Contents

Introduction

This is going to be part of the notes for my Honors thesis independent study, which will be focused on amenability and C^*-algebras. This section of notes will be focused on the essential results in functional analysis, starting from normed vector spaces, working our way up through C^*-algebras.

The primary source for this section is going to be Timothy Rainone's *Functional Analysis-En Route to Operator Algebras*, which has not been published yet.

I do not claim any of this work to be original.

Normed Vector Spaces

Vector Spaces, Norms, and Basic Properties

All vector spaces are defined over $\mathbb{C}$. Most of the information here is in my Real Analysis II notes, so I'm going to skip to some of the more important content.

Definition (Vector Space)**.** A vector space V is a set closed under two operations

$$\begin{aligned} a &: V \times V \to V,\ (v_1, v_2) \mapsto v_1 + v_2 \\ m &: \mathbb{C} \times V \to V,\ (\lambda, v) \mapsto \lambda v. \end{aligned}$$

We refer to a as addition, and m as scalar multiplication; $(V, +)$ is an abelian ring.

Definition (Norm)**.** A norm is a function

$$\|\cdot\| : V \to \mathbb{R}^+,\ x \mapsto \|x\|$$

that satisfies the following properties:

- Positive definiteness: $\|v\| = 0$ if and only if $v = 0_V$.
- Triangle inequality: $\|v + w\| \leqslant \|v\| + \|w\|$.
- Absolute Homogeneity: $\|\lambda v\| = |\lambda|\,\|v\|$, for $\lambda \in \mathbb{C}$.

If a function $p : V \to \mathbb{R}^+$ satisfies the triangle inequality and absolute homogeneity, we say p is a seminorm.

We say the pair $(V, \|\cdot\|)$ is a normed vector space.

Definition (Balls and Spheres)**.** Let X be a normed vector space, $x \in X$, and $\delta > 0$. Then,

$$\begin{aligned} U(x,\delta) &= \{y \in X \mid d(x,y) < \delta\} \\ B(x,\delta) &= \{y \in X \mid d(x,y) \leqslant \delta\} \\ S(x,\delta) &= \{y \in X \mid d(x,y) = \delta\}. \end{aligned}$$

For a normed vector space, we will use the following conventions for common sets:

$$\begin{aligned} U_X &= U(0,1) \\ B_X &= B(0,1) \\ S_X &= S(0,1) \\ \mathbb{D} &= U_{\mathbb{C}} \\ \mathbb{T} &= S_{\mathbb{C}}. \end{aligned}$$

Definition (Equivalent Norms)**.** Two norms on V, $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there are two constants C_1 and C_2 such that

$$\begin{aligned} \|v\|_a &\leqslant C_1 \|v\|_b \\ \|v\|_b &\leqslant C_2 \|v\|_a \end{aligned}$$

for all $v \in V$. We say $\|\cdot\|_a \sim \|\cdot\|_b$.

Examples

Example (Finite-Dimensional Vector Spaces)**.** The vector space $\mathbb{C}^n$ is with the p-norm is denoted ℓ_p^n, where for $p \in [1, \infty]$, the p-norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

In the case with $p = 2$, this gives the traditional Euclidean norm, and with $p = \infty$, this gives the sup norm:

$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|.$$

Example (A Sequence Space)**.** We let $\ell_p = \left\{(x_n)_n \mid x_n \in \mathbb{C}, \|x\|_p < \infty\right\}$ be the collection of sequences in $\mathbb{C}$ with finite p-norm. Here,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

In the case with $p = \infty$, this gives the sequence space ℓ_∞, which has norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Example (A Function Space)**.** We let $\ell^\infty(\Omega)$ denote the set of all bounded functions $f : \Omega \to \mathbb{C}$, equipped with the norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|.$$

If $\Omega = (\Omega, \mathcal{M}, \mu)$ is a measure space, then we let $L^\infty(\Omega)$ be the space of μ-a.e. equal essentially bounded measurable functions, under the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

Series Convergence and Completeness

Proposition (Criteria for Banach Spaces)**:** Let X be a normed vector space. The following are equivalent:

(i) X is a Banach space.[1]

(ii) If $(x_k)_k$ is a sequence of vectors such that $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

(iii) If $(x_k)_k$ is a sequence in X such that $\|x_k\| < 2^{-k}$, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. To show (i) implies (ii), for $n > m > N$, we have

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\leqslant \sum_{k=m+1}^{n} \|x_k\| \\ &< \epsilon, \end{aligned}$$

implying that s_n is Cauchy, and thus converges since X is complete.

Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, it is clear that (ii) implies (iii).

To show (iii) implies (i), we let $(x_n)_n$ be a Cauchy sequence in X. We only need construct a convergent subsequence in order to show that $(x_n)_n$ converges.

Chose $n_1 \in \mathbb{N}$ such that for $n, m \geqslant n_1$, $\|x_m - x_n\| < \frac{1}{2^2}$, and inductively define $n_j > n_{j-1}$ such that $n, m \geqslant n_j$ implies $\|x_m - x_n\| < \frac{1}{2^{j+1}}$.

Let $y_1 = x_{n_1}$, $y_j = x_{n_j} - x_{n_{j-1}}$. Then,

$$\begin{aligned} \|y_j\| &= \left\|x_{n_j} - x_{n_{j-1}}\right\| \\ &< \frac{1}{2^j}, \end{aligned}$$

so $\sum_{j=1}^{\infty} y_j$ converges by our assumption. By telescoping, we see that $\sum_{j=1}^{k} y_j = x_{n_k}$, so $(x_{n_k})_k$ converges. □

Quotient Spaces

Let X be a normed vector space. Then, for $E \subseteq X$ a subspace, there is a quotient space X/E with the projection map $\pi : X \to X/E$, $x \mapsto x + E$. We want to make X/E into a normed space — in order to do this, we use the distance function:

$$\operatorname{dist}_E(x) = \inf_{y \in E} d(x, y),$$

which is uniformly continuous. For E closed, then $\operatorname{dist}_E(x) = 0$ if and only if $x \in E$.

Proposition (Quotient Space Norm)**:** Let X be a normed vector space, and $E \subseteq X$ a subspace. Set

$$\|x + E\|_{X/E} = \operatorname{dist}_E(x).$$

Then,

(1) $\|\cdot\|_{X/E}$ is a well-defined seminorm on X/E.

[1] Complete normed vector space.

(2) If E is closed, then $\|\cdot\|_{X/E}$ is a norm on X/E.

(3) $\|x+E\|_{X/E} \leqslant \|x\|$ for all $x \in X$.

(4) If E is closed, then $\pi : X \to X/E$ is Lipschitz.

(5) If X is a Banach space and E is closed, then X/E is also a Banach space.

Proof.

(1) We will show that $\|\cdot\|_{X/E}$ is well-defined. If $x+E = x'+E$, $x'-x \in E$, so for every $y \in E$, $x'-x+y \in E$. Thus,

$$\begin{aligned}\|x-y\| &= \|x' - (x'-x+y)\|\\ &\geqslant \inf_{z\in E} \|x'-z\|\\ &= \|x'+E\|_{X/E}.\end{aligned}$$

Thus, $\|x+E\|_{X/E} \geqslant \|x'+E\|_{X/E}$, and vice versa.

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and $x \in X$. Then,

$$\begin{aligned}\|\lambda(x+E)\|_{X/E} &= \|\lambda x + E\|_{X/E}\\ &= \inf_{y\in E}\|\lambda x - y\|\\ &= |\lambda| \inf_{y\in E}\left\|x - \lambda^{-1}y\right\|\\ &= |\lambda| \inf_{y'\in E}\|x-y\|\\ &= |\lambda|\,\|x+E\|_{X/E}\end{aligned}$$

Given $x, x' \in X$ and a fixed $\varepsilon > 0$, we have

$$\|x+E\| + \frac{\varepsilon}{2} > \|x-y\|$$

for some $y \in E$, and

$$\|x'+E\| + \frac{\varepsilon}{2} > \|x'-y'\|$$

for some $y' \in E$. Thus,

$$\begin{aligned}\|(x+x')-(y+y')\| &\leqslant \|x-y\| + \|x'-y'\|\\ &< \varepsilon + \|x+E\| + \|x'+E\|.\end{aligned}$$

Since $y+y' \in E$, we have

$$\begin{aligned}\|(x+E)+(x'+E)\|_{X/E} &= \|x+x'+E\|_{X/E}\\ &\leqslant \|(x+x')-(y+y')\|\\ &< \varepsilon + \|x+E\|_{X/E} + \|x'+E\|_{X/E},\end{aligned}$$

meaning

$$\|(x+E)+(x'+E)\| \leqslant \|x+E\| + \|x'+E\|.$$

(2) If E is closed, and $\|x+E\| = 0$, then $x \in E$ so $x+E = 0_{X/E}$.

