Math 395

Homework 4

Due: 2/27/2024

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Collaborators:

Problem 1

Let F be a field, with $F[x]$ denoting the ring of polynomials with coefficients in F. Let $f(x) \in F[x]$ be a monic polynomial. Let $g(x) \in F[x]$ be a nonzero polynomial. We will show that there exist unique $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Consider the ideal generated by $g(x)$, $\langle g(x) \rangle \subseteq F[x]$.

Problem 4

Let $p \in \mathbb{Z}$ be a prime. Let $\mathfrak{m} = \{(pa, b) \mid a, b \in \mathbb{Z}\}$. We will prove that $\mathfrak{m}$ is a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$.

We will do so by showing that $(\mathbb{Z} \times \mathbb{Z})/\mathfrak{m}$ is isomorphic to the field $\mathbb{Z}/p\mathbb{Z}$. Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be defined by $\varphi((i,j)) = [i]_p$. We will show that φ is a surjective homomorphism with kernel $\mathfrak{m}$.
Let $(i,j),(k,\ell) \in \mathbb{Z} \times \mathbb{Z}$. Then,

$$\begin{aligned}
\varphi((i,j) + (k,\ell)) &= \varphi((i+k, j+\ell)) \\
&= [i+k]_p \\
&= [i]_p + [k]_p \\
&= \varphi((i,j)) + \varphi((k,\ell)),
\end{aligned}$$

and

$$\begin{aligned}
\varphi((i,j)(k,\ell)) &= \varphi((ik, j\ell)) \\
&= [ik]_p \\
&= [i]_p[k]_p \\
&= \varphi((i,j))\varphi((k,\ell)).
\end{aligned}$$

Finally, for any $[a]_p \in \mathbb{Z}/p\mathbb{Z}$, we set $(a, 1) \in \mathbb{Z} \times \mathbb{Z}$ such that $\varphi((a,1)) = [a]_p$, meaning φ is surjective.

For $\varphi((x,y)) = [0]_p$, it must be the case that $[x]_p = [0]_p$, meaning $x = pa$ for some $a \in \mathbb{Z}$. Thus, $\ker \varphi = \{(pa, b) \mid a, b \in \mathbb{Z}\} = \mathfrak{m}$. By the first isomorphism theorem, it is the case that $(\mathbb{Z} \times \mathbb{Z})/\mathfrak{m} = \mathbb{Z}/p\mathbb{Z}$. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, $\mathfrak{m}$ must be maximal.

Problem 5

Let p be a prime, and let J be the collection of polynomials in $\mathbb{Z}[x]$ whose constant term is divisible by p. We will show that J is a maximal ideal in $\mathbb{Z}[x]$.