Problem 1

Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a family of subsets satisfying

(i) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;

(ii) If $\{A_k\}_{k\geq 1}$ is a countable family of pairwise disjoint members of $\mathcal{A}$, then $\bigsqcup_{k\geq 1} A_k \in \mathcal{A}$.

Prove that $\mathcal{A}$ is a σ-algebra on Ω.

Proof: We will show that if $\bigsqcup_{k\geq 1} A_k \in \mathcal{A}$ for $\{A_k\}_{k\geq 1}$ pairwise disjoint, then $\bigcup_{n\geq 1} B_n \in \mathcal{A}$ for $\{B_n\}_{n\geq 1}$ any family of elements of $\mathcal{A}$. Without loss of generality, let $\bigsqcup A_k \supseteq \bigcup B_n$.

Define $B_i^* = \left(\bigcup_{n\geq 1} B_n\right) \cap A_i$. Then, the B_i^* are pairwise disjoint, meaning $\bigsqcup_{n\geq 1} B_n^* \in \mathcal{A}$. Notice that

$$\bigsqcup_{i\geq 1} B_i^* = \bigcup_{n\geq 1} B_n.$$

Thus, $\bigcup B_n \in \mathcal{A}$.

Problem 2

Consider the family $\mathcal{E} : \{(-\infty, b) \mid b \in \mathbb{R}\}$. Show that $\sigma(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$.

Proof: Consider the family $\mathcal{E}' := \{[a, b) \mid a, b \in \mathbb{R}\}$. We have established that $\sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

We see that for any element of $\mathcal{E}$, $(-\infty, b) = \bigcup_{n=1}^{\infty}[a-n, b)$, meaning $\mathcal{E} \in \sigma(\mathcal{E}')$, so $\sigma\left(\mathcal{E}\right) \subseteq \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

Additionally, $[a, b) = (-\infty, b) \setminus (-\infty, a)$, meaning $\mathcal{E}' \in \sigma(\mathcal{E})$, so $\sigma(\mathcal{E}') \subseteq \sigma(\mathcal{E})$, so $\sigma\left(\mathcal{E}\right) = \sigma(\mathcal{E}') = \mathcal{B}_{\mathbb{R}}$.

Problem 3

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. We define the product σ-algebra on $\Omega \times \Lambda$ as

$$\mathcal{M} \otimes \mathcal{N} := \sigma\left(\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}\right).$$

Prove that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

Proof: For $a < b$ and $c < d$, it is the case that $(a, b) \times (c, d) \subseteq \mathbb{R}^2$ is open, meaning

$$\begin{aligned}\sigma\left(\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{R}\}\right) &= \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}\\ &\subseteq \mathcal{B}_{\mathbb{R}^2}.\end{aligned}$$

Letting $U \in \mathcal{B}_{\mathbb{R}^2}$, it is the case that $U = \bigcup_{j=1}^{\infty} U(x_j, r_j)$. For each $U(x_j, r_j)$, take $I_j = (x_{jx} - r_j, x_{jx} + r_j) \times (x_{jy} - r_j, x_{jy} + r_j)$, so $U \subseteq \bigcup_{j=1}^{\infty} I_j$. Thus, $U \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, so $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Problem 4

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. A map $f : \Omega \to \Lambda$ is $\mathcal{M}$-$\mathcal{N}$-measurable if $E \in \mathcal{N} \Rightarrow f^{-1}(E) \in \mathcal{M}$.

Let $(\Omega, \mathcal{M})$ be a measurable space and suppose $E \in \mathcal{M}$. Show that $\mathcal{M}_E = \{M \cap E \mid M \in \mathcal{M}\}$ is a σ-algebra on E and the inclusion map $\iota : E \to \Omega$ is $\mathcal{M}_E$-$\mathcal{M}$-measurable.

Proof: Let $M \in \mathcal{M}$. Then, $\iota^{-1}(M) = E \cap M \in \mathcal{M}_E$. Thus, f is $\mathcal{M}_E$-$\mathcal{M}$-measurable.

Problem 5

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces. Suppose $\mathcal{N}$ is generated as a σ-algebra by a family of subsets $\mathcal{E} \subseteq \mathcal{P}(\Lambda)$. Prove that a map $f : \Omega \to \Lambda$ is $\mathcal{M}$-$\mathcal{N}$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$. Conclude that a continuous function $f : X \to Y$ between metric spaces is $\mathcal{B}_X$-$\mathcal{B}_Y$-measurable.

