

Problem (Problem 1): Let R be a ring and M a left R -module.

- (a) Prove that for every $m \in M$, the map $r \mapsto r \cdot m$ from R to M is a homomorphism of R -modules.
- (b) Assume that R is commutative and M an R -module. Prove that there is an isomorphism $\text{hom}_R(R, M) \cong M$ as left R -modules.

Solution:

- (a) Let $m \in M$ be fixed, and define $\varphi_m: R \rightarrow M$ by

$$\varphi_m(r) = r \cdot m.$$

It follows from the axioms of left R -modules that

$$\begin{aligned}\varphi_m(r + s) &= (r + s) \cdot m \\ &= r \cdot m + s \cdot m \\ &= \varphi_m(r) + \varphi_m(s),\end{aligned}$$

and

$$\begin{aligned}\varphi_m(rs) &= (rs) \cdot m \\ &= r \cdot (s \cdot m) \\ &= r \cdot (\varphi_m(s)),\end{aligned}$$

so that φ_m is a homomorphism of left R -modules.

- (b) If $\varphi_m: R \rightarrow M$ is the homomorphism as defined in part (a), we define a map $\varphi: M \rightarrow \text{hom}_R(R, M)$ by

$$\varphi(m)(r) = \varphi_m(r).$$

First, we verify that φ is a homomorphism. If $r \in R$ is arbitrary, then

$$\begin{aligned}\varphi(m + n)(r) &= \varphi_{m+n}(r) \\ &= r \cdot (m + n) \\ &= r \cdot m + r \cdot n \\ &= \varphi_m(r) + \varphi_n(r) \\ &= (\varphi(m) + \varphi(n))(r).\end{aligned}$$

To see that φ is injective, we see that $\ker(\varphi)$ consists of all elements $m \in M$ such that $\varphi(m) = \varphi_0$, where $\varphi_0: R \rightarrow M$ takes $r \mapsto 0$ for all $r \in R$. In particular, since $1 \in R$, it follows that $1 \cdot m = m = 0$, meaning that $\ker(\varphi) = \{0\}$.

To see that φ is surjective, we observe that for any $\psi \in \text{hom}_R(R, M)$, ψ is fully determined by where it maps 1 , as

$$\psi(r) = r \cdot \psi(1).$$

Therefore, if $\psi \in \text{hom}_R(R, M)$, then we may find $m \in M$ corresponding to ψ by taking

$$m := \psi(1).$$

Thus, $M \cong \text{hom}_R(R, M)$.

Problem (Problem 3): Let R be a ring, and M a left R -module.

- (a) Let N be a subset of M . The *annihilator* of N is defined to be the set

$$\text{ann}_R(N) = \{r \in R \mid r \cdot n = 0 \text{ for all } n \in N\}.$$

Prove that $\text{ann}_R(N)$ is a left-ideal of R .

- (b) Show that if N is an R -submodule of M , then $\text{ann}_R(N)$ is a two-sided ideal of R .

- (c) For a subset I of R , the *annihilator* of I in M is defined to be the set

$$\text{ann}_M(I) = \{m \in M \mid x \cdot m = 0 \text{ for all } x \in I\}.$$

Find a natural condition on I that guarantees $\text{ann}_M(I)$ is a submodule of M .

- (d) Let R be an integral domain. Prove that every finitely generated torsion R -module has a nonzero annihilator.

Solution:

- (a) First, we observe that $\text{ann}_R(N)$ is nonempty, as $0 \in \text{ann}_R(N)$. Additionally, if $s, t \in \text{ann}_R(N)$, then for all $n \in N$,

$$\begin{aligned} (s - t) \cdot n &= s \cdot n - t \cdot n \\ &= 0, \end{aligned}$$

so that N is closed under subtraction. Finally, if $r \in R$ and $s \in \text{ann}_R(N)$, then for all $n \in N$,

$$\begin{aligned} (rs) \cdot n &= r \cdot (s \cdot n) \\ &= r \cdot 0 \\ &= 0, \end{aligned}$$

meaning that $rs \in \text{ann}_R(N)$, or that $\text{ann}_R(N)$ is a left-ideal of R .

- (b) Let N be an R -submodule of M , and let $s \in \text{ann}_R(N)$. If $r \in R$, then for all $n \in N$, $r \cdot n \in N$, so that $(sr) \cdot n = s \cdot (r \cdot n) = 0$, meaning that $sr \in \text{ann}_R(N)$. Thus, $\text{ann}_R(N)$ is a right-ideal, hence a two-sided ideal for R .

- (c) We observe to start that $\text{ann}_M(I)$ contains 0 and is additively closed, since if $m, n \in \text{ann}_M(I)$ and $x \in I$ are arbitrary, then

$$\begin{aligned} x \cdot (m + n) &= x \cdot m + x \cdot n \\ &= 0. \end{aligned}$$

Therefore, if we desire for $\text{ann}_M(I)$ to be a submodule of M , we would need $r \cdot m \in \text{ann}_M(I)$ for all $m \in \text{ann}_M(I)$, which would mean $r \cdot m$ would have to satisfy the condition

$$\begin{aligned} 0 &= x \cdot (r \cdot m) \\ &= (xr) \cdot m, \end{aligned}$$

meaning that we would require $xr \in \text{ann}_M(I)$. In other words, this means that $\text{ann}_M(I)$ would have to be a right-ideal for R .

- (d) Let $M = \langle a_1, \dots, a_n \rangle$ be a finitely generated torsion R -module. Since M has torsion, for each a_i , there is some $0 \neq r_i \in R$ such that $r_i \cdot a_i = 0$. The product

$$r = \prod_{i=1}^n r_i$$

is necessarily nonzero as R is an integral domain, and satisfies $r \cdot a_i = 0$ for all i by rearrangement of factors, so that $(r) \subseteq \text{ann}_R(M)$ as $\text{ann}_R(M)$ is an ideal containing r . Thus, $\text{ann}_R(M)$ is a nonzero ideal.