Problem 1

Problem: Use the "contradiction format" of mathematical induction to show that every integer $n \geqslant 2$ is the product of one or more primes.

Solution. Suppose toward contradiction that it is not the case. Let k denote the least element that is not the product of one or more primes. Then, for any $n < k$, n is the product of one or more primes. If, for any $n < k$, $n|k$, then k is the product of at least one prime number, as n is the product of one or more primes. If $n \nmid k$ for all $k < n$, then k is prime, meaning that k is the product of one or more primes. $\perp$

Problem 2

Problem: Prove that $\mathbb{N} \times \mathbb{N}$ is well-ordered by the lexicographical order.

Solution. Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ be distinct. Then, either $a = c$ or $a \neq c$. If $a = c$, then $b \neq d$, and since $\mathbb{N}$ is totally ordered, this means $(a, b) < (c, d)$ or vice versa. If $a \neq c$, then since $\mathbb{N}$ is totally ordered, $(a, b) < (c, d)$ or vice versa via the lexicographical order.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be nonempty. Since A is nonempty, we define the set of distinct first coordinates $S = \{a_i\}_{i \in I}$, which is thus nonempty. We set $A_1 = \left\{(a_j, b_j)\right\}_{j \in J}$ such that a_j are all equal to the least element in $S \subseteq \mathbb{N}$. Following the lexicographical order, we then find the least element in A_1 by selecting the least value of $\left\{b_j\right\}_{j \in J}$, yielding the least value of A in lexicographical order. Thus, $\mathbb{N} \times \mathbb{N}$ under the lexicographical order is well-ordered.

Problem 3

Problem: Prove there exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for $(m, n) \in \mathbb{N}$, we have

- $m \leqslant 1$ or $n = 0$: $f(m, n) = 0$
- m is prime or n is prime: $f(m, n) = f(m - 2, n + 2^n) + 1$
- $m > 1$, $n \neq 0$, and neither m nor n are prime: $f(m, n) = f\left(m, \left\lfloor \frac{n}{2} \right\rfloor\right) + 1$.

Solution. If m is prime and n is not prime, then under the lexicographical ordering, $(m - 2, n + 2^n) \prec (m, n)$, so the function's input "reduces" towards the base case. Similarly, if m is not prime and n is prime, then $(m - 2, n + 2^n) \prec (m, n)$ by the lexicographical order.

If m and n are composite, then the lexicographical order still has $\left(m, \left\lfloor \frac{n}{2} \right\rfloor\right) \prec (m, n)$, meaning the function's input still "reduces" toward the base case.

Since the lexicographical ordering is a well-ordering, the function will necessarily terminate at the base case.

Problem 4

Problem: Let $\sim$ be a relation on $\mathbb{N} \times \mathbb{N}$ under the lexicographical order. We say (a, b) is a child of (c, d) if $(a, b) \sim (c, d)$ and $(a, b) \prec (c, d)$, where $\prec$ is the lexicographical order.

We have two definitions for "descendant" below. Which one is the correct one?

(1) We say (a, b) is a descendant of (c, d) if (a, b) is a child of (c, d) or (a, b) is a descendant of a child of (c, d).

(2) We say (a, b) is a descendant of (c, d) if (a, b) is a child of (c, d) or (a, b) is a child of a descendant of (c, d).

Solution. Definition (1) is the correct definition. We let

$$C((m,n)) = \{(a,b) \mid (a,b) \text{ is a child of } (m,n)\}.$$

Define

$$D : \mathbb{N} \times \mathbb{N} \times P(\mathbb{N} \times \mathbb{N}), D((m,n)) = C((m,n)) \cup \bigcup_{((a,b)) \in C((m,n))} D((a,b)) \tag{$*$}$$

We want to show that there exists a unique function D that satisfies condition ($*$).

If this is not the case, pick the smallest (m,n) for which there is no such D. So, for every $(a,b) \in C(m,n)$, $D(a,b)$ is defined and satisfies ($*$).

Define

$$D(m,n) = C(m,n) \cup \bigcup_{(a,b) \in C((m,n))} D((a,b)).$$

Problem 5

Problem: Let S be well-ordered by $\prec$. Then, for every $x \in S$, if x is non-maximal, then x has a successor. The successor is defined by

$$\exists y \succ x \text{ s.t. } \neg \exists z\, x \prec z \prec y.$$

Solution. Let $x \in S$ be nomaximal. Set

$$T = \{y \in S \mid x \prec y\}.$$

Since x is nonmaximal, T is nonempty, meaning there exists a least element z. Then, z is a successor of x, because for all y, $x \prec y$, then $y \in T$, meaning $y = z$ or $z \prec y$, since z is the least element of T.

Problem 6

Problem: Every $S \subseteq \mathbb{R}$ well-ordered by the traditional $<$ relation is countable.

Solution. Let $S \subseteq \mathbb{R}$ be well-ordered. It is enough to show that $S \cap [z, z+1]$ is countable for every $z \in \mathbb{Z}$, as

$$S = \bigcup_{z \in \mathbb{Z}} S \cap [z, z+1]$$

is a countable union of countable sets.

For every $x \in S$, let $f(x) = x^+ - x$, where x^+ is the successor of x in S. If x has no successor, we let $f(x) = 0$.

It is enough to show that $S_0 = S \cap [0,1]$ is countable. We have S_0 is well-ordered.

For every $k \in \mathbb{Z}_{>0}$, define

$$A_k = \left\{x \in S_0 \mid f(x) > \frac{1}{k}\right\}.$$

Notice that $|A_k| \leqslant k$ for all k, since S is well-ordered by $<$. Since

$$S_0 = \bigcup_{k=1} \qquad\qquad \infty A_k,$$

and each A_k is finite, it is the case that S_0 is countable.

Remark ("Converse" to Problem 6)**:** The previous problem states that we cannot embed an uncountable well-ordered set into $\mathbb{R}$. Here, an embedding means that there is a function $f : S \to \mathbb{R}$ such that f is injective and f preserves order. In other words, S and $f(S) \subseteq \mathbb{R}$ are order-isomorphic.

A question we may be interested in is if every countable ordinal can be embedded into $\mathbb{R}$.