Problem (Problem 1)**:** Let $(a_n)_n$ be a sequence for which $\sum_{n=0}^{\infty}|a_n|^2$ is finite. For each positive N, define $f_N(z) = \sum_{n=0}^{\infty} a_n z^n$, and define $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(a) Show that f is holomorphic on $\mathbb{D}$.

(b) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f_N\left(re^{i\theta}\right)\right|^2 d\theta.$$

(c) For each $r \in (0,1)$, determine in terms of $(a_n)_n$ the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta.$$

(d) Determine in terms of $(a_n)_n$ the limit

$$\lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta.$$

Solution:

(a) Let $0 < r < 1$. Since each f_N is analytic, we can use the Cauchy Integral Formula to compute a_N explicitly, yielding

$$\begin{aligned}|a_N| &= \left|\frac{1}{2\pi i}\int_0^{2\pi}\frac{f_N(\xi)}{\xi^{N+1}}\,d\xi\right|\\ &\leqslant \frac{1}{r^N}\sup_{|z|=r}|f_N(z)|.\end{aligned}$$

Therefore, if we are able to show that the value

$$\sup_{|z|=r}|f_N(z)|$$

is uniformly bounded by a constant for all N, we will be able to use the Cauchy–Hadamard theorem to show that $\limsup_{N\to\infty}|a_N|^{1/N} \leqslant 1$. Toward this end, we use the Cauchy–Schwarz inequality, which yields

$$\begin{aligned}\sup_{|z|=r}|f_N(z)| &= \sup_{|z|=r}\left|\sum_{n=0}^{N} a_n z^n\right|\\ &\leqslant \sup_{|z|=r}\left(\sum_{n=0}^{N}|a_n|^2\right)^{1/2}\left(\sup_{m=0}^{N}|z|^{2m}\right)^{1/2}\\ &\leqslant \sup_{|z|=r}\underbrace{\left(\sum_{n=0}^{\infty}|a_n|^2\right)^{1/2}}_{=:K}\left(\sum_{m=0}^{\infty}|z|^{2m}\right)^{1/2}\\ &= \frac{K}{(1-|r|^2)^{1/2}}.\end{aligned}$$

Since we have established this uniform bound, we thus find that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence at least 1, so f is analytic on $\mathbb{D}$, hence holomorphic on $\mathbb{D}$.

(b) We write out the integral to yield

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|f_N\left(re^{i\theta}\right)\right|^2\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{N}a_nr^ne^{in\theta}\right)\overline{\left(\sum_{m=0}^{N}a_mr^me^{im\theta}\right)}\,d\theta\\ &= \frac{1}{2\pi}\sum_{n=0}^{N}\sum_{m=0}^{N}a_n\overline{a_m}r^{m+n}\int_0^{2\pi}e^{i(n-m)\theta}\,d\theta\\ &= \sum_{n=0}^{N}|a_n|^2r^{2n}.\end{aligned}$$

(c) Since f is holomorphic with radius of convergence at least 1, the series expression on $S(0,r)$ converges uniformly, so that we may exchange sum and integral. This yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n\overline{a_m}r^{m+n}\int_0^{2\pi}e^{i(n-m)\theta}\,d\theta\\ &= \sum_{n=0}^{\infty}|a_n|^2r^{2n}.\end{aligned}$$

(d) Since the sequence $(a_n)_n$ is square-summable, the limit is well-defined, and we get

$$\begin{aligned}\lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta &= \lim_{r\nearrow 1}\sum_{n=0}^{\infty}|a_n|^2r^{2n}\\ &= \sum_{n=0}^{\infty}|a_n|^2.\end{aligned}$$

Problem (Problem 2)**:** Let $\varphi\colon [0,1]\to\mathbb{C}$ be continuous, and define $f\colon \mathbb{C}\setminus[0,1]\to\mathbb{C}$ by

$$f(z) = \int_0^1 \frac{\varphi(t)}{t-z}\,dt.$$

Show that f is holomorphic and determine the derivative of f in terms of φ.

