Quasi-Review: Locally Compact Groups and the Banach $*$-algebra $L_1(G)$

Basic Properties of Topological Groups

A topological group is a group G equipped with a topology such that the operations

$$\begin{aligned}(x, y) &\mapsto xy \\ x &\mapsto x^{-1}\end{aligned}$$

are continuous. In general, we will let 1 denote the identity of G.

We call G a locally compact group if the topology of G is locally compact. Equivalently, the topology of G is locally compact if there there is a neighborhood system about 1 consisting of pre-compact open sets.

We will refer to the following subset operations in G regularly:

$$\begin{aligned}Ax &= \{ax \mid a \in A\} \\ xA &= \{xa \mid a \in A\} \\ A^{-1} &= \left\{a^{-1} \mid a \in A\right\} \\ AB &= \{ab \mid a \in A, b \in B\}.\end{aligned}$$

A subset V is called *symmetric* if $V = V^{-1}$.

These are some useful propositions.

Proposition: Let G be a topological group.

(i) The topology of G is invariant under translations and inversion.

(ii) For every neighborhood U of 1, there is a symmetric neighborhood V of 1 such that $VV \subseteq U$.

(iii) If H is a subgroup of G, then so is $\overline{H}$.

(iv) Every open subgroup of G is closed.

(v) If A and B are compact subsets of G, then so is AB.

Proposition: Suppose H is a subgroup of the topological group G.

(i) If H is closed, then G/H is Hausdorff.

(ii) If G is locally compact, then so is G/H.

(iii) If H is normal, then G/H is a topological group.

We will assume all the time that G is Hausdorff, via the following proposition.

Corollary: If G is a T1 topological group, then G is Hausdorff. If G is not T1, then $\overline{\{1\}}$ is a closed normal subgroup with $G/\overline{\{1\}}$ is a Hausdorff topological group.

Proposition: Every locally compact group G has a subgroup G_0 that is open, closed, and σ-compact.

Considering various functions $f\colon G \to \mathbb{C}$, we define the left and right translates of f as

$$\begin{aligned}L_y f(x) &= f\left(y^{-1}x\right) \\ R_y f(x) &= f(xy),\end{aligned}$$

and say that f is left (right) uniformly continuous if $\|L_y f - f\|_u \to 0$ ($\|R_y f - f\|_u \to 0$) as $y \to 1$.

Proposition: If $f \in C_c(G)$, then f is left and right uniformly continuous.

A left *Haar measure* is a nonzero Radon measure μ on G such that $\mu(xE) = \mu(E)$ for every Borel subset $E \subseteq G$.

Proposition: Every locally compact group G admits a left Haar measure λ. This Haar measure is unique up to a constant multiple.

If we have a left Haar measure λ, then if we define

$$\lambda_x(E) = \lambda(Ex),$$

we have that λ_x is again a left Haar measure, so there is some number $\Delta(x)$ such that $\lambda_x = \Delta(x)\lambda$, where $\Delta(x)$ is independent of the original choice of λ.

The function $\Delta\colon G \to (0, \infty)$ defined as such is known as the *modular function* of G.

Proposition: The function Δ is a continuous homomorphism from G to $\mathbb{R}_{>0}$, and for any $f \in L_1(\lambda)$, we have

$$\int R_y f \, d\lambda = \Delta\left(y^{-1}\right) \int f \, d\lambda.$$

We call G *unimodular* if $\Delta \equiv 1$.

Proposition: If $G/[G, G]$ is compact, then G is unimodular.

Convolutions and $L_1(G)$

If G is a locally compact group, we let $M(G)$ denote the space of complex-valued Radon measures on G. The convolution of two measures $\mu, \nu \in M(G)$ is given as follows. If we let

$$I(\phi) = \iint \phi(xy) \, d\mu(x) d\nu(y),$$

then we observe that $I(\phi)$ is a linear functional on $C_0(G)$ that satisfies

$$|I(\phi)| \leq \|\phi\|_u \|\mu\| \|\nu\|,$$

meaning that it is given by a measure $\mu * \nu \in M(G)$ with $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$. We call $\mu * \nu$ the convolution of μ and ν.

Observe that if $\delta_x \in M(G)$ is the point mass at $x \in G$, then

$$\begin{aligned} \int \phi \, d(\delta_x * \delta_y) &= \iint \phi(uv) \, d\delta_x(u) \delta_y(v) \\ &= \phi(xy) \\ &= \int \phi \, d\delta_{xy}, \end{aligned}$$

meaning that $\delta_x * \delta_y = \delta_{xy}$.

