

This is a set of notes I am taking for my Differential Topology class. The references occasionally used include

- *Geometry of Differential Forms* by Morita;
- *Topology from the Differentiable Viewpoint* by Milnor;
- *Differential Topology* by Hirsch
- *Introduction to Smooth Manifolds* by Lee;
- *A Short Course in Differential Topology* by Dundas.

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## Basic Properties

**Definition:** A topological space  $M$  is called a *manifold* if it satisfies the following:

- $M$  is Hausdorff (points can be separated by open sets);
- $M$  is second countable (the basis for the topology of  $M$  is countable);
- $M$  is locally Euclidean (every point in  $M$  has a neighborhood homeomorphic to  $\mathbb{R}^n$  for some  $n$ ).

In particular, the third condition says that for every  $p \in M$ , there is  $U \in \mathcal{O}_p$  and a homeomorphism  $\varphi: U \rightarrow \mathbb{R}^n$ . The value of  $n$  is called the *dimension* of the manifold  $M$ .

**Definition:** Let  $M$  be an  $n$ -manifold. A *chart* on  $M$  is a pair  $(U, \varphi)$  such that  $U \subseteq M$  is open,  $\varphi: U \rightarrow \mathbb{R}^n$  is a homeomorphism.

A family of charts  $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$  is known as an *atlas* if

$$M = \bigcup_{i \in I} U_i.$$

To understand the smooth structure of a manifold, we consider a point  $p \in M$  and two charts  $(U, \varphi_U)$  and  $(V, \varphi_V)$  such that  $p \in U$  and  $p \in V$ . The functions  $\varphi_U: U \rightarrow \mathbb{R}^n$  and  $\varphi_V: V \rightarrow \mathbb{R}^n$  are homeomorphism,

meaning that  $\varphi_V \circ \varphi_U^{-1}: \varphi_U(U \cap V)^n \rightarrow \mathbb{R}^n$  defined on the (nonempty)  $U \cap V$  is also a homeomorphism.

In particular, we develop the smooth structure by making sure all such pairs  $\varphi_V \circ \varphi_U^{-1}$  are *diffeomorphisms*. To do this, we need to first develop the derivative in  $\mathbb{R}^n$ .

**Definition:** Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a function. We say  $f$  is *differentiable* at  $p \in \mathbb{R}^n$  if there is a linear map  $L \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$  such that

$$\frac{|f(p+h) - f(p) - Lh|}{|h|} \rightarrow 0$$

as  $h \rightarrow 0$ .

The *derivative* of  $f$  is the association  $f \mapsto L$  for each  $p \in \mathbb{R}^n$ . We write  $D_p f$  to denote this map. Note that we consider elements of  $\text{Mat}_n(\mathbb{R})$  as points in  $\mathbb{R}^{n^2}$  with the standard topology on  $\mathbb{R}^{n^2}$ .

A function  $f$  is called a *diffeomorphism* if it is (sufficiently) continuously differentiable and has a (similarly sufficiently) continuously differentiable inverse.

**Definition:** If  $(U, \varphi_U)$  and  $(V, \varphi_V)$  are charts such that  $U \cap V \neq \emptyset$ , the function  $\varphi_V \circ \varphi_U^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is known as the *transition map* between  $\varphi_U$  and  $\varphi_V$ .

A smooth structure for  $M$  is an atlas  $\{(U_i, \varphi_i)\}_{i \in I}$  such that for all  $i, j$ , the transition maps  $\varphi_j \circ \varphi_i^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  are diffeomorphisms where defined.

If  $\{(U_i, \varphi_i)\}_{i \in I}$  is a *maximal* smooth atlas — i.e., any other smooth atlas that contains  $\{(U_i, \varphi_i)\}_{i \in I}$  is equal to  $\{(U_i, \varphi_i)\}_{i \in I}$  — then we call  $\{(U_i, \varphi_i)\}_{i \in I}$  a *smooth structure* for  $M$ .

**Note:** From now on, we use “manifold” to refer to smooth manifolds, and will say *topological* manifolds if the manifold does not necessarily admit a smooth structure.

**Definition:** A *submanifold* of a manifold  $M$  with dimension  $n + m$  is a subset  $N \subseteq M$  such that for each  $p \in N$ , there is a chart  $\varphi: U \supseteq M \rightarrow \mathbb{R}^{n+m}$  with  $p \in U$  and  $\varphi(U \cap N) = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+m}$ .

**Definition:** A map  $f: M \rightarrow N$  between manifolds is called *smooth* if for any chart  $(U, \varphi_U)$  in  $M$  and corresponding chart  $(V, \varphi_V)$  in  $N$ , the map  $\varphi_V \circ f \circ \varphi_U^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is (sufficiently) continuously differentiable.

The function  $f$  is a *diffeomorphism* if  $f$  is a smooth bijection with smooth inverse, and we say the manifolds  $M$  and  $N$  are diffeomorphic if they admit a diffeomorphism.

**Remark:** If  $f: M \rightarrow N$  is smooth, then any representation of  $f$  is smooth. To see this, if  $(U, \varphi_1)$  and  $(U, \varphi_2)$  are charts in  $M$ , with corresponding charts  $(V, \psi_1)$  and  $(V, \psi_2)$ , then

$$\psi_1 \circ f \circ \varphi_1^{-1} = (\psi_1 \circ \psi_2^{-1}) \circ (\psi_2 \circ f \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_1^{-1}),$$

where the transition maps  $\psi_1 \circ \psi_2^{-1}$  and  $\varphi_2 \circ \varphi_1^{-1}$  are smooth.

## More on Smooth Maps

Generally speaking, we will refer to charts on a dimension  $n$  smooth manifold by  $(U, \varphi) = (U; x_1, \dots, x_n)$ , where  $x_i: U \rightarrow \mathbb{R}$  are the coordinates of  $U$ . Additionally, if  $(\mathbb{R}^n; e_1, \dots, e_n)$  are the identity chart on  $\mathbb{R}^n$ , and the  $e_i$  are standard coordinates on  $\mathbb{R}^n$ , then the coordinate maps satisfy

$$x_i = e_i \circ \varphi.$$

**Definition:** Let  $(U; x_1, \dots, x_n)$  be a chart on a manifold  $M$  of dimension  $n$ . If  $f: M \rightarrow \mathbb{R}$  is a  $C^\infty$  function,

we define the *partial derivative* of  $f$  with respect to  $x_i$  at  $p$  to be

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial e_i}(\varphi(p)).$$

In particular,

$$\frac{\partial f}{\partial x_i} \circ \varphi^{-1} = \frac{\partial (f \circ \varphi^{-1})}{\partial e_i}$$

as functions on  $\phi(U)$ .

**Proposition:** The coordinate functions  $x_1, \dots, x_n$  satisfy  $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker delta.

*Proof.* For any  $p \in U$ , we calculate

$$\begin{aligned} \frac{\partial x_i}{\partial x_j}(p) &= \frac{\partial (x_i \circ \varphi^{-1})}{\partial e_j}(\varphi(p)) \\ &= \frac{\partial ((e_i \circ \varphi))}{\partial e_j}(\varphi(p)) \\ &= \frac{\partial e_i}{\partial e_j}(\varphi(p)) \\ &= \delta_{ij}. \end{aligned}$$

□

## Examples

There are a couple special examples of (smooth) manifolds.

- (i) Open subsets of  $\mathbb{R}^n$  are always manifolds.
- (ii) The general linear group,  $GL_n(\mathbb{R})$  of  $n \times n$  invertible matrices, viewed as a subset of  $Mat_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ , is a manifold. Furthermore, it is an open subset of  $\mathbb{R}^{n^2}$ , as considering the map  $\det: Mat_n(\mathbb{R}) \rightarrow \mathbb{R}$  given by  $A \mapsto \det(A)$ , we see that  $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ .
- (iii) The special linear group,  $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ , consisting of  $n \times n$  matrices with determinant 1, is also a smooth manifold. Furthermore, this manifold is a closed subset of  $\mathbb{R}^{n^2}$ , as it is equal to  $\det^{-1}(\{1\})$ .
- (iv) The  $n$ -sphere,  $S^n$ , given by

$$S^n = \left\{ (x_0, \dots, x_n) \mid \sum_{i=0}^n x_i^2 = 1 \right\}$$

is a manifold in  $\mathbb{R}^n$ . That it is a smooth manifold is quite a bit less obvious.

Now, in low dimensions, we know that  $S^2 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , and that the continuously differentiable transformation  $z \mapsto \frac{1}{z}$  takes the neighborhood basis of  $\infty$  to deleted neighborhoods of 0, and takes the neighborhood basis of 0 to the neighborhood basis of  $\infty$ . This is our desired smooth structure.

In the case of the general  $S^n$ , we use two stereographic projections to construct our smooth structure. The first stereographic projection is via the north pole,  $N_p$ , and maps points on  $S^n \setminus \{N_p\}$  bijectively to  $\mathbb{R}^n$ ; this is a chart that is defined everywhere on  $S^n$  except  $N_p$ . Similarly, we may use a stereographic projection originating from the south pole,  $S_p$ , so as to create another chart defined everywhere except  $S_p$ . These two stereographic projections are our desired smooth structure, as these two charts are all that is necessary to cover  $S^n$ .

(v) The real projective plane, consisting of lines through the origin in  $\mathbb{R}^{n+1}$ , can be expressed as

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \mathbb{R}^\times.$$

We will show that this is a manifold by constructing a family of charts mapping to  $\mathbb{R}^n$ .

Consider a point  $(r_0, \dots, r_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ . If  $r_0 \neq 0$ , then by dividing, we may associate this point's equivalence class in  $\mathbb{RP}^n$  to

$$(1, r_1/r_0, \dots, r_n/r_0) \in \{1\} \times \mathbb{R}^n,$$

so we may associate all points of the form  $[(r_0, \dots, r_n)]$  with  $r_0 \neq 0$  with a chart  $(U_0, \varphi_0)$  that maps  $\mathbb{RP}^n$  to  $\mathbb{R}^n$ .