Problem (Problem 3)**:** Let $f\colon\mathbb{C}\to\mathbb{C}$ be entire.

(a) Suppose there exist $C, R > 0$ and $n\in\mathbb{N}$ such that $|f(z)|\leqslant C|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree at most n.

(b) Suppose that $g\colon\mathbb{C}\to\mathbb{C}$ is also entire and $|f(z)|\leqslant|g(z)|$ for all $z\in\mathbb{C}$. Show that there exists some $\alpha\in\mathbb{C}$ with $|\alpha|\leqslant 1$ such that $f(z) = \alpha g(z)$ for all $z\in\mathbb{C}$.

(c) Suppose that there exists some $\theta\in\mathbb{R}$ such that $f(\mathbb{C})\cap\left\{re^{i\theta}\;\middle|\; r>0\right\} = \emptyset$. Show that f is constant.

Solution:

(a) Let $r > R$. Then, by the Cauchy estimate, we get that

$$\begin{aligned}\left|f^{(n+1)}(0)\right| &\leqslant \frac{(n+1)!}{r^{n+1}}\sup_{|z|=r}|f(z)|\\ &\leqslant \frac{(n+1)!}{r^{n+1}}\sup_{|z|=r}\left(C|z|^n\right)\\ &= \frac{C(n+1)!}{r},\end{aligned}$$

so since r is arbitrary and f is entire, we find that $f^{(n+1)}(0) = 0$, so that the power series expansion of f about 0 terminates beyond $n+1$, meaning that f is a polynomial of degree at most n.

(b) If g is 0, then we are done. Else, assume that g is not identically zero. Observe that if g is everywhere non-vanishing, then the function $\frac{f(z)}{g(z)}$ is entire, and satisfies

$$\left|\frac{f(z)}{g(z)}\right| \leqslant 1,$$

hence $\frac{f(z)}{g(z)} = \alpha$ for some α with $|\alpha| \leqslant 1$.

Now, if $g(z)$ does admit zeros, they must be isolated zeros, or else by the identity theorem, we would have that g is identically zero on $\mathbb{C}$. We observe that if $a \in \mathbb{C}$ is a zero for g, we may then write

$$g(z) = (z-a)^n g^*(z),$$

with $g^*(z)$ holomorphic and $g^*(a) \neq 0$. Additionally, since $|f(z)| \leqslant |g(z)|$, we must have $f(a) = 0$, so that, similarly,

$$f(z) = (z-a)^m f^*(z)$$

with $f^*(z)$ holomorphic and $f^*(a) \neq 0$. We observe that, since $|f(z)| \leqslant |g(z)|$, in a sufficiently small deleted neighborhood of a, that $f^*(z)$ and $g^*(z)$ are both approximately constant, meaning that, necessarily, $|z-a|^m \leqslant |z-a|^n \frac{g^*(a)}{f^*(a)}$, so that for sufficiently small $|z-a|$, it follows that $m \geqslant n$.

We define $k(z) = \frac{f(z)}{g(z)}$, and define a holomorphic extension of $k(z)$ by

$$h(z) = \begin{cases} k(z) & g(z) \neq 0 \\ \lim_{z\to a}(z-a)k(z) & g(a) = 0. \end{cases}$$

This is a holomorphic extension of k, as

$$\lim_{z\to a} \frac{h(z)-h(a)}{z-a} = \lim_{z\to a} h'(z),$$

so that we have $|h(z)| \leqslant 1$ for all z. Thus, h is a bounded entire function, hence constant, so $\frac{f(z)}{g(z)} = \alpha$ where defined with $|\alpha| \leqslant 1$, and $f(z) = \alpha g(z)$.

