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## Cardinality and Countability

### Section 1.1: Countable Sets

**Definition** (Denumerable Set). A set  $S$  is denumerable if there exists a function  $f : S \rightarrow \mathbb{N}$  with  $f$  a bijection. We also say  $S$  is countably infinite.

**Definition** (Countable Set). We say  $S$  is countable if  $S$  is either finite or denumerable.

**Theorem** (Countability of Unions): If  $A$  and  $B$  are countable sets, then  $A \cup B$  is countable.

**Theorem** (Countability of Subsets): If  $A \subseteq B$ , then if  $B$  is countable, then  $A$  is countable.

**Theorem** (Union of Finite Sets): If  $A$  and  $B$  are finite, then  $A \cup B$  is finite.

*Proof.* If  $A$  is finite and  $B$  has one element, then we show that  $A \cup B$  is finite (with two cases).

Afterward, for  $|B| > 1$ , we use induction on  $|B|$ . □

**Definition** (Finite Set). A set  $A$  is finite if there exists a bijection  $f : S \rightarrow \{1, 2, \dots, n\}$  for some  $n \in \mathbb{N} = \{0, 1, \dots\}$ .

We write  $|A| = n$ .

**Theorem** (Disjoint Union of Countable Sets): If  $A$  is denumerable,  $B$  is finite, and  $A \cap B = \emptyset$ , then  $A \cup B$  is denumerable.

*Proof.* There exists a bijection  $f : A \rightarrow \mathbb{N}$  (since  $A$  is denumerable), and a bijection  $g : B \rightarrow \{1, 2, \dots, n\}$  for some  $n \in \mathbb{N}$  (since  $B$  is finite).

We create a new bijection  $h : A \cup B \rightarrow \mathbb{N}$  by:

$$h(x) = \begin{cases} g(x) - 1 & x \in B \\ f(x) + n & x \in A \end{cases}.$$

Since  $A \cap B = \emptyset$ , we know that  $h$  is well-defined.

Now, we must show that  $h$  is a bijection.

Suppose  $h(x) = h(y)$ .

**Case 1:** If  $x, y \in B$ , then  $h(x) = g(x) - 1$ , and  $h(y) = g(y) - 1$ , meaning  $g(x) - 1 = g(y) - 1$ , meaning  $g(x) = g(y)$ . Since  $g$  is a bijection,  $x = y$ .

**Case 2:** If  $x, y \in A$ , a similar argument yields that  $x = y$ .

**Case 3:** Without loss of generality, let  $x \in A$  and  $y \in B$ . If  $x \in A$ , then  $h(x) = f(x) + n$  and  $h(y) = g(y) - 1$ . Thus,  $f(x) + n = g(y) - 1$ . However, since  $f(x) + n \geq n$  and  $0 \leq g(y) - 1 \leq n - 1$ . Thus, we get that  $0 \leq n \leq n - 1$ , which is a contradiction.

Thus, we have shown that  $h$  is injective. □

**Theorem** (Cartesian Product of Natural Numbers):  $\mathbb{N} \times \mathbb{N}$  is denumerable.

*Proof.* We consider  $\mathbb{N} \times \mathbb{N}$  as

$$\mathbb{N} \times \mathbb{N} = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \cup \dots,$$

$$\begin{array}{llllll} \mathbb{N} \times \{0\} : & (0, 0) & (1, 0) & (2, 0) & (3, 0) & \dots \\ \mathbb{N} \times \{1\} : & (0, 1) & (1, 1) & (2, 1) & (3, 1) & \dots \\ \mathbb{N} \times \{2\} : & (0, 2) & (1, 2) & (2, 2) & (3, 2) & \dots \\ \mathbb{N} \times \{3\} : & (0, 3) & (1, 3) & (2, 3) & (3, 3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then, we can find an (informal) bijection as follows:

$$\begin{array}{llllll} \mathbb{N} \times \{0\} : & \cancel{(0,0)}^0 & \cancel{(1,0)}^2 & \cancel{(2,0)}^5 & \cancel{(3,0)}^9 & \dots \\ \mathbb{N} \times \{1\} : & \cancel{(0,1)}^1 & \cancel{(1,1)}^4 & \cancel{(2,1)}^8 & (3,1) & \dots \\ \mathbb{N} \times \{2\} : & \cancel{(0,2)}^3 & \cancel{(1,2)}^7 & (2,2) & (3,2) & \dots \\ \mathbb{N} \times \{3\} : & \cancel{(0,3)}^6 & (1,3) & (2,3) & (3,3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can also find a bijection  $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ , with

$$P(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

A fun challenge is to prove that  $P$  is a bijection. □

**Theorem** (Countability of the Rationals):  $\mathbb{Q}$  is denumerable.

**Theorem** (Countability of the Integers): The set  $\mathbb{Z}$  is denumerable.

*Proof.* Let  $f : \mathbb{Z} \rightarrow \mathbb{N}$  be defined by

$$f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

□

**Definition** (Cardinality). We say two sets,  $A$  and  $B$ , have the same cardinality if there exists a bijection  $f : A \rightarrow B$ .

