Basics

Definition: Let A be a C^*-algebra. A *representation* of A is a $*$-homomorphism $\pi\colon A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Definition: Two representations $\pi\colon A \to B(\mathcal{H}_\pi)$ and $\rho\colon A \to B(\mathcal{H}_\rho)$ are called unitarily equivalent if there is a unitary map $U\colon \mathcal{H}_\rho \to \mathcal{H}_\pi$ such that

$$\pi(a) = U\rho(a)U^*$$

for all $a \in A$.

Definition: If $\pi\colon A \to B(\mathcal{H}_\pi)$ and $\rho\colon A \to B(\mathcal{H}_\rho)$ be representations. Then, the formula

$$\pi \oplus \rho(a)(h, k) \coloneqq (\pi(a)h, \rho(a)k)$$

defines the *direct sum* of π and ρ. If π is unitarily equivalent to a direct sum $\rho_1 \oplus \rho_2$, then we consider $\rho_1 \oplus \rho_2$ to be a decomposition of π in terms of the "smaller" representations.

Definition: A closed subspace $\mathcal{K}$ of $\mathcal{H}_\pi$ is *invariant* under π if $\pi(a)k \in \mathcal{K}$ for all $a \in A$ and $k \in \mathcal{K}$.

Observe that if $\mathcal{K}$ is an invariant subspace, then the orthogonal complement $\mathcal{K}^\perp$ is also invariant. This follows from the fact that if $y \in \mathcal{K}^\perp$, then

$$\begin{aligned} \langle k, \pi(a)y \rangle &= \langle \pi(a)^* k, y \rangle \\ &= \langle \pi(a^*)k, y \rangle \\ &= 0 \end{aligned}$$

for all $k \in \mathcal{K}$.

Conversely, if $\mathcal{K}$ is invariant, then we can recover $\pi = \pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp}$, via the canonical unitary isomorphism $U\colon \mathcal{K} \oplus \mathcal{K}^\perp \to \mathcal{H}_\pi$ given by $(k, y) \mapsto k + y$.

Definition: A representation π is *irreducible* if there are no closed invariant subspaces apart from $\{0\}$ and $\mathcal{H}_\pi$.

Lemma: A representation π of a C^*-algebra A is irreducible if and only if $\pi(A)' = \mathbb{C}I_{\mathcal{H}}$, where $\pi(A)'$ denotes the commutant of $\pi(A)$.

Proof. Suppose $\mathcal{V}$ is a nontrivial invariant subspace for π. Then, the orthogonal projection $P_{\mathcal{V}}$ commutes with every $\pi(A)$ and is not a scalar multiple of $I_{\mathcal{H}}$.

Now, suppose there is a non-scalar operator T commuting with $\pi(A)$. Then, either the real or imaginary part of T is a self-adjoint operator S that commutes with $\pi(A)$. From the continuous functional calculus, since $\sigma(S)$ is not one point, there are some nonzero continuous $f, g \in C(\sigma(S))$ such that $fg = 0$. Then, since $f(S), g(S) \in C^*(S)$, and $f(S), g(S)$ commute with $\pi(A)$, it follows that $\overline{f(S)\mathcal{H}}$ and $\overline{g(S)\mathcal{H}}$ are nonzero mutually orthogonal invariant subspaces, so π is reducible. $\square$

Definition: If π is a representation of the C^*-algebra A, then we call the subspace

$$[\pi(A)\mathcal{H}_\pi] = \overline{\operatorname{span}}\{\pi(a)h \mid h \in \mathcal{H}_\pi, a \in A\}$$

the *essential subspace* of $\mathcal{H}_\pi$. The representation π is called *nondegenerate* if the essential subspace $\mathcal{K}$ is equal to $\mathcal{H}_\pi$.

Note that the representation π being nondegenerate is equivalent to $\pi(1) = I_{\mathcal{H}_\pi}$ if A has an identity, or $\pi(e_i) \to I_{\mathcal{H}_\pi}$ strongly for any approximate identity $(e_i)_{i \in I}$.

The essential subspace is always invariant, and π is equivalent to $\pi|_{\mathcal{K}} \oplus 0$. Generally, if I is an ideal in A, then the subspace

$$\mathcal{K} = \overline{\operatorname{span}}\{\pi(a)h \mid h \in \mathcal{H}_\pi, a \in I\}$$

is invariant, but π is not zero on $\mathcal{K}^\perp$ unless I is an essential ideal.[I] Any nondegenerate representation of an ideal I extends canonically to a nondegenerate representation π of A on the same space.

