Math 395

Homework 1

Due: 2/1/2024

Name: ____________________________________

Collaborators: ____

1. Let S be the subset of $\mathrm{Mat}_2(\mathbf{R})$ be the set consisting of matrices of the form $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$.

   (a) Prove that S is a ring.

   (b) Show that $J = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a right identity in S, i.e., $AJ = A$ for all $A \in \mathrm{Mat}_2(\mathbf{R})$.

   (c) Show that J is not a left identity for S, i.e., there is an element $B \in S$ so that $JB \neq B$.

2. Show that the subset $S = \{[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}, [15]_{18}\}$ is a subring of $\mathbf{Z}/18\mathbf{Z}$. Does S have an identity?

3. Define a new addition and multiplication on $\mathbf{Z}$ by

$$\begin{aligned} a \oplus b &= a + b - 1 \\ a \odot b &= ab - (a+b) + 2. \end{aligned}$$

   Prove that under these operations $\mathbf{Z}$ is an integral domain.

4. Let R be a ring and define $Z(R) = \{a \in R : ar = ra \text{ for every } r \in R\}$. Prove that $Z(R)$ is a subring of R. It is referred to as the center of R.

5. Let R be a ring and fix an element $x \in R$. Show that the set $\{rx : r \in R\}$ is a subring of R.

6. Let S and T be subrings of a ring R.

   (a) Is $S \cap T$ a subring of R? Justify your answer with a proof or counterexample.

   (b) Is $S \cup T$ a subring of R? Justify your answer with a proof or counterexample.

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(F)$ where F is a field.

   (a) Prove that A is invertible if and only if $ad - bc \neq 0_F$.

   (b) Prove that A is a zero divisor if and only if $ad - bc = 0_F$.

   (c) If instead we consider a matrix $A \in \mathrm{Mat}_2(\mathbf{Z})$, do the same conclusions hold? If so, prove them. If not, adjust them to true statements and prove those statements.