Excess Area Identities and Operator Symbols in Bergman Spaces: A Multifaceted Analysis

Avinash Iyer

Occidental College

Summary

Contents

Definition and Notations

- Ω : a region in $\mathbb{C}$ e.g. $\mathbb{D}, D(0,r), \mathbb{A}(0,r,1), \mathbb{C}$
- $\lambda(z) = \lambda(|z|) \in C^\infty(\Omega)$: weight function

Definition (λ-weighted Square-Integrable Functions)

$$L^2(\Omega,\lambda) = \left\{ f \,\middle|\, \int_\Omega |f(z)|^2 \lambda(z)\, dA(z) < \infty \right\}$$

- $L^2(\Omega,\lambda)$ forms a Hilbert space with inner product

$$\langle f, g\rangle_{L^2(\Omega,\lambda)} = \int_\Omega f(z)\overline{g(z)}\lambda(z)\, dA(z)$$

inducing the norm

$$\|f\|^2_{L^2(\Omega,\lambda)} = \int_\Omega |f(z)|^2 \lambda(z)\, dA(z)$$

Definitions and Notations

Definition (Holomorphic Function on Ω)

$$h \in \mathcal{O}(\Omega) \iff \frac{\partial h(z)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial h(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}\right) = 0,\ \forall z \in \Omega$$

Definition (λ-weighted Bergman Space)

$$A^2(\Omega, \lambda) := \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda).$$

Definitions and Notations

Definition ($A^{1,2}(\Omega, \lambda)$)

$$A^{1,2}(\Omega, \lambda) = \left\{ h \in A^2(\Omega, \lambda) \;\middle|\; \frac{\partial h}{\partial z} \in A^2(\Omega, \lambda) \right\}$$

Definition (Weighted Image-Area)

Let $h \in A^{1,2}(\Omega, \lambda)$.

$$A_{\Omega,\lambda}(h) = \int_\Omega \left| \frac{\partial h}{\partial z} \right|^2 \lambda(z) \, dA(z)$$

Definitions and Notations

- $A^2(\Omega, \lambda)$ has a reproducing kernel i.e $\exists!\, K_\Omega^\lambda(\cdot, z) \in A^2(\Omega, \lambda)$:

$$h(z) = \left\langle h(\cdot), K_\Omega^\lambda(\cdot, z) \right\rangle_{L^2(\Omega,\lambda)}$$

- $A^2(\Omega, \lambda)$ is a closed subspace of $L^2(\Omega, \lambda)$.

Definition (Orthogonal Projection)

Let $P^{\Omega,\lambda} : L^2(\Omega, \lambda) \longrightarrow A^2(\Omega, \lambda)$

$$\begin{aligned}\left(P^{\Omega,\lambda} h\right)(z) &:= \left\langle h(\cdot), K_\Omega^\lambda(\cdot, z) \right\rangle_{L^2(\Omega,\lambda)} \\ &= \int_\Omega h(w) \overline{K_\Omega^\lambda(w, z)} \lambda(w)\, dA(w)\end{aligned}$$

Definitions and Notations

Definition (Multiplication Operator)

Let $M_\varphi : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda)$ where $\varphi \in L^\infty(\Omega)$

$$M_\varphi(h) := \varphi h$$

Definition (Toeplitz Operator)

$T_\varphi^{\Omega,\lambda} : A^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$, where $\varphi \in L^\infty(\Omega)$

$$T_\varphi^{\Omega,\lambda} := P^{\Omega,\lambda} M_\varphi$$

Definitions and Notations

Definition (Commutator)

Let $\left[P^{\Omega,\lambda}, M_\varphi\right] : L^2(\Omega,\lambda) \to L^2(\Omega,\lambda)$

$$\left[P^{\Omega,\lambda}, M_\varphi\right] := P^{\Omega,\lambda} M_\varphi - M_\varphi P^{\Omega,\lambda}$$

Definition (Hankel Operator)

Let $H_\varphi^{\Omega,\lambda} : A^2(\Omega,\lambda) \to (A^2(\Omega,\lambda))^\perp$

$$\begin{aligned} H_\varphi^{\Omega,\lambda} &:= -\left[P^{\Omega,\lambda}, M_\varphi\right]\Big|_{A^2(\Omega,\lambda)} \\ &= \left(I - P^{\Omega,\lambda}\right) M_\varphi \\ &= M_\varphi - P^{\Omega,\lambda} M_\varphi \\ &= M_\varphi - T_\varphi^{\Omega,\lambda} \end{aligned}$$

