Problem (Problem 1)**:** Let $U \subseteq \mathbb{C}$ be a nonempty open set.

Given a sequence $(z_n)_n \subseteq U$, we write $z_n \to \partial U$ if, for every compact subset $K \subseteq U$, there exists some $N = N(K) \in \mathbb{N}$ such that $z_n \notin K$ whenever $n \geqslant N$.

Given a function $u \colon U \to \mathbb{R}$, define

$$\limsup_{z \to \partial U} u(z) = \inf_{\substack{K \subseteq U \\ K \text{ compact}}} \sup_{z \in U \setminus K} u(z).$$

(a) For each positive integer $n \in \mathbb{N}$, define

$$K_n := \left\{ z \in U \ \middle|\ |z| \leqslant n, \operatorname{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{n} \right\}.$$

Show that:

    (i) each K_n is compact;

    (ii) $K_n \subseteq K_{n+1}^\circ$;

    (iii) $U = \bigcup_{n=1}^\infty K_n$.

(b) Let $L := \limsup_{z \to \partial U} u(z)$.

    (i) Show that for each $S > L$, there is a compact subset $K \subseteq U$ such that $u(z) \leqslant S$ for all $z \in U \setminus K$.

    (ii) Show that there exists a sequence $(z_n)_n$ in U with $z_n \to \partial U$ and $\limsup_{n \to \infty} u(z_n) \leqslant L$.

(c) Prove that

$$\limsup_{z \to \partial U} u(z) = \sup_{\substack{(z_n)_n \subseteq U \\ z_n \to \partial U}} \limsup_{n \to \infty} u(z_n),$$

where the supremum is over all sequences $(z_n)_n$ with $(z_n)_n \to \partial U$.

Solution:

(a) We claim that the set

$$C_n = \left\{ z \in U \ \middle|\ \operatorname{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{n} \right\}$$

is closed. Then, we observe that $K_n = B(0, n) \cap C_n$ would then be an intersection of a closed set with a closed and bounded set, hence a closed and bounded set, hence compact. Towards this end, suppose $(w_k)_k \subseteq C_n$ converges to $w \in \mathbb{C}$. Then, for each k, we have

$$\inf_{z \in \mathbb{C} \setminus U} |w_k - z| \geqslant \frac{1}{n}.$$

Observe then that for any $z \in \mathbb{C} \setminus U$, we have

$$|w_k - z| \geqslant \frac{1}{n}$$

for each k, meaning that

$$\lim_{k \to \infty} |w_k - z| \geqslant \frac{1}{n},$$

or that

$$|w - z| \geqslant \frac{1}{n}.$$

In particular, it must be the case that $w \in U$, and that

$$\inf_{z \in \mathbb{C} \setminus U} |w - z| \geq \frac{1}{n},$$

so that $w \in C_n$, and thus C_n is closed, and K_n is compact.

To see that $K_n \subseteq K_{n+1}^\circ$, we show that $C_n \subseteq C_{n+1}^\circ$ by understanding the picture of C_n°. Towards this end, we see that $z \in C_n^\circ$ if and only if $z \in U$ and there is some $r > 0$ such that $\text{dist}_{\mathbb{C} \setminus U}(w) \geq \frac{1}{n}$ for all $w \in U(z, r)$.

Observe that if $\varepsilon > 0$, then if z satisfies $\text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{n} + \varepsilon$, then if $w \in \mathbb{C} \setminus U$ and $\zeta \in U(z, \varepsilon/2)$, we have

$$\frac{1}{n} + \varepsilon \leq |z - w|$$
$$\leq |z - \zeta| + |\zeta - w|$$
$$< \varepsilon/2 + |\zeta - w|,$$

meaning that $|\zeta - w| \geq \frac{1}{n} + \varepsilon/2$ for all $w \in \mathbb{C} \setminus U$, so that $\text{dist}_{\mathbb{C} \setminus U}(\zeta) \geq \frac{1}{n}$. In particular, this means that C_n° consists of all z for which there exists ε such that $\text{dist}_{\mathbb{C} \setminus U}(z) \geq \frac{1}{n} + \varepsilon$, or more succinctly,

$$C_n^\circ = \left\{ z \in U \;\middle|\; \text{dist}_{\mathbb{C} \setminus U}(z) > \frac{1}{n} \right\}.$$

