

Problem 1

Show that $C_0(\mathbb{R})$ is a Banach space.

Proof: We know that $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$, meaning we need show $C_0(\mathbb{R})$ is closed under the uniform norm.

Let $(f_n)_n \rightarrow f$, with $(f_n)_n \in C_0(\mathbb{R})$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then,

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f_n(x) - f(x)| + |f_n(x)| \\ &\leq \|f_n - f\|_u + |f_n(x)| \end{aligned}$$

By the definition of uniform convergence, for all $n \geq N_\varepsilon$, $\|f_n - f\| < \varepsilon/2$ and by the definition of vanishing at $\pm\infty$, for all $|x| > M$, $|f_n(x)| < \varepsilon/2$. Thus,

$$< \varepsilon,$$

meaning $f(x) \in C_0(\mathbb{R})$, so $C_0(\mathbb{R})$ is closed, so it is complete.

Problem 2

Show that ℓ_2 is a Hilbert space.

Proof: Let $(x_n)_n$ be a Cauchy sequence in ℓ_2 . Let $x_n(k)$ denote the index k of the sequence $x_n \in \ell_2$. Then, by the equivalence of norms, $\exists c \in \mathbb{R}$ such that

$$\begin{aligned} |x_n(k) - x_m(k)| &\leq c \|x_m(k) - x_n(k)\|_2 \\ &\rightarrow 0 \end{aligned} \quad \text{since } (x_n)_n \text{ is Cauchy in } \ell_2.$$

Since \mathbb{R} (or \mathbb{C}) is complete, $x_n(k) \rightarrow x(k)$ as $k \rightarrow \infty$. We set $(x(k))_k$ to be the sequence such that $x_n(k) \rightarrow x(k)$ for each n .

We must show that $\|x_n - x\|_2 \rightarrow 0$ as $n \rightarrow \infty$. This is equivalent to

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{k=1}^N \lim_{m \rightarrow \infty} |x_n(k) - x_m(k)|^2 &\leq \lim_{m \rightarrow \infty} \sup_{m \geq M} \|x_n - x_m\|^2 \\ &\leq \varepsilon^2 \end{aligned} \quad \text{since } (x_n)_n \text{ is Cauchy.}$$

Thus, $\|x_n - x_m\| \rightarrow 0$ as $m \rightarrow \infty$ and $n \rightarrow \infty$, so $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Problem 3

Suppose (X, d) is a complete metric space and $(x_n)_n$ is a sequence in X such that there is a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leq \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof: We will show that $(x_n)_n$ is convergent by showing that $(x_n)_n$ Cauchy. Let m, n be such that $m \geq n$.

Notice that $d(x_n, x_{n-1}) \leq \theta^{n-2} d(x_2, x_1)$. Thus,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq d(x_2, x_1) (\theta^{m-2} + \theta^{m-3} + \cdots + \theta^{n-1}) \\ &= d(x_2, x_1) \theta^{n-1} (1 + \theta + \theta^2 + \cdots + \theta^{p-q-1}) \\ &\leq d(x_2, x_1) \frac{\theta^{n-1}}{1 - \theta}. \end{aligned}$$

Notice that the sequence $\left(\frac{\theta^{n-1}}{1-\theta}\right)_n \rightarrow 0$ in \mathbb{R} , meaning $(x_n)_n$ is Cauchy. Since X is complete, $(x_n)_n$ is convergent.

Problem 4

Let (X, d) be a complete metric space, and suppose $f : X \rightarrow X$ is a contractive map — i.e., there is a $\theta \in (0, 1)$ with

$$d(f(x), f(y)) \leq \theta d(x, y).$$

Prove that f has a unique fixed point.

Proof: Let $x_0 \in X$, and define $x_n = f(x_{n-1})$. For all n , we have

$$d(x_n, x_{n-1}) \leq \theta^n d(x_1, x_0).$$

Therefore, for x_m, x_n arbitrary in X with $m > n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \theta^m d(x_1, x_0) + \cdots + \theta^{n+1} d(x_1, x_0) \\ &= d(x_1, x_0) \theta^{n+1} (1 + \theta + \cdots + \theta^{m-n-1}) \\ &\leq d(x_1, x_0) \frac{\theta^{n+1}}{1 - \theta}. \end{aligned}$$

Since $\left(\frac{\theta^{n+1}}{1-\theta}\right)_n \rightarrow 0$ in \mathbb{R} as $n \rightarrow \infty$, it must be the case that $d(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$. Since X is complete, this means $(x_n)_n \rightarrow x$ for some $x \in X$, meaning $f(x) = x$.

