Contents

Introduction **1**

Hilbert Spaces **1**
Theory of Hilbert Spaces 1
Orthogonal Projections 6
Orthonormal Sets and Orthonormal Bases 11
Tensor Products and Direct Sums of Hilbert Spaces 17
Direct Sums 17

Bounded Operators on Hilbert Spaces **18**
Structure of $\mathcal{B}(\mathcal{H})$ 18
Algebraic-Analytic Structure 19
Topologies on $\mathcal{B}(\mathcal{H})$ 22

Introduction

This is going to be part of my notes for my Honors Thesis independent study, focused on Amenability and C^*-algebras. This set of notes will be focused on the theory of Hilbert spaces and bounded linear operators on Hilbert spaces. The primary source for this section of notes will be Timothy Rainone's *Functional Analysis: En Route to Operator Algebras*.

I do not claim any of this work to be original.

Hilbert Spaces

In quantum mechanics, the state of a non-relativistic particle is given by a vector in some Hilbert space, which evolves by moving around that space. Specifically, the state of such a particle is determined entirely by the wave function $\xi = \xi(x, t)$, where $x \in \mathbb{R}$ is position and t is time. The wave function is a probability distribution satisfying

$$\int_{\mathbb{R}} |\xi(x,t)|^2 \, d\lambda = 1.$$

In particular, ξ is an element of the space $L_2(\mathbb{R}, \lambda)$. The observables on ξ are modeled as operators on $L_2(\mathbb{R}, \lambda)$.

Theory of Hilbert Spaces

In undergraduate linear algebra, the dot product of vectors in $\mathbb{R}^n$, $v \cdot w$, is intimately tied to the geometry of $\mathbb{R}^n$ through the equations

$$\begin{aligned} v \cdot v &= \|v\|^2 \\ v \cdot w &= \|v\| \, \|w\| \cos\theta. \end{aligned}$$

Inner product spaces help generalize these properties.

Definition. Let X be a vector space over a field $\mathbb{F}$.

(1) An inner product on X is a map

$$\begin{aligned} \langle \cdot, \cdot \rangle : X \times X &\to \mathbb{F} \\ (x, y) &\mapsto \langle x, y \rangle \end{aligned}$$

which satisfies the following conditions for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{F}$.

(i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$;
(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
(iii) $\langle x, x \rangle \geqslant 0$;
(iv) $\langle x, x \rangle = 0 \Rightarrow x = 0_X$.

If $\langle \cdot, \cdot \rangle$ satisfies (i)–(iii), but not necessarily (iv), then it is called a semi-inner product.

(2) If $\langle \cdot, \cdot \rangle$ is an inner product on X, the pair $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Remark: A semi inner product also satisfies, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{F}$,

$$\langle x, \lambda y + \mu z \rangle = \overline{\lambda} \langle x, z \rangle + \overline{\mu} \langle y, z \rangle .$$

A semi-inner product is linear in the first variable and conjugate linear in the second variable.

Definition. Let X be a complex vector space. A map

$$F : X \times X \to \mathbb{C}$$

which is linear in the first variable and conjugate linear in the second variable is called a sesquilinear form on X.

A fundamental fact about sesquilinear forms is that for any given sesquilinear form, we are able to pass it into a form that only consists of the same elements in both inputs.

Lemma (Polarization Identity)**:** Let $F : X \times X \to \mathbb{C}$ be a sesquilinear form on X. For all $x, y \in X$, we have

$$\begin{aligned} 4F(x, y) &= F(x+y, x+y) + iF(x+iy, x+iy) - F(x-y, x-y) + iF(x-iy, x-iy) \\ &= \sum_{k=0}^{3} i^k F\left(x + i^k y, x + i^k y\right). \end{aligned}$$

Proof. Taking each expression

$$\begin{aligned} F(x+y, x+y) &= F(x,x) + F(x,y) + F(y,x) + F(y,y) \\ iF(x+iy, x+iy) &= iF(x,x) - F(y,x) + F(x,y) + iF(y,y) \\ -F(x-y, x-y) &= -F(x,x) + F(x,y) + F(y,x) - F(y,y) \\ -iF(x-iy, x-iy) &= -iF(x,x) - F(y,x) + F(x,y) - iF(y,y). \end{aligned}$$

Adding these expressions up, we get the polarization identity. □

The following fact follows from the polarization identity.

Fact. If F and G are two sesquilinear forms that agree on the diagonal — i.e., $F(x, x) = G(x, x)$ — then F and G agree everywhere.

Fact. Let X be an inner product space, and suppose $z_1, z_2 \in X$ are such that $\langle x, z_1 \rangle = \langle x, z_2 \rangle$ for all $x \in X$. Then, $z_1 = z_2$.

Proof. We have $\langle x, z_1 \rangle = \langle x, z_2 \rangle$. Then, $\langle x, z_1 - z_2 \rangle = 0$ for all $x \in X$, so $\langle z_1 - z_1, z_1 - z_2 \rangle = 0$, so $z_1 - z_2 = 0$. □

Let's see some inner product spaces.

Example (Finite-Dimensional Space)**.** The finite dimensional space $\mathbb{C}^n$ admits an inner product space given by

$$\langle \xi, \eta \rangle = \sum_{j=1}^{n} \xi_j \overline{\eta_j},$$

where ξ and η are n dimensional vectors over $\mathbb{C}$.

Example (Sequence Space)**.** The space of square-summable sequences,

$$\ell_2 = \left\{ (\lambda_k)_k \;\middle|\; \sum_{n=1}^{\infty} |\lambda_n|^2 := \|\lambda\|^2 < \infty \right\}$$

is an inner product space with the inner product

$$\langle \lambda, \mu \rangle = \sum_{n=1}^{\infty} \lambda_n \overline{\mu_n}.$$

The Cauchy–Schwarz inequality provides for this to be a well-defined inner product.

$$\begin{aligned} \sum_{n=1}^{N} |\lambda_n \overline{\mu_n}| &\leqslant \left(\sum_{n=1}^{N} |\lambda_n|^2\right)^{1/2} \left(\sum_{n=1}^{N} |\mu_n|^2\right)^{1/2} \\ &\leqslant \|\lambda\|_2 \|\mu\|_2 \\ &< \infty. \end{aligned}$$

Example (Continuous Functions)**.** The space $X = C([0,1])$ admits an inner product given by

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$

Example (Sesquilinear Form on Continuous Function Space)**.** Let Ω be a locally compact Hausdorff space and suppose $\varphi : C_0(\Omega) \to \mathbb{F}$ is a positive linear functional. We know that $\varphi = \varphi_\mu$ for some positive regular finite measure μ on $(\Omega, \mathcal{B}_\Omega)$, and

$$\varphi_\mu(f) = \int_\Omega f\, d\mu.$$

We get a semi inner product on $C_0(\Omega)$ by

$$\begin{aligned} \langle \cdot, \cdot \rangle_\varphi : C_0(\Omega) \times C_0(\Omega) &\to \mathbb{F} \\ (f, g) &\mapsto \int_\Omega f\overline{g}\, d\mu. \end{aligned}$$

We claim that, when μ has full support, $\langle \cdot, \cdot \rangle_\varphi$ is an inner product.

Suppose $g \in C_0(\Omega)$ with $g \geqslant 0$ and $g \neq 0$. Then, there is a nonempty open subset $U \subseteq \Omega$ and $\delta > 0$ such that $g(x) \geqslant \delta$ for all $x \in U$. Since μ has full support, it must be the case that $\mu(U) > 0$, so

$$\begin{aligned} \varphi(g) &= \int_\Omega g\, d\mu \\ &\geqslant \int_\Omega \delta \mathbb{1}_U\, d\mu \\ &= \delta\mu(U) \\ &> 0. \end{aligned}$$

Thus, if $\langle f, f \rangle_\varphi = 0$, then $\varphi\left(|f|^2\right) = 0$, so $f = 0$.

Example (Hilbert–Schmidt Operators)**.** Let $\mathbb{M}_n$ be the $*$-algebra of $n \times n$ matrices over the complex numbers. Let $\operatorname{tr} : \mathbb{M}_n \to \mathbb{C}$ denote the trace. The trace is a linear, positive, faithful functional satisfying $\operatorname{tr}(a^*) = \overline{\operatorname{tr}(a)}$ for all $a \in \mathbb{M}_n$. The trace induces an inner product

$$\langle a, b \rangle_{HS} = \operatorname{tr}(b^* a),$$

where the subscript HS stands for Hilbert–Schmidt.

