Complex Analysis

Analyticity and Path-Independence in the Complex Plane

Baby's First Complex Function Theory

We are interested in functions of the form $f(z)$, where $z = x + iy$ is some complex number. Note that this is specifically different from a function $g\colon \mathbb{R}^2 \to \Omega$ for some domain Ω; in the latter case, we have independent variables x and y, while in the former case, we must express $z = x + iy$.

Now, consider a contour integral

$$\begin{aligned}\oint_C w(z)\,dz &= \oint_C w(z)\,(dx + i dy)\\ &= \oint_C w(z)\,dx + i\oint_C w(z)\,dy.\end{aligned}$$

Taking $A_x = w(z)$ and $A_y = iw(z)$, we have

$$= \oint_C \mathbf{A} \cdot\, d\vec{\ell}.$$

We want to know if this is equal to, by Green's Theorem,

$$= \int_S (\nabla \times \mathbf{A})\,d\mathbf{a},$$

and when this integral is zero. Note that $(\nabla \times \mathbf{A}) \cdot \hat{n} = 0$, so $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$.

Note that we can take

$$w(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$.

After a lot of tedious derivation, we get the Cauchy–Riemann equations.

Theorem (Cauchy–Riemann Equations)**:**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

Furthermore, the Cauchy–Riemann equations guarantee that w is analytic,[1] which leads to Cauchy's theorem.

Theorem (Cauchy's Theorem)**:** If C is a simple closed curve in a simply connected region, then w is analytic if and only if

$$\oint_C w(z)\,dz = 0.$$

Fact. The function $w(z)$ is analytic inside the simply connected region R if any of these hold:

- w satisfies the Cauchy–Riemann equations;

[1] Equal to its Taylor series, also holomorphic.

- $w'(z)$ is unique and exists;
- $\frac{\partial w}{\partial \overline{z}} = 0$.
- w can be expanded in a Taylor series: $w(z) = \sum_{n \geqslant 0} c_n (z - a)^n$;[II]
- $w(z)$ is path-independent everywhere in R: $\oint_C w(z)\, dz = 0$.

Example. Considering $w(z) = z$, we have $u = x$ and $v = y$, so it satisfies the Cauchy–Riemann equations. However, neither $\text{Re}(z)$ nor $\text{Im}(z)$ are analytic, and neither is $\overline{z} = x - iy$.

Remark: Whenever we say "analytic at p," we mean "analytic in a neighborhood of p."

Note that since $\mathbb{C}$ is a non-compact locally compact Hausdorff space, we may carry out a one-point compactification of $\mathbb{C}$, by adjoining a point $\{\infty\}$, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. This compactified $\mathbb{C}^*$ is often represented as a unit sphere with the north pole, determined by $(0, 0, 1)$, is the point at infinity. The correspondence between $\mathbb{C}^* \setminus \{\infty\}$ and $\mathbb{C}$ is evaluated via stereographic projection.

We define $\frac{z}{\infty} = 0$ and $\frac{z}{0} = \infty$ for any $z \neq 0, \infty$. The correspondence between $z = x + iy$ in the plane to Z on the Riemann sphere with $\mathbb{R}^3$ coordinates (ξ_1, ξ_2, ξ_3) is

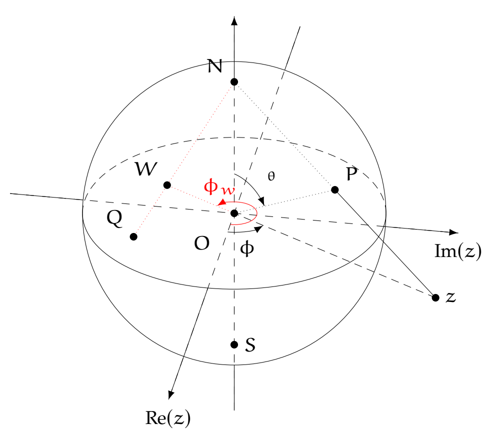
$$\begin{aligned}
\xi_1 &= \frac{2\,\text{Re}(z)}{|z|^2 + 1} \\
\xi_2 &= \frac{2\,\text{Im}(z)}{|z|^2 + 1} \\
\xi_3 &= \frac{|z|^2 - 1}{|z|^2 + 1}.
\end{aligned}$$

Inverting, we may find

$$\begin{aligned}
x &= \frac{\xi_1}{1 - \xi_3} \\
y &= \frac{\xi_2}{1 - \xi_3},
\end{aligned}$$

and with polar coordinates,

$$z = \cot(\theta/2) e^{i\phi}.$$



To determine analyticity at ∞, we set $\zeta = \frac{1}{z}$, and analyze the analyticity of $\tilde{w}(\zeta) = w(1/z)$ at 0.

[II]This is the real definition of analytic.

Cauchy's Integral Formula

Consider the function $w(z) = c/z$, integrated around a circle of radius R.

$$\begin{aligned}\oint_{\Gamma} w(z)\,dz &= C\int_{0}^{2\pi}\frac{e^{-i\varphi}}{R}\underbrace{iRe^{i\varphi}\,d\varphi}_{dz}\\ &= ic\int_{0}^{2\pi}dz\\ &= 2\pi ic.\end{aligned}$$