Problem (Problem 1):

(a) Fix topological spaces X and Y, and consider the set of all continuous maps $X \to Y$. Define a relation on this set by saying that f is related to g whenever f is homotopic to g. Prove that this relation is an equivalence relation.

(b) Prove that any space X is homotopy equivalent to itself, that if X is homotopy equivalent to Y, then Y is homotopy equivalent to X, and that if X is homotopy equivalent to Y and Y is homotopy equivalent to Z, then X is homotopy equivalent to Z.

Solution:

(a) For reflexivity, we may select the identity homotopy $F\colon X{\times}I \to Y$, given by $F(x, t) = f(x)$ for all $t \in I$ and all $x \in X$.

For symmetry, if $F\colon X \times I \to Y$ is a homotopy with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, then we may define the homotopy $G\colon X \times I \to Y$ by taking $G(x, t) = F(x, 1 - t)$. This is a composition of continuous maps, so it is continuous, and has $G(x, 0) = g(x)$ and $G(x, 1) = f(x)$, so the relation is symmetric.

For transitivity, we let $F\colon X{\times}I \to Y$ be a homotopy between f and g, and let $G\colon X{\times} I \to Y$ be a homotopy between g and h. Define a homotopy $H\colon X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \le t \le 1/2 \\ G(x, 2t - 1) & 1/2 \le t \le 1 \end{cases}.$$

This is a well-defined function since $G(x, 0) = F(x, 1)$ by the definition of the homotopies F and G, while it is continuous since both F and G are continuous, the functions $2t$ and $2t - 1$ are continuous, and F and G agree at $t = 1/2$.

Therefore, the relation is transitive, so the relation $f \sim g$ if f is homotopic to g is an equivalence relation.

(b) For the homotopy equivalences between X, Y, and Z, we will define them via the following collection of maps:

$$X \overset{f}{\underset{\overline{f}}{\rightleftarrows}} Y \overset{g}{\underset{\overline{g}}{\rightleftarrows}} Z$$

where

$$\overline{f} \circ f \simeq \mathrm{id}_X$$
$$f \circ \overline{f} \simeq \mathrm{id}_Y$$
$$\overline{g} \circ g \simeq \mathrm{id}_Y$$
$$g \circ \overline{g} \simeq \mathrm{id}_Z.$$

Reflexivity follows from the fact that the identity map is homotopic to itself via the identity homotopy, while symmetry follows from flipping the roles of f and $\overline{f}$ in the definitions of the homotopy equivalence between X and Y.

For transitivity, we claim that the functions $g \circ f$ and $\overline{f} \circ \overline{g}$ are the pair between X and Z that satisfy our desired result. That is, we claim that

$$\left(\overline{f} \circ \overline{g}\right) \circ (g \circ f) \simeq \operatorname{id}_X$$
$$(g \circ f) \circ \left(\overline{f} \circ \overline{g}\right) \simeq \operatorname{id}_Z .$$

We start by claiming that

$$\overline{f} \circ \overline{g} \circ g \circ f \simeq \overline{f} \circ \operatorname{id}_Y \circ f \qquad (*)$$

Let $H \colon Y \times I \to Y$ be the homotopy that maps $\overline{g} \circ g$ to id_Y. Then, if we define

$$F \colon X \times I \to X$$
$$(x, t) \mapsto \overline{f} \circ H_t \circ f,$$

we see that F is continuous with

$$F(x, 0) = \overline{f} \circ \overline{g} \circ g \circ f$$
$$F(x, 1) = \overline{f} \circ \operatorname{id}_Y \circ f.$$

Therefore, the claim $(*)$ is established. Collapsing with id_Y and using the fact that "is homotopic to" is an equivalence relation, we thus establish

$$\overline{f} \circ \overline{g} \circ g \circ f \simeq \overline{f} \circ \operatorname{id}_Y \circ f$$
$$= \overline{f} \circ f$$
$$\simeq \operatorname{id}_X .$$

By a similar process using the homotopy between $f \circ \overline{f}$ and id_Y, we thus establish

$$g \circ f \circ \overline{f} \circ \overline{g} \simeq g \circ \operatorname{id}_Y \circ \overline{g}$$
$$= g \circ \overline{g}$$
$$\simeq \operatorname{id}_Z .$$

Therefore, homotopy equivalence is reflexive, symmetric, and transitive.