The primary text for Algebra II is Dummit and Foote's *Abstract Algebra*, and will cover the following topics:

- modules and advanced linear algebra;
- representation theory of finite groups;
- field theory and Galois theory.

Contents

Modules and Advanced Linear Algebra **1**
Tensor Products of Modules 1

Modules and Advanced Linear Algebra

Tensor Products of Modules

To motivate tensor products, we recall a basic fact from linear algebra. If we assume that R is a field, and M, N are finite-dimensional R-vector spaces, then the following equation necessarily holds:

$$\dim(M \oplus N) = \dim(M) + \dim(N).$$

We want to construct a similar operation on vector spaces, $M \otimes N$, that satisfies

$$\dim(M \otimes N) = \dim(M)\dim(N).$$

For now, we will label this by $M \bar{\otimes} N$, where we use the $\bar{\otimes}$ to refer to the fact that this is a temporary definition. Naively, we might seek to define $M \bar{\otimes} N$ as follows. If we let $\{x_1, \dots, x_k\}$ be a basis for M and $\{y_1, \dots, y_\ell\}$ a basis for N, then we will define $M \bar{\otimes} N$ to be all the formal R-linear combinations over the basis

$$B = \{x_i \otimes y_j \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}.$$

While this is technically correct — as in, this does yield a vector space with

$$\dim(M \bar{\otimes} N) = \dim(M)\dim(N),$$

the issue is that this definition is not canonical, in that it depends on chosen bases for M and N. Furthermore, it is not clear how one may generalize from this definition to modules over arbitrary rings, which do not necessarily have bases. To resolve this issue, we will go about defining a construction that "extends," in a sense, this definition of $M \bar{\otimes} N$.

To start, we define the simple tensor $m \otimes n$ for any $m \in M$ and $n \in N$. If we let

$$m = \sum_{i=1}^{k} \lambda_i x_i$$
$$n = \sum_{j=1}^{\ell} \mu_j y_j,$$

then we will define

$$m \otimes n = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \lambda_i \mu_j (x_i \otimes y_j).$$

We observe that every element of $M\bar{\otimes}N$ is a sum (i.e., an *integral* linear combination) of simple tensors, as by regrouping we may take

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}\lambda_{ij}(x_i\otimes y_j)=\sum_{i=1}^{k}\sum_{j=1}^{\ell}(\lambda_{ij}x_i)\otimes y_j.$$

The simple tensors satisfy the following relations:

(R1) $(m_1+m_2)\otimes n=m_1\otimes n+m_2\otimes n$;

(R2) $m\otimes(n_1+n_2)=m\otimes n_1+m\otimes n_2$;

(R3) $(\alpha m)\otimes n=m\otimes(\alpha n)$

for $m,m_1,m_2\in M$, $n,n_1,n_2\in N$, and $\alpha\in R$.

Proposition: These are the defining relations for $M\bar{\otimes}N$ in the category of abelian groups.

We will simply take this proposition as fact.

Now, let

$$\begin{aligned}Q&=M\times N\\&=\{(m,n)\mid m\in M,n\in N\}\end{aligned}$$

be the Cartesian product of M and N as sets. We will then take $\mathbb{Z}[Q]$ to be the standard free $\mathbb{Z}$-module (i.e., free abelian group) on Q. That is, $\mathbb{Z}[Q]$ is the set of formal linear combinations

$$v=\sum_{q\in Q}\lambda_q q,$$

where $\lambda_q\in\mathbb{Z}$ and only finitely many coefficients are nonzero. By the universal property of free abelian groups, the map $(m,n)\mapsto m\otimes n$ descends to a unique homomorphism $\varphi\colon\mathbb{Z}[Q]\to M\bar{\otimes}N$. Such a homomorphism is necessarily surjective as every element of $M\bar{\otimes}N$ is an integral linear combination of simple tensors, meaning that we have

$$M\bar{\otimes}N\cong\mathbb{Z}[Q]/\ker(\varphi)$$

as abelian groups.

Now, consider the subgroup of $\mathbb{Z}[Q]$, which we denote K, that is generated by the following elements:

(I) $(m_1+m_2,n)-(m_1,n)-(m_2,n)$;

(II) $(m,n_1+n_2)-(m,n_1)-(m,n_2)$;

(III) $(\alpha m,n)-(m,\alpha n)$

for $m_1,m_2,m\in M$, $n_1,n_2,n\in N$, and $\alpha\in R$. Then, from proposition that the relations (R1) through (R3) define $M\bar{\otimes}N$, it follows that $K=\ker(\varphi)$. Thus, we may define the tensor product canonically as follows.

Definition: Letting M,N,Q,K be as above, we define

$$M\otimes N\coloneqq\mathbb{Z}[Q]/K,\tag{†}$$

and define $m\otimes n=(m,n)+K$.

So far, this has only given us an abelian group. We may ask how to define $\mathbb{Z}[Q]/K$ as an R-vector space, which naturally seems to be defined by

$$r\left(\sum_{i=1}^{n}m_i\otimes n_i\right)=\sum_{i=1}^{n}(rm_i)\otimes n_i\tag{$*$}$$

To show that the right-hand side of $(*)$ is well-defined is a very difficult task. We will not do it here.

Now, we can actually quite easily generalize $(\dagger)$ to modules over non-fields.

- If R is a commutative ring with 1, and M and N are left R-modules, the definition in $(\dagger)$ copies over exactly.
- If R is non-commutative with 1, then the definition in $(\dagger)$ makes sense, but the scalar multiplication in $(*)$ does *not* hold.

  In fact, we need to change the assumptions for M and N as R-modules. In particular, we need M to be a *right* R-module, and N to be a left R-module, and take the generators of type (III) for K to be defined by

  (III') $(mr, n) - (m, rn)$

  for $m \in M$, $n \in N$, and $r \in R$. This gives the tensor product $M \otimes_R N$ an abelian group structure, but does not endow it with a R-module structure.