Definition. Let X be an inner product space.

(1) We say two vectors $x, y \in X$ are orthogonal if $\langle x, y\rangle = 0$. We write $x \perp y$.

(2) Let $z \neq 0$ be a fixed vector in X. We define the one dimensional projection

$$P_z(x) = \frac{\langle x, z\rangle}{\langle z, z\rangle} z.$$

Note that P_z is linear and its range is the one-dimensional subspace $\operatorname{span}(z)$.

Note: There are a lot of propositions, lemmas, and exercises in this section of my professor's textbook, but I'm not going to be going through all of them since we learn a lot of this in Real Analysis II.

We can turn any semi-inner product space into a seminormed vector space using the semi-inner product. If the semi-inner product is a true inner product, then we can use the inner product to define a norm.

Definition. Let X be a semi-inner product space. For each $x \in X$, we set

$$\|x\| = \langle x, x\rangle^{1/2}.$$

Theorem (Pythagoras)**:** Let X be a semi-inner product space, and suppose $x_1, x_2, \dots, x_n$ are pairwise orthogonal. Then,

$$\left\|\sum_{j=1}^{n} x_j\right\|^2 = \sum_{j=1}^{n} \|x_j\|^2$$

Corollary: Let X be an inner product space, and fix $z \neq 0$ in X. Then, for all $x, y \in X$, we have

(1) $\|x\|^2 = \|x - P_z(x)\|^2 + \|P_z(x)\|^2$;

(2) $\|P_z(x)\| \leqslant \|x\|$;

(3) $|\langle x, z\rangle| \leqslant \|x\| \|y\|$, with equality if and only if x and y are linearly independent (the Cauchy–Schwarz inequality);

(4) $\|x + y\| \leqslant \|x\| + \|y\|$;

(5) $\|\cdot\|$ is a norm on X.

Proposition: If X is an inner product space, then the inner product

$$\langle \cdot, \cdot\rangle : X \times X \to \mathbb{F}$$

is continuous.

We often start with a semi-inner product, then construct an inner product by quotient out by the null space.

Proposition: Let $\langle \cdot, \cdot\rangle$ be a semi-inner product on X.

(1) The set

$$N = \{x \in X \mid \langle x, x\rangle = 0\}$$

is a subspace of X.

(2) The map

$$\langle x + N, y + N\rangle_{X/N} = \langle x, y\rangle$$

is an inner product on the quotient space X/N.

Proposition (Parallelogram Law)**:** Let X be an inner product space. Then,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Recall that Banach spaces include ideas regarding isometric isomorphisms — however, we cannot immediately assume this extends to inner product spaces since they include an inherent geometric structure as well. As it turns out, this automatically appears from the definition of an isometry.

Proposition: Let X and Y be inner product spaces. Suppose $V : X \to Y$ is a linear transformation. The following are equivalent.

(i) V is an isometry;

(ii) for each $x, x' \in X$, we have $\langle V(x), V(x')\rangle_Y = \langle x, x'\rangle_X$.

Proof. To show that (ii) implies (i), we see that for $x \in X$,

$$\begin{aligned}\|V(x)\|^2 &= \langle V(x), V(x)\rangle \\ &= \langle x, x\rangle \\ &= \|x\|^2.\end{aligned}$$

We define the sesquilinear forms

$$\begin{aligned}F(x, x') &= \langle V(x), V(x')\rangle_Y \\ G(x, x') &= \langle x, x'\rangle.\end{aligned}$$

Since V is norm-preserving, we have

$$\begin{aligned}F(x, x) &= \|V(x)\|^2 \\ &= \|x\|^2 \\ &= G(x, x),\end{aligned}$$

so by the polarization identity, F and G agree everywhere. □

Definition. Let X and Y be inner product spaces. A surjective linear isometry $U : X \to Y$ is called a unitary operator.

Equivalently, a unitary operator is a linear isomorphism $U : X \to Y$ that preserves the inner product. We say X and Y are unitarily isomorphic.

Example (A Nonunitary Isometry)**.** Consider the right shift on ℓ_2, defined by

$$R(\xi_1, \xi_2, \dots,) = (0, \xi_1, \xi_2, \dots).$$

Then, R is not onto, but for each $\xi, \eta \in \ell_2$, we have $\langle R(\xi), R(\eta)\rangle = \langle \xi, \eta\rangle$. Thus, R is isometric but not unitary.

Definition (Hilbert Space)**.** A Hilbert space is an inner product space $\mathcal{H}$ over $\mathbb{C}$ such that the norm $\|x\|^2 = \langle x, x\rangle$ is complete.

Example. The space ℓ_2 of all square-summable sequences is a Hilbert space.

Example. If $(\Omega, \mathcal{M}, \mu)$ is any measure space, then $L_2(\Omega, \mu)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_\Omega f\overline{g}\, d\mu.$$

Orthogonal Projections

Recall that closed subspaces of Banach spaces may not always admit a topological complement (for instance, $c_0 \subseteq \ell_\infty$). However, in a Hilbert space, a closed subspace always admits an orthogonal projection operator (hence a topological complement).

Theorem (Hilbert Projection Theorem)**:** Let $\mathcal{H}$ be a Hilbert space. Suppose $C \subseteq \mathcal{H}$ is a closed and convex set. Given $x \in \mathcal{H}$, there is a unique $y_x \in C$ such that $\operatorname{dist}_C(x) = d(x, y_x)$. We say y_x is the point in C closest to x

Proof. Set $d = \operatorname{dist}_C(x)$. If $x \in C$, we take $y = x$, so we assume $x \notin C$.

We find a sequence $(y_n)_{n\geqslant 1}$ with $d(x, y_n) \to d$ decreasing. Set $z_n = y_n - x$. We have $\|z_n\| \to d$ decreasing, meaning $\|z_n\|^2 \to d^2$ decreasing. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for $n \geqslant N$,

$$\|z_n\|^2 < d^2 + \varepsilon.$$

We claim that $(y_n)_n$ is a Cauchy sequence in C. If $p, q \in \mathbb{N}$, we see that

$$\begin{aligned} y_p - y_q &= z_p - z_q \\ \left\|\frac{1}{2}(z_p + z_q)\right\| &= \left\|\frac{1}{2}(y_p + y_q) - x\right\| \\ &\geqslant d, \end{aligned}$$

as $\frac{1}{2}(y_p + y_q)$ belongs to C. Thus, for $p, q \geqslant N$, we have

$$\begin{aligned} \left\|y_p - y_q\right\|^2 &= \left\|z_p - z_q\right\|^2 \\ &= 2\left\|z_p\right\|^2 + 2\left\|z_q\right\|^2 - \left\|z_p + z_q\right\|^2 \\ &= 2\left\|z_p\right\|^2 + 2\left\|z_q\right\|^2 - 4\left\|\frac{1}{2}(z_p + z_q)\right\|^2 \\ &\leqslant 2d^2 + 2\varepsilon + 2d^2 + 2\varepsilon - 4d^2 \\ &= 4\varepsilon. \end{aligned}$$

Since C is closed, we thus have $d = \lim_{n\to\infty} d(x, y_n) = d(x, y)$ for $(y_n)_n \to y$ for some $y \in C$.

To see uniqueness, suppose $y_1, y_2 \in C$ with $d(x, y_i) = d$. Set $z = y_j - x$ for each j. We have

$$\begin{aligned} 0 &\leqslant \|z_1 - z_2\|^2 \\ &= 2\|z_1\|^2 + 2\|z_2\|^2 - 4\left\|\frac{1}{2}(z_1 + z_2)\right\|^2, \end{aligned}$$

meaning

$$\begin{aligned} 0 &\leqslant \|y_1 - y_2\|^2 \\ &= 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 - 4\left\|\frac{1}{2}(y_1 + y_2) - x\right\|^2 \\ &\leqslant 2d^2 + 2d^2 - 4d^2 \\ &= 0. \end{aligned}$$

Thus, $y_1 = y_2$. □

Definition. Let $\mathcal{H}$ be a Hilbert space, and let $M \subseteq \mathcal{H}$ be a closed subspace. We define

$$P_M : \mathcal{H} \to \mathcal{H}$$

by $P_M(x) = y_x$, where y_x is the unique point from the Hilbert projection theorem.

