

# Quasi-Review: Locally Compact Groups and the Banach $*$ -algebra $L_1(G)$

## Basic Properties of Topological Groups

A topological group is a group  $G$  equipped with a topology such that the operations

$$\begin{aligned}(x, y) &\mapsto xy \\ x &\mapsto x^{-1}\end{aligned}$$

are continuous. In general, we will let  $1$  denote the identity of  $G$ .

We call  $G$  a locally compact group if the topology of  $G$  is locally compact. Equivalently, the topology of  $G$  is locally compact if there is a neighborhood system about  $1$  consisting of pre-compact open sets.

We will refer to the following subset operations in  $G$  regularly:

$$\begin{aligned}Ax &= \{ax \mid a \in A\} \\ xA &= \{xa \mid a \in A\} \\ A^{-1} &= \{a^{-1} \mid a \in A\} \\ AB &= \{ab \mid a \in A, b \in B\}.\end{aligned}$$

A subset  $V$  is called *symmetric* if  $V = V^{-1}$ .

These are some useful propositions.

**Proposition:** Let  $G$  be a topological group.

- (i) The topology of  $G$  is invariant under translations and inversion.
- (ii) For every neighborhood  $U$  of  $1$ , there is a symmetric neighborhood  $V$  of  $1$  such that  $VV \subseteq U$ .
- (iii) If  $H$  is a subgroup of  $G$ , then so is  $\overline{H}$ .
- (iv) Every open subgroup of  $G$  is closed.
- (v) If  $A$  and  $B$  are compact subsets of  $G$ , then so is  $AB$ .

**Proposition:** Suppose  $H$  is a subgroup of the topological group  $G$ .

- (i) If  $H$  is closed, then  $G/H$  is Hausdorff.
- (ii) If  $G$  is locally compact, then so is  $G/H$ .
- (iii) If  $H$  is normal, then  $G/H$  is a topological group.

We will assume all the time that  $G$  is Hausdorff, via the following proposition.

**Corollary:** If  $G$  is a T1 topological group, then  $G$  is Hausdorff. If  $G$  is not T1, then  $\overline{\{1\}}$  is a closed normal subgroup with  $G/\overline{\{1\}}$  is a Hausdorff topological group.

**Proposition:** Every locally compact group  $G$  has a subgroup  $G_0$  that is open, closed, and  $\sigma$ -compact.

Considering various functions  $f: G \rightarrow \mathbb{C}$ , we define the left and right translates of  $f$  as

$$\begin{aligned}L_y f(x) &= f(y^{-1}x) \\ R_y f(x) &= f(xy),\end{aligned}$$

and say that  $f$  is left (right) uniformly continuous if  $\|L_y f - f\|_u \rightarrow 0$  ( $\|R_y f - f\|_u \rightarrow 0$ ) as  $y \rightarrow 1$ .

**Proposition:** If  $f \in C_c(G)$ , then  $f$  is left and right uniformly continuous.

A left *Haar measure* is a nonzero Radon measure  $\mu$  on  $G$  such that  $\mu(xE) = \mu(E)$  for every Borel subset  $E \subseteq G$ .

**Proposition:** Every locally compact group  $G$  admits a left Haar measure  $\lambda$ . This Haar measure is unique up to a constant multiple.

If we have a left Haar measure  $\lambda$ , then if we define

$$\lambda_x(E) = \lambda(Ex),$$

we have that  $\lambda_x$  is again a left Haar measure, so there is some number  $\Delta(x)$  such that  $\lambda_x = \Delta(x)\lambda$ , where  $\Delta(x)$  is independent of the original choice of  $\lambda$ .

The function  $\Delta: G \rightarrow (0, \infty)$  defined as such is known as the *modular function* of  $G$ .

**Proposition:** The function  $\Delta$  is a continuous homomorphism from  $G$  to  $\mathbb{R}_{>0}$ , and for any  $f \in L_1(\lambda)$ , we have

$$\int R_y f \, d\lambda = \Delta(y^{-1}) \int f \, d\lambda.$$

We call  $G$  *unimodular* if  $\Delta \equiv 1$ .

