

Problem 1

Let v_1, \dots, v_n be mutually orthogonal vectors in an inner product space V . Show that

$$\left\| \sum_{k=1}^n v_k \right\|^2 = \sum_{k=1}^n \|v_k\|^2.$$

Proof:

$$\begin{aligned} \left\| \sum_{k=1}^n v_k \right\|^2 &= \left\langle \sum_{k=1}^n v_k, \sum_{k=1}^n v_k \right\rangle \\ &= \sum_{i=1}^n \left\langle \sum_{k=1}^n v_k, v_i \right\rangle \\ &= \sum_{i=1}^n \langle v_i, v_i \rangle && \text{since for } i \neq j, \langle v_i, v_j \rangle = 0 \\ &= \sum_{i=1}^n \|v_i\|^2 \end{aligned}$$

Problem 2

Let V be an inner product space and fix $w \neq 0$ in V . We define the one-dimensional projection

$$P_w : V \rightarrow V; P_w(v) := \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

(i) Prove that $v - P_w(v) \perp P_w(v)$.

(ii) Show that $P_w : V \rightarrow V$ is a linear operator with $\|P_w\|_{\text{op}} = 1$.

(iii) Show that $P_w \circ P_w = P_w$.

Proof of (i):

$$\begin{aligned} \langle v - P_w(v), P_w(v) \rangle &= \langle v, P_w(v) \rangle - \langle P_w(v), P_w(v) \rangle \\ &= \langle v, P_w(v) \rangle - \|P_w(v)\|^2 \\ &= \left\langle v, \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\rangle - \|P_w(v)\|^2 \\ &= \frac{\langle v, w \rangle}{\langle w, w \rangle} \langle v, w \rangle - \|P_w(v)\|^2 \\ &= \frac{|\langle v, w \rangle|^2}{\|w\|^2} - \frac{|\langle v, w \rangle|^2}{\|w\|^2} \\ &= 0 \end{aligned}$$

Proof of (ii):

$$\begin{aligned} \|P_w\|_{\text{op}} &= \sup_{\|v\| \leq 1} \left\| \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\| \\ &= \sup_{\|v\| \leq 1} \frac{|\langle v, w \rangle|}{\|w\|} \\ &\leq \sup_{\|v\| \leq 1} \frac{\|v\| \|w\|}{\|w\|} \\ &= 1 \end{aligned}$$

Proof of (iii):

$$\begin{aligned} P_w(P_w(v)) &= P_w \left(\frac{\langle v, w \rangle}{\langle w, w \rangle} w \right) \\ &= \frac{\left\langle \frac{\langle v, w \rangle}{\langle w, w \rangle} w, w \right\rangle}{\langle w, w \rangle} w \\ &= \frac{\langle v, w \rangle}{\langle w, w \rangle} w \\ &= P_w(v). \end{aligned}$$

Problem 3

Let V be an inner product space. Prove the reverse Cauchy-Schwarz Inequality which states

$$v, w \in V, \text{ and } |\langle v, w \rangle| = \|v\| \|w\| \Rightarrow v = \alpha w.$$

Proof: If $\|w\| = 0$, then $w = 0$, so $\langle v, w \rangle = 0$ and $\alpha = 0$. Suppose $\|w\| \neq 0$. Then,

$$\begin{aligned} |\langle v, w \rangle| &= \|v\| \|w\| \\ \|w\| \left| \frac{\langle v, w \rangle}{\langle w, w \rangle} \right| &= \|v\|, \end{aligned}$$

so $P_w(v) = v$, meaning $w = \alpha v$.

Problem 4

Let V be an inner product space. Then, for any $v, w \in V$, show that

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

Proof:

$$\begin{aligned} \langle v + w, v + w \rangle + \langle v - w, v - w \rangle &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle + \langle v, v \rangle - \langle v, w \rangle - \langle w, v \rangle + \langle -w, -w \rangle \\ &= \langle v, v \rangle + \langle v, v \rangle + \langle w, w \rangle + \langle w, w \rangle \\ &= 2\|v\|^2 + 2\|w\|^2 \end{aligned}$$

Problem 5

Let $\lambda = (\lambda_k)_k$ belong to ℓ_∞ . Show that the map

$$D_\lambda : \ell_2 \rightarrow \ell_2; D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$

Proof:

Well-Defined: Let $(\zeta_k)_k = 0$ for all $k \in \mathbb{N}$. Then,

$$\begin{aligned} D_\lambda((\zeta_k)_k) &= (\lambda_k \zeta_k)_k \\ &= ((\lambda_k)(0))_k \\ &= 0 \end{aligned}$$

