2

Prove that a connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

($\Rightarrow$) Let G be a graph with a non-closed Eulerian trail, $T = v_1..v_k$. Then, every internal vertex in T must be even, as any edge on the trail "entering" an internal vertex v_i must be paired with a different edge that "exits" the vertex.

However, since T is not a circuit, $d(v_1) \equiv 1 \mod 2$, otherwise every edge entering v_1 would be paired with an edge exiting v_1, meaning T would be a circuit, and v_k must have one edge entering v_k after all edges incident on it are paired up.

Finally, G cannot have only one vertex of odd degree, as $E(G) = \frac{\sum d(v)}{2}$, and if only v_1 were odd, $E(G)$ would not be an integer.

($\Leftarrow$) We will prove that if G has exactly two vertices of odd degree, then G contains a non-closed Eulerian trail.

Base Case: The graph K_2 is the smallest graph with two odd vertices, and it contains a non-closed Eulerian trail $v_1 v_2$.

Inductive Hypothesis: Suppose that a graph of n vertices with two vertices of odd degree has an Eulerian trail. Then, the graph with $n-1$ vertices will either be Eulerian or have an Eulerian trail.

Proof: Let $T = v_1..v_k$ be an Eulerian trail where v_1 and v_k are odd. In $G - v_1$. the following cases will occur:

Case 1: One vertex will have its degree reduced — because $\sum d(v)$ must be even, v_k is the vertex that will have its degree reduced, in which case $G - v_1$ is Eulerian, so we can create an Eulerian trail by starting from v_1, entering v_k, and going along the Eulerian circuit of $G - v_k$ before ending back at v_1.

Case 2: $2k+1$ vertices have their degree reduced for some $k > 0$ — In this case, $G - v_1$ must have more than one component. Each component must feature an Eulerian trail, as each component must have two vertices have their degree reduced by 1. Therefore, we start at v_1, go along the Eulerian trail in one component, return to v_1, etc.

In either case, we have that G must yield a component with an Eulerian trail.

5

Show how Theorem 5.1 and Corollary 5.2 can be used to help answer the question on the first page of Chapter 5 concerning the drawing in Figure 5.2

We can draw vertices at each of the intersections of two lines, with each line leaving the vertex signifying an edge. If the graph is Eulerian (or has exactly two odd vertices), then the drawing can be made without lifting one's pencil, otherwise it requires lifting one's pencil.

9

A connected graph G of even size contains exactly four odd vertices. Therefore G contains neither an Eulerian circuit nor an Eulerian trail.

(a) Show that G contains two trails T_1 and T_2 such that every edge of G belongs to exactly one of these trails.

(b) Show that G contains two trails T_1' and T_2', each of even size, such that every edge of G belongs to exactly one of these trails.

I don't know how to do this problem.

11

What is the minimum number of bridges in Königsberg that must be traversed (counting multiplicities) to conduct a round trip that crosses each of the seven bridges at least once.

There must be at least two bridges crossed twice in order to create a round trip in Königsberg.

X1

A connected graph G has an Eulerian circuit if every vertex of G has even degree.

Let C be a maximal trail in G, and let $v \in C$. Suppose toward contradiction that C is not closed. Then, $\exists v_i$ that serves as an endpoint to C: $C = v_i, \dots, v_k$. We know that $d(v_i) \geq 1$. $d(v_i) \neq 2$, as otherwise C would be extendible by adding $v_i v_1$ for $v_1 \leftrightarrow v_i$. Finally, any other vertices in the trail must maintain the parity of v_i, meaning v_i is odd. $\perp$ So, C must be a circuit. In particular, every vertex in C must have even degree (as every incident edge is paired with an outgoing edge).

Consider $G' = (V(G), E(G) - E(C))$. Then, since G is an even graph, and every vertex in C is even, the degree of every vertex in G is reduced by an even number when creating G'. Consider the component of G' that contains v. Since this component is connected, $\exists C'$ maximal circuit such that $v \in C'$.

We combine C' and C into a circuit by starting at v, going along C', then returning to v, then going along C back to v. Since C and C' are maximal, neither cannot be extended, so C' concatenated with C must be such that neither is extendible, so C' concatenated with C contains all edges of G.

X2

(a)

Give an algorithm for finding an Eulerian circuit in a connected graph all of whose vertices have positive even degree.

- Pick a vertex v.
- Trace a circuit C in G along v.
- Delete the edges of this circuit.
- Find components of $G - C$, trace a circuit in each component.
- Concatenate each circuit with C.

(b)

Give an algorithm for finding an Eulerian trail in a connected graph with vertices of odd degree.

- Start at a vertex v with odd degree.
- Trace a circuit returning to v.
- Delete this circuit.
- Trace a trail to the other vertex of odd degree.
- Concatenate.