Contents

Motivations

- $\{z^n\}_{n=0}^{\infty}$ form a *complete orthogonal basis* for $A^2(\mathbb{D})$
- If $h \in \mathcal{O}(\mathbb{D})$, then h is analytic:

$$h(z) = \sum_{n=0}^{\infty} h_n z^n$$

  and

$$S_N := \sum_{n=0}^{N} h_n z^n$$

  converges uniformly on compact subsets.
- Relationship between L^2 norm of h to the ℓ^2 norm of $\{h_k\}_{k=0}^{\infty}$:

$$\|h\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |h(z)|^2 \, dA(z) = \pi \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- $\left[T_{\bar{z}}^{\mathbb{D}} M_z, DM_z\right](z^m) = 0$

Problems

- How can we expand **established identities concerning the area of the image of domains** under a holomorphic map in different Bergman spaces?
- Can we study the **structural properties of integral operators** (such as Toeplitz and Hankel operators) using the properties of Bergman spaces?

Literature Review on Previous Results I

- D'Angelo's excess area identity [D'A19]

Let $h \in A^{1,2}(\mathbb{D})$. Then,

$$\begin{aligned} A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) &= \left\| \frac{\partial (zh)}{\partial z} \right\|^2_{L^2(\mathbb{D})} - \left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\mathbb{D})} \\ &= \frac{1}{2} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta \\ &= \pi \left\| Sh \right\|^2_{L^2(b\mathbb{D})} \end{aligned}$$

where Sh is the restriction of h to the unit circle.

Literature Review on Previous Results II

- Excess area identity with Blaschke product multiplier
- 'Excess area' identity for harmonic functions [BÇGH22]

- Generating symbols for Toeplitz operators for a given initial p and target polynomial q on unit disc and polydisc, $T_\varphi^{\mathbb{D}}(p) = q$ and $T_\varphi^{\mathbb{D}^n}(p) = q$ [ÇDTR+24]

- Substituted derivatives for Toeplitz operators in excess area identity [ÇDTR+24]

Contents

Summary of Results

1. Results and Observations influenced by the Area Difference of the image of $\mathbb{D}$ between zh and h:
   i. On $\mathcal{F}^2 = A^2(\mathbb{C}, e^{-|z|^2})$, $A^2(\mathbb{D}, \lambda)$, $A^2(D(0,r))$
   ii. On convergence of identities on certain weighted discs.
2 Results and Observations influenced by symbol-generating algorithm for Toeplitz Operators
   i. On unweighted and weighted Toeplitz operators relation
   ii. On creating symbols for Unweighted and weighted Hankel operators and commutator operators on $A^2(\mathbb{D})$

Methods Used

- Relation between L^2 norms of functions and ℓ^2 norms of Taylor series:

$$\|h\|^2_{L^2(\mathbb{D})} = \sum_{k=0}^{\infty} \frac{|h_k|^2}{k+1}$$

- Integration by parts via Stokes's theorem on forms:

$$\oint_{b\Omega} f\, dz = \int_{\Omega} \frac{\partial f}{\partial \bar{z}}\, d\bar{z} \wedge dz$$
$$\oint_{b\Omega} f\, d\bar{z} = \int_{\Omega} \frac{\partial f}{\partial z} dz \wedge d\bar{z}.$$

- Inequalities e.g. Cauchy-Schwarz inequality, Hölder's inequality
- Special functions e.g. beta, gamma, hypergeometric

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta = \pi \left\|Sh\right\|^2_{L^2(b\mathbb{D})}$$

