

We recall from linear algebra that a linear operator  $T: V \rightarrow V$  is called diagonalizable if there is an orthonormal basis  $\{e_j\}_{j \in J}$  and a bounded collection of elements  $\{\lambda_j\}_{j \in J}$  such that for every  $x \in V$ , we have

$$Tx = \sum_{j \in J} \lambda_j \langle x, e_j \rangle e_j.$$

When  $V$  is a Hilbert space, there are a variety of generalizations. It will be useful to review the [basic properties](#) of compact and Fredholm operators.

## Spectral Theory for Compact Normal Operators

The first, most basic version of the spectral theorem is the one for compact normal operators. We recall the different types of spectra.

**Definition:** Let  $T \in B(X)$ , where  $X$  is a Banach space.

(i) The *point spectrum* of  $T$  is the set

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \ker(T - \lambda I) \neq \{0\}\},$$

which are the eigenvalues of  $T$ .

(ii) The *approximate point spectrum* of  $T$  is the set

$$\pi(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not bounded below}\}.$$

(iii) The *compression spectrum* of  $T$  is

$$\gamma(T) = \{\lambda \in \mathbb{C} \mid \text{im}(T - \lambda I) \text{ is not dense in } X\}.$$

There is a useful characterization of compact operators as follows.

**Lemma:** The following for  $T \in B(H)$  are equivalent:

(i)  $T$  is compact;

(ii)  $T|_{B_H}$  is a weak–norm continuous function from  $B_H$  into  $H$ .

*Proof.* Suppose  $T$  is compact. Then, if  $(x_i)_{i \in I}$  is a weakly convergent net in  $B_H$  with limit  $x$ , and  $\varepsilon > 0$ , there is some finite-rank  $S \in F(H)$  with  $\|S - T\|_{\text{op}} < \varepsilon/3$ . We have

$$\begin{aligned} \|Tx_i - Tx\| &\leq \|Tx_i - Sx_i\| + \|Sx_i - Sx\| + \|Sx - Tx\| \\ &\leq 2\|T - S\|_{\text{op}} + \|Sx_i - Sx\|. \end{aligned}$$

Every operator in  $B(H)$  is weak–weak continuous, and since  $\text{im}(S)$  is finite-dimensional, all norms coincide, so that  $Sx_i \rightarrow Sx$  in norm, giving that  $\|Tx_i - Tx\| < \varepsilon/3$  for sufficiently large  $i$ . Thus,  $T$  is weak–norm continuous.

If  $T$  is weak–norm continuous, then since  $B_H$  is weakly compact, it follows that  $T(B_H)$  is compact by continuity.  $\square$

**Lemma:** A diagonalizable operator  $T$  in  $B(H)$  is compact if and only if its eigenvalues  $\{\lambda_j \mid j \in J\}$  corresponding to an orthonormal basis  $\{e_j \mid j \in J\}$  belongs to  $c_0(J)$ .

*Proof.* Since  $T$  is diagonalizable, we have

$$Tx = \sum_{j \in J} \lambda_j \langle x, e_j \rangle e_j.$$

If  $T \in K(H)$ , and  $\varepsilon > 0$ , then we set

$$J_\varepsilon = \{j \in J \mid |\lambda_j| \geq \varepsilon\}.$$

If  $J_\varepsilon$  is infinite, then since  $\langle x, e_j \rangle \rightarrow 0$  by Parseval's identity, we have that the net  $(e_j)_{j \in J_\varepsilon}$  converges weakly to zero. Yet, since  $\|Te_j\| = |\lambda_j| \geq \varepsilon$  for any  $j \in J_\varepsilon$ , this contradicts the fact that  $T$  is weak-norm continuous. Thus,  $J_\varepsilon$  is finite for any  $\varepsilon > 0$ , so  $(\lambda_j)_{j \in J}$  vanishes at infinity.

Now, if  $J_\varepsilon$  is finite for every  $\varepsilon > 0$ , we may define  $T_\varepsilon \in F(H)$  by

$$T_\varepsilon = \sum_{j \in J_\varepsilon} \lambda_j \langle \cdot, e_j \rangle e_j,$$

and

$$\begin{aligned} \|(T - T_\varepsilon)x\|^2 &= \left\| \sum_{j \notin J_\varepsilon} \lambda_j \langle x, e_j \rangle e_j \right\|^2 \\ &= \sum_{j \in J_\varepsilon} |\lambda_j|^2 |\langle x, e_j \rangle|^2 \\ &\leq \varepsilon^2 \|x\|^2, \end{aligned}$$

so  $\|T - T_\varepsilon\| \leq \varepsilon$ , meaning that  $T \in \overline{F(H)} = K(H)$ .  $\square$

Note that by some basic computations, if  $T$  is diagonalizable, then we have

$$\begin{aligned} T^* &= \sum_{j \in J} \overline{\lambda_j} \langle \cdot, e_j \rangle e_j \\ T^*T &= \sum_{j \in J} |\lambda_j|^2 \langle \cdot, e_j \rangle e_j \\ &= TT^*. \end{aligned}$$

Thus, in particular, we have that every diagonalizable operator is normal.