Proof: Let $\mathcal{N}$ be generated by $\mathcal{E}$. Then, for any $E_1, E_2 \in \mathcal{E}$, it is the case that $E_1^c \in \mathcal{N}$ or $E_1 \cup E_2 \in \mathcal{N}$.

Let f be measurable. Then, since $\mathcal{E} \subseteq \mathcal{N}$, and for any $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$, it is the case that for any $E \in \mathcal{E}$, $f^{-1}(E) \in \mathcal{M}$.

Let f be a function such that for any $E \in \mathcal{E}$, $f^{-1}(E) \in \mathcal{M}$. So, $f^{-1}(E^c) = \left(f^{-1}(E)\right)^c \in \mathcal{M}$, and $f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2) \in \mathcal{M}$. Therefore, for any $E \in \mathcal{N}$, it must be the case that $f^{-1}(E) \in \mathcal{M}$.

Since the preimage of any element of the topology on Y is the topology on X if f is continuous, it is the case that such a continuous function is $\mathcal{B}_X$-$\mathcal{B}_Y$-measurable.

Problem 6

Suppose $(\Omega, \mathcal{M})$ is a measurable space and $f : \Omega \to \Lambda$ is a map. Show that $\mathcal{N} := \{E \subseteq \Lambda \mid f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra on Λ and f is $\mathcal{M}$-$\mathcal{N}$-measurable. $\mathcal{N}$ is called the σ-algebra produced by f.

Proof: Let $E \in \mathcal{N}$. Then, $\left(f^{-1}(E)\right)^c \in \mathcal{M}$ (since f is $\mathcal{M}$-$\mathcal{N}$-measurable), meaning $f^{-1}(E^c) \in \mathcal{M}$, so $E^c \in \mathcal{N}$.

Let $E_1, E_2 \in \mathcal{N}$. Then, $f^{-1}(E_1) \cup f^{-1}(E_2) \in \mathcal{M}$, so$f^{-1}(E_1 \cup E_2) \in \mathcal{M}$, so $E_1 \cup E_2 \in \mathcal{N}$.

Since $\mathcal{M}$ is a σ-algebra, the above holds for countable unions, meaning $\mathcal{N}$ is a σ-algebra.

Problem 7

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and suppose $\{E_k\}_{k \geq 1}$ is a decreasing sequence of measurable sets with $\mu(E_1) < \infty$. Show that

$$\begin{aligned} \mu\left(\bigcap_{k \geq 1} E_k\right) &= \lim_{k \to \infty} \mu(E_k) \\ &= \inf_{k \geq 1} \mu(E_k). \end{aligned}$$

Proof: We see that for n, $\bigcap_{k=1}^n E_k = E_n$. Therefore, $\mu\left(\bigcap_{k=1}^n E_k\right) = \mu(E_n)$, meaning

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{\infty} E_k\right) &= \lim_{n \to \infty} \mu\left(\bigcap_{k=1}^{n} E_k\right) \\ &= \lim_{n \to \infty} \mu(E_n). \end{aligned}$$

Problem 8

Let $(\Omega, \mathcal{M})$ and $(\Lambda, \mathcal{N})$ be measurable spaces and suppose $f : \Omega \to \Lambda$ is measurable. If μ is a measure on $\mathcal{M}$, show that

$$f_*\mu : \mathcal{N} \to [0, \infty]; \quad f_*\mu(E) := \mu(f^{-1}(E))$$

defines a measure on $(\Lambda, \mathcal{N})$. This is called the pushforward measure.

Proof: Clearly, $f_*\mu(\emptyset) = 0$. Let $E_1, E_2 \in \mathcal{N}$ be disjoint and nonempty. Note that $E_1 \sqcup E_2 \in \mathcal{N}$. Thus,

$$\begin{aligned} f_*\mu(E_1 \sqcup E_2) &= \mu\left(f^{-1}(E_1 \sqcup E_2)\right) \\ &= \mu\left(f^{-1}(E_1) \sqcup f^{-1}(E_2)\right) \\ &= \mu(f^{-1}(E_1)) + \mu(f^{-1}(E_2)) \\ &= f_*\mu(E_1) + f_*\mu(E_2), \end{aligned}$$

meaning $f_*\mu$ is a measure on $(\Lambda, \mathcal{N})$.

