Amenable Discrete Groups

Conditions and Applications

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Paradoxical Decompositions

The primary goal of this section will be to introduce the idea of a paradoxical decomposition (and its effects on the analytic properties of $\mathbb{R}^3$) through the Banach–Tarski Paradox. The ultimate goal is to prove the following statement.

Proposition 2.0.1 (General Banach–Tarski Paradox)**:** If A and B are bounded subsets of $\mathbb{R}^3$ with nonempty interior, there is a partition of A into finitely many disjoint subsets such a sequence of isometries applied to these subsets yields B.

The existence of the Banach–Tarski paradox throws a wrench into a major idea that we may have about subsets of $\mathbb{R}^3$ — namely, that they always have some "volume" to them that is invariant under isometry, similar to how "area" in $\mathbb{R}^2$ is invariant under isometry.

2.1 Prelude: Group Action Essentials

We begin by discussing some of the basic properties of group actions.

Definition (Group Action)**.** Let G be a group, and A be a set. A left group action of G onto A is a map $\alpha : G \times A \to A$ that satisfies

- $\alpha(g_1, (g_2, a)) = \alpha(g_1 g_2, a)$ for all $g_1, g_2 \in G$ and $a \in A$;
- $\alpha(e_G, a) = a$ for all $a \in A$.

For the sake of brevity, we write $(g, a) = g \cdot a$.

Every group action can be represented by a permutation on A.

Definition (Permutation Representation)**.** For each g, the map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$ is a permutation of A. There is a homomorphism associated to these actions, $\varphi : G \to \mathrm{Sym}(A)$, where $\mathrm{Sym}(A)$ is the symmetric group on the elements of A.

The permutation representation can run in the opposite direction in the following sense: given a nonempty set A and a homomorphism $\psi : G \to \mathrm{Sym}(A)$, we can take $g \cdot a = \psi(g)(a)$, where $\psi(g) = \sigma_g \in \mathrm{Sym}(A)$ is a permutation.

Just as we can pass group actions into permutation representations, and discuss ideas like the kernel of homomorphisms, we can also discuss the kernel of ajn action.

Definition (Kernel)**.** The **kernel** of the action of G on A is the set of elements in g that act trivially on A:

$$\{g \in G \mid \forall a \in A,\ g \cdot a = a\}.$$

The kernel of the group action is the kernel of the permutation representation $\varphi : G \to \mathrm{Sym}(A)$.

Definition (Stabilizer)**.** For each $a \in A$, we define the **stabilizer** of a under G to be the set of elements in G that fix a:

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

Remark: The kernel of the group action is the intersection of the stabilizers of every element of A.

For each $a \in A$, G_a is a subgroup of G.

Definition (Faithful Action)**.** An action is **faithful** if the kernel of the action is the identity, e_G. Equivalently, the permutation representation $\varphi : G \to \mathrm{Sym}(A)$ is injective.

The following definition will be useful in the future as we dig deeper into the idea of paradoxical groups.

Definition (Free Action)**.** For a set X with G acting on X, the action of G on X is free if, for every $x \in X$, $g \cdot x = x$ if and only if $g = e_G$.

The most important theorem relating to group actions is the orbit-stabilizer theorem. As we prove the following theorem, we will reveal the definition of an orbit as a type of equivalence class.

Theorem 2.1.1 (Orbit-Stabilizer Theorem)**:** Let G be a group that acts on a nonempty set A. We define a relation $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. This is an equivalence relation, with the number of elements in $[a]_\sim$ found by taking the index of the stabilizer of a in G, $|G : G_a|$.

Proof. We start by seeing that $a \sim a$, as $e_G \cdot a = a$. Similarly, if $a \sim b$, then there exists $g \in G$ such that $a = g \cdot b$. Thus,

$$\begin{aligned} g^{-1} \cdot a &= g^{-1} \cdot (g \cdot b) \\ &= g^{-1} g \cdot b \\ &= e \cdot b \\ &= b, \end{aligned}$$

meaning that $b \sim a$. Finally, if we have $a \sim b$ and $b \sim c$, we have $a = g \cdot b$ and $b = h \cdot c$ for some $g, h \in G$. Therefore,

$$a = g \cdot (h \cdot c)$$

$$= (gh) \cdot c,$$

meaning $a \sim c$. Thus, the relation $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

We claim there is a bijection between the left cosets of G_a and the elements of $[a]_\sim$.

Define $C_a = \{g \cdot a \mid g \in G\}$, which is the set of elements in the equivalence class of a. Define the map $g \cdot a \mapsto gG_a$. Since $g \cdot a$ is always an element of C_a, this map is surjective. Additionally, since $g \cdot a = h \cdot a$ if and only if $\left(h^{-1}g\right) \cdot a = a$, we have $h^{-1}g \in G_a$, which is only true if $gG_a = hG_a$. Thus, the map is injective.

