq x_{n+1}$  for  $n > m$ .

A sequence that is increasing or decreasing is **monotone** (or eventually monotone).

## Example

The sequence

$$a_1 = 1$$

$$a_{n+1} = \frac{1}{2}a_n + 2$$

is increasing and bounded above.

We will prove the first statement via induction:

**Base:**  $a_1 = 1, a_2 = \frac{1}{2} + 2 = \frac{5}{2} \geq 1$

**Inductive Hypothesis**  $a_n \leq a_{n+1} \Rightarrow a_{n+1} \leq a_{n+1}$

**Proof:**

$$a_n \leq a_{n+1}$$

$$\frac{1}{2}a_n \leq \frac{1}{2}a_{n+1}$$

$$\frac{1}{2}a_n + 2 \leq \frac{1}{2}a_{n+1} + 2$$

$$a_{n+1} \leq a_{n+2}$$

To prove the sequence is bounded above, we do the following:

$$a_1 = 1 \leq 4$$

$$\frac{1}{2}a_1 \leq 2$$

$$\frac{1}{2}a_1 + 2 \leq 2$$

$$a_2 \leq 4$$

We claim that  $\forall n, a_n \leq 4 \Rightarrow a_{n+1} \leq 4$ , as we have shown the base case.

$$a_n \leq 4$$

$$\frac{1}{2}a_n \leq 2$$

$$\frac{1}{2}a_n + 2 \leq 4$$

$$a_{n+1} \leq 4$$

## Convergence of Sequences

Let  $L \in \mathbb{R}, \varepsilon > 0$ . Then, the  $\varepsilon$ -neighborhood of  $L$  is  $(L - \varepsilon, L + \varepsilon) = V_\varepsilon(L)$ .

$$x \in V_\varepsilon(L)$$

$$\Leftrightarrow$$

$$|x - L| < \varepsilon$$

$$L - \varepsilon < x < L + \varepsilon$$

With this in mind, we know the following:

## Definition of Convergence

A real sequence  $(x_n)_n$  converges to a number  $x$  if

$$(\forall \varepsilon > 0) (\exists N_\varepsilon \in \mathbb{N}) \ni n \geq N \Rightarrow |x_n - x| < \varepsilon$$

If no such  $L$  exists, then  $(x_n)_n$  is said to **diverge**.

A sequence  $(x_n)_n$  in a metric space  $(X, d)$  converges to a point  $x$  if

$$(\forall \varepsilon > 0) (\exists N_\varepsilon \in \mathbb{N}) \ni d(x_n, x) < \varepsilon$$

Essentially, we want to show that there always exists an  $N$  such that the  $N$ th tail (i.e.,  $x_N, x_{N+1}, \dots$ ) are within  $\varepsilon$  of  $L$  for any  $\varepsilon$ .

**Note:**  $N$  usually depends on  $\varepsilon$  (the smaller the  $\varepsilon$ , the larger the  $N$ ).

## Convergence Proof

$$\left(\frac{1}{n}\right)_n \xrightarrow{n \rightarrow \infty} 0$$

We know that

$$|x_n - L| = \left|\frac{1}{n}\right|$$

Given  $\varepsilon > 0$ , we want  $\frac{1}{n} < \varepsilon$ , meaning  $n > \frac{1}{\varepsilon}$ .

**Proof:** Let  $\varepsilon > 0$ . By the Archimedean property corollary, find  $N \in \mathbb{N}$  large such that  $\frac{1}{N} < \varepsilon$ .

$$\begin{aligned} n &\geq N \\ \frac{1}{n} &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

so, if  $n \geq N$ , then

$$\begin{aligned} |x_n - L| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

## Convergence Proof 2

Show that

$$\left(\frac{5n-1}{3-n}\right)_{n \geq 4} \xrightarrow{n \rightarrow \infty} -5$$

$$\begin{aligned} |x_n - L| &= \left| \frac{5n-1}{3-n} + 5 \right| \\ &= \frac{14}{|3-n|} \\ &= \frac{14}{n-3} < \varepsilon \\ \frac{14}{n-3} &< \varepsilon \\ n &> \frac{14}{\varepsilon} + 3 \end{aligned}$$

**Proof:** Let  $\varepsilon > 0$ . Find  $N' \in \mathbb{N}$  so large that  $\frac{1}{N'} < \frac{\varepsilon}{14}$  (which exists by the Archimedean property corollary).  
Let  $N = N' + 3$ . If  $n \geq N$ , then

$$\begin{aligned} n-3 &\geq \frac{1}{N'} \\ \frac{1}{n-3} &\leq \frac{1}{N'} \\ &< \frac{\varepsilon}{14} \end{aligned}$$

whence

$$\begin{aligned} |x_n - L| &= \frac{14}{n-3} \\ &< \frac{14\varepsilon}{14} \\ &= \varepsilon \end{aligned}$$

## Sequences and their Limits, cont'd

## Convergence and Distance

Let  $(X, d)$  be a metric space, and let  $(x_n)_n$  be a sequence in the metric space. The following are equivalent:

- (i)  $(x_n)_n \rightarrow x$
- (ii)  $(d(x_n, x))_n \rightarrow 0$

(i)  $\Rightarrow$  (b) Let  $\varepsilon > 0$ . Find  $N_\varepsilon \in \mathbb{N}$  so large such that  $d(x_n, x) < \varepsilon$  whenever  $n \geq N_\varepsilon$ .

So,  $|d(x_n, x) - 0| = d(x_n, x) < \varepsilon$  for all  $\varepsilon > 0$ . Whence,  $(d(x_n, x))_n \rightarrow 0$ .

(ii)  $\Rightarrow$  (i) This direction is similar.

In  $\mathbb{R}$ , this implies that

$$\begin{aligned} (x_n)_n &\rightarrow x \\ &\Leftrightarrow \\ (|x_n - x|)_n &\rightarrow 0 \end{aligned}$$



## Comparison Proposition

Let  $(X, d)$  be a metric space and let  $x \in X$ , and suppose  $(x_n)_n$  is a sequence in  $X$ .

If  $\exists c \geq 0$ ,  $m \in \mathbb{N}$ , and a sequence  $(a_n)_n \in \mathbb{R}^+$  with  $(a_n)_n \rightarrow 0$  and  $d(x_n, x) \leq ca_n \forall n > m$ . Then,  $(x_n)_n \rightarrow x$ .

Let  $\varepsilon > 0$ . Note that  $\frac{\varepsilon}{c} > 0$ .

Find  $N_1 \in \mathbb{N}$  large such that  $n \geq N_1 \Rightarrow |a_n - 0| < \frac{\varepsilon}{c}$ , which is always possible since  $(a_n)_n \rightarrow 0$ .

Let  $N = \max(N_1, m)$ . Then,  $n \geq N \Rightarrow n \geq N_1$  and  $n \geq m$ . So,

$$\begin{aligned} d(x_n, x) &\leq ca_n \\ &< c \frac{\varepsilon}{c} \\ &= \varepsilon \end{aligned}$$

so,  $n \geq N \Rightarrow d(x_n, x) < \varepsilon$ , whence  $(x_n)_n \rightarrow x$

## Comparison Proposition, Example 1

Prove

$$\left( \frac{\sin(n^2 - 1)}{n^2 + 3} \right)_n \rightarrow 0$$

$$\begin{aligned} \left| \frac{\sin(n^2 - 1)}{n^2 + 3} - 0 \right| &= \frac{|\sin(n^2 - 1)|}{n^2 + 3} \\ &\leq \frac{1}{n^2 + 3} \\ &\leq \frac{1}{n^2} \\ &\leq \frac{1}{n} \end{aligned}$$

We know that  $a_n = \frac{1}{n}$  converges to 0. So, by our comparison proposition, we are done.

