Problem 1

Using the definition of the derivative find $f'(c)$ where $c \in \mathbb{R}$ and $f(x) = \frac{1}{x}$.

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} \\ &= \lim_{x\to c} \frac{c - x}{(xc)(x-c)} \\ &= \lim_{x\to c} \frac{-1}{xc} \\ &= -\frac{1}{x^2} && c \neq 0 \end{aligned}$$

Problem 2

Let $n \in \mathbb{N}$ and consider the function

$$f(x) = \begin{cases} x^n, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

For which values of n is f differentiable at $x = 0$.

We have that on $(0, \infty)$, $f(x) = x^n$, meaning $f'(x)$ on $(0, \infty)$ is nx^{n-1}. Therefore, as $(x_n)_n \to 0$ for $x_n \in (0, \infty)$, $\left(\frac{f(x_n) - f(0)}{x_n - 0}\right)_n \to 0$, taking $f(0)$ as given above, assuming $n > 1$ — otherwise, $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 1$.

Problem 3

Consider the function

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

Show that f is differentiable at $x = 0$ and find $f'(0)$.

Let $(x_n)_n \to 0$, $x_n \neq 0$. Let $(x_{n_k})_k$ denote the sequence of irrational values of x_n, and let $(x_{m_l})_l$ denote the sequence of rational values of x_n. Then, $(f(x_n))_n \to 0$, regardless of whether $x_n \in (x_{m_l})_l$ or $x_n \in (x_{n_k})_k$. So, having established that the limit exists, we find that

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{x^2 - 0^2}{x - 0} \\ &= \lim_{x\to 0} x \\ &= 0 \end{aligned}$$

Problem 4

Determine the values of x where $f(x) = x|x|$ is differentiable.

We can see that $f(x) = x|x|$ is equivalent to

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}.$$

Since x^2 and $-x^2$ are polynomials, we have that for $c < 0$, f is differentiable, as we evaluate $\frac{d}{dx}(-x^2)\big|_c$ and for $c > 0$, f is also differentiable by evaluating $\frac{d}{dx}(x^2)\big|_c$.

At $x = 0$, we have to evaluate the left-hand and right-hand limits

$$f'(0)^+ = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} \qquad\qquad = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0}$$
$$= 0f'(0)^-$$
$$= 0.$$

Since the left and right-hand derivatives agree with each other, it is the case that f is differentiable at $x = 0$, meaning $f(x) = x|x|$ is differentiable on $\mathbb{R}$.

Problem 5

Let I be an interval and suppose $f : I \to \mathbb{R}$ is differentiable with $f'(x) < 0$ for all $x \in I$. Show that f is strictly decreasing on I.

By a lemma, we know that for $c \in I$ and $f'(c) < 0$, it must be the case that $\exists\delta$ such that for all $x \in (c-\delta, c)$, $f(c) < f(x)$. Since this is the case for all $c \in I$, f is strictly decreasing.

Problem 6

Prove that $f(x) = x^3 + e^x$ has a unique real root.

We know that for $x = -1$, $f(x) < 0$, and for $x = 1$, $f(x) > 0$. By the Intermediate Value Theorem, it must be the case that $\exists c \in [-1, 1]$ such that $f(c) = 0$. Additionally, it is also the case that $f'(x) = 3x^2 + e^x > 0\ \forall x$, meaning that $f(x)$ is strictly increasing on its domain, so f cannot take the value of 0 at any other point d^*, otherwise there would be a point where $f'(k) = 0$ for some k between c and d.

Problem 7

Suppose $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, and satisfies $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$.

(i)

Show that there is a $c_1 \in (0, 1)$ with $f'(c_1) = 1$.

Since f is continuous on $[0, 2]$, f is continuous on $[0, 1]$, and since f is differentiable on $(0, 2)$, f is differentiable on $(0, 1)$. We apply the mean value theorem on $[0, 1]$ to find c_1^*.

(ii)

Show that there is a $c_2 \in (1, 2)$ with $f'(c_2) = 0$.

Since f is continuous on $[0, 2]$, f is continuous on $[1, 2]$, and since f is differentiable on $(0, 2)$, f is differentiable on $(1, 2)$. Apply Rolle's Theorem on $[1, 2]$ to find c_2.

(iii)

Show that there is a $c_3 \in (0, 2)$ with $f'(c_3) = 1/3$.

Letting $c_1 \in (0, 1)$ and $c_2 \in (1, 2)$ be defined as above, we apply Darboux's Theorem on $[c_1, c_2]$ to find c_3 such that $f'(c_3) = 1/3$.

Problem 8

Suppose $f, g : \mathbb{R} \to (0, \infty)$ are everywhere differentiable with $f' = f$ and $g' = g$. Prove that $f = \alpha g$ for some constant $\alpha > 0$.

$$
\begin{aligned}
f &= \alpha g \\
f' &= (\alpha g)' \\
&= \alpha g' \\
&= \alpha g \\
&= f
\end{aligned}
$$

Problem 9

Let $h = \mathbb{1}_{[0,\infty)}$. Prove that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ for which $f' = h$ on $\mathbb{R}$.

Since h is discontinuous at $x = 0$, f must be non-differentiable at $x - 0$; however, since h takes a value at $x = 0$, it must also be the case that f is differentiable at $x = 0$. $\perp$

Problem 10

Let $s > t > 0$ and $n \geq 2$. By analyzing the function $f(x) = x^{1/n} - (x-1)^{1/n}$ on $[1, \infty)$, show that

$$s^{1/n} - t^{1/n} < (s-t)^{1/n}$$

$$
\begin{aligned}
s^{1/n} - t^{1/n} &< (s-t)^{1/n} \\
\left(\frac{s}{t}\right)^{1/n} - 1 &< \left(\frac{s}{t} - 1\right)^{1/n} \\
\left(\frac{s}{t}\right)^{1/n} - \left(\frac{s}{t} - 1\right)^{1/n} &< 1,
\end{aligned}
$$

and

$$
\begin{aligned}
f'(x) &= \frac{1}{n}\left(\frac{1}{x^{1/n}} - \frac{1}{(x-1)^{1/n}}\right) \\
&= \frac{1}{n}\left(\frac{(x-1)^{1/n} - x^{1/n}}{x^{1/n}(x-1)^{1/n}}\right) \\
&< 0,
\end{aligned}
$$

and

$$f(1) = 1,$$

so,

$$f\left(\frac{s}{t}\right) < 1$$

Problem 11

Show that for all $x > 0$,

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$

Apply the Mean value theorem on $[0, x]$: $\exists c \in (0, x)$ such that

$$
\begin{aligned}
\frac{\sqrt{1+x} - 1}{x} &= \frac{1}{2\sqrt{1+c}} \\
\sqrt{1+x} - 1 &= \frac{1}{2\sqrt{1+c}}x \\
&\leq \frac{1}{2}x && c \geq 0 \\
\sqrt{1+x} &\leq 1 + \frac{1}{2}x.
\end{aligned}
$$

I don't know how to show the second part.