## Preliminary Statements

**Theorem** (Definition of Countability). *A set $S$ is countable if and only if there exists an injection $f : S \hookrightarrow \mathbb{N}$.*

*Proof.* Let $S$ be countable.

**Case 1:** We have $S$ is finite if and only if there is a map $f : S \to \{1, 2, \dots, n\}$, where $f$ is a bijection. Letting $\iota : \{1, 2, \dots, n\} \to \mathbb{N}$ be defined by $\iota(n) = n$, it is clear that $\iota$ is an injection.

Considering the map $\iota \circ f : S \to \mathbb{N}$, since $\iota$ is injective and $f$ is injective, so too is $\iota \circ f$, meaning our desired injection is $\iota \circ f$.

**Case 2:** By definition, a set $S$ is countably infinite if and only if there exists a bijection $g : S \to \mathbb{N}$, which is our desired injection.

□

**Theorem** (Injection into a Finite Set). *Let $S$ be a nonempty set. If there exists an injection $S \hookrightarrow \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, then $S$ is finite.*

*Proof.* We begin by showing the reverse direction.

Let $\sigma : S \hookrightarrow \{1, 2, \dots, n\}$ be an injection for some $n \in \mathbb{N}$. Define $s_i$ by $\sigma(s_i) = i$ for $i \in \operatorname{im}(\sigma)$.

Notice that $\sigma' : S \to \sigma(S)$ is a bijection, since $\sigma$ is injective and any map of the form $f : A \to f(A)$ is surjective by definition.

We define $r : \sigma(S) \hookrightarrow \mathbb{N}$ selecting $i_1$ to be the least element in $\sigma(S)$ (which exists by the well-ordering principle since $\{1, 2, \dots, n\} \subseteq \mathbb{N}$ is nonempty), and mapping $r(i_1) = 1$. Similarly, we inductively select $i_k$ to be the least element in $\sigma(S) \setminus \{i_1, i_2, \dots, i_{k-1}\}$, and map $r(i_k) = k$. From this construction, it is clear that $r$ is injective.

Then, defining $r' : \sigma(S) \to r(\sigma(S))$, we can see that $r'$ is a bijection, with $r(\sigma(S)) = \{1, 2, \dots, j\}$ for some $j \leqslant n$ (since, by definition, $\sigma$ is an injection, meaning $\sigma(s_i) \leqslant n$ for all $n$).

Taking $r' \circ \sigma' : S \to \{1, 2, \dots, j\}$, we see that this is a composition of bijections, meaning it is a bijection. Thus, $S$ is finite.

In the forward direction, we can see that if $S$ is finite, then the bijection $h : S \to \{1, 2, \dots, n\}$ is an injection, and we are done. □

## 1.1

**Problem.** Show that the function $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = (-1)^{n+1} \left\lfloor \frac{n+1}{2} \right\rfloor$$

is a bijection.

**Solution.** We begin by showing that f is injective. Let $f(n_1) = f(n_2)$. Then, we have two cases: one if $f(n_1)$ and $f(n_2)$ are positive, and one if $f(n_1)$ and $f(n_2)$ are negative. In the either case, we have

$$\begin{aligned} f(n_1) &= (-1)^{n_1+1}\left\lfloor\frac{n_1+1}{2}\right\rfloor, \\ f(n_2) &= (-1)^{n_2+1}\left\lfloor\frac{n_2+1}{2}\right\rfloor, \end{aligned}$$

meaning

$$\left\lfloor\frac{n_1+1}{2}\right\rfloor = \left\lfloor\frac{n_2+1}{2}\right\rfloor.$$

If $f(n_1)$ and $f(n_2)$ are positive, this implies that $n_1$ and $n_2$ are odd (so that $n_1+1$, $n_2+1$ are even). Since $n_1+1$ and $n_2+1$ are even, this implies

$$\begin{aligned} \left\lfloor\frac{n_1+1}{2}\right\rfloor &= \frac{n_1+1}{2} \\ \left\lfloor\frac{n_2+1}{2}\right\rfloor &= \frac{n_2+1}{2}, \end{aligned}$$

meaning $n_1 = n_2$.

