Notationally, we will use 1 to denote the identity operator.

Contents

Preliminaries

We start by recalling some of the topologies on $B(H)$.

Definition: Let H be a Hilbert space, with $B(H)$ denoting the space of bounded operators on H.

The *strong operator topology*, or SOT, is the locally convex topology generated by the seminorms

$$\{\, \|Tv\| \mid T \in B(H), v \in H \}$$

The *weak operator topology*, or WOT, is the locally convex topology generated by the seminorms

$$\{ |\langle Tv, w\rangle| \mid T \in B(H), v, w \in H \}$$

Theorem: Let $\phi\colon B(H) \to \mathbb{C}$ be a linear functional. The following are equivalent:

(i) there are $\xi_k, \eta_k \in H$ such that $\phi(T) = \sum_{k=1}^{n} \langle T\xi_k, \eta_k\rangle$;

(ii) ϕ is WOT-continuous;

(iii) ϕ is SOT-continuous.

Proof. The directions (i) implies (ii) implies (iii) are pretty much by definition. To see (iii) implies (i), we have $\xi_1, \dots, \xi_n$ such that, for all $T \in B(H)$, $\max\|T\xi_k\| \leq 1$ implies $\phi(T) \leq 1$. Then, we have

$$|\phi(T)| \leq \left(\sum_{k=1}^{n} \|T\xi_k\|^2\right)^{1/2}.$$

Let

$$\begin{aligned} H^{(n)} &:= \bigoplus_{k=1}^{n} H \\ T^{(n)} &:= \operatorname{diag}(T, \dots, T) \in B\bigl(H^{(n)}\bigr), \end{aligned}$$

and let $\xi = (\xi_1, \dots, \xi_n) \in H^{(n)}$. We see then that the linear functional $\psi\colon H \rightarrow \mathbb{C}$ given by

$$\psi\Bigl(T^{(n)}\xi\Bigr) = \phi(T)$$

defines a linear functional on the closed subspace of K spanned by the vectors

$$\Bigl\{T^{(n)}\xi \Bigm| T \in B(H)\Bigr\},$$

and has

$$\Bigl|\psi\Bigl(T^{(n)}\xi\Bigr)\Bigr| \leq \Bigl\|T^{(n)}\xi\Bigr\|,$$

so by the Riesz Representation Theorem for Hilbert Spaces, it follows there is $\eta = (\eta_1, \dots, \eta_n)$ such that

$$\begin{aligned} \phi(x) &= \Bigl\langle T^{(n)}\xi, \eta\Bigr\rangle \\ &= \sum_{k=1}^{n} \langle T\xi_k, \eta_k\rangle. \end{aligned}$$

□

Corollary: Every SOT-closed convex subset of $B(H)$ is WOT-closed.

Proof. The closed convex subsets of a locally convex topological vector space are determined by the continuous linear functionals, as follows from an application of the Hahn–Banach separation. □

Theorem: The unit ball of $B(H)$ is WOT-compact.

Proof. Let $\overline{\mathbb{D}}$ denote the closed unit disk of $\mathbb{C}$, and consider the set

$$K = \prod_{x,y \in B_H} \overline{\mathbb{D}}.$$

This space is compact by Tychonoff's theorem. Define the embedding $\phi\colon B_{B(H)} \rightarrow K$ given by

$$\phi(T) = (\langle Tx, y\rangle)_{x,y}.$$

By Cauchy–Schwarz, we have

$$\begin{aligned} |\langle Tx, y\rangle| &\leq \|T\|_{\text{op}}\|x\|\|y\| \\ &\leq 1, \end{aligned}$$

so ϕ is well-defined. We see that ϕ is WOT-continuous by definition and injective, so we only need to show that $\text{im}(\phi)$ is closed. Let $(T_i)_i \subseteq B_{B(H)}$ be a net with

$$\lim_{i\in I}(\langle T_i x, y\rangle)_{x,y} = (z_{x,y})_{x,y}.$$

We have that $(z_{x,y})_{x,y} \in K$ since K is compact, and since the product topology is the topology of pointwise convergence, we have

$$\lim_{i\in I}\langle T_i x, y\rangle = z_{x,y}$$

defines a sesquilinear form $F(x, y)$. This means we may find $T \in B_{B(H)}$ such that $F(x, y) = \langle Tx, y\rangle$, and so $(T_i)_i \to T$ in WOT. □

Structure of von Neumann Algebras

There are a variety of ways we will understand the structure of von Neumann algebras. We start with discussing the most basic characterization of von Neumann algebras (emerging from the Double Commutant Theorem), then go into more depth into the structure of abelian von Neumann algebras, and end with a discussion of a characterization of a von Neumann algebra as a dual space.

Double Commutant Theorem

Definition: Let $M \subseteq B(H)$. We define the *commutant* to be

$$M' := \{S \in B(H) \mid TS = ST \text{ for all } T \in M\}.$$

The double commutant of M is denoted M'', and has $M \subseteq M''$.

We see that M' is a WOT-closed subalgebra, and if M' is self-adjoint, then M' is a C^*-algebra. Additionally, if $M_1 \subseteq M_2$, then $M_1' \supseteq M_1'$.

Theorem (Double Commutant Theorem)**:** Let M be a unital C^*-subalgebra of $B(H)$. The following are equivalent:

(i) $M = M''$;

(ii) M is WOT-closed;

(iii) M is SOT-closed.

Proof. The implications (i) implies (ii) follows from the discussion above, and (ii) if and only (iii) follow from the definitions (as subalgebras are convex). We focus on showing that (iii) implies (i).

For a fixed $\xi \in H$, let P be the projection onto the closure of the subspace $\{T\xi \mid T \in M\}$. We see that $P\xi = \xi$, since $1 \in M$. Additionally, $PTP = TP$ for each $T \in M$, so $P \in M'$. Letting $V \in M''$, we have that $PV = VP$, so $V\xi \in PH$. In particular, for each $\varepsilon > 0$, there is $S \in M$ such that $\|(V - S)\xi\| < \varepsilon$.

Let $\xi_1, \dots, \xi_n \in H$, and set $\xi = (\xi_1, \dots, \xi_n)$ in $H^{(n)}$. Letting $\rho \colon B(H) \hookrightarrow B\big(H^{(n)}\big)$ be the embedding defined by

$$T \mapsto T^{(n)},$$

we see that

$$\rho(M)' = \{S \in B(K) \mid S_{ij} \in M'\}.$$

Therefore, we have that $\rho(V) \in \rho(M)''$, meaning that using the same process as above in the amplified algebra, we have

$$\begin{aligned}\sum_{k=1}^{n} \|(V - T)\xi_k\|^2 &= \|(\rho(V) - \rho(T))\xi\|^2 \\ &< \varepsilon^2,\end{aligned}$$

meaning that we can approximate V in SOT from M, so $V \in M$. □

Definition: A *von Neumann algebra* is a unital SOT-closed (or WOT-closed) C^*-subalgebra of $B(H)$.

The double commutant theorem says that $M = M''$ is a characterization of a von Neumann algebra.

Observe that if $T \in M$ is a normal operator in a von Neumann algebra M, then if E denotes the spectral measure for T, and $S \in M'$, then $TS = ST$, so by Fuglede's Theorem, $T^*S = ST^*$, meaning that $Sf(T) = f(T)S$ for all $f \in B_\infty(\sigma(T))$. In particular, this means that $E(S) \in M'' = M$. Since the closed linear span of the characteristic functions χ_S is equal to $B_\infty(\sigma(T))$, it follows that, if M is a von Neumann algebra, then M is the (norm)-closed linear span of all of its projections.

To see this another way, let $a \in M_{\text{s.a.}}$, and consider a partition $-\|a\| = t_0 < t_1 < \cdots < t_n = \|a\|$, where $t_{j+1} - t_j < \varepsilon$ for each $j = 0, \dots, n-1$, and define projections

$$P_i = \chi_{[t_{j-1}, t_j)}$$

for $j = 1, \dots, n-1$, and $P_n = \chi_{[t_{n-1}, t_n]}$. Then, we necessarily have

$$\left\| a - \sum_{j=1}^{n} t_j P_j \right\|_{\text{op}} < \varepsilon,$$

so every self-adjoint operator is in the norm-closed linear span of the projections of M. Since every element of M can be written as a decomposition of self-adjoint operators, it follows that M is the norm-closed linear span of its projections.

Proposition: Let M be a von Neumann algebra, and let $A \in M$.

(a) If A is normal, and ϕ is a bounded Borel function on $\sigma(A)$, then $\phi(A) \in M$.

(b) The operator A is the linear combination of four unitaries in M.

(c) If E and F are the projections onto $\overline{\operatorname{im}(A)}$ and $\ker(A)$ respectively, then $E, F \in M$.

(d) If $A = W|A|$ is the polar decomposition for A, then W and $|A|$ are in M.

