Extra Problem 3

Problem:

(a) If T is $\in$-transitive, then $\bigcup T \subseteq T$.

(b) If $\bigcup T \subseteq T$, then T is $\in$-transitive.

Solution.

(b)

(b) Let $\bigcup T \subseteq T$. Let $s \in T$, $x \in s$. Then, $s \subseteq \bigcup T$, so $x \in \bigcup T$, so $x \in T$.

Extra Problem 4

Problem: Prove that ordinal addition is associative.

Solution. Let α, β, γ be ordinals.

$$\alpha + \beta \cong \{0\} \times \alpha \cup \{1\} \times \beta$$

under the lexicographical order.

Additionally,

$$(\alpha + \beta) + \gamma \cong \underbrace{\{0\} \times (\alpha + \beta) \cup \{1\} \times \gamma}_{S},$$

ordered lexicographically.

Finally,

$$\alpha + (\beta + \gamma) \cong \underbrace{\{0\} \times \alpha \cup \{1\} \times (\beta + \gamma)}_{T}$$

ordered lexicographically.

It is enough to show that S is order isomorphic to T, since ordinals are unique up to order isomorphism.

Let $f : S \to T$. Then, for $x \in S$, we have $x \in \{0\} \times (\alpha + \beta)$ or $x \in \{1\} \times \gamma$.

$$f(x) = \begin{cases} (0, a) & x = (0, a);\ \text{for some } a \in \alpha \\ (1, a) & x = (0, a);\ \text{for some } a \in (\alpha + \beta) \setminus \alpha \\ (1, \beta + c) & x = (1, c);\ c \in \gamma \end{cases}$$

We need to show that f is well-defined and order-preserving.