**Theorem:** An operator  $T \in B(H)$  is diagonalizable with eigenvalues vanishing at infinity if and only if it is a compact normal operator.

*Proof.* Now we only need to show that every compact normal operator is diagonalizable. Since  $T$  is compact, we know that the spectrum of  $T$  consists of 0 and a countable set of isolated points, and since  $T$  is normal, its spectral radius is equal to the operator norm, meaning that there is some  $\lambda$  such that  $|\lambda| = \|T\|_{\text{op}}$ . In particular, there is an eigenvector for  $T$ .

Let  $\mathcal{Z}$  be the family of orthonormal systems of eigenvectors of  $T$ , ordered by inclusion. Since we have established that this family is nonempty, and the union provides an upper bound for any chain in  $\mathcal{Z}$ , there is some maximal orthonormal system  $\{e_j\}_{j \in J}$  with corresponding eigenvalues  $\{\lambda_j\}_{j \in J}$ . We let  $P$  be the projection onto the closed subspace spanned by the  $e_j$ . For each  $x \in H$ , we have

$$\begin{aligned} TPx &= T \left( \sum_{j \in J} \langle x, e_j \rangle e_j \right) \\ &= \sum_{j \in J} \lambda_j \langle x, e_j \rangle e_j \\ &= \sum_{j \in J} \langle x, \overline{\lambda_j} e_j \rangle e_j \\ &= \sum_{j \in J} \langle x, T^* e_j \rangle e_j \\ &= \sum_{j \in J} \langle Tx, e_j \rangle e_j \\ &= PTx. \end{aligned}$$

Thus, the operator  $(I - P)T$  is normal, and is also compact. If  $P \neq I$ , then either  $(I - P)T = 0$ , and every unit vector in  $(I - P)(H)$  is an eigenvector for  $T$  (contradicting maximality), or else  $(I - P)T \neq 0$ , in which case there is  $e_0 \in (I - P)(H)$  with  $Te_0 = \lambda e_0$  and  $|\lambda| = \|(I - P)T\|_{\text{op}}$ , which once again contradicts maximality.

Thus,  $P = I$ , and we are done.  $\square$

## Spectral Theory for Normal Operators

We now generalize from the special case of compact operators. Here, we cannot use the convenient properties of compact operators with respect to finite dimensionality/codimensionality.

First, we notice that if  $T \in B(H)$  is a normal operator, then  $C^*(T)$ , the  $C^*$ -algebra generated by  $T$ , is abelian, so from [the Gelfand isomorphism](#), we have that  $C^*(T) \cong C(\sigma(T))$  are isometrically  $*$ -isomorphic.

We will generalize this in a moment, but first we will apply the continuous functional calculus to show an important commutation relation. In  $M_n(\mathbb{C})$ , we know that an operator  $S$  commutes with a normal operator  $T$  if and only if all the eigenspaces for  $T$  are invariant under  $S$ ; since  $T$  and  $T^*$  commute, it then follows that  $S$  commutes with  $T^*$ .

It turns out that this generalizes to infinite-dimensional spaces, but the proof requires the use of the continuous functional calculus.

**Proposition** (Fuglede's Theorem): If  $S$  and  $T$  are operators in  $B(H)$ , and  $T$  is normal, then  $ST = TS$  implies  $ST^* = T^*S$ .

*Proof.* Define

$$e^{\lambda T} = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}.$$

This is an element of  $C^*(T)$  by the continuous functional calculus, and similarly,  $e^{\lambda T^*} \in C^*(T)$ , with

$$e^{\lambda T^*} = e^{\lambda T^* - \bar{\lambda}T} e^{\bar{\lambda}T}.$$

There is some self-adjoint operator  $R$  such that  $\lambda T^* - \bar{\lambda}T = iR$ , meaning that

$$U(\lambda) = e^{\lambda T^* - \bar{\lambda}T}$$

is a unitary operator in  $C^*(T)$  with  $U(\lambda)^* = U(-\lambda)$ .

It follows from the expression for  $e^{\lambda T}$  that  $S$  commutes with  $e^{\lambda T}$  for every  $\lambda$ , so that

$$e^{-\lambda T^*} S e^{\lambda T^*} = U(-\lambda) S U(\lambda),$$

with the operators uniformly bounded in norm by  $\|S\|$ .

