Problem 1

Recall that a subset $U \subseteq \mathbb{R}$ is **open** if

$$(\forall x \in U)(\exists \varepsilon > 0) \ni V_{\varepsilon}(x) \subseteq U.$$

Prove that a mapping $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U) \subseteq \mathbb{R}$ is open for every open $U \subseteq \mathbb{R}$.

(⇒) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall c \in \mathbb{R}$, $x \in V_{\delta}(c) \Rightarrow f(x) \in V_{\varepsilon}(f(c))$. Let U be an open set such that $f(c) \in U$. Then, $\exists \varepsilon_0$ such that $V_{\varepsilon_0}(f(c)) \subseteq U$. So, $\exists \delta_0$ such that $V_{\delta_0}(c) \subseteq f^{-1}(V_{\varepsilon_0}(f(c))) \subseteq f^{-1}(U)$. So, $f^{-1}(U)$ is open.