Tensor Products, Bilinear Maps, and Linear Maps

First, we review some definitions of algebraic tensor products.

Definition of Tensor Products

From linear algebra, we know that if X and Y are vector spaces, then the *tensor product* of X and Y, denoted $X \otimes Y$, is the universal object such that if $t\colon X \times Y \to Z$ is a bilinear map, then there is a unique linear map $T\colon X \otimes Y \to Z$, alongside an injection $(x, y) \overset{\iota}{\mapsto} x \otimes y$ such that $T \circ \iota = t$.

$$\begin{tikzcd} X \times Y \arrow[r, "\iota"] \arrow[dr, "t"'] & X \otimes Y \arrow[d, "T"] \\ & Z \end{tikzcd}$$

Elements of the tensor product are linear combinations of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

Identifying Tensor Products as Spaces of Maps

We observe that if t is a bilinear *form* — i.e., a bilinear map $t\colon X \times Y \to \mathbb{F}$ for some field $\mathbb{F}$ — then there is a unique linear map $T\colon X \otimes Y \to \mathbb{F}$, meaning that the space of bilinear forms, $\hom(X \times Y, \mathbb{F})$, is in one-to-one correspondence with elements of the algebraic dual of the tensor product, $(X \otimes Y)'$.

We can in fact view tensors in and of themselves as bilinear forms. For any $x \in X$ and $y \in Y$, we may define a bilinear form $B_{x,y}\colon X' \times Y' \to \mathbb{F}$ by $B_{x,y}(\varphi, \psi) = \varphi(x)\psi(y)$. Thus, we have a unique linear map $X \otimes Y \to \hom(X' \times Y', \mathbb{F})$ taking $x \otimes y \mapsto B_{x,y}$.

The map is injective, since if

$$\sum_{i=1}^{n} B_{x_i, y_i} = 0,$$

then for any $\varphi \in X'$ and $\psi \in Y'$, we have

$$\sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = 0.$$

Since X' separates the points of X and similarly for Y' and the points of Y, we must have that

$$\sum_{i=1}^{n} x_i \otimes y_i = 0.$$

Thus, we have an embedding $X \otimes Y \hookrightarrow \hom(X' \times Y', \mathbb{F})$; if X and Y are dual spaces, then there is a corresponding embedding $X' \otimes Y' \hookrightarrow \hom(X \times Y, \mathbb{F})$ by identifying

$$\sum_{i=1}^{n} \varphi_i \otimes \psi_i \mapsto \left((x, y) \mapsto \sum_{i=1}^{n} \varphi_i(x)\psi_i(y) \right).$$

For any bilinear form B, there are two associated linear maps, $L_B\colon X \to Y'$ and $R_B\colon Y \to X'$, given by

$$\begin{aligned} B(x, y) &= \langle y, L_B(x) \rangle \\ &= \langle x, R_B(y) \rangle, \end{aligned}$$

where we let $\langle \cdot, \cdot \rangle \colon X \times X' \to \mathbb{F}$ denote the canonical duality given by

$$\langle x, \varphi \rangle = \varphi(x).$$

Thus, we see that every element of the tensor product,

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

gives us two linear maps

$$L_u(\varphi) = \sum_{i=1}^{n} \varphi(x_i) y_i$$
$$R_u(\psi) = \sum_{i=1}^{n} \psi(y_i) x_i,$$

giving two more identifications $X \otimes Y \hookrightarrow \hom(X', Y)$ and $X \otimes Y \hookrightarrow \hom(Y', X)$.

If one of X or Y is a dual space, we get $X' \otimes Y \hookrightarrow \hom(X, Y)$ and $X \otimes Y' \hookrightarrow \hom(Y, X)$.

Specifically, the elements of $\hom(X, Y)$ that correspond to elements of $X' \otimes Y$ are the finite-rank linear maps.