Fixing  $x, y \in H$ , define  $f: \mathbb{C} \rightarrow \mathbb{C}$  by

$$f(\lambda) = \langle e^{-\lambda T^*} S e^{\lambda T^*} x, y \rangle.$$

It follows that  $f$  is an entire function with  $|f(\lambda)| \leq \|S\|$  for all  $\lambda$ , so that

$$\begin{aligned} \langle e^{-\lambda T^*} S e^{\lambda T^*} x, y \rangle - \langle Sx, y \rangle &= f(\lambda) - f(0) \\ &= 0, \end{aligned}$$

so that

$$e^{-\lambda T^*} S e^{\lambda T^*} - S = 0.$$

Thus,  $ST^* - T^*S = 0$ .  $\square$

## Spectral Theorem I

In order to prove the spectral theorem for normal operators, we use the concept of a spectral measure.

**Definition:** Let  $\Omega$  be a compact Hausdorff space, and  $H$  a Hilbert space. A *spectral measure*  $E$  relative to  $(\omega, H)$  is a map  $E$  from the Borel  $\sigma$ -algebra of  $\Omega$  to the set of projections on  $B(H)$  satisfying

- (i)  $E(\emptyset) = 0, E(\Omega) = I_H;$
- (ii)  $E(S_1 \cap S_2) = E(S_1)E(S_2);$
- (iii) for all  $x, y \in H$ , the map  $E_{x,y}: S \rightarrow \langle E(S)x, y \rangle$  is a regular complex Borel measure on  $\Omega$ .

We will let  $B_\infty(\Omega)$  be the set of bounded Borel functions on  $\Omega$ , and  $M(\Omega)$  the space of regular Borel complex measures with the total variation norm.

**Example:** Let  $\Omega$  be a compact Hausdorff space,  $\mu$  a positive regular Borel measure on  $\Omega$ . Let  $M_\varphi \in B(L_2(\Omega, \mu))$  be defined by

$$M_\varphi f = \varphi f.$$

We observe that

$$\begin{aligned} \|M_\varphi f\|^2 &= \int |\varphi f|^2 d\mu \\ &\leq \|\varphi\|_{L_\infty} \int |f|^2 d\mu. \end{aligned}$$

In particular, this means that  $\|M_\varphi\|_{\text{op}} \leq \|\varphi\|_{L_\infty}$ .

The map  $L_\infty(\Omega, \mu) \rightarrow B(L_2(\Omega, \mu))$  is thus a  $*$ -homomorphism of  $C^*$ -algebras, where  $M_\varphi^* = M_{\bar{\varphi}}$ .

In fact, since this map is injective, it is in fact an isometric  $*$ -homomorphism of  $C^*$ -algebras, following from the continuous functional calculus.

**Lemma:** Let  $\Omega$  be a compact Hausdorff space,  $H$  a Hilbert space. Let  $\mu_{x,y} \in M(\Omega)$  for each  $x, y \in H$ . Suppose that for each Borel set  $S$  in  $\Omega$ , the function  $\pi_S: H \times H \rightarrow \mathbb{C}$  given by  $(x, y) \mapsto \mu_{x,y}(S)$  is a sesquilinear form.

Then, for any  $f \in B_\infty(\Omega)$ , the function

$$\begin{aligned} \pi_f: H \times H &\rightarrow \mathbb{C} \\ (x, y) &\mapsto \int f d\mu_{x,y} \end{aligned}$$

is a sesquilinear form.

*Proof.* This is a standard bootstrapping argument. We start by letting  $f$  be a simple function, so we may write

$$f = \sum_{j=1}^n \lambda_j \mathbf{1}_{S_j}$$

for pairwise disjoint Borel subsets  $S_1, \dots, S_n$  of  $\Omega$  and complex numbers  $\lambda_1, \dots, \lambda_n$ . Then,

$$\int f d\mu_{x,y} = \sum_{j=1}^n \lambda_j \mu_{x,y}(S_j),$$

and since the (bounded) sesquilinear forms on  $H$  are in one to one correspondence with  $B(H)$ , it follows that the case for simple functions follows.

If  $f \in B_\infty(\Omega)$  is arbitrary, then there is a sequence  $(f_n)_n \rightarrow f$  of simple functions converging in the uni-

form norm. We observe that

$$\int |f_n - f| d|\mu_{x,y}| \leq \|f_n - f\|_{L_\infty} |\mu_{x,y}|(\Omega),$$

so we may exchange limit and integral by dominated convergence, giving

$$\int f d\mu_{x,y} = \lim_{n \rightarrow \infty} \int f_n d\mu_{x,y}$$

for every  $x, y \in H$ . Thus,  $\pi_f$  is a sesquilinear form on  $H$ .  $\square$

**Theorem:** Let  $\Omega$  be a compact Hausdorff space,  $H$  a Hilbert space, and  $E$  a spectral measure on  $(\Omega, H)$ . Then, for any  $f \in B_\infty(\Omega)$ , the map  $\pi_f: H \times H \rightarrow \mathbb{C}$  given by

$$(x, y) \mapsto \int f dE_{x,y}$$

is a bounded sesquilinear form, with  $\|\pi_f\| \leq \|f\|_{L_\infty}$ .