The Gelfand–Naimark–Segal Construction

Definition: An element a of a C^*-algebra A is called *positive* if there is $b \in A$ with $a = b^*b$. Equivalently, a is positive if and only if $\sigma(a) \subseteq [0,\infty)$.

There are a few useful identities for positive elements. Specifically, the following hold:

$$\|a\|^2 1_A \geq a^*a$$
$$\|a\|^2 b^*b - b^*a^*ab \geq 0.$$

Definition: A linear functional $\rho\colon A \to \mathbb{C}$ is called *positive* if $\rho(a) \geq 0$ whenever $a \geq 0$. A positive linear functional of norm 1 is called a *state*.

Lemma: Let f be a positive linear functional on a C^*-algebra A. Then, for all $a, b \in A$, we have

$$f(b^*a) = \overline{f(a^*b)}$$

and

$$|f(b^*a)|^2 \leq f(b^*b)f(a^*a).$$

Proof. To see the first identity, we let $\lambda \in \mathbb{C}$, and observe that

$$\begin{aligned} 0 &\leq f\big((\lambda a + b)^*(\lambda a + b)\big) \\ &= |\lambda|^2 f(a^*a) + \overline{\lambda} f(a^*b) + \lambda f(b^*a) + f(b^*b). \end{aligned}$$

Now, since $|\lambda|^2 f(a^*a) + f(b^*b)$ is always real, we must have

$$\operatorname{Im}\Big(\overline{\lambda} f(a^*b) + \lambda f(b^*a)\Big) = 0$$

for all λ. By taking $\lambda = 1$ and $\lambda = i$, we get equality of imaginary and real parts of $f(a^*b)$ and $\overline{f(b^*a)}$.

As for the Cauchy–Schwarz inequality, we observe that if $\lambda = x\overline{f(b^*a)}$ for some $x \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leq x^2|f(b^*a)|^2 f(a^*a) + x|f(a^*b)|^2 + x|f(b^*a)|^2 + f(b^*b) \\ &= x^2|f(b^*a)|^2 f(a^*a) + 2x|f(b^*a)|^2 + f(b^*a). \end{aligned}$$

The right-hand side is a quadratic in x that is always greater than or equal to 0, so

$$4|f(b^*a)|^4 - 4|f(b^*a)|^2 f(a^*a)f(b^*b) \leq 0.$$

□

To understand the GNS construction, we start by taking a state τ on a C^*-algebra A. Then, defining

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\},$$

we observe that $\tau(b^*a) = 0$ if either a or b are in N_τ. In particular, we get the inner product on A/N_τ given by

$$\langle a + N_\tau, b + N_\tau\rangle = \tau(b^*a).$$

Define $\mathcal{H}_\tau$ to be the Hilbert space completion of A/N_τ. Since $\|a\|^2 b^*b - b^*a^*ab$ is of the form c^*c, we have

$$\|a(b + N_\tau)\|^2 = \tau(b^*a^*ab)$$

[I] An essential ideal is one that has nonzero intersection with any other closed ideal of A.

$$
\begin{aligned}
&= \|a\|^2\tau(b^*b) - \tau(c^*c)\\
&\leq \|a\|^2\tau(b^*b)\\
&= \|a\|^2\|b + N_\tau\|^2.
\end{aligned}
$$

In particular, this means that the elements of A act as bounded operators on A/N_τ, which we extend to operators $\pi_\tau(a)$ in the completion. This gives a nondegenerate representation π_τ of A on the Hilbert space $\mathcal{H}_\tau$.

Lemma: Suppose A is a non-unital C^*-algebra, and $\rho \in S(A)$. Then, if $(e_i)_{i\in I}$ is an approximate identity for A, $\rho(e_i) \to 1$. Furthermore, the formula $\tau(a + \lambda 1) = \rho(a) + \lambda$ defines a state τ on the unitization $\widetilde{A}$.