Since there is a one to one map between the equivalence classes of a under the action of G, and the number of left cosets of G_a, we know that the number of equivalence classes of a under the action of G is $|G : G_a|$. □

Definition (Orbit)**.** Let G act on A, and let $a \in A$. The **orbit** of a under G is the set

$$G \cdot a = \{g \cdot a \in A \mid g \in G\}$$

2.2 Paradoxical Decompositions in $\mathbb{R}^3$

With the essential facts about group actions in mind, we can turn our attention to "paradoxical" actions that seem to recreate a set by using some of its disjoint proper subsets.

Definition (Paradoxical Decompositions and Paradoxical Groups)**.** Let G be a group that acts on a set X, with $E \subseteq X$. We say E is G**-paradoxical** if there exist pairwise disjoint proper subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$ and group elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j$$

and

$$E = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

If G acts on itself by left-multiplication, and G satisfies these conditions, we say G is a **paradoxical group**.

Example 2.2.1. The free group on two generators, $F(a, b)$, is a paradoxical group.

The free group is defined to be the set of all reduced words over the set $\left\{a, b, a^{-1}, b^{-1}, e_{F(a,b)}\right\}$, where aa^{-1}, $a^{-1}a$, bb^{-1}, and $b^{-1}b$ are replaced with the identity $e_{F(a,b)}$.

To see that $\mathsf{F}(a, b)$ is a paradoxical group, we let $W(x) = \{w \in \mathsf{F}(a, b) \mid w \text{ starts with } x\}$. For instance, $ba^2ba^{-1} \in W(b)$.

Since every word in F is either the empty word, or starts with one of a, b, a^{-1}, b^{-1}, we see that

$$\mathsf{F}(a, b) = \left\{e_{\mathsf{F}(a,b)}\right\} \sqcup W(a) \sqcup W(b) \sqcup W\left(a^{-1}\right) \sqcup W\left(b^{-1}\right).$$

For $w \in \mathsf{F}(a, b) \setminus W(a)$, it is the case that $a^{-1}w \in W\left(a^{-1}\right)$, so $w \in aW\left(a^{-1}\right)$. Thus, for any $t \in \mathsf{F}(a, b)$, $t \in W(a)$ or $t \in \mathsf{F}(a, b) \setminus W(a) = aW\left(a^{-1}\right)$, so $\mathsf{F}(a, b) = W(a) \sqcup aW\left(a^{-1}\right)$. Similarly, for any $w \in \mathsf{F}(a, b) \setminus W(b)$, it is the case that $b^{-1}w \in W\left(b^{-1}\right)$, so $w \in bW\left(b^{-1}\right)$. Thus, for any $t \in \mathsf{F}(a, b)$, $t \in W(b)$ or $t \in \mathsf{F}(a, b) \setminus W(b) = bW\left(b^{-1}\right)$. Thus, $\mathsf{F}(a, b) = W(b) \sqcup bW\left(b^{-1}\right)$.

We have thus constructed

$$\begin{aligned} \mathsf{F}(a, b) &= W(a) \sqcup aW\left(a^{-1}\right) \\ &= W(b) \sqcup bW\left(b^{-1}\right), \end{aligned}$$

a paradoxical decomposition of $\mathsf{F}(a, b)$ with the action of left-multiplication.

Now that we understand a little more about paradoxical groups, we now want to understand the actions of paradoxical groups on sets.

Proposition 2.2.1: Let G be a paradoxical group that acts freely on X. Then, X is G-paradoxical.

Proof. Let $A_1, \dots, A_n, B_1, \dots, B_m \subset G$ be pairwise disjoint, and let $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$\begin{aligned} G &= \bigcup_{j=1}^{n} g_j A_j \\ &= \bigcup_{j=1}^{m} h_j B_j. \end{aligned}$$

Let $M \subseteq X$ contain exactly one element from every orbit in X.

Claim. The set $\{g \cdot M \mid g \in G\}$ is a partition of X

Proof of Claim. Since M contains exactly one element from every orbit in X, it is the case that $G \cdot M = X$, so

$$\bigcup_{g \in G} g \cdot M = X$$

Additionally, for $x, y \in M$, if $g \cdot x = h \cdot y$, then $\left(h^{-1}g\right) \cdot x = y$, meaning y is in the orbit of x and vice versa, implying $x = y$. Since G acts freely on X, we must have $h^{-1}g = e_G$.

