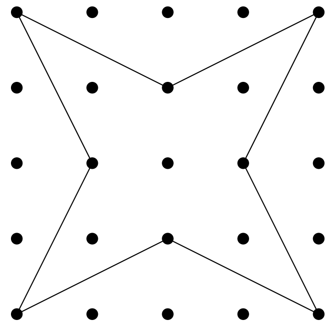
6a

Show that there is no knight's tour on the 5×5 chessboard.

In order to visit all the corners of the 5×5 chessboard, one needs to go along the following cycle:



If the knight were to visit any of the other squares, it would have to visit one of the center four squares more than once, meaning a knight's tour is impossible.

9

A graph G of order 20 has the integers $1, 2, \ldots, 20$ as its vertices. Two vertices i and j are adjacent if $i + j$ is odd. Is G Hamiltonian?

G is a complete bipartite graph with $2, 4, \ldots, 20$ as one partite set and $1, 3, \ldots 19$ as another partite set, meaning that a potential Hamiltonian cycle is as follows: $1, 2, 3, \ldots, 20, 1$.

10

A graph G of order $n \geq 4$ has $d(v) \geq (n+1)/2$ for each vertex v of G.

(a) Show that G is Hamiltonian.

(b) If v is a vertex of G, is $G - v$ Hamiltonian?

(a)

Let $u, v \in V(G)$. If $u \not\leftrightarrow v$, then $d(u)+d(v) \geq n+1 \geq n$, so by Ore's Theorem, G must be Hamiltonian.

If $u \leftrightarrow v$, then the following cases must hold:

(i) WLOG, $\exists c \not\leftrightarrow v$, so $d(c) + d(v) \geq n + 1 \geq n$, so by Ore's Theorem, G must be Hamiltonian.

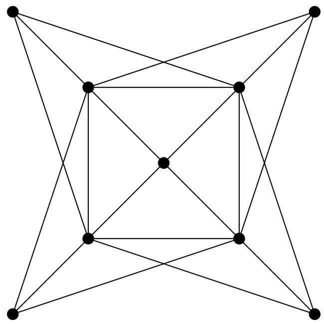
(ii) $\forall c \leftrightarrow u,\ c \leftrightarrow v$, meaning G is complete, and so G is Hamiltonian.

(b)

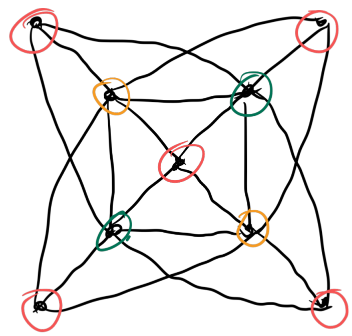
If v is deleted from G, the degree of any remaining vertex is either reduced by 1 or not reduced at all. In either case, the conditions of Ore's Theorem must hold, so G would still be Hamiltonian.

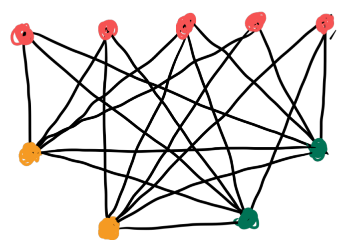
11

Determine whether the following graph is Hamiltonian:



As depicted below, the graph is tripartite.





If we delete the orange and green vertices, we will be left with five components (the five solitary vertices above), so by Theorem 6.5, the graph is not Hamiltonian.

Extra Problem 1

Let G be a graph of order $n \geq 3$, where $\forall v \in G,\ d(v) \geq \frac{n}{2}$. Let P be a path in G with longest length.

(a) Show that P has at least $n/2$ vertices.

(b) Assume there is a cycle C with the same vertices as P. Show that P has n vertices.

(a)

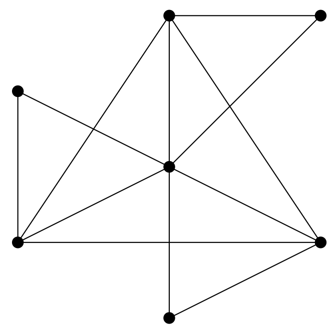
Let $P = (v_1, v_2, \dots, v_k)$ be a path in G with maximum length. Suppose toward contradiction that $|P| < n/2$, meaning $k < n/2$. Then, $\nexists v_i$ such that v_i is connected to any of $v_1, \dots, v_k$, or else we would be able to extend P. Thus, $\forall v \in \{v_1, \dots, v_k\}$, v is only adjacent to other members in $v_1, \dots, v_k$. However, this means that the maximum value v can take is $k-1$, and since $k < n/2$, this means $k - 1 < n/2$, or that v would not satisfy one of the conditions of G. $\perp$

(b)

Let P is a path of maximum length in G, and C be a cycle in G such that $V(C) = V(P)$. Suppose toward contradiction that $|P| < n$. Then, $\exists v \in G$ such that $\forall v_i \in P$, $v \not\leftrightarrow v$. However, since $|P| \geq n/2$, as shown earlier, this means $d(v) < n/2$, which is a contradiction.

Extra Problem 2

Prove that the following graph is not Hamiltonian:



All of the degree 2 vertices in the graph must pass through the center vertex, meaning that a path through this graph cannot pass through all three of the degree 2 vertices. Thus, the graph is not Hamiltonian.

Extra Problem 3

The statement of Theorem 6.5 is ambiguous. In the theorem, where it says "deleting k vertices," there are two ways to interpret it:

(i) Deleting *any* k vertices.

(ii) Deleting *some* k vertices.

Which one is the correct interpretation?

Given that Theorem 6.4 states that for a Hamiltonian graph, deleting *any* k vertices yields a graph with at most k components. The opposite of the statement would most likely be deleting *some* k vertices in a non-Hamiltonian graph yields a graph with more than k vertices.

Extra Problem 4

The contradiction within the proof is that there are at most $n/2 - 1$ vertices that are adjacent to v_k (I agree that it could have been phrased better).