Proof. Since approximate identities in C^*-algebras are increasing, ρ is positive, and $\rho(e_i) \leq 1$ for each e_i, it follows that $(\rho(e_i))_{i\in I}$ converges to L for some $L \leq 1$. Now, since

$$
\begin{aligned}
e_i^2 &= \left(e_i^{1/2}\right)^* e_i e_i^{1/2}\\
&\leq \left(e_i^{1/2}\right)^* e_i^{1/2}\\
&= e_i.
\end{aligned}
$$

Thus, if $a \in A$, Cauchy–Schwarz gives

$$
\begin{aligned}
|\rho(e_i a)|^2 &\leq \rho\left(e_i^2\right)\rho(a^*a)\\
&\leq \rho(e_i)\|a\|^2\\
&\leq L\|a\|^2.
\end{aligned}
$$

This holds for all $a \in A$ if and only if $L \geq 1$ as $\|\rho\| = 1$, so since $L \leq 1$ it follows that $\rho(e_i) \to 1$.

By the earlier equation, we also have that $|\rho(a)|^2 \leq \rho(a^*a)$. Combining with the identity $\rho(a^*) = \overline{\rho(a)}$, we get

$$
\begin{aligned}
\tau\left((\lambda 1 + a)^*(\lambda 1 + a)\right) &= \tau\left(|\lambda|^2 1 + \overline{\lambda}a + \lambda a^* + a^*a\right)\\
&= |\lambda| + 2\operatorname{Re}\left(\overline{\lambda}\rho(a)\right) + \rho(a^*a)\\
&\geq |\lambda| - 2|\lambda||\rho(a)| + \rho(a^*a)\\
&\geq |\lambda|^2 - 2|\lambda||\rho(a)| + |\rho(a)|^2\\
&\geq (|\lambda| - |\rho(a)|)^2\\
&\geq 0.
\end{aligned}
$$

Thus, τ is positive and has $\tau(1) = 1$. □

Definition: If $\pi\colon A \to B(\mathcal{H}_\pi)$ is a representation, then a vector $h \in \mathcal{H}_\pi$ is called cyclic for π if the set

$$
[\pi(A)h] \coloneqq \{\pi(a)h \mid a \in A\}
$$

spans a dense subspace of $\mathcal{H}_\pi$.

Proposition: If ρ is a state on a C^*-algebra A, then there is a unit cyclic vector h_ρ in $\mathcal{H}_\rho$ such that

$$
\rho(a) = \langle \pi_\rho(a)h_\rho, h_\rho\rangle
$$

for all $a \in A$.

Conversely, if h is a unit cyclic vector for a representation $\pi\colon A \to B(\mathcal{H}_\pi)$, then the map $\tau\colon A \to \mathbb{C}$ given by

$$
\tau(a) = \langle \pi(a)h, h\rangle
$$

is a state on A. The map $a \mapsto \pi(a)h$ induces a unitary isomorphism between $\mathcal{H}_\tau$ and $\mathcal{H}_\pi$ such that $\pi(a) = U\pi_\tau(a)U^*$.

Proof. If A is unital, then we set $h_\rho = 1 + N_\rho$ as our cyclic vector. It then follows that

$$\begin{aligned}[\pi(A)h_\rho] &= \{\pi(a) + N_\rho \mid a \in A\}\\ &= A/N_\rho,\end{aligned}$$

which is norm-dense in $\mathcal{H}_\rho$.

If A is non-unital, we may use the lemma to extend ρ to $\tau\colon \widetilde{A} \to \mathbb{C}$; the inclusion $A \hookrightarrow \widetilde{A}$ induces an isometry V of $\mathcal{H}_\rho$ into $\mathcal{H}_\tau$ mapping $a + N_\rho$ to $a + N_\tau$. This isometry is such that

$$V\pi_\rho(a) = \pi_\tau(a)V$$

for all $a \in A$. We may identify $\mathcal{H}_\rho$ with the subspace $V\mathcal{H}_\rho$ of $\mathcal{H}_\tau$, since $\mathcal{H}_\rho$ is then essential subspace $[\pi_\tau(a)\mathcal{H}_\tau]$ of $\pi_\tau|_A$, with $\pi_\tau|_A = \pi_\rho \oplus 0$. The projection of $1 + N_\tau$ onto $\mathcal{H}_\rho$ satisfies

$$\begin{aligned}\pi_\rho(a)h_\rho &= \pi_\tau(a)(1 + N_\tau)\\ &= a + N_\tau.\end{aligned}$$