Examples

> **Example** (Matrices)**:** If we let $\mathbb{F}^n$ and $\mathbb{F}^m$ be endowed with the standard bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$, we may identify $\mathbb{F}^n \otimes \mathbb{F}^m \cong \mathbb{M}_{m,n}(\mathbb{F})$.
>
> We may identify $e_i \otimes f_j$ with the matrix unit e_{ij}.

> **Example** (Vector-Valued Functions)**:** Let $\mathcal{F}(S)$ denote the vector space of functions from a set S into the field $\mathbb{F}$; if X is a vector space, then $\mathcal{F}(S, X)$ may denote the vector space of all functions from S to X with pointwise operations. Given $f \in \mathcal{F}(S)$ and any $x \in X$, we may define a function from S to X by taking $s \mapsto f(s)x$. We may write this as $f \cdot x$
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> This defines a bilinear map $\mathcal{F}(S) \times X \to \mathcal{F}(S, X)$ taking $(f, x) \mapsto f \cdot x$.
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> We may thus find a linear map
>
> $$\sum_{i=1}^{n} f_i \otimes x_i \mapsto \sum_{i=1}^{n} f_i \cdot x_i.$$
>
> We show that this map is injective. Suppose we have
>
> $$\sum_{i=1}^{n} f_i \cdot x_i = 0,$$
>
> so that
>
> $$\sum_{i=1}^{n} f_i(s) x_i = 0.$$
>
> Yet, since the evaluation functionals $f \mapsto f(s)$ are a separating subset of $\mathcal{F}(S)$, we have
>
> $$\sum_{i=1}^{n} f_i \otimes x_i = 0.$$
>
> This gives an embedding of $\mathcal{F}(S) \otimes X \hookrightarrow \mathcal{F}(S, X)$.

> **Example** (Vector-Valued Measures)**:** Let $\mathcal{A}$ be an algebra of subsets of S. The vector space $M_{\text{f.a.}}(S)$ denotes the space of all finitely additive scalar-valued measures on $\mathcal{A}$. An element of $M_{\text{f.a.}}(S)$ is a function from $\mathcal{A}$ to $\mathbb{F}$ such that $\mu(\emptyset) = 0$ and
>
> $$\mu\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$
>
> for every finite collection of pairwise disjoint sets in $\mathcal{A}$.
>
> For every vector space X, we can similarly define $M_{\text{f.a.}}(S, X)$ to be the space of finitely additive X-valued measures on on $\mathcal{A}$, and obtain an embedding of the tensor product $M_{\text{f.a.}}(S) \otimes X \hookrightarrow M_{\text{s.a.}}(S, X)$.

Summary

- If X and Y are vector spaces and $t\colon X \times Y \to Z$ is a bilinear map, then there is a unique linear map $T\colon X \otimes Y \to Z$ such that $T \circ \iota = t$.
- Letting $\hom(X \times Y, \mathbb{F})$ denote the space of bilinear forms on $X \times Y$ to an underlying field $\mathbb{F}$, there are identifications

$$\begin{aligned} (X \otimes Y)' &\leftrightarrow \hom(X \times Y, \mathbb{F}) \\ X \otimes Y &\hookrightarrow \hom(X' \times Y', \mathbb{F}) \\ X' \otimes Y &\hookrightarrow \hom(X, Y) \\ X \otimes Y' &\hookrightarrow \hom(Y, X). \end{aligned}$$

- We may identify vector-valued functions and measures as

$$\begin{aligned} \mathcal{F}(S) \otimes X &\hookrightarrow \mathcal{F}(S, X) \\ M_{\text{f.a.}}(S) \otimes X &\hookrightarrow M_{\text{f.a.}}(S, X). \end{aligned}$$

Projective Tensor Products

If X and Y are Banach spaces, there are a variety of ways we may seek to norm the tensor product $X \otimes Y$. The basic requirement we have is that we want

$$\|x \otimes y\| \leq \|x\| \|y\|,$$

and for any representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

we want

$$\|u\| \leq \sum_{i=1}^{n} \|x_i\| \|y_i\|.$$

Therefore, we must have

$$\|u\| \leq \inf\left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \;\middle|\; u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

The latter value is thus the largest possible candidate for a norm on $X \otimes Y$ that has these desired qualities. We thus define

$$\|u\|_{\wedge} = \inf\left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \;\middle|\; u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

to be the *projective norm* on $X \otimes Y$.