## Comparison Proposition, Example 2

Prove

$$\left( \frac{1}{2^n} \right)_n \rightarrow 0$$

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &\geq 1 + n &> n && \text{Bernoulli's Inequality} \end{aligned}$$

so,

$$\frac{1}{2^n} < \frac{1}{n}$$

Since  $a_n = \frac{1}{n}$  converges, we know that  $\frac{1}{2^n}$  must converge.

## Sequence Divergence

A sequence  $(x_n)_n$  is **divergent** if it does not converge.  $(x_n)_n \rightarrow 0$  if and only if

$$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) \ni (\forall n \geq N_\varepsilon) d(x_n, x) < \varepsilon$$

So,  $(x_n)_n$  diverges if and only if

$$(\exists \varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \rightarrow d(x_n, x) \geq \varepsilon_0$$

#### Diverging Sequence Proof

Show that the following sequence diverges:

$$a_n = (-1)^n$$

##### Step 1

$$((-1)^n)_n \not\rightarrow 1$$

Take  $\varepsilon_0 = 1/2$ , given any  $N \in \mathbb{N}$ , we will find  $n \geq N$  odd:

$$\begin{aligned} n &= 2N + 1 \\ d((-1)^n, 1) &= 2 \\ &\geq \varepsilon_0 \end{aligned}$$

##### Step 2

$$((-1)^n)_n \not\rightarrow -1$$

by letting  $\varepsilon_0 = 1/2$  and  $n = 2N$ .

#### Diverging Sequence Proof 2

Does

$$a_n = (\sin(n))_n$$

converge?

It is not the case that  $(\sin(n))_n \rightarrow L$  for any  $L \in \mathbb{R}$ .

**Case 1** If  $L > 1$ , set  $\varepsilon_0 = \frac{L-1}{2}$ . Then, given any  $N \in \mathbb{N}$ , pick  $n = N$ .

$$\begin{aligned} |\sin(n) - L| &= L - \sin(n) \\ &\geq L - 1 \\ &> \frac{L-1}{2} \\ &= \varepsilon_0 \end{aligned}$$

**Case 2** Similarly for  $L < -1$

**Case 3** Suppose  $-1 < L < 1$ .

**Case 3.1** Suppose  $L > 0$ . Set  $\varepsilon_0 = \frac{L}{2}$ . Given any  $N$ , find  $n \geq N$  with  $\sin(n) < 0$ .

We find  $k$  large such that  $N < (2k+1)\pi$ , which we can always do because we are finding  $k > \frac{1}{2} \left( \frac{N}{\pi} - 1 \right)$ , which is always possible by the Archimedean property.

Note that  $N < (2k+1)\pi < (2k+2)\pi$ . Note that  $\sin(x) < 0$  on the interval  $((2k+1)\pi, (2k+2)\pi)$ . Note that  $|(2k+1)\pi - (2k+2)\pi| = \pi$ . Let  $n = \lceil (2k+1)\pi \rceil$ . Then,  $|L - \sin(n)| \geq \frac{L}{2} = \varepsilon_0$

**Case 3.2** Suppose  $L < 0$ , set  $\varepsilon_0 = \frac{-L}{2}$ . Given  $N$ , find  $n \geq N$  with  $\sin(n) > 0$ . Using the same strategy as above, we find  $n$  such that  $|L - \sin(n)| > \varepsilon_0$

**Case 3.3** Suppose  $L = 0$ . Set  $\varepsilon_0 = 1/2$ . Given  $N \in \mathbb{N}$ , find  $n \geq N$  with  $\sin(n) \geq \frac{1}{2}$ . Then,  $|\sin(n) - 0| = \sin(n) \geq \varepsilon_0$ .

Showing that a sequence diverges is not easy — later, we will divergence with non-uniqueness of convergent subsequences.

### Alternating Series

Consider again

$$((-1)^n)_{n \geq 0} = (1, -1, 1, -1, \dots)$$

The even entries converge to 1:

$$((-1)^n)_{2n} = (1, 1, 1, \dots)$$

Similarly, the odd entries converge to  $-1$ :

$$((-1)^n)_{2n+1} = (-1, -1, -1, \dots)$$

Both of these subsequences of the same sequence converge to different values, meaning that the alternating series diverges.

### Uniqueness of Limits

A sequence  $(x_n)_n$  can converge to at most one limit.

Suppose toward contradiction that  $(x_n)_n$  converges to  $L_1$  and  $L_2$  with  $L_1 \neq L_2$ .

WLOG, let  $L_2 > L_1$ . Take  $\varepsilon = \frac{L_2 - L_1}{3}$ .

Since  $(x_n)_n$  converges to  $L_1$ ,  $\exists N_1 \in \mathbb{N}$  such that  $|x_n - L_1| < \varepsilon$ . Similarly, since  $(x_n)_n$  converges to  $L_2$ ,  $\exists N_2 \in \mathbb{N}$  such that  $|x_n - L_2| < \varepsilon$ .

Let  $N = \max N_1, N_2$ . If  $n \geq N$ , then  $n \geq N_1$  and  $n \geq N_2$ .

So,  $|x_n - L_1| < \varepsilon$  and  $|x_n - L_2| < \varepsilon$ . So,  $x_n \in V_\varepsilon(L_1)$ , and  $x_n \in V_\varepsilon(L_2)$ , meaning  $x_n \in V_\varepsilon(L_1) \cap V_\varepsilon(L_2)$ , but  $V_\varepsilon(L_1) \cap V_\varepsilon(L_2) = \emptyset$ .  $\perp$

### Useful Lemmas for Convergence

#### Absolutely Convergent Sequences

Let  $(x_n)_n$  be a real sequence. If  $x_n$  converges to  $x$ , then  $|x_n| \rightarrow |x|$ . However, the converse is not the case.

Note that since  $(x_n)_n \rightarrow x$ ,  $d(x_n, x) \rightarrow 0$ .

By the reverse triangle inequality, we have

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &\leq 0 \end{aligned}$$

## Convergence to Zero

Let  $a_n$  be a sequence.

$$(a_n)_n \rightarrow 0$$

$$\Leftrightarrow$$

$$|(a_n)| \rightarrow 0$$

$(\Rightarrow)$  If  $(a_n)_n \rightarrow 0$ , then we showed previously that  $|(a_n)_n| \rightarrow |0| = 0$

$(\Leftarrow)$  Suppose  $|(a_n)_n| \rightarrow 0$ . Given  $\varepsilon > 0$ , then  $\exists N$  such that  $n \geq N$  implies

$$||a_n| - 0| < \varepsilon$$

$$||a_n|| < \varepsilon$$

$$|a_n| < \varepsilon$$

$$|a_n - 0| < \varepsilon$$

So,  $(a_n)_n \rightarrow 0$

## Geometric Sequence

Let  $b \in \mathbb{R}$ . Consider

$$(b^n)_{n \geq 0} = (1, b, b^2, \dots)$$

We claim the sequence is convergent provided  $-1 < b \leq 1$ . Otherwise, the sequence is divergent.