Thus, we can see that $g_1 \cdot M \neq g_2 \cdot M$, implying $\{g \cdot M \mid g \in G\}$ is a partition of X. □

We define

$$A_j^* = \bigcup_{g \in A_j} g \cdot M,$$

and similarly define

$$B_j^* = \bigcup_{h \in B_j} h \cdot M.$$

As a useful shorthand, we can also write $A_j^* = A_j \cdot M$, and similarly, $B_j^* = B_j \cdot M$, to denote the union of the elements of A_j and B_j respectively acting on M.

Since $\{g \cdot M \mid g \in G\}$ is a partition of X, and $A_1, \ldots, A_n, B_1, \ldots, B_m \subset G$ are pairwise disjoint, it must be the case that $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^* \subset X$ are also pairwise disjoint.

For the original $g_1, \ldots, g_n, h_1, \ldots, h_m$ that defined the paradoxical decomposition of G, we thus have

$$\begin{aligned} \bigcup_{j=1}^{n} g_j \cdot A_j^* &= \bigcup_{j=1}^{n} \left(g_j A_j\right) \cdot M \\ &= G \cdot M \\ &= X, \end{aligned}$$

and

$$\begin{aligned} \bigcup_{j=1}^{m} h_j \cdot B_j^* &= \bigcup_{j=1}^{m} \left(h_j B_j\right) \cdot M \\ &= G \cdot M \\ &= X. \end{aligned}$$

Thus, X is G-paradoxical. □

Remark: This proof requires the axiom of choice, as we invoked it to define M to contain exactly one element from every orbit in X.

Now that we have established $F(a, b)$ as being a paradoxical group, we wish to use it to construct paradoxical decompositions of the unit sphere $S^2 \subseteq \mathbb{R}^3$.

Definition (Special Orthogonal Group)**.** For $n \in \mathbb{N}$, we define $SO(n)$ to be the group of all real $n \times n$ matrices A such that $A^T = A^{-1}$ and $\det(A) = 1$.

In terms of an isometry of $\mathbb{R}^3$, the group $SO(3)$ denotes the set of all rotations about any line through the origin.

Fact 2.2.1. If H is a paradoxical group, and $H \leqslant G$, then G is a paradoxical group.

With this fact in mind, we will show that $SO(3)$ is a paradoxical group.

Theorem 2.2.1: There are rotations A and B that about lines through the origin in $\mathbb{R}^3$ that generate a subgroup of $SO(3)$ isomorphic to $F(a, b)$

Proof. We take

$$A = \begin{bmatrix} 1/3 & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} 1/3 & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$
$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

We let $A^{\pm}$ denote A and A^{-1} respectively, and similarly for $B^{\pm}$.

Let w be a reduced word in $\{A, A^{-1}, B, B^{-1}\}$ which is not the empty word. We claim that w cannot be the identity.

Without loss of generality, we assume that w ends in A or A^{-1} — this is because if w is the identity, then AwA^{-1} and $A^{-1}wA$ are also the identity.

We will show that there exist $a, b, c \in \mathbb{Z}$ with $b \not\equiv 0 \bmod 3$ such that

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}.$$

If $b \not\equiv 0 \bmod 3$, and w is not empty, then w cannot act as the identity.

We induct on the length of w. For $w = A^{\pm}$, we have

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix},$$

proving the base case.

Let $k > 0$, meaning $w = A^{\pm}w'$, or $w = B^{\pm}w'$, with w' not equal to the empty. The inductive hypothesis says

$$w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}$$

for some $a', b', c' \in \mathbb{Z}$, and $b' \not\equiv 0 \bmod 3$. In particular,

$$A^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \mp 4b \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix}$$
$$B^{\pm}w' \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} 3a' \\ (b' \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix}.$$

Now, we set

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix},$$

meaning

$$a = \begin{cases} a' \mp 4b', & w = A^{\pm}w' \\ 3a', & w = B^{\pm}w' \end{cases}$$
$$b = \begin{cases} b' \pm 2a', & w = A^{\pm}w' \\ b' \mp 2c', & w = B^{\pm}w' \end{cases}$$
$$c = \begin{cases} 3c', & w = A^{\pm}w' \\ c' \pm 4b', & w = B^{\pm}w' \end{cases}$$

Let w^* denote the word such that $w' = A^{\pm}w^*$ or $w' = B^{\pm}w^*$. We write

$$w^* = \frac{1}{3^{k-2}} \begin{pmatrix} a'' \\ b''\sqrt{2} \\ c'' \end{pmatrix},$$

where $a'', b'', c'' \in \mathbb{Z}$. Note that it may not be the case that w^* is a non-empty word. We examine the following four cases.

