Here, we overview and discuss some of the most important results related to projections in von Neumann algebras.

Comparison of Projections

Recall that if H is a Hilbert space, an element $w \in B(H)$ is called a partial isometry if, for any $h \in \ker(w)^{\perp}$, we have $\|Wh\| = \|h\|$. We call $\ker(w)^{\perp}$ the initial space of W and $\operatorname{im}(w)$ the final space of W.

There are a variety of equivalent definitions for partial isometries.

Proposition: If $w \in B(H)$, then the following are equivalent:

  (i) w is a partial isometry;

  (ii) w^* is a partial isometry;

  (iii) w^*w is a projection onto the initial space of w;

  (iv) ww^* is a projection onto the final space of w;

  (v) $ww^*w = w$;

  (vi) $w^*ww^* = w^*$.

Theorem (Polar Decomposition)**:** Let $a \in B(H)$. Then, there is a partial isometry $w \in B(H)$ with initial space $\ker(a)^{\perp}$ and final space $\overline{\operatorname{im}(a)}$ such that $a = w|a|$.

If $a \in M \subseteq B(H)$, where M is a von Neumann algebra, then both $|a|$ and w are in M.

Equivalence of Projections

If $M \subseteq B(H)$ is a von Neumann algebra, then we say two projections $p, q \in P(M)$, where $P(M)$ denotes the space of projections of M, are (Murray–von Neumann) *equivalent* in M if there is a partial isometry $v \in P(M)$ such that $v^*v = p$ and $vv^* = q$. We will write $p \sim q$.

Note that projections have an ordering by saying that $p \leq q$ if $pq = qp = p$, or $\operatorname{im}(p) \subseteq \operatorname{im}(q)$. This allows us to say that p is *sub-equivalent* to q (in M), written $p \preceq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$.[I]

The sub-equivalence relation in fact forms a partial order, and equivalence as projections forms an equivalence relation. We will first show that it is a preorder.

Proposition: In a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\preceq$ is a preorder.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry follows from the fact that if v is a partial isometry, then so is v^*.

Now, we will show transitivity for $\preceq$, from which we will see that $\sim$ is transitive. Letting $p, q, r \in P(M)$ be such that $p \preceq q$ and $q \preceq r$, we have partial isometries $u, v \in M$ with

---

[I]We will say that the projection q majorizes p if $p \preceq q$, and we will say that q dominates p if $p \leq q$.

$u^*u = p$, $uu^* \le q$, $v^*v = q$, and $vv^* \le r$. Then, we have

$$
\begin{aligned}
qu &= quu^*u \\
&= (quu^*)u \\
&= uu^*u \\
&= u,
\end{aligned}
$$

so that

$$
\begin{aligned}
(vu)^*(vu) &= u^*v^*vu \\
&= u^*qu \\
&= u^*u \\
&= p \\
(vu)(vu)^* &= vuu^*v^* \\
&\le vqv^* \\
&= vv^*vv^* \\
&= vv^* \\
&\le r.
\end{aligned}
$$

Therefore, $p \preceq r$, so $\preceq$ is a transitive relation. $\qquad\square$

To see that $\preceq$ is a partial order, we need an analogue of the Cantor–Schröder–Bernstein theorem for projections. In fact, it can be proven in a similar manner. First, we discuss a simple lemma.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are pairwise orthogonal families of projections with $p_i \preceq q_i$, then $\sum_{i \in I} p_i \preceq \sum_{i \in I} q_i$.

Proof. Let u_i be the partial isometries with $u_i^*u_i = p_i$ and $r_i := u_iu_i^* \le q_i$. Then, the r_i are pairwise orthogonal since the q_i are pairwise orthogonal, and for any $i \ne j$,

$$
\begin{aligned}
u_i^*u_j &= u_i^*u_iu_i^*u_ju_j^*u_j \\
&= u_ir_ir_ju_j \\
&= 0 \\
u_iu_j^* &= u_iu_i^*u_iu_j^*u_ju_j^* \\
&= u_ip_ip_ju_j^* \\
&= 0.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
\left(\sum_{i \in I} u_i^*\right)\left(\sum_{j \in I} u_j\right) &= \sum_{i \in I} u_i^*u_i \\
&= \sum_{i \in I} p_i
\end{aligned}
$$

$$\left(\sum_{i \in I} u_i\right)\left(\sum_{j \in I} u_j^*\right) = \sum_{i \in I} u_i u_i^*$$

$$\leq \sum_{i \in I} q_i.$$

This gives $\sum_{i \in I} p_i \preceq \sum_{i \in I} q_i$. □

Theorem: If $e \preceq f$ and $f \preceq e$, then $e \sim f$.