If  $b = 0$ , then the sequence  $(b^n)_n = (0, 0, 0, \dots)$  is convergent.

If  $b = 1$ , then the sequence  $(b^n)_n = (1, 1, 1, \dots)$  is convergent.

If  $b = -1$ , then the sequence  $(b^n)_n = (1, -1, 1, \dots)$  is divergent.

**Case 1** Suppose  $0 < b < 1$ . Then,  $\frac{1}{b} > 1$ , so  $\frac{1}{b} = 1 + a$ .

So, by Bernoulli's Inequality,  $(\frac{1}{b})^n = (1 + a)^n \geq 1 + na > na$ , so  $b^n < \frac{1}{na}$ .

$$|b^n - 0| = b^n$$

$$< \frac{1}{na}$$

$$= \frac{1}{a} \frac{1}{n}$$

$$\rightarrow 0$$

So,  $(b^n)_n \rightarrow 0$ .

**Case 2** Suppose  $-1 < b < 0$ . If we look at  $|b^n| = |b|^n$ , we know that  $(|b|^n)_n \rightarrow 0$  by our work above. By the previous lemma, we know that since  $|b^n| \rightarrow 0$ ,  $b^n \rightarrow 0$ .

**Case 3** Suppose  $b > 1$ . Then,  $b = 1 + a$  where  $a > 0$ .

$$b^n = (1 + a)^n$$

$$\geq 1 + na$$

$$> na$$

Bernoulli's Inequality

Suppose toward contradiction that  $(b^n)_n \rightarrow L$ . Let  $\varepsilon_0 = 1$ . Find  $N \in \mathbb{N}$  such that  $N > \frac{L+1}{a}$ .  $N$  must exist by the Archimedean property.

Therefore,  $(N)(a) > L + 1$ . If  $n \geq N$ , then  $(n)(a) > (N)(a) > L + 1$ , so  $|b^n - L| \geq na - L \geq \varepsilon_0$ .  $\perp$

**Case 4** Suppose  $b < -1$ , and suppose toward contradiction that  $(b^n)_n \rightarrow L$ . By the previous lemma, we know that  $|b^n| \rightarrow |L|$ . So,  $|b|^n \rightarrow |L|$ . But,  $|b| > 1$ , which means our assumption contradicts the result from above.  $\perp$

## Sequences and Limits, Cont'd

*n*th Root Convergence

If  $c > 0$ , then  $(c^{1/n})_n \rightarrow 1$ .

**Case 1:** If  $c = 1$ , then we get  $(c^{1/n})_n = (1, 1, 1, \dots)$ , which clearly converges to one.

**Case 2:** Assume that  $c > 1$ . Then,  $c^{1/n} > 1$ , because if  $d = c^{1/n} \leq 1$ , then  $d^n \leq 1$ , so  $c \leq 1$ . We can write  $c^{1/n} = (1 + d_n)$ , where  $d_n > 0$ .

$$\begin{aligned} c &= c^n \\ &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned} \quad \text{Bernoulli's Inequality}$$

So,  $d_n \leq \frac{c}{n}$ . Remember,  $c^{1/n} = 1 + d_n$ .

$$\begin{aligned} |c^{1/n} - 1| &= c^{1/n} - 1 \\ &= d_n \\ &\leq c \cdot \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $c^{1/n} \rightarrow 1$ .

**Case 3:** Assume  $0 < c < 1$ . Then,  $c^{1/n} < 1$ , so  $\frac{1}{c^{1/n}} > 1$ .

So, we can write  $\frac{1}{c^{1/n}} = (1 + d_n)$ , where  $d_n > 0$ .

$$\begin{aligned} c^{1/n} &= \frac{1}{1 + d_n} \\ 1 - c^{1/n} &= 1 - \frac{1}{1 + d_n} \\ &= \frac{d_n}{1 + d_n} \\ &\leq d_n \end{aligned}$$

Remember,  $\frac{1}{c^{1/n}} = 1 + d_n$

$$\begin{aligned} \frac{1}{c} &= (1 + d_n)^n \\ &\geq 1 + nd_n \\ &> nd_n \end{aligned}$$

So,  $d_n \leq \frac{1}{cn}$

$$\begin{aligned} |1 - c^{1/n}| &= 1 - c^{1/n} \\ &\leq d_n \\ &\leq \frac{1}{c} \frac{1}{n} \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $(c^{1/n})_n \rightarrow 1$ .

## Positive Sequence Convergence

Let  $(x_n)_n$  be a sequence with  $x_n \in \mathbb{R}^+ \forall n \in \mathbb{N}$ , with  $(x_n)_n \rightarrow x$ . Then,  $x$  is also positive, and  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ .

Suppose toward contradiction that  $x < 0$ . Let  $\varepsilon = \frac{|0-x|}{2}$ . Since  $(x_n)_n$  converges to  $x$ , we know that  $x_n \in V_\varepsilon(x)$  for large  $n$ . However, every member of  $V_\varepsilon(x) < 0$ , and  $x_n > 0$ .  $\perp$

**Case 1:** If  $x = 0$ , we will show that  $(\sqrt{x_n})_n \rightarrow 0$ .

Let  $\varepsilon > 0$ , find  $N \in \mathbb{N}$  large such that if  $n \geq N$ , we have

$$\begin{aligned} |x_n - 0| &< \varepsilon^2 \\ x_n &< \varepsilon^2 \\ \sqrt{x_n} &< \varepsilon \\ |\sqrt{x_n} - 0| &< \varepsilon \end{aligned}$$

**Case 2:** If  $x > 0$ , we will show that  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ .

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{1}{\sqrt{x}} |x_n - x| \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $|\sqrt{x_n} - \sqrt{x}| \rightarrow 0$ , so  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ .

**$n$ th Root of  $n$  Convergence**

$$\left(n^{1/n}\right)_n \rightarrow 1$$

We will make use of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Note that  $n^{1/n} > 1$  for  $n$  past 1. So, we write

$$n^{1/n} = 1 + d_n \quad d_n > 0$$

$$n = (1 + d_n)^n$$

$$= \sum_{k=0}^n \binom{n}{k} d_n^k$$

$$= \binom{n}{0} + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n$$

$$\geq \binom{n}{0} + \binom{n}{2} d_n^2$$

as all terms are positive

$$= 1 + \frac{n(n-1)}{2} d_n^2$$

so

$$n - 1 \geq \frac{n(n-1)}{2} d_n^2$$

$$\frac{2}{n} \geq d_n^2$$

$$\frac{\sqrt{2}}{\sqrt{n}} \geq d_n$$

So, we have

$$|n^{1/n} - 1| = n^{1/n} - 1$$

$$= d_n$$

$$\leq \sqrt{2} \frac{1}{\sqrt{n}}$$

$$\rightarrow 0$$

by previous corollary

So,  $(n^{1/n})_n \rightarrow 1$ .

### Multiplication by Geometric Sequence

Let  $0 \leq b < 1$ . Show that

$$(nb^n)_n \rightarrow 0$$

If  $0 < b < 1$  (the 0 case is trivial). So,  $\frac{1}{b} > 1$ , meaning  $\frac{1}{b} = 1 + d$  for some  $d > 0$ .

$$\begin{aligned} \frac{1}{b^n} &= (1 + d)^n \\ &\geq \frac{n(n-1)}{2} d^2 \\ \frac{2}{d^2(n)(n-1)} &\geq b^n \\ nb^n &\leq \frac{2}{d^2(n-1)} \\ &\rightarrow 0 \end{aligned} \quad \text{by previous corollary}$$

Therefore,  $(nb^n)_n \rightarrow 0$ .

