Problem (Problem 1)**:** Show that a sequence of R-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

is exact if and only if the sequence

$$0 \longrightarrow \hom(P, L) \xrightarrow{f_*} \hom(P, M) \xrightarrow{g_*} \hom(P, N)$$

is exact.

Solution: Suppose that the sequence of R-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

is exact. That is, f is injective, and $\operatorname{im}(f) = \ker(g)$. Now, let $\varphi \in \hom(P, L)$, and suppose $\varphi \in \ker(f_*)$. Then, it follows that $f_*(\varphi) \equiv 0$, whence for all $v \in P$, we have

$$f_*(\varphi)(v) = f(\varphi(v)).$$

for all $v \in L$. Yet, since f has kernel equal to 0, this means that $\varphi(v) = 0$ for all $v \in L$, so that $\varphi \equiv 0$.

Now, we consider the relationship between $\operatorname{im}(f_*)$ and $\ker(g_*)$. First, we observe that $g_* \circ f_*(\varphi) = (g \circ f)_*(\varphi)$, but since $\ker(g) = \operatorname{im}(f)$, it follows that $g \circ f$ is 0, as the original sequence is exact. Therefore, $\operatorname{im}(f_*) \subseteq \ker(g_*)$. Now, suppose $\psi \in \ker(g_*)$. That is, for all $v \in P$, we have $g_*(\psi)(v) = 0$. In particular, this means that we have

$$g(\psi(v)) = 0.$$

It follows then that $\psi(v) \in \operatorname{im}(f)$, as we assume that the original sequence of R-modules is exact. In particular, there is some $w \in L$ such that $\psi(v) = f(w)$. Note that since f is injective, such a w is uniquely determined, whence the map $\tau \colon P \to L$ defined by $v \mapsto w$ is well-defined. In particular, we also have

$$\begin{aligned} f_*(\tau)(v) &= f(\tau(v)) \\ &= f(w) \\ &= \psi(v) \end{aligned}$$

for all $v \in P$, so that $f_*(\tau) = \psi$. In particular, this gives $\operatorname{im}(f_*) = \ker(g_*)$.

Now, let the hom sequence

$$0 \longrightarrow \hom(P, L) \xrightarrow{f_*} \hom(P, M) \xrightarrow{g_*} \hom(P, N)$$

be exact. Since the hom sequence is exact, it follows that $f_*(\varphi) = 0$ if and only if $\varphi = 0$. In particular, if $v \in P$, then $f(\varphi(v)) = 0$ if and only if $\varphi(v) = 0$, whence $\ker(f) = 0$. Thus, f is injective.

We start by showing that $\operatorname{im}(f) \subseteq \ker(g)$. If $q \in \operatorname{im}(f)$, then there is some $r \in L$ such that $f(r) = q$. Now, let $P = \langle r \rangle$ be the R-module generated by r, and let $\varphi \colon P \hookrightarrow L$ be the inclusion of P into L. Then, we observe that $f(\iota(r)) = q$, whence $g_* \circ f_*(\iota)(r) = 0 = g(q)$, by the exactness of the hom sequence.

Finally, let $q \in \ker(g)$. We observe that $\iota \colon \langle q \rangle \hookrightarrow M$ is an inclusion of R-modules, so that there is some $\varphi \colon \langle q \rangle \to L$ such that $f_*(\varphi) = \iota$. In particular, this means that, as $\varphi(q) \in L$, we have

$$f(\varphi(q)) = q,$$

whence $q \in \operatorname{im}(f)$. Thus, $\operatorname{im}(f) = \ker(g)$, so the original sequence of R-modules is exact.

Problem (Problem 2)**:** Let R be a local ring with maximal ideal M. Show that every finitely generated projective R-module P is free.

Solution: Let P be a finitely generated projective module. We observe that P/MP can be viewed as a vector space over $K := R/M$, and since P is a finitely generated module, so too is P/MP, whence there is a basis $\overline{x_1}, \dots, \overline{x_n}$ for P/MP. By Nakayama's Lemma, it follows that $P = \langle x_1, \dots, x_n \rangle$ as an R-module.

In particular, we have the module homomorphism $f\colon F = R^n \to P$ given by

$$(a_1, \dots, a_n) \mapsto a_1 \cdot x_1 + \cdots + a_n \cdot x_n$$

is surjective. Now, we see then that

$$P \cong F/\ker(f)$$

by the First Isomorphism Theorem. Since P is projective, it follows that the sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r] & F \arrow[r, "f"] & P \arrow[r] \arrow[l, bend left, dashed, "q"] & 0 \end{tikzcd}$$

admits a section $q\colon P \to F$, whence $F \cong P \oplus Q$ where $Q = \ker(f)$.