We call P_M the orthogonal projection of $\mathcal{H}$ onto M.

Fact. There are some facts about the orthogonal projection that are useful for us to know.

- $P_M(x) = x \Leftrightarrow x \in M$;
- $\operatorname{Ran}(P_M) = M$;
- $P_M \circ P_M = P_M$ (i.e., that P_M is idempotent).

Definition. Let X be an inner product space, and suppose $S \subseteq X$ is an arbitrary subset. We define the perp of S, $S^\perp$, to be

$$S^\perp = \{x \in X \mid \langle x, y\rangle = 0 \text{ for all } y \in S\}.$$

Exercise: Let $S \subseteq \mathcal{H}$ be an arbitrary subset. Prove the following.

(1) $S^\perp$ is always a closed subspace of $\mathcal{H}$.

(2) $S \subseteq \left(S^\perp\right)^\perp$.

(3) $S \cap S^\perp = \{0\}$.

Solution:

(1) For $x, x' \in S^\perp$ and $\alpha \in \mathbb{C}$, we have for all $y \in S$,

$$\begin{aligned}\left\langle x + \alpha x', y\right\rangle &= \langle x, y\rangle + \alpha\left\langle x', y\right\rangle\\ &= 0,\end{aligned}$$

so $S^\perp$ is a subspace. Additionally, for any sequence $(x_n)_n \subseteq S^\perp$ with $(x_n)_n \to x$ in X, the continuity of the inner product gives

$$\begin{aligned}\langle x_n, y\rangle &\to \langle x, y\rangle\\ &= 0.\end{aligned}$$

(2) For $t \in S$, we have, for all $x \in S^\perp$,

$$\begin{aligned}\langle x, t\rangle &= 0\\ &= \langle t, x\rangle,\end{aligned}$$

meaning $t \in \left(S^\perp\right)^\perp$.

(3) If $t \in S \cap S^\perp$, then $t \in S$ and $t \in S^\perp$, so

$$\langle t, t\rangle = 0,$$

so $t = 0$.

One of the features of Hilbert spaces is that closed subspaces are always complemented.

Theorem: Let $M \subseteq \mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then, the following are true.

(1) $x - P_M(x) \in M^\perp$ for all $x \in \mathcal{H}$.

(2) $\mathcal{H} = M \oplus M^\perp$.

(3) $(M^\perp)^\perp = M$.

(4) Let P and Q denote the projection operators onto M and $M^\perp$ according to the decomposition $\mathcal{H} = M \oplus M^\perp$. Then, $P = P_M$ and $Q = P_{M^\perp}$.

(5) P_M is linear, $P_M^2 = P_M$, $\operatorname{Ran}(P_M) = M$, $\|P_M\| = 1$, and $\ker(P_M) = M^\perp$.

(6) $\mathcal{H}/M \cong M^\perp$ are isometrically isomorphic.

Proof.

(1) Let $y = P_M(x)$, and set $z = x - y$. We know that $\|z\| = \operatorname{dist}_M(x) = d$. Let $0 \neq \xi \in M$. Set $\zeta = P_\xi(z) = \frac{\langle z,\xi\rangle}{\langle \xi,\xi\rangle}\xi$.

We claim that $\zeta = 0$. Note that

$$\begin{aligned}\|z - \zeta\| &= \|x - y - \zeta\|\\ &= \|x - (y + \zeta)\|\\ &\geqslant d,\end{aligned}$$

as $y + \zeta \in M$.

On the other hand, we have

$$\begin{aligned}\|z - \zeta\|^2 + \|\zeta\|^2 &= \|z\|^2\\ &= d^2.\end{aligned}$$

Thus, $\|z - \zeta\| \leqslant d$. With $\|z - \zeta\| = d$, we have $\|x - y - \zeta\| = d$. Thus, we must have $y + \zeta = y$, so $\zeta = 0$.

(2) If $x \in \mathcal{H}$, we have

$$x = P_M(x) + x - P_M(x),$$

and since $M \cap M^\perp = \{0\}$, we have $\mathcal{H} = M \oplus M^\perp$.

(3) It is the case that $M \subseteq (M^\perp)^\perp$. Let $x \in (M^\perp)^\perp$. Write $x = y + z$ according to the decomposition $\mathcal{H} = M \oplus M^\perp$. Then, $z = x - y \in (M^\perp)^\perp \cap (M^\perp) = \{0\}$, so $x = y \in M$, so $M = (M^\perp)^\perp$.

(4) By the way we have defined P and Q, we must have $P(x) = P_M(x)$ for every $x \in \mathcal{H}$. Let $\widetilde{P}$ and $\widetilde{Q}$ be the bounded linear projections according to the decomposition $\mathcal{H} = M^\perp \oplus (M^\perp)^\perp$. Since $M = (M^\perp)^\perp$, we have $\widetilde{Q} = P$. Additionally, we must have $\widetilde{P} = P_{M^\perp}$. Thus,

$$\begin{aligned}Q &= I - P\\ &= I - \widetilde{Q}\\ &= \widetilde{P}\\ &= P_{M^\perp}.\end{aligned}$$

(5) By the Pythagorean theorem, we have

$$\|x\|^2 = \|P_M(x)\|^2 + \|x - P_M(x)\|^2$$

for every $x \in \mathcal{H}$, so $\|P_M(x)\| \leqslant \|x\|$, meaning $\|P_M\| \leqslant 1$. Since $P_M^2 = P_M$, we also have $\|P_M\| \geqslant 1$.

(6) Notice that $P_{M^\perp} : \mathcal{H} \to M^\perp$ is a 1-quotient map with the kernel $\ker(P_{M^\perp}) = M$. Thus, we have $\mathcal{H}/M \cong M^\perp$.

□

Corollary: The following are true.

(1) The quotient of a Hilbert space is a Hilbert space.

(2) If $M \subsetneq \mathcal{H}$, then $M^\perp \neq \{0\}$. Additionally, if $M^\perp = \mathcal{H}$, then $M = \{0\}$.

(3) For any subset $S \subseteq \mathcal{H}$, we have $(S^\perp)^\perp = \overline{\operatorname{span}}(S)$.

Exercise: Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, and let $E \subseteq \mathcal{M}$ be measurable. We look at the set of essentially E-supported square-integrable functions:

$$M_E = \{\xi \in L_2(\Omega, \mu) \mid \xi|_{E^c} = 0 \ \mu\text{-a.e.}\}.$$

(1) Show that M_E is a closed subspace of $L_2(\Omega, \mu)$, and prove that the orthogonal projection onto M_E is given by

$$P_{M_E}(\xi) = \xi \mathbb{1}_E.$$

(2) Note that the restriction $(E, \mathcal{M}|_E, \mu_E)$ is a measure space, where

$$\begin{aligned} \mathcal{M}_E &= \{F \cap E \mid F \in \mathcal{M}\} \\ \mu_E &= \mu|_{\mathcal{M}_E}. \end{aligned}$$

Prove that $L_2(E, \mu_E)$ and M_E are unitarily isomorphic.

Solution:

(1) If ξ and η are two functions that are essentially E-supported, then the sum $\xi + \alpha\eta$, where $\alpha \in \mathbb{C}$, is also essentially E-supported. Similarly, if $(\xi_n)_n \to \xi$ is a sequence of essentially E-supported functions converging in norm to ξ, then we have $(\xi_m - \xi_n)|_{E^c} = 0$ for each ξ_m, ξ_n, so ξ is also essentially E-supported.

To show that P_{M_E} defined by $P_{M_E}(\xi) = \xi\mathbb{1}_E$ is the orthogonal projection onto M_E, we show that P_{M_E} is idempotent and maps all members of M_E to themselves. For $\xi \in L_2(\Omega, \mu)$, we see that

$$\begin{aligned} P_{M_E}^2(\xi) &= P_{M_E}(\xi\mathbb{1}_E) \\ &= \xi(\mathbb{1}_E)(\mathbb{1}_E) \\ &= \xi\mathbb{1}_E \\ &= P_{M_E}(\xi). \end{aligned}$$

Additionally, for any $\xi \in M_E$, we have that $\xi\mathbb{1}_E \equiv \xi$ since $\xi|_{E^c} = 0$ μ-a.e. Thus, P_{M_E} is an idempotent operator that preserves the closed subspace M_E, so by the Hilbert projection theorem, it is necessarily the only (up to μ-a.e. equivalence) orthogonal projection onto M_E.