### Boundedness and Convergence

If  $(x_n)_n$  is a convergent sequence,  $x_n$  is bounded. The converse is false in general.

Suppose  $(x_n)_n \rightarrow x$ . Let  $\varepsilon = 1$ .

Then,  $\exists N \in \mathbb{N}$  such that  $x_n \in V_\varepsilon(x)$  for all  $n \geq N$ .

Let  $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x-1|, |x+1|\}$ . If  $n \geq N$ , then  $|x_n| \leq c$ , because  $x_n \in V_\varepsilon(x)$ . If  $n < N$ , then  $|x_n| \leq c$ .

Together, we have  $|x_n| \leq c$  for all  $n$ .

To show the converse is not true, consider  $((-1)^n)_n$ . This sequence is bounded but not convergent.

### Algebraic Operations on Sequences

Let  $(x_n)_n \rightarrow x$ ,  $(y_n)_n \rightarrow y$ , and  $(z_n)_n \rightarrow z$  be convergent sequences. Let  $t \in \mathbb{R}$ . Then, the following are all true:

- (1)  $(x_n \pm y_n)_n \rightarrow x \pm y$
- (2)  $(tx_n)_n \rightarrow tx$
- (3)  $(x_n y_n)_n \rightarrow xy$
- (4) Assume  $z_n \neq 0 \forall n$ , and  $z \neq 0$ . Then,  $\left(\frac{1}{z_n}\right)_n \rightarrow \frac{1}{z}$ , and  $\left(\frac{x_n}{z_n}\right)_n \rightarrow \frac{x}{z}$ .

**Proof of (1)** Let  $\varepsilon > 0$ . Since  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $\exists N_1, N_2 \in \mathbb{N}$  such that  $n \geq N_1 \rightarrow |x_n - x| < \frac{\varepsilon}{2}$ , and  $n \geq N_2 \rightarrow |y_n - y| < \frac{\varepsilon}{2}$ .

Let  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then  $n \geq N_1$  and  $n \geq N_2$ .

$$\begin{aligned} |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$



**Proof of (3)** We have  $(x_n)_n \rightarrow x$  and  $(y_n)_n \rightarrow y$ .

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |x_n - x||y| \end{aligned}$$

Since convergent sequences are bounded,  $\exists c \in \mathbb{R}$  such that  $|x_n| < c$ ,  $\forall n$

$$\begin{aligned} &\leq c|y_n - y| + |x_n - x||y| \\ &\rightarrow 0 \end{aligned}$$

Therefore,  $|x_n y_n - xy| \rightarrow 0$ , so  $x_n y_n \rightarrow xy$ .

**Proof of (4)** We have  $z_n \neq 0$  and  $z \neq 0$ . Let  $\varepsilon > 0$ .

$$\begin{aligned} \left| \frac{1}{z_n} - \frac{1}{z} \right| &= \frac{|z - z_n|}{|z_n z|} \\ &= |z_n - z| \frac{1}{|z|} \frac{1}{|z_n|} \end{aligned}$$

Let  $\varepsilon = \frac{|z|}{2}$ . Since  $(z_n)_n \rightarrow z$ , we know that  $z_n \in V_\varepsilon(z)$  for  $n \geq N \in \mathbb{N}$ . For  $n \geq N$ ,  $|z_n| > \frac{|z|}{2}$ , so  $\frac{1}{|z_n|} < \frac{2}{|z|}$ .

$$\begin{aligned} &\leq |z_n - z| \frac{2}{|z|^2} \\ &\rightarrow 0 \end{aligned}$$

So,  $\left| \frac{1}{z_n} - \frac{1}{z} \right| \rightarrow 0$ , so  $\frac{1}{z_n} \rightarrow \frac{1}{z}$

### Ordering of Limits

Let  $(x_n)_n \rightarrow x$  and  $(y_n)_n \rightarrow y$ . If  $x_n \leq y_n$  for all  $n$ , then  $x \leq y$ .

Suppose toward contradiction that  $x > y$ .

Let  $\varepsilon = \frac{x-y}{2}$ .

So,  $\exists N_1 \in \mathbb{N}$  such that  $n \geq N_1 \Rightarrow y_n \in V_\varepsilon(y)$ , and  $\exists N_2 \in \mathbb{N}$  such that  $n \geq N_2 \Rightarrow x_n \in V_\varepsilon(x)$ .

Let  $N = \max\{N_1, N_2\}$ . Then,  $x_N \in V_\varepsilon(x)$  and  $y_N \in V_\varepsilon(y)$ . But that means  $x_N > y_N$ .  $\perp$

In particular, if  $(x_n)_n \rightarrow x$ , and  $a \leq x_n \leq b$ , then  $a \leq x \leq b$ .

### Squeeze Theorem

Let  $(x_n)_n \rightarrow x$ ,  $(y_n)_n \rightarrow y$ , and  $(z_n)_n \rightarrow z$ , where  $x_n \leq y_n \leq z_n$  for all  $n$ .

If  $L = x = z$ , then  $y = L$ .

Let  $\varepsilon > 0$ . Find  $N_1, N_2 \in \mathbb{N}$  such that  $n \geq N_1 \Rightarrow V_\varepsilon(L)$ , and  $n \geq N_2 \Rightarrow V_\varepsilon(L)$ .

Let  $N = \max\{N_1, N_2\}$ . Then,  $n \geq N \Rightarrow x_n, z_n \in V_\varepsilon(L)$ . Thus,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

so  $y_n \in V_\varepsilon(L)$ , so  $(y_n)_n \rightarrow L$ .

For example, let  $a_n = \frac{\sin(n)}{n}$ . Then, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and both sides of the inequality go to zero,  $a_n \rightarrow 0$

As another example, consider  $a_n = (2^n + 3^n)^{1/n}$ . Then,

$$\begin{aligned} 3^n &\leq 2^n + 3^n \leq 2 \cdot 3^n \\ 3 &\leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3 \end{aligned}$$

Since  $2^{1/n} \rightarrow 1$ , we have  $a_n \rightarrow 3$ .

### Ratio Test

Let  $(x_n)$  be a sequence of strictly positive numbers, with  $\left(\frac{x_{n+1}}{x_n}\right)_n \rightarrow r < 1$ . Then,  $(x_n)_n \rightarrow 0$ .

Since  $r < 1$ , then  $1 - r > 0$ . Let  $\rho = r + \frac{1-r}{2}$ , and  $\varepsilon = \rho - r = \frac{1-r}{2}$ .

Since the sequence converges,  $\exists N \in \mathbb{N}$  such that for  $n \geq N$ ,

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} - r \right| &< \varepsilon \\ \frac{x_{n+1}}{x_n} &< \rho \\ x_{n+1} &< \rho x_n \end{aligned}$$

In particular,  $x_{N+1} < \rho x_N$ , and  $x_{N+2} < \rho x_{N+1} < \rho^2 x_N$ . Inductively, one can show that  $\forall k \geq 1$ ,  $x_{N+k} < \rho^k x_N$ .

$$0 < x_{N+k} < \rho^k x_N$$

In particular, as  $k \rightarrow \infty$ , both sides of the inequality go to 0, implying that  $x_n \rightarrow 0$

### Monotone Convergence Theorem

Let  $(x_n)_n$  be a monotone sequence. Then,  $(x_n)_n$  is convergent if and only if it is bounded.