**Theorem** (Finite Subset Cardinality): If  $m, n \in \mathbb{N}$  and  $m \neq n$ , then  $\{1, 2, \dots, m\}$  and  $\{1, 2, \dots, n\}$  do not have the same cardinality.

**Theorem** (Infinitude of the Natural Numbers):  $\mathbb{N}$  is not finite.

**Example.** If  $A \subsetneq B$  and  $|A| = |B|$ , then both  $A$  and  $B$  are infinite.

In order to prove this, we need to show that every injection from a finite set to itself is a bijection.

## Section 1.2

**Definition** (Uncountable Set). A set is uncountable if it is not countable.

**Theorem** (Uncountability of  $\mathbb{R}$ ):  $\mathbb{R}$  is uncountable.

*Proof.* For all  $x \in \mathbb{R}$ , and for all  $j \in \mathbb{N}$ , we define  $[x]_j$  to denote the  $j + 1$ -th digit after the decimal point in the decimal expansion of  $x$ .

For example,  $[\pi]_0 = 1$ ,  $[\pi]_1 = 4$ , etc.

Let  $f : \mathbb{N} \rightarrow \mathbb{R}$ . We will show that  $f$  is not surjective.

Let  $y \in [0, 1) \subseteq \mathbb{R}$  defined by  $\forall j \in \mathbb{N}$ ,

$$[y]_j = \begin{cases} 0 & [f(j)]_j = 1 \\ 1 & [f(j)]_j \neq 1 \end{cases}.$$

We claim that  $y \notin f(\mathbb{N})$ . We will show that  $\forall j \in \mathbb{N}$ ,  $f(j) \neq y$ .

We can see that if  $[f(j)]_j = 1$ , then  $[y]_j = 0$ . Similarly, if  $[f(j)]_j \neq 1$ , then  $[y]_j = 1$ . Either way,  $[f(j)]_j \neq [y]_j$  for all  $j \in \mathbb{N}$ .  $\square$

**Remark:** The above proof is an example of a diagonalization proof. It can be imagined as

$$\begin{array}{c|c} f(0) & *.a_1^{\neq} a_2^{\neq} a_3 \dots \\ f(1) & *.b_1 b_2^{\neq} b_3 \dots \\ f(2) & *.c_1 c_2 c_3^{\neq} \dots \\ \vdots & \vdots \end{array}$$

**Note:** A substantial problem that we might need to deal with is that a real number does not necessarily have a unique decimal representation. For instance,  $3.999 \dots = 4.000 \dots$ .

In order to resolve this issue, we can default to the option with trailing 0 over trailing 9.

**Definition** (Power Set). The power set of a set  $S$  is

$$P(S) = \{A \mid A \subseteq S\}.$$

**Theorem** (Power Set Surjection): Let  $f : S \rightarrow P(S)$ . Then,  $f$  is not surjective.

*Proof.* Let  $T = \{x \in S \mid x \notin f(x)\}$ . Then,  $T \notin f(S)$ .

Let  $y \in S$ . We want to show that  $f(y) \neq T$ . Suppose toward contradiction that  $f(y) = T$ . Then, if  $y \in T$ , then  $y \in f(y)$ , which implies that  $y \notin T$ .

If  $y \notin T$ , then  $y \notin f(y)$ , which implies that  $y \in T$ .

Thus, it cannot be the case that  $f(y) = T$ .  $\square$

**Definition** (Cardinality Comparison). Let  $A$  and  $B$  be sets. Then, we write  $\text{card}(A) \leq \text{card}(B)$  if there exists an injective map  $f : A \hookrightarrow B$ .

We write  $\text{card}(A) < \text{card}(B)$  if there exists an injection  $f : A \hookrightarrow B$  but no bijection.

**Example** (Cardinality of the Power Set). For every set,

$$\text{card}(S) < \text{card}(P(S)).$$

- (1) We know that  $\text{card}(S) \leq \text{card}(P(S))$ , defining  $f : S \hookrightarrow P(S)$ ,  $f(a) = \{a\}$ , since if  $f(x) = f(y)$ , then  $\{x\} = \{y\}$ , meaning  $x \in \{y\}$ , so  $x = y$ .