In particular, since $\frac{1}{n} > \frac{1}{n+1}$, it follows that $C_n \subseteq C_{n+1}^\circ$. Paired with the fact that $B(0, n) \subseteq U(0, n+1)$, we obtain that

$$K_n = B(0, n) \cap C_n$$
$$\subseteq U(0, n+1) \cap C_{n+1}^\circ$$
$$= (B(0, n+1) \cap C_n)^\circ$$
$$= K_{n+1}^\circ.$$

Finally, to show that $U = \bigcup_{n=1}^\infty K_n$, we write

$$\bigcup_{n=1}^\infty K_n = \bigcup_{n=1}^\infty (B(0, n) \cap C_n)$$
$$= \left(\bigcup_{n=1}^\infty B(0, n) \right) \cap \left(\bigcup_{n=1}^\infty C_n \right),$$

and since $\bigcup_{n=1}^\infty B(0, n) = \mathbb{C}$, it follows that we must show that

$$\bigcup_{n=1}^\infty C_n = U.$$

Towards this end, we prove that if $A \subseteq \mathbb{C}$ is any subset, then $\text{dist}_A(z) = 0$ if and only if $z \in \overline{A}$. Towards this end, if $\text{dist}_A(z) = 0$, then for any k, there is $w \in A$ such that $|w - z| < \frac{1}{n}$, so that we may construct a sequence $(w_n)_n$ in A such that $(w_n)_n \to z$, or that $z \in \overline{A}$. Similarly, if $z \in \overline{A}$, then if $(w_n)_n$ is a sequence in A converging to z, and $\varepsilon > 0$, it follows that $|w_n - z| < \varepsilon$ for sufficiently large n, so that $\inf_{w \in Z} |w - z| = 0$.

Since U is open, it follows that for any $z \in \mathbb{C} \setminus U$, since $\mathbb{C} \setminus U = \overline{\mathbb{C} \setminus U}$, $\text{dist}_{\mathbb{C} \setminus U}(z) = 0$. Equivalently, if $z \in U$, we must have $\text{dist}_{\mathbb{C} \setminus U}(z) > 0$, so that there exists n sufficiently large such that $\text{dist}_{\mathbb{C} \setminus U}(z) \geqslant 1/n$; this means $z \in C_n$, so that

$$U \subseteq \bigcup_{n=1}^{\infty} C_n.$$

Meanwhile, if $z \in \bigcup_{n=1}^{\infty} C_n$, then there is some N such that $\text{dist}_{\mathbb{C} \setminus U}(z) \geqslant 1/N$, meaning that $\text{dist}_{\mathbb{C} \setminus U}(z) > 0$, meaning $z \notin \mathbb{C} \setminus U$, so that $z \in U$.

(b)

   (i) If $S = L + \varepsilon$ for $\varepsilon > 0$, it follows by the definition of the infimum that there exists a compact subset $K \subseteq U$ such that $\sup_{z \in U \setminus K} u(z) \leqslant S$. Therefore, for all $z \in U \setminus K$, $u(z) \leqslant S$.

  (ii) Let $L_n = L + \frac{1}{n}$. We find $K_{j_n} \subseteq U$ that satisfies

- $u(z) \leqslant L_n$ for all $z \in U \setminus K_{j_n}$;
- $|z| \leqslant j_n$ for all $z \in K_{j_n}$;
- $\text{dist}_{\mathbb{C} \setminus U}(z) \geqslant \frac{1}{j_n}$.

    The existence of such a K_{j_n} follows from the proof in (i) and the definitions in part (a). We may find $z_n \in U \setminus K_{j_n}$, so that $u(z_n) \leqslant L_n$.

    The sequence $(z_n)_n$ thus escapes all the K_{j_n}, and since any $K \subseteq U$ is contained in some sufficiently large K_{j_n}, it follows that $(z_n)_n \to \partial U$. Furthermore, since $u(z_n) \leqslant L_n$ for each n, we have

$$\limsup_{n \to \infty} u(z_n) \leqslant \limsup_{n \to \infty} L_n$$
$$= L.$$

(c)

Problem (Problem 2): Let

$$U = \{z \in \mathbb{C} \mid |z| < 1, \text{Im}(z) > 0\}.$$

(a) Construct a conformal map from U to $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

(b) Construct an unbounded harmonic function $u \colon U \to (0, \infty)$ such that for all $(x_0, y_0) \in \partial U \setminus \{(1, 0)\}$, we have that $\lim_{(x,y) \to (x_0, y_0)} u(x, y) = 0$.

