Solution (12.4, Problem 6)**:** Upon separation of variables, we get

$$\begin{aligned}\frac{1}{a^2T}\frac{d^2T}{dt^2} &= \frac{1}{X}\frac{d^2X}{dx^2}\\ &\begin{cases}k^2\\ 0\\ -k^2\end{cases}.\end{aligned}$$

Using some black magic, we get the cases of

$$\begin{aligned}T(x) &= \begin{cases}Ae^{akt} & k^2\\ At+B & 0\\ A\cos(akt)+B\sin(akt) & -k^2\end{cases}\\ X(x) &= \begin{cases}Ce^{kx} & k^2\\ Cx+D & 0\\ C\cos(kx)+D\sin(kx) & -k^2\end{cases}.\end{aligned}$$

By plugging in the boundary conditions of $u(0,t) = u(1,t) = 0$, we quickly remove the former two cases, we are of the form

$$\begin{aligned}T(t) &= A\cos(akt)+B\sin(akt)\\ X(x) &= C\cos(kx)+D\sin(kx).\end{aligned}$$

Since $X(0) = 0$, we must have $C = 0$, and since $X(1) = 0$, we have $k = n\pi$, $n \in \mathbb{Z}$. Thus, we have functions of the form

$$u_n(x,t) = (A_n\cos(n\pi at)+B_n\sin(n\pi at))\sin(n\pi x),$$

and the general solution of

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(n\pi at)+B_n\sin(n\pi at))\sin(n\pi x).$$

Plugging in the initial condition, we have

$$\begin{aligned}u(x,0) &= \sum_{n=1}^{\infty}A_n\sin(n\pi x)\\ &= \frac{1}{100}\sin(3\pi x),\end{aligned}$$

so that $A_n = \frac{1}{100}$ at $x = 3$ and 0 elsewhere. Writing our amended solution, we have

$$u(x,0) = \left(\frac{1}{100}\cos(3\pi at)+B_3\sin(3\pi at)\right)\sin(3\pi ax).$$

Taking derivatives, we have

$$\begin{aligned}\left.\frac{\partial u}{\partial t}\right|_{(x,0)} &= B_3\sin(3\pi ax)\\ &= 0,\end{aligned}$$

so $B_3 = 0$, and we arrive at the solution

$$u(x,t) = \frac{1}{100}\cos(3\pi at)\sin(3\pi x).$$

Solution (12.4, Problem 8)**:** Upon separation of variables, we get

$$\frac{1}{a^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$
$$\begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}.$$

Using some black magic, we get the cases of

$$T(x) = \begin{cases} Ae^{akt} & k^2 \\ At + B & 0 \\ A\cos(akt) + B\sin(akt) & -k^2 \end{cases}$$
$$X(x) = \begin{cases} Ce^{kx} & k^2 \\ Cx + D & 0 \\ C\cos(kx) + D\sin(kx) & -k^2 \end{cases}.$$

We plug in the boundary conditions of $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$ to obtain

$$X_n(x) = \begin{cases} C_n\cos\left(\frac{n\pi}{L}x\right) & -k^2 \\ Cx + D & 0 \end{cases}$$
$$T_n(t) = \begin{cases} B_n\cos\left(\frac{n\pi a}{L}t\right) & -k^2 \\ At + B & 0 \end{cases}$$

We may evaluate the solution

$$u(x,t) = X_0(x)T_0(t) + \sum_{n=1}^{\infty} D_n\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right).$$

To do this, we start with the initial condition, giving $T_0(t) = 1$ and $X_0(x) = x$. Taking the partial derivative with respect to t, we get

$$\frac{\partial u}{\partial t} = X_0(x)\frac{dT_0}{dt} - \sum_{n=1}^{\infty} D_n\cos\left(\frac{n\pi}{L}x\right)\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi a}{L}t\right).$$

Therefore,

$$u(x,t) = x$$

Solution (12.5, Problem 2)**:** Separating variables, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$
$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}.$$

Thus, we have

$$X_n = A_n\cos(\lambda x) + B_n\sin(\lambda x).$$

Using the boundary conditions of $X_n(a) = X_n(0) = 0$, we simplify to

$$X_n = B_n\sin\left(\frac{n\pi}{a}x\right).$$

Similarly, we have

$$Y_n(y) = C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right).$$

Applying the boundary condition of $\frac{\partial u}{\partial y}\Big|_{(x,0)} = 0$, we have $D_n = 0$, and

$$u(x,y) = \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right).$$

We have

$$\begin{aligned} f(x) &= u(x,b) \\ &= \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi}{a}x\right). \end{aligned}$$

Using the expansion of Fourier coefficients, we have

$$K_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)\,dx.$$

Solution (12.5, Problem 4)**:** Separating variables, we get

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= -\frac{1}{Y}\frac{d^2Y}{dy^2} \\ &= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases}. \end{aligned}$$

This evaluates to

$$\begin{aligned} X &= A\cos(\lambda x) + B\sin(\lambda x) \\ Y &= C\cosh(\lambda y) + B\sinh(\lambda y). \end{aligned}$$

