Math 395: Homework 8
Due: November 26, 2024
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Problem 1

Problem: Let V_1, V_2 be subspaces of V. Show that $V = V_1 \perp V_2$ if

(i) $V = V_1 \oplus V_2$;

(ii) given any $v, v' \in V$, when we write $v = v_1 + v_2$ and $v' = v_1' + v_2'$, for $v_i, v_i' \in V_i$, we have

$$\varphi\left(v, v'\right) = \varphi_1\left(v_1, v_1'\right) + \varphi_2\left(v_2, v_2'\right),$$

where $\varphi_i = \varphi|_{V_i \times V_i}$.

Solution: By condition (i), since $V = V_1 \oplus V_2$, it is the case that $V_1 + V_2 = V$ and $V_1 \cap V_2 = \{0\}$, meaning that for any $v \in V$, we can write $v = v_1 + v_2$ for unique $v_1 \in V_1$ and $v_2 \in V_2$. Thus, we must show that $\varphi(v, w) = 0$ for any $v \in V_1$ and $w \in V_2$.

From condition (ii), we know that

$$\begin{aligned}
\varphi\left(v, v'\right) &= \varphi\left(v_1 + v_2, v_1' + v_2'\right)\\
&= \varphi\left(v_1, v_1'\right) + \varphi\left(v_1, v_2'\right) + \varphi\left(v_2, v_1'\right) + \varphi\left(v_2, v_2'\right)\\
&= \varphi_1\left(v_1, v_1'\right) + \varphi_2\left(v_2, v_2'\right).
\end{aligned}$$

Since, by definition, we have $\varphi_i = \varphi$ for $v_i \in V_i$, we have $\varphi\left(v_1, v_1'\right) = \varphi_1\left(v_1, v_1'\right)$ and $\varphi\left(v_2, v_2'\right) = \varphi_2\left(v_2, v_2'\right)$. Thus, by this equality, we have

$$\varphi\left(v_1, v_2'\right) + \varphi\left(v_2, v_1'\right) = 0.$$

Considering $\varphi\left(v_1, v_2'\right)$, we uniquely decompose $v_1 = v_1 + 0$, where $0 \in V_2$, and $v_2' = 0 + v_2'$, where $0 \in V_1$, yielding

$$\begin{aligned}
\varphi\left(v_1, v_2'\right) &= \varphi_1\left(v_1, 0\right) + \varphi_2\left(0, v_2'\right)\\
&= 0.
\end{aligned}$$

Similarly, we have $\varphi\left(v_2, v_1'\right) = 0$, meaning that for any $v \in V_1$ and $w \in V_2$, we must have

$$\varphi(v, w) = 0.$$

Thus, V_1 and V_2 are orthogonal complements, yielding $V = V_1 \oplus V_2$.

Problem 2

Problem: Let $T \in \operatorname{Hom}_F(V, V)$, and let φ be a bilinear form on V. Prove that $\psi(v, w) = \varphi(T(v), w)$ is a bilinear form on V.

Solution: Let $v, v_1, v_2, w, w_1, w_2 \in V$ and $\alpha \in F$. Then,

$$\begin{aligned}
\psi\left(\alpha v_1 + v_2, w\right) &= \varphi\left(T\left(\alpha v_1 + v_2\right), w\right)\\
&= \varphi\left(\alpha T\left(v_1\right) + T\left(v_2\right), w\right)\\
&= \alpha\varphi\left(T\left(v_1\right), w\right) + \varphi\left(T\left(v_2\right), w\right)\\
&= \alpha\psi\left(v_1, w\right) + \psi\left(v_2, w\right)
\end{aligned}$$

$$\begin{aligned}
\psi\left(v, \alpha w_1 + w_2\right) &= \varphi\left(T(v), \alpha w_1 + w_2\right)\\
&= \alpha\varphi\left(T(v), w_1\right) + \varphi\left(w_2\right)
\end{aligned}$$

$$= \alpha\varphi(T(v), w_1) + \varphi(T(v), w_2)$$
$$= \alpha\psi(v, w_1) + \psi(v, w_2).$$

Thus, ψ is a bilinear form.