(3) For $x \in X$,

$$\begin{aligned} \|x + E\|_{X/E} &= \inf_{y \in E} \|x - y\| \\ &\leqslant \|x\|. \end{aligned}$$

(4) We have

$$\begin{aligned} \|(x + E) - (x' + E)\|_{X/E} &= \|x - x' + E\|_{X/E} \\ &\leqslant \|x - x'\|. \end{aligned}$$

(5) Let X be complete and $E \subseteq X$ be closed. Let $(x_k + E)_k$ be a sequence in X/E with $\|x_k + E\| < 2^{-k}$. We want to show that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

For each k, since $\|x_k + E\| < 2^{-k}$, there exists $y_k \in E$ such that $\|x_k - y_k\| < 2^{-k}$. Since X is complete, $\sum_{k=1}^{\infty} x_k - y_k$ converges.

Let $\left(\sum_{k=1}^{n} x_k - y_k\right)_n \to x$ in X. Applying the canonical projection map, π, to both sides, we get

$$\begin{aligned} \sum_{k=1}^{n} (x_k + E) &= \sum_{k=1}^{n} \pi(x_k) \\ &= \pi\left(\sum_{k=1}^{n} (x_k - y_k)\right) \\ &\to \pi(x), \end{aligned}$$

implying that $\sum_{k=1}^{\infty} (x_k + E)$ converges.

□

Exercise: Consider ℓ_∞ and its closed subspace c_0. If $\pi : \ell_\infty \to \ell_\infty / c_0$ denotes the canonical quotient map, with $(z_k)_k \in \ell_\infty$, show that

$$\|(z_k)_k + c_0\| = \limsup_{k \to \infty} |z_k|$$

Solution. By the definition of the quotient norm, we have

$$\begin{aligned} \|(z_k)_k + c_0\|_{\ell_\infty / c_0} &= \inf_{(a_k)_k \in c_0} \|(z_k)_k - (a_k)_k\| \\ &= \inf_{(a_k)_k \in c_0} \sup_{k \in \mathbb{N}} |z_k - a_k| \\ &= \limsup_{k \to \infty} |z_k|. \end{aligned}$$

Bounded Linear Operators

Definition (Continuous Functions)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is called Lipschitz if there is a constant $C > 0$ such that

$$d_Y(f(x), f(x')) \leqslant C d_x(x, x')$$

for all $x, x' \in X$.

If $C \leqslant 1$, a Lipschitz map is known as a contraction.

If

$$d_Y(f(x), f(x')) = d_X(x, x')$$

for all $x, x' \in X$, then f is known as an isometry.

Proposition (Categorization of Continuous Linear Maps)**:** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map. The following are equivalent:

(i) T is continuous at 0.

(ii) T is continuous.

(iii) T is uniformly continuous.

(iv) T is Lipschitz.

(v) There exists a constant $C > 0$ such that $\|T(x)\| \leqslant C\|x\|$ for all $x \in X$.

Definition (Bounded Linear Operator)**.** Let X and Y be normed vector spaces, and let $T : X \to Y$ be a linear map.

(1) T is bounded if $T(B_X)$ is bounded in Y. Equivalently, T is bounded if and only if

$$\sup_{x \in B_X} \|T(x)\| < \infty,$$

or that $\exists r > 0$ such that $T(B_X) \subseteq B_Y(0, r)$.

