Introduction

Oh hey, it's another one of those textbook notes that I never complete. I've decided to try something different in order to develop my understanding of measure theory. One of the primary for understanding measure theory is Gerald B. Folland's *Real Analysis and Applications* — and one of the benefits it has over a lot of other texts is that it has a significant number of exercises. I'm going to try to do them all — I'll start with Chapters 1–3, and if that goes well enough, continue up through whatever chapter I end up having to tap out at. Interspersed, I will include various notes. I figure that in order to make a subject like measure theory really stick, I need to deal with it consistently.

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Exercise (Exercise 1)**:** A family of sets $\mathcal{R} \subseteq P(X)$ is called a ring if it is closed under finite unions and differences. A ring that is closed under countable unions is called a σ-ring.

(a) Rings (σ-rings) are closed under finite (countable) intersections.

(b) If $\mathcal{R}$ is a ring (σ-ring), then $\mathcal{R}$ is an algebra (σ-algebra) if and only if $X \in \mathcal{R}$.

(c) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

(d) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in R\}$ is a σ-algebra.

Proposition (Proposition 1.2)**:** The Borel σ-algebra, $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following:

(a) the open intervals, $\mathcal{E}_1 = \{(a, b) \mid a < b\}$;

(b) the closed intervals, $\mathcal{E}_2 = \{[a, b] \mid a < b\}$;

(c) the half-open intervals, $\mathcal{E}_3 = \{(a, b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a, b) \mid a < b\}$;

(d) the open rays, $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$;

(e) the closed rays, $\mathcal{E}_7 = \{[a, \infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] \mid a \in \mathbb{R}\}$.

Proof. The elements for $\mathcal{E}_j$ for $j \neq 3, 4$ are open or closed, and the elements of $\mathcal{E}_3$, $\mathcal{E}_4$ are G_δ sets — for instance,

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right).$$

Thus, $\sigma(\mathcal{E}_j) \subseteq \mathcal{B}_{\mathbb{R}}$ for each j. On the other hand, every open set in $\mathbb{R}$ is a countable union of open intervals, so $\mathcal{B}_{\mathbb{R}} \subseteq \sigma(\mathcal{E}_1)$. Thus, $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{E}_1)$. □