Problem (Problem 1)**:** Let $0 \leqslant r < R \leqslant \infty$. Suppose $(a_n)_n, (b_n)_n \subseteq \mathbb{C}$ are such that the series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ converge in $A(z_0, r, R)$, and are such that

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$$

for all $z \in A(z_0, r, R)$. Show that $a_n = b_n$ for all n.

Solution: Suppose we have the functions

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \\ &= f_1(z) + f_2(z) \\ g(z) &= \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n \\ &= g_1(z) + g_2(z) \end{aligned}$$

are written so that f_1, g_1 are holomorphic defined on $U(z_0, R)$ while f_2, g_2 are holomorphic defined on $\mathbb{C} \setminus B(z_0, r)$. The assumption that $f(z) = g(z)$ on $A(z_0, r, R)$ gives $f_1(z) - g_1(z) = g_2(z) - f_2(z)$, or

$$\sum_{n=0}^{\infty}(a_n - b_n)(z-z_0)^n = \sum_{n=-\infty}^{-1}(b_n - a_n)(z-z_0)^n$$

on $A(z_0, r, R)$. This means that we may define a function $h(z)$ by letting $r < \rho < R$ and taking

$$h(z) = \begin{cases} \sum_{n=0}^{\infty}(a_n - b_n)(z-z_0)^n & |z-z_0| \leqslant \rho \\ \sum_{n=-\infty}^{-1}(b_n - a_n)(z-z_0)^n & |z-z_0| > \rho \end{cases},$$

which we observe is holomorphic on the entirety of $\mathbb{C}$ as a result of the fact that the separate power series expansions $\sum_{n=0}^{\infty}(a_n - b_n)(z-z_0)^n$ and $\sum_{n=-\infty}^{-1}(b_n - a_n)(z-z_0)^n$ are holomorphic on their respective domains of definition, while they are equal on $A(z_0, r, R)$.

Furthermore, we see that $\lim_{z\to\infty}|h(z)| = 0$, whence h is a bounded entire function, so $h \equiv K$ for some constant K. This means that, for $|z - z_0| < \rho$,

$$\sum_{n=0}^{\infty} a_n - b_n(z-z_0)^n = K,$$

meaning that $a_0 - b_0 = K$ and $a_{n\geqslant 1} - b_{n\geqslant 1} = 0$. Yet, for $|z - z_0| > \rho$, we must have

$$\sum_{n=1}^{\infty}(a_{-n} - b_{-n})(z-z_0)^{-n} = K,$$

but there are no constant terms in this series expansion, meaning that $a_{n\leqslant -1} - b_{n\leqslant -1} = 0$, and that $K = 0$. Thus, we have $a_0 - b_0 = 0$, and we are done.

Problem (Problem 2)**:**

(a) Determine the Laurent series expansion of the function

$$f(z) = \frac{z}{(z-3)^2(z-4)}$$

that converges on $A(0, 3, 4)$.

(b) Show that there does not exist a holomorphic function $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ satisfying $|f(z)| \geqslant |z|^{-2/3}$.

Solution:

(a) We start by taking a partial fraction decomposition of f to yield

$$\begin{aligned} f(z) &= \frac{4}{z-4} - \frac{4}{z-3} - \frac{3}{(z-3)^2} \\ &= \frac{4}{z-4} - \frac{4}{z-3} + 3\frac{d}{dz}\left(\frac{1}{z-3}\right) \end{aligned}$$

We seek to expand about $z = 0$ within the ball $U(0,4)$ and outside the closed ball $B(0,3)$. This means that the first term in our partial fraction expansion becomes

$$\begin{aligned} a_1(z) &= -\frac{1}{1-\frac{z}{4}} \\ &= -\sum_{n=0}^{\infty}\frac{z^n}{4^n}. \end{aligned}$$

The expansion in the second and third terms will require a little bit more work. Dividing out by z, we find that the second term becomes

$$\begin{aligned} a_2(z) &= -\frac{4}{z\left(1-\frac{3}{z}\right)} \\ &= -\frac{4}{z}\sum_{n=0}^{\infty}\frac{3^n}{z^n} \\ &= -\sum_{n=1}^{\infty}\frac{4\cdot 3^{n-1}}{z^n} \\ &= -\sum_{n=-\infty}^{-1} 12(3^{-n})z^n, \end{aligned}$$

which converges outside the closed ball $B(0,3)$. Finally, for the third term, we observe that, using term-by-term differentiation (allowable as the series is uniformly convergent), we have

$$\begin{aligned} 3\frac{d}{dz}\left(\frac{1}{z-3}\right) &= 3\frac{d}{dz}\left(\sum_{n=1}^{\infty} 3^{n-1}z^{-n}\right) \\ &= \sum_{n=1}^{\infty} -n3^n z^{-(n+1)} \\ &= \sum_{n=-\infty}^{-1} n3^{-n}z^{n-1}. \end{aligned}$$