Abelian von Neumann Algebras

Definition: Two subsets $M_1 \subseteq B(H_1)$ and $M_2 \subseteq B(H_2)$ are said to be *spatially isomorphic* if there is an isomorphism $U\colon H_1 \to H_2$ such that $UM_1U^{-1} = M_2$.

Definition: A vector e_0 is said to be separating for $S \subseteq B(H)$ if the only operator $T \in S$ for which $Te_0 = 0$ is the 0 operator.

Proposition: If S is a subspace of $B(H)$, then every cyclic vector for S is separating for S'. If A is a C^*-algebra of operators, then a vector is cyclic for A if and only if it is separating for A'.

Proof. If e_0 is cyclic for S, and $T \in S'$ with $Te_0 = 0$, then for every $L \in S$, we have $TLe_0 = LTe_0 = 0$, meaning that $T[Se_0] = 0$. Since e_0 is cyclic, this means $T = 0$.

If A is a unital C^*-subalgebra of $B(H)$, with e_0 separating for A', we let P be the projection onto $N = [Ae_0]^{\perp}$. Since N reduces A, it follows that $P \in A'$, but since $e_0 \perp N$, we have $Pe_0 = 0$. Since e_0 is separating for A', it follows that $P = 0$, so e_0 is cyclic for A. $\square$

Corollary: If A is an abelian algebra of operators, every cyclic vector for A is separating.

Theorem: If H is separable, and A is an unital, abelian C^*-subalgebra of $B(H)$, then the following are equivalent:

(a) A is a maximal abelian von Neumann algebra;

(b) $A = A'$;

(c) A is SOT-closed with a cyclic vector;

(d) there is a compact metric space X, a regular Borel measure μ supported on X, and an isomorphism $U\colon L_2(X,\mu) \to H$ such that $UA_\mu U^{-1} = A$, where A_μ is the representation of $L_\infty(X,\mu)$ as the space of multiplication operators acting on $L_2(X,\mu)$.

Proof. If A is a maximal abelian von Neumann algebra, then $A = A''$ and $A \subseteq A'$, or that $A' \supseteq A'' = A$, so $A = A'$. Similarly, if $A = A'$, then $A = A' = A''$, so that A is a maximal abelian von Neumann algebra. Thus, (a) and (b) are equivalent.

Now, assume $A = A'$, it follows that $A = A''$, so that A is SOT-closed and contains the identity. Let $\{e_n\}_{n\geq 1}$ be a maximal sequence of unit vectors with $[Ae_n] \perp [Ae_m]$ whenever

$n \leq m$. Then, by maximality, we have

$$H = \bigoplus_{n \geq 1} [Ae_n].$$

Let $P_n = [Ae_n]$, and set $e_0 = \sum_{n=1}^{\infty} 2^{-n} e_n$. Since P_n reduces A, $P_n \in A'$, so from (b), $P_n \in A$, meaning that $e_n = 2^n P e_0 \in [Ae_0]$, and thus $[Ae_n] \subseteq [Ae_0]$ for each n. Thus, e_0 is cyclic for A. This shows (b) implies (c).

Now, since H is separable, B_A is WOT-compact, meaning there is a countable WOT-dense subset. Let A_1 be the C^*-algebra generated by this WOT-dense subset; then, A_1 is a separable C^*-algebra that is WOT-dense in A. Let X be the character space of A_1; since A_1 is separable, X is metrizable, and let $\rho \colon C(X) \to A_1 \subseteq A \subseteq B(H)$ be the inverse Gelfand transform. Then, ρ is a representation of $C(X)$, so there is a spectral measure E on X such that

$$\rho(f) = \int f \, dE.$$

For every bounded Borel function, we then have

$$\begin{aligned} \widetilde{\rho}(\phi) &= \int \phi \, dE \\ &\in A_1'' \\ &= A'' \\ &= A \end{aligned}$$

by the Double Commutant Theorem.

Letting e_0 be a cyclic vector for A, set $\mu(B) = \langle E(B)e_0, e_0 \rangle$ for any Borel $B \subseteq X$. We have

$$\langle \widetilde{\rho}(\phi) e_0, e_0 \rangle = \int \phi \, d\mu$$

for every $\phi \in B_\infty(X)$, and

$$\begin{aligned} \|\widetilde{\rho}(\phi) e_0\|^2 &= \langle \widetilde{\rho}(\phi)^* \widetilde{\rho}(\phi) e_0, e_0 \rangle \\ &= \int |\phi|^2 \, d\mu. \end{aligned}$$

Therefore, $B_\infty(X)$, considered as a dense subspace of $L_2(X, \mu)$, admits the well-defined isometry $U \colon B_\infty(X) \to H$ given by $U\phi = \widetilde{\rho}(\phi) e_0$. We may extend U to be an isometry on all of $L_2(X, \mu)$.

Now, if $\phi \in B_\infty(X)$ and $\psi \in L_\infty(X, \mu)$, then

$$\begin{aligned} UM_\psi \phi &= U(\psi\phi) \\ &= \widetilde{\rho}(\psi\phi) e_0 \end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\rho}(\psi)\widetilde{\rho}(\phi)e_0 \\
&= \widetilde{\rho}(\psi)U\phi.
\end{aligned}
$$

That is, $UA_\mu U^{-1} = \widetilde{\rho}(L_\infty(X,\mu))$. Yet, since A_μ is WOT-closed in $B(L_2(X,\mu))$, we have $\widetilde{\rho}(L_\infty(X,\mu))$ is WOT-closed in $B(H)$. Furthermore, since $\widetilde{\rho}(L_\infty(X,\mu)) \supseteq \rho(C(X)) = A_1$, we have $UA_\mu U^{-1} = A$. This shows (c) implies (d).

Finally, to show (d) implies (b), we show that $A_\mu = A_\mu'$. Let $T \in A_\mu'$. Since X is compact and μ is regular, it follows that $\mu(X) < \infty$. Then, $1 \in L_2(X,\mu)$, so we may set $L_2(X,\mu) \ni \phi = T(1)$. For any $\psi \in L_\infty(X,\mu)$, then $\psi \in L_2(X,\mu)$, with $T\psi = TM_\psi 1 = M_\psi T(1) = \psi\phi$, with

$$
\begin{aligned}
\|\phi\psi\| &= \|T\psi\| \\
&\leq \|T\|_{\mathrm{op}}\|\psi\|.
\end{aligned}
$$

Set $\Delta_n = \{x \in X \mid |\phi(x)| \geq n\}$. Setting $\psi = \chi_{\Delta_n}$, we have

$$
\begin{aligned}
\|T\|_{\mathrm{op}}^2\mu(\Delta_n) &= \|T\|_{\mathrm{op}}^2\|\psi\|^2 \\
&\geq \|\phi\psi\|^2 \\
&= \int_{\Delta_n} |\phi|^2\,d\mu \\
&\geq n^2\mu(\Delta_n).
\end{aligned}
$$

Yet, since T is bounded, for sufficiently large n it follows that $\mu(\Delta_n) = 0$, meaning $\phi \in L_\infty(\mu)$, and since $T = M_\phi$ on $L_\infty(\mu)$, we have $T = M_\phi$. □

Trace-Class Operators and the σ-Weak Operator Topology

In order to discuss a further structural characterization of von Neumann algebras, we start by discussing trace-class operators and characterizing $B(H)$ as a dual space.

An operator $T \in B(H)$ is called *trace-class* if there exists an orthonormal basis $(e_i)_{i\in I}$ such that the quantity

$$
\begin{aligned}
\operatorname{tr}(|T|) &\coloneqq \sum_{i\in I}\langle |T|e_i, e_i\rangle \\
&< \infty.
\end{aligned}
$$

Similarly, an operator $T \in B(H)$ is called *Hilbert–Schmidt* if the quantity $\operatorname{tr}(T^*T) < \infty$. The set of all trace-class operators is denoted $L_1(B(H))$, while the set of Hilbert–Schmidt operators is denoted $L_2(B(H))$. We list some essential properties of trace-class operators. The proofs can be found in [Con00, Ch. 3, §18].

Proposition (Properties of trace-class and Hilbert–Schmidt operators)**:** Let $T_1 \in L_1(B(H))$ and $T_2 \in L_2(B(H))$. The following properties hold.

(i) The quantities

$$\|T_1\|_1 \coloneqq \operatorname{tr}(|T_1|)$$
$$\|T_2\|_2 \coloneqq \operatorname{tr}(T_2^* T_2)$$

define norms for T_1 and T_2 respectively.

(ii) For any $A \in B(H)$, we have $\operatorname{tr}(AT_1) = \operatorname{tr}(T_1 A)$, and $|\operatorname{tr}(AT_1)| \leq \|A\|_{\text{op}} \|T_1\|_1$.

