

This is a collection of old complex analysis qualifier exam solutions, as well as some notes on useful results and proofs.

Useful Results and Proofs

Analytic Functions

Definition: Let $U \subseteq \mathbb{C}$ be an open set. A function $f: U \rightarrow \mathbb{C}$ is called *analytic* if, for any $z_0 \in U$, there is $r > 0$ and $(a_k)_k \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in U(z_0, r)$.

Analytic functions form a \mathbb{C} -algebra.

Theorem (Identity Theorem): Let $f, g: U \rightarrow \mathbb{C}$ be analytic functions defined on a connected open set (also known as a region). If

$$A = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

admits an accumulation point in U , then $f = g$ on U .

Proof. To begin, we show that if $f: U \rightarrow \mathbb{C}$ is an analytic function that is not uniformly zero, then for any $z_0 \in U$, there is $\rho > 0$ such that f is nonzero on $\dot{U}(z_0, \rho) \subseteq U$. Towards this end, we may write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

for all $z \in U(z_0, r)$, some $r > 0$, and since f is not uniformly zero, there is some minimal ℓ such that $a_\ell \neq 0$. This yields

$$f(z) = (z - z_0)^\ell \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k;$$

the function $h: U(z_0, r) \rightarrow \mathbb{C}$ given by

$$h(z) = \sum_{k=0}^{\infty} a_{k+\ell} (z - z_0)^k$$

then has the same radius of convergence as f and is not zero at z_0 , so that g is not zero on some $U(z_0, \rho)$ as g is continuous.

Now, we let V_1 be the set of accumulation points of A in U , and let $V_2 = U \setminus V_1$.

If $z \in V_2$, then there is some $r_1 > 0$ such that $\dot{U}(z_0, r_1) \cap A = \emptyset$, or that $\dot{U}(z_0, r_1) \subseteq A^c$. Meanwhile, since U is open, there is some $r_2 > 0$ such that $U(z_0, r_2) \subseteq U$, meaning that if $r = \min\{r_1, r_2\}$, then $U(z_0, r) \subseteq U \setminus A$. Thus, V_2 is open.

Meanwhile, if $z \in V_1$, then since $V_1 \subseteq U$, it follows that there is $r > 0$ such that $U(z, r)$ and $(a_k)_k$ such that

$$f(w) - g(w) = \sum_{k=0}^{\infty} a_k (w - z)^k$$

for all $w \in U(z, r)$. We claim that $f(w) - g(w)$ is uniformly zero on $U(z, r)$. Else, if there were $w_0 \in U(z, r)$ such that $f(w_0) \neq g(w_0)$, then it would follow that there is $0 < s \leq r$ such that $f(w) \neq g(w)$ for all $w \in U(w_0, s)$. Yet, this would contradict the assumption that z is an accumulation point, meaning that V_1 is open.

Since V_1 and V_2 are disjoint open sets whose union is equal to U , it follows that either $V_1 = U$ or $V_2 = U$. If $A \neq \emptyset$, then the identity theorem follows. \square

Differentiability

Definition: If $U \subseteq \mathbb{C}$ is an open set, then we say f is differentiable at $z_0 \in U$ if

$$\lim_{w \rightarrow z_0} \frac{f(w) - f(z_0)}{w - z_0}$$

exists. We call this value the *derivative* of f at z_0 , and usually write $f'(z_0)$.

If f is differentiable at every $z_0 \in U$, we say f is differentiable on U .

If f is continuous and admits a continuous derivative, then we say f is *holomorphic*.

Note that the limit must be independent of direction. That is, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(w) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Now, given $U \subseteq \mathbb{C}$, write $z = x + iy$ and

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$. Observe then that if f is differentiable at $x_0 + iy_0 \in U$, then since the limit is independent of path, by taking the limit over real numbers, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(u(x+h, y) + iv(x+h, y)) - (u(x, y) + iv(x, y))}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \end{aligned}$$

and by taking over the imaginary numbers,

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{(u(x, y+h) + iv(x, y+h)) - (u(x, y) + iv(x, y))}{ih} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, we obtain the following.

Definition: The system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

is known as the *Cauchy–Riemann Equations*.

Observe that if f is differentiable, then the u and v in the definition of f satisfy the Cauchy–Riemann equations. Yet, we desire to understand a bit more about when exactly f is differentiable or holomorphic.

Proposition: If $f = u + iv$ is a holomorphic function such that u, v are in $C^2(U)$, then u and v are harmonic. That is, u and v satisfy Laplace’s equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We call u and v *harmonic conjugates* for each other. That is, if $u: U \rightarrow \mathbb{R}$ is a harmonic function, then $v \in C^1(U)$ is called a harmonic conjugate if the Cauchy–Riemann equations hold for u and v .

Theorem: Let $U \subseteq \mathbb{R}^2$ be a ball or all of \mathbb{R}^2 . Then, every harmonic function on U has a harmonic conjugate. If $u \in C^3(U)$, then this conjugate is itself harmonic.

