

These are some notes on generating functions I'm taking, primarily from *The Art of Computer Programming*, Volume I, by Donald Knuth.

Definition: If (a_0, a_1, \dots) is a sequence of numbers, then the infinite series

$$\begin{aligned} G(z) &= a_0 + a_1 z + \dots \\ &= \sum_{n=0}^{\infty} a_n z^n \end{aligned}$$

is known as the *generating function* for the sequence.

We can try to understand some basic operations for the generating function.

To start, if $G(z)$ is the generating function for $(a_n)_n$, then the generating function for $(a_{n-m})_n$ for a fixed m is given by $z^m G(z)$, as

$$\begin{aligned} z^m G(z) &= z^m \sum_{n=0}^{\infty} a_n z^n \\ &= \sum_{n=m}^{\infty} a_{n-m} z^n. \end{aligned}$$

Similarly, the generating function for $(a_{n+m})_n$ is given by $z^{-m} \left(G(z) - \sum_{k=0}^{m-1} a_k z^k \right)$, as

$$z^{-m} \sum_{n=m}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_{n+m} z^n.$$

Example: Our first example is finding a generating function for the Fibonacci sequence. Letting $G(z)$ be the generating function for $(F_n)_n$, we have that $zG(z)$ is the generating function for $(F_{n-1})_n$ and $z^2 G(z)$ is the generating function for $(F_{n-2})_n$. In particular, since $F_n = F_{n-1} + F_{n-2}$, we have that

$$(1 - z - z^2)G(z) = p(z),$$

where $p(z)$ is some polynomial satisfying $p(0) = 0$ and $p(1) = 1$. In particular, we can show that this yields

$$G(z) = \frac{z}{1 - z - z^2}.$$

In general, for a linear recurrence $a_n = c_1 a_{n-1} + \dots + c_m a_{n-m}$, the corresponding generating function will be of the form

$$G(z) = \frac{p(z)}{1 - c_1 z - \dots - c_m z^m}$$

for some polynomial $p(z)$. In particular, for the simplest case, this yields

$$\frac{1}{1 - z} = 1 + z + z^2 + \dots$$

is the generating function for the sequence $(1, 1, 1, \dots)$.

Next, we discuss multiplication. If we let

$$G(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$H(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then we observe that

$$G(z)H(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) z^n.$$

In the particular case where b is the sequence of all 1s, then

$$\frac{1}{z}G(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k \right) z^n.$$

When we want binomial coefficients of the form

$$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k},$$

we usually use the generating functions for $\left(\frac{a_n}{n!}\right)_n$ and $\left(\frac{b_n}{n!}\right)_n$, which yields the generating function for $\left(\frac{c_n}{n!}\right)_n$.

Next, to extract particular terms of a generating function, we let $\omega = e^{2\pi i/m}$ be a primitive m th root of unity, and find

$$\sum_{n \equiv r \pmod m} a_n z^n = \frac{1}{m} \sum_{k=0}^{m-1} \omega^{-kr} G(\omega^k z).$$

For instance, if $m = 3$ and $r = 1$, then

$$a_1 z + a_4 z^4 + \dots = \frac{1}{3} (G(z) + \omega^{-1} G(\omega z) + \omega^{-2} G(\omega^2 z)).$$

Differentiation and integration yield similar results:

$$\begin{aligned} zG'(z) &= \sum_{n=0}^{\infty} n a_n z^n \\ \int_0^z G(t) dt &= \sum_{k=1}^{\infty} \frac{1}{k} a_{k-1} z^k \end{aligned}$$

We next write down some generating functions.

- Binomial theorem:

$$(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} z^k.$$

- Exponents:

$$\begin{aligned} e^z &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \\ (e^z - 1)^n &= n! \sum_{k=0}^{\infty} S(n, k) \frac{z^k}{k!}, \end{aligned}$$

where $S(n, k)$ denotes the Stirling numbers of the second kind. The Stirling numbers of the second kind denote the number of ways to partition a set of n elements into k disjoint (nonempty) subsets.

- Logarithms:

$$\ln(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k$$

$$\frac{1}{(1-z)^{m+1}} \ln\left(\frac{1}{1-z}\right) = \sum_{k=1}^{\infty} (H_{m+k} - H_m) \binom{m+k}{k} z^k,$$

where H_k denotes the k th harmonic number.

Example: Consider the elementary symmetric functions on n variables, given by

$$e_m = \sum_{1 \leq j_1 < \dots < j_m \leq n} x_{j_1} \cdots x_{j_m}.$$

Observe that the e_m appear as the coefficient of z^m in the polynomial

$$G_n(z) = \prod_{i=1}^n (1 + x_i z).$$