(iii) Both $L_1(B(H))$ and $L_2(B(H))$ are ideals in $B(H)$ satisfying

$$\|AT_{1,2}\|_{1,2} \leq \|A\|_{\text{op}} \|T_{1,2}\|_{1,2}.$$

Furthermore, both $L_1(B(H))$ and $L_2(B(H))$ are subsets of $K(H)$.

(iv) The operator T_1 is the product of two Hilbert–Schmidt operators, and any operator S is trace-class if and only if it is the product of two Hilbert–Schmidt operators.

(v) The pairing $\langle A, B\rangle = \operatorname{tr}(B^* A)$ defines an inner product on $L_2(B(H))$, and $L_2(B(H))$ is a Hilbert space with respect to this inner product.

The main thing we are interested in is understanding the duality properties of trace-class operators. We observe that the following is an analogue of the duality $(c_0)^* = \ell_1$.

Theorem: For any $T \in L_1(B(H))$, define the linear functional $\phi_T \colon K(H) \to \mathbb{C}$ by $\phi_T(A) = \operatorname{tr}(TA) = \operatorname{tr}(AT)$. Then, the map $T \mapsto \phi_T$ is an isometric isomorphism between $L_1(B(H))$ and $(K(H))^*$.

Proof. We observe that

$$\sup\left\{ |\operatorname{tr}(AC)| \;\middle|\; C \in K(H), \|C\|_{\text{op}} \leq 1 \right\} \leq \|A\|_1,$$

so that Φ_A is a bounded linear functional on $K(H)$ satisfying $\|\Phi_A\| \leq \|A\|_1$. Defining $\rho \colon L_1(B(H)) \to K(H)$ by $\rho(A) = \Phi_A$, we have that ρ is a linear map with $\|\rho(A)\| \leq \|A\|_1$ for all $A \in L_1(B(H))$.

Now, we will show that ρ is surjective with $\|\rho(A)\| \geq \|A\|_1$ for any $A \in L_1(B(H))$. Define a sesquilinear form for $\Phi \in K(H)^*$ by $[g,h] = \Phi(\theta_{g,h})$, where $\theta_{g,h}$ is the rank-one bounded operator given by

$$\theta_{g,h}(k) = \langle k, h\rangle g.$$

We have that $|[g,h]| \leq \|\Phi\|\|g\|\|h\|$ for all g and h, so $[\cdot,\cdot]$ is bounded, so there is $A \in B(H)$ such that $[g,h] = \langle Ag, h\rangle$. We will show that $A \in L_1(B(H))$ with $\Phi = \Phi_A$.

Let $C \in F(H)$ be given by

$$C = \sum_{k=1}^{n} \theta_{g_k,h_k},$$

Then,

$$\Phi(C) = \Phi\left(\sum_{k=1}^{n} \theta_{g_k \otimes h_k}\right)$$

$$\begin{aligned}
&= \sum_{k=1}^{n} \langle Ag_k, h_k\rangle \\
&= \sum_{k=1}^{n} \operatorname{tr}(A\theta_{g_k,h_k}) \\
&= \operatorname{tr}(AC).
\end{aligned}$$

If we can show that $A \in L_1(B(H))$, then both Φ and Φ_A are bounded linear functionals on $K(H)$ that agree on $F(H)$.

For this, let $A = W|A|$ be the polar decomposition of A, and let $(e_i)_{i\in I}$ be an orthonormal basis. For any finite subset $F \subseteq I$, we have

$$C_F \coloneqq \left(\sum_{i\in F} \theta_{e_i,e_i}\right) W^*$$

is a contraction in $F(H)$ with

$$\begin{aligned}
\|\Phi\| &\geq |\Phi(C_F)| \\
&= \left|\Phi\left(\sum_{i\in F} e_i \otimes We_i\right)\right| \\
&= \sum_{i\in F} |\langle Ae_i, We_i\rangle| \\
&= \sum_{i\in F} \langle |A|e_i, e_i\rangle.
\end{aligned}$$

Letting F grow arbitrarily gives $\|\Phi\| \geq \|A\|_1$, so $A \in L_1(B(H))$, and $\Phi = \Phi_A$. Yet, this means $\|\Phi_A\| \geq \|A\|_1$, so ρ is an isometry. □

Similarly, just as $(\ell_1)^* = \ell_\infty$, the following holds.

Theorem: Let $\Psi\colon L_1(B(H)) \to \mathbb{C}$ be given by

$$\Phi_B(A) = \operatorname{tr}(AB).$$

Then, the map $B \mapsto \Phi_B$ defines an isometric isomorphism of $B(H)$ onto $\ell_1(B(H))^*$.

Proof. That $\|\Psi_B\| \leq \|B\|$ follows from the fact that $|\operatorname{tr}(AB)| \leq \|A\|_1 \|B\|_{\text{op}}$. Defining $\rho(B) = \Psi_B$, we have ρ is linear. If $\varepsilon > 0$, we use the Riesz lemma to find a unit vector g such that $\|Bg\| > \|B\|_{\text{op}} - \varepsilon$. Find a unit vector h such that $\langle Bg, h\rangle = \|Bg\|$. Then, letting $C = \theta_{g,h}$, we have $C \in L_1(B(H))$ with $\|C\|_1 = 1$, with

$$\begin{aligned}
\|\Psi_B\| &\geq |\operatorname{tr}(BC)| \\
&= \langle Bg, h\rangle \\
&= \|Bg\| \\
&> \|B\|_{\text{op}} - \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we have $\|\Psi_B\| = \|B\|_{\text{op}}$, and ρ is an isometry.

Now, let $\Psi \in L_1(B(H))^*$. Then, there is an operator $B \in B(H)$ such that $\langle Bg, h\rangle = \Psi(\theta_{g,h})$ for all $g, h \in H$. Then, it follows that $\Psi(T) = \Psi_B(T)$ for every finite-rank operator T, so since $F(H)$ is dense in $L_1(B(H))$, we have that both Ψ and Ψ_B are bounded linear functionals with $\Psi = \Psi_B$. □

Therefore, we can talk about the weak* topology on $B(H)$ induced by $L_1(B(H))$. We discuss an alternative form of convergence known as σ-WOT and σ-SOT convergence.

Definition: Let H be a Hilbert space. The σ-strong operator topology on $B(H)$ is the locally convex topology defined by the family of seminorms

$$p_\xi(T) = \|(T \otimes 1)\xi\|$$

for all $\xi \in H \otimes \ell_2$. The norm is defined by

$$\|(T \otimes 1)\xi\| = \left(\sum_{k=1}^{\infty} \|T\xi_k\|^2\right)^{1/2}.$$

The σ-weak operator topology on $B(H)$ is the locally convex topology defined by the family of seminorms

$$q_{\xi,\eta} = |\langle (T \otimes 1)\xi, \eta\rangle|$$

for all $\xi, \eta \in H \otimes \ell_2$. The inner product is defined by

$$|\langle (T \otimes 1)\xi, \eta\rangle| = \left|\sum_{k=1}^{\infty} \langle T\xi_k, \eta_k\rangle\right|.$$

We note that σ-WOT and WOT are equal on bounded subsets of $B(H)$. Furthermore, the following holds.

Proposition: The weak* topology on $B(H)$ induced by $L_1(B(H))$ and the σ-WOT are identical.

Proof. First, we observe that for any sequences $\xi, \eta \in H \otimes \ell_2$, we have that the operator

$$T = \sum_{k=1}^{\infty} \theta_{\xi_k, \eta_k} \tag{$*$}$$

is trace-class. Since multiplication by an element of $B(H)$ is continuous with respect to the trace-class norm, it follows that, whenever $(S_i)_i \to S$ is a w^*-convergent net, then

$$\begin{aligned}\sum_{k=1}^{\infty} \langle S_i\xi_k, \eta_k\rangle &= \operatorname{tr}\left(\sum_{k=1}^{\infty} \theta_{S_i\xi_k, \eta_k}\right)\\ &= \operatorname{tr}(S_iT)\\ &\to \operatorname{tr}(ST)\end{aligned}$$

$$= \sum_{k=1}^{\infty} \langle S\xi_k, \eta_k \rangle.$$

Therefore, we have that each seminorm tends to 0 for all $\xi, \eta \in H \otimes \ell_2$, meaning $(S_i)_i \to S$ in σ-WOT.

Now, if $(S_i)_i \to S$ in σ-WOT, then since every trace-class operator is of the form in $(*)$, it follows that $\mathrm{tr}(S_i T) \to \mathrm{tr}(ST)$ for every $T \in L_1(B(H))$, so $(S_i)_i \to S$ is w^*-convergent. □

Preduals and Normal Linear Maps

The existence of a predual for $B(H)$ extends to all von Neumann algebras.

Theorem: Suppose V is a Banach space. Then, there exists a vector space W such that $V \cong W^*$ if and only if there exists a locally convex topology τ on V such that B_V is τ-compact. Furthermore, if we set

$$W = \{\varphi \in V^* \mid \varphi|_{B_V} \text{ is } \tau\text{-continuous}\},$$

then $\Phi \colon V \to W^*$ defined by $v \mapsto \hat{v}$ is an isometric isomorphism.

