

This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

## Useful Results and Proofs

### Analytic Functions

**Definition:** Let  $U \subseteq \mathbb{C}$  be an open set. A function  $f: U \rightarrow \mathbb{C}$  is called *analytic* if, for any  $z_0 \in U$ , there is  $r > 0$  and  $(a_k)_k \subseteq \mathbb{C}$  such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all  $z \in U(z_0, r)$ .

Analytic functions form a  $\mathbb{C}$ -algebra.

**Theorem (Identity Theorem):** Let  $f, g: U \rightarrow \mathbb{C}$  be analytic functions defined on a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in  $U$ , then  $f = g$  on  $U$ .

*Proof.* To begin, we show that if  $f: U \rightarrow \mathbb{C}$  is an analytic function that is not uniformly zero, then for any  $z_0 \in U$ , there is  $\rho > 0$  such that  $f$  is nonzero on  $\dot{U}(z_0, \rho) \subseteq U$ . Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all  $z \in U(z_0, r)$ , some  $r > 0$ , and since  $f$  is not uniformly zero, there is some minimal  $\ell$  such that  $a_\ell \neq 0$ . This yields

$$f(z) = (z - z_0)^\ell \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function  $h: U(z_0, r) \rightarrow \mathbb{C}$  given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as  $f$  and is not zero at  $z_0$ , so that  $h$  is not zero on some  $U(z_0, \rho)$  as  $h$  is continuous.

Now, we let  $V_1$  be the set of accumulation points of  $A$  in  $U$ , and let  $V_2 = U \setminus V_1$ .

If  $z \in V_2$ , then there is some  $r_1 > 0$  such that  $\dot{U}(z, r_1) \cap A = \emptyset$ , or that  $\dot{U}(z, r_1) \subseteq A^c$ . Meanwhile, since  $U$  is open, there is some  $r_2 > 0$  such that  $U(z, r_2) \subseteq U$ , meaning that if  $r = \min\{r_1, r_2\}$ , then  $U(z, r) \subseteq U \setminus A$ . Thus,  $V_2$  is open.

Meanwhile, if  $z \in V_1$ , then since  $V_1 \subseteq U$ , it follows that there is  $r > 0$  such that  $U(z, r)$  and  $(a_k)_k$  such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all  $w \in U(z, r)$ . We claim that  $f(w) - g(w)$  is uniformly zero on  $U(z, r)$ . Else, if there were  $w_0 \in U(z, r)$  such that  $f(w_0) \neq g(w_0)$ , then it would follow that there is  $0 < s \leq r$  such that  $f(w) \neq g(w)$  for all  $w \in U(w_0, s)$ . Yet, this would contradict the assumption that  $z$  is an accumulation point, meaning that  $V_1$  is open.

Since  $V_1$  and  $V_2$  are disjoint open sets whose union is equal to  $U$ , it follows that either  $V_1 = U$  or  $V_2 = U$ . If  $A \neq \emptyset$ , then the identity theorem follows.  $\square$

## Differentiability

**Definition:** If  $U \subseteq \mathbb{C}$  is an open set, then we say  $f$  is differentiable at  $z_0 \in U$  if

$$\lim_{w \rightarrow z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of  $f$  at  $z_0$ , and usually write  $f'(z_0)$ .

If  $f$  is differentiable at every  $z_0 \in U$ , we say  $f$  is differentiable on  $U$ .

If  $f$  is continuous and admits a continuous derivative, then we say  $f$  is *holomorphic*.

Note that the limit must be independent of direction. That is, for all  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$\left| \frac{f(w) - f(z_0)}{w - z_0} - f'(z_0) \right| < \varepsilon$$

whenever  $0 < |z - z_0| < \delta$ .

Now, given  $U \subseteq \mathbb{C}$ , write  $z = x + iy$  and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where  $u = \operatorname{Re}(f)$  and  $v = \operatorname{Im}(f)$ . Observe then that if  $f$  is differentiable at  $x_0 + iy_0 \in U$ , then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(u(x+h, y) + iv(x+h, y)) - (u(x, y) + iv(x, y))}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(u(x, y+h) + iv(x, y+h)) - (u(x, y) + iv(x, y))}{ih} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

**Definition:** The system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if  $f$  is differentiable, then the  $u$  and  $v$  in the definition of  $f$  satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly  $f$  is differentiable or holomorphic.

**Proposition:** If  $f = u + iv$  is a holomorphic function such that  $u, v$  are in  $C^2(U)$ , then  $u$  and  $v$  are harmonic. That is,  $u$  and  $v$  satisfy Laplace’s equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We call  $u$  and  $v$  *harmonic conjugates* for each other. That is, if  $u: U \rightarrow \mathbb{R}$  is a harmonic function, then  $v \in C^1(U)$  is called a harmonic conjugate if the Cauchy–Riemann equations hold for  $u$  and  $v$ .

**Theorem:** Let  $U \subseteq \mathbb{R}^2$  be a ball or all of  $\mathbb{R}^2$ . Then, every harmonic function on  $U$  has a harmonic conjugate. If  $u \in C^3(U)$ , then this conjugate is itself harmonic.

**Lemma:** Let  $g: U((x_0, y_0), R) \rightarrow \mathbb{R}$  be such that  $g$  and  $\frac{\partial g}{\partial x}$  are continuous. Then,  $G: U((x_0, y_0), R) \rightarrow \mathbb{R}$ , given by

$$G(x, y) = \int_{y_0}^y g(x, t) dt$$

satisfies

$$\frac{\partial G}{\partial x} = \int_{y_0}^y \frac{\partial g}{\partial x}(x, t) dt.$$

*Proof of Lemma.* Write

$$\frac{G(x+h, y) - G(x, y)}{h} - \int_{y_0}^y \frac{\partial g}{\partial x}(x, t) dt = \int_{y_0}^y \left( \frac{g(x+h, t) - g(x, t)}{h} - \frac{\partial g}{\partial x}(x, t) \right) dt.$$

By mean value theorem, the first term is equal to  $\frac{\partial g}{\partial x}(x_1, t)$  for some  $x_1$  between  $x$  and  $x+h$ . As  $h \rightarrow 0$ ,  $x_1 \rightarrow x$ , as  $\frac{\partial g}{\partial x}$  is uniformly continuous on a compact subset that contains  $x$  and  $x+h$ . We may exchange limit and integral to obtain the desired result.  $\square$

*Proof of Theorem.* We prove for the case of  $U = U((x_0, y_0), R)$ . Define

$$v(x, y) = \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt + \phi(x),$$

with  $\phi(x)$  to be determined later. By the fundamental theorem of calculus, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

while by differentiating under the integral sign, and using the fact that  $u$  is harmonic, we have

$$\begin{aligned} \frac{\partial v}{\partial x} &= \int_{y_0}^y \frac{\partial^2 u}{\partial x^2}(x, t) dt + \frac{d\phi}{dx} \\ &= - \int_{y_0}^y \frac{\partial^2 u}{\partial y^2}(x, t) dt + \frac{d\phi}{dx} \\ &= - \frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \frac{d\phi}{dx}. \end{aligned}$$

Defining  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi(x) = - \int_{x_0}^x \frac{\partial u}{\partial y}(s, y_0) ds,$$

we see that  $v$  thus satisfies all the necessary requirements to be a harmonic conjugate.

Now, if  $u$  is  $C^3$ , then we defined  $v$  via the derivative of  $u$ , so that  $v$  is  $C^2$ , and thus  $v$  is harmonic.  $\square$

## Cauchy's Integral Formula

**Proposition:** Fix  $z_0 \in \mathbb{C}$ ,  $R > 0$ , and  $f: U(z_0, R) \rightarrow \mathbb{C}$  holomorphic. For all  $z \in U(z_0, R)$ , we have

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

*Proof.* It suffices to show that

$$\frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = 0.$$

By using the chain rule and fundamental theorem of calculus, we find

$$\begin{aligned} \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{\int_0^1 f'((1-t)z + t\zeta)(\zeta - z) dt}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{S(z_0, R)} \int_0^1 f'((1-t)z + t\zeta) dt d\zeta \\ &= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{d}{d\zeta} \left( \frac{1}{t} f((1-t)z + t\zeta) \right) d\zeta \\ &= 0. \end{aligned}$$

$\square$

**Proposition:** Let  $f: U \rightarrow \mathbb{C}$  be a holomorphic function. The following all hold:

- (i)  $f$  is analytic;
- (ii)  $f$  is smooth with  $f^{(n)}$  holomorphic;
- (iii) for all  $z_0 \in U$ , if we let  $R = \sup\{r > 0 \mid U(z_0, r) \subseteq U\}$ , then there is  $(a_n)_n \subseteq \mathbb{C}$  such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the power series has radius of convergence  $R$ .

