Chapter 4 Problems

4.7

Cylindrical Coordinates

In cylindrical coordinates, we have

Quantity	Value
$\mathbf{r}$	$\rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} + z\hat{k}$
u_1	ρ
u_2	ϕ
u_3	z
$\hat{e}_1$	$\hat{\rho}$
$\hat{e}_2$	$\hat{\phi}$
$\hat{e}_3$	$\hat{k}$
$\frac{\partial \mathbf{r}}{\partial u_1}$	$\hat{\rho}$
$\frac{\partial \mathbf{r}}{\partial u_2}$	$\rho\hat{\phi}$
$\frac{\partial \mathbf{r}}{\partial u_3}$	$\hat{k}$
h_1	1
h_2	ρ
h_3	1

- Line element:

$$\begin{aligned}(ds)^2 &= \sum_i h_i^2 (du_i)^2 \\ &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2 .\end{aligned}$$

- Area element:

$$\begin{aligned}d\mathbf{a} &= \left(\sum_k \epsilon_{ijk}\hat{e}_k\right) h_i h_j du_i du_j \\ &= \rho\hat{k}\, d\rho d\phi \\ &= \rho\hat{\rho}\, d\phi dz \\ &= \hat{\phi}\, dz d\rho\end{aligned}$$

- Volume element:

$$\begin{aligned}d\tau &= h_1 h_2 h_3\, du_1 du_2 du_3 \\ &= \rho\, d\rho d\phi dz\end{aligned}$$

Spherical Coordinates

Quantity	Value
$\mathbf{x}$	$r\sin\theta\cos\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath} + r\cos\theta\hat{k}$
u_1	r
u_2	ϕ
u_3	θ
$\hat{e}_1$	$\hat{\rho}$
$\hat{e}_2$	$\hat{\phi}$
$\hat{e}_3$	$\hat{\theta}$
$\frac{\partial \mathbf{x}}{\partial u_1}$	$\hat{r}$
$\frac{\partial \mathbf{x}}{\partial u_2}$	$r\sin\theta\hat{\phi}$
$\frac{\partial \mathbf{x}}{\partial u_3}$	$r\hat{\theta}$
h_1	1
h_2	$r\sin\theta$
h_3	r

- Line element:

$$\begin{aligned}(ds)^2 &= \sum_i h_i^2 (du_i)^2\\ &= (dr)^2 + r^2\sin^2\theta\,(d\phi)^2 + r^2\,(d\theta)^2\end{aligned}$$

- Area element:

$$\begin{aligned}d\mathbf{a} &= \left(\sum_k \epsilon_{ijk}\hat{e}_k\right) h_i h_j\, du_i\, du_j\\ &= r\sin\theta\hat{\theta}\ dr d\phi\\ &= r^2\sin\theta\hat{r}\ d\phi d\theta\\ &= r\hat{\phi}\ d\theta dr\end{aligned}$$

- Volume element:

$$\begin{aligned}d\tau &= h_1 h_2 h_3 du_1 du_2 du_3\\ &= r^2\sin\theta\ dr d\phi d\theta\end{aligned}$$

4.9

We have

$$\begin{aligned}\epsilon_{ijk} &= \left(\hat{e}_i \times \hat{e}_j\right)\cdot\hat{e}_k\\ &= \det\begin{pmatrix}\hat{e}_i & \hat{e}_j & \hat{e}_k\end{pmatrix}\\ &= \det\begin{pmatrix}\hat{e}_i^T\\ \hat{e}_j^T\\ \hat{e}_k^T\end{pmatrix}.\end{aligned}$$

