HW 9

Problem 1

Is the open unit ball in $\mathbb{R}^3$ compact? Prove your answer.

The open unit ball in $\mathbb{R}^3$ is not compact, as the following is an open cover with no finite subcover.

$$B_1(0,0,0) = \bigcup_{n\in\mathbb{Z}^+} B_{1-\frac{1}{n}}(0,0,0)$$

For any finite n, we get that the set on the right side is lacking some component of $B_1(0,0,0)$.

Problem 2

Every discrete topological space is compact.

Consider the topological space of $\mathbb{R}$ under the discrete metric. Then, $\mathbb{R}_d = \bigcup_{x\in\mathbb{R}}\{x\}$. Since every set in $\mathbb{R}$ is open, this is an open cover of $\mathbb{R}$. However, any finite subset of the quantity on the right hand side does not cover $\mathbb{R}_d$. Therefore, $\mathbb{R}_d$ is not compact.

No discrete topological space is compact.

Let X be a finite discrete topological space. Then, $X = \bigcup_{x\in X}\{x\}$, is an open cover, which is also finite, so X is compact.

Problem 3

Prove that the union of two compact subsets of a topological space is compact.

Let A and B be compact subsets of X. Then, for some open cover of $A = \bigcup_{i\in I} C_i, C_i, \exists F \underset{\text{finite}}{\subseteq} I$ such that $A = \bigcup_{i\in F} C_i$. Similarly, for $B = \bigcup_{j\in J} D_j, D_j \underset{\text{open}}{\subseteq} B, \exists G \underset{\text{finite}}{\subseteq} J$ such that $B = \bigcup_{j\in G} D_j$. Then, $A \cup B = \bigcup_{C_i\in X, D_j\in Y} C_i \cup D_j$. Let $E_{ij} = C_i \cup D_j$. Then, since A and B are compact, we have $A \cup B = \bigcup_{i\in F, j\in G} E_{ij}$, which is a finite number of E_{ij} as F and G are finite. Therefore, $A \cup B$ is finite.

Problem 4

Prove that the union of infinitely many compact subsets of a topological space need not be compact.

Consider the following union of compact subsets in $\mathbb{R}$.

$$[0,\infty) = \bigcup_{n\in\mathbb{Z}^+} [n, n+1]$$

The set $[0,\infty)$ is not compact, yet each of the subsets $[n, n+1]$ is compact.

Problem 5

Prove that the continuous image of a compact set is compact.

Let $f : X \to Y$ be a continuous function. Let $A \subseteq X$ be compact. Let $f(A) = \bigcup_{i\in I} C_i$ be an open cover of $f(A)$. Then, $f^{-1}(f(A)) = A = f^{-1}\left(\bigcup_{i\in I} C_i\right) = \bigcup_{i\in I} f^{-1}(C_i)$ by a previous result. Since f is continuous, $f^{-1}(C_i)$ is open, meaning that $A = \bigcup_{i\in I} f^{-1}(C_i)$ is an open cover of A, and since A is compact, there is a finite subcover $F \in I$, so $f(A)$ has a finite subcover as A has a finite subcover. Therefore, $f(A)$ is compact.

Problem 6

Prove that the following two definitions of a compact space are correct:

- A set A is compact if for every collection F of sets open in X with $A \subseteq \bigcup_{W\in F} W$, there is a finite $F' \subseteq F$ with $A \subseteq \bigcup_{W\in F'} W$.
- A is compact if A with the subspace topology is a compact topological space.

Suppose A is compact with the first definition. Then, $A = A \cap \left(\bigcup_{W\in F} W\right) = \bigcup_{V\in F} V$ for $V = W \cap A$. Since F' is finite and a subset of F, this means $A = A \cap \left(\bigcup_{W\in F'} W\right) = \bigcup_{V\in F'} V$. So A is a compact topological space, as $V \underset{\text{open}}{\subseteq} A$. Since all these steps are reversible, we get that the two definitions are equal.

HW 10

Problem 1

Prove that every compact subset of a nonempty metric space is bounded.

Let $A \subseteq X$ be a compact set and $x \in X$. Since $X = \bigcup_{k\in\mathbb{N}} B_k(x)$, we can find $A = A \cap \left(\bigcup_{k\in\mathbb{N}} B_k(x)\right) = \bigcup_{k\in\mathbb{N}}(A \cap B_k(x))$. Since this is an open cover of A as $B_k(x) \underset{\text{open}}{\subseteq} X$ and $A \cap B_k(x) \underset{\text{open}}{\subseteq} A$ by the subspace topology, this open cover has a finite subcover as A is compact. This means there is a maximum k' such that $A = \bigcup_1^{k'}(A \cap B_k(x))$. So, $A \subset B_{k'+1}(x)$, so A is bounded.

