Normed Vector Spaces

Vector Spaces

Throughout, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **vector space** over $\mathbb{F}$ is a nonempty set V equipped with two operations: vector addition and scalar multiplication.

$$
\begin{aligned}
V \times V &\xrightarrow{+} V \\
(v, w) &\mapsto v + w && \text{Vector Addition} \\
F \times V &\to V \\
(\alpha, v) &\mapsto \alpha v && \text{Scalar Multiplication}
\end{aligned}
$$

The vector space is an Abelian group, where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have:

(i) $u + (v + w) = (u + v) + w$

(ii) $\exists 0_v \in V$ with $\forall v \in V,\ 0_v + v = v + 0_v = v$

(iii) $(\forall v \in V)(\exists w \in V)$ with $v + w = 0_v$

(iv) $\forall v, w \in V,\ v + w = w + v$

(v) $\alpha(v + w) = \alpha v + \alpha w,\ (\alpha + \beta)v = \alpha v + \beta v$

(vi) $\alpha(\beta w) = (\alpha\beta)w$

(vii) $1 \cdot v = v$

Remarks:

(a) 0_v is unique and known as the zero vector.

(b) The vector w in (iii) is unique, and denoted $-v$.

(c) $0 \cdot v = 0_v$

(d) $(-1) \cdot v = -v$

(e) Property (iv) follows from all the other axioms.

(f) For $n \in \mathbb{N}$, $n \cdot v = \underbrace{v + v + \cdots + v}_{n \text{ times}}$

Subspaces

Let V be a vector space over $\mathbb{F}$. A **subspace** is a nonempty subset $W \subseteq V$ satisfying the following:

(i) $w \in W, \alpha \in \mathbb{F} \to \alpha w \in W$.

(ii) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

Remark: 0_v is always a member of any subspace; a subspace is also a vector space.

Proposition: Intersection of Subspaces

If $\{W_i\}_{i \in I}$ is a family of subspaces of V, then, $\bigcap W_i$ is a subspace of V.

Proposition: Union of Subspaces

It is not the case that the union of subspaces of V also a subspace. For example, consider $\mathbb{R}^2$ with the traditional vector space operations:

$$
\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} x + x' \\ y + y' \end{pmatrix} \\
\alpha \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}
\end{aligned}
$$

If $W_1, W_2 \in V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Generated Subspaces

Let $S \subseteq V$ be any subset of a vector space V. Then,

$$\operatorname{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, v_1, \dots, v_n \in S \right\}$$

Remarks:

- $\operatorname{span}(S) \subseteq V$ is a subspace.
- $\operatorname{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. Thus, $\operatorname{span}(S)$ is the "smallest" subspace containing S, or the subspace generated by S.

Proposition: Quotient Group on Vector Space

Let V be a vector space, and let $W \subseteq V$ is a subspace. Define $u \sim_W v \leftrightarrow u - v \in W$.

(1) $\sim_W$ is an equivalence relation.

(2) If $[v]_W$ denotes the equivalence class of v, then $[v]_W = v + W = \{v + w | w \in W\}$.

(3) $V/W := \{[v]_W | v \in V\}$ is a vector space with $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ and $\alpha[v]_W = [\alpha v]_W$.

Proof of (1):

- Reflexive: $u \sim_W u$, since $u - u = 0 \in W$.
- Transitive: Suppose $u \sim_W v$, and $v \sim_W z$. Then, $u - v \in W$, and $v - z \in W$. So, $(u - v) + (v - z) \in W$, so $u - z \in W$. Whence, $u \sim_W z$.
- Symmetric: If $u \sim_W v$, then $u - v \in W$, so $-1 \cdot (u - v) \in W$, so $v - u \in W$. Whence, $v \sim_W u$.

Proof of (2):

$$\begin{aligned}
[v]_W &= \{u \in V \mid u \sim_W v\} \\
&= \{u \in V \mid u - v \in W\} \\
&= \{u \in V \mid u = v + w \text{ some } w \in W\} \\
&= \{v + w \mid w \in W\} \\
&= v + W
\end{aligned}$$

Proof of (3): Prove that the operation is well-defined.

Bases

Let V be a vector space and $S \subseteq V$ be a subset.

(1) S is said to be spanning for V if $\operatorname{span}(S) = V$.