Note that $\hat{e}_i\hat{e}_j^T = \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$. Thus,

$$\begin{aligned}
\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ij\ell} &= \sum_{\ell} \det\begin{pmatrix} \hat{e}_m & \hat{e}_n & \hat{e}_\ell \end{pmatrix} \det\begin{pmatrix} \hat{e}_i & \hat{e}_j & \hat{e}_\ell \end{pmatrix} \\
&= \sum_{\ell} \det\begin{pmatrix} \hat{e}_m & \hat{e}_n & \hat{e}_\ell \end{pmatrix} \det\begin{pmatrix} \hat{e}_i^T \\ \hat{e}_j^T \\ \hat{e}_\ell^T \end{pmatrix} \\
&= \sum_{\ell} \det\begin{pmatrix} \hat{e}_m\hat{e}_i^T & \hat{e}_m\hat{e}_j^T & \hat{e}_m\hat{e}_\ell^T \\ \hat{e}_n\hat{e}_i^T & \hat{e}_n\hat{e}_j^T & \hat{e}_n\hat{e}_\ell^T \\ \hat{e}_\ell\hat{e}_i^T & \hat{e}_\ell\hat{e}_j^T & \hat{e}_\ell\hat{e}_\ell^T \end{pmatrix} \\
&= \sum_{\ell} \det\begin{pmatrix} \delta_{mi} & \delta_{mj} & \delta_{m\ell} \\ \delta_{ni} & \delta_{nj} & \delta_{n\ell} \\ \delta_{\ell i} & \delta_{\ell j} & \delta_{\ell\ell} \end{pmatrix} \\
&= \sum_{\ell} \det\begin{pmatrix} \delta_{mi} & \delta_{mj} & \delta_{m\ell} \\ \delta_{ni} & \delta_{nj} & \delta_{n\ell} \\ \delta_{\ell i} & \delta_{\ell j} & 1 \end{pmatrix} \\
&= \det\begin{pmatrix} \delta_{mi} & \delta_{mj} \\ \delta_{ni} & \delta_{nj} \end{pmatrix} \\
&= \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}.
\end{aligned}$$

4.11

(a)

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k \\
&= -\sum_{i,j,k} \epsilon_{jik} B_j A_i \hat{e}_k \\
&= -(\mathbf{B} \times \mathbf{A})
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= \sum_{i,j,k} \left(\epsilon_{ijk} A_i B_j \hat{e}_k\right) \cdot A_i \hat{e}_i \\
&= \sum_{i,j,k} \delta_{ik} \left(\epsilon_{ijk} A_i^2 B_j\right) \\
&= 0.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i,j,\ell} \epsilon_{ij\ell} A_\ell B_i C_j \\
&= \sum_{i,j,\ell} \left(\epsilon_{\ell ij} A_\ell B_i\right) C_j
\end{aligned}$$

$$= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

and

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j,\ell}\epsilon_{ij\ell}A_\ell B_i C_j\\ &= \sum_{i,j,\ell}\left(\epsilon_{j\ell i}C_j A_i\right)B_i\\ &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}).\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \sum_{i,j}\epsilon_{ijk}A_i\left(\sum_{\alpha,\beta}\epsilon_{\alpha\beta j}B_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\epsilon_{ijk}\epsilon_{\alpha\beta j}A_iB_\alpha C_\beta\\ &= -\left(\sum_{i,j,\alpha,\beta}\epsilon_{ikj}\epsilon_{\alpha\beta j}A_iB_\alpha C_\beta\right)\\ &= -\left(\sum_{i,j,\alpha,\beta}\left(\delta_{i\alpha}\delta_{k\beta}-\delta_{i\beta}\delta_{k\alpha}\right)A_iB_\alpha C_\beta\right)\\ &= \sum_{i,j,\alpha,\beta}\left(\delta_{k\alpha}\delta_{i\beta}-\delta_{i\alpha}\delta_{k\beta}\right)A_iB_\alpha C_\beta\\ &= \sum_{i,j,\alpha,\beta}\left(B_\alpha\delta_{k\alpha}\right)\left(A_iC_\beta\delta_{i\beta}\right)-\left(C_\beta\delta_{k\beta}\right)\left(A_iB_\alpha\delta_{i\alpha}\right)\\ &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}).\end{aligned}$$

(e)