Similarly, we may define  $U_k$  via

$$U_k = \{[(r_0, \dots, r_n)] \mid r_k \neq 0\}$$

with corresponding chart

$$\begin{aligned} \varphi_k: U_k &\rightarrow \mathbb{R}^n \\ [(r_0, \dots, r_n)] &\mapsto \frac{1}{r_k}(r_0, \dots, \widehat{r_k}, \dots, r_n), \end{aligned}$$

where  $\widehat{r_k}$  denotes the exclusion of the  $r_k$  coordinate. Varying  $k$  from 0 to  $n$ , we see that

$$\mathbb{RP}^n = \bigcup_{k=0}^n U_k,$$

the chart functions  $\varphi_k: U_k \rightarrow \mathbb{R}^n$  are homeomorphisms (as they are just division and projections). Furthermore, the transition maps  $\varphi_j \circ \varphi_i^{-1}$  are coordinate-wise rational functions defined by

$$(u_1, \dots, u_n) \mapsto \left( \frac{u_1}{u_i}, \dots, \frac{1}{u_i}, \dots, \frac{u_n}{u_i} \right),$$

where the  $\frac{1}{u_i}$  is at position  $j$ .

(vi) We now turn to a very important example from algebraic geometry: the Grassmannian,  $\text{Gr}(k, n)$ , consisting of all the  $k$ -dimensional subspaces of  $\mathbb{R}^n$ .

This is a  $k(n - k)$ -dimensional manifold; we need to understand what the smooth structure is. To do this, we let  $\langle \cdot, \cdot \rangle$  be an inner product on  $\mathbb{R}^n$ , and for any  $V \in \text{Gr}(k, n)$ , we consider maps in  $\text{Hom}(V, V^\perp)$ , where  $V^\perp$  denotes the orthogonal complement of  $V$ .

Now, we see that if  $W \in \text{Gr}(k, n)$  is any other  $k$ -dimensional subspace, the orthogonal  $P_V: \mathbb{R}^n \rightarrow V$  restricted to  $W$  is a linear isomorphism if and only if  $W \not\subseteq V^\perp$ , or that  $W \cap V^\perp = \{0\}$ .

We see that if  $W$  is such that  $P_V|_W: W \rightarrow V$  is a linear isomorphism, the inverse  $(P_V|_W)^{-1}: V \rightarrow W$  is well-defined; so, we may make a correspondence between  $\text{Hom}(V, V^\perp)$  and  $\text{Gr}(k, n)$  by noting that any such  $T \in \text{Hom}(V, V^\perp)$  has a corresponding graph  $(v, T(v))$ , so we take  $v \mapsto P_V|_W^{-1}(v)$ , then project onto  $V^\perp$  by taking  $T(P_{V^\perp}(P_V|_W^{-1}(v))) = T(v)$ . We depict it as a diagram below.

$$\begin{array}{ccccc}
 & & \text{id} & \xrightarrow{\quad} & V \\
 & & \nearrow & & \uparrow P_V \\
 V & \xrightarrow{P_V|_W^{-1}} & W & \xrightarrow{\iota} & \mathbb{R}^n \\
 & \searrow & & & \downarrow P_{V^\perp} \\
 & & T & \xrightarrow{\quad} & V^\perp
 \end{array}$$

Defining  $\mathcal{U}_V = \{W \in \text{Gr}(k, n) \mid W \cap V^\perp = \{0\}\}$ , we may define the chart from  $\mathcal{U}_V$  onto  $\text{Hom}(V, V^\perp)$  by  $\varphi_V = P_{V^\perp} \circ P_V|_W^{-1}$ . The family  $\{(\mathcal{U}_V, \varphi_V) \mid V \in \text{Gr}(k, n)\}$  is our smooth atlas.

## Inverse and Implicit Function Theorems

In order to replace manifolds with linear maps, we need to understand smooth maps on  $\mathbb{R}^n$ . The most important theorems in this regard are the inverse function theorem and the implicit function theorem.

**Theorem (Inverse Function Theorem):** Let  $f: \mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuously differentiable function. If  $D_p f$  is invertible as a linear map, then  $f$  has a local, continuously differentiable inverse  $f^{-1}: V \rightarrow W$ , where  $p \in W \subseteq \mathcal{U}$  and  $f(p) \in V \subseteq \mathbb{R}^n$ .

Additionally,  $D_p(f^{-1})$  is given by the inverse of the derivative's corresponding linear map evaluated at  $p$ ,  $D_p(f^{-1}) = (D_p f(f^{-1}(p)))^{-1}$ .

The proof uses the contraction mapping theorem. Recall that if  $X$  is a complete metric space, and  $f: X \rightarrow X$  is a strict uniform contraction — that is, there exists  $0 \leq \lambda < 1$  such that  $d(f(x), f(y)) \leq \lambda d(x, y)$  for all  $x, y \in X$  — then  $f$  has a unique fixed point.

We begin with a technical lemma.

**Lemma:** If  $\mathcal{U}(0, r) \subseteq V$  for some  $r > 0$  where  $V$  is a normed vector space,  $g: V \rightarrow V$  is a uniform contraction, and  $f = \text{id} + g$ , then the following hold:

- $(1 - \lambda)\|x - y\| \leq \|f(x) - f(y)\|$  (in particular,  $f$  is injective);
- if  $g(0) = 0$ , then

$$\mathcal{U}(0, (1 - \lambda)r) \subseteq f(\mathcal{U}(0, r)) \subseteq \mathcal{U}(0, (1 + \lambda)r).$$

*Proof of Lemma.* To see the first item, we notice that by the triangle inequality,

$$\begin{aligned}
 \|x - y\| - \|f(x) - f(y)\| &\leq \|x - y\| - \|x - y\| + \|g(x) - g(y)\| \\
 &\leq \lambda\|x - y\|,
 \end{aligned}$$

so  $(1 - \lambda)\|x - y\| \leq \|f(x) - f(y)\|$ , and  $f$  is injective. Furthermore, we see that if  $g(0) = 0$ , then

$$\begin{aligned}
 f(\mathcal{U}(0, r)) &= \mathcal{U}(0, r) + g(\mathcal{U}(0, r)) \\
 &\subseteq \mathcal{U}(0, r) + \lambda\mathcal{U}(0, r) \\
 &= \mathcal{U}(0, (1 + \lambda)r).
 \end{aligned}$$

Finally, if  $y \in \mathcal{U}(0, (1 - \lambda)r)$ , then we want to find  $x$  such that  $y = f(x) = x + g(x)$ ; equivalently, we see that we want  $x$  such that  $x = y - g(x)$ . Since the function  $F(x) = y - g(x)$  is a translation of a uniform contraction,  $F(x)$  is a contraction, so there is a fixed point, meaning  $y \in f(\mathcal{U}(0, r))$ .  $\square$

**Note:** We will use  $|\cdot|$  to denote the norm on  $\mathbb{R}^n$ .

*Proof of the Inverse Function Theorem.* By using a series of affine maps — first by translating  $p$  to 0, then translating  $f(p)$  to 0, then inverting  $D_0 f$  as per our assumption, we may safely assume that  $p = f(p) = 0$  and  $D_0 f = \text{Id}$ .

Set  $g = f - \text{Id}$ . We will show that  $g$  is a contraction in a sufficiently small ball. Fixing  $x, y \in \mathbb{R}^n$ , consider the map  $\mathbb{R} \rightarrow \mathbb{R}^n$  given by  $t \mapsto g(x + t(y - x))$ . Notice that by the Fundamental Theorem of Calculus,

$$|g(y) - g(x)| \leq |y - x| \sup_{0 \leq t \leq 1} |g'(x + t(y - x))|.$$

Furthermore, since  $g'(0) = 0$  by the fact that  $D_0 f = \text{Id}$  and  $(\text{Id})' = \text{Id}$ , and since  $f$  is continuously differentiable, there is  $r > 0$  such that

$$|g(y) - g(x)| \leq \frac{1}{2}|y - x|$$

for all  $x, y \in U(0, r)$ . Thus,  $g$  is a strict contraction on  $U(0, r)$ . By the previous lemma, we see that

$$U(0, r/2) \subseteq f(U(0, r));$$

by setting  $U = U(0, r) \cap f^{-1}(U(0, r))$ , we see that the map  $f|_U: U \rightarrow V := U(0, r/2)$  is a bijection. The inverse function  $f^{-1}: V \rightarrow U$  thus exists.

Now, we let  $h = f^{-1}$ ,  $x \in U$ ,  $y \in V$  such that  $h(x) = y$ , and  $A = D_x f$ . We will show that  $A^{-1} = D_y h$ , which is enough to show that  $h$  is continuously differentiable, as we assume the map  $x \mapsto D_x f$  is continuous, and inversion is continuous in  $GL_n(\mathbb{R})$ .

For sufficiently small vectors  $s$  and  $k$ , since  $f$  and  $h$  are bijections, we have

$$h(y + k) = x + s,$$

so

$$f(x + s) = y + k.$$

Furthermore, by unraveling the definitions of  $f = g + \text{Id}$ ,  $s$ , and  $k$ , and the fact that  $g$  is a uniform contraction on  $U$ , we get

$$\begin{aligned} |s - k| &= |(f(x + s) - f(x)) - s| \\ &= |(x + s + g(x + s)) - (x + g(x)) - s| \\ &= |g(x + s) - g(x)| \\ &\leq \frac{|s|}{2}. \end{aligned}$$

In particular, since

$$\begin{aligned} |s| &\leq |s - k| + |k| \\ &\leq |k| + \frac{|s|}{2}, \end{aligned}$$

we see that  $|s|/2 \leq |k|$ . We calculate

$$\begin{aligned} |h(y + k) - h(y) - A^{-1}k| &= |x + s - x - A^{-1}(f(x + s) - f(x))| \\ &= |s - A^{-1}(f(x + s) - f(x))| \\ &\leq \|A^{-1}\|_{\text{op}} |As - f(x + s) - f(x)|. \end{aligned}$$

Thus, since  $|s|/2 \leq |k|$ ,

$$\begin{aligned} \frac{|h(y + k) - h(y) - A^{-1}k|}{|k|} &\leq \frac{2\|A^{-1}\|_{\text{op}} |As - f(x + s) - f(x)|}{|s|} \\ &\rightarrow 0, \end{aligned}$$

so  $D_y h = A^{-1}$ . □

One of the primary uses of the inverse function theorem is to prove the implicit function theorem.

