Problem 1

If F is a finite set and $k : F \to F$ is a self-map, prove that k is injective if and only if k is surjective.

Let k be injective.

$$\begin{aligned} \text{card}(F) &= \text{card}(k(F)) && \text{definition of injection} \\ k(F) &\subseteq F && \text{definition of function} \\ k(F) &= F \end{aligned}$$

Let k be surjective.

$$\begin{aligned} k \circ k^{-1}(F) &= F && \text{definition of surjection} \\ (k \circ k^{-1}) \circ k(F) &= k(F) && \text{apply } k \text{ on the right} \\ k \circ (k^{-1} \circ k)(F) &= k(F) && \text{associative property} \end{aligned}$$

Therefore, $k^{-1} \circ k = \text{id}_F$, meaning k is injective.

Problem 2

Prove that a set A is infinite if and only if there is a non-surjective injection $f : A \to A$.

Problem 3

Let A, B, and C be sets and suppose $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$. Prove that $\text{card}(A) < \text{card}(C)$.

Problem 4

If $A \subseteq B$ is an inclusion of sets with A countable and B uncountable, show that $B \setminus A$ is uncountable.

Let A be countable. Then, $A = \emptyset$, A is finite, or $\exists f : \mathbb{N} \mapsto A$.

Let $g : B \to \mathbb{N}$. By the definition of countability, g is not injective.

Let $k : B \setminus A \to \mathbb{N}$. We will show that for the above three cases, k is not injective.

Case 1 If A is the empty set, then we know that $B \setminus A = B$, and since g is not injective, k must not be injective.

Case 2 Let $L = \{1, 2, \dots, n\}$ where n is the cardinality of A. Since A is finite, $\exists h : A \to L$. Let $s : \mathbb{N} \setminus L \to \mathbb{N}$, $s(l) = l + (n + 1)$. Finally, let $t : B \setminus A \to \mathbb{N} \setminus L$. s is a bijection (as it is a linear function), and t is not injective, as $\text{card}(B \setminus A) = \text{card}(B) - n$. Thus, $s \circ t$ is not injective, so $B \setminus A$ is not countable.

Case 3

Problem 5

Is the set $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ countable?

Problem 6

Consider the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$. Is $\mathcal{F}(\mathbb{N})$ countable?

Problem 7

Let $k \in \mathbb{N}$.

(i) Prove that $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \mathbb{N}}_{k \text{ times}}$ is countable.

(ii) Show that the set $\mathbb{N}^\infty := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}\}$ consisting of all sequences of natural numbers is uncountable.

(iii) Prove that the set of **finitely-supported** natural sequences $c_c(\mathbb{N}) := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}, n_k = 0 \text{ for all but finitely many } k\}$ is countable.

Problem 8

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range $\operatorname{ran}(f)$ cannot contain any interval.

Problem 9

Prove that the set

$$\mathcal{P} := \left\{ \sum_{k=0}^{n} a_k x^k \mid n \subseteq \mathbb{N}_0, a_k \in \mathbb{Q} \right\}$$

consisting of all polynomials with rational coefficients, is countable.

Problem 10

A real number t is called **algebraic** if there is a nonzero polynomial p with rational coefficients such that $p(t) = 0$. If $t \in \mathbb{R}$ is not algebraic, then it is called **transcendental**. For example, $\sqrt{2}$ is algebraic, but π is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.