(c) By adding a sufficient multiple of $2\pi k$ to θ, we may assume that $\theta > 0$. In particular, this means that

$$\log_\theta : \mathbb{C} \setminus \left\{ re^{i\theta} \;\middle|\; r > 0 \right\} \to \{z \mid \theta < \operatorname{Im}(z) < \theta + 2\pi\}$$

is holomorphic. Finally, we observe that the Cayley Transform,

$$\varphi(z) = \frac{z-i}{z+i}$$

takes the upper half-plane to the unit disk. Therefore, the composition $\varphi \circ \log_\theta \circ f : \mathbb{C} \to \mathbb{D}$ is an entire function that is bounded, hence constant. Since φ and $\log_\theta$ are non-constant, it follows that f is constant.

Problem (Problem 4)**:** Let $U = \{z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1\}$. Suppose $f : U \to \mathbb{C}$ is holomorphic, and there exists $C > 0$ and $\eta \in \mathbb{R}$ such that

$$|f(z)| \leqslant C(1+|z|)^{\eta}$$

for all $z \in U$. Show that for each $n \geqslant 0$, there exists a constant $C_{n,\eta} \geqslant 0$ dependent only on n and η such that

$$\left|f^{(n)}(x)\right| \leqslant C_{n,\eta}(1+|x|)^{\eta}$$

for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$, $0 < r < 1$, and to start, assume $\eta \geqslant 0$. Then, from Cauchy's estimate, a bunch of triangle inequalities, and the fact that $\eta \geqslant 0$ and $r < 1$, we find that

$$\begin{aligned}
\left|f^{(n)}(x)\right| &\leqslant \frac{n!}{r^n} \sup_{|w-x|=r} |f(w)| \\
&\leqslant \frac{n!}{r^n} \sup_{|w-x|=r} \left(C(1+|w|)^{\eta}\right) \\
&\leqslant \frac{Cn!}{r^n} \sup_{|w-x|=r} \left(1 + \left|w - \frac{3}{2}x\right| + \frac{3}{2}|x|\right)^{\eta} \\
&\leqslant \frac{Cn!}{r^n} \sup_{|w-x|=r} (1 + |w-x| + 2|x|)^{\eta} \\
&\leqslant \frac{Cn!}{r^n} \sup_{|w-x|=r} (1 + r + 2|x|)^{\eta} \\
&\leqslant \frac{Cn!}{r^n}(2+2|x|)^{\eta} \\
&\leqslant \frac{C2^{\eta}n!}{r^n}(1+|x|)^{\eta}.
\end{aligned}$$

In particular, since this inequality holds for every $0 < r < 1$, it necessarily for $r = 1$, so that $C_{n,\eta} = C2^{\eta}n!$.

Now, if $\eta < 0$.

Problem (Problem 5)**:** Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geqslant 1$, where $a_0, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$.

(a) Show that there exist n complex numbers $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ not necessarily distinct such that $P(z) = a_n(z-\alpha_1)\cdots(z-\alpha_n)$.

(b) Suppose $|\alpha_0| > |\alpha_n|$. Show that there exists some $\alpha \in \mathbb{C}$ for which $|\alpha| > 1$ and $P(\alpha) = 0$.

Solution:

(a) Dividing out by a_n, we take

$$P(z) = a_n\left(z^n + \frac{a_{n-1}}{a_n}z^{n-1} + \cdots + \frac{a_1}{a_n}z + \frac{a_0}{a_n}\right).$$

By the fundamental theorem of algebra, we can find some α_1 such that $P(\alpha_1) = 0$. Therefore, by polynomial division, we have a monic polynomial $q(z)$ with degree $n-1$ such that

$$P(z) = a_n q(z)(z-\alpha_1).$$

If $q(z)$ is a constant polynomial, it is necessarily equal to 1 and we are done. Else, inductively, we may find $\alpha_2, \dots, \alpha_n \in \mathbb{C}$ such that $q(z) = (z-\alpha_2)\cdots(z-\alpha_n)$, meaning that

$$P(z) = a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

(b)