*Proof.*

- (i) There exists  $r < s$  with  $U(z_0, s) \subseteq U$  and  $r < r_1 < s$  such that  $S(z_0, r_1) \subseteq U$ . By Cauchy's Integral Formula, and a power series expansion of  $\frac{1}{\xi - z}$  about  $z_0$ , this gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{\xi - z} d\xi \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left( \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right)}_{=: a_n} \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n. \end{aligned}$$

- (ii) Analytic functions are automatically smooth, hence complex-differentiable with continuous

derivative.

(iii) If  $r < r_1 < R$ , then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left( \frac{1}{2\pi i} \int_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \right),$$

and since the series converges uniformly, we have

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

Since  $r$  was arbitrary, this holds for any  $0 < r_1 < R$ , whence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for all  $z \in U(z_0, R)$ .

□

**Corollary:** Let  $U \subseteq \mathbb{C}$  be open, let  $z_0 \in U$ , and  $r > 0$  with  $B(z_0, r) \subseteq U$ . The following hold:

(i) for all  $z \in U(z_0, r)$ ,

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi;$$

(ii) for all  $n > 0$ ,

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{\zeta \in S(z_0, r)} |f(\zeta)|.$$

This particular result is known as the *Cauchy Estimate*.

**Theorem** (Liouville's Theorem): If  $f: \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic and bounded in modulus, then  $f$  is constant.

Liouville's Theorem follows from applying Cauchy's estimate to  $f$  and using the fact that  $f$  is bounded to find that all higher derivatives of  $f$  vanish.

**Theorem** (Fundamental Theorem of Algebra): If  $p(z) = a_n z^n + \cdots + a_1 z + a_0$  has  $n \geq 1$  and  $a_n \neq 0$ , then there is at least one  $z_0$  such that  $p(z_0) = 0$ .

*Proof.* Suppose  $p(z)$  were never zero. It would follow then that  $\frac{1}{p(z)}$  is also an entire function.

Since  $\lim_{|z| \rightarrow \infty} |p(z)| = \infty$ , it follows that  $\lim_{|z| \rightarrow \infty} \frac{1}{|p(z)|} = 0$ , whence  $\left| \frac{1}{p(z)} \right|$  is an entire function that is bounded (as all functions that vanish at infinity are bounded). This means that  $\frac{1}{p(z)}$  is constant, so  $p(z)$  is constant. □

**Corollary:** Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be a nonconstant entire function. Then,  $f(\mathbb{C})$  is dense in  $\mathbb{C}$ .

*Proof.* Suppose there were  $w \in \mathbb{C}$  and  $r > 0$  such that  $U(w, r) \cap f(\mathbb{C}) = \emptyset$ . Then,  $|f(z) - w| \geq r$  for all  $z \in \mathbb{C}$ , meaning that

$$g(z) = \frac{1}{f(z) - w}$$

is bounded and entire (the entirety following from the fact that  $f(z) - w$  is nonvanishing). □

## Cycles, Winding Numbers, and Homology

Now, we may generalize some of these results related to Cauchy's Integral Formula.

**Proposition:** Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a piecewise  $C^1$  loop. For all  $z \in \mathbb{C} \setminus \text{im}(\gamma)$ , we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{\xi - z} d\xi \in \mathbb{Z}.$$

*Proof.* Let  $\phi: [a, b] \rightarrow \mathbb{C}$  be defined by

$$\phi(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} ds.$$

Then, we observe

$$\phi(b) = \oint_{\gamma} \frac{1}{\xi - z} d\xi.$$

Then, define  $\psi: [a, b] \rightarrow \mathbb{C}$  by

$$\psi(t) = \frac{e^{\phi(t)}}{\gamma(t) - z}.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned} \phi'(t) &= \frac{\gamma'(t)}{\gamma(t) - z} \\ \psi'(t) &= \frac{\phi'(t)e^{\phi(t)}}{\gamma(t) - z} - \frac{e^{\phi(t)}\gamma'(t)}{(\gamma(t) - z)^2} \\ &= 0, \end{aligned}$$

whence  $\psi(t)$  is constant, and  $\psi(t) = \psi(a)$ , so

$$\psi(a) = \frac{1}{\gamma(a) - z}.$$

In particular,  $\psi(b) = \psi(a)$ , so

$$\begin{aligned} e^{\phi(b)} &= \psi(b)(\gamma(b) - z) \\ &= \psi(a)(\gamma(a) - z) \\ &= 1, \end{aligned}$$

so  $\phi(b) = 2\pi i k$  for some  $k \in \mathbb{Z}$ . □

**Definition:** Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a piecewise  $C^1$  loop. For all  $z \in \mathbb{C} \setminus \text{im}(\gamma)$ , define

$$n(\gamma; z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{\xi - z} d\xi$$

to be the *winding number* of  $\gamma$  about  $z$ .

**Definition:** A piecewise  $C^1$  *cycle* is a formal sum

$$\Gamma = \gamma_1 + \cdots + \gamma_n,$$

where the  $\gamma_j: [a_j, b_j] \rightarrow \mathbb{C}$  are piecewise  $C^1$  loops. The *length* of  $\Gamma$  is the sum of the lengths of the respective  $\gamma_j$ .

Given a piecewise  $C^1$  cycle  $\Gamma$ , define

$$\oint_{\Gamma} f(z) dz = \sum_{j=1}^n \oint_{\gamma_j} f(z) dz,$$

and

$$n(\Gamma; z) = \sum_{j=1}^n n(\gamma_j; z).$$

**Proposition:** The following hold for the winding number  $n(\gamma; z)$ :

- (i) the function  $n(\Gamma; \cdot): \mathbb{C} \setminus \text{im}(\Gamma) \rightarrow \mathbb{Z}$  is continuous;
- (ii)  $n(\Gamma; z)$  is constant on each connected component of  $\mathbb{C} \setminus \text{im}(\Gamma)$ ;
- (iii) there exists a unique unbounded connected component with  $n(\Gamma; z) = 0$  for all  $z$  in this unbounded connected component.

*Proof.*

- (i) Since  $\text{im}(\Gamma)$  is compact, any  $z \notin \text{im}(\Gamma)$  admits a strictly positive

$$\text{dist}_{\text{im}(\Gamma)}(z) = \inf_{w \in \text{im}(\Gamma)} |w - z|.$$

Let  $w \in \mathbb{C}$  be such that

$$|w - z| < \frac{1}{2} \text{dist}_{\text{im}(\Gamma)}(z),$$

so that  $w \in \mathbb{C} \setminus \text{im}(\Gamma)$ . Observe then that

$$\begin{aligned} |n(\Gamma; z) - n(\Gamma; w)| &= \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\xi - z} - \frac{1}{\xi - w} d\xi \right| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^n \oint_{\gamma_j} \left| \frac{1}{\xi - z} - \frac{1}{\xi - w} \right| |d\xi| \\ &= \frac{1}{2\pi} \sum_{j=1}^n \oint_{\gamma_j} \left| \frac{z - w}{(\xi - z)(\xi - w)} \right| |d\xi| \\ &\leq \frac{1}{2\pi} \left( \frac{2}{\text{dist}_{\text{im}(\Gamma)}(z)} \right)^2 \ell(\Gamma) |z - w|, \end{aligned}$$

whence  $|n(\Gamma; z) - n(\Gamma; w)|$  is sufficiently small whenever  $|z - w|$  is sufficiently small.