Excess Area on Fock Space

Given $0 < \rho < 1$, $h \in \mathcal{F}^2$, let $h_\rho(z) := h(\rho z)$

$$\begin{aligned} &A_{\mathcal{F}^2}\left(zh_\rho\right) - A_{\mathcal{F}^2}\left(h_\rho\right) \\ &= \pi \left\|zT_{\overline{z}}^{\mathcal{F}^2}\left(h_\rho\right)\right\|^2_{\mathcal{F}^2} + \pi \left\|T_{\overline{z}}^{\mathcal{F}^2}\left(h_\rho\right)\right\|^2_{\mathcal{F}^2} + \pi \left\|H_{\overline{z}}^{\mathcal{F}^2}\left(h_\rho\right)\right\|^2_{\mathcal{F}^2} \\ &= \pi \left\|z^2 h_\rho\right\|^2_{\mathcal{F}^2} - 2\pi \left\|zh_\rho\right\|^2_{\mathcal{F}^2} + \pi \left\|h_\rho\right\|^2_{\mathcal{F}^2} \end{aligned}$$

Here, the restriction of h to the unit circle in D'Angelo's identity is replaced with the Bergman projection on $\mathbb{C}$.

Using Integration by Parts to find Excess Area Identity: Wedge Product I

We want to use Integration by Parts and Stokes's Theorem to prove this excess area identity.

The area is integrated with respect to $dA = dx \wedge dy$. The wedge product has the following properties:

$$
\begin{aligned}
(a + b) \wedge c &= a \wedge c + b \wedge c \\
a \wedge b &= -b \wedge a \\
a \wedge a &= 0.
\end{aligned}
$$

Using Integration by Parts to find Excess Area Identity: Wedge Product II

With $z = x + iy$, $\overline{z} = x - iy$, the substitution $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$ yields

$$\begin{aligned} dx \wedge dy &= \frac{1}{2i}\left(d\overline{z} \wedge dz\right) \\ &= -\frac{1}{2i}\left(dz \wedge d\overline{z}\right). \end{aligned}$$

Using Integration by Parts to find Excess Area Identity: Calculation I

Set $\lambda = e^{-|z|^2} = e^{-z\bar{z}}$. Then, for $h, \frac{\partial h}{\partial z} \in A^2\left(\mathbb{C}, e^{-|z|^2}\right)$, we have

$$\begin{aligned}\left\|\frac{\partial h}{\partial z}\right\|^2_{A^2\left(\mathbb{C},e^{-|z|^2}\right)} &= \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{\partial h}{\partial z}\right)\overline{\left(\frac{\partial h}{\partial z}\right)} e^{-|z|^2}\, dx \wedge dy \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\left(\frac{\partial h}{\partial z}\right)} e^{-|z|^2}\, d\bar{z} \wedge \left(\frac{\partial h}{\partial z}\right) dz\end{aligned}$$

We now use integration by parts to move the derivative from $\overline{\left(\frac{\partial h}{\partial z}\right)}$ to $e^{-|z|^2}$.

$$\begin{aligned}\overline{\frac{\partial}{\partial z}}\left(\bar{h} e^{-|z|^2}\frac{\partial h}{\partial z}\right) = {}&\left(\left(\overline{\frac{\partial h}{\partial z}}\right)e^{-|z|^2}\right)\left(\frac{\partial h}{\partial z}\right) + \left(\bar{h}\,\overline{\frac{\partial}{\partial z}}\left(e^{-|z|^2}\right)\right)\left(\frac{\partial h}{\partial z}\right) \\ &+ \cancelto{0}{\left(\bar{h} e^{-|z|^2}\right)\overline{\frac{\partial}{\partial z}}\left(\frac{\partial h}{\partial z}\right)}.\end{aligned}$$

$$\left(\frac{\partial h}{\partial z}\right)\overline{\left(\frac{\partial h}{\partial z}\right)}e^{-|z|^2} = \overline{\frac{\partial}{\partial z}}\left(\bar{h} e^{-|z|^2}\frac{\partial h}{\partial z}\right) + \left(\bar{h}\right)\left(z\frac{\partial h}{\partial z}\right)\left(e^{-|z|^2}\right)$$

Using Integration by Parts to find Excess Area Identity: Calculation II

Rewriting the integral, we now have

$$\left\|\frac{\partial h}{\partial z}\right\|^2_{A^2\left(\mathbb{C},e^{-|z|^2}\right)} = \underbrace{\frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\frac{\partial}{\partial z}}\left(\overline{h}e^{-|z|^2}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz}_{\text{Integral } A} + \frac{1}{2\pi i}\int_{\mathbb{C}} \left(\overline{h}\right)\left(z\frac{\partial h}{\partial z}\right)\left(e^{-|z|^2}\right)\, d\overline{z}\wedge dz$$

Let's turn our attention to Integral A.

