The Extended Complex Plane and Linear Fractional Transformations

Definition: Let $\mathbb{C}$ be the complex plane. We define $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ to be the one-point compactification of $\mathbb{C}$.

Open sets in $\hat{\mathbb{C}}$ look like either:

- open sets $U \subseteq \mathbb{C}$;
- sets $U \subseteq \hat{\mathbb{C}}$ where $\infty \in U$, and $U \setminus \{\infty\} \subseteq \mathbb{C}$ is open, where additionally, there exists R such that the set $\{z \in \mathbb{C} \mid |z| > R\} \subseteq U \setminus \{\infty\}$.

Proposition: There is a continuous bijection between $\hat{\mathbb{C}}$ and the unit sphere S^2 in $\mathbb{R}^3$ given by

$$z \mapsto \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

if $z \neq \infty$, and $z \mapsto (0,0,1)$ if $z = \infty$.

Proposition: Given the metric ρ on S^2 given by

$$\rho((x_1,x_2,x_3),(y_1,y_2,y_3)) = \sqrt{2(1 - x_1y_1 - x_2y_2 - x_3y_3)},$$

we have a corresponding natural metric on $\hat{\mathbb{C}}$ given by

$$d(z,w) = \begin{cases} \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}} & z,w \in \mathbb{C} \\ \frac{2}{\sqrt{1+|z|^2}} & w = \infty, z \in \mathbb{C} \\ \frac{2}{\sqrt{1+|w|^2}} & z = \infty, w \in \mathbb{C} \\ 0 & w = z = \infty. \end{cases}$$

Furthermore, the open sets as defined above are exactly the ones induced by this metric.

Definition: A subset $C \subseteq \hat{\mathbb{C}}$ is called a *Riemann Circle* if

- C is a union of a straight line with ∞,

$$C = \{z \in \mathbb{C} \mid a\operatorname{Re}(z) + b\operatorname{Im}(z) = c\} \cup \{\infty\};$$

- C is a Euclidean circle,

$$C = \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

A subset $D \subseteq \hat{\mathbb{C}}$ is called a *Riemann disc* if either

- D is an open half-plane in $\mathbb{C}$;
- D is an open disc in $\mathbb{C}$;
- D is the complement of a closed disc in $\mathbb{C}$.

Note: The Riemann circles correspond to circles drawn on S^2, while Riemann discs correspond to circular caps on S^2.