Solution (8.3, Problem 12)**:** Expressing in matrix form, we have

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ 4t \end{pmatrix}.$$

Solving for the fundamental matrix, we have

$$\begin{aligned}\det\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} &= \lambda^2 - 4 + 3 \\ &= \lambda^2 - 1,\end{aligned}$$

so we have eigenvalues of ± 1, with eigenvectors

$$\begin{aligned}\mathbf{v}_1 &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.\end{aligned}$$

Thus, our fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ 3e^t & e^{-t} \end{pmatrix}$$

Using the power of Mathematica, we may find the inverse of Φ to be

$$\Phi^{-1}(t) = \begin{pmatrix} -\frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t & -\frac{1}{2}e^t \end{pmatrix}.$$

Using yet more of the power of Mathematica, we evaluate

$$\Phi^{-1}(t)\begin{pmatrix} 0 \\ 4t \end{pmatrix} = \begin{pmatrix} 2te^{-t} \\ -2te^t \end{pmatrix},$$

with integrals of

$$\begin{aligned}\int \Phi^{-1}(t)\mathbf{F}(t)\,dt &= \begin{pmatrix} -2\left(te^{-t} + e^{-t}\right) \\ -2\left(te^t - e^t\right) \end{pmatrix} \\ \begin{pmatrix} e^t & e^{-t} \\ 3e^t & e^{-t} \end{pmatrix}\begin{pmatrix} -2\left(te^{-t} + e^{-t}\right) \\ -2\left(te^t - e^t\right) \end{pmatrix} &= \begin{pmatrix} -4t \\ -8t-4 \end{pmatrix}\end{aligned}$$

Thus, the general solution to the system is

$$\mathbf{x}(t) = \begin{pmatrix} e^t & e^{-t} \\ 3e^t & e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -4t \\ -8t-4 \end{pmatrix}.$$

Solution (8.3, Problem 16)**:** We have

$$\det\begin{pmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2,$$

so $\lambda = 1, 2$. These have respective eigenvectors of

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, we have the solution matrix of

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}.$$

Using the power of Mathematica, we find the inverse and calculate

$$\Phi^{-1}(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$
$$\Phi^{-1}(t)\begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ 2e^{-5t} - 2e^{-2t} \end{pmatrix}$$
$$\begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\begin{pmatrix} -e^{-4t} + 2e^{-t} \\ 2e^{-5t} - 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}.$$

Thus, we have the solution of

$$\mathbf{x}(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}.$$

Solution (8.3, Problem 20)**:** We have

$$\det\begin{pmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1,$$

giving eigenvalues of $1, 1$. We need to use generalized eigenvectors here, meaning that we have

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\mathbf{w} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, we have the solution matrix of

$$\Phi(t) = \begin{pmatrix} e^t & \frac{1}{4}te^t + e^t \\ -e^t & \frac{1}{4}te^t - e^t \end{pmatrix}.$$

This solution matrix has the inverse of

$$\Phi^{-1}(t) = \begin{pmatrix} \frac{1}{2t}\left(-4e^{-t} + t\right) & -\frac{1}{2t}\left(4e^{-t} + t\right) \\ \frac{2e^{-t}}{t} & \frac{2e^{-t}}{t} \end{pmatrix}.$$

Multiplying by $\mathbf{F}$, we have

$$\Phi^{-1}(t)\mathbf{F}(t) = \begin{pmatrix} -\frac{4e^{-t}}{t} \\ \frac{4e^{-t}}{t} \end{pmatrix}.$$

This integral cannot be solved analytically. Thus, we have the solution of

$$\mathbf{x} = \begin{pmatrix} e^t & \frac{1}{4}te^t + e^t \\ -e^t & \frac{1}{4}te^t - e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & \frac{1}{4}te^t + e^t \\ -e^t & \frac{1}{4}te^t - e^t \end{pmatrix}\int \begin{pmatrix} -\frac{4e^{-t}}{t} \\ \frac{4e^{-t}}{t} \end{pmatrix} dt.$$