Case 1: Suppose $w = A^{\pm}B^{\pm}w^*$. Then, $b = b' \mp 2a'$, where $a' = 3a''$. Since $b' \not\equiv 0 \bmod 3$ (by the inductive hypothesis), it is also the case $b \equiv 0 \bmod 3$.

Case 2: Suppose $w = B^{\pm}A^{\pm}w^*$. Then, $b = b' \mp 2c'$, where $c' = 3c''$. Since $b' \not\equiv 0 \bmod 3$ (by the inductive hypothesis), it is also the case that $b \not\equiv 0 \bmod 3$.

Case 3: Suppose $w = A^{\pm}A^{\pm}w^{*}$. Then, we have

$$\begin{aligned} b &= b' \pm 2a' \\ &= b' \pm 2\left(a'' \pm 4b''\right) \\ &= b' + \left(b'' \pm 2a''\right) - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Thus, regardless of the value of b'', since $b' \not\equiv 0 \bmod 3$ by the inductive hypothesis, it is the case that $b \not\equiv 0 \bmod 3$.

Suppose $w = B^{\pm}B^{\pm}w^{*}$. Then, we have

$$\begin{aligned} b &= b' \mp 2c' \\ &= b' \mp 2\left(c'' \pm 4b''\right) \\ &= b' + \left(b'' \mp 2c''\right) - 9b'' \\ &= 2b' - 9b''. \end{aligned}$$

Thus, regardless of the value of b'', since $b' \not\equiv 0 \bmod 3$ by the inductive hypothesis, it is the case that $b \not\equiv 0 \bmod 3$.

We have thus shown that any non-empty reduced word over $\left\{A, A^{-1}, B, B^{-1}\right\}$ does not act as the identity. The subgroup of $SO(3)$ generated by $\left\{A, A^{-1}, B, B^{-1}\right\}$ is isomorphic to $F(a, b)$. □

Remark: Since $SO(n)$ contains a subgroup isomorphic to $SO(3)$ for all $n \geqslant 3$, it is the case that $SO(n)$ also contains a subgroup isomorphic to $F(a, b)$.

Since we have shown that $SO(3)$ is paradoxical, as it contains a paradoxical subgroup, we can now begin to examine the action of $SO(3)$ on subsets of $\mathbb{R}^3$.

Theorem 2.2.2 (Hausdorff Paradox)**:** There is a countable subset D of S^2 such that $S^2 \setminus D$ is $SO(3)$-paradoxical.

Proof. Let A and B be the rotations in $SO(3)$ that serve as the generators of the subgroup isomorphic to $F(a, b)$. Since A and B are rotations, so too is any reduced word over $\left\{A, A^{-1}, B, B^{-1}\right\}$. Thus, any such non-empty word contains two fixed points.

We let

$$F = \left\{x \in S^2 \mid s \text{ is a fixed point for some word } w\right\}.$$

Since the set of all reduced words in $\left\{A, A^{-1}, B, B^{-1}\right\}$ (henceforth $F(A, B)$) is countably infinite, so too is F. Thus, the union of all these fixed points under the action of all such words w is countable.

$$D = \bigcup_{w \in F(A,B)} w \cdot F.$$

Therefore, $F(A, B)$ acts freely on $S^2 \setminus D$, so $S^2 \setminus D$ is $SO(3)$-paradoxical. □

Unfortunately, the Hausdorff paradox is not enough for us to be able to prove the Banach–Tarski paradox. In order to do this, we need to be able to show that two sets are "similar" under the action of a group.

Definition (Equidecomposable Sets)**.** Let G act on X, and let $A, B \subseteq X$. We say A and B are G-**equidecomposable** if there are partitions $\{A_j\}_{j=1}^{n}$ of A and $\{B_j\}_{j=1}^{n}$ of B, and elements $g_1, \dots, g_n \in G$, such that for all j,

$$B_j = g_j \cdot A_j.$$

We write $A \sim_G B$ if A and B are G-equidecomposable.

Fact 2.2.2. The relation $\sim_G$ is an equivalence relation.

Proof. Let A, B, and C be sets.

To show reflexivity, we can select $g_1 = g_2 = \cdots = g_n = e_G$. Thus, $A \sim_G A$.

To show symmetry, let $A \sim_G B$. Set $\{A_j\}_{j=1}^{n}$ to be the partition of A, and set $\{B_j\}_{j=1}^{n}$ to be the partition of B, such that there exist $g_1, \dots, g_n \in G$ with $g_j \cdot A_j = B_j$. Then,

$$\begin{aligned} g_j^{-1} \cdot (g_j \cdot A_j) &= g_j^{-1} \cdot B_j \\ A_j &= g_j^{-1} \cdot B_j, \end{aligned}$$

so $B_j \sim_G A_j$.