**Theorem (Implicit Function Theorem):** Let  $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be continuously differentiable, and let  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ . Assume

- $f(a, b) = 0$ ;
- the map  $y \mapsto f(a, y)$  defined on  $\mathbb{R}^m \rightarrow \mathbb{R}^m$  is invertible in a neighborhood of  $b$  — i.e.,  $D_b(y \mapsto f(a, y))$  as a linear map has rank  $m$ .

Then, there exists a continuously differentiable function  $g: U \rightarrow V$ , where  $U \in \mathcal{O}_a$  and  $V \in \mathcal{O}_b$  such that  $f(x, g(x)) = 0$  on  $U$ .

Essentially, the theorem says that we can solve  $f(x, y) = 0$  on a neighborhood of  $(a, b)$  by a function only depending on  $x$ . This means that about  $(a, b)$  in the graph  $\Gamma(f)$ , there is a coordinate representation as an  $m$ -manifold given by  $g$ .

*Proof of the Implicit Function Theorem.* Define a function  $F: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$  by

$$F(x, y) = (x, f(x, y)).$$

Since  $f$  is continuously differentiable, this function  $F$  is also continuously differentiable, so we may define  $U \in \mathcal{O}_a$ ,  $V \in \mathcal{O}_b$ , and  $W \in \mathcal{O}_{F(a,b)}$  such that

$$F: U \times V \rightarrow W$$

is continuously differentiable with continuously differentiable inverse (owing to the Inverse Function Theorem), so that  $G = F^{-1} = (G_1, G_2)$  is defined on  $W$ . We see that

$$(x, y) = F(G_1(x, y), f(G_1(x, y), G_2(x, y))),$$

meaning that  $G_1(x, y) = x$ , and  $y = f(x, G_2(x, y))$ . Since at  $b$ ,  $f(a, b) = 0$ , we have that  $g(x) = G_2(x, 0)$  is the desired function.  $\square$

## Constructing $C^\infty$ Maps on Manifolds

**Definition:** A function  $f: U \rightarrow \mathbb{R}$ , where  $U \subseteq \mathbb{R}^n$  is open, is called  $C^\infty$  if the partial derivatives of all orders,

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

are continuous. Here,  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a *multi-index*, where the  $\alpha_i$  are positive integers for each  $i$ , and  $|\alpha|$  is defined by  $|\alpha| = \sum_{i=1}^n \alpha_i$ .

We are concerned now with constructing  $C^\infty$  functions on  $C^\infty$ -manifolds.<sup>I</sup> In order to do this, we introduce the bump functions.

**Definition:** The *bump function* that is equal to 1 on  $B(0, 1)$  and is zero outside  $U(0, 2)$  is given by

$$h(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$b(x) = \frac{h(4 - |x|^2)}{h(4 - |x|^2) + h(|x|^2 - 1)}. \quad (*)$$

<sup>I</sup>A  $C^\infty$  manifold is one where all the transition functions  $\varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$  are  $C^\infty$  functions.

**Lemma:** Let  $M$  be a  $C^\infty$  manifold. Let  $U \in \mathcal{O}_p$ , and let  $f: U \rightarrow \mathbb{R}$  be an arbitrary  $C^\infty$  function defined on  $U$ .

Then, there exists  $V \in \mathcal{O}_p$  with  $\bar{V} \subseteq U$ , and a  $C^\infty$  function  $\tilde{f}$  defined on  $M$  such that

$$\tilde{f}(q) = \begin{cases} f(q) & q \in V \\ 0 & q \notin U. \end{cases}$$

*Proof.* Let  $(W, \varphi)$  be a chart centered at  $p$  with  $\varphi(p) = 0$  and  $U(0, 3) \subseteq \varphi(W)$ . Let  $\bar{b} = b \circ \varphi$ , where  $b$  is the bump function defined in (\*). Then,  $\bar{b}$  is a  $C^\infty$  function on  $W$ , and is 0 outside  $\varphi^{-1}(U(0, 2)) \subseteq W$ .

We define  $\bar{b}$  to be equal to zero on  $W^c$ . Thus, if we define  $V = \varphi^{-1}(U(0, 1))$ , then  $V \in \mathcal{O}_p$ ,  $\bar{V} \subseteq U$ , and  $\bar{b}$  is equal to 1 on  $V$ . Letting

$$\tilde{f}(q) = \begin{cases} \bar{b}(q)f(q) & q \in W \\ 0 & q \notin W, \end{cases}$$

we see that  $\tilde{f}$  satisfies the required property.  $\square$

Given an atlas  $\{(U_i, \varphi_i)\}$ , we want to be able to “glue” functions together by using these charts. A fundamental construction for this purpose is known as a partition of unity.

**Definition:** Let  $X$  be a topological space.

- An open cover  $\{U_i\}_{i \in I}$  is called *locally finite* if, for every  $x \in X$ , there is some  $V \in \mathcal{O}_x$  such that  $V \cap U_i = \emptyset$  for all but finitely many  $i$ .
- Another open cover  $\{V_j\}_{j \in J}$  is called a *refinement* of another open cover  $\{U_i\}_{i \in I}$  if for all  $j \in J$ , there exists some  $i \in I$  such that  $V_j \subseteq U_i$ .
- We say  $X$  is *paracompact* if, for any open cover of  $X$ , there is a locally finite refinement.

**Proposition:** Let  $M$  be a topological manifold. Then, for any open cover  $\{U_i\}_{i \in I}$  of  $M$ , there is a countable, locally finite refinement  $\{V_k\}_{k=1}^\infty$  with the  $\bar{V}_k$  compact. In particular,  $M$  is paracompact.

Additionally, we may select the coordinate maps  $\psi_k: V_k \rightarrow \mathbb{R}^n$  such that  $\psi_k(V_k) = U(0, 3)$ , and  $\{\psi_k^{-1}(U(0, 1))\}_{k=1}^\infty$  is an open cover of  $M$ .

*Proof.* Since  $M$  is a locally Euclidean and second countable, there is a countable basis of pre-compact open sets  $\{O_\ell\}_{\ell=1}^\infty$ . In particular, we may select an exhaustion of  $M$  by pre-compact sets by defining

$$\begin{aligned} E_1 &= O_1 \\ E_k &= O_1 \cup O_2 \cup \cdots \cup O_{\ell_k}, \end{aligned}$$

where  $\ell_k$  is some sufficiently large index as follows. Since  $\bar{E}_k$  is compact, there is a sufficiently large  $\ell$  such that  $\bar{E}_k \subseteq O_1 \cup \cdots \cup O_\ell$ . Defining  $\ell_{k+1}$  to be the smallest index greater than  $\ell_k$  that satisfy this property, we define

$$E_{k+1} = O_1 \cup \cdots \cup O_{\ell_{k+1}}.$$

For arbitrary  $k$ , each  $\bar{E}_k$  is compact, and  $\bar{E}_k \subseteq E_{k+1}$ , and  $\bigcup_{k=1}^\infty E_k = M$ . Note that if  $M$  is compact, this process terminates in a finite number of steps.

Now, let  $\{U_i\}_{i \in I}$  be an arbitrary open cover of  $M$ , and fix  $k \geq 1$ . For each  $p \in \bar{E}_k \setminus E_{k-1}$ , select  $i_p$  such that  $p \in U_{i_p}$ , and select a chart  $(V_p, \psi_p)$  about  $p$  that satisfies  $\psi_p(p) = 0$ ,  $\psi_p(V_p) = U(0, 3)$ , and  $V_p \subseteq U_{i_p} \cap (E_{k+1} \setminus \bar{E}_{k-2})$ , where we set  $E_{-1} = E_0 = \emptyset$ . Finally, set  $W_p = \psi_p^{-1}(U(0, 1))$ .



Since  $\overline{E_k} \setminus E_{k-1}$  is compact, we may select a finite number of such  $p$  such that the open sets  $W_p$  cover  $\overline{E_k} \setminus E_{k-1}$ . Applying this process to all  $k$ , and lining up the charts  $(V_p, \psi_p)$  corresponding to the finite number of points  $p$  chosen at each stage, we have the locally finite refinement of  $\{U_i\}_{i \in I}$  with each  $\overline{V_k}$  compact,  $\psi_k(V_k) = U(0, 3)$ , and  $\{\psi_k^{-1}(U(0, 1))\}$  an open cover of  $M$ .  $\square$

**Definition:** Let  $M$  be a  $C^\infty$  manifold. A family  $\{f_k\}_{k=1}^\infty$  of at most countably many  $C^\infty$  functions on  $M$  is called a *partition of unity* on  $M$  if it satisfies:

- for each  $k$ ,  $f_k(p) \geq 0$  for all  $p \in M$ , and the family  $\{\text{supp}(f_k)\}_{k=1}^\infty$  is locally finite;
- at all points  $p$  on  $M$ ,  $\sum_{k=1}^\infty f_k(p) = 1$ .

If  $\{\text{supp}(f_k)\}_{k=1}^\infty$  is a refinement of an open cover  $\{U_i\}_{i \in I}$ , then we say the partition of unity is *subordinate* to the open cover.

**Theorem:** Let  $M$  be a  $C^\infty$  manifold, and let  $\{U_i\}_{i \in I}$  be an open cover of  $M$ . Then, there exists a partition of unity  $\{f_k\}_{k=1}^\infty$  that is subordinate to  $\{U_i\}_{i \in I}$ .

*Proof.* Let  $\{V_k\}_{k=1}^\infty$  be a locally finite refinement of  $\{U_i\}_{i \in I}$  such that the charts  $(V_k, \psi_k)$  have  $\psi_k(V_k) = U(0, 3)$ .