*Proof.* The previous lemma shows that  $\pi_f$  is a sesquilinear form. We only need to show that  $\|\pi_f\| \leq \|f\|_{L_\infty}$ . Let  $\Omega = S_1 \cup \dots \cup S_n$ , with  $S_1, \dots, S_n$  pairwise disjoint Borel sets. Then,

$$\begin{aligned} \sum_{j=1}^n |\langle E(S_j)x, y \rangle| &= \sum_{j=1}^n |\langle E(S_j)x, E(S_j)y \rangle| \\ &\leq \left( \sum_{j=1}^n \|E(S_j)x\|^2 \right)^{1/2} \left( \sum_{j=1}^n \|E(S_j)y\|^2 \right)^{1/2} \\ &= \|E(\Omega)x\| \|E(\omega)y\| \\ &= \|x\| \|y\|. \end{aligned}$$

Thus,  $\|E_{x,y}\| \leq \|x\| \|y\|$ , so

$$\begin{aligned} \left| \int f dE_{x,y} \right| &\leq \|f\|_{L_\infty} \|E_{x,y}\| \\ &\leq \|f\|_{L_\infty} \|x\| \|y\|. \end{aligned}$$

Thus,  $\|\pi_f\| \leq \|f\|_{L_\infty}$ .  $\square$

Thus, paired with the correspondence of sesquilinear forms and bounded operators on a Hilbert space, we obtain the following result.

**Theorem:** Let  $\Omega$  be a compact Hausdorff space,  $H$  a Hilbert space, and  $E$  a spectral measure on  $(\Omega, H)$ . Then, for each  $f \in B_\infty(\Omega)$ , there is a unique bounded operator  $T$  on  $H$  such that

$$\langle Tx, y \rangle = \int f dE_{x,y}.$$

We will define the *integral* of  $f \in B_\infty(\Omega)$  to be the (unique) operator such that for all  $x, y \in H$ ,

$$\left\langle \left( \int f dE \right) x, y \right\rangle = \int f dE_{x,y}.$$

**Proposition:** If  $E$  is a spectral measure for  $(\Omega, H)$ , and we define

$$\begin{aligned} \rho: B_\infty(\Omega) &\rightarrow B(H) \\ f &\mapsto \int f dE, \end{aligned}$$

then  $\rho$  is a representation for the  $C^*$ -algebra  $B_\infty(\Omega)$ . That is,  $\rho$  is a unital  $*$ -homomorphism.

*Proof.* Linearity follows from a bootstrapping argument, and boundedness from the definition of the sesquilinear form.

Thus, we only need to show multiplicativity. Similarly from bootstrapping, we only need to show the case when  $f$  and  $g$  are simple. Suppose  $f = \mathbb{1}_S$  and  $g = \mathbb{1}_{S'}$ . Then,

$$\begin{aligned}\rho(fg) &= \int \mathbb{1}_S \mathbb{1}_{S'} dE \\ &= E(S \cap S') \\ &= E(S)E(S') \\ &= \left( \int \mathbb{1}_S dE \right) \left( \int \mathbb{1}_{S'} dE \right) \\ &= \rho(f)\rho(g)\end{aligned}$$

and similarly projections are self-adjoint.  $\square$

Now, we've elucidated a lot of properties of spectral measures, but we still have not answered the question of their existence. This is the spectral theorem.

**Theorem** (Spectral Theorem for Bounded Normal Operators): Let  $\Omega$  be a compact Hausdorff space,  $H$  a Hilbert space, and let  $\varphi: C(\Omega) \rightarrow B(H)$  be a unital  $*$ -homomorphism. Then, there is a unique spectral measure  $E$  with respect to  $(\Omega, H)$  such that

$$\varphi(f) = \int f dE$$

for all  $f \in C(\Omega)$ . Moreover, if  $T \in B(H)$ , then  $T$  commutes with  $\varphi(f)$  for all  $f \in C(\Omega)$  if and only if  $T$  commutes with  $E(S)$  for all Borel  $S \subseteq \Omega$ .

*Proof.* For any  $x, y \in H$ , the function  $\tau_{x,y}: C(\Omega) \rightarrow \mathbb{C}$  given by

$$f \mapsto \langle \varphi(f)x, y \rangle$$

is linear with  $\|\tau_{x,y}\|_{\text{op}} \leq \|x\|\|y\|$ . From the Riesz Representation Theorem, there is a unique measure  $\mu_{x,y} \in M(\Omega)$  such that

$$\tau_{x,y}(f) = \int f d\mu_{x,y}$$

for all  $f \in C(\Omega)$ . We also have that  $\|\mu_{x,y}\| = \|\tau_{x,y}\|_{\text{op}}$ . The function

$$(x, y) \mapsto \langle \varphi(f)x, y \rangle$$

is a sesquilinear map from  $H$  to  $M(\Omega)$  such that  $x \mapsto \mu_{x,y}$  is linear and  $y \mapsto \mu_{x,y}$  is conjugate-linear. Thus, for all  $f \in B_\infty(\Omega)$ , the map

