

Problem (Problem 1): Prove that our cell complex structure for T^2 coincides with a product cell complex structure on $S^1 \times S^1$ for some cell complex structure on S^1 .

Solution: We consider the cell complex structure on S^1 given by one 0-cell and one 1-cell with characteristic map $\Phi: [0, 1] \hookrightarrow S^1$ identifying $0 \sim 1$. If we consider two copies of S^1 in this fashion, this gives rise to a CW complex structure with

- 1 0-cell, which we label as e^0 ;
- 2 1-cells, which we label as e_1^1 and e_2^1 ;
- and 1 2-cell, which we label as e^2 .

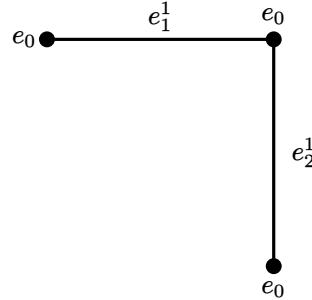
Since the attaching maps (viewed as restrictions of the characteristic maps at the boundary) for both e_1^1 and e_2^1 coincide at e^0 , it follows that there are characteristic maps $\Phi_{1,2}|_{\partial e_{1,2}^1} = e^0$

$$\Phi_{1,2}: e_{1,2}^1 \hookrightarrow S^1 \times S^1$$

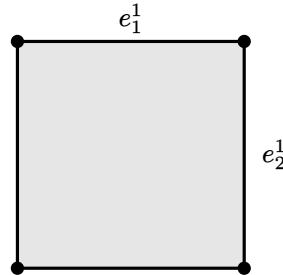
such that

$$\Phi_{1,2}|_{\partial e_{1,2}^1}(\partial e_{1,2}^1) = e^0.$$

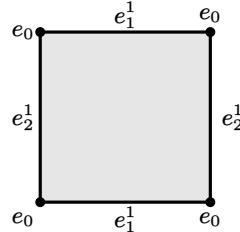
Therefore, we may draw the 1-skeleton as the following identification.



We consider the 2-cell e^2 as a square given by $e_1^1 \times e_2^1$. That is, it is given by the following diagram.



We observe that the characteristic map for e^2 , $\Psi: e^2 \hookrightarrow S^1 \times S^1$, is such that ∂e^2 is identified with the 1-skeleton. Therefore, we obtain the following diagram.



This is precisely the cell complex structure of the torus.

Problem (Problem 2): Prove that if X is a cell complex, then so is the suspension SX .

Solution: We observe that the product $X \times [0, 1]$ is a cell complex as it is a Cartesian product of a cell complex with one 1-cell, $e^1 \cong [0, 1]$, and two 0-cells, which are the endpoints of the interval. Similarly, $X \times \{0\}$ and $X \times \{1\}$ are cell complexes as they are the Cartesian product of a cell complex with the 0-cell, $e^0 \cong \{0\} = \{1\}$.

Since $X \times \{0\}$ is a subcomplex of $X \times [0, 1]$ with the attaching map given by restricting the product to $\{0\}$, it follows that $X \times [0, 1]/(X \times \{0\})$ is a cell complex, so $SX = X \times [0, 1]/(X \times \{0\})/(X \times \{1\})$ is a cell complex.

Problem (Problem 3): Consider the infinite-dimensional sphere $S^\infty = \bigcup_n S^n$, given by attaching two cells of arbitrarily high dimension.

- (a) Describe all the subcomplexes of S^∞ with this cell complex structure.
- (b) Prove that S^∞ is contractible.

Solution:

- (a) The n -skeleton for X is given by

$$X^n = 2e^n \cup 2e^{n-1} \cup \dots \cup 2e^1 \cup 2e^0,$$

where we let e^i represent the i -cell. Each of the n -skeleta is a subcomplex of S^∞ . Additionally, we observe that within the n -skeleton, the attaching maps for each e^n have it such that ∂e^n is identified with X^{n-1} , so there are two sub-skeleta defined by

$$\begin{aligned} A_1 &= e^n \cup 2e^{n-1} \cup \dots \cup 2e^1 \cup 2e^0 \\ A_2 &= e^n \cup 2e^{n-1} \cup \dots \cup 2e^1 \cup 2e^0. \end{aligned}$$

This holds for each n -skeleton, so these are the subcomplexes of S^∞ .