## Dijkstra's Algorithm Modification

In class, we gave a version of Dijkstra's algorithm for finding the distance between a vertex $u$ and all the other vertices in a weighted graph. Appropriately modify that algorithm such that it not only outputs $d(u, v)$, but also outputs the shortest $u, v$ path.

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We will commence the algorithm as follows:

**Input** A weighted graph, $G$, and $u \in V(G)$

**Output** For each $z \in V(G)$, the distance $d(u, z)$, and the $u, v$-geodesic.

**Initialization** Extend the weight function such that if $xy \notin E(G)$, then $w(xy) = \infty$. Create $S$ that contains all vertices whose distances from $u$ are known. Let $S := \{u\}$. Let $t : V \to \mathbb{R}^+ \cup \{0, \infty\}$ which will keep track of the tentative distance between $u$ and $z$. Let $t(z) := w(uz)$ for all $z \neq u$, and $t(u) := 0$. Additionally, create a path $P_z$ for $z$, starting at $u$ and ending at $z$. We will let $P'_z$ be the tentative path.

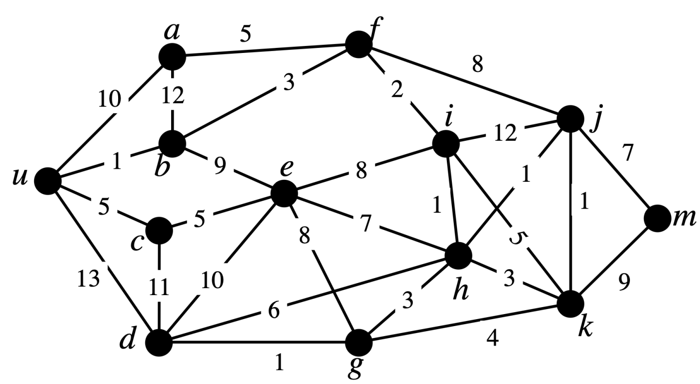
**Condition to Terminate Loop** If $t(z) = \infty$ for all $z \notin S$ or $S = V$, then go to end.

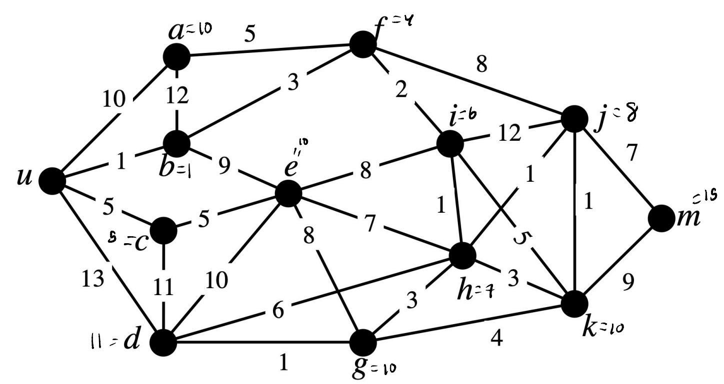
**Loop** Else, pick $v \in V - S$ such that $t(v) = \min_{z \notin S} t(z)$. Add $v$ to $S$. Explore the edges from $v$ to update tentative distances; for each edge $vz$ with $z \notin S$, $t(z) := \min\{t(z), t(v) + w(vz)\}$. Add $z$ to $P'_v$.

**End** Set $d(u, v) = t(v)\ \forall v \in V$, and set $P_v$ to $P'_v$ for all $v \in V$.

## Modified Dijkstra's Algorithm, Example

Perform the modified Dijkstra's algorithm on the following graph:






Paths:

- $P_a = u, a$
- $P_b = u, b$
- $P_c = u, c$
- $P_d = u, b, f, i, h, g, d$
- $P_e = u, b, e$
- $P_f = u, b, f$
- $P_g = u, b, f, i, h, g$
- $P_h = u, b, f, i, h$
- $P_i = u, b, f, i$
- $P_j = u, b, f, j$
- $P_k = u, b, f, j, k$
- $P_m = u, b, f, j, m$

## 2.3.1

Assign integer weights to the edges of $K_n$. Prove that the total weight of every cycle in $K_n$ is even if and only if the total weight of each triangle is even.

($\Rightarrow$) Suppose there exists a triangle in $K_n$ whose weight is odd. Then, this triangle is a cycle in $K_n$ whose weight is odd, meaning that it is not the case that the total weight of every cycle in $K_n$ is odd.

($\Leftarrow$) Suppose toward contradiction that $K_n$ has a cycle of odd weight. Then, we will show that there exists a triangle of odd weight in $K_n$.

Base Case: If $n = 3$, then $K_n$ containing a cycle of odd weight means $K_3$, the triangle graph, is of odd weight, so $K_3$ contains a triangle of odd weight.

