Abstract

We discuss the much celebrated Regular Value Theorem and Sard's Theorem, and discuss some of the consequences and applications of these results.

A smooth map between manifolds $f: M \to N$ includes a certain family of local information; for instance, the derivative $D_p f: T_p M \to T_{f(p)} N$, which is a linear map between tangent spaces at p and q, is defined on a coordinate chart $U \subseteq M$ for p and a corresponding coordinate chart $V \subseteq N$ for $f(p)$. Yet, the properties of this linear map can give us information about the underlying map f.

To understand this, we need to dive into the world of regular and critical values.

Much of this document is based on the book *Topology from the Differentiable Viewpoint* and assorted notes from my Differential Topology class.

Sard's Theorem

Definition: Let $f: M \to N$ be a smooth map, and let $p \in M$. We say p is a *critical point* for f if $D_p f$ does not have the same rank as the dimension of $T_{f(p)}N$. If $D_p f$ has the same rank as the dimension of $T_{f(p)}N$, then we say that p is a *regular point* of f.

We say $q \in N$ is a *critical value* for f if $f^{-1}(\{q\})$ contains a critical point for f. Else,we say that q is a *regular value*.

We start with the case of Sard's Theorem on $\mathbb{R}^n$. Then, we will expand this to the case of any arbitrary manifold by means of a technical lemma.

Theorem (Sard's Theorem)**:** Let $f: \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be a smooth map. Then, if C is the set of critical points for f, we have $f(C) \subseteq \mathbb{R}^m$ has measure zero.

Proof. We use induction on n. The statement only makes sense for $n \geq 0$ and $p \geq 1$. Clearly, the theorem is true for $n = 0$.

Let $C_1 \subseteq C$ be the set of all $x \in U$ such that $D_x f$ is zero, and similarly, let C_i be the set of all x such that $(D_x)^j f$ is zero for all $j \leq i$. We obtain a descending sequence of closed sets $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$.

We start by showing that $f(C \setminus C_1)$ has measure zero. For each $x \in C \setminus C_1$, we find an open neighborhood $V \subseteq \mathbb{R}^n$ such that $f(V \cap C)$ has measure zero. Since $\mathbb{R}^n$ is second countable, $C \setminus C_1$ is covered by countably many such open neighborhoods, it follows that $f(C \setminus C_1)$ has measure zero.

Since $x \notin C_1$, there is some partial derivative, which we use change of coordinates to write as $\frac{\partial f}{\partial x_1}$, that is not zero at x. Let

$$h(x) = (f_1(x), x_2, \dots, x_n).$$

Then, since $D_x h$ is nonsingular, by the inverse function theorem, h maps some neighborhood V of x diffeomorphically onto an open set $V' \subseteq \mathbb{R}^n$. The composition $f \circ h^{-1}$ then maps V' to $\mathbb{R}^m$ then maps V' to $\mathbb{R}^m$.

Observe that the set of critical points of g is precisely $h(V \cap C)$, so the set $g(C')$ is equal to $f(V \cap C)$.

For each hyperplane $(t, x_2, \dots, x_n) \in V'$, we observe that $g(t, x_2, \dots, x_n)$ is contained in $t \times \mathbb{R}^{m-1} \subseteq \mathbb{R}^m$, meaning that g maps hyperplanes to hyperplanes. Let

$$g^t: \left(t \times \mathbb{R}^{n-1}\right) \cap V' \to t \times \mathbb{R}^{m-1}$$

be the restriction of g. A point in $t \times \mathbb{R}^{n-1}$ is a critical value for g^t if and only if it is critical for g, since the matrix of first derivatives for g is of the form

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} 1 & 0 \\ * & \left(\frac{\partial g_i^t}{\partial x_j}\right) \end{pmatrix}.$$

From the induction hypothesis, it follows that the critical values of g^t has measure zero in $t \times \mathbb{R}^{m-1}$. In particular, the critical values of g intersects each hyperplane in $t \times \mathbb{R}^{m-1}$ in a set of measure zero, meaning that by Fubini's theorem, $g(C') = f(V \cap C)$ has measure zero.

Now, for each $x_0 \in C_k \setminus C_{k+1}$, there is some $(k+1)$-th derivative that is not zero, which we write

$$\frac{\partial^{k+1} f_r}{\partial x_{s_1} \cdots \partial x_{s_{k+1}}}.$$

Then, writing

$$w(x) = \frac{\partial^k f_r}{\partial x_{s_2} \cdots \partial x_{s_{k+1}}},$$

we observe that $w(x)$ vanishes at x_0, but $\frac{\partial w}{\partial x_{s_1}}$ does not. For definiteness, we let $s_1 = 1$. The map $h\colon U \to \mathbb{R}^n$, defined by

$$h(x) = (w(x), x_2, \dots, x_n)$$

then carries a neighborhood V of x_0 diffeomorphically onto an open set V'. Then, h carries $C_k \cap V$ into the hyperplane $\{0\} \times \mathbb{R}^{n-1}$. Again, we consider the map $g = f \circ h^{-1}$, and define

$$g_0\colon \left(\{0\} \times \mathbb{R}^{n-1}\right) \cap V' \to \mathbb{R}^m$$

to be the restriction of g. Inductively, the critical values of g_0 has measure zero in $\mathbb{R}^m$. Yet, each point in $h(C_k \cap V)$ is a critical point of g_0 as all derivatives of order $\leq k$ vanish, meaning that

$$g_0 \circ h(C_k \cap V) = f(C_k \cap V)$$

has measure zero. $\square$