Representations

> **Definition:** If A is a C^*-algebra, a representation of A is a pair (π, H) where H is a Hilbert space and $\pi\colon A \to B(H)$ is a $*$-homomorphism. If A is unital, then we require $\pi(1) = I$.

Note that if A does not have an identity, we can extend to the unitization $A_1 = A \oplus \mathbb{C}$ and define $\widetilde{\pi}(a, \lambda) = \pi(a) + \lambda I$ for any $a \in A$ and $\lambda \in \mathbb{C}$.

Note that every representation is contractive and the range of any representation is closed.

> **Example:**
>
> (a) If A is a C^*-subalgebra of $B(H)$, then the inclusion map $A \hookrightarrow B(H)$ is a representation.
>
> (b) If (X, Ω, μ) is a σ-finite measure space, then $\pi\colon L_\infty(\mu) = B(L_2(\mu))$, where $\pi(\phi) = M_\phi$, is a representation.
>
> (c) If X is compact, and μ is a positive Borel measure on X, then $\pi_\mu\colon C(X) \to B(L_2(\mu))$ defined by $\pi_\mu(f) = M_f$ is a representation of $C(X)$.

States

For now, we will assume that M is a unital self-adjoint subspace of a C^*-algebra A. If ρ is a linear functional on M, then the equation

$$\rho^*(a) = \overline{\rho(a^*)}$$

defines another linear functional; if $\rho = \rho^*$, then we call ρ hermitian. Equivalently, ρ is hermitian if $\rho(a^*) = \overline{\rho(a)}$. If ρ is a bounded hermitian functional on M, then we claim that

$$\|\rho\| = \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\}.$$

This follows from the fact that if $\varepsilon > 0$, then from the Riesz lemma, we may find a in the unit ball of M with $|\rho(a)| > \|\rho\| - \varepsilon$. For a suitable λ with $|\lambda| = 1$, we have

$$\begin{aligned} \|\rho\| - \varepsilon &< |\rho(a)| \\ &= \rho(\lambda a) \\ &= \overline{\rho(\lambda a)} \\ &= \rho\big((\lambda a)^*\big). \end{aligned}$$

If $a_0 = \operatorname{Re}(\lambda a)$, we have $\|a_0\| \leq 1$ with $\rho(a_0) > \|\rho\| - \varepsilon$. Thus,

$$\|\rho\| \leq \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\},$$

with the reverse inequality being true by definition.

We say the linear functional ρ is *positive* if for any $a \in M_+$, $\rho(a) \geq 0$; if $\rho(1) = 1$, then we say ρ is a state. In fact, if ρ is positive, then ρ is hermitian, since if $a = a^*$, then

We start by considering a version of the Cauchy–Schwarz inequality for states.

> **Proposition:** If ρ is a positive linear functional on a C^*-algebra A, then
>
> $$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b).$$

> *Proof.* With $a \in A$, we have $a^*a \in A_+$, so $\rho(a^*a) \geq 0$. Then, since ρ is hermitian, we have that
>
> $$\langle a, b\rangle = \rho(b^*a)$$
>
> defines a positive sesquilinear form on A, so the traditional Cauchy–Schwarz inequality gives the desired result. ☐

References

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