Representations

Definition: If A is a C^*-algebra, a representation of A is a pair (π, H) where H is a Hilbert space and $\pi\colon A \to B(H)$ is a $*$-homomorphism. If A is unital, then we require $\pi(1) = I$.

Note that if A does not have an identity, we can extend to the unitization $A_1 = A \oplus \mathbb{C}$ and define $\widetilde{\pi}(a, \lambda) = \pi(a) + \lambda I$ for any $a \in A$ and $\lambda \in \mathbb{C}$.

Note that every representation is contractive and the range of any representation is closed.

Example:

(a) If A is a C^*-subalgebra of $B(H)$, then the inclusion map $A \hookrightarrow B(H)$ is a representation.

(b) If (X, Ω, μ) is a σ-finite measure space, then $\pi\colon L_\infty(\mu) = B(L_2(\mu))$, where $\pi(\phi) = M_\phi$, is a representation.

(c) If X is compact, and μ is a positive Borel measure on X, then $\pi_\mu\colon C(X) \to B(L_2(\mu))$ defined by $\pi_\mu(f) = M_f$ is a representation of $C(X)$.

Definition: Let A be a C^*-algebra.

(i) If d is a cardinal number, H a Hilbert space, we let $H^{(d)}$ denote the direct sum of H with itself over d. If $T \in B(H)$, we let $T^{(d)}$ be the direct sum of T with itself over d, which is known as the d-fold inflation of T.

Given a representation $\pi\colon A \to B(H)$, we have $\pi^{(d)}\colon A \to B\big(H^{(d)}\big)$, defined by $\pi^{(d)}(a) = \pi(a)^{(d)}$ is a representation, which is known as the inflation of π.

If $d = \aleph_0$, we will denote their respective inflations as $H^{(\infty)}$ and $\pi^{(\infty)}$.

(ii) If $\{(\pi_i, H_i)\}_{i\in I}$ is a collection of representations of A, the direct sum of these representations is the representation

$$\begin{aligned} \bigoplus_{i\in I} \pi_i\colon A &\to B\left(\bigoplus_{i\in I} H_i\right) \\ a &\mapsto \bigoplus_{i\in I} \pi_i(a). \end{aligned}$$

Note that since all representations are contractive, the direct sum is in fact a bounded operator. Furthermore, if π is isometric (hence injective), then so too is its inflation.

Example: If X is a compact topological space, and $(\mu_n)_n$ is a sequence of positive Borel measures for X, with corresponding representations $\pi_n\colon C(X) \to B(L_2(\mu_n))$ taking $f \mapsto M_f$, then $\bigoplus_{n\geq 1} \pi_n$ is also a representation.

States

For now, we will assume that M is a unital self-adjoint subspace of a C^*-algebra A. If ρ is a linear functional on M, then the equation

$$\rho^*(a) = \overline{\rho(a^*)}$$

defines another linear functional; if $\rho = \rho^*$, then we call ρ hermitian. Equivalently, ρ is hermitian if $\rho(a^*) = \overline{\rho(a)}$. If ρ is a bounded hermitian functional on M, then we claim that

$$\|\rho\| = \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\}.$$

This follows from the fact that if $\varepsilon > 0$, then from the Riesz lemma, we may find a in the unit ball of M with $|\rho(a)| > \|\rho\| - \varepsilon$. For a suitable λ with $|\lambda| = 1$, we have

$$\|\rho\| - \varepsilon < |\rho(a)|$$

$$
\begin{aligned}
&= \rho(\lambda a)\\
&= \overline{\rho(\lambda a)}\\
&= \rho\bigl((\lambda a)^*\bigr).
\end{aligned}
$$

If $a_0 = \operatorname{Re}(\lambda a)$, we have $\|a_0\| \leq 1$ with $\rho(a_0) > \|\rho\| - \varepsilon$. Thus,

$$\|\rho\| \leq \sup\{\rho(a) \mid a \in M_{\text{s.a.}}, \|a\| \leq 1\},$$

with the reverse inequality being true by definition.

We say the linear functional ρ is *positive* if for any $a \in M_+$, $\rho(a) \geq 0$; if $\rho(1) = 1$, then we say ρ is a state. In fact, if ρ is positive, then ρ is hermitian, since if $a = a^*$, then

We start by considering a version of the Cauchy–Schwarz inequality for states.

Proposition: If ρ is a positive linear functional on a C^*-algebra A, then

$$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b).$$

Proof. With $a \in A$, we have $a^*a \in A_+$, so $\rho(a^*a) \geq 0$. Then, since ρ is hermitian, we have that

$$\langle a, b\rangle = \rho(b^*a)$$

defines a positive sesquilinear form on A, so the traditional Cauchy–Schwarz inequality gives the desired result. □

References

[KR97] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I.* Vol. 15. Graduate Studies in Mathematics. Elementary theory, Reprint of the 1983 original. American Mathematical Society, Providence, RI, 1997, pp. xvi+398. ISBN: 0-8218-0819-2. DOI: 10.1090/gsm/015. URL: https://doi.org/10.1090/gsm/015.

[RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^*-algebras.* Vol. 60. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. xiv+327. ISBN: 0-8218-0860-5. DOI: 10.1090/surv/060. URL: https://doi.org/10.1090/surv/060.

[Con00] John B. Conway. *A course in operator theory.* Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: 10.1090/gsm/021. URL: https://doi.org/10.1090/gsm/021.