Proof. The forward direction follows from the Banach–Alaoglu theorem, so we only need to show the reverse direction.

To start, we observe that $W^* \subseteq V^{**}$, so in particular, if $v \neq 0$, we must have that $\|\hat{v}|_{W^*}\| \leq \|\hat{v}\| = \|v\|$, where the latter equivalence follows from the fact that the canonical embedding of V into its double dual is isometric. Now, by Hahn–Banach separation, there is some τ-continuous linear functional $\varphi \colon V \to \mathbb{C}$ such that $\varphi(v) \neq 0$. Now, since the unit ball of V is compact with respect to τ, it follows that

$$\begin{aligned} C &= \sup_{v \in B_V} |\varphi(v)| \\ &< \infty, \end{aligned}$$

so that Φ is injective into W^*.

Now, if $\psi \in W^*$, then by the Hahn–Banach extension, there is some norm-preserving extension $\overline{\psi} \in V^{**}$. By Hahn–Banach separation, there are $v_i \in V$ such that $\|v_i\| \leq \|\psi\|$ and the w^*-convergent net $\hat{v}_i \to \overline{\psi}$. Since the unit ball of V is compact, it follows that there is a subnet $v_{i_j} \to v$. In particular, we have $\Phi(v) = \psi$, and $\|v\| \leq \|\psi\|$, so that Φ is surjective with Φ^{-1} contractive. □

If M is a von Neumann algebra, then we get the predual

$$M_* = \{\varphi \in M^* \mid \varphi|_{B_M} \text{ is WOT-continuous}\}.$$

A theorem of Sakai gives that this is, in fact, a characterization of von Neumann algebras.

Theorem (Sakai)**:** Let A be a C^*-algebra. Then, A is a von Neumann algebra if and only if there is a Banach space V such that $A \cong V^*$ as Banach spaces.

Now, we discuss normal maps between von Neumann algebras, and a characterization.

Theorem (Krein–Smulian)**:** Let V be a Banach space, and let $C \subseteq V^*$ be a convex subset. Then, C is w^*-closed if and only if the intersection of C with any ball of any radius in V^* is w^*-closed.

Definition: Let M, N be von Neumann algebras. Then, a map $T\colon M \to N$ is called *normal* if it is w^*-continuous.

Theorem: Consider the following conditions for a map $T\colon M \to N$ between von Neumann algebras:

(i) T is normal;

(ii) $T|_{B_M}$ is WOT-WOT continuous;

(iii) $T|_{B_M}$ is SOT-SOT continuous.

Then, (i) and (ii) are equivalent, and (iii) implies (i). Furthermore, if $N = \mathbb{C}$ or T is a $*$-homomorphism, then (ii) implies (iii).

Proof. We start by showing (i) implies (ii). Let T be normal. Then, using the principle of uniform boundedness, it follows that T is norm-continuous, with $C = \|T\| < \infty$. Since the weak* topology on B_M is equal to the WOT on B_M, it follows that this holds on CB_M, meaning that T is WOT-WOT continuous when restricted to B_M.

Now, we show (ii) implies (i). It suffices to show $T^\dagger(N_*) \subseteq M_*$, meaning that if $\varphi \in N_*$, then we will show that $\varphi \circ T \in M_*$; for this, it suffices to show that $\ker(\varphi \circ T)$ is w^*-closed, so we only need to show that $\ker(\varphi \circ T) \cap B_M$ is w^*-closed by the Krein–Smulian theorem. Yet, since the weak* topology and WOT coincide on B_M, it follows that T is normal. □

Kaplansky Density Theorem and Pedersen's Up-Down Theorem

We will now discuss two very useful theorems.

Kaplansky's Density Theorem

Lemma: Let $(T_i)_{i\in I}, (S_i)_{i \in I} \subseteq B(H)$ be nets with $(T_i)_i \to T$, $(S_i)_i \to S$ in SOT. If $\sup_{i\in I}\|T_i\| < \infty$, then $(T_iS_i)_i \to TS$ in SOT.

Proof. Set $R = \sup_i\|T_i\|$. Then, for any $\xi \in H$,

$$\begin{aligned}
\|TS\xi - T_iS_i\xi\| &\leq \|(T-T_i)S\xi\| + \|T_i(S-S_i)\xi\| \\
&\leq \|(T-T_i)\xi\| + R\|(S-S_i)\xi\| \\
&\to 0.
\end{aligned}$$

□

Proposition: Let $f \in C(\mathbb{C})$. Then, the map $T \mapsto f(T)$ on normal operators in $B(H)$ is SOT-continuous on bounded subsets of $B(H)$.

Proof. Let $(T_i)_i$ be a uniformly bounded net of operators converging to T in SOT, with $R = \sup_i\|T_i\|$. By Stone–Weierstrass, we are able to approximate f uniformly $B(0, R)$ by a sequence of polynomials $(p_n)_n \subseteq \mathbb{C}[z, \overline{z}]$. Since multiplication is SOT-continuous on bounded subsets, it follows that $(p_n(T_i, T_i^*))_i \to p_n(T_i, T_i^*)$ in SOT.

Fix $\xi \in H$, $\varepsilon > 0$, and set N to be such that

$$\sup_{z \in B(0,R)} |f(z) - p_N(z, \overline{z})| < \frac{\varepsilon}{3\|\xi\|},$$

and i_0 to be such that for all $i \geq i_0$,

$$\|(p_N(T_i, T_i^*) - p_N(T, T^*))\xi\| < \varepsilon/3.$$

Then,

$$\begin{aligned} \|(f(T) - f(T_i))\xi\| &\leq \|(f(T) - p_N(T, T^*))\|\|\xi\| + \|(p_N(T, T^*) - p_N(T_i, T_i^*))\xi\| + \|(p_N(T_i, T_i^*) - f(T_i))\xi\| \\ &< \varepsilon. \end{aligned}$$

□

Now, we observe that if $T \in B(H)_{\text{s.a.}}$, then $\sigma(T) \subseteq \mathbb{R}$, meaning that $T + z1$ is invertible for any $z \in \mathbb{C}$ with $\text{Im}(z) \neq 0$.

Definition: Let $T \in B(H)_{\text{s.a.}}$. Then, the *Cayley transform* of T is given by the operator

$$c(T) := (T - i1)(T + i1)^{-1}.$$

Observe that the Cayley transform emerges from the continuous functional calculus on $c(z) = \frac{z-i}{z+i}$, meaning that $c(T)$ is a unitary operator, and $(T - i1)(T + i1)^{-1} = (T + i1)^{-1}(T - i1)$. This gives the following.

Proposition: The Cayley Transform is SOT-continuous on $B(H)_{\text{s.a.}}$.

Proof. Let $(T_j)_j \to T$ be a net of self-adjoint operators. By continuous functional calculus, we have $\left\|(T_j + i1)^{-1}\right\| \leq 1$ for all $i \in I$. If $\xi \in H$, we have

$$\begin{aligned} \|c(T)\xi - c(T_j)\xi\| &= \left\|c(T)\xi - (T_j + i1)^{-1}(T_j - i1)\xi\right\| \\ &= \left\|2i(T_j + i1)^{-1}(T - T_j)(T - i1)^{-1}\xi\right\| \\ &\leq 2\left\|(T - T_j)(T - i1)^{-1}\xi\right\|. \end{aligned}$$

Thus, SOT-convergence of $(T_j)_j$ to T implies SOT convergence of the Cayley transform. □

Corollary: If $f \in C_0(\mathbb{R})$, then the map $T \mapsto f(T)$ is SOT-continuous on $B(H)_{\text{s.a.}}$.

Proof. Since f vanishes at infinity, it follows that the function

$$g(z) := \begin{cases} 0 & z = 1 \\ f\left(i\frac{1+z}{1-z}\right) & \text{else} \end{cases}$$

defines a continuous function on S^1. Since any continuous function on $\mathbb{C}$ is SOT-continuous on bounded sets, it follows that g is SOT-continuous on unitary operators, so by composing g with the Cayley transform, it follows that f is SOT-continuous. □

For any subset $S \subseteq B(H)$, we define

$$(S)_1 \coloneqq \{T \in S \mid \|T\| \leq 1\}.$$

Theorem (Kaplansky Density Theorem)**:** Let $A \subseteq B(H)$ be a $*$-subalgebra. Then,

$$\overline{A_{\text{s.a.}}}^{\text{SOT}} = \left(\overline{A}^{\text{SOT}}\right)_{\text{s.a.}}$$

and

$$\overline{(A)_1}^{\text{SOT}} = \left(\overline{A}^{\text{SOT}}\right)_1.$$

Proof. Denote $B = \overline{A}^{\text{SOT}}$. We start by showing that it suffices to show that A is (operator) norm-closed. This follows from the fact that norm convergence implies SOT convergence, meaning that if C denotes the norm closure of A, then $\overline{C}^{\text{SOT}} = \overline{A}^{\text{SOT}}$.