(2)

Proposition: Let $\mathcal{H}$ be a Hilbert space, and suppose $\{M_i\}_{i=1}^n$ is a finite family of mutually orthogonal closed subspaces. Write $M = \sum_{i=1}^n M_i$ for the internal sum.

(1) $M \subseteq \mathcal{H}$ is a closed subspace, and $M = \bigoplus_{i=1}^n M_i$ is the internal direct sum.

(2) $P_M = \sum_{i=1}^n P_{M_i}$.

Proof. To see (1), we know that since $M_i \perp M_j$ for each $i \neq j$, it is the case that $M_i \cap M_j = \{0\}$ for each $i \neq j$, so it is indeed a direct sum.

To see (2), let $x \in \mathcal{H}$, and write $x = y + z$ according to the decomposition $\mathcal{H} = M \oplus M^\perp$. Since $M_j \subseteq M$, we have $\ker\left(P_{M_j}\right) \supseteq M^\perp$ for each j. Thus, $P_{M_j}(z) = 0$ for every j.

Since $M = \bigoplus_{i=1}^n M_i$, we write $y = \sum_{i=1}^n y_i$, with $y_i \in M_i$ uniquely. Since M_i are mutually orthogonal, we know that $M_i \subseteq M_j^\perp = \ker\left(P_{M_j}\right)$ for each $i \neq j$. We compute

$$\begin{aligned} P_{M_j}(x) &= P_{M_j}(y + z) \\ &= P_{M_j}(y) \\ &= P_{M_j}\left(\sum_{i=1}^n y_i\right) \end{aligned}$$

$$= \sum_{i=1}^{n} P_{M_j}(y_i)$$
$$= y_j.$$

Thus, we get

$$\begin{aligned}\left(\sum_{i=1}^{n} P_{M_i}\right)(x) &= \sum_{i=1}^{n} P_{M_i}(x)\\ &= \sum_{i=1}^{n} y_i\\ &= y\\ &= P_M(x).\end{aligned}$$

□

We can now turn our attention to understanding the continuous dual of Hilbert spaces.

Definition. Let X be an inner product space, and fix $z \in X \setminus \{0\}$. We define $\varphi_z : X \to \mathbb{F}$ by $\varphi_z(x) = \langle x, z\rangle$.

Proposition: Let X be an inner product space. Each $\varphi_z \in X^*$, and the map $X \to X^*$ defined by $z \mapsto \varphi_z$ is a conjugate linear isometry.

Proof. We see that φ_z is linear. We have

$$\begin{aligned}|\varphi_z(x)| &= |\langle x, z\rangle|\\ &\leqslant \|x\| \|z\|,\end{aligned}$$

with

$$\begin{aligned}\varphi_z\left(\frac{z}{\|z\|}\right) &= \frac{1}{\|z\|}\langle z, z\rangle\\ &= \|z\|,\end{aligned}$$

so $\|\varphi_z\|_{\text{op}} = \|z\|$. For every $x \in X$, we also have

$$\begin{aligned}\varphi_{z_1+\alpha z_2}(x) &= \langle x, z_1 + \alpha z_2\rangle\\ &= \langle x, z_1\rangle + \overline{\alpha}\langle x, z_2\rangle\\ &= (\varphi_{z_1} + \overline{\alpha}\varphi_{z_2})(x).\end{aligned}$$

□

If $\mathcal{H}$ is a Hilbert space, then the map $\mathcal{H} \to \mathcal{H}^*$ given by $z \mapsto \varphi_z$ is a bijection. This is known as the Riesz Representation Theorem (not to be confused for the Riesz representation Theorem for measures on $C_c(\Omega)$).

Theorem (Riesz Representation Theorem)**:** Let $\mathcal{H}$ be a Hilbert space. If $\varphi \in H^*$, then there exists a unique $z \in \mathcal{H}$ such that $\varphi = \varphi_z$.

Proof. We assume $\varphi \neq 0$. We have $M = \ker(\varphi) \subseteq \mathcal{H}$ is a proper closed subspace, so we can choose $w \in M^\perp$ such that $w \neq 0$.

We see that $\ker(\varphi) \subseteq \ker(\varphi_w)$, meaning that $\varphi = \lambda\varphi_w$ for some $\lambda \in \mathbb{F}$. We compute

$$\begin{aligned}\varphi(x) &= \lambda\varphi_w(x)\\ &= \lambda\langle x, w\rangle\end{aligned}$$

$$= \left\langle x, \overline{\lambda} w \right\rangle.$$

Set $z = \overline{\lambda} w$.

To show uniqueness, if $\varphi = \varphi_{z_1} = \varphi_{z_2}$, then $\langle x, z_1 - z_2 \rangle = 0$ for all $x \in \mathcal{H}$, so $z_1 - z_2 \in \mathcal{H}^\perp = \{0\}$, so $z_1 = z_2$. □

Theorem: Every Hilbert space is reflexive.

Proof. Let $\iota : \mathcal{H} \to \mathcal{H}^{**}$ be the canonical embedding. Let $f \in \mathcal{H}^{**}$, and define $\psi : \mathcal{H} \to \mathbb{C}$ by $\psi(x) = \overline{f(\varphi_x)}$. For all $x_1, x_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\psi(x_1 + \lambda x_2) &= \overline{f(\varphi_{x_1 + \lambda x_2})} \\
&= \overline{f\left(\varphi_{x_1} + \overline{\lambda}\varphi_{x_2}\right)} \\
&= \overline{f(\varphi_{x_1}) + \overline{\lambda} f(\varphi_{x_2})} \\
&= \overline{f(\varphi_{x_1})} + \lambda \overline{f(\varphi_{x_2})} \\
&= \psi(x_1) + \lambda \psi(x_2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\psi(x)| &= \left|\overline{f(\varphi_x)}\right| \\
&= |f(\varphi_x)| \\
&\leqslant \|f\| \, \|\varphi_x\| \\
&= \|f\| \, \|x\|.
\end{aligned}$$

Thus, $\psi \in \mathcal{H}^*$, so we know that $\psi = \varphi_z$ for some $z \in H$. Thus,

$$\begin{aligned}
\overline{f(\varphi_x)} &= \psi(x) \\
&= \varphi_z(x) \\
&= \langle x, z \rangle \\
&= \overline{\langle z, x \rangle},
\end{aligned}$$

so $f(\varphi_x) = \langle z, x \rangle = \varphi_x(z) = \hat{z}(\varphi_x)$, so $f = \hat{z}$, so ι is surjective. □

Orthonormal Sets and Orthonormal Bases

Definition. Let X be an inner product space, and let A be an indexing set.

(1) A subset $\{x_\alpha\}_{\alpha \in A}$ is called orthogonal if $\left\langle x_\alpha, x_\beta \right\rangle = 0$ for $\alpha \neq \beta$.

(2) An orthonormal set is an orthogonal set consisting of unit vectors. The set $\{e_\alpha\}_{\alpha \in A}$ is orthonormal if

$$\left\langle e_\alpha, e_\beta \right\rangle = \begin{cases} 1 & \alpha = \beta \\ 0 = \alpha \neq \beta \end{cases}.$$

Exercise: Show every orthogonal set is linearly independent.

Solution: Let $\{x_\alpha\}_{\alpha \in A}$ be an orthogonal set. Then, for

$$\sum_{i=1}^{n} a_i x_{\alpha_i} = 0,$$

we take

$$\begin{aligned}\left\langle x_{\alpha_j}, \sum_{i=1}^{n} a_i x_{\alpha_i}\right\rangle &= a_j \left\|x_{\alpha_j}\right\|^2\\ &= 0,\end{aligned}$$

so $a_i = 0$ for all i.

