Solution (11.2, Problem 2)**:** We evaluate

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)f(x)\,dx \\ &= \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)f(x)\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx \\ &= \frac{1}{\pi}\left(\frac{1}{n}\cos(nx)\Big|_{0}^{\pi}\right) \\ &= \frac{1}{n\pi}\left((-1)^n - 1\right). \end{aligned}$$

Therefore, our Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi}\sin(nx).$$

Solution (11.2, Problem 8)**:** We evaluate

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)(3-2x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} 3\cos(nx) - 2x\cos(nx)\,dx \\ &= \begin{cases} 3 & n = 0 \\ 0 & \text{else.} \end{cases} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)(3-2x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} 3\sin(nx) - 2x\sin(nx)\,dx \\ &= \frac{1}{\pi}\left(\frac{3}{n}\cos(nx)\Big|_{-\pi}^{\pi} - 2\left(\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right)\Big|_{-\pi}^{\pi}\right) \\ &= \frac{4(-1)^n}{n}. \end{aligned}$$

Thus, our Fourier series is

$$f(x) = 3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nx).$$

Solution (11.2, Problem 10)**:** Using integration by parts, we evaluate

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi/2} \cos\left(\frac{n}{2}x\right)\cos(x)\,dx \\ &= \begin{cases} \frac{8}{\pi(4-n^2)}\cos\left(\frac{n\pi}{4}\right) & n \neq 2 \\ \frac{1}{2} & n = 2 \end{cases}. \\ b_n &= \frac{2}{\pi}\int_{0}^{\pi/2} \sin\left(\frac{n}{2}x\right)\cos(x)\,dx \\ &= \begin{cases} \frac{8}{\pi(4-n^2)}\left(\sin\left(\frac{n\pi}{4}\right) - \frac{n}{2}\cos\left(\frac{n\pi}{4}\right)\right). & n \neq 2 \\ \frac{1}{\pi} & n = 2 \end{cases}. \end{aligned}$$

Thus, with $a_0 = \frac{2}{\pi}$, we have the Fourier series

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{2}x\right) + b_n \sin\left(\frac{n}{2}x\right).$$

Solution (11.2, Problem 17)**:** We first start by finding the series expansion. Evaluating, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx \\ &= \begin{cases} \frac{2(-1)^n}{n^2} & n > 0 \\ \frac{\pi^2}{3} & n = 0 \end{cases} \\ b_n &= \frac{1}{\pi}\int_0^{\pi} x^2 \sin(nx)\,dx \\ &= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right). \end{aligned}$$

Thus, we have the Fourier series

$$x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos(nx) + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right)\sin(nx).$$

Using the input $x = 0$, we get

$$\begin{aligned} 0 &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos(nx) \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}. \end{aligned}$$

Meanwhile, using the input of $-\pi$, we have $f(-\pi) = 0$, and

$$\begin{aligned} 0 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} \\ \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Solution (11.2, Problem 18)**:** Adding, we get

$$\frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}.$$

Solution (11.3, Problem 6)**:** The function

$$f(x) = e^x - e^{-x}$$

is odd.

Solution (11.3, Problem 10)**:** The function

$$f(x) = \left|x^5\right|$$

is even.

Solution (11.3, Problem 12)**:** This function is even, so we expand in the cosine series. This gives

$$a_n = \int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)\,dx$$

$$\begin{aligned} &= \int_1^2 \cos\left(\frac{n\pi}{2}x\right)\,dx \\ &= \begin{cases} \frac{2}{n\pi} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}. \end{aligned}$$

Solution (11.3, Problem 18)**:** This function is odd, so we expand in the sine series. This gives

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x^3 \sin(nx)\,dx \\ &= \frac{(-1)^{n+1}\pi^3}{n} + \frac{6\pi(-1)^n}{n^3}. \end{aligned}$$

Solution (11.3, Problem 20)**:** This function is odd, so we expand in a sine series.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi (x+1)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x\sin(nx) + \sin(nx)\,dx \\ &= \frac{2\left(1 + (-1)^{n+1} + (-1)^{n+1}\pi\right)}{n\pi}. \end{aligned}$$

Solution (11.3, Problem 34)**:** Since $f(0) = 0$, we expand in a sine series. This gives

$$b_n = \int_0^2 x(2-x)\sin\left(\frac{n\pi}{2}x\right)\,dx \qquad = \frac{16}{n^3\pi^3}\left(1 + (-1)^{n+1}\right).$$

Solution (11.4, Problem 2)**:** We have the cases of $\lambda = \alpha^2, 0, -\alpha^2$, with corresponding solution forms of

$$\begin{aligned} y &= c_1\cos(\alpha x) + c_2\sin(\alpha x) \\ y &= c_1 + c_2 x \\ y &= c_1\cosh(\alpha x) + c_2\sinh(\alpha x). \end{aligned}$$

