Alternating Series and Conditional Convergence

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A Series

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A Series

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

This series appears to be related to the harmonic series, but also very different:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{Harmonic Series}$$

Divergence of the Harmonic Series

We can show that the harmonic series is divergent as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

Differences

However, our alternating harmonic series is different. Taking partial sums, we get the following sequence:

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{(-1)^{n+1}}{n} \\
s_1 &= 1 \\
s_2 &= \frac{1}{2} \\
s_3 &= \frac{5}{6} \\
s_4 &= \frac{7}{12} \\
&\vdots
\end{aligned}
$$

Clearly, this sequence does not grow without bound — it is bounded above by 1, and doesn't seem to dip below $\frac{1}{2}$

Convergence

The alternating harmonic does converge. Specifically,

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Convergence

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...or does it?

Rearranging the Alternating Harmonic Series

Rearrange the series as follows:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots$$

Rearranging the Alternating Harmonic Series

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Rearranging the Alternating Harmonic Series

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\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\cdots \\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \\
&= \frac{1}{2}\ln 2
\end{aligned}$$