Remark: Given an inner product space X and a finite linearly independent subset $F = \{x_1, \dots, x_n\}$, we can always use the Gram–Schmidt process to generate an orthonormal subset $G = \{u_1, \dots, u_n\} \subseteq X$ with $\operatorname{span}(G) = \operatorname{span}(F)$. Inductively, we take

$$\begin{aligned}v_1 &= x_1\\ v_k &= x_k - \sum_{j=1}^{k-1} \frac{\langle x_k, v_j\rangle}{\langle v_j, v_j\rangle} v_j\\ u_k &= \frac{1}{\|v_k\|} v_k.\end{aligned}$$

Exercise: Let A be an arbitrary set, and consider the Hilbert space $\ell_2(A)$. Show that $\{e_\alpha\}_{\alpha\in A} \subseteq \ell_2(A)$ is an orthonormal set.

Example. The family of continuous functions $(e_n : \mathbb{T} \to \mathbb{C})_{n\in\mathbb{Z}}$ is an orthonormal basis for the arc length measure space $(\mathbb{T}, \mathcal{L}_{\mathbb{T}}, \nu)$.

$$\begin{aligned}\langle e_n, e_m\rangle &= \int_{\mathbb{T}} e_n \overline{e_m}\, d\nu\\ &= \int_{\mathbb{T}} e_n e_{-m}\, d\nu\\ &= \int_{\mathbb{T}} e_{n-m}\, d\nu\\ &= \delta_{mn}.\end{aligned}$$

Theorem: Let $\mathcal{H}$ be a Hilbert space, and suppose $(e_\alpha)_{\alpha\in A}$ is an orthonormal family in $\mathcal{H}$.

(1) If $(c_\alpha)_{\alpha\in A} \in \ell_2(A)$, then $\sum_{\alpha\in A} c_\alpha e_\alpha$ is summable in $\mathcal{H}$, and

$$\left\|\sum_{\alpha\in A} c_\alpha e_\alpha\right\| = \|(c_\alpha)_\alpha\|.$$

(2) The map $T : \ell_2(A) \to \mathcal{H}$ defined by $T(\xi) = \sum_{\alpha\in A} \xi(\alpha) e_\alpha$ is a linear isometry.

(3) If $x \in \mathcal{H}$, then $\sum_{\alpha\in A} |\langle x, e_\alpha\rangle|^2 \leqslant \|x\|^2$. This is known as Bessel's inequality.

(4) If $M = \overline{\operatorname{span}}(\{e_\alpha\}_{\alpha\in A})$, then $P_M(x) = \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha$ is the orthogonal projection onto M.

Proof.

(1) We let $\mathcal{F}$ be the collection of finite subsets of A directed by inclusion. For $F \in \mathcal{F}$, we define

$$\begin{aligned}s_F &= \sum_{\alpha\in F} c_\alpha e_\alpha\\ c_F &= \sum_{\alpha\in F} |c_\alpha|^2.\end{aligned}$$

By the Pythagorean theorem, we have

$$\|s_F\|^2 = \left\|\sum_{\alpha\in F} c_\alpha e_\alpha\right\|^2$$

$$\begin{aligned}
&= \sum_{\alpha\in F} \|c_\alpha e_\alpha\|^2\\
&= \sum_{\alpha\in F} |c_\alpha|^2\\
&= c_F.
\end{aligned}$$

We claim the net $(s_F)_{F\in\mathcal{F}}$ is Cauchy in $\mathcal{H}$. For F and G in $\mathcal{F}$, we set

$$d_\alpha = \begin{cases} c_\alpha & \alpha\in F\\ -c_\alpha & \alpha\in G\end{cases}.$$

Then,

$$\begin{aligned}
\|s_F - s_G\|^2 &= \left\|\sum_{\alpha\in F} c_\alpha e_\alpha - \sum_{\alpha\in G} c_\alpha e_\alpha\right\|^2\\
&= \left\|\sum_{\alpha\in F\triangle G} d_\alpha e_\alpha\right\|\\
&= \sum_{\alpha\in F\triangle G} |d_\alpha|^2\\
&= c_{F\triangle G}.
\end{aligned}$$

Let $\varepsilon > 0$. Since $\sum_{\alpha\in A} |c_\alpha|^2$ is summable, there is a finite $F_0 \subseteq A$ such that for all $F \in \mathcal{F}$ with $F\cap F_0 = \emptyset$, we have $c_F \leqslant \varepsilon^2$.

If F and G are finite subsets of A with $F\supseteq F_0$ and $G\supseteq F_0$, then $F_0\subseteq F\cap G$, so $(F\triangle G)\cap F_0 = \emptyset$, so

$$\begin{aligned}
\|s_F - s_G\|^2 &= c_{F\triangle G}\\
&< \varepsilon^2.
\end{aligned}$$

We define $s = \sum_{\alpha\in A} c_\alpha e_\alpha$. This limit exists since $\mathcal{H}$ is complete and Cauchy nets converge. The norm of s is computed as

$$\|s\|^2 = \sum_{\alpha\in A} |c_\alpha|^2.$$

(2) This follows directly from (1).

(3) Let $F \subseteq A$ be finite, and set $M_F = \operatorname{span}\{e_\alpha \mid \alpha\in F\}$. Since M_F is finite-dimensional, M_F is closed. For $x\in\mathcal{H}$ and $\beta\in A\setminus F$, the orthogonality of $(e_\alpha)_{\alpha\in A}$ provides

$$\left\langle x - \sum_{\alpha\in F}\langle x, e_\alpha\rangle e_\alpha, e_\beta\right\rangle = 0.$$

Thus, we write

$$x = x - \sum_{\alpha\in F}\langle x, e_\alpha\rangle e_\alpha + \sum_{\alpha\in F}\langle x, e_\alpha\rangle e_\alpha,$$

which gives $P_{M_F} = \sum_{\alpha\in F}\langle x, e_\alpha\rangle e_\alpha$. Using the Pythagorean theorem, we get

$$\sum_{\alpha\in F} |\langle x, e_\alpha\rangle|^2 = \|P_{M_F}(x)\|^2$$

$$\leqslant \|x\|^2.$$

The inequality follows by taking the supremum,

$$\begin{aligned}\sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2 &= \sup_{F \subseteq A} \left(\sum_{\alpha \in F} |\langle x, e_\alpha \rangle|^2 \right) \\ &\leqslant \|x\|^2.\end{aligned}$$

(4) Fix $x \in \mathcal{H}$, and for each $\alpha \in A$, we set $c_\alpha = \langle x, e_\alpha \rangle$. We have $(c_\alpha)_{\alpha \in A} \in \ell_2(A)$, so $\sum_{\alpha \in A} c_\alpha e_\alpha$ is norm-summable in $\mathcal{H}$. Continuity of the inner product yields

$$\left\langle x - \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha, e_\beta \right\rangle = 0,$$

so $x - \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha \in M^\perp$, meaning $P_M(x) = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$.

□

Corollary: If $\mathcal{H}$ is a Hilbert space, and $\{x_\alpha\}_{\alpha \in A}$ is an orthogonal set such that $\sum_{\alpha \in A} \|x_\alpha\|^2$ is summable, then $\sum_{\alpha \in A} x_\alpha$ is summable and

$$\left\| \sum_{\alpha \in A} x_\alpha \right\|^2 = \sum_{\alpha \in A} \|x_\alpha\|^2.$$

Proof. Set $e_\alpha = \frac{1}{\|x_\alpha\|} x_\alpha$, and $c_\alpha = \|x_\alpha\|$ in the proof of the theorem above. □

Example. If $(e_n)_{n \geqslant 1}$ is the set of standard coordinate vectors in ℓ_2, then $\sum_{n \in \mathbb{N}} \frac{1}{n} e_n$ is summable, but the series does not converge absolutely.

Definition. Let $\mathcal{H}$ be a Hilbert space. An orthonormal basis in $\mathcal{H}$ is a maximal orthonormal set E. That is, if $E \subsetneq E'$, then E' is not an orthonormal basis.

Lemma: Let $\mathcal{H}$ be a Hilbert space. Every orthonormal set in $\mathcal{H}$ is contained in an orthonormal basis.

Proof. Let $F \subseteq \mathcal{H}$ be an orthonormal set. Let

$$\mathcal{E} = \{E \subseteq \mathcal{H} \mid F \subseteq E,\ E \text{ orthonormal}\},$$

and order $\mathcal{E}$ by inclusion. For any chain $\mathcal{C}$ in $\mathcal{E}$, then $U = \bigcup_{C \in \mathcal{C}} C$ is an upper bound for $\mathcal{C}$, as for any two vectors $e_\alpha, e_\beta \in U$, both e_α and e_β are contained in some $C \in \mathcal{C}$, so $\left\langle e_\alpha, e_\beta \right\rangle = \delta_{\alpha\beta}$. Applying Zorn's lemma, we get the desired result. □

Orthonormal bases, like Schauder bases, have dense linear span in a Hilbert space.

