Problem 1

Show that a discrete metric space is compact if and only if it is finite.

Proof: Let (X, d) be a discrete metric space. Suppose (X, d) is not finite. Then, we can create an open cover of X defined by

$$X = \bigcup_{x \in X} \{x\}.$$

Since every subset of X is finite, this is an open cover, but this does not contain a finite subcover as X is infinite.

Suppose (X, d) is not compact. Then, there is an open cover of X

$$X \subseteq \bigcup_{i \in I} U_i$$

with no finite subcover. Specifically this means that for each $i \in I$, there is some $x_i \in U_i$ such that $x_i \notin \bigcup U_{-i}$. Therefore, we have $\{x_i\}_{i=1}^{\infty} \subseteq X$, so X is infinite.

Problem 2

Let X be a metric space and suppose $Y \subseteq X$. Show that $K \subseteq Y$ is compact in Y with the relative topology if and only if K is compact in X.

Problem 5

Let V be a finite-dimensional normed space. Show that the unit ball $B := \{v \in V \mid \|v\| \leq 1\}$ is compact.

Proof: Having shown that all norms on V are equivalent, we can create a homeomorphism $f : V \to \ell_2^n$, where $\dim(V) = n$.