Problem 1.1.13

Let G be the graph whose vertex set is the set of k-tuples with coordinates $\{0, 1\}$, with x adjacent to y if x and y differ by exactly one position. Determine whether G is bipartite.

G is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

Problem 1.1.26

Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least $2k$ vertices. Determine all such graphs with $2k$ vertices.

Suppose G is a graph with girth 4 with every vertex of degree k. Let $v_i \in V(G)$. Then, there must be k vertices which v_i is adjacent to. However, none of these vertices can be adjacent to themselves or G would have girth 3. Thus, we can form a bipartition such that v_i is in a set of at least k vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to k vertices in a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least $2k$ vertices.

The graphs with exactly $2k$ vertices are the $K_{n,n}$ complete bipartite graphs.

Problem 1.1.27

Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with $k^2 + 1$ vertices.

Let G be a simple graph with girth 5. Suppose that every vertex of G has degree k. Let $u \in V(G)$. Then, u has k adjacent vertices, each of which is not adjacent to each other (or else the girth of G would be 3). Let this set be N. The elements of N cannot have any other common neighbors aside from u, or else the girth of G would be 4, meaning each has $k - 1$ distinct neighbors. Therefore, the total number of vertices in our graph includes u, the elements of N that are the k distinct neighbors of u, and the $k(k - 1)$ distinct vertices for each vertex in N. Therefore, our total is $1 + k + k(k - 1) = k^2 + 1$.

If there were any vertex with degree greater than k, then there would be additional vertices beyond the k^2+1 vertices necessary for a k-regular graph.

For $k = 2$, we have the graph C_5 for an example of a graph with $k^2 + 1$ vertices, and for $k = 3$ we have the Petersen graph.

Problem 1.1.30

Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry of A^2 and MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

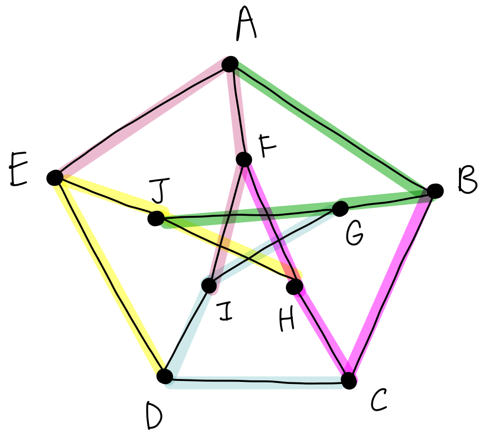
Let A be the adjacency matrix for a simple graph G. In A, every vertex's corresponding row and column are identical, meaning that the entry $A^2_{i,i}$ will be equal to $r_i c_i$ for row i and column i corresponding to v_i. Thus, $r_i c_i$ is equal to $|c_i|^2$, which is equal to the sum of the elements of c_i, which is equal to the degree of v_i.

Let M be the incidence matrix for a simple graph G. In MM^T, the diagonal element $MM^T_{i,i}$ will be equal to $r_i r_i^T$, where r_i represents the edge incidence row of v_i. This is equal to $\left|r_i^T\right|^2$, which is equal to the sum of the elements of r_i, which is equal to the number of edges incident on v_i, which is equal to the degree of v_i.

The entry in position (i, j) in both A^2 and MM^T shows whether vertices v_i and v_j are adjacent to each other.

Problem 1.1.34

Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4.



A – B – G – J
B – C – H – F
C – D – I – G
D – E – J – H
E – A – F – I