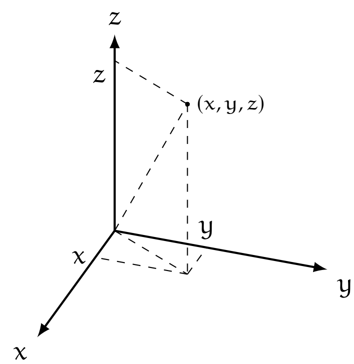
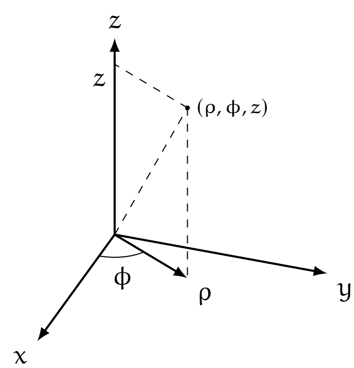
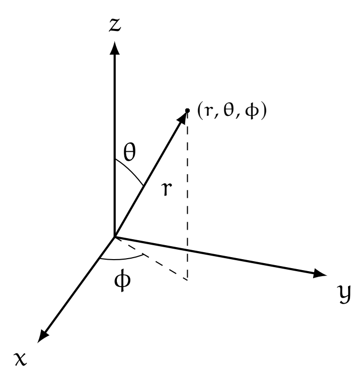
Things You Just Gotta Know

Coordinate Systems





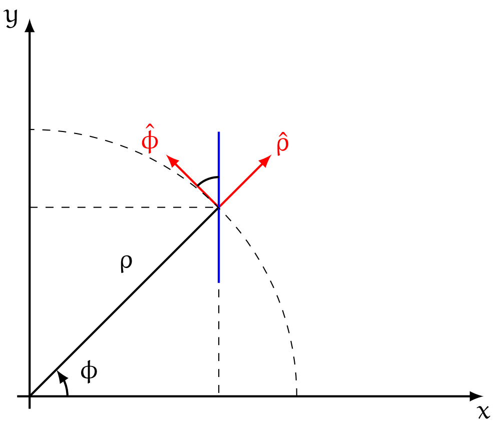


We want to focus on vector-valued functions of coordinates.

$$\vec{V}(\mathbf{r}) = V_x(x,y)\hat{\imath} + V_y(x,y)\hat{\jmath}.$$

Notice that a vector function uses the coordinate system twice. Once for the function's inputs, once for the vectors themselves.

Polar Coordinates



We can also express the inputs to $\vec{V}$ in polar coordinates, (ρ, ϕ).

$$\vec{V}(\mathbf{r}) = V_\rho(\rho,\phi)\,\hat{\imath} + V_\phi(\rho,\phi)\,\hat{\jmath}.$$

To extract the input functions, we take

$$\begin{aligned} V_x &= \hat{\imath}\cdot\vec{V} \\ V_y &= \hat{\jmath}\cdot\vec{V}. \end{aligned}$$

Alternatively, we can project $\vec{V}$ onto the $\hat{\rho}, \hat{\phi}$ axis:

$$\vec{V}(\mathbf{r}) = V_\rho(\rho,\phi)\,\hat{\rho} + V_\phi(\rho,\phi)\,\hat{\phi},$$

and we extract

$$\begin{aligned} V_\rho &= \hat{\rho}\cdot\vec{V} \\ V_\phi &= \hat{\phi}\cdot\vec{V}. \end{aligned}$$

Notice that $\mathbf{r}$ is an abstract vector; we need to project it onto a basis.

For instance, we can take the position vector and project it onto the cartesian and polar axes:

$$\begin{aligned} \mathbf{s} &= x\hat{\imath} + y\hat{\jmath} \\ &= \rho\cos\phi\hat{\imath} + \rho\sin\phi\hat{\jmath} \\ &= \rho\hat{\rho} \\ &= \sqrt{x^2+y^2}\hat{\rho} \end{aligned}$$

The main reason we avoided using the $\hat{\rho}, \hat{\phi}$ axis up until this point is that ρ and ϕ are *position-dependent*, while the $\hat{\imath}, \hat{\jmath}$ axis is position-independent.

