Complex Numbers

A complex number is an ordered pair of real numbers, $(a, b) = a + bi$. A vector in $\mathbb{R}^2$ is also an ordered pair, (a, b) of real numbers.

Indeed, vector addition and scalar multiplication on complex numbers are defined just as with $\mathbb{R}^2$. However, unlike vectors in $\mathbb{R}^2$, there is also an operation $\cdot$. We desire for $(0, 1) \cdot (0, 1) = (-1, 0)$; essentially, $i^2 = -1$. We say that i is a square foot of -1; every complex number except 0 has two square roots.

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + bi) + (c + di)\\ &:= a(c) + adi + bci + bd(i^2)\\ &:= (ac - bd) + (ad + bc)i\\ &= (ac - bd, ad + bc)\end{aligned}$$

Thus, $\mathbb{R}^2$ with the operations $+$ and the above defined complex multiplication is known as $\mathbb{C}$. We write as $a + bi$ instead of (a, b).

Given $z = (a + bi) \in \mathbb{C}$, we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$. If $\text{Im}(z) = 0$, then $z \in \mathbb{R} \times \{0\} \subset \mathbb{C}$. However, many people say that $\mathbb{R} \subseteq \mathbb{C}$, even if $\mathbb{C}$ isn't defined as such.

Reciprocals of Complex Numbers

Let $z \in \mathbb{C}$, where $z \neq 0$. Then, $\exists w \in C$ such that $zw = 1$.

Let $w = c + di$. We want to show that $zw = 1$.

$$(a + bi) + (c + di) = (ac - bd) + (ad + bc)i$$

with the condition that

$$\begin{aligned}ac - bd &= 1\\ ad + bc &= 0.\end{aligned}$$

Thus, let $w = c + di$, with $a, b \neq 0$

$$\begin{aligned}c &= \frac{a}{a^2 + b^2}\\ d &= \frac{-b}{a^2 + b^2}\end{aligned}$$

For every $z \neq 0$, with $z = a + bi$, the *reciprocal* of z is defined as $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Then, for $w \in \mathbb{C}$, we define

$$\frac{w}{z} := w\left(\frac{1}{z}\right).$$

Properties of Complex Numbers

Let $z = a + bi \in C$. Then, the (Euclidean) norm (or absolute value) of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

(i) $z\overline{z} = |z|^2$

(ii) $\overline{(\overline{z})} = z$

(iii) $\overline{(z+w)} = \overline{z} + \overline{w}$

(iv) $\overline{zw} = \overline{z} \cdot \overline{w}$

(v) $z + \overline{z} = 2\text{Re}(z)$, so $\text{Re}(z) = \frac{z+\overline{z}}{2}$

(vi) $z - \overline{z} = 2\text{Im}(z)i$, so $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

Polar Representation

Let $z = a+bi$ (or $z = (a,b)$). Then, $|z| = \sqrt{a^2+b^2}$ is the *radius*, and the *argument* is found by $\theta = \arctan(b/a)$ for $a \neq 0$. Therefore, the full polar representation is as follows:

$$z = |z|\left(\cos\theta + i\sin\theta\right). \qquad \theta \in [0, 2\pi)$$

If $z = 0$, then $|z| = 0$, and $\arg z$ is undefined.

For example, we can find $\arg i$ in $[\pi, 3\pi)$ as $\frac{5\pi}{2}$.

For z_1 and z_2 in polar form, we have:

$$\begin{aligned} |z_1z_2| &= |z_1||z_2| && (1)\\ \arg(z_1z_2) &= \arg z_1 + \arg z_2 \mod 2\pi && (2) \end{aligned}$$

Proof of (1):

$$\begin{aligned} |z_1z_2|^2 &= (z_1z_2)\overline{(z_1z_2)}\\ &= z_1z_2\overline{z_1}\,\overline{z_2}\\ &= z_1\overline{z_1}z_2\overline{z_2}\\ &= |z_1|^2|z_2|^2 \end{aligned}$$

Since $|z| \geq 0$, we get $|z_1z_2| = |z_1||z_2|$.

Let $z = 2(\cos\pi/6 + i\sin\pi/6)$, and let $f : \mathbb{C} \to \mathbb{C}$ defined as $f(w) = zw$. Then, f rotates w by $\pi/6$ and scales w by 2.

Theorem: For $n \in \mathbb{N}$, if $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Proof: Induct on n. For the base case, we know that $n = 1$ satisfies this property. For $n > 1$, we have:

$$\begin{aligned} z^{n+1} &= (z^n)(z)\\ &= \left(r^n(\cos(n\theta) + i\sin(n\theta))\right) r(\cos\theta + i\sin\theta)\\ &= (r^n)(r)\left(\cos(n\theta+\theta) + i\sin(n\theta+\theta)\right) && \text{Polar Representation Definition}\\ &= r^{n+1}(\cos\left((n+1)\theta\right) + i\sin((n+1)\theta)) \end{aligned}$$

We can use this technique to find the "roots of unity." For example, to find all z such that $z^3 = 1$, we use our technique:

$$\begin{aligned}
z^3 &= 1\\
|z| &= 1\\
\arg z^3 &= 0\\
3\arg z &= 0 \mod 2\pi\\
\arg z &= \frac{k2\pi}{3}\\
&= 0, \frac{2\pi}{3}, \frac{4\pi}{3}\\
z_1 &= 1\\
z_2 &= (\cos 2\pi/3 + i\sin 2\pi/3)\\
z_3 &= (\cos 4\pi/3 + i\sin 4\pi/3)
\end{aligned}$$

We can see that $z_2^2 = z_3$.

For the n case, we find $z_2 = \cos(2\pi/n) + i\sin(2\pi/n)$, and $z_k = z_2^{k-1}$.