Math 395

Homework 2
Due: 2/8/2024

Name: Avinash Iyer

Collaborators:

Problem 2

Let I, J be ideals in ring R. Define $I + J = \{i + j \mid i \in I, j \in J\}$. This is referred to as the sum of the ideals.

(a) We will prove that $I + J$ is an ideal in R that contains I and J.

To start, since I and J are ideals in R, I and J are each subrings of R, meaning both I and J contain 0_R. Therefore, taking $j = 0_R$, we find that $\{i + 0_R \mid i \in I\} \subseteq I + J$, and similarly, taking $i = 0_R$, we find that $\{0_R + j \mid j \in J\} \subseteq I + J$. These sets are, respectively, I and J, meaning I and J are both subsets of $I + J$.

We will show that $I + J$ is an ideal in R by showing that $I + J$ is a subring that is closed under multiplication by all elements of R. Firstly, $I + J$ is non-empty since, as exhibited earlier, both I and J are subrings, meaning $0_R \in I$ and $0_R \in J$, so $0_R + 0_R = 0_R \in I + J$. Let $x, y \in I + J$. Then, $x = x_i + x_j$ and $y = y_i + y_j$ for some $x_i, y_i \in I$ and $x_j, y_j \in J$. Then,

$$\begin{aligned} x - y &= (x_i + x_j) - (y_i + y_j) \\ &= (x_i - y_i) + (x_j - y_j), \end{aligned}$$

which is an element of $I + J$. Similarly,

$$\begin{aligned} xy &= (x_i + x_j) + (y_i + y_j) \\ &= (x_i y_i) + (x_j y_j + x_i y_j + x_j y_i). \end{aligned}$$

Since $x_i y_i \in I$, as I is a subring, and $x_j y_j \in J$, as J is a subring, as well as $x_i y_j \in J$ and $x_j y_i \in J$ as J is an ideal, we have that $x_j y_j + x_i y_j + x_j y_i \in J$, so $xy \in I + J$.

Finally, we will show that $I + J$ is closed under multiplication by elements from R. Let $r \in R$, $a \in I + J$. Then, $a = a_i + a_j$ for $a_i \in I$ and $a_j \in J$. So,

$$\begin{aligned} ra &= r(a_i + a_j) \\ &= ra_i + ra_j, \end{aligned}$$

and

$$\begin{aligned} ar &= (a_i + a_j)r \\ &= a_i r + a_j r, \end{aligned}$$

and since I and J are both ideals, $ra_i, a_i r \in I$ and $ra_j, a_j r \in J$, so $ar, ra \in I + J$.

Therefore, $I + J$ is an ideal that contains I and J.

(b) Let $a, b \in \mathbf{Z}$. We will show that $a\mathbf{Z} + b\mathbf{Z} = \gcd(a, b)\mathbf{Z}$.

By Bezout's identity, it is the case that there are integers x and y such that $xa + yb = \gcd(a, b)$. Since $xa \in a\mathbf{Z}$, and $yb \in b\mathbf{Z}$, as $a\mathbf{Z}$ and $b\mathbf{Z}$ are ideals in $\mathbf{Z}$, it is the case that $xa + yb \in a\mathbf{Z} + b\mathbf{Z}$. Therefore, $\gcd(a, b)\mathbf{Z} \subseteq a\mathbf{Z} + b\mathbf{Z}$.