Problem 1

True or false: If H is a minor of G, then H is a contraction of a subgraph of G.

True.

Problem 2

Prove each of the following.

(a) There exists an infinite family F of graphs such that no graph in F is a subgraph of another graph in F.

(b) There exists an infinite family F of graphs such that no graph in F is a contraction of another graph in F.

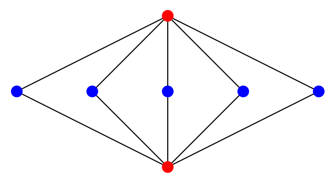
(c) There exists an infinite family F of graphs such that no graph in F is a subgraph or a contraction of another graph in F.

(a)

Cycles; if we delete any vertex or edge of any cycle, then the degree of at least two vertices is reduced by 1, while all vertices in every cycle are of degree 2.

(b)

The family of graphs $K_{2,n}$:



Problem 3

Prove that the set of all planar graphs is minor-closed.

Let G be any planar graph. Then, by Wagner's theorem, it must be the case that neither K_5 nor $K_{3,3}$ are minors of G. Therefore, any minor of G, G', must also not have K_5 nor $K_{3,3}$ as a minor — otherwise, we would take the steps to create G', then the steps to create one of the forbidden minors, and G would have the forbidden minors as a minor.

Thus, since no minor of any planar graph can be non-planar, it must be the case that planarity is minor-closed.

Problem 4

Let P be an arbitrary set of graphs. Let P' be the set of all graphs not in P. By the Graph Minor Theorem, P has a finite subset F of graphs that are minor-minimal in P. Similarly, P' has a finite subset F' of graphs that are minor minimal in P'. Prove that if P is minor-closed, then a graph G is in P' if and only if G has a minor in F'. So, if P is minor-closed, then P and P' are both "characterized" by F'. In fact, if P is minor-closed, then F consists of only one graph, namely the graph with only one vertex. Why?

Suppose P is minor-closed. Then, if $H \in F$, then no minor of H is in P or F.

($\Rightarrow$) Let $G \in P$. Suppose toward contradiction that there is a graph $H \in F'$ that is a minor of G.

Since P is minor-closed, and H is a minor of $G \in P$, $H \in P$.

However, $H \in F' \subseteq P'$. So, $H \in P$ and $H \in P'$. $\perp$

($\Leftarrow$) Suppose that no graph in F' is a minor of G. Suppose toward contradiction that $G \notin P$.

Then, $G \in P'$.

So, by definition of F', G must have a minor in F'.

So, $\exists H \in F'$ such that H is a minor of G. $\perp$

Since any graph that is in F will be

Problem 5

A graph G is apex if $G - v$ is planar for some vertex v of G. Prove that the set of apex graphs is minor-closed.

Problem 6

Prove that if G is a connected graph, then for every edge e, $G - e$ has at most two connected components.

Let $e = ab$.

If $G - e$ is connected, then we are done. Otherwise, if $G - e$ is not connected, $\exists v, w \in V(G - e)$ such that there is no path in $G - e$ from v to w.

Since G is connected, $\exists$ a path $P = (v, v_1, \dots, a, b, \dots, v_n, w)$ in G. It must be the case that $e \in P$, or else $P \in G - e$.

Let $x \in V(G - e)$ such that $x \neq v$ and $x \neq w$. We will show that $\exists$ a path in $G - e$ from x to v or x to w.

Let P_{xa} be a path in G from x to a. If $e \notin P_{xa}$, then $(v, \dots, a) \cup P_{xa}$ is a path from v to x without e.

If $e \in P_{xa}$, then $P_{xa} = (x, \dots, b, a)$. So, $(x, \dots, b) \cup (b, \dots, w)$ is a path from x to w without e.