Introduction

Oh hey, it's another one of those textbook notes that I never complete. I've decided to try something different in order to develop my understanding of measure theory. One of the primary for understanding measure theory is Gerald B. Folland's *Real Analysis and Applications* — and one of the benefits it has over a lot of other texts is that it has a significant number of exercises. I'm going to try to do them all — I'll start with Chapters 1–3, and if that goes well enough, continue up through whatever chapter I end up having to tap out at. Interspersed, I will include various notes. I figure that in order to make a subject like measure theory really stick, I need to deal with it consistently.

Contents

Chapter 1

Section 1.2

Definition (σ-Algebra)**.** An algebra of sets on X is a nonempty collection $\mathcal{A}$ of X that is closed under finite unions and complements.

A σ-algebra is an algebra that is closed under countable unions.

Exercise (Exercise 1)**:** A family of sets $\mathcal{R} \subseteq P(X)$ is called a ring if it is closed under finite unions and differences. A ring that is closed under countable unions is called a σ-ring.

(a) Rings (σ-rings) are closed under finite (countable) intersections.
(b) If $\mathcal{R}$ is a ring (σ-ring), then $\mathcal{R}$ is an algebra (σ-algebra) if and only if $X \in \mathcal{R}$.
(c) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
(d) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Solution:

(a) Note that for any $E, F \in \mathcal{R}$, that $E \cap F = E \cup F \setminus ((E \setminus F) \cup (F \setminus E))$.
(b) Let $\mathcal{R}$ be a σ-ring. Then, $\mathcal{R}$ is a σ-algebra if for some $E \in \mathcal{R}$, $E^c \in \mathcal{R}$. Since $E^c = X \setminus E \in \mathcal{R}$, we have $X \setminus E \cup E \in \mathcal{R}$ as $\mathcal{R}$ is closed under (countable) unions. Hence, $X \in \mathcal{R}$.

    If $X \in \mathcal{R}$, then for any $E \in \mathcal{R}$, $E^c = X \setminus E \in \mathcal{R}$. Thus, $\mathcal{R}$ is closed under intersections.
(c) Since $\mathcal{R}$ is a σ-ring, we only need show that the set $\mathcal{A} = \{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is closed under complements. We see that for any $E \in \mathcal{A}$, it is the case that either $E \in \mathcal{R}$ or $E^c \in \mathcal{R}$, so $E^c \in \mathcal{A}$ if and only if $E^c \in \mathcal{R}$ or $E \in \mathcal{R}$, so $\mathcal{A}$ is closed under complements.
(d) Let $\mathcal{R}$ be a σ-ring, and let $\mathcal{A} = \{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. We will show that $\mathcal{A}$ is closed under unions and complements.

    Let $E, F \in \mathcal{A}$. Then, for all $S \in \mathcal{R}$, we have $E \cap S \in \mathcal{R}$ and $F \cap S \in \mathcal{R}$. Since $\mathcal{R}$ is closed under unions, we must have $(E \cup F) \cap S = (E \cap S) \cup (F \cap S) \in \mathcal{R}$, so $E \cup F \in \mathcal{A}$.

Let $E \in \mathcal{A}$.

Proposition (Proposition 1.2)**:** The Borel σ-algebra, $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following:

(a) the open intervals, $\mathcal{E}_1 = \{(a, b) \mid a < b\}$;

(b) the closed intervals, $\mathcal{E}_2 = \{[a, b] \mid a < b\}$;

(c) the half-open intervals, $\mathcal{E}_3 = \{(a, b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a, b) \mid a < b\}$;

(d) the open rays, $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$;

(e) the closed rays, $\mathcal{E}_7 = \{[a, \infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] \mid a \in \mathbb{R}\}$.