Using the Neumann boundary condition, we get

$$\begin{aligned} X_n &= A_n \cos\left(\frac{n\pi}{a}x\right) \\ Y_n &= C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right). \end{aligned}$$

Therefore, $u(x,y)$ is of the form

$$u(x,y) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{a}x\right)\cosh\left(\frac{n\pi}{a}y\right) + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right).$$

We may plug this into an expression for $u(x,0)$ to get

$$x = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{a}x\right),$$

meaning

$$\begin{aligned} B_0 &= a \\ B_n &= \frac{2\left((-1)^n - 1\right)a}{n^2\pi^2}, \end{aligned}$$

and plugging in the condition that $u(x,b) = 0$, we have

$$D_n = -\coth\left(\frac{n\pi b}{a}\right)B_n.$$

Solution (12.5, Problem 6)**:** Separating variables, we have

$$\begin{aligned}-\frac{1}{X}\frac{d^2X}{dx^2} &= \frac{1}{Y}\frac{d^2Y}{dy^2}\\ &= \begin{cases}-\lambda^2\\ 0\\ \lambda^2\end{cases}.\end{aligned}$$

We thus have

$$Y = A\cos(\lambda y) + B\sin(\lambda y).$$

Using the Neumann boundary condition in y, we may simplify this to

$$Y_n = A_n\cos(ny).$$

This gives

$$X_n = B_n\cosh(nx) + C_n\sinh(nx).$$

We thus have

$$u(x,y) = \sum_{n=0}^{\infty} A_n\cosh(nx)\cos(ny) + \sum_{n=1}^{\infty} C_n\sinh(nx)\cos(ny).$$

Evaluating the boundary condition at $u(0,y)$, we have

$$g(y) = \sum_{n=0}^{\infty} A_n\cos(ny),$$

so that

$$A_n = \frac{2}{\pi}\int_0^{\pi} g(y)\cos(ny)\,dy.$$

Evaluating the derivative at $x = 1$, we have

$$\begin{aligned}\left.\frac{\partial u}{\partial x}\right|_{(1,y)} &= \sum_{n=1}^{\infty} nA_n\sinh(n)\cos(ny) + \sum_{n=1}^{\infty} nC_n\cosh(n)\cos(ny)\\ &= 0.\end{aligned}$$

Therefore, $C_n = -A_n\tanh(n)$.

Solution (12.5, Problem 8)**:** Separating variables, we have

$$\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2} &= -\frac{1}{Y}\frac{d^2Y}{dy^2}\\ &= \begin{cases}-\lambda^2\\ 0\\ \lambda^2\end{cases}\end{aligned}$$

Using the Dirichlet boundary condition on X, we get

$$\begin{aligned}X_n &= A_n\sin(n\pi x)\\ Y_n &= B_n\cosh(n\pi y) + C_n\sinh(n\pi y).\end{aligned}$$

Thus, we have $u(x,y)$ of the form

$$u(x,y) = \sum_{n=1}^{\infty} A_n\sin(n\pi x)\cosh(n\pi y) + \sum_{n=1}^{\infty} B_n\sin(n\pi x)\sinh(n\pi y).$$

Setting

$$\begin{aligned}\left.\frac{\partial u}{\partial y}\right|_{x=0} &= \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x)\\ &= 0,\end{aligned}$$

we have $B_n = 0$, so

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cosh(n\pi y).$$

Solving for the Fourier coefficients, we get

$$A_n = \frac{2}{\cosh(n\pi)}\int_0^1 f(x)\sin(n\pi x)\,dx.$$

Solution (12.6, Problem 2)**:** We may homogenize the boundary condition by letting $u(x,t) = v(x,t) + \psi(x)$, and solving

$$k\frac{d^2\psi}{dx^2} = 0,$$

where $\psi(0) = u_0$ and $\psi(1) = 0$, so $\psi = -\frac{1}{u_0}x + u_0$. This gives the boundary value problem

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = 0$$

with Dirichlet boundary of $v(0) = v(1) = 0$. Separating variables as $v(x,t) = X(x)T(t)$, we then get

$$\begin{aligned}\frac{1}{T}\frac{dT}{dt} &= \frac{k}{X}\frac{d^2X}{dx^2}\\ &= \begin{cases}-\alpha^2\\ 0\\ \alpha^2\end{cases},\end{aligned}$$

so that

$$\begin{aligned}X_n &= A_n \sin\left(n\pi\sqrt{k}x\right)\\ T_n &= e^{-kn^2\pi^2 t},\end{aligned}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi\sqrt{k}x\right)e^{-kn^2\pi^2 t}.$$