(2) The operator norm of T is the value

$$\|T\|_{op} = \sup_{x \in B_X} \|T(x)\|.$$

Lemma: Let $T : X \to Y$ be a linear map between normed vector spaces. Then,

$$\|T\|_{op} = \sup_{x \in S_X} \|T(x)\|$$

and for all $x \in X$,

$$\|T(x)\| \leqslant \|T\|_{op} \|x\|.$$

Lemma: Let $T : X \to Y$ be a bounded linear map between normed vector spaces. Then, for any $x \in X$ and $r > 0$,

$$r\|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|$$

Proof. Let $C = \sup_{y \in B(x,r)} \|T(y)\|$. If $z \in B(0, r)$, then $z + x, z - x \in B(x, r)$, meaning

$$2T(z) = T(z + x) + T(z - x),$$

so by the triangle inequality, we get

$$\begin{aligned} 2\|T(z)\| &\leqslant \|T(z + x)\| + \|T(z - x)\| \\ &\leqslant 2\max\{\|T(z + x)\|, \|T(z - x)\|\} \\ &\leqslant 2C. \end{aligned}$$

Thus,

$$\|T(z)\| \leqslant \sup_{y \in B(x,r)} \|T(y)\|,$$

meaning

$$r\|T\|_{op} \leqslant \sup_{y \in B(x,r)} \|T(y)\|.$$

∎

Remark: For a linear map $T : X \to Y$, the following are equivalent:

(1) T is continuous.

(2) T is bounded.

(3) $\|T\|_{op} < \infty$.

Definition. Let X and Y be normed spaces, $T : X \to Y$ a linear map.

(1) T is bounded below if there exists C_2 such that $\|T(x)\| \geqslant C_2 \|x\|$ for all $x \in X$.

(2) T is bicontinuous if T is bounded and bounded below.

$$C_2 \|x\| \leqslant \|T(x)\| \leqslant C_1 \|x\|$$

(3) T is a bicontinuous isomorphism if T is bijective, linear, and bicontinuous. We say X and Y are bicontinuously isomorphic.

(4) We say T is an isometric isomorphism if T is bijective, linear, and an isometry.

Example. Let ρ be the continuous surjective wrapping function $\rho : [0, 2\pi] \to \mathbb{T}$, $\rho(t) = e^{it}$. There is an induced isometry

$$T_\rho : C(\mathbb{T}) \to C([0, 2\pi]),$$

defined by $T_\rho (f)(t) = f \circ \rho(t) = f\left(e^{it}\right)$.

The range of T_ρ is $C = \{G \in C([0, 2\pi]) \mid g(0) = g(2\pi)\}$, which means that $C(\mathbb{T})$ and C are isometrically isomorphic Banach spaces.

Proposition: Let X and Y be normed spaces, and $T : X \to Y$ be a linear map. The following are equivalent.

(i) T is bicontinuous.

(ii) $T : X \to \operatorname{Ran}(T)$ is a linear isomorphism and homeomorphism.

Proof. Let T be bicontinuous. Then, T is linear, injective, and surjective onto $\operatorname{Ran}(T)$. Since T is continuous, T is bounded. Let $S : \operatorname{Ran}(T) \to X$ be defined by $S(T(x)) = x$. We can see that S is well-defined, since $T : X \to \operatorname{Ran}(T)$ is surjective, and so has a left inverse. Similarly, since $\|S(T(x))\| = \|x\| \leqslant \frac{1}{C_2} \|T(x)\|$, S is continuous.

Let $S : \operatorname{Ran}(T) \to X$ be defined by $S(T(x)) = x$. Since T is continuous, it is bounded, so

$$\|T(x)\| \leqslant \|T\|_{op} \|x\|.$$

Since S is bounded,

$$\begin{aligned} \|x\| &= \|S(T(x))\| \\ &= \|S\|_{op} \|T(x)\|, \end{aligned}$$

so $\frac{1}{\|S\|_{op}} \|x\| \leqslant \|T(x)\|$. □

Corollary: Let X be a vector space with $\|\cdot\|$ and $\|\cdot\|'$ two norms. The following are equivalent:

(i) The norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

(ii) The map $\mathrm{id}_X : (X, \|\cdot\|) \to \left(X, \|\cdot\|'\right)$.

Proposition (Properties of Bounded Linear Operators)**:** Let X, Y, Z be normed spaces, $T : X \to Y$, $S : X \to Y$, and $R : Y \to Z$ be linear maps.