Thus, h_ρ is cyclic for π_ρ, and has

$$\begin{aligned}\langle \pi_\rho(a)h_\rho, h_\rho\rangle &= \langle \pi_\tau(a)(1 + N_\tau), 1 + N_\tau\rangle\\ &= \tau(a)\\ &= \rho(a).\end{aligned}$$

Now, suppose h is a unit cyclic vector for $\pi\colon A \to B(\mathcal{H}_\pi)$. Let τ be a positive functional of norm at most 1; then, τ has norm equal to $\|h\|^2 = 1$ since $\pi(e_i)h \to h$ for any approximate identity $(e_i)_{i\in I}$ of A. Next,

$$\begin{aligned}N_\tau &= \{a \in A \mid \langle \pi(a^*a)h, h\rangle = 0\}\\ &= \{a \in A \mid \pi(a)h = 0\},\end{aligned}$$

meaning there is a well-defined linear map $U_0\colon A/N_\tau \to \mathcal{H}_\pi$ given by $U_0(a + N_\tau) = \pi(a)h$. The map U_0 is isometric, since

$$\begin{aligned}\langle U_0(a + N_\tau), U_0(b + N_\tau)\rangle &= \langle \pi(b^*a)h, h\rangle\\ &= \tau(b^*a)\\ &= \langle a + N_\tau, b + N_\tau\rangle.\end{aligned}$$

Thus, U_0 extends to an isometry U on the completion $\mathcal{H}_\tau$ of A/N_τ, and maps onto $\overline{\operatorname{span}}\{\pi(a)h \mid a \in A\}$, which is equal to $\mathcal{H}_\pi$ since h is cyclic. Thus, U is unitary.

We find that

$$\begin{aligned}U\pi_\tau(a)(b + N_\tau) &= U(ab + N_\tau)\\ &= \pi(ab)h\\ &= \pi(a)(\pi(b)h)\\ &= \pi(a)U(b + N_\tau).\end{aligned}$$

Thus, $\pi_\tau(a)$ is unitarily equivalent to $\pi(a)$ for each a. □

Remark: The completion of A/N_ϕ, which we denoted $\mathcal{H}_\phi$, is often denoted as $L_2(A, \phi)$.

Proposition: Let ϕ and ψ be positive linear functionals with $\psi \leq \phi$ inducing representations π_ϕ and π_ψ.

Then, there is a unique operator $T \in \pi_\phi(A)' \subseteq B(H_\phi)$, with $0 \leq T \leq I$, such that

$$\begin{aligned} \psi(x) &= \phi(T\pi_\phi(x)) \\ &= \langle T\pi_\phi(x)\xi_\phi, \xi_\phi \rangle_\phi, \end{aligned}$$

where ξ_ϕ denotes the cyclic vector for ϕ.

Proof. We define a sesquilinear form F on $\mathcal{H}_\phi$ such that

$$\psi(y^*x) = F(\pi_\phi(x)\xi_\phi, \pi_\phi(y)\xi_\phi).$$

This is a bounded sesquilinear form that is necessarily positive. Thus, there is some $T \in B(\mathcal{H}_\phi)$ with $0 \leq T \leq I$ such that $F(\eta, \zeta) = \langle T\eta, \zeta \rangle_\phi$ for all $\eta, \zeta \in \mathcal{H}_\phi$.

Now, for $x, y, z \in A$, we have

$$\begin{aligned} \langle T\pi_\phi(x)\pi_\phi(z)\xi_\phi, \pi_\phi(y)\xi_\phi \rangle_\phi &= \psi(y^*(xz)) \\ &= \psi\big((x^*y)^*z\big) \\ &= \langle T\pi_\phi(z)\xi_\phi, \pi_\phi(x^*)\pi_\phi(y)\xi_\phi \rangle_\phi \\ &= \langle \pi_\phi(x)T\pi_\phi(z)\xi_\phi, \pi_\phi(y)\xi_\phi \rangle_\phi. \end{aligned}$$

Therefore, we have $T\pi_\phi(x) = \pi_\phi(x)T$. □

Notice that if $\psi \leq \phi$, then $\omega = \phi - \psi$ is positive linear functional such that $\phi = \psi + \omega$, meaning that π_ϕ is unitarily equivalent to

$$\mathcal{H}_\phi = [(\pi_\psi \oplus \pi_\omega)(A)(\xi_\psi \oplus \xi_\omega)],$$

and T is the compression of the projection onto the first coordinate.

