Homeomorphism of the Open-Ball Topology on $\mathbb{R}^2$ and the Rectangle Topology on $\mathbb{R}^2$

Avinash Iyer

Occidental College

Abstract

In this paper, I introduce the concept of metric spaces (including the definition of distance metrics, open sets, and continuity), topologies on sets, and homeomorphisms. With this knowledge, we can prove that two particular topologies on $\mathbb{R}^2$ are homeomorphic.

Sets and Metrics

Consider a set X. If we let $a, b \in X$, we might ask what the "distance" between a and b, as doing so might help tease out properties of X. We want the distance function to be defined for every pair of points, and to yield a positive number.

$$d : X \times X \to \mathbb{R}^+$$

In order to maintain a coherent idea of distance, we also need the following to hold:

- Commutativity: $d(a, b) = d(b, a)$.
- Zero distance condition: $d(x, x) = 0$, and if $d(a, b) = 0$, then $a = b$.
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$. This property is important as it means that two points have arbitrarily small distance if they have arbitrarily small distance with any intermediate point.

We now have a well-defined distance metric. Every pair of points must have a unique distance, and no pair of points can map to two distance points. We call X equipped with $d : X \times X \to \mathbb{R}$ a **metric space**.