Solution (29.5)**:**

(a) We have

$$\left(\vec{w}\cdot\overleftrightarrow{T}\right)_k = \sum_{i,j,k} w_i T_{jk}\delta_{ij} = \sum_{i,k} w_i T_{ik},$$

which is a first-rank tensor.

(b) Since $\vec{w}\cdot\overleftrightarrow{T}$ is a first-rank tensor, and we are taking the dot product of two first rank tensors the expression $\vec{w}\cdot\overleftrightarrow{T}\cdot\vec{v}$ is a scalar (or rank zero tensor).

(c) We have

$$\overleftrightarrow{T}\cdot\overleftrightarrow{U} = \left(\sum_{i,j} T_{ij} e_i\otimes e_j\right)\cdot\left(\sum_{k,\ell} U_{k\ell} e_k\otimes e_\ell\right) = \sum_{i,j,k,\ell} T_{ij}U_{k\ell}(e_k\cdot e_i)(e_j\cdot e_\ell),$$

which is a scalar.

(d) The expression $\overleftrightarrow{T}\vec{v}$ expresses the operation of the linear map

$$\overleftrightarrow{T} = \sum_{i,j} T_{ij} e_i\otimes e_j$$

on

$$\vec{v} = \sum_i v_i e_i,$$

meaning that $\overleftrightarrow{T}\vec{v}$ is a vector.

(e) The expression $\overleftrightarrow{T}\overleftrightarrow{U}$ is a composition of two linear maps on $V\otimes V$, so it is a rank 2 tensor (or another linear map on $V\otimes V^*$).

Solution (29.7)**:** We have 2^4 or 16 components in A_{ijkl}.

Solution (29.10)**:** We have

$$T_{ij} = \sum_{k,\ell} R_{ik}R_{j\ell}T_{k\ell},$$

so that

$$\sum_{ij}\epsilon_{ij}T_{ij} = \sum_{i,j,k,\ell}\epsilon_{ij}R_{ik}R_{j\ell}T_{k\ell} = \sum_{i,j,k,\ell} R_{ik}R_{j\ell}\epsilon_{k\ell}T_{k\ell} = \sum_{k,\ell}\epsilon_{k\ell}T_{k\ell}.$$

Solution (29.11)**:**

(a) We may write T_{ij} as $T = \frac{1}{2}\left(T + T^T\right) + \frac{1}{2}\left(T - T^T\right)$, which are the symmetric and antisymmetric components.

(b) Taking

$$S_{ij} = \sum_{k,\ell} R_{ik} R_{j\ell} S_{k\ell},$$

we have

$$\begin{aligned}
S_{ji} &= \sum_{k,\ell} R_{jk} R_{i\ell} S_{k\ell} \\
&= \sum_{k,\ell} R_{j\ell} R_{ik} S_{\ell k} \\
&= \sum_{k,\ell} R_{ik} R_{j\ell} S_{\ell k} \\
&= \sum_{k,\ell} R_{ik} R_{j\ell} S_{k\ell} \\
&= S_{ij}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_{ij} &= \sum_{k,\ell} R_{ik} R_{j\ell} A_{k\ell} \\
A_{ji} &= \sum_{k,\ell} R_{jk} R_{i\ell} A_{k\ell} \\
&= -\sum_{k,\ell} R_{ik} R_{j\ell} A_{k\ell} \\
&= -A_{ij}.
\end{aligned}$$

In matrix form, we have

$$\begin{aligned}
S_{ji} &= S_{ij}^{T} \\
&= \left(R S_{k\ell} R^{T} \right)^{T} \\
&= R S_{k\ell} R^{T}.
\end{aligned}$$

and similarly,

$$\begin{aligned}
-A_{ji} &= \left(A_{ij} \right)^{T} \\
&= \left(R A_{k\ell} R^{T} \right)^{T} \\
&= R A_{k\ell} R^{T}.
\end{aligned}$$

Solution (29.12)**:**

(a) Let $\mathbf{v} = R\mathbf{v}'$, where R is a rotation matrix. Then, we must have

$$\begin{aligned}
\frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y &= \nabla' \cdot \mathbf{v} \\
&= \nabla \cdot R\mathbf{v}' \\
&= \nabla \cdot \begin{pmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) \cos(\theta) & \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \end{pmatrix} \\
&= \nabla \cdot \begin{pmatrix} v'_x \cos(\theta) - v'_y \sin(\theta) \\ v'_x \sin(\theta) + v'_y \cos(\theta) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left(v_x' \cos(\theta) - v_y' \sin(\theta)\right) + \frac{\partial}{\partial y}\left(v_x' \sin(\theta) + v_y' \cos(\theta)\right)\\
&= \left(\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}\right)v_x' + \left(\cos(\theta) - \sin(\theta)\frac{\partial}{\partial y}\right)v_y'\\
&= R^T \nabla \cdot \mathbf{v}'
\end{aligned}
$$

(b) Let $\mathbf{v} = R\mathbf{v}'$, where R is a rotation matrix. Then, we must have

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= \sum_{i} \frac{\partial}{\partial x_i} v_i\\
&= \sum_{i,j} \frac{\partial}{\partial x_i} R_{ij} v_j'\\
&= \sum_{i,j} R_{ji} \frac{\partial}{\partial x_j} v_j'\\
&= \nabla' \cdot \mathbf{v}',
\end{aligned}
$$

where we were able to switch the indices on R by the fact that it is not a function of x_i.