Proof. We will let $e_0 := e$ and $f_0 := f$. Let v and w be partial isometries with $v^* v = e$, $vv^* = f_1 \leq f$, $w^* w = f$, $ww^* = e_1 \leq e$. Inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of some projection dominated by f_1, it follows that we may write $f_2 := ve_1(ve_1)^*$ with $f_2 \leq f_1$. Since w maps the range of f_1 onto the range of some projection dominated by e_1, it follows that we may write $wf_1(wf_1)^* =: e_2$. Observe also that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

Inductively, we obtain decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1}, and w maps the range of f_n isometrically onto that of e_{n+1}. Defining $e_\infty := \inf_n e_n$ and $f_\infty = \inf_n f_n$, we have that v maps the range of e_∞ onto that of f_∞, and w that of f_∞ onto the range of e_∞. Note that we have $e_\infty \sim f_\infty$.

As discussed earlier, we have that $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$, so since sums of pairwise orthogonal families of projections respects equivalence, we have

$$\sum_{n=0}^{\infty} (e_{2n} - e_{2n+1}) \sim \sum_{n=0}^{\infty} (f_{2n+1} - f_{2n+2})$$

$$\sum_{n=0}^{\infty} (e_{2n+1} - e_{2n+2}) \sim \sum_{n=0}^{\infty} (f_{2n} - f_{2n+1}).$$

Therefore, we get

$$e = e_\infty + \sum_{n=0}^{\infty} (e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty} (e_{2n+1} - e_{2n+2})$$

$$\sim f_\infty + \sum_{n=0}^{\infty} (f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty} (f_{2n} - f_{2n+1})$$

$$= f.$$

□

Central Projections and the Comparison Theorem

The projections in a von Neumann algebra form a complete lattice, as the collection of closed subspaces of H form a complete lattice under the operations

$$\bigvee_{i \in I} X_i := \overline{\sum_{i \in I} X_i}$$

$$\bigwedge_{i \in I} X_i := \bigcap_{i \in I} X_i.$$

If $S \subseteq H$ is any subset, then we will define the range projection of S by

$$[S] := P_{\overline{\operatorname{span}(S)}}.$$

Proposition: If $M \subseteq B(H)$ is a von Neumann algebra, and $x \in M$, then $[xH]$ and $[x^*H]$ are in M, with $[xH] \sim [x^*H]$ in M.

Proof. Let $x = v|x|$ be the polar decomposition. Note that $v \in M$. Now, vv^* is the projection onto $\overline{xH}$ and v^*v is the projection onto $\ker(x)^{\perp} = \overline{x^*H}$. Thus, these projections are equivalent in M. $\qquad\square$

Definition: Let $x \in M$. We define the *central support* to be the projection

$$z(x) = \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say p and q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $M \subseteq B(H)$ is a von Neumann algebra, then the central support of any $p \in P(M)$ is given by

$$z(p) = [MpH].$$

Let $w = [MpH]$. Since M is unital, it follows that $p \leq w$, and since $\overline{MpH}$ is a reducing subspace for both M and M', we have $w \in M \cap M'$, so $z(p) \leq w$.

Conversely, if $x \in M$, then

$$xpH = xz(p)pH$$
$$= z(p)xpH,$$

meaning that $[xpH] \leq z(p)$, so $w \leq z(p)$ as x was arbitrary.

Proposition: Let M be a von Neumann algebra, and let $p, q \in P(M)$ be projections. The following are equivalent:

(i) p and q are centrally orthogonal;

(ii) $pMq = \{0\}$;

(iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ with $p_0 \sim q_0$.

Proof. Let p and q be centrally orthogonal. Then, for any $x \in M$, we have

$$pxq = pz(p)xz(q)q$$

$$= pxz(p)z(q)q$$
$$= 0.$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, so $p \leq 1 - z(q)$. Since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, meaning that $z(p)z(q) = 0$. Therefore, (i) and (ii) are equivalent.

Suppose (ii) is not the case. Let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$. Defining

$$p_0 = [pxqH]$$
$$q_0 = [qx^*pH],$$

we have that $p_0 \leq p$, $q_0 \leq q$, and since $(pxq)^* = qx^*p$, we have $p_0 \sim q_0$.

Now, if there are nonzero projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim q_0$, then if v is a partial isometry with $v^*v = p_0$, $vv^* = q_0$, then $v^* = p_0 v^* q_0$, meaning

$$pv^*q = pp_0 v^* q_0 q$$
$$= p_0 v^* q_0$$
$$= v^*$$
$$\neq 0,$$

meaning $pMq \neq \{0\}$. $\qquad \square$

Theorem (Comparison Theorem): Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there is a central projection $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1 - z) \preceq p(1 - z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ of pairwise orthogonal projections with $p_i \sim q_i$ and, setting

$$p_0 = \sum_{i \in I} p_i$$
$$q_0 = \sum_{i \in I} q_i,$$

we have $p_0 \preceq q_0$. From above, we have that $p_0 \sim q_0$.