$$(x, y) \mapsto \int f d\mu_{x,y}$$

is a sesquilinear form, with

$$\begin{aligned}\left| \int f d\mu_{x,y} \right| &\leq \|f\|_{L_\infty} \|\mu_{x,y}\| \\ &\leq \|f\|_{L_\infty} \|x\| \|y\|,\end{aligned}$$

so there is a unique bounded operator,  $\psi(f) \in B(H)$  such that

$$\langle \psi(f)x, y \rangle = \int f d\mu_{x,y}$$

for all  $x, y \in H$ . If  $f \in C(\Omega)$ , then we have that

$$\begin{aligned}\langle \psi(f)x, y \rangle &= \int f d\mu_{x,y} \\ &= \tau_{x,y}(f) \\ &= \langle \varphi(f)x, y \rangle,\end{aligned}$$

so  $\psi(f) = \varphi(f)$ .

We now show that  $\psi$  is a  $*$ -homomorphism. If  $f \in C(\Omega)$  with  $\bar{f} = f$ , then  $\varphi(f)$  is self-adjoint, meaning that

$$\int f d\mu_{x,x} = \langle \varphi(f)x, x \rangle$$

is a real number, so  $\mu_{x,x}$  is a real measure. Thus, if  $f \in B_\infty(\Omega)$  is arbitrary, dominated convergence gives that

$$\langle \psi(f)x, x \rangle = \int f d\mu_{x,x}$$

is real. Thus,  $\psi(f)$  is self-adjoint, so  $\psi$  preserves involutions.

If  $f \in B_\infty$  and  $x \in H$ , then we claim that it is enough to show that

$$\langle \psi(fg)x, x \rangle = \langle \psi(f)\psi(g)x, x \rangle \quad (*)$$

holds for any  $g \in C(\Omega)$ . A way to rewrite  $(*)$  is by

$$\int gf d\mu_{x,x} = \int g d\mu_{x,\psi(\bar{f})x},$$

so if  $(*)$  holds for all  $g \in C(\Omega)$ , then the measures  $fd\mu_{x,x}$  and  $\mu_{x,\psi(\bar{f})x}$  are equal since their corresponding linear functionals are necessarily equal. In particular, this holds for all such  $g$ .

Since  $\varphi$  is a  $*$ -homomorphism, the equation  $(*)$  holds for all  $f, g \in C(\Omega)$ , so it holds if  $f \in C(\Omega)$  and  $g \in B_\infty(\Omega)$  by density. Similarly, by replacing  $f$  and  $g$  with their conjugates, we have

$$\langle \psi(\bar{f}g)x, x \rangle = \langle \psi(\bar{f})\psi(\bar{g})x, x \rangle,$$

so by taking conjugates and using the fact that  $\psi$  is a homomorphism, we get

$$\langle \psi(gf)x, x \rangle = \langle \psi(g)\psi(f)x, x \rangle \quad (**)$$

for all  $g \in B_\infty(\Omega)$ . Using  $(*)$  by interchanging  $g$  and  $f$ , we obtain that  $(**)$  holds for all  $f, g \in B_\infty(\Omega)$ . Since  $x \in H$  was arbitrary, we have  $\psi(gf) = \psi(g)\psi(f)$ , so  $\psi$  is a homomorphism.

Now, if  $S$  is a Borel subset of  $\Omega$ , we let  $E(S) = \psi(\mathbb{1}_S)$ . We see that  $E(S)$  is a projection on  $H$ , and that the map  $E: S \rightarrow E(S)$  is a spectral measure, with  $E_{x,y} = \mu_{x,y} \in M(\Omega)$ , as

$$\begin{aligned}E_{x,y}(S) &= \langle E(S)x, y \rangle \\ &= \langle \psi(\mathbb{1}_S)x, y \rangle \\ &= \int \mathbb{1}_S d\mu_{x,y}.\end{aligned}$$

If  $f \in B_\infty(\Omega)$ , then from a bootstrapping argument, we have

$$\left\langle \left( \int f dE \right) x, y \right\rangle = \int f dE_{x,y}$$

$$= \int f \, d\mu_{x,y} \\ = \langle \psi(f)x, y \rangle,$$

so that

$$\psi(f) = \int f \, dE,$$

and in particular, for all  $f \in C(\Omega)$ ,

$$\varphi(f) = \int f \, dE.$$

Additionally, for all  $x, y \in H$ , if  $E'$  is another spectral measure that satisfies

$$\varphi(f) = \int f \, dE',$$

then we have for all  $x, y \in H$ ,

$$\begin{aligned} \int f \, dE'_{x,y} &= \langle \varphi(f)x, y \rangle \\ &= \int f \, dE_{x,y}, \end{aligned}$$

so  $E'_{x,y} = E_{x,y}$  for all  $x, y$ , meaning that for all Borel  $S \subseteq \Omega$ ,

$$\langle E'(S)x, y \rangle = \langle E(S)x, y \rangle,$$

meaning  $E = E'$ .