- (a) If  $(x_n)_n$  is increasing and bounded above, then  $(x_n)_n \rightarrow \sup(\{x_n \mid n \in \mathbb{N}\})$ .
- (b) If  $(x_n)_n$  is decreasing and bounded below, then  $(x_n)_n \rightarrow \inf(\{x_n \mid n \in \mathbb{N}\})$ .

We have already shown that all convergent sequences are bounded.

Assume that  $(x_n)_n$  is monotonic and bounded.

**Case 1:** Suppose  $(x_n)_n$  is increasing. Let  $\sup\{x_n \mid n \in \mathbb{N}\} := u$ . We claim that  $(x_n)_n \rightarrow u$ .

Let  $\varepsilon > 0$ . By the definition of supremum,  $\exists N \in \mathbb{N}$  such that  $u - \varepsilon < x_N$ . Note that  $\forall n \geq N$ ,  $u - \varepsilon < x_N \leq x_n \leq u$ .

Therefore, if  $n \geq N$ , then  $|x_n - u| < \varepsilon$ .

**Case 2:** Suppose  $(x_n)_n$  is decreasing. Let  $\ell := \inf\{x_n \mid n \in \mathbb{N}\}$ . We claim that  $(x_n)_n \rightarrow \ell$ .

Let  $\varepsilon > 0$ . By the definition of infimum,  $\exists N \in \mathbb{N}$  such that  $\ell + \varepsilon > x_N$ . Additionally,  $\forall n \geq N$ ,  $\ell \leq x_n \leq x_N < \ell + \varepsilon$ .

Therefore, if  $n \geq N$ ,  $|x_n - \ell| < \varepsilon$ .

## Applications of the Monotone Convergence Theorem

## Lemma

If  $(x_n)_n$  is a convergent sequence, and  $m \in \mathbb{N}$ , the  $m$ -th tail,  $x_{(m)} = (x_{m+k})_{k=1}^\infty$  is also convergent. If  $(x_n)_n \rightarrow L$  then  $x_{(m)} \rightarrow L$ .

Let  $\varepsilon > 0$ . Find  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |x_n - L| < \varepsilon$ . If  $k \geq N$ , then  $m + k \geq N$ , so  $|x_{m+k} - L| < \varepsilon$ .

Thus,  $(x_{m+k})_k \rightarrow L$

Consider the inductively defined sequence

$$\begin{aligned} x_1 &= 8 \\ x_{n+1} &= \frac{1}{2}x_n + 2 \\ (x_n)_n &= (8, 6, 5, 9/2, 17/4, \dots) \end{aligned}$$

We claim that  $x_n \geq 4 \forall n$ .

$$x_1 = 8 \geq 4$$

Suppose  $x_k \geq 4$ . We will show that  $x_{k+1} \geq 4$ .

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4 \end{aligned}$$

Therefore,  $(x_n)_n$  is bounded below by 4.

We claim that  $(x_n)_n$  is decreasing.  $\forall n \in \mathbb{N}$ ,

$$\begin{aligned} x_{n+1} &\leq x_n \Leftrightarrow \\ \frac{1}{2}x_n + 2 &\leq x_n \\ \Leftrightarrow 4 &\leq x_n \end{aligned}$$

By the monotone convergence theorem, we know that  $(x_n)_n \rightarrow L$ .

To find  $L$ , we use the recursive relationship and the lemma.

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2}x_n + 2\right)_{n=1}^\infty \\ L &= \frac{1}{2}L + 2 \\ L &= 4 \end{aligned}$$

Consider the following sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{4} \\ x_3 &= 1 + \frac{1}{4} + \frac{1}{9} \\ x_k &= \sum_{k=1}^n \frac{1}{k^2} \end{aligned}$$

We will show that  $(x_n)_n$ , the sequence of partial sums, converges.

Clearly, these partial sums form an increasing sequence. We only need to show that the sequence is bounded above.

$$\begin{aligned}
 k^2 &\geq k(k-1) & k \geq 2 \\
 \frac{1}{k^2} &\leq \frac{1}{k(k-1)} \\
 &= \frac{1}{k-1} - \frac{1}{k} \\
 \sum_{k=2}^n \frac{1}{k^2} &\leq \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) \\
 \sum_{k=1}^n \frac{1}{k^2} &\leq 1 + \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right)
 \end{aligned}$$

But

$$1 + \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{n}$$

so, we have

$$\begin{aligned}
 \sum_{k=1}^n \frac{1}{k^2} &\leq 2 - \frac{1}{n} \\
 &< 2
 \end{aligned}$$

So,  $(x_n)_n$  is bounded above.

#### Nested Intervals Theorem, Alternative Proof

Let  $I_n = [a_n, b_n]$  be a countable family of nested intervals. Then,

$$\bigcap I_n \neq \emptyset$$

Since the intervals are nested, it must be the case that  $a_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq b_1$ .

Similarly,  $a_1 \leq a_n \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$ .

So,  $(a_n)_n$  is an increasing sequence bounded above by  $b_1$  and  $(b_n)_n$  is a decreasing sequence bounded below by  $a_1$ .

So,  $(b_n)_n \rightarrow r$  and  $(a_n)_n \rightarrow \ell$

Note that  $\ell = \sup(a_n)$  and  $r = \inf(b_n)$ .

Fix  $n \in \mathbb{N}$ , then for any  $m \geq n$ ,  $a_n \leq a_m \leq b_m \leq b_n$ . So,  $\sup(a_m) = \ell \leq b_n$ . Unlocking  $n$ , we get that  $\ell \leq \inf(b_n) = r$ .

#### Calculating Square Roots

Let  $a \in \mathbb{R}^+$ . We will construct a sequence  $(x_n)_n \rightarrow \sqrt{a}$ .

Let

$$x_1 = 1$$

Define

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right).$$

We will prove that  $x_n^2 \geq a$ .

$$\begin{aligned}
 2x_{n+1} &= x_n + \frac{a}{x_n} \\
 2x_{n+1}x_n &= x_n^2 + a \\
 0 &= x_n^2 - 2x_{n+1}x_n + a
 \end{aligned}$$

So,  $x_n$  is a real root, meaning

$$\begin{aligned}
 \Delta &= 4x_{n+1}^2 - 4a \\
 x_{n+1}^2 &\geq a & \forall n
 \end{aligned}$$

So,  $\forall n \geq 2$

$$x_n^2 \geq a$$

We will show that  $x_n$  is ultimately decreasing.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \\ &= \frac{1}{2} \underbrace{\left( \frac{x_n^2 - a}{x_n} \right)}_{\geq 0 \quad \forall n \geq 2} \end{aligned}$$

So, we have that  $(x_n)_n$  is decreasing and bounded below, meaning  $(x_n)_n \rightarrow x$  for some  $x \in \mathbb{R}$ .