Problem 2

Prove every compact subset of a metric space is closed.

Let A be a compact subset of a metric space X. Suppose towards contradiction that A is not closed. Then, $\exists x \in X$ such that x is a limit point of A but $x \notin A$. Then, consider the set $K = \bigcup_{n\in\mathbb{R}} \overline{\mathrm{cl}\left(B_{1/n}(x)\right)}$. This is a union of open sets as it is a union of complements of closed sets, and since $K = X$, $A \subseteq K$. However, since x is a limit point, $\forall r > 0$, $B_r(x) \cap A - \{x\} \neq \emptyset$. Therefore, K is an open cover of A but does not have a finite subcover, so A is not compact. Therefore, we have reached a contradiction, so A is closed.

Problem 3

Let $I = [0, 1]$. Show every continuous map $f : I \to \mathbb{R}$ is bounded.

Since $I = [0, 1]$, I is closed and bounded, so I is compact. Since f is continuous, this means that $f(I)$ must also be compact. So, by the Heine-Borel theorem, $f(I)$ must be closed and bounded. So f is bounded.

Give an example of an unbounded continouous function $f : (0, 1) \to \mathbb{R}$

$$f(x) = \tan\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)$$

Problem 4

Suppose X is a discrete topological space with at least two points. Show X is disconnected.

Let $a, b \in X$ be nonequal points in X. Then, $\{a\} \underset{\text{open}}{\subseteq} X$ and $\{b\} \underset{\text{open}}{\subseteq} X$, but $\{a\} \cap \{b\} = \emptyset$. Therefore, X is disconnected.

Problem 5

Show that if X has the discrete topology, then X is totally disconnected.

We want to show the following:

- If $a \in X$, $\{a\} \underset{\text{open}}{\subseteq} X$ and $\{a\}$ is connected: Since X has the discrete topology, $\{a\}$ is open, and every element of $\{a\}$ is just a, which is equal to itself, so by vacuous truth, $\{a\}$ is connected.
- Any subset $A \subseteq X$ where $|A| \geq 2$ is disconnected: by the previous problem, we have that any two non-equal elements of A are themselves disconnected sets, so A is disconnected.

The converse does hold.

Problem 6

Prove that the Cantor set as a subset of $\mathbb{R}$ is totally disconnected.

For any $a, b \in C$ where $a < b$, then $a = \frac{k}{3^n}$ and $b = \frac{l}{3^n}$, but the set $[a, b]$ is missing its middle third, so $c = \frac{a+b}{2} \notin C$. Since for any non-singleton set $[a, b] \in C$, the set is disconnected. So, C is totally disconnected.

HW 11

Problem 1

Let X, Y be sets with $A, C \subseteq X$, $B, D \subseteq Y$. Show $(A \times B) \cap (C \times D) = E \times F$ for some $E \subseteq X$, $F \subseteq Y$

Let $(a, b) \in (A \times B) \cap (C \times D)$. Then, $a \in A \cap C$ and $b \in B \cap D$. Let $E = A \cap C$ and $F = B \cap D$. Then, $(a, b) \in E \times F \subseteq X$. So $(A \times B) \cap (C \times D) \subseteq E \times F$. Similarly, if $(a, b) \in E \times F$, we have that $a \in A \cap C$, $b \in B \cap D$, so $(a, b) \in (A \times B) \cap (C \times D)$, so $E \times F = (A \times B) \cap (C \times D)$.

Show this means the product topology is close under finite intersections.

Let $X \times Y$ be a topological space under the product topology. Then, any open set can be expressed as $M = \bigcup_{i \in I, j \in J} A_i \times B_j$ for $A_i \underset{\text{open}}{\subseteq} X$, $B_i \underset{\text{open}}{\subseteq} Y$. Similarly, another open set in $X \times Y$ can be expressed as $N = \bigcup_{k \in K, l \in L} C_k \times D_l$ where $C_k \underset{\text{open}}{\subseteq} X, D_l \underset{\text{open}}{\subseteq} Y$. Then, $M \cap N = \bigcup_{o \in O, p \in P} E_o \times P_p$ by the previous result. Since the finite intersection of open sets is open, $M \cap N$ is open as it is the union of open sets $E_o \underset{\text{open}}{\subseteq} X$ and $P_p \underset{\text{open}}{\subseteq} Y$.

Problem 2

Prove every open rectangle is a union of open balls.

Let $(a, b) \times (c, d)$ be an open rectangle in $\mathbb{R}^2$. Then, for some point $(x, y) \in (a, b) \times (c, d)$, we can find a radius $r = \min\{d(x, a), d(x, b), d(y, c), d(y, d)\}$. Then, $B_r(x, y) \subseteq (a, b) \times (c, d)$, so $A = (a, b) \times (c, d) = \bigcup_{(x,y) \in A} B_r(x, y)$.