This yields a Laurent series expansion of

$$f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{4^n} + \sum_{n=-\infty}^{-1}\left(-12(3^{-n})z^n + n3^{-n}z^{n-1}\right).$$

(b) Suppose toward contradiction that there were such an $f(z)$. Since $|z|^{-2/3}$ is strictly greater than zero along its domain, it would follow that $|f(z)|$ would not have any zero along its domain. This means that $g(z) = \frac{1}{f(z)} \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ would be defined on its entire domain. Furthermore, we would have

$$|g(z)| \leqslant |z|^{2/3},$$

and on $U(0, \varepsilon)$, we know that $|z|^{2/3}$ is bounded above by $\varepsilon^{2/3}$ as $|z|^{2/3} \colon \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ is an increasing function. Thus, since g would be locally bounded around 0, it would follow that g has a removable singularity at 0. This means that there is a holomorphic extension $h \colon \mathbb{C} \to \mathbb{C}$ that agrees with g on $\mathbb{C} \setminus \{0\}$. In particular, we would have $|h(z)| \leqslant |z|^{2/3}$ for all $z \in \mathbb{C} \setminus \{0\}$.

Now, let $R > 0$. Using the Cauchy estimate on $S(0, R)$, we have, for any fixed $n > 0$,

$$\begin{aligned}\left|h^{(n)}(z)\right| &\leqslant \frac{n!}{R^n} \sup_{|z|=R} |h(z)| \\ &\leqslant \frac{n!}{R^n} \sup_{|z|=R} |z|^{2/3} \\ &= \frac{n!}{R^{n-2/3}}.\end{aligned}$$

Yet, since R is arbitrary, it follows that $\left|h^{(n)}(z)\right| = 0$ for all $n > 0$, whence h is constant. Yet, since $|h(z)| \leqslant |z|^{2/3}$ for all $z \in \mathbb{C} \setminus \{0\}$, it follows that $|h(z)| \leqslant \varepsilon^{2/3}$ for any $\varepsilon > 0$, whence $|h(z)| = 0$ for all $z \in \mathbb{C}$. At the same time, we explicitly defined $g(z)$ in a manner such that it could never equal zero, meaning that such an f cannot exist.

Problem (Problem 3)**:** Let $0 < r < R$. Show that there does not exist a holomorphic bijection $f \colon \mathbb{D} \setminus \{0\} \to A(0, r, R)$.

Solution: Suppose there were a holomorphic bijection $f \colon \mathbb{D} \setminus \{0\} \to A(0, r, R)$. Since $|f(z)| \leqslant R$ for all $z \in \mathbb{D} \setminus \{0\}$, it follows that the singularity at 0 is removable, so there is a holomorphic function $g \colon \mathbb{D} \to A(0, r, R)$.

Considering $g(0)$, we observe that $g(0) = \lim_{z\to 0} f(z)$, meaning that $g(0) \in \overline{A}(0, r, R)$ as $g(0)$ is a limit point of the image $f(\mathbb{D} \setminus \{0\})$, where f is continuous. However, it cannot be the case that $g(0) \in \partial A(0, r, R)$, as g is holomorphic so this would contradict the open mapping principle. Thus, we must have $g(0) \in A(0, r, R)$, meaning that there is some $z_0 \in \mathbb{D} \setminus \{0\}$ such that $f(z_0) = g(0)$.

Let $(z_n)_n \subseteq \mathbb{D} \setminus \{0\}$ be a sequence with $z_n \to 0$. Observe then that $\lim_{n\to\infty} f(z_n) = g(0)$ as g is the unique holomorphic extension of f. However, since f is a holomorphic bijection, the open mapping principle means that f has a continuous inverse, meaning that $f^{-1}(f(z_n)) = z_n$ is continuous, with $\lim_{n\to\infty} f^{-1}(f(z_n)) = f^{-1}(g(0)) = z_0$, but $(z_n)_n \to 0$, meaning that by uniqueness of limits, $z_0 = 0$. Therefore, it cannot be the case that such a holomorphic f exists.