Proof. The elements for $\mathcal{E}_j$ for $j \neq 3, 4$ are open or closed, and the elements of $\mathcal{E}_3$, $\mathcal{E}_4$ are G_δ sets — for instance,

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right).$$

Thus, $\sigma(\mathcal{E}_j) \subseteq \mathcal{B}_{\mathbb{R}}$ for each j. On the other hand, every open set in $\mathbb{R}$ is a countable union of open intervals, so $\mathcal{B}_{\mathbb{R}} \subseteq \sigma(\mathcal{E}_1)$. Thus, $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{E}_1)$. □

Section 1.3

Theorem (Theorem 1.9)**:** Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$, and let $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$. Then, $\mathcal{M}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, so is $\overline{\mathcal{M}}$. If $E \cup F \in \overline{\mathcal{M}}$, with $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{N}$, we may assume $E \cap N = \varnothing$ — else, we replace F with $F \setminus E$ and N with $N \setminus E$. Then, $E \cup F = (E \cup N) \cap (N^c \cup F)$, so $(E \cup F)^c = (E \cup N)^c \cup (N \setminus F)$. Since $(E \cup N)^c \in \mathcal{M}$ and $N \setminus F \subseteq N$, we have $(E \cup F)^c \in \overline{\mathcal{M}}$, so $\overline{M}$ is a σ-algebra.

If $E \cup F \in \overline{\mathcal{M}}$ as above, we set $\overline{\mu}(E \cup F) = \mu(E)$. This is well-defined, since if $E_1 \cup F_1 = E_2 \cup F_2$, with $F_j \subseteq N_j \in \mathcal{N}$, then $E_1 \subseteq E_2 \cup N_2$, so $\mu(E_1) \leqslant \mu(E_2) + \mu(N_2) = \mu(E_2)$. Similarly, $\mu(E_2) \subseteq \mu(E_1)$. □

Exercise (Exercise 6)**:** Complete the proof of Theorem 1.9.

Solution: We now wish to show that every subset of a null set in $\mathcal{M}$ is an element of $\overline{\mathcal{M}}$. This can be seen by the fact that for some $F \subseteq N \in \mathcal{N}$, we have $F = \varnothing \cup F \in \overline{\mathcal{M}}$.

To show uniqueness, we suppose there is some other measure $\nu \colon \overline{\mathcal{M}} \to [0, \infty)$ such that ν agrees with μ on $\mathcal{M}$. For some $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{N}$, we have

$$\begin{aligned} \nu(E \cup F) &= \mu(E) \\ &= \overline{\mu}(E \cup F). \end{aligned}$$

Exercise (Exercise 7)**:** If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, and $a_1, \ldots, a_n \in [0, \infty)$, then $\mu = \sum_{j=1}^{n} a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Solution: It is clear that $\mu(\varnothing) = \varnothing$. If we have a sequence of disjoint sets $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, then

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \sum_{j=1}^{n} a_j \mu_j \left(\bigcup_{i=1}^{\infty} E_i\right) \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{\infty} \mu_j(E_i) \end{aligned}$$

$$= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} a_j \mu_j\right)(E_i)$$
$$= \sum_{i=1}^{\infty} \mu(E_i).$$

Exercise (Exercise 8)**:** If $(X, \mathcal{M}, \mu)$ is a measure space, and $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, then $\mu(\liminf E_j) \leqslant \liminf \mu(E_j)$. Additionally, if $\mu\left(\bigcup_{j\geqslant 1} E_j\right) < \infty$, then $\mu(\limsup E_j) \geqslant \limsup \mu(E_j)$.