Prove every open ball is a union of open rectangles.

Let $B_r(x, y)$ be an open ball in $\mathbb{R}^2$. Then, for any $(a, b) \in B_r(x, y)$, we can find k such that $B_{k\sqrt{2}}(a, b) \subseteq (x, y)$. Then, the set $(a - k, a + k) \times (b - k, b + k)$ is an open rectangle which is a subset of the open ball $B_{k\sqrt{2}}(a, b)$, which is a subset of $B_r(x, y)$, so $(a - k, a + k) \times (b - k, b + k)$ is an open rectangle subset of $B_r(x, y)$. So, $B = B_r(x, y) = \bigcup_{(a,b) \in B} (a - k, a + k) \times (b - k, b + k)$.

Problem 3

Let $\mathcal{T}_1 = \mathbb{R}^2$ under the Euclidean metric and $\mathcal{T}_2$ be the product topology on $\mathbb{R} \times \mathbb{R}$. Show these two topologies are equivalent.

First, we will show that every element of $\mathcal{T}_2$ is a union of open rectangles. Let $A, B \underset{\text{open}}{\subseteq} \mathbb{R}$. Then, $A = \bigcup_{a,b \in E} (a, b)$ and $B = \bigcup_{c,d \in F} (c, d)$ by the definition of open sets in $\mathbb{R}$. So, $A \times B$, which is open in $\mathcal{T}_2$, is equal to $\bigcup_{a,b \in E} (a, b) \times \bigcup_{c,d \in F} (c, d)$. Using a rule we can take for granted, we have that $A \times B = \bigcup_{a,b \in E, c,d \in F} (a, b) \times (c, d)$, which is a union of open rectangles.

Let $A \in \mathcal{T}_1$. Then, A is an open set in $\mathbb{R}^2$, so by a previous result, A is a union of open balls. So, A is a union of open rectangles in $\mathbb{R}^2$, so $A \in \mathcal{T}_1$. Similarly, if $B \in \mathcal{T}_2$, then B is a union of open rectangles in $\mathbb{R}^2$, so B is in $\mathcal{T}_1$, so $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 4

Let V, W, X, Y be topological spaces, $V \simeq X$, $W \simeq Y$. Show $V \times W \simeq X \times Y$.

Let $f : V \to X$, $g : W \to Y$ be homeomorphisms. Then, f and g are continuous bijections with continuous inverses. Let $h : (V \times W) \to (X \times Y)$ be defined as $h(v, w) = (f(v), g(w))$. We want to show that h is a homeomorphism.

- Since f and g are bijections and are the "constituent functions" of h, we know that h is a bijection.
- Let $A \underset{\text{open}}{\subseteq} X \times Y$. Then, $A = \bigcup_{i \in I} A_i \times B_i$ for $A_i \underset{\text{open}}{\subseteq} X, B_i \underset{\text{open}}{\subseteq} Y$. So, $h^{-1}(A) = h^{-1}\left(\bigcup_{i \in I} A_i \times B_i\right) = \bigcup_{i \in I} h^{-1}(A_i \times B_i) = \bigcup_{i \in I} f^{-1}(A) \times g^{-1}(B)$ by rules of discrete math. So, since f^{-1} and g^{-1} are homeomorphisms, $f^{-1}(A_i) \underset{\text{open}}{\subseteq} V$ and $g^{-1}(B_i) \underset{\text{open}}{\subseteq} W$, so h is continuous by the product topology.
- Similarly, if $C \underset{\text{open}}{\subseteq} V \times W$, then $h(C) \underset{\text{open}}{\subseteq} X \times Y$, so h^{-1} is continuous.
- Therefore, since h is a continuous bijection with a continuous inverse, h is a homeomorphism, so $V \times W \simeq X \times Y$.

HW 12

Problem 1

Prove that the projection map $\pi_i : X_1 \times X_2 \to X_i$, defined as $\pi_1(x_1, x_2) = x_1$ and similarly for π_2, is continuous.

Let $A_1 \underset{\text{open}}{\subseteq} X_1$ and $A_2 \underset{\text{open}}{\subseteq} X_2$. Then, $\pi_1^{-1}(A_1) = A_1 \times X_2$, and $\pi_2^{-1}(A_2) = X_1 \times A_2$. By the product topology, we know that $A_1 \times X_2 \underset{\text{open}}{\subseteq} X_1 \times X_2$ because $A_1 \underset{\text{open}}{\subseteq} X_1$ and $X_2 \underset{\text{open}}{\subseteq} X_2$, and similarly $X_1 \times A_2 \underset{\text{open}}{\subseteq} X_1 \times X_2$, so π_1 and π_2 are continuous.