If $f(n_1)$ and $f(n_2)$ are odd, this implies that $n_1$ and $n_2$ are even, so

$$\begin{aligned} \left\lfloor\frac{n_1+1}{2}\right\rfloor &= \frac{n_1}{2} \\ \left\lfloor\frac{n_2+1}{2}\right\rfloor &= \frac{n_2}{2}, \end{aligned}$$

once again implying that $n_1 = n_2$.

To show surjectivity, let $z \in \mathbb{Z}$. Suppose $z < 0$. Then, we find $n \in \mathbb{N}$ by taking $n = -2z$. If $z > 0$, we take $n = 2z - 1$, and if $z = 0$, we take $n = 0$.

## 1.2

**Problem.** Given bijections $f : \mathbb{N} \to \mathbb{Z}$ and $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, show that the function $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ defined by $h(x, y) = P\left(f^{-1}(x), f^{-1}(y)\right)$ is bijective.

**Solution.** We begin by showing injectivity. Since f is bijective, so too is $f^{-1}$, meaning that for

$$h(x, y) = h(x', y'),$$

we have

$$\begin{aligned} P\left(f^{-1}(x), f^{-1}(y)\right) &= P\left(f^{-1}(x'), f^{-1}(y')\right) \\ f^{-1}(x) &= f^{-1}(x') \end{aligned}$$

$$f^{-1}(y) = f^{-1}(y') \qquad \text{since } P \text{ is bijective}$$

meaning

$$\begin{aligned} x &= x' \\ y &= y' && \text{since } f^{-1} \text{ is bijective.} \end{aligned}$$

Thus, $h$ is injective.

Let $n \in \mathbb{N}$. Since $P$ is surjective, there exist $a, b$ such that $P(a, b) = n$. Since $f^{-1}$ is surjective, there exists $x, y \in \mathbb{Z}$ such that $f^{-1}(x) = a$ and $f^{-1}(y) = b$. Thus, there exist $x, y \in \mathbb{Z}$ such that $h(x, y) = n$.

## 1.3

**Problem.** If $A$ and $B$ are countably infinite, show that $A \times B$ is countably infinite.

**Solution.** By the definition of countably infinite sets, there exist bijections $\alpha : A \to \mathbb{N}$ and $\beta : B \to \mathbb{N}$. Additionally, we know that there exists a bijection $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Define $h : A \times B \to \mathbb{N}$ by $h(a, b) = P(\alpha(a), \beta(b))$. Then, since $h$ is a composition of bijections, $h$ is a bijection between $A \times B$ and $\mathbb{N}$.

## 1.5

**Problem.** If $A_1, A_2, \dots$ is an infinite sequence of (pairwise) disjoint finite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

**Solution.** For all $i \in \mathbb{N}$, there exists $f_i : A_i \to \{1, 2, \dots, n_i\}$ such that $f_i$ is a bijection, by the definition of finitude.

Let $x \in \bigcup_{i=1}^{\infty} A_i$. Then, $x \in A_i$ for exactly one value of $i$, since the sets $A_i$ are pairwise disjoint.

Define

$$p(x) = f_i(x) - 1 + \sum_{j=1}^{i-1} n_j.$$

Then, $p$ is a bijection, meaning $\bigcup_{i=1}^{\infty} A_i$ is denumerable.

## 1.6

**Problem.** If $A_1, A_2, \dots$ is an infinite sequence of disjoint countably infinite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

**Solution.** We define $\chi_n : A_n \to \mathbb{N}$ to be bijections that define the cardinality of $A_n$, and let $a_{i,n} \in A_n$ be defined by $\chi_n(a_{i,n}) = i$. We let $p_n$ denote the $n$th prime number.