For each  $k$ , using the bump function  $(*)$ , define

$$\widetilde{b}_k(q) = \begin{cases} b \circ \psi_k(q) & q \in V_k \\ 0 & q \notin V_k. \end{cases}$$

Then,  $\widetilde{b}_k$  is a  $C^\infty$  function defined on  $M$ , and since  $\text{supp}(\widetilde{b}_k) \subseteq V_k$ , we may set

$$f = \sum_{k=1}^\infty \widetilde{b}_k.$$

The function  $f$  is a  $C^\infty$  function defined on the whole of  $M$ . If we let  $W_k = \psi_k^{-1}(U(0, 1))$ , then since  $\{W_k\}_{k \geq 1}$  is an open cover of  $M$ , for any  $q \in M$ , there exists  $j$  such that  $\widetilde{b}_j(q) = 1$ . Thus,  $f$  never equals 0, so we if we set

$$f_k = \frac{\widetilde{b}_k}{f},$$

the family  $\{f_k\}_{k \geq 1}$  is a partition of unity subordinate to  $\{U_i\}_{i \in I}$ .  $\square$

## Tangent Spaces and Vector Fields

Smooth manifolds are able to be embedded into some Euclidean space,<sup>II</sup> so we start by considering them as such.

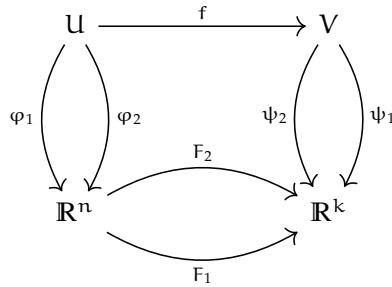
**Definition:** If  $f: M \rightarrow N$  is a smooth map between an  $n$ -dimensional manifold  $M$  and a  $k$ -dimensional manifold  $N$  that are embedded into some Euclidean space  $\mathbb{R}^\ell$ , the *derivative* of  $f$  at  $p$ , defined for charts  $(U, \varphi)$  and  $(V, \psi)$ , where  $f(U) \subseteq V$ , is defined by

$$D_p f = D_p (\psi^{-1} \circ f \circ \varphi),$$

where  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is defined to be a map such that  $f = \psi^{-1} \circ F \circ \varphi$ .

<sup>II</sup>This is actually a very deep theorem.

**Remark:** This definition is independent of the chart representation. To see this, notice that as we have embedded both  $M$  and  $N$  into Euclidean space, the maps  $\varphi: U \rightarrow \mathbb{R}^n$  and  $\psi: V \rightarrow \mathbb{R}^k$  are diffeomorphisms, hence their derivatives are invertible linear maps.



Using some coordinate changes, we see that

$$F_1 = (\psi_1 \circ \psi_2^{-1}) \circ F_2 \circ (\varphi_2 \circ \varphi_1^{-1})$$

so by the chain rule,

$$\begin{aligned}
 Df &= D(\psi_1^{-1} \circ F_1 \circ \varphi_1) \\
 &= D(\psi_1)^{-1} \circ D(F_1) \circ D(\varphi_1) \\
 &= D(\psi_1^{-1}) \circ D((\psi_1 \circ \psi_2^{-1}) \circ F_2 \circ (\varphi_2 \circ \varphi_1^{-1})) \\
 &= D(\psi_1)^{-1} \circ D(\psi_1) \circ D(\psi_2^{-1} \circ F_2 \circ \varphi_2) \circ D(\varphi_1)^{-1} \circ D(\varphi_1) \\
 &= D(\psi_2^{-1} \circ F_2 \circ \varphi_2).
 \end{aligned}$$

Note here that the chain rule is being used in  $\mathbb{R}^\ell$ , which Dundas calls the “flat chain rule,”<sup>III</sup> rather than the general case on a manifold.

One of the issues with this strategy, though, is that embeddings may carry different properties (though at high enough dimensions, any two embeddings are diffeomorphic to each other). For instance, embeddings  $S^1 \hookrightarrow \mathbb{R}^3$  form the field of knot theory, which is a very intricate field.

As a result, we want to be able to define tangent spaces, derivatives, and the like without having to refer to coordinates. In order to do this, we need to discuss germs of functions.

**Definition:** Let  $g: M \rightarrow N$  map  $p \mapsto g(p)$ . We define an equivalence relation on the space of functions  $f: M \rightarrow N$  with  $f(p) = g(p)$  by taking  $f_1 \sim f_2$  whenever  $f_1 = f_2$  on some open neighborhood  $A \in \mathcal{O}_p$ . The equivalence class  $[g]_p$  is known as the *germ* of  $g$  at  $p$ .

We denote by  $\mathcal{C}_p$  the space of germs of  $C^\infty$  functions  $f: M \rightarrow \mathbb{R}$  at  $p$ .

**Remark:** Often, books will use  $\mathcal{O}_p$  to refer to the space of germs at  $p$ . We will use  $\mathcal{C}_p$  for this purpose though, as we have defined  $\mathcal{O}_p$  to refer to the system of open neighborhoods at  $p$ .

**Proposition:** The space  $\mathcal{C}_p$  with the operations

- $[g] + [h] = [g + h]$ ;
- $\alpha[g] = [\alpha g]$ ;
- $[g] \cdot [h] = [g \cdot h]$

forms an algebra over  $\mathbb{R}$ .

<sup>III</sup>Flatness is always relative.

**Definition:** Let  $W_p$  be the space of germs of smooth maps  $\gamma: \mathbb{R} \rightarrow M$  that send  $0 \mapsto p$ . The *tangent space*  $T_p M$  is defined by  $W_p / \sim$ , where we define the equivalence relation  $[g_1] \sim [g_2]$  for two germs at  $p$   $g_1: M \rightarrow \mathbb{R}$  and  $g_2: M \rightarrow \mathbb{R}$  by

$$(\varphi \circ g_1)'(0) = (\varphi \circ g_2)'(0)$$

for all  $\varphi \in \mathcal{C}_p$ .

**Definition:** If  $f: M \rightarrow N$  is a smooth map, we define the map  $T_p f: T_p M \rightarrow T_{f(p)} N$  to act via

$$T_p f([\gamma]) = [f \circ \gamma]$$

for all  $\gamma \in W_p$ .

**Proposition (Chain Rule):** If  $f: M \rightarrow N$  and  $g: N \rightarrow L$  are smooth maps, then

$$T_{f(p)} g \circ T_p f = T_p (g \circ f).$$

*Proof.* If  $\gamma \in W_p$ , then

$$\begin{aligned} T_{f(p)} g \circ T_p f([\gamma]) &= T_{f(p)} g([f \circ \gamma]) \\ &= [g \circ f \circ \gamma] \\ &= T_p (g \circ f)([\gamma]). \end{aligned}$$

□

A terribly kept secret is that this function  $T_p f$  is actually the differential  $D_p f$ . This requires us to prove that this definition comports with the definition for the case of  $M$  as an embedded manifold. We require a few basic propositions for this purpose whose proofs follow from the inverse function theorem and various definitions.

**Proposition:** If  $f: M \rightarrow N$  is a locally invertible smooth map about  $p \in M$ , then  $T_p f: T_p M \rightarrow T_{f(p)} N$  is an isomorphism of vector spaces.

**Proposition:** If  $0 \in \mathbb{R}^k$ , then  $T_0 \mathbb{R}^k$  is represented by linear maps  $t \mapsto \lambda t$  for some vector  $\lambda \in \mathbb{R}^k$ . Therefore, if  $M$  is  $k$ -dimensional,  $T_p M \cong \mathbb{R}^k \cong \text{Hom}(\mathbb{R}, \mathbb{R}^k)$ .

**Proposition:** If  $\varphi: U \rightarrow \mathbb{R}^k$  is a local diffeomorphism about  $p$  such that  $\varphi(p) = 0$ , then if  $f \in C^\infty(\mathbb{R}^k)$ ,  $f \circ \varphi \in C^\infty(U)$ , which induces an algebra homomorphism

$$\begin{aligned} \varphi^*: \mathcal{C}_{0, \mathbb{R}^k} &\rightarrow \mathcal{C}_{p, M} \\ f &\mapsto f \circ \varphi. \end{aligned}$$

**Proposition:** If  $M$  and  $N$  are embedded submanifolds of  $\mathbb{R}^n$  with dimensions  $m$  and  $k$  respectively, and  $f: M \rightarrow N$  is a smooth map, then

$$D_p f \equiv T_p f.$$

*Proof.* Let  $(U, \varphi)$  and  $(V, \psi)$  be charts for  $M$  and  $N$  respectively with  $p \in U$  and  $f(p) \in V$ . Then, we may consider the coordinate maps  $\varphi: U \rightarrow \mathbb{R}^m$  and  $\psi: V \rightarrow \mathbb{R}^k$  to be such that  $p \mapsto 0$  and  $f(p) \mapsto 0$  respectively.

Now, we see that  $T_p f$  and  $D_p f$  can be written as

$$\begin{aligned} T_p f &= T_{f(p)} \psi^{-1} \circ T_0 F \circ T_{f(p)} \varphi \\ D_p f &= D_p (\psi^{-1} \circ F \circ \varphi). \end{aligned}$$

Yet, since  $T_0 F = D_0 F$ ,  $T_{f(p)} \psi^{-1} = D_{f(p)} \psi^{-1}$ , and  $T_p \varphi = D_p \varphi$ , the chain rule gives

$$\begin{aligned} T_p f &= T_p (\psi^{-1} \circ F \circ \varphi) \\ &= D_p (\psi^{-1} \circ F \circ \varphi) \end{aligned}$$

$$= D_p f,$$

implying that  $T_p f = D_p f$ .  $\square$

Now that we have established that we can consider manifolds as either standalone entities or as submanifolds of  $\mathbb{R}^n$ , we now shift our focus to understanding what information the derivative map  $D_p f: T_p M \rightarrow T_{f(p)} M$  gives us about the underlying topology of  $M$  and  $N$ .

**Definition:** Let  $f: M \rightarrow N$  be a smooth map, and let  $p \in M$ . We say  $p$  is a *critical point* for  $f$  if  $D_p f$  does not have maximal rank.

If  $D_p f$  has maximal rank, then we say  $p$  is a *regular point* of  $f$ .

We say  $q \in N$  is a *critical value* if  $f^{-1}(\{q\})$  contains a critical point for  $f$ . Else, we say  $q$  is a *regular value*.