(2) S is linearly independent if, for $\sum_{j=1}^{n} \alpha_j v_j = 0_v$ with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

(3) S is a basis for V if S is linearly independent and spanning for V.

Proposition: Existence of Basis

Every vector space admits a basis. If $B_0 \subseteq V$ is linearly independent, $\exists B \subseteq V$ such that B is a basis and $B \supseteq B_0$.

Background: A relation on a set X is a subset $R \subseteq X \times X$. If R is reflexive ($x \sim x$), transitive ($x \sim y, y \sim z \to x \sim z$), and antisymmetric ($x \sim y, y \sim x \to x = y$), then R is an ordering, and we write $x \leq y$.

If $\leq$ is an ordering of X such that $\forall x, y \in X,\ x \leq y$ or $y \leq x$, then $\leq$ is a total (or linear) ordering.

Let $\leq$ be an ordering of X, let $Y \subseteq X$. An upper bound for Y is an element $u \in X$ such that $y \leq u\ \forall y \in Y$. A maximal element in X is an element $m \in X$ such that $x \in X,\ x \geq m \to x = m$.

Example: $\mathbb{N}$ under the division ordering defines $a \leq b \Leftrightarrow a|b$. If we want to find the maximal elements of $A = \{2, 6, 9, 12\}$, we would see that they are 9 and 12 (since no element of A can be divided by 9 and 12). Meanwhile, $\mathbb{N}$ itself has no maximal elements.

This leads us to ask: given an ordered set, $(X, \leq)$, does X admit maximal elements.

Zorn's Lemma (or Axiom): Let $(X, \leq)$ be an ordered set. Suppose that every totally ordered subset, $Y \subseteq X$ has an upper bound in X. Then, X admits at least one maximal element.

The proof of Zorn's Lemma relies on the Axiom of Choice (and Zorn's Lemma is equivalent to the Axiom of Choice).

Proof: Let $X = \{D \mid B_0 \subseteq D \subseteq V\}$ with D linearly independent. Since $B_0 \subseteq X$, $X \neq \emptyset$. Define $D, E \in X$, $D \leq E \Leftrightarrow D \subseteq E$. We will show that X has a maximal element.

Consider any totally ordered subset, $Y = \{D_i\}_{i \in I}$. Consider $D = \bigcup D_i$. Clearly, $B_0 \subseteq D \subseteq V$. Suppose $\sum \alpha_k v_k = 0_v$ with $v_1, \dots, v_n \in D$. Therefore, $\exists D_j$ with $v_1, \dots, v_n \in D_j$ because Y is totally ordered. However, by definition, D_j is a linearly independent set — therefore, $\alpha_k = 0$. Thus, D is linearly independent.

Since D is linearly independent, and $B_0 \subseteq D$, it must be the case that $D \in X$. D is also an upper bound for Y. So, by Zorn's Lemma, X has a maximal element, B.

So, $B_0 \subseteq B \subseteq V$, B is independent, and B is maximal in X. We claim that B is a basis for V. Suppose toward contradiction that $\exists v \in V$ such that $v \notin \text{span}(B)$. Consider $B' = B \cup \{v\}$.

Then, $B_0 \subseteq B'$, and B' is linearly independent — if $\sum \alpha_k v_k + \alpha v = 0$, where $v_1, \dots, v_n \in B$, then either:

- If $\alpha = 0$, then $\alpha_k v_k = 0 \Rightarrow \alpha_k = 0$.
- If $\alpha \neq 0$, then $\sum \alpha_k v_k = -\alpha v$, which means $v \in \text{span}(B)$. $\perp$

Thus, we have a linearly independent set, B', with $B \subseteq B'$, and $B_0 \subseteq B'$. Therefore, $B' \in X$. However, this contradicts the maximality of B. Therefore, $\text{span}(B) = V$, and B is a basis for V.

Examples: Vector Spaces

(1) n-Dimensional Vectors:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_j \in \mathbb{F} \right\}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y+n \end{pmatrix} = \begin{pmatrix} x_1 + y + 1 \\ \vdots \\ x_n + y + n \end{pmatrix}$$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$B = \{e_1, \dots, e_n\}$$

where e_i denotes the unit vector at position i.