Solution: Note that

$$\liminf E_j = \bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} E_j.$$

Labeling

$$F_n = \bigcap_{j=n}^{\infty} E_j,$$

we have a sequence of inclusions

$$F_1 \subseteq F_2 \subseteq \cdots,$$

meaning that

$$\mu(\limsup E_j) = \inf_{n\geqslant 1} \mu(F_n).$$

Note that we have

$$\mu(F_n) = \mu\left(\bigcup_{n=j}^{\infty} E_j\right).$$

Exercise (Exercise 9)**:** If $(X, \mathcal{M}, \mu)$ is a measure space, and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Solution: We have

$$\begin{aligned} \mu(E) &= \mu(((E \cup F) \setminus F) \sqcup E \cap F) \\ \mu(E) &= \mu(E \cup F) - \mu(F) + \mu(E \cap F) \\ \mu(E) + \mu(F) &= \mu(E \cup F) + \mu(E \cap F). \end{aligned}$$

Exercise (Exercise 12)**:** Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ with $\mu(E \triangle M) = 0$, then $\mu(E) = \mu(F)$.

(b) Let $E \sim F$ if $\mu(E \triangle F) = 0$. Then, $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then, $\rho(E, G) \leqslant \rho(E, F) + \rho(F, G)$, hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Solution:

(a) Note that $E = (E \setminus F) \sqcup (E \cap F)$, and $F = (F \setminus E) \sqcup (F \cap E)$. We also have $\mu(E \triangle F) = \mu(E \setminus F) + \mu(F \setminus E) = 0$, so $\mu(F \setminus E) = \mu(E \setminus F) = 0$. Thus,

$$\begin{aligned} \mu(F) &= \mu(F \cap E) \\ &= \mu(E \cap F) \\ &= \mu(E). \end{aligned}$$

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space.

- If $\mu(X) < \infty$, then μ is called finite.
- If $X = \bigcup_{j\geqslant 1} E_j$, where $E_j \in \mathcal{M}$ for each j and $\mu(E_j) < \infty$, then μ is called σ-finite.
- If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$, then we say μ is semifinite.

Exercise (Exercise 13)**:** Every σ-finite measure is semifinite.

Solution: Let $(X, \mathcal{M}, \mu)$ be a measure space such that $X = \bigcup_{j\geqslant 1} E_j$, where $\{E_j\}_{j\geqslant 1} \subseteq \mathcal{M}$ and $\mu(E_j) < \infty$ for each j.

Suppose $\mu(E) = \infty$. Then, we may find $F \subseteq E$ by finding j such that $\mu(E_j) > 0$, and taking $F = E_j \cap E$. Then, it must be the case that $0 < \mu(F) \leqslant \mu(E_j) < \infty$.

Exercise (Exercise 14)**:** If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subseteq E$ such that $C < \mu(F) < \infty$.

Solution: By the definition of a semifinite measure, there exists $F_1 \subseteq E$ such that $0 < \mu(F_1) < \infty$. We let $\delta_1 = \mu(F_1)$.

Now, it must be the case that $\mu(E \setminus F_1) = \infty$, else $\infty = \mu(E) = \mu(E \setminus F_1) + \mu(F_1) < \infty$, a contradiction.

Hence, there exists $F_2 \subseteq E \setminus F_1$ with $0 < \mu(F_2) < \infty$. We let $\delta_2 = \mu(F_2)$. Similarly, we find $\delta_n = \mu(F_n)$, where $F_n \subseteq E \setminus (F_1 \cup \cdots \cup F_{n-1})$.

Now, consider the series $\sum_{n\geqslant 1} \delta_n = \sum_{n\geqslant 1} \mu(F_n) = \mu\left(\bigsqcup_{n\geqslant 1} F_n\right)$. This series must diverge, as otherwise we would have $\mu(E) = \mu\left(\bigsqcup_{n\geqslant 1} F_n\right) < \infty$, which is yet again a contradiction.

Thus, for a given $C > 0$, we find N so large such that $\sum_{n=1}^{N} \delta_n > C$. Then, $F = \bigsqcup_{n=1}^{N} F_n$ is our desired set.

Exercise (Exercise 15)**:** Let μ be a measure on $(X, \mathcal{M})$. Define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) \mid F \subseteq E \text{ and } \mu(F) < \infty\}$.

(a) μ_0 is a semifinite measure It is called the semifinite part of μ.