Now, we must figure out the position-dependence of $\hat{\rho}$ and $\hat{\phi}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho}d\rho + \frac{\partial \mathbf{r}}{\partial \phi}d\phi.$$

If we hold ϕ constant, it must be the case that any change in ρ is in the $\hat{\rho}$ direction. Therefore,

$$\hat{\rho} = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|}$$

$$
\begin{aligned}
&= \frac{\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}}{\left|\cos\phi\hat{\imath} + \sin\phi\hat{\jmath}\right|}\\
&= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\hat{\phi} &= \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left\|\frac{\partial \mathbf{r}}{\partial \rho}\right\|}\\
&= \frac{-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}}{\left\|-\rho\sin\phi\hat{\imath} + \rho\cos\phi\hat{\jmath}\right\|}\\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}.
\end{aligned}
$$

Thus, we can see that the $\hat{\rho}, \hat{\phi}$ axis is orthogonal.

$$
\begin{aligned}
\frac{\partial\hat{\rho}}{\partial\phi} &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\\
&= \hat{\phi},\\
\frac{\partial\hat{\phi}}{\partial\phi} &= -\hat{\rho},\\
\frac{\partial\hat{\phi}}{\partial\rho} &= 0,
\end{aligned}
$$

and

$$
\frac{\partial\hat{\rho}}{\partial\rho} = 0
$$

Example (Velocity)**.**

$$
\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{s}}{dt}\\
&= \frac{d}{dt}\left(x\hat{\imath}\right) + \frac{d}{dt}\left(y\hat{\jmath}\right).
\end{aligned}
$$

In the case of cartesian coordinates, $\hat{\imath}$ and $\hat{\jmath}$ are constants.

$$
= v_x\hat{\imath} + v_y\hat{\jmath}
$$

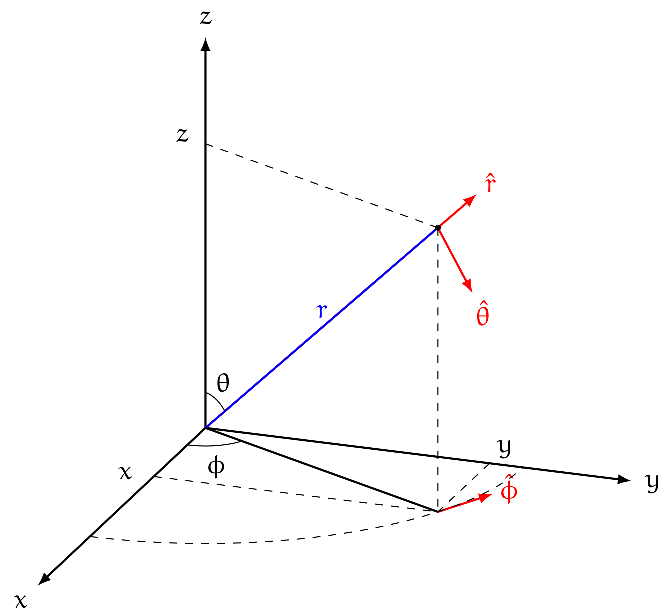
When we examine polar coordinates, since $\hat{\rho}$ and $\hat{\phi}$ are position-dependent, we must use the chain rule.[1]

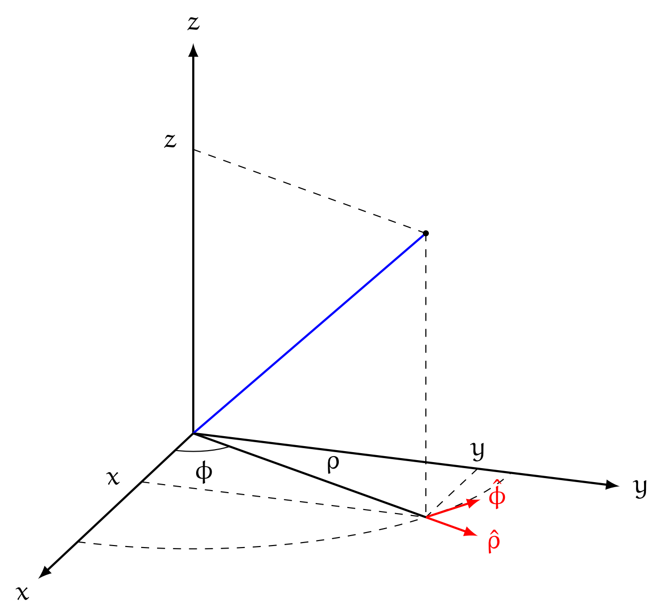
$$
\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{s}}{dt}\\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt}\\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\cancelto{0}{\frac{\partial\hat{\rho}}{\partial\rho}}\frac{d\rho}{dt} + \underbrace{\frac{\partial\hat{\rho}}{\partial\phi}}_{=\hat{\phi}}\frac{d\phi}{dt}\right)\\
&= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi}\\
&= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}.
\end{aligned}
$$

Notice that $\dot{\rho}$ is the radial velocity and $\dot{\phi} = \omega$ is the angular velocity.

