

Amenability: A (Somewhat) Brief Introduction

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Outline

- 1 Definitions
- 2 Paradoxical Decompositions
- 3 From Paradoxical Decompositions to Amenability
- 4 Equivalent Definitions and Other Criteria
- 5 Remarks and Acknowledgments

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Groups

If A is a set, and $\star: A \times A \rightarrow A$ is an operation such that

- $a \star (b \star c) = (a \star b) \star c$;
- there exists e_A such that $a \star e_A = e_A \star a = a$;
- for each a there exists a^{-1} such that $a \star a^{-1} = a^{-1} \star a = e_A$,

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then we call the pair (A, \star) a *group*.

We abbreviate $a \star b$ as ab .

Subgroups, Quotient Groups

Let G be a group.

- If $H \subseteq G$ is a subset that satisfies, for all $a, b \in H$, $ab^{-1} \in H$, then we say H is a *subgroup*.

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- The equivalence classes under the relation $g \sim_N g'$ if $g^{-1}g' \in N$ form a group $gN := [g]_{\sim}$ known as the *quotient group* G/N .

Some Groups

- The integers \mathbb{Z} are a group under addition.
- The group of invertible $n \times n$ matrices over \mathbb{C} , $GL_n(\mathbb{C})$, is a group under matrix multiplication.
- The subgroup $SO(n) \subseteq GL_n(\mathbb{R})$ consisting of $n \times n$ orthogonal matrices with determinant 1 is a group under multiplication.

Group Actions

Let G be a group, and X a set. Let $\rho: G \times X \rightarrow X$ be a function that satisfies, for all $g, h \in G$ and $x \in X$,

- $\rho(e_G, x) = x$;
- $\rho(g, \rho(h, x)) = \rho(gh, x)$.

Then, we say ρ is an *action* of G on X . We write $\rho(g, x) = g \cdot x$.

σ -Algebras and Measures

If X is a set, then a collection of subsets $\{A_i\}_{i \in I} = \mathcal{A} \subseteq P(X)$ is known as an *algebra* of subsets if

- ① $\emptyset, X \in \mathcal{A}$;
- ② for any $A_i \in \mathcal{A}$, $A_i^c \in \mathcal{A}$;
- ③ for any $A_i, A_j \in \mathcal{A}$, $A_i \cup A_j \in \mathcal{A}$.

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If, for any countable collection, $\{A_n\}_{n \geq 1} \subseteq \mathcal{A}$, condition (3) holds, then we say \mathcal{A} is a σ -*algebra* of subsets.

σ -Algebras and Measures, Cont'd

If X is a set and \mathcal{A} is a σ -algebra, then a map $\mu: \mathcal{A} \rightarrow [0, \infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for disjoint sets $A, B \in \mathcal{A}$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,

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σ -Algebras and Measures, Cont'd

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then we say μ is a *finitely additive* measure. If $\{A_n\}_{n \geq 1}$ is a countable collection of disjoint sets, then if μ satisfies

- $\mu\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} \mu(A_n)$,

we say μ is a measure.

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Questions?

- If G is a group, is it possible to reconstruct G by using some subset of G ?
- When may we find a finitely additive probability measure $\mu: P(G) \rightarrow [0, 1]$ such that $\mu(E) = \mu(tE)$ for all $E \subseteq G$?
- Are these questions even related?

Free Groups

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- We begin by considering a special group, known as $F(a, b)$ or the *free group on two generators*.
- We define $F(a, b)$ to be the set of all “words” in the alphabet $\{a, b, a^{-1}, b^{-1}\}$, subject to the condition that, for $w, w' \in F(a, b)$,

$$\begin{aligned} waa^{-1}w' &\sim wa^{-1}aw' \sim ww' \\ wbb^{-1}w' &\sim wb^{-1}bw' \sim ww'. \end{aligned}$$

- Examples: $a^2bab^{-1}, b^{-1}a^2b^2ab \in F(a, b)$.

A Curiosity

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A Curiosity, Cont'd

Similarly, we can do this for a , giving a decomposition of $F(a, b)$ in two separate ways:

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Furthermore, note that $W(a), W(b), W(a^{-1}), W(b^{-1})$ are disjoint.

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Furthermore, note that $W(a), W(b), W(a^{-1}), W(b^{-1})$ are disjoint.

These decompositions seem to be downright paradoxical — we take a part of the group, translate some of it, and get the whole group back!

Defining Paradoxical Decompositions

Let G be a group. A *paradoxical decomposition* of G consists of

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$; and
- elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$;

such that

$$\begin{aligned} G &= \bigcup_{i=1}^n g_i A_i \\ &= \bigcup_{j=1}^m h_j B_j. \end{aligned}$$

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If G admits a paradoxical decomposition, we say G is *paradoxical*.

Paradoxical Actions

If G acts on a set X , then a subset $A \subseteq X$ is G -*paradoxical* if there exist

- pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq A$; and
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A paradoxical group is a paradoxical set under the action of left-multiplication.

Examples

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- The free group $F(a, b)$ is paradoxical.
- Any group that contains a paradoxical subgroup is paradoxical.
- $F(S)$, where S is any nonempty set with more than two elements, is paradoxical.

A Paradoxical Subgroup of $SO(3)$

The following two matrices (and their inverses) generate a subgroup of $SO(3)$ that is isomorphic to $F(a, b)$.

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

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$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

This is proven using the Ping-Pong lemma.

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of \mathbb{R}^3 with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B .

Introducing the Banach–Tarski Paradox

Theorem (The Banach–Tarski Paradox)

Let A and B be bounded subsets of \mathbb{R}^3 with nonempty interior. There is a partition of A into finitely many disjoint subsets such that a sequence of isometries applied to these subsets yields B .

- In other words, not all subsets of \mathbb{R}^3 have a definite “volume” invariant under isometry.

Equidecomposability

Let G be a group that acts on a set X , and let $A, B \subseteq X$. If there exist

- finite partitions, $A_1, \dots, A_n \subseteq A$, $B_1, \dots, B_n \subseteq B$
- group elements $g_1, \dots, g_n \in G$

such that $g_i \cdot A_i = B_i$, then we say A and B are G -*equidecomposable*.

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Effectively, A and B are “equal” to each other up to the group action.

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such that $g_i \cdot A_i = B_i$, then we say A and B are *G -equidecomposable*.

Effectively, A and B are “equal” to each other up to the group action.

If A is G -paradoxical, then so too is B .

The Banach–Tarski Paradox: Proof Outline I

- 1 We use the two matrices

$$A = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}.$$

to generate a subgroup of $\mathrm{SO}(3)$ isomorphic to $F(a, b)$.

The Banach–Tarski Paradox: Proof Outline II

- ② We use the decomposition

$$\begin{aligned} F(a, b) &= a^{-1}W(a) \cup W(a^{-1}) \\ &= b^{-1}W(b) \cup W(b^{-1}) \end{aligned}$$

to duplicate the unit sphere in \mathbb{R}^3 , S^2 , except for a countable subset D . (The *Hausdorff Paradox*.)

The Banach–Tarski Paradox: Proof Outline III

- ⑤ Define a relation $A \leq B$ if A is G -equidecomposable with a subset of B , and show that if $A \leq B$ and $B \leq A$, then A and B are G -equidecomposable.
- ⑥ Show that $A \subseteq \mathbb{R}^3$ is equidecomposable with a subset of $B \subseteq \mathbb{R}^3$.

We're done...or are we?

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Ill-Behaved Groups

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Some Recent Developments