(b) If μ is semifinite, then $\mu = \mu_0$.

(c) There is a measure ν on $\mathcal{M}$ which only assumes the values 0 and ∞ such that $\mu = \mu_0 + \nu$.

Solution:

(a) Let $E \in \mathcal{M}$ be such that $\mu_0(E) = \infty$. Suppose toward contradiction that μ_0 is not semifinite. Then, for any $F \subseteq E$, it is the case that $\mu(F) = 0$ or $\mu(F) = \infty$, so it would then be the case that $\mu_0(E) = 0$, a contradiction.

(b) If $\mu(E) < \infty$, then $\mu_0(E) = \mu(E)$, as $E \subseteq E$ and $\mu(E) < \infty$.

    If $\mu(E) = \infty$, then it is clear that $\mu_0(E) = \infty$, as for each $C > 0$ there is some $F \subseteq E$ such that $C < \mu(F) < \infty$.

    Thus, $\mu = \mu_0$.

(c) We define the measure ν on $\mathcal{M}$ by taking $\nu(E) = 0$ whenever $\mu(E) < \infty$ and $\nu(E) = \infty$ whenever $\mu(E) = \infty$.

Exercise: Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subseteq X$ is called locally measurable if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets.

It is obvious that $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$. If $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called saturated.

(a) If μ is σ-finite, then μ is saturated.

(b) $\widetilde{\mathcal{M}}$ is a σ-algebra.

(c) Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then, $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ called the saturation of μ.

(d) If μ is complete, so is $\widetilde{\mu}$.

(e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}$. Then, $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

(f) Let X_1 and X_2 be disjoint uncountable sets, $X = X_1 \sqcup X_2$, and $\mathcal{M}$ the σ-algebra of countable and cocountable sets in X. Let μ_0 be the counting measure on $P(X_1)$ and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then,

- μ is a measure on $\mathcal{M}$;
- $\widetilde{\mathcal{M}} = P(X)$;
- and $\widetilde{\mu} \neq \underline{\mu}$.

Solution:

(a) Let μ be σ-finite, and let $E \in \widetilde{\mathcal{M}}$. We know that $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. In particular, we can select a disjoint collection $\{A_j\}_{j=1}^{\infty}$ such that $\mu(A_j) < \infty$ and $\bigsqcup_{j\geqslant 1} A_j = X$. Thus, since $E = X \cap E$, we must have $E \in \mathcal{M}$ as E is locally measurable.

Section 1.4

Definition. An outer measure on a nonempty set X is a function $\mu^*\colon P(X) \to [0,\infty]$ such that

- $\mu^*(\emptyset) = 0$;
- $\mu^*(A) \leqslant \mu^*(B)$ if $A \subseteq B$;
- $\mu^*\left(\bigcup_{j\geqslant 1} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu^*(A_j)$.

Proposition: Let $\mathcal{E} \subseteq P(X)$, and $\rho\colon \mathcal{E} \to [0,\infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\emptyset) = 0$. For any $A \subseteq X$, define

$$\mu^*(A) = \inf\left\{ \sum_{j\geqslant 1} \rho(E_j) \,\middle|\, E_j \in \mathcal{E} \text{ and } A \subseteq \bigcup_{j\geqslant 1} E_j \right\}.$$

Then, μ^* is an outer measure.

Proof. For any $A \subseteq X$, there exists $\{E_j\}_{j\geqslant 1} \subseteq \mathcal{E}$ such that $A \subseteq \bigcup_{j\geqslant 1} E_j$ (taking $E_j = X$). Clearly, $\mu^*(\emptyset) = \emptyset$.

Additionally, since $A \subseteq B$, we the infimum taken to define $\mu^*(A)$ includes the corresponding set in the definition of $\mu^*(B)$, so $\mu^*(A) \leqslant \mu^*(B)$.