- (ii) If  $C$  is a connected component of  $\mathbb{C} \setminus \text{im}(\Gamma)$ , and  $n(\Gamma; \cdot): C \rightarrow \mathbb{Z}$  is continuous, then since  $\mathbb{Z}$  is discrete,  $n(\Gamma; \cdot)$  is constant on  $C$ .
- (iii) For uniqueness, if there are unbounded connected components  $C_1$  and  $C_2$  of  $\mathbb{C} \setminus \text{im}(\Gamma)$ , then there exists  $M > \sup_{z \in \text{im}(\Gamma)} |z|$  and  $w_1 \in C_1, w_2 \in C_2$  such that  $|w_1| > 2M$  and  $|w_2| > 2M$ . Since  $\mathbb{C} \setminus \overline{U(0, 2M)}$  is path connected, there exists  $\gamma: [0, 1] \rightarrow \mathbb{C}$  with  $|\gamma(t)| \geq 2M$  and  $\gamma(0) = w_1, \gamma(1) = w_2$ . Therefore,  $w_1$  and  $w_2$  are in the same connected component.

Existence then follows from  $\text{im}(\Gamma)$  being compact.

Finally, let  $(z_n)_n \subseteq \mathbb{C}$ , where  $\mathbb{C}$  is the unbounded connected component, be such that  $\lim_{n \rightarrow \infty} |z_n| = \infty$ . For  $M > \sup_{z \in \text{im}(\gamma)} |z|$ , there exists  $m \in \mathbb{N}$  such that  $|z_m| > M$ . Then, we have

$$\begin{aligned} |n(\Gamma; z_m)| &= \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\xi - z} d\xi \right| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^k \oint_{\gamma_j} \frac{1}{|\xi - z|} |d\xi| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^k \oint_{\gamma_j} \frac{1}{|z_m| - M} |d\xi| \\ &= \frac{\ell(\Gamma)}{2\pi(|z_m| - M)}, \end{aligned}$$

whence  $\lim_{m \rightarrow \infty} n(\Gamma; z_m) = 0$ , meaning that there exists  $N$  such that  $|n(\Gamma; z_m)| < 1$  for all  $m \geq N$ , meaning  $n(\Gamma; z_m) = 0$  for all sufficiently large  $m$ . Since  $\mathbb{C}$  is connected, it thus follows that  $n(\Gamma; z) = 0$  for all  $z \in \mathbb{C}$ . □

**Definition:** Let  $U \subseteq \mathbb{C}$  be open. A cycle  $\Gamma$  is *homologous to zero in  $U$*  if  $\text{im}(\Gamma) \subseteq U$  and for all  $z \in \mathbb{C} \setminus U$ ,  $n(\Gamma; z) = 0$ .

**Theorem** (Cauchy's Integral Formula, General Case): Let  $\Gamma = \gamma_1 + \cdots + \gamma_k$  be a piecewise  $C^1$  cycle homologous to zero in  $U$ , and  $f: U \rightarrow \mathbb{C}$  holomorphic. Then, for all  $z \in U \setminus \text{im}(\Gamma)$ ,

$$n(\Gamma; z)f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi$$

**Theorem** (Cauchy's Integral Theorem): Let  $U \subseteq \mathbb{C}$  be open,  $f: U \rightarrow \mathbb{C}$  holomorphic, and  $\Gamma$  homologous to zero in  $U$ . Then,

$$\oint_{\Gamma} f(z) dz = 0.$$

**Definition:** A region  $U \subseteq \mathbb{C}$  is called *simply connected* if its complement in the extended complex plane is connected.

**Theorem:** If  $U \subseteq \mathbb{C}$  is simply connected, then every loop in  $U$  is homologous to zero.

*Proof.* Extend the function  $n(\gamma; \cdot)$  to the extended complex plane by defining  $n(\gamma; \infty) = 0$ . This extended function is continuous on  $\hat{\mathbb{C}} \setminus U$ , as  $n(\gamma; \cdot)$  is zero on the unique unbounded connected component of  $\mathbb{C} \setminus \text{im}(\gamma)$ . It follows that  $n(\gamma; z)$  is equal to zero on  $\hat{\mathbb{C}} \setminus U$ , whence  $\gamma$  is homologous to zero in  $U$ . □

**Proposition:** Let  $U \subseteq \mathbb{C}$  be a region,  $f: U \rightarrow \mathbb{C}$  holomorphic. The following are equivalent:

- (i) there exists a holomorphic function  $F: U \rightarrow \mathbb{C}$  such that  $F'(z) = f(z)$ ;
- (ii) for every piecewise  $C^1$  loop  $\gamma$  with  $\text{im}(\gamma) \subseteq U$ , we have

$$\oint_{\gamma} f(z) dz = 0.$$

*Proof.* The direction (i)  $\Rightarrow$  (ii) follows immediately from the fundamental theorem of calculus. In the reverse direction, we define  $F: U \rightarrow \mathbb{C}$  by

$$f(z) = \int_{\sigma(z_0, z)} f(\xi) d\xi,$$

where  $\sigma(z_0, z): [0, 1] \rightarrow U$  is a piecewise  $C^1$  curve with  $\sigma(0) = z_0$  and  $\sigma(1) = z$ . Such a curve always



exists as  $U$  is open and connected (hence path-connected). The integral is well-defined, since if  $\gamma_1$  and  $\gamma_2$  are any two such paths, then  $\Gamma = \gamma_1 \setminus \gamma_2$  is a piecewise  $C^1$  loop. Additionally,  $F$  is continuous.

Now, we evaluate the derivative of  $F$ . Let  $z \in U$ ,  $r > 0$  such that  $U(z, r) \subseteq U$ , and  $h \in \mathbb{C}$  be such that  $z + h \in U(z, r)$ . Then,

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{\sigma(z_0, z_0+h)} f(\xi) d\xi - \frac{1}{h} \int_{\sigma(z_0, z)} f(\xi) d\xi \\ &= \frac{1}{h} \int_{\sigma(z, z+h)} f(\xi) d\xi. \end{aligned}$$

We may assume that  $\sigma(z, z+h)$  is a straight line, so that

$$\int_{\sigma(z, z+h)} f(\xi) d\xi = hf(z),$$

meaning that

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \frac{1}{|h|} \left| \int_{\sigma(z, z+h)} f(\xi) d\xi - f(z) \right| \\ &\leq \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)|. \end{aligned}$$

Since  $f$  is continuous, it follows that the right hand side goes to zero as  $|h|$  becomes small. Thus,  $F'$  is continuous, so  $f$  is holomorphic.  $\square$

Observe that  $\mathbb{C} \setminus \{0\}$  is not simply connected, meaning that, for instance, the function

$$f(z) = \frac{1}{z}$$

does not have a holomorphic antiderivative defined on the entirety  $\mathbb{C} \setminus \{0\}$ , as

$$\int_{S^1} f(z) dz = 2\pi i.$$

Yet, if we restrict  $f(z)$  to a simply connected subset of  $\mathbb{C}$ , there is a holomorphic antiderivative. Choosing such a simply connected subset of  $\mathbb{C}$  is known as choosing a *branch* of the logarithm. In fact, there is more that we can say.

**Corollary:** Let  $U \subseteq \mathbb{C}$  be simply connected, and let  $f: U \rightarrow \mathbb{C} \setminus \{0\}$  be a nonvanishing holomorphic function. For each fixed pair  $z_0 \in U$  and  $w_0 \in \mathbb{C}$  for which  $e^{w_0} = f(z_0)$ , there exists a unique holomorphic function  $g: U \rightarrow \mathbb{C}$  for which  $g(z_0) = w_0$  and  $e^{g(z)} = f(z)$ .

We call  $g$  the logarithm of  $f$ , written  $g(z) = \log(f(z))$ .