Proposition: Let X and Y be Banach spaces. Then, $\|\cdot\|_{\wedge}$ is a norm on $X \otimes Y$ with $\|x \otimes y\|_{\wedge} = \|x\|\|y\|$.

Proof. We start by showing homogeneity. Assume $\lambda \neq 0$. Then, if

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

we have

$$\lambda u = \sum_{i=1}^{n} (\lambda x_i) \otimes y_i,$$

we have

$$\begin{aligned} \|\lambda u\|_{\wedge} &\leq \sum_{i=1}^{n} \|\lambda x_i\|\|y_i\| \\ &= |\lambda| \sum_{i=1}^{n} \|x_i\|\|y_i\| \\ &= |\lambda|\|u\|_{\wedge}. \end{aligned}$$

Similarly,

$$\|u\|_{\wedge} \leq |\lambda|^{-1}\|\lambda u\|_{\wedge},$$

whence $\|\lambda u\|_{\wedge} = |\lambda|\|u\|_{\wedge}$.

Now, let $\varepsilon > 0$ and let $u, v \in X \otimes Y$ have representations

$$\begin{aligned} u &= \sum_{i=1}^{n} x_i \otimes y_i \\ v &= \sum_{j=1}^{m} w_j \otimes z_j \end{aligned}$$

such that

$$\begin{aligned} \sum_{i=1}^{n} \|x_i\|\|y_i\| &\leq \|u\|_{\wedge} + \varepsilon/2 \\ \sum_{j=1}^{m} \|w_j\|\|z_j\| &\leq \|v\|_{\wedge} + \varepsilon/2. \end{aligned}$$

Then, we have a representation

$$u + v = \sum_{i=1}^{n} x_i \otimes y_i + \sum_{j=1}^{m} w_j \otimes z_j,$$

so that

$$\begin{aligned} \|u + v\|_{\wedge} &\leq \sum_{i=1}^{n} \|x_i\|\|y_i\| + \sum_{j=1}^{m} \|w_j\|\|z_j\| \\ &\leq \|u\|_{\wedge} + \|v\|_{\wedge} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we obtain the triangle inequality.

Now, we show that the norm is definite. Let $\|u\|_\wedge = 0$. Then, for any $\varepsilon > 0$, there is a representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

such that

$$\sum_{i=1}^{n} \|x_i\| \|y_i\| \leq \varepsilon.$$

In particular, for any $\varphi \in X^*$ and $\psi \in Y^*$, we have

$$\left|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)\right| \leq \varepsilon \|\varphi\| \|\psi\|.$$

Since the quantity $\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)$ is independent of the representation of u, it follows that this sum equals zero. Yet, since X^* and Y^* separate the points of X and Y, it follows that $u = 0$.

Finally, we know that $\|x \otimes y\|_\wedge \leq \|x\| \|y\|$, so we let $\varphi \in B_{X^*}$ and $\psi \in B_{Y^*}$ such that $\varphi(x) = \|x\|$ and $\psi(y) = \|y\|$. We let $b\colon X \times Y \to \mathbb{F}$ be given by $B(w, z) = \varphi(w)\psi(z)$. The linearization B is a linear functional on $X \otimes Y$ with

$$\begin{aligned} \left|B\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right| &\leq \sum_{i=1}^{n} |B(x_i \otimes y_i)| \\ &= \sum_{i=1}^{n} |\varphi(x_i)\psi(y_i)| \\ &\leq \sum_{i=1}^{n} \|x_i\| \|y_i\|, \end{aligned}$$

so that $|B(u)| \leq \|u\|_\wedge$ for every $u \in X \otimes Y$. In particular, this means that B is a bounded linear functional on the normed space $(X \otimes Y, \|\cdot\|_\wedge)$ with norm at most 1, whence $\|x\| \|y\| = B(x \otimes y) \leq \|x \otimes y\|_\wedge$. □

We may thus complete $(X \otimes Y, \|\cdot\|_\wedge)$ with respect to the projective norm to obtain the *projective tensor product* of the Banach spaces X and Y, which we denote $X \hat{\otimes} Y$.

