Solution (11.2, Problem 2)**:** We evaluate

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) \, dx$$

$$= \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(nx) \, dx$$

$$= \frac{1}{\pi} \left(\frac{1}{n} \cos(nx) \Big|_0^{\pi} \right)$$

$$= \frac{1}{n\pi} \left((-1)^n - 1 \right).$$

Therefore, our Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi} \sin(nx).$$

Solution (11.2, Problem 8)**:** We evaluate

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)(3 - 2x) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \cos(nx) - 2x \cos(nx) \, dx$$

$$= \begin{cases} 3 & n = 0 \\ 0 & \text{else.} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx)(3 - 2x) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \sin(nx) - 2x \sin(nx) \, dx$$

$$= \frac{1}{\pi} \left(\frac{3}{n} \cos(nx) \Big|_{-\pi}^{\pi} - 2 \left(\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right) \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{4(-1)^n}{n}.$$

Thus, our Fourier series is

$$f(x) = 3 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx).$$

Solution (11.2, Problem 10)**:** Using integration by parts, we evaluate

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} \cos\left(\frac{n}{2} x \right) \cos(x) \, dx$$

$$= \begin{cases} \frac{8}{\pi(4 - n^2)} \cos\left(\frac{n\pi}{4} \right) & n \neq 2 \\ \frac{1}{2} & n = 2 \end{cases}.$$

$$b_n = \frac{2}{\pi} \int_0^{\pi/2} \sin\left(\frac{n}{2} x \right) \cos(x) \, dx$$

$$= \begin{cases} \frac{8}{\pi(4 - n^2)} \left(\sin\left(\frac{n\pi}{4} \right) - \frac{n}{2} \cos\left(\frac{n\pi}{4} \right) \right). & n \neq 2 \\ \frac{1}{\pi} & n = 2 \end{cases}.$$

Thus, with $a_0 = \frac{2}{\pi}$, we have the Fourier series

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{2}x\right) + b_n \sin\left(\frac{n}{2}x\right).$$

Solution (11.2, Problem 17): We first start by finding the series expansion. Evaluating, we have

$$a_n = \frac{1}{\pi} \int_0^\pi x^2 \cos(nx) \, dx$$

$$= \begin{cases} \frac{2(-1)^n}{n^2} & n > 0 \\ \frac{\pi^2}{3} & n = 0 \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^\pi x^2 \sin(nx) \, dx$$

$$= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right).$$

Thus, we have the Fourier series

$$x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right) \sin(nx).$$

Using the input $x = 0$, we get

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Meanwhile, using the input of $-\pi$, we have $f(-\pi) = 0$, and

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution (11.2, Problem 18): Adding, we get

$$\frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}.$$

Solution (11.3, Problem 6): The function

$$f(x) = e^x - e^{-x}$$

is odd.

Solution (11.3, Problem 10): The function

$$f(x) = \left|x^5\right|$$

is even.

Solution (11.3, Problem 12): This function is even, so we expand in the cosine series. This gives

$$a_n = \int_0^2 f(x) \cos\left(\frac{n\pi}{2}x\right) dx$$

$$= \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx$$

$$= \begin{cases} \frac{2}{n\pi} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

Solution (11.3, Problem 18): This function is odd, so we expand in the sine series. This gives

$$b_n = \frac{2}{\pi} \int_0^\pi x^3 \sin(nx)\, dx$$

$$= \frac{(-1)^{n+1}\pi^3}{n} + \frac{6\pi(-1)^n}{n^3}.$$

Solution (11.3, Problem 20): This function is odd, so we expand in a sine series.

$$b_n = \frac{2}{\pi} \int_0^\pi (x+1)\sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_0^\pi x\sin(nx) + \sin(nx)\, dx$$

$$= \frac{2\left(1 + (-1)^{n+1} + (-1)^{n+1}\pi\right)}{n\pi}.$$

Solution (11.3, Problem 34): Since $f(0) = 0$, we expand in a sine series. This gives

$$b_n = \int_0^2 x(2-x)\sin\left(\frac{n\pi}{2}x\right) dx \qquad\qquad = \frac{16}{n^3\pi^3}\left(1 + (-1)^{n+1}\right).$$

Solution (11.4, Problem 2): We have the cases of $\lambda = \alpha^2, 0, -\alpha^2$, with corresponding solution forms of

$$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$
$$y = c_1 + c_2 x$$
$$y = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

Substituting our boundary conditions for each of these cases, where the first listed equation is the case $y(1) = 0$ and the second listed equation is the case $y(0) + y'(0) = 0$, we have

$$c_1 \cos(\alpha) + c_2 \sin(\alpha) = 0$$
$$\alpha(c_2 - c_1) = 0$$

$$c_1 + c_2 = 0$$
$$c_1 + c_2 = 0$$

$$c_1 \cosh(\alpha) + c_2 \sinh(\alpha) = 0$$
$$\alpha(c_1 + c_2) = 0.$$

In the case with $\lambda = \alpha^2$, we have $c_1 = c_2$ (as $\alpha \neq 0$), giving $y(1) = c_1 \cos(\alpha) + c_1 \sin(\alpha)$, which simplifies to $\sqrt{2}c_1 \sin\left(\alpha + \frac{\pi}{4}\right) = 0$. This has solutions of $\alpha = n\pi - \frac{\pi}{4}$, where $n \in \mathbb{Z}$.

In the case with $\lambda = 0$, we have the case $y = c_1 - c_1 x$.

The case with $\lambda = -\alpha^2$ simplifies to $c_1 = -c_2$, and $c_1 \cosh(\alpha) + c_1 \sinh(\alpha) = 0$, which is only true if $c_1 = c_2 = 0$. This is a trivial solution.

Therefore, we have eigenvalues $\lambda = 0, \left(\pi n - \frac{\pi}{4}\right)^2$.

$$\lambda_1 = 0$$

$$\lambda_2 = \frac{\pi^2}{16}$$
$$\lambda_3 = \frac{9\pi^2}{16}$$
$$\lambda_4 = \frac{25\pi^2}{16}.$$

Solution (11.4, Problem 4)**:** We have the periodic Sturm–Liouville boundary condition of

$$y(-L) - y(L) = 0$$
$$y'(-L) - y'(L) = 0.$$

Letting $\lambda = \alpha^2$, we have solutions of the form

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Using the first boundary condition, we have

$$2c_2 \sin(\alpha L) = 0,$$

so $\alpha = \frac{n\pi}{L}$, where $n \in \mathbb{Z}$. Varying our coefficients and our values of n, we recover the family

$$y = \left\{ 1, \sin\left(\frac{n\pi}{L}\right), \cos\left(\frac{n\pi}{L}\right) \,\middle|\, n \in \mathbb{Z} \right\}.$$

Solution (11.4, Problem 8)**:**

(a)

Solution (11.4, Problem 10)**:**

Solution (12.3, Problem 2)**:**

Solution (12.3, Problem 4)**:**