To show transitivity, let $A \sim_G B$ and $B \sim_G C$. Let $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ be the partitions of A and B respectively and $g_1, \dots, g_n \in G$ such that $g_i \cdot A_i = B_i$. Let $\{B_j\}_{j=1}^{m}$ and $\{C_j\}_{j=1}^{m}$ be partitions of B and C, and $h_1, \dots, h_m \in G$, such that $h_j \cdot B_j = C_j$.

We refine the partition of A to A_{ij} by taking $A_{ij} = g_i^{-1}(B_i \cap B_j)$, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Then, $(h_j g_i) \cdot A_{ij}$ maps the refined partition of A to a refined partition of C, meaning A and C are G-equidecomposable. □

Fact 2.2.3. For $A \sim_G B$, there is a bijection $\phi : A \to B$ by taking $C_i = C \cap A_i$, and mapping $\phi(C_i) = g_i \cdot C_i$.

We can now use this equidecomposability to glean information about the existence of paradoxical decompositions.

Proposition 2.2.2: Let G act on X, with $E, E' \subseteq X$ such that $E \sim_G E'$. Then, if E is G-paradoxical, then so too is E'.

. Let $A_1, \dots, A_n, B_1, \dots, B_m \subset E$ be pairwise disjoint, with $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$E = \bigcup_{i=1}^{n} g_i \cdot A_i$$

$$= \bigcup_{j=1}^{m} h_j \cdot B_j.$$

We let

$$A = \bigsqcup_{i=1}^{n} A_i$$
$$B = \bigsqcup_{j=1}^{m} B_j.$$

It follows that $A \sim_G E$ and $B \sim_G E$, since we can take the partition of A to be $A_1, \ldots, A_n$, and partition E by taking $g_i \cdot A_i$ for $i = 1, \ldots, n$, and similarly for B.

Since $E \sim_G E'$, and $\sim_G$ is an equivalence relation, it follows that $A \sim_G E'$ and $B \sim_G E'$. Thus, there is a paradoxical decomposition of E' in $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. □

We will now show that S^2 is $SO(3)$ paradoxical.

Proposition 2.2.3: Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \backslash D$ are $SO(3)$-equidecomposable.

. Let L be a line in $\mathbb{R}^3$ such that $L \cap D = \varnothing$. Such an L must exist since S^2 is uncountable.

Define $\rho_\theta \in SO(3)$ to be a rotation about L by an angle of θ. For a fixed $n \in \mathbb{N}$ and fixed $\theta \in [0, 2\pi)$, define $R_{n,\theta} = \left\{x \in D \mid \rho_\theta^n \cdot x \in D\right\}$. Since D is countable, $R_{n,\theta}$ is necessarily countable.

We define $W_n = \{\theta \mid R_{n,\theta} \neq \varnothing\}$. Since the map $\theta \mapsto \rho_\theta^n \cdot x$ into D is injective, it is the case that W_n is countable. Therefore,

$$W = \bigcup_{n \in \mathbb{N}} W_n$$

is countable.

Thus, there must exist $\omega \in [0, 2\pi) \setminus W$. We define ρ_ω to be a rotation about L by ω. Then, for every $n, m \in \mathbb{N}$, we have

$$\rho_\omega^n \cdot D \cap \rho_\omega^m \cdot D = \varnothing.$$

We define $\widetilde{D} = \bigsqcup_{n=0}^{\infty} \rho_\omega^n D$. Note that

$$\rho_\omega \cdot \widetilde{D} = \rho_\omega \cdot \bigsqcup_{n=0}^{\infty} \rho_\omega^n \cdot D$$
$$= \bigsqcup_{n=1}^{\infty} \rho_\omega^n \cdot D$$

$$= \widetilde{D} \setminus D,$$

meaning $\widetilde{D}$ and D are $SO(3)$-equidecomposable.

Thus, we have

$$\begin{aligned} S^2 &= \widetilde{D} \sqcup \left(S^2 \setminus \widetilde{D}\right) \\ &\sim_{SO(3)} \left(\rho_\omega \cdot \widetilde{D}\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\ &= \left(\widetilde{D} \setminus D\right) \sqcup \left(S^2 \setminus \widetilde{D}\right) \\ &= S^2 \setminus D, \end{aligned}$$

establishing S^2 and $S^2 \setminus D$ as $SO(3)$-equidecomposable.

In particular, this means S^2 is also $SO(3)$-paradoxical. $\square$

Chapter 3

Tarski's Theorem

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