Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a bounded region, $f\colon \overline{U} \to \mathbb{C}$ continuous such that $f|_U$ is holomorphic. Suppose f is nonvanishing in U, and that there exists $c > 0$ such that $|f(z)| = c$ for all $z \in \partial U$. Prove that there exists some $\theta \in \mathbb{R}$ such that $f(z) = ce^{i\theta}$ for all $z \in \overline{U}$.

Solution: Since f is holomorphic on the connected, bounded, open set U, it follows from the maximum modulus principle that for all $z \in U$, we have $|f(z)| \leqslant |f(w)|$ for all $w \in \partial U$. In particular, we must have $|f(z)| \leqslant c$ for all $z \in U$. Since $|f(z)| \neq 0$ for all $z \in U$, it follows that $\frac{1}{|f(z)|} \geqslant \frac{1}{c}$ for all $z \in U$. Yet, at the same time, since $\frac{1}{f(z)}$ is holomorphic, we must have $\frac{1}{|f(z)|} \leqslant \frac{1}{|f(w)|}$ for all $w \in \partial U$, meaning that $\frac{1}{|f(z)|} \leqslant \frac{1}{c}$, so that $\left|\frac{1}{f}\right| = \frac{1}{c}$, or that $|f(z)| = c$ for all $z \in U$.

In particular, for all $z \in U$, we have $|f(z)| \geqslant |f(w)|$ for all $z \in U$, the maximum modulus principle gives that f is constant. Since $|f(z)| = c$, we thus have $f(z) = ce^{i\theta}$ for some $\theta \in \mathbb{R}$.

Problem (Problem 2)**:** For $0 < r < R$, let $A(z_0, r, R) = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. Suppose that there exists a continuous $f\colon \overline{A(z_0, r, R)} \to \mathbb{C}$ such that $f|_{A(z_0,r,R)}$ is holomorphic, and that there exist constants C_r and C_R in $\mathbb{R}$ such that $\operatorname{Re}(f(z)) = C_r$ on $S(z_0, r)$, and $\operatorname{Re}(f(z)) = C_R$ on $S(z_0, R)$> Show that $C_r = C_R$, and that f is constant for all $z \in \overline{A(z_0, r, R)}$.

Solution: Without loss of generality, since we may take $g(z) = f(z - z_0)$, we may assume that $z_0 = 0$, so that we let $u(x, y)\colon \overline{A(0, r, R)} \to \mathbb{R}$ be given by $u(x, y) = \operatorname{Re}(f(x - x_0 + i(y - y_0)))$. Since u is the real part of a holomorphic function, u is necessarily harmonic, so by the extended maximum modulus principle, u takes on its maximum modulus on either $S(0, r)$ or $S(0, R)$. In other words, it is the case that the maximum modulus for u is either $|C_r|$ or $|C_R|$.

Now, consider the function

$$w(x, y) = u(x, y) - C_r - (C_R - C_r)\frac{\ln\left(x^2 + y^2\right) - \ln\left(r^2\right)}{\ln(R^2) - \ln(r^2)}.$$

We start by verifying that w is harmonic. Towards this end, since Laplace's equation is linear, we only need to evaluate the expression in the fraction free of the constants.

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)} \frac{2x}{x^2 + y^2} \\
\frac{\partial^2 w}{\partial x^2} &= \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)} \left(\frac{2}{x^2 + y^2} - 2x\left(\frac{2x}{\left(x^2 + y^2\right)^2}\right)\right) \\
&= \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)} \frac{2y^2 - 2x^2}{x^2 + y^2} \\
\frac{\partial^2 w}{\partial y^2} &= \frac{C_R - C_r}{\ln(R^2) - \ln(r^2)} \frac{2x^2 - 2y^2}{x^2 + y^2},
\end{aligned}$$

which means that the sum is zero, and thus w is harmonic. In particular, it also satisfies the extended maximum modulus principle, meaning that w attains its maxima on the boundary of the annulus. Since, for $x + iy \in S(0, r)$, we have $u(x, y) = C_r$ and $x^2 + y^2 = r^2$, we thus get that $w = 0$, and similarly, $w = 0$ on $S(0, R)$, meaning that w is identically zero on $\overline{A(0, r, R)}$.

Thus, we find that

$$u(x, y) = C_r + (C_R - C_r)\frac{\ln\left(x^2 + y^2\right) - \ln\left(r^2\right)}{\ln(R^2) - \ln(r^2)}.$$