Linear:

$$\begin{aligned} D_\lambda((\alpha \xi_k)_k + (\beta \zeta_k)_k) &= D_\lambda((\alpha \xi_k + \beta \zeta_k)_k) \\ &= (\lambda_k(\alpha \xi_k + \beta \zeta_k))_k \\ &= (\alpha \lambda_k \xi_k + \alpha \lambda_k \zeta_k)_k \\ &= (\alpha \lambda_k \xi_k)_k + (\beta \lambda_k \zeta_k)_k \\ &= \alpha(\lambda_k \xi_k)_k + \beta(\lambda_k \zeta_k)_k \\ &= \alpha D_\lambda((\xi_k)_k) + \beta D_\lambda((\zeta_k)_k) \end{aligned}$$

Bounded:

$$\begin{aligned} \|D_\lambda\|_{\text{op}} &= \sup_{\|\xi_k\|_k \leq 1} \|D_\lambda((\xi_k)_k)\| \\ \|D_\lambda((\xi_k)_k)\| &= \left(\sum_{k=1}^{\infty} |\lambda_k \xi_k|^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} \left(\sup_{k \in \mathbb{N}} |\lambda_k| |\xi_k| \right)^2 \right)^{1/2} \\ &= \|\lambda\|_\infty \left(\sum_{k=1}^n |\xi_k|^2 \right)^{1/2} \\ &= \|\lambda\|_\infty \|\xi_k\| \end{aligned}$$

Therefore,

$$\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty.$$

Problem 6

Consider the vector space $C([0, 2\pi])$ equipped with

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt.$$

(i) Show that this pairing defines an inner product on $C([0, 2\pi])$.

Proof: We will show that $\langle f, g \rangle$ satisfies the axioms of the inner product.

Addition:

$$\begin{aligned} \langle f_1 + f_2, g \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (f_1(t) + f_2(t)) \overline{g(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (f_1(t) \overline{g(t)} + f_2(t) \overline{g(t)}) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f_1(t) \overline{g(t)} dt + \frac{1}{2\pi} \int_0^{2\pi} f_2(t) \overline{g(t)} dt \\ &= \langle f_1, g \rangle + \langle f_2, g \rangle. \end{aligned}$$

Scalar Multiplication:

$$\begin{aligned} \langle \alpha f, g \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (\alpha f(t)) \overline{g(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \alpha (f(t) \overline{g(t)}) dt \\ &= \alpha \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt \right) \\ &= \alpha \langle f, g \rangle. \end{aligned}$$

Conjugation:

$$\begin{aligned} \overline{\langle g, f \rangle} &= \frac{1}{2\pi} \int_0^{2\pi} \overline{g(t) \overline{f(t)}} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt \\ &= \langle f, g \rangle. \end{aligned}$$

Positive Definition:

$$\begin{aligned} \langle f, f \rangle &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{f(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt \\ &\geq 0. \end{aligned}$$

For $\langle f, f \rangle = 0$, we have that the integral equals zero — since f is continuous, it means that if $|f(t)|^2 > 0$ for some $t_0 \in [0, 2\pi]$, then $|f(t)|^2 \neq 0$ on some interval $[t_0 - \delta, t_0 + \delta]$, meaning the integral can only equal zero if f is 0_f on $[0, 2\pi]$.

(ii) For $n \in \mathbb{Z}$, set $e_n(t) = \cos(nt) + i \sin(nt)$. Show that the family $\{e_n\}_{n \in \mathbb{Z}}$ is orthonormal.

Proof: We will show that $\{e_n\}_{n \in \mathbb{Z}}$ is orthonormal by showing that $\langle e_n, e_n \rangle = 1$ and $\langle e_n, e_m \rangle = 0$ for $m \neq n$.

$$\begin{aligned}
 \langle e_n, e_n \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(nt) + i \sin(nt))(\cos(nt) - i \sin(nt)) dt \\
 &= \frac{1}{2\pi} \int_0^{2\pi} (\cos^2(nt) + \sin^2(nt)) dt \\
 &= \frac{1}{2\pi} \int_0^{2\pi} dt \\
 &= 1 \\
 \langle e_n, e_m \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(nt) + i \sin(nt))(\cos(mt) - i \sin(mt)) dt \\
 &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(mt) \cos(nt) + i \sin(nt) \cos(mt) - i \sin(mt) \cos(nt) + \sin(nt) \sin(mt)) dt \\
 &= \frac{1}{2\pi} \left(\int_0^{2\pi} (\cos(mt) \cos(nt) + \sin(nt) \sin(mt)) dt + i \int_0^{2\pi} (\sin(nt) \cos(mt) - \sin(mt) \cos(nt)) dt \right) \\
 &= 0.
 \end{aligned}$$

Problem 7

Let V be any normed space, $p \in [1, \infty]$, and suppose $T : \ell_p^n \rightarrow V$ is linear. Show that T is bounded.