Suppose it were the case that there existed s, t distinct with $f(s) = s$ and $f(t) = t$. Then, $d(f(s), f(t)) = d(s, t) \leq \theta d(s, t)$, but $\theta < 1$, which is a contradiction. Thus, x is unique.

Problem 5

If $(f_k)_k$ is an orthonormal sequence in a Hilbert space \mathcal{H} , show that the map

$$\begin{aligned} \varphi : \ell_2 &\rightarrow \mathcal{H} \\ \varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k \end{aligned}$$

is a well-defined isometry.

Proof: Let $\xi, \eta \in \ell_2$. Then,

$$\begin{aligned} d(\xi, \eta) &= \|\xi - \eta\|_2 \\ \varphi(\xi) &= \sum_{k=1}^{\infty} \xi(k) f_k \\ \varphi(\eta) &= \sum_{k=1}^{\infty} \eta(k) f_k \\ d(\varphi(\xi), \varphi(\eta)) &= \left(\sum_{k=1}^{\infty} \langle \xi(k) f_k - \eta(k) f_k, \xi(k) f_k - \eta(k) f_k \rangle \right)^{1/2} \\ &= \|\xi - \eta\|_2 \end{aligned}$$

Parseval's Identity.

Problem 6

Let $T : V \rightarrow W$ be a continuous linear map between normed spaces which is bounded below; that is, there is a $C > 0$ such that $\|T(v)\| \geq C \|v\|$ for all $v \in V$. If V is complete, show that $\text{Ran}(T) \subseteq W$ is a closed subspace, and that $V \cong \text{Ran}(T)$ are uniformly isomorphic.

Proof: Let $(v_n)_n$ be a Cauchy sequence in V .

Problem 7

Let X and Y be metric spaces with completions (\tilde{X}, ι_X) and (\tilde{Y}, ι_Y) respectively. If $f : X \rightarrow Y$ is an isometry, show that there is a unique isometry $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ that extends f . That is, the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \iota_X \uparrow & & \uparrow \iota_Y \\ X & \xrightarrow{f} & Y \end{array}$$

Proof:

Problem 10

Let X be a metric space. Show that the following are equivalent:

- (i) Every meager set has empty interior.
- (ii) The complement of a meager set is dense.

Moreover, show that these equivalent statements hold true if the metric space is complete.

Proof: Let $A = \bigcup_{i \geq 1} A_i$ be a meager subset of X . Suppose $A^\circ = \emptyset$. Then, $\overline{A^c} = (A^\circ)^c = X$, so A^c is dense in X .

Suppose $\overline{A^c} = X$. Then, $(A^\circ)^c = X$, so $A^\circ = \emptyset$.

Let X be a complete metric space. Let $A \subseteq X$ be meager.

Problem 10

Let V be an infinite-dimensional normed space with linear basis B .

- (i) If $W \subset V$ is a proper subspace, show that $W^\circ = \emptyset$.
- (ii) If V is a Banach space, show that B is uncountable. You may use the fact that finite-dimensional subspaces are always closed.

Proof of (i): Let $W \subset V$ be proper. Suppose $U(x, \varepsilon) \subseteq W$ for some $x \in V$ and $\varepsilon > 0$. Then, for $v \in V$, we have that $\frac{\varepsilon}{2} \frac{v}{\|v\|} + x \in U(x, \varepsilon)$, meaning $v \in W$, so $V \subseteq W$, which is a contradiction. Thus, $W^\circ = \emptyset$.

Proof of (ii): Let $\{e_n\}_{n \geq 1}$ be a countable, linearly independent set. Let $W_1 = \text{span}\{e_1\}$, $W_2 = \text{span}\{e_1, e_2\}$, and so on. We have that each $W_n \subseteq V$ is closed (by assumption), and $W_1 \subseteq W_2 \subseteq \dots$. Since each W_n has empty interior, it cannot be the case that $V = \bigcup W_n$ by Baire's Theorem.