Problem (Problem 1)**:** Let R be a ring and M a left R-module.

(a) Prove that for every $m \in M$, the map $r \mapsto r \cdot m$ from R to M is a homomorphism of R-modules.

(b) Assume that R is commutative and M an R-module. Prove that there is an isomorphism $\hom_R(R, M) \cong M$ as left R-modules.

Solution:

(a) Let $m \in M$ be fixed, and define $\varphi_m \colon R \to M$ by

$$\varphi_m(r) = r \cdot m.$$

It follows from the axioms of left R-modules that

$$\begin{aligned}\varphi_m(r+s) &= (r+s) \cdot m\\ &= r \cdot m + s \cdot m\\ &= \varphi_m(r) + \varphi_m(s),\end{aligned}$$

and

$$\begin{aligned}\varphi_m(rs) &= (rs) \cdot m\\ &= r \cdot (s \cdot m)\\ &= r \cdot (\varphi_m(s)),\end{aligned}$$

so that φ_m is a homomorphism of left R-modules.

(b) If $\varphi_m \colon R \to M$ is the homomorphism as defined in part (a), we define a map $\varphi \colon M \to \hom_R(R, M)$ by

$$\varphi(m)(r) = \varphi_m(r).$$

First, we verify that φ is a homomorphism. If $r \in R$ is arbitrary, then

$$\begin{aligned}\varphi(m+n)(r) &= \varphi_{m+n}(r)\\ &= r \cdot (m+n)\\ &= r \cdot m + r \cdot n\\ &= \varphi_m(r) + \varphi_n(r)\\ &= (\varphi(m) + \varphi(n))(r).\end{aligned}$$

To see that φ is injective, we see that $\ker(\varphi)$ consists of all elements $m \in M$ such that $\varphi(m) = \varphi_0$, where $\varphi_0 \colon R \to M$ takes $r \mapsto 0$ for all $r \in R$. In particular, since $1 \in R$, it follows that $1 \cdot m = m = 0$, meaning that $\ker(\varphi) = \{0\}$.