**Remark:** To show that  $D_p f: T_p M \rightarrow T_{f(p)} N$  is surjective, we take an element of  $T_{f(p)} N$  (i.e., a smooth map  $\gamma: \mathbb{R} \rightarrow N$ ) and show that there is a smooth map  $\delta: \mathbb{R} \rightarrow M$  such that  $f \circ \delta = \gamma$  on a neighborhood of  $f(p)$ .

The study of critical points is actually very vital in understanding the underlying manifold's global topology. This is the field known as Morse theory, and we will discuss it later in the course.

**Definition:** Let  $f: M \rightarrow \mathbb{R}$  be a smooth function, with  $M$  a manifold. We say  $f$  is *Morse* if all the critical points of  $f$  are isolated, and the critical points are nondegenerate, in the sense that the Hessian matrix, given by

$$H_p f = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{i,j=1}^n$$

has nonzero determinant, where  $(x_1, \dots, x_n)$  is a coordinate system about the critical point  $p$ .

Morse functions allow us to calculate a quantity known as the index of the manifold at any given value of  $\mathbb{R}$ , thereby allowing us to reconstruct the manifold from the information that the functions give us.

There are two important theorems related to critical points/values and regular points/values.

**Theorem (Sard's Theorem):** Let  $f: M \rightarrow N$  be a smooth map of manifolds. The set of critical values in  $N$  is of measure zero (where measure is defined by the measure of the image under a coordinate map).

We will not prove Sard's Theorem now, but we will prove a very useful result that is often used in conjunction with Sard's Theorem.

**Theorem (Regular Value Theorem):** Let  $f: M \rightarrow N$  be a smooth map of manifolds with dimensions  $m \geq n$  respectively. If  $q \in N$  is a regular value, then  $f^{-1}(\{q\}) \subseteq M$  is a submanifold of dimension  $m - n$ .

*Proof.* Let  $p \in f^{-1}(\{q\})$ , and let  $(U, \varphi)$  be a chart of  $p$  where  $\varphi: U \rightarrow \mathbb{R}^m \cong T_p M$  are identified as such. Since  $D_p f$  is of full rank, we know that  $K = \ker(D_p f)$  is of dimension  $m - n$ , so that  $K \cong \mathbb{R}^{m-n}$ .

Let  $L: \mathbb{R}^m \rightarrow \mathbb{R}^{m-n} \cong K$  be a projection, and define

$$F: \mathbb{R}^m \supseteq U \rightarrow N \times \mathbb{R}^{m-n} \\ x \mapsto (f(x), L(x)).$$

Notice then that  $D_p F = (D_p f, L)$ , where the latter comes about from the fact that  $L$  is a linear map. Then, we have that  $D_p f$  is of rank  $n$ , and  $L$  is of rank  $m - n$ , meaning that  $D_p F: \mathbb{R}^m \rightarrow \mathbb{R}^m$  is of full rank, hence is invertible on a neighborhood  $V \times W \subseteq N \times \mathbb{R}^{m-n}$ , where  $W$  is a neighborhood of  $0 \in \mathbb{R}^{m-n}$ , so we may identify  $U \cong V \times W$ .

By composing with the projection  $\pi: N \times \mathbb{R}^{m-n} \rightarrow N$  given by  $(q, W) \rightarrow q$ , we have that  $f = \pi \circ F$ , so that  $f^{-1}(\{q\}) = F^{-1} \circ \pi^{-1}(\{q\})$ , meaning that  $f^{-1}(\{q\}) \cong \mathbb{R}^{m-n}$ .  $\square$

One of the central uses of the regular value theorem is the fact that it allows us to prove a smooth version of Brouwer's Fixed Point Theorem for the general case of manifolds, known as the No Retraction Theorem.

**Theorem (No Retraction Theorem):** Let  $M$  be a compact smooth  $n$ -dimensional manifold with boundary, and let  $N = \partial M$  be the boundary. There does not exist any smooth surjective function  $r: M \rightarrow N$  such that  $r(x) = x$  for every  $x \in N$ .

*Proof.* Suppose toward contradiction that there were such a retraction, which we call  $r$ . Let  $X$  be the set of critical points for  $r$  in  $M$ ; then, by Sard's Theorem,  $r(X) \subseteq N$  has measure zero, so there exists a regular value  $y \in N$ .

By the Regular Value Theorem,  $r^{-1}(\{y\})$  is a smooth 1-dimensional manifold, as  $N$  is a  $n - 1$  dimensional manifold, meaning that  $r^{-1}(\{y\})$  is either  $S^1$  or an open interval. If  $r^{-1}(\{y\})$  is a circle, then  $r^{-1}(\{y\})$  necessarily must be contained in the interior of  $M$ , which would contradict the fact that  $y \in \partial M$ . Therefore,  $r^{-1}(\{y\})$  is an interval, and specifically is one that has both of its endpoints on  $N$ , as on the interior of  $M$ , such an interval must be identified to a 1-dimensional subspace of  $M$ , so there is some  $z \neq y \in N$  such that  $z \in r^{-1}(\{y\})$ . Yet, that means that  $r(z) = y \neq z$ , which is a contradiction.  $\square$

## The Tangent Bundle

Recall that we defined the differential  $D_p f$  via the action on the manifold about the point  $p$ . Unfortunately, the issue with this formulation is that it is purely local — the main reason we study manifolds is that we want to be able to use local information about the function to obtain insights about the global topology of the manifold. We need a construction that allows us to collect information about all the differentials at points of  $M$  together. This is the tangent bundle.

**Definition:** Let  $M$  be a manifold. The *tangent bundle*  $TM$  is the disjoint union

$$TM = \bigsqcup_{p \in M} T_p M.$$

Now, if  $M$  is a manifold of dimension  $m$ , then  $TM$  is a manifold of dimension  $2m$ . To see this, observe that if  $p \in \mathbb{R}^m$ , then  $T_p \mathbb{R}^m \cong \mathbb{R}^m$ .

Therefore, if at each point in  $\mathbb{R}^m$ , we assign a copy of the tangent space, we have that

$$T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m.$$

If  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is smooth, we get the map  $Tf: T\mathbb{R}^m \rightarrow T\mathbb{R}^n$  given by

$$(x, v) \mapsto (f(x), D_x f(v)).$$

Now, given a coordinate map  $\varphi: M \supseteq U \rightarrow \mathbb{R}^m$ , we may define

$$T\varphi(U) = \varphi(U) \times \mathbb{R}^m.$$

Thus, if  $\{(U_i, \varphi_i)\}_{i \in I}$  is an atlas for  $M$ , we have transition maps

$$\begin{aligned} T\psi(U \cap V) &\rightarrow T\varphi(U \cap V) \\ (x, v) &\mapsto (\varphi \circ \psi^{-1}(x), D_x(\varphi \circ \psi^{-1})(v)). \end{aligned}$$

Thus, if  $f: M \rightarrow N$  is a smooth map, it induces a differential map on the tangent bundles  $Df: TM \rightarrow TN$ .

**Remark:** If  $M$  and  $N$  only have  $C^1$  structures, it turns out that there is a compatible  $C^\infty$  structure, meaning that we may safely assume that any  $C^1$  manifold is  $C^\infty$ .

## Vector Fields

**Definition:** If  $M$  is a manifold, then a *vector field* on  $M$  is a smooth right-inverse of the projection map

$$\begin{aligned}\pi: TM &\rightarrow M \\ (x, v) &\mapsto x.\end{aligned}$$

When we consider vector fields on manifolds, some basic questions crop up. The most basic of them all is the following: does there exist a nowhere-vanishing vector field on  $M$ ?

- In any Euclidean space, we may take a constant nonzero vector as our assignment, so the answer is yes.
- For  $S^1$ , we can embed it into  $\mathbb{R}^2$ , then take the map  $(x, y) \mapsto ((x, y), (-y, x))$ , which is the family of tangent vectors to the unit circle in  $\mathbb{R}^2$ .
- For  $S^2$ , the answer is no. This is the much-celebrated “hairy ball theorem.”
- For  $S^3$ , the answer is yes. In fact, for  $S^{(2n-1)}$  where  $n$  is a natural number, the answer is yes, while for  $S^{2n}$ , the answer is no.

Now, if  $M$  is a manifold, with  $(U, \varphi)$  a chart on  $M$  with  $p \in U$ , identification  $\varphi: U \rightarrow \mathbb{R}^n$ , and coordinate representation  $(x_1, \dots, x_n)$ . Since  $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ , we may write the local coordinates for  $T_p M$  formally as  $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ . In other words, we consider the  $\frac{\partial}{\partial x_i}$  as a basis for the vector space  $T_p M$ .

If  $X$  is a vector field on  $M$ , we may express  $X$  locally about  $p$  formally as

$$X = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i},$$

where each  $a_i: U \rightarrow \mathbb{R}$  is a  $C^\infty$  function.

The first question we have is whether this is well-defined. To do this, we consider another chart,  $(V, \psi)$  with  $p \in V$ , identification  $\psi: V \rightarrow \mathbb{R}^n$ , and coordinate representation  $(y_1, \dots, y_n)$ . Given the coordinate change  $\psi \circ \varphi^{-1}$ , we want to consider a corresponding coordinate change  $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \mapsto \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right)$ . Via the chain rule on  $\mathbb{R}^n$ , we find that this corresponding coordinate change is

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j},$$

which emerges from applying the differential to  $\psi \circ \varphi^{-1}$ .

The space of vector fields on  $M$  has algebraic structure.

- If  $X$  and  $Y$  are vector fields, then so is  $\alpha X + \beta Y$  for all  $\alpha, \beta \in \mathbb{R}$ .
- If  $f \in C^\infty(M)$ , then  $f \cdot X$  is also a vector field on  $X$ , which is locally represented by

$$f \cdot X = \sum_{i=1}^n (f a_i) \frac{\partial}{\partial x_i}.$$

Furthermore,  $X$  acts on the space  $C^\infty(M)$  via differentiation. If  $p \in U$  with local coordinates  $(x_1, \dots, x_n)$ , we have  $X(f)$  locally defined by

$$X(f)(p) := \sum_{i=1}^n a_i(p) \frac{\partial f}{\partial x_i}(p).$$

This action has two properties:

- Linearity:  $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$ ;
- Leibniz Rule:  $X(fg) = f \cdot X(g) + g \cdot X(f)$ .