Theorem: Let $\mathcal{H}$ be a Hilbert space, and $E = (e_\alpha)_{\alpha \in A}$ be an orthonormal set. Let $M = \overline{\operatorname{span}} \{e_\alpha \mid \alpha \in A\}$. The following are equivalent:

(i) E is an orthonormal basis;

(ii) $M^\perp = \{0\}$;

(iii) $M = \mathcal{H}$;

(iv) for each $x \in \mathcal{H}$, we have $x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$;

(v) for each $x \in \mathcal{H}$, we have $\|x\|^2 = \sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2$ (known as Parseval's identity);

(vi) for each $x, y \in \mathcal{H}$, we have $\langle x, y \rangle = \sum_{\alpha \in A} \langle x, e_\alpha \rangle \overline{\langle y, e_\alpha \rangle}$.

Proof. To see (i) implies (ii), we suppose there is $v \in M^{\perp}$ with $\|v\| = 1$. Then, $\{v\} \cup E$ is an orthonormal set containing E, which contradicts the maximality of E.

The equivalence of (ii) and (iii) follows from the fact that $\mathcal{H}^{\perp} = \{0\}$ and $\{0\}^{\perp} = \mathcal{H}$.

To see that (iii) implies (i), we suppose there is $v \in \mathcal{H}$ such that $v \notin E$ and $\{v\} \cup E$ is an orthonormal set. THen, for each $\alpha \in A$, we have $\langle v, e_\alpha \rangle = 0$, so $\langle v, x \rangle = 0$ for each $x \in \operatorname{span}\{e_\alpha \mid \alpha \in A\}$. Since the inner product is continuous, we have $\langle v, x \rangle = 0$ for each $x \in \overline{\operatorname{span}}\{e_\alpha \mid \alpha \in A\} = M = \mathcal{H}$, implying that $\|v\| = 0$.

To see that (iii) implies (iv), recall that we proved that $P_M(x) = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$, but since $M = \mathcal{H}$, we have $P_M(x) = x$.

We see that (v) follows from (iv) by the previous theorem.

To see that (v) implies (i), if $\langle v, e_\alpha \rangle = 0$ for each $\alpha \in A$, we must have $\|v\| = 0$, so E is a maximal orthonormal set.

To see that (vi) implies (v), we let $x = y$ in the hypothesis of (vi).

To see that (iv) implies (vi), we let $x, y \in \mathcal{H}$. We let $x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$, and $y = \sum_{\beta \in A} \left\langle y, e_\beta \right\rangle e_\beta$. By the continuity of the inner product and the orthonormality of E, we have

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha, \sum_{\beta \in A} \left\langle y, e_\beta \right\rangle e_\beta \right\rangle \\
&= \sum_{\alpha, \beta \in A} \langle x, e_\alpha \rangle \overline{\left\langle y, e_\beta \right\rangle} \left\langle e_\alpha, e_\beta \right\rangle \\
&= \sum_{\alpha \in A} \langle x, e_\alpha \rangle \overline{\left\langle y, e_\beta \right\rangle}.
\end{aligned}$$

□

For an orthonormal basis $\{e_\alpha\}_{\alpha \in A}$ and a given $x \in \mathcal{H}$, we often refer to the terms $\langle x, e_\alpha \rangle$ as the Fourier coefficients of x with respect to the basis $\{e_\alpha\}_{\alpha \in A}$.

Recall that any two vector spaces X and Y are isomorphic if and only if $\dim(X) = \dim(Y)$. A similar idea holds for Hilbert spaces.

Proposition: Let $\mathcal{H}$ be a Hilbert space. Any two orthonormal bases for $\mathcal{H}$ have the same cardinality.

Proof. Let $E = \{e_\alpha\}_{\alpha \in A}$ and $F = \left\{f_\beta\right\}_{\beta \in B}$ be two orthonormal bases for $\mathcal{H}$. If E is finite, then it must be a Hamel basis as orthogonal sets are independent and finite orthonormal bases are spanning by Parseval's identity. Thus, $\dim(\mathcal{H}) < \infty$, and since F is independent, F is finite, so it must be a Hamel basis, with $\operatorname{card}(E) = \operatorname{card}(F)$.

Suppose A and B are both infinite. For each $\beta \in B$, consider

$$A_\beta := \left\{\alpha \,\middle|\, \left\langle f_\beta, e_\alpha \right\rangle \neq 0\right\}.$$

Since

$$\begin{aligned}
\left\|f_\beta\right\|^2 &= \sum_{\alpha \in A} \left|\left\langle f_\beta, e_\alpha \right\rangle\right|^2 \\
&= 1
\end{aligned}$$

is summable, A_β must be countable. Additionally, $A \subseteq \bigcup_{\beta \in B} A_\beta$, since

$$\|e_\alpha\|^2 = \sum_{\beta \in B} \left|\left\langle e_\alpha, f_\beta \right\rangle\right|^2.$$

Since card $\left(A_\beta\right) \leqslant \aleph_0 \leqslant$ card(B), we get

$$\text{card}(A) \leqslant \text{card}\left(\bigcup_{\beta \in B} A_\beta\right) \leqslant \text{card}(B).$$

Similarly, card(B) $\leqslant$ card(A), so card(A) = card(B) by Cantor–Schröder–Bernstein. □

Definition. Let $\mathcal{H}$ be a Hilbert space. The Hilbert dimension of $\mathcal{H}$, written hdim($\mathcal{H}$), is the cardinality of E for any orthonormal basis E of $\mathcal{H}$.

We can characterize all Hilbert spaces with countable Hilbert dimension.

Proposition: Let $\mathcal{H}$ be a Hamel basis with $\dim(H) = n < \infty$. Then, $\mathcal{H} \cong \ell_2^n$ are unitarily isomorphic.

Proof. Let $\{v_1, \dots, v_n\}$ be a Hamel basis for $\mathcal{H}$. Applying the Gram–Schmidt process, we obtain an orthonormal set $\{u_1, \dots, u_n\}$ with the same span as $\{v_1, \dots, v_n\}$. The map $T \colon \ell_2^n \to \mathcal{H}$ given by $T\left(e_j\right) = u_j$ is a surjective isometry. □

Proposition: Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. The following are equivalent.

(i) $\mathcal{H}$ is separable;

(ii) $\text{hdim}(\mathcal{H}) = \aleph_0$;

(iii) $\mathcal{H} \cong \ell_2$.

Proof. Let $\{x_k\}_{k=1}^\infty \subseteq \mathcal{H}$ be norm-dense, and let $(e_\alpha)_\alpha$ be an orthonormal basis. Note that for $\alpha \neq \beta$, we have

$$\begin{aligned}\left\|e_\alpha - e_\beta\right\|^2 &= \left\langle e_\alpha - e_\beta, e_\alpha - e_\beta\right\rangle\\ &= 2,\end{aligned}$$

so $\left\|e_\alpha - e_\beta\right\| = \sqrt{2}$. For each $\alpha \in A$, the density of $\{x_k\}_{k=1}^\infty$ allows us to find $J(\alpha) \in \mathbb{N}$ such that

$$\|e_\alpha - x_{J_\alpha}\| < \frac{1}{2}.$$

We have a map $J : A \to \mathbb{N}$. We claim that J is injective. If not, then there are $\alpha, \beta \in A$ with $\alpha \neq \beta$, $J(\alpha) = J(\beta)$. We then have

$$\begin{aligned}\sqrt{2} &= \left\|e_\alpha - e_\beta\right\|\\ &\leqslant \left\|e_\alpha - x_{J(\alpha)}\right\| + \left\|x_{J(\alpha)} - e_\beta\right\|\\ &= \left\|e_\alpha - x_{J(\alpha)}\right\| + \left\|x_{J(\beta)} - e_\beta\right\|\\ &< 1.\end{aligned}$$

Thus, J is injective, so A is countable.

If $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then we have $\mathcal{H} \cong \ell_2(\mathbb{N}) = \ell_2$.