In the case of  $f : \emptyset \rightarrow \{\emptyset\}$ , we define  $\emptyset = f \subseteq \emptyset \times \{\emptyset\}$ .

- (2) Since there exists no bijection  $f : S \rightarrow P(S)$ , it is the case that  $\text{card}(S) \neq \text{card}(P(S))$ .

**Example** (Decimal Expansion). We know that for some decimal expansion

$$\begin{aligned} 3.14159 \dots &= 3 + \frac{1}{10} + \frac{4}{100} + \dots \\ &= \sum_{i=0}^{\infty} \frac{n_i}{10^i}, \end{aligned}$$

with  $0 \leq n_i \leq 9$  for  $i \geq 1$ .

However, we can also write any real number as

$$\sum_{i=0}^{\infty} \frac{n_i}{3^i}$$

with  $0 \leq n_i \leq 2$  for all  $i \geq 1$ .

**Example** (Finite Strings). Let  $S$  be the set of all finite strings of 0 and 1.  $S$  is countable.

**Proof 1:** We define  $f : S \rightarrow \mathbb{N}$  by, for a string  $x \in S$ ,  $x$  starts with  $n_1$  zeroes, then has  $n_2$  ones, then  $n_3$  zeroes, etc. We define  $f(x) := 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times 7^{n_4} \times 11^{n_5} \dots$ , or

$$f(x) = \prod_i p_i^{n_i},$$

where  $p_i$  denotes the  $i$ th prime number. We can see that  $f$  is an injection.

Since  $S$  is infinite (proof omitted), we can see that  $f(S)$  is also infinite.<sup>1</sup> Since  $f(S)$  is an infinite subset of  $\mathbb{N}$ ,  $f(S)$  is denumerable, meaning there exists a bijection  $q : f(S) \rightarrow \mathbb{N}$ . Therefore, we have  $q \circ f : S \rightarrow \mathbb{N}$  is a bijection, meaning  $S$  is denumerable.

**Proof 2:** List the elements of  $S$  by length and lexicographic order: short strings come before long strings, and 0s come before 1s.

Rank	String
0	0
1	1
2	00
3	01
4	10
5	11
$\vdots$	$\vdots$

<sup>1</sup>If  $f(S)$  is finite, then there exists a bijection  $g : f(S) \rightarrow \{1, \dots, n\}$ . Composing  $g$  and  $f$ , we find  $S$  is finite as  $g \circ f|_S$  is a bijection.

This pattern yields a systematic way to map  $S$  to the natural numbers.

**Proof 3:** We can see that

$$S = \bigcup_{i=1}^{\infty} S_i,$$

where  $S_i$  is the set of all strings of length  $i$ , each of which contains  $2^i$  elements.

Since each  $S_i$  is finite, and  $S_i \cap S_j = \emptyset$  (by definition). Thus,  $S$  is a countable union of pairwise disjoint countable sets, so  $S$  is countable.

**Example** (All Possible Writings). Let  $W$  be the set of all possible writings in English. We let  $W_n$  denote the writing with  $n$  characters. Then,

$$W = \bigcup_{n=1}^{\infty} W_n,$$

which is a countable union of disjoint finite sets, which is countable.

Similarly, we can list all the writings by length and lexicographic order.

This result implies that “almost all” real numbers, in a sense, are unable to be described.

### Section 1.3: Cantor–Schröder–Bernstein Theorem

**Example.** If we have  $|A| \leq |B|$  and  $|B| \leq |A|$ , it does not necessarily imply  $|A| = |B|$ .

This is because the  $\leq$  in the cardinality comparison implies there exist injections  $f : A \hookrightarrow B$  and  $g : B \hookrightarrow A$ , not that the cardinalities are necessarily “less than or equal to” each other.

However, at the same time, this fact is true — this is what is known as the Cantor–Schröder–Bernstein Theorem.

**Theorem** (Cantor–Schröder–Bernstein): Let  $f : C \hookrightarrow D$  and  $g : D \hookrightarrow C$  be injective maps. Then,  $|C| = |D|$ .

*An Informal Proof Sketch.* Consider  $C$  to be a set of cats and  $D$  to be a set of dogs. Every cat chases a dog, and every dog chases a cat, with different cats chasing different dogs and vice versa.

There are four potential arrangements:

- (1) A set of cats and dogs are chasing each other in a circle.
- (2) A chain of dogs chasing cats that starts with a dog.
- (3) A chain of cats chasing dogs that starts with a cat.
- (4) An endless chain of cats chasing dogs with no discernible start or end point.