Finally, if  $T$  is an operator on  $H$  commuting with all the elements of the range of  $\varphi$ , then if  $f \in C(\Omega)$ , we have

$$\begin{aligned} \int f \, d\mu_{Tx,y} &= \langle \psi(f)Tx, y \rangle \\ &= \langle T\psi(f)x, y \rangle \\ &= \langle \psi(f)x, T^*y \rangle \\ &= \int f \, d\mu_{x,T^*y}, \end{aligned}$$

so that  $E_{Tx,y} = E_{x,T^*y}$ , and  $E(S)T = TE(S)$  for all Borel  $S \subseteq \Omega$ . Conversely, if  $T$  commutes with all the projections  $E(S)$ , then we have

$$\begin{aligned} \langle E(S)Tx, y \rangle &= \langle TE(S)x, y \rangle \\ &= \langle E(S)x, T^*y \rangle, \end{aligned}$$

or that  $E_{Tx,y} = E_{x,T^*y}$ , so for all  $f \in C(\Omega)$ ,

$$\int f \, dE_{Tx,y} = \int f \, dE_{x,T^*y},$$

or that

$$\begin{aligned} \langle \varphi(f)Tx, y \rangle &= \langle \varphi(f)x, T^*y \rangle \\ &= \langle T\varphi(f)x, y \rangle, \end{aligned}$$

and since this holds for all  $x, y \in H$ ,  $\varphi(f)T = T\varphi(f)$ .  $\square$

The most important case is when the  $*$ -homomorphism in question is a representation of the  $C^*$ -algebra generated by a normal operator, and is often known as *the* spectral theorem.

**Theorem:** Let  $T$  be a normal operator on a Hilbert space  $H$ . There is a unique spectral measure  $E$  relative to  $(\sigma(T), H)$  such that

$$T = \int \iota \, dE,$$

where  $\iota$  is the inclusion map of  $\sigma(T)$  into  $\mathbb{C}$ .

*Proof.* Let  $\varphi: C(\sigma(T)) \rightarrow B(H)$  be the functional calculus at  $T$ . There is then a unique spectral measure  $E$  relative to  $(\sigma(T), H)$  such that

$$\varphi(f) = \int f \, dE$$

for all  $f \in C(\sigma(T))$ . In particular, we have

$$\begin{aligned} T &= \varphi(\iota) \\ &= \int \iota \, dE, \end{aligned}$$

and uniqueness following from the fact that 1 and  $\iota$  generate  $C(\sigma(T))$  as a  $C^*$ -algebra.  $\square$

We call the spectral measure in this special case the *resolution of the identity* for  $T$ . We have that for all  $f \in B_\infty(\sigma(T))$ , we may unambiguously define

$$f(T) = \int f \, dE.$$

We call the unital  $*$ -homomorphism taking  $f \mapsto f(T)$  the *Borel functional calculus* at  $T$ .

## Examples

**Example:** Let  $\mu$  be a regular compactly supported Borel measure on  $\mathbb{C}$ . Define  $M_z$  on  $L_2(\mu)$  by  $M_z f = zf$  for each  $f \in L_2(\mu)$ . Then,  $M_z^* f = \bar{z}f$ , and  $M_z$  is normal.

Now, we claim that  $\sigma(M_z) = \text{supp}(\mu)$ . This follows from the fact that if  $\lambda \in \mathbb{C} \setminus \text{supp}(\mu)$ , then the operator  $S$  defined by

$$Sf = (z - \lambda)^{-1}f$$

has that  $\|Sf\| < \infty$  for all  $f \in L_2(\mu)$ .

In particular, this means that for any bounded Borel function  $\phi$ , we may define  $M_\phi f = \phi f$  and we have  $\phi(M_z) = M_\phi$ .

**Example:** If  $(X, \Omega, \mu)$  is a  $\sigma$ -finite measure space, and  $H = L_2(X, \mu)$ , we may define, for any  $\phi \in L_\infty(\mu)$ , the operator  $M_\phi f = \phi f$ . Then,  $M_\phi$  is normal with  $M_\phi^* = M_{\bar{\phi}}$ .

The *essential range* of  $\phi$  is defined as

$$\text{ess ran}(\phi) = \bigcap \left\{ \overline{\phi(S)} \mid S \in \Omega, \mu(X \setminus S) = 0 \right\}.$$

Then, we have that  $\sigma(M_\phi) = \text{ess ran}(\phi)$ . We see that if  $\lambda \notin \text{ess ran}(\phi)$ , then there is a set  $S$  in  $\Omega$  with  $\mu(X \setminus S) = 0$  and  $\lambda \notin \phi(S)$ , so there is  $\delta > 0$  so  $|\phi(x) - \lambda| \geq \delta$  for all  $x \in S$ . Therefore, we may define

$$M_\psi = (M_\phi - \lambda)^{-1}$$

with  $\psi \in L_\infty(\mu)$ .