If $A \subseteq X$ and $B \subseteq Y$ are subsets, then we will let

$$A \otimes B \coloneqq \{x \otimes y \mid x \in A, y \in B\}.$$

Proposition: The closed unit ball of $X \hat{\otimes} Y$ is the closed convex hull of $B_X \otimes B_Y$.

Proof. Since the closed unit ball is the closure of the unit ball in $X \otimes Y$, it suffices to prove the proposition for the space $(X \otimes Y, \|\cdot\|_\wedge)$. Let u be an element of the open unit ball of $(X \otimes Y, \|\cdot\|_\wedge)$.

By the definition of the projective norm, there is a representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

such that

$$\sum_{i=1}^{n} \|x_i\| \|y_i\| < 1.$$

Let

$$w_i = \|x_i\|^{-1} x_i$$

$$\begin{aligned} z_i &= \|y_i\|^{-1} y_i \\ \lambda_i &= \|x_i\|\|y_i\|. \end{aligned}$$

Then, we have

$$u = \sum_{i=1}^{n} \lambda_i w_i \otimes z_i$$

with $w_i \in B_X$, $z_i \in B_Y$, $\lambda_i \geq 0$, and $\sum_{i=1}^{n} \lambda_i < 1$. Thus, $u \in \operatorname{conv}(B_X \otimes B_Y)$, meaning that the closed unit ball of $X \otimes Y$ is contained in $\overline{\operatorname{conv}}(B_X \otimes B_Y)$. Yet, we must also have $B_X \otimes B_Y$ contained in the closed unit ball of $(X \otimes Y, \|\cdot\|_\wedge)$, so it holds for $\overline{\operatorname{conv}}(B_X \otimes B_Y)$. □

Tensor Products of Linear Operators on Banach Spaces

In general, if we have two linear maps $S\colon X \to E$ and $T\colon Y \to F$, we have a bilinear map $X \times Y \to E \otimes F$ given by $(x, y) \mapsto (Sx) \otimes (Ty)$. From the universal property, we get a linear map $S \otimes T\colon X \otimes Y \to E \otimes F$ such that $(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$.

When X and Y have norms, and we are concerned with continuity, we observe that if $u \in X \otimes Y$ has representation

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

we have

$$\begin{aligned} \|(S \otimes T)(u)\|_\wedge &= \left\| \sum_{i=1}^{n} (Sx_i) \otimes (Ty_i) \right\| \\ &\leq \|S\|_{\mathrm{op}} \|Y\|_{\mathrm{op}} \sum_{i=1}^{n} \|x_i\|\|y_i\| <> \end{aligned}$$

whence

$$\|(S \otimes T)(u)\|_\wedge \leq \|S\|_{\mathrm{op}} \|T\|_{\mathrm{op}} \|u\|_\wedge.$$

This gives that $\|S \otimes T\|_{\mathrm{op}} \leq \|S\|_{\mathrm{op}} \|T\|_{\mathrm{op}}$. Meanwhile, since $\|x \otimes y\|_\wedge = \|x\|\|y\|$, we have $\|S \otimes T\|_{\mathrm{op}} \geq \|S\|_{\mathrm{op}} \|T\|_{\mathrm{op}}$, so that $\|S \otimes T\|_{\mathrm{op}} = \|S\|_{\mathrm{op}} \|T\|_{\mathrm{op}}$.

