Problem 4

Problem: Let $\sim$ be a relation on $\mathbb{N} \times \mathbb{N}$ under the lexicographical order. We say (a, b) is a child of (c, d) if $(a, b) \sim (c, d)$ and $(a, b) \prec (c, d)$, where $\prec$ is the lexicographical order.

We have two definitions for "descendant" below. Which one is the correct one?

(1) We say (a, b) is a descendant of (c, d) if (a, b) is a child of (c, d) or (a, b) is a descendant of a child of (c, d).
(2) We say (a, b) is a descendant of (c, d) if (a, b) is a child of (c, d) or (a, b) is a child of a descendant of (c, d).

Solution. Definition (1) is the correct definition. We let

$$C((m, n)) = \{(a, b) \mid (a, b) \text{ is a child of } (m, n)\}.$$

Define

$$D : \mathbb{N} \times \mathbb{N} \times P(\mathbb{N} \times \mathbb{N}), D((m, n)) = C((m, n)) \cup \bigcup_{((a,b))\in C((m,n))} D((a, b)) \qquad (*)$$

We want to show that there exists a unique function D that satisfies condition $(*)$.

If this is not the case, pick the smallest (m, n) for which there is no such D. So, for every $(a, b) \in C(m, n)$, $D(a, b)$ is defined and satisfies $(*)$.

Define

$$D(m, n) = C(m, n) \cup \bigcup_{(a,b)\in C((m,n))} D((a, b)).$$

Problem 5

Problem: Let S be well-ordered by $\prec$. Then, for every $x \in S$, if x is non-maximal, then x has a successor. The successor is defined by

$$\exists y \succ x \text{ s.t. } \neg\exists z\ x \prec z \prec y.$$

Solution. Let $x \in S$ be nomaximal. Set

$$T = \{y \in S \mid x \prec y\}.$$

Since x is nonmaximal, T is nonempty, meaning there exists a least element z. Then, z is a successor of x, because for all y, $x \prec y$, then $y \in T$, meaning $y = z$ or $z \prec y$, since z is the least element of T.

Problem 6

Problem: Every $S \subseteq \mathbb{R}$ well-ordered by the traditional $<$ relation is countable.

Solution. Let $S \subseteq \mathbb{R}$ be well-ordered. It is enough to show that $S \cap [z, z + 1]$ is countable for every $z \in \mathbb{Z}$, as

$$S = \bigcup_{z\in\mathbb{Z}} S \cap [z, z + 1]$$

is a countable union of countable sets.

For every $x \in S$, let $f(x) = x^+ - x$, where x^+ is the successor of x in S. If x has no successor, we let $f(x) = 0$.

It is enough to show that $S_0 = S \cap [0, 1]$ is countable. We have S_0 is well-ordered.

For every $k \in \mathbb{Z}_{>0}$, define

$$A_k = \left\{x \in S_0 \mid f(x) > \frac{1}{k}\right\}.$$

Notice that $|A_k| \leqslant k$ for all k, since S is well-ordered by $<$.

Remark ("Converse" to Problem 6)**:** The previous problem states that we cannot embed an uncountable well-ordered set into $\mathbb{R}$. Here, an embedding means that there is a function $f : S \to \mathbb{R}$ such that f is injective and f preserves order. In other words, S and $f(S) \subseteq \mathbb{R}$ are order-isomorphic.

A question we may be interested in is if every countable ordinal can be embedded into $\mathbb{R}$.