Problem 2

Let Y, X_1, X_2 be topological spaces. For each $i = 1, 2$, let $f_i : Y \to X_i$ be a map. Prove $f : Y \to X_1 \times X_2$ defined as $f(y) = (f_1(y), f_2(y))$ is continuous iff f_1 and f_2 are continuous.

Let f be continuous. Then, for any set open in the product topology $X_1 \times X_2$, the inverse image of that set is open in Y. So, if $A = \bigcup_{i \in I} A_i \times B_i$ where $A_i \underset{\text{open}}{\subseteq} X_1$ and $B_i \underset{\text{open}}{\subseteq} X_2$, $f^{-1}(A) = f^{-1}\left(\bigcup_{i \in I} A_i \times B_i\right) = \bigcup_{i \in I} f^{-1}(A_i \times B_i)$. Since f is continuous and $A_i \times B_j \underset{\text{open}}{\subseteq} X_1 \times X_2$ by the definition of product topology, we have that $f^{-1}(A_i \times B_i) \underset{\text{open}}{\subseteq} Y$. Therefore, $f^{-1}(A_i \times B_i) = f_1^{-1}(A_i) \cap f_2^{-1}(B) \underset{\text{open}}{\subseteq} Y$. Since the intersection of two open sets is open, we have that $f_1^{-1}(A_i)$ and $f_2^{-1}(B_i)$ are open in Y, so f_1 and f_2 are continuous.

Let f_1 and f_2 be continuous. Then, for all $A_i \underset{\text{open}}{\subseteq} X_1$, $f_1^{-1} \underset{\text{open}}{\subseteq} Y$, and for all $B_i \underset{\text{open}}{\subseteq} X_2$, $f_2^{-1}(B_i) \underset{\text{open}}{\subseteq} Y$. So, for any $A \underset{\text{open}}{\subseteq} X_1 \times X_2$, we have $A = \bigcup_{i \in I} A_i \times B_i$ for $A_i \underset{\text{open}}{\subseteq} X_1$, $B_i \underset{\text{open}}{\subseteq} X_2$. Then, $f^{-1}(A) = \bigcup_{i \in I} f^{-1}(A_i \times B_i)$. Similarly to the previous result, we have $f^{-1}(A_i \times B_i) = f_1^{-1}(A_i) \cap f_2^{-1}(B_i)$. Since $f_1^{-1}(A_i) \underset{\text{open}}{\subseteq} Y$ and $f_2^{-1}(B_i) \underset{\text{open}}{\subseteq} Y$, we have that $f^{-1}(A_i \times B_i) \underset{\text{open}}{\subseteq} Y$, so $f^{-1}(A)$ is a union of open sets, which is open. So, f is continuous.

Problem 3

For nonempty topological spaces X, Y, show that $\forall x \in X, \{x\} \times Y \simeq Y$ under subspace topology.

Let $f : \{x\} \times Y \to Y$ be defined as $f(x, y) = y$. Since x is non-changing, we know that $\forall y \in Y, \exists (x, b) \in \{x\} \times Y$ such that $f(x, b) = y$, namely setting y to be the second coordinate, meaning f is surjective. Similarly, if $f(x, a) = f(x, b)$, then we have that $a = b$, $(x, a) = (x, b)$, meaning f is injective. Therefore, f is a bijection.

To prove continuity, we must show that $A \underset{\text{open}}{\subseteq} Y \to f^{-1}(A) \underset{\text{open}}{\subseteq} \{x\} \times Y$. For any $A \underset{\text{open}}{\subseteq} Y$, we have that $f^{-1}(A) = \{x\} \times A$. Since $\{x\} \underset{\text{open}}{\subseteq} \{x\}$ by subspace topology and $A \underset{\text{open}}{\subseteq} Y$, we have that $\{x\} \times A \underset{\text{open}}{\subseteq} \{x\} \times Y$ by the product topology, so f is continuous. Similarly, let $\{x\} \times A \underset{\text{open}}{\subseteq} \{x\} \times Y$. Then, $f(\{x\}, A) = A$, and since $\{x\} \underset{\text{open}}{\subseteq} \{x\}$, we have that $A \underset{\text{open}}{\subseteq} Y$ by the product topology, so f^{-1} is continuous.

Problem 4

Consider S^1 as a subpsace of $\mathbb{R}^2$. For each of the following, determine whether $f : X \to S^1$ is a homeomorphism for $f(x) = (\cos(2\pi x), \sin(2\pi x))$, and justify reasoning.