[1]Note that $\hat{\rho} = \hat{\rho}\left(\rho, \phi\right)$ and $\hat{\phi} = \hat{\phi}(\rho, \phi)$.

Spherical and Cylindrical Coordinates





Polar	Cylindrical	Spherical
$\mathbf{s} = s(\rho, \phi)$	$\mathbf{s} = s(\rho, \phi, z)$	$\mathbf{s} = s(r, \phi, \theta)$
$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$	$\mathbf{s} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$

Here,[II] ϕ denotes the polar angle and θ denotes the azimuthal angle. Notice that $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

[II]Physicists amirite?

We can see that $\hat{\rho}$, $\hat{\phi}$, and $\hat{\theta}$ in spherical coordinates are also position-dependent.

$$\begin{aligned}
\hat{r} &= \frac{\frac{\partial \mathbf{s}}{\partial r}}{\left\|\frac{\partial \mathbf{s}}{\partial r}\right\|} \\
&= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k} \\
\hat{\phi} &= \frac{\frac{\partial \mathbf{s}}{\partial \phi}}{\left\|\frac{\partial \mathbf{s}}{\partial \phi}\right\|} \\
&= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
\hat{\theta} &= \frac{\frac{\partial \mathbf{s}}{\partial \theta}}{\left\|\frac{\partial \mathbf{s}}{\partial \theta}\right\|} \\
&= \cos\phi\cos\theta\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}
\end{aligned}$$

Scale Factors and Jacobians

Coordinate System	Line Element	Area Element	Volume Element
Polar	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi$	$d\mathbf{a} = r\ drd\phi$	—
Cylindrical	$d\mathbf{s} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz$	—	$d\mathbf{v} = r\ drd\phi dz$
Spherical	$d\mathbf{s} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta$	$d\mathbf{a} = r^2\sin\theta d\phi d\theta$	$d\mathbf{v} = r^2\sin\theta\ drd\phi d\theta$

In cylindrical coordinates, we can use the chain rule to find the value of $d\mathbf{r}$:

$$d\mathbf{r} = \hat{\rho}d\rho + \rho\hat{\phi}d\phi + \hat{k}dz.$$

The extra factor of ρ in the expression of $\rho\hat{\phi}d\phi$ is the *scale factor* on ϕ.

Similarly, in spherical coordinates, we have

$$d\mathbf{r} = \hat{r}dr + r\sin\theta\hat{\phi}d\phi + r\hat{\theta}d\theta,$$

with scale factors of $r\sin\theta$ on $\hat{\phi}d\phi$ and r on $\hat{\theta}d\theta$.

When we go from line elements (of the form $d\mathbf{r}$) to area elements (of the form $d\mathbf{a}$), we can see that the area element in polar coordinates is $d\mathbf{a} = \rho d\rho d\phi$ — we need the extra factor of ρ to account for the fact that the magnitude of the area element scales with the radius.

Similarly, the volume element in cylindrical coordinates is $d\mathbf{v} = r dr d\phi dz$ and the volume element in spherical coordinates is $r^2\sin\theta dr d\phi d\theta$.

Recall that the definition of an angle ϕ that subtends an arc length s is $\phi\frac{s}{r}$, where r is the radius of a circle. We can imagine a similar concept on a sphere — a *solid angle* measured in steradians is of the form $\Omega = \frac{A}{r^2}$, where A denotes the surface area subtended by the angle Ω. In particular, since $d\Omega = \frac{dA}{r^2}$, we find that $d\Omega = \sin\theta d\phi d\theta$.

