Amenability: A Not-Particularly-Brief Introduction

Avinash Iyer

Occidental College

February 2, 2025

Outline

Contents

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

then we call the pair $(A, \star)$ a *group*.

Groups

If A is a set, and $\star\colon A \times A \to A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

then we call the pair $(A, \star)$ a *group*.

We abbreviate $a \star b$ as ab.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.
- If $N \subseteq G$ is a subgroup that satisfies, for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say N is a *normal subgroup*.

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.
- If $N \subseteq G$ is a subgroup that satisfies, for all $g \in G$ and $h \in N$, $ghg^{-1} \in N$, then we say N is a *normal subgroup*.
- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_\sim$ known as the *quotient group* G/N.

Some Groups

- The integers $\mathbb{Z}$ are a group under addition.
- The group of invertible $n \times n$ matrices over $\mathbb{C}$, $\mathrm{GL}_n(\mathbb{C})$, is a group under matrix multiplication.
- The subgroup $\mathrm{SO}(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ consisting of orthogonal matrices is a group under multiplication.

Group Actions

Let G be a group, and X a set. Let $\rho \colon G \times X \to X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X. We write $\rho(g, x) = g \cdot x$.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathtt{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathtt{A}$;
2. for any $A_i \in \mathtt{A}$, $A_i^c \in \mathtt{A}$;
3. for any $A_i, A_j \in \mathtt{A}$, $A_i \cup A_j \in \mathtt{A}$.

σ-Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i\in I} = \mathtt{A} \subseteq P(X)$ is known as an *algebra* of subsets if

1. $\emptyset, X \in \mathtt{A}$;
2. for any $A_i \in \mathtt{A}$, $A_i^c \in \mathtt{A}$;
3. for any $A_i, A_j \in \mathtt{A}$, $A_i \cup A_j \in \mathtt{A}$.

If, for any countable collection, $\{A_n\}_{n\geq 1} \subseteq \mathtt{A}$, condition (3) holds, then we say $\mathtt{A}$ is a *σ-algebra* of subsets.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathtt{A}$ is a σ-algebra, then a map $\mu\colon \mathtt{A} \to [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathtt{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.

σ-Algebras and Measures, Cont'd

If X is a set and $\mathtt{A}$ is a σ-algebra, then a map $\mu \colon \mathtt{A} \to [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathtt{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

then we say μ is a *finitely additive* measure.If $\{A_n\}_{n \geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n)$,

we say μ is a measure.

Contents

Questions?

- If G is a group, is it possible to reconstruct G by using some subset of G?
- When may we find a finitely additive probability measure $\mu\colon P(G) \to [0,1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Contents

Contents