Since SOT convergence implies WOT convergence, it follows that $\overline{A_{\text{s.a.}}}^{\text{SOT}} \subseteq B_{\text{s.a.}}$. If $T \in B_{\text{s.a.}}$, then there exists a net $(T_i)_i \to T$ in SOT. Taking adjoints is WOT-continuous, so $\left(\frac{T_i+T_i^*}{2}\right)_i \subseteq A_{\text{s.a.}}$ converges to T in WOT. Therefore, $T \in \overline{A_{\text{s.a.}}}^{\text{WOT}}$, but since $A_{\text{s.a.}}$ is convex, $\overline{A_{\text{s.a.}}}^{\text{WOT}} = \overline{A_{\text{s.a.}}}^{\text{SOT}}$, meaning $B_{\text{s.a.}} = \overline{A_{\text{s.a.}}}^{\text{SOT}}$.

Now, to show that $\overline{(A)_1}^{\text{SOT}} = (B)_1$, we start by showing that the SOT closure of $(A_{\text{s.a.}})_1$ and $(B_{\text{s.a.}})_1$ coincide. Let $x \in (B_{\text{s.a.}})_1$, and let $(T_i)_i \subseteq A_{\text{s.a.}}$ converge to T in SOT. Let $f \in C_0(\mathbb{R})$ be a function with $\|f\|_u = 1$ and $f(t) = t$ for $|t| \leq 1$. Then, $(f(T_i))_i \subseteq (A_{\text{s.a.}})_1$, converging to $f(T) = T$ in SOT, meaning $(A_{\text{s.a.}})_1$ is SOT dense in $(B_{\text{s.a.}})_1$.

Next, we show that $\overline{\mathbb{M}_2(A)}^{\text{SOT}} = \mathbb{M}_2(B)$. Fixing elements

$$\begin{aligned}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix} &\in \mathbb{M}_2(B)\\ \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} &\in H \oplus H,\end{aligned}$$

we use the fact that $B = \overline{A}^{\text{SOT}}$, so for each i,j, we can find $T_{ij} \in A$ such that $\|(T_{ij} - S_{ij})\xi_j\| < \varepsilon$. In particular, this gives

$$\begin{aligned}\left\|\begin{pmatrix} T_{11} - S_{11} & T_{12} - S_{12} \\ T_{21} - S_{21} & T_{22} - S_{22}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\|^2 &= \sum_{i=1}^{2}\|(T_{i1} - S_{i1})\xi_1 + (T_{i2} - S_{i2})\xi_2\|^2\\ &< 8\varepsilon^2.\end{aligned}$$

Now, since we have $\overline{(A)_1}^{\text{SOT}} \subseteq (B)_1$, we then select $S \in (B)_1$, and consider

$$\begin{aligned}\overline{S} &= \begin{pmatrix} 0 & S \\ S^* & \end{pmatrix}\\ &\in (\mathbb{M}_2(B))_1,\end{aligned}$$

which is self-adjoint. Therefore, by applying the earlier result replacing A and B with $\mathbb{M}_2(A)$ and $\mathbb{M}_2(B)$, we have a net $\left(\overline{S_i}\right)_i \subseteq (\mathbb{M}_2(A)_{\text{s.a.}})_1$ converging to $\overline{S}$ in SOT.

Now, if S_i denotes the $(1,2)$ entry of $\overline{S_i}$, then we observe that $\|S_i\| \leq 1$ and converges to S in SOT upon application to the vector $(0, \xi)$. □

Note that the choice of 1 for the operator norm bound in the KDT is arbitrary; by introducing some factors, we find that for any R, we have $\overline{(A)_R}^{\text{SOT}} = (B)_R$. The primary case will find use for is where $R = \|T\|$ for some $T \in B$.

Corollary: If $M \subseteq B(H)$ is a unital $*$-subalgebra, then the following are equal to each other:

- $\overline{M}^{\sigma\text{-SOT}}$;
- $\overline{M}^{\sigma\text{-WOT}}$;
- $\overline{M}^{\text{SOT}}$;
- $\overline{M}^{\text{WOT}}$;
- M''.

In particular, this means that M is a von Neumann algebra if and only if it is σ-SOT or σ-WOT closed.

Proof. The latter three equivalences follow from the Double Commutant Theorem. Now, since σ-SOT convergence implies σ-WOT convergence, which implies WOT convergence, it follows that all we need to show that $\overline{M}^{\text{SOT}} \subseteq \overline{M}^{\sigma\text{-SOT}}$. For $T \in \overline{M}^{\text{SOT}}$, we may find $(T_i)_i \to T$, where the net is contained in $(M)_{\|T\|}$, convergent in SOT. Since the net is uniformly bounded, and the σ-SOT and SOT coincide on bounded subsets, it follows that $\overline{M}^{\text{SOT}} \subseteq \overline{M}^{\sigma\text{-SOT}}$. □

Pedersen's Up-Down Theorem

If $M \subseteq B(H)$, we let M_σ/M_δ be the set of operators in $B(H)_{\text{s.a.}}$ that can be obtained as SOT limits of monotone increasing/decreasing sequences from M. Similarly, let M^m and M_m be the sets obtained by monotone increasing/decreasing *nets* from M. We have that $M \subseteq M_\sigma \subseteq M^m$, and $M_\delta = -(-M_\sigma)$. Furthermore, if M is SOT-closed, then $M^m = M_m = M$.

We investigate the converse.

Lemma: Let A be a C^*-subalgebra of $B(H)$ with SOT closure M. If p is a projection in M, then for any sequence $(\xi_n)_n$ of unit vectors in H, there is an element y in $\left(\left(A_+^1\right)_\sigma\right)_\delta$ such that $y(1-p)\xi_n = 0$, and $(1-y)p\xi_n = 0$ for all n. Here, A^1 denotes the unit ball of A.

Proof. We will approximate p by vectors of the form $p\xi_n$ and $(1-p)\xi_n$. By the Kaplansky density theorem, we can find $(x_k)_k$ in A_+^1 such that $\|p\xi_n - x_k p\xi_n\| < 1/k$, and $\|x_n(1-p)\xi_i\| < 2^{-n}/n$ for all $i \leq n$.

For any $n < m$, define

$$y_{n,m} = \left(1 + \sum_{k=n}^{m} kx_k\right)^{-1}\left(\sum_{k=n}^{m} kx_k\right).$$

From results in spectral theory, we see that $y_{nm} \in A_+^1$ with $y_{nm} \leq \sum_{k=n}^{m} kx_k$. Therefore, for any $i \leq n$, we have

$$\begin{aligned}\langle y_{nm}(1-p)\xi_i, (1-p)\xi_i\rangle &\leq \sum_{k=n}^{m} 2^{-k}\\ &< 2^{-n+1}.\end{aligned}$$

Now, since $mx_m \leq \sum_{k=n}^{m} kx_k$, we have $(1 + mx_m)^{-1} mx_m \leq y_{nm}$, and thus

$$\begin{aligned}1 - y_{nm} &\leq (1 + mx_m)^{-1}\\ &\leq \frac{1}{1+m}(1 + m(1 - x_m)),\end{aligned}$$

so for $i \leq m$, we have $\langle p\xi_i, p\xi_i\rangle \leq \frac{2}{1+m}$. For fixed n, the sequence $(y_{nm})_m$ is monotone increasing and SOT-convergent to an element $y_n \in \left(A_+^1\right)_\sigma$. Furthermore, since $y_{n+1,m} \leq y_{nm}$, we have that $y_{n+1} \leq y_n$, and so $(y_n)_n$ is monotone decreasing to an element y in $\left(\left(A_+^1\right)_\sigma\right)_\delta$. This gives

$$\begin{aligned}\langle y_n(1-p)\xi_i, (1-p)\xi_i\rangle &\leq 2^{-n+1}\\ \langle (1-y_n)p\xi_i, p\xi_i\rangle &\leq 0,\end{aligned}$$

so since $0 \leq y \leq 1$, we have $y(1-p)\xi_i = 0$ and $(1-y)p\xi_i = 0$ for all i. □

Theorem: Let A be a C^*-subalgebra of $B(H)$ with SOT closure M. If H is separable, then $M_+^1 = \left(\left(A_+^1\right)_\sigma\right)_\delta$ and $M_{\text{s.a.}} = ((A_{\text{s.a.}})_\sigma)_\delta$.

Proof. Let $(\xi_i)_i$ be a dense subsequence of the unit ball of H. Then, we have that each projection in M belongs to $\left(\left(A_+^1\right)_\sigma\right)_\delta$. If A acts non-degenerately on H, we have that 1 is the largest element in M_+^1, meaning that $1 \in \left(A_+^1\right)_\sigma$.