When we are dealing with products of scale factors, we need to use the Jacobian to determine the proper scale factor on any given element:

$$\begin{aligned}
d\mathbf{a} &= dxdy \\
&= |J|\ dudv,
\end{aligned}$$

where $|J|$ denotes the determinant of the Jacobian matrix. We write the Jacobian as follows:

$$\begin{aligned} J &= \frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}\\ &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{pmatrix}. \end{aligned}$$

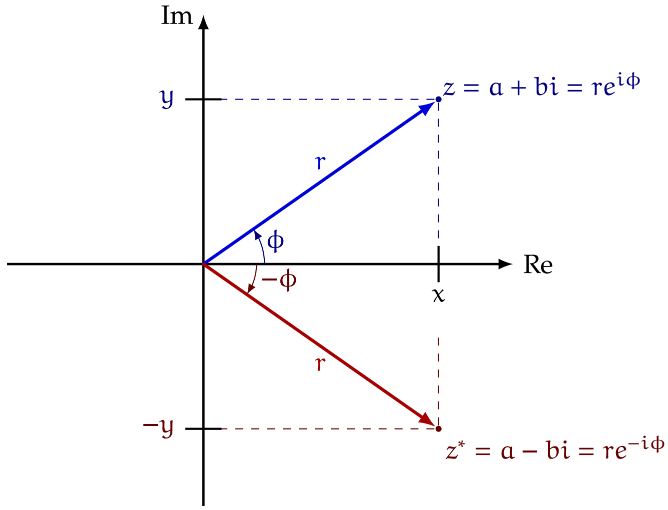
We specifically desire the determinant:

$$|J| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Complex Numbers

Quantity	Expression and/or Criterion
Cartesian form	$z = a + bi$
Polar form	$z = re^{i\phi}$
r	$\sqrt{a^2 + b^2}$
ϕ	$\arg z = \arctan\left(\frac{b}{a}\right)$
Cartesian z^*	$z^* = a - bi$
Polar z^*	$z = re^{-i\phi}$
$\lvert z\rvert$	$\sqrt{zz^*}$
$\text{Re}(z)$	$\text{Re}(z) = \frac{z+z^*}{2}$
$\text{Im}(z)$	$\text{Im}(z) = \frac{z-z^*}{2i}$
$\cos\phi$	$\frac{e^{i\phi}+e^{-i\phi}}{2}$
$\sin\phi$	$\frac{e^{i\phi}-e^{-i\phi}}{2i}$
$e^{i\phi}$	$\cos\phi + i\sin\phi$
$e^{in\phi}$	$\cos\left(n\phi\right) + i\sin\left(n\phi\right)$

Introduction



A complex number is denoted

$$z = a + bi$$

where $i^2 = -1$ and $a, b \in \mathbb{R}$. This is known as the cartesian representation. However, we can also imagine z as the polar representation:

$$z = re^{i\phi},$$

where $\phi = \arg z$ is known as the argument, and $r = |z|$ is the modulus. We can see the relation between the cartesian and polar representations through Euler's identity:[III]

$$r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

We denote the conjugate of z as z^*[IV], found by $z^* = a - bi = re^{-i\phi}$.

We find $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, the real and imaginary parts of z, by

$$\mathrm{Re}(z) = \frac{z + z^*}{2}$$
$$\mathrm{Im}(z) = \frac{z - z^*}{2i}.$$

We say that a complex number of the form $e^{i\phi}$ is a *pure phase*, as $\left|e^{i\phi}\right| = 1$.

To find if some complex number z is purely real or purely imaginary, we can use the following criterion:

$$z \in \mathbb{R} \Leftrightarrow z = z^*$$
$$z \in i\mathbb{R} \Leftrightarrow z = -z^*.$$

Example (Real, Imaginary, or Complex?)**.** Consider

$$z_1 = i^i.$$

To find if this is purely real or complex, we take

$$z_1^* = (-i)^{-i}$$

[III]This can be proven relatively easily through substitution into the Taylor series, which is allowed because e^z is entire.
[IV]Physicists amirite?

$$
\begin{aligned}
&= \left(\frac{1}{-i}\right)^{i} \\
&= i^{i}.
\end{aligned}
$$

Thus, $z_1 \in \mathbb{R}$. In order to determine the value of i^i, we substitute the polar form:

$$
\begin{aligned}
z_1 &= \left(e^{i\frac{\pi}{2}}\right)^{i} \\
&= e^{-\frac{\pi}{2}}.
\end{aligned}
$$

Some Trigonometry with Complex Exponentials

Consider $z = \cos\phi + i\sin\phi$. We can see that

$$
\begin{aligned}
\mathrm{Re}(z) &= \cos\phi \\
&= \frac{(\cos\phi + i\sin\phi) + (\cos\phi - i\sin\phi)}{2} \\
&= \frac{e^{i\phi} + e^{-i\phi}}{2} \\
\mathrm{Im}(z) &= \sin\phi \\
&= \frac{(\cos\phi + i\sin\phi) - (\cos\phi - i\sin\phi)}{2i} \\
&= \frac{e^{i\phi} - e^{-i\phi}}{2i}.
\end{aligned}
$$

We can actually define $\sin\phi$ and $\cos\phi$ with the above derivation.