Recall that the state space $S(A)$ is a w^*-compact convex subset of A^*, so it is equal to the closed convex hull of its extreme points by Krein–Milman. These extreme points are known as pure states.

Proposition: If ϕ is a state on A, then π_ϕ is irreducible if and only if ϕ is a pure state.

Proof. If π_ϕ is irreducible, then for any positive linear functional $\psi \leq \phi$, it must be the case that ψ is a multiple of ϕ since there are no sub-representations for π_ϕ. In particular, this means that $\psi = \phi$, so ϕ is pure.

Now, let ϕ be pure. Suppose there is a projection P in $\pi_\phi(A)'$ with $P \neq 0, I$. Then, $P\xi_\phi \neq 0$, as else we would have

$$\begin{aligned} 0 &= \pi_\phi(x)P\xi_\phi \\ &= P[\pi_\phi(x)\xi_\phi] \end{aligned}$$

for all $x \in A$, whence P would be equal to zero. Similarly, we must have $(I - P)\xi_\phi \neq 0$. For each $x \in A$, let

$$\begin{aligned} \phi_1(x) &= \langle \pi_\phi P\xi_\phi, P\xi_\phi \rangle_\phi \\ &= \phi(P\pi_\phi(x)) \\ \phi_2(x) &= \langle \pi_\phi(I-P)\xi_\phi, (I-P)\xi_\phi \rangle \\ &= \phi((I-P)\pi_\phi(x)). \end{aligned}$$

Then, ϕ_1, ϕ_2 are positive linear functionals on A, with $\phi = \phi_1 + \phi_2$, $\phi_1 = \lambda\phi$, and $\phi_2 = (1-\lambda)\phi$ for some $0 < \lambda < 1$, following from the fact that ϕ is pure. For any $\varepsilon > 0$, there is $x \in A$ with $\|\pi_\phi\xi_\phi - P\xi_\phi\| < \varepsilon$, and we have

$$\begin{aligned} \phi(x^*x) &= \|\pi_\phi(x)\xi_\phi\|^2 \\ &= \|P\xi_\phi\|^2, \end{aligned}$$

whence

$$\|(I-P)\pi_\phi\xi_\phi - P\xi_\phi\| < \varepsilon,$$

and

$$\begin{aligned}(1-\lambda)\|P\xi_\phi\|^2 &= \phi_2(x^*x)\\ &= \|(I-P)\pi_\phi\xi_\phi\|^2\\ &< \varepsilon^2,\end{aligned}$$

and since ε is arbitrary, we have $\lambda = 1$, which means no such P exists. Thus, $\pi_\phi(A)' = \mathbb{C}I$, and so π_ϕ is irreducible. □

Corollary: If A is a C^*-algebra and $x \in A$, then there is an irreducible representation π of A with $\|\pi(x)\| = \|x\|$.

Proof. Let π be represented via the pure state ϕ. There is a pure state on A with $\phi\Big((xx^*)^2\Big) = \Big\|(xx^*)^2\Big\|$, emerging from extending the isomorphism $C^*\Big((xx^*)^2\Big) \cong C_0\Big(\sigma\Big((xx^*)^2\Big)\Big)$. Thus,

$$\begin{aligned}\|x\|^2 &= \langle xx^*, xx^*\rangle^{1/2}\\ &= \|xx^*\|\\ &= \|\pi(x)x^*\|\\ &\leq \|\pi(x)\|\|x\|\\ &\leq \|x\|^2,\end{aligned}$$

so the inequalities are equalities, and $\|\pi(x)\| = \|x\|$. □

Spectrum of a C^*-Algebra

Definition: If A is a C^*-algebra, the *spectrum* of A, denoted $\hat{A}$, is the set of unitary equivalence classes of irreducible representations of A.