Lemma: Let $g: U((x_0, y_0), R) \rightarrow \mathbb{R}$ be such that g and $\frac{\partial g}{\partial x}$ are continuous. Then, $G: U((x_0, y_0), R) \rightarrow \mathbb{R}$, given by

$$G(x, y) = \int_{y_0}^y g(x, t) dt$$

satisfies

$$\frac{\partial G}{\partial x} = \int_{y_0}^y \frac{\partial g}{\partial x}(x, t) dt.$$

Proof of Lemma. Write

$$\frac{G(x+h, y) - G(x, y)}{h} - \int_{y_0}^y \frac{\partial g}{\partial x}(x, t) dt = \int_{y_0}^y \left(\frac{g(x+h, t) - g(x, t)}{h} - \frac{\partial g}{\partial x}(x, t) \right) dt.$$

By mean value theorem, the first term is equal to $\frac{\partial g}{\partial x}(x_1, t)$ for some x_1 between x and $x+h$. As $h \rightarrow 0$, $x_1 \rightarrow x$, as $\frac{\partial g}{\partial x}$ is uniformly continuous on a compact subset that contains x and $x+h$. We may exchange limit and integral to obtain the desired result. \square

Proof of Theorem. We prove for the case of $U = U((x_0, y_0), R)$. Define

$$v(x, y) = \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt + \phi(x),$$

with $\phi(x)$ to be determined later. By the fundamental theorem of calculus, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

while by differentiating under the integral sign, and using the fact that u is harmonic, we have

$$\begin{aligned} \frac{\partial v}{\partial x} &= \int_{y_0}^y \frac{\partial^2 u}{\partial x^2}(x, t) dt + \frac{d\phi}{dx} \\ &= - \int_{y_0}^y \frac{\partial^2 u}{\partial y^2}(x, t) dt + \frac{d\phi}{dx} \\ &= - \frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \frac{d\phi}{dx}. \end{aligned}$$

Defining $\phi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(x) = - \int_{x_0}^x \frac{\partial u}{\partial y}(s, y_0) ds,$$

we see that v thus satisfies all the necessary requirements to be a harmonic conjugate.

Now, if u is C^3 , then we defined v via the derivative of u , so that v is C^2 , and thus v is harmonic. \square

Cauchy's Integral Formula and Homology of Cycles

Proposition: Fix $z_0 \in \mathbb{C}$, $R > 0$, and $f: U(z_0, R) \rightarrow \mathbb{C}$ holomorphic. For all $z \in U(z_0, R)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. It suffices to show that

$$\frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = 0.$$

By using the chain rule and fundamental theorem of calculus, we find

$$\begin{aligned} \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{\int_0^1 f'((1-t)z + t\zeta)(\zeta - z) dt}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{S(z_0, R)} \int_0^1 f'((1-t)z + t\zeta) dt d\zeta \\ &= \frac{1}{2\pi i} \int_{S(z_0, R)} \frac{d}{d\zeta} \left(\frac{1}{t} f((1-t)z + t\zeta) \right) d\zeta \\ &= 0. \end{aligned}$$

\square

Proposition: Let $f: U \rightarrow \mathbb{C}$ be a holomorphic function. The following all hold:

- (i) f is analytic;
- (ii) f is smooth with $f^{(n)}$ holomorphic;
- (iii) for all $z_0 \in U$, if we let $R = \sup\{r > 0 \mid U(z_0, r) \subseteq U\}$, then there is $(a_n)_n \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the power series has radius of convergence R .

Proof.

- (i) There exists $r < s$ with $U(z_0, s) \subseteq U$ and $r < r_1 < s$ such that $S(z_0, r_1) \subseteq U$. By Cauchy's Integral Formula, and a power series expansion of $\frac{1}{\xi - z}$ about z_0 , this gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{\xi - z} d\xi \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right)}_{=: a_n} \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n. \end{aligned}$$

- (ii) Analytic functions are automatically smooth, hence complex-differentiable with continuous

derivative.

(iii) If $r < r_1 < R$, then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \right),$$

and since the series converges uniformly, we have

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0, r_1)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

Since r was arbitrary, this holds for any $0 < r_1 < R$, whence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for all $z \in U(z_0, R)$.

□

Maximum Modulus Principle

Theorem (Mean Value Property): Let $U \subseteq \mathbb{C}$ be open, $f: U \rightarrow \mathbb{C}$ holomorphic, with $z_0 \in U$ and $r > 0$ such that $B(z_0, r) \subseteq U$. Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Proof. By the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(\xi)}{\xi - z_0} d\xi.$$

Parametrizing $\gamma(\theta) = z_0 + re^{i\theta}$, we get

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

□

Corollary: If $u: \mathbb{R}^2 \supseteq U \rightarrow \mathbb{R}$ is harmonic, $(x_0, y_0) \in U$, and $r > 0$ is such that $B((x_0, y_0), r) \subseteq U$, then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos(\theta), y_0 + r \sin(\theta)) d\theta.$$

Proof. Take real parts of the mean value property for holomorphic $f = u + iv$.

□

Old Exams

Notation

- $U(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$

- $S(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$
- $\dot{U}(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$
- $A(z_0, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$