These four cases create a bijection from  $C$  to  $D$ :

- (1) Pair each cat with the dog that it is chasing.
- (2) Pair each cat with the dog that it is chasing.
- (3) Pair each cat with the dog that *is chasing it*.
- (4) Pair each cat with the dog that it is chasing.

□

*A More Formal Proof Sketch.* For  $C = \{c_i\}_{i \in I}$  and  $D = \{d_i\}_i$ , we have four types of sequences.

- (i) Circular sequence: for some  $m \in \mathbb{N}$ , there exist  $c_1, \dots, c_m$  and  $d_1, \dots, d_m$  such that  $f(c_i) = d_i$  and  $g(d_i) = c_{i+1}$ , where  $c_{m+1} = c_1$ .
- (ii) Cat sequence: there is  $c_1, c_2, \dots$  and  $d_1, d_2, \dots$  such that  $f(c_i) = d_i$  and  $g(d_i) = c_{i+1}$ .
- (iii) Dog sequence: there is  $c_1, c_2, \dots$  and  $d_1, d_2, \dots$  such that  $f(c_i) = d_{i+1}$  and  $g(d_i) = c_i$ .
- (iv) Bi-infinite sequence:  $\{c_i\}_{i \in \mathbb{Z}}$  and  $\{d_i\}_{i \in \mathbb{Z}}$  such that  $f(c_i) = d_i$  and  $g(d_i) = c_{i+1}$ .

**Claim 1:** For every  $c \in C$ ,  $c$  is in exactly one sequence that is either a circular sequence, a cat sequence, a dog sequence, or a bi-infinite sequence.

We define our bijection  $h : C \rightarrow D$  by

$$h(c) = \begin{cases} g^{-1}(c) & c \text{ in a dog sequence} \\ f(c) & \text{else} \end{cases}.$$

**Claim 2:**  $h$  is well-defined.

**Claim 3:**  $h$  is a bijection.

□

**Theorem:** For every set  $A, B$ , either  $|A| \leq |B|$  or  $|B| \leq |A|$ .

In order to prove this, we need the axiom of choice.

**Example (Cardinality of the Reals).** Recall that  $|\mathbb{N}| < |P(\mathbb{N})|$  and  $|\mathbb{N}| < |\mathbb{R}|$ . According to the previous theorem, it is the case that either  $|P(\mathbb{N})| \leq |\mathbb{R}|$  or  $|\mathbb{R}| \leq |P(\mathbb{N})|$ .

In particular,  $|P(\mathbb{N})| = |\mathbb{R}|$ .

*An Informal Proof.* Let  $S$  be the set of all functions  $f : \mathbb{N} \rightarrow \{0, 1\}$ . We will show that  $|S| = |P(\mathbb{N})|$  and  $|S| = |\mathbb{R}|$ . This will show that  $|P(\mathbb{N})| = |\mathbb{R}|$  (by composing bijections).

To show that  $|S| = |P(\mathbb{N})|$ , define a subset of  $\mathbb{N}$  by the support<sup>ii</sup> of some element of  $S$ . This is a bijection between  $P(\mathbb{N})$  and  $S$ .

To show  $|S| = |\mathbb{R}|$ , we place a decimal point in front of the string, and consider it as a real number in base 2, which yields a bijection between  $S$  and  $[0, 1]$ .

Next, we show that  $|(0, 1]| = |(0, 1)|$ .

Finally, we show that  $|(0, 1)| = \mathbb{R}$ . Take  $f : (0, 1) \rightarrow \mathbb{R}$  to be  $\cot(\pi x)$  — or  $\tan(\pi x - \pi/2)$ . These are bijections from  $(0, 1)$  to  $\mathbb{R}$ . □

**Definition (Continuum Hypothesis).** We are aware that

$$|\mathbb{N}| < |\mathbb{R}| = |P(\mathbb{N})|.$$

The continuum hypothesis states that there exists no set  $S$  such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$

The continuum hypothesis is independent of the ZFC axioms.<sup>iii</sup>

<sup>ii</sup>The elements that  $f$  does not map to 0 for some  $f \in S$ .

<sup>iii</sup>Zermelo–Fraenkel Axioms with the Axiom of Choice.

