Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a nonempty open set. Given a sequence $(z_n)_n \subseteq U$, we write $z_n \to \partial U$ if, for every compact subset $K \subseteq U$, there exists some $N = N(K) \in \mathbb{N}$ such that $z_n \notin K$ whenever $n \geqslant N$.

Given a function $u \colon U \to \mathbb{R}$, define

$$\limsup_{z \to \partial U} u(z) = \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{z \in U \setminus K} u(z).$$

(a) For each positive integer $n \in \mathbb{N}$, define

$$K_n := \left\{ z \in U \;\middle|\; |z| \leqslant n, \operatorname{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{n} \right\}.$$

Show that:

(i) each K_n is compact;

(ii) $K_n \subseteq K_{n+1}^{\circ}$;

(iii) $U = \bigcup_{n=1}^{\infty} K_n$.

(b) Let $L := \limsup_{z \to \partial U} u(z)$.

(i) Show that for each $S > L$, there is a compact subset $K \subseteq U$ such that $u(z) \leqslant S$ for all $z \in U \setminus K$.

(ii) Show that there exists a sequence $(z_n)_n$ in U with $z_n \to \partial U$ and $\limsup_{n \to \infty} u(z_n) \geqslant L$.

(c) Prove that

$$\limsup_{z \to \partial U} u(z) = \inf_{\substack{(z_n)_n \subseteq U \\ z_n \to \partial U}} \limsup_{n \to \infty} u(z_n).$$

Solution:

(a) We claim that the set

$$C_n = \left\{ z \in U \;\middle|\; \operatorname{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{n} \right\}$$

is closed. Then, we observe that $K_n = B(0, n) \cap C_n$ would then be an intersection of a closed set with a closed and bounded set, hence a closed and bounded set, hence compact. Towards this end, suppose $(w_k)_k \subseteq C_n$ converges to $w \in \mathbb{C}$. Then, for each k, we have

$$\inf_{z \in \mathbb{C} \setminus U} |w_k - z| \geqslant \frac{1}{n}.$$

Observe then that for any $z \in \mathbb{C} \setminus U$, we have

$$|w_k - z| \geqslant \frac{1}{n}$$

for each k, meaning that

$$\lim_{k \to \infty} |w_k - z| \geqslant \frac{1}{n},$$

or that

$$|w - z| \geqslant \frac{1}{n}.$$

In particular, it must be the case that $w \in U$, and that

$$\inf_{z \in \mathbb{C} \setminus U} |w - z| \geqslant \frac{1}{n},$$

so that $w \in C_n$, and thus C_n is closed, and K_n is compact.

To see that $K_n \subseteq K_{n+1}^{\circ}$, we show that $C_n \subseteq C_{n+1}^{\circ}$ by understanding the picture of C_n°. Towards this end, we see that $z \in C_n^{\circ}$ if and only if $z \in U$ and there is some $r > 0$ such that $\operatorname{dist}_{\mathbb{C}\setminus U}(w) \geqslant \frac{1}{n}$ for all $w \in U(z, r)$.

Observe that if $\varepsilon > 0$, then if z satisfies $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant \frac{1}{n} + \varepsilon$, then if $w \in \mathbb{C} \setminus U$ and $\zeta \in U(z, \varepsilon/2)$, we have

$$\begin{aligned}
\frac{1}{n} + \varepsilon &\leqslant |z - w| \\
&\leqslant |z - \zeta| + |\zeta - w| \\
&< \varepsilon/2 + |\zeta - w|,
\end{aligned}$$

meaning that $|\zeta - w| \geqslant \frac{1}{n} + \varepsilon/2$ for all $w \in \mathbb{C} \setminus U$, so that $\operatorname{dist}_{\mathbb{C}\setminus U}(\zeta) \geqslant \frac{1}{n}$. In particular, this means that C_n° consists of all z for which there exists ε such that $\operatorname{dist}_{\mathbb{C}\setminus U}(z) \geqslant \frac{1}{n} + \varepsilon$, or more succinctly,

$$C_n^{\circ} = \left\{ z \in U \;\middle|\; \operatorname{dist}_{\mathbb{C}\setminus U}(z) > \frac{1}{n} \right\}.$$

In particular, since $\frac{1}{n} > \frac{1}{n+1}$, it follows that $C_n \subseteq C_{n+1}^{\circ}$. Paired with the fact that $B(0, n) \subseteq U(0, n+1)$, we obtain that

$$\begin{aligned}
K_n &= B(0, n) \cap C_n \\
&\subseteq U(0, n+1) \cap C_{n+1}^{\circ} \\
&= (B(0, n+1) \cap C_n)^{\circ} \\
&= K_{n+1}^{\circ}.
\end{aligned}$$