The function $h : \bigcup_{n=1}^{\infty} A_n \to \mathbb{N}$ defined by $h(a_{i,k}) = p_k^{\chi_k(a_{i,k})}$ is an injection, as each $A_k$ is disjoint and prime numbers do not divide each other. Thus, we know that $\bigcup_{n=1}^{\infty} A_n$ is countable.

## 1.7

**Problem.** Construct an explicit polynomial bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

**Solution.** Let $Q : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $Q(x, y, z) = P(P(x, y), z)$, where $P(x, y) = \frac{(x+y)(x+y+1)}{2} + x$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

We know that $Q$ is a bijection since it is a composition of bijections. I do not want to expand this expression.

## Extra Problem 1

**Problem.** Prove that if $A$ and $B$ are finite sets, then $A \cup B$ is finite.

**Solution.** We know $A \setminus B \subseteq A$; since $A$ is finite, so too is $A \setminus B$ (by Extra Problem 3).

Since $A \cup B = (A \setminus B) \cup B$ is a disjoint union of finite sets, $A \cup B$ is finite.

**Remark** (Disjoint Union of Finite Sets is Finite)**:** Let $A, B$ be disjoint finite sets. Then, $A \cup B$ is finite.

To prove this, by the definition of finitude, there exist $\alpha : A \to \{1, 2, \dots, m\}$ and $\beta : B \to \{1, 2, \dots, n\}$ bijections for some $m, n \in \mathbb{N}$.

We can create a new function $f : A \cup B \to \{1, 2, \dots, m + n\}$ by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) + m & x \in B \end{cases}.$$

We can see that $h$ is a well-defined bijection since $A \cap B = \emptyset$.

## Extra Problem 2

**Problem.** Prove that for every $n \in \mathbb{N}$, every subset of $\{0, 1, \dots, n\}$ is finite.

**Solution.** For any subset $P \subseteq \{0, 1, \dots, n\}$, the inclusion map is an injection into $\{0, 1, \dots, n\}$; composing the inclusion map with the bijection $a : \{0, 1, \dots, n\} \to \{1, 2, \dots, n + 1\}$ defined by $a(m) = m + 1$, we see that there is an injection $a \circ \mathrm{id} : P \hookrightarrow \{1, 2, \dots, n + 1\}$, meaning $P$ is finite by the theorem above.

## Extra Problem 3

**Problem.** Prove that every subset of a finite set is finite.

**Solution.** Since every empty set is finite, so too is every subset of the empty set. Similarly, any empty subset of a given finite set is also finite.

Let $A$ be a nonempty finite set. Then, there exists a bijection $\alpha : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

Let $B \subseteq A$ be nonempty. The inclusion map $\iota : B \hookrightarrow A$ defined by $\iota(x) = x$ is an injection.

Thus, $\alpha \circ \mathrm{id} : B \hookrightarrow \{1, 2, \dots, n\}$ is an injection, as it is a composition of injections. By the established theorem above, this means $B$ is finite.

## Extra Problem 4

**Problem.** Prove that every infinite subset of $\mathbb{N}$ is denumerable.

**Solution.** Let $A \subseteq \mathbb{N}$ be infinite.

Since $A$ is nonempty, by the well-ordering principle, there must exist a least element of $A$, which we label as $a_0$.

Consider $A \setminus \{a_0\}$. Since $A$ is infinite, $A \setminus \{a_0\}$ must also be infinite, meaning there is a least element of $A \setminus \{a_0\}$ by the well-ordering principle. We label this element as $\{a_1\}$.

Now, we consider $A \setminus \{a_0, a_1\}$, and use the well-ordering principle to extract $a_2$, and inductively extract $a_i$ by using the well ordering principle on $A \setminus \{a_0, a_1, \dots, a_{i-1}\}$.

The function $f : A \to \mathbb{N}$ defined by $f(a_i) = i$ is a bijection, since $f(a_i) = f\left(a_j\right)$ if and only if $i = j$.

Thus, $f$ is a denumeration of $A$.