**Exercise (Challenge Problem).** Let  $T = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{N}; \text{ finitely many nonzero } a_i\}$ . Is  $T$  countable? We also write

$$T = \bigoplus_{i=0}^{\infty} \mathbb{N}.$$

## Axiomatic Set Theory

**Question.** Is there a set  $A$  such that  $A \in A$ ?

**Answer.** Yes! There is the set  $\{\dots\{\}\dots\}$ , which contains infinitely many sets in itself. Additionally, there is the set  $A = \{x \mid x \text{ is a set}\}$ .

**Example (Russell's Paradox).** Consider the set

$$R = \{x \mid x \notin x\}.$$

The question is if  $R \in R$ . However, this cannot be true, because if  $R \in R$ , then  $R \notin R$  and vice versa.

## Axioms of Set Theory

We cannot just say

$$S = \{x \mid x \text{ is blah}\},$$

as evidenced by Russell's paradox. We need to carefully construct rules to create a rigorous description of formal set theory.

**Axiom (Existence).** The existence axiom states that there exists a set:

$$\exists a (a = a).$$

**Axiom (Empty Set).** The empty set axiom states that there exists a set with no elements:

$$\exists a \forall x (x \notin a).$$

**Axiom (Pairing).** The pairing axiom states that, given any sets  $a$  and  $b$ , there is a set  $c$  such that the only elements of  $c$  are  $a$  and  $b$ :

$$\forall a \forall b \exists c \forall x (x \in c \Leftrightarrow x = a \vee x = b)$$

**Axiom (Extensionality).** The axiom of extensionality states that if two sets have the same elements, they are the same sets:

$$\forall a \forall b (\forall x (x \in a \Leftrightarrow x \in b) \Rightarrow a = b)$$

**Question.** What is a set?

**Answer.** The unsatisfying answer is that "set" and "element" have no meaning *per se*. The main reason we define these axioms is to define relationships between objects (rather than objects themselves).

**Example.** We want to prove that for every set  $b$ , there exists a set  $\{b\}$ .

Symbolically, we want to show

$$\forall b \exists c \forall x (x \in c \Leftrightarrow x = b).$$

In particular, we can see that, in the pairing axiom, there is no requirement that  $a$  and  $b$  be distinct. Therefore, we can use the pairing axiom of  $a = b$  and  $b = b$ . Therefore, the pairing axiom becomes

$$\forall b \forall b \exists c \forall x (x \in c \Leftrightarrow x = b \vee x = b),$$

which reduces to

$$\forall b \exists c \forall x (x \in c \Leftrightarrow x = b).$$

In particular, if  $b = \{\}$  in the previous example, then the pairing axiom implies the uniqueness of the empty set. We will denote  $\{\} = \emptyset$ . We can create a tower

$$\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \dots,$$

entirely consisting of the empty set.

**Axiom (Union).** The axiom of union states that if  $a$  and  $b$  are sets, there exists a set  $c$  whose elements are either elements of  $a$  or elements of  $b$ , and every element of  $a$  is in  $c$  and every element of  $b$  is in  $c$ :

$$\forall a \forall b \exists c \forall x (x \in c \Leftrightarrow x \in a \vee x \in b)$$

**Definition.** The string  $a \subseteq b$  is shorthand for

$$\forall x (x \in a \Rightarrow x \in b).$$

**Axiom (Power Set).** The power set axiom states that for all  $a$ , there is a set  $b$  such that all elements of  $b$  are subsets of  $a$  and all subsets of  $a$  are contained in  $b$ :

$$\forall a \exists b \forall y (y \in b \Leftrightarrow y \subseteq a).$$

**Definition.** We let  $(a, b)$  be shorthand for the set

$$\{a, \{a, b\}\}.$$

**Exercise.** If  $\{a, \{a, b\}\} = \{c, \{c, d\}\}$ , it is the case that  $a = c$  and  $b = d$ .

Recall that

$$c = \{x \mid x \text{ is blah}\}$$

is a problematic definition of a set. However, if  $a$  is a set, we can define

$$c = \{x \mid x \in a \wedge x \text{ is blah}\},$$

which does not cause any contradictions. The following axiom schema formalizes this fact.

**Axiom (Comprehension schema).** The comprehension schema says that, given any formula  $\varphi(x)$ , in which  $x$  is a free variable, there exists a set  $c$  whose elements are those in  $a$  that satisfy  $\varphi$ :

$$\forall a \exists c \forall x (x \in c \Leftrightarrow x \in a \wedge \varphi(x)).$$

**Remark:** There are infinitely many axioms in the comprehension schema, one for each formula  $\varphi$ . This is why it is known as a schema rather than an axiom.

**Remark:** Since we can specify a formula  $\varphi(x) : x \neq x$ , the comprehension schema obviates the empty set axiom.