Theorem (De Moivre)**.**

$$
\begin{aligned}
e^{inx} &= \cos(nx) + i\sin(nx) \\
&= \left(e^{ix}\right)^{n} \\
&= (\cos x + i\sin x)^{n}.
\end{aligned}
$$

Example (Finding $\cos(2x)$ and $\sin(2x)$)**.**

$$
\begin{aligned}
\cos(2x) + i\sin(2x) &= (\cos x + i\sin x)^2 \\
&= \left(\cos^2 x - \sin^2 x\right) + i\left(2\sin x\cos x\right).
\end{aligned}
$$

Since the real parts and imaginary parts have to be equal, this means

$$
\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
\sin^2 x &= 2\sin x\cos x.
\end{aligned}
$$

In particular, we can see that $e^{in\phi} = (-1)^n$ and $e^{in\frac{\pi}{2}} = i^n$.[v]

Additionally, we can see that for $z = re^{i\phi}$,

$$
\begin{aligned}
z^{1/m} &= \left(re^{i\phi + 2\pi n}\right)^{1/m} \\
&= r^{1/m}e^{i\frac{1}{m}(\phi + 2\pi n)},
\end{aligned}
$$

where $n \in \mathbb{N}$ and m is fixed. For $r = 1$, we call these values the m roots of unity.

[v] This will be especially useful when we get to Fourier series.

Example (Waves and Oscillations)**.** Recall that for a wave with spatial frequency k, angular frequency ω, and amplitude A, the wave is represented by

$$f(x,t) = A\cos(kx - \omega t).$$

The speed of a wave v is equal to $\frac{\omega}{k}$.

Simple harmonic motion is characterized by the solution to the differential equation $\ddot{\mathbf{x}} = -\omega^2\mathbf{x}$, where $\mathbf{x}$ denotes position. In simple harmonic motion, there is no spatial motion, meaning our function is only of time:

$$\begin{aligned} f(t) &= A\cos\omega t \\ &= \mathrm{Re}\left(Ae^{i\omega t}\right). \end{aligned}$$

As a result of the representation of complex numbers in polar form, we can do math entirely in exponentials, then take the real part of our solution to find $f(t)$.

Unfortunately, in the real world, there is friction; as a result, our oscillation is damped by an exponential factor.

Example (Hyperbolic Sine and Hyperbolic Cosine)**.** We wish to calculate $\cos ix$ and $\sin ix$.

$$\begin{aligned} \cos ix &= \frac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right) \\ &= \frac{e^{-x} + e^{x}}{2} \end{aligned}$$

We define $\cosh x = \cos(ix)$. Additionally,

$$\begin{aligned} -i\sin ix &= -i\frac{1}{2i}\left(e^{i(ix)} - e^{-i(ix)}\right) \\ &= i\frac{e^{ix} - e^{-ix}}{2i}. \\ &= \frac{e^{x} - e^{-x}}{2}. \end{aligned}$$

We define $\sinh x = -i\sin(ix)$.

Similar to how $\cos^2 x + \sin^2 x = 1$, we can find that $\cosh^2 x - \sinh^2 x = 1$.

Index Algebra

We usually denote vectors by either $\vec{A}$, $\mathbf{A}$, or

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is defined by a basis.

If we imagine we are in n-dimensional space, we can let A_i where $i = 1, 2, \dots, n$ denote both

- the ith component of $\vec{A}$;
- the entire vector $\vec{A}$ (since i can be arbitrary).

Contractions and Dummy Indices

Consider $C = AB$, where A, B are $n \times m$ and $m \times p$ matrices respectively.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}.$$

Definition (Matrix Multiplication in Index Notation)**.** For matrices A and B, where A is an $m \times n$ and B is a $n \times p$ matrix, we write

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

We say that k is a dummy index, since k takes values from 1 to n. Note that the value we calculate is C_{ij}; in other words, in the sum $\sum_k A_{ik}B_{kj}$, the indices of the form ij are the "net indices" from the multiplication.

Note that if $C = BA$, then

$$\begin{aligned} C_{ij} &= \sum_{k=1}^{n} B_{ik}A_{kj} \\ &= \sum_{k=1}^{n} A_{kj}B_{ik} \\ &\neq \sum_{k=1}^{n} A_{ik}B_{kj}. \end{aligned}$$

The corresponding fact is that $AB \neq BA$ necessarily.