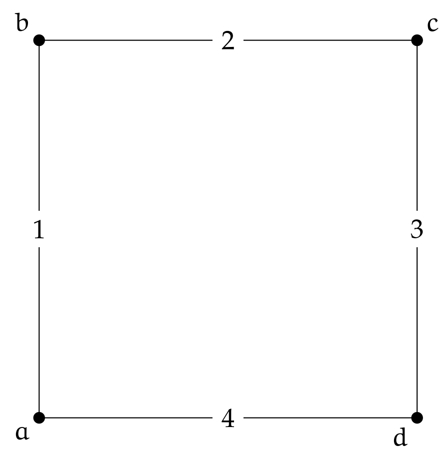
Inductive Hypothesis: If $K_n$ contains a cycle of odd weight, then $K_{n-1}$ for some deleted vertex contains a cycle of odd weight, meaning that $K_{n-1}$ has a triangle of odd weight.

Proof: For any cycle with odd weight, there must be an odd number of edges in the cycle of odd weight. Thus, for any vertex with an *even* number of edges with odd weight incident on it, deletion maintains the parity of the total number of edges with odd weight. So, $K_{n-1}$ for the deleted vertex will still contain an odd cycle, and thus by the inductive hypothesis, $K_{n-1}$ will have a triangle of odd weight.

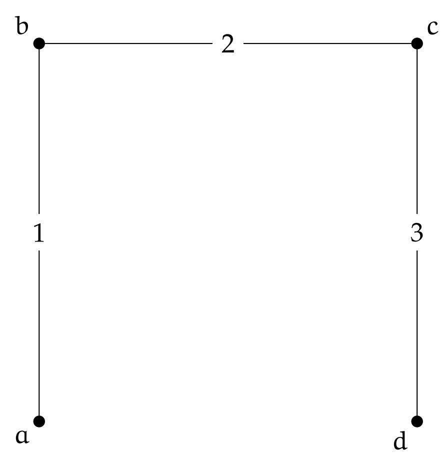
## 2.3.2

Prove or disprove: If $T$ is a minimum weight spanning tree of a weighted graph $G$, then the $u, v$ path in $T$ is the minimum weight $u, v$ path in $G$.

This assertion is **false**, as we can see in the following graph:




The minimum weight spanning tree is as follows:




But, we can see that the shortest weight path between $a$ and $d$ in $G$ is of weight 4, while it is of weight 6 in the minimum weight spanning tree.

## 2.3.5

There are five cities in a network. The travel time for travelling directly from $i$ to $j$ is the entry $a_{i,j}$ in the matrix below. The matrix is not symmetric, and $a_{i,j} = \infty$ indicates there is no direct route. Determine the least travel time and quickest route for each pair $i, j$.

$$\begin{pmatrix} 0 & 10 & 20 & \infty & 17 \\ 7 & 0 & 5 & 22 & 33 \\ 14 & 13 & 0 & 15 & 27 \\ 30 & \infty & 17 & 0 & 10 \\ \infty & 15 & 12 & 8 & 0 \end{pmatrix}$$

Using Dijkstra's algorithm, we are able to find the optimal travel time as seen in the following table (where the column entry represents $i$ and the row entry represents $j$).

| $(i, j)$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 10 | 15 | 25 | 17 |
| 2 | 7 | 0 | 5 | 14 | 24 |
| 3 | 14 | 13 | 0 | 15 | 25 |
| 4 | 30 | 25 | 17 | 0 | 17 |
| 5 | 22 | 15 | 12 | 8 | 0 |

## 2.3.7

Let G be a weighted connected graph with distinct edge weights. Without using Kruskal's algorithm, prove that G has only one minimum weight spanning tree.

Let $E_T = \{e_1, \dots, e_k\}$ be the edge set of the minimum weight spanning tree of $T \subseteq G$. Suppose there exists a $T'$ such that $w(T') \leq w(T)$. Then, by the result from Exercise 2.1.37, we know that we can create $T'$ from $T$ by subtracting one edge in $G$ and adding a different edge in $G$, or that $T' = T - e + e'$.

Since all the edges of $G$ are of distinct weight, then $w(e) \neq w(e')$, meaning that $w(T') \neq w(T)$. Therefore, it must be the case that the inequality is sharp, or that $w(T') < w(T)$ — however, since we had assumed that $T$ was a minimum weight spanning tree, this means $T'$ cannot exist (or else it would be a minimum weight spanning tree), so there is only one minimum weight spanning tree in $G$.

## 2.3.13

Let $T$ be a minimum weight spanning tree and let $T'$ be another spanning tree in $G$. Prove that $T'$ can be turned into $T$ by a list of steps that exchange one edge of $T'$ for one edge of $T$ such that the edge set is always a spanning tree and the total weight never increases.

Let $E(T')$ denote the edge list in $T'$ and let $E(T)$ denote the edge list in $T$. By Theorem 2.1.7, we know that for any edge $e' \in E(T') - E(T)$, there exists an edge $e \in E(T) - E(T')$ such that $T' + e - e'$ is also a spanning tree of $G$. In the case where $T$ is a minimum weight spanning tree of $G$, we know that $T'$ must have a weight greater than or equal to $T$, and that any edge $e'$ that is exchanged for $e$, $w(e') \geq w(e)$. So, at the end of the program (where we have exchanged every $e' \in E(T') - E(T)$ for every $e \in E(T) - E(T')$), we know that any edge exchange must either reduce the total weight of $T'$ or keep the total weight the same.