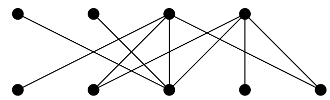
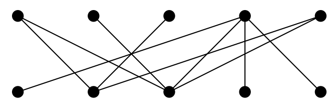
3.1.1

Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.

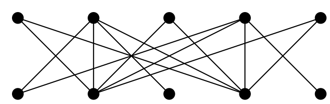
Graph 1:



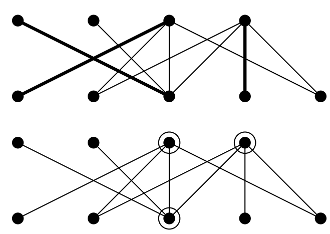
Graph 2:



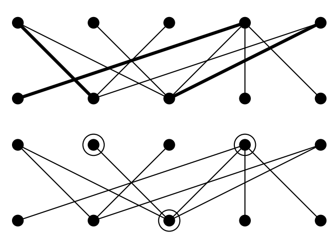
Graph 3:



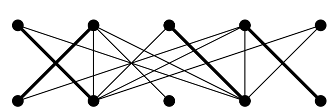
Graph 1:

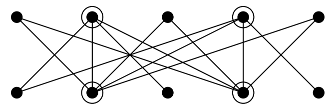


Graph 2:



Graph 3:





In all of these cases, we have solved the dual problem and found that the size of the minimum vertex cover is equal to the size of the maximum matching. Because all of these graphs are bipartite, we know that $\alpha'(G) = \beta(G)$.

3.1.2

Determine the minimum size of a maximal matching in C_n.

The minimum size of a maximal matching in C_n is $\left\lceil \frac{n}{3} \right\rceil$.

Base Case For $n = 3$, the minimum size of a maximal matching is 1, and in $n = 4$, the minimum size of a maximal matching is 2, which are $\left\lceil \frac{n}{3} \right\rceil$ for $n = 3$ and $n = 4$ respectively.

Inductive Step In C_k, we assume that the minimum size of a maximal matching is $\left\lceil \frac{k}{3} \right\rceil$. For $k+1$, we have the following cases that we will prove:

- $k + 1 \equiv 1 \bmod 3$: If $k + 1$ is congruent to 1 modulo 3, then C_k is of the form C_{3n}, with a minimum-size maximal matching of $\left\lceil \frac{k}{3} \right\rceil$. The edge pattern of a maximal matching M in a $3n$-gon is $m_1, e_1, e_2, m_2, e_3, e_4, \dots$ for $m_i \in M$ and $e_i \in E(C_k) - M$. Insertion of an edge e (and its requisite vertices) to create C_{k+1} would yield an edge pattern of $m_1, e_1, e, e_2, \dots$, meaning that M' would need to include e to be a maximal matching. Therefore $|M'| = |M| + 1 = \left\lceil \frac{k+1}{3} \right\rceil$.
- $k + 1 \equiv 2 \bmod 3$: If $k+1$ is congruent to 2 modulo 3, then C_k must contain at least one example of an edge pattern that is m_1, e_1, m_2. If we insert an an edge to create the case of C_{k+1}, we can insert it into our cycle as m_1, e_1, e, m_2, meaning that the size of M' does not change. Therefore $|M'| = |M| = \left\lceil \frac{k+1}{3} \right\rceil$.
- $k + 1 \equiv 0 \bmod 3$: If $k+1$ is congruent to 0 modulo 3, then k is congruent to 2 modulo 3, meaning there must be one edge pattern of m_1, e_1, m_2 for $m_i \in M$ and $e_i \in E(C_k) - M$. So, we are able to insert e to create C_{k+1} as m_1, e_1, e, m_2 as with the previous case, meaning $|M'| = |M| = \left\lceil \frac{k+1}{3} \right\rceil$.

3.1.5

Prove that $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ for every graph G.

$$\begin{aligned}
\alpha(G) + \beta(G) &= n(G) \\
\alpha(G) &= n(G) - \beta(G) \\
\alpha(G) &\geq n(G) - \alpha(G)\Delta(G) && \text{maximum theoretical size of a minimum edge cover} \\
\alpha(G) &\geq \frac{n(G)}{\Delta(G) + 1}
\end{aligned}$$

3.1.9

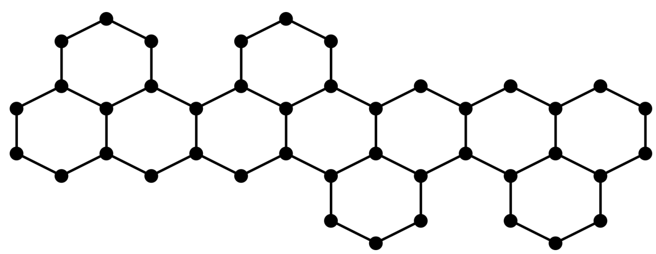
Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Let M be a maximal matching with fewer than $\alpha'(G)/2$ edges. Since M is maximal, we are assuming that every vertex in G is either in M or is adjacent to a vertex in M. Since M is a maximal matching that is *not* a maximum matching, then there must be an M-augmenting path.

Let v be the start of this M-augmenting path, and let w be its final vertex. Because v and w cannot have any other edges incident on them, we must assume that $\alpha'(G) = |M| + 1$, as we can exchange the matchings within the M-augmenting path to get a maximum matching. However, because $|M| < \alpha'(G)/2$, we must get that $\alpha'(G) < 1 + \alpha'(G)/2$, or that $\alpha'(G) < 2$. However, $\alpha'(G)/2$ must be an integer, meaning that the original assumption (that M is a maximal matching) must be false.

3.1.28

Exhibit a perfect matching in the graph below or give a short proof that it has none.



The minimum edge cover is as follows, and is of size 22, meaning that the maximum matching is of size 20 (as this graph is bipartite), which is not a perfect matching.