

## Textbook Problems

**Solution** (8.1, Problem 6): We have

$$\frac{dx}{dt} = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + e^{-t} \begin{pmatrix} \sin(2t) \\ 4\cos(2t) \\ -1 \end{pmatrix}.$$

**Solution** (8.1, Problem 12): We have

$$\begin{aligned} \frac{dx}{dt} &= 5e^t \cos(t) - 5e^t \sin(t) \\ &= -10e^t \cos(t) + (15e^t \cos(t) - 5e^t \sin(t)) \\ &= -2x + 5y \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= 3e^t \cos(t) - e^t \sin(t) - 3e^t \sin(t) - e^t \cos(t) \\ &= 2e^t \cos(t) - 4e^t \sin(t) \\ &= (-10e^t \cos(t)) + 12e^t \cos(t) - 4e^t \sin(t) \\ &= -2x + 4y. \end{aligned}$$

**Solution** (8.1, Problem 20): Writing our Wronskian, and using the power of Mathematica, we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} 1 & e^{-4t} & 2e^{3t} \\ 6 & -2e^{-4t} & 3e^{3t} \\ -13 & -e^{-4t} & -2e^{3t} \end{pmatrix} \\ &= -4e^{-t}, \end{aligned}$$

which is zero nowhere along  $(-\infty, \infty)$ , so the vectors  $x_1, x_2, x_3$  form a fundamental set of solutions.

**Solution** (8.1, Problem 26): We start with the homogeneous equation:

$$\frac{dx}{dt} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} x.$$

Finding the eigenvalues and eigenvectors, we have

$$\begin{aligned} \det \begin{pmatrix} -1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} &= \lambda^2 - 2 \\ \lambda_1 &= \sqrt{2} \\ \lambda_2 &= -\sqrt{2}, \end{aligned}$$

with associated eigenvectors of

$$\begin{aligned} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \pm \sqrt{2} \begin{pmatrix} a \\ b \end{pmatrix} \\ v_1 &= \begin{pmatrix} 1 \\ -1 - \sqrt{2} \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ -1 + \sqrt{2} \end{pmatrix}. \end{aligned}$$

Therefore, we have the homogeneous solution of

$$x_h(t) = c_1 \begin{pmatrix} -1 \\ -1 - \sqrt{2} \end{pmatrix} e^{\sqrt{2}t} + c_2 \begin{pmatrix} -1 \\ -1 + \sqrt{2} \end{pmatrix} e^{-\sqrt{2}t}.$$

Note that the Wronskian gives

$$\begin{aligned} W(t) &= \det \begin{pmatrix} e^{\sqrt{2}t} & e^{-\sqrt{2}t} \\ (-1 - \sqrt{2})e^{\sqrt{2}t} & (-1 + \sqrt{2})e^{-\sqrt{2}t} \end{pmatrix} \\ &= 2\sqrt{2}, \end{aligned}$$

meaning that these solutions are linearly independent. Next, we examine the particular solution.

$$\begin{aligned} \mathbf{x}_p(t) &= \begin{pmatrix} t^2 - 2t + 1 \\ 4t \end{pmatrix} \\ \frac{d\mathbf{x}_p}{dt} &= \begin{pmatrix} 2t - 2 \\ 4 \end{pmatrix}. \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} t^2 - 2t + 1 \\ 4t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} &= \begin{pmatrix} -t^2 - 2t - 1 + t^2 + 4t - 1 \\ -t^2 + 2t - 1 + 4t - 6t + 5 \end{pmatrix} \\ &= \frac{d\mathbf{x}_p}{dt}. \end{aligned}$$

Thus, this is the general solution of our equation on  $(-\infty, \infty)$ .

**Solution** (8.2, Problem 12): Computing the characteristic polynomial, we get

$$p(x) = (6 - \lambda)(3 - \lambda)(-5 - \lambda),$$

giving

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -5 \\ \lambda_3 &= 6. \end{aligned}$$

Plugging these into

$$\begin{pmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda_i \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

we get

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{v}_3 &= \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{-5t} + c_3 \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} e^{6t}.$$

**Solution** (8.2, Problem 16): Using the power of Mathematica, we find the eigenvalues of  $\{5.05452, 4.09561, -2.92362, 2.02882, -0.155338\}$ , with corresponding eigenvectors yielding the general solution of

$$\mathbf{x}(t) = c_1 e^{5.05452t} \begin{pmatrix} -0.0312209 \\ -0.949058 \\ -0.239535 \\ -0.195825 \\ -0.0508861 \end{pmatrix}$$

$$\begin{aligned}
& + c_2 e^{4.09561t} \begin{pmatrix} -0.280232 \\ -0.836611 \\ -0.275304 \\ 0.176045 \\ 0.338775 \end{pmatrix} \\
& + c_3 e^{-2.92362t} \begin{pmatrix} 0.262219 \\ = 0.162664 \\ -0.826218 \\ -0.346439 \\ 0.31957 \end{pmatrix} \\
& + c_4 e^{2.02882t} \begin{pmatrix} -0.313235 \\ -0.64181 \\ -0.31754 \\ -0.173787 \\ 0.599018 \end{pmatrix} \\
& + c_5 e^{-0.155338t} \begin{pmatrix} -0.301294 \\ 0.466599 \\ 0.222136 \\ 0.0534311 \\ -0.799567 \end{pmatrix}.
\end{aligned}$$

**Solution** (8.2, Problem 24): We start by finding the eigenvalues of the matrix of the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}.$$

Using computational assistance, we find eigenvalues of  $\lambda_1 = 8$ ,  $\lambda_2 = -1$ , and  $\lambda_3 = -1$ . By solving the eigenvector equation for  $\lambda_1 = 8$ , we obtain the eigenvector of

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Meanwhile, the eigenvector equation for  $\lambda_2, \lambda_3$  gives eigenvectors of

$$\begin{aligned}
\mathbf{v}_2 &= \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \\
\mathbf{v}_3 &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.
\end{aligned}$$

Therefore, we get the general solution of

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{8t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

**Solution** (8.2, Problem 28): The matrix in the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \mathbf{x}$$

is in Jordan canonical form, meaning that we know that its eigenvalues are 4, 4, 4, and the generalized eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

subject to the chain  $\mathbf{v}_1 \rightarrow \mathbf{w}_2 \rightarrow \mathbf{w}_3$ . Thus, our general solution is

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t} \left( t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) + c_3 e^{4t} \left( \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

**Solution** (8.2, Problem 40): We start by taking the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{pmatrix} \mathbf{x},$$

and finding the eigenvalues. With computational assistance, we find the characteristic polynomial of

$$0 = \lambda^3 + \lambda^2 - \lambda + 15.$$

This has roots of

$$\lambda = 3, 1 \pm 2i.$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_{2,3} = \begin{pmatrix} 1 \\ -9/2 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we have

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + e^t \left( c_2 \cos\left(\frac{1}{2}t\right) + c_3 \sin\left(\frac{1}{2}t\right) \right) \begin{pmatrix} 1 \\ -9/2 \\ 1 \end{pmatrix} + e^t \left( c_3 \cos\left(\frac{1}{2}t\right) - c_2 \sin\left(\frac{1}{2}t\right) \right) \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.$$

**Solution** (8.2, Problem 42): We start by taking the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{pmatrix} \mathbf{x}$$

**Solution** (8.2, Problem 46):

## Extra Problems