Proof: Let T be a linear transformation from ℓ_p^n to V . Then,

$$\|T\|_{\text{op}} = \sup_{\substack{v \in \ell_p^n \\ \|v\| \leq 1}} \|T(v)\|$$

Problem 8

Let $\mathbb{P}[0, 1] = \{\sum_0^n a_k x^k \mid a_k \in \mathbb{C}\} \subseteq C([0, 1])$ denote the linear subspace of all polynomial functions equipped with the uniform norm $\|\cdot\|_u$ inherited from $C([0, 1])$. We define the map

$$\begin{aligned}
 D : \mathbb{P}[0, 1] &\rightarrow \mathbb{P}[0, 1] \\
 D(p(x)) &= p'(x).
 \end{aligned}$$

Show that D is unbounded.

Proof: Let $p(x) = x^n$. Then, in $\mathbb{P}[0, 1]$,

$$\begin{aligned}
 \|p\|_u &= 1 \\
 \|D(p)\|_u &= n.
 \end{aligned}$$

For any $L \in \mathbb{R}$, we can find a $n \in \mathbb{N}$ sufficiently large such that $\|D(p)\|_u = n > L$, by the Archimedean property. Therefore, D is unbounded.

Problem 9

Let V be an infinite-dimensional normed space. Show that there is a linear functional $\varphi : V \rightarrow \mathbb{F}$ that is unbounded.

Proof: Let $B = \{x_n\}$ be the basis for V . We define $\varphi : V \rightarrow \mathbb{F}$ as $\varphi(x) = \sum_n n \alpha_n$ for the $\alpha_n x_n$ component in x . Then, φ is linear and unbounded, as the values n takes are not bounded, seeing as V is infinite-dimensional.

Problem 10

Let $a, b \in \mathbb{M}_n$. Show the following properties of the operator norm.

- (i) $\|a\|_{\text{op}} = \sup \{ |\langle a\xi, \eta \rangle| \mid \xi, \eta \in B_{\ell_2^n} \}$
- (ii) $\|a^*\|_{\text{op}} = \|a\|_{\text{op}}$
- (iii) $\|ab\|_{\text{op}} \leq \|a\|_{\text{op}} \|b\|_{\text{op}}$
- (iv) $\|a^*a\|_{\text{op}} = \|a\|_{\text{op}}^2$

Proof:

(i)

$$\begin{aligned}
\langle a\xi, \eta \rangle &\leq \|a\xi\| \|\eta\| \\
&= \|a\xi\| \\
&\leq \sup_{\xi \in B_{\ell_2^n}} \|a\xi\| \\
&= \|a\|_{\text{op}}. \\
\|a\|_{\text{op}} &= \sup_{\xi \in B_{\ell_2^n}} \|a\xi\|
\end{aligned}$$

Set $\eta = \frac{a\xi}{\|a\xi\|}$. Then,

$$\begin{aligned}
&= \sup_{\xi \in B_{\ell_2^n}} \frac{1}{\|a\xi\|} \langle a\xi, \eta \rangle \\
&= \sup \left\{ \langle a\xi, \eta \rangle \mid \xi, \eta \in B_{\ell_2^n} \right\}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\|a^*\|_{\text{op}} &= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle a^*\xi, \eta \rangle| \\
&= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle \xi, a^{**}\eta \rangle| \\
&= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle a\xi, \eta \rangle| \\
&= \|a\|_{\text{op}}.
\end{aligned}$$

definition of conjugate transpose

by absolute value

(iii)

$$\begin{aligned}
\|ab\|_{\text{op}} &= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle (ab)\xi, \eta \rangle| \\
&= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle a(b\xi), \eta \rangle| \\
&= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle b\xi, a^*\eta \rangle| \\
&\leq \sup_{\xi \in B_{\ell_2^n}} \|b\xi\| \sup_{\eta \in B_{\ell_2^n}} \|a^*\eta\| \\
&= \|b\|_{\text{op}} \|a^*\|_{\text{op}} \\
&= \|a\| \|b\|.
\end{aligned}$$

(iv)

$$\begin{aligned}
\|a^*a\|_{\text{op}} &= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle (a^*a)\xi, \eta \rangle| \\
&= \sup_{\xi, \eta \in B_{\ell_2^n}} |\langle a\xi, a^{**}\eta \rangle| \\
&= \sup_{\xi \in B_{\ell_2^n}} \|a\xi\|^2 \\
&= \|a\|_{\text{op}}^2
\end{aligned}$$