Solution (Special Case)**:** Suppose there were a holomorphic bijection $f \colon \mathbb{D} \setminus \{0\} \to A(0, r, R)$ with holomorphic inverse. Notice that for all $z \in \mathbb{D} \setminus \{0\}$, we would then have $|f(z)| < R$, meaning that f is necessarily locally bounded close to 0. Thus, the singularity at 0 is removable, so there is a unique holomorphic function $g \colon \mathbb{D} \to \mathbb{C}$ with $g|_{\mathbb{D}\setminus\{0\}} = f$.

We notice that g is an injection, as $g|_{\mathbb{D}\setminus\{0\}}$ is a bijection and $g(0)$ is uniquely defined. As a result, it follows that the restriction $g \colon \mathbb{D} \to \operatorname{im}(g)$ is a holomorphic bijection. Furthermore, we also notice that

$$\begin{aligned}\lim_{z\to 0} |g(z)| &= \lim_{z\to 0} |f(z)| \\ &\geqslant r \\ &> 0,\end{aligned}$$

meaning that g is nonvanishing on $\mathbb{D}$. In particular, there is a logarithm $h(z)\colon \mathbb{D} \to \mathbb{C}$ such that

$$g(z) = e^{h(z)},$$

and $f(z) = e^{h(z)}$ when restricted to $\mathbb{D} \setminus \{0\}$. Now, since the identity map $\mathrm{id}\colon A(0,r,R) \longrightarrow A(0,r,R)$ is a bijective holomorphic map with holomorphic inverse, it follows that

$$e^{h(z)} = \mathrm{id}(f(z)).$$

Yet, this means that

$$\mathrm{id}(z) = e^{h\left(f^{-1}(z)\right)},$$

meaning that id admits a logarithm. Yet, $A(0,r,R)$ is not simply connected, while id is nonvanishing, which is a contradiction. Thus, no such f exists.

Problem (Problem 4)**:** Show that if f is entire and satisfies $\lim_{z\to\infty} f(z) = \infty$, then f is a polynomial.

Solution: Consider the function $g\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $g(z) = f\left(\frac{1}{z}\right)$. Since f is entire and $\frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, it follows that, given the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have the Laurent series expansion

$$g(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

Observe that the limit $\lim_{z\to\infty} f(z)$ is equivalent to $\lim_{z\to 0} f\left(\frac{1}{z}\right)$, whence $\lim_{z\to 0} g(z) = \infty$. Therefore, g has a pole of order k at 0, whence

$$g(z) = \sum_{n=0}^{k} a_n z^{-n}.$$

Since $g\left(\frac{1}{z}\right) = f(z)$, it then follows that

$$f(z) = \sum_{n=0}^{k} a_n z^n.$$

Problem (Problem 5)**:** Let $r > 0$, $f, g\colon \dot{U}(0,r) \longrightarrow \mathbb{C}$ be holomorphic functions such that $g(z) \neq 0$ for all $z \in \dot{U}(0,r)$. Show that the singularity at 0 is essential for f if and only if the singularity for $h := \frac{f}{g}$ at 0 is essential.

Solution: Since $g \neq 0$ on $\dot{U}(0,r)$ and g does not have an essential singularity at 0, it follows that that the singularity for $g(z)$ at 0 is either a pole or removable. This allows us to write $g(z) = z^{-m}\widetilde{g}(z)$, where $m \geqslant 0$ is a positive integer and $\widetilde{g}(z)$ is holomorphic (and necessarily nonzero) on $U(0,r)$. Note that if $m = 0$, then the singularity at 0 is removable, and if $m > 0$, then the singularity at 0 is a pole of order m.

Now, we may write

$$h(z) = z^m \frac{f(z)}{\widetilde{g}(z)},$$

where $\widetilde{g}(z)$ is never zero, hence $h(z)\colon \dot{U}(0,r) \longrightarrow \mathbb{C}$ is holomorphic. In particular, since f is also holomorphic, it follows that f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

so we may write

$$\begin{aligned} h(z) &= \frac{1}{\widetilde{g}(z)} \sum_{n=-\infty}^{\infty} a_n z^{m+n} \\ &= \frac{1}{\widetilde{g}(z)} \sum_{n=-\infty}^{\infty} a_{n-m} z^n \end{aligned}$$

Observe then that the singularity at 0 for f is essential if and only if the set of all $n < 0$ for which $a_n \neq 0$ is unbounded below. Since m is constant, it follows that the set of n for which $a_{n-m} \neq 0$ is unbounded below, meaning that the singularity at 0 for h is essential, and vice versa.