Let $w := z(q - q_0)$. Since $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are maximal, it follows that $z(q - q_0)$ and $z(p - p_0)$ are centrally orthogonal, yielding $(p - p_0)w = 0$, meaning $pw = p_0 w$.

If we let v be a partial isometry implementing the equivalence $p_0 \sim q_0$, then we have that vw is a partial isometry implementing the equivalence $p_0 w \sim q_0 w$. Therefore, we have

$$pw = p_0 w$$
$$\sim q_0 w$$
$$\leq q.$$

Similarly, $p_0(1 - w) \sim q_0(1 - w)$, so since $q - q_0 \leq w$, we have $q(1 - w) \preceq p(1 - w)$. $\quad \square$

Recall that a factor is a von Neumann algebra M such that $Z(M) = \mathbb{C}1$.

Corollary: If M is a factor, then any two projections in M can be compared.

The Type Decomposition

Definition: Let M be a von Neumann algebra, and $p \in B(H)$ a projection not necessarily in M. The algebra pMp is known as a corner (or compression) of M.

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra, and let $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, and $(pMp)' = M'p$, $(M'p)' = pMp$.

Corollary: If M is a factor and $p \in P(M)$, then pMp and $M'p$ are both factors.

Definition: Let $M \subseteq B(H)$ be a von Neumann algebra. We say a projection $p \in P(M)$ is

- minimal if $p \neq 0$ and the only subprojections of p are 0 and p;

- abelian if pMp is abelian;

- finite if $q \leq p$ and $q \sim p$ implies $q = p$;

- semifinite if there are pairwise orthogonal finite projections $p_i \in P(M)$ such that $p = \sum_{i \in I} p_i$;

- purely infinite if $p \neq 0$ and there is no nonzero finite projection $q \leq p$;

- properly infinite if $p \neq 0$ and zp is not finite for any nonzero central projection $z \in Z(P(M))$.

We say that the von Neumann algebra M is finite/semifinite/purely infinite/properly infinite if the projection 1 satisfies its respective condition.

What we will be working towards is known as the type decomposition, which forms the basis for the dimension theory of von Neumann algebras. Eventually, we will show that every von Neumann algebra M can be decomposed as

$$M = M_{\text{sf}} \oplus M_{\text{III}},$$

where M_{sf} is a semifinite von Neumann algebra and M_{III} is a type III von Neumann algebra, a definition we will discuss shortly. First, we must expand on some of the ways that comparison of projections interacts with these properties of those projections.

Lemma: Let $\{p_i\}_{i \in I}$ be a family of centrally orthogonal projections in a von Neumann algebra $M \subseteq B(H)$. If each p_i is abelian (finite), then the sum $\sum_{i \in I} p_i$ is also abelian (finite).

Proof. If each p_i is abelian, then since they are centrally orthogonal for any $i \neq j$, then for any $x, y \in M$, we have $p_i x p y p_j = 0$. Therefore, we have

$$(pxp)(pyp) = \sum_{i \in I} p_i x p_i y p_i$$
$$= (pyp)(pxp),$$

so p is abelian.

Now, if each p_i is finite, and $u \in M$ is such that $uu^* \leq u^*u = p$, then for all i we have $z(p_i)u^*uz(p_i) = p$, and $uz(p_i)u^* = z(p_i)uu^* \leq p_i$, meaning $uz(p_i)u^* = p_i$, and

$$uu^* = uz(p)u^*$$
$$= \sum_{i \in I} uz(p_i)u^*$$
$$= p.$$

$\square$

Proposition: Let p, q be nonzero projections in a von Neumann algebra with $p \preceq q$. If q is finite (purely infinite), then p is also finite (purely infinite).

Proof. Suppose q is finite, and $p \sim q$, with $v \in M$ implementing the equivalence. Let $u \in M$ be such that $u^*u = p$ and $uu^* \leq p$. Then,

$$(vuv^*)^*(vuv^*) = q$$
$$(vuv^*)(vuv^*)^* \leq q,$$

so since q is finite, $(vuv^*)(vuv^*)^* = q$, so $uu^* = p$.

Now, if $p \leq q$, then if $u^*u = p$ with $uu^* \leq p$, then by setting $w = u + (q - p)$, we have that $w^*w = q$ and $ww^* \leq q$, so $u^*u + (q - p) = ww^* = q$, meaning $uu^* = p$.

