Problem (Problem 1)**:** Let R be a Euclidean domain with norm N, and let

$$m = \min\{N(x) \mid x \in R \setminus \{0\}\}.$$

Show that any $u \in R \setminus \{0\}$ satisfying $N(u) = m$ is invertible.

Solution: Let u satisfy $N(u) = m$. Applying the division algorithm, we find that

$$1 = uq + r,$$

where $r = 0$ or $N(r) < N(u)$. In the former case, we find that $q = u^{-1}$, while the latter case violates the assumption that $N(u)$ is of minimal value.

Problem (Problem 2)**:** Show that in a UFD every irreducible element is prime. Conclude that if R is a Noetherian domain, then R is a UFD if and only if every irreducible element is prime.

Solution: Let R be a UFD, and let h be an irreducible element such that $h|ab$ for some $a, b \in R$.

Write the unique (up to associates) factorizations into irreducibles for a and b, giving

$$\begin{aligned} a &= a_1 a_2 \cdots a_r \\ b &= b_1 b_2 \cdots b_s. \end{aligned}$$

Therefore, for some $k \in R$, we have

$$hk = (a_1 a_2 \cdots a_r)(b_1 b_2 \cdots b_s).$$

Since h is irreducible, and the factorizations for a and b are unique up to associates, there is some $u_j \in R^\times$ such that $h = u_j a_j$ or some $v_k \in R^\times$ such that $h = v_k b_k$ (else we would have a different factorization for ab into irreducibles). Thus, $h|a$ or $h|b$ depending on which of these hold, so that h is prime.

Since we already know that primes are irreducible, it follows that, in a Noetherian domain, since every element has at least one factorization into irreducibles, such a factorization is unique if and only if every irreducible element is prime.

Problem (Problem 4)**:** Let R be a domain in which every prime ideal is principal. Show that R is a PID by using the following suggestions.

(i) Assume that the set of nonprincipal ideals is nonempty. Then, use Zorn's Lemma to find a maximal element I in it.

(ii) Since I is not prime, there exist $a, b \in R$ such that $ab \in I$ but $a, b \notin I$. Let $I_a = I + (a)$, and let J be defined by

$$J = \{x \in R \mid xI_a \subseteq I\}.$$

Verify that J is an ideal of R. Deduce a contradiction by showing that $I = I_a J$.

Solution: Let $\mathcal{X}$ be the set of all nonprincipal ideals of R, ordered by inclusion. Suppose toward contradiction that $\mathcal{X}$ were nonempty. Let $\{K_\alpha\}_{\alpha \in A} = \mathcal{C} \subseteq \mathcal{X}$ be a chain in $\mathcal{X}$, and let $I = \bigcup_{\alpha \in A} K_\alpha$, which is an upper bound for $\mathcal{C}$. We claim that I is nonprincipal.

Suppose not. Then, $I = (v)$ for some $v \in R$; since $v \in I$, it follows that $v \in K_\alpha$ for some $\alpha \in A$, meaning that $(v) \subseteq K_\alpha$, or that $K_\alpha = I = (v)$, which would contradict the assumption that K_α is nonprincipal.

Since I is nonprincipal, I is not prime, so there exists some $ab \in I$ with $a \notin I$ and $b \notin I$. Letting $I_a = I + (a)$, since $I \subsetneq I_a$, we must $I_a = (u)$ for some $u \in R$.

Let

$$J = \{x \in R \mid x(I + (a)) \subseteq I\}.$$

Observe that J is closed under subtraction, since if $x, y \in J$, we have

$$\begin{aligned}(x - y)(I + (a)) &= x(I + (a)) - y(I + (a))\\ &\subseteq I,\end{aligned}$$

since I is closed under subtraction. Similarly, if $r \in R$, then

$$\begin{aligned}rx(I + (a)) &= r(x(I + (a)))\\ &\subseteq I,\end{aligned}$$

since I is closed under multiplication by elements from R. Thus, J is an ideal. In particular, since J contains I and $b \notin I$, J must be a principal ideal of the form (v), so that $I_a J = (uv)$ is principal as well.

Now, we observe that elements of $I_a J$ are of the form

$$\begin{aligned}\sum_{k=1}^{n} (x_k + r_k a)(s_k v) &= \sum_{k=1}^{n} x_k(s_k v) + s_k v(r_k a)\\ &\in I,\end{aligned}$$

so that $I_a J \subseteq I$.

If $x \in I$, then since $x \in I_a$, and $I_a = (u)$, it follows that $x = \ell u$ for some $\ell \in R$. Additionally, since $rx \in I$ for arbitrary $r \in R$, it follows that $r\ell u = \ell r u \in I$, meaning that $\ell(u) \subseteq I$, meaning that $\ell \in J$. Thus, $x \in I_a J$, implying that $I = I_a J$, meaning I is principal, which is a contradiction of the fact that I is (allegedly) not principal.