Note that the index that is summed over always appears exactly twice.

Definition (Symmetric Matrix)**.** Let C be a matrix. Then, we say C is symmetric if

$$C_{ij} = C_{ji}$$

Definition (Antisymmetric Matrix)**.** Let C be a matrix. We say C is antisymmetric if

$$C_{ij} = -C_{ji}.$$

We can always decompose a random matrix into the sum of a symmetric matrix and an antisymmetric matrix.

Two Special Tensors

Name	Notation	Definition
Kronecker Delta	δ_{ij}	$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
Levi–Civita Symbol	ϵ_{ijk}	$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3) \text{ cyclically} \\ -1 & (i,j,k) = (2,1,3) \text{ cyclically} \\ 0 & \text{else} \end{cases}$

Order of (i, j, k)	Value of ϵ_{ijk}
$1, 2, 3$	1
$3, 1, 2$	1
$2, 3, 1$	1
$1, 3, 2$	-1
$2, 1, 3$	-1
$3, 2, 1$	-1
else	0

Value	Index Notation
$\mathbf{A} \times \mathbf{B}$	$\sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{e}_k$
$(\mathbf{A} \times \mathbf{B})_\ell$	$\sum_{i,j} \epsilon_{ij\ell} A_i B_j$
$\left(\hat{e}_i \times \hat{e}_j\right) \cdot \hat{e}_k$	ϵ_{ijk}
B_i	$\sum_{\alpha} B_\alpha \delta_{\alpha i}$
$\mathbf{A} \cdot \mathbf{B}$	$\sum_{i,j} A_i B_j \delta_{ij}$
$\sum_{j,k} \epsilon_{mjk} \epsilon_{njk}$	$2\delta_{mn}$
$\sum_{\ell} \epsilon_{mn\ell} \epsilon_{ij\ell}$	$\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$

Definition (Kronecker Delta)**.** The Kronecker Delta, δ_{ij}, is the tensor that denotes the identity matrix.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example (Extracting an Index)**.** Consider A as vector. Then,

$$\sum_i A_i \delta_{ij} = A_j.$$

In other words, the Kronecker Delta collapses the sum to the jth index.

Example (Orthonormal Basis from Kronecker Delta)**.** Let $\{\hat{e}_i\}_{i=1}^n$ be a basis for some vector space V. If

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

for every i, j, then $\{\hat{e}_i\}_{i=1}^n$ is an orthonormal basis for V.

Definition (Levi–Civita Symbol)**.** In two dimensions, as a matrix, we write

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

meaning

$$\epsilon_{ij} = \begin{cases} 1 & i = 1, j = 2 \\ -1 & i = 2, j = 1 \\ 0 & \text{else} \end{cases}.$$

The Levi–Civita Symbol is antisymmetric, just as the Kronecker Delta is symmetric.

In three dimensions, we define

$$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3) \text{ cyclically} \\ -1 & (i,j,k) = (2,1,3) \text{ cyclically} \\ 0 & \text{else} \end{cases}.$$

In other words, $\epsilon_{ijk} = -\epsilon_{jik}$.

Exercise (Relations between δ_{ij} and ϵ_{ijk})**.**

$$\sum_{j,k} \epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn}$$
$$\sum_{\ell} \epsilon_{mn\ell}\epsilon_{ijl} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$$

Definition (Dot Product)**.** Let $\{\hat{e}_i\}_{i=1}^{n}$ be an orthonormal basis for V. Let $\mathbf{A} = \sum_i A_i\hat{e}_i$ and $\mathbf{B} = \sum_i B_i\hat{e}_i$. Then,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \sum_{i,j} (A_i\hat{e}_i)\cdot\left(B_j\hat{e}_j\right)\\ &= \sum_{i,j} A_iB_j\left(\hat{e}_i\cdot\hat{e}_j\right)\\ &= \sum_{i,j} A_iB_j\delta_{ij}\\ &= \sum_{i} A_iB_i\end{aligned}$$

Definition (Cross Product)**.** Let $\{\hat{e}_i\}_{i=1}^{3}$ be the standard basis over $\mathbb{R}^3$. Let $\mathbf{A} = \sum_i A_i\hat{e}_i$ and $\mathbf{B} = \sum_i B_i\hat{e}_i$. Then,

