3

The graph G in Figure 50 is connected and contains no bridges. Find a strong orientation of G.

4

Suppose that D is an orientation of a connected graph G such that for each vertex v of G, some edge is directed toward v and some edge is directed away from v. Is D a strong orientation on G.

Since G is a connected graph, there must be a path P between any two vertices v_1 and v_2. Call this path P, travelling along the orientation D. Then, the path "exits" v_1 and "enters" v_2. Suppose that there is no path from v_2 back to v_1.

Then, within $G - P$ it must be the case that either v_1 or v_2 are of degree 0, or there is a point in $G - P$ wherein the interior vertices have no edges directed "out" both of which would contradict the assumptions. Additionally, since there is at least one edge directed "out" from v_2 and directed "in" v_1.

Extra Problem 1

Determine whether each of the following statements is equivalent to Robbin's Theorem.

(a) A graph G has a strong orientation if and only if G is connected and every pair of distinct vertices in G is in a directed cycle.

(b) A graph G has a strong orientation if and only if G is connected and every pair of distinct vertices in G is in a directed circuit.

(c) A graph G has a strong orientation if and only if G is connected and every pair of distinct vertices in G is in a directed closed walk.

Extra Problem 2

Let K_n be a strong tournament with $n \geq 3$.

(a) Prove that for every j in $\{2, \ldots, n-2\}$, K_n has a directed cycle of length $1 + j$ or $1 + n - j$.

(b) Prove that for every j in $\{2, \ldots, n-2\}$, K_n has n distinct directed cycles $C_1, \ldots, C_n$ such that each C_i has length $1 + j$ or $1 + n - j$.