(c) Suppose that $v \colon U \to (0, \infty)$ is an unbounded harmonic function such that for all $(x_0, y_0) \in \partial U \setminus \{(1, 0)\}$, we have that $\lim_{(x,y) \to (x_0, y_0)} v(x, y) = 0$. Show that there exists a sequence $((x_n, y_n))_n$ in U converging to $(1, 0)$ and $\lim_{n \to \infty} v(x_n, y_n) = \infty$.

Solution:

(a) We start by taking the Cayley transform, mapping $\mathbb{H}$ to $\mathbb{D}$, given by $\frac{z-i}{z+i}$. The inverse Cayley transform, which maps $\mathbb{D}$ to $\mathbb{H}$, is then given by the inverse transform, which takes

$$Q(z) = i\frac{1+z}{1-z}.$$

By taking $a + bi \in U$ with $b > 0$ and $a^2 + b^2 \leqslant 1$, we find that

$$i\frac{1 + (a + bi)}{1 - a - bi} = \frac{1}{(1-a)^2 + b^2}\left(-2b + i(1 - a^2 - b^2)\right).$$

Therefore, we observe that the inverse transform maps U to the second quadrant, admitting arguments between $\frac{\pi}{2}$ and π. By squaring, we have

$$(Q(z))^2 = -\left(\frac{z+1}{1-z}\right)^2,$$

which maps to complex numbers with arguments between π and 2π. Multiplying by -1, we get

$$H(z) = \left(\frac{z+1}{1-z}\right)^2$$

mapping from U to the upper half-plane. Since we composed a series of bijective holomorphic maps (and, within a correct domain for the case of square root, ones that have holomorphic inverse), it follows that H is a bijective holomorphic map with holomorphic inverse, hence conformal.

(b) Consider the function

$$u(x,y) = \mathrm{Im}(H(x+yi)).$$

We observe that u is the imaginary part of a holomorphic function, so it is harmonic. Since H maps U conformally to the upper half-plane, it follows that u maps U to $(0, \infty)$, and that u is unbounded, as H is unbounded. It remains to show that u maps ∂U to 0 in limit save for $(1, 0)$. Towards this end, we split the case into two parts.

If $x_0 + iy_0 = e^{i\theta}$ for some $0 < \theta_0 < \pi$, then

$$\frac{e^{i\theta}+1}{1-e^{i\theta}} = \frac{(1+\cos(\theta)+i\sin(\theta))(1-\cos(\theta)+i\sin(\theta))}{2-2\cos(\theta)}$$
$$= \frac{1}{2-2\cos(\theta)}\left(1-\cos^2(\theta)-\sin^2(\theta)+2i\sin(\theta)\right)$$
$$= \frac{2i\sin(\theta)}{2-2\cos(\theta)}$$
$$= i\cot(\theta/2).$$

Squaring, we then get

$$\left(\frac{e^{i\theta}+1}{1-e^{i\theta}}\right)^2 = -\cot^2(\theta/2)$$
$$\in \mathbb{R},$$

so that $u(x_0, y_0) = 0$ whenever $x_0 + iy_0 = e^{i\theta}$ for some $0 < \theta_0 < \pi$.

Meanwhile, if $x_0 + iy_0 = x_0$ with $x_0 \neq 1$, then

$$H(x_0 + iy_0) = \left(\frac{x_0+1}{1-x_0}\right)^2$$
$$\in \mathbb{R},$$

so that $u(x_0, y_0) = 0$ yet again.

(c) We let $v \equiv u$, where u is defined as above. Since u is unbounded, it follows that for each $N \geqslant 1$, there is $(x_N, y_N) \in U$ such that $u(x_N, y_n) \geqslant N$. Inductively, this allows us to construct a sequence

$(x_n, y_n) \subseteq U$ such that $u(x_n, y_n) \geqslant n$, meaning that $\lim_{n \to \infty} u(x_n, y_n) = \infty$.

Since u is harmonic, it is subharmonic, so by a previously established theorem, since $\sup(u) = \infty$, it follows that $((x_n, y_n))_n \to \partial U$. Yet, this sequence cannot converge to any element of $\partial U \setminus \{(1, 0)\}$, as otherwise, we would have $u(x_n, y_n) \to 0$, which would contradict the fact that u is continuous as it is harmonic. Therefore, we have $((x_n, y_n))_n \to (1, 0)$.