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \sum_{i,j} (A_i\hat{e}_i)\times\left(B_j\hat{e}_j\right)\\ &= \sum_{i,j} A_iB_j\left(\hat{e}_i\times\hat{e}_j\right)\\ &= \sum_{i,j,k} A_iB_j\left(\epsilon_{ijk}\hat{e}_k\right).\end{aligned}$$

Instead of asking about $\mathbf{A}\times\mathbf{B}$, we ask about $(\mathbf{A}\times\mathbf{B})_\ell$, yielding

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})_\ell &= (\mathbf{A}\times\mathbf{B})\cdot\hat{e}_\ell\\ &= \left(\sum_{i,j,k} A_iB_j\left(\epsilon_{ijk}\hat{e}_k\right)\right)\cdot\hat{e}_\ell\\ &= \sum_{i,j} \epsilon_{ij\ell}A_iB_j.\end{aligned}$$

Remark: This notation for $\mathbf{A}\times\mathbf{B}$ automatically shows us that

$$(\mathbf{B}\times\mathbf{A})_\ell = \sum_{i,j} \epsilon_{ij\ell}B_iA_j$$

$$\begin{aligned}
&= -\sum_{i,j} \epsilon_{ji\ell} B_i A_j \\
&= -\sum_{i,j} \epsilon_{ji\ell} A_j B_i \\
&= -\sum_{i,j} \epsilon_{ij\ell} A_i B_j && i,j \text{ are dummy indices} \\
&= -\left(\mathbf{A}\times\mathbf{B}\right)_\ell .
\end{aligned}$$

Example (Central Force and Angular Momentum)**.** A central force is defined by

$$\mathbf{F} = f(r)\hat{r},$$

where $\hat{r}$ is a radial vector.

Angular momentum is defined by

$$\mathbf{L} = \mathbf{r}\times\mathbf{p},$$

where $\mathbf{r}$ denotes position and $\mathbf{p}$ denotes momentum. Then,

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\mathbf{r}\times\mathbf{p}\right) \\
&= \left(\frac{d}{dt}\mathbf{r}\times\mathbf{p}\right) + \mathbf{r}\times\left(\frac{d\mathbf{p}}{dt}\right) \\
&= m\left(\frac{d}{dt}\mathbf{r}\times\frac{d}{dt}\mathbf{r}\right) + \mathbf{r}\times\left(f(r)\hat{r}\right) \\
&= f(r)\left(\mathbf{r}\times\hat{r}\right).
\end{aligned}$$

This implies that $\frac{d\mathbf{L}}{dt} = 0$ under a central force.

Example (Determinant)**.** Let $\mathbf{M} = M_{ij}$ be square. We denote $\mathbf{M}_i$ to be the vector denoting the ith-row. Then,

$$\begin{aligned}
m &= |\mathbf{M}| \\
&= \mathbf{M}_1\cdot\left(\mathbf{M}_2\times\mathbf{M}_3\right) \\
&= \mathbf{M}_3\cdot\left(\mathbf{M}_1\times\mathbf{M}_2\right) \\
&= \mathbf{M}_2\cdot\left(\mathbf{M}_3\times\mathbf{M}_1\right).
\end{aligned}$$

Example (Trace)**.** Let $\mathbf{M} = M_{ij}$ be a square matrix. We define $\operatorname{tr}(\mathbf{M}) = \sum_i M_{ii}$. Equivalently,

$$\begin{aligned}
\operatorname{tr}(\mathbf{M}) &= \sum_{ij} M_{ij}\delta_{ij} \\
&= \sum_i M_{ii}.
\end{aligned}$$

Note that

$$\begin{aligned}
\operatorname{tr}(I_n) &= \sum_i \delta_{ii} \\
&= n.
\end{aligned}$$

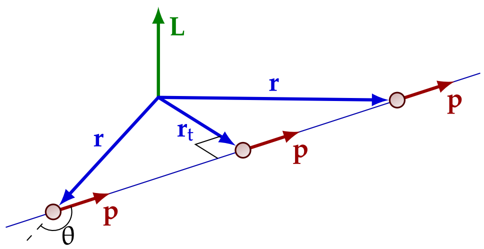
When we upgrade to 3 matrices, we take

$$\operatorname{tr}(ABC) = \sum_{i,j}\left(\sum_{k,\ell} A_{ik}B_{k\ell}C_{\ell j}\right)\delta_{ij}$$

$$
\begin{aligned}
&= \sum_{i,k,\ell} A_{ik} B_{k\ell} C_{\ell i} \\
&= \sum_{i,k,\ell} C_{\ell i} A_{ik} B_{k\ell} \\
&= \operatorname{tr}(CAB).
\end{aligned}
$$

In other words, the trace is invariant under cyclic permutations.