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\cdot\mathbf{D}) &= \sum_{\alpha,\beta}(\mathbf{A}\times\mathbf{B})_\alpha(\mathbf{C}\times\mathbf{D})_\beta\,\delta_{\alpha\beta}\\ &= \sum_{\substack{\alpha,\beta,\\ i,j,\\ m,n}}\epsilon_{ij\alpha}\epsilon_{mn\beta}A_iB_jC_mD_n\delta_{\alpha\beta}\\ &= \sum_{\substack{\alpha,\\ i,j,\\ m,n}}\epsilon_{ij\alpha}\epsilon_{mn\alpha}A_iB_jC_mD_n\\ &= \sum_{\substack{i,j,\\ m,n}}A_iB_jC_mD_n\left(\delta_{mi}\delta_{nj}-\delta_{mj}\delta_{ni}\right)\\ &= \sum_{\substack{i,j,\\ m,n}}\left(\left(A_iC_m\delta_{mi}\right)\left(B_jD_n\delta_{nj}\right)\right)-\left(\left(B_jC_m\delta_{mj}\right)-\left(A_iD_n\delta_{ni}\right)\right)\\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D}).\end{aligned}$$

Chapter 5 Problems

5.1

Let $f(x) = x^n$. We use linearity for the general case.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\
&= \lim_{h\to 0}\frac{x^n + n\left(hx^{n-1}\right) + \cdots + nh^{n-1}x + h^n - x^n}{h}\\
&= \lim_{h\to 0}\left(nx^{n-1} + \cdots + nh^{n-2}x + h^{n-1}\right)\\
&= nx^{n-1}.
\end{aligned}$$

5.6

$$\begin{aligned}
\cos(N\phi) + i\sin(N\phi) &= (\cos\phi + i\sin\phi)^N\\
&= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k\left(\sin\phi^{N-k}\right)\left(e^{i\frac{\pi}{2}}\right)^{N-k}\\
&= \sum_{k=0}^{N}\binom{N}{k}(\cos\phi)^k(\sin\phi)^{N-k}\left(\cos\left((N-k)\frac{\pi}{2}\right) + i\sin\left((N-k)\frac{\pi}{2}\right)\right)\\
&= \sum_{k=0}^{N}\binom{N}{k}\cos\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\\
&\quad + i\left(\sum_{k=0}^{N}\binom{N}{k}\sin\left((N-k)\frac{\pi}{2}\right)(\cos\phi)^k(\sin\phi)^{N-k}\right).
\end{aligned}$$

We get the final answer by equating real and imaginary parts.

Chapter 6 Problems

6.3

(a) Looking at the ratio test first, we find

- Ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right|\\
&= 1,
\end{aligned}$$

  which is an inconclusive result.

- Comparison test:

$$\frac{1}{\sqrt{n}} > \frac{1}{n} \qquad \forall n \geqslant 1.$$

  Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

(b)

- Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\left(\frac{1}{2}\right)\right| \\ &= \frac{1}{2} \\ &< 1,\end{aligned}$$

  meaning the series converges by the ratio test.

- $$\frac{1}{n2^n} < \frac{1}{2^n} \qquad \text{for all } n \geqslant 1,$$

  and since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, it must be the case that $\sum_{n=1}^{\infty}\frac{1}{n2^n}$ converges.