For each $x \in M_+^1$, there is a sequence of spectral projections $(p_k)_k$ such that x is the norm limit of $\sum_{k=1}^{n} 2^{-k} p_k$. This is given by letting $p_1 = (1/2, 1]$, $p_2 = (1/4, 1/2) \cup (3/4, 1]$, etc.

Let $(z_{km})_m$ be a sequence in $\left(A_+^1\right)_\sigma$ decreasing to p_k, and define

$$x_n = \sum_{k=1}^{n} 2^{-k} z_{kn} + 2^{-n}.$$

Since $\left(A_+^1\right)_\sigma$ is convex, it follows that $x_n \in \left(A_+^1\right)_\sigma$, and we have

$$x_n - x_{n+1} = \sum_{k=1}^{n} 2^{-k}(z_{kn} - z_{k,n+1}) + 2^{-n} - \left(2^{-n-1} z_{n+1,n+1} + 2^{-n-1}\right)$$

$$\geq 0,$$

so that $(x_n)_n$ is decreasing. We have

$$x_n - x \leq \sum_{k=1}^{n} 2^{-k}(z_{kn} - p_k) + 2^{-m}$$

for any $n > m$, so $(x_n)_n \to x$ and $x \in \left(\left(A_+^1\right)_\sigma\right)_\delta$.

To show that $M_{\text{s.a.}} = ((A_{\text{s.a.}})_\sigma)_\delta$, note that any $x \in M_{\text{s.a.}}$ can be written as $\alpha y - \beta$ for α, β positive and $y \in M_+^1$. □

Theorem: A C^*-subalgebra M of $B(H)$ is a von Neumann algebra if and only if $(M_{\text{s.a.}})^m = M_{\text{s.a.}}$.

Proof. The forward direction is clear from the fact that SOT closure includes SOT closure includes the monotone nets.

Now, in the reverse direction, suppose $M_{\text{s.a.}}$ is monotone closed. By cutting with a projection, we may assume that $1 \in M$. To show that M is a von Neumann algebra, it suffices to show that any projection in the SOT closure of M belongs to M. Let $\xi \in pH$ and $\eta \in (1-p)H$. Then, there is an element $y \in M_+$ such that $y\xi = \xi$ and $y\eta = 0$. The range projection $p_{\xi\eta}$ of y (emerging from the polar decomposition) belongs to M, and has $p_{\xi\eta}\xi = \xi$ and $p_{\xi\eta}\eta = 0$.

The projections $\inf\{p_{\xi\eta_1}, \dots, p_{\xi\eta_n}\}$ forms a decreasing net in M_+ as $\{\eta_1, \dots, \eta_n\}$ runs through the finite subsets of $(1-p)H$. Therefore, we have the limit projection p_ξ is less than or equal to p.

We have that p is the limit of the increasing net of projections $\sup\{p_{\xi_1}, \dots, p_{\xi_n}\}$ as the collection $\{\xi_1, \dots, \xi_n\}$ runs through finite subsets of pH. Thus, $p \in M$. □

Two Fundamental von Neumann Algebras

We focus now on two special von Neumann algebras.

Group von Neumann Algebras

We start by discussing a little bit of theory of unitary representations.

Let Γ be a discrete group. A unitary representation of Γ is a homomorphism $\pi\colon \Gamma \to U(H)$. The trivial representation of Γ is given by $\pi(g) = 1$. The left regular representation is $\lambda\colon \Gamma \to U(\ell_2(\Gamma))$ given by $(\lambda(g)\xi)(x) = \xi(g^{-1}x)$. The right regular representation is $\rho\colon \Gamma \to U(\ell_2(\Gamma))$, given by $(\rho(g)\xi)(x) = \xi(xg)$.

If $\Lambda < \Gamma$ is a subgroup, then the representation $\pi\colon \Gamma\colon \ell_2(\Gamma/\Lambda)$, given by $(\pi(g)\xi)(x) = \xi(g^{-1}x)$ is a *quasi-regular* representation.

We say two representations $\pi_i\colon \Gamma \to U(H_i)$, for $i = 1, 2$ are *equivalent* if there exists a unitary $U\colon H_1 \to H_2$ such that $U\pi_1(g) = \pi_2(g)U$ for all $g \in \Gamma$. Note that the left and right regular representations are equivalent under the unitary $U\colon \ell_2(\Gamma) \to \ell_2(\Gamma)$ given by $U\xi(x) = \xi(x^{-1})$.

Given a unitary representation $\pi\colon \Gamma \to U(H)$, the adjoint representation $\overline{\pi}\colon \Gamma \to U(\overline{H})$ is given by $\overline{\pi(g)}\overline{\xi} = \overline{\pi(g)\xi}$. Note that $\overline{\xi} = \langle \cdot, \xi \rangle$ is the linear functional.

If $\pi_i\colon \Gamma \to U(H_i)$, we define the direct sum representation by

$$\left(\bigoplus_{i\in I} \pi_i\right)(g) = \bigoplus_{i\in I} \pi_i(g),$$

and if I is finite, we have

$$\left(\bigotimes_{i\in I} \pi_i\right)(g) = \bigotimes_{i\in I} \pi_i(g).$$

Lemma (Fell's Absorption Principle)**:** Let $\pi\colon \Gamma \to U(H)$ be a unitary representation of a discrete group Γ, and let 1_H be the trivial representation of Γ on H. Then, $\lambda \otimes \pi$ and $\lambda \otimes 1_H$ are equivalent.

Proof. Consider the unitary in $U(\ell_2(\Gamma) \otimes H)$ given by $U(\delta_g \otimes \xi) = \delta_g \otimes \pi(g)\xi$ for all $g \in \Gamma$ and $\xi \in H$. Then, for all $h, g \in \Gamma$ and $\xi \in H$, we have

$$\begin{aligned}
(U^*(\lambda \otimes \pi)(h)U)(\delta_g \otimes \xi) &= (U^*(\lambda \otimes \pi)(h))(\delta_g \otimes \pi(g)\xi) \\
&= U^*(\delta_{hg} \otimes \pi(h)\pi(g)\xi) \\
&= \delta_{hg} \otimes \pi(hg)^{-1}\pi(h)\pi(g)\xi \\
&= (\lambda \otimes 1_H)(h)(\delta_g \otimes \xi).
\end{aligned}$$

□

For any $\xi, \eta \in \ell_2(\Gamma)$, the convolution of ξ with η is

$$\begin{aligned}
\xi \cdot \eta(x) &= \sum_{g\in\Gamma} \xi(g)\eta\left(g^{-1}x\right) \\
&= \sum_{g\in\Gamma} \xi\left(xg^{-1}\right)\eta(g).
\end{aligned}$$

By Cauchy–Schwarz, we have $\xi \cdot \eta \in \ell_\infty(\Gamma)$ with $\|\xi \cdot \eta\|_{\ell_\infty} \leq \|\xi\|_{\ell_2}\|\eta\|_{\ell_2}$. If $\xi, \eta \in \ell_1(\Gamma)$, we have $\|\xi \cdot \eta\|_{\ell_1} \leq \|\xi\|_{\ell_1}\|\eta\|_{\ell_1}$.

Given $\xi \in \ell_2(\Gamma)$, we set

$$\begin{aligned}
D_\xi &= \{\eta \in \ell_2(\Gamma) \mid \xi \cdot \eta \in \ell_2(\Gamma)\}; \\
D'_\xi &= \{\eta \in \ell_2(\Gamma) \mid \eta \cdot \xi \in \ell_2(\Gamma)\},
\end{aligned}$$

with

$$\begin{aligned}
L_\xi\eta &= \xi \cdot \eta \\
R_\xi\eta &= \eta \cdot \xi
\end{aligned}$$

acting on D_ξ and D'_ξ respectively.

Lemma: The operators L_ξ and R_ξ have closed graphs in $\ell_2(\Gamma) \oplus \ell_2(\Gamma)$.

Proof. Let $(\eta_n)_n \to \ell_2(\Gamma)$ be a sequence such that $\eta_n \to \eta \in \ell_2(\Gamma)$, and $L_\xi \eta_n \to \zeta \in \ell_2(\Gamma)$. Then, for any $x \in \Gamma$, we have $|\zeta(x) - (\xi \cdot \eta)(x)| \leq \|\xi\|_{\ell_2} \|\eta_n - \eta\|_{\ell_2}$, so $\xi \cdot \eta = \zeta \in \ell_2(\Gamma)$, meaning $\eta \in D_\xi$ and $L_\xi \eta = \zeta$. □

A left convolver is a vector $\xi \in \ell_2(\Gamma)$ such that $\xi \cdot \ell_2(\Gamma) \subseteq \ell_2(\Gamma)$. If ξ is a left convolver, then by the closed graph theorem, we have $L_\xi, R_\xi \in B(\ell_2(\Gamma))$ whenever ξ is a left(/right resp.) convolver. The space of left/right convolvers contains δ_g for each $g \in \Gamma$.