We had

$$\begin{aligned} x_{n+1} &= \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \\ x &= \frac{1}{2} \left( x + \frac{a}{x} \right) \\ x &= \frac{a}{x} \\ x^2 &= a \\ x &= \sqrt{a} \end{aligned}$$

remember that  $x > 0$

### Euler's Number

Consider

$$\begin{aligned} (e_n)_n &= \left( 1 + \frac{1}{n} \right)^n \\ &= \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} \end{aligned}$$

Similarly,

$$\begin{aligned} e_{n+1} &= \sum_{k=0}^{\infty} \left( \frac{1}{k!} \prod_{j=1}^{k-1} \left( 1 - \frac{j}{n+1} \right) \right) \\ e_{n+1} &\geq e_n \end{aligned}$$

$\forall n$

We claim that  $(e_n)_n$  is bounded above.

$$e_1 = \left(1 + \frac{1}{1}\right)^1$$

$$2 \leq e_n$$

$$e_n = \sum_{k=0}^n \left( \frac{1}{k!} \underbrace{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}_{\leq 1} \right)$$

$$2^{k-1} \leq k!$$

$$k \geq 2$$

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}}$$

$$e_n = \sum_{k=0}^n \frac{1}{k!} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)$$

$$\leq \sum_{k=0}^n \frac{1}{k!}$$

$$\leq 2 + \sum_{\ell=1}^{n-1} \frac{1}{2^\ell}$$

$$< 3$$

so, we have

$$2 \leq e_n \leq 3$$

so, by the monotone convergence theorem,  $(e_n)_n$  converges

$$e := \sup_n \left(1 + \frac{1}{n}\right)^n$$

### Monotone Divergence

A sequence that is increasing and *unbounded* diverges to infinity.

Let  $M > 0$ . Since  $(x_n)_n$  is unbounded,  $\exists N \in \mathbb{N}$  such that  $x_N > M$

Thus, if  $n \geq N$ , then  $x_n \geq x_N > M$ .

Consider

$$h_n = \sum_{k=1}^n \frac{1}{k}$$

We can see that  $h_n < h_{n+1}$ . The primary question is as to whether  $(h_n)_n$  is bounded.

$$h_1 = 1$$

$$\geq 1$$

$$h_2 = 1 + \frac{1}{2}$$

$$\geq 1 + \frac{1}{2}$$

$$h_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2}$$

$$h_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

so, we have

$$h_{2^k} \geq 1 + \sum_{i=1}^k \frac{1}{2}$$

Let  $M$  be large. Find  $n$  such that  $n > 2(M - 1)$ . In this case,  $n/2 + 1 > M$ . Let  $N = 2^n$ . Then, for  $m \geq N$ ,  $h_m > M$ .

Thus,  $(h_n)_n$  diverges to infinity.

### Natural Sequences

A **natural sequence** is a strictly increasing sequence of natural numbers,  $(n_k)_{k=1}^\infty$

$$n_1 < n_2 < n_3 < \dots$$

where  $\forall k \in \mathbb{N}, n_k \in \mathbb{N}$ .

#### Natural Sequence Property

Given  $(n_k)_k$  natural sequence, show that  $(n_k) \geq k$ .

**Base Case:** We know that  $n_1 \leq 1$ , as  $n_1 \in \mathbb{N}$ .

**Inductive Step:** To be continued...

### Subsequences

Let  $(x_n)_n$  be a sequence. A subsequence  $(x_{n_k})_{k=1}^\infty$ , where  $(n_k)_k$  is a natural sequence.

For example, if  $(x_n)_n = (-1)^n$ . If  $(n_k) = 2k$ , then,  $(x_{n_k}) = ((-1)^{2k})_k = (1, 1, 1, \dots)$ . But, if  $(n_k) = 2k + 1$ , then  $(x_{n_k}) = (-1, -1, -1, \dots)$ .

If  $(x_n) = (1/n)_n$ , and  $(n_k)_k = k^2$ , then  $(x_{n_k})_k = (1/k^2)_k = (1, 1/4, 1/9, \dots)$ .

If  $(x_n)_n$  is a sequence, its  $m$ -th **tail** is  $(x_{m+k}) = (x_m, x_{m+1}, x_{m+2}, \dots)$ , where  $n_k = m + k$ .

### Convergence of Subsequence

If  $(x_n)_n \rightarrow x$ , then for any natural sequence  $(n_k)_k$ ,

$$(x_{n_k})_k \rightarrow x$$

Let  $\varepsilon > 0$ . Find  $N \in \mathbb{N}$  large such that  $n \geq N$ ,  $|x_n - x| < \varepsilon$ .

Take  $K = N$ . Then,

$$\begin{aligned} n_k &\geq k \\ &\geq K \\ &= N \\ \Rightarrow |x_{n_k} - x| &< \varepsilon \end{aligned}$$

### Corollary to Convergence of Subsequences

Given a sequence  $(x_n)_n$ , if there are two subsequences  $(x_{n_k})_k \rightarrow x$ ,  $(x_{n_\ell})_\ell \rightarrow x'$ , where  $x \neq x'$ , then  $(x_n)_n$  is divergent.

Recall the geometric sequence

$$(b^n)_{n=1}^\infty \rightarrow 0$$

if  $0 < b < 1$ .

The sequence  $(1, b, b^2, \dots)$  is decreasing and bounded below (as all elements are positive), meaning that by the monotone convergence theorem,  $(b^n)_n \rightarrow \ell$ .

Given  $n = 2k$ , we know that  $(b^{2k})_k \rightarrow \ell$ .

$$\begin{aligned} b^{2k} &= (b^k)^2 \\ (b^k)^2 &\rightarrow \ell^2 \\ \ell^2 &= \ell \\ \ell &= \{0, 1\} \end{aligned}$$

since  $b < 1$

$$\ell = 0$$

### Divergence and Subsequence

If  $(x_n)_n \nrightarrow x$ , then

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) (\exists n \geq N) \ni |x_n - x| \geq \varepsilon_0$$

We can use this to construct a sequence to show divergence.

Let  $(x_n)_n$  be a sequence, and  $x \in \mathbb{R}$ .

$$\begin{aligned} (x_n)_n &\nrightarrow x \\ &\Leftrightarrow \\ &(\exists \varepsilon_0 > 0) (\exists (x_{n_k})_k) \end{aligned}$$

with

$$|x_{n_k} - x| \geq \varepsilon_0$$

( $\Rightarrow$ ) We know  $\exists \varepsilon_0 > 0$  as above. We construct the sequence as follows:

$$N = 1 \Rightarrow \exists n_1 \geq 1$$

with

$$|x_{n_1} - x| \geq \varepsilon_0$$

$$N = n_1 + 1 \Rightarrow \exists n_2 \geq n_1 + 1$$

with

$$|x_{n_2} - x| \geq \varepsilon_0$$

$$N = n_2 + 1 \Rightarrow \exists n_3 \geq n_2 + 1$$

with

$$|x_{n_3} - x| \geq \varepsilon_0$$

Assume we have  $n_1 < n_2 < \dots, n_k$  with

$$|x_{n_j} - x| \geq \varepsilon_0$$

$$j = 1, 2, \dots, k$$

$$N = n_k + 1 \Rightarrow \exists n_{k+1} \geq n_k + 1$$

with

$$|x_{n_{k+1}} - x| \geq \varepsilon_0$$

Iteratively, we have our desired subsequence  $(x_{n_k})_k$ .

( $\Leftarrow$ ) If  $(x_n)_n \rightarrow x$ , any subsequence converges to  $x$ .

By assumption,  $(\exists \varepsilon_0 > 0) (\exists (n_k)_k)$  with  $|x_{n_k} - x| \geq \varepsilon_0$ . Thus,  $(x_{n_k})_k \nrightarrow x$ .