*Proof.* Since  $f$  is nonvanishing and  $U$  is simply connected, it follows that  $\frac{f'}{f}$  is holomorphic on  $U$ , meaning there is  $\tilde{g}: U \rightarrow \mathbb{C}$  such that  $\tilde{g}'(z) = \frac{f'(z)}{f(z)}$ . Thus, there is some constant  $K$  such that

$$f(z) = Ke^{\tilde{g}(z)}.$$

Define

$$g(z) = \log(K) + \tilde{g}(z).$$

$\square$

**Theorem** (Morera's Theorem): Let  $U \subseteq \mathbb{C}$  be open,  $f: U \rightarrow \mathbb{C}$  continuous. Suppose

$$\oint_T f(z) dz = 0$$

for all triangles  $T \subseteq U$  homologous to zero. Then,  $f$  is holomorphic.

*Proof.* Since  $U$  is open, if  $z_0 \in U$ , there is  $r$  such that  $U(z_0, r) \subseteq U$ . Define  $F: U(z_0, r) \rightarrow \mathbb{C}$  by

$$F(z) = \int_{\sigma(z_0, z)} f(\xi) d\xi,$$

where  $\sigma$  is the straight line from  $z_0$  to  $z$ . For  $0 < |h| < r - |z - z_0|$ , we construct the straight lines  $\sigma(z, z + h)$  and  $\sigma(z_0, z + h)$ , such that

$$T = \sigma(z_0, z) + \sigma(z, z + h) - \sigma(z_0, z + h),$$

and

$$\begin{aligned} \oint_T f(z) dz &= 0 \\ &= \int_{\sigma(z_0, z)} f(\xi) d\xi + \int_{\sigma(z, z+h)} f(\xi) d\xi - \int_{\sigma(z_0, z+h)} f(\xi) d\xi \\ &= F(z) - F(z+h) + \int_{\sigma(z, z+h)} f(\xi) d\xi, \end{aligned}$$

meaning

$$\begin{aligned} F(z+h) - F(z) &= \int_{\sigma(z, z+h)} f(\xi) d\xi \\ \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{\sigma(z, z+h)} f(\xi) d\xi \\ \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \frac{1}{h} \int_{\sigma(z, z+h)} (f(\xi) - f(z)) d\xi \right| \\ &\leq \frac{1}{|h|} |h| \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)| \\ &= \sup_{w \in \text{im}(\sigma(z, z+h))} |f(w) - f(z)|. \end{aligned}$$

Since  $f$  is continuous, it follows that for sufficiently small  $|h|$ , the right-hand-side goes to zero, whence  $F'(z) = f(z)$ , meaning  $F$  is holomorphic, so  $F$  is analytic, meaning  $f$  is analytic, so  $f$  is holomorphic.  $\square$

**Corollary:** Let  $U \subseteq \mathbb{C}$  be open,  $\gamma: [a, b] \rightarrow U$  a piecewise  $C^1$  curve, and  $g: U \times \text{im}(\gamma) \rightarrow \mathbb{C}$  continuous. Suppose that for each  $w \in \text{im}(\gamma)$ , the function  $g(\cdot, w)$  is holomorphic. Then,

$$f(z) := \int_{\gamma} g(z, w) dw$$

is holomorphic.

*Proof.* Let  $T$  be a triangle in  $U$  homologous to zero. Then, by Fubini's Theorem,

$$\oint_T f(z) dz = \oint_T \int_{\gamma} g(z, w) dw dz$$

$$= \int_{\gamma} \oint_{\Gamma} g(z, w) dz dw.$$

The interior integral vanishes for every  $w$  as  $g(\cdot, w)$  is holomorphic. Thus,  $f$  is holomorphic.  $\square$

**Definition:** Let  $U \subseteq \mathbb{C}$  be open,  $\gamma_1, \gamma_2$  piecewise  $C^1$  loops in  $U$ . We say  $\gamma_1$  and  $\gamma_2$  are homotopic in  $U$  if there is a continuous function

$$H: [a, b] \times [0, 1] \rightarrow U$$

such that

$$H(s, 0) = \gamma_1(s)$$

$$H(s, 1) = \gamma_2(s)$$

$$H(a, t) = H(b, t).$$

For each  $t$ ,  $H(\cdot, t)$  is a continuous loop. We call  $H$  a homotopy between  $\gamma_0$  and  $\gamma_1$ .

**Theorem:** If  $\gamma_0$  and  $\gamma_1$  are homotopic in  $U$ , then  $\Gamma = \gamma_1 - \gamma_0$  is homologous to zero in  $U$ .

**Theorem:** If  $K \subseteq U$  is compact and  $U$  is connected, then there is some cycle  $\Gamma$  homologous to zero in  $U$  such that  $n(\Gamma; z) = 1$  for all  $z \in K$ .

**Corollary:** Let  $U$  be a region. The following are equivalent:

- (i)  $U$  is simply connected;
- (ii) for every nonvanishing holomorphic function  $f: U \rightarrow \mathbb{C} \setminus \{0\}$ , there is a holomorphic function  $g: U \rightarrow \mathbb{C}$  such that  $f(z) = e^{g(z)}$ ;
- (iii) for all cycles  $\Gamma$  with  $\text{im}(\Gamma) \subseteq U$ ,  $\Gamma$  is homologous to zero in  $U$ .

## Maximum Modulus Principle

**Theorem (Mean Value Property):** Let  $U \subseteq \mathbb{C}$  be open,  $f: U \rightarrow \mathbb{C}$  holomorphic, with  $z_0 \in U$  and  $r > 0$  such that  $B(z_0, r) \subseteq U$ . Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

*Proof.* By the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{\xi - z_0} d\xi.$$

Parametrizing  $\gamma(\theta) = z_0 + re^{i\theta}$ , we get

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

$\square$

**Corollary:** If  $u: \mathbb{R}^2 \supseteq U \rightarrow \mathbb{R}$  is harmonic,  $(x_0, y_0) \in U$ , and  $r > 0$  is such that  $B((x_0, y_0), r) \subseteq U$ , then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos(\theta), y_0 + r \sin(\theta)) d\theta.$$

*Proof.* Take real parts of the mean value property for holomorphic  $f = u + iv$ .  $\square$

Observe then that the triangle inequality implies that

$$|u(x_0, y_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |u(x_0 + r \cos(\theta), y_0 + r \sin(\theta))| d\theta.$$

Functions that satisfy this weaker criterion are known as *subharmonic*. It is subharmonic functions for which the most general case of the *maximum modulus principle* hold.

**Theorem (Maximum Modulus Principle):** Let  $U \subseteq \mathbb{R}^2$  be open and connected, and let  $u: U \rightarrow \mathbb{R}$  be subharmonic. Suppose there exists  $(x_0, y_0) \in U$  such that  $u(x_0, y_0) \geq u(x, y)$  for all  $x, y \in U$ . Then,  $u$  is constant.

*Proof.* Let  $\lambda = u(x_0, y_0)$ , and let  $E = \{(x, y) \mid u(x, y) = \lambda\} = u^{-1}(\{\lambda\})$ . We see immediately that  $E$  is closed; we claim that  $E$  is also open.

Fix  $(x_1, y_1) \in E$ . Then,  $u(x_1, y_1) = \lambda$ . Take  $r > 0$  such that  $U((x_1, y_1), r) \subseteq U$ . Then, for all  $0 < s < r$ , we have  $S((x_1, y_1), s) \subseteq U$ , meaning that

$$\begin{aligned} \lambda &= u(x_1, y_1) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + s \cos(\theta), y_1 + s \sin(\theta)) d\theta \\ &\leq \lambda, \end{aligned}$$

with the latter inequality following from the fact that  $\lambda$  is a local maximum. Therefore,  $u(x_1 + s \cos(\theta), y_1 + s \sin(\theta)) = \lambda$  for all  $0 < s < r$ , whence  $U((x_1, y_1), r) \subseteq E$ . Thus,  $E$  is open, so since  $U$  is connected, it follows that  $E$  is all of  $U$ , meaning  $u$  is constant.  $\square$

**Corollary:** If  $U \subseteq \mathbb{R}^2$  is bounded and  $u: \bar{U} \rightarrow \mathbb{R}$  is continuous with  $u|_U$  subharmonic, then there exists  $(x_0, y_0) \in \partial U$  such that  $u(x_0, y_0) = \sup_{(x,y) \in U} u(x, y)$ .