Substituting our boundary conditions for each of these cases, where the first listed equation is the case $y(1) = 0$ and the second listed equation is the case $y(0) + y'(0) = 0$, we have

$$\begin{aligned} c_1\cos(\alpha) + c_2\sin(\alpha) &= 0 \\ \alpha(c_2 - c_1) &= 0 \\ \\ c_1 + c_2 &= 0 \\ c_1 + c_2 &= 0 \\ \\ c_1\cosh(\alpha) + c_2\sinh(\alpha) &= 0 \\ \alpha(c_1 + c_2) &= 0. \end{aligned}$$

In the case with $\lambda = \alpha^2$, we have $c_1 = c_2$ (as $\alpha \neq 0$), giving $y(1) = c_1\cos(\alpha) + c_1\sin(\alpha)$, which simplifies to $\sqrt{2}c_1\sin\left(\alpha + \frac{\pi}{4}\right) = 0$. This has solutions of $\alpha = n\pi - \frac{\pi}{4}$, where $n \in \mathbb{Z}$.

In the case with $\lambda = 0$, we have the case $y = c_1 - c_1 x$.

The case with $\lambda = -\alpha^2$ simplifies to $c_1 = -c_2$, and $c_1\cosh(\alpha) + c_1\sinh(\alpha) = 0$, which is only true if $c_1 = c_2 = 0$. This is a trivial solution.

Therefore, we have eigenvalues $\lambda = 0, \left(\pi n - \frac{\pi}{4}\right)^2$.

$$\lambda_1 = 0$$

$$\lambda_2 = \frac{\pi^2}{16}$$
$$\lambda_3 = \frac{9\pi^2}{16}$$
$$\lambda_4 = \frac{25\pi^2}{16}.$$

Solution (11.4, Problem 4)**:** We have the periodic Sturm–Liouville boundary condition of

$$y(-L) - y(L) = 0$$
$$y'(-L) - y'(L) = 0.$$

Letting $\lambda = \alpha^2$, we have solutions of the form

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Using the first boundary condition, we have

$$2c_2 \sin(\alpha L) = 0,$$

so $\alpha = \frac{n\pi}{L}$, where $n \in \mathbb{Z}$. Varying our coefficients and our values of n, we recover the family

$$y = \left\{1, \sin\left(\frac{n\pi}{L}\right), \cos\left(\frac{n\pi}{L}\right) \;\middle|\; n \in \mathbb{Z}\right\}.$$

Solution (11.4, Problem 8)**:**

(a) Using the guess $y = e^{kt}$, we get the characteristic equation of

$$k^2 + k + \lambda = 0,$$

which has solutions

$$k = -\frac{1}{2} \pm \sqrt{\lambda}.$$

This splits into three cases.

If $\lambda = 0$, then $y = Ae^{-\frac{1}{2}t} + Bte^{-\frac{1}{2}t}$. Plugging in the boundary conditions, we get that $A = B = 0$, so this cannot be a solution.

Similarly, if $\lambda > 0$, then $y = A\cosh\left(\left(-\frac{1}{2} + \sqrt{\lambda}\right)t\right) + B\sinh\left(\left(-\frac{1}{2} + \sqrt{\lambda}\right)t\right)$, which once again yields $A = B = 0$ when plugging in the boundary conditions.

Thus, we are left with $\lambda < 0$, which has solutions of the form

$$y = e^{-\frac{1}{2}t}\left(A\cos\left(\sqrt{\lambda}t\right) + B\sin\left(\sqrt{\lambda}t\right)\right).$$

Plugging in the boundary conditions, we get that $A = 0$, and with the other boundary condition, we get

$$B\sin\left(2\sqrt{\lambda}\right) = 0,$$

meaning that since $B \neq 0$, we have $\lambda = \frac{n^2\pi^2}{4}$. The corresponding eigenvectors are

$$y = e^{-\frac{1}{2}t}\sin\left(\frac{n\pi}{2}t\right),$$

where $n = 1, 2, \ldots$.

(b) We may put the equation in self-adjoint form by multiplying by e^x, giving

$$\frac{d}{dx}\left(e^x \frac{dy}{dx}\right) + e^x \lambda y = 0.$$

(c) We have the orthogonality relation

$$\int_0^2 y_n(x) y_m(x) e^x \, dx = k_n \delta_{mn}.$$

Solution (11.4, Problem 10)**:** We multiply out by e^{-x^2} to get

$$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right) + 2ne^{-x^2}y = 0,$$

with orthogonality relation

$$\int_{-\infty}^{\infty} y_n(x)y_m(x)e^{-x^2}\,dx = \delta_{mn}.$$

Solution (12.3, Problem 2)**:**

Solution (12.3, Problem 4)**:**