Let $L(\Gamma)$ be the set of left convolvers, and $R(\Gamma)$ the space of right convolvers. Setting $\overline{\xi}(x) = \overline{\xi(x^{-1})}$, we have $L_\xi^* = L_{\overline{\xi}}$, with $L_{\xi \cdot \eta} = L_\xi L_\eta$. This gives the structure of unital $*$-subalgebras for both $L(\Gamma)$ and $R(\Gamma)$. In fact, we have something more.

Theorem: Let Γ be a discrete group. Then, $L(\Gamma)$ and $R(\Gamma)$ are von Neumann algebras. Furthermore, $L(\Gamma) = \operatorname{im}(\rho)'$ and $R(\Gamma) = \operatorname{im}(\lambda)'$.

Proof. It is enough to show that $L(\Gamma) = R(\Gamma)' = \operatorname{im}(\rho)'$. Note that we automatically have the inclusions $L(\Gamma) \subseteq R(\Gamma)' \subseteq \operatorname{im}(\rho)'$, so we only need to show that $\operatorname{im}(\rho)' \subseteq L(\Gamma)$. Let $T \in \operatorname{im}(\rho)'$, and set $\xi = T\delta_e$. Then, for all $g \in \Gamma$, we have

$$\begin{aligned} \xi \cdot \delta_g &= \rho\left(g^{-1}\right)(T\delta_e) \\ &= T\rho\left(g^{-1}\right)\delta_e \\ &= T\delta_g. \end{aligned}$$

By linearity and continuity, this extends to all $\eta \in L(\Gamma)$, so $T = L_\xi \in L(\Gamma)$. □

Proposition: Let Γ be a discrete group. Then, $\tau(x) = \langle x\delta_e, \delta_e \rangle$ defines a normal faithful trace on $L(\Gamma)$.

If Γ is abelian, then the dual group $\hat{\Gamma} := \operatorname{Hom}(\Gamma, S^1)$ becomes a compact group when endowed with the topology of pointwise convergence. This group then admits a Haar measure, which we may normalize by setting $\mu\left(\hat{\Gamma}\right) = 1$.

Define the Fourier transform $\mathcal{F} \colon \ell_2(\Gamma) \to L_2\left(\hat{\Gamma}\right)$ by

$$(\mathcal{F}\xi)(\chi) = \sum_{g \in \Gamma} \xi(g)\langle \chi, g \rangle,$$

where $\langle \chi, g \rangle$ implements the Pontryagin duality between Γ and $\hat{\Gamma}$. This map is a unitary between $\ell_2(\Gamma)$ and $L_2\left(\hat{\Gamma}\right)$.

If ξ is a left convolver, then $L_\xi = \mathcal{F}^{-1} M_{\mathcal{F}\xi} \mathcal{F}$, so we obtain a canonical isomorphism between $L(\Gamma)$ and $L_\infty\left(\hat{\Gamma}\right)$, where

$$\tau(L_\xi) = \int \mathcal{F}\xi \, d\mu$$

for every $L_\xi \in L(\Gamma)$.

If $x = \sum_{g\in\Gamma} \alpha_g \delta_g \in \ell_2(\Gamma)$, then we will also write x, or we will write $\sum_{g\in\Gamma} \alpha_g u_g$, to denote the operator $L_x \in L(\Gamma)$. The set $\{\alpha_g\}_{g\in\Gamma}$ are called the Fourier coefficients of x.

Definition: A discrete group Γ is said to be infinite conjugacy class (icc) if every nontrivial conjugacy class of Γ is infinite.

Theorem: Let Γ be a discrete group. Then, $L(\Gamma)$ is a factor if and only if Γ is icc.

Proof. Let $h \in \Gamma \setminus \{e\}$, and suppose $h^\Gamma \coloneqq \{ghg^{-1} \mid g \in \Gamma\}$ is finite. Setting

$$x = \sum_{k\in h^\Gamma} u_k,$$

we have that $x \notin \mathbb{C}1$, and for all $g \in g$, we have

$$\begin{aligned} u_g x u_g^* &= \sum_{k\in h^\Gamma} u_{gkg^{-1}} \\ &= x, \end{aligned}$$

so $x \in \{u_g\}_{g\in G}' \cap L(\Gamma) = Z(L(\Gamma))$.

Conversely, if Γ is icc, and $x = \sum_{g\in\Gamma} \alpha_g u_g \in Z(L(\Gamma)) \setminus \mathbb{C}1$, then for all $h \in \Gamma$, we have

$$\begin{aligned} x &= u_h x u_h^* \\ &= \sum_{g\in\Gamma} \alpha_g u_{hgh^{-1}} \\ &= \sum_{g\in\Gamma} \alpha_{h^{-1}gh} u_g, \end{aligned}$$

so all the Fourier coefficients for x are constant on conjugacy classes. Since $x \in L(\Gamma) \subseteq \ell_2(\Gamma)$, we have that $\alpha_g = 0$ for all $g \neq e$, meaning $x = \tau(x) \in \mathbb{C}$. $\square$

Group Measure Space

Let Γ be a discrete group acting on a finite measure space (X, μ). We say the action is *quasi-invariant* if, for each $g \in \Gamma$ and every measurable $E \subseteq X$, we have that gE is measurable, and $\mu(gE) = 0$ if and only if $\mu(E) = 0$. If we have that $\mu(gE) = \mu(E)$ for all $g \in \Gamma$ and measurable $E \subseteq X$, then we say the action is *measure-preserving*.

A quasi-invariant action of a group induces an action on the space of measurable functions of the group given by $g \mapsto \alpha_g$, where $\alpha_g(f)(x) = f(g^{-1}x)$. Furthermore, if $f \in L_\infty(X, \mu)$, then $\|\alpha_g(f)\|_{L_\infty} = \|f\|_{L_\infty}$.

Now, for each $g \in \Gamma$, the pushforward $g_*\mu$ is given by $g_*\mu(E) = \mu(g^{-1}E)$. If the action is quasi-invariant, then $g_*\mu \ll \mu$, so there is a Radon–Nikodym derivative $\frac{dg_*\mu}{d\mu}$ that is positive and integrable, with

$$\int \sigma_{g^{-1}}(f)\, d\mu = \int f\, dg_*\mu$$

$$= \int f \frac{dg_* \mu}{d\mu}\, d\mu.$$

The *Koopman representation* of the action is the representation $\pi\colon \Gamma \to U(L_2(X,\mu))$ given by

$$\pi_g(\xi) = \left(\frac{dg_*\mu}{d\mu}\right)^{1/2} \alpha_g(\xi).$$

We observe that if $f \in L_\infty(X,\mu)$, $\xi \in L_2(X,\mu)$, and $g \in \Gamma$, then

$$\begin{aligned} \pi_g M_a \pi_{g^{-1}}(\xi) &= \pi_g\left(a\left(\frac{dg_*^{-1}\mu}{d\mu}\right)^{1/2} \alpha_{g^{-1}}(\xi)\right) \\ &= \alpha_g(a)\left(\frac{dg_*\mu}{d\mu}\alpha_g\left(\frac{dg_*^{-1}\mu}{d\mu}\right)\right)^{1/2}\xi \\ &= M_{\alpha_g(a)}\xi \end{aligned}$$

If we let $H = L_2(X,\mu)\bar{\otimes}\ell_2(\Gamma)$, then we have the normal representation of $L_\infty(X,\mu)$ on H given by $a \mapsto M_a \otimes 1 \in B(H)$. Furthermore, we have the diagonal action of Γ on H given by $u_g = \pi_g \otimes \lambda_g \in U(H)$.

> **Remark:** Note that we can also view H as $\ell_2(\Gamma, L_2(X,\mu))$ — that is, the space of square-summable sequences whose entries are elements of $L_2(X,\mu)$.

The *group measure space construction* associated to the action of Γ is the von Neumann algebra $L_\infty(X,\mu) \rtimes \Gamma$ generated by the operators $M_a \otimes 1$ and u_g. We may consider $L_\infty(X,\mu)$ as a von Neumann subalgebra of $L_\infty(X,\mu) \rtimes \Gamma$, and $u_g a u_{g^{-1}} = \alpha_g(a)$ under this identification. Furthermore, by Fell's absorption principle, we have $\pi \otimes \lambda \cong 1 \otimes \lambda$, meaning that the map $\lambda_g \mapsto u_g$ extends to $L(\Gamma)$, so we also have an inclusion $L(\Gamma) \subseteq L_\infty(X,\mu) \rtimes \Gamma$.

We will consider $L_2(X,\mu)$ as a subspace of $L_2(X,\mu)\bar{\otimes}\ell_2(\Gamma)$ given by $U\xi = \xi \otimes \delta_e$. Then, $e\colon L_2(X,\mu)\bar{\otimes}\ell_2(\Gamma) \to L_2(X,\mu)$ will be the orthogonal projection, and we let $E\colon L_\infty(X,\mu) \rtimes \Gamma \to B(L_2(X,\mu))$ be given by $E(x) = exe$.