**Proposition:** If  $G/[G, G]$  is compact, then  $G$  is unimodular.

## Convolutions and $L_1(G)$

If  $G$  is a locally compact group, we let  $M(G)$  denote the space of complex-valued Radon measures on  $G$ . The convolution of two measures  $\mu, \nu \in M(G)$  is given as follows. If we let

$$I(\phi) = \iint \phi(xy) \, d\mu(x) d\nu(y),$$

then we observe that  $I(\phi)$  is a linear functional on  $C_0(G)$  that satisfies

$$|I(\phi)| \leq \|\phi\|_u \|\mu\| \|\nu\|,$$

meaning that it is given by a measure  $\mu * \nu \in M(G)$  with  $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ . We call  $\mu * \nu$  the convolution of  $\mu$  and  $\nu$ .

Observe that if  $\delta_x \in M(G)$  is the point mass at  $x \in G$ , then

$$\begin{aligned} \int \phi \, d(\delta_x * \delta_y) &= \iint \phi(uv) \, d\delta_x(u) d\delta_y(v) \\ &= \phi(xy) \\ &= \int \phi \, d\delta_{xy}, \end{aligned}$$

meaning that  $\delta_x * \delta_y = \delta_{xy}$ .

The estimate  $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$  gives that convolution makes  $M(G)$  a Banach algebra, which we call the *measure algebra* of  $G$ . Furthermore,  $M(G)$  admits an involution

$$\mu^*(E) = \overline{\mu(E^{-1})},$$

so that

$$\int \phi \, d\mu^* = \int \phi(x^{-1}) \, d\overline{\mu(x)}.$$

We may identify the space  $L_1(G)$  to be the subspace of  $M(G)$  where a function  $f$  is identified with the measure  $f(x)dx$ . If  $f, g \in L_1(G)$ , then the convolution of  $f$  and  $g$  is the function

$$f * g(x) = \int f(y)g(y^{-1}x) \, dy.$$

With convolution and the involution given by

$$\begin{aligned} f^*(x)dx &= \overline{f(x^{-1})}d(x^{-1}) \\ f^*(x) &= \Delta(x^{-1})\overline{f(x^{-1})}, \end{aligned}$$

we have that  $L_1(G)$  is a Banach  $*$ -algebra known as the *group algebra* of  $G$ .

Now, we observe that if  $G$  is discrete, then if  $\delta_e$  is the point mass at 1, we have that  $f * \delta = \delta * f = f$  for any function  $f$ . If  $G$  is not discrete, we must use an *approximate identity* for  $G$ . In particular, we can select a family of mollifiers  $\{\psi_U\}_{U \in \mathcal{U}}$  such that

$$\begin{aligned} \|\psi_U * f - f\| &\rightarrow 0 \\ \|f * \psi_U - f\| &\rightarrow 0 \end{aligned}$$

if  $f$  is uniformly continuous and  $U \rightarrow \{1\}$  in a neighborhood system  $\mathcal{U}$  of 1.

## Homogeneous Spaces

If  $G$  is a locally compact group, then  $G$  can act on a locally compact Hausdorff space by homeomorphisms. Recall from algebra that the group action is transitive if there is one orbit. We call  $S$  a  $G$ -space.

The standard example of a transitive  $G$ -space is the quotient space  $G/H$  for some closed subgroup  $H$  of  $G$ . These are, to an extent, the only  $G$ -spaces, as follows from the orbit-stabilizer theorem. If  $S$  is a  $G$ -space, then we may define a map  $\phi: G \rightarrow S$  by  $\phi(x) = x \cdot s_0$ , and take the quotient by the stabilizer subgroup

$$H = \{x \in G \mid x \cdot s_0 = s_0\},$$

so that  $\Phi: G/H \rightarrow S$  has  $\Phi \circ q = \phi$  for the quotient map  $q: G \rightarrow G/H$  is a continuous bijection.