Taking residues modulo M, we have that $F/MF \cong P/MP \cong Q/MQ$, and since P/MP and F/MF are finite-dimensional K-vector spaces with the same dimension, it follows that $Q/MQ \cong \{0\}$ by the invariance of dimension.

Finally, we observe that any element $v \in F$ has a decomposition $v = p + q$ for a unique $q \in Q$ and unique $p \in P$. In particular, we have the surjection $\pi\colon F \to Q$ given by $v \mapsto q$. In particular, we observe that $\ker(\pi) \cong P$, meaning that $Q = F/P$ is a quotient of two finitely generated modules, hence finitely generated. Nakayama's Lemma thus gives $Q = \{0\}$.

Problem (Problem 5)**:** Prove that if

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is a short exact sequence of R-modules, and Q is an arbitrary R-module, then the sequence

$$0 \longrightarrow \hom(N, Q) \xrightarrow{g^*} \hom(M, Q) \xrightarrow{f^*} \hom(L, Q)$$

is exact.

Solution: Let $\varphi \in \ker(g^*)$, so that $g^*(\varphi) = 0$, or $\varphi \circ g = 0$. For any $n \in N$, since g is surjective, there is $m \in M$ such that $g(m) = n$, whence $\varphi(g(m)) = \varphi(n) = 0$. Thus, $\varphi \equiv 0$, meaning g^* is injective.

Now, we observe that since $g \circ f = 0$, we have $f^* \circ g^*(\varphi) = g \circ f \circ \varphi \equiv 0$, meaning that $\operatorname{im}(g^*) \subseteq \ker(f^*)$.

Now, let $\varphi \in \hom(M, Q)$ be such that $\varphi \in \ker(f^*)$. We seek to find $\psi \in \hom(N, Q)$ such that $g^*(\psi) = \varphi$. Since g is surjective, g admits a right inverse, so that if $n \in N$, then there is $m \in M$ such that $g(m) = n$. Define $\psi\colon N \to Q$ by $\psi(n) = \varphi(h(n))$, where $h\colon N \to M$ is a (necessarily injective) right inverse for g. We observe that ψ is defined on all of N as g is surjective, and that ψ is a homomorphism since φ is a homomorphism. Additionally, since h is injective, it follows that ψ is well-defined. Thus, the sequence is exact at $\hom(M, Q)$.

Problem (Problem 8)**:**

(a) Let G be a group such that $|G| = p^n$ for some prime p and some $n \geqslant 1$. Let X be a finite G-set, and let X^G be the set of all fixed points of the action. Show that $|X| = \left|X^G\right| \bmod p$.

(b) Show that every group G of order p^2 is abelian.

Solution:

(a) We observe that, by definition, X^G is the set of all elements of X with trivial orbit. That is, $X \setminus X^G$ consists of all the nontrivial orbits of X. Letting $x_1, \dots, x_\ell$ be representatives for each of these orbits, we observe that

$$|X| = \left|X^G\right| + \sum_{k=1}^{\ell} |G \cdot x_k|.$$

From the orbit–stabilizer theorem, it follows that

$$|X| = \left|X^G\right| + \sum_{k=1}^{\ell} [G : \operatorname{stab}_G(x_k)].$$

Since each of the $G \cdot x_k$ are nontrivial orbits, it follows that $[G : \operatorname{stab}_G(x_k)] \neq 1$, whence each index is a power of p. Thus, we obtain

$$|X| = \left|X^G\right| \text{ modulo } p.$$

(b) Let G act on itself via conjugation, so that $Z(G)$ is the set of fixed points under this action. Therefore, we get the equation

$$|G| = |Z(G)| + \sum_{k=1}^{\ell} |G \cdot x_k|$$

for some orbit representatives $x_1, \dots, x_\ell$. We claim that there is no nontrivial orbit.

First, we observe that $|Z(G)| \geqslant 1$ as $Z(G)$ is a subgroup and thus contains the identity element. Now, if $Z(G) = 1$, then the sum of the sizes of the orbits $|G \cdot x_k|$ is $p^2 - 1$, implying

$$\sum_{k=1}^{\ell} [G : \operatorname{stab}_G(x_k)] = p^2 - 1,$$

but p divides each nontrivial index, implying that $p \mid p^2 - 1$, which is a contradiction as $p^2 - 1$ and p^2 are coprime. Next, if $|Z(G)| = p$, we see that $G/Z(G)$ has order p, whence $G/Z(G)$ is cyclic, contradicting the result from Problem 7. Therefore, $|Z(G)| = p^2$, whence $ghg^{-1} = h$ for all $g \in G$ and all $h \in G$, or that $gh = hg$, so G is abelian.