Problem 5

Problem: Let $V = \mathbb{R}^2$, and set $\varphi((x_1, y_1), (x_2, y_2)) = x_1 x_2$.

(a) Show this is a bilinear form. Give a matrix representing this form. Is this form nondegenerate?

(b) Let $W = \text{span}_{\mathbb{R}}(e_1)$, where e_1 is the standard basis element. Show that $V = W \perp W^\perp$.

(c) Calculate $\left(W^\perp\right)^\perp$.

Solution:

(a) We have, for $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\varphi(\alpha(x_1, y_1) + (x_2, y_2), (x_3, y_3)) &= \varphi((\alpha x_1 + x_2, y_1 + y_2), (x_3, y_3)) \\
&= (\alpha x_1 + x_2)(x_3) \\
&= \alpha(x_1 x_3) + (x_2 x_3) \\
&= \alpha\varphi((x_1, y_1), (x_3, y_3)) + \varphi((x_2, y_2), (x_3, y_3))
\end{aligned}$$

$$\begin{aligned}
\varphi((x_1, y_1), \alpha(x_2, y_2) + (x_3, y_3)) &= \varphi((x_1, y_1), (\alpha x_2 + x_3, y_2 + y_3)) \\
&= x_1(\alpha x_2 + x_3) \\
&= \alpha(x_1 x_2) + (x_1 x_3) \\
&= \alpha\varphi((x_1, y_1), (x_2, y_2)) + \varphi((x_1, y_1), (x_3, y_3)).
\end{aligned}$$

Using the basis $\mathcal{B} = \{(1,0), (0,1)\}$, where $e_1 = (1,0)$ and $e_2 = (0,1)$, we have the matrix representation of

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a degenerate bilinear form, since, for instance, taking $(x_1, y_1) = (0,5)$ and $(x_2, y_2) = (1,8)$, we have

$$\varphi((x_1, y_1), (x_2, y_2)) = 0,$$

despite $(x_1, y_1), (x_2, y_2) \neq (0,0)$.

(b) Letting $W = \{\alpha e_1 \mid \alpha \in \mathbb{R}\}$, we see that for any $(x_2, y_2) \in \mathbb{R}^2$, that

$$\varphi((\alpha, 0), (x_2, y_2)) = \alpha x_2,$$

which equals zero whenever $\alpha = 0$ or $x_2 = 0$. Since we can select $\alpha \neq 0$, if we want $(x_2, y_2) \in W^\perp$, we need $x_2 = 0$. Thus, $W^\perp = \text{span}(e_2)$.

Additionally, since W and $W^\perp$ are subspaces, $W \cap W^\perp = \{0\}$, and for any $v = (x_1, y_1) \in \mathbb{R}$, we have $(x_1, y_1) = (x_1, 0) + (0, y_2) \in W_1 + W_2$, we have that $W \oplus W^\perp = V$.

Therefore, we must have $W_1 \perp W_2 = V$.

(c) We know that $W^\perp = \text{span}(e_2)$. Thus, we see that $\left(W^\perp\right)^\perp$ is the set of all $(x, y) \in \mathbb{R}^2$ such that

$$\varphi((x, y), (0, \alpha)) = 0.$$

Since this holds for all $(x, y) \in \mathbb{R}^2$, we have that $\left(W^\perp\right)^\perp = V$.

Exercise

Problem: If $\operatorname{char}(\mathbb{F}) = 2$, show that $\varphi(v, v) = 0$ is a redundant condition provided $\varphi(w, v) = -\varphi(v, w)$ for all $v, w \in V$.

Solution: Since $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$, this applies in particular for $v = w$. Thus, we have

$$
\begin{aligned}
\varphi(v, v) &= -\varphi(v, v)\\
2\varphi(v, v) &= 0 \qquad (*)\\
\varphi(v, v) &= 0;
\end{aligned}
$$

where we used the property that $\operatorname{char}(\mathbb{F}) \neq 2$ to move from the line in $(*)$ to the final line.