More generally, we say that an  $\mathbb{R}$ -linear map with these two properties is a *derivation*. It turns out that the span of the set of derivations on  $M$  is actually equal to the space of vector fields of  $M$ .

Finally, vector fields on  $M$  admit an intrinsic multiplication.

**Definition:** If  $X$  and  $Y$  are vector fields on  $M$ , the *Lie Bracket* of  $X$  and  $Y$  is defined for any  $f \in C^\infty(M)$  by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

**Proposition:** Let  $p \in M$  have local chart  $(U, \varphi)$ , where  $\varphi = (x_1, \dots, x_n)$  is the coordinate map, and let  $X$  and  $Y$  be vector fields on  $M$  with local representation on  $(U, \varphi)$  given by

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$$

$$Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}.$$

Then,  $[X, Y]$  has local representation given by

$$[X, Y] = \sum_{i=1}^n \sum_{j=1}^n \left( a_i \frac{\partial b_j}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

**Proposition:** The Lie Bracket  $[X, Y]$  of vector fields on  $M$  satisfies three properties:

- bilinearity:  $[\alpha X_1 + \beta X_2, Y] = \alpha[X_1, Y] + \beta[X_2, Y]$  and  $[X, \alpha Y_1 + \beta Y_2] = \alpha[X, Y_1] + \beta[X, Y_2]$  for all  $\alpha, \beta \in \mathbb{R}$ ;
- alternating:  $[X, Y] = -[Y, X]$ ;
- Jacobi identity:  $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ ;
- extended bilinearity for  $C^\infty(M)$ :  $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$ .

## Submanifolds of Dimension 1

**Definition:** Let  $X$  be a vector field on  $M$ , with  $p \in M$ . An *integral curve* for  $X$  through  $p$  is a  $C^\infty$  map  $c: \mathbb{R} \rightarrow M$  such that  $0 \mapsto p$  and  $Dc: T\mathbb{R} \rightarrow TM$  maps  $\frac{\partial}{\partial t} \mapsto X$ , where  $(t, \frac{\partial}{\partial t})$  are the global coordinates for the tangent bundle  $T\mathbb{R}$ .

In local coordinates, we may express  $c$  and  $Dc$  via

$$c(t) = (x_1(t), \dots, x_n(t))$$

$$Dc\left(\frac{\partial}{\partial t}\right) = \sum_{i=1}^n \frac{dx_i}{dt} \frac{\partial}{\partial x_i}$$

$$= \left( \frac{dx_1}{dt}, \dots, \frac{dx_n}{dt} \right)$$

when  $p \in U \subseteq M$  is a chart with local coordinates  $(x_1, \dots, x_n)$ . We then say that  $c$  is an integral curve if, for a local representation

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i},$$

we have

$$Dc\left(\frac{\partial}{\partial t}\right) = \sum_{i=1}^n \frac{dx_i}{dt} \frac{\partial}{\partial x_i},$$

or that

$$\frac{dx_i}{dt} = a_i(x_1(t), \dots, x_n(t))$$

for all  $i$ , and  $(x_1(0), \dots, x_n(0)) = p$ .

One can imagine an integral curve as a “flow” following a vector field traced out by a particle.<sup>IV</sup>

**Theorem:** Given a vector field  $X$  on  $M$ , there is a unique integral curve passing through every  $p \in M$ .

In order to prove this theorem, we need to recall the Picard–Lindelöf theorem from ordinary differential equations that gives us a sufficient condition for existence and uniqueness of solutions to initial value problems.

**Theorem (Picard–Lindelöf):** Let  $x: \mathbb{R} \rightarrow \mathbb{R}^n$  and  $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by

$$\begin{aligned} \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}\right) &= \dot{x} \\ &= f(t, x), \end{aligned}$$

with  $x(0) = x_0$ . If  $f$  is Lipschitz, then the initial value problem has a unique solution defined on  $(-\varepsilon, \varepsilon)$  for a sufficiently small  $\varepsilon$ .

Recall that  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is Lipschitz if there is  $L < \infty$  such that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq L.$$

Notice that if  $f$  is continuously differentiable and defined on a compact domain, this follows immediately from the extreme value theorem.

The proof of the Picard–Lindelöf theorem follows from a technique known as Picard iteration. Essentially, we rewrite the differential equation in integral form

$$x(t) = x_0 + \int_0^t f(s, x(s)) ds,$$

and start by setting

$$x_0(t) = x_0.$$

Then, we inductively define

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s)) ds.$$

Defining the integral operator

$$K(y(t)) = x_0 + \int_0^t f(s, y(s)) ds,$$

we essentially desire to show that  $K$  has a fixed point for all continuous  $y$  on a sufficiently small neighborhood of 0. This will follow from showing that  $K$  is a contraction in the supremum metric on this sufficiently small neighborhood of 0 whenever  $f$  is Lipschitz, which will allow us to use the contraction mapping theorem, as continuous functions on a compact set are complete under the supremum metric.

<sup>IV</sup>This is where the connection between differential topology and partial differential equations begins to appear.



**Corollary:** If  $M$  is a manifold, and  $X$  a vector field on  $M$  with  $p \in M$ , then there exists an integral curve  $c: (-\varepsilon, \varepsilon) \rightarrow M$  such that  $c(0) = p$ .

**Definition:** We call the vector field  $X$  *complete* if every integral curve along every point of  $p$  can be extended to all of  $\mathbb{R}$ .

There are many vector fields on manifolds that aren't complete. For instance, if  $M = \mathbb{R}^2 \setminus \{0\}$ , and  $X = \frac{\partial}{\partial x_1}$ , then an integral curve through  $(1, 0)$  cannot be extended to all of  $\mathbb{R}$ , as it would hit the missing point at the origin.

## Flows and Diffeomorphism Groups

Complete vector fields on manifolds enable us to create diffeomorphisms. Furthermore, it can be shown that if  $M$  has dimension greater than or equal to 2, then  $\text{diff}(M)$  is  $k$ -transitive, in that any  $k$ -tuple of distinct elements can be mapped to any other  $k$ -tuple of distinct elements.

**Definition:** A *flow* on  $M$  is a one-parameter group of diffeomorphisms of  $M$ , defined by

$$\begin{aligned}\varphi: \mathbb{R} &\rightarrow \text{diff}(M) \\ t &\mapsto \varphi_t,\end{aligned}$$

where  $\varphi_t(p) := c_p(t)$  when  $c_p$  is the integral curve through  $p$ .

In particular,  $\text{im}(\varphi)$  is the flow.

**Proposition:** If  $M$  is a connected manifold with  $\dim(M) \geq 2$ , then  $\text{diff}(M)$  is  $k$ -transitive.

**Remark:** The reason this does not work if  $M$  is a 1-dimensional manifold is that  $\mathbb{R}$  is linearly ordered and  $S^1$  is cyclically ordered.

*Proof.* We start with the case of  $k = 1$ .

Let  $p$  and  $q$  be in the same chart,  $(U, \varphi)$ , where  $\varphi = (x_1, \dots, x_n)$  is the coordinate map. By composing with a series of affine transformations of  $\mathbb{R}^n$ , we may assume that  $\varphi(p)$  is the origin and  $q$  is on the coordinate axis  $x_1$ . Furthermore, let  $N$  be a compact subset of  $\mathbb{R}^n$  such that  $p, q \in N$ .

Define  $f$  to be a smooth bump function on  $M$  such that  $f$  is 1 on  $N$  and 0 outside a neighborhood of  $N$ . Then, if  $X = f \frac{\partial}{\partial x_1}$  on  $U$  and zero outside, we observe that the integral curve through  $p$  passes through  $q$ , and is a flow with  $\varphi(t) = \psi_t(p) = q$  for some  $t \in \mathbb{R}$ .

Thus  $\text{diff}(M)$  acts transitively on points in the same chart. Meanwhile, if  $p$  and  $q$  are not in some chart, we find a finite length "chain" of intersecting charts that move from the chart at  $p$  to the chart at  $q$ , then by composing a finite collection of diffeomorphisms, we find our desired diffeomorphism.  $\square$

## Differential Forms

Now that we have a (reasonable) understanding of the tangent spaces of a manifold, we now concern ourselves with the dual to the tangent space, which are known as the *cotangent spaces*. Similar to how vector fields emerge from the tangent bundle and its projection onto  $M$ , the "dual" to vector fields, known as differential forms, emerge from the cotangent bundle.

Note that if  $T_p M$  is a tangent space that is isomorphic to  $\mathbb{R}^n$ , then the corresponding cotangent space, denoted  $T_p^* M$  is isomorphic to  $(\mathbb{R}^n)^* := \text{Hom}(\mathbb{R}^n, \mathbb{R})$ .

Recall that the basis for  $T_p M$  is given by  $\left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$  for local coordinates  $(x_1, \dots, x_n)$ . The corresponding dual basis for  $T_p^* M$  is established from the dual basis for a vector space  $V$ . Recall that if  $(v_1, \dots, v_n)$  is a

basis for  $V$ , then  $(v_1^*, \dots, v_n^*)$  is a basis for  $V^*$ , where

$$v_i^*(v_j) = \delta_i^j \\ := \begin{cases} 1 & i = j \\ 0 & \text{else.} \end{cases}$$

The corresponding (formal) basis for  $T_p^*M$  is given by  $(dx_1, \dots, dx_n)$ .

Now, the tangent bundle for  $M$  is given by

$$TM = \bigcup_{p \in M} T_p M,$$

and similarly, the cotangent bundle for  $M$  is given by

$$T^*M = \bigcup_{p \in M} T_p^*M.$$

Next, we concern ourselves with the manifold structure of  $T^*M$ . To start, if  $U \subseteq M$  is a chart, then  $T^*U := U \times (\mathbb{R}^n)^*$ . Now we concern ourselves with the transition maps.