**Corollary:** If  $U \subseteq \mathbb{C}$  is open and connected, with  $f: U \rightarrow \mathbb{C}$  holomorphic, then if  $|f|: U \rightarrow \mathbb{R}$  has a local maximum at  $z_0 \in U$ , then  $f$  is constant.

*Proof.* Let  $r > 0$  be such that  $U(z_0, r) \subseteq U$ . Then, restricting  $|f|$  to  $U(z_0, r)$ , we see that  $|f|$  restricted to  $U(z_0, r)$  is subharmonic viewed as a function on  $U(z_0, r)$ , hence  $|f|$  is constant on  $U(z_0, r)$ .

Now, by the mean value property and triangle inequality, it follows that for all  $0 < s < r$ , we have

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + se^{i\theta})| d\theta \\ &= |f(z_0)|, \end{aligned}$$

meaning that these are equalities. In particular, there exists some  $\theta_s$  such that  $e^{i\theta_s} f(z_0 + se^{i\theta}) \geq 0$ , meaning that for this value of  $s$ , we have

$$\begin{aligned} |f(z_0)| &= e^{i\theta_s} \int_0^{2\pi} f(z_0 + se^{i\theta}) d\theta \\ &= e^{i\theta_s} f(z_0), \end{aligned}$$

with the latter equality following from the mean value property. Since this holds for any  $s$ , it follows that  $\theta_s$  is independent of  $s$ , meaning that  $f(z)e^{i\theta_s} \geq 0$  for all  $z \in U(z_0, r)$ , meaning that  $\text{Im}(e^{i\theta_s} f(z)) = 0$  on  $U(z_0, r)$ , whence  $f(z)e^{i\theta_s}$  is constant, meaning  $f$  is constant on  $U(z_0, r)$ .

Finally, by the identity theorem, it follows that  $f$  is constant on  $U$ .  $\square$

**Definition:** Let  $U \subseteq \mathbb{R}^2$  be an open set. We say a sequence  $U \supseteq ((x_n, y_n))_n \rightarrow \partial U$  if, for every compact  $K \subseteq U$ , the set  $\{n \in \mathbb{N} \mid (x_n, y_n) \in K\}$  is finite.

**Definition:** Let  $U \subseteq \mathbb{R}^2$  be an open set. Given a function  $u: U \rightarrow \mathbb{R}$ , define

$$\limsup_{(x,y) \rightarrow \partial U} u(x,y) := \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{(x,y) \in U \setminus K} u(x,y).$$

These definitions allow us to extend the maximum modulus principle for subharmonic functions even further.

**Theorem:** Let  $U \subseteq \mathbb{C}$  be a region,  $u: U \rightarrow \mathbb{R}$  a nonconstant subharmonic function. If  $((x_n, y_n))_n \subseteq U$  is such that  $u(x_n, y_n) \rightarrow \sup_{(x,y) \in U} u(x,y)$ , then  $((x_n, y_n))_n \rightarrow \partial U$ . Moreover,  $\limsup_{(x,y) \rightarrow \partial U} u(x,y) = \sup_{(x,y) \in U} u(x,y)$ .

*Proof.* Suppose toward contradiction that  $((x_n, y_n))_n \not\rightarrow \partial U$ , so there exists a compact subset  $K \subseteq U$  and a subset  $((x_{n_k}, y_{n_k}))_k$  wholly contained in  $K$ . Since  $K$  is compact, there is a subsequence of  $((x_{n_k}, y_{n_k}))_k$  converging to  $(x_0, y_0) \in U$ . Therefore,  $u(x_0, y_0) = \sup_{(x,y) \in U} u(x,y)$ , so  $u$  is constant by the maximum modulus principle, which is a contradiction.

Finally,  $\limsup_{(x,y) \rightarrow \partial U} u(x,y) \leq \sup_{(x,y) \in U} u(x,y)$ , while if  $((x_n, y_n))_n \rightarrow \partial U$  is such that  $u(x_n, y_n)$  converges to  $\sup_{(x,y) \in U} u(x,y)$ , then  $\sup_{(x,y) \in U} u(x,y) = \lim_{n \rightarrow \infty} u(x_n, y_n) \leq \limsup_{(x,y) \rightarrow \partial U} u(x,y)$ .  $\square$

**Theorem (Open Mapping Principle):** Let  $U \subseteq \mathbb{C}$  be a region, and let  $f: U \rightarrow \mathbb{C}$  be a nonconstant holomorphic function. Then,  $f(U) \subseteq \mathbb{C}$  is open.

*Proof.* Let  $z_0 \in U$  and  $r > 0$  be such that  $B(z_0, r) \subseteq U$ . We will show that there exists  $R$  such that  $U(f(z_0), R) \subseteq f(U(z_0, r)) \subseteq U$ , whence  $f(U)$  is open.

Since  $U$  is a region and  $f$  is nonconstant, the zeros of  $g(z) := f(z) - f(z_0)$  are isolated, so there exists some  $0 < s < r$  such that

$$\delta = \inf_{|z-z_0|=s} |f(z) - f(z_0)|$$

is strictly greater than zero. We claim that  $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$ . Suppose this were not the case, meaning there would be some  $\xi \in U(f(z_0), \delta/2) \setminus f(U(z_0, r))$ , and define  $h: B(z_0, s) \rightarrow \mathbb{C}$  by

$$h(z) = \frac{1}{f(z) - \xi}.$$

Since  $\xi \notin f(U(z_0, r))$ , this is holomorphic, while  $\xi \in U(f(z_0), \delta/2)$  implies

$$\begin{aligned} \sup_{|z-z_0|=s} |h(z)| &= \sup_{|z-z_0|=s} \frac{1}{|f(z) - \xi|} \\ &\leq \sup_{|z-z_0|=s} \frac{1}{|f(z) - f(z_0)| - |f(z_0) - \xi|} \\ &\leq \frac{1}{\delta - \delta/2} \\ &= \frac{2}{\delta}. \end{aligned}$$

Yet,

$$\begin{aligned} |h(z_0)| &= \frac{1}{|f(z_0) - \xi|} \\ &> \frac{2}{\delta}, \end{aligned}$$

contradicting the maximum modulus principle. Thus,  $U(f(z_0), \delta/2) \subseteq f(U(z_0, r))$ .  $\square$

In the proof of the Hadamard Three-Lines Theorem, we used the function  $h_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}$  for this purpose.

## Classification of Singularities

The classification of singularities seeks to answer two fundamental questions: if  $U \subseteq \mathbb{C}$  is open,  $z_0 \in U$ , and  $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$  is holomorphic,

- does  $f$  have a holomorphic extension to  $U$  including  $z_0$ ;
- and what else can we say about the behavior of  $f$  at  $z_0$ ?

**Definition:** Let  $U \subseteq \mathbb{C}$  be open,  $z_0 \in U$ ,  $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$  holomorphic.

- If there exists a holomorphic  $g: U \rightarrow \mathbb{C}$  with  $g = f$  on  $U \setminus \{z_0\}$ , then we say  $z_0$  is a *removable singularity*.
- If  $\lim_{z \rightarrow z_0} |f(z)| = \infty$ , then we say  $f$  has a *pole* at  $z_0$ .
- Else, we say  $f$  has an *essential singularity* at  $z_0$ .

**Theorem (Riemann's Theorem on Removable Singularities):** Let  $U \subseteq \mathbb{C}$  be open,  $z_0 \in U$ , and  $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$  holomorphic. Then,  $z_0$  is a removable singularity if and only if  $\lim_{z \rightarrow z_0} f(z) = 0$ .

*Proof.* If  $z_0$  is removable, then  $g(z)$  is a holomorphic function with  $g(z) = f(z)$  on  $U \setminus \{z_0\}$ , and since  $g$  is continuous, it follows that  $\lim_{z \rightarrow z_0} g(z) = g(z_0)$ , whence  $\lim_{z \rightarrow z_0} (z - z_0)g(z) = \lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$ .