Solution (8.3, Problem 28)**:** We have

$$\det\begin{pmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{pmatrix} = \lambda^2 + 1,$$

so there are eigenvalues of $\lambda = \pm i$, which admit eigenvectors of

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm i\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, obtain solution vectors and fundamental matrix of

$$\mathbf{x}_1 = \begin{pmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$$

$$\mathbf{x}_2 = \begin{pmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} \cos(t) - \sin(t) & \cos(t) + \sin(t) \\ \cos(t) & \sin(t) \end{pmatrix}.$$

Using the power of Mathematica, we find the inverse

$$\Phi^{-1}(t) = \begin{pmatrix} -\sin(t) & \cos(t) + \sin(t) \\ \cos(t) & -\cos(t) + \sin(t) \end{pmatrix}$$

$$\Phi^{-1}(t)\begin{pmatrix} \tan(t) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(t) + \sin(t) - \sin(t)\tan(t) \\ -\cos(t) + 2\sin(t) \end{pmatrix}$$

$$\Phi(t)\int \Phi^{-1}(t)\mathbf{F}(t)\,dt = \begin{pmatrix} -3\cos^2(t) + \cos(t)(\ln(\sec(t) + \tan(t)) - \sin(t)) - \sin(t)(\ln(\sec(t) + \tan(t)) + 2\sin(t)) \\ -1 + \cos(t)(\ln(\sec(t) + \tan(t)) - \sin(t)) \end{pmatrix}.$$

Thus, our general solution is

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} \cos(t) - \sin(t) & \cos(t) + \sin(t) \\ \cos(t) & \sin(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} -3\cos^2(t) + \cos(t)(\ln(\sec(t) + \tan(t)) - \sin(t)) - \sin(t)(\ln(\sec(t) + \tan(t)) + 2\sin(t)) \\ -1 + \cos(t)(\ln(\sec(t) + \tan(t)) - \sin(t)) \end{pmatrix}. \end{aligned}$$

Solution (8.3, Problem 30)**:** Finding the eigenvectors and eigenvalues, we have $\lambda = 2, 2, 1$, and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives the fundamental solution matrix of

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{2t} & 0 & e^{t} \end{pmatrix}.$$

We calculate the inhomogeneous term using Mathematica to find

$$\Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)\,dt = \begin{pmatrix} \frac{1}{4}\left(-3 - 2t + 8e^{t}(1+t)\right) \\ (1+t)\left(-1 + 2e^{t}\right) \\ \frac{1}{4}\left(-3 + \left(-2 + 8e^{t}\right)t\right) \end{pmatrix},$$

giving the general solution of

$$\mathbf{x}(t) = \begin{pmatrix} e^{2t} & e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{2t} & 0 & e^{t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} \frac{1}{4}\left(-3 - 2t + 8e^{t}(1+t)\right) \\ (1+t)\left(-1 + 2e^{t}\right) \\ \frac{1}{4}\left(-3 + \left(-2 + 8e^{t}\right)t\right) \end{pmatrix}$$

Solution (8.3, Problem 32)**:** From inspection, the matrix has an eigenvalue of 1 with eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and since the matrix has linearly dependent rows, 0 is an eigenvalue for the homogeneous system, with eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, we have a fundamental solution matrix of

$$\Phi(t) = \begin{pmatrix} e^{t} & 1 \\ 0 & 1 \end{pmatrix}.$$

Finding Ψ_A, we have

$$\Phi(t)\Phi^{-1}(0) = \begin{pmatrix} e^{t} & 1 - e^{t} \\ 0 & 1 \end{pmatrix}.$$

We see that

$$\Psi_A(t)\int \Psi_A(-t)\mathbf{F}(t)\,dt = \begin{pmatrix}\ln(t)\\ \ln(t)\end{pmatrix}.$$

Now, finding the initial conditions, we have

$$\begin{pmatrix}2\\ -1\end{pmatrix} = \begin{pmatrix}e & 1-e\\ 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$
$$\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}\frac{3}{e}-1\\ -1\end{pmatrix}.$$