Recall from linear algebra that if  $A: V \rightarrow W$  is a linear map, then  $A^*: W^* \rightarrow V^*$  is the dual map (or transpose) given by  $A^*\varphi = \varphi \circ A$ . Similarly, recall that for two charts  $(U, \varphi)$  and  $(V, \psi)$ , the tangent bundle admits the map

$$T(\varphi(U \cap V)) \rightarrow T(\psi(U \cap V)), \\ (x, v) \mapsto (\psi \circ \varphi^{-1}(x), D_x(\psi \circ \varphi^{-1})).$$

Thus, we may consider the transition map for the cotangent bundle  $T^*(\psi(U \cap V)) \rightarrow T^*(\varphi(U \cap V))$  to be the dualization of the transition map for the tangent bundle.

**Definition:** A differential (1-)form on  $M$  is a smooth section of the projection map  $\pi^*: T^*M \rightarrow M$ .

Locally, if  $U \subseteq M$  has coordinates  $(x_1, \dots, x_n)$ , then  $(dx_1, \dots, dx_n)$  are coordinates for the cotangent space over  $U$ , with forms denoted

$$\omega = \sum_{i=1}^n f_i dx_i,$$

where  $f_i \in C^\infty(M)$  for all  $i$ . The differentials  $dx_i$  satisfy

$$dx_i \left( \frac{\partial}{\partial x_j} \right) = \delta_i^j.$$

## Some Exterior Algebra

One of the most important use cases for differential forms is that they enable us to perform integration on manifolds. This requires a construction known as the exterior algebra.

If  $\mathbb{R}^n$  admits a basis  $(v_1, \dots, v_n)$ , we define the exterior algebra

$$\Lambda(\mathbb{R}^n) = \bigoplus_{i=1}^n \Lambda^i(\mathbb{R}^n),$$

where

$$\Lambda^0(\mathbb{R}^n) \cong \mathbb{R}$$

$$\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$$

$$\Lambda^i(\mathbb{R}^n) = \text{span}\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k, v_i \wedge v_i = 0\}.$$

In other words, the family of  $\Lambda^k(\mathbb{R}^n)$  are alternating and multilinear. In fact, the exterior algebra is the “universal” algebra for alternating multilinear maps.

**Theorem:** If  $t: V \times \cdots \times V \rightarrow W$  is an alternating  $k$ -multilinear map, then if we define

$$\begin{aligned} \iota: V \times \cdots \times V &\hookrightarrow \Lambda^k(V) \\ (v_1, \dots, v_k) &\mapsto v_1 \wedge \cdots \wedge v_k, \end{aligned}$$

then there is a unique linear map  $T: \Lambda^k(V) \rightarrow W$  such that  $T \circ \iota = t$ .

$$\begin{array}{ccc} V \times \cdots \times V & \xrightarrow{\iota} & \Lambda^k(V) \\ & \searrow t & \downarrow T \\ & & W \end{array}$$

The multiplication in  $\Lambda(\mathbb{R}^n)$  is given by the exterior product,

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_\ell) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_\ell.$$

Additionally, we see that for  $\sigma \in S_k$ ,

$$v_{i_{\sigma(1)}} \wedge \cdots \wedge v_{i_{\sigma(k)}} = \text{sgn}(\sigma) v_{i_1} \wedge \cdots \wedge v_{i_k}.$$

**Definition:** A  $k$ -form on a finite-dimensional real vector space  $V$  is an alternating  $k$ -multilinear map

$$\omega: V^k \rightarrow \mathbb{R}.$$

We denote the set of all  $k$ -forms on  $V$  by  $\mathcal{A}^k(V)$ .

**Proposition:**

$$\mathcal{A}^k(V) \cong \Lambda^k(V^*).$$

*Proof.* Apply the universal property. □

## A Coordinate-Free Definition of the Cotangent Space

Technically, the “cotangent space” that we defined above with the formal coordinates  $(dx_1, \dots, dx_n)$  is not the real, coordinate-free definition of the cotangent space. In order to define the cotangent space in a coordinate-free fashion, we need to go back to the definition of the tangent space using germs of functions  $\mathcal{C}_p$ .

Recall that the tangent space of  $M$  is defined by an equivalence relation on  $W_p$ , which is the space of germs of smooth maps  $\gamma: \mathbb{R} \rightarrow M$  where  $\gamma(0) = p$ . Specifically, the equivalence relation  $[g_1] \sim [g_2]$  for two germs  $g_1, g_2: \mathbb{R} \rightarrow M$  with  $0 \mapsto p$  is given by  $(\varphi \circ g_1)'(0) = (\varphi \circ g_2)'(0)$  for all  $\varphi \in \mathcal{C}_p$ .

Inside  $\mathcal{C}_p$ , we can look at the ideal  $J_p$  defined by

$$J_p = \{\bar{\varphi} \in \mathcal{C}_p \mid \varphi(p) = 0\}.$$

Notice that  $J_p$  is a maximal ideal in  $\mathcal{C}_p$ , as every germ in  $\mathcal{C}_p$  differs from an element of  $J_p$  by a real number. We then define  $J_p^2$  to be functions of the form  $\varphi\psi$ , where  $\varphi, \psi \in J_p$ . In essence, elements of  $J_p^2$  vanish to (at least) order 2 at  $p$ .

**Definition:** The *cotangent space* of  $p$  at  $M$  is given by

$$T_p^*M = J_p/J_p^2.$$

**Definition:** If  $\varphi \in \mathcal{C}_p$ , then the *exterior derivative* of  $\varphi$  is defined by

$$d\varphi = (\varphi - \varphi(p)) + J_p^2.$$

In order to see that this is actually the derivative (i.e., the best linear approximation), we note that by Taylor's Theorem, if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function, we have

$$f(t) \approx f(0) + c_1 t + c_2 t^2 + O(t^3)$$

so

$$\begin{aligned} f(t) - f(0) &\approx c_1 t + \underbrace{c_2 t^2 + O(t^3)}_{\in J_p^2} \\ f(t) - f(0) + J_p^2 &= c_1 t + J_p^2. \end{aligned}$$

**Proposition:** The exterior derivative satisfies  $d(\varphi\psi) = d(\varphi)\psi + \varphi d(\psi)$ .

*Proof.* We must show that

$$d(\varphi)\psi + \varphi d(\psi) - d(\varphi\psi) \in J_p^2.$$

Expanding definitions, we have

$$\begin{aligned} d(\varphi)\psi + \varphi d(\psi) - d(\varphi\psi) &= \psi(\varphi - \varphi(p)) + \varphi(\psi - \psi(p)) - (\varphi\psi - \varphi(p)\psi(p)) \\ &= (\varphi - \varphi(p))(\psi - \psi(p)) \\ &\in J_p^2. \end{aligned}$$

□

We now encounter a very important question: does this definition of the cotangent space comport with the definition above as the dual of the tangent space? The answer is yes.

**Proposition:** Let  $d\varphi \in T_p^*(M)$ . Define the map

$$\begin{aligned} \alpha: T_p^*(M) &\rightarrow (T_p M)^* \\ d\varphi &\mapsto \{[\gamma] \mapsto (\varphi \circ \gamma)'(0)\}. \end{aligned}$$

Furthermore, the map  $\alpha$  satisfies the following.

- (a)  $\alpha$  is linear and well-defined.
- (b)  $\alpha$  is natural, in the sense that if  $f: (M, p) \rightarrow (N, q)$  is a smooth map, then the following diagram commutes:

$$\begin{array}{ccc} T_q^*N & \xrightarrow{T_f^*} & T_p^*M \\ \alpha_{N,q} \downarrow & & \downarrow \alpha_{M,p} \\ (T_q N)^* & \xrightarrow{(T_f)^*} & (T_p M)^* \end{array}$$

Here,  $T_f^*(d\varphi) = d(\varphi \circ f)$ .

(c)  $\alpha$  is an isomorphism.

(d) In local coordinates  $(x_1, \dots, x_n)$  about  $p$ , we have

$$d\varphi = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i} dx_i.$$

*Proof.*

(a) This follows from the definition of the tangent space (and specifically, the equivalence relation between germs  $\gamma: \mathbb{R} \rightarrow M$  that defined the tangent space).

(b) Set  $\omega_\varphi = \{[\gamma] \mapsto (\varphi \circ \gamma)'(0)\}$ . Then, we observe that, if  $d\varphi \in T_q^*N$  and  $[\gamma] \in T_pM$ , that

$$\begin{aligned} (Tf)^* \circ \alpha_{N,q}(d\varphi)[\gamma] &= (Tf)^* \omega_\varphi([\gamma]) \\ &= \omega_\varphi(Tf([\gamma])) \\ &= \omega_\varphi([f \circ \gamma]) \\ &= (\varphi \circ f \circ \gamma)'(0). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \alpha_{M,p} \circ T^*f(d\varphi)([\gamma]) &= \alpha_{M,p}(d(\varphi \circ f))([\gamma]) \\ &= \omega_{\varphi \circ f}([\gamma]) \\ &= (\varphi \circ f \circ \gamma)'(0). \end{aligned}$$

□

## Derivations and the Dual of the Cotangent Space

We review the different spaces that we have discussed thus far.

- (i) The tangent space,  $T_pM$ , is defined to be equivalence classes of germs of smooth curves  $\gamma: \mathbb{R} \rightarrow M$  with  $\gamma(0) = p$ , where two curves  $\gamma_1$  and  $\gamma_2$  are set to be equal if  $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$  for all  $\varphi \in C^\infty(M)$ .
- (ii) The cotangent space  $T_p^*M$  is defined to be  $J_p/J_p^2$ , where  $J_p$  is the ideal in  $\mathcal{C}_{p,M}^\vee$  consisting of all functions  $\varphi$  such that  $\varphi(p) = 0$ . If  $\varphi \in \mathcal{C}_{p,M}$ , we define the exterior derivative  $d\varphi$  by  $(\varphi - \varphi(p)) + J_p^2$  — i.e., taking the residue class modulo  $J_p^2$  of  $\varphi - \varphi(p)$ .
- (iii) The relationship between the cotangent space and the dual of the tangent space,  $(T_pM)^*$ , is given by the map  $\alpha_{p,M}(d\varphi)([\gamma]) = (\varphi \circ \gamma)'(0)$ . This  $\alpha_{p,M}$  is a well-defined natural isomorphism of vector spaces.