Problem (Problem 3)**:** Let

$$U = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1\}.$$

Let $f \colon \overline{U} \to \mathbb{C}$ be a continuous bounded function for which $f|_U$ is holomorphic. Suppose there exist constants $M_0 \geqslant 0$ and $M_1 \geqslant 0$ such that

$$\sup_{\mathrm{Re}(z)=0} |f(z)| \leqslant M_0$$

$$\sup_{\mathrm{Re}(z)=1} |f(z)| \leqslant M_1.$$

Show that for all $r \in [0, 1]$,

$$\sup_{\mathrm{Re}(z)=r} |f(z)| \leqslant M_0^{1-r} M_1^r.$$

Solution: Let $\varepsilon > 0$ be fixed. Define

$$f_\varepsilon(z) = f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)}.$$

We will show that $\sup_{z \in \overline{U}} |f_\varepsilon(z)| \leqslant 1$. Towards this end, if $\mathrm{Re}(z) = 0$, we have $z = bi$ for some $b \in \mathbb{R}$; since $M_0, M_1 \in \mathbb{R}_{\geqslant 0}$, we then get

$$\left| f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)} \right| = \left| f(z) M_0^{bi-1} M_1^{-bi} e^{-\varepsilon(b^2+1)} \right|$$

$$= \left| f(z) M_0^{-1} e^{-\varepsilon(b^2+1)} \right|$$

$$\leqslant \left| f(z) M_0^{-1} \right|$$

$$\leqslant 1.$$

Similarly, if $\mathrm{Re}(z) = 1$, then we have $z = 1 + bi$ for some $b \in \mathbb{R}$, and since $M_0, M_1 \in \mathbb{R}_{\geqslant 0}$, we have

$$\left| f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)} \right| = \left| f(z) M_0^{bi} M_1^{-bi-1} e^{\varepsilon(-2bi-b^2)} \right|$$

$$= \left| f(z) M_1^{-1} e^{-b^2 \varepsilon} \right|$$

$$\leqslant \left| f(z) M_1^{-1} \right|$$

$$\leqslant 1.$$

Since $|f_\varepsilon(z)| \leqslant 1$ holds on both $\mathrm{Re}(z) = 0$ and $\mathrm{Re}(z) = 1$, it follows by the maximum modulus principle that we must have $|f_\varepsilon(z)| \leqslant 1$ on the interior. In particular, this means that

$$\sup_{z \in \overline{U}} |f_\varepsilon(z)| \leqslant 1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\left| f(z) M_0^{z-1} M_1^{-z} \right| \leqslant 1$$

for all $z \in \overline{U}$, so that

$$|f(z)| \leqslant \left| M_0^{1-z} \right| \left| M_1^z \right|$$

$$= M_0^{1-\mathrm{Re}(z)} M_1^{\mathrm{Re}(z)}.$$

In particular, this means that for $\mathrm{Re}(z) = r$, we have

$$|f(z)| \leqslant M_0^{1-r} M_1^r,$$

meaning this holds for the supremum over all z with $\mathrm{Re}(z) = r$, yielding

$$\sup_{\mathrm{Re}(z)=r} |f(z)| \leqslant M_0^{1-r} M_1^r.$$

Problem: Let $U \subseteq \mathbb{C}$ be a region, $f \colon U \to \mathbb{C} \setminus \{0\}$ a holomorphic function. We say that f has a logarithm in U if there exists a holomorphic function $g \colon U \to \mathbb{C}$ satisfying $f(z) = e^{g(z)}$ for all $z \in U$.

(a) Show that f has a logarithm in U if and only if for every piecewise C^1 cycle Γ in U, we have

$$\oint_\Gamma \frac{f'}{f}\, dz = 0.$$

(b) If γ is a piecewise C^1 loop in U, show that

$$\oint_\gamma \frac{f'}{f}\, dz = 2\pi i n(f \circ \gamma; 0).$$

(c) Let $J \subseteq \mathbb{N}$ be a countably infinite set. Show that f has a logarithm in U if and only if it has kth roots for every $k \in J$, so that for each $k \in J$, there is a holomorphic function $h \colon U \to \mathbb{C} \setminus \{0\}$ satisfying $h(z)^k = f(z)$ for all $z \in U$.