The estimate $\|\mu * \nu\| \leq \|\mu\| \|\mu\|$ gives that convolution makes $M(G)$ a Banach algebra, which we call the *measure algebra* of G. Furthermore, $M(G)$ admits an involution

$$\mu^*(E) = \overline{\mu(E^{-1})},$$

so that

$$\int \phi \, d\mu^* = \int \phi\left(x^{-1}\right) \, d\overline{\mu(x)}.$$

We may identify the space $L_1(G)$ to be the subspace of $M(G)$ where a function f is identified with the measure $f(x)dx$. If $f, g \in L_1(G)$, then the convolution of f and g is the function

$$f * g(x) = \int f(y) g\left(y^{-1}x\right) \, dy.$$

With convolution and the involution given by

$$f^*(x)dx = \overline{f(x^{-1})}d\left(x^{-1}\right)$$
$$f^*(x) = \Delta\left(x^{-1}\right)\overline{f(x^{-1})},$$

we have that $L_1(G)$ is a Banach $*$-algebra known as the *group algebra* of G.

Now, we observe that if G is discrete, then if δ_e is the point mass at 1, we have that $f * \delta = \delta * f = f$ for any function f. If G is not discrete, we must use an *approximate identity* for G. In particular, we can select a family of mollifiers $\{\psi_U\}_{U\in\mathcal{U}}$ such that

$$\|\psi_U * f - f\| \to 0$$
$$\|f * \psi_U - f\| \to 0$$

if f is uniformly continuous and $U \to \{1\}$ in a neighborhood system $\mathcal{U}$ of 1.

Homogeneous Spaces

If G is a locally compact group, then G can act on a locally compact Hausdorff space by homeomorphisms. Recall from algebra that the group action is transitive if there is one orbit. We call S a G-space.

The standard example of a transitive G-space is the quotient space G/H for some closed subgroup H of G. These are, to an extent, the only G-spaces, as follows from the orbit-stabilizer theorem. If S is a G-space, then we may define a map $\phi\colon G \to S$ by $\phi(x) = x \cdot s_0$, and take the quotient by the stabilizer subgroup

$$H = \{x \in G \mid x \cdot s_0 = s_0\},$$

so that $\Phi\colon G/H \to S$ has $\Phi \circ q = \phi$ for the quotient map $q\colon G \to G/H$ is a continuous bijection.

Proposition: If G is σ-compact, then Φ is a homeomorphism.

Proof. It suffices to show that ϕ maps open sets in G to open sets in S. Suppose U is open in G, $x_0 \in U$. Pick a compact symmetric neighborhood V of 1 such that $x_0VV \subseteq U$. Since G is σ-compact, there is a countable $\{y_n\}_{n\geq 1} \subseteq G$ such that $\{y_nV\}_{n\geq 1}$ covers G. Then, we have

$$S = \bigcup_{n=1}^{\infty} \phi(y_nV).$$

The sets $\phi(y_nV)$ are homeomorphic to $\phi(V)$ since the map $s \mapsto y_n \cdot s$ is a homeomorphism of S, and all the y_nV are compact, hence closed.

By Baire Category Theorem for LCH spaces, it follows that $\phi(V)$ has an interior point, which we call $\phi(x_1)$ for $x_1 \in V$. Then, $\phi(x_0)$ is an interior point of $\phi\left(x_0x_1^{-1}V\right)$, and $x_0x_1^{-1}V \in x_0VV \subseteq U$, so that $\phi(x_0)$ is an interior point of $\phi(U)$. Thus $\phi(U)$ is open. □

If S is a transitive G-space that is isomorphic to a quotient space G/H, then will write $S \cong G/H$, and call S a *homogeneous space*. The identification is dependent on the choice of base point, but the identity $s_0' = x_0 \cdot x_0$ induces a map $H' = x_0Hx_0^{-1}$, inducing a G-equivariant homeomorphism $G/H \cong G/H'$.

We will address the question of whether there is a G-invariant Radon measure on G/H — that is, a radon measure μ such that $\mu(xE) = \mu(E)$ for every $x \in G$.

We assume that G is a locally compact group with left Haar measure dx, a H is a closed subgroup of G with left Haar measure $d\xi$, and $q\colon G \to G/H$ is the quotient map $q(x) = xH$, and Δ_G, Δ_H the corresponding modular functions.

Let $P\colon C_c(G) \to C_c(G/H)$ be defined by

$$Pf(xH) = \int_H f(x\xi)\,d\xi.$$

This is well-defined by left-invariance of $d\xi$. If $\phi \in C(G/H)$, then

$$P((\phi \circ q) \cdot f) = \phi \cdot Pf.$$

Lemma: If $E \subseteq G/H$ is compact, then there is a compact $K \subseteq G$ with $q(K) = E$.

Proof. Let V be an open neighborhood of 1 in G with compact closure. Since q is an open map, $q(xV)$ is an open cover of E, so there is a finite subcover $q(x_j V)$. Let $K = q^{-1}(E) \cap \bigcup_{j=1}^{n} x\overline{V}$. Then, since $q^{-1}(E)$ is closed, K is compact with $q(K) = E$. □

Lemma: If $F \subseteq G/H$ is compact, then there is $f \geq 0$ in $C_c(G)$ with $Pf = 1$ on F.