We start by understanding the irreducible representations of $K(\mathcal{H})$. Recall the characterization of $K(\mathcal{H})$ as

$$K(\mathcal{H}) = \overline{\operatorname{span}}\Big\{h \otimes \overline{k} \;\Big|\; h, k \in \mathcal{H}\Big\},$$

where

$$h \otimes \overline{k}(\ell) = \langle \ell, k\rangle h.$$

Let $\pi\colon K(\mathcal{H}) \to B(\mathcal{H}_\pi)$ be an irreducible representation, and let $e \in \mathcal{H}$ be a unit vector. Then, $e \otimes \overline{e}$ is the projection of $\mathcal{H}$ onto the span of e, meaning $P = \pi(e \otimes \overline{e})$ is the projection onto the closed subspace $P\mathcal{H}_\pi$. Let $\xi \in B(\mathcal{H}_\pi)$ be fixed, and consider

$$[\pi(K(\mathcal{H}))\xi] = \overline{\operatorname{span}}\{\pi(T)\xi \mid T \in K(\mathcal{H})\}.$$

This is a nonzero invariant subspace, so by irreducibility, $[\pi(K(\mathcal{H}))\xi]$ is all of $\mathcal{H}_\pi$. Define $U\colon \mathcal{H} \to \mathcal{H}_\pi$ by

$$Uh = \pi(h \otimes \overline{e})\xi.$$

Then, we have

$$\begin{aligned}\langle Ug, Uh\rangle &= \langle \pi(g \otimes \overline{e})\xi, \pi(h \otimes \overline{e})\xi\rangle\\ &= \big\langle \xi, \pi\big((g \otimes \overline{e})^*(h \otimes \overline{e})\big)\xi\big\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \xi, \pi((e \otimes \overline{g})(h \otimes \overline{e}))\xi \rangle \\
&= \langle \xi, \pi(\langle h, g \rangle (e \otimes \overline{e}))\xi \rangle \\
&= \langle g, h \rangle \langle \xi, \pi(e \otimes \overline{e})\xi \rangle \\
&= \langle g, h \rangle \|\xi\|^2 \\
&= \langle g, h \rangle.
\end{aligned}
$$

Furthermore, since

$$
\begin{aligned}
\pi\left(h \otimes \overline{k}\right)\xi &= \pi\left(h \otimes \overline{k}\right)\pi(e \otimes \overline{e})\xi \\
&= \langle e, k \rangle \pi(h \otimes \overline{e})\xi \\
&= \langle e, k \rangle Uh,
\end{aligned}
$$

we have that every element of the spanning set is in the range of U. Thus, U is a unitary, and

$$
\begin{aligned}
\pi\left(h \otimes \overline{k}\right)(Ug) &= \pi\left(\left(h \otimes \overline{k}\right)(g \otimes \overline{e})\right)\xi \\
&= \langle g, k \rangle \pi(h \otimes \overline{e})\xi \\
&= \langle g, k \rangle Uh \\
&= U(\langle g, k \rangle h) \\
&= U\left(\left(h \otimes \overline{k}\right)g\right).
\end{aligned}
$$

In particular, this means we have $\pi(T)U = UT$ for all $T \in K(\mathcal{H})$. Thus, π is unitarily equivalent to the identity, meaning that the spectrum of $K(\mathcal{H})$ is simply the identity.

Similar to how the analysis of commutative C^*-algebras depends on the identification between characters and maximal ideals on the C^*-algebra, given by $\phi \mapsto \ker(\phi)$. Similarly, we are interested in the ideals that are the kernels of irreducible representations, known as primitive ideals.

Proposition: Let A be a C^*-algebra. Then,

(a) every closed ideal in A is the intersection of the primitive ideals that contain it;

(b) if I is a primitive ideal, and J, K are ideals such that $J \cap K \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.

Proof.

(a) Let $a \in A$ with $a \notin I$. Then, consider $a + I \in A/I$. There is an irreducible representation of A/I such that $\|\pi(a + I)\| = \|a + I\| \neq 0$. Yet, composing with the quotient map, this yields an irreducible representation $\pi \circ q$ with $a \notin \ker(\pi \circ q)$.

(b) Let π be an irreducible representation with $\ker(\pi) = I$. If $J \nsubseteq I$, then $\pi(J) \neq 0$, so $[\pi(J)\mathcal{H}] =: \mathcal{V}$ is nonzero. Since J is an ideal, $\mathcal{V}$ is invariant, but this means $[\pi(J)\mathcal{H}] = \mathcal{H}$. Yet, this yields

$$
\begin{aligned}
\pi(K)(\pi(J)\mathcal{H}) &\subseteq \pi(K \cap J)\mathcal{H} \\
&\subseteq \pi(I) \\
&= 0,
\end{aligned}
$$

whence $K \subseteq \ker(\pi)$.

□

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