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\frac{\partial}{\partial z}}\left(\overline{h}e^{-|z|^2}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz &= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{D(0,r)} \overline{\frac{\partial}{\partial z}}\left(\overline{h}e^{-|z|^2}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz \\ &= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{D(0,r)} d\left(\overline{h}e^{-|z|^2}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz.\end{aligned}$$

Using Integration by Parts to find Excess Area Identity: Calculation III

We can now use Stokes's Theorem to move the integral in the limit from the interior of the disc to the boundary of the disc (dropping a derivative in the process).

$$\begin{aligned}\frac{1}{2\pi i}\lim_{r\to\infty}\int_{D(0,r)} d\left(\overline{h}e^{-|z|^2}\frac{\partial h}{\partial z}\right)\, d\overline{z}\wedge dz &= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{bD(0,r)} \overline{h}e^{-|z|^2}\frac{\partial h}{\partial z}\, dz \\ &= \frac{1}{2\pi i}\lim_{r\to\infty} e^{-r^2}\int_{bD(0,r)} \overline{h}\frac{\partial h}{\partial z}\, dz \\ &= 0\end{aligned}$$

Returning to the original area integral, we now have

$$\left\|\frac{\partial h}{\partial z}\right\|^2_{A^2\left(\mathbb{C},e^{-|z|^2}\right)} = \frac{1}{2\pi i}\int_{\mathbb{C}}\left(\overline{h}\right)\left(z\frac{\partial h}{\partial z}\right)e^{-|z|^2}\, d\overline{z}\wedge dz$$

Excess Area on $A^2(\mathbb{D},\lambda)$

D'Angelo's Identity:

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \frac{1}{2}\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta = \pi \left\|Sh\right\|^2_{L^2(b\mathbb{D})}$$

Excess Area on $A^2(\mathbb{D},\lambda)$

Let $h \in A^{1,2}(\mathbb{D},\lambda)$, $\lambda(z) = 1 - |z|^2$. Then,

$$A_{\mathbb{D},\lambda}\left(z^{m+1}h\right) - A_{\mathbb{D},\lambda}\left(z^m h\right) = \pi \left\|z^m h\right\|^2_{L^2(\mathbb{D},\lambda)}.$$

Here, the restriction of h to the unit circle is replaced with the function itself.

Excess Area on $A^2(D(0,r))$

Excess Area on $A^2(D(0,r))$, $0 < r < 1$

Let $f_{r,a_k}(\zeta) = (rf_{a_k}) \circ (rf_b)^{-1}(\zeta)$ and $|f_{r,a_k}(\zeta)| = r$ when $|\zeta| = r$, $a_k \neq b$.

Let $B_r = \prod_{k=1}^{n} f_{r,a_k}$ be a modified finite Blaschke product. Then,

$$A_{D(0,r)}(B_r h) - r^{2N} A_{D(0,r)}(h) = \pi r^{2(N-1)} \sum_{k=1}^{n} m_k \left\| Sh\left(f_{r,a_k}^{-1}\right) \right\|_{L^2(bD(0,r))}^2,$$

where m_k is the multiplicity of f_{r,a_k} and $N = \sum_{k=1}^{n} m_k$.

Excess Area on $A^2(D(0,r))$, cont'd

Excess Area Identity for Harmonic Functions on $D(0,r)$, $0<r<1$

For a harmonic function $u \in L^2(D(0,r))$, $\exists v \in L^2\,(D(0,r))$ harmonic conjugate [BÇGH22]. Let $h = u + iv$ be the corresponding holomorphic function. Then,

$$\left\|\frac{\partial(zu)}{\partial z}\right\|^2_{L^2(D(0,r))} - r^2\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(D(0,r))}$$
$$= \frac{1}{4}\left(\underbrace{r^2\pi\,\|Sh\|^2_{L^2(bD(0,r))}}_{A_{D(0,r)}(zh)-r^2A_{D(0,r)}(h)} + 2r^2\pi\Re(h_0^2) + \|h\|^2_{L^2(D(0,r))}\right).$$

