Math 395

Homework 8
Due: 4/30/2024

Name: Avinash Iyer

Collaborators:

Problem 1

Let K/F be a Galois extension with $\text{Gal}(K/F)$ abelian of order 10. We will compute the intermediate fields between F and K, and their dimensions over F.

Since $\text{Gal}(K/F)$ is abelian and of order 10, it must be the case that $\text{Gal}(K/F) \cong \mathbb{Z}/10\mathbb{Z}$.

The subgroups of $\text{Gal}(K/F)$ are isomorphic to the subgroups of $\mathbb{Z}/10\mathbb{Z}$; since $10 = 2 \cdot 5$, it must be the case that $\langle 2 \rangle$, with order 5 and $\langle 5 \rangle$, with order 2, are the two proper subgroups of $\mathbb{Z}/10\mathbb{Z}$ (by Lagrange's Theorem). We will let $H_1 \leq \text{Gal}(K/F)$ be isomorphic to $\langle 2 \rangle$, and $H_2 \leq \text{Gal}(K/F)$ be isomorphic to $\langle 5 \rangle$.

Let $A = K^{H_1}$. Then, since $[\mathbb{Z}/10\mathbb{Z} : \langle 2 \rangle] = 2$, it is the case that $[A : F] = 2$. Similarly, for $B = K^{H_2}$, it is the case that $[\mathbb{Z}/10\mathbb{Z} : \langle 5 \rangle] = 5$, so $[B : F] = 5$.

Problem 3

We will find $\text{Gal}(x^4 - 5x^2 + 6)$ over $\mathbb{Q}$.

To start, factoring $x^4 - 5x^2 + 6$, we find it is equal to $(x^2 - 3)(x^2 - 2) = (x - \sqrt{3})(x + \sqrt{3})(x - \sqrt{2})(x + \sqrt{2})$ in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Since $x^4 - 5x^2 + 6$ is separable in $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \text{Spl}(x^4 - 5x^2 + 6)$, it must be the case that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension.

We know that the basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$, meaning that for $\sigma \in \text{Gal}(K/F)$, we have $\sigma(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) + a + b\sigma(\sqrt{2}) + c\sigma(\sqrt{3}) + d\sigma(\sqrt{2})\sigma(\sqrt{6})$. Thus, the possible elements of $\text{Gal}(K/F)$ are

$$
\begin{aligned}
\sigma_0 &:= \text{id} \\
\sigma_1 &:= \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \end{cases} \\
\sigma_2 &:= \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases} \\
\sigma_3 &:= \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases} .
\end{aligned}
$$

Notice that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0$, meaning we have $\text{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z}$.

Problem 6

We will prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of $\mathbb{Q}(\zeta_n)$ for any $n \geq 1$.

It is known that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.