Now, we consider a new space,  $\mathcal{D}_pM$ , consisting of all derivations at  $p$ .

**Definition:** Let  $p \in M$ . A *derivation* of  $p$  at  $M$  is a linear map  $X_p: \mathcal{C}_{p,M} \rightarrow \mathbb{R}$  satisfying the Leibniz rule:

$$X_p(f \cdot g) = X_p(f)g(p) + f(p)X_p(g),$$

The space of derivations at  $p$  is denoted  $\mathcal{D}_pM$ .

In order to understand  $\mathcal{D}_pM$  for a general manifold, we start by examining the case of  $\mathbb{R}^n$ ; specifically,  $\mathcal{D}_0\mathbb{R}^n$ . The linear maps  $\mathcal{C}_{0,\mathbb{R}^n} \rightarrow \mathbb{R}$  are spanned by partial derivatives — i.e., the maps  $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$ ,

<sup>V</sup>Recall that this is the set of germs of smooth functions  $\varphi: M \rightarrow \mathbb{R}$ .

where we use the derivation

$$\left. \frac{\partial}{\partial x_i} \right|_0 (f) = \frac{\partial f}{\partial x_i}(0).$$

The partial derivatives are linearly independent and satisfy the relations

$$\frac{\partial}{\partial x_i}(x_j) = \delta_i^j.$$

Our hope is that a general derivation on a manifold is similar — i.e., spanned by partial derivatives in local coordinates. Toward this end, we start by understanding two facts about derivations at 0.

- These derivations annihilate constant functions, which follows from the Leibniz rule. If  $X$  is such a derivation, then

$$\begin{aligned} X_0(1) &= X_0(1 \cdot 1) \\ &= 2X_0(1), \end{aligned}$$

so that  $X(1) = 0$ .

- Derivations annihilate functions of higher order than the constants — if  $X$  is a derivation, and we have coordinates  $x_1, \dots, x_n$ , then

$$\begin{aligned} X_0(x_i x_j) &= X_0(x_i) x_j(0) + x_i(0) X_0(x_j) \\ &= 0. \end{aligned}$$

Thus, if we take a Taylor expansion of  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  about 0, which we write as

$$f(x) \approx c + \sum_{i=1}^n a_i x_i + O(x^2),$$

then

$$\begin{aligned} D_0(f) &= \sum_{i=1}^n a_i D_0(x_i) \\ &=: \sum_{i=1}^n a_i c_i, \end{aligned}$$

so that

$$D = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}.$$

**Definition:** Let  $f: M \rightarrow N$  be a smooth map, such that  $f(p) = q$ . Then, there is an induced map  $f_*: \mathcal{D}_p M \rightarrow \mathcal{D}_q N$  given by

$$f_* X(\varphi) = X(\varphi \circ f)$$

for all  $\varphi \in \mathcal{C}_{q,N}$  and all  $X \in \mathcal{D}_p M$ . In particular, if we let  $X_p$  denote the evaluation of the derivation at  $p$ , then we have

$$(f_* X)_q(\varphi) = X_p(\varphi \circ f).$$

This is known as the *pushforward* of the derivation.

We know from linear algebra that if we have a vector space  $V$ , there is a canonical map from  $V$  to  $V^{**}$  given by  $x \mapsto \hat{x}$ , where  $\hat{x}(\varphi) = \varphi(x)$ . If  $V$  is finite-dimensional, then this is an isomorphism. This isomorphism is known as *canonical evaluation*, and is often denoted  $\text{ev}$ .

$$\begin{array}{ccccc}
 V & \xrightarrow{\quad} & V^* & \xrightarrow{\quad} & V^{**} \\
 & \searrow & & \nearrow & \\
 & & \text{---} \hat{x} \mapsto \hat{\hat{x}} \text{---} & & 
 \end{array}$$

Just as we showed that  $T_p^*M \cong (T_pM)^*$  are canonically isomorphic, we would ideally like a similar canonical isomorphism between  $(T_pM)^*$  and  $\mathcal{D}_pM$ , so that we may use a variety of compositions to establish that  $\mathcal{D}_pM$  is canonically isomorphic to  $T_pM$ , and let derivations be considered as equivalent to vector fields.

Towards this end, we observe that the exterior derivative,  $d: \mathcal{C}_{p,M} \rightarrow T_p^*M$ , is a derivation. Define now the map  $\beta: (T_pM^*)^* \rightarrow \mathcal{D}_pM$  by

$$g \mapsto \left\{ \mathcal{C}_{p,M} \xrightarrow{d} T_p^*M \xrightarrow{g} \mathbb{R} \right\}.$$

**Proposition:** The map  $\beta$  is a natural isomorphism, in the sense that if  $f: M \rightarrow N$  is a smooth map, then the following diagram commutes.

$$\begin{array}{ccc}
 (T_p^*M)^* & \xrightarrow{\beta_{p,M}} & \mathcal{D}_pM \\
 (D^*f)^* \downarrow & & \downarrow f_* \\
 (T_q^*N)^* & \xrightarrow{\beta_{q,N}} & \mathcal{D}_qN
 \end{array}$$

Here, recall that  $T^*f(\varphi) \equiv D^*f(d\varphi) = d(\varphi \circ f)$ , as we defined for the map  $\alpha_{p,M}$ , while  $(D^*f)^*$  is the dualization of  $D^*f$ .

The proposition then gives us the family of canonical isomorphisms given by the following.

$$\begin{array}{ccccccc}
 T_pM & \xrightarrow{\text{ev}} & (T_pM)^{**} & \xrightarrow{\alpha^*} & (T_p^*M)^* & \xrightarrow{\beta} & \mathcal{D}_pM \\
 & \searrow & & & \nearrow & & \\
 & & \text{---} [\gamma] \mapsto X_\gamma \text{---} & & & & 
 \end{array}$$

Here,  $X_\gamma$  is the derivation given by  $X_\gamma(\varphi) = (\varphi \circ \gamma)'(0)$  for any  $\varphi \in C^\infty(M)$ . Note that  $\alpha^*$  is the dualization of the linear map  $\alpha: (T_pM)^* \rightarrow T_p^*M$ .

## Operations on Differential Forms

Recall that  $\mathcal{A}^1(M)$  is the set of differential 1-forms on  $M$ , which is equivalent to the space of smooth sections of  $T^*M$ . These forms have local expansions of the form

$$\omega = \sum_{i=1}^n f_i dx_i,$$

where the  $f_i \in C^\infty(M)$ . The exterior derivative of  $f \in \mathcal{C}_{p,M}$  is then given by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

We may then view  $C^\infty(M) = \mathcal{A}^0(M)$ . This follows from the fact that  $\lambda^0(T_p^*M) = \mathbb{R}$ , implying that  $M \times \mathbb{R} = T^*M$ , so that smooth sections are smooth functions  $f: M \rightarrow \mathbb{R}$ .

The exterior derivative is then a linear map  $d: \mathcal{A}^0(M) \rightarrow \mathcal{A}^1(M)$ . Similarly, if  $X$  is a vector field on  $M$ , and  $\omega \in \mathcal{A}^1(M)$ , then we may consider  $\omega(X) \in C^\infty(M)$  to be defined as follows. In local coordinates, if we write

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$$

$$\omega = \sum_{i=1}^n f_i dx_i,$$

then

$$\begin{aligned} \omega(X)(p) &= \sum_{i=1}^n f_i(p) dx_i \left( \sum_{j=1}^n a_j(p) \frac{\partial}{\partial x_j} \right) \\ &= \left( \sum_{i=1}^n f_i a_i \right)(p). \end{aligned}$$

We may then expand to differential  $k$ -forms, denoted  $\mathcal{A}^k(M)$ . These are smooth sections of  $\wedge^k(T^*M)$ , which are the  $k$ -multilinear (as  $C^\infty$  modules) alternating maps  $(\mathcal{X}(M))^k \rightarrow \mathbb{R}$ , where  $\mathcal{X}(M)$  denotes the vector fields on  $M$ .

A  $k$ -form evaluates on a  $k$ -tuple of vector fields and outputs a  $C^\infty$  function. In local coordinates,  $k$ -forms appear as

$$\omega = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

**Definition:** We define the *exterior product*

$$\begin{aligned} \mathcal{A}^k(M) \times \mathcal{A}^\ell(M) &\rightarrow \mathcal{A}^{k+\ell}(M) \\ (\omega, \eta) &\mapsto \omega \wedge \eta, \end{aligned}$$

where in local coordinates, if we have

$$\begin{aligned} \omega &= \sum_I f_I dx_I \\ \eta &= \sum_J g_J dx_J, \end{aligned}$$

then

$$\omega \wedge \eta = \sum_{I, J} f_I g_J dx_I \wedge dx_J.$$

We observe that the exterior product has the property that, for arbitrary vector fields  $X_1, \dots, X_{k+\ell} \in \mathcal{X}(M)$ ,

$$\omega \wedge \eta(X_1, \dots, X_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} \text{sgn}(\sigma) \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}).$$

**Definition:** We define the *exterior derivative* to be a linear map

$$d: \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}(M)$$



given by, if

$$\omega = \sum_I f_I dx_I,$$

the expression

$$d\omega = \sum_{j=1}^n \sum_I \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I.$$

**Remark:** Note that  $d$  may not be injective nor surjective. The nature of the map  $d$  is a part of de Rham cohomology.

## Notations

- A general normed space  $V$  will have its norm denoted by  $\|\cdot\|$ . If  $V = \mathbb{R}^n$ , then we denote the norm by  $|\cdot|$ .
- We denote topological spaces by  $(X, \tau)$ .
- $U(x, r) = \{y \in V \mid \|x - y\| < r\}$ .
- $B(x, r) = \{y \in V \mid \|x - y\| \leq r\}$ .
- $\mathcal{N}_p$ : neighborhood system centered at  $p \in X$ .
- $\mathcal{O}_p$ : system of *open* neighborhoods centered at  $p \in X$ .
- When we say a number  $n$  is positive, we mean that  $n \geq 0$ . Similarly, a sequence  $(a_n)_n$  is decreasing (increasing) if  $a_n \geq a_{n+1}$  ( $a_n \leq a_{n+1}$ ).