In the general case, we have some $q_0 \leq q$ such that $p \sim q_0 \leq q$.

Since projections are purely infinite when they have no nonzero finite subprojections, the purely infinite case follows from the finite case. $\square$

Proposition: A projection $p \in P(M)$ is semifinite if and only if p is the supremum of finite projections. In particular, a supremum of semifinite projections is also semifinite.

Proof. If p is semifinite, then p is the sum (and hence the supremum) of a family of pairwise orthogonal finite projections.

Now, suppose $p = \bigvee_\alpha p_\alpha$, where each p_α is finite. Let $\{q_\beta\}_\beta$ be a maximal family of pairwise orthogonal finite subprojections of p. If we set

$$q_0 = p - \sum_\beta q_\beta,$$

and suppose that $q_0 \neq 0$, then there exists some p_α such that p_α and q_0 are not orthogonal (else it would contradict maximality), so they are not centrally orthogonal. Therefore, we have a nonzero subprojection $q_1 \leq q_0$ such that $q_1 \preceq p_\alpha$, so it is finite by what we showed previously; this contradicts the maximality of the set $\{q_\beta\}_\beta$. $\square$

Corollary: Let p be a projection in a von Neumann algebra M. If p is semifinite (purely infinite), then the central support $z(p)$ is also semifinite (purely infinite).

Proof. The central support is the supremum over all equivalent projections, so since the supremum of semifinite projections is again semifinite, it follows from the previous proposition.

Furthermore, a nonzero projection is purely infinite if and only if it is centrally orthogonal to every semifinite projection, so we obtain the corollary in this case. $\qquad\square$

Corollary: Let p, q be nonzero projections in a von Neumann algebra such that $p \preceq q$. If q is semifinite, then so is p.

Proof. It is enough to consider the case when q is central, in which case we may take $p \leq q$. Let p_0 be the maximal semifinite subprojection of p. Since q is semifinite, it is the supremum of its finite subprojections. Since $z(p - p_0) \leq q = z(q)$, it follows that if $p - p_0 \neq 0$, then there would exist a nonzero finite subprojection that would be equivalent to a subprojection of $p - p_0$, which contradicts the definition of p_0. Thus, p is the supremum of its finite subprojections, so it is semifinite. $\qquad\square$

Lemma: Let M be a properly infinite von Neumann algebra. Then, there exists a projection $p \in P(M)$ such that $p \sim 1 - p \sim 1$.

Proof. Since M is properly infinite, there exists $u \in M$ with $uu^* \leq u^*u = 1$. Set $p_0 = 1 - uu^*$. Then, $p_n = u^n p_0 (u^n)^*$ is a pairwise orthogonal family of equivalent projections. Let $\{q_i\}_{i \in I}$ be a maximal family of pairwise orthogonal equivalent projections in M extending $\{p_n\}_{n \in \mathbb{N}}$, and set $q_0 = 1 - \sum_{i \in I} q_i$.

By the comparison theorem, there is $z \in Z(P(M))$ such that $q_0 z \leq q_{i_0} z$ and $q_{i_0}(1 - z) \leq q_0(1 - z)$. If it were the case that $z = 0$, then we would have $q_{i_0} \leq q_0$, contradicting maximality of $\{q_i\}_{i \in I}$, so $z \neq 0$, and we have

$$
\begin{aligned}
z = q_0 z + \sum_{i \in I} q_i z \\
\preceq q_{i_0} z + \sum_{i \neq i_0} q_i z \\
= \sum_{i \in I} q_i z \leq z,
\end{aligned}
$$

so $z \sim \sum_{i \in i} q_i z$ by Cantor–Schröder–Bernstein for projections. Decomposing $\{q_i\}_{i \in I}$ into infinite subsets, we may construct two projections p and $z - p$ such that $p \sim z - p \sim z$.

Now, let $\{r_j\}_{j \in J}$ be a maximal family of centrally orthogonal projections with $r_j \sim z(r_j) - r_j \sim z(r_j)$. Then, the argument above shows that $\sum_{j \in J} z(r_j) = 1$, so by setting $p = \sum_{j \in J} r_j$, we obtain our desired result. $\qquad\square$

References

[Tak79] Masamichi Takesaki. *Theory of operator algebras. I*. Springer-Verlag, New York-Heidelberg, 1979, pp. vii+415. ISBN: 0-387-90391-7.

[Con00] John B. Conway. *A course in operator theory*. Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: 10.1090/gsm/021. URL: https://doi.org/10.1090/gsm/021.

[Dav25] Kenneth R. Davidson. *Functional analysis and operator algebras*. Vol. 13. CMS/CAIMS Books in Mathematics. Springer, Cham, [2025] ©2025, pp. xiv+797.