- $X_1 = [0, 1] \subset \mathbb{R}$
- $X_2 = [0, 1) \subset \mathbb{R}$

$f : X_1 \to S^1$ is **not** a homeomorphism, as the point at $(0, 1)$ can be mapped to both $\{0\}$ and $\{1\}$. Meanwhile, $f : X_2 \to S^1$ is a homeomorphism, because every open interval that passes through the point at $(0, 1)$ can be split into two intervals that are open in $[0, 1)$.

Problem 5

Find an equivalence relation $\sim$ on I^2 such that $I^2/\sim$ is homeomorphic to $S^1 \times I$, and find a homeomorphism and prove it is well-defined.

Let $\sim = \{(0, a) \sim (1, a)\}$. Then, we can find $f : I^2/\sim S^1 \times I$ by doing $f(x, y) = (\cos(2\pi x), \sin(2\pi x), y)$. Since $\cos, \sin$, and y are all well-defined functions, we have that f is a well-defined function for any $x, y \in [0, 1]$.

Find an equivalence relation such that $I^2/\sim$ is homeomorphic to a torus.

Let $\sim := \{(0, a) \sim (1, a), (b, 0) \sim (b, 1)\}$. Then, $I^2/\sim$ is a torus.

HW 13

Problem 1

Let $Q = \mathbb{R}/\{x \sim (x+1)\}$. What familiar topological space is Q homeomorphic to?

$Q \simeq S^1$, as for any element $x \in [0,1)$, $x+n \in Q$ for all $n \in \mathbb{Z}$. Therefore, Q can be represented as the real line wrapping around itself an infinite number of times, making it homeomorphic to a circle. For the homeomorphism, let $f : Q \to S^1$ be defined as $f(p) = (\cos(2\pi p), \sin(2\pi p))$ where $[p] \in Q$ and $p \in [0,1)$.

Problem 2

Let $Q = \mathbb{R}^2/\{(x,y) \sim (x+1,y) \sim (x,y+1)\}$. Which familiar topological space is Q homeomorphic to?

Since the equivalence relation on $\mathbb{R}^2$ is the two dimensional analog to S^1 in the previous example, we have that $Q \simeq S^1 \times S^1$, which is also homeomorphic to $[0,1) \times [0,1)$. We can find a homeomorphism $f : Q \to [0,1) \times [0,1)$ defined as $f([a],[b) = (a,b)$where $a, b \in [0,1)$.

Problem 3

Let $P = S^1/\{(x,y) \sim (-x,-y)\}$. Show that P is homeomorphic to S^1, and find a homeomorphism without proof.

When looking at S^1 with polar coordinates, we have that θ ranges from $[0,\pi)$, and that when $\theta = \pi$, $\theta = 0$ as well, which means P is a semicircle, with $f : P \to S^1$ defined as $f(r,\theta) = (r, 2\theta)$ for $\theta \in [0,\pi)$.

Problem 4

Do problem 5.6 on page 37 in *Intuitive Topology*.

HW 14

Problem 1

State which of the following spaces are homeomorphic to each other.

Answer is omitted due to inability to draw.

Problem 2

Which of the following letters considered as subspaces of $\mathbb{R}^2$ are manifolds?

ABCDEFGHIJKLMNOPQRSTUVWXYZ

The letters that in sans serif are manifolds are ones which do not contain a tripoint or quadripoint, meaning that the following are manifolds:

CDIJLMNOSUVWZ

Problem 3

How many connected non-homeomorphic 1-manifolds are there?

- $[0, 1]$
- $[0, 1)$
- $(0, 1)$
- S^1

Problem 4

Is an open ball in $\mathbb{R}^n$ minus its center a manifold? More precisely, let $x \in \mathbb{R}^n$. Is $B_r(x) - \{x\}$ a manifold? Prove your answer.

Since for every $y \in B_r(x) - \{x\}$, we can find an open ball by letting $s = \min(d(y, x), r - d(y, x))$ under the Euclidean metric, and letting $B_s(y)$ be our point. Since for every point we can find an open ball, we have that every point is locally homeomorphic to $\mathbb{R}^n$, so $B_r(x) - \{x\}$ is a manifold.

Problem 5

Is $S^1/\{(1, 0) \sim (-1, 0)\}$ a manifold?

Since $S^1/\{(1, 0) \sim (-1, 0)\}$ can be expressed as a lemniscate, which is not locally homeomorphic at the quadripoint center, the set is **not** a manifold.

Is $S^1/\{(x, y) \sim (-x, -y)\}$ a manifold?

Since $S^1/\{(x, y) \sim (-x, -y)\}$ is homeomorphic to S^1, and since S^1 is a manifold, so is $S^1/\{(x, y) \sim (-x, -y)\}$.