Dilation and Contraction from $A^2(D(0,r))$ to $A^2(\mathbb{D})$

Contracting $h \in A^{1,2}(\mathbb{D})$ by taking $h_r = h(rz)$ for some $0 < r < 1$,

$$A_{\mathbb{D}}\left(zh_r\right) - A_{\mathbb{D}}\left(h_r\right) = \pi \left\|Sh_r\right\|^2_{L^2(b\mathbb{D})} \tag{1}$$

$$A_{D(0,r)}(zh) - r^2 A_{D(0,r)}(h) = \pi r^2 \left\|Sh\right\|^2_{L^2(bD(0,r))}. \tag{2}$$

Dilating $h \in A^{1,2}(D(0,r))$ by taking $h_{\frac{1}{r}} = h(\frac{z}{r})$ for some $0 < r < 1$

$$A_{D(0,r)}\left(zh_{1/r}\right) - r^2 A_{D(0,r)}\left(h_{1/r}\right) = \pi r^2 \left\|Sh_{1/r}\right\|^2_{L^2(bD(0,r))} \tag{3}$$

$$A_{\mathbb{D}}(zh) - A_{\mathbb{D}}(h) = \pi \left\|Sh\right\|^2_{L^2(b\mathbb{D})} \tag{4}$$

Dilation from $A^2(D(0,r),\lambda_r)$ to $\mathcal{F}^2$

Weighted Area on $D(0,r)$

Let $\lambda_r(z) = \chi_{D(0,r)}\left(1 - \frac{|z|^2}{r^2}\right)^{r^2}$ where $r > 0$. Then,

$$A_{D(0,r),\lambda_r}(h) = \int_{D(0,r)} |h'(z)|^2 \left(1 - \frac{|z|^2}{r^2}\right)^{r^2} dA(z)$$

We find that, as $r \to \infty$, $A_{D(0,r),\lambda_r}(h) \to A_{\mathcal{F}^2}(h)$.

Additionally, we know that

$$A_{\mathcal{F}^2}\left(h_\rho\right) = \left\|T_{\underline{z}}^{\mathcal{F}^2} h_\rho\right\|_{\mathcal{F}^2}^2$$

Berezin Transform Convergence

Reproducing Kernel on $A^2(D(0,r), \lambda_r)$

$$K^{\lambda_r}_{D(0,r)}(w,z) = \frac{1}{\left(1 - \frac{\bar{z}w}{r^2}\right)^{r^2+2}}$$

$K^{\lambda_r}_{D(0,r)}(w,z)$ uniformly converges on compact subsets of $D(0,r)$.

Reproducing Kernel on Fock Space

$$K_{\mathcal{F}^2}(w,z) = e^{\bar{z}w}$$

Berezin Transform Convergence, Cont'd

Definition (Berezin Transform ([Zhu07])

Let

$$k_z^{\Omega,\lambda}(w) := \frac{K_\Omega^\lambda(w,z)}{\sqrt{K_\Omega^\lambda(z,z)}}$$

Then, for some bounded operator T on $L^2(\Omega,\lambda)$, define $\mathcal{B}^{\Omega,\lambda} : B(L^2(\Omega,\lambda)) \longrightarrow L^2(\Omega,\lambda)$

$$(\mathcal{B}^{\Omega,\lambda}T)(z) := \left\langle Tk_z^{\Omega,\lambda}, k_z^{\Omega,\lambda} \right\rangle_{L^2(\Omega,\lambda)}$$

Berezin Transform Convergence, Cont'd

- For $\varphi \in L^\infty(\Omega, \lambda)$, $\mathcal{B}^{\Omega,\lambda} T_\varphi = \mathcal{B}^{\Omega,\lambda} M_\varphi$. (see Axler and Zheng, [AZ98a]).
- φ is harmonic if and only if $\mathcal{B}^{\Omega,\lambda} M_\varphi = \varphi$ (proof by Engliš, [Eng94]).
- We find that, for $T_\varphi^{D(0,r),\lambda_r} = P^{D(0,r),\lambda_r} M_\varphi$, the Berezin transform $\mathcal{B}^{D(0,r),\lambda_r} T_\varphi^{D(0,r),\lambda_r}$ converges pointwise to $\mathcal{B}^{\mathcal{F}^2} T_\varphi^{\mathcal{F}^2}$ as $r \to \infty$ from Göğüş and Şahutoğlu ([Gc20])
- This convergence is uniform on compact subsets of $\mathbb{C}$ (proof inspired by Göğüş and Şahutoğlu in [Gc20]).