Now, if  $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$ , then there is  $r$  such that  $B(z_0, r) \subseteq U$ , and since  $f$  is locally bounded around  $z_0$ , it follows that

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for all  $z \in U(z_0, r)$ . Yet, the formula extends to  $z_0$  as it is bounded, whence we may define the holomorphic extension for  $f$  by

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\zeta)}{\zeta - z} d\zeta & z = z_0 \end{cases}.$$

□

**Proposition (Existence of Laurent Series):** Suppose  $f: A(z_0, r, R) \rightarrow \mathbb{C}$  is holomorphic, with  $0 \leq r < R$ . Then, there exist holomorphic functions

$$\begin{aligned} g_1: U(z_0, R) &\rightarrow \mathbb{C} \\ g_2: \mathbb{C} \setminus B(z_0, r) &\rightarrow \mathbb{C} \end{aligned}$$

such that  $f = g_1 + g_2$  on  $A(z_0, r, R)$ . Moreover, there exists  $(a_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}$  such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for all  $z$ , and the series converges uniformly on  $A(z_0, \rho, s)$  with  $r < \rho < s < R$ .

*Proof.* Fix  $z \in A(z_0, r, R)$ . Then, for  $r < \rho_1, \rho_2 < |z - z_0|$ , the cycle

$$\Gamma_1 = S(z_0, \rho_1) - S(z_0, \rho_2)$$

is homologous to zero in  $A(z_0, r, |z - z_0|)$ . By Cauchy's Integral Theorem, it then follows that

$$\oint_{S(z_0, \rho_1)} \frac{f(\xi)}{\xi - z} d\xi = \oint_{S(z_0, \rho_2)} \frac{f(\xi)}{\xi - z} d\xi.$$

Similarly, for  $|z - z_0| < s_1, s_2 < R$ , we have

$$\oint_{S(z_0, s_1)} \frac{f(\xi)}{\xi - z} d\xi = \oint_{S(z_0, s_2)} \frac{f(\xi)}{\xi - z} d\xi.$$

Define  $g_1: U(z_0, R) \rightarrow \mathbb{C}$  by

$$g_1(z) = \frac{1}{2\pi i} \oint_{S(z_0, s)} \frac{f(\xi)}{\xi - z} d\xi,$$

where  $|z - z_0| < s < R$ . This function is holomorphic by Morera's Theorem. Similarly, we may define  $g: \mathbb{C} \setminus B(z_0, r) \rightarrow \mathbb{C}$  by

$$g_2(z) = -\frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{\xi - z} d\xi,$$

where  $r < \rho < |z - z_0|$ . We claim that  $f = g_1 + g_2$  on  $A(z_0, r, R)$ .

For  $z \in A(z_0, r, R)$ , we may find, for any  $r < \rho < |z - z_0| < s < R$ , we let

$$\Gamma = S(z_0, s) - S(z_0, \rho),$$

homologous to zero in  $A(z_0, r, R)$ , whence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left( \oint_{S(z_0, s)} \frac{f(\xi)}{\xi - z} d\xi - \int_{S(z_0, \rho)} \frac{f(\xi)}{\xi - z} d\xi \right) \\ &= g_1(z) + g_2(z). \end{aligned}$$

□

**Theorem:** Let  $U \subseteq \mathbb{C}$ ,  $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$  be holomorphic with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on  $\dot{U}(z_0, R)$  for some  $R$  with  $U(z_0, R) \subseteq U$ . Then,

- (i)  $f$  has a removable singularity at  $z_0$  if and only if  $a_n = 0$  for all  $n < 0$ ;
- (ii)  $f$  has a pole at  $z_0$  if and only if

$$1 \leq |\{n < 0 \mid a_n \neq 0\}| < \infty.$$

- (iii)  $f$  has an essential singularity at  $z_0$  if and only if

$$|\{n < 0 \mid a_n \neq 0\}| = \infty.$$

*Proof.*

- (i) If  $a_n = 0$  for all  $n < 0$ , then  $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$ , so  $f$  has a removable singularity at  $z_0$ .

Conversely, if  $f$  has a removable singularity at  $z_0$ , then for  $n < 0$ , we have

$$a_n = \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

for any  $0 < \rho < R$ . Since  $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$ , then for any  $\varepsilon > 0$ , there is sufficiently small  $\rho$  such that

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right| \\ &\leq \rho^{-1-n} \sup_{|\xi - z_0| = \rho} |(\xi - z_0)f(z)| \\ &\leq \varepsilon. \end{aligned}$$

Thus,  $|a_n| = 0$  for all  $n < 0$ .

- (ii) If  $a_n \neq 0$  for a nonempty finite collection of  $n < 0$ , we let  $m$  be the largest number such that  $a_{-m} < 0$ , so that  $f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$ . It follows that  $\lim_{z \rightarrow z_0} (z - z_0)^m f(z) = a_{-m} \neq 0$ . In particular, there is some small  $\delta$  such that

$$|(z - z_0)^m f(z) - a_{-m}| < \frac{|a_{-m}|}{2},$$

whence for a sufficiently small  $\varepsilon$ ,

$$\begin{aligned} |f(z)| &> \frac{|a_{-m}| - |(z - z_0)^m f(z) - a_{-m}|}{|z - z_0|^m} \\ &> \frac{|a_{-m}|}{2|z - z_0|^m} \\ &> \frac{1}{\varepsilon}, \end{aligned}$$

so  $f$  has a pole at  $z_0$ .

Conversely, if  $f$  has a pole at  $z_0$ , there is some  $r > 0$  such that  $|f(z)| \geq 1$  whenever  $z \in \dot{U}(z_0, r)$ . Define  $g: \dot{U}(z_0, r) \rightarrow \mathbb{C}$  by

$$g(z) = \frac{1}{f(z)},$$

whence  $g$  is holomorphic and bounded on  $\dot{U}(z_0, r)$ . By the classification of singularities,  $z_0$  is a removable singularity of  $g$ , so there exists a holomorphic function  $h: U(z_0, r) \rightarrow \mathbb{C}$  that is equal to  $\frac{1}{f(z)}$  for  $z \neq z_0$  and  $\lim_{z \rightarrow z_0} \frac{1}{f(z)}$  if  $z = z_0$ . We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

where we must have  $b_0 = 0$ . Let  $m$  be the smallest positive integer such that  $h(z) = (z - z_0)^m \tilde{h}(z)$ , where  $\tilde{h}(z) = \sum_{n=0}^{\infty} b_{n+m} (z - z_0)^n$  and  $b_m \neq 0$ . The function  $\tilde{h}$  is holomorphic on  $U(z_0, r)$  and nonzero on some  $U(z_0, \rho)$  with  $0 < \rho \leq r$ , so that  $\tilde{f}(z) = \frac{1}{\tilde{h}(z)}$  is holomorphic on  $U(z_0, \rho)$ , so

$$f(z) = (z - z_0)^{-m} \tilde{f}(z)$$



on  $\dot{U}(z_0, \rho)$ . Writing

$$\tilde{f}(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

we deduce that

$$f(z) = \sum_{n=-m}^{\infty} c_{n+m} (z - z_0)^n.$$

(iii) Follows from (i) and (ii). □

**Definition:** Let  $U \subseteq \mathbb{C}$  be open.

- If  $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$  has a pole or a removable singularity at  $z_0$ , then the order of the pole or singularity is the smallest  $m \geq 0$  such that

$$f(z) = (z - z_0)^{-m} g(z)$$

with  $g: U \rightarrow \mathbb{C}$  is holomorphic and has  $g(z_0) \neq 0$ .

- If  $f: U \rightarrow \mathbb{C}$  has a zero at  $z_0 \in U$ , then the order of the zero at  $z_0$  is the smallest  $m \geq 0$  such that

$$f(z) = (z - z_0)^m g(z)$$

where  $g: U \rightarrow \mathbb{C}$  is holomorphic and has  $g(z_0) \neq 0$ .

If  $z_0 \in U$  is either a pole or a zero (or a removable singularity that is a zero when  $f$  is extended), then the *order* of  $f$  is the unique  $m \in \mathbb{Z}$  such that

$$f(z) = (z - z_0)^m g(z)$$

with  $g(z_0) \neq 0$ .