Solution:

(a) Let $f(z)$ have the logarithm $e^{g(z)}$. If γ is any piecewise C^1 loop in U, we wish to show that

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = 0,$$

meaning that for any piecewise C^1 cycle $\Gamma = \gamma_1 + \cdots + \gamma_n$, we would have this sum add up to zero.

Toward this end, we observe that

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = \oint_\gamma g'(z)\, dz,$$

meaning that this integral is equal to zero over γ.

Now, if

$$\int_\Gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for all piecewise C^1 cycles Γ in U, then it certainly follows for a cycles consisting of a single C^1 loop, γ, where $\mathrm{im}(\gamma) \subseteq U$. Therefore, by an established proposition, there is some holomorphic function $g \colon U \to \mathbb{C}$ such that

$$g'(z) = \frac{f'(z)}{f(z)}.$$

Following a similar argument to the specific case of U being a simply connected region and f nonvanishing, we see that if we define

$$h(z) = f(z)e^{-g(z)},$$

that

$$h'(z) = f'(z)e^{-g(z)} - f(z)g'(z)e^{-g(z)}$$

$$= f'(z)e^{-g(z)} - f(z)\frac{f'(z)}{f(z)}e^{-g(z)}$$

$$= 0,$$

so that $h(z)$ is some constant $k \in \mathbb{C}$, following from the fundamental theorem of calculus as, if $z_0 \in U$, $r > 0$, and $z \in U(z_0, r)$, we get

$$h(z) - h(z_0) = \int_0^1 h'((1-t)z_0 + tz)(z - z_0)\, dt$$

$$= 0,$$

meaning h is constant on $U(z_0, r)$, hence constant by the identity theorem.

Since f is never zero, it follows that $k \neq 0$, so that there is some $k^* \in \mathbb{C}$ with $e^{k^*} = k$, and

$$f(z) = e^{k^* g(z)}.$$

Since $k^* g(z)$ is also holomorphic, we thus find that f has a logarithm.

(b) Let $\gamma \colon [a, b] \to U$ be a piecewise C^1 loop. We observe then that

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = \int_a^b \frac{\gamma'(t)f'(\gamma(t))}{f(\gamma(t))}\, dt$$

$$= \int_a^b \frac{1}{(f \circ \gamma)(t)}(f \circ \gamma)'(t)\, dt$$

$$= \oint_{f \circ \gamma} \frac{1}{w}\, dw$$

$$= 2\pi i n(f \circ \gamma; 0).$$

(c) If f has a logarithm in U, then we have a holomorphic $g(z)$ such that $f(z) = e^{g(z)}$. Then, if $k \in J$, we have

$$h(z) = e^{\frac{1}{k} g(z)}$$

satisfies $h(z)^k = \left(e^{\frac{1}{k} g(z)}\right)^k = e^{g(z)}$, so that $f(z) = h(z)^k$.

Suppose that for all $k \in J$, there is $h_k(z)$ such that $h_k(z)^k = f(z)$. Now, since J is a countably infinite subset of N, J is necessarily unbounded. Letting $\gamma \colon [a, b] \to U$ be a satisfactory piecewise C^1 loop, we then observe that for arbitrary $k \in J$, we must have

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = k \oint_\gamma \frac{h_k'(z)}{h_k(z)}\, dz.$$

Since the integral on the left is single-valued, and k is arbitrary and unbounded, it follows that both sides are necessarily equal to zero, so that

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for any piecewise C^1 loop. Thus, by part (a), we see that f has a logarithm.

Problem (Problem 5):

(a) Let $U \subseteq \mathbb{C}$ be a region, $f \colon U \to \mathbb{C} \setminus \{0\}$ a holomorphic function, and let g be a logarithm of f. Suppose there exist $z, w \in U$ such that $f(z) = f(w)$. Show that for any piecewise C^1 curve $\gamma \colon [a, b] \to U$ for which $\gamma(a) = z$ and $\gamma(b) = w$, we have $g(z) - g(w) = 2\pi i n(f \circ \gamma; 0)$.

