Problem 1

True or false: If H is a minor of G, then H is a contraction of a subgraph of G.

True.

Problem 2

Prove each of the following.

(a) There exists an infinite family F of graphs such that no graph in F is a subgraph of another graph in F.

(b) There exists an infinite family F of graphs such that no graph in F is a contraction of another graph in F.

(c) There exists an infinite family F of graphs such that no graph in F is a subgraph or a contraction of another graph in F.

Problem 3

Prove that the set of all planar graphs is minor-closed.

Let G be any planar graph. Then, by Wagner's theorem, it must be the case that neither K_5 nor $K_{3,3}$ are minors of G. Therefore, any minor of G, G', must also not have K_5 nor $K_{3,3}$ as a minor — otherwise, we would take the steps to create G', then the steps to create one of the forbidden minors, and G would have the forbidden minors as a minor.

Thus, since no minor of any planar graph can be non-planar, it must be the case that planarity is minor-closed.

Problem 4

Let P be an arbitrary set of graphs. Let P' be the set of all graphs not in P. By the Graph Minor Theorem, P has a finite subset F of graphs that are minor-minimal in P. Similarly, P' has a finite subset F' of graphs that are minor minimal in P'. Prove that if P is minor-closed, then a graph G is in P' if and only if G has a minor in F'. So, if P is minor-closed, then P and P' are both "characterized" by F'. In fact, if P is minor-closed, then F consists of only one graph, namely the graph with only one vertex. Why?

Problem 5

A graph G is apex if $G - v$ is planar for some vertex v of G. Prove that the set of apex graphs is minor-closed.