## Bolzano-Weierstrass Theorem

If  $(x_n)_n$  is a bounded sequence, then  $(x_n)_n$  admits a convergent subsequence.

## Lemma

Let  $(x_n)_n$  be any real sequence. Then,  $\exists n_k$  such that  $(x_{n_k})_k$  is monotone.

A **peak** of a sequence  $(x_n)_n$  is an  $x_m$  such that  $x_m \geq x_n \forall n \geq m$ .

**Case 1:** There are infinitely many peaks,  $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ , where  $n_1 < n_2 < \dots$ . Then,  $(x_{n_k})_k$  is decreasing.

**Case 2:** There are finitely many peaks. Let these peaks be  $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ .

Let  $n_1 = m_r + 1$ . Since  $x_{n_1}$  is not a peak,  $\exists n_2 > n_1$  such that  $x_{n_2} > x_{n_1}$ . Since  $x_{n_2}$  is not a peak,  $\exists n_3 > n_2$  such that  $x_{n_3} > x_{n_2}$ .

Iteratively, we have an increasing sequence of non-peaks  $(x_{n_k})_k$ .

Since  $(x_n)_n$  admits a monotone subsequence, and  $(x_{n_k})_k$  is bounded as  $(x_n)_n$  is bounded, this monotone, bounded subsequence must converge by the monotone convergence theorem.

## Limit Superior and Limit Inferior

Let  $X = (x_n)_n$  be a bounded real sequence. By Bolzano-Weierstrass,  $(x_n)_n$  admits at least one convergent subsequence.

Let

$$\overline{X} := \left\{ t \mid t \in \mathbb{R}, t = \lim_{k \rightarrow \infty} x_{n_k} \right\} \quad \text{for any subsequence } (x_{n_k})_k$$

Then,  $t \in \overline{X}$  is called a **limit point** of  $X$ .

Let  $u_1 = \sup_{n \geq 1}(x_n)$ ,  $\ell_1 = \inf_{n \geq 1}(x_n)$ . Clearly,  $\ell_1 \leq u_1$ , and  $\overline{X} \subseteq [\ell_1, u_1]$ .

Let  $u_2 = \sup_{n \geq 2}(x_n)$  and  $\ell_2 = \inf_{n \geq 2}(x_n)$ .

Since  $u_1$  is an upper bound for  $(x_n)_n$ , it is an upper bound for  $(x_n)_{n \geq 2}$ , so  $u_2 \leq u_1$ . Similarly, since  $\ell_1$  is a lower bound for  $(x_n)_n$ , it is a lower bound for  $(x_n)_{n \geq 2}$ , so  $\ell_2 \geq \ell_1$ .

As a result, we can see that  $\overline{X} \subseteq [\ell_2, u_2]$ .

We continue, letting  $u_m = \sup_{n \geq m}(x_n)$ , and  $\ell_m = \inf_{n \geq m}(x_n)$ . We get  $\ell_1 \leq \ell_2 \leq \dots$ , and  $u_1 \geq u_2 \geq \dots$ , and  $\overline{X} \subseteq [\ell_m, u_m]$ ,  $\forall m$ .

We get a nested sequence of intervals  $[\ell_1, u_1] \supseteq [\ell_2, u_2] \supseteq \dots$ . By the Nested Intervals Theorem, we know that

$$\begin{aligned} \overline{X} &\subseteq \bigcap_{m \geq 1} [\ell_m, u_m] \\ &= [\ell, u] \end{aligned}$$

where  $\ell = \sup(\ell_m)$  and  $u = \inf(u_m)$ .

Given a bounded sequence  $(x_n)_n = X$ ,

$$\begin{aligned} u &= \inf_{m \geq 1} (u_m) \\ &= \inf_{m \geq 1} \left( \sup_{n \geq m} x_n \right) \end{aligned}$$

called the **limit superior** of  $(x_n)_n$

$$u = \limsup_{n \rightarrow \infty} x_n$$

and

$$\begin{aligned} \ell &= \sup_{m \geq 1} (\ell_m) \\ &= \sup_{m \geq 1} \left( \inf_{n \geq m} (x_n) \right) \end{aligned}$$

called the **limit inferior** of  $(x_n)_n$

$$\ell = \liminf_{n \rightarrow \infty} x_n$$

### Applications of Limit Superior and Limit Inferior

Let  $(x_n)_n$  be bounded. Then,

- (1)  $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$
- (2)  $(x_n)_n \rightarrow x \Leftrightarrow \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$

- (1) This was proven with the Nested Intervals Theorem
- (2) Let  $\varepsilon > 0$ . Then,  $\exists N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$ .

We know that  $u_m = \sup_{n \geq m} x_n$ . If  $m \geq N$ , then  $u_m \in [x - \varepsilon/2, x + \varepsilon/2]$ . Therefore,  $|u_m - x| \leq \varepsilon/2 < \varepsilon$ , so  $(u_m)_m \rightarrow x = \limsup_{n \rightarrow \infty} x_n$ .

Similarly, we know that  $\ell_m = \inf_{n \geq m} x_n$ . If  $m \geq N$ , then  $\ell_m \in [x - \varepsilon/2, x + \varepsilon/2]$ . So,  $|\ell_m - x| \leq \varepsilon/2 < \varepsilon$ , so  $(\ell_m)_m \rightarrow x = \liminf_{n \rightarrow \infty} x_n$ .

Consider the sequence

$$\begin{aligned} x_n &= \begin{cases} 2 + \frac{1}{n} & n \in 2\mathbb{N} \\ -\frac{1}{n} & n \in 2\mathbb{N} - 1 \end{cases} \\ &= (-1, 5/2, -1/3, 9/4, -1/5, \dots) \end{aligned}$$

We begin by constructing the  $u_m$  sequence:  $(5/2, 5/2, 9/4, 9/4, \dots)$ . We can see that  $u_m \rightarrow 2$ .

Then, we construct the  $\ell_m$  sequence:  $(-1, -1/3, -1/3, -1/5, -1/5, \dots)$ . We can see that  $\ell_m \rightarrow 0$ .

**Exercise:** If  $(x_n)_n$  and  $(y_n)_n$  are sequences with  $x_n \leq y_n \forall n$ , then  $\limsup x_n \leq \limsup y_n$  and  $\liminf x_n \leq \liminf y_n$ .