HW 15

Problem 1

For each of the following manifolds, state without proof (i) its dimension; (ii) its boundary (if it has any); (iii) whether it is compact; (iv) whether or not it's closed.

Answer is omitted.

Problem 2

State, without proof, whether or not each of the following topological spaces is a manifold (with or without boundary)

Answer is omitted.

Problem 3

Show for all $x \in S^1$, $S^1 - \{x\} \simeq (0,1) \subset \mathbb{R}$.

We can express $x \in S^1$ as $(1, \theta)$ for some $\theta \in [0, 2\pi)$. So, $S^1 - \{x\} = [0, \theta) \cup (\theta, 2\pi)$. We can transform x so $\theta = 0$ by taking $f : S^1 - \{x\} \to S^1 - \{(1,0)\}$ by taking $f(1,\theta) = \theta - \theta'$ for $x = (1, \theta')$. Then, we can find $g : S^1 - \{(1,0)\}$ by taking $g(1,\theta) = \frac{\theta}{2\pi}$.

Problem 4

Let $X = \text{cl}(B_1(0,0)) - B_{0.5}(0,0)$, $Y = S^1 \times [0,1]$. Find a homeomorphism without proof $f : X \to Y$

By letting $f : X \to Y$ by taking $f(r,\theta) = (\cos(\theta), \sin(\theta), 2r)$ for $\theta \in [0, 2\pi)$ and $r \in [0.5, 1]$ in polar coordinates, and $(\sin\theta, \cos\theta, 2r)$ in cartesian coordinates.

Problem 5

Prove that every closed subset of a compact topological space is compact.

Let X be a compact topological space and let $A \underset{\text{closed}}{\subseteq} X$. Let C be an open cover of A. Then, $C := \{V_\alpha \mid \alpha \in I\}$, where $V_\alpha \underset{\text{open}}{\subseteq} A$. So, $V_\alpha = U_\alpha \cap A$, where $U_\alpha \underset{\text{open}}{\subseteq} X$. Then, $A = \bigcup_{\alpha \in I} V_\alpha \subseteq \bigcup U_\alpha$. So, since $X = A \cup \overline{A}$, we have $X = \bigcup U_\alpha \cup \overline{A}$. Since A is closed, $F := \{U_\alpha \mid \alpha \in I\} \cup \overline{A}$ is an open cover of X. This means $\exists F' \underset{\text{finite}}{\subseteq} F$ as X is compact. So, $X = \bigcup_{\alpha \in I'} U_\alpha \cup \overline{A}$ where I' is finite. So, since $A \subseteq X$, we have $A \subseteq \bigcup_{i \in I'} U_\alpha \cup \overline{A}$. Since $A \not\subseteq \overline{A}$ by definition of complement, we have $A \subseteq \bigcup_{i \in I'} U_\alpha$. So, $A = \left(\bigcup_{\alpha \in I} U_\alpha\right) \cap A$, so $A = \bigcup \alpha \in I' V_\alpha$. So, C has a finite subcover $I' := \{V_\alpha \mid \alpha \in I'\}$, so A is compact.

HW 16

Problem 1

Let $A \subseteq M$. Prove that $\text{int}(A)$ is the union of all subsets $B \subseteq A$ such that $B \underset{\text{open}}{\subseteq} M$.

For the forward direction, let $x \in \text{int}(A)$. Then, $\exists B \underset{\text{open}}{\subseteq} M$ such that $x \in B$, by the topological definition of interior. So, for all $x \in A$, we have that $x \in \bigcup_{i \in I} B_i$ for some index set I, $B_i \subseteq A$, and $B_i \underset{\text{open}}{\subseteq} M$. So $\text{int}(A) \subseteq \bigcup_{i \in I} B_i$.

In the reverse direction, let $x \in \bigcup_{i \in I} B_i$ where $B_i \subseteq A$ and $B_i \underset{\text{open}}{\subseteq} M$. Then, $\exists B_k$ such that $x \in B_k$, and since $B_k \subseteq A$ and $B_k \underset{\text{open}}{\subseteq} M$, we have that $x \in A$ and $\exists B_k \underset{\text{open}}{\subseteq} M$ such that $x \in B_k$, so $x \in \text{int}(A)$. So $\bigcup_{i \in I} B_i \subseteq \text{int}(A)$

So, by the definition of set equality, $\text{int}(A) = \bigcup_{i \in I} B_i$.

Prove that $\text{cl}(A) = \bigcap_{i \in I} B_i$ where $A \subseteq B_i$ and $B_i \underset{\text{closed}}{\subseteq} M$.