Suppose $\{A_j\}_{j\geqslant 1} \subseteq P(X)$, and let $\varepsilon > 0$. For each j, there exists $\{E_{j,k}\}_{k\geqslant 1} \subseteq \mathcal{E}$ such that $A_j \subseteq \bigcup_{k\geqslant 1} E_{j,k}$ and $\sum_{k\geqslant 1} \rho(E_{j,k}) \leqslant \mu^*(A_j) + \varepsilon 2^{-j}$. Thus, if $A = \bigcup_{j\geqslant 1} A_j$, we have $A \subseteq \bigcup_{j,k\geqslant 1} E_{j,k}$, and $\sum_{j,k\geqslant 1} \rho(E_{j,k}) \leqslant \sum_{j\geqslant 1} \mu^*(A_j) + \varepsilon$, meaning $\mu^*(A) \leqslant \sum_{j\geqslant 1} \mu^*(A_j) + \varepsilon$. Sine this holds for all $\varepsilon > 0$, we must have $\mu^*\left(\bigcup_{j\geqslant 1} A_j\right) \leqslant \sum_{j\geqslant 1} \mu^*(A_j)$. □

Definition. If μ^* is an outer measure, a set $A \subseteq X$ is called μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \subseteq X$. In other words, A is measurable if it serves as a well-behaved "cookie cutter" for any subset of X.

Note that it suffices to show that

$$\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition. If $\mathcal{A} \subseteq P(X)$ is an algebra, a function $\mu_0\colon \mathcal{A} \to [0,\infty]$ is called a premeasure if $\mu_0(\emptyset) = 0$ and, for any sequence of disjoint sets $\{A_j\}_{j=1}^{\infty}$ in $\mathcal{A}$ such that $\bigsqcup_{j=1}^{\infty} A_j \in \mathcal{A}$, we have

$$\mu_0\left(\bigsqcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

A premeasure induces an outer measure on X by

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) \;\middle|\; A_j \in \mathcal{A}, E \subseteq \bigcup_{j=1}^{\infty} A_j\right\}.$$

Exercise (Exercise 17)**:** If μ^* is an outer measure on X and $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*\left(E \cap \left(\bigsqcup_{j=1}^{n} A_j\right)\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$.

Solution: By the definition of measurability, we have

$$\begin{aligned}
\mu\left(E \cap \left(\bigsqcup_{j=1}^{\infty} A_j\right)\right) &= \mu\left(\left(E \cap \left(\bigsqcup_{j=1}^{\infty} A_j\right)\right) \cap A_1\right) + \mu\left(\left(E \cap \left(\bigsqcup_{j=1}^{\infty} A_j\right)\right) \cap A_1^c\right) \\
&= \mu(E \cap A_1) + \mu\left(E \cap \left(\bigsqcup_{j=2}^{\infty} A_j\right)\right).
\end{aligned}$$

Continuing in this pattern, we get

$$\mu\left(E \cap \left(\bigsqcup_{j=1}^{\infty} A_j\right)\right) = \sum_{j=1}^{\infty} \mu(E \cap A_j).$$

Exercise (Exercise 18)**:** Let $\mathcal{A} \subseteq P(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$, and let μ^* be the induced outer measure.

(a) For any $E \subseteq X$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$, $\mu^*(A) \leqslant \mu^*(E) + \varepsilon$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

(c) If μ_0 is σ-finite, then the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Solution:

(a) We know that

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) \;\middle|\; A_j \in \mathcal{A}, E \subseteq \bigcup_{j=1}^{\infty} A_j\right\},$$

meaning that, by the definition of infimum, for any $\varepsilon > 0$, there exists some sequence $\{A_j\}_{j=1}^{\infty}$ in $\mathcal{A}$ such that

$$\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \mu^*(E) + \varepsilon.$$

Defining $A = \bigcup_{j=1}^{\infty} A_j$, we have $A \in \mathcal{A}_\sigma$.

(b) Let $\mu^*(E) < \infty$.