Problem 9

A group G is paradoxical if there are pairwise disjoint subsets of G; $E_1, \dots, E_n, F_1, \dots, F_m$ and group elements $t_1, \dots, t_n, s_1, \dots, s_m$ such that

$$\begin{aligned} G &= \bigsqcup_{j=1}^{n} t_j E_j \\ &= \bigsqcup_{k=1}^{m} s_k F_k. \end{aligned}$$

A mean on a group G is a finitely additive probability measure $\nu : \mathcal{P}(G) \to [0,1]$ that is translation invariant; that is, $\nu(tE) = \nu(E)$ for all $E \subseteq G$ and $t \in G$. A group is said to be amenable if it admits a mean.

Show that a paradoxical group is nonamenable.

Proof: Let G be paradoxical. Suppose toward contradiction that there existed such a ν. Then, $\nu(G)$, and

$$\begin{aligned} \nu(G) &= \nu\left(\bigsqcup_{j=1}^{n} t_j E_j\right) \\ &= \sum_{j=1}^{n} \nu(t_j E_j) \\ &= \sum_{j=1}^{n} \nu(E_j). \end{aligned}$$

We know that $G \cup s_1 F_1 = G$, meaning $\nu(G) = \nu(G \cup s_1 F_1)$. However,

$$\begin{aligned} \nu(G \cup s_1 F_1) &= \nu\left(\bigsqcup_{j=1}^{n} t_j E_j \sqcup s_1 F_1\right) \\ &= \sum_{j=1}^{n} \nu(t_j E_j) + \nu(s_1 F_1) \\ &= \nu(G) + \nu(s_1 F_1) \\ &= \nu(G) + \nu(F_1) \\ &> \nu(G). \end{aligned}$$

Problem 10

Let Δ be a totally disconnected compact metric space (for example, the Cantor set). Suppose $\varphi : C(\Delta) \to \mathbb{R}$ is a state — φ is linear, continuous, positive, and $\varphi(\mathbb{1}_\Delta) = 1$.

(i) Show that $\mathcal{C} := \{E \mid E \subseteq \Delta\}$ is an algebra of subsets on Δ.

(ii) Show that

$$\mu_0 : \mathcal{C} \to [0,1]; \quad \mu_0(E) = \varphi\left(\mathbb{1}_E\right)$$

is a well-defined finitely additive measure.

(iii) If $\{E_k\}_{k\geq 1}$ is a countable family of members of $\mathcal{C}$ such that $\bigsqcup_{k\geq 1} E_k \in \mathcal{C}$, show that

$$\mu_0\left(\bigsqcup_{k\geq 1} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

Proof:

(i) If $E \in \mathcal{C}$, then $E \subseteq \Delta$, so $E^c \subseteq \Delta$, and for $E_1, E_2 \in \mathcal{C}$, $E_1 \cup E_2 \in \Delta$.

(ii) Let $E, F \in \mathcal{C}$ with $E \cap F = \emptyset$. Then,

$$\begin{aligned}
\mu_0(E \sqcup F) &= \varphi\left(\mathbb{1}_{E\sqcup F}\right) \\
&= \varphi\left(\mathbb{1}_E + \mathbb{1}_F\right) \\
&= \varphi\left(\mathbb{1}_E\right) + \varphi\left(\mathbb{1}_F\right) \\
&= \mu_0(E) + \mu_0(F).
\end{aligned}$$

(iii) Let $\{E_k\}_{k\geq 1}$ be a countable family of members of $\mathcal{C}$ with $\displaystyle\bigsqcup_{k\geq 1} E_k \in \mathcal{C}$. We see that for any $n \in N$, $\displaystyle\bigsqcup_{k=1}^{n} E_k \in \mathcal{C}$, since $\mathcal{C}$ is an algebra of subsets.

Therefore,

$$\mu_0\left(\bigsqcup_{k=1}^{n}\right) = \sum_{k=1}^{n} \mu_0(E_k),$$

for any $n \in \mathbb{N}$, as μ_0 is finitely additive. Since $\displaystyle\bigsqcup_{k\geq 1} E_k \in \mathcal{C}$, it is then the case that

$$\begin{aligned}
\mu_0\left(\bigsqcup_{k=1}^{\infty}\right) &= \lim_{n\to\infty} \mu_0\left(\bigsqcup_{k=1}^{n}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mu_0(E_k) \\
&= \sum_{k=1}^{\infty} \mu_0(E_k).
\end{aligned}$$