**Theorem (Casorati–Weierstrass):** Let  $U \subseteq \mathbb{C}$  be an open set, and let  $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$  be a holomorphic function. If  $z_0$  is an essential singularity of  $f$ , then for any  $r > 0$  with  $U(z_0, r) \subseteq U$ ,  $f(\dot{U}(z_0, r)) \subseteq \mathbb{C}$  is dense.

*Proof.* Suppose  $f(\dot{U}(z_0, r))$  is not dense in  $\mathbb{C}$ . Then, there exists  $\varepsilon > 0$  and  $w \in \mathbb{C}$  such that  $|f(z) - w| \geq \varepsilon$  for all  $z \in \dot{U}(z_0, r)$ . We will show that  $z_0$  is either removable or a pole.

Define

$$g: \dot{U}(z_0, r) \rightarrow \mathbb{C}$$

by

$$g(z) = \frac{1}{f(z) - w}.$$

Observe that  $g$  is definitionally bounded, so  $z_0$  is a removable singularity of  $g$ . Thus, there is a holomorphic function  $h: U(z_0, r) \rightarrow \mathbb{C}$  such that  $h(z) = g(z)$  for  $z \neq z_0$  and  $h(z) = \lim_{z \rightarrow z_0} g(z)$ . We write

$$h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

and take

$$h(z) = (z - z_0)^m \tilde{h}(z),$$

which exists as  $g$  is not uniformly zero, and where  $\tilde{h}(z_0) \neq 0$ . Consequently, there is  $0 < \rho \leq r$  such that  $\tilde{f}: U(z_0, \rho) \rightarrow \mathbb{C}$  given by

$$\tilde{f}(z) = \frac{1}{\tilde{h}(z)},$$

which is holomorphic. Thus,

$$f(z) = w + (z - z_0)^{-m} \tilde{f}(z).$$

Thus, we get

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n,$$

whence  $z_0$  is either removable or a pole. □

## The Argument Principle and Rouché's Theorem

**Definition:** If  $U \subseteq \mathbb{C}$  is open, and  $V \subseteq U$  is an open subset such that  $U \setminus V$  consists solely of isolated points, then a function  $f: V \rightarrow \mathbb{C}$  is *meromorphic* if it is holomorphic on  $V$  and every  $z_0 \in U \setminus V$  is either a pole or a removable singularity. We say  $f$  is meromorphic on  $U$ .

**Theorem:** Let  $U \subseteq \mathbb{C}$  be an open set,  $\Gamma$  a piecewise  $C^1$  cycle homologous to zero in  $U$ . Let  $f$  be meromorphic on  $U$  with no poles or zeros on  $\text{im}(\Gamma)$ .

- (i) The set  $\{z_0 \in U \mid \text{ord}_{z_0}(f) \neq 0, n(\Gamma; z_0) \neq 0\}$  is finite.
- (ii) We have

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = \sum_{\substack{z_0 \in U \\ \text{ord}_{z_0}(f) \neq 0 \\ n(\Gamma; z_0) \neq 0}} n(\Gamma; z_0) \text{ord}_{z_0}(f).$$

*Proof.*

- (i) Let  $K = \{z \in U \mid n(\Gamma; z) \neq 0\} \cup \text{im}(\Gamma)$ . We know that for  $R > \sup_{w \in \text{im}(\Gamma)} |w|$ , we have  $n(\Gamma; z) = 0$  for all  $z \in \mathbb{C} \setminus B(0, R)$ , meaning that  $K$  is bounded. Furthermore, if  $(z_n)_n \rightarrow z \in \mathbb{C}$ , then either  $z \in \text{im}(\Gamma) \subseteq K$  or  $z \in \mathbb{C} \setminus \text{im}(\Gamma)$ , so that  $n(\Gamma; z) \neq 0$  by the continuity of the map  $w \mapsto n(\Gamma; w)$ . Thus,  $z \in K$  as  $\Gamma$  is homologous to zero in  $U$ . It follows thus that  $K$  is compact.

Let  $E = \{z_0 \in K \mid \text{ord}_{z_0}(f) \neq 0\}$ , meaning that  $E$  consists of poles and zeros of  $f$ . These points are isolated, meaning  $E$  is a closed subset of  $K$ , hence compact. Since  $E$  is compact and contains isolated points only, it follows that  $E$  is finite.

- (ii) For each  $z \in K$ , select  $\delta_z > 0$  such that  $U(z, \delta_z) \subseteq U$ , and  $U(z, \delta_z) \cap E \subseteq \{z\}$ . Such a  $\delta_z$  exists since  $z$  is isolated in  $E$  and  $U$  is open. The collection  $\{U(z, \delta_z) \mid z \in K\}$  is an open cover of  $K$ , so there is a finite subcover  $\{U(z_1, \delta_1), \dots, U(z_m, \delta_m)\}$ . Define

$$V = \bigcup_{j=1}^m U(z_j, \delta_j),$$

so that  $V$  is open with  $K \subseteq V \subseteq U$ . Since  $\{z \in U \mid n(\Gamma; z) \neq 0\} \subseteq K \subseteq V$ , it follows that  $\Gamma$  is homologous to zero in  $V$ .

Define

$$g(z) = f(z) \prod_{z \in E} (z - z_0)^{-\text{ord}_{z_0}(f)}.$$

Since  $\text{ord}_z(g) = 0$  for all  $z \in V$ , all the singularities of  $g$  are removable, and  $\frac{g'}{g}$  is holomorphic on  $V$  and satisfies

$$\frac{g'}{g} = \frac{f'}{f} - \sum_{z_0 \in E} \frac{1}{z - z_0} \text{ord}_{z_0}(f).$$

Cauchy's Integral theorem provides the desired result.

□

**Theorem (Rouché's Theorem):** Let  $U \subseteq \mathbb{C}$  be an open set,  $\Gamma$  a piecewise  $C^1$  cycle homologous to zero in  $U$ . Let  $f$  and  $g$  be meromorphic on  $U$  with no poles or zeros on  $\text{im}(\Gamma)$ . If  $|f(z) - g(z)| < |f(z)| + |g(z)|$  for all  $z \in \text{im}(\Gamma)$ , then

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{\Gamma} \frac{g'(z)}{g(z)} dz.$$

*Proof.* Since  $|f(z) - g(z)| < |f(z)| + |g(z)|$  on  $\text{im}(\Gamma)$ , coupled with the fact that  $\text{ord}_z(f) = \text{ord}_z(g) = 0$  on  $\text{im}(\Gamma)$  implies that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1$$

for all  $z \in \text{im}(\Gamma)$ . This only holds if  $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$  for  $z \in \text{im}(\Gamma)$ . Since  $\text{im}(\Gamma)$  is compact, there exists some  $\varepsilon > 0$  such that

$$\text{dist}_{(-\infty, 0]} \left| \frac{f}{g}(\text{im}(\Gamma)) \right| \geq \varepsilon.$$

Since  $\frac{f}{g}$  is continuous and  $\text{im}(\Gamma)$  is compact, there also exists some  $\delta > 0$  such that whenever  $\text{dist}_{\text{im}(\Gamma)}(z) < \delta$ , we have  $\frac{f(z)}{g(z)} \in \mathbb{C} \setminus (-\infty, 0]$ .

Setting  $V = \{z \in U \mid \text{dist}_{\text{im}(\Gamma)}(z) < \delta\}$ , we let  $h: V \rightarrow \mathbb{C}$  be defined by

$$h(z) = \log \left( \frac{f(z)}{g(z)} \right)$$

for the branch of the logarithm that excludes  $(-\infty, 0]$ , which is well-defined as  $\frac{f}{g} \notin (-\infty, 0]$  on  $V$ . This satisfies

$$\frac{h'}{h} = \frac{f'}{f} - \frac{g'}{g},$$

whence by Cauchy's Integral Theorem,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{h'(z)}{h(z)} dz \\ &= \frac{1}{2\pi i} \left( \oint_{\Gamma} \frac{f'(z)}{f(z)} dz - \oint_{\Gamma} \frac{g'(z)}{g(z)} dz \right). \end{aligned}$$

□

**Remark:** Most use cases for Rouché's Theorem involve finding  $g(z)$  such that  $|f(z) - g(z)| < |g(z)|$  on  $\text{im}(\Gamma)$ , where both  $f(z)$  and  $g(z)$  have no zeros or poles on  $\text{im}(\Gamma)$ .