Proof. Let E be a compact neighborhood of F in G/H. We find $K \subseteq G$ compact such that $q(K) = E$. Select positive $g \in C_c(G)$ with $g > 0$ on K, and $\phi \in C_c(G/H)$ supported in E with $\phi = 1$ on F. Set

$$f = \frac{\phi \circ q}{Pg \circ q} g,$$

with the fraction equal to zero whenever the numerator is zero. We have f is continuous, since $Pg > 0$ on $\operatorname{supp}(\phi)$, has support contained in $\operatorname{supp}(g)$, and $Pf = \frac{\phi}{Pg} Pg = \phi$. □

Proposition: If $\phi \in C_c(G/H)$, then there exists $f \in C_c(G)$ with $Pf = \phi$ and $q(\operatorname{supp}(f)) = \operatorname{supp}(\phi)$, and has $f \geq 0$ if $\phi \geq 0$.

Proof. If $\phi \in C_c(G)$, then by the previous lemma, then there exists $g \geq 0$ in $C_c(G)$ with $Pg = 1$ on $\operatorname{supp}(\phi)$. Letting $f = (\phi \circ q)g$, then $Pf = \phi(Pg) = \phi$. □

Theorem: Let G be a locally compact group, H a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, we have

$$\begin{aligned}
\int_G f(x)\, dx &= \int_{G/H} Pf\, d\mu \\
&= \int_{G/H} \int_H f(x\xi)\, d\xi\, d\mu(xH)
\end{aligned}$$

for any $f \in C_c(G)$.

Proof. Suppose there is a G-invariant measure μ. Then, $f \mapsto \int Pf\, d\mu$ is a nonzero left-invariant positive linear functional on $C_c(G)$, so by the uniqueness of Haar measure, we have $\int Pf\, d\mu = c \int f(x) dx$ for some c.

This formula fully determines μ, meaning that μ is unique up to the arbitrary constant factor in Haar measure. We may assume that $c = 1$, so we have for any $\eta \in H$,

$$\begin{aligned}
\Delta_G(\eta) \int_G f(x)\, dx &= \int_G f\left(x\eta^{-1}\right) dx \\
&= \int_{G/H} \int_H f\left(x\xi\eta^{-1}\right) d\xi\, d\mu(xH) \\
&= \Delta_H(\eta) \int_{G/H} \int_H f(x\xi)\, d\xi\, d\mu(xH) \\
&= \Delta_H(\eta) \int_G f(x)\, dx,
\end{aligned}$$

so that $\Delta_G(\eta) = \Delta_H(\eta)$. □

Representations of a Group and its Group $*$-Algebra

If G is a locally compact group, then a *unitary representation* of G is a homomorphism $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$, where $\mathcal{U}(\mathcal{H}_\pi)$ denotes the unitary group of a Hilbert space $\mathcal{H}_\pi$. We call $\mathcal{H}_\pi$ the *representation space* of π, and the dimension of $\mathcal{H}_\pi$ is called the dimension (or degree) of the representation.

We do not require π to be continuous in the norm topology of $\mathcal{B}(\mathcal{H}_\pi)$, but as it turns out, both weak and strong continuity are equivalent as the WOT and SOT coincide on $\mathcal{U}(\mathcal{H}_\pi)$. If $(T_\alpha)_\alpha \to T$ is a net of unitary operators converging in WOT, then for any $u \in \mathcal{H}_\pi$, we have

$$\begin{aligned} \|(T_\alpha - T)u\|^2 &= \|T_\alpha u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu\rangle + \|Tu\|^2 \\ &= \|u\|^2 - 2\operatorname{Re}\langle T_\alpha u, Tu\rangle, \end{aligned}$$

and the latter term converges to $2\|Tu\|^2 = 2\|u\|^2$, so that $\|T_\alpha u - Tu\| \to 0$.

If G acts on a locally compact Hausdorff space S, then G acts on $C(S)$ by $(\pi(g)f)s = f\left(g^{-1} \cdot s\right)$. If S has a G-invariant Radon measure, then π defines a unitary representation on $L_2(\mu)$.

The most important representation is the representation on $L_2(G)$ induced by the action of G on itself by left-multiplication. This defines $(\pi_L(g)f)(y) = f\left(x^{-1}y\right)$.

If π_1 and π_2 are unitary representations of G, then an intertwining operator for π_1 and π_2 is a bounded linear map $T\colon H_1 \to H_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. We say that π_1 and π_2 are *unitarily equivalent* if the set of intertwiners admits a unitary map.

Functions of Positive Type

References

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