I have not shown most of the extraneous work because it is tedious to show.

Solution (12.1, Problem 2)**:** Separating with $u = X(x)Y(y)$, we have

$$Y\frac{dX}{dx} + 3X\frac{dY}{dy} = 0,$$

so that

$$\begin{aligned}\frac{dX}{dx} &= CX\\ \frac{dY}{dy} &= -\frac{C}{3}Y,\end{aligned}$$

meaning

$$u(x,y) = Ke^{Cx-\frac{C}{3}y}.$$

Solution (12.1, Problem 4)**:** Separating by taking $u(x,y) = X(x)Y(y)$, we have

$$\frac{1}{X}\left(\frac{dX}{dx}\right) = \frac{1}{Y}\left(\frac{dY}{dy}\right) + 1.$$

Therefore, this equation splits into

$$\begin{aligned}\frac{dX}{dx} &= CX\\ \frac{dY}{dy} &= (C-1)Y,\end{aligned}$$

yielding the solution of

$$u(x,y) = Ke^{Cx+(C-1)y}.$$

Solution (12.1, Problem 10)**:** Separating with $u(x,t) = X(x)T(t)$, we have

$$kT(t)\frac{d^2X}{dx^2} = X(t)\frac{dT}{dt},$$

so that

$$\frac{k}{X}\left(\frac{d^2X}{dx^2}\right) = \frac{1}{T}\left(\frac{dT}{dt}\right).$$

Setting these quantities equal to C, we have

$$u(x,t) = \begin{cases} e^{Ct}\left(A\cos\left(\sqrt{\frac{-C}{k}}x\right) + B\cos\left(\sqrt{\frac{-C}{k}}x\right)\right) & C < 0\\ e^{Ct}\left(Ae^{\sqrt{\frac{C}{k}}x} + Be^{-\sqrt{\frac{C}{k}}x}\right) & C > 0\\ Ax + B & C = 0.\end{cases}$$

Solution (12.1, Problem 12)**:** Separating with $u(x,t) = X(x)T(t)$, we get

$$\frac{a^2}{X}\left(\frac{d^2X}{dx^2}\right) = \frac{1}{T}\left(\frac{d^2T}{dt^2} + 2k\frac{dT}{dt}\right).$$

Setting equal to C and going through tedious algebra, we have the solution

$$u(x,t) = \begin{cases} \left(a_1 e^{\left(-k+\sqrt{k^2+C}\right)t} + a_2 e^{\left(-k+\sqrt{k^2+C}\right)t}\right)\left(b_1 e^{\frac{\sqrt{C}}{a}x} + b_2 e^{-\frac{\sqrt{C}}{a}x}\right) & c > 0 \\ \left(a_1 e^{\left(-k+\sqrt{k^2+C}\right)t} + a_2 e^{\left(-k+\sqrt{k^2+C}\right)t}\right)(Ax+B) & C = 0 \\ \left(a_1 e^{\left(-k+\sqrt{k^2+C}\right)t} + a_2 e^{\left(-k+\sqrt{k^2+C}\right)t}\right)\left(b_1 \cos\left(\sqrt{\frac{-C}{a}}x\right) + b_2 \sin\left(\sqrt{\frac{-C}{a}}x\right)\right) & -k^2 < C < 0 \\ \left(a_1 e^{-kt} + a_2 t e^{-kt}\right)\left(b_1 \cos\left(\sqrt{\frac{-C}{a}}x\right) + b_2 \sin\left(\sqrt{\frac{-C}{a}}x\right)\right) & C = -k^2 \\ e^{-kt}\left(a_1 \cos\left(\sqrt{\left|k^2+c\right|}x\right) + a_2 \sin\left(\sqrt{\left|k^2+c\right|}x\right)\right)\left(b_1 \cos\left(\sqrt{\frac{-C}{a}}x\right) + b_2 \sin\left(\sqrt{\frac{-C}{a}}x\right)\right) & C < -k^2 \end{cases}$$

Solution (12.1, Problem 18)**:** Since $B = 5$, $A = 3$, and $C = 1$, this is a hyperbolic PDE.

Solution (12.2, Problem 2)**:** The boundary value problem is

$$\begin{aligned} u(x,0) &= 0 \\ u(0,t) &= u_0 \\ u(L,t) &= u_1. \end{aligned}$$

Solution (12.2, Problem 4)**:** The boundary value problem is

$$\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_{(0,t)} &= 0 \\ \left.\frac{\partial u}{\partial x}\right|_{(L,t)} &= 0 \\ u(x,0) &= 100 \\ \left.\frac{\partial u}{\partial t}\right|_{(x,t)} &= -50. \end{aligned}$$

Solution (12.2, Problem 6)**:** The boundary value problem is

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sin(\pi x/L) \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= 0. \end{aligned}$$

Solution (11.1, Problem 2)**:** We evaluate

$$\begin{aligned} \langle f_1, f_2 \rangle &= \int_{-1}^{1} \left(x^3\right)\left(x^2+1\right)\,dx \\ &= \int_{-1}^{1} x^5 + x^3\,dx \\ &= 0, \end{aligned}$$

by even/odd rules.