Unweighted and Weighted Toeplitz Operators
Relation I

Using an extension of [ÇDTR+24, Lemma 2.1]

For weight $\lambda(z) = \left(1 - |z|^2\right)^{\alpha}$ ($\alpha \geqslant 0$) on the unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

$$\frac{T^{\mathbb{D},\lambda_\alpha}_{\bar{z}^m}(z^n)}{T^{\mathbb{D}}_{\bar{z}^m}(z^n)} = \begin{cases} \frac{\Gamma(m-n+\alpha-2)\Gamma(n+1)(m+1)}{\Gamma(m-n+2)\Gamma(n+\alpha+2)} & \text{if } m \leqslant n \\ \text{indeterminate} & \text{else} \end{cases}$$

$$T^{\mathbb{D},\lambda_\alpha}_{\bar{z}^m}(z^n) = s_{n,m,\alpha} T^{\mathbb{D}}_{\bar{z}^m}(z^n), \text{ and } \lim_{n\to\infty} s_{n,m,\alpha} = 1$$

Existence of Commutator Symbols

Given p and q are harmonic polynomials and $\frac{\partial}{\partial z}(p) \neq 0$, there does not exist a polynomial symbol ϕ, such that $\left[P^{\mathbb{D}}, M_\phi\right](p) = q$ or $\left[P^{\mathbb{D},\lambda}, M_\phi\right](p) = q$.

Compare to [ÇDTR+24], who worked on constructing Toeplitz symbols mapping between holomorphic polynomials.

Unweighted and Weighted Hankel Operator on $A^2(\mathbb{D})$

Existence of Hankel Operator Symbols

Given some holomorphic polynomials p, q where p is not constant, there does not exist a polynomial symbol ϕ such that $H_{\phi}^{\mathbb{D}}(p) = \overline{q}$ or $H_{\phi}^{\mathbb{D},\lambda}(p) = \overline{q}$

Contents

Remarks on the Annulus

Toeplitz Operator on Monomials on $A^2(\mathbb{A}(0,r,1))$

For all integers m and n,

$$T_{\bar{z}^m}^{\mathbb{A}(0,r,1)}(z^n) = \begin{cases} \frac{2mr^{2m}\ln(r)}{\left(r^{2m}-1\right)} z^{-m-1} & \text{if } n = -1 \\ \frac{r^{2m}-1}{2m\ln(r)} z^{-1} & \text{if } n = m-1 \\ \frac{(n-m+1)\left(1-r^{2n+2}\right)}{(n+1)\left(1-r^{2n-2m+2}\right)} z^{n-m} & \text{else} \end{cases}.$$

We attempted to find an algorithm to generate $\varphi \in L^\infty(\mathbb{A}(0,r,1))$ such that $T_\varphi^{\mathbb{A}(0,r,1)}(p) = q$ for given holomorphic Laurent polynomials p and q, but ran into trouble beyond the case where p has roots outside $\overline{\mathbb{A}(0,r,1)}$.

Future Directions

- Existence (or lack thereof) of bounded symbols for Toeplitz operators for a given initial polynomial p and target polynomial q on $\mathbb{A}(0, r, 1)$, $T_{\varphi}^{\mathbb{A}(0,r,1)}(p) = q$
- Extension of 'excess area' identity to harmonic functions in $L^2\left(\mathbb{C}, e^{-|z|^2}\right)$.
- Connection between non-weighted and weighted Toeplitz operators when the weight is exponential, $(1 - |z|^2)^A e^{\frac{-B}{(1-|z|^2)^{\alpha}}}$ $(A \geqslant 0, B > 0, \alpha > 0)$.

Contents

Experience at the REU

- Group consisted of Sakia Akamah (Rose–Hulman Institute of Technology), Jennifer Yuan (NYU Abu Dhabi), and myself.
- Spent 7 weeks doing various calculations, final week spent preparing the final report and presentation.
- Other projects at the REU included machine learning, coding theory, and mathematical biology,

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- Work was not all that we did, though.

Pictures

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