## Worked Examples and Problem-Solving Methods

**Example:** Suppose  $U$  is a region in  $\mathbb{C}$  that contains 0, and suppose  $f: U \rightarrow \mathbb{C}$  is a holomorphic function satisfying

$$\left| f\left(\frac{1}{n}\right) \right| < e^{-n}$$

for sufficiently large  $n$ . We will show that this means  $f$  is 0 everywhere.

Toward this end, since  $U$  is open, there is some  $r > 0$  such that  $U(0, r) \subseteq U$ . Since  $f$  is holomorphic, on  $U(0, r)$ , we may write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for some sequence  $(a_n)_n \subseteq \mathbb{C}$ . Now, we also observe that

$$\begin{aligned} |f(0)| &= \lim_{n \rightarrow \infty} \left| f\left(\frac{1}{n}\right) \right| \\ &\leq \lim_{n \rightarrow \infty} e^{-n} \\ &= 0. \end{aligned}$$

Suppose toward contradiction that  $f$  were nonconstant. Then, there would be some minimal positive value  $\ell$  such that

$$f(z) = z^\ell \sum_{n=0}^{\infty} a_{n+\ell} z^n$$

has  $a_\ell \neq 0$ . Thus, defining

$$g(z) = \sum_{n=0}^{\infty} a_{n+\ell} z^n,$$

we observe that  $g(0) \neq 0$ , meaning that on some sufficiently small ball  $U(0, \delta) \subseteq U(0, r)$ , we have  $|g(z)| > \left| \frac{a_\ell}{2} \right|$  for all  $z \in U(0, \delta)$ . In particular, this means that for  $n$  with  $\frac{1}{n} < \delta$ ,

$$\begin{aligned} e^{-n} &\geq \left| f\left(\frac{1}{n}\right) \right| \\ &= n^{-\ell} \left| g\left(\frac{1}{n}\right) \right| \\ &\geq \frac{|a_\ell|}{2n^\ell}, \end{aligned}$$

whence

$$|a_\ell| \leq \frac{2n^\ell}{e^n}.$$

Yet, since  $n$  is arbitrary and  $\ell$  is constant, this implies that  $|a_\ell| = 0$ , contradicting the assumption that there were such a  $g$ . Thus, in particular, we have that  $f(z) = 0$  on  $U(0, r)$ , whence  $f$  is zero everywhere by the identity theorem.

## Cauchy Estimate Problems

**Example:** Suppose  $f$  is an entire function, and suppose there exists a constant  $C$  such that for all  $z \in \mathbb{C}$ ,

$$|f(z)| \leq C(1 + |z|)^{1/2}.$$

We will show that  $f$  is then constant. Toward this end, we will be able to use the Cauchy estimate by taking

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{R^n} \sup_{|z|=R} |f(z)| \\ &\leq \frac{Cn!}{R^n} \sup_{|z|=R} (1 + |z|)^{1/2} \\ &= \frac{Cn!}{R^n} (1 + R)^{1/2}, \end{aligned}$$

whence for all  $n \geq 1$ , since  $R$  is arbitrary, we have  $|f^{(n)}(z)| = 0$ , so  $f$  is constant.

## Maximum Modulus Principle Problems

### The Phragmén–Lindelöf Method

The maximum modulus principle is primarily useful in the case where  $f$  is continuous on the closure of a bounded open set  $U$  and holomorphic on the interior. Yet, this fails to be true if  $U$  is unbounded.

For instance, if

$$U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \right\},$$

and  $f(z) = e^{e^z}$ , then

$$f\left(x \pm \frac{\pi}{2}i\right) = e^{\pm ie^x},$$

whence  $|f(z)| = 1$  for  $z \in \partial U$ . Yet,  $f(z) \rightarrow \infty$  very rapidly along the positive real axis, which is contained in  $U$ .

Yet, all hope is not lost in the case that  $U$  is unbounded. If  $U$  is unbounded and there is  $g: U \rightarrow \mathbb{C}$  such that  $|f| < |g|$ , and  $g \rightarrow \infty$  “slowly” (so to speak) as  $z \rightarrow \infty$ , then it turns out that  $f$  is actually bounded in  $U$ , and we can use the maximum modulus principle to obtain other conclusions about  $f$ .

Finding such a  $g$  is part of the *Phragmén–Lindelöf* method, which we expand upon here.

**Example:** From the Cauchy estimates, we know that if  $f$  is entire and

$$|f(z)| \leq C(1 + |z|)^{1/2},$$

then  $f$  is constant.

**Theorem (Hadamard Three-Lines Theorem):** Let  $a, b \in \mathbb{R}$  be fixed with  $a < b$ . Let  $U = \{z \mid a < \operatorname{Re}(z) < b\}$ . Suppose  $|f(z)| < B$  for all  $z \in U$  and some fixed  $B < \infty$ . Define

$$M(x) = \sup\{|f(z)| \mid z \in \bar{U}\}.$$

Then,

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}.$$

*Proof.* Suppose  $M(a) = M(b) = 1$ . Our task now is to show that  $|f(z)| \leq 1$  for all  $z \in U$ . Toward this end, define

$$h_\varepsilon(z) = \frac{1}{1 + \varepsilon(z - a)}$$

for  $z \in \bar{U}$ . We have  $|h_\varepsilon| \leq 1$  in  $\bar{U}$ , so that

$$|f(z)h_\varepsilon(z)| \leq 1$$

for all  $z \in \partial U$ . Furthermore, since  $|1 + \varepsilon(z - a)| \geq \varepsilon|\operatorname{Im}(z)|$ , we have

$$|f(z)h_\varepsilon(z)| \leq \frac{B}{\varepsilon|\operatorname{Im}(z)|}$$

for all  $z \in \bar{U}$ . Cut out a (closed) rectangle  $R$  from  $\bar{U}$  via the lines  $\operatorname{Im}(z) = \pm \frac{B}{\varepsilon}$ . Thus, along  $\partial R$ , we have  $|f(z)h_\varepsilon(z)| \leq 1$ , so that  $|f(z)h_\varepsilon(z)| \leq 1$  on  $R$  by the maximum modulus principle.

Yet, since  $|f(z)h_\varepsilon(z)| \leq \frac{B}{\varepsilon|\operatorname{Im}(z)|}$  on the entirety of  $\bar{U}$ , and  $\frac{B}{\varepsilon|\operatorname{Im}(z)|} < 1$  outside  $R$ , it follows that  $|fh_\varepsilon| \leq 1$  on  $\bar{U}$ , so  $|f(z)h_\varepsilon(z)| \leq 1$  for all  $z \in U$  and all  $\varepsilon > 0$ . Taking the limit as  $\varepsilon \rightarrow 0$ , we obtain the desired result, that  $|f(z)| \leq 1$ .

In the general case, we define

$$g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)},$$

where for all  $M > 0$  and complex  $w$ , we have  $M^w = e^{w \ln(M)}$ . Then,  $g$  is entire,  $g$  is always nonzero,  $\frac{1}{g}$  is bounded on  $\bar{U}$ , and has

$$\begin{aligned} |g(a + iy)| &= M(a) \\ |g(b + iy)| &= M(b), \end{aligned}$$

meaning that  $\frac{f}{g}$  satisfies the previous assumptions, so that  $|f/g| \leq 1$  in  $U$ . □

In the Phragmén–Lindelöf method, we seek to find a particular  $\varepsilon$ -dependent function  $h_\varepsilon : U \rightarrow \mathbb{C}$  such that the following hold:

- $|fh_\varepsilon(z)| \leq M$  for all  $z \in \partial U$ ;
- $\lim_{\varepsilon \rightarrow 0} h_\varepsilon(z) = 1$ ;
- there exists a *bounded*  $V \subseteq U$  such that  $|fh_\varepsilon| \leq M$  on  $\partial V$  and on  $U \setminus \bar{V}$ .

## Old Exams

### Notation

- $U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$
- $S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$
- $\dot{U}(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$
- $A(z_0, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$