Problem (Problem 1)**:** A topological group is a group which is also a Hausdorff topological space where the group operations are continuous.

Recall the definition of the concatenation operation on the fundamental group. Now, let G be a path-connected topological group, and let $\pi_1(G, e)$ be the fundamental group of G with base point e. Use the Hilton–Eckmann argument to prove that the concatenation operation on the fundamental group is commutative.

Solution: Define two operations, $*$ and $\cdot$, on the homotopy-classes of functions $f\colon S^1 \to (G, e)$, where $S^1 \cong [0,1]/(\{0\} \sim \{1\})$ given by

$$\begin{aligned} f * g &= \begin{cases} f(2t) & 0 \leqslant t \leqslant 1/2 \\ g(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases} \\ f \cdot g &= f(t)g(t), \end{aligned}$$

where the latter is multiplication within the group and the former is concatenation. We see that the identity map

$$\begin{aligned} \mathrm{id}\colon S^1 &\to (G, e) \\ t &\mapsto e \end{aligned}$$

is an identity for both $*$ and $\cdot$. Our task now is to show that the Hilton–Eckmann condition holds. That is, let $a, b, c, d\colon S^1 \to (G, e)$ be continuous maps with base point e. Then,

$$\begin{aligned} (a * b) \cdot (c * d) &= (a * b)(t) \cdot (c * d)(t) \\ &= \begin{cases} a(2t)c(2t) & 0 \leqslant t \leqslant 1/2 \\ b(2t-1)d(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases} \\ &= (a \cdot c) * (b \cdot d), \end{aligned}$$

whence $\cdot = *$ and the concatenation operation is commutative.

Problem (Problems 2–4)**:**

(2) Let M and N be smooth, orientable, closed manifolds of the same dimension n, and let $f\colon M \to N$ be a smooth function. Show that f induces a map $f^*\colon H^n_{DR}(N) \to H^n_{DR}(M)$ which is multiplication by an integer. This is called the degree of f and is written $\deg(f)$.

(3) Recall the definition of the degree of f from one of the previous problem sets, counting the sums of signs of determinants of the derivative of f over the preimage of a regular value of f. Prove that the two definitions of the degree agree.

(4) With the setup of the previous exercises, prove that if ω is an arbitrary n-form on N, then

$$\int_M f^*\omega = \deg(f)\int_N \omega.$$

Solution: Letting $\omega \in H^n_{DR}(N)$ be a nonvanishing top-dimensional form. By the naturality of the de Rham isomorphism, it follows that there is some $\delta \in \mathbb{R}$ such that.

$$\int_M f^*\omega = \delta \int_N \omega$$

Our task now is to show that $\delta \in \mathbb{Z}$. In particular, we will show that $\delta = \deg(f)$, where $\deg(f)$ is defined as before.

Toward this end, let q be a regular value of f. We may use a smooth bump function to restrict ω to a small open neighborhood V of q. It follows then that $f^{-1}(q) = \{p_1, \dots, p_\ell\}$ for some ℓ, with corresponding disjoint open neighborhoods $U_1, \dots, U_\ell$ locally diffeomorphic to V, whence the support of $f^*\omega$ is contained in the union of $U_1, \dots, U_\ell$. If $f^{-1}(q) = \emptyset$, then

$$\begin{aligned}\int_M f^*\omega &= \int_\emptyset f^*\omega\\ &= \delta \int_N \omega\\ &= 0,\end{aligned}$$

whence $\delta = 0$. If $f^{-1}(q) \neq \emptyset$, then we see that

$$\int_M f^*\omega = \sum_{k=1}^{\ell} \int_{U_k} f^*\omega.$$

Now, since f is a local diffeomorphism on each of the U_k, it follows that

$$\begin{aligned}\int_{U_k} f^*\omega &= \operatorname{sgn}\bigl(\det\bigl(D_{p_k} f\bigr)\bigr) \int_V \omega\\ &= \operatorname{sgn}\bigl(\det\bigl(D_{p_k} f\bigr)\bigr) \int_N \omega.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}\int_M f^*\omega &= \sum_{k=1}^{\ell} \operatorname{sgn}\bigl(\det\bigl(D_{p_k} f\bigr)\bigr) \int_N \omega\\ &= \deg(f) \int_N \omega,\end{aligned}$$

giving that $\deg(f)$ as defined via cohomology and as defined via summation over neighborhoods of preimages of a regular value are equal to each other.