Suppose E is measurable. Then, for any $T \subseteq X$, we have

$$\mu^*(T) = \mu^*(E \cap T) + \mu^*(E^c \cap T).$$

Chapter 3

Section 3.5

Definition. A function $F\colon \mathbb{R} \to \mathbb{C}$ is called *absolutely continuous* if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any finite set of disjoint open intervals $\left\{\left(a_j, b_j\right)\right\}_{j=1}^{N}$ with

$$\sum_{j=1}^{N}\left(b_j - a_j\right) < \delta,$$

we have

$$\sum_{j=1}^{N}\left|F\left(b_j\right) - F\left(a_j\right)\right| < \varepsilon.$$

Remark: All absolutely continuous functions are uniformly continuous.

Exercise (Exercise 36)**:** Let G be a continuous, increasing function on $[a, b]$, and let $G(a) = c$, $G(b) = d$.

(a) If $E \subseteq [c, d]$ is a Borel set, then $m(E) = \mu_G\left(G^{-1}(E)\right)$.

(b) If f is a Borel-measurable and integrable function on $[c, d]$, then

$$\int_c^d f(y)\,dy = \int_a^b f(G(x))\,dG(x).$$

If G is absolutely continuous, then

$$\int_a^b f(y)\,dy = \int_a^b f(G(x))G'(x)\,dx$$

(c) The validity of (b) may fail if G is merely right-continuous.

Solution:

(a) We may start by assuming that E is a closed subinterval of $[c, d]$, which we call $[\alpha, \beta]$, with $\alpha \geqslant c$ and $\beta \leqslant d$. Then, $m(E) = \beta - \alpha$, and

$$\begin{aligned} \mu_G\left(G^{-1}[\alpha, \beta]\right) &= G\left(G^{-1}(\beta)\right) - G\left(G^{-1}(\alpha)\right) \\ &= \beta - \alpha. \end{aligned}$$

Using countability, we may apply this to all Borel sets.

(b) We start with the borel set $E \subseteq [c, d]$, and its corresponding indicator function, giving

$$\begin{aligned} \int_c^d \mathbb{1}_E(y)\,dy &= m(E) \\ &= \mu_G\left(G^{-1}(E)\right) \\ &= \int_a^b \mathbb{1}_E(G(x))\,dG(x). \end{aligned}$$

By linearity, this applies to simple functions, and by Monotone Convergence, to all integrable $f\colon [c, d] \to \mathbb{C}$.

Furthermore, by Lebesgue differentiation and the Lebesgue–Radon–Nikodym theorem, we also have

$$\int_a^b f(G(x))\,dG(x) = \int_a^b f(G(x))\frac{dG}{dx}\,dx.$$

Exercise (Exercise 37)**:** Let $F\colon \mathbb{R} \to \mathbb{C}$. There is a constant M such that $|F(x) - F(y)| \leqslant M|x - y|$ for all $x, y \in \mathbb{R}$ (i.e., F is Lipschitz) if and only if F is absolutely continuous and $|F'| \leqslant M$ almost everywhere.

Solution: Let F be Lipschitz. Then, setting $\delta = \varepsilon/M$, we see that F is absolutely continuous, and since

$$\sup_{x,y\in\mathbb{R}} \frac{|F(y) - F(x)|}{|y - x|} \leqslant M,$$

with the left-hand side including $|F'(x)|$, we have that $|F'| \leqslant M$ almost everywhere.

Meanwhile, if F is absolutely continuous with bounded derivative, we see that for almost every $x < y$, there is $c \in (x, y)$ such that $F'(c) = \frac{F(y)-F(x)}{y-x}$, and

$$\begin{aligned}\left|F'(c)\right| &= \frac{|F(y) - F(x)|}{|y - x|}\\ &\leqslant M,\end{aligned}$$

so by taking suprema, we have

$$\sup_{x,y\in\mathbb{R}} \frac{|F(y) - F(x)|}{y - x} \leqslant M.$$