### Ratio Test and Root Test Equivalent Convergence

If  $(a_n)_n$  is a sequence of strictly positive terms such that

$$\left( \frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$$

then,

$$\left( a_n^{1/n} \right)_{n=1}^{\infty} \rightarrow \rho$$

Let  $\varepsilon > 0$ . Then,  $\exists N$  large such that  $\forall n \geq N$ ,

$$\begin{aligned}
 \left| \frac{a_{n+1}}{a_n} - \rho \right| &< \varepsilon & \forall n \geq N \\
 \Rightarrow \frac{a_{n+1}}{a_n} &< \rho + \varepsilon & \forall n \geq N \\
 a_{n+1} &< a_n(\rho + \varepsilon) & \forall n \geq N \\
 a_n &< a_N(\rho + \varepsilon)^{n-N} & \forall n \geq N \\
 a_n &< (\rho + \varepsilon)^n \cdot \frac{a_N}{(\rho + \varepsilon)^N} \\
 a_n^{1/n} &< (\rho + \varepsilon) \left( \frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\
 \limsup a_n^{1/n} &\leq \limsup (\rho + \varepsilon) \left( \frac{a_N}{(\rho + \varepsilon)^N} \right)^{1/n} \\
 \limsup_{n \rightarrow \infty} a_n^{1/n} &\leq \rho + \varepsilon
 \end{aligned}$$

**Case 1:** If  $\rho = 0$ , the case is trivial.

**Case 2:** Suppose  $\rho > 0$ . Find  $\varepsilon > 0$  small such that  $0 < \varepsilon < \rho$ .

Since  $\left( \frac{a_{n+1}}{a_n} \right)_n \rightarrow \rho$ , find  $N$  large such that  $\frac{a_{n+1}}{a_n} > \rho - \varepsilon$ . So,  $\forall n \geq N$ ,

$$\begin{aligned}
 a_{n+1} &\geq a_n(\rho - \varepsilon) \\
 a_n &\geq a_N(\rho - \varepsilon)^{n-N} \\
 a_n^{1/n} &\geq (\rho - \varepsilon) \left( \frac{a_N}{(\rho - \varepsilon)^N} \right)^{1/n} \\
 \liminf a_n^{1/n} &\geq \rho - \varepsilon
 \end{aligned}$$

thus,

$$\rho \leq \liminf a_n^{1/n}$$

Together,  $\rho \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \rho$ , so  $\liminf a_n^{1/n} = \limsup a_n^{1/n} = \rho$ , whence  $(a_n^{1/n}) \rightarrow \rho$

### Properties of $\overline{X}$

We found earlier that  $\overline{X} \subseteq [\ell, u]$ . We claim that

$$\begin{aligned}
 \sup \overline{X} &= u \\
 \inf \overline{X} &= \ell
 \end{aligned}$$

We have shown that  $u$  is an upper bound for  $\overline{X}$ . The goal is to show that  $u$  is the least upper bound.

Let  $\varepsilon > 0$ . We need to find a  $t \in \overline{X}$  with  $u - \varepsilon < t$ . Note that  $u - \varepsilon < u_m \forall m$ .

We know that  $u - \varepsilon < u_1$ . Since  $u_1 = \sup_{n \geq 1} x_n$ , we know  $\exists n_1 \in \mathbb{N}$  with  $u - \varepsilon < x_{n_1} < u_1$ .

Consider  $u_{n_1+1} = \sup_{n > n_1} x_n$ . We know that  $u - \varepsilon < u_{n_1+1}$ . Therefore,  $\exists x_{n_2}$  with  $n_2 > n_1$  and  $u - \varepsilon < x_{n_2} < u_{n_1+1}$ .

Then, we use  $u_{n_2+1}$ . Then,  $\exists n_3 > n_2$  with  $u - \varepsilon < x_{n_3} < u_{n_2+1}$ .

We get a subsequence from the natural sequence  $n_1, n_2, \dots$ , where  $u - \varepsilon < x_{n_k} \forall k$ .

Also,  $x_{n_k} < u_1 \forall k$ . Therefore,  $(x_{n_k})_k$  is a bounded sequence. By Bolzano-Weierstrass,  $\exists$  a convergent subsequence

$$(x_{n_{k_j}})_j \rightarrow t$$

We know that  $u - \varepsilon \leq t$ . Note that  $t \in \overline{X}$ .

**Exercise:** Show that  $\inf \overline{X} = \ell$ .

### Cauchy Sequences

A sequence  $(x_n)_n$  in a metric space  $(X, d)$  is Cauchy if

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \ni p, q \geq N \Rightarrow d(x_p, x_q) < \varepsilon$$

if  $(X, d) = (\mathbb{R}, |\cdot|)$ :

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) \ni p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon$$

Consider the sequence  $(x_n)_n = \frac{1}{n}$ . Then,

$$\begin{aligned} |x_p - x_q| &= \left| \frac{1}{p} - \frac{1}{q} \right| \\ &= \frac{1}{q} - \frac{1}{p} \\ &\leq \frac{1}{q} \end{aligned}$$

Given  $\varepsilon > 0$ , find  $N$  large such that  $\frac{1}{N} < \varepsilon$ . Then,  $p, q \geq N$  implies

$$\begin{aligned} \left| \frac{1}{p} - \frac{1}{q} \right| &< \frac{1}{q} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

Show that  $(-1)^n$  is not Cauchy.

$$(\exists \varepsilon_0 > 0) (\forall N \in \mathbb{N}) \ni p, q \geq N \Rightarrow |x_p - x_q| \geq \varepsilon_0$$

### Boundedness of Cauchy Sequences

Cauchy sequences are bounded.

Let  $\varepsilon = 1$ . Then, by the Cauchy criterion,  $\exists N \in \mathbb{N}$  such that  $p, q \geq N \Rightarrow |x_p - x_q| < 1$ .

In particular,  $\forall n \geq N$ ,

$$\begin{aligned} |x_n| &= |x_n - x_N + x_N| \\ &\leq |x_n - x_N| + |x_N| \\ &< 1 + |x_N| \end{aligned} \quad \text{Triangle Inequality}$$

Let  $c = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$ . Then,  $x_n \leq c \forall n \geq 1$ . Thus,  $x_n$  is bounded.

### Convergent Subsequences of Cauchy Sequences

If  $(x_n)_n$  is Cauchy and  $(x_n)_n$  admits a convergent subsequence, then  $(x_n)_n$  is convergent.

Say  $(x_{n_k})_k \rightarrow x$  for some natural sequence  $(n_k)_k$ . We claim that  $(x_n)_n \rightarrow x$ .

Let  $\varepsilon > 0$ . Since  $(x_n)_n$  is Cauchy,  $\exists N \in \mathbb{N}$  such that  $p, q \geq N \Rightarrow |x_p - x_q| < \varepsilon/2$ .

Also, since  $(x_{n_k})_k \rightarrow x$ , then  $\exists K \in \mathbb{N}$  and  $K \geq N$  with  $k \geq K \Rightarrow |x_{n_k} - x| < \varepsilon/2$ .

For all  $k \geq K$ ,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \end{aligned}$$

Let  $N_1 = \max\{N, K\}$ . Then,

$$\begin{aligned} n \geq N_1 &\Rightarrow n \geq N && \text{by max} \\ &\Rightarrow n_k \geq k \geq K \geq N && \text{def. of natural sequence} \\ |x_n - x| &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

### Cauchy Sequence Convergence in the Reals

Let  $(x_n)_n$  be any sequence in  $\mathbb{R}$ . The following are equivalent:

- (1)  $(x_n)_n$  converges.
- (2)  $(x_n)_n$  is Cauchy.

(1)  $\Rightarrow$  (2) (Holds in any metric space). Suppose  $(x_n)_n \rightarrow x$ . Find  $N$  large such that  $n \geq N \Rightarrow d(x_n, x) < \varepsilon/2$ .

Then,  $p, q \geq N \Rightarrow$

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

(2)  $\Rightarrow$  (1) If  $(x_n)_n$  is Cauchy, then  $(x_n)_n$  is bounded.

By Bolzano-Weierstrass,  $(x_n)_n$  admits a convergent subsequence, so by our previous lemma,  $(x_n)_n$  must converge.

**Note:** To show (2)  $\Rightarrow$  (1), we used Bolzano-Weierstrass, which requires the monotone convergence theorem, which itself requires the completeness axiom. This is why we cannot show (2)  $\Rightarrow$  (1) converges.