(1) $\|\alpha T\|_{op} = |\alpha| \|T\|_{op}$

(2) $\|T + S\|_{op} \leqslant \|T\|_{op} + \|S\|_{op}$

(3) $\|T\|_{op} = 0$ if and only if $T = 0$

(4) $\|R \circ T\|_{op} \leqslant \|R\|_{op} \|T\|_{op}$

(5) $\|\mathrm{id}_X\|_{op} = 1$

(6) If $E \subseteq X$ is a subspace, then $\|T|_E\|_{op} \leqslant \|T\|_{op}$

Proof. We will prove (4) here. For $x \in B_X$, we have

$$\begin{aligned}\|R \circ T(x)\| &= \|R(T(x))\| \\ &\leqslant \|R\|_{op} \|T(x)\| \\ &\leqslant \|R\|_{op} \|T\|_{op}.\end{aligned}$$

Taking the supremum, we obtain $\|R \circ T\|_{op} \leqslant \|R\|_{op} \|T\|_{op}$. □

Recall: $\mathcal{L}(X, Y)$ is the set of all linear operators with domain X and codomain Y.

Proposition: Let X and Y be normed spaces.

(1) The collection $\mathcal{B}(X, Y) = \left\{T \in \mathcal{L}(X, Y) \mid \|T\|_{op} < \infty\right\}$ equipped with the operator norm is a normed space known as the space of bounded linear operators between X and Y.

(2) If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

(3) The continuous dual space, $X^* = \mathcal{B}(X, \mathbb{C})$ is a Banach space.

Proof. We will prove (2). Let $(T_n)_n$ be Cauchy under $\|\cdot\|_{op}$. Since Cauchy sequences are bounded, there is some $C > 0$ such that $\|T_n\|_{op} \leqslant C$ for all $n \geqslant 1$. For $x \in X$,

$$\|T_n(x) - T_m(x)\| \leqslant \|T_n - T_m\|_{op} \|x\|,$$

meaning $(T_n(x))_n$ is Cauchy in Y. Since Y is complete, we define

$$T(x) = \lim_{n\to\infty} T_n(x)$$

in Y. If $x \in B_X$, we have

$$\begin{aligned}\|T(x)\| &= \left\|\lim_{n\to\infty} T_n(x)\right\| \\ &= \lim_{n\to\infty} \|T_n(x)\| \\ &\leqslant \limsup_{n\to\infty} \|T_n(x)\| \\ &\leqslant C\|x\|,\end{aligned}$$

meaning $\|T\|_{op} \leqslant C$.

Let $\varepsilon > 0$, and $N \in \mathbb{N}$ large such that $n, m \geqslant N$, $\|T_n - T_m\|_{op} \leqslant \varepsilon$. For $x \in B_X$,

$$\begin{aligned}\|T_n(x) - T(x)\| &= \lim_{m\to\infty} \|T_n(x) - T_m(x)\| \\ &\leqslant \limsup_{m\to\infty} \|T_n - T_m\|_{op} \|x\| \\ &< \varepsilon.\end{aligned}$$

Thus, $\|T - T_n\|_{op} < \varepsilon$ for all $n \geqslant N$. □

Definition (Algebras)**.** Let A be an algebra over $\mathbb{C}$.

(1) If A admits a norm $\|\cdot\|$ satisfying $\|ab\| \leqslant \|a\| \|b\|$, then A is a normed algebra. If A is unital, then $\|1_A\| = 1$.

(2) If A is complete with respect to its norm, then A is called a Banach algebra, and if A is unital, then A is a unital Banach algebra.

Lemma: In a normed algebra A, the map $\cdot : A \times A \to A, (a, b) \mapsto ab$ is continuous.

Proposition: Let X be a normed space. The set of bounded operators $\mathcal{B}(X, X) = \mathcal{B}(X)$ is a unital normed algebra. Moreover, if X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

Proposition: Let A be a unital Banach algebra, $a \in A$. The series

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges absolutely in A. We call $\exp(a)$ the exponential of a.

(1) $\exp(0) = 1_A$

(2) If A is commutative, then $\exp(a + b) = \exp(a)\exp(b)$.

(3) We have $\exp(a) \in GL(A)$ with $\exp(a)^{-1} = \exp(-a)$.

(4) $\left\|\exp(a)\right\| \leqslant \exp\left(\|a\|\right)$.

Quotient Maps

Definition. A map $f : X \to Y$ is called open if $U \subseteq X$ is open implies $f(U) \subseteq Y$ is open.