Solution (29.14)**:** We have

$$
\begin{aligned}
\sigma_{ij}' &= \sum_{k,\ell} R_{ik} R_{j\ell} \sigma_{k\ell}\\
&= \sum_{k,\ell} R_{ik} R_{j\ell} (Y_{k\ell mn} \varepsilon_{mn})\\
&= \sum_{k,\ell} R_{ik} R_{j\ell} \left(\sum_{o,p} R_{mo} R_{np}' \epsilon_{op}'\right)\\
&= \sum_{k,\ell,o,p} R_{ik} R_{j\ell} R_{mo} R_{np} Y_{k\ell op}' \epsilon_{op}',
\end{aligned}
$$

so that

$$
Y_{ijmn} = \sum_{k,\ell,o,p} R_{ik} R_{j\ell} R_{mo} R_{np} Y_{k\ell op}',
$$

meaning the elastic modulus is a rank 4 tensor.

Solution (29.23)**:** We have

$$
\begin{aligned}
T_{ij} &= \sum_{k,\ell} R_{ik} R_{j\ell} \left(\frac{1}{3}\operatorname{tr}(T)\delta_{k\ell} + \frac{1}{2}(T_{k\ell} - T_{\ell k}) + \frac{1}{2}\left(T_{k\ell} + T_{\ell k} - \frac{2}{3}\operatorname{tr}(T)\delta_{ij}\right)\right)\\
&= \frac{1}{3}\delta_{ij} + \frac{1}{2}(T_{ij} - T_{ji}) + \frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3}\delta_{ij}\right).
\end{aligned}
$$

Solution (29.24)**:**

(a) Letting $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be vectors, we note that the quantity

$$
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \sum_{i,j,k,\ell} \epsilon_{ijk} \delta_{k\ell} A_i B_j C_\ell
$$

is a scalar quantity, so it is invariant under rotation. In particular, this means that, since $\mathbf{C}$ transforms as a vector, so too does $\mathbf{A} \times \mathbf{B}$.

(b) Writing

$$
(\mathbf{A} \times \mathbf{B})_k = \sum_{i,j} \epsilon_{ijk} A_i B_j,
$$

we may write

$$C_{ik} = \sum_j \epsilon_{ijk}\epsilon_{ijk}B_j,$$

to yield the expression

$$(\mathbf{A}\times\mathbf{B})_k = \sum_i C_{ik}A_i.$$

The reason we can't expand this beyond three dimensions is that, in four dimensions, we would need a second-rank tensor to be of the form

$$C_{i\ell} = \sum_{i,j} \epsilon_{ijk\ell}B_iD_k,$$

so that this second-rank tensor would act on a vector and yield another vector; however, this means the second-rank tensor requires two vectors as an "input."

Solution (29.25)**:**

(a) We verify this by plugging in the value of B_k to attempt to recover i, j.

$$\begin{aligned}
T_{ij} &= \sum_k \epsilon_{ijk}B_k \\
&= \sum_k \epsilon_{ijk}\left(\frac{1}{2}\sum_{i,j}\epsilon_{kij}T_{ij}\right) \\
&= \frac{1}{2}\sum_{i,j,k}\epsilon_{ijk}\epsilon_{ijk}T_{ij} \\
&= \frac{1}{2}\left(2T_{ij}\right) \\
&= T_{ij}.
\end{aligned}$$

(b) Separating $T_{jk} = \frac{1}{2}\left(T_{jk} + T_{kj}\right) + \frac{1}{2}\left(T_{jk} - T_{kj}\right)$, we take

$$\begin{aligned}
B_i &= \frac{1}{2}\sum_{j,k}\epsilon_{ijk}\left(\frac{1}{2}\left(T_{jk}+T_{kj}\right) + \frac{1}{2}\left(T_{jk}-T_{kj}\right)\right) \\
&= \underbrace{\frac{1}{2}\sum_{j,k}\frac{1}{2}\epsilon_{ijk}\left(T_{jk}+T_{kj}\right)}_{=0} + \frac{1}{2}\sum_{j,k}\frac{1}{2}\epsilon_{ijk}\left(T_{jk}-T_{kj}\right).
\end{aligned}$$

(c) We take

$$\begin{aligned}
T_{i'j'} &= \sum_{i,j} R_{ii'}R_{jj'}T_{ij} \\
&= \sum_{i,j,k} R_{ii'}R_{jj'}\epsilon_{ijk}B_k \\
&= \sum_{i',j'} \epsilon_{i'j'k}R_{i'k}B_k,
\end{aligned}$$

meaning that rotating the dual tensor gives the dual of a rotated vector.