Thus, our solution is

$$\mathbf{x}(t) = \begin{pmatrix}e^t & 1-e^t\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{e}-1\\ -1\end{pmatrix} + \begin{pmatrix}\ln(t)\\ \ln(t)\end{pmatrix}.$$

Solution (8.3, Problem 35)**:**

(a) We find the eigenvalues $\lambda = 4, 3, 1, 0$, with respective eigenvectors of

$$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\\ 0\\ 0\end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix}3\\ 1\\ 2\\ 1\end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix}2\\ 1\\ 0\\ 0\end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix}-6\\ -4\\ 1\\ 2\end{pmatrix}.$$

(b) These give the fundamental solution matrix of

$$\Phi(t) = \begin{pmatrix}-e^{4t} & 3e^{3t} & 2e^t & -6\\ e^{4t} & e^{3t} & e^t & -4\\ 0 & 2e^{3t} & 0 & 1\\ 0 & e^{3t} & 0 & 2\end{pmatrix}$$

$$\Phi^{-1}(t) = \begin{pmatrix} -\frac{1}{3}e^{-4t} & \frac{2}{3}e^{-4t} & 0 & \frac{1}{3}e^{-4t} \\ 0 & 0 & \frac{2}{3}e^{-3t} & -\frac{1}{3}e^{-3t} \\ \frac{1}{3}e^{-t} & \frac{1}{3}e^{-t} & -2e^{-t} & \frac{8}{3}e^{-2t} \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

(c)

$$\Phi^{-1}\mathbf{F} = \begin{pmatrix} \frac{1}{3}e^{-5t}\left(-e^{2t}t + e^{t} + 2\right) \\ \frac{1}{3}e^{-3t}\left(2e^{2t} - 1\right) \\ \frac{1}{3}\left(t + e^{-2t} + 8e^{-t} - 6e^{t}\right) \\ \frac{1}{3}\left(2 - e^{2t}\right) \end{pmatrix}$$

$$\int \Phi^{-1}\mathbf{F}\,dt = \begin{pmatrix} \frac{1}{540}e^{-5t}\left(20e^{2t}(3t+1) - 45e^{t} - 72\right) \\ \frac{1}{9}e^{-3t}\left(1 - 6e^{2t}\right) \\ \frac{1}{6}\left(t^2 - e^{-2t} - 16e^{-t} - 12e^{t}\right) \\ \frac{1}{6}\left(4t - e^{2t}\right) \end{pmatrix}$$

$$\Phi\int \Phi^{-1}\mathbf{F}\,dt = \begin{pmatrix} \frac{1}{27}e^{t}\left(9t^2 - 3t - 1\right) - 4t - \frac{e^{-t}}{5} - 5e^{2t} - \frac{59}{12} \\ \frac{1}{54}e^{t}\left(9t^2 + 6t + 2\right) - \frac{8t}{3} - \frac{3e^{-t}}{10} - 2e^{2t} - \frac{95}{36} \\ \frac{2t}{3} - \frac{3e^{2t}}{2} + \frac{2}{9} \\ \frac{4t}{3} - e^{2t} + \frac{1}{9} \end{pmatrix}$$

Thus, we have

$$\mathbf{x} = \begin{pmatrix} -e^{4t} & 3e^{3t} & 2e^{t} & -6 \\ e^{4t} & e^{3t} & e^{t} & -4 \\ 0 & 2e^{3t} & 0 & 1 \\ 0 & e^{3t} & 0 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} + \begin{pmatrix} \frac{1}{27}e^{t}\left(9t^2 - 3t - 1\right) - 4t - \frac{e^{-t}}{5} - 5e^{2t} - \frac{59}{12} \\ \frac{1}{54}e^{t}\left(9t^2 + 6t + 2\right) - \frac{8t}{3} - \frac{3e^{-t}}{10} - 2e^{2t} - \frac{95}{36} \\ \frac{2t}{3} - \frac{3e^{2t}}{2} + \frac{2}{9} \\ \frac{4t}{3} - e^{2t} + \frac{1}{9} \end{pmatrix}$$

