Problem (Problem 1)**:** Let I, J, K be ideals of R.

(a) Show that $(IJ)K = I(JK)$.

(b) Show that $(I + J)K = IK + JK$.

Problem (Problem 4)**:** Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and let $\iota_{S_i}\colon R \to S_i^{-1}R$ be the corresponding localization homomorphisms. Use the universal property of localization to show that there exists a unique ring homomorphism $\iota'\colon S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. Provide an explicit description of this ring homomorphism. Use this to show that if R is an integral domain and S an arbitrary multiplicative subset of R, then $S^{-1}R$ injects into the fraction field $K = \operatorname{frac}(R)$.

Solution: We observe that $\iota_{S_2}\colon R \to S_2^{-1}R$ maps elements of S_1 to units in $S_2^{-1}R$, as the units in $S_2^{-1}R$ are elements of the form $\frac{s}{s'}$ with $s, s' \in S_2$, so by the universal property, there is a unique ring homomorphism $\iota'\colon S_1^{-1}R \to S_2^{-1}R$ such that $\iota' \circ \iota_{S_1} = \iota_{S_2}$. In particular, this is the map $\left[\frac{r}{1}\right]_{S_1^{-1}R} \mapsto \left[\frac{r}{1}\right]_{S_2^{-1}R}$.

Since any arbitrary multiplicative subset $S \subseteq R$ of an integral domain is contained in $R \setminus \{0\}$, it follows that $S^{-1}R$ injects into $(R \setminus \{0\})^{-1}R =: \operatorname{frac}(R)$.

Problem (Problem 5)**:** Let $R = \mathbb{Q} \times \mathbb{Q}$ and $S = \{(1,1)\} \cup (\mathbb{Q}^\times \times \{0\})$. The goal of this problem is to identify the localization $S^{-1}R$.

(a) Describe explicitly when $\frac{(a_1,a_2)}{(s_1,s_2)}$ is equal to $\frac{(b_1,b_2)}{(t_1,t_2)}$ in $S^{-1}R$.

(b) Use your result from part (a) to show that the localization $S^{-1}R$ is isomorphic to the localization $T^{-1}\mathbb{Q}$, where $T = \mathbb{Q} \setminus \{0\}$, hence is isomorphic to $\mathbb{R}$.

(c) Find the kernel of the localization homomorphism $\iota_S\colon R \to S^{-1}R$.

Solution:

(a) By the definition of the equivalence relation, we must have an element $(r_1, r_2) \in S$ such that

$$(r_1(a_1t_1 - b_1s_1), r_2(a_2t_2 - b_2s_2)) = (0,0).$$

In particular, since $r_1 \in \mathbb{Q}^\times$, and we may always select $r_2 = 0$, it follows that

$$r_1(a_1t_1 - b_1s_1) = 0,$$

so that $a_1t_1 - b_1s_1 = 0$ (as $\mathbb{Q}$ is an integral domain).

(b) We consider the map $\pi_1\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, which maps $(a_1, a_2) \mapsto a_1$. Observe then that $S^{-1}R$ satisfies the universal property for localization, as we may write $S = (\mathbb{Q}^\times \times \{0\}) \cup (\mathbb{Q}^\times \cup \{1\})$, which maps to $\mathbb{Q}^\times \subseteq \mathbb{Q}$ under this projection map.

In particular, we see that the induced map $\widetilde{\pi_1}\colon S^{-1}R \to \mathbb{Q}$ is given by

$$\widetilde{\pi_1}\left(\frac{(a_1, a_2)}{(s_1, s_2)}\right) = a_1s_1^{-1}$$

for $s_1 \in \mathbb{Q}^\times$ and $a_1 \in \mathbb{Q}$.

Now, we observe that the map $\operatorname{id} \circ \pi_1 = \pi_1$, and that $T^{-1}\mathbb{Q}$ satisfies the universal property for localization with respect to id, inducing the homomorphism $\widetilde{\operatorname{id}}$ that takes

$$\widetilde{\operatorname{id}}\left(\frac{a}{s}\right) = as^{-1}$$

for $s \in \mathbb{Q}^\times$. Yet, we also observe that, if we set $\iota_T' = \iota_T \circ \widetilde{\pi_1} \circ \iota_S$, that

$$\widetilde{\operatorname{id}} \circ \iota_T'(a_1, a_2) = \widetilde{\operatorname{id}} \circ \iota_T \circ \widetilde{\pi_1} \circ \iota_S(a_1, a_2)$$

$$
\begin{aligned}
&= \widetilde{\mathrm{id}} \circ \iota_T \circ \widetilde{\pi}_1\left(\frac{(a_1, a_2)}{(1,1)}\right) \\
&= \widetilde{\mathrm{id}} \circ \iota_T(a_1) \\
&= \widetilde{\mathrm{id}}\left(\frac{a_1}{1}\right) \\
&= a_1 \\
&= \pi_1(a_1, a_2).
\end{aligned}
$$

Thus, $T^{-1}\mathbb{Q}$ also satisfies the universal property for localization, implying that $T^{-1}\mathbb{Q}$ and $S^{-1}R$ are isomorphic.

$$
\begin{tikzcd}
& S^{-1}R \arrow[dr, "\widetilde{\pi}_1"] & \\
\mathbb{Q}\times\mathbb{Q} \arrow[ur, "\iota_S"] \arrow[rr, "\pi_1"'] \arrow[drr, "\iota'_T"'] & & \mathbb{Q} \arrow[loop right, "\mathrm{id}"] \arrow[d, bend right, "\iota_T"'] \\
& & T^{-1}\mathbb{Q} \arrow[u, bend right, "\widetilde{\mathrm{id}}"']
\end{tikzcd}
$$

(c)

Problem (Problem 7)**:** Let $S \subseteq R$ be a multiplicative subset, and let $\iota_S \colon R \to S^{-1}R$ be the corresponding localization homomorphism. Consider the map

$$
\begin{aligned}
\alpha \colon \left\{P' \,\middle|\, P' \text{ is a prime ideal of } S^{-1}R\right\} &\to \left\{P \,\middle|\, P \text{ is a prime ideal of } R \text{ such that } S \cap P = \emptyset\right\} \\
P' &\mapsto \iota_S^{-1}(P').
\end{aligned}
$$

(a) Verify that α is well-defined.

(b) Define an inverse map β by $\beta(P) = P \cdot S^{-1}R$. Show that β is well-defined. That is, $\beta(P)$ is a prime ideal of $S^{-1}R$.

(c) Show that α and β are mutual inverses.

Solution:

(a) We observe that ι_S takes 1_R to $\frac{1}{1} \equiv 1_{S^{-1}R}$, the latter equality coming from the fact that $\frac{a}{1} \cdot \frac{1}{1} = \frac{a}{1}$, so that if P' is a prime ideal in $S^{-1}R$, then $\iota_S^{-1}(P')$ is a prime ideal in $S^{-1}R$.

(b) Let P be a prime ideal in R, and let $\frac{a}{s} \cdot \frac{b}{t} \in P \cdot S^{-1}R$. We desire to show that either $\frac{a}{s}$ or $\frac{b}{t}$ are in $P \cdot S^{-1}R$. Since elements of the form $\frac{s}{t}$ are units in $S^{-1}R$, it follows that $\frac{a \cdot b}{1} \in P \cdot S^{-1}R$.