Forward direction: Let $x \in \text{cl}(A)$. Then, if $x \in A$, we know that $x \in \text{cl}(B_i)$ by assumption. Otherwise, if $x \in \text{bd}(A)$, we have that $\exists C \underset{\text{open}}{\subseteq} M$ such that $x \in C \to y \in A$ where $y \neq x$, and since $y \in A$, $y \in \text{cl}(B_i)$, So, $x \in \text{cl}(B_i)$ by the definition of closure. Since B_i is a closed set, we know that $\text{cl}(B_i) = B_i$, so $x \in B_i$, meaning that $A \subseteq \bigcap_{i \in I} B_i$.

Reverse direction: By definition of closure, we know that $A \subseteq \text{cl}(A)$, and $\text{cl}(A)$ is closed. So, if $x \in \bigcap_{i \in I} B_i$, then $x \in \text{cl}(A)$ because, if x is in every closed superset of A, and $A \subseteq \text{cl}(A)$, then $x \in \text{cl}(A)$. So, $\bigcap_{i \in I} B_i \subseteq \text{cl}(A)$.

So, by the definition of set equality, $\text{cl}(A) = \bigcap_{i \in I} B_i$.

Problem 2

Find an example of nested, nonempty, closed subsets $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathbb{R}$ such that $\bigcap B_i = \emptyset$.

Let $B_i = [i, \infty)$. Each of these sets is closed as their complement is $(-\infty, i)$, but their intersection is $\emptyset$.

Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ be nonempty closed subsets of a compact topological space X. Prove that their intersection $\bigcap B_i$ is nonempty.

Suppose that $\bigcap B_i = \emptyset$. Then, $X = \overline{\bigcap B_i} = \bigcup \overline{B_i}$. Since $\overline{B_i} \underset{\text{open}}{\subseteq} X$ by the definition of a closed set, $X = \bigcup \overline{B_i}$ is a union of open sets, meaning that $F := \{\overline{B}_i \mid i \in I\}$ is an open cover of X. Since X is compact, we have $F' \underset{\text{finite}}{\subseteq} F$, meaning $F = \{B_1, \ldots, B_n\}$. So, $\bigcap_{i \in F'} B_i = \emptyset$ as $\bigcup_{i \in F'} B_i = X$. Since B_i are nested and $\bigcap_{i \in F'} B_i = \emptyset = B_n$ where n is the largest element of F'. So, $B_n = \emptyset$, which yields a contradiction. Therefore, $\bigcap B_i \neq \emptyset$.

Let $B_1 \supseteq B_2 \supseteq \cdots$ be nested, nonempty, closed, and compact subsets of $\mathbb{R}$. Is $\bigcap B_i$ necessarily nonempty?

Since B_1 is compact in $\mathbb{R}$, we have that B_1 is closed and bounded, and since each $B_i \underset{\text{closed}}{\subseteq} \mathbb{R}$, we have $\overline{B_i} \underset{\text{open}}{\subseteq} \mathbb{R}$, so $\overline{B_i} \cap B_1 \underset{\text{open}}{\subseteq} B_1$, so $B_i \underset{\text{closed}}{\subseteq} B_1$. Applying 2(b), we let $X = B_1$, meaning that $\bigcap_{i \geq 2} B_i$ is nonempty. Since B_1 is a nonempty superset of $\bigcap_{i \geq 2}$, $\bigcap_{i \geq 1} B_i$ is nonempty.

Problem 3

Using the Invariance of Domain Theorem, show that every compact 3-manifold embedded in $\mathbb{R}^3$ has boundary in both senses of the term.

Suppose M is a compact 3-manifold embedded in $\mathbb{R}^3$ that does not have boundary. So, $\forall x \in M$, x has a neighborhood that is homeomorphic to $\mathbb{R}^3$. So, $\exists U \underset{\text{open}}{\subseteq} M$ such that $x \in U$ and $U \simeq \mathbb{R}^3$, which means $\exists h : U \to \mathbb{R}^3$ where h is a homeomorphism. Since h is a homeomorphism, then $h^{-1} : \mathbb{R}^3 \to U \subseteq M \subseteq \mathbb{R}^3$ is a homeomorphism. By the invariance of domain theorem, we have that $h^{-1}\left(\mathbb{R}^3\right) \underset{\text{open}}{\subseteq} \mathbb{R}^3$. Let $W = h^{-1}\left(\mathbb{R}^3\right)$. Then, $W \underset{\text{open}}{\subseteq} \mathbb{R}^3$. For all $x \in W_x$ where W_x denotes the open set that contains x, we have that $x \in M$, meaning $M = \bigcup W_x$, so M is open as it is the union of open sets. Meanwhile, M is also closed as it is compact, and every compact set is closed and bounded in $\mathbb{R}^n$ by the Heine-Borel theorem. So, since M is clopen, we get that $\mathbb{R}^3$ is not connected by a previous result, which is a contradiction. Therefore, M has boundary.

