Problem (Problem 1)**:** Let $f\colon M \to N$ be a smooth map of manifolds. Prove that the graph of f is a smooth submanifold of $M \times N$.

Solution: Let (U, φ) be a chart on M with $\varphi(U) \cong \mathbb{R}^m$, and (V, ψ) a chart on N with $\psi(V) \cong \mathbb{R}^n$ and $f(U) \subseteq V$.

Define a chart on $M \times N$ corresponding to $U \times V$, and notice that the graph of $f|_U$ is a subset of $U \times V$.

Problem (Problem 2)**:** Let $U(n)$ be the set of unitary complex $n \times N$ matrices. Write $SU(n) \leqslant U(n)$ for the kernel of the determinant map.

(a) Show that $U(1)$ is diffeomorphic to the circle, so that $SU(1)$ is a point.

(b) Prove that $U(n)$ is a smooth manifold.

(c) Prove that $SU(2)$ is diffeomorphic to S^3, the three-sphere.

(d) Prove that $U(2)$ is diffeomorphic to $S^1 \times S^3$.

Solution:

(a) Since complex 1×1 matrices are diffeomorphic to $\mathbb{C}$, we see that $x \in U(1)$ if and only if $|x|^2 = 1$, meaning $|x| = 1$, so $x = e^{i\theta}$ for some θ. In particular, this means that the assignment $x \mapsto e^{i\theta}$ gives a diffeomorphism between S^1 and $U(1)$.

(b) Consider the self-map $f\colon \operatorname{Mat}_n(\mathbb{C}) \to \operatorname{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ given by $f(A) = A^*A$. Note that this maps $\operatorname{Mat}_n(\mathbb{C})$ to positive semi-definite matrices $\operatorname{Mat}_n(\mathbb{C})^+$.

We want to calculate the derivative of f by taking

$$\begin{aligned} f(A+H) - f(A) &= (A+H)^*(A+H) - A^*A \\ &= (A^* + H^*)(A+H) - A^*A \\ &= A^*A + H^*A + A^*H + H^*H - A^*A \\ &= H^*A + A^*H + H^*H. \end{aligned}$$

Dividing out by $\|H\|_{op}$, we find that $D_A(f) = A + A^*$. Now, since I is of full rank, so too is $\frac{1}{2}I$, meaning that $D_{\frac{1}{2}I}(f) = I$, and thus f has a locally defined inverse about I. In particular, this means that $f^{-1}(\{I\})$ consists entirely of regular points, or that I is a regular value for f. Thus, $U(n)$ is a smooth manifold.

(c) We view S^3 as a subset of $\mathbb{C}^2$, so that S^3 consists of all (z_1, z_2) such that

$$|z_1|^2 + |z_2|^2 = 1.$$

We claim that the matrix

$$A_{z_1,z_2} = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}$$

is an element of $SU(2)$. Since it is uniquely determined by z_1 and z_2 in S^3, it follows that $SU(2)$ is diffeomorphic to S^3.

To see this, observe that

$$\begin{aligned} \det(A) &= 1 \\ A^*A &= \begin{pmatrix} \overline{z_1} & -z_2 \\ \overline{z_2} & z_1 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} |z_1|^2 + |z_2|^2 & z_2\overline{z_1} - z_2\overline{z_1} \\ z_1\overline{z_2} - z_1\overline{z_2} & |z_1|^2 + |z_2|^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, SU(3) is diffeomorphic to S^3, with the diffeomorphism given by coordinate assignment.

(d) Observe that if $(z_1, z_2) = z \in S^3 \subseteq \mathbb{C}^2$, then if $a \in U(2)$, we have $az \in S^3$. In particular, since unitary matrices are invertible, the operation of $a \in U(2)$ on $z \in S^3$ by multiplication is a group action.

We observe now that the action of U(2) on $S^3 \subseteq \mathbb{C}^2$ by matrix multiplication is transitive, since for any element $(w_1, w_2) \in S^3$, the matrix

$$\begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

and

$$\begin{pmatrix} \overline{w_1} & \overline{w_2} \\ -w_2 & w_1 \end{pmatrix}\begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Additionally, we observe that for any θ,

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

meaning

$$S^3 \cong U(2)/P,$$

where P consists of all matrices of the form

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

We observe that P is diffeomorphic to S^1 via a coordinate assignment, so $U(2) \cong S^3 \times S^1$.