Interactions

A simple question we may be interested in is understanding the various properties of the action of Γ on (X,μ), and how they are reflected in terms of $L(\Gamma)$ or $L_\infty(X,\mu) \rtimes \Gamma$.

> **Definition:** Let $A \subseteq M$ be von Neumann algebras, with A abelian. We say A is a masa (maximal abelian self-adjoint subalgebra) if $A' \cap M = A$.

> **Definition:** If G is a probability measure preserving (pmp) action on (X,μ), we say the action is *essentially free* if for all $g \in G \setminus \{e\}$, we have that
>
> $$\mu(\{x \in X \mid gx = x\}) = 0.$$
>
> Equivalently, the action is free if, for all $E \subseteq X$ with $\mu(E) > 0$, we have $a \in L_\infty(X,\mu)$ such that
>
> $$(a - \alpha_g(a))\chi_E \neq 0$$

Definition: We say an action of G on (X,μ) is *ergodic* if, whenever $E \subseteq X$ is a measurable subset with $gE = E$, then $\mu(E) = 0$ or $\mu(E) = 1$.

Definition: We say a Borel probability measure space (X,μ) is *standard* if there exists a compact metric space Y and Borel probability measure ν on Y such that $(Y,\nu) \cong (X,\mu)$ as measure spaces, modulo null sets.

Proposition: Suppose G acts via probability measure preserving transformations on a standard measure space (X,μ). If the action of G is essentially free, then $L_{\infty}(X,\mu)$ is a masa in $L_{\infty}(X,\mu) \rtimes G$.

Proof. Without loss of generality, we may assume that X is a compact metric space. Let $x \in L_{\infty}(X,\mu)' \cap L_{\infty}(X,\mu) \rtimes G$. We may write

$$x = \sum_{g\in G} x_g u_g,$$

where $x_g \in L_{\infty}(X,\mu)$.

Then, for all $f \in L_{\infty}(X,\mu)$, we have

$$\begin{aligned} xf &= \sum_{g\in G} x_g \alpha_g(f) u_g \\ fx &= \sum_{g\in G} f x_g u_g. \end{aligned}$$

Therefore, we have that $fx_g = \alpha_g(f)x_g$ for all $g \in G$ and all $f \in L_{\infty}(X\mu)$. Therefore, all we need to show is that $g = e$, meaning that we are only dealing with the projection onto the subalgebra $L_{\infty}(X,\mu) \otimes \delta_e \cong L_{\infty}(X,\mu)$.

Fix $g \in G$ with $g \neq e$, so that $\mu(\{x \in X \mid gx = x\}) = 0$. Suppose toward contradiction that $x_g \neq 0$. Let d be a metric on X. Then, for all $\varepsilon > 0$, we have that

$$\mu(\{p \in X \mid d(gp,p) \geq \varepsilon, x_g(p) \neq 0\}) > 0.$$

Set $A = \{p \in X \mid d(gp,p) \geq \varepsilon, x_g(p) \neq 0\}$. Then, there exists $p \in A$ such that $\mu(U(p,\varepsilon/3) \cap A) > 0$. For all $z, y \in U(p,\varepsilon/3) \cap A$, we have

$$\begin{aligned} d(gz,y) &\geq \varepsilon - d(z,y) \\ &> \varepsilon - \frac{2\varepsilon}{3} \\ &= \frac{\varepsilon}{3} \\ &> 0. \end{aligned}$$

Set $E = U(p,\varepsilon/3) \cap A$. Then, $\chi_E x_g = \chi_{gE} x_g$ by our finding above, yet since $gE \cap E = \emptyset$, $\mu(E) \neq 0$, and $x_g(p) \neq 0$ for all $p \in E$, we have $\langle \chi_E x_g, \chi_{gE} x_g \rangle = 0$ while $\|\chi_E x_g\|^2 > 0$, which cannot happen.

Therefore, we must have $x = x_e \in L_{\infty}(X,\mu)$. $\square$

Proposition: Let G act on a standard Borel probability space (X,μ) by pmp transformations. The following are equivalent:

(i) the action is ergodic;

(ii) if $\xi \in L_2(X,\mu)$, and $\alpha_g\xi = \xi$ for all $g \in G$, then $\xi \in \mathbb{C}1$;

(iii) if $\varphi\colon X \to \mathbb{C}$ is measurable, and $\alpha_g\varphi = \varphi$ for all $g \in G$, then there exists λ such that $\varphi = \lambda$ almost everywhere.

Proof. The direction (iii) implies (ii) is clear from the fact that L_2 functions are measurable.

Next, we show (ii) implies (i). If we let $f = \chi_E$, then

$$\begin{aligned} \|\alpha_g\chi_E - \chi_E\|_{L_2}^2 &= \mu(gE \triangle E) \\ \|\chi_E - \mu(E)1\|_{L_2}^2 &= \mu(E)\mu(E^c). \end{aligned}$$

Since both of these measures are finite, the fact that the action of G is ergodic follows.

Finally, we show (i) implies (iii). Given $\varepsilon > 0$, there exists z such that $\mu(\varphi^{-1}(U(z,\varepsilon))) > 0$, so that $\varphi^{-1}(U(z,\varepsilon))$ is G-invariant.

In particular, since the action of G is ergodic, $\mu(\varphi^{-1}(U(z,\varepsilon))) = 1$, so $\varphi \in L_\infty(X,\mu)$. Let C be an essential bound for φ.

Consider the set

$$E = \bigcap \left\{ B(p,\varepsilon) \mid \varepsilon > 0, |p| \leq C, \mu\left(\varphi^{-1}(U(p,\varepsilon))\right) = 1 \right\}.$$

Then, since X is compact, it follows that $E = \{\lambda\}$, and for all $\varepsilon > 0$, we have $\mu(\varphi^{-1}(U(\lambda,\varepsilon))) = 1$, so that $\varphi = \lambda$ almost everywhere. $\square$

Corollary: Let (X,μ) be a standard probability space equipped with an essentially free pmp G-action. Then,

(i) $Z(L_\infty(X) \rtimes G) = \{f \in L_\infty(X) \mid \alpha_g f = f \text{ almost everywhere for all } g \in G\}$;

(ii) $L_\infty(X) \rtimes G$ is a factor if and only if the action is ergodic.

Proof.

(i) If $\alpha_g f = f$ almost everywhere for all g, then $f \in L_\infty(X)'$, so $u_g f = (\alpha_g f)u_g = f u_g$, meaning that for all $g \in G$, we have $f \in Z(L_\infty(X) \rtimes G)$.

Conversely, if $f \in Z(L_\infty(X) \rtimes G)$, then $f \in (L_\infty(X) \rtimes G)' \cap L_\infty(X) = L_{\infty(X)}$ as the action of G is essentially free. Therefore, $\alpha_g f = u_g f u_g^{-1} = f$.

(ii) This follows from (i) and the previous proposition.

$\square$

Note that if the action is not essentially free, then it is not necessarily possible to show that $(L_\infty(X,\mu))' \cap (L_\infty(X) \rtimes G) = L_\infty(X,\mu)$. For instance, the action of G is trivial, then

$$L_\infty(X) \rtimes G \cong L_\infty(X) \bar{\otimes} L(G).$$

In particular, we cannot say that the action of G is ergodic if and only if $L_\infty(X) \rtimes G$ is a factor. We will seek to resolve this issue by "modding out" by stabilizers in some reasonable fashion.

Equivalence Relations and von Neumann Algebras

Definition: Let (X, μ) be a standard Borel probability space. A discrete pmp equivalence relation on (X, μ) is a Borel subset $R \subseteq X \times X$ satisfying

(i) R is an equivalence relation;

(ii) for almost every $x \in X$, $[x]_R$ is countable;

(iii) for all $f\colon R \to [0, +\infty]$ Borel, we have

$$\int_X \sum_{y \in [x]_R} f(x, y)\, d\mu(x) = \int_X \sum_{y \in [x]_R} f(y, x)\, d\mu(x).$$

Condition (iii) is often known as the *mass transport principle*.

The best example is that of a group G acting on a standard probability space (X, μ) by pmp transformations. The set

$$R_{G,X} = \{(x, g \cdot x) \mid x \in X, g \in G\}$$

is then a pmp equivalence relation.

Theorem (Feldman–Moore Theorem)**:** Let R be a pmp equivalence relation on (X, μ). Then, there exists a group G acting on (X, μ) such that $R = R_{G,X}$.

Note that the action of G may not be free.

Furthermore, we may define the measure $\overline{\mu}$ on R by

$$\overline{\mu} = \int_X |\{y \mid (x, y) \in E\}|\, d\mu(x).$$

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