(2) $m \times n$ Matrices:

$$\mathbb{M}_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{F} \right\}$$

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$\alpha(a_{ij}) = (\alpha a_{ij})$$

$$B = \{e_{ij}\}$$

where e_{ij} denotes a matrix of 0 everywhere except column i and row j.

(3) Functions with domain Ω:

$$\begin{aligned}\mathcal{F}(\Omega,\mathbb{F}) &= \{f \mid f : \Omega \to \mathbb{F}\}\\ (f+g)(x) &= f(x)+g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

(4) Bounded functions with domain Ω:

$$\begin{aligned}\ell_\infty(\Omega,\mathbb{F}) &= \{f \in \mathcal{F}(\Omega,\mathbb{F}) \mid \|f\|_u \leq \infty\}\\ \|f\|_u &= \sup_{x\in\Omega}|f(x)|\end{aligned}$$

Exercises:

- Triangle Inequality: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$
- Scalar Multiplication/Absolute Homogeneity: $\|\alpha f\|_u = |\alpha|\|f\|_u$
- Positive Definite: $\|f\|_u = 0 \Rightarrow f = \mathbb{0}$

Proof of Triangle Inequality: Given $x \in \Omega$,

$$\begin{aligned}|(f+g)(x)| &= |f(x)+g(x)|\\ &\leq |f(x)|+|g(x)|\\ &\leq \|f\|_u + \|g\|_u\end{aligned}$$

Therefore,

$$\begin{aligned}\sup|(f+g)(x)| &\leq \|f\|_u + \|g\|_u\\ \|f+g\|_u &\leq \|f\|_u + \|g\|_u\end{aligned}$$

(5) Continuous functions on closed and bounded intervals:

$$C([a,b],\mathbb{F}) = \{f : [a,b] \to \mathbb{F} \mid f \text{ continuous}\}$$

Check that $C([a,b],\mathbb{F}) \subseteq \ell_\infty([a,b],\mathbb{F})$ is a subspace.

(6) Let $f : [a,b] \to \mathbb{R}$ be any function. Let $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

$$\begin{aligned}\operatorname{var}(f;\mathcal{P}) &:= \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\\ \operatorname{var}(f) &= \sup_{\mathcal{P}}\operatorname{var}(f;\mathcal{P})\\ \mathrm{BV}([a,b]) &= \{f : [a,b]\to\mathbb{R} \mid \operatorname{var}(f) < \infty\}\\ \|f\|_{\mathrm{BV}} &= |f(a)| + \operatorname{var}(f)\end{aligned}$$

$\mathrm{BV}([a,b])$ is a vector space.

Question: Is $\mathbb{1}_{\mathbb{Q}} \in \mathrm{BV}([0,1])$?

(7) Suppose $K \subseteq V$ is a *convex* subset of a vector space: $v,w \in K, t \in [0,1] \Rightarrow (1-t)v + tw \in K$. Let $\operatorname{Aff}(K) = \{f : K \to \mathbb{R} \mid f \text{ is affine}\}$, where f is affine if $\forall v,w \in K, t \in [0,1], f((1-t)v+tw) = (1-t)f(v)+tf(w)$.

Exercise: Show that $\operatorname{Aff}(K) \subseteq \mathcal{F}(K,\mathbb{R})$ is a subspace.

(8) Let S be defined as

$$S = \{(a_k)_{k=1}^{\infty} \mid a_k \in \mathbb{F}\}.$$

Under pointwise operations, S is a vector space.

$$\begin{aligned}(a_k)_k + (b_k)_k &= (a_k+b_k)_k\\ \alpha(a_k)_k &= (\alpha a_k)_k\end{aligned}$$

Note 1: $S = \mathcal{F}(\mathbb{N},\mathbb{F})$.

Note 2: $c_{00} \subseteq \ell_1 \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq S$.

- $c_{00} = \{(a_k)_k \mid \text{finitely many } a_k \neq 0\}$
- $c_0 = \{(a_k)k \mid (a_k)_k \to 0\}$

- $c = \{(a_k)_k \mid (a_k)_k \to a < \infty\}$
- $\ell_\infty = \{(a_k)_k \mid \|(a_k)_k\|_u < \infty\}$
- $\ell_1 = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k| = a < \infty\right\}$

(9) $C_C(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subspaces.