Solution (11.1, Problem 4)**:** We evaluate

$$\begin{aligned} \langle f_1, f_2 \rangle &= \int_0^{\pi} \cos(x)\sin^2(x)\,dx \\ &= -\int_0^0 u^2\,dx && u = \sin(x) \\ &= 0. \end{aligned}$$

Solution (11.1, Problem 10)**:** We evaluate

$$\begin{aligned}\left\langle \sin\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right)\right\rangle &= \int_0^p \sin\left(\frac{n\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)\,dx\\ &= \int_0^\pi \sin(nt)\sin(mt)\,dt\\ &= \begin{cases}0 & m \neq n\\ \frac{\pi}{2} & m = n\end{cases}.\end{aligned}$$

Solution (11.1, Problem 12)**:** For two separate "classes" of functions, we have

$$\begin{aligned}\int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)(1)\,dx &= 0\\ \int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right)(1)\,dx &= 0\\ \int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx &= 0\\ \int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx &= 0.\end{aligned}$$

Furthermore, for two members of the same "class" of functions with different m, n, we know that

$$\begin{aligned}\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\cos\left(\frac{m\pi x}{p}\right)\,dx &= \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx\\ &= 0\\ \int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)\,dx &= \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx\\ &= 0.\end{aligned}$$

Evaluating norms, we get

$$\begin{aligned}\int_{-p}^{p} \sin^2\left(\frac{n\pi x}{p}\right)\,dx &= p\\ \int_{-p}^{p} \cos^2\left(\frac{n\pi x}{p}\right)\,dx &= p\\ \int_{-p}^{p} \,dx &= 2p\end{aligned}$$

Solution (Extra Problem)**:**

(i) We recognize this as the transport equation with $a = -3$, so the solution is

$$u(x,t) = \ln(x + 3t - 1),$$

with

$$\begin{aligned}u(3,40) &= \ln(122)\\ u(40,3) &= \ln(48).\end{aligned}$$

(ii) We use separation of variables to solve the heat equation, taking $u(x,t) = X(x)T(t)$. After some tedious algebra, we get

$$\frac{1}{T}\left(\frac{dT}{dt}\right) = \frac{2}{X}\left(\frac{d^2X}{dx^2}\right)$$

$$= \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}.$$

In the case with λ^2, we get $u = e^{\lambda^2 t}\left(Ae^{\lambda/\sqrt{2}x} + Be^{-\lambda/\sqrt{2}x}\right)$, which does not satisfy the boundary conditions.

Similarly, in the case with 0, we get $u = Ax + B$, which only satisfies the boundary conditions when $u = 0$, and does not satisfy the initial conditions.

Therefore, taking the case of $-\lambda^2$, we have

$$\begin{aligned} X &= A\sin\left(\frac{\lambda}{\sqrt{2}}x\right) + B\cos\left(\frac{\lambda}{\sqrt{2}}x\right) \\ T &= Ce^{-\lambda^2 t}. \end{aligned}$$

Plugging in our boundary conditions, we get that $\lambda \in \frac{1}{\sqrt{2}}\mathbb{Z}^+$ and $B = 0$, yielding

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2/2t}\sin\left(\frac{n}{2}x\right).$$

Finally, plugging in our initial condition, we get

$$\sin(2x) = \sum_{n=1}^{\infty} C_n e^{-n^2/2t}\sin\left(\frac{n}{2}x\right),$$

or that

$$u(x,t) = e^{-8t}\sin(2x).$$

(iii) Using separation of variables to solve the heat equation, we take $u(x,t) = X(x)T(t)$. After some algebra, we get

$$\begin{aligned} \frac{1}{T}\left(\frac{dT}{dt}\right) &= \frac{1}{X}\left(\frac{d^2X}{dx^2}\right) \\ &= \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}. \end{aligned}$$

Using a similar method as with (ii) to narrow down our possibilities, we get that $\lambda \in \pi\mathbb{Z}^+$, and

$$\begin{aligned} X_n &= A_n\cos(\pi n x) + B_n\sin(\pi n x) \\ T_n &= Ce^{-\pi^2 n^2 t}. \end{aligned}$$

Using the Neumann boundary condition, we get that $B_n = 0$ for all n, meaning

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-\pi^2 n^2 t}\cos(\pi n x).$$

Plugging in our initial condition, we get that $C_0 = 8$, $C_3 = -4$, and everything else is 0, so

$$u(x,t) = 8 - 4e^{-9\pi^2 t}\cos(3\pi x).$$

(iv) Using separation of variables on the wave equation, we write $u(x,t) = X(x)T(t)$, and get

$$\frac{1}{T}\left(\frac{d^2T}{dt^2}\right) = \frac{1}{X}\left(\frac{d^2X}{dx^2}\right)$$

$$= \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}.$$

As in (ii) and (iii), both λ^2 and 0 yield trivial solutions when we plug in the boundary conditions $u(0,t) = u(2,t) = 0$, so we are left with the form

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$
$$T(t) = C\cos(\lambda t) + D\sin(\lambda t).$$

By plugging in the boundary condition $u(0,t) = 0$, we get that $A = 0$, and by plugging in $u(2,t) = 0$, we get that $B\sin(2\lambda) = 0$, so $\lambda = \frac{\pi}{2}n$. Our solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{\pi}{2}nt\right) + D_n \sin\left(\frac{\pi}{2}nt\right)\right) B_n \sin\left(\frac{\pi}{2}nx\right).$$

Rewriting with different constants, we get

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi}{2}nx\right)\cos\left(\frac{\pi}{2}nt\right) + B_n \sin\left(\frac{\pi}{2}nx\right)\sin\left(\frac{\pi}{2}nt\right).$$

Now, plugging in our first initial condition, with $u(x,0) = \sin(2\pi x)$, we get $A_4 = 1$ and $A_{n\neq 4} = 0$. This gives the narrowed expression

$$u(x,t) = \sin(2\pi x)\cos(2\pi t) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi}{2}nx\right)\sin\left(\frac{\pi}{2}nt\right).$$

Using the second initial condition of $\frac{\partial u}{\partial t}\Big|_{(x,0)} = 0$, we get $B_n = \frac{1}{3\pi}$, so our particular solution is

$$u(x,t) = \sin(2\pi x)\cos(2\pi t) + \frac{1}{3\pi}\sin(3\pi x)\sin(3\pi t).$$