Now, if  $\lambda \in \text{ess ran}(\phi)$ , then for every  $n$ , there is  $S_n \in \Omega$  with  $0 < \mu(S_n) < \infty$  and  $|\phi(x) - \lambda| < 1/n$  for all  $x \in S_n$ . Set

$$f_n = (\mu(S_n))^{-1/2} \mathbb{1}_{S_n},$$

so  $f_n \in L_2(\mu)$  and  $\|f_n\| = 1$ . Yet, we have

$$\begin{aligned} \|(M_\phi - \lambda)f_n\|^2 &= \frac{1}{\mu(S_n)} \int_{S_n} |\phi - \lambda|^2 d\mu \\ &< \frac{1}{n^2}, \end{aligned}$$

meaning that  $\lambda$  is an element of the approximate point spectrum of  $M_\phi$ .

**Proposition:** Let  $N_k$  be a normal operator on  $H_k$  for each  $k \geq 1$ , and assume that  $\sup_k \|N_k\|_{\text{op}} < \infty$ . Let  $E_k$  be the spectral measure for  $N_k$ , and define

$$N = \bigoplus_{k=1}^{\infty} N_k$$

acting on

$$H = \bigoplus_{k=1}^{\infty} H_k.$$

Then,

(a) we have

$$\sigma(N) = \overline{\bigcup_{k=1}^{\infty} \sigma(N_k)};$$

(b) and if  $E$  is the spectral measure for  $N$ , for any Borel  $S \subseteq \sigma(N)$ ,

$$E(S) = \bigoplus_{k=1}^{\infty} E_k(S \cap \sigma(N_k)).$$

*Proof.*

(a) We observe that  $\lambda \in \rho(N)$  if and only if

$$N - \lambda I_H = \bigoplus_{k=1}^{\infty} (N_k - \lambda I_k)$$

is invertible, meaning that we must have  $N_k - \lambda I_k$  is invertible for each  $k$ . In particular, this means that

$$\rho(N) = \left( \bigcap_{k=1}^{\infty} \rho(N_k) \right)^\circ,$$

so that

$$\sigma(N) = \overline{\bigcup_{k=1}^{\infty} \sigma(N_k)}.$$

(b) Any projection  $P: H \rightarrow H$  necessarily decomposes as

$$P = \bigoplus_{k=1}^{\infty} P_k P,$$

where  $P_k$  denotes the projection onto  $H_k$ . Therefore, we have

$$\begin{aligned} E(S) &= \bigoplus_{k=1}^{\infty} P_k E(S) \\ &= \bigoplus_{k=1}^{\infty} E_k(\sigma(N_k)) E(S) \\ &= \bigoplus_{k=1}^{\infty} E_k(S \cap \sigma(N_k)). \end{aligned}$$

□

## Spectral Theorem II

We can now reformulate the spectral theorem for the special case where the normal operator  $N$  admits a special vector. Recall that a *reducing subspace* for an operator  $T$  on a Hilbert space  $H$  is an invariant subspace  $W$  such that both  $W$  and  $W^\perp$  are invariant.

**Definition:** A vector  $e_0 \in H$  is *star-cyclic* for an operator  $A$  if  $H$  is the smallest reducing subspace for  $A$  that contains  $e_0$ . We say  $A$  is star-cyclic if  $A$  admits a star-cyclic vector.

A vector  $e_0$  is cyclic for  $A$  if the smallest invariant subspace for  $A$  that contains  $e_0$  is  $H$ . We say  $A$  is cyclic if  $A$  has a cyclic vector.

**Proposition:** If  $A$  has either a cyclic or star-cyclic vector, then  $H$  is separable.

*Proof.* By definition, both  $C^*(A)$  and  $\{p(A) \mid p \in \mathbb{C}[x]\}$  are separable subalgebras of  $B(H)$ . □

If  $\mu$  is a compactly supported measure on  $\mathbb{C}$ , and  $M_z$  is the multiplication by  $z$  as defined above, then if we let  $K = \text{supp}(\mu)$ , we have  $C^*(M_z) = \{M_u \mid u \in C(K)\}$ . Since  $C(K)$  is dense in  $L_2(\mu)$ , it follows that 1 is a star-cyclic vector for  $M_z$ .

As it turns out, the converse is also true.

**Theorem:** A normal operator  $N$  is star-cyclic if and only if  $N$  is unitarily equivalent to  $M_z$  for some compactly supported measure  $\mu$  on  $\mathbb{C}$ . If  $e_0$  is a star-cyclic vector for  $N$ , then we may choose  $\mu$  to be such that there is an isomorphism  $V: H \rightarrow L_2(\mu)$  with  $Ve_0 = 1$  and  $VNV^{-1} = M_z$ .

*Proof.* If  $N \cong M_z$ , then we already see that  $N$  is star-cyclic.