(d)

$$\mathbf{x}(t) = c_1e^{4t}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} + c_3e^{t}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} -6 \\ -4 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{27}e^{t}\left(9t^2 - 3t - 1\right) - 4t - \frac{e^{-t}}{5} - 5e^{2t} - \frac{59}{12} \\ \frac{1}{54}e^{t}\left(9t^2 + 6t + 2\right) - \frac{8t}{3} - \frac{3e^{-t}}{10} - 2e^{2t} - \frac{95}{36} \\ \frac{2t}{3} - \frac{3e^{2t}}{2} + \frac{2}{9} \\ \frac{4t}{3} - e^{2t} + \frac{1}{9} \end{pmatrix}$$

Solution (8.4, Problem 2)**:** We see that $A^2 = I$, so that

$$\begin{aligned} e^{At} &= \left(I + \frac{1}{2!}It^2 + \frac{1}{4!}It^4 + \cdots\right) + \left(At + \frac{1}{3!}At^3 + \frac{1}{5!}At^5 + \cdots\right) \\ &= \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \end{aligned}$$

with

$$e^{-At} = \begin{pmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix}.$$

Solution (8.4, Problem 4)**:** Calculating, we have

$$\begin{aligned} A^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \\ A^3 &= \mathbf{0}. \end{aligned}$$

Thus, we have the matrix of

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 0 & 0 \\ 5t + \frac{3}{2}t^2 & t & 0 \end{pmatrix}.$$

Solution (8.4, Problem 6)**:** Now that we know

$$e^{At} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix},$$

we have the solution of

$$\mathbf{x} = \begin{pmatrix} c_1 \cosh(t) + c_2 \sinh(t) \\ c_1 \sinh(t) + c_2 \cosh(t) \end{pmatrix}.$$

Solution (8.4, Problem 8)**:** We have the general solution of

$$\mathbf{x}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 0 & 0 \\ 5t + \frac{3}{2}t^2 & t & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Solution (8.4, Problem 26)**:** We calculate

$$\begin{aligned} A^2 &= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ A^3 &= \mathbf{0}, \end{aligned}$$

so A is nilpotent. Thus, we may calculate

$$e^{At} = \begin{pmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{pmatrix},$$

and find the solution

$$\mathbf{x} = \begin{pmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Solution (8.4, Problem 27)**:**

(a) Using Mathematica, we find the matrix exponential of

$$e^{At} = \begin{pmatrix} \frac{2}{5}e^{t} + \frac{3}{5}e^{6t} & -\frac{2}{5}e^{t} + \frac{2}{5}e^{6t} \\ -\frac{3}{5}e^{t} + \frac{3}{5}e^{6t} & \frac{3}{5}e^{t} + \frac{2}{5}e^{6t} \end{pmatrix}.$$

Thus, we find the solution of

$$\mathbf{x} = \begin{pmatrix} \frac{2}{5}e^{t} + \frac{3}{5}e^{6t} & -\frac{2}{5}e^{t} + \frac{2}{5}e^{6t} \\ -\frac{3}{5}e^{t} + \frac{3}{5}e^{6t} & \frac{3}{5}e^{t} + \frac{2}{5}e^{6t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Using some tedious algebra, we are able to convert this solution to the form

$$\mathbf{x} = k_1 e^{t}\begin{pmatrix} 2 \\ -3 \end{pmatrix} + k_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

by setting $k_1 \coloneqq \frac{1}{5}(c_1 - c_2)$ and $k_2 \coloneqq \frac{1}{5}(c_1 + c_2)$.

(b) We find the matrix exponential of

$$e^{At} = \begin{pmatrix} e^{-2t}(\cos(t) - \sin(t)) & -e^{-2t}\sin(t) \\ 2e^{-2t}\sin(t) & e^{-2t}(\sin(t) + \cos(t)) \end{pmatrix}$$

and general solution of

$$\mathbf{x} = \begin{pmatrix} e^{-2t}(\cos(t) - \sin(t)) & -e^{-2t}\sin(t) \\ 2e^{-2t}\sin(t) & e^{-2t}(\sin(t) + \cos(t)) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$