We have the initial condition $v(x,0) = f(x) - \psi(x)$, so

$$A_n = 2\int_0^1 \left(f(x) + \frac{1}{u_0}x - u_0\right)\sin(n\pi x)\,dx,$$

and

$$u(x,t) = v(x,t) - \frac{1}{u_0}x + u_0.$$

Solution (12.6, Problem 4)**:** Homogenizing the boundary conditions, we have

$$u(x,t) = v(x,t) + \psi(x),$$

where

$$k\frac{d^2\psi}{dx^2} = -r.$$

This gives

$$\psi(x) = -\frac{r}{2k}x^2 + Bx + C.$$

Plugging in the boundary conditions, we get

$$\psi(x) = -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0.$$

The homogeneous heat equation is given by

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = 0,$$

where $v(x,0) = f(x) - \psi(x)$, $v(0,t) = v(1,t) = 0$. By separating variables, we get

$$\begin{aligned}\frac{1}{T}\frac{d^2T}{dt^2} &= \frac{k}{X}\frac{d^2X}{dx^2}\\ &= \begin{cases}-\alpha^2\\ 0\\ \alpha^2\end{cases}\end{aligned}$$

Using the Dirichlet boundary conditions, we get

$$\begin{aligned}X_n &= A_n \sin\left(n\pi\sqrt{k}x\right)\\ T_n &= e^{-kn^2\pi^2 t}.\end{aligned}$$

This gives

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi\sqrt{k}x\right)e^{-kn^2\pi^2 t}.$$

The Fourier coefficients are

$$A_n = 2\int_0^1 (f(x) - \psi(x))\sin(n\pi x)\,dx,$$

and we get the solution of $u(x,t) = v(x,t) + \psi(x)$.

Solution (12.6, Problem 10)**:** We start with $u(x,t) = v(x,t) + \psi(x)$. Then,

$$a\frac{d^2\psi}{dx^2} - g = 0.$$

Thus,

$$\psi = \frac{g}{2a}x^2 + Bx + C,$$

giving $C = 0$ and $B = -\frac{g}{2a}$.

We then have the homogeneous heat equation of

$$a\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}$$

$$= \begin{cases} -\lambda^2 \\ 0 \\ \lambda^2 \end{cases},$$

with $v(x,0) = -\psi(x)$. Using the Dirichlet boundary conditions, this gives

$$\begin{aligned} X_n &= A_n \sin(n\pi x) \\ T_n &= B_n \cos\left(n\pi\sqrt{a}t\right) + C_n \sin\left(n\pi\sqrt{a}t\right). \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cos\left(n\pi\sqrt{a}t\right) + \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sin\left(n\pi\sqrt{a}t\right).$$

We have

$$-\frac{g}{2a}x(x-1) = \sum_{n=1}^{\infty} A_n \sin(n\pi x),$$

and

$$A_n = 2\int_0^1 -\frac{g}{2a}x(x-1)\sin(n\pi x)\,dx.$$

I do not know how to deal with the B_n.

Solution (Extra Problems)**:**

(i) In standard form, we have

$$\frac{\partial u}{\partial t} + \frac{x+1}{3}\frac{\partial u}{\partial x} = 0,$$

so that

$$\frac{dx}{dt} = \frac{x+1}{3},$$

and

$$x = ke^{t/3} - 1.$$

With $x(0) = k - 1$, we calculate

$$\begin{aligned} k &= (x+1)e^{-t/3} \\ k - 1 &= (x+1)e^{-t/3} - 1. \end{aligned}$$

Thus, via the method of characteristics, we have

$$u(x,t) = \left((x+1)e^{-t/3} - 1\right)^2.$$

(ii) In standard form, we have

$$\frac{\partial u}{\partial t} + 2(t+1)\frac{\partial u}{\partial x} = 0,$$

so

$$\begin{aligned} \frac{dx}{dt} &= 2(t+1) \\ x &= (t+1)^2 + C. \end{aligned}$$

Finding $x(0) = C + 1$, we get

$$C + 1 = x - (t+1)^2 + 1,$$

and

$$\begin{aligned} u(x,t) &= u_0(x_0) \\ &= \left(x - (t+1)^2 + 1\right)^3. \end{aligned}$$

(iii) In standard form, we have

$$\frac{\partial u}{\partial x} + \frac{4}{x}\frac{\partial u}{\partial t} = 0.$$

Thus, solving

$$\frac{dx}{dt} = \frac{4}{x},$$

we have

$$\frac{x^2}{8} = t + C,$$

so that

$$x(0) = 2\sqrt{2C}.$$

Thus,

$$x_0 = \sqrt{x^2 - 8t},$$

and

$$\begin{aligned} u(x,t) &= u_0(x_0) \\ &= 2\sqrt{x^2 - 8t} - 5. \end{aligned}$$

(iv) Evaluating the homogeneous equation, we have

$$\frac{\partial u}{\partial t} + 7\frac{\partial u}{\partial x} = 0,$$

with solution

$$u_h(x,t) = (x - 7t)^2.$$

Solving

$$\frac{du}{dt} = u^2,$$

we have

$$u_p(x,t) = -\frac{1}{t}.$$

Thus,

$$u(x,t) = -\frac{1}{t} + (x - 7t)^2.$$

I am aware that this is probably wrong.