Problem (Problem 3)**:** In this exercise, we will prove the Frobenius theorem.

Let M be a smooth manifold of dimension n, and let D be an r-dimensional distribution on M, where $r \leqslant n$. That is, D picks out an r-dimensional D_p of T_pM for each $p \in M$. In other words, at every point, there are r distinguished, linearly independent vector fields defined in a neighborhood of the point.

A submanifold $N \subseteq M$ is called an *integral submanifold* for D if $T_pN = D_p$ for each $p \in M$. We say D is *completely integrable* if an integral submanifold exists through every point. Integral curves of a vector field are integral submanifolds of a vector field.

We call a distribution that is closed under taking Lie brackets involutive. That is, for any vector fields $X, Y \in D$ (i.e., local 1-distributions that lie in D), then $[X, Y] \in D$.

The Frobenius Theorem says that a distribution D on M is completely integrable if and only if it is involutive.

(a) Show that if D is a completely integrable distribution, then D is involutive.

(b) We say vector fields X and Y commute if $[X, Y] = 0$. For fixed vector fields X and Y, write ϕ_t and

ψ_t for the corresponding flows. Show that the following are equivalent:

(i) X and Y commute;

(ii) Y is invariant under ϕ_t;

(iii) the flows ϕ_t and ψ_t commute, so that $\psi_s \circ \phi_t = \phi_t \circ \psi_s$ for all t and s where defined.

(c) Assume D is r-dimensional. Choose local coordinates $\{x_1, \dots, x_n\}$ near a point p and r-linearly independent vector fields $Y_1, \dots, Y_r$ near p. Write Y_i as

$$\sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j},$$

and show that there is some permutation of the coordinates such that the $r \times r$ matrix $(a_{ij})_{1 \leqslant i,j \leqslant r}$ is invertible near p.

(d) Let $(b_{ij})_{1 \leqslant i,j \leqslant r}$ be the inverse of the smoothly varying family of matrices $(a_{ij})_{1 \leqslant i,j \leqslant r}$ from the previous part, and let $X_i = \sum_j b_{ij} Y_j$. Show that

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j>r} c_{ij} \frac{\partial}{\partial x_j}$$

for some suitable smooth functions. Show that $X_1, \dots, X_r$ form a basis for D at every point.

(e) Show that $\left[X_i, X_j\right] = 0$ for $1 \leqslant i, j \leqslant r$.

(f) Use the flows generated by $\{X_1, \dots, X_n\}$ to define a smooth map $\phi\colon V \to U$ where V is a neighborhood of $0 \in \mathbb{R}^r$ and U is a neighborhood of $p \in M$.

(g) Choose coordinates $\{t_1, \dots, t_r\}$ on $\mathbb{R}^r$ such that $\phi_*\left(\frac{\partial}{\partial t_i}\right) = X_i$. Argue by shrinking V and U if necessarily that V is a submanifold of U. Use the fact that the flows generated by $X_1, \dots, X_r$ commute to prove that at an arbitrary point $q \in \phi(V)$, we have $D_q = T_q\phi(V)$. Conclude that $\phi(V)$ locally defines an integral submanifold N of the distribution D.

Solution:

(a) Let $p \in N \subseteq M$, and let $(U; x_1, \dots, x_r)$ be a chart in N such that $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$ forms a basis for T_qN for any $q \in U$. Letting $X_1, \dots, X_r$ be linearly independent vector fields for D, we observe that, for any $f \in C^\infty(N)$,

$$[X_k, X_\ell](f) = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} a_{ki}(q) \frac{\partial a_{\ell i}}{\partial x_j}(q) - a_{\ell i}(q) \frac{\partial a_{ki}}{\partial x_j}(q) \right) \frac{\partial f}{\partial x_i}(q)$$

Observe that if we extend $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$ to a basis for T_qM, then via this computation, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$ remains invariant under the calculation of the Lie bracket, so D_p is involutive.