6.9

$$\begin{aligned}\sum_{n=-N}^{N} e^{inx} &= 1 + \sum_{n=1}^{N} e^{-inx} + \sum_{n=1}^{N} e^{inx} \\ &= 1 + e^{-ix}\sum_{n=1}^{N} e^{i(n-1)x} + e^{ix}\sum_{n=1}^{N} e^{i(n-1)x} \\ &= 1 + e^{-ix}\sum_{n=0}^{N-1} e^{-inx} + e^{ix}\sum_{n=0}^{N-1} e^{inx} \\ &= 1 + e^{-ix}\frac{1-e^{-iNx}}{1-e^{-ix}} + e^{ix}\frac{1-e^{iNx}}{1-e^{ix}} \\ &= 1 + \frac{e^{-ix}-e^{-i(N+1)x}}{1-e^{-ix}} + \frac{1-e^{iNx}}{e^{-ix}-1} \\ &= 1 + \frac{\left(e^{-ix}-1\right) + e^{iNx} - e^{-i(N+1)x}}{1-e^{-ix}} \\ &= \frac{e^{iNx}-e^{-i(N+1)x}}{1-e^{-ix}} \\ &= \frac{e^{iNx}-e^{-i(N+1)x}}{e^{-i\left(\frac{x}{2}\right)}\left(e^{i\left(\frac{x}{2}\right)}-e^{-i\left(\frac{x}{2}\right)}\right)} \\ &= \frac{e^{i\left(N+\frac{1}{2}\right)x}-e^{-i\left(N+\frac{1}{2}\right)x}}{e^{-i\left(\frac{x}{2}\right)}-e^{-i\left(\frac{x}{2}\right)}} \\ &= \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}.\end{aligned}$$

6.13

(a)

$$\begin{aligned}\frac{d}{dz}\left(\arctan(z)\right) &= 1 - z^2 + z^4 - z^6 + \cdots \\ &= \sum_{i=0}^{\infty}(-1)^n z^{2n} \\ &= \frac{1}{1+z^2}.\end{aligned}$$

(b)

$$\begin{aligned}
\rho &= \limsup_{k\to\infty} \sqrt[k]{\left|(-1)^k\right|}\\
&= 1\\
r &= \frac{1}{\rho}\\
&= 1.
\end{aligned}$$

(c)

$$\begin{aligned}
\rho &= \limsup_{k\to\infty} \left(\left|\frac{(-1)^k}{2k+1}\right|\right)^{1/k}\\
&= \limsup_{k\to\infty} \frac{1}{(2k+1)^{1/k}}\\
&= 1\\
r &= \frac{1}{\rho}\\
&= 1.
\end{aligned}$$

6.25

$$\begin{aligned}
e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!}\\
&= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k\,\theta^{2k+1}}{(2k+1)!}\\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

6.37

$$\begin{aligned}
F &= GmM_E\,(R_E + h)^{-2}\\
&= \frac{GmM_E}{R_E^2}\left(1 + \frac{h}{R_E}\right)^{-2}\\
&= \frac{GmM_E}{R_E^2}\left(1 + (-2)\frac{h}{R_E} + \frac{(-2)(-3)}{2!}\left(\frac{h}{R_E}\right)^2 + \cdots\right)\\
&\approx \frac{GmM_E}{R_E^2}.
\end{aligned}$$

It would be the case that for $h = (0.05)\,R_E$, there would need to be a 10% negative correction to the estimated force.

6.42

$$\left(1 - k^2\sin^2\phi\right)^{-1/2} = \left(1 + \left(-k^2\sin^2\phi\right)\right)^{-1/2}$$

$$
\begin{aligned}
&= 1 + \left(-\frac{1}{2}\right)\left(-k^2 \sin^2 \phi\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-k^2 \sin^2 \phi\right)^2 + \cdots \\
&\approx 1.
\end{aligned}
$$

The integrand then expands to

$$
\begin{aligned}
T &\approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \, d\phi \\
&= 2\pi\sqrt{\frac{L}{g}}.
\end{aligned}
$$

With a first-order correction and $\alpha = \pi/3$, the fractional change in T is

$$
\begin{aligned}
T &\approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} 1 + k^2 \sin^2 \phi \, d\phi \\
&= 4\sqrt{\frac{L}{g}}\left(\pi/2 + \frac{1}{4}\left(\frac{\pi}{4}\right)\right) \\
&= \left(2 + \frac{1}{4}\right)\pi\sqrt{\frac{L}{g}}.
\end{aligned}
$$

Thus, the error in $T \approx 2\pi\sqrt{\frac{L}{g}}$ with $\alpha = \pi/3$ is an underestimate of 11%.