If $(e_n)_{n \geqslant 1}$ is the canonical orthonormal basis for ℓ_2, then we know that span(E) is dense in ℓ_2, so E is a countable total subset of ℓ_2, so $\mathcal{H}$ is separable. □

Tensor Products and Direct Sums of Hilbert Spaces

We have shown that closed subspaces and quotient spaces of Hilbert spaces are Hilbert spaces. Now, we turn our attention to external direct sums and tensor products.

Direct Sums

In linear algebra, we learn that, for a normal $n \times n$ matrix, we can decompose ℓ_2^n into orthogonal pieces that the matrix acts on by scalar multiplication. In order to understand the spectral theorem for normal operators on Hilbert spaces, we need to understand such a decomposition.

Proposition: Let $\{\mathcal{H}_i\}_{i\in I}$ be a family of Hilbert spaces. The set

$$\bigoplus_{i\in I}\mathcal{H}_i = \left\{(x_i)_{i\in I}\ \middle|\ x_i \in \mathcal{H}_i \text{ and } \sum_{i\in I}\|x_i\|^2 \text{ is summable}\right\}$$

equipped with pointwise operations is a vector space, with inner product

$$\langle x, y\rangle := \sum_{i\in I}\langle x_i, y_i\rangle$$

for $(x_i)_{i\in I}, (y_i)_{i\in I} \in \bigoplus_{i\in I}\mathcal{H}_i$ that induces the complete norm

$$\|(x_i)_i\| := \left(\sum_{i\in I}\|x_i\|^2\right)^{1/2}.$$

The Hilbert space $\bigoplus_{i\in I}\mathcal{H}_i$ is known as the external direct sum of the family $\{\mathcal{H}_i\}_{i\in I}$.

Example. If I is a set, and for each $i \in I$, we have $\mathcal{H}_i = \mathbb{C}$, then $\bigoplus_{i\in I}\mathcal{H}_i = \ell_2(I)$.

Example. If we fix a Hilbert space $\mathcal{H}$, the external direct sum $\bigoplus_{n\geqslant 1}\mathcal{H}$ is often denoted by $\mathcal{H}^\infty$ or $\ell_2(\mathcal{H})$.

Example. Let $\{(\Omega_n, \mathcal{M}_n, \mu_n)\}_n$ be a countable family of measure spaces, and let $(\Sigma, \mathcal{M}, \mu)$ be the coproduct of these spaces, defined by

$$\begin{aligned}\Sigma &:= \coprod_{n=1}^{\infty}\Omega_n\\ \mathcal{M} &:= \left\{E \subseteq \Sigma\ \middle|\ \iota_n^{-1}(E) \in \mathcal{M}_n \text{ for all } n\right\}\\ \mu(E) &= \sum_{n=1}^{\infty}\mu_n\left(\iota_n^{-1}(E)\right).\end{aligned}$$

Then, the map

$$V : L_2(\Sigma, \mu) \to \bigoplus_{n\geqslant 1}L_2(\Omega_n, \mu_n),$$

defined by

$$V(\xi) = (\xi \circ \iota_n)_n$$

is a well-defined unitary isomorphism.

Let $\xi \in L_2(\Sigma, \mu)$. Since each $\iota_n : \Omega_n \to \Sigma$ is measurable, $\xi \circ \iota_n : \Omega_n \to \mathbb{C}$ is also measurable. Additionally, if ξ is 0 μ-a.e., then so is $\xi \circ \iota_n$. Moreover, we have

$$\|V(\xi)\|^2 = \|(\xi \circ \iota_n)_n\|^2$$

$$
\begin{aligned}
&= \sum_{n\geqslant 1} \|\xi \circ \iota_n\|^2 \\
&= \sum_{n\geqslant 1} \int_{\Omega_n} |\xi \circ \iota_n(x)|^2 \, d\mu_n \\
&= \sum_{n\geqslant 1} \int_{\Omega_n} |\xi|^2 \circ \iota_n(x) \, d\mu_n \\
&= \int_{\Sigma} |\xi|^2 \, d\mu \\
&= \|\xi\|^2 .
\end{aligned}
$$

This shows V is a well-defined linear map. Our calculation shows that V is an isometry.

We only need to write an inverse, for which which we define

$$W : \bigoplus_{n\geqslant 2} L_2(\Omega_n, \mu_n) \to L_2(\Sigma, \mu),$$

defined by

$$W((\xi_n)_n) = \xi,$$

where

$$
\begin{aligned}
\xi &\coloneqq \coprod_{n\geqslant 1} (\xi_n : \Omega \to \mathbb{C}) \\
\xi(x, n) &= \xi_n(x).
\end{aligned}
$$

Bounded Operators on Hilbert Spaces

Hilbert spaces are, according to John Conway, anyway, "boring," so we are interested in understanding the effects of operators on Hilbert spaces.

In the case of quantum mechanics, a particle with wave function ξ moving along the x axis has position equivalent to its expected value,

$$\int_{\mathbb{R}} x\,|\xi(x)|^2 \, d\lambda = \langle \mathrm{id}_{\mathbb{R}}\, \xi, \xi \rangle,$$

where the x coordinate is now an observable of an operator $\xi \overset{Q}{\mapsto} \mathrm{id}_{\mathbb{R}}\, \xi$, which is known as position. This operator is only defined on its domain, as it is not bounded.

Similarly, linear momentum is the map $\xi \overset{P}{\mapsto} \xi'$ (on the domain that it is defined), yielding

$$\langle P(\xi), \xi \rangle = \int_{\mathbb{R}} \frac{d\xi}{dx} \overline{\xi(x)} \, d\lambda,$$

from which we get the uncertainty principle

$$PQ(\xi) = I(\xi) + QP(\xi).$$

Structure of $\mathcal{B}(\mathcal{H})$

If X is a Banach space, then $\mathcal{B}(X)$, the space of bounded linear operators on X, is a unital Banach algebra. We will study the structure of $\mathcal{B}(\mathcal{H})$, which is the space of bounded linear operators on a Hilbert space.

Algebraic-Analytic Structure

Fact. Let $T, S\colon \mathcal{H} \to \mathcal{K}$ be linear maps between Hilbert spaces.

(1) We have $T = S$ if and only if $\langle T(x), y\rangle = \langle S(x), y\rangle$ for all $x \in \mathcal{H}, y \in \mathcal{K}$.

(2) If $\mathcal{H} = \mathcal{K}$, then $T = S$ if and only if $\langle T(x), x\rangle = \langle S(x), x\rangle$ for all $x \in \mathcal{H}$.

Proof.

(1) This follows from the fact that $\langle x, z_1\rangle = \langle x, z_2\rangle$ for all x if and only if $z_1 = z_2$.

(2) We define the sesquilinear forms $F\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $G\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$\begin{aligned} F(x, y) &= \langle T(x), y\rangle \\ G(x, y) &= \langle S(x), y\rangle . \end{aligned}$$

We see that $T = S$ if and only if $F = G$, if and only if F and G agree on the diagonal, meaning $\langle T(x), x\rangle = \langle S(x), x\rangle$ for all $x \in \mathcal{H}$.

□

Fact. If $T\colon \mathcal{H} \to \mathcal{K}$ is a linear map, then

$$\|T\|_{\text{op}} = \sup\left\{|\langle T(x), y\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\}.$$

Proof. By the Riesz Representation theorem, we have that $B_{K^*} = \{\langle\cdot, y\rangle \mid y \in B_{\mathcal{K}}\}$, meaning we have

$$\|T(x)\| = \sup_{y \in B_{\mathcal{K}}} |\langle T(x), y\rangle| .$$

Taking the supremum over $x \in B_{\mathcal{Y}}$ yields

$$\begin{aligned} \|T\|_{\text{op}} &= \sup_{x \in B_{\mathcal{H}}} \|T(x)\| \\ &= \sup\left\{|\langle T(x), y\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\}. \end{aligned}$$

□

Definition. Let $F : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ be a sesquilinear form. We define the norm

$$\|F\| := \sup\left\{|F(x, y)| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\right\}.$$

We say F is bounded if $\|F\| < \infty$.