(b) Let $V \subseteq \mathbb{C}$ be a simply connected region. Fix a finite collection of points $a_1, \ldots, a_k \in V$ and define $U = V \setminus \{a_1, \ldots, a_k\}$. Let $\delta > 0$ be sufficiently small such that $B(a_i, \delta) \cap B(a_j, \delta) = \emptyset$ for $i \neq j$, and let $\gamma_j \colon [0, 2\pi] \to U$ be given by $\gamma_j(\theta) = a_j + \delta e^{i\theta}$.

Show that a holomorphic function $f \colon U \to \mathbb{C} \setminus \{0\}$ admits a logarithm if and only if $n(f \circ \gamma_j; 0) = 0$ for all j.

Solution:

(a) Let $\gamma \colon [a, b] \to U$ be a piecewise C^1 curve with $\gamma(a) = z$ and $\gamma(b) = w$. Writing $f(z) = e^{g(z)}$, we observe that, by the fundamental theorem of calculus,

$$\int_\gamma \frac{f'(z)}{f(z)} \, dz = \int_\gamma g'(z) \, dz$$
$$= \int_a^b g'(\gamma(t))\gamma'(t) \, dt$$
$$= \int_a^b (g \circ \gamma)'(t) \, dt$$
$$= (g \circ \gamma)(b) - (g \circ \gamma)(a)$$
$$= g(w) - g(z).$$

Meanwhile, by direct substitution, and using the fact that $f(z) = f(w)$, we obtain

$$\int_\gamma \frac{f'(z)}{f(z)} \, dz = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t) \, dt$$
$$= \int_a^b \frac{1}{f(\gamma(t))}(f \circ \gamma)'(t) \, dt$$
$$= \oint_{f \circ \gamma} \frac{1}{\zeta} \, d\zeta$$
$$= 2\pi i n(f \circ \gamma; 0).$$

(b) We use the condition in Problem 4 that f has a logarithm if and only if for every piecewise C^1 cycle in U, we have

$$\oint_\gamma \frac{f'(z)}{f(z)} \, dz = 0.$$

Since each of the γ_j are piecewise C^1, it follows that if f has a logarithm in U, then

$$\oint_{\gamma_j} \frac{f'(z)}{f(z)} \, dz = 2\pi i n(f \circ \gamma_j; 0)$$
$$= 0.$$

meaning that $n(f \circ \gamma_j; 0) = 0$ for each γ_j.

Now, suppose f satisfies $n(f \circ \gamma_j; 0) = 0$ for each j. Let γ be any piecewise C^1 loop in U, and define a piecewise C^1 cycle

$$\Gamma = \gamma - \sum_{j=1}^{k} n(\gamma; a_j)\gamma_j.$$

We claim that Γ is homologous to zero in U. Toward this end, we observe that $\mathbb{C} \setminus U$ consists of two disjoint sets: $\mathbb{C} \setminus V$ and $\{a_1, \ldots, a_k\}$. We observe that Γ is a cycle in V as Γ is a cycle in U, so $n(\Gamma; z) = 0$ for any $z \in \mathbb{C} \setminus V$ as V is simply connected. Meanwhile, we have

$$n(\Gamma; a_i) = \oint_\Gamma \frac{1}{z - a_i}\, dz$$

$$= \oint_\gamma \frac{1}{z - a_i}\, dz - \sum_{j=1}^{k} n(\gamma; a_j) \oint_{\gamma_j} \frac{1}{z - a_i}\, dz$$

$$= n(\gamma; a_i) - n(\gamma; a_i)$$

$$= 0,$$

where we use the fact that $n(\gamma_j; a_j) = 1$ by a previously established result.

Thus, we see that Γ is homologous to zero in U. By the generalization of Cauchy's Integral Theorem, it follows that, since $f(z) \neq 0$ for all $z \in U$, we have

$$0 = \oint_\Gamma \frac{f'(z)}{f(z)}\, dz$$

$$= \oint_\gamma \frac{f'(z)}{f(z)}\, dz - \sum_{j=1}^{k} n(\gamma; a_j) \oint_{\gamma_j} \frac{f'(z)}{f(z)}\, dz$$

$$= \oint_\gamma \frac{f'(z)}{f(z)}\, dz - \sum_{j=1}^{k} n(\gamma; a_j) n(f \circ \gamma_j; 0)$$

$$= \oint_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Since γ is an arbitrary piecewise C^1 loop in U, this holds for all such formal linear combinations of piecewise C^1 loops, hence for all piecewise C^1 cycles in U. Thus, f admits a logarithm in U.