- $C_C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ compactly supported}\}$: $f : \mathbb{R} \to \mathbb{F}$ is compactly supported if $\exists [a, b]$ such that $x \notin [a, b] \Rightarrow f(x) = 0$.
- $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ continuous, } \lim_{x \to \pm\infty} f(x) = 0\}$

(10) Let S be any non-empty set.

$$\mathbb{F}(S) := \{f : S \to \mathbb{F} \mid f \text{ finitely supported}\}$$
$$\text{supp}(f) = \{x \in S \mid f(x) \neq 0\}$$

We claim that $\mathbb{F}(S) \subseteq \mathcal{F}(S, \mathbb{F})$ is a subspace. Consider $e_t : S \to \mathbb{F}$ defined as follows:

$$e_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}.$$

We claim that $\xi = \{e_t\}_{t \in S}$ is a basis for $\mathbb{F}(S)$.

Indeed, given $f \in \mathbb{F}(S)$, we know that $\text{supp}(f) = \{t_1, \dots, t_n\} \subseteq S$. Therefore, $f = \sum_{k=1}^{n} f(t_k) e_{t_k} \in \text{span}(\xi)$. Therefore, ξ is spanning for $\mathbb{F}(S)$. Suppose $\sum_{k=1}^{n} \alpha_{t_k} e_{t_k} = \mathbb{0}$ for some $\alpha_k \in \mathbb{F}$, $t_k \in S$.

$$\left(\sum_{k=1}^{\alpha_{t_k}} e_{t_k}\right) = \mathbb{0}(t_1)$$
$$\alpha_{t_1} = 0.$$

Similarly, $\alpha_{t_j} = 0$ for $j = 1, \dots, n$. Therefore, ξ is linearly independent. Since ξ is linearly independent and spanning, ξ forms a basis for $\mathbb{F}(S)$.

Note: The free vector space, $\mathbb{F}(S)$, displays the universal property.

> There are functions $\iota : S \to \mathbb{F}(S)$, where $\iota(t) = e_t$, and given any map $\varphi : S \to V$ for V a vector space over $\mathbb{F}$, $\exists!$ linear map $T_\varphi : \mathbb{F}(S) \to V$ such that $\iota \circ T_\varphi = \varphi$.

Proof: Every $f \in \mathbb{F}(S)$ has a unique expression $f = \sum_{k=1}^{n} f(t_k) e_{t_k}$, where $\text{supp}(f) = \{t_1, \dots, t_n\}$. Therefore,

$$T_\varphi(f) := \sum_{k=1}^{n} f(t_k) \varphi(t_k)$$

Exercise: Show T_φ is linear and unique.

Exercise 2: Suppose V is a vector space over $\mathbb{F}$ with basis B. Show that $\mathbb{F}(B) \cong V$. Remember that $V \cong W$ if $\exists\, T : V \to W$ such that T is bijective and linear.

Normed Spaces

To every vector $v \in V$, we want to assign a length to v, $\|v\|$.

A **norm** on a vector space V is a map

$$\|\cdot\| : V \to \mathbb{R}^+$$
$$v \mapsto \|v\| \geq 0$$

such that

(i) Homogeneity: $\|\alpha v\| = |\alpha| \|v\|$

(ii) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

(iii) Positive definiteness: $\|v\| = 0 \Rightarrow v = \mathbb{0}_V$.

If $p : V \to \mathbb{R}^+$ satisfies (i) and (ii), then p is a **seminorm**.

The pair $(V, \|\cdot\|)$ is called a normed space.

Two norms, $\|\cdot\|$ and $\|\cdot\|'$ are called **equivalent** if $\exists c_1, c_2 \geq 0$ with, $\forall v \in V$,

$$\|v\| \leq c_1\|v\|'$$
$$\|v\|' \leq c_2\|v\|$$

Note: On $\mathbb{R}^n$, all norms are equivalent.

Exercise: If p is any seminorm on V, then $|p(v) - p(w)| \leq p(v - w)$.

Notation: If V is a normed space, then $B_V = \{v \in V \mid \|v\| \leq 1\}$, and $U_V = \{v \in V \mid \|v\| < 1\}$ are the closed and open unit ball respectively.