Proposition: If $F : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a bounded sesquilinear form, then there exists a unique $S \in B(\mathcal{K}, \mathcal{H})$ such that

$$F(x, y) = \langle x, S(y)\rangle .$$

. Fix $y \in \mathcal{K}$, and consider the linear functional $\varphi : \mathcal{H} \to \mathbb{C}$ given by $\varphi(x) = F(x, y)$. Since φ is linear, we have

$$\begin{aligned} |\varphi(x)| &= |F(x, y)| \\ &\leqslant \|F\|\,\|y\| \end{aligned}$$

for all $x \in B_{\mathcal{H}}$, meaning $\varphi \in \mathcal{H}^*$. Thus, there is a unique $z \in \mathcal{H}$ such that $\varphi = \varphi_z$. We define $S(y) := z$. Doing this for each $y \in \mathcal{K}$, we get a map $S : \mathcal{K} \to \mathcal{H}$ such that

$$F(x, y) = \langle x, S(y)\rangle .$$

We show that S is linear and bounded. Let $y_1, y_2 \in \mathcal{K}$ and $\alpha \in \mathcal{C}$. For all $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle x, S(y_1 + \alpha y_2)\rangle &= F(x, y_1 + \alpha y_2)\\ &= F(x, y_1) + \overline{\alpha} F(x, y_2)\\ &= \langle x, S(y_1)\rangle + \overline{\alpha}\langle x, S(y_2)\rangle\\ &= \langle x, S(y_1) + \alpha S(y_2)\rangle .\end{aligned}$$

Thus, S is linear. We also have

$$\begin{aligned}\|S\|_{op} &= \sup\{|\langle x, S(y)\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\}\\ &= \sup\{|F(x,y)| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\}\\ &= \|F\| .\end{aligned}$$

For uniqueness, we see that if $\langle x, S_1(y)\rangle = F(x,y) = \langle x, S_2(y)\rangle$, then $S_1 = S_2$ necessarily. □

Theorem: Let $\mathcal{H}, \mathcal{K}, \mathcal{L}$ be Hilbert spaces. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then there is a unique bounded operator $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We call T^* the Hilbert space adjoint of T. Moreover, the following are true for $T, S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $R \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, and $\lambda \in \mathbb{C}$:

(1) $(T + \lambda S)^* = T^* + \overline{\lambda} S^*$;

(2) $T^{**} = T$;

(3) $(R \circ T)^* = T^* \circ R^*$;

(4) if T is invertible, then $\left(T^{-1}\right)^* = (T^*)^{-1}$;

(5) $\|T^*\| = \|T\|$;

(6) $\|T^*T\| = \|T\|^2$ (known as the C^*-property).

Proof. We define $F : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ by $F(x,y) = \langle T(x), y\rangle$. We have F is a sesquilinear form, and

$$\begin{aligned}\|F\| &= \sup\{|F(x,y)| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\}\\ &= \sup\{|\langle T(x), y\rangle| \mid x \in B_{\mathcal{H}}, y \in B_{\mathcal{K}}\}\\ &= \|T\|_{op} .\end{aligned}$$

Thus, there is a unique operator $S_T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\langle T(x), y\rangle = \langle x, S_T(y)\rangle$, with $\|S_T\| = \|T\|$. We define $T^* = S_T$.

We will show (6).

$$\begin{aligned}\|T^*T\| &= \sup_{\substack{x \in B_{\mathcal{H}}\\ y \in B_{\mathcal{K}}}} |\langle T^*T(x), y\rangle|\\ &\geqslant \sup_{x \in B_{\mathcal{H}}} |\langle T^*T(x), x\rangle|\\ &= \sup_{x \in B_{\mathcal{H}}} |\langle T(x), T(x)\rangle|\\ &= \sup_{x \in B_{\mathcal{H}}} \|T(x)\|^2\\ &= \left(\sup_{x \in B_{\mathcal{H}}} \|T(x)\|\right)^2\end{aligned}$$

$$\begin{aligned}
&= \|T\|^2 \\
&= \|T\| \, \|T\| \\
&= \|T^*\| \, \|T\| \\
&\geqslant \|T^*T\|.
\end{aligned}$$

□

Exercise: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Write $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ to be the (Hilbert space) adjoint, and $T^\dagger : K^* \to H^*$ to be the Banach space adjoint. Let $\rho_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^*$ be the conjugate linear isometry $x \mapsto \varphi_x$, and let $\rho_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^*$ to be the conjugate linear isometry $y \mapsto \varphi_y$. Show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{K}^* \arrow[r, "T^\dagger"] & \mathcal{H}^* \\
\mathcal{K} \arrow[u, "\rho_{\mathcal{K}}"] \arrow[r, "T^*"'] & \mathcal{H} \arrow[u, "\rho_{\mathcal{H}}"']
\end{tikzcd}$$

Proof. Let $x \in \mathcal{H}$, $y \in \mathcal{K}$. By the Riesz representation theorem, we have $\varphi_x = \langle \cdot, x \rangle$ and $\varphi_y = \langle \cdot, y \rangle$. Thus, we have

$$\begin{aligned}
T^\dagger \left(\varphi_y\right)((x)) &= \varphi_y \left(T(x)\right) \\
&= \langle T(x), y \rangle \\
&= \langle x, T^*(y) \rangle \\
&= \varphi_x \left(T^*(y)\right).
\end{aligned}$$

□

Corollary: The adjoint map $* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $T \mapsto T^*$ is an involution, meaning $\mathcal{B}(\mathcal{H})$ is a unital $*$-algebra. If $\dim(\mathcal{H}) > 1$, then $\mathcal{B}(\mathcal{H})$ is noncommutative.

Definition. A Banach $*$-algebra is a Banach algebra A with an involution satisfying

$$\|a^*\| = \|a\|$$

for all $a \in A$. If A is a Banach $*$-algebra that satisfies the C^*-property, then A is called a C^*-algebra.

We can now look at some examples of operators and adjoints.

Example. Let $a = \left(a_{ij}\right)_{i,j} \in \mathrm{Mat}_{m,n}(\mathbb{C})$, with the linear operator

$$T_a : \ell_2^n \to \ell_2^m$$

defined by $T_a(\xi) = a\xi$. Since ℓ_2^n is finite-dimensional, T_a is bounded. The conjugate transpose $a^* = \left(\overline{a_{ji}}\right)_{i,j}$ is an $n \times m$ matrix satisfying

$$\begin{aligned}
\langle T_a(\xi), \eta \rangle &= \langle a\eta, \xi \rangle \\
&= (a\xi)^* \eta \\
&= \xi^* a^* \eta \\
&= \langle \xi, a^* \eta \rangle \\
&= \langle \xi, T_{a^*}(\eta) \rangle,
\end{aligned}$$

meaning $T_a^* = T_{a^*}$.

Topologies on $\mathcal{B}(\mathcal{H})$

Given a Banach space X, we can introduce two locally convex topologies on $\mathcal{B}(X)$ — namely, the weak operator topology and the strong operator topology, both of which are weaker than the norm topology.

Lemma: Let $\mathcal{H}$ be a Hilbert space, and let $(T_\alpha)_\alpha$ be a net in $\mathcal{B}(\mathcal{H})$. The following are equivalent:

(i) $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$;

(ii) for all $\xi, \eta \in \mathcal{H}$, $\langle T_\alpha(\xi), \eta \rangle \to \langle T(\xi), \eta \rangle$;

(iii) for all $\xi, \eta \in B_{\mathcal{H}}$, $\langle T_\alpha(\xi), \eta \rangle \to \langle T(\xi), \eta \rangle$;

(iv) for all $\xi \in \mathcal{H}$, $\langle T_\alpha(\xi), \xi \rangle \to \langle T(\xi), \xi \rangle$;

(v) for all $\xi \in B_{\mathcal{H}}$, $\langle T_\alpha(\xi), \xi \rangle \to \langle T(\xi), \xi \rangle$.

Proof. We only need to prove the equivalence between (i) and (ii). The rest follow from scaling or the polarization identity.

We know that $(T_\alpha)_\alpha \xrightarrow{\text{WOT}} T$ in $\mathcal{B}(\mathcal{H})$ if and only if $\varphi(T_\alpha(\xi)) \to \varphi(T(\xi))$ for each $\xi \in \mathcal{H}$ and $\varphi \in \mathcal{H}^*$. By the Riesz representation theorem, each $\varphi \in \mathcal{H}^*$ is of the form $\varphi(\cdot) = \langle \cdot, \eta \rangle$. $\square$