**Proposition:** If  $G$  is  $\sigma$ -compact, then  $\Phi$  is a homeomorphism.

*Proof.* It suffices to show that  $\phi$  maps open sets in  $G$  to open sets in  $S$ . Suppose  $U$  is open in  $G$ ,  $x_0 \in U$ . Pick a compact symmetric neighborhood  $V$  of 1 such that  $x_0 V V \subseteq U$ . Since  $G$  is  $\sigma$ -compact, there is a countable  $\{y_n\}_{n \geq 1} \subseteq G$  such that  $\{y_n V\}_{n \geq 1}$  covers  $G$ . Then, we have

$$S = \bigcup_{n=1}^{\infty} \phi(y_n V).$$

The sets  $\phi(y_n V)$  are homeomorphic to  $\phi(V)$  since the map  $s \mapsto y_n \cdot s$  is a homeomorphism of  $S$ , and all the  $y_n V$  are compact, hence closed.

By Baire Category Theorem for LCH spaces, it follows that  $\phi(V)$  has an interior point, which we call  $\phi(x_1)$  for  $x_1 \in V$ . Then,  $\phi(x_0)$  is an interior point of  $\phi(x_0 x_1^{-1} V)$ , and  $x_0 x_1^{-1} V \in x_0 V V \subseteq U$ , so that  $\phi(x_0)$  is an interior point of  $\phi(U)$ . Thus  $\phi(U)$  is open.  $\square$

If  $S$  is a transitive  $G$ -space that is isomorphic to a quotient space  $G/H$ , then will write  $S \cong G/H$ , and call  $S$  a *homogeneous space*. The identification is dependent on the choice of base point, but the identity  $s'_0 = x_0 \cdot x_0$  induces a map  $H' = x_0 H x_0^{-1}$ , inducing a  $G$ -equivariant homeomorphism  $G/H \cong G/H'$ .

We will address the question of whether there is a  $G$ -invariant Radon measure on  $G/H$  — that is, a radon measure  $\mu$  such that  $\mu(xE) = \mu(E)$  for every  $x \in G$ .

We assume that  $G$  is a locally compact group with left Haar measure  $dx$ , a  $H$  is a closed subgroup of  $G$  with left Haar measure  $d\xi$ , and  $q: G \rightarrow G/H$  is the quotient map  $q(x) = xH$ , and  $\Delta_G, \Delta_H$  the corresponding modular functions.

Let  $P: C_c(G) \rightarrow C_c(G/H)$  be defined by

$$Pf(xH) = \int_H f(x\xi) d\xi.$$

This is well-defined by left-invariance of  $d\xi$ . If  $\phi \in C(G/H)$ , then

$$P((\phi \circ q) \cdot f) = \phi \cdot Pf.$$

**Lemma:** If  $E \subseteq G/H$  is compact, then there is a compact  $K \subseteq G$  with  $q(K) = E$ .

*Proof.* Let  $V$  be an open neighborhood of 1 in  $G$  with compact closure. Since  $q$  is an open map,  $q(xV)$  is an open cover of  $E$ , so there is a finite subcover  $q(x_j V)$ . Let  $K = q^{-1}(E) \cap \bigcup_{j=1}^n x_j \bar{V}$ . Then, since  $q^{-1}(E)$  is closed,  $K$  is compact with  $q(K) = E$ .  $\square$

**Lemma:** If  $F \subseteq G/H$  is compact, then there is  $f \geq 0$  in  $C_c(G)$  with  $Pf = 1$  on  $F$ .

*Proof.* Let  $E$  be a compact neighborhood of  $F$  in  $G/H$ . We find  $K \subseteq G$  compact such that  $q(K) = E$ . Select positive  $g \in C_c(G)$  with  $g > 0$  on  $K$ , and  $\phi \in C_c(G/H)$  supported in  $E$  with  $\phi = 1$  on  $F$ . Set

$$f = \frac{\phi \circ q}{Pg \circ q} g,$$

with the fraction equal to zero whenever the numerator is zero. We have  $f$  is continuous, since  $Pg > 0$  on  $\text{supp}(\phi)$ , has support contained in  $\text{supp}(g)$ , and  $Pf = \frac{\phi}{Pg} Pg = \phi$ .  $\square$

**Proposition:** If  $\phi \in C_c(G/H)$ , then there exists  $f \in C_c(G)$  with  $Pf = \phi$  and  $q(\text{supp}(f)) = \text{supp}(\phi)$ , and has  $f \geq 0$  if  $\phi \geq 0$ .

