Problem: Let Y be the quasicircle (or Warsaw circle), the closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph of $y = \sin(1/x)$, the segment $[-1,1]$ of the y axis, and an arc connecting these two pieces. Collapsing the segment of Y in the y-axis gives a quotient map $f\colon Y \to S^1$.

Prove that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$.

Solution: Suppose there were a lift $\widetilde{f}\colon Y \to \mathbb{R}$ such that if p is the covering map $t \mapsto e^{2\pi i t}$, then $p \circ \widetilde{f} = f$.

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"'] & \\ S^1 \cong Y/\{0\} \times [-1,1] & Y \arrow[l, "f"] \arrow[ul, "\widetilde{f}"'] \end{tikzcd}$$

If we let 1 be the value that $\{0\}\times[-1,1]$ is mapped to under f, then we observe that $\widetilde{f} = p^{-1}\circ f$ will map $Y\setminus\{0\}\times[-1,1]$ homeomorphically to $(n, n+1)$ for some $n \in \mathbb{Z}$; in particular, since $\{0\} \times [-1,1]$ is connected, it must be the case that p^{-1} gives an injective map from $S^1 \cong f(Y)$ to $\mathbb{R}$ — specifically, a bijective map from S^1 to $(n, n+1]$ or $[n, n+1)$. Yet, applying the intermediate value theorem by approaching $1 \in S^1$ from either direction yields a contradiction, meaning such a lift cannot exist.