Now, suppose  $N$  has a star-cyclic vector  $e_0$ . Letting  $E$  be the spectral measure for  $N$ , we define

$$\mu(S) = \langle E(S)e_0, e_0 \rangle$$

for every Borel subset  $S$  of  $\mathbb{C}$ , and let  $K = \text{supp}(\mu)$ . If  $\phi \in B_\infty(K)$ , then we have

$$\begin{aligned} \|\phi(N)e_0\|^2 &= \langle \phi(N)e_0, \phi(N)e_0 \rangle \\ &= \langle |\phi|^2(N)e_0, e_0 \rangle \\ &= \int |\phi|^2 dE_{e_0, e_0} \\ &= \int |\phi|^2 d\mu, \end{aligned}$$

so, considering  $B_\infty(K)$  as a subspace of  $L_2(\mu)$ , we may define an isometry  $U\phi = \phi(N)e_0$  between  $B_\infty(K)$

and  $\{\phi(N)e_0 \mid \phi \in B_\infty(K)\}$ . Yet, since  $e_0$  is star-cyclic, the range of  $U$  is dense in  $H$ , so  $U$  extends to an isometric isomorphism  $V: L_2(\mu) \rightarrow H$ .

Now, if  $\phi \in B(K)$ , then

$$\begin{aligned} VM_z V^{-1}(\phi(N)e_0) &= VM_z(\phi) \\ &= V(z\phi) \\ &= N\phi(N)e_0, \end{aligned}$$

so that  $VM_z V^{-1} = N$ . By exchanging  $V$  for  $V^{-1}$ , we get our desired result.  $\square$

It turns out then that any theorem about the multiplication operators applies to any star-cyclic normal operator. We may now give a complete unitary invariant of star-cyclic normal operators.

**Definition:** Two measures  $\mu_1$  and  $\mu_2$  defined on the measurable space  $(X, \Omega)$  are *mutually absolutely continuous* if  $\mu_1(S) = 0$  if and only if  $\mu_2(S) = 0$ . We write  $[\mu_1] = [\mu_2]$ .

**Theorem:** If we let  $M_{\mu_1} = M_z \in B(L_2(\mu_1))$  and  $M_{\mu_2} = M_z \in B(L_2(\mu_2))$ , then we have  $M_{\mu_1} \cong M_{\mu_2}$  are unitarily equivalent if and only if  $[\mu_1] = [\mu_2]$ .

*Proof.* Let  $[\mu_1] = [\mu_2]$ , and take  $\phi = \frac{d\mu_1}{d\mu_2}$ . If  $g \in L_1(\mu_1)$ , then  $g\phi \in L_1(\mu_2)$ , with

$$\int g\phi \, d\mu_2 = \int g \, d\mu_1.$$

Therefore, if we let  $\phi \in L_2(\mu_1)$ , we have  $\sqrt{\phi}f \in L_2(\mu_2)$  with  $\|\sqrt{\phi}f\| = \|f\|$ . That is,  $U: L_2(\mu_1) \rightarrow L_2(\mu_2)$  given by  $Uf = \sqrt{\phi}f$  is an isometry. For any  $g \in L_2(\mu_2)$ , we may define  $f = \phi^{-1/2}g \in L_2(\mu_1)$  with  $Uf = g$ , so  $U$  is surjective, with  $UM_{\mu_1}U^{-1} = M_{\mu_2}$ .

Now, if we let  $V: L_2(\mu_1) \rightarrow L_2(\mu_2)$  be an isomorphism with  $VM_{\mu_1}V^{-1} = M_{\mu_2}$ , we take  $\psi = V(1)$ , where we use the fact that  $\mu_1$  and  $\mu_2$  are compactly supported measures. Therefore,  $\psi \in L_2(\mu_2)$ .  $\square$

## References

- [Ped89] Gert K. Pedersen. *Analysis Now*. Vol. 118. Graduate Texts in Mathematics. Springer-Verlag, New York, 1989, pp. xiv+277. ISBN: 0-387-96788-5. DOI: [10.1007/978-1-4612-1007-8](https://doi.org/10.1007/978-1-4612-1007-8). URL: <https://doi.org/10.1007/978-1-4612-1007-8>.
- [Con90] John B. Conway. *A Course in Functional Analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.
- [Bla06] B. Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of  $C^*$ -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4; 3-540-28486-9. DOI: [10.1007/3-540-28517-2](https://doi.org/10.1007/3-540-28517-2). URL: <https://doi.org/10.1007/3-540-28517-2>.
- [Dav25] Kenneth R. Davidson. *Functional analysis and operator algebras*. Vol. 13. CMS/CAIMS Books in Mathematics. Springer, Cham, [2025] ©2025, pp. xiv+797. ISBN: 978-3-031-63664-6; 978-3-031-63665-3. DOI: [10.1007/978-3-031-63665-3](https://doi.org/10.1007/978-3-031-63665-3). URL: <https://doi.org/10.1007/978-3-031-63665-3>.