*Proof.* If  $\phi \in C_c(G)$ , then by the previous lemma, then there exists  $g \geq 0$  in  $C_c(G)$  with  $Pg = 1$  on  $\text{supp}(\phi)$ . Letting  $f = (\phi \circ q)g$ , then  $Pf = \phi(Pg) = \phi$ .  $\square$

**Theorem:** Let  $G$  be a locally compact group,  $H$  a closed subgroup. There is a  $G$ -invariant Radon measure  $\mu$  on  $G/H$  if and only if  $\Delta_G|_H = \Delta_H$ . In this case, we have

$$\begin{aligned} \int_G f(x) dx &= \int_{G/H} Pf d\mu \\ &= \int_{G/H} \int_H f(x\xi) d\xi d\mu(xH) \end{aligned}$$

for any  $f \in C_c(G)$ .

*Proof.* Suppose there is a  $G$ -invariant measure  $\mu$ . Then,  $f \mapsto \int Pf d\mu$  is a nonzero left-invariant positive linear functional on  $C_c(G)$ , so by the uniqueness of Haar measure, we have  $\int Pf d\mu = c \int f(x) dx$  for some  $c$ .

This formula fully determines  $\mu$ , meaning that  $\mu$  is unique up to the arbitrary constant factor in Haar measure. We may assume that  $c = 1$ , so we have for any  $\eta \in H$ ,

$$\begin{aligned} \Delta_G(\eta) \int_G f(x) dx &= \int_G f(x\eta^{-1}) dx \\ &= \int_{G/H} \int_H f(x\xi\eta^{-1}) d\xi d\mu(xH) \\ &= \Delta_H(\eta) \int_{G/H} \int_H f(x\xi) d\xi d\mu(xH) \\ &= \Delta_H(\eta) \int_G f(x) dx, \end{aligned}$$

so that  $\Delta_G(\eta) = \Delta_H(\eta)$ .  $\square$

## Representations of a Group and its Group $*$ -Algebra

If  $G$  is a locally compact group, then a *unitary representation* of  $G$  is a homomorphism  $\pi: G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ , where  $\mathcal{U}(\mathcal{H}_\pi)$  denotes the unitary group of a Hilbert space  $\mathcal{H}_\pi$ . We call  $\mathcal{H}_\pi$  the *representation space* of  $\pi$ , and the dimension of  $\mathcal{H}_\pi$  is called the dimension (or degree) of the representation.

We do not require  $\pi$  to be continuous in the norm topology of  $\mathcal{B}(\mathcal{H}_\pi)$ , but as it turns out, both weak and strong continuity are equivalent as the WOT and SOT coincide on  $\mathcal{U}(\mathcal{H}_\pi)$ . If  $(T_\alpha)_\alpha \rightarrow T$  is a net of unitary operators converging in WOT, then for any  $u \in \mathcal{H}_\pi$ , we have

$$\begin{aligned}\|(T_\alpha - T)u\|^2 &= \|T_\alpha u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu \rangle + \|Tu\|^2 \\ &= \|u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu \rangle,\end{aligned}$$

and the latter term converges to  $2\|Tu\|^2 = 2\|u\|^2$ , so that  $\|T_\alpha u - Tu\| \rightarrow 0$ .

If  $G$  acts on a locally compact Hausdorff space  $S$ , then  $G$  acts on  $C(S)$  by

## Functions of Positive Type

## References

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