Example (Angular Momentum, Revisited)**.**



Recall that

$$
\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p}, \\
&= I\boldsymbol{\omega}.
\end{aligned}
$$

where $\mathbf{p} = m\dot{\mathbf{x}}$, and I denotes the moment of inertia. Note that $I \sim mr^2$. On a more fundamental level, it is the case that the first equation, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is the "true" definition of $\mathbf{L}$.
Consider a small portion m_α about some axis at radius $\mathbf{r}_\alpha$ and momentum $\mathbf{p}_\alpha$. Then, we have

$$
\begin{aligned}
\mathbf{L}_\alpha &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\
&= \sum_\alpha m_\alpha \left(\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\right).
\end{aligned}
$$

In the infinitesimal case (i.e., as $\alpha \to 0$), we get

$$
\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, \rho \, d\tau,
$$

where ρ denotes volume density. Applying the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, we find

$$
\mathbf{L} = \int \left(\boldsymbol{\omega}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})\right) \rho \, d\tau.
$$

Switching to index notation, we have

$$
\begin{aligned}
L_i &= \int \left(\omega_i r^2 - r_i \sum_j r_j \omega_j\right) \rho \, d\tau \\
&= \sum_j \int \omega_j \left(\delta_{ij} r^2 - r_i r_j\right) \rho \, d\tau \\
&= \sum_j \omega_j \underbrace{\left(\int \left(\delta_{ij} r^2 - r_i r_j\right) \rho \, d\tau\right)}_{\text{moment of inertia tensor}} \\
&= \sum_j I_{ij} \omega_j.
\end{aligned}
$$

Binomial Theorem

The binomial theorem allows us to calculate the expansion

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m.$$

In the case of $(x+y)^2 = x^2y^0 + 2x^1y^1 + x^0y^2 = x^2 + 2xy + y^2$. Recall that

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}.$$

Recall that $0! = 1$.

Infinite Series

Let

$$S = \sum_{k=0}^{\infty} a_k$$

be an infinite series. We are often curious as to the convergence of this sum (for a variety of reasons). Formally, we have to invoke partial sums

$$S_N = \sum_{k=0}^{N} a_k,$$

and see if the sequence of partial sums is convergent. However, we will prefer to use series convergence tests.

Example (Geometric Series)**.** Let

$$\begin{aligned} S &= \sum_{k=0}^{\infty} r^k \\ &= 1 + r + r^2 + \cdots \end{aligned}$$

Then, we have

$$\begin{aligned} S_N &= \sum_{k=0}^{N} r^k \\ rS_N &= \sum_{k=0}^{N} r^k. \end{aligned}$$

Subtracting, we get

$$\begin{aligned} (1-r)S_N &= 1 - r^{N+1} \\ S_N &= \frac{1-r^{N+1}}{1-r}. \end{aligned}$$

In the limit, we expect that if $r \to \infty$, and $r < 1$, then $r^{N+1} \to 0$. In the infinite case, we have

$$\begin{aligned} S &= \sum_{k=0}^{\infty} r^k \\ &= \frac{1}{1-r}, \end{aligned}$$

if $r < 1$.

There are a few prerequisites for series convergence:

- there exists some K for which for all $k \geqslant K$, $a_{k+1} \leqslant a_k$;
- $\lim_{k\to\infty} < \infty$;
- we need the series to reduce "quickly" enough.

Example (Ratio Test)**.** A series $S = \sum_k a_k$ converges if the ratio of consecutive terms is (eventually) less than 1:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1.$$

Example (Applying the Ratio Test)**.** Consider $S = \sum_k \frac{1}{k!}$. Then,

$$\begin{aligned} r &= \lim_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} \\ &= \lim_{k\to\infty} \frac{1}{k+1} \\ &= 0 < 1 \end{aligned}$$

Example (Riemann Zeta Function)**.** We write

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

In order to evaluate the convergence of the Riemann zeta function. We have

$$\begin{aligned} r &= \lim_{k\to\infty} \frac{\frac{1}{(k+1)^s}}{\frac{1}{k^s}} \\ &= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^s \\ &= 1. \end{aligned}$$

Unfortunately, this means the ratio test is inconclusive.

For examples of evaluations of the zeta function, we have

$$\begin{aligned} \zeta(1) &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ \zeta(2) &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots \\ &= \frac{\pi^2}{6}. \end{aligned}$$