Let $x \in \partial M$. Since M is closed, we have that $\text{cl}(M) = M$, and $x \notin \text{int}(M)$ since there would exist a neighborhood around x homeomorphic to $\mathbb{R}^3$, which violates the assumption that $x \in \partial M$. So, $x \in \text{bd}(M)$.

HW 17

Problem 1

Which of the surfaces are homeomorphic? Which are isotopic?

Answer is omitted.

Problem 2

Each of the surfaces (a) and (b) is a closed disk with two flat "strips" glued as "handles." (The one in (b) can also be described as a closed disk minus two open disks.) Give an argument to prove that (a) and (b) are not homeomorphic. Then, draw a series of pictures to show that a torus minus an open disk is homeomorphic to the surface given in (a).

Answer is omitted.

Problem 3

Use the fact that $\mathbb{R}$ is connected to show that S^1 is connected.

Since $\mathbb{R}$ is connected, and the map $f : \mathbb{R} \to S^1, f(x) = (\cos(2\pi x), \sin(2\pi x))$ is continuous, we have that S^1 must be connected.

Problem 4

Show S^1 cannot be embedded in $\mathbb{R}$.

Let $f : S^1 \to \mathbb{R}$ be a continuous injective function, where $f(S^1) \subset \mathbb{R}$ is an embedding. So, $f(S^1)$ is connected because S^1 is connected and f is continuous. Let $x, y, z \in S^1$, $x \neq y \neq z$. Because f is injective, $f(x), f(y), f(z) \in f(S^1) \subseteq \mathbb{R}$. WLOG, let $f(x) < f(y) < f(z)$. Since $S^1 - \{y\} \simeq (0, 1)$ which is connected, we should have $f(S^1 - \{y\})$ also be connected. Let $A = f(S^1 - \{y\}) = f(S^1) - f(y)$. So, $A = (A \cap (-\infty, f(y))) \cup (A \cap (f(y), \infty))$. The intervals are disjoint, non-empty ($f(x) \in (-\infty, f(y))$ and $f(z) \in (f(y), \infty)$), and open in $\mathbb{R}$ meaning $A \cap \{\text{the intervals}\} \underset{\text{open}}{\subseteq} A$, so this is disconnected. So, we have reached a contradiction, so $f(S^1)$ cannot be an embedding.

HW 18

Problem 1

Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1,$ and at least two of x, y, z are in the set $\{0, 1\}\}$ Let $F = \text{bd}(\text{cl}(N_{0.1}(A)))$. Draw a picture of F, and find n such that $F \simeq nT^2$.

Picture is omitted. $F \simeq T^2$ as one can expand the size of the bottom square and push the top end down, creating a square within a square that is connected with diagonal lines. Each of these holes can be rounded, creating a 5 holed torus. Therefore, $F \simeq 5T^2$.

Problem 2

Let (X, d) be a metric space. Prove or disprove: $\forall \epsilon > 0, N_\epsilon(A) = \bigcup_{a \in A} B_\epsilon(A)$.

Let $x \in N_\epsilon(A)$. Then, $d(x, a) < \epsilon$ for some $a \in A$. So, $x \in B_\epsilon(a)$, so $x \in \bigcup_{a \in A} B_\epsilon(a)$, so $N_\epsilon(A) \subseteq \bigcup_{a \in A} B_\epsilon(a)$. Similarly, let $x \in \bigcup_{a \in A} B_\epsilon(a)$. Then, $\exists a$ such that $d(x, a) < \epsilon$, so $x \in N_\epsilon(A)$, so $\bigcup_{a \in A} B_\epsilon(a) \subseteq N_\epsilon(A)$, so they are equal.

Problem 3

Problem 4

Let $f : X \to Y$ be a homeomorphism between topological spaces. Prove that A separates X iff $f(A)$ separates Y.

Suppose A separates X. Then, $X - A$ is disconnected. So, $f(X - A)$ is disconnected, as connectedness is an invariant. So, $f(X) - f(A)$ is disconnected since $X - A = X \cap \overline{A}$ and $f(X \cap \overline{A}) = f(X) \cap \overline{f(A)} = f(X) - f(A)$. Since $f(X) = Y$ as f is a homeomorphism, we have $Y - f(A)$ is disconnected. So $f(A)$ separates Y.

Suppose $f(A)$ separates Y. Then, $Y - f(A)$ is disconnected. So, $f^{-1}(Y - f(A))$ is disconnected, as connectedness is an invariant. So, $f^{-1}(Y) - f^{-1}(f(A))$ is disconnected, meaning $X - A$ is disconnected, so A separates X.