Proposition: Let X and Y be normed spaces, $T : X \to Y$ a linear map. The following are equivalent:

(i) T is surjective and open.

(ii) $T(U_X) \subseteq Y$ is open.

(iii) There exists $\delta > 0$ such that $\delta U_Y \subseteq T(U_X)$.

(iv) There exists δ such that $\delta B_Y \subseteq T(B_X)$.

(v) There exists $M > 0$ such that for all $y \in Y$, there exists $x \in X$ with $T(x) = y$ and $\|x\| \leqslant M\|y\|$.

Proof. To see (i) implies (ii), if T is surjective and open, then it is clear that $T(U_X)$, which is the image of an open set, is open.

To see (ii) implies (iii), if $T(U_X)$ is open, we have $0_Y \in T(U_X)$, so there is some δ such that $U(0, \delta) \subseteq T(U_X)$, meaning $\delta U_Y \subseteq T(U_X)$.

Assuming (iii), we see that $\frac{\delta}{2} B_Y \subseteq \delta U_Y \subseteq T(U_X) \subseteq T(B_X)$.

To see (iv) implies (v), let δ be such that $\delta B_Y \subseteq T(B_X)$, and set $M = \frac{1}{\delta}$. Note that for $y \in Y, y \neq 0$, $\frac{\delta}{\|y\|} y \in \delta B_Y$, meaning $\frac{\delta}{\|y\|} y = T(x)$ for some $x \in B_X$, implying that $T\left(\frac{\|y\|}{\delta} x\right) = y$. Finally, since $x \in B_X$, $\frac{\|y\|}{\delta}\|x\| \leqslant \frac{1}{\delta}\|y\| = M\|y\|$.

To see (v) implies (i), we can see that T is surjective by the assumption. Let $U \subseteq X$ be open, $y_0 \in T(U)$. Then, there exists x_0 such that $T(x_0) = y_0$, and $\delta > 0$ such that $U(x_0, \delta) \subseteq U$. Note that $U(x_0, \delta) = x_0 + \delta U_X$, so $x_0 + \delta U_X \subseteq U$. Applying T, we get $T(x_0 + \delta U_X) \subseteq T(U)$, or $y_0 + \delta T(U_X) \subseteq T(U)$. By assumption, since given $y \in U_Y$, there exists $x \in X$ such that $\|x\| \leqslant M\|y\|$, meaning $\|x\| \leqslant M$, we have $U_Y \subseteq T(MU_X)$. Thus, $\frac{1}{M} U_Y \subseteq T(U_X)$, meaning $y_0 + \frac{\delta}{M} U_Y \subseteq y_0 \delta T(U_X) \subseteq T(U)$, so $U_Y\left(y_0, \frac{\delta}{M}\right) \subseteq T(U)$. ◻

Definition. Let X and Y be normed vector spaces.

(1) A bounded linear map $T : X \to Y$ that is surjective and open is known as a quotient map.

(2) If $T(U_X) = U_Y$, then T is called a 1-quotient map.

Exercise: If $T(B_X) = B_Y$, show that $T(U_X) = U_Y$.

Solution. Since $T(B_X) = B_Y$, it is the case that $(T(B_X))^\circ = B_Y^\circ$. Since T is an open map, T is continuous, meaning $(T(B_X))^\circ = T\left(B_X^\circ\right)$. Thus, $T(U_X) = U_Y$.

Proposition: Let X and Y be normed vector spaces with $T : X \to Y$ a quotient map. If X is a Banach space, then Y is a Banach space.

Proof. We will show that Y is complete by showing that an absolutely convergent series converges.

Let $(y_k)_k$ be a sequence in Y with $\sum_{k=1}^{\infty} \|y_k\| < \infty$. Since T is a quotient map, there is a universal $M > 0$ such that for all k, there is $x_k \in X$ such that $T(x_k) = y_k$ and $\|x_k\| \leqslant M \|y_k\|$. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} &\leqslant M \sum_{k=1}^{\infty} \|y_k\| \\ &< \infty. \end{aligned}$$

Since X is complete, $\sum_{k=1}^{\infty} x_k$ converges. Let $\sum_{k=1}^{\infty} x_k = x$. Then, $\left(T\left(\sum_{k=1}^{n} x_k\right)\right)_n \xrightarrow{n\to\infty} T(x)$, meaning $\sum_{k=1}^{\infty} y_k = T(x)$. Thus, $\sum_{k=1}^{\infty} y_k$ converges in Y, so Y is a Banach space. ◻

Proposition: Let X be a normed vector space, $E \subseteq X$ a closed subspace. The canonical quotient map, $\pi : X \to X/E$ is a 1-quotient map.