Finally, we may extend $S \otimes T$ to the completions $X \hat{\otimes} Y$ and $E \hat{\otimes} F$. This gives the following proposition.

Proposition: Let $S\colon X \to E$ and $T\colon Y \to F$ be bounded linear operators. Then, there is a unique operator

$$S \hat{\otimes} T\colon X \hat{\otimes} Y \to W \hat{\otimes} Z(x \otimes y) \qquad \mapsto (Sx) \otimes (Ty).$$

Furthermore, $\left\|S \hat{\otimes} T\right\|_{\mathrm{op}} = \|S\|_{\mathrm{op}} \|T\|_{\mathrm{op}}$.

Inheritance of the Projective Norm

In general, the projective tensor product does not respect subspaces, in the sense that if $W \leq X$ is a subspace, so that $W \otimes Y \leq X \otimes Y$ is an algebraic subspace, the norm on $W \otimes Y$ induced by $(X \otimes Y, \|\cdot\|_\wedge)$ is not necessarily the same as the projective norm on $W \otimes Y$.

This follows from the fact that the definition of the norm $(W \otimes Y, \|\cdot\|_\wedge)$ is restricted to all representations in $W \otimes Y$, and since there are more representations for u in $X \otimes Y$, it follows that the norm of u in $(X \otimes Y, \|\cdot\|_\wedge)$ is lesser than or equal to the norm of u in $(W \otimes Y, \|\cdot\|_\wedge)$.

We start by discussing the special case of complemented subspace. Recall that a closed subspace $E \leq X$ is called *complemented* if there is another closed subspace W such that $X = E \oplus W$. An equivalent characterization of a complemented subspace is that there is a continuous projection $P_E \colon X \to E$ such that $X = E \oplus \ker(P_E)$.

Proposition: Let E and F be complemented subspaces of X and Y respectively. Then, $E \otimes F$ is complemented in $X \otimes Y$, and the norm on $E \otimes F$ induced by the projective norm on $X \otimes Y$ is equivalent to the projective norm (in the sense of norm equivalence) on $E \otimes F$.

If E and F are complemented by projections of norm 1, then $E \otimes F$ is a subspace of $X \otimes Y$ that is also complemented by a projection of norm 1.

Proof. Let P and Q be projections from X, Y onto E, F respectively. Then, $P \otimes Q$ is a projection of $X \otimes Y$ onto $E \otimes F$.

Let $u \in E \otimes F$. We have that $\|u\|_{\wedge, X\otimes Y} \leq \|u\|_{\wedge, E \otimes F}$, so we let

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

be a representation of u in $X \otimes Y$. Then,

$$\begin{aligned} u &= P \otimes Q(u) \\ &= \sum_{i=1}^{n} (Px_i) \otimes (Qy_i) \end{aligned}$$

is a representation of u in $E \otimes F$, whence

$$\begin{aligned} \|u\|_{\wedge, E\otimes F} &\leq \sum_{i=1}^{n} \|Px_i\| \|Qy_i\| \\ &\leq \|P\|_{\mathrm{op}} \|Q\|_{\mathrm{op}} \sum_{i=1}^{n} \|x_i\| \|y_i\|, \end{aligned}$$

so it follows that for every representation of u in $X \otimes Y$, we have

$$\begin{aligned} \|u\|_{\wedge, X\otimes Y} &\leq \|u\|_{\wedge, E\otimes F} \\ &\leq \|P\|_{\mathrm{op}} \|Q\|_{\mathrm{op}} \|u\|_{\wedge, X\otimes Y}. \end{aligned}$$

If E and F are complemented by projections of norm 1, we have $\|u\|_{\wedge, X\otimes Y} = \|u\|_{\wedge, E\otimes F}$ for every $u \in E \otimes F$, as $\|P \otimes Q\|_{\mathrm{op}} = \|P\|_{\mathrm{op}} \|Q\|_{\mathrm{op}}$. □

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