Proof. We know that $\|\pi(x)\| \leqslant \|x\|$, meaning $\pi(U_X) \subseteq U_{X/E}$.

Let $\pi(x) = x + E \subseteq U_{X/E}$. Then, $\inf_{y\in E} \|x - y\| \leqslant 1$, meaning there exists some y such that $\|x - y\| < 1$, meaning $\pi(x - y) = \pi(x)$. ◻

Corollary: If X is a Banach space, $E \subseteq X$ a closed subspace, then X/E is a Banach space.

Corollary: Let X be a normed vector space and $E \subseteq X$ be closed. If two of $X, E, X/E$ are complete, the third is also complete.

Proof. We have shown that if X is complete, then E is necessarily complete (since E is closed) and X/E is complete as shown above.

Let E and X/E be complete. We now want to show that X is complete. Let $(x_k)_k$ be Cauchy in X.

For each k, let $x_k = s_k + y_k$, where $y_k \in E$ and $s_k + E = \pi(x_k)$. Notice that, since x_k is Cauchy, so too is s_k, as $\|s_k\| \leqslant \|x_k\|$ for all k. Additionally, for $m, n \geqslant N$, we have

$$\begin{aligned} \|x_m - x_n\| &= \|s_m + y_m - (s_n + y_n)\| \\ &\leqslant \|s_m - s_n\| + \|y_m - y_n\| \\ &< \varepsilon, \end{aligned}$$

implying that $(y_k)_k$ is Cauchy in E. Since X/E and E are complete, we define $x = \lim_{k\to\infty} s_k + \lim_{k\to\infty} y_k$. Finally, for $m, n \geqslant N$, we have

$$\begin{aligned} \|x - x_n\| &= \lim_{m\to\infty} \|x_m - x_n\| \\ &\leqslant \varepsilon, \end{aligned}$$

meaning $(x_k)_k \xrightarrow{k\to\infty} x$, so X is complete. ◻

Proposition: Let X and Y be normed spaces, $E \subseteq X$ a closed subspace, and $T : X \to Y$ bounded linear with $E \subseteq \ker(T)$. Then, there exists a unique bounded linear map $\overline{T} : X/E \to Y$ such that $\overline{T} \circ \pi = T$. Moreover, $\overline{T}$ is injective if and only if $E = \ker(T)$ and $\left\|\overline{T}\right\| = \|T\|$.

Proof. The existence and uniqueness of $\overline{T} : X/E \to Y$ such that $\overline{T} \circ \pi = T$ follows from the First Isomorphism Theorem for vector spaces, as does the fact that $\overline{T}$ is injective and only if $\ker(T) = E$.

Let $x + E \in X/E$. For $y \in E$, we have

$$\begin{aligned}\left\|\overline{T}(x+E)\right\| &= \left\|\overline{T}(x-y+E)\right\|\\ &= \|T(x-y)\|\\ &\leqslant \|T\|\,\|x-y\|.\end{aligned}$$

Taking infimum over all $y \in E$, we get $\left\|\overline{T}(x+E)\right\| \leqslant \|T\|\,\|x+E\|$, meaning $\left\|\overline{T}\right\| \leqslant \|T\|$. Additionally,

$$\begin{aligned}\|T\| &= \left\|\overline{T} \circ \pi\right\|\\ &\leqslant \left\|\overline{T}\right\| \|\pi\|\\ &= \left\|\overline{T}\right\|.\end{aligned}$$

□

Theorem (First Isomorphism Theorem for Normed Vector Spaces)**:** Let X and Y be normed vector spaces, $T \in \mathcal{B}(X, Y)$.

(1) T is a quotient map if and only if $\overline{T} : X/\ker(T) \to Y$ is a bicontinuous isomorphism.

(2) T is a 1-quotient map if and only if $\overline{T} : X/\ker(T) \to Y$ is an isometric isomorphism.